The Classification of Unitary Reflection Groups

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Contents

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Introduction

This work is based on the book by Lehrer and Taylor [**9**]. The ideas of this text are followed and theorems are developed to understand the classification of finite reflection groups. The classification was first completed by Shephard and Todd in 1954 [**13**] and later re-verified by Cohen in 1976 [**3**] using alternative methods. The classification was achieved through the contributions of several mathematicians, such as Bagnera, Blichfeldt, Klein, Jordan, Valentiner, Mitchell, and others, who used both projective and geometric techniques to classify the linear reflection groups. This classification is closely related to the classification of finite Coxeter groups.

The classification of finite reflection groups, which are subgroups of the unitary group $U(V)$ that are generated by reflections (linear transformations that fix a hyperplane pointwise and negate the normal vector). The classification is done by analyzing the action of the group on the underlying vector space V, and breaking it down into irreducible components.

The first step is to decompose the space V into the direct sum of mutually orthogonal subspaces V_1, $V_2, \dots, V_m$ such that the restriction of G to each subspace acts irreducibly. This means that the group G can be written as the direct product of the subgroups G_i that act on each V_i. If the dimension of V is 1, then G is cyclic and the classification is trivial. If V is irreducible and imprimitive, then G is conjugate to the group $G(m, p, n)$. If V is primitive and has dimension 2, then G is of tetrahedral, octahedral or icosahedral type, and there are 19 possibilities.

If V is primitive and has dimension greater than 2, and G contains reflections of order 3, then there are 3 possibilities: $W(\mathcal{L}_3)$, $W(\mathcal{L}_4)$ or $W(\mathcal{M}_3)$. If G contains only reflections of order 2, then the set of lines spanned by the roots of the reflections in G is an indecomposable star-closed line system and $G = W(\mathfrak{L})$. The classification then proceeds by analyzing the properties of this line system, such as its dimension and the presence of certain types of lines.

If the line system is a 3-system, then it is equivalent to $\mathcal{A}_n$ or to one of $\mathcal{K}_5, \mathcal{K}_6, \mathcal{E}_6, \mathcal{E}_7$ or $\mathcal{E}_8$. If the line system is a 4-system, then it is equivalent to one of $\mathcal{J}_3^{(4)}$, $\mathcal{F}_4, \mathcal{N}_4$ or $\mathcal{O}_4$. And if the line system is a 5-system, then it is equivalent to one of $\mathcal{H}_3, \mathcal{J}_3^{(5)}$ or $\mathcal{H}_4$.

CHAPTER 1

Preliminaries

1. Background in groups

Let G be a group and X be a set on which G acts by $gx \in X$.The G-orbit of element x is $\{gx \mid g \in G\}$ and the set of all G-orbits on X is denoted by X/G. If $A \subseteq X$ the pointwise stabilizer of A is the subgroup

$$G_A := \{g \in G \mid ga = a \ \text{ for all } \ a \in A\}.$$

If X is subset of G, the normaliser $N_G(X)$ and centraliser $C_G(X)$ are the subgroups

$$N_G(X) := \{g \in G \mid gXg^{-1} = X\} \quad \text{and} \quad C_G(X) := \{g \in G \mid gx = xg \ \text{ for all } \ x \in X\}.$$

The center of G is the subgroup $Z(G) := C_G(G)$. The cyclic group of order n is denote by $\mathcal{C}_n$. Given a prime p, the direct product of n copies of $\mathcal{C}_p$ is known as the elementary abelian group of oder p^n.

The commutator of $x, y \in G$ is $[x, y] := xyx^{-1}y^{-1}$. If X and Y are subset of G, the subgroup generate by the commutators $[x, y]$ is

$$\left[X, Y\right] = \langle [x, y] \mid x \in X, \ y \in Y \rangle .$$

The commutator subgroup or derived group of G is $G' := [G, G]$.

A group G is called the central product of subgroups H and K, written $G = H \circ K$, if $G = HK$ and $[H, K] = 1$. In this case $H \cap K \subseteq Z(G)$ and in particular, for $H \cap K = 1$ the central product turns out to be the direct product $H \times K$.

The normal closure of a subset X in a group G is the smallest normal subgroup of G that contains X. This means it is the smallest subgroup that contains X and has the property that all its elements are conjugates of each other. In particular, if we take an element g of G, then the normal closure of g is the subgroup generated by the set of all its conjugates. That is, the subgroup consisting of all elements of the form hgh^{-1}, where h is any element of G. This subgroup is represented as $\left\langle g^G \right\rangle$.

For example, if we consider the group of permutations of a finite set and take a specific permutation, its normal closure would be the subgroup generated by all permutations obtained through conjugations of the original permutation.

A group G is extension of a group N by a group H if N is normal subgroup of G such that $G/N \cong H$. This information can be encoded into a short exact sequence of groups

$$1 \longrightarrow N \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1$$

where $\alpha : N \longrightarrow G$ is injective and $\beta : G \longrightarrow H$ is surjective.

1.1. Representation theory. Now we will a review some of the basic facts about modules and representation theory. This review is from [**8, 11**].

Let V a vector space of dimension n over the field $\mathbf{C}$ and let

$$\mathrm{GL}(V) := \{\varphi : V \longrightarrow V \mid \varphi \text{ invertible linear transformation}\}.$$

A lineal representation of a group G with representation space V is a homomorphism

$$\rho : G \longrightarrow \mathrm{GL}(V).$$

If $\varphi \in \mathrm{GL}(V)$ and V has a finite basis (e_i) of n elements, the linear map φ is defined by square matrix $A = (a_{ij})$ of order n. The coefficients a_{ij} are complex numbers, they are obtained by expressing the images $a(e_i)$ in terms of the basis (e_i)

$$a(e_i) := \sum_i a_{ij} e_i.$$

If B is the matrix of $\psi \in \mathrm{GL}(V)$, then AB is the matrix of $\varphi\psi$.

The representation is called faithful if ρ is injective. If $\rho : G \longrightarrow \mathrm{GL}(V)$ is a representation, we say that G acts on V and we call V a G-module. The action of $g \in \mathrm{GL}(V)$ on $v \in V$ is defined by $gv := \rho(g)v$.

DEFINITION 1.1. *The dual space of V is the space V^* of all linear maps $\varphi : V \longrightarrow \mathbf{C}$ with*

$$(\varphi + \psi)(v) := \varphi(v) + \psi(v)$$

$$(\alpha\varphi)(v) := \alpha\varphi(v)$$

The contragredient representation of ρ is the homomorphism $\rho^* : G \longrightarrow \mathrm{GL}(V^*)$ defined by $(\rho^*(g)\varphi)(v) = \varphi(\rho(g)^{-1}v)$ for all $g \in G$, $\varphi \in V^*$ and $v \in V$.

A G-submodule of V is a vector subspace U such that $gu \in U$ for all $g \in G$ and all $u \in U$. The G-module V is irreducible if 0 and V are its only G-submodules. If U is a G-submodule of V, then V/U is a G-module with action $g(v+U) := gv + U$. We say that V is an extension of U by V/U. This extension is said to split if there is a G-submodule W of V such that $V = U \oplus W$.

If V and W are G-modules, a G-homomorphism is a linear transformation $\alpha : V \to W$ such that $\alpha(gv) = g\alpha(v)$ for all $g \in G$ and $v \in V$. We denote the set of all G-homomorphisms from V to W by $\mathrm{Hom}_G(V, W)$; it is a vector space over $\mathbf{C}$. A linear transformation $\alpha : V \to V$ is called an endomorphism and the set $\mathrm{End}_G(V) := \mathrm{Hom}_G(V,V)$ of all endomorphisms is a ring in which multiplication is defined to be composition of functions.

THEOREM 1.1 (Maschke). *If G is a finite group and if U is a submodule of the G-module V, then the extension V of U by V/U splits. In other words, every submodule of V is a direct summand.*

THEOREM 1.2 (Schur's Lemma). *If V and W are irreducible G-modules, every non-zero homomorphism from V to W is an isomorphism. In addition,* $\mathrm{End}_G(V) \simeq \mathbf{C}$.

2. Hermitian Form

DEFINITION 1.2. *Given a vectorial space V of dimension n over $\mathbf{C}$, an* **hermitian form** *on V is a mapping $(-,-) : V \times V \longrightarrow \mathbf{C}$ such that*

$$\big(v_1 + v_2, w\big) = \big(v_1, w\big) + \big(v_2, w\big)$$
$$\big(v, w_1 + w_2\big) = \big(v, w_1\big) + \big(v, w_2\big)$$
$$(av, w) = a(v, w)$$
$$\overline{(v, w)} = (w, v)$$

for all $v, w, v_1, v_2 \in V$ y $a \in \mathbf{C}$.

If $(v,v) > 0$ for all $v \neq 0$, $(-,-)$ is called positive definete and $(v,v) = 0$ if only if $v = 0$.

If $e_1, e_2, \dots e_n$, is basis of V, we may define a positive definite hermitian form on V by

$$(u, v) = \sum_{i=1}^{n} u_i \overline{v_i}$$

where $u := u_1 e_1 + u_2 e_2 + \cdots + u_n e_n$ and $v := v_1 e_1 + v_2 e_2 + \cdots + v_n e_n$.

If $(-,-)$ and $[-,-]$ are two positive definite hermitian forms on V then there exists an invertible linear transformation $\varphi : V \longrightarrow V$ such that if $\{e_i\}_{i=1}^n$ and $\{\alpha_i\}_{i=1}^n$ are orthonormal bases [1] of $(-,-)$ and $[-,-]$ respectively and $\varphi(e_i) = \alpha_i$ so

$$\begin{aligned}[\varphi(u), \varphi(v)] = \left[\sum_{i=1}^{n} u_i \varphi(e_i), \sum_{j=1}^{n} v_j \varphi(e_j)\right] &= \sum_{i=1}^{n}\sum_{j=1}^{n} u_i \overline{v_j} [\varphi(e_i), \varphi(e_j)] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} u_i \overline{v_j} [\alpha_i, \alpha_j] = \sum_{i=1}^{n} u_i \overline{v_i} = (u, v)\end{aligned}$$

for all $u, v \in V$. In other words, the positive definite hermitian forms $(-,-)$ and $[-,-]$ are equivalent.

[1] A bases $e_1, \dots, e_n$ for V is orthogonal if $(e_i, e_j) = 0$ for all $i \neq j$; it is orthonormal if in addition $(e_i, e_i) = 1$ for all i.

Let $\mathrm{GL}(V) := \{\varphi : V \longrightarrow V \mid \varphi \ \text{ invertible linear transformation}\}$ the group of invertible linear transformation on V. A subgroup G of $\mathrm{GL}(V)$ is said to leave the form $(-,-)$ invariant if

$$(gv, gw) = (v, w), \quad \text{for all } g \in G, \ \text{ for all } \ u, v \in V.$$

We also say tha $(-,-)$ is a G-invariant form.

Let M be the matrix of $g \in \mathrm{GL}(V)$ with respect to an orthonormal basis of V. Then g is unitary if only if M is unitary matrix, i.e. $M\overline{M}^t = I$, where $\overline{M}^t$ denotes the transpose of the complex conjugate of M and I is the indentity matrix. If $(-,-)$ is a positve definite hermitian form on V, we say that $g \in \mathrm{GL}(V)$ is **unitary** (or **isometry**) if $(gv, gw) = (v, w)$ for all $v, w \in V$; that is, $(-,-)$ is $\langle x \rangle$-invariant.

LEMMA 1.1. *If G is a finite subgroup of* $\mathrm{GL}(V)$*, there exists a G-invariant positive definite hermitian form on V.*

PROOF. Let a positive definite hermitian form $[-,-]$ on V and define a new form by

$$(v, w) := \sum_{g \in G} [gv, gw]$$

Let $v_1, v_2, w \in V$ and $g \in G$ and note that

$$\begin{aligned}(v_1 + v_2, w) = \sum_{g \in G} [g(v_1 + v_2), gw] = \sum_{g \in G} [gv_1 + gv_2, gw] &= \sum_{g \in G} [gv_1, gw] + \sum_{g \in G} [gv_2, gw] \\ &= (v_1, w) + (v_2, w).\end{aligned}$$

Let $\alpha \in \mathbf{C}$ and $u, v \in V, \ g \in G$ then

$$\begin{aligned}(\alpha u, v) = \sum_{g \in G} [g\alpha u, gv] = \sum_{g \in G} [\alpha gu, gv] &= \sum_{g \in G} \alpha [gu, gv] \\ = \alpha \sum_{g \in G} [gu, gv] &= \alpha (u, v).\end{aligned}$$

It is easy to see that $\overline{(v, w)} = (w, v)$. It is positive definite since it is clear that $(v, w) \geq 0$ and that $(v, w) = 0$ if and only if $[gv, gw] = 0$. Let us now see that it is G-invariant. If $g, h \in G$, then $gh \in G$ and

$$(hv, hw) = \sum_{g \in G} [ghv, ghw] = \sum_{g \in G} [gv, gw] = (v, w)$$

and effect the form $(-,-)$ is G-invariant.

□

The group of all isometries of V is denoted by $\mathrm{U}(V)$ is called **unitary group** of the form. Its subgruop of transformation of determinant 1 is called **special unitary group**. The corresponding groups of unitary matrices on V will be denoted by $\mathrm{U}_n(\mathbf{C})$ and $\mathrm{SU}_n(\mathbf{C})$, where $n := \dim V$. The group $\mathrm{U}(V)$ depends on the form but as any two positive definite hermitian forms on V are equivalent, $U(V)$ is unique up to conjugacy in $\mathrm{GL}(V)$; if $(-,-)$ and $[-,-]$ are two positive definite hermitian form on V then $\mathrm{U}_{(-,-)}(V)$ and $\mathrm{U}_{[-,-]}(V)$ are conjugate in

GL(V).

If U is a subset of V define the orthogonal complement of U by

$$U^{\perp} := \{v \in V \mid (u, v) = \sum_{i=1}^{n} u_i \overline{v_i} = 0 \quad \text{for all} \quad u \in U\}.$$

This subset satisfies well-known properties from algebra courses.

The following corollary is deduced from Schur's Lemma.

COROLLARY 1.1. *If V is an irreducible G-module and if $(-,-)$ and $[-,-]$ are positive definite G-invariant hermitian forms on V, then for some real number $\lambda > 0$ we have $(u, v) = \lambda[u, v]$ for all $u, v \in V$.*

DEFINITION 1.3. *Let 1 be the identity element of* $\mathrm{GL}(V)$*. For $g \in \mathrm{GL}(V)$ and $H \subseteq \mathrm{GL}(V)$ put*

(i) $\mathrm{Fix}\, g := \mathrm{Ker}(1 - g) = \{v \in V \mid gv = v\}$
(ii) $V^H := \mathrm{Fix}_V(H) := \{v \in V \mid hv = v \ \text{ for all } \ h \in H\}$*, and*
(iii) $[V, g] := \mathrm{Im}(1 - g)$.

3. Reflections and reflection groups

DEFINITION 1.4. *A linear transformation g is a* **reflection** *(or pseudo-reflection) if the order of g is finite and if* $\dim[V, g] = 1$.

In other words, a reflection in V is a linear transformation of V of finite order with exactly $n-1$ eigenvalues equal to 1. If g is a reflection, the subspace $\mathrm{Fix}\, g$ is a hyperplane, called the reflective hyperplane of g. And we can say that a reflection over V is a diagonalizable linear isomorphism $V \to V$, which is not the identity, but leaves a pointwise invariant hyperplane $H \subseteq V$.

If a spans $[V, g]$, that is, $[V, g] = \mathbf{C}a$, then for all $v \in V$, there exists $\varphi(v) \in \mathbf{C}$ such that

$$v - gv = \varphi(v)a,$$

where $\varphi : V \to C$ is a linear functional such that $\mathrm{Fix}\, g = \mathrm{Ker}\, \varphi$.

We call g a unitary reflection if it preserves the Hermitian form $(-,-)$. In this case, $\mathrm{Fix}\, g$ is orthogonal to $[V, g]$ and $V = [V, g] \perp \mathrm{Fix}\, g$, because $[V, g] = (\mathrm{Fix}\, g)^{\perp}$.

Suppose $g \in GL(V)$ is a reflection of the order m. Then the cyclic group $\langle g \rangle$ has the order m and therefore, leaves a Hermitian positive definite form invariant. Therefore, every reflection g is a unitary reflection with respect to some form.

If $H = \mathrm{Fix}\, g$, then g leaves the line (one-dimensional subspace) $H^{\perp}$ invariant. Therefore, with respect to a decomposition-adapted basis $V = H^{\perp} \perp H$, g has matrix $\mathrm{diag}[\zeta, 1, \ldots, 1]$, where ζ is a primitive m^{th}-root of unity.

DEFINITION 1.5. (a) *A root of line ℓ of V is any non-zero vector of ℓ. If g is a unitary reflection, a root of g is a root of the line $[V, g]$.*

(b) *A root a is short, long or tall if (a, a) is 1,2 or 3, respectively.*

Every line in $\mathbf{C}$ contains long, short and tall roots, each of which is unique except for multiplication by an element of $\mathbf{S}^1 := \{z \in \mathbf{C} | |z| = 1\}$.

LEMMA 1.2. *If $g, h \in GL(V)$, then* $\mathrm{Fix}\left(ghg^{-1}\right) = g\,\mathrm{Fix}\, h$.

PROOF. It can be observed that $\mathrm{Fix}(ghg^{-1}) = \left\{v \in V \mid (1 - ghg^{-1})v = 0\right\}$ and since $(1 - ghg^{-1})v = 0$ we get $g(1-h)g^{-1}v = 0$ so

$$\mathrm{Fix}(ghg^{-1}) = g\{v \in V \mid (1-h)v = 0\} = g\,\mathrm{Fix}(h).$$

□

In particular, if r is a reflection with reflecting hyperplane $H := \mathrm{Fix}\, r$, then grg^{-1} is a reflection with reflecting hyperplane $gH = \mathrm{Fix}(grg^{-1})$.

DEFINITION 1.6. *A* **unitary reflection group** *is a finite subgroup of* $\mathrm{U}(V)$ *that is generated by reflections.*

Some autors call these **complex reflection groups** [**2**]. Every finite subgroup of $\mathrm{GL}(V)$ generated by reflections is a unitary reflection group with respect to some positive definite Hermitian form on V.

Remark 1. *In other words, if G are finite subgroup of* $\mathrm{GL}(V)$ *a reflection in G is an element $r \in G$ such that*

$$\dim[V, r] = \dim \mathrm{Im}(1 - r) = \mathrm{codim}_V \mathrm{Fix}\, r = 1$$

and if R is the set of reflections of G it is said that G is a reflection group if $G = \langle R \rangle$.

An observation, is that a unitary reflection group is not an abstract finite group, but actually one of a group together with a faithful representation given by matrices. On the complex numbers, all finite reflection groups have been classified by Shephard and Todd [**13**]. A given group can act as a reflection group or in some other way. For example, for $\zeta := \exp(2\pi i/m)$, the element $\mathrm{diag}[\zeta, 1, \ldots, 1]$ generates a cyclic reflection group of order m, but the (isomorphic) group generated by $\mathrm{diag}[\zeta, \zeta, 1, \ldots, 1]$ is not a reflection group.

EXAMPLE 1. *Let ω a cube root of unity, the matrices*

$$r := \begin{bmatrix} \omega & 0 \\ -\omega^2 & 1 \end{bmatrix} \quad y \quad s := \begin{bmatrix} 1 & \omega^2 \\ 0 & \omega \end{bmatrix}$$

are reflections of order 3 and generate a group of order 24.

EXAMPLE 2. *Now let's look at an example of a unitary reflection group that is the symmetric group* $\mathrm{Sym}(n)$ *of all permutations of* $\{1,2,\ldots,n\}$. *In order to express* $\mathrm{Sym}(n)$ *as a group of reflections, we must take a basis* $e_1, e_2, \ldots, e_n$ *of* V *and let* $x_1, x_2, \ldots, x_n$ *be the dual base for* V^**;*

$$x_i(e_j) = \delta_{ij} \quad \forall i, j \quad 1 \le i, j \le n.$$

If we take $\pi \in \mathrm{Sym}(n)$ *so that* $\pi(e_i) = e_{\pi(i)}$ *which is the permutation matrix associated with* π. *In particular, if the transposition* (i, j) *corresponds to the reflection that exchanges* e_i *with* e_j *and fixes the other elements, we have that*

$$\begin{aligned} \mathrm{Fix}((i,j)) &= \left\{ v \in V \mid (i,j)v = v \right\} \\ &= \left\{ v \in V \mid (v_1, v_2, \ldots, v_i, \ldots, v_j, \ldots, v_n) = (v_1, v_2, \ldots, v_j, \ldots, v_i, \ldots, v_n) \right\} \\ &= \left\{ v \in V \mid v_i = v_j \right\} = \left\{ v \in V \mid v_i - v_j = 0 \right\} = \mathrm{Ker}(x_i - x_j) \end{aligned}$$

It can be seen that $\dim(\mathrm{Fix}((i,j))) = n-1$ *so the transpositions* (i,j) *are reflections and as* $\mathrm{Sym}(n)$ *is generated by transpositions* (i,j) *is a group generated by reflections so it is a unitary reflection group.*

If we have a hyperplane H in V, we can define $L_H : V \to \mathbf{C}$ a linear mapping such that $H = \mathrm{Ker}(L_H)$. The element $L_H \in V^*$ is determined by H except for multiplication by non-zero scalars.

LEMMA 1.3. *Suppose that* r *is a reflelction of order* m *in* $\mathrm{GL}(V)$, *let* $H := \mathrm{Fix}\, r$ *and suppose that* $[V, r] = \mathbf{C}a$. *Then there exists a primitive* m^{th}*-root of unity* α *such that*

$$rv = v - (1-\alpha)\frac{L_H(v)}{L_H(a)}a$$

PROOF. By definition of reflection we have that $rv = v - \varphi(v)a$ where φ is a lineal functional such that $H = \mathrm{Ker}\,\varphi$ and $a \notin H$. So $ra = \alpha a$ where $\alpha \in \mathbf{C}$ of order m and $\varphi(a) = 1-\alpha$ well $ra = a - \varphi(a)a = a\alpha$ so $1 - \varphi(a) = \alpha \quad (a \neq 0)$. Now be $\varphi = \lambda L_H$ with $\lambda \neq 0$

$$rv = v - \varphi(v)a = v - \lambda L_H(v)a$$

if $\lambda = \frac{1-\alpha}{L_H(a)}$ then $rv = v - (1-\alpha)\frac{L_H(v)}{L_H(a)}a$.

□

COROLLARY 1.2. *Suppose that* r *is a unitary reflection of order* m *in* $\mathrm{U}(V)$ *and that* a *is root of* r *of length 1. Then there exists a primitive* m^{th}*-root of unity* α *such that for all* $v \in V$ *we have* $rv = v - (1-\alpha)(v,a)a$

DEFINITION 1.7. *Given a non-zero vector $a \in V$ and m^{th}-root of unity $\alpha \neq 1$, define the reflection $r_{a,\alpha}$ by*

$$r_{a,\alpha}(v) := v - (1-\alpha)\frac{(v,a)}{(a,a)}a$$

From this definition it follows that $r_{a,\alpha}$ is of order m.

LEMMA 1.4. *Also if we have two reflections $r_{a,\alpha}$ and $r_{a,\beta}$ the following properties are satisfied*

(i) $r_{a,\alpha} r_{a,\beta} = r_{a,\alpha\beta}$.

(ii) *For* $g \in \mathrm{U}(V)$, $g r_{a,\alpha} g^{-1} = r_{ga,\alpha}$.

(iii) *For* $\lambda \in \mathbf{C}$ *such that* $\lambda \neq 0$, $r_{\lambda a,\alpha} = r_{a,\alpha}$.

PROOF. To verify (i) this, let us use the definition

$$\begin{aligned}
r_{a,\alpha} r_{a,\beta}(v) &= r_{a,\beta}(v) - (1-\alpha)\frac{(r_{a,\beta}(v),a)}{(a,a)}a \\
&= v - (1-\beta)\frac{(v,a)}{(a,a)}a - (1-\alpha)\frac{\left(v-(1-\beta)\frac{(v,a)}{(a,a)}a, a\right)}{(a,a)}a \\
&= v - (1-\beta)\frac{(v,a)}{(a,a)}a - (1-\alpha)\left[\frac{(v,a)-(1-\beta)\frac{(v,a)}{(a,a)}(a,a)}{(a,a)}a\right] \\
&= v - (1-\beta)\frac{(v,a)}{(a,a)}a - (1-\alpha)\left[\frac{(v,a)}{(a,a)} - (1-\beta)\frac{(v,a)}{(a,a)}\right]a \\
&= v - (1-\beta)\frac{(v,a)}{(a,a)}a - (1-\alpha)\frac{(v,a)}{(a,a)}a + (1-\alpha)(1-\beta)\frac{(v,a)}{(a,a)}a \\
&= v + \left[-(1-\beta)-(1-\alpha)+(1-\alpha)(1-\beta)\right]\frac{(v,a)}{(a,a)}a \\
&= v + \left[-1+\beta-1+\alpha+1-\beta-\alpha+\alpha\beta\right]\frac{(v,a)}{(a,a)}a \\
&= v + (-1+\alpha\beta)\frac{(v,a)}{(a,a)}a = v - (1-\alpha\beta)\frac{(v,a)}{(a,a)}a = r_{a,\alpha\beta}(v)
\end{aligned}$$

Similary form prove (ii);

$$\begin{aligned}
g r_{a,\alpha} g^{-1}(v) &= g\left[g^{-1}v - (1-\alpha)\frac{(g^{-1}v,a)}{(a,a)}a\right] = gg^{-1}v - (1-\alpha)\frac{g(g^{-1}v,a)}{(a,a)}a \\
&= v - (1-\alpha)\frac{(gg^{-1}v,a)}{(a,a)}a = v - (1-\alpha)\frac{(gg^{-1}v,ga)}{(ga,ga)}ga \\
&= v - (1-\alpha)\frac{(v,ga)}{(ga,ga)}ga = r_{ga,\alpha}(v)
\end{aligned}$$

The part (iii) is similary. □

THEOREM 1.3. (a) *If $g \in \mathrm{U}(V)$ and $g r_{a,\alpha} g^{-1} = r^k_{a,\alpha}$ for some k, then $k = 1$.*

(b) *The unitary reflectfion $r_{a,\alpha}$ and $r_{b,\beta}$ commute if and only if $\mathbf{C}a = \mathbf{C}b$ or $(a,b) = 0$.*

The proof of these theorems is in [**9**, pag. 12]. Part (a) tells us that a reflection is not conjugated to a proper power to it.

THEOREM 1.4. *A subspace W of V is invariant with respect to the reflection r if and only if $W \subseteq \operatorname{Fix} r$ or $[V,r] \subseteq W$.*

COROLLARY 1.3. *If r is a unitary reflection with root a, then the subspace W is invariant with respect to r if and only if $a \in W$ or $a \in W^{\perp}$.*

3.1. Irreducible unitary reflection groups. A complex reflection group $G \subseteq \mathrm{GL}(V)$ is said to be irreducible if V is an irreducible representation of G. If V is an irreducible G-module, we say that G is an irreducible unitary reflection group.

THEOREM 1.5. *Suppose that G is a finite group generated by reflection on V, which leaves the positive definite hermitian form $(-,-)$ invariant. Then V is the direct sum of pairwise orthogonal subspaces $V_1, V_2, \dots, V_m$ such that the restriction G_i of G to V_i acts irreducibly on V_i and $G \cong G_1 \times G_2 \times \cdots \times G_m$*

See [**9**, Theorem 1.27, pag. 15]

DEFINITION 1.8. *The support of a unitary reflection group $G \subseteq \mathrm{U}(V)$ is the subspace M of V spanned by roots of the reflections in G.*

Equivalently, the support of G is the orthogonal complement of the subspace V^G of fixed points of G.

COROLLARY 1.4. *Suppose that G is a unitary reflection group and that H is a reflection subgroup of G acting irreducibly on its support W. If $r \in G$ is a reflection with root a such that $a \notin W \cup W^{\perp}$, then $\langle H, r\rangle$ acts irreducibly on $W \oplus \mathbf{C}a$.*

COROLLARY 1.5. *Suppose that G is a unitary reflection group of rank n that acts irreducibly on its support. Then there is a chain of subgroups*

$$1 = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_n \subseteq G_\ell = G$$

where $n \le \ell$, such that for $1 \le i \le n$, there are reflections r_i such that $G_i = \left\langle G_{i-1}, r_i \right\rangle$ and for $1 \le i \le n, G_i$ is generated by i reflections, has rank i, and acts irreducibly on its support.

See [**9**].

