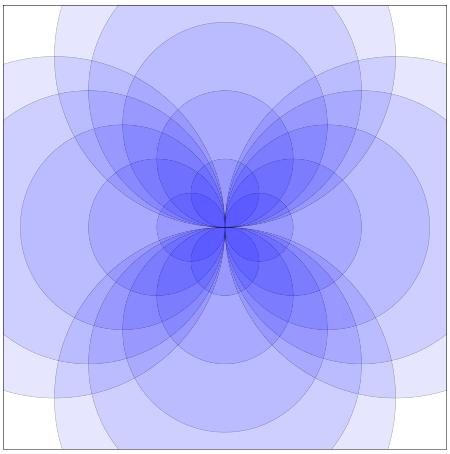
1495 UNIVERSITY OF ABERDEEN

MX4557 Complex Analysis

F. Olukoya



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Module Handbook

Students are asked to make themselves familiar with the information on key institutional policies which have been made available within MyAberdeen or on the university website.

These policies are relevant to all students and will be useful to you throughout your studies. They contain important information and address issues such as what to do if you are absent, how to raise an appeal or a complaint and how seriously the University takes your feedback.

These institutional policies should be read in conjunction with this Course Description Form, in which School-specific policies are detailed. Further information can be found on the University's Infohub webpage or by visiting the Infohub.

Welcome to *MX4557 Complex Analysis*. The aim of the module is to further develop understanding of the concepts, techniques, and tools of calculus. Calculus is the mathematical study of variation. This course emphasises differential and integral calculus in one variable, and sequences and series of functions.

Intended Learning Outcomes (ILO's)

By the end of the course, you should be able

i. to state and illustrate the definitions of the concepts introduced in the course,
ii. to state the theorems of the course, to explain their significance, and to give examples to indicate the role of the hypotheses,
iii. to demonstrate knowledge and understanding of proof techniques used in the course,
iv. to use the methods and results of the course to solve problems at levels similar to those seen in the course.

In particular, you should:

- Be able to perform routine calculations with complex functions.
- Understand the concept of analyticity and be familiar with the Cauchy-Riemann equations.
- Know the necessary background and development of an elementary version of Cauchy's theorem.
- Know the important consequences of Cauchy's theorem: Cauchy's integral formulae, Liouville's Theorem and Taylor Series.
- Know about Laurent Series and the classification of isolated singularities.
- Know the Cauchy residue theorem and some of its applications.

Lecture notes and recommended reading

These are based on Markus Upmeier's excellent set of notes and a full-set of gapped notes is available on MyAberdeen. The gaps will be revealed in due course as the term progress.

Most library books dealing with elementary complex analysis are useful for this course.

Possible codes in QML are 515.30-01 (Functions of a Complex Variable), 515-01 (Mathematical Analysis) and 620.001 51 (Engineering Mathematics).

For specific recommendations, see the Leganto Reading on MyAberdeen (click Books & Tools). Further references are mentioned in the lecture notes.

Logistics

Lectures

This course is taught in person in the Fraser Noble Building, FN156.
Lectures take place on Tuesdays 14:00–15:00 and Thursdays 13:00–14:00, weeks 27–34, 38-40, with the first lecture being on January 28, 2024.

Tutorials

Tutorials are on Fridays 9:00–10:00 in the Fraser Noble Building, FN156, and will start during the second week of lectures with the first tutorial being on February 4, 2024. Students should work through the current problem sheet (Table Table 1 indicates which problem sheet a tutorial covers.) and attempt as many exercises as possible.

Timetable

Details of all sessions (lectures and tutorials) will be available on your timetable. Please make sure that you have the correct day, time and room for each session. You should check this regularly as there are occasionally changes, particularly in the first couple of weeks of the semester.

Table 1 displays a detailed schedule for the semester.

Week	Day	Chapter	Material
27	T	1	Complex numbers and functions
	Th	1	Complex numbers and functions
	F		
28	T	2	Complex-valued functions
	Th	2	Complex-valued functions
	F		Tutorial 1: Problem Sheet 1
29	T	3	Topology of the complex plane
	Th	3	Topology of the complex plane ***(Assignment)***
	F		Tutorial 2: Problem Sheet 2
30	T	4	Holomorphic functions
	Th	4	Holomorphic functions
	F		Tutorial 3: Problem Sheet 3 ***(Assignment due)***
31	T	5	Power series
	Th	5	Power series
	F		Tutorial 4: Problem Sheet 4
32	T	6	Countour integrals
	Th	6	Countour integrals
	F		Tutorial 5: Problem Sheet 5
33	T	7	Cauchy's Theorem
	Th	7	Cauchy's Theorem
	F		Tutorial 6: Problem Sheet 6
34	T		Applications of Cauchy's Theorem
	Th	8	Applications of Cauchy's Theorem ***(Assignment)***
	F	8	Tutorial 7: Problem Sheet 7
35-37			***Spring Break (Assignment due)***
	T	9	Laurent series and singularities

Week	Day	Chapter	Material
38	Th	9	Laurent series and singularities
	F		Tutorial 8: Problem Sheet 8
	T	10	Residue Theorem
39	Th	10	Residue Theorem
	F		Tutorial 9: Problem Sheet 9
	T	11	Applications to integrals
40	Th	11	Applications to integrals
	F		Tutorial 10: Problem Sheet 10
41			***Exams***

Table 1: Detailed Schedule

Attendance requirements

You should attend classes regularly and do the work of the course. If you fail to do this, you may be asked to discuss your reasons with the Course Organiser and possibly be reported to the Registry as an unauthorised withdrawal from the course.
https://www.abdn.ac.uk/students/academic-life/monitoring-and-student-progress.php

Private Study

In addition to lectures and tutorials, you should aim to spend at least **six** hours a week working on the course. During this time you should

(a) work through the example sheets,

(b) study your lecture notes and related texts,

(c) prepare in-course assignments.

Lecturers

This semester Dr. Feyisayo (Shayo) Olukoya will be the lecturer on the module. As mentioned above you can reach me by email at feyisayo.olukoya1@abdn.ac.uk; you should also feel free to arrange an in-person meeting my office is FN163 in the Fraser Noble; meeting virtually over teams is also an option.

MyAberdeen

All resources for the module (lecture notes, problem sheets, solutions e.t.c) will be made available on MyAberdeen at the appropriate time.

Occasionally, important information will be distributed to your university email account.

Assessment

Continuous Assessment

There are two Continuous Assessments and a final exam for this course.

- Assignment 1 (15%): Released Thursday 13 February. The deadline is Friday 21 February at 3pm.
- Assignment 2(15%): Released Thursday 20 March. The deadline is Friday 28 March at 3pm.

Assignments will be released on MyAberdeen and you should submit your solutions as a pdf file on MyAberdeen by the due date. Only in exceptional circumstances will work handed in after the due date be accepted for formal assessment.

Individual work

Your submission for Continuous Assessment must be your own individual work unless stated otherwise. Further information and a reference to the University Code of Conduct on Student Discipline can be found in the Information Booklet.

You are encouraged to discuss exercises for this course with other students, for example in small groups. You may shop around for ideas but it is expected that you possess and use your own critical faculties, to write up your own solutions. You must not copy sentences from fellow students. If you do not understand the solution of another student, do not attempt to use it as your own solution.

Formative assessment

At the end of each chapter in the notes, you will find problem sheets with questions addressing the content covered in that chapter. Although these sheets do not contribute to the continuous assessment component, you are ***strongly encouraged*** to attempt them as they are designed to consolidate your understanding and enhance your problem-solving skills. Full solutions are provided and will appear after the sheet or sheets have been covered in example classes.

Final Exam

The final exam will be during Exam Week and comprises 70% of the module mark. It is an unseen, closed-book examination. The examination will require you to state definitions, state (and possibly prove) results, and apply these to solving problems. You should be able to state every definition and result in the module unless they are marked in the lecture notes as non-examinable.

Resits

Resit In the summer there will be a resit examination consisting of a single two-hour exam paper. The CGS mark awarded for the resit will be the maximum of

(a) the mark obtained by using the resit examination result combined with the carried-forward CA mark as in the original diet and

(b) the mark based on the resit examination performance alone.

Calculators in exams

Calculators will **not** be allowed in the main examination or the resit examination.

Student Support

For advice on academic and non-academic issue (resonable adjutsments, financial, international, personal or health matters) please contact Student Services.

You can find the University's education policy for students by following this link.

If you have any problems with the course—mathematical or organisational—please contact me, or your class representative, or Mark Grant (who is in charge of undergraduate teaching).

Report a bug

If you encounter any issues with these notes or would like to leave feedback regarding lectures, tutorials, assessments, etc., please feel free to report a bug. I will review submissions periodically, so do share your thoughts.

Part I

Complex numbers and functions

Chapter 1

Geometry of the Complex plane

This section is a brief reminder of Sections 3 and 4 of MA1006 Algebra.

Definition 1.1. The *complex numbers* $\mathbb{C}$ are the set of all pairs $z = (x, y) \in \mathbb{R}^2$ of real numbers with the addition

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) \tag{1.1}$$

and the multiplication

$$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1), \tag{1.2}$$

where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. We call $x = \Re(z)$ the *real part* and $y = \Im(z)$ the *imaginary part* of z.

From Section 3.8 in MA1006 Algebra we recall:

Proposition 1.1. *The complex numbers are a field.*

Remark 1.1.

We view the real numbers as a subset of $\mathbb{C}$ by identifying $x \in \mathbb{R}$ with $(x, 0) \in \mathbb{C}$. The **imaginary unit** is $i = (0, 1)$. With these conventions, a calculation using Equation 1.2 shows that

$$z = x + iy. \tag{1.3}$$

Using this notation, we can manipulate complex numbers in the same way as real numbers, keeping in mind the identity

$$i^2 = -1. \tag{1.4}$$

Of course, $\mathbb{C}$ is a one-dimensional vector space over itself. Restricting the scalar multiplication to $\mathbb{R}$ makes $\mathbb{C}$ into a vector space over $\mathbb{R}$, isomorphic to $\mathbb{R}^2$, of dimension two with standard basis $1, i \in \mathbb{C}$.

Proposition 1.2.

a. *In the standard basis, every* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_{2\times 2}(\mathbb{R})$ *corresponds uniquely to an* $\mathbb{R}$*-linear map* $T_A \colon \mathbb{C} \to \mathbb{C}$, *namely*

$$T_A(x + iy) = (ax + by) + i(cx + dy). \tag{1.5}$$

b. T_A *is* $\mathbb{C}$*-linear* $\iff$ $a = d$ *and* $b = -c$. *In this case,*

$$T_A(z) = \alpha \cdot z, \qquad \alpha = a + ic. \tag{1.6}$$

Proof.

(a) This is a recap from linear algebra. An $\mathbb{R}$-linear map $T\colon \mathbb{C} \to \mathbb{C}$ is uniquely determined by the image of the basis vectors $T(1), T(i)$ which, conversely, may be prescribed arbitrarily. If we write $T(1) = a+ic, T(i) = b+id$, then the linear transform T is described entirely by $a, b, c, d \in \mathbb{R}$. The expression Equation 1.5 is obtained by expanding the left hand side by $\mathbb{R}$-linearity.

(b) In the same way, $\mathbb{C}$-linear maps correspond to $\alpha \in \mathrm{M}_{1\times 1}(\mathbb{C})$ as in Equation 1.6. An $\mathbb{R}$-linear map T_A is $\mathbb{C}$-linear $\iff$ $T_A(i) = T_A(i1) = iT_A(1)$ $\iff$

$$\begin{pmatrix} b \\ d \end{pmatrix} = T_A(i) = iT_A(1) = i\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} -c \\ a \end{pmatrix}.$$

In this case, $T_A(x+iy) = (ax-cy) + i(cx+ay) = \alpha(x+iy)$.

$\square$

Definition 1.2. The *conjugate* of $z \in \mathbb{C}$ is the complex number

$$\overline{z} = (x, -y),$$

and the *modulus* (also called *absolute value*) is

$$|z| = \sqrt{x^2+y^2} \geqslant 0.$$

Proposition 1.3. *The following formulas hold for* $z, w \in \mathbb{C}$:

$$\overline{z \cdot w} = \overline{z} \cdot \overline{w} \qquad \overline{z + w} = \overline{z} + \overline{w} \tag{1.7}$$

$$\overline{\overline{z}} = z \qquad \overline{i} = -i,\ \overline{1} = 1 \tag{1.8}$$

$$z \cdot \overline{z} = |z|^2 \qquad |z \cdot w| = |z| \cdot |w| \tag{1.9}$$

$$\Re(z) = \frac{z + \overline{z}}{2} \qquad \Im(z) = \frac{z - \overline{z}}{2i} \tag{1.10}$$

$$z^{-1} = \frac{\overline{z}}{|z|^2} \quad \textit{if } z \neq 0 \tag{1.11}$$

Proposition 1.4. *The following inequalities hold for* $z, w \in \mathbb{C}$ *:*

$$|z + w| \leqslant |z| + |w| \qquad |z - w| \geqslant \big||z| - |w|\big| \tag{1.12}$$

$$|\Re(z)| \leqslant |z| \qquad |\Im(z)| \leqslant |z| \tag{1.13}$$

Thinking of complex numbers as points in the plane, we can use **polar coordinates** to represent them (see Figure 1.1).

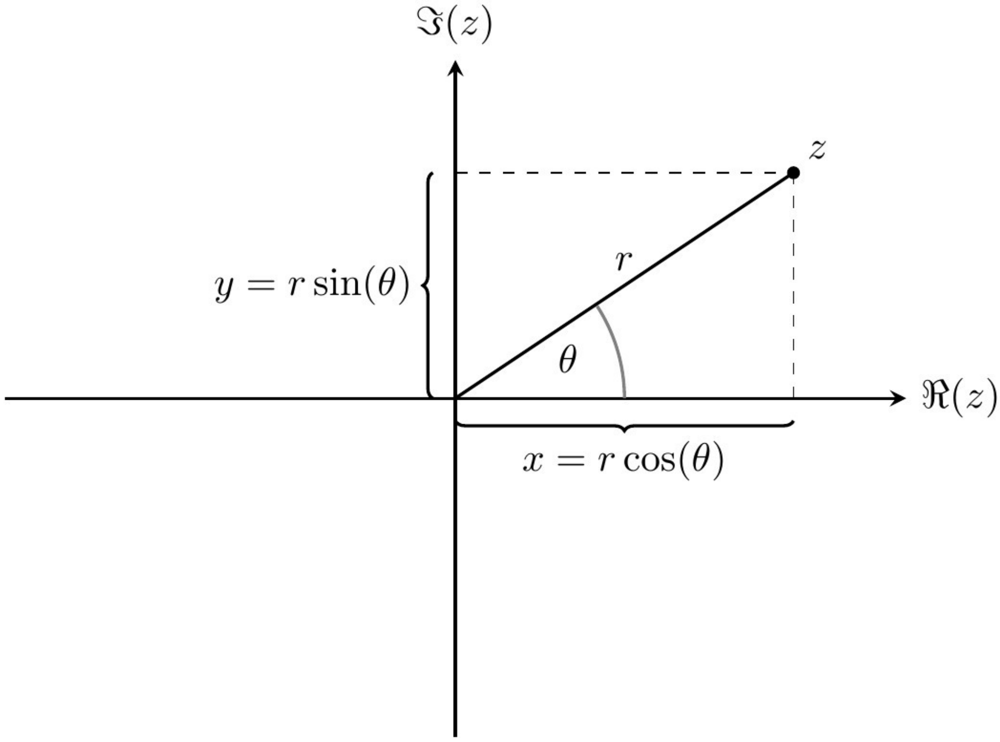


Figure 1.1: Polar coordinates

Proposition 1.5. *For every non-zero complex number $z = (x, y)$ there is an* ***argument*** *$\theta \in \mathbb{R}$ and a* ***radius*** *$r = |z| > 0$ such* ***that***

$$x = r\cos(\theta), y = r\sin(\theta). \tag{1.14}$$

This representation is unique up to replacing θ by $\theta + 2\pi k$ for any $k \in \mathbb{Z}$.

Proof. (omitted)

Since $x^2 + y^2 = r^2(\cos(\theta)^2 + \sin(\theta)^2) = r^2$, the radius must be $r = |z|$. Define the complex number $w = r^{-1}z$ and write $w = u + iv$ for its real and imaginary parts.

We will prove the existence of $\theta \in \mathbb{R}$ with $u = \cos(\theta)$, $v = \sin(\theta)$. This also proves the existence of a representation Equation 1.14, by multiplying by r. Since $u^2 + v^2 =$

$|w|^2 = r^{-2}|z|^2 = 1$, we know $|u| \leqslant 1$, $|v| \leqslant 1$. Recall that $\cos\colon [0,\pi] \to [-1,1]$ and $\sin\colon [-\pi/2,\pi/2] \to [-1,1]$ are bijections. Hence

$$u = \cos(\alpha) \qquad \text{for some } \alpha \in [0,\pi],$$
$$v = \sin(\beta) \qquad \text{for some } \beta \in [-\pi/2,\pi/2].$$

As $\sin(\beta)^2 = v^2 = 1 - u^2 = 1 - \cos(\alpha)^2 = \sin(\alpha)^2$, we have $\sin(\alpha) = \pm\sin(\beta) = \sin(\pm\beta)$. To produce the correct θ, we distinguish two cases.

Case 1 $\alpha \in [0,\pi/2]$. Then $\alpha = \pm\beta$ by the injectivity of the sine function on the interval $[-\pi/2,\pi/2]$. Setting $\theta = \pm\alpha = \beta$, we find that $u = \cos(\theta)$ and $v = \sin(\theta)$, as required.

Case 2 $\alpha \in [\pi/2,\pi]$. Then $\pi - \alpha, \beta \in [-\pi/2,\pi/2]$ and $\sin(\pi-\alpha) = \sin(\alpha) = \pm\sin(\beta)$, so $\pi - \alpha = \pm\beta$ by injectivity. Setting $\theta = \pm\alpha = \pm\pi - \beta$, we find $u = \cos(\theta), v = \sin(\theta)$, using trigonometric identities.

This completes the existence part of the proof. For uniqueness, we already know that $r = |z| > 0$ is unique, so it remains to consider

$$x = r\cos(\theta_1) = r\cos(\theta_2), \qquad y = r\sin(\theta_1) = r\sin(\theta_2).$$

To translate the situation into an interval that we understand, pick $k_1, k_2 \in \mathbb{Z}$ so that $\theta_1 + 2\pi k_1, \theta_2 + 2\pi k_2 \in [-\pi,\pi)$. Then

$$\cos(\theta_1 + 2\pi k_1) = \cos(\theta_1) = \cos(\theta_2) = \cos(\theta_2 + 2\pi k_2).$$

Using the injectivity of the cosine function and considering cases as above, we find that $\theta_1 + 2\pi k_1 = \pm(\theta_2 + 2\pi k_2)$. If the sign is '+' we get $\theta_1 - \theta_2 = 2\pi(k_2 - k_1)$

and we are done, so suppose $\theta_1 + 2\pi k_1 = -(\theta_2 + 2\pi k_2)$. Then

$$\sin(\theta_1) = \sin(\theta_2) = \sin(\theta_2 + 2\pi k_1) = -\sin(\theta_1 + 2\pi k_1) = -\sin(\theta_1)$$

implies $\sin(\theta_1) = 0$. Therefore θ_1 is a multiple of 2π, which implies that $\theta_1 + 2\pi k_1 = -(\theta_2 + 2\pi k_2) = 0$ since these numbers were chosen in $[-\pi, \pi)$ and we have $2\pi\mathbb{Z} \cap [-\pi, \pi) = \{0\}$. Hence $\theta_1 - \theta_2 = 2\pi(k_2 - k_1)$. $\square$

To get around the non-uniqueness of the argument in polar coordinates, we restrict θ to lie in a half-open interval of length 2π. Here is the most common convention.

Definition 1.3. The **principal argument** of a non-zero $z \in \mathbb{C}$ is the unique $\theta \in (-\pi, \pi]$ such that Equation 1.14 holds, and we write $\arg(z) = \theta$.

Definition 1.4. The value of the **exponential function** at the complex number $z = x + i\theta$, where $x, \theta \in \mathbb{R}$, is defined as

$$e^{x+i\theta} = e^x(\cos(\theta) + i\sin(\theta)). \tag{1.15}$$

Proposition 1.5 implies that every complex number can be represented in **polar form**

$$z = re^{i\theta}. \tag{1.16}$$

The addition of complex numbers is the usual addition of vectors in $\mathbb{R}^2$. To visualize multiplication, the polar form is useful. Combining Equation 1.2 and Equation 1.15, we find that

$$
\begin{aligned}
e^{i\theta_1} \cdot e^{i\theta_2} &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \\
&\quad + i\big[\cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1)\big] \\
&= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = e^{i(\theta_1+\theta_2)}.
\end{aligned}
$$

Here we have used the trigonometric addition formulas. Hence

$$
z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2} \implies z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}. \tag{1.17}
$$

Complex multiplication adds the angles and multiplies the radii.

Proposition 1.6. $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$ *for all* $z_1, z_2 \in \mathbb{C}$. *Moreover, we have* $(e^z)^n = e^{nz}$ *for all* $z \in \mathbb{C}$, $n \in \mathbb{Z}$.

Proof.

The first assertion follows from Equation 1.17 combined with the identity $e^{x_1}e^{x_2} = e^{x_1+x_2}$ for $x_1, x_2 \in \mathbb{R}$ from MA1005 Calculus. The second claim follows from this by induction.

□

The polar form can be applied to the construction of n^{th} roots.

For example, the **n^th^ root of unity** is $\zeta_n = e^{i\frac{2\pi}{n}}$ and satisfies

$$
(\zeta_n)^n = (e^{i\frac{2\pi}{n}})^n = e^{2\pi i} = 1.
$$

Proposition 1.7. *Every complex number* $z \neq 0$ *has an* n^{th} *root* w *satisfying* $w^n = z$.

If w is an n^{th} root of z, the set of all n^{th} roots of z is

$$\{w, \zeta_n \cdot w, \zeta_n^2 \cdot w, \dots, \zeta_n^{n-1} \cdot w\}.$$

Proof.

Write $z = re^{i\theta}$ and $w = se^{i\varphi}$ for $\theta, \varphi \in [0, 2\pi)$ and $r, s > 0$. By the uniqueness of the polar form, the equation $w^n = z$ is equivalent to $s^n = r$ and $n\varphi = \theta + 2\pi k$ for some $k \in \mathbb{Z}$. Of course, $s = \sqrt[n]{r}$ is the unique positive n^{th} root from MA1005 Calculus. From $0 \leqslant n\varphi < 2\pi n$ and $0 < -\theta \leqslant 2\pi$ we get

$$0 < k = \frac{n\varphi - \theta}{2\pi} < n.$$

Since k is an integer, this implies $k = 0, \dots, n-1$ and hence $\varphi = \frac{\theta + 2\pi k}{n}$ for such k are the only possible solutions for φ. In summary,

$$w_k = \sqrt[n]{r}e^{i\varphi} = \sqrt[n]{r}e^{i\frac{\theta+2\pi k}{n}} = \zeta_n^k \cdot w_0, k = 0, \dots, n-1,$$

are all the possible n^{th} roots of z.

$\square$

Questions for further discussion

- The complex numbers are obtained by 'adjoining' a symbol i with $i^2 = -1$. If instead we would have adjoined a different symbol ϵ with $\epsilon^2 = -1$, would the set of elements $x + \epsilon y$ still define a field?
- The real numbers have a total order '$\leqslant$'. Why doesn't it make sense to extend this definition to the complex numbers?

- Describe geometrically the set $R_n = \{1, \zeta_n, \dots, (\zeta_n)^{n-1}\}$ of n^{th} roots of unity. Find a connection between R_n and the cyclic group $C_n = \{\overline{0}, \dots, \overline{n-1}\}$ of integers modulo n from MX3020 Group Theory.

1.1 Exercises

Note

This problem sheet is intended as a recap and contains more problems than can be discussed during the tutorials.

Exercise 1.1

Verify

$$z = x + iy$$

and

$$i^2 = -1$$

straight from the definition Equation 1.2.

Show Solution 1.1 on P144

Exercise 1.2

How many real solutions x does $x^2 + 1 = 0$ have? Show that the polynomial equation $z^2 + 1 = 0$ has exactly two solutions $z \in \mathbb{C}$.

Show Solution 1.2 on P145

Exercise 1.3

Give examples of complex numbers $z, w \neq 0$ such that $z^2 + w^2 = 0$.

Show Exercise 1.3 on P146

Exercise 1.4

Sketch the position of the complex numbers $i, 1+i, \frac{3+2i}{4}$ in the plane.

Show Solution 1.4 on P147

Exercise 1.5

Express the following complex numbers z in the form $x+iy$ with $x, y \in \mathbb{R}$.

$$(1+i)^{20},\ (5+3i)(1+2i),\ (1-i)(2+3i),\ (1-i)i(1+i),\ \frac{2+i}{1-i}$$

Show Solution 1.5 on P147

Exercise 1.6

Express the following complex numbers z in the form $x+iy$ with $x, y \in \mathbb{R}$.

$$1/i,\ \frac{1}{1+i},\ \frac{3+i}{3-i}$$

Show Solution 1.6 on P148

Exercise 1.7

Find the modulus and the conjugate of the following complex numbers.

$$2+i,\ i,\ 5-3i,\ \frac{1+i}{2+i}$$

Show Solution 1.7 on P148

Exercise 1.8

Describe the sets $A = \{z \in \mathbb{C} \mid \Im(z) > 0\}$, $B = \{z \in \mathbb{C} \mid \Re(z) \leqslant 1\}$, $C = \{z \in \mathbb{C} \mid \Re((1+i)z) = 0\}$, and $A \cap B$ geometrically.

Show Solution 1.8 on P149

Exercise 1.9

Describe the set $D = \{z \in \mathbb{C} \mid z \cdot \overline{z} = 1\}$ geometrically.

Hint: Write $z = re^{i\theta}$ in polar form.

Show Solution 1.9 on P152

Exercise 1.10

Draw all nine sets described by the following conditions on the complex number z.

$$\begin{array}{ccc} |z| = 1, & |z| < 1, & 1 < |z| < 2, \\ |1+z| > 1, & |2-z| < 2, & 3 < |z+i| < 4, \\ |z-1| < |z+1|, & |z| = |z+1|, & |z-1| = |z+i|. \end{array}$$

Show Solution 1.10 on P153

Exercise 1.11

Let $S = \{x + iy \in \mathbb{C} \mid 0 \leqslant x, y \leqslant 1\}$. Draw S and the sets

$$\begin{array}{ll} A = \{2z \mid z \in S\}, & B = \{\overline{z} \mid z \in S\}, \\ C = \{-z \mid z \in S\}, & D = \{z^2 \mid z \in S\}. \end{array}$$

Show Solution 1.11 on P158

Exercise 1.12

Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disk. Draw the sets

$$A = \{2z \mid z \in D\}, \quad B = \{z^2 \mid z \in D\}, \quad C = \{|z| \mid z \in D\}.$$

Show Solution 1.12 on P160

Exercise 1.13

Show that $i = e^{i\pi/2}$ and $-1 = e^{i\pi}$.