THEOREM 1.6. *If G_1 and G_2 are finite irreducible unitary reflection subgroups of $U(V)$, then G_1 and G_2 are conjugate in $GL(V)$ if and only if they are conjugate in $U(V)$.*

4. Imprimitivity and primitivity of a unitary reflection group

DEFINITION 1.9. *The G-module V is imprimitive if for some $m > 1$ it is a direct sum $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ of non-zero subspace V_i for $1 \leq i \leq m$ such that the action of G on V permutes the subspace $V_1, V_2, \ldots, V_m$ among themselves. A subgroup G of $\mathrm{GL}(V)$ is called imprimitive if there exists a decomposition of vector space $V = \bigoplus_{i=1}^{m} V_i = V_1 \oplus \cdots \oplus V_m$ of nontrivial proper linear subspaces of V for $m > 1$ and are permuted transitively by G. The set $\{V_1, V_2, \ldots V_m\}$ is called a system of imprimitivity for V. Otherwise V is primitive.*

Similarly define the decomposition $V = \bigoplus_{i=1}^{m} V_i$ to be imprimitive if for all $g \in G$ and for all V_i we get $gV_i = V_j$ for some j, and for all V_i and V_j there exists a $h \in G$ such that $hV_i = V_j$.

In other words, for a subgroup G of $\mathrm{GL}(V)$ and ρ a linear representation of G is called imprimitive if there a descomposition of subspaces of the space V of the representation ρ into a direct sum of proper subspaces $V_1, V_2, \ldots, V_m$ such that for $g \in G$ and $1 \leq i \leq m$ there exists a $1 \leq j \leq m$ such that $\rho(g)V_i = V_j$.

A example is considered a representation ρ of symmetry group $\mathrm{Sym}(n)$ in $\mathbf{C}^n$ with basis $e_1, e_2, \ldots, e_n$ is imprimitive (transitive) and the 1-dimensional subspaces $\{\mathbf{C}e_1, \mathbf{C}e_2, \ldots, \mathbf{C}e_n\}$ form a system of imprimitivity for ρ.

DEFINITION 1.10. *Given an irreducible G-module I, the isotypic component V_I of V corresponding to I is the sum of all G-submodules of V isomorphic to I.*

THEOREM 1.7. *If G is a finite primitive subgroup of $GL(V)$ and if A is an abelian normal subgroup of G, then A is cyclic and contained in the centre $Z(G)$ of G.*

PROOF. Since A is abelian, its elements have a common eigenvector $w \in V$. For each $a \in A$, there must be number $\chi(a) \in \mathbf{C}^\times$ such that is a eigenvalue, i.e., the map

$$\begin{gathered} \chi : A \longrightarrow \mathbf{C}^\times, \\ aw := \chi(a)w \end{gathered}$$

is a homomorphism (character)

$$\chi(ab)w = abw = a(bw) = a(\chi(b)w) = \chi(b)aw = \chi(b)\chi(a)w = \chi(a)\chi(b)w, \quad a, b \in A$$

and for a a, b in A such that $\chi(a)w = \chi(b)w$ then $aw = bw$ hence $a = b$, is adjective, therefore A is Cyclic.

Now define the isotypic component of V corresponding to χ by

$$V_\chi := \{v \in V \mid av = \chi(a)v, \ \text{ for all } \ a \in A\}$$

For $g \in G$ define the character $g\chi$ of A by $(g\chi)(a) := \chi(g^{-1}ag)$. The normality of A implies that gV_χ is the isotypic component corresponding to character $g\chi$. Since

$$V = \bigoplus_{\chi} V_\chi = \bigoplus_{\chi} \{v \in V \mid av = \chi(a)v, \ \text{ with } \ a \in A\}.$$

And G permutes the component isotypic of A them among themselves. the fact that G is primitive implies that there only one isotypic component $V = V_\chi$. This shows that A acts on V by scalars, so $A \subseteq Z(G)$. □

THEOREM 1.8. *If $G \subseteq U(V)$ is a finite primitive unitary reflection group and if N is a normal subgroup of G, then either V is an irreducible N-module or $N \subseteq Z(G)$.*

PROOF. Proof. Suppose that V is reducible as an N-module. Then $V = V_1 \oplus \cdots \oplus V_k$, where $k > 1$ and the V_i are proper irreducible N-submodules. For each i there exists a reflection $r \in G$ with root a such that $rV_i \neq V_i$. Since N is normal in G, rV_i is an irreducible N-submodule and hence $rV_i \cap V_i = \{0\}$. On the other hand if $a^\perp \cap V_i \neq \{0\}$, then $rV_i = V_i$ and to avoid a contradiction we must have $a^\perp \cap V_i = \{0\}$. Since $a^\perp$ is a hyperplane in V it follows that $\dim V_i = 1$ for all i. But now N is abelian and by the previous theorem we have $N \subseteq Z(G)$. □

LEMMA 1.5. *If G is an irreducible imprimitive unitary reflection group and if $\{V_1, V_2, \ldots, V_k\}$ is a system of imprimitivity for G, then* $\dim V_i = 1$ *for all i.*

PROOF. For a contradiction fix i such that $\dim V_i > 1$. Then since G is irreducible there exists a j such that $j \neq i$ and a reflection $r \in G$ such that $rV_i = V_j$. Since $V_i \cap \mathrm{Fix}\, r$ has positive dimension, which contradicts the fact that $V_i \cap V_j = \{0\}$, so $\dim V_i = 1$. □

LEMMA 1.6. *Let V a vector space of dimension n over* $\mathbf{C}$ *with positive definite hermitian form $(-,-)$ and that $V_1, V_2, \ldots, V_k$ are 1-dimensional subspaces spanned by linearly independent vectors $e_1, e_2, \ldots, e_k$. If r is a reflection that permutes the V_i among themselves and if $rV_i = V_j$ for some $i \neq j$, then the order of r is 2 and $V_h \subseteq \mathrm{Fix}\, r$ for all $h \neq i, j$.*

PROOF. We have $re_i = \theta e_j$ for some θ. If a is a short root of r, then

$$re_i = \theta e_j = e_i - (1-\alpha)(e_i, a)a = e_i - \varphi(e_i)a$$

for some α. And note that $e_i - re_i = e_i - \theta e_j = (1-\alpha)(e_i, a)a = \varphi(e_i)a.$

And we see that the short root a is a linear combination of the basis vectors e_i and e_j, which means that a is in the sum of the subspaces V_i and V_j, $a \in V_i + V_j$. Since $a \in V_i + V_j$, and a is the short root of r, we have $[V, r] \subseteq V_i + V_j$, that means the sum of subspaces $V_i + V_j$ is r-invariant by Theorem 1.4.

If suppose that $(a, V_i) \neq 0$ and $a \in V_i$ such that $rV_i = V_j$. The idea is that if r permutes two subspaces V_i and V_j, it must preserve the sum $V_i + V_j$, and since the short root of r is in the sum $V_i + V_j$, it follows that all other subspaces V_h are in the fixed set of r.

Now consider $v_i \in V_i$ where $rv_i = v_j$. Then there exists a $h \in \{1, \ldots, k\}$ such that $r^2 v_i \in V_h$ then $r^2 v_i \in (\mathbf{C}a + \mathbf{C}v_i) \cap V_h = (V_i + V_j) \cap V_h$ then $h = i$ or $h = j$. Since $a \notin V_i$ we have $r^2 v_i = v_i$ and $\theta^2 = 1$ which $\theta = -1$ then r is of order two $(a \in V_i + V_j)$.

Now consider that $(a, V_h) \neq 0$ for some $h \neq i, j$ then there exists $t \neq i, j$ such that $rV_h = V_t$, this force $a \in (V_h + V_t) \cap (V_i + V_j) = \{0\}$ a contradiction. Then $(a, V_h) = 0$ for $h \neq i, j$. Therefore $rV_h = V_h$, i.e., $V_h \subseteq \mathrm{Fix}\, r$. □

5. Blichfeldt's Theorem

An important and fundamental theorem to the classification of primitive unitary reflection groups is Blichfeldt's theorem [**1, 9, 10**]. This theorem helps in limiting the orders of the reflections of the groups.

If $v \in \mathbf{C}^n$ the length is $||v|| := \sqrt{(v,v)}$ and the norm of a linear transformation $t : \mathbf{C}^n \longrightarrow \mathbf{C}^n$ is definited by

$$||t|| := \sup\{||tv|| \mid ||v|| = 1\}.$$

Then $||tv|| \leq ||t||||v||$ for all v and $\|t\| = \max\{|\lambda| \mid \lambda$ is an eigenvalue of $t\}$. If $s \in \mathrm{U}(V)$ has finite order, the its eigenvalues are roots of unity.

For $s \in \mathrm{U}(V)$, $\lambda - s$ denote $\lambda I_n - s$. The condition that s has an eigenvalue λ such that $|\lambda - \mu| \leq 1$ for all eigenvalues μ of s is equivalent to saying that all eigenvalues of s lie within $\pi/3$ of λ on the unit circle.

THEOREM 1.9. *Let G be a finite primitive subgroup of $U_n(\mathbf{C})$ and suppose that $s \in G$ has an eigenvalue λ such that $|\lambda - \mu| \leq 1$ for all eigenvalues μ of s. Then $s \in Z(G)$.*

PROOF. Suppose, by way of contradiction, that $s \notin Z(G)$ and let $H := \langle s^G \rangle = \langle g^{-1}sg \mid g \in G\rangle$ be the normal closure of s on G. Let G acts on $V := \mathbf{C}^n$ and define $V_\lambda := \{v \in V \mid sv = \lambda v\}$.

Sice $s \notin Z(G)$, we have V_λ is not H-invariant ($V_\lambda \neq V$). If it were H-invariant, then its image under the action of G would form an system of imprimitivity for G since H is normal subgroup of G, s would act as scalar multiplication by λ in V (contrary to hypothesis).

Let $v_1, v_2, \ldots, v_r$ be an orthonormal basis for V_λ and choose a conjugate t of s such that $tV_\lambda \neq V_\lambda$ and such that $\sum_{i=1}^{r} \left\|(t-\lambda)v_i\right\|$ is minimal. Suppose first that $t^{-1}stV_\lambda = V_\lambda$. If $t^{-1}stv = \lambda v$ for all $v \in V_\lambda$, then $tv \in V_\lambda$ for all $v \in V_\lambda$, contrary to the choice of t.

Hence $t^{-1}st$ has an eigenvector $v \in V$, then $tv \in V_\lambda$ with is an s-eigenvalue $\mu \neq \lambda$. That is, $t^{-1}stv = \mu v$ and so $stv = \mu tv$. Since v and tv are eigenvectors of s corresponding to different eigenvalues, we have $(v, tv) = 0$ and consequently

$$\begin{aligned}\|(t-\lambda)v\|^2 &= (tv - \lambda v, tv - \lambda v) = (tv, tv - \lambda v) - \lambda(v, tv - \lambda v)\\ &= (tv, tv) - (tv, \lambda v) - \lambda(v, tv) + \lambda(v, \lambda v) \leq \|t\|^2\|v\|^2 + |\lambda|^2\|v\|^2 = 2\|v\|^2\end{aligned}$$

so $\|t - \lambda\| \leq \sqrt{2}$ it's a contradiction, because for θ eigenvalue of t have that $|\theta - \lambda| \leq 1$ and hence $|t - \lambda| \leq 1$. This proves that V_λ is not $t^{-1}st$-invariant.

For any $\mu_1, \mu_2, \ldots, \mu_r \in \mathbf{C}$ we have

$$\begin{aligned}\sum_{i=1}^{r}\left\|\left(t^{-1}st-\lambda\right)v_i\right\| &= \sum_{i=1}^{r}\left\|\left(t^{-1}st-s\right)v_i\right\| = \sum_{i=1}^{r}\|(st-ts)v_i\| \\ &= \sum_{i=1}^{r}\left\|\left((s-\lambda)\left(t-\mu_i\right)-\left(t-\mu_i\right)(s-\lambda)\right)v_i\right\| \\ &= \sum_{i=1}^{r}\left\|(s-\lambda)\left(t-\mu_i\right)v_i\right\| \\ &\le \sum_{i=1}^{r}\left\|\left(t-\mu_i\right)v_i\right\|, \quad \text{because } \|s-\lambda\|\le 1\end{aligned}$$

For each i, choose μ_i so that $\left\|\left(t-\mu_i\right)v_i\right\|$ is minimal. Then $\sum_{i=1}^{r}\left\|\left(t-\mu_i\right)v_i\right\| \le \sum_{i=1}^{r}\|(t-\lambda)v_i\|$, whence

$$\sum_{i=1}^{r}\left\|\left(t^{-1}st-\lambda\right)v_i\right\| \le \sum_{i=1}^{r}\left\|\left(t-\mu_i\right)v_i\right\| \le \sum_{i=1}^{r}\|(t-\lambda)v_i\|$$

and our initial choice of t forces equality throughout, and implies that $\mu_i=\lambda$ realises the minimum value of $\left\|\left(t-\mu_i\right)v_i\right\|$. But now

$$\begin{aligned}\left\|\left(t-\mu_i\right)v_i\right\|^2 &= \|tv_i\|^2-\bar{\mu}_i\left(tv_i,v_i\right)-\mu_i\overline{\left(tv_i,v_i\right)}+\left|\mu_i\right|^2\|v_i\|^2 \\ &= 1+\left|\mu_i\right|^2-\bar{\mu}_i\left(tv_i,v_i\right)-\mu_i\overline{\left(tv_i,v_i\right)} \\ &= 1+\left|\mu_i-\left(tv_i,v_i\right)\right|^2-\left|\left(tv_i,v_i\right)\right|^2\end{aligned}$$

and so the minimum of this expression is 0 and occurs when $\mu_i=\left(tv_i,v_i\right)=\lambda$ for all i. Hence $tv_i=\lambda v_i$ for all i, a contradiction. This completes the proof. □

COROLLARY 1.6. *Let be a finite primitive subgroup of* $\mathrm{U}_n(\mathbf{C})$ *y suppose that for* $s\in G$*, all eigenvalues of* s *lie on arc of length less than* $2\pi/5$ *on the unit circle. Then* $s\in Z(G)$.

PROOF. The eigenvalus of s have the form $e^{i\theta}\lambda$ for some eigenvalue λ, where $0\le\theta\le 2\pi/5$. If there is an eigenvalue $e^{i\theta}\lambda$ such that $\pi/15\le\theta\le\pi/3$, the all eigenvalues μ of s is such that $|e^{i\theta}\lambda-\mu|\le\pi/3$ and by Blichfeldt's theorem so $s\in Z(G)$.

Now suppose tha t there is no such eigenvalue. It follows that every eigenvalues of s^5 lies whithin $\pi/3$ of λ^5 and Blichfeldt's theorem $s^5\in Z(G)$. So $s^5=\lambda^5 I$. But all eigenvalues of s are strictly within $2\pi/5$ of λ therefore $s=\lambda I$ and hence $s\in Z(G)$. □

COROLLARY 1.7. *If* $n>1$ *and* G *is a finite primitive subgroup of* $\mathrm{U}_n(\mathbf{C})$*, then every reflection in* G *has order 2,3,4 or 5.*

PROOF. If $t\in G$ is a reflection of order m, then the eigenvalues of t are 1 and $e^{2\pi ik/m}$ for some k coprime with m. Replacing t by a power we can suppose that $k=1$. And by Blichfeldt's theorem, all eigenvalues of t lie within $\pi/3$ of 1, so we have $2\pi/m>\pi/3$; that is, $m<6$. Theorefore $m=2,3,4$ or 5. □

CHAPTER 2

Finite subgroups of Quaternions and subgroups of $\mathrm{SU}_2(\mathbf{C})$

This chapter presents the classification of the subgroups of the special unitary group $\mathrm{SU}_2(\mathbf{C})$. This classification is the key to obtain the classification of the finite reflection unitary groups of rank 2.

1. The Quaternions

The algebra $\mathbf{H}$ of quaternions is a 2-dimensional and 4-dimensional vector space over $\mathbf{C}$ and over $\mathbf{R}$ respectively. $\mathbf{H}$ as vector space over $\mathbf{C}$ has basis 1 and j with a multiplication where 1 is the identity element and j satisfies $j^2 = -1$ and $ij = -ji$. If we restrict the scalar to $\mathbf{R}$, the algebra $\mathbf{H}$ has basis $1, i, j, k$ where $k = ij$. This is the usual description of $\mathbf{H}$ and the generators satisfy the symmetrical relations $i^2 = j^2 = k^2 = ijk = -1$. The quaternion algebra, $\mathbf{H}$, is usually written as

$$\mathbf{H} := \{a + ib + jc + kd \mid a, b, c, d \in \mathbf{R}\}.$$

Given $q = a + ib + jc + kd$ define the conjugate of q to be $\overline{q} = a - ib - jc - kd$. Define the norm of q by $\mathrm{N}(q) := q\overline{q} = a^2 + b^2 + c^2 + d^2$ and the trace $\mathrm{Tr}(q) = q + \overline{q}$. Thus, for $q \neq 0$ the inverse of q is $\mathrm{N}(q)^{-1}\overline{q}$.

The set $\mathbf{S}^3 := \{q \in \mathbf{H} \mid \mathrm{N}(q) = 1\}$ is the both the unit 3-sphere in $\mathbf{R}^4$ and a subgroup of the group $\mathbf{H}^*$ of non-zero quaternions (unit quaternions).

For $q \in \mathbf{H}$, multiplication by q on the left defines the $\mathbf{R}$-linear transformation

$$\begin{aligned} L(q) : \mathbf{H} &\longrightarrow \mathbf{H} \\ h &\longmapsto qh \end{aligned}$$

and multiplication right by $\overline{q}$ defines the $\mathbf{R}$-linear transformation

$$\begin{aligned} R(q) : \mathbf{H} &\longrightarrow \mathbf{H} \\ h &\longmapsto h\overline{q}. \end{aligned}$$

THEOREM 2.1. *For non-zero elements $q, r \in \mathbf{H}$, the following ere equivalent*

(i) *q and r have the same norm and trace,*

(ii) *$q = hrh^{-1}$ for some $h \in \mathbf{S}^3$.*

1.1. The groups $\mathrm{SU}_2(\mathbf{C})$. Remember that the special unitary group of degree 2 over field is the set of 2×2 unitary matrices with determinant 1 and complex entries,

$$\mathrm{SU}_2(\mathbf{C}) := \left\{ M \in \mathrm{SL}_2(\mathbf{C}) \mid M\overline{M}^t = \overline{M}^t M = I_2 \right\}.$$

Performing some calculations, we obtain

$$\mathrm{SU}_2(\mathbf{C}) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \middle| \ a, b \in \mathbf{C}, |a|^2 + |b|^2 = 1 \right\}$$

to see this let's consider $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SU}_2(\mathbf{C})$, then $\det M = 1$ and $\overline{M}^t = M^{-1}$, so

$$\begin{cases} ad - bc = 1 \\ a\bar{a} + c\bar{c} = 1 \\ b\bar{b} + d\bar{d} = 1 \\ a\bar{b} + c\bar{d} = 1 \end{cases}$$

solving these system equations we get $c = -\overline{b}$ and $d = \overline{a}$. Since $\det M = 1$ so $a\bar{a} + b\bar{b} = |a|^2 + |b|^2 = 1$.

THEOREM 2.2. *Every finite subgroup G of* $\mathrm{SL}_2(\mathbf{C})$ *is conjugate to a finite subgroup of* $\mathrm{SU}_2(\mathbf{C})$.

By Lemma 1.1, it leaves invariant a positive definite Hermitian form. Let $G \subset \mathrm{SL}_2(\mathbf{C})$ be finite and let $\langle \cdot, \cdot \rangle : \mathbf{C}^2 \to \mathbf{C}$ be a positive definite Hermitian form on $\mathbf{C}$. Then define

$$\begin{aligned} \langle \cdot, \cdot \rangle_G : \mathbf{C}^2 &\to \mathbf{C} \\ (u, v) \longmapsto \langle u, v \rangle_G &:= \frac{1}{|G|} \sum_{g \in G} \langle gu, gv \rangle. \end{aligned}$$

PROOF. Note that $\langle \cdot, \cdot \rangle_G$ is well defined since $\langle \cdot, \cdot \rangle_G$ is so and G is finite, so $|G| < \infty$ and furthermore has a finite number of elements $g \in G$. Then $\langle \cdot, \cdot \rangle_G$ is a positive definite hermitian form (see lemma 1.1). It satisfies $\langle Mu, Mv \rangle_G = \langle u, v \rangle_G$ for $u, v \in \mathbf{C}$ and $M \in \mathrm{SU}_2(\mathbf{C})$, so G is a unitary group with respect to $\langle \cdot, \cdot \rangle_G$, and from this fact, $\langle \cdot, \cdot \rangle_G$ has an orthonormal basis (taking the columns or the rows of some $g \in G$).

Let $M : \left(\mathbf{C}^2, \langle \cdot, \cdot \rangle_G\right) \to (\mathbf{C}, \langle \cdot, \cdot \rangle)$ such that M maps the orthonormal basis to the canonical basis. Then $\langle u, v \rangle_G = \langle Mu, Mv \rangle = \langle Mgu, Mgv \rangle = \left\langle MgM^{-1}u, MgM^{-1}v \right\rangle = \langle u, v \rangle$, for $g \in G$ and $u, v \in \mathbf{C}$. So we are done.

□

THEOREM 2.3. $\mathrm{SU}_2(\mathbf{C}) \simeq S^3 \simeq \mathbf{S}^3$, *where* $S^3 = \left\{ \left(x_1, x_2, x_3, x_4\right) \in \mathbf{R}^4 | x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \right\}$.

PROOF. By definition,

$$\mathrm{SU}_2(\mathbf{C}) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \mid a, b \in \mathbf{C}, |a|^2 + |b|^2 = 1 \right\},$$

$$S^3 = \left\{ \left(x_1, x_2, x_3, x_4\right) \in \mathbf{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \right\},$$

$$\mathbf{S}^3 = \left\{ q \in \mathbf{H} \mid q\overline{q} = 1 \right\}.$$

As $a, b \in \mathbf{C}$, we can write them as $a = (u_1, v_1), b = (u_2, v_2)$, and using $|a|^2 = (u_1 + iv_1)(u_1 - iv_1) = u_1^2 + v_1^2, |b|^2 = (u_2 + iv_2)(u_2 - iv_2) = u_2^2 + v_2^2$, we have

$$\mathrm{SU}_2(\mathbf{C}) \xrightarrow{\simeq} S^3$$

$$\begin{bmatrix} u_1 + iv_1 & u_2 + iv_2 \\ -u_2 + iv_2 & u_1 - iv_1 \end{bmatrix} \longmapsto (u_1, v_1, u_2, v_2)$$

And using that $ij = k$, we construct the isomorphism

$$\mathrm{SU}_2(\mathbf{C}) \xrightarrow{\simeq} \mathbf{S}^3.$$

$$\begin{bmatrix} u_1 + iv_1 & u_2 + iv_2 \\ -u_2 + iv_2 & u_1 - iv_1 \end{bmatrix} \longmapsto (u_1 + iv_1) + (u_2 + iv_2)\, j = u_1 + iv_1 + ju_2 + kv_2$$

□

2. The binary dihedral groups

For each positive integer m, let us take

$$\zeta_m := \exp\frac{2\pi i}{m} = \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}.$$

Recall that $\mathrm{SU}_2(\mathbf{C}) \xrightarrow{\simeq} \mathbf{S}^3$ so

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \longmapsto a + bj$$

then

$$\begin{bmatrix} \zeta_m^k & 0 \\ 0 & \zeta_m^{-k} \end{bmatrix} \longmapsto a + bj$$

so

$$\begin{bmatrix} \zeta_m^k & 0 \\ 0 & \zeta_m^{-k} \end{bmatrix} \longmapsto \left(\cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}\right)^k = \left(\exp\frac{2\pi i}{m}\right)^k = \cos\frac{2k\pi}{m} + i\sin\frac{2k\pi}{m}$$

with $k = 1, \ldots, n$ and $n \in \mathbf{N}$. Then $\mathcal{C}_m := \langle \cos\frac{2k\pi}{m} + i\sin\frac{2k\pi}{m} \rangle$, $n \in \mathbf{N}$, be the cyclic subgroup of order m generated by ζ_m.

Let $\mathcal{D}_m$ be the subgroup of $\mathbf{H}^*$ generated by ζ_m and j. Note that for some basis

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \longmapsto j$$

so $\mathcal{D}_m := \langle \zeta_m, j \rangle$. For $\alpha \in \mathcal{C}_m$ we have $j\alpha j^{-1} = \bar{\alpha} = \alpha^{-1}$ and so $\mathcal{C}_m$ is a normal subgroup of $\mathcal{D}_m$.

If m is odd, $\mathcal{D}_m = \mathcal{D}_{2m}$ and if m is even, $\mathcal{C}_m$ is a subgroup of index 2 in $\mathcal{D}_m$. Therefore, the order of $\mathcal{D}_{2m}$ is $4m$ and $x^2 = -1$ for all $x \in \mathcal{D}_{2m} \backslash \mathcal{C}_{2m}$. The quotient $\mathcal{D}_{2m}/\langle -1 \rangle$ is the dihedral group of order $2m$ and any group isomorphic to $\mathcal{D}_{2m}$ is called a binary dihedral group. The group $\mathcal{Q} := \mathcal{D}_4 = \{\pm 1, \pm i, \pm j, \pm k\}$ is often referred to as the quaternion group.

3. The binary polyhedral groups

Remember that two elements of $\mathbf{S}^3$ are conjugate if and only if they have the same trace. In fact, knowledge of the trace of small order elements will be very useful to you, so it is good to have knowledge of the trace of some elements:

order	3	4	5	6	8	10
trace	-1	0	$-\tau, \tau^{-1}$	1	$\pm\sqrt{2}$	$\tau, -\tau^{-1}$

where $\tau = \frac{1}{2}\left(1+\sqrt{5}\right)$.

From this table we see that $\varpi := 1/2(-1+i+j+k)$ is an element of order 3 . Direct calculation shows that ϖ normalises $\mathcal{Q}$.

Binary tetrahedral group. Thus $\mathcal{T} := \mathcal{Q}\langle\varpi\rangle$ is a group of order 24. Note that for some basis

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longmapsto 1, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \longmapsto j, \quad \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \longmapsto i, \quad \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \longmapsto k$$

it is easily seen that $\mathcal{T}\backslash\mathcal{Q}$ consists of the 16 elements of the form $1/2(\pm 1 \pm i \pm j \pm k)$. The group $\mathcal{T}$ is generate by i and ϖ.

Binary octahedral group. The element $\gamma := \frac{1}{\sqrt{2}}(1+i)$ has order 8 . From the description of $\mathcal{T}$ in the previous paragraph we see that γ normalises both $\mathcal{Q}$ and $\mathcal{T}$. Since $\gamma^2 = i, \mathcal{O} := \mathcal{T}\langle\gamma\rangle$ has order 48 . The group $\mathcal{O}$ is the binary octahedral group; it can be generated by ϖ and γ.

Note that

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \longmapsto \frac{1}{\sqrt{2}}(1+i).$$

The set $\mathcal{O}\backslash\mathcal{T}$ consists of the 24 elements of the form $\frac{1}{\sqrt{2}}(\pm u \pm v)$, where u and v are distinct elements of $\{1, i, j, k\}$. We see from the table that 12 of these elements have order 4 and 12 have order 8.

Binary icosahedral group. The element $\sigma := 1/2\left(\tau^{-1}+i+\tau j\right)$ has order 5 and acts on the space V of pure quaternions via the homomorphism $B : \mathbf{S}^3 \to SO_3(\mathbf{R})$. The 12 vectors $\pm\tau i \pm j, \pm\tau j \pm k$ and $\pm i \pm \tau k$ form the vertices of a regular icosahedron in V and they are permuted among themselves by $B(q)$, for $q \in \mathcal{T}$, and by $B(\sigma)$. It is clear that only ± 1 can fix all six lines spanned by these vectors and thus the group $\mathcal{I}$ generated by $\mathcal{T}$ and σ is finite. The group $\mathcal{I}$ is the binary icosahedral group; it can be generated by σ and i.

3.1. The subgroups of the quaternions.

THEOREM 2.4. *Every finite subgroup of* $\mathbf{H}^*$ *is conjugate in* $\mathbf{S}^3$ *to one of the following groups:*

(i) *the cyclic group* $\mathcal{C}_m$,
(ii) *the binary dihedral group* $\mathcal{D}_{2m}$,
(iii) *the binary tetrahedral group* $\mathcal{T}$ *of order 24 ,*
(iv) *the binary octahedral group* $\mathcal{O}$ *of order 48 ,*
(v) *the binary icosahedral group* $\mathcal{I}$ *of order 120 .*

4. The subgrops of $\mathrm{SU}_2(\mathbf{C})$

THEOREM 2.5. *Every finite subgroup of* $SU_2(\mathbf{C})$ *is conjugate in* $SU_2(\mathbf{C})$ *to one of the following groups:*

(i) *the cyclic group* $\mathcal{C}_m$ *of order m generated by*

$$\begin{bmatrix} \exp(2\pi i/m) & 0 \\ 0 & \exp(-2\pi i/m) \end{bmatrix}$$

(ii) *the binary dihedral group* $\mathcal{D}_{2m}$ *of order* $4m$ *generated by*

$$\begin{bmatrix} \exp(\pi i/m) & 0 \\ 0 & \exp(-\pi i/m) \end{bmatrix} \text{ and } \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(iii) *the binary tetrahedral group* $\mathcal{T}$ *of order 24 generated by*

$$\frac{1}{2}\begin{bmatrix} -1-i & 1-i \\ -1-i & -1+i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

(iv) *the binary octahedral group* $\mathcal{O}$ *of order 48 generated by*

$$\frac{1}{2}\begin{bmatrix} -1-i & 1-i \\ -1-i & -1+i \end{bmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix}$$

(v) *the binary icosahedral group* $\mathcal{I}$ *of order 120 generated by*

$$\frac{1}{2}\begin{bmatrix} \tau^{-1}-\tau i & 1 \\ -1 & \tau^{-1}+\tau i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

5. Finite unitary reflection groups of rank two

In this section describe that there are 19 primitive reflection subgroups of $\mathrm{U}_2(C)$.

Let G a finite subgroup of $\mathrm{U}_2(\mathbf{C})$ and if $g \in G$, we choose $\lambda_g \in \mathbf{C}$ such that

$$\lambda_g^2 = \det g.$$

Then $\widehat{G} = \left\{\pm\lambda_g^{-1} g \mid g \in G\right\}$ is a finite subgroup of $\mathrm{SU}_2(\mathbf{C})$ and $G \subseteq \mathcal{C}_m \circ \widehat{G}$, where m is the exponent of G. By construction we have

$$G/Z(G) \simeq \widehat{G}/Z(\widehat{G}).$$

If G is primitive, then (up to conjugacy in $\mathrm{U}_2(\mathbf{C})$) $\widehat{G}$ is a binary polyhedral group $(\mathcal{T},\mathcal{O},\mathcal{I})$, in its two-dimensional representation.

Define of type of G to be $\mathcal{T},\mathcal{O}$ or $\mathcal{I}$ according to whether $\widehat{G}$ is isomorphic to $\mathcal{T},\mathcal{O}$ or $\mathcal{I}$, respectively.

THEOREM 2.6. *Suppose G is $\mathcal{T},\mathcal{O}$ or $\mathcal{I}$ and let $W=\mathcal{C}_n\circ G$ where $n=\exp(G)$. Let $W'=W\cap G$. Then W is a reflection group and $W'=G$.*

PROOF. IF $G=\mathcal{T}$, consider $x\in\mathcal{T}$ with $\mathrm{odr}(x)=m>2$ and let $\zeta\in\mathbf{C}^\times$ order $(m\in\{3,4,6\})$. In this case $n=\exp(\mathcal{T})=12$ and so $\mathcal{C}_{12}=\left\langle e^{2\pi i/3},i\right\rangle$. For somo basis $x=\begin{bmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{bmatrix}$ is reflection, then $\zeta x=\begin{bmatrix}\zeta^2 & 0\\ 0 & 1\end{bmatrix}$ and $\zeta^{-1}x=\begin{bmatrix}1 & 0\\ 0 & \zeta^{-2}\end{bmatrix}$ are reflections, thus $\zeta x\zeta^{-1}x=x^2$ is reflection. We have $\langle x^2\mid x\in\mathcal{T},\mathrm{ord}(x)>2\rangle\subseteq W\cap G$. And like for all $x\in\mathcal{T}$ we has $x^2\in\mathcal{T}$ and $\langle x^2\mid x\in\mathcal{T}\rangle=\mathcal{T}$, then

$$\langle \zeta x\mid x\in\mathcal{T},\ \mathrm{ord}(x)=m,\ \zeta\in\mathcal{C}_{12}\ \text{ of same order}\rangle=W.$$

W is a group of reflections as it is generated by reflections of the form ζx where $\zeta\in C_{12}$ and $x\in\mathcal{T}$ whit $\mathrm{ord}(x)>2$. The same argument for $\mathcal{C}_{60}\circ\mathcal{I}$.

Now suppose that $G=\mathcal{O}$ and $W=\mathcal{C}_{24}\circ\mathcal{I}$ where $24=\exp(G)$. Let $x,y\in G$ of order $\mathrm{ord}(x)=\mathrm{ord}(y)=m>2$ and $x\neq y$. If $\zeta_m\in\mathcal{C}_{24}=\langle e^{2\pi i/3}e^{2\pi i/4}\rangle$ of order m. Then $\zeta_m x$ and $\zeta_m^{-1}y$ are reflections and so $\zeta_m x\zeta_m^{-1}y=xy$ is in subgroup generated by reflections.

Since $\langle xy\mid x,y\in\mathcal{O},x\neq y\ \text{ and }\ \mathrm{ord}(x)=\mathrm{ord}(y)>2\rangle=\mathcal{O}$, then $W\cap\mathcal{O}=\mathcal{O}$. Therefore

$$W=\mathcal{C}_{24}\circ\mathcal{O}=\langle\zeta_m x\mid x\in\mathcal{O},\ \mathrm{ord}(x)=m>2,\ \zeta_m\in\mathcal{C}_{24}\rangle$$

□

We define

$$\mathbf{T}:=\mathcal{C}_{12}\circ\mathcal{T}=\mathcal{C}_{12}\circ\langle i,\varpi\rangle,$$

$$\mathbf{O}:=\mathcal{C}_{24}\circ\mathcal{O}=\mathcal{C}_{24}\circ\langle\gamma,\varpi\rangle\quad\text{and}$$

$$\mathbf{I}:=\mathcal{C}_{60}\circ\mathcal{I}=\mathcal{C}_{60}\circ\langle\sigma,i\rangle$$

By Theorem 2.6, these are reflection groups.

Remark 2. *If G is a finite primitive reflection subgroup of $U_2(\mathbf{C})$ then (up to conjugacy) G is a normal subgroup of $\mathbf{T},\mathbf{O}$ or $\mathbf{I}$ according to whether G is of type $\mathcal{T},\mathcal{O}$ or $\mathcal{I}$, respectively.*

G	Structure	$\lvert G\rvert$	$\lvert Z(G)\rvert$
G_4	$\left\langle r_1, r_1' \right\rangle \simeq SL_2(\mathbf{F}_3)$	24	2
G_5	$\left\langle r_1, r_2' \right\rangle = \mathcal{C}_3 \times \mathcal{T}$	72	6
G_6	$\left\langle r, r_1 \right\rangle = \mathcal{C}_4 \circ G_4$	48	4
G_7	$\left\langle r, r_1, r_2 \right\rangle = \mathcal{C}_3 \times (\mathcal{C}_4 \circ \mathcal{T})$	144	12

TABLE 1. The rank 2 reflection groups of type $\mathcal{T}$ where $r_1' = r r_1 r$ and $r_2' = r r_2 r$

6. The reflection groups of type $\mathcal{T}$, $\mathcal{O}$ and $\mathcal{I}$

Remember that the generators of $\mathcal{T}$ are

$$a := \frac{1}{2}\begin{bmatrix} -1-i & 1-i \\ -1-i & -1+i \end{bmatrix} \quad \text{and} \quad b := \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

of order 3 and 4 respectively. And has trace -1 and 0. Since two elements are conjugate if and only if they have the same trace, then b and $-b$ are conjugated in $\mathcal{T}$. Since $Z(\mathrm{SU}_2(\mathbf{C})) = \{I, -I\}$ we have $\mathcal{T}/Z(\mathcal{T}) \simeq \mathrm{Alt}(4)$ with $\overline{a}, \overline{a}^2$ and $\overline{b}$ as representatives. And also $\mathbf{T} = \mathcal{C}_{12} \circ \mathcal{T}$ and we have that ib and $-ib$ are conjugated in $\mathbf{T}$. Define

$$r = ib = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad r_1 = \omega a = \frac{\omega}{2}\begin{bmatrix} -1-i & 1-i \\ -1-i & -1+i \end{bmatrix} \quad \text{and} \quad r_2 = \omega a^2 = \frac{\omega}{2}\begin{bmatrix} -1+i & -1+i \\ 1+i & -1-i \end{bmatrix}$$

which have order 2, 3 and 3 (traces $0, -1$ and -1) respectively. Now the groups are defined

$$G_4 := \langle r_1^{\mathbf{T}} \rangle, \quad G_5 := \langle r_1^{\mathbf{T}}, r_2^{\mathbf{T}} \rangle, \quad G_6 := \langle r^{\mathbf{T}}, r_1^{\mathbf{T}} \rangle \quad \text{and} \quad G_7 := \langle r^{\mathbf{T}}, r_1^{\mathbf{T}}, r_2^{\mathbf{T}} \rangle$$

the groups of type $\mathcal{T}$ where $r^{\mathbf{T}}$ denote the conjugacy class of $r \in \mathbf{T}$.

For groups of type $\mathcal{O}$. The generators of $\mathcal{O}$ are

$$a := \frac{1}{2}\begin{bmatrix} -1-i & 1-i \\ -1-i & -1+i \end{bmatrix} \text{ and } c := \frac{1}{\sqrt{2}}\begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix}$$

of orders 3 and 8 (traces -1 and $2/\sqrt{2}$) respectively. Note that $\mathcal{O}/Z(\mathcal{O}) \simeq \mathrm{Sym}(4)$ has as representatives of the conjugation classes to $\overline{a}, \overline{b}, \overline{c}$ and $\overline{d}$ where $b := c^2$ and $d := c^3 a^2 c^2$ (d so that $d^{-1} a d = a^{-1}$). The elements $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ correspond to the permutations $(2,3,4), (1,3)(2,4), (1,2,3,4)$ and $(2,3)$ in $\mathrm{Sym}(4)$. Note that ib and $-ib$ are conjugated in $\mathbf{O} = \mathcal{C}_{24} \circ \mathcal{O}$. The representatives of the conjugation classes of $\mathbf{O}$ reflections are as follows $ib, \omega a, \omega^2 a, id, (1-i)c\sqrt{2}$, and $(1+i)c/\sqrt{2}$.

The conjugacy classes of cyclic subgroups of $\mathbf{O}$ generated by reflections are represented by $r := ib, r_1 := \omega a, r_3 := id$ and $r_4 := (1+i)c/\sqrt{2}$ of orders $2,3,2$ and 4 , respectively. And we have

$$r_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad r_4 = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

G	Structure	$\lvert G\rvert$	$\lvert Z(G)\rvert$
G_8	$\left\langle r_4, r_4' \right\rangle = \mathcal{T}\mathcal{C}_4$	96	4
G_9	$\langle r_3, r_4\rangle = \mathcal{C}_8 \circ \mathcal{O}$	192	8
G_{10}	$\left\langle r_1, r_4' \right\rangle = \mathcal{C}_3 \times \mathcal{T}\mathcal{C}_4$	288	12
G_{11}	$\langle r_1, r_3, r_4\rangle = \mathcal{C}_3 \times (\mathcal{C}_8 \circ \mathcal{O})$	576	24
G_{12}	$\left\langle r_3, r_3', r_3'' \right\rangle \simeq GL_2(\mathbf{F}_3)$	48	2
G_{13}	$\left\langle r, r_3, r_3'' \right\rangle = \mathcal{C}_4 \circ \mathcal{O}$	96	4
G_{14}	$\left\langle r_1, r_3' \right\rangle = \mathcal{C}_3 \times G_{12}$	144	6
G_{15}	$\langle r, r_1, r_3\rangle = \mathcal{C}_3 \times (\mathcal{C}_4 \circ \mathcal{O})$	288	12

TABLE 2. The rank 2 reflection groups of type $\mathcal{O}$

Define $r_3' = r r_3 r, r_3'' = \left(r_1^2 r_4\right)^{-1} r_3 \left(r_1^2 r_4\right)$ and $r_4' = r_3^{-1} r_4 r_3$. That is, we put

$$r_3' := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad r_3'' := \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1+i \\ 1-i & 0 \end{bmatrix}$$

and

$$r_4' := \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ -1+i & 1+i \end{bmatrix}.$$

For groups of type $\mathcal{I}$. The generators of $\mathcal{I}$ are

$$e := \frac{1}{2}\begin{bmatrix} \tau^{-1}-\tau i & 1 \\ -1 & \tau^{-1}+\tau i \end{bmatrix} \quad \text{and} \quad b := \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

of orders 5 and 4 respectively. Thus the non-trivial conjugacy classes of the group $\overline{\mathcal{I}} := \mathcal{I}/Z(\mathcal{I}) \simeq \mathrm{Alt}(5)$ are represented by $\bar{a}, \bar{b}, \bar{e}$ and $\bar{e}^2$, where $a := (be)^3 e^2 b^3$. The elements a and b are the previous generators of $\mathcal{T}$ and the elements $\bar{a}, \bar{b}$ and $\bar{e}$ correspond to the permutations $(2,4,3), (2,3)(4,5)$ and $(1,2,4,3,5)$ in Alt(5). Let ζ be an eigenvalue of e. Then ζ is a fifth root of unity and we may take $\zeta := \exp(2\pi i/5)$ so that $\tau = \zeta + \zeta^{-1} + 1$. The group $\mathbf{I}$ has seven conjugacy classes of reflections with representatives $ib, \omega a, \omega^2 a, \zeta e, \zeta^{-1} e, \zeta^2 e^2$ and $\zeta^{-2} e^2$.

As representatives for the conjugacy classes of cyclic subgroups of $\mathbf{I}$ we take $r := ib, r_1 := \omega a$ and $r_5 := \zeta^2 e^3$. Their orders are 2,3 and 5 respectively and their matrices are

$$r = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad r_1 = \frac{\omega}{2}\begin{bmatrix} -1-i & 1-i \\ -1-i & -1+i \end{bmatrix}$$

$$\text{and} \quad r_5 = \frac{\zeta^2}{2}\begin{bmatrix} -\tau+i & -\tau+1 \\ \tau-1 & -\tau-i \end{bmatrix}.$$

G	Structure	$\lvert G\rvert$	$\lvert Z(G)\rvert$
G_{16}	$\left\langle r_5, r_5' \right\rangle = \mathcal{C}_5 \times \mathcal{I}$	600	10
G_{17}	$\langle r, r_5 \rangle = \mathcal{C}_5 \times (\mathcal{C}_4 \circ \mathcal{I})$	1200	20
G_{18}	$\left\langle r_1^2, r_5 \right\rangle = \mathcal{C}_{15} \times \mathcal{I}$	1800	30
G_{19}	$\langle r, r_1, r_5 \rangle = \mathcal{C}_{15} \times (\mathcal{C}_4 \circ \mathcal{I})$	3600	60
G_{20}	$\left\langle r_1, r_1'' \right\rangle = \mathcal{C}_3 \times \mathcal{I}$	360	6
G_{21}	$\left\langle r, r_1'' \right\rangle = \mathcal{C}_3 \times (\mathcal{C}_4 \circ \mathcal{I})$	720	12
G_{22}	$\langle r, r', r'' \rangle = \mathcal{C}_4 \circ \mathcal{I}$	240	4

TABLE 3. The rank 2 reflection groups of type $\mathcal{I}$

are distinct. Now define $r' := r_1^{-2} r r_1^2$, $r'' := r_5^{-1} r r_5$, $r_1'' := r_5 r_1 r_5^{-1}$ and $r_5' := r^{-1} r_5 r$. That is,

$$r' := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad r'' := \frac{1}{2} \begin{bmatrix} \tau & (\tau-1)i+1 \\ (-\tau+1)i+1 & -\tau \end{bmatrix},$$

$$r_1'' := \frac{\omega}{2} \begin{bmatrix} -\tau i - 1 & (-\tau+1)i \\ (-\tau+1)i & \tau i - 1 \end{bmatrix} \quad \text{and } r_5' := -\frac{\zeta^2}{2} \begin{bmatrix} \tau - i & -\tau+1 \\ \tau-1 & \tau+i \end{bmatrix}.$$

The group $\mathcal{I} = 3600$ and as for the reflection subgroups of **T** and **O**.

CHAPTER 3

The groups $G(m,p,n)$

This chapter introduces the family of unitary reflection groups known as $G(m,p,n)$ which is the notation of Shephard and Todd [**13**]. The complex irreducible complex reflection groups were classified by Geoffrey Shephard and John Todd [**13**] and then Arjeh Cohen [**3**] gives a classification using more modern methods.

1. Definition and properties of the groups $G(m,p,n)$

For $m,p,n \geq 1$ and $p|m$ we define

$$A(m,p,n) := \left\{ \begin{bmatrix} \zeta_1 & 0 & \cdots & 0 & 0 \\ 0 & \zeta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \zeta_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \zeta_n \end{bmatrix} \;\middle|\; \begin{array}{l} \zeta_i^m = 1 \text{ and} \\ \left(\zeta_1 \cdots \zeta_n\right)^{m/p} = 1 \end{array} \right\}$$

the group of all diagonal matrix $n \times n$ such that each nonzero entries are mth root of unit with the condition that $\det(A)^{m/p} = 1$ where $A \in A(m,p,n)$. Note that there are m^n possibilities for $n \times n$ diagonal matrices with m^{th} roots of unity on the diagonal such that $(\det A)^{\frac{m}{p}} = 1$ so $|A(m,p,n)| = \frac{m^n}{p}$.

EXAMPLE 3. *The groups*

$$A(2,1,2) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

$$A(2,2,2) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

Let $\mathrm{Sym}(n)$ the subgroup of permutation matrices in $\mathrm{GL}(\mathbf{C}^n)$.

DEFINITION 3.1. *For $m,p,n \geq 1$ and $p|m$, we define the group*

$$G(m,p,n) := A(m,p,n) \rtimes \mathrm{Sym}(n).$$

The group $G(m,p,n)$ is the group of monomial $n \times n$ matrices with only one nonzero entry in each row and column whose nonzero entries is a m^{th} roots of unity and the product of nonzero entries are m/p-root of unity. Note that $|G(m,p,n)| = |A(m,p,n)||\mathrm{Sym}(n)| = m^n n!/p$. The elements of $G(m,p,n)$ are of the form $\zeta^\lambda w$ where $\zeta^\lambda \in A(m,p,n)$ is a diagonal matrix with $\det \zeta^\lambda = \zeta_1^{\lambda_1}\zeta_2^{\lambda_2}\cdots\zeta_n^{\lambda_n}$ and $w \in \mathrm{Sym}(n)$ is a permutation matrix.

EXAMPLE 4. *The groups*

$$G(2,1,2)=\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}-1&0\\0&-1\end{bmatrix},\begin{bmatrix}-1&0\\0&1\end{bmatrix},\begin{bmatrix}1&0\\0&-1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&-1\\-1&0\end{bmatrix},\begin{bmatrix}0&-1\\1&0\end{bmatrix},\begin{bmatrix}0&1\\-1&0\end{bmatrix}\right\}$$

$$G(2,2,2)=\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}-1&0\\0&-1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&-1\\-1&0\end{bmatrix}\right\}$$

$$\begin{bmatrix}0&0&i\\-i&0&0\\0&1&0\end{bmatrix}\in G(4,2,3)$$

From now on we will always consider that $G(m,p,n)$ is defined with respect to the orthonormal basis $e_1,e_2,\dots,e_n$ for V.

LEMMA 3.1. *The element $r\in G(m,p,n)$ is a reflection if and only if r has one of the following forms:*

(i) *For some i and for some $(m/p)^{th}$ root of unity $\theta\neq 1$, $r=r_{e_i,\theta}$.*
(ii) *For some $i\neq j$ and for some m^{th} root of unity θ, $r=r_{e_i-\theta e_j,-1}$. In this case the order of r is two, $re_i=\theta e_j$, $re_j=\theta^{-1}e_i$ and $re_k=e_k$ for all $k\neq i,j$.*

PROOF. Suppose $r\in G(m,p,n)$ is a reflection. If r is in $A(m,p,n)$, then it must be of the form $r_{e_i,\theta}$ for some i and $(m/p)^{\text{th}}$ root of unity $\theta\neq 1$, which is item (i). If r is not in $A(m,p,n)$, then for some $i\neq j$, we have $re_i=\theta e_j$ for some m^{th} root of unity θ. From the Lemma 1.6, we know that the order of r must be 2. It can be shown that $e_i-\theta e_j\in[V,r]$, and this forms item (ii) of the Lemma, where $r=r_{e_i-\theta e_j,-1}$.

Conversely. In the case (i), for some i and for some (m/p)-root of 1, $\theta\neq 1$

$$r=r_{e_i,\theta}=\begin{bmatrix}1&&&&&&\\&1&&&&&\\&&\ddots&&&&\\&&&\theta&&&\\&&&&\ddots&&\\&&&&&1&\\&&&&&&1\end{bmatrix}=\theta_i$$

that is a reflection contained in $A(m,p,n)\subseteq G(m,p,n)$. In the case (ii), for some $i\neq j$ and for some m-root of unity θ,

$$r = r_{e_i - \theta e_j, -1} = \theta (i\,j) \theta^{-1} = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & \theta^{-1} & & \\ & & & \ddots & & & \\ & & \theta & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

i.e.

$$e_i \longmapsto \theta e_j$$

$$e_j \longmapsto \theta^{-1} e_i$$

for the lemma 1.6 it a reflection of order two.

□

THEOREM 3.1. *The group $G(m,m,n)$ contains $m\binom{n}{2}$ reflections and the order of every reflection is two.*

PROOF. For the group $G(m,m,n)$, its follows that is generate for reflection of type (ii) from lemma 3.1 and the reflection are the form $\zeta(i\,j)\zeta^{-1}$ where ζ is a m-root of 1. This reflections are of order two $|\zeta(i\,j)\zeta^{-1}| = |(i\,j)| = 2$. □

EXAMPLE 5. *Note in the example 4 that all reflection in $G(2,2,2)$ are of order two.*

THEOREM 3.2. *For the group $G(m,m,n)$.*

(1) *If $n > 2$ or if m is odd, the reflections form a single conjugacy class.*
(2) *If m is even, $G(m,m,2)$ contains two conjugacy class of reflections.*

PROOF. (1) The reflections on $G(m,m,n)$ are of the form $r = \zeta_i (i\,j) \zeta_i^{-1}$ where ζ is a m^{th}-root of 1. And we can conjugate by $w \in \mathrm{Sym}(n)$ with $w(i\,j)w^{-1} = (12)$

$$w r w^{-1} = w\zeta_i (i\,j)\zeta_i^{-1} w^{-1} = w\zeta_i w^{-1} w(i\,j) w^{-1} w \zeta_i^{-1} w^{-1} = \zeta_1 (12) \zeta_1^{-1}.$$

Now, suppose that $n > 2$, and let $k > 2$, then

$$(\zeta_k \zeta_1^{-1})\zeta_1(12)\zeta_1^{-1}(\zeta_k\zeta_1^{-1})^{-1} = \zeta_k(12)\zeta_k^{-1} = (12).$$

An only conjugacy class. And if $n = 2$ and m is odd, $m = 2\lambda - 1$, $\lambda \in \mathbf{N}$. Note that $\zeta^m = \zeta^{2\lambda - 1} = \zeta^{2\lambda}\zeta^{-1} = 1$, so $\zeta^{2\lambda} = \zeta$, let's consider the reflection $\zeta_1(12)\zeta_1^{-1}$, then

$$(\zeta_2^{2\lambda}\zeta_1^{-1})\zeta_1(12)\zeta_1^{-1}(\zeta_2^{2\lambda}\zeta_1^{-1})^{-1} = \zeta_2^{2\lambda}(12)\zeta_2^{-2\lambda} = \zeta_2(12)\zeta_2^{-1}.$$

Has only conjugacy class.

(2) $G(m,m,2) = A(m,m,2) \rtimes \mathrm{Sym}(2) = \mathcal{C}_m \rtimes \mathcal{C}_2 = \mathcal{D}_{2m}$ has a two conjugacy class.

□

THEOREM 3.3. *If $m > 1$, then the group $G(m,p,n)$ is an imprimitive unitary reflection group irreducible if and only if $(m,p,n) \neq (2,2,2)$.*

PROOF. ($\Longrightarrow$) Suppose that $G(m,p,n)$ is reducible. Then $G(m,p,n)$ has a proper invariant subspace W. Then W is fixed by each reflection $r_{e_i-e_j,-1} := (ij)$, then $e_i - e_j \in W$ or $e_i - e_j \in W^{\perp}$. If $e_i - e_j \in W$ and $e_j - e_k \in W^{\perp}$ for distinct indices i, j and k, then $e_i - e_k \in W$ or $e_i - e_k \in W^{\perp}$, that is a contradiction (W is Sym(n) invariant). We may suppose without loss of generality that $e_i - e_j \in W^{\perp}$ for all i, j and $W = \mathbf{C}(e_1 + e_2 + \cdots + e_n)$. If $\zeta^{\lambda} = \mathrm{diag}(\zeta^{\lambda_1}, \zeta^{\lambda_2}, \ldots, \zeta^{\lambda_n}) \in A(m,p,n)$, then ζ^{λ} fixed W and so $\zeta^{\lambda_1} = \zeta^{\lambda_2} = \cdots = \zeta^{\lambda_n}$. Therefore $m = 1$ or $m = p = n = 2$.

($\Longleftarrow$) For $m = 1$, then $G(1,1,n) = \mathrm{Sym}(n)$ that is reducible. The group $G(2,2,2) = A(2,2,2) \rtimes \mathrm{Sym}(2) = \mathcal{C}_2 \times \mathcal{C}_2$ is reducible, because acts on $\mathbf{C}^2$ that has basis e_1, e_2 and $e_1 + e_2$ are orthogonal to $e_1 + e_2$ therefore $\mathbf{C}^2 = \mathbf{C}(e_1 - e_2) \oplus \mathbf{C}(e_1 + e_2)$.

$\square$

LEMMA 3.2. *If G is a transitive group of permutations of the finite set Ω, and if G is generated by transpositions, then $G = \mathrm{Sym}(\Omega)$.*

For the proof see [**9**, Lemma 2.13, pag. 28].

THEOREM 3.4. *If V is a vector space of dimension $n \geq 2$ over $\mathbf{C}$ with a positive definite hermitian form $(-,-)$ and if G is an irreducible imprimitive finite subgroup of $U(V)$, which is generate by reflections. Then G is conjugate to $G(m,p,n)$ for $m > 1$ and $p|m$.*

PROOF. Let G a imprimitive subgroup, then consider $\Omega := \{V_1, V_2, \ldots, V_k\}$ be system of imprimitivity for G. We have $\dim V_i = 1$ by lemma 1.5 for all i and $k = n$ because dime $V = n$. Now, consider a basis $\{e_1, \ldots, e_n\}$ for V such that $V_i = \mathbf{C}e_i$ and $(e_i, e_i) = 1$.

If the reflection r fixes every subspace V_i, then $r = r_{e_i,\theta}$ (by (i) of lemma 3.1). Otherwise $rV_i = V_j$ for $i \neq j$, so r is of order 2 by lemma 1.6 and acts on Ω by permuting the subspaces. Thus the group G^{Ω} of permutations of Ω induced by the action of G is transitive and generated by transpositions. Now lemma 3.2 shows that $G^{\Omega} = \mathrm{Sym}(\Omega)$.

For $i = 1,2,...,n$ choose a reflection $r_i \in G$ such that $r_i V_1 = V_i$, i.e., $r_i e_1 = e_i$. The subgroup $\sum := \langle r_2, r_3, \ldots, r_n \rangle$ is isomorphic to Sym(n). And like this Sym(n) is subgroup of G.

With respect to the basis $e_1, e_2, \ldots, e_n$ all element $g \in G$ has the form

$$ge_i := \theta_i e_{\pi(i)} \qquad \text{for} 1 \leq i \leq n$$

where π is a permutation of $\{1,2,...,n\}$ and $\theta_1, \theta_2, ..., \theta_n \in \mathbf{C}$. If $ge_i = \theta e_j$ for some θ, then

$$1 = (e_i, e_i) = (ge_i, ge_i) = \theta\overline{\theta}(e_j, e_j) = \theta\overline{\theta}$$

Therefore, if $[-,-]$ is the hermitian form on V such that $\left[e_i, e_j\right] = \delta_{ij}$ for all i, j, then $[-,-]$ is G-invariant. By corollary 1.1 to Schur's Lemma $[-,-]$ is a scalar multiple of $(-,-)$

and thus $\left(e_i, e_j\right) = 0$ for all $i \neq j$.

Let $\Theta := \left\{\theta \in \mathbf{C} \mid re_i = \theta e_j \text{ for some reflection } r \in G \text{ and some } i \neq j\right\}$. Let $\theta \in \Theta$ and suppose that there a reflection $r \in G$ such that $re_1 = e_2$ because the subgroup $\sum$ acts doubly transitively on the basis $e_1, e_2, \ldots, e_n$.