Show Solution 1.13 on P160

Exercise 1.14

Express the following complex numbers z in the form $x + iy$ with $x, y \in \mathbb{R}$.

$$e^{i\pi/4}, \; e^{i\pi}, \; e^{i\frac{2\pi}{3}}$$

Show Solution 1.14 on P160

Exercise 1.15

Write each of the following complex numbers in polar form $re^{i\theta}$ with $r > 0$ and $-\pi < \theta \leqslant \pi$.

$$i, \quad -1, \quad -i, \quad 1+i, \quad 1-i, \quad i-1, \quad \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

Draw each of these numbers in the complex plane.

Show Solution 1.15 on P161

Exercise 1.16

Calculate i^{2021} and $(1+i)^{20}$.

Show Solution 1.16 on P162

Exercise 1.17

Solve the equation $(1-i)^n - 2075 = 2021$ and find $n \in \mathbb{N}$.

Show Solution 1.17 on P163

Exercise 1.18

Prove that for $z \in \mathbb{R}$ we have

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Use these equations to extend the definition of the functions $\cos(z), \sin(z)$ to complex arguments $z \in \mathbb{C}$. Find $z \in \mathbb{C}$ with $\sin(z) = 2$.

Hint: Put $w = e^{iz}$ and reduce to a quadratic equation.

Show Solution 1.18 on P163

Exercise 1.19

Prove the following statements.

a. $\overline{zw} = \overline{z} \cdot \overline{w}$ for all $z, w \in \mathbb{C}$

b. $\overline{z_1 z_2 \cdots z_n} = \overline{z}_1 \overline{z}_2 \cdots \overline{z}_n$ for all $z_1, z_2, \dots, z_n \in \mathbb{C}$ (use induction)

c. $\overline{(z^n)} = (\overline{z})^n$ for all $z \in \mathbb{C}$

d. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a polynomial with real coefficients $a_0, \dots, a_n \in \mathbb{R}$. Prove that $\overline{p(z)} = p(\overline{z})$. Deduce that all roots of $p(z)$ occur in complex conjugate pairs.

Show Solution 1.19 on P165

Exercise 1.20

Show that $|z| = |-z|$ and $|\overline{z}| = |z|$. Prove also that $|\lambda z| = \lambda |z|$ for all $\lambda \geqslant 0$.

Show Solution 1.20 on P166

Exercise 1.21

Prove that $\mathfrak{R}(z) = \frac{1}{2}(z + \overline{z})$, $\mathfrak{I}(z) = \frac{1}{2i}(z - \overline{z})$.

Show Solution 1.21 on P166

Exercise 1.22

Prove that $\overline{e^z} = e^{\overline{z}}$. Deduce that $|e^z| = e^{\mathfrak{R}(z)}$.

Show Solution 1.22 on P167

Exercise 1.23

Prove that $|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2$.

Show Solution 1.23 on P167

Exercise 1.24

Show that $|z+w|^2 = |z|^2+2\Re(z\overline{w})+|w|^2$. Use this to determine the conditions on z, w for $|z+w| = |z| + |w|$ to hold.

Show Solution 1.24 on P168

Exercise 1.25

Assuming we know the triangle inequality $|z + w| \leqslant |z| + |w|$ for all $z, w \in \mathbb{C}$, prove the reverse triangle inequality

$$|z - w| \geqslant \big||z| - |w|\big|.$$

Show Solution 1.25 on P169

Exercise 1.26

Let K be a field with $\mathbb{R} \subset K \subset \mathbb{C}$. Prove that $K = \mathbb{R}$ or $K = \mathbb{C}$.

Show Solution 1.26 on P170

Further resources

- Freitag–Busam (Freitag and Busam 2009, chap. 1) for additional exercises and historical background.
- https://en.wikipedia.org/wiki/Complex_number for overview and history
- https://youtu.be/T647CGsuOVU for some visualization

Chapter 2

Complex-valued functions

Definition 2.1. A *complex function* $f\colon D \to \mathbb{C}$ is a map with *domain of definition* $D \subset \mathbb{C}$ and codomain the complex plane. Thus, f assigns to each $z = x + iy \in D$ in the domain a complex number

$$f(z) = u(z) + iv(z).$$

We call $u\colon D \to \mathbb{R}$ the *real part* and $v\colon D \to \mathbb{R}$ the *imaginary part* of the complex function f.

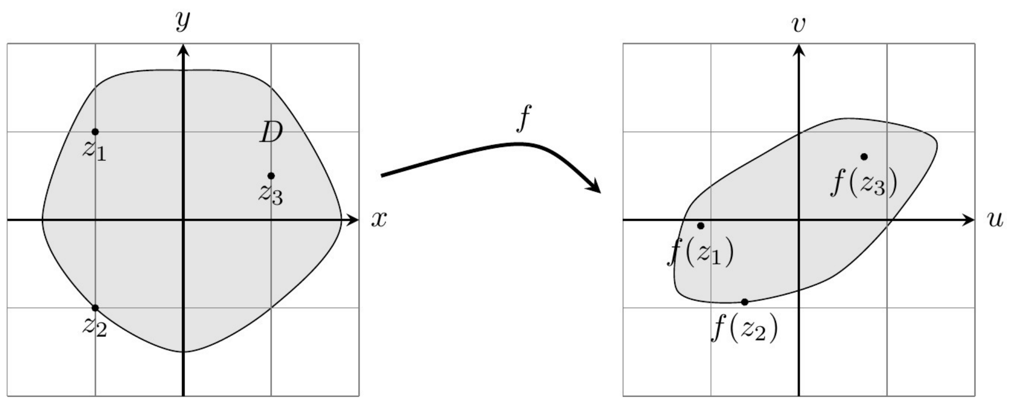


Figure 2.1: Schematic picture of a complex function

We can visualize complex functions as in Figure 2.1. In practice, most complex functions are defined by a formula.

Example 2.1.

Let $a, b \in \mathbb{C}$ be fixed complex numbers. Then $f(z) = az + b$ is a complex function with domain $D = \mathbb{C}$, written $f\colon \mathbb{C} \to \mathbb{C}$. For instance,

$$f(z) = iz, f(z) = 2z, f(z) = z + 3.$$

Example 2.2.

A *polynomial* is a complex function $f\colon \mathbb{C} \to \mathbb{C}$ of the form

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0, \tag{2.1}$$

with complex *coefficients* $a_n, \ldots, a_0 \in \mathbb{C}$ and with domain $D = \mathbb{C}$.

More generally, a *rational function* is a complex function for which there are polynomials f, g with $g \neq 0$ so that

$$h(z) = \frac{f(z)}{g(z)}. \tag{2.2}$$

The domain of h is $D = \{z \in \mathbb{C} \mid g(z) \neq 0\}$. However, if the polynomials f, g have a common factor, we can cancel that factor in the fraction Equation 2.2 and regard h as a complex function on a larger domain.

For example, $h(z) = \frac{z+1}{z^2-1}$ has domain $\mathbb{C} \setminus \{\pm 1\}$, but we can rewrite the fraction as $h(z) = \frac{1}{z-1}$, which makes sense on the extended domain $\mathbb{C} \setminus \{+1\}$.

We have already met several complex functions in the previous section.

Example 2.3.

The *exponential function* $\exp\colon \mathbb{C} \to \mathbb{C}$ is defined by $\exp(z) = e^z = e^x(\cos(y) + i\sin(y))$ and has domain $D = \mathbb{C}$.

Example 2.4.

The *sine function* $\sin\colon \mathbb{C} \to \mathbb{C}$ and the *cosine function* $\cos\colon \mathbb{C} \to \mathbb{C}$ are the complex functions defined by

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}. \tag{2.3}$$

(See the corresponding exercise in Section 1.1)

Definition 2.2. We define the following subsets of the complex plane:

$$\begin{aligned}
\mathbb{C}^\times &= \{z \in \mathbb{C} \mid z \neq 0\} && \textbf{punctured plane}\\
\mathbb{C}^- &= \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leqslant 0\} && \textbf{slit plane}\\
S &= \{z = x + iy \mid y \in (-\pi, \pi)\} && \textbf{principal strip}\\
\mathbb{H} &= \{z = x + iy \mid y > 0\} && \textbf{upper half-plane}
\end{aligned}$$

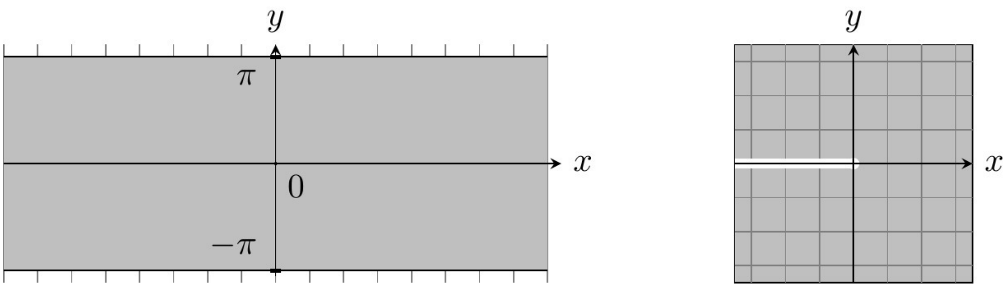


Figure 2.2: The principal strip S and the slit plane $\mathbb{C}^-$

Example 2.5.

The *principal branch* of the *complex logarithm* is the complex function

$$\mathsf{log}\colon \mathbb{C}^- \longrightarrow S \subset \mathbb{C}$$

defined by

$$\mathsf{log}(w) = \mathsf{log}(r) + i\theta \iff w = re^{i\theta}, r > 0, \theta \in (-\pi, \pi). \tag{2.4}$$

Here $\mathsf{log}(r)$ denotes the (real) logarithm from MA1005 Calculus. In other words, the real part of $\mathsf{log}(w)$ is the logarithm of the modulus $r = |w|$ and the imaginary part of $\mathsf{log}(w)$ is the argument function $\mathsf{arg}(w)$ from Definition 1.3.

To evaluate Equation 2.4, write w in polar coordinates, ensuring that $\theta \in (-\pi, \pi)$. For example, $i = e^{i\pi/2}$ and so $\mathsf{log}(i) = \mathsf{log}(1) + i\frac{\pi}{2} = i\frac{\pi}{2}$.

Proposition 2.1.

a. *The exponential function is surjective onto* $\mathbb{C}^\times$.

b. *We have*

$$\mathsf{exp}(\mathsf{log}(w)) = w \ (\forall w \in \mathbb{C}^-) \tag{2.5}$$

$$\mathsf{log}(\mathsf{exp}(z)) = z \ \ (\forall z \in S). \tag{2.6}$$

Hence the restriction $\mathsf{exp}\,|_S$ *of the exponential to the principal strip is a bijection* $\mathsf{exp}\,|_S\colon S \to \mathbb{C}^-$ *onto the slit plane, with inverse* $\mathsf{log}(w)$.

Proof.

a. Recall from MA1005 Calculus that $e^x \neq 0$ for all $x \in \mathbb{R}$. As $|\,\mathsf{exp}(x + iy)| = e^x \neq 0$, the image of exp is contained in $\mathbb{C}^\times$. For proving that exp is onto $\mathbb{C}^\times$, recall that every non-zero number can be written in polar form $w = re^{i\theta}$, $\theta \in (-\pi, \pi]$. Then $\mathsf{exp}(z) = w$ for $z = \mathsf{log}(r) + i\theta$.

b. Equations Equation 2.5, Equation 2.6 are straightforward to verify using Equation 2.4.

□

Since the *graph* $\Gamma(f) = \{(z, w) \in D \times \mathbb{C} \mid f(z) = w\}$ of a complex function is a subset of four-dimensional space, we cannot visualize complex functions as easily as real functions. We will now discuss some alternatives.

Image grid

A useful way to picture a complex function is to sketch its values on a grid G. The image grid $f(G)$ is a distorted version of the original grid which can be navigated easily using the grid lines. For example, to determine $f(1 + 2i)$, take one step in x-direction and two steps in y-direction on the distorted grid. Formally, let $G = \{z = x + iy \in \mathbb{C} \mid x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$ be the *unit grid* and define the *image grid* as (see Figure 2.3)

$$f(G) = \{w = u + iv \in \mathbb{C} \mid \exists z \in G : f(z) = w\}.$$

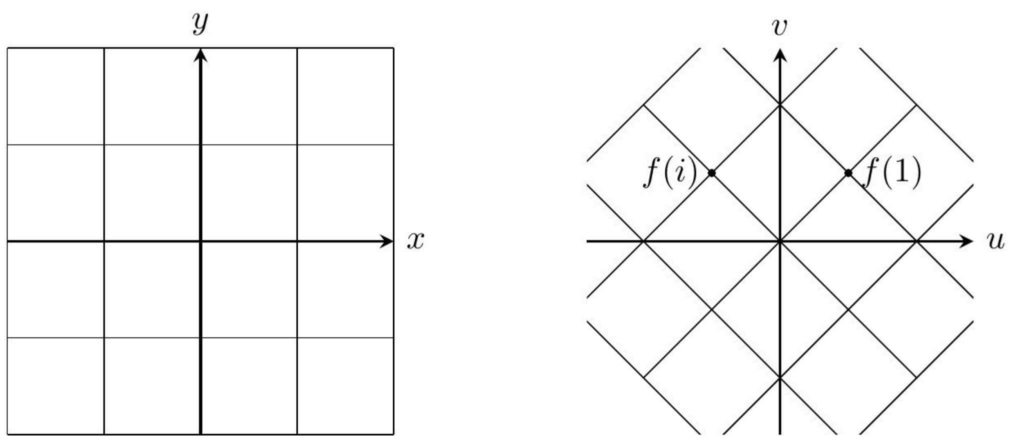


Figure 2.3: The unit grid G and the image grid $f(G)$ for $f(z) = \frac{1+i}{\sqrt{2}} z$

In practice, the image grid can often be described by finding a familiar equation that all of its members $u + iv$ satisfy. When this is not possible, a computer will help sketching an approximate image grid.