If s is a reflection such that $se_1 = \eta e_2$, then rr_2s is a reflection,

$$rr_2se_1 = rr_2(\eta e_2) = r\eta e_1 = \theta\eta e_2$$

$$rr_2se_2 = rr_2(\eta^{-1}e_1) = r\eta^{-1}e_2 = \theta^{-1}\eta^{-1}e_1$$

then $\theta\eta \in \Theta$, therefore Θ is a finite subgroup of $\mathbf{C}$. Thus for some m, Θ is the group of m^{th} roots of unity.

If $r \in G$ is a reflection such that $re_1 = \theta e_1$, where $\theta \neq 1$, then $r^{-1}r_2r$ is a reflection such that $r^{-1}r_2re_1 = r^{-1}r_2\theta e_1 = r^{-1}\theta e_2 = \theta e_2$. Therefore the set

$$\left\{\theta \in \mathbf{C} \mid ge_1 = \theta e_1 \text{ for some } g \in G \text{ such that } V_1^{\perp} = \operatorname{Fix} g\right\}$$

is a subgroup of Θ and hence the group of q^{th} roots of unity for some divisor q of m. We have $G(q,1,n) \subseteq G$ and $m > 1$, otherwise $G = \Sigma$ would be reducible.

If $\theta \in \Theta$ and r is a reflection such that $re_1 = \theta e_2$, then $r_2re_1 = \theta e_1, r_2re_2 = \theta^{-1}e_2$ and $r_2r \in A(m,m,n)$. It follows that $A(m,m,n)$ is a subgroup of G and therefore $G(m,p,n) \subseteq G$, where $p = m/q$. On the other hand, $G(m,p,n)$ contains all reflections of G and so $G = G(m,p,n)$. □

The group $G(1,1,n+1) = \operatorname{Sym}(n+1)$ that act in the hyperplane $\mathbf{C}(e_1 + \cdots + e_n)^{\perp}$ is denote by $W(A_n)$.

THEOREM 3.5. *Suppose that V is a vector space of dimension n over $\mathbf{C}$ with positive definite hermitian form $(-,-)$ and that G is an irreducible imprimitive finite subgroup of $U(V)$, which is generated by reflections. Suppose that H is a subgroup of G generated by reflections with roots in a subspace M of V. If $m := \dim M > 1$ and if the action of H on M is primitive, then H is conjugate to $W\left(A_m\right)$.*

PROOF. Let G an irreducible imprimitive finite group and let $\Omega := \{V_1, V_2, \ldots, V_n\}$ a system of imprimitivity for G. Let H be the subgroup of G that satisfies the hypotheses, since H is primitive, then it is irreducible on M $(1 < \dim M = m < \dim V = n)$.

Observe that if it is assumed that for some i it is assumed that $V_i \subseteq M$, then HV_i form a system of imprimitivity for M, a contradiction, thus $V_i \nsubseteq M$ for all i (or $V_i = M$ contrary to $\dim M = m > 1$).

Suppose that $r \in H$ is a reflection with root a. If r fixes every V_i, then $a \in V_k$ for some k, then $V_k \subseteq M$ a contradiction. Therefore, it is necessary to $rV_i = V_j$ for some $i \neq j$, so r is of order 2 and thus act by transposition over Ω. Thus $a \in V_i + V_j$, then $a \in (V_i + V_j) \cap M$ (by lemma 1.6) then $V_i+M=V_j+M$, so $V_i,V_j\subseteq M$ a contradiction to $V_i\nsubseteq M$ for all i.

If Γ is an orbit of H in Ω so Γ are of the form

$$\Gamma:=\{V_1,V_2,\dots,V_k\}.$$

Put $N:=V_1+V_2+\cdots+V_k$ and H is generate for the reflections with roots in M, then $N\cap M\neq\{0\}$ is H-invariant, thus coincides with M. It clear that $N:=M+V_1$ and so M is a subspace of codimension 1 on N. Therefore $k=m+1$. If $i,j>k$ and if $r\in H$ is a reflection such that $rV_i=V_j$, then $(V_i+V_j)\cap M\neq\{0\}$, thus $(V_i+V_j)\cap N\neq\{0\}$ a contradiction. If follows that H fixes $N^{\perp}$ and acts on N as $G(1,1,k)=\mathrm{Sym}(k)$. Thus it acts on M as $W(A_m)=G(1,1,m+1)$.

□

Remark 3. *The set of scalar matrixes of* $G(m,p,n)$ *are*

$$\left\{\zeta\in\mathbf{C}|\zeta^{nm/p}=1\right\}\cap\{\zeta\in\mathbf{C}|\zeta^{m}=1\}=\left\{\zeta\in\mathbf{C}|\zeta^{d}=1\right\}$$

where $d=\gcd(nm/p,m)$.

COROLLARY 3.1. *If* $G=G(m,p,n)$*, then* $|Z(G)|=\frac{m}{p}\gcd(p,n)$.

PROOF. If $x\in Z(G)$, then x is a scalar $n\times n$ matrix of the form

$$x=\zeta I=\begin{bmatrix}\zeta&0&\cdots&0&0\\0&\zeta&\cdots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&\zeta&0\\0&0&\cdots&0&\zeta\end{bmatrix}$$

so $\det x=\zeta^n$, since $x\in G$, then $(\det x)^{m/p}=(\zeta^n)^{m/p}=1$. The number of soluction of its equation is

$$\gcd\left(\frac{mn}{p},m\right)=\gcd\left(\frac{mn}{p},\frac{pm}{p}\right)=\frac{m}{p}\gcd(n,p).$$

Thus $|Z(G)|=\frac{m}{p}\gcd(n,p)$. □

THEOREM 3.6. *The only conjugates in the set of groups* $G(m,p,n)$ *is* $G(2,1,2)$ *conjugate to* $G(4,4,2)$.

See [**3**, Remarks (2.5)-(iv), pag. 388]. To prove this, it is necessary to make use of the invariant polynomials.

Remark 4. *Let* G *be a nonabelian finite group is called extra-special* p*-group if* $|Z(G)|=p$ *and* $[G,G]=Z(G)=\Phi(G)$*. We are interested in the fact that there are two extra special groups of order 8 and that the non-abelian groups of order* p^3*, with* p *a prime number, are extra special* p*-groups.* [**4**]

(1) *For te case* G *is a nonabelian group of order* $8=2^3$*, then it is isomorphic to either* $\mathcal{D}_4$ *and quaternion group. The group* $\mathcal{D}_4$ *has two normal subgroups of order 4.*

(2) *If group G of order p^3 with p a prime number. Then $|Z(G)| = p$ and $[G,G] = Z(G)$. And has in the subgroup lattice, $p+1$ subgroups of order p^2.*

Remark 5. *Let a group $G = G(3,3,3)$ with order $54 = 3^3 \cdot 2$. This group contains one 3-subgroup of Sylow order 27 (first Sylow Theorem),*

$$E = \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle$$

*were ω is a primitive cube root of 1, this subgroup E are nonabelian group of order p^3 for the prime $p = 3$. The center of E are $Z(E) = \{I, \omega I, \omega^2 I\}$ that is of order 3. The subgroup E has a normal (abelian) 3-subgroup and note that $E' = [E,E] = Z(E)$. Then E have $p+1 = 3+1 = 4$ subgroups of order $p^2 = 3^2 = 9$ (extraspecial p-groups, see [**4**]) and are normal subgroups in E by first Sylow Theorem.*

THEOREM 3.7. *Suppose that $G := G(m,p,n)$ is irreducible and has more than one system of imprimitivity. Then G is one of the following groups.*

(i) *$G(2,1,2) \simeq G(4,4,2)$ or $G(4,2,2)$, each of which has the same three systems of imprimitivity: $\{\mathbf{C}e_1, \mathbf{C}e_2\}$, $\left\{\mathbf{C}\left(e_1+e_2\right), \mathbf{C}\left(e_1-e_2\right)\right\}$ and $\left\{\mathbf{C}\left(e_1+ie_2\right), \mathbf{C}\left(e_1-ie_2\right)\right\}$.*

(ii) *$G(3,3,3)$ with four systems of imprimitivity: $\Pi_0 := \{\mathbf{C}e_1, \mathbf{C}e_2, \mathbf{C}e_3\}$ and, for a fixed primitive cube root of unity ω,*

$$\Pi_i := \left\{\mathbf{C}\left(e_1+\omega_2 e_2+\omega_3 e_3\right) \mid \omega_2, \omega_3 \in \left\{1,\omega,\omega^2\right\} \text{ and } \omega_2\omega_3 = \omega^i\right\},$$

for $i := 1, 2$ and 3 .

(iii) *$G(2,2,4)$ with three systems of imprimitivity:*

$$\Lambda_0 := \{\mathbf{C}e_1, \mathbf{C}e_2, \mathbf{C}e_3, \mathbf{C}e_4\} \text{ and}$$

$$\Lambda_i := \left\{\mathbf{C}\left(e_1+\varepsilon_2 e_2+\varepsilon_3 e_3+\varepsilon_4 e_4\right) \mid \varepsilon_j = \pm 1 \text{ and } \varepsilon_2\varepsilon_3\varepsilon_4 = (-1)^i\right\},$$

for $i := 1$ and 2 .

PROOF. Note that if $\{\mathbf{C}u_1\mathbf{C}u_2, \ldots, \mathbf{C}u_n\}$ is a system of imprimitivity for G and B is a subgroup of G that fixes all $\mathbf{C}u_i$, then for $g \in G$ such that $gu_i = u_j$ and $b \in B$ we have $gbg^{-1}u_i = gg^{-1}u_i = u_i$, then $gbg^{-1} \in B$ and B is normal subgroup. Now let's see what is abelian, let $b, b' \in B$ so $bb'u_i = bu_i = b'bu_i$, then $bb' = b'b$. The system of imprimitivity $\{\mathbf{C}e_1, \mathbf{C}e_2, \ldots, \mathbf{C}e_n\}$ is the set of isotypic component of $A = A(m,p,n)$. Since A permute the u_i, then $A \neq B$. Then AB/A is a normal subgroup of $G/A \simeq \mathrm{Sym}(n)$, then $n = 2, 3$ or 4 and $|AB/A| = n$. In each case, B acts transitively on $\{\mathbf{C}e_1, \mathbf{C}e_2, \ldots, \mathbf{C}e_n\}$ and since $A \cap B$ consists of diagonal matrices, which commute with all elements of B, it follows that $A \cap B$ is the group of scalar matrices θI, where $\theta^{nm/p} = 1$, then $|A \cap B| = \frac{m}{p}\gcd(n,p)$.

We also have $G/B \simeq \mathrm{Sym}(n)$ and so $|B| = |A| = m^m/p$. Thus

$$n = \left|AB/A\right| = |B/A \cap B| = \frac{m^n/p}{\gcd(n,p)m/p} = \frac{m^n}{\gcd(n,p)m}$$

$$m^{n-1} = n\gcd(n,p)$$

where $p|m$ and $n \in \{2,3,4\}$. The posible solutions are

(a) for $n=2$; we have $m = 2\gcd(2,p)$, then

$$p=1 \quad \text{and} \quad m=2 \quad \text{we have} \quad (2,1,2)$$

$$p=2 \quad \text{and} \quad m=4 \quad \text{we have} \quad (4,2,2)$$

$$p=4 \quad \text{and} \quad m=4 \quad \text{we have} \quad (4,4,2)$$

Note that for $p=6$ we have $m=4$ and $6 \not| 4$. Similar for $p = 3,7,8,9,\ldots$.

(b) If $n=3$, then $m^2 = 3\gcd(3,p)$, note that only possibility such that $p|m$ is $p=3$. Thus we have $(3,3,3)$.

(c) If $n=4$, then $m^3 = 4\gcd(4,p)$, the only possibility is $(2,2,4)$.

Let us look at the cases separately. For $n=2$ we have the groups $G(2,1,2), G(4,2,2)$ and $G(4,4,2)$.

Suppose that $n=2$ and that $u_1 := e_1 + \eta e_2$ where $\eta \neq 0$. If $\mathbf{C}u_1$ is fixed by $x := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $\eta = \pm 1$. And we have the system of imprimitivity $\{\mathbf{C}(e_1 - e_2), \mathbf{C}(e_1 + e_2)\}$.

If $\mathbf{C}u_1$ and $\mathbf{C}u_2$ are interchange by x, we have may take $u_1 := e_1 + \eta e_2$ and $u_2 := e_1 + \eta^{-1} e_2$ where $\eta \neq 0, \pm 1$. If $G = G(2,1,2)$ or $G(4,2,2)$, then $y := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \in G$ and interchange $\mathbf{C}u_1$ and $\mathbf{C}u_2$ and hence $\eta = \pm i$. In the case of $G(4,4,2)$, we have $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \in G$ and then $\eta = \pm i$. Therefore, the groups G(2, 1, 2), G(4, 2, 2) and G(4, 4, 2) have the same three priming systems

$$\{\mathbf{C}e_1, \mathbf{C}e_2\}, \left\{\mathbf{C}\left(e_1 + e_2\right), \mathbf{C}\left(e_1 - e_2\right)\right\} \quad \text{and} \quad \left\{\mathbf{C}\left(e_1 + ie_2\right), \mathbf{C}\left(e_1 - ie_2\right)\right\}.$$

and this completes the proof of (i).

Suppose $n=3$. In this case $G(3,3,3)$ has order $54 = 3^3 \cdot 2$, so it has a Sylow 3-subgroup and if ω is a primitive cube root of unity, this subgroup E of the remark 5 which has order 27. The center of E has order 3 and every distinct element of the identity of E has order 3.

Therefore, E has four (abelian and normal) subgroups of order 9 (by first Sylow's Theorem and remark 5). Thus each of the sets of subspaces listed in part (ii), for

$$\Pi_1 := \left\{ \mathbf{C}\begin{bmatrix} 1 \\ 1 \\ \omega \end{bmatrix}, \mathbf{C}\begin{bmatrix} 1 \\ \omega \\ 1 \end{bmatrix}, \mathbf{C}\begin{bmatrix} 1 \\ \omega^2 \\ \omega^2 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ \omega^2 \end{bmatrix} = \begin{bmatrix} 1 \\ \omega \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}\begin{bmatrix} 1 \\ \omega \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \omega^2 \\ \omega^2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} 1 \\ \omega^2 \\ \omega^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \omega \end{bmatrix}$$

(similar for Π_2, Π_3) of the statement of the theorem provide systems of imprimitivity associated to each normal subgroup and the conjugation is given by $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which takes each element of E to its inverse (modulo to the center). Therefore $G(3,3,3)$ has four imprimitivity systems.

Finally, suppose that $n = 4$. Then the order of $G := G(2,2,4)$ is $192 = 3 \cdot 2^6$. The order of the subgroup E generated by

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

is 32 and $E = O_2(G)$, the largest normal 2-subgroup of G. The center $Z(E)$ has order tow.

There are six normal abelian subgroups of E of order 8 (extraspecial p-groups) but only three of these are normal in G ($n_2 = 3$, first Sylow Theorem); they correspond to the three systems of imprimitivity in part (iii) of the theorem.

□

LEMMA 3.3. *$A(m,p,n) \subseteq A(m,q,n)$ if and only if $q|p$.*

PROOF. If $A(m,p,n) \subseteq A(m,q,n)$ and if $x = \mathrm{diag}(e^{2\pi i p/m}, 1, \ldots, 1) \in A(m,p,n)$, then

$$(e^{2\pi i p/m})^{m/q} = 1, \quad \text{and} \quad e^{2\pi i p/q} = 1$$

then $p/q \in \mathbf{Z}$. Therefore $q|p$.

If $q|p$ and for $x = \mathrm{diag}(e^{2\pi i p/m}, 1, \ldots, 1) \in A(m,p,n)$, then

$$e^{2\pi i p/m} = e^{2\pi i k q/m} = (e^{2\pi i k})^{q/m} = 1.$$

Therefore $A(m,p,n) \subseteq A(m,q,n)$. □

Remark 6. *Note that the group $G(m,p,n)$ is a normal subgroup of $G(m,1,n)$. Consider $\rho : G(m,p,n) \to G(m,1,n)$.*

$$\begin{aligned} G(m.p,n) &= \{\zeta^\lambda w \in G(m,1,n) \ : \ (m/p)(\lambda_1 + \cdots + \lambda_n) = 0 \mod m\} \\ &= \{\zeta^\lambda w \in G(m,1,n) \ : \ \lambda_1 + \cdots + \lambda_n = 0 \mod p\} \\ &= \{\zeta^\lambda w \in G(m,1,n) \ : \ e^{(2\pi i/p)(\lambda_1 + \cdots + \lambda_n)} = 1\} \\ &= \ker(\zeta^\lambda w \longmapsto e^{(2\pi i/p)(\lambda_1 + \cdots + \lambda_n)}) \end{aligned}$$

Por tanto $G(m,p,n) = \ker\rho$.

COROLLARY 3.2. *Suppose that $m \geq 2$ and that G is a finite reflection subgroup of $U_n(\mathbf{C})$ which contains $G(m,p,n)$ as a normal subgroup.*

(i) *If $n \geq 3$ and $(m,p,n) \notin \{(3,3,3),(2,2,4)\}$, then for some divisor q of p we have $G = G(m,q,n)$.*

(ii) *If $n = 2$ and $(m,p,2) \neq (4,2,2)$, then G is contained in $G(2m,2,2)$.*

PROOF. (i) Supposed that $n \geq 3$. Then the group $G(m,p,n)$ have a only system of imprimitivity $\{\mathbf{C}e_1,\dots,\mathbf{C}e_n\}$ that is preserved by G (Theorem 3.7). And $G = G(k,q,n)$ for some k and q with q divisor of k by Theorem 3.4. If θ is a k^{th} root of 1, then $h := \operatorname{diag}(\theta^{-1},\theta,1,\dots,1) \in G$ and if we take $r \in G(m,p,n)$ to be the reflection that interchanges e_1 and e_3. Then

$$hrh^{-1}(e_1) = hr(\theta e_1) = h(\theta e_3) = \theta e_3 \quad \text{and} \quad hrh^{-1}(e_3) = hr(e_3) = h(e_1) = \theta^{-1}e_1$$

thus $hrh^{-1} \in G(m,p,n)$. It follows that θ is a m^{th} root of 1 and hence $k = m$. Note that by Lemma 3.3 we have that $A(m,p,n) \subseteq A(m,q,n)$. Therefore $G(m,p,n) \trianglelefteq G(m,q,n)$.

(ii) Firts, note that $G(2m,2,2) = \left\langle r, \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix} \mid r \in G(m,1,2) \right\rangle$ where θ is a $2m^{th}$ root of 1. Let's see this, if $\zeta_i^m = 1$ where $i = 1,2$, then

$$\begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix} = \begin{bmatrix} \zeta_1\theta & 0 \\ 0 & \zeta_2\theta \end{bmatrix}$$

then $(\zeta_i\theta)^{2m} = 1$ and $(\zeta_1\zeta_2\theta^2)^m = 1$, then $\left\langle r, \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix} \mid r \in G(m,1,2) \right\rangle \subseteq G(2m,2,2)$.

Let $x \in A(2m,2,2)$, because of that $x = \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix}$ and note that x can be written as

$$x = \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} = \begin{bmatrix} \zeta_1' & 0 \\ 0 & \zeta_2' \end{bmatrix} \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}$$

where $(\zeta_i')^m = 1$, $(\zeta_1'\zeta_2')^m = 1$ and $\theta^{2m} = 1$. Then $G(2m,2,2) \subseteq \left\langle r, \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix} \mid r \in G(m,1,2) \right\rangle$.

Let $r = \operatorname{diag}(e^{2\pi i p/m},1) \in G(m,p,1)$ and $h = \operatorname{diag}(e^{2\pi i/m},1) \in G(2m,2,2)$, then $hrh^{-1} \in G(m,p,2)$.

□

Remark 7.

2. Imprimitive subgroups of primitive reflection groups

THEOREM 3.8. *Suppose that G is a finite primitive group of rank at least 3 . If H is a reflection subgroup of rank 2 that acts irreducibly and imprimitively on its support, then H is conjugate to one of* $G(3,1,2)$, $G(3,3,2)$, $G(4,2,2)$, $G(2,1,2) \simeq G(4,4,2)$ *or* $G(5,5,2)$. *If H contains a reflection of order 3 , then* $H \simeq G(3,1,2)$.

PROOF. By the Theorem 3.4 it follows that H is conjugate to $G(m,p,2)$ for some p divisor of m. Thus H contains an element with eigenvalue $\exp(2\pi i/m), \exp(-2\pi i/m)$ and 1. By Blichfeldt's Theorem we have $m \leq 5$. And if $p \neq m$, then H contains a reflection of order m/p and its order is at most 3, then $m/p \leq 3$, thus $m \leq 3p$. The group $G(m,p,2)$ with $m \leq 5$ and $m \leq 3p$ is irreducible except for $G(2,2,2)$. □

THEOREM 3.9. *Suppose that G is a primitive unitary reflection group in* $U(V)$ *and that W is a proper subspace of V of dimension at least two. Suppose that H is a subgroup of G that fixes* $W^{\perp}$ *pointwise and which acts on W as an irreducible imprimitive reflection group not conjugate to* $G(2,1,2), G(3,3,2), G(4,4,2)$, $G(4,2,2), G(2,2,3), G(3,3,3)$ *nor* $G(2,2,4)$. *Then there is a reflection* $r \in G$ *with root* $a \notin W$ *such that the action of* $\langle H, r\rangle$ *on* $W \oplus \mathbf{C}a$ *is primitive.*

PROOF. Let $m := \dim W$ with $1 < m < n$. Due to the restrictions in the hypotheses about H and the theorem 3.7 there is only one system of imprimitivity $W_1, \ldots, W_m$ for H acting on W.

Suppose that $r \in G$ is a reflection, with root a, that does not fix W and suppose that the action of $\langle H, r\rangle$ on $W \oplus \mathbf{C}a$ is imprimitive .

Let $\Omega := \{V_1, \ldots, V_{m+1}\}$ be the system of impritmitivity for $\langle H, r\rangle$ and suppose first that $V_i \nsubseteq W$ for all i. Then by the lemma 1.6 the reflections in H act in Ω as transpositions and by the theorems 3.4-3.5 we have that H acts on W as $W(A_m) = G(1,1,m+1)$. Since we suppose that the action of H on W is primitive, we have that $m \leq 3$. But then H is conjugated to $G(3,3,2)$ or $G(2,2,3)$ which is a contradiction by Theorem 3.8. Therefore $V_i \subseteq W$ for all i.

Since W is an irreducible H-module, the images of V_i under H span W and form a system of imprimitivity for H which, by uniqueness, must be $\{W_1, \ldots, W_m\}$. Thus $\Omega = \{W_1, \ldots, W_m, W_{m+1}\}$, where $W_{m+1} := W^{\perp} \cap rW$ and we may choose the notation so that $rW_m = W_{m+1}$. In particular, $\langle H, r\rangle$ acts irreducibly on $W \oplus \mathbf{C}a$ and we note that by Lemma 1.6, $r^2 = 1$. If $W_m = \mathbf{C}w_m$, where $(w_m, w_m) = 1$, then $(w_m, rw_m) = 0$ and

$$\left(w_m, rw_m\right) = \left(w_m, w_m - 2(w_m, a)a\right) = \left(w_m, w_m\right) - 2\overline{(w_m, a)}(w_m, a)$$

$$0 = 1 - 2\left|(w_m, a)\right|^2$$

therefore $\left|(w_m, a)\right|^2 = \frac{1}{2}$.

If $\dim V = m+1$, then the primitivity of G ensures that there is a reflection r that does not fix the set $\left\{W_1, W_2, \ldots, W_m, W^{\perp}\right\}$. The considerations of the previous paragraph show that $\langle H, r\rangle$ must be primitive in this case.

Thus from now on we may suppose that $\dim V > m+1$. By induction on $\dim V - m$ we may choose a reflection s with root b such that $\langle H, r, s\rangle$ is primitive and, in particular, such that s does not fix $W \oplus \mathbf{C}a$. The group $\langle H, r\rangle$ acts transitively on $\{W_1, \ldots, W_{m+1}\}$ and so, on replacing s by a suitable conjugate, we may suppose that $sW_m \nsubseteq W \oplus \mathbf{C}a$. If $\langle H, s\rangle$ is not primitive on $W \oplus \mathbf{C}b$, then the results obtained above for r and a imply that $\left|(w_m, b)\right|^2 = 1/2$. Similarly, if $\langle H, rsr\rangle$ is not primitive on $W \oplus \mathbf{C}rb$ and if $rsrW_m \neq W_m$, then $\left|(w_m, rb)\right|^2 = 1/2$. On the other hand, if $rsrw_m = w_m$, then $(w_m, rb) = 0$. Putting $w_{m+1} := rw_m$ we may write $b := \kappa w_m + \lambda w_{m+1} + \mu v$, where $(v, v) = 1$ and v is orthogonal to w_i for $1 \leq i \leq m+1$. Then

$$1 = (b, b) = |\kappa|^2 + |\lambda|^2 + |\mu|^2.$$

Now $\kappa = (b, w_m)$ and $\lambda = (b, w_{m+1}) = (rb, w_m)$. Thus $\left|(w_m, rb)\right|^2 = 1/2$ implies $\mu = 0$ and hence $b \in W \oplus \mathbf{C}a$, which is a contradiction. It follows that $(w_m, rb) = 0$. But now $sw_m = v$ and hence $\langle H, r, s\rangle$ is imprimitive, contradicting the choice of s. Thus there is a reflection r such that $\langle H, r\rangle$ is primitive.

□

3. Generators for $G(m,p,n)$

In this section determined the minimal set of generating reflection for the groups $G(m,p,n)$.

For $i := 1, \ldots, n-1$ let $r_i := r_{e_i - e_{i+1}, -1}$ be the reflection of order 2 that interchanges the basis vectors e_i and e_{i+1} and fixes e_j for $j \neq i, i+1$. Let $t := r_{e_1, \zeta_m}$ be the reflection of order m that fixes $e_2, \ldots, e_n$ and sends e_1 to $\zeta_m e_1$ and let $s := t^{-1} r_1 t$ be the reflection of order 2 that interchanges e_1 and $\zeta_m e_2$. For $m > 1$, the groups $G(m,m,n)$ and $G(m,1,n)$ can be generated by n reflections; however, the minimum number of reflections required to generate $G(m,p,n)$, for $p \neq 1, m$ is $n+1$. In particular, we have

$$G(1,1,n) = \langle r_1, r_2, \ldots, r_{n-1}\rangle \simeq \mathrm{Sym}(n)$$

$$G(m,m,n) = \langle s, r_1, r_2, \ldots, r_{n-1}\rangle$$

$$G(m,1,n) = \langle t, r_1, r_2, \ldots, r_{n-1}\rangle, \quad \text{and}$$

$$G(m,p,n) = \langle s, t^p, r_1, r_2, \ldots, r_{n-1}\rangle \quad \text{for } p \neq 1, m.$$

EXAMPLE 6. *consider the group $G(2,1,3)$, we have*

$$G(2,1,3) = \langle t, r_1, r_2\rangle = \left\langle \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle$$

3.1. Consequences of Blichfeldt's theorem.

LEMMA 3.4. *Suppose that G is a finite primitive group acting on the space V and that $\dim V \geq 3$. If H is reflection subgroup of rank 2 and if the action of H on its support is primitive, then H may be identified with the group $G_4 \simeq \mathrm{SL}_2(\mathbf{F}_3)$ or $G_6 \simeq \mathcal{C}_4 \circ \mathrm{SL}_2(\mathbf{F}_3)$.*

PROOF. Let W a subspace of V spanned by the roots of the reflection in H.

H is a subgroup of rank 2 so the dimension of the support M is 2 where $M = \mathbf{C}\alpha \oplus \mathbf{C}\beta$ with α, β are root of tow reflections in H and H acting on M is primitive. Suppose that $\mathcal{T}$ are subgroup of H, we have $a = (1/2)(1+i+j+k) \in \mathcal{T}$ is of order 6 and $\det a = 1$, and for some basis

$$a = \begin{bmatrix} -\omega & 0 & 0 \\ 0 & -\omega^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with eigenvalues $-\omega$ and $-\omega^2$ show that $|-\omega - 1| \leq \pi/3$ and $|-\omega^2 - 1| \leq \pi/3$ and so by Blichfeld's theorem $a \in Z(G)$, and therefore $a \in \mathcal{T}$, the which is impossible sice $\mathcal{T}$ has center $\{1, -1\}$.

From the list of the 19 primitive reflection subgroup of $\mathrm{U}_2(\mathbf{C})$ this leaves only $G_4 \simeq \mathrm{SL}_2(\mathbf{F}_3)$ and $G_6 \simeq \mathcal{C}_4 \circ \mathrm{SL}_2(\mathbf{F}_3)$ are candidates for H by are the only primitive groups that non containing a $\mathcal{T}$. $\square$

Remark 8. *Note that G_{12} no containing $\mathcal{T}$ but containing a conjugates to $\mathcal{T}$. Remember that G_{12} are of order 48 and have $|Z(G_{12})| = 2$. We have $G_{12} := \langle r_3, r_3', r_3'' \rangle$ where*

$$r_3 := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}, \quad r_3' := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad r_3'' := \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1-i \\ 1-i & 0 \end{bmatrix}$$

and $\mathcal{T} := \langle a, b \rangle$ where

$$a := \frac{1}{2} \begin{bmatrix} -1-i & 1-i \\ -1-i & -1+i \end{bmatrix} \longmapsto \varpi,$$

LEMMA 3.5. *Consider G as in the lemma 3.4. Then the reflections in G have order 2 or 3.*

PROOF. Suppose that $t \in G$ is a reflection of order at least 4 and α be a root of t. Since G is primitive, the subspace $\mathbf{C}g\alpha$ (for $g \in G$) they cannot all be orthogonal to each, if so to form a system of imprimitivity for G.

Without loss of generality, there is $g \in G$ such that v and $w := gv$ are linearly independent and no orthogonal. Let K the subgroup generate by t and gtg^{-1} (of order 4). Also K acting on $\mathbf{C}\{v, w\}$ irreducibly.

K is not primitive, if it were it would be a group generated by two reflections of order 4.

Then K is primitive in its support W, then for lemma 3.4, K is conjugated to G_4 or G_6 and these only contain reflections of order at most 3. $\square$

THEOREM 3.10. *Let G and H be as in the lemma 3.4. Then H is conjugate to G_4.*

PROOF. Consider Lemmas 3.4-3.5. Let W be the subspace spanned by the roots of the reflections in H. And suppose that s is a reflection with root $a \in W \cup W^{\perp}$. The group $K = \langle H, s\rangle$ acts irreducibly on $W \oplus \mathbf{C}a$ and then K is primitive (Theorem3.5). If $H \simeq \mathcal{C}_4 \circ \mathrm{SL}_2(\mathbf{F}_3)$, then K contains an elements conjugate to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}$. But then the group $\langle K, iI\rangle$ is primitive and contains a reflection of order 4 contrary to Lemma 3.5. Therefore $H \simeq \mathrm{SL}_2(\mathbf{F}_3)$. □

COROLLARY 3.3. *Suppose that G is a finite primitive unitary reflection group of rank at least 3 . If G contains no reflections of order 3 , then the set of lines spanned by the roots of the reflections is an m-system for some $m \leq 5$.*

See chapter 4 section 3.

THEOREM 3.11. *If H is a primitive reflection subgroup of the finite primitive group G acting on the space V of dimension $n \geq 3$ and if the rank of H is less than n, then $|Z(H)| \leq 3$*

CHAPTER 4

Line systems for imprimitive and primitive reflection groups

1. Bounds on line systems

This section presents a bound on the size of a line system in $\mathbf{C}^n$ that has a prescribed number of angles. An important consequence of the theorem is that if the set of angles between the lines of a system is finite, then the system itself is finite.

THEOREM 4.1. *Let Σ be a set of unit vectors that span distinct one-dimensional subspaces in $\mathbf{C}^n$. If $A := \left\{|(u,v)|^2 \mid u,v \in \Sigma, u \neq v\right\}$ and $s := |A|$, then*

$$|\Sigma| \leq \begin{cases} \begin{pmatrix} n+s-1 \\ s \end{pmatrix}\begin{pmatrix} n+s-2 \\ s-1 \end{pmatrix} & \text{if } 0 \in A \\ \begin{pmatrix} n+s-1 \\ s \end{pmatrix}^2 & \text{if } 0 \notin A \end{cases}$$

2. Star-closed line systems

A **line system** is a collection of lines through the origin of $\mathbf{C}^n$. Given $a \in \mathbf{C}^n$ we say that is a long root of the line $\ell := \mathbf{C}a$ if $(a,a) = 2$. Define the angle between two lines $\ell := \mathbf{C}a$ and $m := \mathbf{C}b$ as the angle θ such that

$$\cos\theta := \frac{|(a,b)|}{\sqrt{(a,a)(b,b)}}$$

The lines ℓ and m are said at

$$45^\circ \quad \text{if only if} \quad |(a,b)| = \sqrt{2},$$

$$60^\circ \quad \text{if only if} \quad |(a,b)| = 1,$$

$$90^\circ \quad \text{if only if} \quad (a,b) = 0$$

If $\ell := \mathbf{C}a$ is a line, there is a unique reflection r_ℓ of order two that fixes $\ell^\perp$ pointwise and the action on $v \in \mathbf{C}^n$ is given by $r_\ell(v) := v - (v,a)a$ whit a a long root.

DEFINITION 4.1. (1) *A line system $\mathcal{L}$ is said* **star-closed** *if $r_\ell(m) \in \mathcal{L}$ for all lines ℓ, m in $\mathcal{L}$.*

(2) *A* **star** *in $\mathbf{C}^n$ is finite, coplanar star-closed line system. A star of k lines is called a k-star.*

LEMMA 4.1. *Given lines $\ell = \mathbf{C}a$ and $m = \mathbf{C}b$ in $\mathbf{C}^n$ at 60°, there is a unique line n in the plane of ℓ and m at 60° to both ℓ and m. The $r_b(a) = -(a,b)r_b(a)$ is a root of n and hence $\{\ell, m, n\}$ are 3-star.*

PROOF. Firt, let us see that $r_b(a) = -(a,b)r_a(b)$. We have the action is given by

$$r_a(b) = b - (b,a)a$$

then $(a,b)r_b(a) = (a,b) - a$ from which we get $a - (a,b)b = r_b(a) = -(a,b)r_a(b)$.

Let's see now that $c = \mathbf{C}r_b(a)$ at 60° to both ℓ and m. Note that

$$(r_a(b), a) = (b - (b,a)a, a) = -(b,a)$$

and $(r_b(a), a) = (-(a,b)r_a(b), a) = 1$. Similar $(r_b(a), b) = -(a,b)$. Therefore, there is a line n that is at 60° to both ℓ and m.

Finally, the uniqueness. Suppose that $c = \alpha a + \beta b$ is a root and that c at 60° to both a and b. We assume that $(a,b) = -1$, $(a,c) = -1$ and $(b,c) = -1$. Then

$$(a,c) = (a, \alpha a + \beta b) = 2\overline{\alpha} - \overline{\beta} = -1$$

$$(b,c) = (b, \alpha a + \beta b) = \overline{\alpha} - 2\overline{\beta} = -1$$

It follows that $\overline{\alpha} = \overline{\beta} = -1$ so $\alpha = \beta = -1$. Note that

$$c = -a - b = -r_a(b) = -(b - ((b,a)a) = -a - b.$$

□

DEFINITION 4.2. *A line system $\mathcal{L}$ is a k-**system** if*

(1) *$\mathcal{L}$ is star-closed,*
(2) *for all $\ell, m \in \mathcal{L}$ the order of $r_\ell r_m$ is at most k, and*
(3) *there exist $\ell, m \in \mathcal{L}$ such that the order of $r_\ell r_m$ equals k.*

LEMMA 4.2. *If a and b are long roots such that the $|r_a r_b| = m \in \mathbf{Z}$, then there a finite number of possible values for m. In particular,*

(1) *$\left|r_a r_b\right| = 2$ if and only if $(a,b) = 0$,*
(2) *$\left|r_a r_b\right| = 3$ if and only if $|(a,b)| = 1$*
(3) *$\left|r_a r_b\right| = 4$ if and only if $|(a,b)| = \sqrt{2}$, and*
(4) *$\left|r_a r_b\right| = 5$ if and only if $|(a,b)| = \tau$ or τ^{-1}, where $\tau = \frac{1}{2}(1+\sqrt{5})$.*

PROOF. Suppose that a and b are linearly independent. The reflections r_a and r_b fixes the subspace $\langle a, b\rangle := \mathbf{C}a \oplus \mathbf{C}b$ and act as the identity on $\langle a, b\rangle^\perp$.

The matrices of r_a and r_b restricted to $\langle a, b\rangle$ are

$$A := \begin{bmatrix} -1 & -(b,a) \\ 0 & 1 \end{bmatrix} \text{ and } B := \begin{bmatrix} 1 & 0 \\ -(a,b) & -1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} |(a,b)|^2 - 1 & (b,a) \\ -(a,b) & -1 \end{bmatrix}$$

and $|r_a r_b| = \mathrm{ord}(AB) = m$. The eigenvalues of AB are m-roots of unit of the form $\exp(\pm 2\pi hi/m)$ for some h coprime to m and $1 \le h < m$. Also

$$\mathrm{Tr}(AB) = |(a,b)|^2 - 2 = \exp(2\pi hi/m) + \exp(-2\pi hi/m) = 2\cos\left(\frac{2\pi h}{m}\right).$$

and $2\cos\left(\frac{2\pi h}{m}\right) = 2\left(2\cos^2(h\pi/m) - 1\right) = 4\cos^2(h\pi/m) - 2$, therefore $|(a,b)| = 2|\cos(h\pi/m)|$. This gives the result.

□

From the previous lemma it can be seen that a 3-system is composed of reflections with roots that are either orthogonal or at 60°.

In the same way, as should be familiar, we will say that, a line system $\mathcal{L}$ is **decomposable** if there tow subset A and B non-empty such that every line in A is orthogonal to every line in B. We write $\mathcal{L} = A \perp B$. It is said that $\mathcal{L}$ is **indecomposable** if it is not decomposable. To indicate that $\mathcal{L}$ are the orthogonal sum k copies of of the system line $\mathfrak{M}$ we write $\mathcal{L} = k\mathfrak{M}$.

The following $W(\mathcal{L})$ denote the group generate by reflections r_ℓ, where $\ell \in \mathcal{L}$. All line system $\mathcal{L}$ is an orthogonal sum

$$\mathcal{L} = \mathcal{L}_1 \perp \mathcal{L}_2 \perp \ldots \perp \mathcal{L}_k$$

of indecomposable line systems indecomponibles $\mathcal{L}_j$. The line subsystems $\mathcal{L}_j$ characterised as indecomposable maximal subsystems of $\mathcal{L}$ and $\mathcal{L}$ is star-closed if only if $\mathcal{L}_j$ is star-closed for all $1 \le j \le k$. To see this we can consider the case $k = 2$, i.e., $\mathcal{L} = \mathcal{L}_1 \perp \mathcal{L}_2$. If we take lines $\ell, \ell' \in \mathcal{L}$ such that $\ell, \ell' \in \mathcal{L}_1$ where $\ell = \mathbf{C}\alpha$ and $\ell' = \mathbf{C}\alpha'$ so

$$r_\ell(\alpha') = \alpha' - (\alpha', \alpha)\alpha \in \mathcal{L}$$

and all lines of $\mathcal{L}_1$ are orthogonal to $\mathcal{L}_2$ this gives $r_\ell(\ell') \in \mathcal{L}_1$ then $\mathcal{L}_1$ is star-closed. Symmetrically, we can see that $\mathcal{L}_2$ are star-closed and this argument works for $1 \le j \le k$. If considering $W(\mathcal{L}) = \langle r_\ell \mid \ell \in \mathcal{L} \rangle$ where $\mathcal{L} = \mathcal{L}_1 \perp \mathcal{L}_2 \perp \ldots \perp \mathcal{L}_k$ and if we take $W(\mathcal{L}_j) = \langle r_{\ell_j} | \ell_j \in \mathcal{L}_j \rangle$ are the restriction of $W(\mathcal{L})$ to $\mathcal{L}_j$ and and acts irreducibly and so $W(\mathcal{L}) = W(\mathcal{L}_1) \times W(\mathcal{L}_2) \times \cdots \times W(\mathcal{L}_k)$, which is equivalent to the theorem 1.5 see [**9**, Th. 1.27].

THEOREM 4.2. *If $\mathcal{L}$ is a k-system, then $W(\mathcal{L})$ is a finite unitary reflection group.*

PROOF. Let $\mathcal{L}$ a k-system in $\mathbf{C}^n$, we know that it is a finite line system. The group $W(\mathcal{L}) = \langle r_\ell | \ell \in \mathcal{L} \rangle$ act Permuting the lines of $\mathcal{L}$ that is finite number, so it is sufficient to prove that the subgroup that fixes $\mathcal{L}$ is finite. For this let us assume that $\mathcal{L}$ is indecomposable. Let $r_\ell \in W(\mathcal{L})$ such that if $m \in \mathcal{L}$ then $r_\ell(m) = m$, as $r_\ell r_m r_\ell^{-1} = r_{\ell m} = r_m$ which is the case if and only if r_ℓ is a scalar matrix. Now if $r \in W(\mathcal{L})$ is a reflection, then $\det(r) = -1$ and as r_ℓ is a scalar matrix, for some basis

$$r_\ell = \begin{bmatrix} \zeta & & & \\ & \zeta & & \\ & & \ddots & \\ & & & \zeta \end{bmatrix}$$

then $\det(r_\ell) = \zeta^n = \pm 1$ so $|r|$ divide $2n$ therefore $W(\mathcal{L})$ are finite. □

THEOREM 4.3. *If G is a finite unitary reflection group generate by reflection of order two and k is the maximum of the order of products of pairs of reflections of G, then $G = W(\mathcal{L})$ for some k-system $\mathcal{L}$.*

PROOF. Let G a finite unitary reflection group generate by reflection of order two and k the maximum order of the product of two reflections. Let

$$\mathcal{L} = \langle \mathbf{C}\alpha \mid \alpha \text{ is root of a the reflections in } G \rangle.$$

First, let's see that $\mathcal{L}$ is star-closed. Let $\ell = \mathbf{C}a$ and $m = \mathbf{C}b$ then r_a y r_b are in G because they are reflections, Thus we have that $r_a r_b r_a^{-1} = r_{ab} \in G$ (by (ii) of lemma 1.4) then $n = \mathbf{C}ab \in \mathcal{L}$ therefore $\mathcal{L}$ is a line system star-closed. Furthermore is a k-system for $\mathcal{L}$ is spanned by the roots the reflections in the G, Thus, the maximum order of $r_\ell r_m$ for $\ell, m \in \mathcal{L}$ is k where $|r_\ell r_m| = k$. By construction $G = W(\mathcal{L})$. □

THEOREM 4.4. *A 3-system is indecomposable if and only if $W(\mathcal{L})$ is transitive on $\mathcal{L}$.*

PROOF. Suppose that ℓ and m are lines of $\mathcal{L}$ with roots a y b respectability. Since $\mathcal{L}$ is indecomposable we can assume $(a, b) \neq 0$, then a and b span a third line of the 3-star with root that is at 60° of a and of b and that permute ℓ with m.

If $\mathcal{L}$ is indescomposable, there are roots $a = a_0, a_1, \ldots, a_k = b$ such that $(a_j, a_{j+1}) \neq 0$. Then there a element of $W(\mathcal{L})$ that take ℓ to m and it tells us that $W(\mathcal{L})$ permute the elements of $\mathcal{L}$, i.e., is transitive.

Supposed that $\mathcal{L} = A \perp B$ where A and B are star-closed and not empty. If a is a root in A then r_a fixed A and B therefore $W(\mathcal{L})$ not is transitive on $\mathcal{L}$.

□

LEMMA 4.3. *Let $\mathcal{L}$ a star-closed line system on $\mathbf{C}^n$ and ℓ a line in $\mathbf{C}^n$. Then the set of lines of $\mathcal{L}$ orthogonal to ℓ are star-closed.*

PROOF. Let set $\mathcal{L}' := \{m \in \mathcal{L} \mid r_m(\ell) = \ell\}$ and let $\ell = \mathbf{C}a$, $m_1 = \mathbf{C}b_1$ and $m_2 = \mathbf{C}b_2$ then $(b_1, a) = (b_2, a) = 0$ so

$$r_{b_1}(a) = a, \qquad r_{b_2}(a) = a$$

then $r_{b_2}(r_{b_1}(a)) = a$. Therefore $\mathcal{L}'$ is star-closed. □

DEFINITION 4.3. *Given a star-closed line system $\mathcal{L}$, the star-closure X^* of a subset $X \subset \mathcal{L}$ is definite by $X^* := \mathcal{L}_1 \cap \mathcal{L}_2 \cap \ldots \cap \mathcal{L}_k$ where each $\mathcal{L}_j$ all star-closed line system that contain X for $1 \le j \le k$.*

Given a 3-system $\mathcal{L}$ and $X \subseteq \mathcal{L}$, to obtain the star-closed X^* a third line of 3-star is added for each pair of lines that are at 60°. The group $W(X^*)$ is generate for the reflections r_a where a runs through the roots of X.

DEFINITION 4.4. (1) *Given star-closed line systems $\mathcal{L} \subseteq \mathfrak{M}$ in $\mathbf{C}^n$. We say that $\mathfrak{M}$ is a simple extension of $\mathcal{L}$ if $\mathcal{L} \neq \mathfrak{M}$ and $\mathfrak{M}$ is the star-closure of $\mathcal{L} \cup \{\ell\}$ for some $\ell \in \mathfrak{M}$.*
(2) *We say that $\mathfrak{M}$ is the minimal extension of $\mathcal{L}$ if for each line systems $\mathfrak{U}$ such that $\mathcal{L} \subseteq \mathfrak{U} \subsetneq \mathfrak{M}$ we have $\mathcal{L} = \mathfrak{U}$.*

THEOREM 4.5. *If $\mathfrak{L}$ and $\mathfrak{M}$ are indecomposable k-systems and $\mathfrak{L} \subset \mathfrak{M}$, then there is a sequence $\mathfrak{L} = \mathfrak{L}_0 \subset \mathfrak{L}_1 \subset \cdots \subset \mathfrak{L}_h = \mathfrak{M}$ of indecomposable star-closed line systems such that for $1 \le j \le h, \mathfrak{L}_j$ is a simple indecomposable extension of $\mathfrak{L}_{j-1}$.*

3. Line system for imprimitive reflection groups

In this define the line systems associated to the complex reflection groups $G(m, p, n)$, where $n > 1$. For this we consider $e_1, e_2, \ldots, e_n$ an orthonormal basis for $\mathbf{C}^n$. Let μ_m denote the group of m^{th} roots of unity in $\mathbf{C}$.

DEFINITION 4.5. *For $n \ge 2$ we define*

$$\mathcal{D}_n^{(m)} := \{\mathbf{C}(e_i - \zeta e_j) \mid \zeta \in \mu_m, 1 \le i < j \le n\}$$

as the line system associated to the group $G(m, m, n)$.

Remember that the reflections on $G(m, p, n)$ have order 2 if only if $m = p$ or $m = 2p$. And the product of any two of these reflections has at most order three if and only if $m = p$ and $m \le 3$. Note that the roots of the lines of $\mathcal{D}_n^{(m)}$ are of order m and contains $m\binom{n}{2}$ roots of the form $e_i - \zeta e_j$ with $1 \le i < j \le n$.

The line system associated to $G(2, 2, n)$ is $\mathcal{D}_n^{(2)} := \{\mathbf{C}(e_i \pm e_j) \mid 1 \le i < j \le n\}$ which consists of $n(n-1)$ roots. And $\mathcal{D}_n^{(2)}$ is indecomposable for $n \ge 3$. We first see the case $n = 2$ is decomposable.

$$\mathcal{D}_2^{(2)} = \{\mathbf{C}(e_1 \pm e_2)\}$$

if we choose $\ell = \mathbf{C}(e_1+e_2)$ and $\ell' = \mathbf{C}(e_1-e_2)$ then $r_\ell(\ell') = \ell'$, this is due to

$$r_{e_1+e_2}(e_1-e_2) = e_1-e_2-(e_1-e_2, e_1+e_2)(e_1+e_2) = e_1-e_2$$

and also r_ℓ fixed a subspace $W \subseteq \mathcal{D}_2^{(2)}$ iff $r_\ell \in W$ or $r_\ell \in W^\perp$ and as $(e_1-e_2, e_1+e_2) = 0$ then

$$\mathcal{D}_2^{(2)} = \{\mathbf{C}(e_1+e_2)\} \perp \{\mathbf{C}(e_1-e_2)\}.$$

See the case $n \geq 3$. Consider $\ell = \mathbf{C}(e_i \pm e_j)$ and $\ell' = \mathbf{C}(e_h \pm e_k)$ then

$$\begin{aligned} r_{e_i \pm e_j}(e_h \pm e_k) &= e_h \pm e_k - (e_h \pm e_k, e_i \pm e_j)(e_i \pm e_j) \\ &= e_h \pm e_k - \Big[(e_h, e_i) \pm (e_h, e_j) \pm (e_k, e_i) \pm (e_k, e_j)\Big] \end{aligned}$$

where $1 \leq i < j \leq n$ y $1 \leq h < k \leq n$. There are the following cases

(1) If $i = h$ and $j = k$

$$r_{e_i \pm e_j}(e_h \pm e_k) = e_i \pm e_j - [1 \pm 1](e_i \pm e_j)$$

then fix it or change sign.

(2) If $i = h$ and $j \neq k$

$$r_{e_i \pm e_j}(e_h \pm e_k) = e_i \pm e_k - (e_i \pm e_j) = \pm e_k \mp e_j$$

changes coordinate and sign. It is similar for $i \neq h$ and $j = k$.

(3) If $i \neq h$ and $j \neq k$

$$r_{e_i \pm e_j}(e_h \pm e_k) = e_i \pm e_j$$

the fixed.

Therefore $\mathcal{D}_n^{(2)}$ is indecomposable and also $\mathcal{D}_n^{(2)}$ is star-closed for $n \geq 3$. We can also see that $\mathcal{D}_n^{(2)}$ for $n \geq 3$ is a 3-system.

LEMMA 4.4. *The line system $\mathcal{D}_n^{(2)}$ are indecomposable and star-closed for $n \geq 3$.*

Remark 9. *The line systems $\mathcal{D}_n^{(2)}$ also known as lattice (see [**5, 7**]). The elements of $\mathcal{D}_n^{(2)}$ are such that $(x_1, \ldots, x_n) \in \mathbf{Z}^n$ and $\sum_i x_i \in 2\mathbf{Z}$, i.e., they are of the form*

$$v = (0, \ldots, 0, 1, 0, \ldots, 0, \pm 1, 0, \ldots, 0)$$

and $(v, v) = 2$. If $v = e_i + e_j$ with $i \neq j$ so

$$r_v(e_i) = -e_j, \qquad r_v(e_j) = -e_i \quad \text{and} \quad r_v(e_k) = e_k \quad k \neq i, j.$$

Then $\langle r_{e_i \pm e_j} \rangle \simeq W(\mathcal{D}_n^{(2)})$ permute $\{e_i \pm e_j \mid 1 \leq i < j \leq n\}$.

DEFINITION 4.6. *The associate line system to $G(1,1,n) = \mathrm{Sym}(n)$ is*

$$\mathcal{A}_{n-1} := \{\mathbf{C}(e_i - e_j) \mid i < j\}$$

that contains $\binom{n}{2} = n(n-1)$ roots.

It is clear that the elements of $\mathcal{A}_{n-1}$ sastisfy the conditions set forth in the remark 9 and if $v = e_i - e_j$ then

$$r_v(e_i) = e_j, \qquad r_v(e_j) = e_i \quad \text{y} \quad r_v(e_k) = e_k \quad k \neq i, j.$$

where do you have to $W(\mathcal{A}_{n-1}) = \langle r_{e_i - e_j} \rangle$ acts with $\mathrm{Sym}(n)$ over $\langle e_1 + \cdots + e_n \rangle$.

As they are transitive line systems, we have the following Lemma of the theorem 4.4.

LEMMA 4.5. *The line systems $\mathcal{D}_n^{(2)}$ $(n \geq 3)$, $\mathcal{D}_n^{(3)}$ $(n \geq 2)$ and $\mathcal{A}_n$ $(n \geq 1)$ are 3-systems.*

PROOF. Remember that line systems $\mathcal{D}_n^{(2)}$ $(n \geq 3)$, $\mathcal{D}_n^{(3)}$ $(n \geq 2)$ and $\mathcal{A}_n$ $(n \geq 1)$ are associate to reflection groups $G(2,2,n)$, $G(3,3,n)$ and $\mathrm{Sym}(n+1)$ respectively.

Let ζ be an m root of unity. Consider the reflections $\zeta_i^k (ij) \zeta_i^{-k}$ and $\zeta_a^e (ab) \zeta_a^{-e}$

$$\zeta_i^k (ij) \zeta_i^{-k} \zeta_a^e (ab) \zeta_a^{-e} = \zeta_i^k \zeta_a^e (ij)(ab) \zeta_i^{-k} \zeta_a^{-e}$$

If (ij) and (ac) are disjoint, then $\zeta_i^k (ij) \zeta_i^{-k} \zeta_a^e (ab) \zeta_a^{-e}$ has order two. And if $(ij) = (ac)$, then

$$\zeta_i^k (ij) \zeta_i^{-k} \zeta_i^e (ij) \zeta_j^{-e} = \zeta_i^k \zeta_j^{-k} (ij)(ij) \zeta_j^e \zeta_i^{-e} = \zeta_i^k \zeta_j^{-k} \zeta_j^e \zeta_i^{-e} = \zeta_i^{k-e} \zeta_j^{-k+e}$$

this reflection has to most order m and if $\gcd(m, k-e) = 1$ has order m. Therefore for $m = 2$ or 3, then $\mathcal{D}_n^{(m)}$ is a 3-system. And if $m \geq 4$, then $\mathcal{D}_n^{(m)}$ is a m-system. $\square$

EXAMPLE 7. *Note that*

$$\zeta_1^k (12) \zeta_1^{-k} \zeta_2^a (23) \zeta_2^{-a} = \zeta_1^k \zeta_2^{-k} \zeta_1^a \zeta_3^{-a} (12)(23) = \zeta_1^k \zeta_2^{-k} \zeta_1^a \zeta_3^{-a} (123) = \zeta_1^k \zeta_3^{-a} (123) \zeta_1^{-k} \zeta_3^a$$

has order 3.

DEFINITION 4.7. *Two line systems $\mathcal{L}_1$ y $\mathcal{L}_2$ we say to be equivalent if there is a unitary transformation $\psi : \langle \mathcal{L}_1 \rangle \longrightarrow \langle \mathcal{L}_2 \rangle$ such that $\psi(\mathcal{L}_1) = \mathcal{L}_2$ and we denote for $\mathcal{L}_1 \simeq \mathcal{L}_2$.*

EXAMPLE 8. *Observe that $\mathcal{A}_2 = \{e_1 - e_2, e_1 - e_3, e_2 - e_3\}$ and $\mathcal{D}_2^{(3)} = \{e_1 - e_2, e_1 - \omega e_2, e_1 - \omega^2 e_2\}$ we can define $\psi : \mathcal{A}_2 \longrightarrow \mathcal{D}_2^{(3)}$ such that*

$$\psi(e_1 - e_2) = e_1 - e_2, \quad \psi(e_1 - e_3) = e_1 - \omega e_2 \quad \text{and} \quad \psi(e_2 - e_3) = e_1 - \omega^2 e_2$$

then $\mathcal{A}_2 \simeq \mathcal{D}_2^{(3)}$. Similarly, it can be seen that $\mathcal{A}_3 \simeq \mathcal{D}_3^{(2)}$ and $\mathcal{D}_2^{(2)} \simeq 2\mathcal{A}_1$.

DEFINITION 4.8. *The associate line systems to $G(2p, p, n)$ we define by*

$$\mathcal{B}_n^{(2p)} := \mathcal{D}_n^{(2p)} \cup \{\sqrt{2} e_i \mid 1 \leq i \leq n\}$$

EXAMPLE 9.

$$\mathcal{B}_2^2 = \mathcal{D}_2^2 \cup \{\sqrt{2} e_1, \sqrt{2} e_2\} = \{\sqrt{2} e_1, \sqrt{2} e_2, e_1 \pm e_2\} \simeq \mathcal{D}_2^{(4)}$$

Note that $\mathcal{B}_n^{(2)} = \mathcal{D}_n^{(2)} \cup \{\sqrt{2} e_i \mid 1 \leq i \leq n\}$

4. Line systems for primitive reflection groups

4.1. The line system $\mathcal{E}_8$. In this section we would like to describe the line system $\mathcal{E}_8$. First considered the line systems $\mathcal{D}_8^{(2)}$ consisting of 56 roots of the form $\{e_i \pm e_j \mid 1 \le i < j \le 8\}$. If x is root of $\mathcal{D}_8^{(2)}$ then it is such that

$$2x_i \in \mathbf{Z}, \quad \sum x_i \in 2\mathbf{Z}, \quad x_i - x_j \in \mathbf{Z} \tag{4.1}$$

and $(x,x) = 2$. An example of a root of $\mathcal{D}_8^{(2)}$ are $(0,1,0,0,0,-1,0,0)$. Now considered the set

$$X := \left\{ \frac{1}{2}(e_1 \pm e_2 \pm \cdots \pm e_8) \mid \text{ the number of positive coefficients is even} \right\}$$

that spanned 64 lines and we observe that its roots are satisfies 4.1 and each root are of length 2. We define $\mathcal{E}_8$ what the union of $\mathcal{D}_8^{(2)}$ with the lines spanned by X, i.e., $\mathcal{E}_8 := \mathcal{D}_8^{(2)} \cup \{\mathbf{C}x | x \in X\}$ (See [**5, 6, 7, 9, 12**]).