Example 2.6.

Consider $f(z) = z^2$. Then

$$u = x^2 - y^2, \qquad v = 2xy. \tag{2.7}$$

To determine $f(G)$, first fix $x = \pm 1, \pm 2, \pm 3, \ldots$ to be a non-zero integer. Using Equation 2.7 we find $u = \frac{-1}{4x^2}v^2 + x^2$. This is a downward parabola in the (u, v)-plane rotated by 90 degrees with vertex at $(u, v) = (x^2, 0) = (1, 0), (4, 0), (9, 0), \ldots$. Similarly, if we fix y to be a non-zero integer, then $u = \frac{1}{4y^2}v^2 - y^2$ is an upward parabola in the (u, v)-plane rotated by 90 degrees. For $x = 0$, we get $(u, v) = (-y^2, 0)$ which parameterizes the negative u-axis. Similarly, for $y = 0$ we get the positive u-axis. All this is summarized in Figure 2.4.

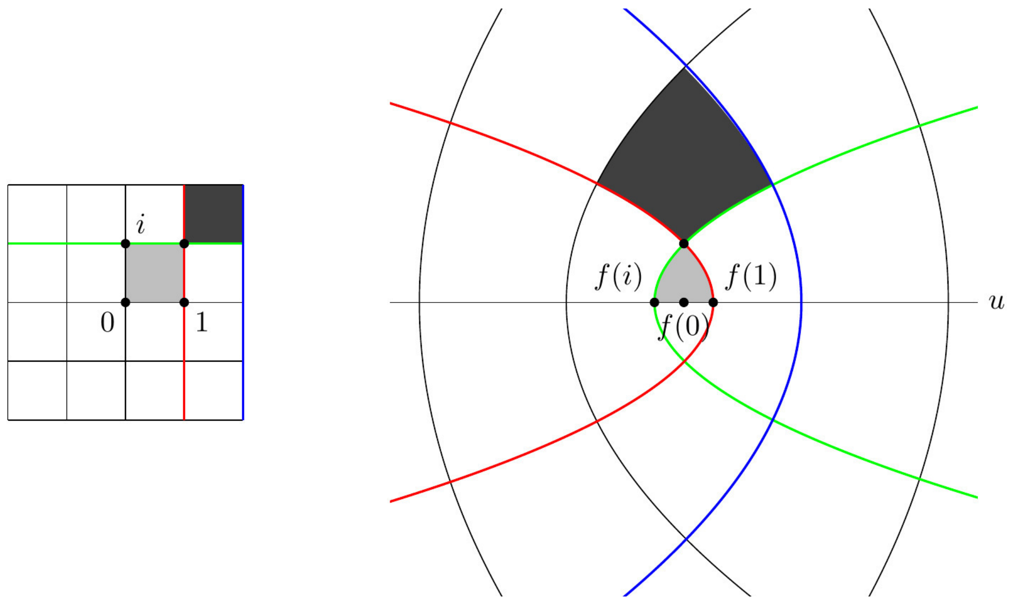


Figure 2.4: Unit grid and image grid of $f(z) = z^2$

Example 2.7.

Consider $f(z) = 1/z$. Then

$$u = \frac{x}{x^2 + y^2}, \qquad v = \frac{-y}{x^2 + y^2}. \tag{2.8}$$

Fixing $x = \pm 1, \pm 2, \dots,$ we have $(u - \frac{1}{2x})^2 + v^2 = \frac{1}{4x^2}$ (verify by substituting Equation 2.8 into this equation). This is a circle of radius $\frac{1}{2x}$ and with center $(u, v) = (\frac{1}{2x}, 0)$. For $x = 0$, equations Equation 2.8 become $(u, v) = (0, -1/y)$ which parameterizes the v-axis. Similarly, for $0 \neq y \in \mathbb{Z}$ we find $u^2 + (v + \frac{1}{2y})^2 = \frac{1}{4y^2}$ and for $y = 0$ we obtain a parametrization of the u-axis. This is summarized in Figure 2.5 (and on the title page).

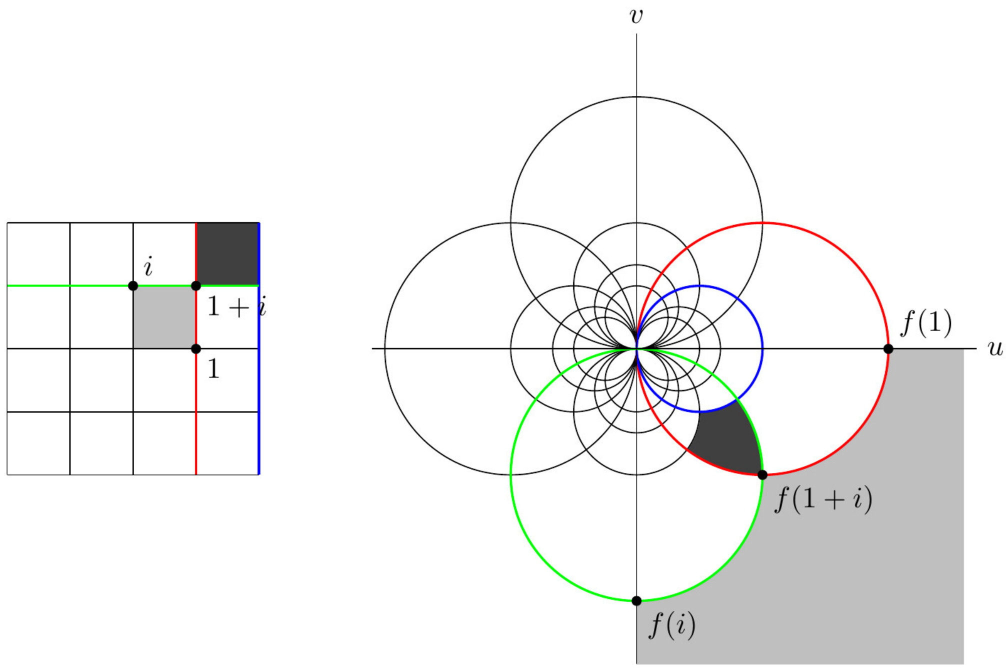


Figure 2.5: Unit grid and image grid of $f(z) = 1/z$. Notice that $z = 0$ gets sent to a 'point at infinity' that is imagined to surround the complex plane.

The previous example can be generalized.

Example 2.8.

Rational functions Equation 2.2 with $f(z) = az + b$ and $g(z) = cz + d$ affine linear, where $a, b, c, d \in \mathbb{C}$, are *Möbius transformations*. Thus

$$f(z) = \frac{az + b}{cz + d}$$

with domain $D = \mathbb{C} \setminus \{-d/c\}$ if $c \neq 0$ and domain $D = \mathbb{C}$ if $c = 0$. To exclude constant functions, we also assume that $ad - bc \neq 0$.

Domain colouring

We represent each unit complex number $e^{i\theta}$ by a color on the color wheel. The modulus r of an arbitrary complex number $re^{i\theta}$ will be represented by the lightness of the color. This assigns a unique color to each complex number, see Figure 2.6. Pure white is never used and would correspond to infinity. Pure black corresponds to the origin.

This is less useful for calculations by hand but generates artistic images using a computer.

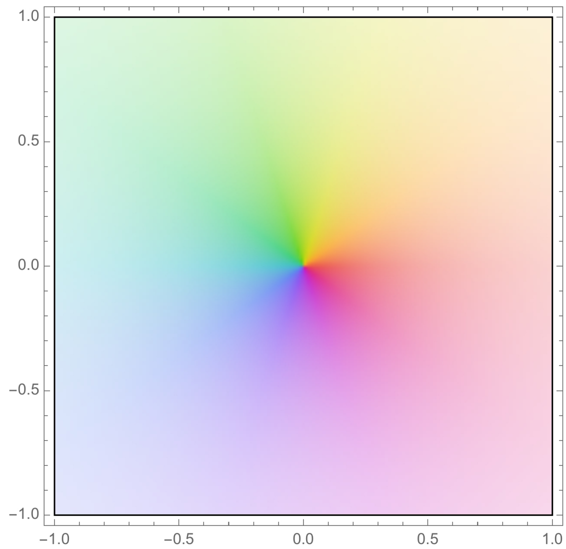


Figure 2.6: Representing complex numbers z by color

This can be used for visualizing complex functions. Draw each point z in the domain of $w = f(z)$ using the color for w.

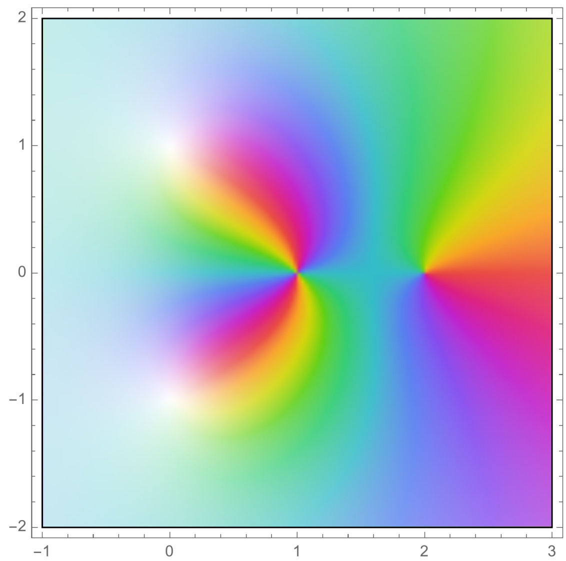


Figure 2.7: Domain colouring of $f(z) = \frac{(z-2)(z-1)^2}{z^2+1}$. Notice that near $z = \pm i$, where f is undefined, the function tends to infinity (pure white). The zeros of f at $z = 1, 2$ can also be seen.

3-dimensional graphs

Another approach is to plot the 3-dimensional graph of any of the following real-valued functions $D \to \mathbb{R}$

$$u, v, |f| = \sqrt{u^2 + v^2}.$$

Again, the missing information can be color-coded (see Figure 2.8).

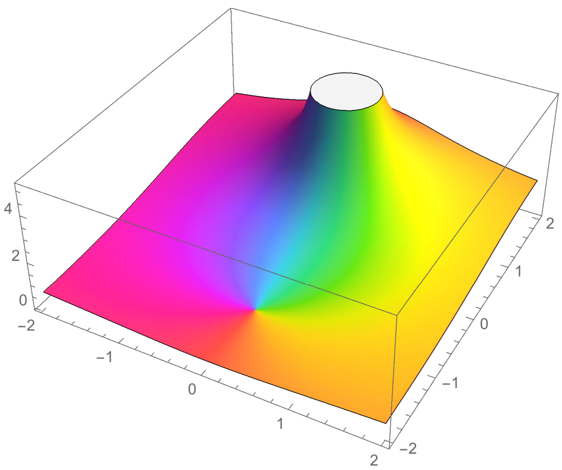


Figure 2.8: The graph of $|f(z)|$ colored by arg(z)

Questions for further discussion

- What are the functions in Example 2.1 geometrically?
- Does Equation 2.6 remain valid for all $z \in \mathbb{C}$? What is the correct modification?
- The two zeros in Figure 2.7 have a slightly different character. What is the difference between the zeros that might account for this?
- In Example 2.6 and Example 2.7} almost all the image gridlines meet at a right angle. The only exception is at $f(0)$ in Example 2.6. Use polar coordinates to explain this behavior of $f(z) = z^2$ at $z = 0$.
- Is it always possible to extend the domain of definition of a complex function?

Is this always sensible?

- Can you think of other branches of the logarithm?

2.1 Exercises

Exercise 2.1

Describe the image set of the complex function $f(z) = \frac{1+z}{1-z}$ with domain $D = \mathbb{C} \setminus \{1\}$. In other words, determine the set of all $w \in \mathbb{C}$ for which $w = \frac{1+z}{1-z}$ has a solution $z \in D$.

Show Solution 2.1 on P170

Exercise 2.2

Sketch the following curves $z(t)$ in the complex plane, where t is a real parameter.

a. $z(t) = t(1+i)$ for $0 \leqslant t \leqslant 1$
b. $z(t) = \cos(t) + i \sin(t)$ for $0 \leqslant t \leqslant \pi$
c. $z(t) = \cos(t) - i \sin(t)$ for $0 \leqslant t \leqslant \pi/2$
d. $z(t) = \frac{1}{1+it}$ for $t \in \mathbb{R}$

Show Solution 2.2 on P171

Exercise 2.3

Let $\log(z)$ be the principal branch of the logarithm. Compute

$$\log(2i), \log(1+i), \log(-3i), \log(5).$$

Show Solution 2.3 on P174

Exercise 2.4

Describe the following complex functions geometrically.

$$f(z) = 3z, f(z) = iz, f(z) = \frac{(1+i)}{\sqrt{2}}z$$

Show Solution 2.4 on P175

Exercise 2.5

Determine a domain of definition for the following complex functions.

$$f(z) = \frac{1}{z}, f(z) = \frac{1+z}{z-1}, f(z) = \frac{z^2-4}{z^2+2z}, f(z) = \frac{1}{\mathsf{exp}(z)}.$$

Show Solution 2.5 on P175

Exercise 2.6

Determine the domain of definition for $f(z) = \frac{1}{\sin(z)}$.