LEMMA 4.6. *The line system $\mathcal{E}_8$ are star-closed.*

PROOF. If we assume that a y b are roots in $\mathcal{E}_8$ belonging to $\mathcal{D}_8^{(2)}$ with $\ell := \mathbf{C}a$ y $\ell' := \mathbf{C}b$ then $r_\ell(\ell') \in \mathcal{E}_8$ by the lemma 4.4.

If we consider that a are root in $\mathcal{D}_8^{(2)}$ and b are root in $\mathbf{C}X$; and suppose that $a = e_i \pm e_j$ then $r_a(b)$ changes sign in the coordinates i and j of b or fixes b then are line of the lines spanned by X. And if a are root of $\mathbf{C}X$ and b are root of $\mathcal{D}_8^{(2)}$, then $r_a(b)$ are root on $\mathcal{D}_8^{(2)}$.

Now considered $\xi := \frac{1}{2}(e_1 + e_2 + \cdots + e_8)$ and b roots in $\mathbf{C}X$. Then

$$r_\xi(b) = b - (b,\xi)\xi = \frac{1}{2}(e_1 \pm e_2 \pm \cdots \pm e_8) - \frac{1}{4}(e_1 \pm e_2 \pm \cdots \pm e_8, e_1 + e_2 + \cdots + e_8)\xi$$

$$r_\xi(b) = \frac{1}{2}(e_1 \pm e_2 \pm \cdots \pm e_8) - \frac{1}{8}(k-l)(e_1 + e_2 + \cdots + e_8)$$

where k is the number of positive coefficients and l the number of negatives and so $k = 8 - l$ then $k - l = k - 8 + k = 2k - 8$, then

$$r_\xi(b) = \frac{1}{2}(e_1 \pm e_2 \pm \cdots \pm e_8) - \frac{1}{8}(2k-8)\xi = \frac{1}{2}(e_1 \pm e_2 \pm \cdots \pm e_8) - \frac{1}{4}(k-4)(e_1 + e_2 + \cdots + e_8).$$

The possibility for k are $2, 4, 6, 8$

- If $k = 2$ then

$$r_\xi(b) = \frac{1}{2}(e_1 \pm e_2 \pm \cdots \pm e_8) - \frac{1}{4}(-2)(e_1 + e_2 + \cdots + e_8)$$

$$r_\xi(b) = \frac{1}{2}(e_1 \pm e_2 \pm \cdots \pm e_8) + \frac{1}{2}(e_1 + e_2 + \cdots + e_8)$$

$$r_\xi(b) = \frac{1}{2}(e_1 + e_1 \pm e_2 + e_2 \pm \cdots \pm e_8 + e_8)$$

  then is of form

$$r_\xi(b) = 1/2(2, 0, \ldots, \underbrace{2}_{2 \le j \le 8}, 0, \ldots, 0)$$

  that is a root in the span of roots in $\mathcal{D}_8^{(2)}$.

- If $k=4$ then

$$r_\xi(b) = \frac{1}{2}(e_1 \pm e_2 \pm \cdots \pm e_8) - \frac{1}{4}(0)(e_1 + e_2 + \cdots + e_8)$$
$$r_\xi(b) = \frac{1}{2}(e_1 \pm e_2 \pm \cdots \pm e_8) = b$$

a root in $\mathbf{C}X$.

- If $k=6$ then

$$r_\xi(b) = \frac{1}{2}(e_1 \pm e_2 \pm \cdots \pm e_8) - \frac{1}{4}(2)(e_1 + e_2 + \cdots + e_8)$$
$$r_\xi(b) = \frac{1}{2}(e_1 \pm e_2 \pm \cdots \pm e_8) + \frac{1}{2}(e_1 + e_2 + \cdots + e_8)$$
$$r_\xi(b) = \frac{1}{2}(e_1 + e_1 \pm e_2 + e_2 \pm \cdots \pm e_8 + e_8)$$

- If $k=8$ then

$$r_\xi(b) = \frac{1}{2}(e_1 \pm e_2 \pm \cdots \pm e_8) - \frac{1}{4}(4)(e_1 + e_2 + \cdots + e_8)$$
$$r_\xi(b) = \frac{1}{2}(e_1 \pm e_2 \pm \cdots \pm e_8) = 2\left(\frac{1}{2}(e_1 + e_2 + \cdots + e_8)\right)$$

□

The line system $\mathcal{E}_7$. The line system $\mathcal{E}_7$ is the subsystem of $\mathcal{E}_8$ of lines orthogonal to ξ. The group $W(\mathcal{E}_8)$ is transitive on lines of $\mathcal{E}_7$ therefore $\mathcal{E}_7$ are a 3-system indecomposable.

EXAMPLE 10. *The root $y = 1/2(e_1 - e_2 - e_3 - e_4 - e_5 + e_6 + e_7 + e_8)$ is orthogonal to ξ.*

The line system $\mathcal{E}_6$. The set of 36 lines of $\mathcal{E}_8$ orthogonal to the star with roots $e_1 - e_2$, $e_2 - e_3$ and $e_1 - e_3$ is the line system $\mathcal{E}_6$.

EXAMPLE 11. *The roots $e_4 - e_8$ and $1/2(e_1 + e_2 + e_3 - e_4 - e_5 + e_6 + e_7 + e_8)$ are roots in the line system $\mathcal{E}_6$.*

DEFINITION 4.9. *The line system $\overline{\mathcal{E}}_8$ consisting of the 36 lines of $\mathcal{A}_8$ together with the lines spanned by the 84 images of $\frac{1}{3}(2,2,2,-1,-1,-1,-1,-1,-1)$ under the action of $W(\mathcal{A}_8)$.*

The line system $\mathcal{K}_6$. Let $\omega := \frac{1}{2}(-1+i\sqrt{3})$ and $\theta := \omega - \omega^2 = \omega - \overline{\omega} = i\sqrt{3}$ and so $\theta^{-1} = -\frac{i\sqrt{3}}{3}$. Take the set

$$X = \left\{ \theta^{-1}(1, \omega^{\alpha_2}, \omega^{\alpha_3}, \dots, \omega^{\alpha_6}) \mid \prod_{i=2}^{6} \alpha_i = 1 \right\}$$

The line system $\mathcal{K}_6$ is the union of the line system $\mathcal{D}_6^{(3)}$ with the set of lines spanned by X.

Fix the root $\xi := \theta^{-1}(1,1,1,1,1,1)$ and considered a root x of a line in $\mathbf{C}X$. The root its of the form

$$x = \theta^{-1}(1, \omega^{\alpha_2}, \omega^{\alpha_3}, \dots, \omega^{\alpha_6})$$

such that $\sum_{i=2}^{6} \alpha_i \equiv 0 \mod 3$.

Observe that

$$(x, \xi) = |\theta^{-1}|^2 \left(1 + \sum_{i=2}^{6} \omega^{\alpha_i}\right) = \frac{1}{3}\left(1 + \sum_{i=2}^{6} \omega^{\alpha_i}\right)$$

and then you have the following cases

- considere that $\alpha_i = 0$ for all i, $\alpha_2 + \cdots + \alpha_6 = 0 \equiv 0 \mod 3$, then

$$(x,\xi) = \frac{1}{3}(1+5) = \frac{6}{3} = 2$$

then $r_\xi(x) = x - 2\xi = \xi - 2\xi = \theta^{-1}(1,1,1,1,1,1) - 2\theta^{-1}(1,1,1,1,1,1) = -\xi$.

- Considered $\alpha_i, \alpha_j \neq 0$ and $\alpha_k = 0$ for $k \neq i,j$, i.e., $\alpha_i + \alpha_j \equiv 0 \mod 3$ (the power possibility are $1+2$ and $2+1$) and note that $\omega^2 + \omega = \omega + \omega^2 = -1$, then

$$(x,\xi) = \frac{1}{3}(4-1) = 1$$

and

$$\begin{aligned} r_\xi(x) &= x - \xi = \theta^{-1}(1,\dots,\underbrace{\omega}_{i},\dots,\underbrace{\omega^2}_{j},\dots,1) - \theta^{-1}(1,1,1,1,1,1) \\ &= \theta^{-1}(0,\dots,\underbrace{\omega-1}_{i},\dots,\underbrace{\omega^2-1}_{j},\dots,0) = (\omega^2-1)\theta^{-1}(0,\dots,\underbrace{-\omega}_{i},\dots,\underbrace{1}_{j},\dots,0) \end{aligned}$$

- Supposed that $\alpha_i \neq 0$ for three distinct i and supposed that $x = \theta^{-1}(1,\omega,\omega,\omega,1,1)$ or $x = \theta^{-1}(1,\omega^2,\omega^2,\omega^2,1,1)$ then

$$(x,\xi) = \frac{1}{3}(3+3\omega) = 1+\omega = -\omega^2 \qquad \text{or} \quad (x,\xi) = \frac{1}{3}(3+3\omega^2) = 1+\omega^2 = -\omega$$

and

$$r_\xi(x) = x - (x,\xi)\xi = \begin{cases} \theta^{-1}(1+\omega^2,\omega+\omega^2,\omega+\omega^2,\omega+\omega^2,1+\omega^2,1+\omega^2) & if \quad (x,\xi) = -\omega^2 \\ \theta^{-1}(1+\omega,\omega^2+\omega,\omega^2+\omega,\omega^2+\omega,1+\omega,1+\omega) & if \quad (x,\xi) = -\omega \end{cases}$$

$$r_\xi(x) = x - (x,\xi)\xi = \begin{cases} \theta^{-1}(-\omega,-1,-1,-1,-\omega,-\omega) & if \quad (x,\xi) = -\omega^2 \\ \theta^{-1}(1+\omega,-1,-1,-1,1+\omega,1+\omega) & if \quad (x,\xi) = -\omega \end{cases}$$

$$r_\xi(x) = x - (x,\xi)\xi = \begin{cases} -\omega\theta^{-1}(1,\omega^2,\omega^2,\omega^2,1,1) & if \quad (x,\xi) = -\omega^2 \\ (1+\omega)\theta^{-1}(1,\omega^2,\omega^2,\omega^2,1,1) & if \quad (x,\xi) = -\omega \end{cases}$$

- Supposed that $\alpha_i \neq 0$ for four distinct i and supposed that $x = \theta^{-1}(1,\omega,\omega,\omega^2,\omega^2,1)$ then

$$(x,\xi) = \frac{1}{3}(2+2\omega^2+2\omega) = 0$$

and $r_\xi(x) = x$.

- Supposed that $\alpha_i \neq 0$ for all i, and supposed that $x = \theta^{-1}(1,\omega,\omega,\omega,\omega,\omega^2)$ then

$$(x,\xi) = \frac{1}{3}(1+4\omega+\omega^2) = \frac{1}{3}(3\omega) = \omega$$

and

$$\begin{aligned}
r_\xi(x) &= x - \omega\xi = \theta^{-1}(1,\omega,\omega,\omega,\omega,\omega^2) - \omega\theta^{-1}(1,1,1,1,1,1)\\
&= \theta^{-1}(1-\omega,0,0,0,0,\omega^2-\omega)\\
&= (1-\omega)\theta^{-1}(1,0,0,0,0,-\omega^2)
\end{aligned}$$

Now consider a root $y = e_i - \omega^\alpha e_j$ of $\mathcal{D}_6^{(3)}$ where $1 \le i < j \le 6$ and $\alpha \in \{0,1,2\}$. Note the following

$$(y,\xi) = (e_i - \omega^\alpha e_j, \xi) = (e_i,\xi) - \omega^\alpha (e_j,\xi) = \overline{\theta^{-1}} - \omega^\alpha \overline{\theta^{-1}} = \overline{\theta^{-1}}(1-\omega^\alpha)$$

$$(y,\xi) = \begin{cases} \overline{\theta^{-1}}(0) & if \quad \alpha = 0\\ \overline{\theta^{-1}}(1-\omega) & if \quad \alpha = 1\\ \overline{\theta^{-1}}(1-\omega^2) & if \quad \alpha = 2 \end{cases}$$

then

$$r_\xi(y) = y - (y,\xi)\xi = \begin{cases} y & if \quad \alpha = 0\\ y + \overline{\theta^{-1}}(1-\omega)\xi & if \quad \alpha = 1\\ y + \overline{\theta^{-1}}(1-\omega^2)\xi & if \quad \alpha = 2 \end{cases}$$

while $\alpha = 1$ so

$$\begin{aligned}
r_\xi(y) &= y - (y,\xi)\xi = y - \overline{\theta^{-1}}(1-\omega)\theta^{-1}(1,1,1,1,1)\\
&= y - \frac{1}{3}(1-\omega)(1,1,1,1,1,1)\\
&= (0,\ldots,\underbrace{1}_{i},\ldots,\underbrace{-\omega}_{j},\ldots,0) - \frac{1}{3}(1-\omega)(1,1,1,1,1,1)\\
&= \frac{1}{3}(-1+\omega,\ldots,\underbrace{2+\omega}_{i},\ldots,\underbrace{-1-2\omega}_{j},\ldots,-1+\omega)\\
&= \frac{-1+\omega}{3}(1,\ldots,\underbrace{\omega^2}_{i},\ldots,\underbrace{\omega}_{j},\ldots,1)
\end{aligned}$$

and if $\alpha = 2$

$$\begin{aligned}
r_\xi(y) &= y - (y,\xi)\xi = y - \overline{\theta^{-1}}(1-\omega^2)\theta^{-1}(1,1,1,1,1)\\
&= y - \frac{1}{3}(1-\omega^2)(1,1,1,1,1,1)\\
&= (0,\ldots,\underbrace{1}_{i},\ldots,\underbrace{-\omega^2}_{j},\ldots,0) - \frac{1}{3}(1-\omega^2)(1,1,1,1,1,1)\\
&= \frac{1}{3}(-1+\omega^2,\ldots,\underbrace{2+\omega^2}_{i},\ldots,\underbrace{-1-2\omega^2}_{j},\ldots,-1+\omega^2)\\
&= \frac{-1+\omega^2}{3}(1,\ldots,\underbrace{\omega}_{i},\ldots,\underbrace{\omega^2}_{j},\ldots,1)
\end{aligned}$$

On the other side $r_y(\xi) = \xi - (\xi,y)y = \xi - \theta^{-1}(1-\overline{\omega}^\alpha)y$

$$r_y(\xi) = \begin{cases} \xi & if \quad \alpha = 0 \\ \xi + \theta^{-1}(1-\omega^2)y & if \quad \alpha = 1 \\ \xi + \theta^{-1}(1-\omega)y & if \quad \alpha = 2 \end{cases}$$

while $\alpha = 1$

$$r_y(\xi) = \xi - \theta^{-1}(1-\omega^2)y = \theta^{-1}(1,\dots,\underbrace{1-(1+\omega^2)}_{i},\dots,\underbrace{1+(1-\omega^2)\omega}_{j},\dots,1)$$

$$= \theta^{-1}(1,\dots,\underbrace{\omega^2}_{i},\dots,\underbrace{\omega}_{j},\dots,1)$$

and if $\alpha = 2$ then

$$r_y(\xi) = \xi - \theta^{-1}(1-\omega)y = \theta^{-1}(1,\dots,\underbrace{1-(1+\omega)}_{i},\dots,\underbrace{1+(1-\omega)\omega^2}_{j},\dots,1)$$

$$= \theta^{-1}(1,\dots,\underbrace{\omega}_{i},\dots,\underbrace{\omega^2}_{j},\dots,1)$$

In effect, it has to.

LEMMA 4.7. *The line system $\mathcal{K}_6$ is star-closed.*

DEFINITION 4.10. *Let $\overline{\mathcal{K}}_6$ denote the line system which is the union of the 30 lines of $\mathcal{D}_6^{(2)}$ with the 96 lines spanned by the images of $\frac{1}{2}(1,1,1,1,1,\theta)$ under the action of $W\left(\mathcal{D}_6^{(2)}\right)$.*

The line system $\mathcal{K}_5$. The line system $\mathcal{K}_5$ is the set of line orthogonal to $\xi = \theta^{-1}(1,1,1,1,1,1)$. Note that $\mathcal{K}_5$ is the union of $\mathcal{A}_5$ and 30 lines spanned by the images of $\theta^{-1}(1,\omega,\omega^2,1,\omega,\omega^2)$ under the action of $W(\mathcal{A}_5)$.

In $\mathcal{K}_6$, the line system $\mathcal{A}_4$ is contained in three line systems of type $\mathcal{A}_5$. These systems represent the orbits of $W(\mathcal{K}_6)$ in subsystems of type $\mathcal{A}_5$ and are obtained by taking the star-closures of $\mathcal{A}_4$ with $\langle e_5 - e_6\rangle, \langle e_5 - \omega e_6\rangle$ and $\left\langle e_5 - \omega^2 e_6\right\rangle$, respectively. Remeber that the minimal extension of $\mathcal{A}_4$ are $\mathcal{D}_5^{(2)}$ and $\mathcal{D}_5^{(3)}$, then the subsystem of $\mathcal{K}_6$ that contained to $\mathcal{A}_4 \perp \langle e_5 - e_6\rangle$ are $\mathcal{A}_5$ and $\mathcal{D}_6^{(3)}$, thus $(\mathcal{A}_4 \perp \langle e_5 - e_6\rangle)^* = \mathcal{A}_5$.

5. The indecomposable 4-systems

To construct a indecomposable 4-system, in each case we begin with the line system $\mathfrak{L}$ of an imprimitive reflection group and a line ℓ not in $\mathfrak{L}$, then adjoin the images of ℓ under the action of $W(\mathfrak{L})$.

The line system $\mathcal{J}_3^{(4)}$. Let $\lambda := -\frac{1}{2}(1+i\sqrt{7})$. Then $\lambda^2+\lambda+2=0$ and $|\lambda| = \sqrt{2}$. As roots of the line system $\mathfrak{L} = \mathcal{B}_3^{(2)}$ we may take $\frac{1}{2}\left(\lambda^2, \pm\lambda^2, 0\right)$, $\frac{1}{2}\left(\lambda^2, 0, \pm\lambda^2\right)$, $\frac{1}{2}\left(0, \lambda^2, \pm\lambda^2\right)$, $(\lambda,0,0)$, $(0,\lambda,0)$ and $(0,0,\lambda)$. The line system $\mathcal{J}_3^{(4)}$ is the union of the 9 lines of $\mathcal{B}_3^{(2)}$ with the set X of 12 lines spanned by the vectors $\frac{1}{2}(\pm\lambda,\pm\lambda,2)$, $\frac{1}{2}(\pm\lambda,2,\pm\lambda)$ and $\frac{1}{2}(2,\pm\lambda,\pm\lambda)$. And we have $\left|\mathcal{J}_3^{(4)}\right| = 21$.

$\overline{\mathcal{J}}_3^{(4)}$, is the union of $\mathcal{B}_3^{(2)}$ with the 12 lines spanned by the vectors $\frac{1}{2}(\pm\bar{\lambda}, \pm\bar{\lambda}, 2)$, $\frac{1}{2}(\pm\bar{\lambda}, 2, \pm\bar{\lambda})$ and $\frac{1}{2}(2, \pm\bar{\lambda}, \pm\bar{\lambda})$.

The line system $\mathcal{F}_4$. In this case $\mathfrak{L} = \mathcal{B}_4^{(2)}$. As roots of $\mathfrak{L}$ we may take the 12 vectors $e_h \pm e_j (1 \leq h < j \leq 4)$ and the 4 vectors $\sqrt{2}e_h (1 \leq h \leq 4)$. The 8 lines $\left\langle \frac{1}{\sqrt{2}}(1, \pm 1, \pm 1, \pm 1) \right\rangle$ form a single orbit under the action of $G(2,1,4)$ and $\mathcal{F}_4$ is the union of these 8 lines with the 16 lines of $\mathcal{B}_4^{(2)}$.

The line system $\mathcal{N}_4$. In this case $\mathfrak{L} = \mathcal{D}_4^{(4)}$. The 24 lines of $\mathfrak{L}$ are spanned by the roots $e_j \pm e_k$ and $e_j \pm ie_k (1 \leq j < k \leq 4)$. The line system $\mathcal{N}_4$ contains 40 lines: it is the union of $\mathcal{D}_4^{(4)}$ and the 16 images of $\left\langle \frac{1+i}{2}(1,1,1,1) \right\rangle$ under the action of $G(4,4,4) = W\left(\mathcal{D}_4^{(4)}\right)$.

The line system $\overline{\mathcal{N}}_4$, is the union of the 16 lines of $\mathcal{B}_4^{(2)}$ and the 24 images of the line $\left\langle \frac{1+i}{2}(1,1,i,i) \right\rangle$ under the action of $G(2,1,4) = W\left(\mathcal{B}_4^{(2)}\right)$ The set of 6 lines of $\mathcal{N}_4$ spanned by the roots

$$(1, i, 0, 0), \quad (1, 0, i, 0), \quad (1, 0, 0, i),$$

$$(0, 1, -1, 0), \quad (0, 1, 0, -1) \text{ and } (0, 0, 1, -1)$$

is a line system of type $\mathcal{A}_3$. It has a unique extension to a line system $\mathfrak{M}$ of type $\mathcal{A}_4$.

The line system $\mathcal{O}_4$. The line system $\mathcal{B}_4^{(4)}$ contains 28 lines: it is the union of $\mathcal{D}_4^{(4)}$ with the coordinate axes (with roots $(1+i)e_j, 1 \leq j \leq 4$). The orbit X of $\left\langle \frac{1+i}{2}(1,1,1,1) \right\rangle$ under the action of $G(4,2,4) = W\left(\mathcal{B}_4^{(4)}\right)$ contains 32 lines. The line system $\mathcal{O}_4$ is the union of $\mathcal{B}_4^{(4)}$ and X; it contains 60 lines.

The line system $\overline{\mathcal{O}}_4$ obtained as the union of $\mathcal{B}_4^{(4)}$ with the 32 images of $\left\langle \frac{1+i}{2}(1,1,1,i) \right\rangle$ under the action of $G(4,2,4)$.

6. The indecomposable 5-systems

The line system $\mathcal{H}_3$. Let $\tau := \frac{1}{2}(1+\sqrt{5})$; then $\tau^2 = \tau + 1$. The line system $\mathcal{H}_3$ is the set of 15 lines joining the mid-points of opposite edges of a regular dodecahedron. The vectors $(1,0,0)$, $\frac{1}{2}\left(1, \pm\tau, \pm\tau^{-1}\right)$ and their cyclic shifts are roots of these lines.

The line system $\mathcal{H}_4$. The wreath product $A := C_2 \wr \mathrm{Alt}(4)$ acts on $\mathbf{C}^4$ by even permutations of the coordinates and arbitrary sign changes. The line system $\mathcal{H}_4$ is the collection of 60 lines spanned by the images of $(1,0,0,0)$, $\frac{1}{2}\left(1, \tau, \tau^{-1}, 0\right)$ and $\frac{1}{2}(1,1,1,1)$ under the action of A. The subsystem of lines orthogonal to any given line of $\mathcal{H}_4$ is equivalent to $\mathcal{H}_3$.

The line system $\mathcal{J}_3^{(5)}$. This is the union of $\mathcal{H}_3$ and the 30 lines spanned by the images of $\frac{1}{2}\left(\tau+\omega, \tau^{-1}\omega - 1, 0\right)$ under the action of $W(\mathcal{H}_3)$. One root for each line of $\mathcal{J}_3^{(5)}$ not in $\mathcal{H}_3$ can be obtained by applying cyclic shifts and sign changes to the vectors $\frac{1}{2}\left(\tau+\omega, \tau^{-1}\omega - 1, 0\right)$, $\frac{1}{2}\left(\tau\omega, 1, \tau^{-1}\omega^2\right)$ and $\frac{1}{2}\left(\omega^2, 1, \tau\omega + 1\right)$.

7. The Goethals-Seidel decomposition for 3-systems

Let $\mathfrak{L}$ is an indecomposable 3-system in $\mathbf{C}^n$ and that Σ is a set of long roots, one for each line of $\mathfrak{L}$. Choose long roots a and b corresponding to a pair of lines at 60° and scale b so that $(a,b)=-1$. Then $c=-a-b$ is a root that spans the third line of the 3-star of a and b and we may suppose that $a,b,c\in\Sigma$ (see Lemma 4.1). The Goethals–Seidel decomposition of Σ has components:

$$\begin{aligned}
\Gamma_a &= \{x\in\Sigma \mid (a,x)=0 \text{ and } (b,x)\neq 0\},\\
\Gamma_b &= \{x\in\Sigma \mid (b,x)=0 \text{ and } (c,x)\neq 0\},\\
\Gamma_c &= \{x\in\Sigma \mid (c,x)=0 \text{ and } (a,x)\neq 0\},\\
\Delta &= \{x\in\Sigma \mid (a,x)=0 \text{ and } (b,x)=0\}, \quad \text{and}\\
\Lambda &= \{x\in\Sigma\backslash\{a,b,c\} \mid (a,x)(b,x)(c,x)\neq 0\}.
\end{aligned}\tag{7.1}$$

Scale the roots $x\in\Gamma_a$ so that $(b,x)=1$, and hence $(c,x)=-1$. Similarly, scale $x\in\Gamma_b$ so that $(c,x)=1$ and scale $x\in\Gamma_c$ so that $(a,x)=1$

LEMMA 4.8. *We have*

$$\Gamma_b = r_c\left(\Gamma_a\right) = \left\{x+c \mid x\in\Gamma_a\right\} \quad \textit{and} \quad \Gamma_c = r_b\left(\Gamma_a\right) = \left\{x-b \mid x\in\Gamma_a\right\}$$

PROOF. If $x\in\Gamma_a$, then $(x,c)=-1$ and thus $r_c(x)=x-(x,c)c=x+c$. Furthermore, $(b,x+c)=(b,x)+(b,c)=1-1=0$ and $(c,x+c)=(c,x)+(c,c)=-1+2=1$, hence $x+c\in\Gamma_b$. Similarly $r_b(x)=x-b\in\Gamma_c$. Conversely, for $y\in\Gamma_b$ we have $r_c(y)=y-c\in\Gamma_a$ and for $z\in\Gamma_c$ we have $r_b(z)=z+b$. Thus $\Gamma_b=r_c\left(\Gamma_a\right)$ and $\Gamma_c=r_b\left(\Gamma_a\right)$ □

LEMMA 4.9. *Suppose that σ is a 3-star and that $\ell\notin\sigma$ is a line at 60° to a line of σ with root a. Let v be the root of ℓ such that $(a,v)=1$ and let V be the subspace spanned by σ and ℓ*

(i) *If ℓ is orthogonal to a line of σ with root b, where $(a,b)=-1$, then there is an orthonormal basis of V with respect to which a, b and v have coordinates $(1,-1,0),(0,1,-1)$ and $(0,-1,-1)$. In this case the star-closure of σ and ℓ is $\mathcal{D}_3^{(2)}$.*

(ii) *If ℓ is at 60° to all three lines $\langle a\rangle,\langle b\rangle$ and $\langle c\rangle$ of σ, where $a+b+c=0$ and $(a,b)=-1$, then $(b,v)\in\left\{\omega,\omega^2 \mid \omega \text{ is a 3-root of } 1\right\}$ and there is an orthonormal basis of V with respect to which a,b and v have coordinates $(1,-1,0),(0,1,-1)$ and $(0,-1,(b,v))$. In this case the star-closure of σ and ℓ is $\mathcal{D}_3^{(3)}$.*

COROLLARY 4.1. *If $\mathfrak{L}$ is a line system in $\mathbf{R}^k$ in which the lines are at 60° or 90°, then $\mathfrak{L}$ is a subset of a 3-system and every line of $\mathfrak{L}$ not in a given 3 -star σ of $\mathfrak{L}$ is orthogonal to at least one line of σ.*

Let us take $v\in\Lambda$ such that $(a,v)=1$. Then, by Lemma 4.9, $(b,v)=\omega$ and $(c,v)=\omega^2$ or else $(b,v)=\omega^2$ and $(c,v)=\omega$. In both cases the star-closure of the lines spanned by a,b and v is equivalent to $\mathcal{D}_3^{(3)}$.

THEOREM 4.6. *Let $\mathfrak{L}$ be a 2- or 3-system of dimension at most three.*

(i) *If* $\dim \mathfrak{L} = 1$*, then* $\mathfrak{L} \simeq \mathcal{A}_1$ *and* $\mathfrak{L}$ *consists of a single line.*
(ii) *If* $\dim \mathfrak{L} = 2$*, then* $\mathfrak{L}$ *is equivalent to* $\mathcal{D}_2^{(2)} \simeq 2\mathcal{A}_1$ *or to a 3 -star* $\mathcal{D}_2^{(3)} \simeq \mathcal{A}_2$.
(iii) *If* $\dim \mathfrak{L} = 3$*, then* $\mathfrak{L}$ *is equivalent to* $3\mathcal{A}_1, \mathcal{A}_1 \perp \mathcal{A}_2, \mathcal{A}_3 \simeq \mathcal{D}_3^{(2)}$ *or* $\mathcal{D}_3^{(3)}$.

PROOF. (i) Is clear.

(ii) In $\mathbf{C}^2$, there is a unique 3-star containing a pair of lines at 60°, $\ell = \mathbf{C}(e_1 - e_2)$ and $m = \mathbf{C}(e_2 - e_3)$ and has the 3-star $\mathcal{A}_2 \simeq \mathcal{D}_2^{(3)}$ by adding the third line at 60 of both. And a unique line orthogonal to a given line, thus $\mathfrak{L}$ is either a pair of orthogonal lines $2\mathcal{A}_1 = \mathcal{A}_1 \perp \mathcal{A}_1 \simeq \mathcal{D}_2^{(2)}$.

(iii) If $\mathfrak{L}$ is decomposable, then $\mathfrak{L} = \mathcal{A}_1 \perp \mathfrak{M}$, where $\mathfrak{M}$ is a star-closed line system of dimension two. Then $\mathfrak{L}$ is equivalent to $3\mathcal{A}_1$ or $\mathcal{A}_1 \perp \mathcal{A}_2$.

If $\mathfrak{L}$ is indecomposable and a simple extension of $\mathcal{A}_2$, then by Lemma 4.9 $\mathfrak{L}$ is equivalent to $\mathcal{D}_3^{(2)}$ or $\mathcal{D}_3^{(3)}$. The line system $\mathcal{A}_3$ is a simple extension of $\mathcal{A}_2$ and therefore it is equivalent to $\mathcal{D}_3^{(2)}$.

It follows from Theorem 4.5 that in order to complete the proof, it is sufficient to prove that neither $\mathcal{D}_3^{(2)}$ nor $\mathcal{D}_3^{(3)}$ have simple extensions in $\mathbf{C}^3$.

Suppose that $\mathfrak{L}$ contains a line system $\mathfrak{M}$ and a root $v := (\alpha, \beta, \gamma)$ such that $\langle v \rangle \notin \mathfrak{M}$, where $\mathfrak{M}$ is either $\mathcal{D}_3^{(2)}$ or $\mathcal{D}_3^{(3)}$. If v is orthogonal to $a := (1, -1, 0)$ and $b := (0, 1, -1)$ then

$$(a, v) = \alpha - \beta = 0, \quad (b, v) = \beta - \gamma = 0$$

then $\alpha = \beta = \gamma$ and on taking the inner product of v with $(1, 1, 0)$ or $(1, -\omega, 0)$ we reach a contradiction, i.e.

$$((1, 1, 0), v) = 2\alpha \quad \text{and} \quad ((1, -\omega, 0), v) = \alpha(1 - \omega)$$

both are not at 60° of v.

Thus we may suppose that $(a, v) = 1$ and hence $\beta = \alpha - 1$. If $(b, v) = 0$, then $\gamma = \alpha - 1$. If $\theta := (b, v) \neq 0$ and if ℓ is at 60° to b and $c = -a - b$, then $|\theta| = |1 + \theta| = 1$ and hence θ is ω or ω^2. Thus $v = (\alpha, \alpha - 1, \alpha + \theta)$, where θ is $-1, \omega$ or ω^2.

$$(v, v) = |\alpha|^2 + (\alpha - 1)(\overline{\alpha} - 1) + (\theta - 1)(\overline{\theta} - 1) = 2$$

$$3|\alpha|^2 + (\bar{\theta} - 1)\alpha + (\theta - 1)\bar{\alpha} + |\theta|^2 + 1 = 2$$

we have $3|\alpha|^2 + (\bar{\theta} - 1)\alpha + (\theta - 1)\bar{\alpha} = 0$.

If $\mathfrak{M} = \mathcal{D}_3^{(2)}$, then $d := (1, 1, 0)$ is a root of $\mathfrak{M}$, which is not orthogonal to v. Therefore $|(d, v)| = 1$ and hence $2|\alpha|^2 - \alpha - \bar{\alpha} = 0$. Combined with the previous equation this implies that α is 0 or 1 . But then $\mathfrak{L}$ contains $\mathcal{D}_3^{(3)}$ and this is a contradiction since the lines $\langle (1, 1, 0) \rangle$ and $\langle (1, -\omega, 0) \rangle$ are neither at 60° nor 90°. If $\mathfrak{M} = \mathcal{D}_3^{(3)}$, then

$(1,-\omega,0)$ and $\left(1,-\omega^2,0\right)$ are roots of $\mathfrak{M}$. If both these roots are at $60°$ to v, then

$$3|\alpha|^2+(\omega-1)\alpha+\left(\omega^2-1\right)\bar{\alpha}=0 \quad \text{and}$$
$$3|\alpha|^2+\left(\omega^2-1\right)\alpha+(\omega-1)\bar{\alpha}=0.$$

On adding these equations we find that $2|\alpha|^2-\alpha-\bar{\alpha}=0$. As before $\alpha=0$ or $\alpha=1$ and again we arrive at a contradiction. The remaining possibility is that v is orthogonal to exactly one of $(1,-\omega,0)$ or $\left(1,-\omega^2,0\right)$. But in neither case is there a solution for α.

□

LEMMA 4.10. *Suppose that $x,y\in\Gamma_a$, where $x\neq y$, and set $\mu=(x,y)$. Then $\mu\in\left\{0,1,-\omega,-\omega^2\right\}$ and the vectors a,b,x and y are linearly independent.*

(i) *If $\mu=0$, then $z=b-c-x-y\in\Gamma_a$ and $(x,z)=(y,z)=0$. The star-closure of a,b,x and y is equivalent to $\mathcal{D}_4^{(2)}$. Furthermore, if $w\in\Gamma_a$, then*

$$(w,x)+(w,y)+(w,z)=2$$

(ii) *If $\mu=1$, then a multiple of $x-y$ belongs to Δ. The star-closure of a,b,x and y is equivalent to $\mathcal{A}_4$.*

(iii) *If $\mu=-\omega$ or $\mu=-\omega^2$, then $\mu x+\bar{\mu}y\in\Gamma_a$ and $a+b-\mu x-y\in\Lambda$. The star-closure of a,b,x and y is equivalent to $\mathcal{D}_4^{(3)}$.*

8. Extensions of $\mathcal{D}_n^{(2)}$ and $\mathcal{D}_n^{(3)}$

In this section we characterise certain simple extensions of $\mathcal{D}_n^{(k)}$, where $k\leq 5$ and $n\geq 2$. Let $V=\mathbf{C}^{n+1}$ and let $e_1,e_2,\dots,e_{n+1}$ be an orthonormal basis of V. The roots of $\mathcal{D}_n^{(k)}$ are vectors of the form $e_h-\zeta e_j$, where ζ is a k^{th} root of unity and $1\leq h<j\leq n$.

If $\mathfrak{L}$ is a simple extension of $\mathcal{D}_n^{(k)}$, and suppose that $\mathfrak{L}$ is the star-closure in V of $\mathcal{D}_n^{(k)}$ and a line ℓ with long root $x=\left(\alpha_1,\alpha_2,\dots,\alpha_{n+1}\right)$ respect to the basis $e_1,e_2,\dots,e_{n+1}$.

LEMMA 4.11. *Suppose that $2\leq k\leq m\leq 5$ and that $\mathfrak{L}$ is an indecomposable m-system and the star-closure in V of $\mathcal{D}_n^{(k)}$ and a line $\ell\notin\mathcal{D}_n^{(k)}$ with root $x=\left(\alpha_1,\alpha_2,\dots,\alpha_{n+1}\right)$ as above. If $\alpha_s=0$ for some $s\leq n$, then either $\left|\alpha_h\right|=\sqrt{2}$ for some $h\leq n$ and the extension is equivalent to*

$$\mathcal{D}_n^{(2)}\subset\mathcal{B}_n^{(2)} \text{ or } \mathcal{D}_n^{(4)}\subset\mathcal{B}_n^{(4)}$$

or $\left|\alpha_h\right|=1$ for two values of $h\leq n$ and the extension is equivalent to

$$\mathcal{D}_n^{(k)}\subset\mathcal{D}_{n+1}^{(k)} \text{ or } \quad \mathcal{D}_n^{(2)}\subset\mathcal{D}_n^{(4)}$$

In the proof of Lemma 4.11, see [**9**, Lemma 7.20, p. 115], it is first established that the group $G(k,1,n)$ acts on $\mathbf{C}^{n+1}$ by permuting the first n coordinates and multiplying their values by k^{th} roots of unity, thus fixing the line system $\mathcal{D}_n^{(k)}$. The proof then supposes that $\alpha_1=0$ and since $\mathfrak{L}$ is indecomposable, we have $\alpha_j\neq 0$ for some $j\leq n$.

Then, the proof proceeds to consider the vectors $a := e_1 - e_j$ and $b := e_1 - \zeta e_j$, where ζ is a k^{th} root of unity. Using these vectors, the proof derives that $\left|\alpha_j\right| \in \left\{1, \sqrt{2}, \tau^{-1}\right\}$ and $\left|\left(b, r_x(a)\right)\right| \in \left\{0, 1, \sqrt{2}, \tau, \tau^{-1}, 2\right\}$. From this, it is concluded that $\left|\alpha_j\right| = 1$ or $\left|\alpha_j\right| = \sqrt{2}$.

If $\left|\alpha_j\right| = \sqrt{2}$, the proof shows that the extension is equivalent to $\mathcal{D}_n^{(2)} \subset \mathcal{B}_n^{(2)}$ or $\mathcal{D}_n^{(4)} \subset \mathcal{B}_n^{(4)}$. If $\left|\alpha_j\right| = 1$, the proof shows that the extension is equivalent to $\mathcal{D}_n^{(k)} \subset \mathcal{D}_{n+1}^{(k)}$ or $\mathcal{D}_n^{(2)} \subset \mathcal{D}_n^{(4)}$.

THEOREM 4.7. *If $\mathfrak{L}$ is an indecomposable 3-system and a simple extension of $\mathcal{D}_n^{(2)}$ for $n \geq 3$, the extension is equivalent to one of*

$$\mathcal{D}_n^{(2)} \subset \mathcal{D}_{n+1}^{(2)}, \quad \mathcal{D}_3^{(2)} \subset \mathcal{A}_4, \quad \mathcal{D}_3^{(2)} \subset \mathcal{D}_4^{(3)}, \quad \mathcal{D}_4^{(2)} \subset \mathcal{K}_5, \quad \mathcal{D}_5^{(2)} \subset \mathcal{E}_6,$$
$$\mathcal{D}_5^{(2)} \subset \mathcal{K}_6, \quad \mathcal{D}_6^{(2)} \subset \mathcal{K}_6, \quad \mathcal{D}_6^{(2)} \subset \mathcal{E}_7, \quad \mathcal{D}_7^{(2)} \subset \mathcal{E}_8 \quad \text{or} \quad \mathcal{D}_8^{(2)} \subset \mathcal{E}_8.$$

These extensions are minimal except for

$$\mathcal{D}_4^{(2)} \subset \mathcal{D}_4^{(2)} \perp \mathcal{A}_1 \subset \mathcal{K}_5, \quad \mathcal{D}_5^{(2)} \subset \mathcal{D}_6^{(2)} \subset \mathcal{K}_6, \quad \mathcal{D}_5^{(2)} \subset \mathcal{E}_6 \subset \mathcal{K}_6$$
$$\mathcal{D}_6^{(2)} \subset \mathcal{D}_6^{(2)} \perp \mathcal{A}_1 \subset \mathcal{E}_7 \quad \text{and } \mathcal{D}_7^{(2)} \subset \mathcal{D}_8^{(2)} \subset \mathcal{E}_8$$

The proof of Theorem proceeds by first assuming that $\mathfrak{L}$ is an indecomposable 3-system and a simple extension of $\mathcal{D}_n^{(2)}$ for $n \geq 3$. Using the notation introduced earlier, we suppose that $\mathfrak{L}$ is a line system in $\mathbf{C}^{n+1}$ and that $\mathfrak{L}$ is the star-closure of $\mathcal{D}n^{(2)}$ and a line ℓ with root $x = \left(\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\right)$.

The first step in the proof is to show that there exist α and β with $\alpha \neq \pm\beta$ such that for all $h \leq n, \alpha_h \in \{\alpha, -\alpha, \beta, -\beta\}$. This is done by assuming the existence of $h, j, k \leq n$ such that the six quantities $\pm\alpha_h, \pm\alpha_j$ and $\pm\alpha_k$ are distinct, and then using inner products with the roots $e_h \pm e_j$ and $e_h \pm e_k$ to arrive at a contradiction.Once it has been established that there exist α and β such that for all $h \leq n, \alpha_h \in \{\alpha, -\alpha, \beta, -\beta\}$, the proof proceeds by considering different cases based on the values of k, the number of values of h such that $\alpha_h = \alpha$. The case $k = n$ is dealt with first, and it is shown that in this case the extension is equivalent to one of the systems listed in the statement of the theorem. The cases $k \neq n$ are then considered, and it is shown that in these cases k must be equal to $n - 1$, and that the extension is equivalent to one of the systems listed in the statement of the theorem.

The proof then goes on to consider the case $k = 4, n = 5$ specifically. In this case, it is shown that the vectors x and $\frac{1}{2}(-1, -1, -1, 1, -i\sqrt{3}, 1)$ are roots of $\mathfrak{L}$, and the third line of their 3-star is spanned by $(0, 0, 0, 1, 0, 1)$. The star-closure of this vector with $\mathcal{D}_5^{(2)}$ is $\mathcal{D}_6^{(2)}$. It follows that $\mathfrak{L}$ is the line system $\overline{\mathcal{K}}_6$ and thus $\mathfrak{L}$ is equivalent to $\mathcal{K}_6$. Thus, the extension is simple but not minimal.

THEOREM 4.8. *If $\mathfrak{L}$ is an indecomposable 3 -system and a simple extension of $\mathcal{D}_n^{(3)}$, where $n \geq 2$, the extension is equivalent to one of*

$$\mathcal{D}_n^{(3)} \subset \mathcal{D}_{n+1}^{(3)}, \quad \mathcal{D}_2^{(3)} \subset \mathcal{D}_3^{(2)}, \quad \mathcal{D}_4^{(3)} \subset \mathcal{K}_5, \quad \mathcal{D}_5^{(3)} \subset \mathcal{K}_6 \quad \text{or} \quad \mathcal{D}_6^{(3)} \subset \mathcal{K}_6.$$

The extensions are minimal except for $\mathcal{D}_5^{(3)} \subset \mathcal{D}_6^{(3)} \subset \mathcal{K}_6$.

LEMMA 4.12. *If $\Gamma_a \neq \emptyset$, then $\left\{\langle x - y\rangle \mid x, y \in \Gamma_a \text{ and } (x, y) = 1\right\}$ is the set of lines spanned by the elements of Δ.*

COROLLARY 4.2. *If $\Gamma_a \neq \emptyset$, we may normalise the roots Σ of $\mathfrak{L}$ so that every root is a linear combination of elements of $\{a,b,c\} \cup \Gamma_a$ with coefficients from $\left\{\pm 1, \pm\omega, \pm\omega^2\right\}$*

THEOREM 4.9. *If $\Gamma_a = \emptyset$, then $\mathfrak{L}$ is $\mathcal{D}_n^{(3)}$ for some n.*

9. Extensions of Euclidean line systems

Let $\mathfrak{L}$ an indecomposable 3-system in $\mathbf{C}^n$ with a 3-star σ whose roots a,b and c have been chosen so that $a+b+c=0$ and $(a,b)=(b,c)=(c,a)=-1$. Furthermore, $\Sigma := \sigma \cup \Gamma_a \cup \Gamma_b \cup \Gamma_c \cup \Delta \cup \Lambda$ is the Goethals-Seidel decomposition for a set Σ of representatives of the roots of $\mathfrak{L}$.

THEOREM 4.10. *The 3-system $\mathfrak{L}$ is the complexification of a Euclidean line system if and only if $\Lambda = \emptyset$.*

THEOREM 4.11. *If $\mathfrak{L}$ is an indecomposable 3-system such that $(x,y)=1$ for all $x,y \in \Gamma_a$, whenever $x \neq y$, then $\mathfrak{L}$ is equivalent to $\mathcal{A}_n$.*

THEOREM 4.12. *Suppose that $\mathfrak{L}$ is an indecomposable 3-system in a real vector space and a simple extension of $\mathcal{A}_n (n \geq 2)$. Then the extension is equivalent to one of*

$$\mathcal{A}_n \subset \mathcal{A}_{n+1}, \quad \mathcal{A}_n \subset \mathcal{D}_{n+1}^{(2)}, \quad \mathcal{A}_5 \subset \mathcal{E}_6, \qquad \mathcal{A}_6 \subset \mathcal{E}_7,$$
$$\mathcal{A}_7 \subset \mathcal{E}_7, \quad \mathcal{A}_7 \subset \mathcal{E}_8 \quad \text{or } \mathcal{A}_8 \subset \mathcal{E}_8.$$

The extensions are minimal except for

$$\mathcal{A}_5 \subset \mathcal{A}_5 \perp \mathcal{A}_1 \subset \mathcal{E}_6, \quad \mathcal{A}_6 \subset \mathcal{A}_7 \subset \mathcal{E}_7$$
$$\mathcal{A}_7 \subset \mathcal{A}_8 \subset \mathcal{E}_8 \quad \text{and} \quad \mathcal{A}_7 \subset \mathcal{D}_8^{(2)} \subset \mathcal{E}_8$$

This completes the all Euclidean extensions of line systems of type $\mathcal{A}_n$ and we now turn our to the line systems $\mathcal{E}_6, \mathcal{E}_7$ and $\mathcal{E}_8$.

THEOREM 4.13. *Suppose that $\mathfrak{L}$ is an indecomposable 3-system in a real vector space and a simple extension of $\mathfrak{M}$, where $\mathfrak{M}$ is one of $\mathcal{E}_n (n = 6,7,8)$. Then the extension $\mathfrak{M} \subset \mathfrak{L}$ is equivalent to $\mathcal{E}_6 \subset \mathcal{E}_7$ or $\mathcal{E}_7 \subset \mathcal{E}_8$. The extension $\mathcal{E}_6 \subset \mathcal{E}_7$ is minimal but $\mathcal{E}_7 \subset \mathcal{E}_7 \perp \mathcal{A}_1 \subset \mathcal{E}_8$.*

10. Extensions of $\mathcal{A}_n, \mathcal{E}_n$ and $\mathcal{K}_n$

THEOREM 4.14. *Suppose that $\mathfrak{L}$ is an indecomposable 3-system and a simple extension of $\mathfrak{M}$, where $\mathfrak{M}$ is one of $\mathcal{A}_n (n \geq 2)$, $\mathcal{E}_n (n = 6,7,8)$ or $\mathcal{K}_n (n = 5,6)$. Then the extension $\mathfrak{M} \subset \mathfrak{L}$ is equivalent to one of*

$$\mathcal{A}_n \subset \mathcal{A}_{n+1}, \quad \mathcal{A}_n \subset \mathcal{D}_{n+1}^{(2)}, \quad \mathcal{A}_n \subset \mathcal{D}_{n+1}^{(3)},$$
$$\mathcal{A}_4 \subset \mathcal{K}_5, \quad \mathcal{A}_5 \subset \mathcal{K}_5, \quad \mathcal{A}_5 \subset \mathcal{E}_6, \quad \mathcal{A}_5 \subset \mathcal{K}_6,$$
$$\mathcal{A}_6 \subset \mathcal{K}_6, \quad \mathcal{A}_6 \subset \mathcal{E}_7, \quad \mathcal{A}_7 \subset \mathcal{E}_7, \quad \mathcal{A}_7 \subset \mathcal{E}_8, \quad \mathcal{A}_8 \subset \mathcal{E}_8,$$
$$\mathcal{E}_6 \subset \mathcal{E}_7, \quad \mathcal{E}_6 \subset \mathcal{K}_6, \quad \mathcal{E}_7 \subset \mathcal{E}_8 \quad \text{or} \quad \mathcal{K}_5 \subset \mathcal{K}_6.$$

The extensions are minimal except that for suitable choices of $\mathcal{A}_5$ subsystems we have

$$\mathcal{A}_4 \subset \mathcal{A}_5 \subset \mathcal{K}_5,$$

$$\mathcal{A}_5 \subset \mathcal{A}_5 \perp \mathcal{A}_1 \subset \mathcal{E}_6 \subset \mathcal{K}_6, \quad \mathcal{A}_5 \subset \mathcal{K}_5 \subset \mathcal{K}_5 \perp \mathcal{A}_1 \subset \mathcal{K}_6,$$

$$\mathcal{A}_5 \subset \mathcal{A}_6 \subset \mathcal{K}_6, \quad \mathcal{A}_5 \subset \mathcal{D}_6^{(2)} \subset \mathcal{K}_6, \quad \mathcal{A}_5 \subset \mathcal{D}_6^{(3)} \subset \mathcal{K}_6,$$

$$\mathcal{A}_6 \subset \mathcal{A}_7 \subset \mathcal{E}_7, \quad \mathcal{A}_7 \subset \mathcal{A}_8 \subset \mathcal{E}_8 \textit{ and } \mathcal{E}_7 \subset \mathcal{E}_7 \perp \mathcal{A}_1 \subset \mathcal{E}_8.$$

THEOREM 4.15. *If $\mathfrak{L}$ is an indecomposable 3-system, then for some n, $\mathfrak{L}$ is equivalent to $\mathcal{A}_n, \mathcal{D}_n^{(2)}$, or $\mathcal{D}_n^{(3)}$, or to $\mathcal{E}_6, \mathcal{E}_7, \mathcal{E}_8, \mathcal{K}_5$ or $\mathcal{K}_6$.*

The proof of Theorem starts by noting that since $\mathfrak{L}$ is a 3-system, it contains a pair of lines at 60° and hence a 3-star σ. The proof then uses Theorem 4.5 , which states that $\mathfrak{L}$ is the end of a chain of simple indecomposable extensions, beginning at σ.

The proof then makes use of the combined results of Theorems 4.6, 4.7 , 4.8 , and 4.14 to show that $\mathfrak{L}$ must be equivalent to one of the five types of line systems: $\mathcal{A}_n$, $\mathcal{D}_n^{(2)}$, $\mathcal{D}_n^{(3)}$, $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$, $\mathcal{K}_5$ or $\mathcal{K}_6$.

In more detail, Theorem 4.6 shows that if the 3-star is of type $\mathcal{A}_n$, then $\mathfrak{L}$ is equivalent to one of the line systems $\mathcal{A}_n, \mathcal{D}_n^{(2)}, \mathcal{D}_n^{(3)}$. Theorems 4.7 and 4.8 show that if the 3-star is of type $\mathcal{D}_n^{(2)}$ or $\mathcal{D}_n^{(3)}$, then $\mathfrak{L}$ is equivalent to one of the line systems $\mathcal{D}_n^{(2)}, \mathcal{D}_n^{(3)}$ or $\mathcal{K}_5, \mathcal{K}_6$. Finally, Theorem 4.14 shows that if the 3-star is of type $\mathcal{A}_n$, $\mathcal{D}_n^{(2)}$, $\mathcal{D}_n^{(3)}$, $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$, $\mathcal{K}_5$, or $\mathcal{K}_6$, then $\mathfrak{L}$ is equivalent to one of the line systems listed in the statement

11. Extensions of 4-systems

Let $\mathfrak{L}$ is an indecomposable 4-system in $\mathbf{C}^n$. Then $\mathfrak{L}$ contains a pair of lines at 45°. If a and b are long roots corresponding to these lines we can take b so that $(a, b) = -\sqrt{2}$. Then $c := r_a(b) = \sqrt{2}a + b$ and $d := r_b(a) = a + \sqrt{2}b$ are roots that span the other two lines of the 4-star of a and b.

A 4-star can be represented either as the line system $\mathcal{B}_2^{(2)}$ with roots $(1, -1)$, $(1, 1)$, $(\sqrt{2}, 0)$ and $(0, \sqrt{2})$ or as the line system $\mathcal{D}_2^{(4)}$ with roots $(1, -1), (1, 1)$, $(1, i)$ and $(1, -i)$. The equivalence is given by the unitary transformation with matrix $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$

LEMMA 4.13. *Suppose that σ is a 4-star and that $\mathfrak{L}$ is an indecomposable 4-system which is the star-closure of σ and a line $\ell \notin \sigma$.*

(i) *If ℓ is orthogonal to a line of σ then $\mathfrak{L}$ is equivalent to $\mathcal{B}_3^{(2)}$.*
(ii) *If ℓ is not orthogonal to any line of σ, then $\mathfrak{L}$ is equivalent to one of*

$$\mathcal{B}_2^{(4)}, \quad \mathcal{D}_3^{(4)} \textit{ or } \mathcal{J}_3^{(4)}.$$

The extensions are minimal except for $\sigma \subset \mathcal{B}_3^{(2)} \subset \mathcal{J}_3^{(4)}$.

LEMMA 4.14. *Let $\mathfrak{L}$ be an indecomposable 4-system and a simple extension of $\mathcal{B}_n^{(2)}$, where $n \geq 2$. Then $\mathfrak{L}$ may be represented as the star-closure in $\mathbf{C}^{n+1}$ of $\mathcal{B}_n^{(2)}$ and a line ℓ with root*

$x = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$. Suppose that $\alpha_j = 0$ for some $j \le n$. If $\alpha_{n+1} = 0$, then $\mathfrak{L} = \mathcal{B}_n^{(4)}$, otherwise $\mathfrak{L}$ is equivalent $\mathcal{B}_{n+1}^{(2)}$.

LEMMA 4.15. *Suppose that $\mathfrak{L}$ is an indecomposable 4-system in $\mathbf{C}^3$ and a simple extension of $\mathcal{B}_3^{(2)}$. Then $\mathfrak{L}$ is $\mathcal{B}_3^{(4)}, \mathcal{J}_3^{(4)}$ or $\overline{\mathcal{J}}_3^{(4)}$.*

LEMMA 4.16. *Suppose that $\mathfrak{L}$ is an indecomposable 4-system in $\mathbf{C}^4$ and a simple extension of $\mathcal{B}_4^{(2)}$. Then $\mathfrak{L}$ is $\mathcal{B}_4^{(4)}, \mathcal{F}_4, \mathcal{N}_4$ or $\overline{\mathcal{O}}_4$. Furthermore, $\mathcal{O}_4$ is the union of $\mathcal{B}_4^{(4)}, \mathcal{F}_4$ and $\mathcal{N}_4$*

THEOREM 4.16. *Suppose that $\mathfrak{L}$ is an indecomposable 4 -system and a simple extension of $\mathcal{B}_n^{(2)}$, where $n \ge 2$. Then the extension is equivalent to one of*

$$\mathcal{B}_n^{(2)} \subset \mathcal{B}_n^{(4)}, \quad \mathcal{B}_n^{(2)} \subset \mathcal{B}_{n+1}^{(2)}, \quad \mathcal{B}_2^{(2)} \subset \mathcal{D}_3^{(4)}, \quad \mathcal{B}_2^{(2)} \subset \mathcal{J}_3^{(4)}$$
$$\mathcal{B}_3^{(2)} \subset \mathcal{J}_3^{(4)}, \quad \mathcal{B}_3^{(2)} \subset \mathcal{F}_4, \quad \mathcal{B}_3^{(2)} \subset \mathcal{N}_4$$
$$\mathcal{B}_4^{(2)} \subset \mathcal{F}_4, \quad \mathcal{B}_4^{(2)} \subset \mathcal{N}_4 \quad \text{or} \quad \mathcal{B}_4^{(2)} \subset \mathcal{O}_4$$

COROLLARY 4.3. *No indecomposable 4-system is a simple extension of $\mathcal{J}_3^{(4)}$.*

The proof of Corollary shows that no indecomposable 4-system can be a simple extension of $\mathcal{J}_3^{(4)}$. It does this by assuming that such a 4-system exists and then reaching a contradiction. Specifically, it supposes that $\mathfrak{L}$ is an indecomposable 4-system and is the star-closure of $\mathcal{J}_3^{(4)}$ and a line ℓ with root x. It then shows that for this to be true, $\mathfrak{L}$ would have to be equivalent to $\mathcal{F}_4$ or $\mathcal{N}_4$, which would contain a line system equivalent to $\mathcal{B}_4^{(2)}$, but this contradicts the fact that $\mathcal{J}_3^{(4)}$ is contained in $\mathbf{C}^3$ and not in $\mathbf{C}^4$. Thus it concludes that no indecomposable 4-system can be a simple extension of $\mathcal{J}_3^{(4)}$.