Show Solution 2.6 on P176

Exercise 2.7

Find all solutions to the following equations:

a. $e^z = -1$,
b. $\sin(z) = -i$

Show Solution 2.7 on P177

Exercise 2.8

Prove that the composition $f \circ f'$ of two Möbius transformations is again a Möbius transformation. If we associate to f, f' the matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, A' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathsf{M}_{2\times 2}(\mathbb{C}),$$

show that $f \circ f'$ is associated to the matrix product AA'.

Show Solution 2.8 on P177

Exercise 2.9

Draw the image grid for $\exp\colon \mathbb{C} \to \mathbb{C}^\times$.

Show Solution 2.9 on P178

Further resources

- https://youtu.be/NtoIXhUgqSk 5 ways to visualize a complex function
- Animation of Möbius transformations: https://youtu.be/0z1fIsUNhO4 and https://www-users.cse.umn.edu/~arnold/moebius/

Chapter 3

Topology of the complex plane

In this section we explain that most of the facts about limits, series, and continuity carry over from real analysis essentially without change.

The modulus of complex numbers defines a *distance* $d(z, w) = |z - w|$ on the plane (this is the usual Euclidean distance), which determines the following standard terminology for metric spaces.

Definition 3.1. The *open disc* of radius $0 \leqslant r \leqslant +\infty$ centered at $z_0 \in \mathbb{C}$ is

$$D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

The *closed disc* $\overline{D}_r(z_0)$ is the set of all $z \in \mathbb{C}$ with $|z - z_0| \leqslant r$.

Definition 3.2. A subset $O \subset \mathbb{C}$ is *open* if for every point $z_0 \in O$ there exists $r > 0$ such that $D_r(z_0) \subset O$ (see Figure 3.1). A subset $C \subset \mathbb{C}$ is called *closed* if the complement $O = \mathbb{C} \setminus C$ is an open subset.

A subset $B \subset \mathbb{C}$ is *bounded* if $B \subset D_r(0)$ for some $0 < r < \infty$.

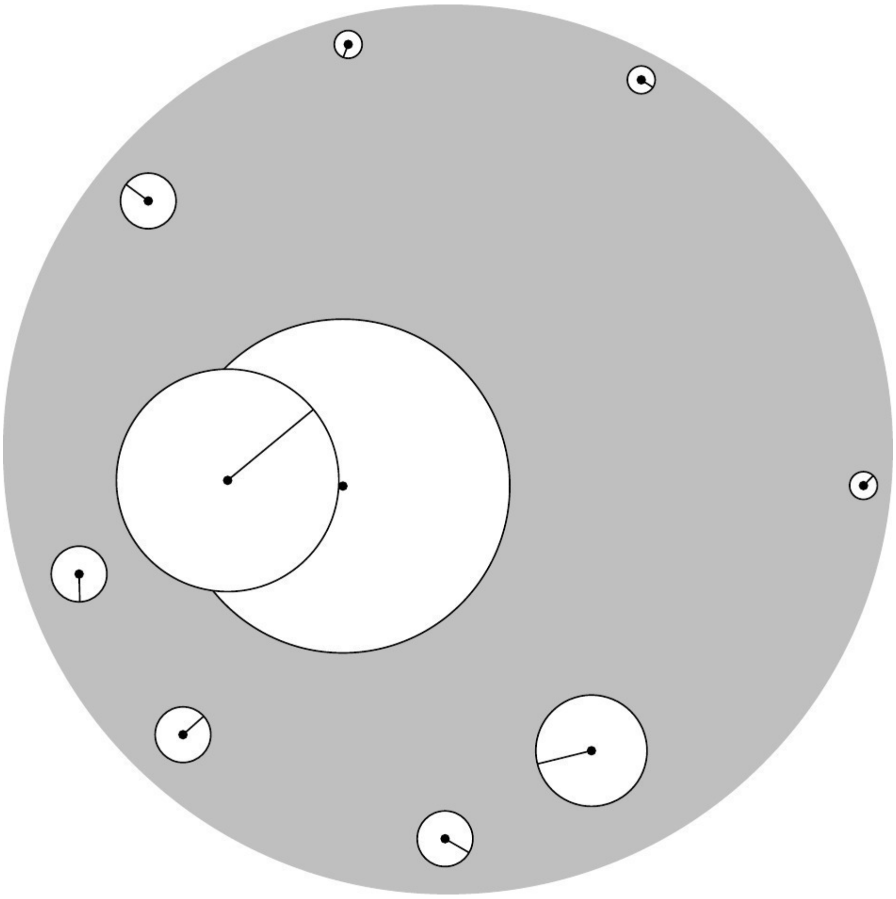


Figure 3.1: The gray open disc is an open subset of $\mathbb{C}$.

Remark 3.1. The above notion of open set determines a topology on $\mathbb{C}$.

Definition 3.3. A sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ has the *limit* $\zeta \in \mathbb{C}$ (or is *convergent* to ζ), written $\lim_{n\to\infty} z_n = \zeta$, if

$$\forall\epsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geqslant n_0 : |z_n - \zeta| < \epsilon.$$

Equivalently, $\lim_{n\to\infty} |z_n - \zeta| = 0$. We call $(z_n)_{n\in\mathbb{N}}$ a *Cauchy sequence* if

$$\forall\epsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n, m \geqslant n_0 : |z_n - z_m| < \epsilon.$$

The same argument as in real analysis shows that the limit w is unique and that every convergent sequence is a Cauchy sequence. The converse is also true by the completeness of real numbers.

Proposition 3.1. *For a sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$, the following are equivalent:*

a. There exists $\zeta \in \mathbb{C}$ such that $\zeta = \lim_{n\to\infty} z_n$.

b. $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Proof.

To prove (b) $\Longrightarrow$ (a) write $z_n = x_n + iy_n$ and notice that

$$|z_n - z_m| = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2} \geqslant |x_n - x_m|, \| y_n - y_m|$$

implies that both $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathbb{R}$. By the completeness of $\mathbb{R}$, these sequences have limits $\chi, \upsilon \in \mathbb{R}$. Set $\zeta = \chi + i\upsilon$ and pick $n_0 \in \mathbb{N}$ such that $|x_n - \chi| < \frac{\epsilon}{\sqrt{2}}$ and $|y_n - \upsilon| < \frac{\epsilon}{\sqrt{2}}$ for all $n \geqslant n_0$. Then

$$|z_n - \zeta| = \sqrt{(x_n - \chi)^2 + (y_n - \upsilon)^2} < \sqrt{\epsilon^2/2 + \epsilon^2/2} = \epsilon$$

for all $n \geqslant n_0$. The converse, (a) $\Longrightarrow$ (b), is left as an exercise.

□

The advantage of Cauchy sequences is that one does not need to know the value of the limit in advance.

Definition 3.4. A *series* of complex numbers $(z_k)_{k\in\mathbb{N}}$ converges to the *limit* ζ, written $\zeta = \sum_{k=0}^{\infty} z_k$, if the sequence of partials sums $\left(w_n = \sum_{k=0}^{n} z_k\right)_{n\in\mathbb{N}}$ converges to ζ. We call a series *absolutely convergent* if the series $\sum_{k=0}^{\infty} |z_k|$ is convergent.

As in real analysis, the Cauchy criterion implies that every absolutely convergent series is convergent. Absolutely convergent series may be rearranged and orders of summation may be exchanged.

Although $\infty \notin \mathbb{C}$, it will be convenient to define $\lim_{n\to\infty} z_n = \infty$ to mean that the sequence $(z_n)_{n\in\mathbb{N}}$ eventually leaves every disk. Symbolically,

$$\forall r > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geqslant n_0 : |z_n| > r.$$

We call $\mathbb{C} \cup \{\infty\}$ the *extended complex plane*.

Definition 3.5. A point $\zeta \in \mathbb{C} \cup \{\infty\}$ is in the *closure* of $D \subset \mathbb{C}$ if there exists a sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in D$ and $\lim_{n\to\infty} z_n = \zeta$.

Let $f\colon D \to \mathbb{C}$ be a complex function and let $\zeta \in \mathbb{C}\cup\{\infty\}$ be in the closure of D. The function $f(z)$ has the *limit* $w \in \mathbb{C} \cup \{\infty\}$ as $z \to \zeta$, written $\lim_{z\to\zeta} f(z) = w$, if for *every* sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in D$ and $\lim_{n\to\infty} z_n = \zeta$ we have $\lim_{n\to\infty} f(z_n) = w$. An equivalent ε-δ-definition is

$$\forall \varepsilon > 0\ \exists \delta > 0 : 0 < |z - \zeta| < \delta, z \in D \implies |f(z) - w| < \varepsilon.$$

A complex function $f\colon D \to \mathbb{C}$ is *continuous at* $\zeta \in D$ if $\lim_{z\to\zeta} f(z) = f(\zeta)$. We call f *continuous on* D if f is continuous at every $\zeta \in D$.

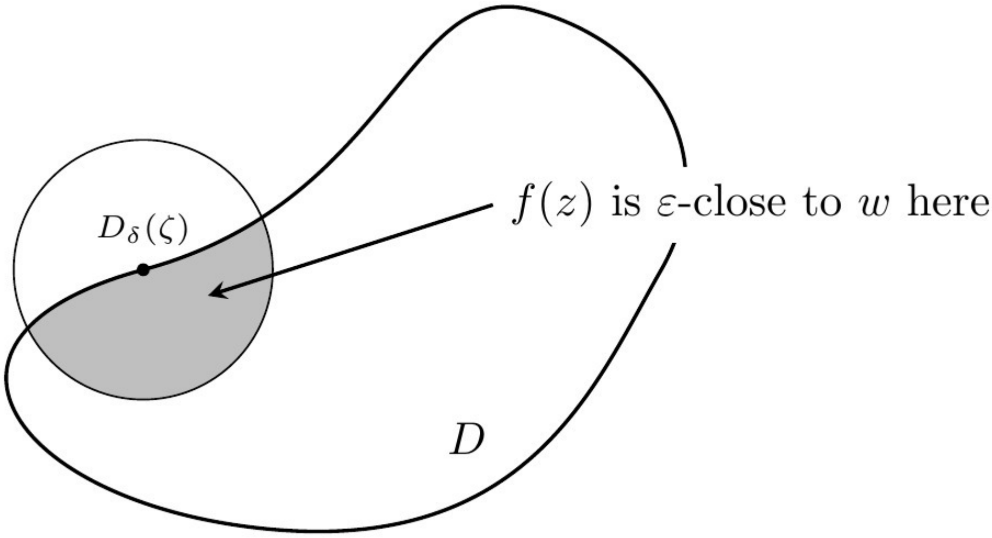


Figure 3.2: The ε-δ-definition of $\lim_{z\to\zeta} f(z) = w$.

> **Convention**
>
> There is a some ambiguity in writing an expression of the form $\lim_{n\to\infty} a_n = f(n)$ as this could be interpreted either as a limit as a complex number z tends to inifinity or as a limit over a sequence a_n.
> In order to resolve this, unless explicitly stated otherwise, in this context, n shall refer to a natural number and a limit $\lim_{n\to\infty}$ should be interpreted as a limit over a sequence indexed by the natural numbers.

Questions for further discussion

- Explain the difference between the notions 'limit of a function', 'limit of a sequence', and 'limit of a series'.

3.1 Exercises

Exercise 3.1

Recall the ratio and root test for series of real numbers.

Show Solution 3.1 on P179

Exercise 3.2

For which $z \in \mathbb{C}$ do the following limits exist?

$$\lim_{n\to\infty} n^{1/n} z, \lim_{n\to\infty} z^n, \lim_{n\to\infty} \frac{z^n}{n}, \lim_{n\to\infty} \frac{z^n}{n!}, \lim_{n\to\infty} \frac{z^n}{n^n}, \lim_{n\to\infty} n! z.$$

Show Solution 3.2 on P180

Exercise 3.3

Show that every convergent sequence $(z_n)_{n\in\mathbb{N}}$ of complex numbers is bounded.

Show Solution 3.3 on P181

Exercise 3.4

Let $\sum_{k=0}^{\infty} z_k$ be a convergent series of complex numbers. Show that $\lim_{k\to\infty} z_k = 0$.

Show Solution 3.4 on P181

Exercise 3.5

For which $z \in \mathbb{C}$ do the following series converge?

$$\sum_{k=0}^{\infty} k z^k, \quad \sum_{k=0}^{\infty} (kz)^k$$

Show Solution 3.5 on P182

Exercise 3.6

Let $f\colon \mathbb{C} \to \mathbb{C}$ be a complex function. Show that

$$\lim_{z\to\infty} f(z) = w \iff \lim_{z\to 0} f\left(\frac{1}{z}\right) = w.$$

Show Solution 3.6 on P183

Exercise 3.7

The Riemann sphere is $\mathbb{S} = \{(a,b,c) \in \mathbb{R}^3 \mid a^2+b^2+c^2 = 1\}$. Show that the stereographic projection

$$F\colon \mathbb{S} \longrightarrow \mathbb{C} \cup \{\infty\}, F(a,b,c) = \begin{cases} \frac{a+ib}{1-c} & \text{if } c \neq 1, \\ \infty & \text{if } c = 1. \end{cases}$$

is a bijection between the Riemann sphere and the extended complex plane. Find a formula for the inverse function.

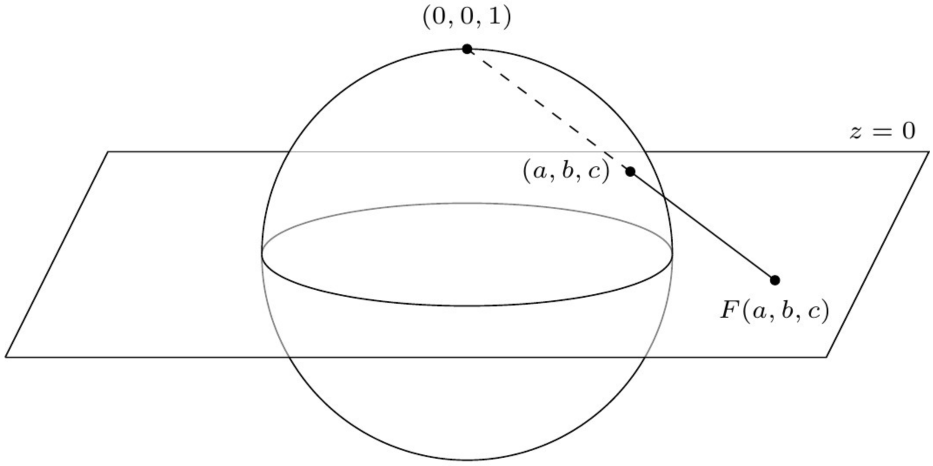


Figure 3.3: Stereographic projection from the north pole

Show Solution 3.7 on P183

Further resources

Part II

Differentiation and contour integrals

Chapter 4

Holomorphic functions

The following definition is a key concept for this course.

Definition 4.1. A complex function $f\colon U \to \mathbb{C}$ with domain an open set $U \subset \mathbb{C}$ is *complex differentiable* at a point $z_0 \in U$ if the limit

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{4.1}$$

exists. We call f *holomorphic* on U if f is complex differentiable at every $z_0 \in U$. The set of all holomorphic functions on U is denoted $\mathcal{O}(U)$. A function that is holomorphic[1] on $U = \mathbb{C}$ is *entire*.

This definition is structurally the same as for real functions. For example, f is continuous at z_0 since multiplying Equation 4.1 by $0 = \lim_{z \to z_0}(z - z_0)$ gives

$$0 = \lim_{z \to z_0} f(z) - f(z_0) \implies f(z_0) = \lim_{z \to z_0} f(z).$$

[1]From Greek *holos* 'whole, complete' and *morphē* 'form', shape

Other formal proofs carry over as well and show the following.

Proposition 4.1. *Let $f, g\colon U \to \mathbb{C}$ be complex differentiable at z_0. Then:*

a. $f + g$ is complex differentiable at z_0 with

$$(f+g)'(z_0) = f'(z_0) + g'(z_0).$$

b. fg is complex differentiable at z_0 with

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0).$$

c. If $g(z_0) \neq 0$, then f/g is complex differentiable at z_0 with

$$(f/g)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}.$$

Proof.

Exercise, using the algebra of limits.

□

Example 4.1.

Directly from Equation 4.1 we find that the constant function $f(z) = c$ is entire with $f'(z) = 0$. The identity function $f(z) = z$ is entire with $f'(z) = 1$. Proposition 4.1 and induction then imply that all polynomials

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

are entire with

$$f'(z) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + ... + a_1.$$

It then follows from Proposition 4.1 part c. that rational functions are holomorphic on their domain of definition.

Proposition 4.2. *Let $f\colon U \to \mathbb{C}$, $g\colon V \to \mathbb{C}$ be complex functions with $f(U) \subset V$, and U, V open. Suppose that f is complex differentiable at $z_0 \in U$ and that g is complex differentiable at $w_0 = f(z_0) \in V$.*

Then $g \circ f$ is complex differentiable at z_0 and

$$(g \circ f)'(z_0) = g'(w_0)f'(z_0). \tag{4.2}$$

Proof.

We can equivalently rewrite Equation 4.1 as

$$f(z_0 + h) = f(z_0) + hf'(z_0) + h\rho(h), \qquad \lim_{h\to 0} \rho(h) = 0, \tag{4.3}$$

where the function $\rho(h)$, $h \neq 0$, is defined by this equation. Similarly,

$$g(w_0 + k) = g(w_0) + kg'(w_0) + k\sigma(k), \qquad \lim_{k\to 0} \sigma(k) = 0. \tag{4.4}$$

Then

$$\frac{(g \circ f)(z_0 + h) - (g \circ f)(z_0)}{h} = \frac{g(w_0 + \overbrace{hf'(z_0) + h\rho(h)}^{k=}\,) - g(w_0)}{h}$$
$$= f'(z_0)g'(w_0) + \left[\rho(h)g'(w_0) + (f'(z_0) + \rho(h))\,\sigma(hf'(z_0) + h\rho(h))\right],$$

where the first equality follows from Equation 4.3, the second from Equation 4.4 and the square bracket tends to zero as $h \to 0$, using the chain rule for limits to see that $\lim_{h\to 0} \sigma(hf'(z_0) + h\rho(h)) = \lim_{k\to 0} \sigma(k) = 0$.

□

One should view Equation 4.3 as a short Taylor series expansion: near the point z_0 the function f is given to zero$^{\text{th}}$ order by a constant $f(z_0)$, with first order correction given by a *complex linear* function $h \mapsto hf'(z_0)$, plus a higher order term $h\rho(h)$ which vanishes infinitesimally (after dividing by h).

We next discuss how complex differentiability compares to real differentiability. Regard a complex function as a real multivariable map

$$f = (u, v)\colon U \longrightarrow \mathbb{R}^2, \qquad U \subset \mathbb{R}^2 \text{ open set.}$$

Recall that f is *real differentiable* at $z_0 = (x_0, y_0)$ if there exists a matrix $J_f(z_0) \in \mathsf{M}_{2\times 2}(\mathbb{R})$, called the *Jacobian*, so that Equation 4.3 holds with the complex multiplication $hf'(z_0)$ replaced by the matrix vector multiplication $J_f(z_0)h$. Real differentiability at z_0 implies that the partial derivatives at z_0 exist and

$$J_f(z_0) = \begin{pmatrix} \frac{\partial u}{\partial x}(z_0) & \frac{\partial u}{\partial y}(z_0) \\ \frac{\partial v}{\partial x}(z_0) & \frac{\partial v}{\partial y}(z_0) \end{pmatrix}. \tag{4.5}$$

Recall that the converse requires the additional assumptions that the partial derivatives exist *in a neighborhood* of z_0 and are *continuous* at z_0.

Proposition 4.3. *For a complex function $f = u + iv$ the following are equivalent:*

a. *f is complex differentiable at z_0*

b. f is real differentiable at z_0 and the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0), \qquad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0), \tag{4.6}$$

hold.

In this case, $f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

Proof.

The two definitions are identical except that the complex multiplication $f'(z_0)h$ by $h \in \mathbb{C}$ is replaced by the matrix vector multiplication $J_f(z_0)h$ by $h \in \mathbb{R}^2$. Our task is to compare these two notions. For this we simply observe that by Proposition 1.2 $J_f(z_0)$ is $\mathbb{C}$-linear if and only if Equation 4.6 holds.

□

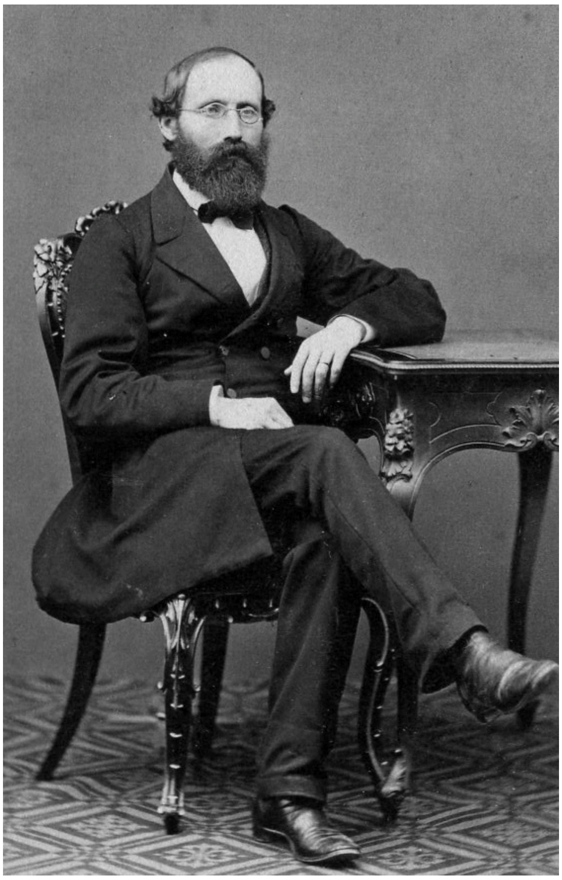


Figure 4.1: Bernhard Riemann. Familienarchiv Thomas Schilling, ca. 1862. WikiMedia Commons. Public Domain

Example 4.2.

For the exponential function, $u = e^x \cos(y)$, $v = e^x \sin(y)$. Then the partial derivatives exist

$$\frac{\partial u}{\partial x} = e^x \cos(y), \quad \frac{\partial u}{\partial y} = -e^x \sin(y), \quad \frac{\partial v}{\partial x} = e^x \sin(y), \quad \frac{\partial v}{\partial y} = e^x \cos(y),$$

are continuous functions on $\mathbb{R}^2$, and satisfy the Cauchy–Riemann equations. It follows that exp is entire with

$$\exp'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \exp(z). \tag{4.7}$$

Example 4.3.

The *hyperbolic cosine* $\cosh(z) = \frac{e^z+e^{-z}}{2}$ and the *hyperbolic sine* function $\sinh(z) = \frac{e^z-e^{-z}}{2}$ are entire with

$$\sinh'(z) = \cosh(z), \qquad \cosh'(z) = \sinh(z).$$

Since $\cosh(z) = 0 \iff e^{2z} = -1 \iff z \in \frac{\pi i}{2} + \pi i\mathbb{Z}$ the *hyperbolic tangent* $\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$ is a holomorphic function

$$\tanh\colon \mathbb{C} \setminus \left(\frac{\pi i}{2} + \pi i\mathbb{Z}\right) \longrightarrow \mathbb{C},$$

with

$$\tanh'(z) = \frac{\cosh(z)^2 - \sinh(z)^2}{\cosh(z)^2} = 1 - \tanh(z)^2.$$

Similarly, the *hyperbolic cotangent* $\coth(z) = \frac{\cosh(z)}{\sinh(z)}$ is a holomorphic function $\coth\colon \mathbb{C} \setminus \pi i\mathbb{Z} \to \mathbb{C}$ with

$$\coth'(z) = 1 - \coth(z)^2.$$

Example 4.4.

The trigonometric sine $\sin(z) = -i\sinh(iz)$ and the cosine $\cos(z) = \cosh(iz)$ are entire with $\sin'(z) = \cos(z)$, $\cos'(z) = -\sin(z)$. The tangent $\tan(z) = \frac{\sin(z)}{\cos(z)}$ is a holomorphic function on $\mathbb{C} \setminus \left(\frac{\pi}{2} + \pi\mathbb{Z}\right)$ with $\tan'(z) = 1 + \tan(z)^2$. The cotangent

$\cot(z) = \frac{\cos(z)}{\sin(z)}$ is a holomorphic function on $\mathbb{C} \setminus \pi\mathbb{Z}$ with $\cot'(z) = 1 - \cot(z)^2$.

Example 4.5.

$f(z) = \bar{z} = x - iy$ has all partial derivatives of all orders, but is *not* complex differentiable at any $z_0 \in \mathbb{C}$ since $\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$.

Theorem 4.1. *Let $f\colon U \to \mathbb{C}$ be a holomorphic function and $z_0 \in U$ with $f'(z_0) \neq 0$. Then there exist open sets $V, W \subset \mathbb{C}$ with $z_0 \in V \subset U$ and $w_0 = f(z_0) \in W$ with the property that the restriction $f|_V$ becomes a bijection $V \to W$ with holomorphic inverse function $(f|_V)^{-1}\colon W \to V$ and*

$$\frac{\partial (f|_V)^{-1}}{\partial w}(w) = \frac{1}{f'((f|_V)^{-1}(w))}. \tag{4.8}$$

Proof.

As in the proof of Proposition 4.3, we view f as a differentiable real multivariable map $U \to \mathbb{R}^2$. Since $f'(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial u}{\partial y}(z_0) \neq 0$ we have

$$\begin{aligned} \det J_f(z_0) &= \frac{\partial u}{\partial x}(z_0)\frac{\partial v}{\partial y}(z_0) - \frac{\partial v}{\partial x}(z_0)\frac{\partial u}{\partial y}(z_0) \\ &= \left(\frac{\partial u}{\partial x}(z_0)\right)^2 + \left(\frac{\partial u}{\partial y}(z_0)\right)^2 \neq 0. \end{aligned}$$

where the first equality follows from Equation 4.5 and the second from Equation 4.6. We may thus apply the inverse function theorem from multivariable calculus. This gives open sets with $z_0 \in V \subset U, w_0 \in W$ such that $f|_V\colon V \to W$ is a bijection and $(f|_V)^{-1}\colon W \to V$ is real differentiable. In particular, $\det J_f(z) \neq 0$ for all $z \in V$ (chain rule). Let $w \in W$ and $z = f^{-1}(w)$. Then

$$J_{(f|_V)^{-1}}(w) = J_f(z)^{-1} = \frac{1}{\det J_f(z)} \begin{pmatrix} \frac{\partial v}{\partial y}(z) & -\frac{\partial u}{\partial y}(z) \\ -\frac{\partial v}{\partial x}(z) & \frac{\partial u}{\partial x}(z) \end{pmatrix},$$

so the Cauchy–Riemann equations are satisfied at every $w \in W$ for the inverse function, so $(f|_V)^{-1}$ is holomorphic. Finally, Equation 4.8 follows by applying the chain rule Equation 4.2 to $f|_V \circ (f|_V)^{-1} = \mathrm{id}_W$ which gives

$$(f|_V)'((f|_V)^{-1}(w)) \cdot \frac{\partial (f|_V)^{-1}}{\partial w}(w) = 1.$$

□

Remark 4.1. Our proof is a simple application of the ordinary inverse function theorem. We will see two different proofs later in the course.