COROLLARY 4.4. *If $\mathfrak{L}$ is an indecomposable 4-system and a simple extension of $\mathcal{F}_4$ or $\mathcal{N}_4$, then $\mathfrak{L} \simeq \mathcal{O}_4$.*

THEOREM 4.17. *Suppose that $\mathfrak{L}$ is an indecomposable 4-system and a simple extension of $\mathcal{D}_n^{(4)}$, where $n \ge 3$. Then the extension is equivalent to one of*

$$\mathcal{D}_n^{(4)} \subset \mathcal{B}_n^{(4)}, \quad \mathcal{D}_n^{(4)} \subset \mathcal{D}_{n+1}^{(4)}, \quad \mathcal{D}_3^{(4)} \subset \mathcal{N}_4 \quad \text{or} \quad \mathcal{D}_4^{(4)} \subset \mathcal{N}_4.$$

Furthermore, there are exactly four extensions of $\mathcal{D}_4^{(4)}$ in $\mathbf{C}^4$ equivalent to $\mathcal{N}_4$.

THEOREM 4.18. *Suppose that $\mathfrak{L}$ is an indecomposable 4-system and a simple extension of $\mathcal{B}_n^{(4)}$, where $n \ge 3$. Then the extension is equivalent to one of*

$$\mathcal{B}_n^{(4)} \subset \mathcal{B}_{n+1}^{(4)}, \quad \mathcal{B}_3^{(4)} \subset \mathcal{O}_4 \quad \text{or} \quad \mathcal{B}_4^{(4)} \subset \mathcal{O}_4.$$

Furthermore, there are exactly two extensions of $\mathcal{B}_4^{(4)}$ in $\mathbf{C}^4$ equivalent to $\mathcal{O}_4$.

COROLLARY 4.5. *No indecomposable 4-system is a simple extension of $\mathcal{O}_4$.*

The Corollary states that there is no indecomposable 4-system that is a simple extension of $\mathcal{O}_4$. This is proven by contradiction. The proof assumes that there exists an indecomposable 4-system, $\mathfrak{L}$ which is a simple extension of $\mathcal{O}_4$ and a line ℓ with root x. It further assumes that $\mathfrak{L}$ is contained in $\mathbf{C}^5$. From Theorem 4.18, it is known that the star-closure of $\mathcal{B}_4^{(4)}$ and ℓ is equivalent to $\mathcal{B}_5^{(4)}$ or $\mathcal{O}_4$. If the star-closure is $\mathcal{B}_5^{(4)}$, then $\mathfrak{L}$ would be equal to $\mathcal{B}_5^{(4)}$ which would be a contradiction because it is known that there are no extensions of $\mathcal{B}_5^{(4)}$ in $\mathbf{C}^5$. Therefore, the assumption is that $\mathfrak{L} \subset \mathbf{C}^4$. In this case, $\mathfrak{L}$ would contain both $\mathcal{O}_4$ and $\overline{\mathcal{O}}_4$, which would imply that it contains lines with roots $\frac{1}{\sqrt{2}}(1,1,1,1)$ and $\frac{1}{\sqrt{2}}(1,1,1,i)$. This is also a contradiction and hence it is concluded that no indecomposable 4-system is a simple extension of $\mathcal{O}_4$.

THEOREM 4.19. *If $\mathfrak{L}$ is an indecomposable 4-system, then for some $n, \mathfrak{L}$ is equivalent to $\mathcal{B}_n^{(2)}, \mathcal{B}_n^{(4)}$, or $\mathcal{D}_n^{(4)}$, or to $\mathcal{F}_4, \mathcal{J}_3^{(4)}, \mathcal{N}_4$ or $\mathcal{O}_4$.*

This theorem states that any indecomposable 4-system can be represented by a simple extension of one of the 8 listed 4-systems, namely $\mathcal{B}_n^{(2)}, \mathcal{B}_n^{(4)}, \mathcal{D}_n^{(4)}, \mathcal{F}_4, \mathcal{J}_3^{(4)}, \mathcal{N}_4$ or $\mathcal{O}_4$. The proof of the theorem is based on the fact that any indecomposable 4-system contains a 4-star, and by using Theorem 4.5, which states that any 4-system is the end of a chain of simple indecomposable extensions beginning at a 4-star. The combined results of Theorems 4.16, 4.17, 4.18, and their corollaries are used to show that any simple extension of these 8 listed 4-systems will result in an indecomposable 4-system.

12. Extensions of 5-systems

In this section we classification of the indecomposable 5-systems is presented in order to exhibit the complete classification of those of the finite reflection groups generated by reflections of order 2.

LEMMA 4.17. *Suppose that $\mathfrak{L}$ is an indecomposable 5-system and the star-closure in $\mathbf{C}^3$ of $\sigma := \mathcal{D}_2^{(5)}$ and a line ℓ. Then $\mathfrak{L}$ is equivalent to $\mathcal{H}_3$ if and only if ℓ is orthogonal to a line of σ.*

If $\mathfrak{L}$ is a 5-system, then by definition $\mathfrak{L}$ contains a 5-star $\sigma \simeq \mathcal{D}_2^{(5)}$. The representation of $\mathcal{D}_2^{(5)}$ has long roots $\left(1, -\zeta^i\right)$ where $0 \le i \le 4$ and $\zeta := \exp(2\pi/5)$. In what follows we shall represent the lines of $\mathfrak{L}$ by long roots, that is vectors of length $\sqrt{2}$, and use the embedding of $\mathcal{D}_2^{(5)}$ in $\mathbf{C}^3$ with long roots $a_i := \left(1, -\zeta^i, 0\right)$. Note that $\tau = 1 + \zeta + \zeta^4 = -\zeta^2 - \zeta^3$ and so

$$\left|\left(a_i, a_j\right)\right| = \begin{cases} \tau & \text{if } j \equiv \pm i + 1 \pmod 5 \\ \tau^{-1} & \text{if } j \equiv \pm i + 2 \pmod 5 \end{cases}$$

LEMMA 4.18. *Let $\sigma = \mathcal{D}_2^{(5)}$ and suppose that $\mathfrak{L}$ is an indecomposable 5 -system and the star-closure in $\mathbf{C}^3$ of σ and a line ℓ spanned by a (long) root x. For $0 \le i < 5$ put $n_i := \left|\left(x, a_i\right)\right|^2$. If ℓ is not orthogonal to any line of σ, then either*

(i) *$n_i = 1$ for all i and $\mathfrak{L}$ is equivalent to $\mathcal{D}_3^{(5)}$ or*
(ii) *we may choose x so that $n_0 = \tau^2, n_1 = n_4 = 2, n_2 = n_3 = 1$ and $\mathfrak{L}$ properly contains $\mathcal{H}_3$.*

THEOREM 4.20. *If $\mathfrak{L}$ is an indecomposable 5-system in $\mathbf{C}^3$, then one of the following holds:*

(i) *$\mathfrak{L}$ is a simple extension of $\mathcal{D}_2^{(5)}$ equivalent to $\mathcal{D}_3^{(5)}, \mathcal{H}_3$ or $\mathcal{J}_3^{(5)}$, or*

(ii) *$\mathfrak{L}$ is a simple extension of $\mathcal{H}_3$ and $\mathfrak{L} = \mathcal{J}_3^{(5)}$ or $\mathfrak{L} = \overline{\mathcal{J}}_3^{(5)}$.*

COROLLARY 4.6. *No indecomposable 5-system is a proper extension of $\mathcal{J}_3^{(5)}$.*

PROOF. Suppose that $\mathfrak{L}$ is an indecomposable 5-system and a simple extension of $\mathcal{J}_3^{(5)}$. It follows from the theorem that $\dim \mathfrak{L} = 4$ and then from Theorem 3.5, $W(\mathfrak{L})$ is primitive. However, the centre of $W\left(\mathcal{J}_3^{(5)}\right)$ contains an element of order 6 , contradicting Theorem 3.11 (It cannot have reflections of order greater than 3). □

THEOREM 4.21. *Suppose that $\mathfrak{L}$ is an indecomposable 5-system and a simple extension of $\mathcal{H}_3$. Then $\mathfrak{L}$ is equivalent to $\mathcal{J}_3^{(5)}$ or $\mathcal{H}_4$.*

COROLLARY 4.7. *No indecomposable 5-system is a proper extension of $\mathcal{H}_4$.*

The proof of Corollary 8.16 shows that if $\mathfrak{L}$ is an indecomposable 5-system and a proper extension of $\mathcal{H}_4$, then there exists a vector x in $\mathfrak{L}$ that is orthogonal to a line in a copy of $\mathcal{H}_3$ contained in $\mathcal{H}_4$. The proof then shows that the only possibility for the coordinates of x is that $x = \left(\frac{1}{2}, 0, -\frac{1}{2}\tau, 0, \frac{1}{2}\tau^{-1}\right)$. However, this leads to a contradiction because the inner product between x and $\frac{1}{2}(1,1,1,1,0) \in \mathcal{H}_4$ is $-\frac{1}{4}\tau^{-1}$, which is not equal to 1. This contradiction implies that no indecomposable 5-system can be a proper extension of $\mathcal{H}_4$.

THEOREM 4.22. *If $\mathfrak{L}$ is an indecomposable 5-system, then $\mathfrak{L}$ is equivalent to $\mathcal{H}_3, \mathcal{H}_4$, $\mathcal{J}_3^{(5)}$ or $\mathcal{D}_n^{(5)}$ for $n \geq 2$*

The proof of Theorem is a culmination of all the results obtained in the previous theorems and corollaries in this section. It starts by considering the simple extensions of $\mathcal{D}_2^{(5)}$, which by Lemma 4.17, are equivalent to $\mathcal{H}_3$ if and only if the line is orthogonal to a line of $\mathcal{D}_2^{(5)}$, or it is a simple extension of $\mathcal{D}_2^{(5)}$ equivalent to $\mathcal{D}_3^{(5)}$ or $\mathcal{J}_3^{(5)}$ if it is not orthogonal to any line of $\mathcal{D}_2^{(5)}$ by Lemma 4.18.

Next, it considers the simple extensions of $\mathcal{H}_3$. By Theorem 4.20, these are equivalent to $\mathcal{J}_3^{(5)}$ or $\mathcal{H}_4$, and by Corollary 4.7, no indecomposable 5-system is a proper extension of $\mathcal{H}_4$.

Finally, it notes that no indecomposable 5-system is a proper extension of $\mathcal{J}_3^{(5)}$ by Corollary 4.6, and thus any indecomposable 5-system that is not a simple extension of $\mathcal{D}_2^{(5)}, \mathcal{H}_3$, or $\mathcal{J}_3^{(5)}$ is equivalent to $\mathcal{D}_n^{(5)}$ for $n \geq 2$.

Therefore, the theorem states that all possible indecomposable 5-systems are $\mathcal{H}_3, \mathcal{H}_4$, $\mathcal{J}_3^{(5)}$ or $\mathcal{D}_n^{(5)}$ for $n \geq 2$.

13. Line systems and reflections of order three

In this section we study line systems adapted to groups generated by reflections of order three for the 'ternary' case and to groups generated by reflections of order two and three for the 'mixed' case.

Let $\ell = \mathbf{C}a$ a line in $\mathbf{C}^n$. Recall that a root a of ℓ is short, long or tall according to whether (a, a) is 1,2 or 3.

DEFINITION 4.11. *Given a tall root a, the reflection t_a of order three is defined by*

$$t_a(v) := v - \frac{1}{3}(1-\omega)(v,a)a = v + \theta^{-1}\omega^2(v,a)a,$$

where $\omega = \frac{1}{2}(-1+i\sqrt{3})$ *and* $\theta := \omega - \omega^2 = i\sqrt{3}$.

Note that $t_a(a) = a - \frac{1}{3}(1-\omega)(a,a)a = a-(1-\omega)a = \omega a$ and $t_a^2(v) = v - \theta^{-1}\omega(v,a)a$. Since $t_a = t_b$ if and only if a and b span the same line ℓ, this reflection may be denoted unambiguously by t_ℓ.

The following lemma is an extension of Lemma 4.2 for the possible values of $|(a,b)|$ where one of the roots a or b is a tall.

LEMMA 4.19. *Suppose that s and t are non-commuting reflections in a finite primitive reflection group of rank at least 3 and suppose that the order of s is 3 . Let $s = \mathbf{C}a$ where a is a tall root.*

(i) *If the order of $t = \mathbf{C}b$ is 3 and if $(b,b) = 3$, then $\langle s,t\rangle \simeq SL_2(\mathbf{F}_3)$ and $|(a,b)| = \sqrt{3}$.*
(ii) *If the order of $t = \mathbf{C}b$ is 2 and if $(b,b) = 2$, then $\langle s,t\rangle \simeq G(3,1,2)$ and $|(a,b)| = \sqrt{3}$.*

If $a := (1,1,1)$ and $b := (\omega,1,1)$, then a and b are tall roots, $|(a,b)| = \sqrt{3}$ and $G := \langle t_a, t_b\rangle$ is the primitive reflection group $G_4 \simeq SL_2(\mathbf{F}_3)$.

DEFINITION 4.12. *A line system $\mathfrak{L}$ is a* **ternary k-system** *if*

(i) *for all lines $\ell \in \mathfrak{L}$, $t_\ell(\mathfrak{L}) \subseteq \mathfrak{L}$,*
(ii) *for all $\ell, m \in \mathfrak{L}$ the order of $t_\ell t_m$ is at most k, and*
(iii) *there exist $\ell, m \in \mathfrak{L}$ such that the order of $t_\ell t_m$ equals k.*

Similarly for k-system and star-closure for a subset X of a k-system we has the follow. Given a ternary k-system $\mathfrak{L}, W(\mathfrak{L})$ denotes the group generated by the reflections t_ℓ, for all $\ell \in \mathfrak{L}$. The star-closure X^* of a subset X of $\mathfrak{L}$ is the intersection of all ternary k-systems $\mathfrak{K}$ such that $X \subseteq \mathfrak{K}$. Evidently $X^* \subseteq \mathfrak{L}$.

Let $\mathcal{L}_1$ denote the unique ternary k-system of dimension 1 . Then $W(\mathcal{L}_1) = \mathcal{C}_3$ is a cyclic group of order 3 . If $\mathfrak{L}$ is the line system of a primitive reflection group generated by reflections of order 3 and of rank $n \geq 3$, it follows from Lemma 4.19 (i) that either $\mathfrak{L} = n\mathfrak{L}_1$ or $\mathfrak{L}$ is a ternary 6-system. Furthermore, if a and b are tall roots that span lines of $\mathfrak{L}$, then $(a,b) = 0$ or $|(a,b)| = \sqrt{3}$.

DEFINITION 4.13. *If $\mathfrak{K}$ is a k-system and $\mathfrak{M}$ is a ternary h-system disjoint from $\mathfrak{K}$ such that $r_\ell(\mathfrak{M}) \subseteq \mathfrak{M}$ and $t_m(\mathfrak{K}) \subseteq \mathfrak{K}$ for all $\ell \in \mathfrak{K}$ and all $m \in \mathfrak{M}$, the pair $\mathfrak{L} = (\mathfrak{K}, \mathfrak{M})$ is called a mixed (k,h)-system. The set of lines of $\mathfrak{L}$ is defined to be the union $\mathfrak{K} \cup \mathfrak{M}$. We put $W(\mathfrak{L}) := W(\mathfrak{K})W(\mathfrak{M})$ and call $W(\mathfrak{L})$ the Weyl group of $\mathfrak{L}$. Note that $W(\mathfrak{K})$ and $W(\mathfrak{M})$ are normal subgroups of $W(\mathfrak{L})$.*

13.1. The mixed line systems $\mathcal{B}_n^{(3)}$. If G is an imprimitive irreducible reflection group that contains reflections of order 3, then $G = G(3p, p, n)$ for some n and some p. The reflections of order 2 in G generate the subgroup $G(3p, 3p, n)$, whose line system is $\mathcal{D}_n^{(3p)}$. The reflections of order 3 commute and their line system is the ternary 3-system $n\mathcal{L}_1$, to wit the coordinate axes. If G is a subgroup of a primitive reflection group, it follows from Theorem 4.21 that $p = 1$. In this case, we denote the mixed $(3,6)$-system $\left(\mathcal{D}_n^{(3)}, n\mathcal{L}_1\right)$ of $G(3,1,n)$ by $\mathcal{B}_n^{(3)}$.

13.2. The line system $\mathcal{L}_2$. Suppose that a and b are tall roots such that $|(a,b)| = \sqrt{3}$. If we scale b so that $(a,b) = \theta$, where $\theta = i\sqrt{3}$, then $t_a(b) = -\omega^2 a + b$ and $t_a^2(b) = \omega t_b(a) = \omega a + b$. The ternary 6-system $\mathcal{L}_2$ is defined to be the set of four lines spanned by $a, b, t_a(b)$ and $t_a^2(b)$. Thus $W\left(\mathcal{L}_2\right) = \left\langle t_a, t_b\right\rangle$ is $G_4 \simeq SL_2\left(\mathbf{F}_3\right)$ and it acts on the lines of $\mathcal{L}_2$ as Sym(4).

13.3. The line system $\mathcal{L}_3$. The group $G(3,1,3)$ acts on $\mathbf{C}^3$ by permuting the coordinates and multiplying their values by ω and ω^2. Thus there are 9 lines in the $G(3,1,3)$ orbit of $\langle(1,1,1)\rangle$. The ternary 6 -system $\mathcal{L}_3$ is the union of these 9 lines with the coordinate axes.

13.4. The line system $\mathcal{L}_4$. We may regard $\mathcal{L}_3$ as a subset of $\mathbf{C}^4$ via the embedding that sends the vector $\left(x_1, x_2, x_3\right)$ to $\left(x_1, x_2, x_3, 0\right)$. The $W\left(\mathcal{L}_3\right)$ orbit Γ of $\langle(0,1,-1,-1)\rangle$ contains 27 lines and the roots of these lines are obtained by cyclically permuting the first three coordinates and multiplying the coordinate values by ω or ω^2. The ternary 6 -system $\mathcal{L}_4$ has 40 lines and consists of the 27 lines of Γ, the 12 lines of $\mathcal{L}_3$ and the line $\langle(0,0,0,\theta)\rangle$.

13.5. The mixed line system $\mathcal{M}_3$. The group $G(3,1,3)$ acts on the line system $\mathcal{L}_3$ and normalises $W\left(\mathcal{L}_3\right)$. It follows that the product $W\left(\mathcal{L}_3\right)G(3,1,3)$ is a finite reflection group, which contains reflections of order 2 as well as reflections of order 3.

The 3 reflections of order 3 in $G(3,1,3)$ belong to $W\left(\mathcal{L}_3\right)$ and the 9 reflections of order 2 in $G(3,1,3)$ generate the group $G(3,3,3)$ whose line system is $\mathcal{D}_3^{(3)}$. The line system $\mathcal{M}_3$ is the mixed $(3,6)$-system $\left(\mathcal{D}_3^{(3)}, \mathcal{L}_3\right)$.

14. Extensions of ternary 6-systems

LEMMA 4.20. *Suppose that r and s are non-commuting reflections of order 3 in primitive reflection group G. If $t \in G$ is a reflection of order 2 or if t is a reflectio of order 3 such that $t \notin \langle r, s\rangle$, then t commutes with a reflection in $\langle r, s\rangle$.*

LEMMA 4.21. *If $\mathfrak{L}$ is an indecomposable ternary 6-system and a simple extension of $\mathcal{L}_2$, then $\mathfrak{L}$ is equivalent to $\mathcal{L}_3$.*

PROOF. From Lemma 4.20 we may suppose that $\mathfrak{L}$ is the star-closure of tall roots a, b and c such that $(a,b) = (b,c) = \theta$ and $(a,c) = 0$, where $\theta = i\sqrt{3}$. Then

$$t_b(a) = a + \theta^{-1}\omega^2(a,b)b = a + \omega^2 b,$$

$$t_b(c) = c + \theta^{-1}\omega^2(c,b)b = c + \theta^{-1}\overline{\theta^{-1}}\omega^2 b = c - \omega^2 b$$

and

$$
\begin{aligned}
t_c^2(b) &= t_c(b) + \theta^{-1}\omega^2 (t_c(b), c)c = b + \omega^2 c + \theta^{-1}\omega^2 (b + \omega^2 c, c)c \\
&= b + \omega^2 c + \theta^{-1}\omega^2 \left[(b,c) + \omega^2 (c,c)\right] c \\
&= b + \omega^2 c + \theta^{-1}\omega^2 \left[\theta + 3\omega^2\right] c \\
&= b + 2\omega^2 c + 3\theta^{-1}\omega c = b + \omega c(2\omega + 3\theta^{-1}) \\
&= b - \omega c
\end{aligned}
$$

Thus $d := t_b t_a^2 t_b(a) = a - \theta b - c$ and $(a,d) = (c,d) = 0$. That is, $\left\{\theta^{-1}a, -\theta^{-1}c, -\theta^{-1}d\right\}$ is an orthonormal basis for $\mathbf{C}^3$ such that, with respect to this basis, we have $a = (\theta, 0, 0), b = (1,1,1)$, $c = (0, -\theta, 0)$ and $d = (0,0,-\theta)$. It follows that $\mathfrak{L}$ is equivalent to $\mathcal{L}_3$. □

LEMMA 4.22. *If $\mathfrak{L}$ is an indecomposable ternary 6-system and a simple extension of $\mathcal{L}_3$, then $\mathfrak{L}$ is equivalent to $\mathcal{L}_4$.*

LEMMA 4.23. *No indecomposable ternary 6-system is a proper extension of $\mathcal{L}_4$.*

THEOREM 4.23. *Suppose that G is a finite primitive reflection group of rank at least 3 .*

(i) *If G is generated by reflections of order 3 , then G is conjugate to $W(\mathcal{L}_3)$ or $W(\mathcal{L}_4)$.*
(ii) *If G contains reflections of orders 2 and 3 , then G is conjugate to $W(\mathcal{M}_3)$.*

COROLLARY 4.8. *A finite primitive reflection group of rank at least 4 does not contain both a reflection of order 2 and a reflection of order 3 .*

COROLLARY 4.9. *If G is a finite primitive reflection group of rank at least 5 , then all reflections in G have order 2 .*

Remark 10. *In summary, these lemmas and theorems state that for primitive reflection groups of rank at least 3, if the group is generated by reflections of order 3, then the group is either equivalent to the Weyl group of the ternary 6-system $\mathcal{L}_3$ or $\mathcal{L}_4$. If the group contains reflections of both orders 2 and 3, then it is equivalent to the Weyl group of the mixed $(3,6)$-system $\mathcal{M}_3$. Additionally, it is stated that for a primitive reflection group of rank at least 4, it cannot contain both reflections of order 2 and 3 and for a primitive reflection group of rank at least 5, all reflections in the group have order 2.*

CHAPTER 5

The classification

The Blichfeldt's theorem tells us that the order of the product of two reflections is a maximum of 5. (see Corollary 1.7, Theorem 3.8). In the primitive reflection groups with $n = 2$ all cases of the Corollary 1.7 occur. But for higher dimensions there are fewer possibilities (see Lemmas 3.4-3.5-3.10). And from this it follows that there are not many primitive reflection groups with reflections of order two.

If $n \geq 3$, when G is primitive has reflection of order 2 or 3. And if has reflections of order 3, there are three possibilities for G: $W(\mathcal{L}_3), W(\mathcal{L}_4)$ or $W(\mathcal{M}_3)$.

And if G does not contain reflections of order 3, then as a consequence of the Blichfeldt's theorem the product of two reflections of G is at most 5. That is, if G is generated by reflections of order 2. Then the line system $\mathcal{L}$ spanned by the roots of these reflections in G is an indecomposable star-closed line system and $G = W(\mathcal{L})$. The possibilities for $\mathcal{L}$ are

(1) If $\mathcal{L}$ is an 3-sistema, then $\mathcal{L}$ it is equivalent to $\mathcal{A}_n$ or to one of $\mathcal{K}_5, \mathcal{K}_6, \mathcal{E}_6, \mathcal{E}_7$ or $\mathcal{E}_8$.
(2) If $\mathfrak{L}$ is a 4-system, then $\mathfrak{L}$ is equivalent to one of $\mathcal{J}_3^{(4)}$, $\mathcal{F}_4, \mathcal{N}_4$ or $\mathcal{O}_4$.
(3) If $\mathfrak{L}$ is a 5-system, then $\mathfrak{L}$ is equivalent to one of $\mathcal{H}_3, \mathcal{J}_3^{(5)}$ or $\mathcal{H}_4$.

1. The classification

THEOREM 5.1 (Shephard and Todd). *Suppose that G is a finite irreducible unitary reflection group of rank n. Then, up to conjugacy in $U_n(\mathbf{C}), G$ belongs to precisely one of the following ten classes:*

(i) $n = 1$ *and G is a cyclic group;*
(ii) $n \geq 2$ *and G is the imprimitive group $G(m, p, n)$ for some $m > 1$ and some divisor p of m;*
(iii) $n = 2$ *and G is one of the 19 primitive unitary reflection groups listed in Table;*
(iv) $n = 3$ *and G is* $W(\mathcal{H}_3), W\left(\mathcal{J}_3^{(4)}\right), W\left(\mathcal{J}_3^{(5)}\right), W(\mathcal{L}_3)$ *or* $W(\mathcal{M}_3)$
(v) $n = 4$ *and G is* $W(\mathcal{F}_4), W(\mathcal{H}_4), W(\mathcal{L}_4), W(\mathcal{N}_4)$ *or* $W(\mathcal{O}_4)$
(vi) $n = 5$ *and G is* $W(\mathcal{K}_5)$;
(vii) $n = 6$ *and G is* $W(\mathcal{E}_6)$ *or* $W(\mathcal{K}_6)$;
(viii) $n = 7$ *and G is* $W(\mathcal{E}_7)$;
(ix) $n = 8$ *and G is* $W(\mathcal{E}_8)$;
(x) $n \geq 4$ *and G is* $W(\mathcal{A}_n) \simeq \mathrm{Sym}(n+1)$.

PROOF. If G is imprimitive, then by Theorem 3.4, G is conjugate to $G(m,p,n)$ for some $m>1$ and some divisor p of m. If $n=1$ then G is cyclic. If $n=2$ and G is primitive, there are 19 possibilities for G see chapter 2 and section 6.

Therefore, from now on we may suppose that G is primitive and $n \geq 3$.

If G contains reflections of order three, it follows from Theorems 4.23 that G is $W\left(\mathcal{L}_3\right), W\left(\mathcal{L}_4\right)$ or $W\left(\mathcal{M}_3\right)$.

Thus we may suppose that the order of every reflection in G is two.

By Theorems 4.2 and 4.3 the roots of the reflections in G span the lines of an indecomposable k-system $\mathfrak{L}$ such that $G = W(\mathfrak{L})$. From consequences of Blichfeldt's theorem $k \leq 5$ by Theorem 1.7 and $\mathfrak{L}$ contains at least one of the line systems $\mathcal{A}_2 \simeq \mathcal{D}_2^{(3)}, \mathcal{B}_2^{(4)}, \mathcal{B}_2^{(2)} \simeq \mathcal{D}_2^{(4)}$ or $\mathcal{D}_2^{(5)}$. So we have the following cases.

If $\mathfrak{L}$ is a 3-system, then $\mathfrak{L}$ is an extension of a 3-star $\mathcal{A}_2$. From Theorem 4.15, $\mathfrak{L}$ is equivalent to one of the line systems $\mathcal{A}_n, \mathcal{E}_6, \mathcal{E}_7, \mathcal{E}_8, \mathcal{K}_5$ or $\mathcal{K}_6$. The group $W\left(\mathcal{A}_1\right) \simeq \mathrm{Sym}(2)$ is cyclic, $W\left(\mathcal{A}_2\right) \simeq G(3,3,2)$ and $W\left(\mathcal{A}_3\right) \simeq G(2,2,3)$.

If $\mathfrak{L}$ is a 4-system, then $\mathfrak{L}$ is an extension of a 4-star $D_2^{(4)}$. From Theorem 4.19, $\mathfrak{L}$ is equivalent to $\mathcal{J}_3^{(4)}, \mathcal{F}_4, \mathcal{N}_4$ or $\mathcal{O}_4$.

If $\mathcal{L}$ is a 5-system, then $\mathfrak{L}$ is an extension of a 5 -star $\mathcal{D}_2^{(5)}$ and G contains $G(5,5,2)$. If $\dim \mathfrak{L} = 3$, then from Theorem 4.20, $\mathfrak{L}$ is equivalent to $\mathcal{H}_3$ or $\mathcal{J}_3^{(5)}$. If $\dim \mathfrak{L} > 3$, then from Theorem 3.9, G contains a primitive reflection subgroup of rank 3 . Thus $\mathfrak{L}$ is an extension of $\mathcal{H}_3$ or $\mathcal{J}_3^{(5)}$ and it follows from Theorem 4.21 and Corollaries 4.6 and 4.7 that $\mathfrak{L}$ is equivalent to $\mathcal{H}_4$. $\square$

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