Example 4.6.

Recall from Proposition 2.1 that $\exp|_S \colon S \to \mathbb{C}^-$ is a bijection with the logarithm as inverse function. As $\exp'(z) = \exp(z) \neq 0$, we conclude from Theorem 4.1 that $\log \colon \mathbb{C}^- \to S$ is holomorphic with

$$\log'(z) = 1/z. \tag{4.9}$$

Questions for further discussion

- $x \mapsto x^3$ is a real differentiable bijection whose inverse function is not differentiable at $x = 0$. What about the complex analogue $z \mapsto z^{1/3}$?

4.1 Exercises

Exercise 4.1

Write the following complex functions in the form $f = u + iv$:

$$f_1(z) = \sin(z), \qquad f_2(z) = e^{z^2}, \qquad f_3(z) = \cosh(z)$$

Show Solution 4.1 on P185

Exercise 4.2

At which points $z \in \mathbb{C}$ are the following functions complex differentiable? At which points are they real differentiable?

$$\begin{aligned} f_1(z) &= z, & f_2(z) &= \overline{z}, & f_3(z) &= z^3 + z, \\ f_4(z) &= \frac{1}{2iz}, & f_5(z) &= |z|^2, & f_6(z) &= \frac{|z|^2}{\overline{z}} \end{aligned}$$

Show Solution 4.2 on P186

Exercise 4.3

Let $f\colon V \to W$ be a bijection of open sets $V, W \subset \mathbb{C}$. Assume that f is complex differentiable at z_0 and that f^{-1} is complex differentiable at $w_0 = f(z_0)$. Show that $f'(z_0) \neq 0$. (The same result holds for real differentiable maps to show $\det J_f(z_0) \neq 0$.)

Show Solution 4.3 on P187

Exercise 4.4

Let $f\colon U \to \mathbb{C}$ be a holomorphic function with domain an open set $U \subset \mathbb{C}$. Suppose also that $f'(z)$ is holomorphic. Write $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$. As we will see, a consequence of f being holomorphic, is that u and v have continuous second order derivatives. Show that:

a. $|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$

b. Both u, v satisfy the Laplace equation $\Delta(u) = \Delta(v) = 0$, where the Laplace operator is defined as $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

c. Fix $n \in \mathbb{N}$. Use b. to find a solution to $\Delta(u) = 0$ that satisfies $u(z) = \cos(n\theta)$ for all $|z| = 1$, $z = e^{i\theta}$ on the unit circle.

d. Does the converse to b. hold?

Show Solution 4.4 on P187

Exercise 4.5

Find subdomains where sin, cos, sinh, cosh, tan, tanh are injective. Compute the derivative of an inverse using the chain rule. Deduce an explicit formula for the derivative of an inverse.

Show Solution 4.5 on P189

Exercise 4.6

Show that the set $\mathcal{O}(U)$ of holomorphic functions on U becomes a ring under the operations of point-wise addition and multiplication:

$$(f + g)(z) = f(z) + g(z), \qquad (fg)(z) = f(z)g(z).$$

Show Solution 4.6 on P190

Exercise 4.7

Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function with image $f(\mathbb{C}) \subset \mathbb{R}$. Prove that f is constant.

Show Solution 4.7 on P191

Exercise 4.8

Let U be an open set and assume that U is *path-connected*, meaning that for all $z_0, z_1 \in U$ there exists a continuously differentiable map $\gamma\colon [0,1] \to U$ such that $\gamma(0) = z_0$, $\gamma(1) = z_1$.

a. Let $f\colon U \to \mathbb{C}$ be a holomorphic function such that $f'(z) = 0$ for all $z \in U$. Prove that $f(z)$ is a constant function.

   Hint: Consider the real and imaginary parts of $f \circ \gamma$.

b. Show that the only entire functions $f(z)$ satisfying $f'(z) = Cf(z)$ for a constant $C \in \mathbb{C}$ are the functions $f(z) = De^{Cz}$, $D \in \mathbb{C}$.

c. Find all entire solutions $f(z)$ of $f''(z) = f(z)$.

   Hint: Consider $g = f + f'$, $h = f - f'$

Show Solution 4.8 on P191

Chapter 5

Power Series

Definition 5.1. A *formal power series* is an expression of the form

$$P = \sum_{n=0}^{\infty} a_n T^n \tag{5.1}$$

with *coefficents* $a_n \in \mathbb{C}$. As no convergence is required, this is really just a sequence $(a_n)_{n\in\mathbb{N}}$ of complex numbers.

We call a_0 the *constant term* of P. The *order* of P is the smallest $n \in \mathbb{N}$ such that $a_n \neq 0$.

Example 5.1.

Every polynomial is a formal power series.

Definition 5.2. Let $\mathbb{C}[\![T]\!]$ be the set of all formal power series. Define the addition and multiplication of $P = \sum_{n=0}^{\infty} a_n T^n$, $Q = \sum_{n=0}^{\infty} b_n T^n \in \mathbb{C}[\![T]\!]$ by

$$P + Q = \sum_{n=0}^{\infty} (a_n + b_n) T^n, \tag{5.2}$$

$$PQ = \sum_{n=0}^{\infty} c_n T^n, c_n = \sum_{i+j=n} a_i b_j. \tag{5.3}$$

Proposition 5.1.

a. *These operations make* $\mathbb{C}[\![T]\!]$ *a commutative ring with unit* 1, *the series with constant term* 1 *and all higher coefficients zero.*

b. $P \in \mathbb{C}[\![T]\!]$ *has a multiplicative inverse* $\iff$ *the constant term of* P *is non-zero.*

Proof.

a. is straightforward and left as an exercise.

b. Having an inverse amounts to the existence of $Q = \sum_{n=0}^{\infty} b_n T^n$ such that $PQ = 1$. By Equation 5.3 this means

$$a_0 b_0 = 1 \qquad \sum_{i+j=n} a_i b_j = 0 (\forall n > 0). \tag{5.4}$$

The first equations shows that a_0 is invertible, proving '$\implies$'. For '$\impliedby$', the idea is to use Equation 5.4 to construct the coefficients b_n of the inverse inductively. Of course, $b_0 = a_0^{-1}$. For $n = 1$, Equation 5.4 gives $a_0 b_1 + a_1 b_0 = 0$ so

$$b_1 = -a_0^{-2} a_1.$$

For $n = 2$, $a_0 b_2 + a_1 b_1 + a_2 b_0 = 0$. Inserting the known b_0, b_1 implies

$$b_2 = a_0^{-3}(a_1^2 - a_0 a_2).$$

For general n, given $b_0, \dots, b_{n-1}$ we rearrange Equation 5.4 as

$$b_n = -a_0^{-1} \sum_{j=0}^{n-1} a_{n-j} b_j,$$

which defines b_n by recursion.

$\square$

It is unnecessary to remember Equation 5.4. It is enough to recall the ansatz $PQ = 1$ and the strategy of computing the coefficients of Q inductively.

Example 5.2.

Let $P = 1 - T$. Then $P^{-1} = \sum_{n=0}^{\infty} T^n$ is the geometric series. We verify $(1 - T)\sum_{n=0}^{\infty} T^n = \sum_{n=0}^{\infty} T^n - \sum_{n=1}^{\infty} T^n = 1$ (telescope sum).

Definition 5.3. The *domain* $D(P)$ of a formal power series Equation 5.1 is the set of all $z \in \mathbb{C}$ such that the series of complex numbers

$$P(z) = \sum_{n=0}^{\infty} a_n z^n,$$

obtained by substituting T by z, converges. We obtain a complex function

$$D(P) \longrightarrow \mathbb{C}, z \longmapsto P(z).$$

More generally, fix a *center* $z_0 \in \mathbb{C}$. We then have a complex function

$$D(P, z_0) := z_0 + D(P) \longrightarrow \mathbb{C}, z \longmapsto P(z - z_0) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (5.5)$$

which differs from $P(z)$ only by a translation.

> **Notational clash**
>
> There is a small clash in notation. Recall that for a complex number z and a positive real number r, we write $D_r(z)$ for the open disc of radius r about z, while for a formal power series P, $D(P)$ represents its domain of convergence. You can tell these apart by whether or not there is a subscript and also by referring to the context.
>
> So for instance in interpreting a chain of the form:
>
> $$D_\rho(z_0) \subset D(P, z_0) \subset \overline{D}_\rho(z_0)$$
>
> - the first term refers to the open disc of radius ρ about the z_0,
> - the second is the domain of the power series P with centre z_0, and,
> - the last is the closed disc of radius ρ about z_0.

Example 5.3.

The geometric series $P = \sum_{n=0}^{\infty} T^n$ converges for all $|z| < 1$ since the partial sums $P_n(z) = \frac{z^{n+1}-1}{z-1}$ tend to $\frac{1}{1-z}$ as $n \to \infty$. The geometric series diverges for all $|z| \geqslant 1$, since in this case $(z^n)_{n\in\mathbb{N}}$ is not a null sequence. Hence $D(P) = D_1(0)$.

Recall the following concept from analysis.

Definition 5.4. For a complex function $f\colon D \to \mathbb{C}$, the *uniform norm* is

$$\|f\|_{\infty,D} = \sup_{z\in D} |f(z)|. \tag{5.6}$$

A sequence of functions $(f_n)_{n\in\mathbb{N}}$ is *uniformly convergent* to f on D if

$$\|f - f_n\|_{\infty,D} \to 0 \text{ as } n \to \infty.$$

A series $\sum_{n=0}^{\infty} f_n(z)$ is a *uniformly convergent series* on D if the sequence of *partial sums* $P_n(z) = \sum_{k=0}^{n} f_k(z)$ converges uniformly.

Uniform convergence on a subset $C \subset D$ refers to the uniform convergence of the functions restricted to C.

Theorem 5.1. *Every formal power series P has a unique* radius of convergence $0 \leqslant \rho \leqslant +\infty$ *such that for every center z_0*

$$D_\rho(z_0) \subset D(P, z_0) \subset \overline{D}_\rho(z_0). \tag{5.7}$$

In fact,

$$\rho = \sup\left\{r \geqslant 0 \;\middle|\; (|a_k| r^k)_{k\in\mathbb{N}} \textit{ is a bounded sequence}\right\}. \tag{5.8}$$

Moreover, $P(z - z_0)$ converges absolutely and uniformly on every smaller disk $D_r(z_0)$ with $0 < r < \rho$ and diverges at every point $z \in \mathbb{C}$ with $|z - z_0| > \rho$.

Proof.

For uniqueness, suppose $D_{\rho_\ell}(z_0) \subset D(P, z_0) \subset \overline{D}_{\rho_\ell}(z_0)$ for $\ell = 1, 2$ and $\rho_1 \neq \rho_2$, say $0 \leqslant \rho_1 < \rho_2$. Pick $z \in \mathbb{C}$ with $\rho_1 < |z - z_0| < \rho_2$. Then $z \in D_{\rho_2}(z_0) \subset D(P, z_0) \subset \overline{D}_{\rho_1}(z_0)$, so $|z - z_0| \leqslant \rho_1$, a contradiction.

It suffices to prove the last statement in the theorem for ρ defined by Equation 5.8, because this implies Equation 5.7. For $z \in \mathbb{C}$ with $|z - z_0| > \rho$ the terms $a_k(z - z_0)^k$

do not tend to zero (they are unbounded), so the series $P(z-z_0)$ diverges.

Let $0 < r < \rho$. Using that ρ is the *least* upper bound, there exists $r < s < \rho$ with $|a_k|s^k < B$ bounded. Pick $w \in \mathbb{C}$ with $s = |w - z_0|$. For all $z \in D_r(z_0)$ we then have

$$\sum_{k=0}^{\infty} |a_k(z-z_0)^k| = \sum_{k=0}^{\infty} \Big| a_k(w-z_0)^k \Big(\underbrace{\frac{z-z_0}{w-z_0}}_{q=}\Big)^k \Big| \leqslant B \sum_{k=0}^{\infty} q^k. \tag{5.9}$$

Since $|q| < 1$, the geometric series on the right converges. Hence the comparison test implies that $P(z-z_0)$ converges absolutely. As for uniform convergence, let P_n be the n-th partial sum. As in Equation 5.9, for all $z \in D_r(z_0)$

$$|P(z-z_0) - P_n(z-z_0)| = \left| \sum_{k=n}^{\infty} a_k(z-z_0)^k \right| \leqslant B \sum_{k=n}^{\infty} q^k.$$

The right hand side is independent of z, so $\|P - P_n\|_{\infty, D_r(z_0)} \xrightarrow{n\to\infty} 0$.

□

Remark 5.1. The domain $D(P)$ of a formal power series is roughly a disk of radius ρ, with uncertain behavior on the boundary circle $|z| = \rho$. Determining the behavior on the boundary can be very difficult. On the other hand, the radius of convergence is easy to compute. If you find $z \in \mathbb{C}$ for which $P(z)$ converges (for example, using the ratio or the root test), you can deduce $|z| < \rho$. If $P(z)$ diverges at $z \in \mathbb{C}$, you know $|z| \geqslant \rho$.

Example 5.4 (optional)**.**

For $P = \sum_{n=1}^{\infty} \frac{z^n}{n}$ we have $\rho = 1$ by Equation 5.8. We investigate the behavior on the boundary circle $|z| = 1$. For $z = 1$, this is the divergent harmonic series. We claim that P converges for all other boundary points, so $D(P) = \overline{D}_1(0) \setminus \{1\}$. Consider

$$
\begin{aligned}
(z-1)\sum_{k=1}^{n}\frac{z^k}{k} &= \sum_{k=2}^{n+1}\frac{z^k}{k-1} - \sum_{k=1}^{n}\frac{z^k}{k} \\
&= \frac{z^{n+1}}{n} - z + \sum_{k=2}^{n}\left(\frac{z^k}{k-1} - \frac{z^k}{k}\right) \\
&= \frac{z^{n+1}}{n} - z + \sum_{k=2}^{n}\frac{z^k}{k(k-1)}.
\end{aligned}
$$

The series $\sum_{k=2}^{\infty}\frac{z^k}{k(k-1)}$ is (absolutely) convergent for $|z| \leqslant 1$. It follows that the left hand sequence of partial sums converges as $n \to \infty$. Dividing by $z - 1 \neq 0$ implies that P converges.

Corollary 5.1. *The restriction of the complex function $f(z) = P(z - z_0)$ defined in Equation 5.5 to $D_\rho(z_0) \subset D(P, z_0)$ is continuous.*

Proof.

Since the translation $z \mapsto z - z_0$ is a homeomorphism, it suffices to consider the case $z_0 = 0$. We prove continuity at each $w \in D_\rho(0)$. Let $\epsilon > 0$. By uniform convergence, we can pick $n \in \mathbb{N}$ with $\|P_n - P\|_{\infty, D_\rho(0)} < \epsilon/3$. Since the polynomial $P_n(z)$ is continuous, there is $\delta > 0$ such that $|P_n(z) - P_n(w)| < \epsilon/3$ for all $z \in D_\delta(w)$. By shrinking δ, we may suppose $D_\delta(w) \subset D_\rho(0)$. Then have

$$
|P(z) - P(w)| \leqslant |P(z) - P_n(z)| + |P_n(z) - P_n(w)| + |P_n(w) - P(w)| < \epsilon.
$$

for all $z \in D_\delta(w)$.

$\square$

Remark 5.2. Continuity need not hold on $D(P, z_0)$ (Sierpinski, 1916).

Example 5.5.

The series $P = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$ has $\rho = 1$. On the boundary circle, $|z^n/n^2| \leqslant 1/n^2$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and is independent of z. This implies uniform convergence and the proof of Corollary 5.1 shows that P is continuous on its domain $D(P) = \overline{D}_1(0)$.

When the radius of convergence is positive, we drop the word 'formal' and simply speak of a (convergent) *power series*.

Example 5.6.

The *exponential series*

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

has radius of convergence $+\infty$, since $\frac{z^n}{n!}$ is bounded (even tends to zero), as factorials grow faster than powers. Alternatively, for each $z \in \mathbb{C}$ we have absolute convergence by the ratio test

$$\left| \frac{z^{n+1}/(n+1)!}{z^n/n!} \right| = \frac{|z|}{n+1} \longrightarrow 0 \text{ as } n \to \infty.$$

Therefore the power series defines a complex function $\exp\colon \mathbb{C} \to \mathbb{C}$. Mathematically, this is a good definition for the exponential function. Using Equation 2.3 we obtain from this the series expansions of the sine and the cosine:

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \qquad \sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

Example 5.7 (optional)**.**

We first generalize the definition of *binomial coefficients* to complex numbers $\alpha \neq 0$

and $k \in \mathbb{N}$. Set

$$\binom{\alpha}{k} = \begin{cases} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

The *binomial series* is

$$B_\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k.$$

For $\alpha \in \mathbb{C} \setminus \mathbb{N}$ we have

$$\left| \frac{\binom{\alpha}{k+1} z^{k+1}}{\binom{\alpha}{k} z^k} \right| = \left| z \frac{\alpha - k}{k+1} \right| \longrightarrow |z| \text{ as } k \to \infty.$$

Therefore, the ratio test implies $\rho = 1$ when $\alpha \notin \mathbb{N}$. When $\alpha \in \mathbb{N}$ only finitely many binomial coefficients $\binom{\alpha}{k}$ are non-zero, so B_α is a polynomial and $\rho = +\infty$ and indeed by the binomial theorem we have

$$B_\alpha(z) = (z+1)^\alpha, \qquad \forall \alpha \in \mathbb{N}.$$

Theorem 5.2. *Let P be a formal power series. Assume the radius of convergence $\rho > 0$ is positive. Then for each center $z_0 \in \mathbb{C}$ the function*

$$P \colon D_\rho(z_0) \longrightarrow \mathbb{C}, z \longmapsto P(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{5.10}$$

is holomorphic with derivative given by termwise differentiation,

$$P'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}. \tag{5.11}$$

Hence P is infinitely complex differentiable, by induction.

Proof.

We show that P is complex differentiable at every $w \in D_\rho(z_0)$. As $\sum_{k=0}^\infty kq^k$ converges for $|q| < 1$, the same argument given in Equation 5.9 shows that Equation 5.11 is a convergent series of complex numbers. For simplicity of notation, suppose $z_0 = 0$. We have absolutely convergent series

$$\begin{aligned}
\frac{P(w+h)-P(w)}{h} - P'(w) &= \sum_{n=0}^\infty a_n \frac{(w+h)^n - w^n}{h} - P'(w) \\
&= \sum_{n=0}^\infty a_n \left(\sum_{k=1}^n \binom{n}{k} w^{n-k} h^{k-1} \right) - P'(w) \\
&= \sum_{n=0}^\infty \sum_{k=2}^n a_n \binom{n}{k} w^{n-k} h^{k-1} \\
&= \sum_{k=2}^\infty \sum_{n=0}^\infty a_n \binom{n}{k} w^{n-k} h^{k-1},
\end{aligned}$$

where we use $\binom{n}{k} = 0$ for $k > n$ and where we have exchanged the order of summation, by absolute convergence. The above is a power series $R(z) = \sum_{k=0}^\infty c_k z^k$ with $c_{k-1} = \sum_{n=0}^\infty a_n \binom{n}{k} w^{n-k}$ for $k \geqslant 2$ and $c_0 = 0$. Our calculation shows that $R(h)$ converges, so the radius of convergence of R is at least h. Therefore $R(h)$ is continuous on $D_h(z_0)$, so $\lim_{h\to 0} R(h) = R(0) = c_0 = 0$. In other words, $\frac{P(w+h)-P(w)}{h} - P'(w) = R(h) \longrightarrow 0$ as $h \to 0$.

$\square$

The *identity theorem* for power series is the following result.

Corollary 5.2. *Let $P = \sum_{n=0}^\infty a_n z^n$, $Q = \sum_{n=0}^\infty b_n z^n$ be formal power series with positive radius of convergence. Let $0 \neq z_\ell \in D(P) \cap D(Q)$ be a null sequence,* $\lim_{\ell\to\infty} z_\ell = 0$, *of non-zero complex numbers in the common domain. If $P(z_\ell) = Q(z_\ell)$ for all ℓ, then all coefficients $a_n = b_n$ agree.*

Proof.

We prove this by induction.

Base case $n = 0$. By continuity of P and Q at 0,

$$\begin{aligned} a_0 = P(0) = P\left(\lim_{\ell\to\infty} z_\ell\right) &= \lim_{\ell\to\infty} P(z_\ell) \\ = \lim_{\ell\to\infty} Q(z_\ell) = Q\left(\lim_{\ell\to\infty} z_\ell\right) &= Q(0) = b_0. \end{aligned}$$

Inductive step $n+1$. Assume that $a_0 = b_0, \dots, a_n = b_n$. By subtracting $\sum_{k=0}^n a_k z_\ell^k = \sum_{k=0}^n b_k z_\ell^k$ from $P(z_\ell)$ and $Q(z_\ell)$ we find

$$\sum_{k=n+1}^\infty a_k z_\ell^k = \sum_{k=n+1}^\infty b_k z_\ell^k.$$

As $z_\ell \neq 0$, we can divide this equation by z_ℓ^{n+1} and get $\tilde{P}(z_\ell) = \tilde{Q}(z_\ell)$ for the formal power series $\tilde{P} = \sum_{k=0}^\infty a_{k+n+1} z_\ell^k$ and $\tilde{Q} = \sum_{k=0}^\infty b_{k+n+1} z_\ell^k$, which have the same (positive) radius of convergence. An application of the base case to $\tilde{P}$ and $\tilde{Q}$ then shows that $a_{n+1} = b_{n+1}$ for the constant terms, as required.

$\square$

Questions for further discussion

- Give precise statements of the comparison test, the ratio test and root test. Recall how the ratio test is proven by comparison with the geometric series.
- Find a power series P with $D(P) = \overline{D}_1(0) \setminus \{\pm 1, \pm i\}$.
- Explain why 'is a bounded sequence' in Equation 5.8 can be replaced by 'is a null sequence'.

5.1 Exercises

Exercise 5.1

Find the radius of convergence for the following power series centered at the origin.

i. $\sum_{n=0}^{\infty} \frac{z^n}{n^3}$,
ii. $\sum_{n=0}^{\infty} z^{3n}$,
iii. $\sum_{n=0}^{\infty} \frac{z^n}{n^n}$.

Hint: Recall the ratio and root tests for series of complex numbers

Show Solution 5.1 on P193

Exercise 5.2

Treating e^z, $\sin(z)$, $\cos(z)$ as formal power series with their usual Taylor expansion, find the terms of order ≤ 3 of the following power series:

i. $e^z \sin(z)$,
ii. $\sin(z)\cos(z)$,
iii. $1/\cos(z)$

Show Solution 5.2 on P194

Exercise 5.3

Let $P = \sum_{n=0}^{\infty} a_n z^n$, $Q = \sum_{n=0}^{\infty} b_n z^n$ be power series with positive radii of convergence $\rho_P, \rho_Q > 0$. Show that:

a. $P+Q = \sum_{n=0}^{\infty}(a_n+b_n)z^n$ has radius of convergence $\rho \geqslant \min(\rho_P, \rho_Q)$.
b. $PQ = \sum_{n=0}^{\infty}\left(\sum_{i+j=n} a_i b_j\right) z^n$ has radius of convergence $\rho \geqslant \min(\rho_P, \rho_Q)$.

Show Solution 5.3 on P195

Exercise 5.4

Find a solution to the non-linear differential equation

$$f'(z) + f(z)^2 = 0, \qquad f(0) = 1$$

on a disk centered at $z_0 = 0$ by making the ansatz $f(z) = \sum_{n=0}^{\infty} a_n z^n$, inductively determining the coefficients a_n, and finding the radius of convergence.

Show Solution 5.4 on P195

Exercise 5.5

Let $P(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series centered at $z_0 = 0$ and assume that the radius of convergence $\rho > 0$ is positive. Suppose that $P(z) \in \mathbb{R}$ for all $z \in D_\rho(0) \cap \mathbb{R}$. Prove that all coefficients a_n, $n \in \mathbb{N}$, must be real numbers. Deduce that $\overline{P(z)} = P(\overline{z})$ for all $z \in D_\rho(0)$.

Show Solution 5.5 on P197

Exercise 5.6

Prove that the ring of formal power series $\mathbb{C}[\![T]\!]$ is an *integral domain*. In other words, show that $PQ = 0 \implies P = 0$ or $Q = 0$.

Show Solution 5.6 on P198

Exercise 5.7

The *binomial coefficient* of a complex number $\alpha \neq 0$ and $k \in \mathbb{N}$ is defined as

$$\binom{\alpha}{k} = \begin{cases} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

The *binomial series* is the formal power series

$$B_\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} T^k.$$

a. Determine the radius of convergence of B_α.

b. Prove that if $\alpha \in \mathbb{N}$, then $B_\alpha(z) = (z+1)^\alpha$.

c. Show the *generalized Vandermonde* identity for $\alpha, \beta, \alpha+\beta \in \mathbb{C}^\times$,

$$\sum_{k=0}^{n} \binom{\alpha}{k} \binom{\beta}{n-k} = \binom{\alpha+\beta}{n}. \tag{$\star$}$$

d. Prove that for $\alpha = 1/k$ the complex function $B_\alpha(z)$ satisfies $B_\alpha(z)^k = z+1$ on its domain. Hence $B_\alpha(z)$ is a k-th root of the function $z+1$.

Show Solution 5.7 on P199

Exercise 5.8

For $a > 0$ and $z \in \mathbb{C}$ define $a^z = \exp(\log(a)z)$. Show that:

a. $a^z b^z = (ab)^z$ for all $a, b > 0,\ z \in \mathbb{C}$

b. $a^z a^w = a^{z+w}$ for all $a > 0,\ z, w \in \mathbb{C}$

c. $|a^z| = a^{\Re(z)}$ for all $a > 0,\ z \in \mathbb{C}$

The *Riemann ζ-function* is defined as

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad \text{for } \Re(z) > 1. \tag{$\star$}$$

d. Prove that the series ($\star$) converges absolutely for all $z \in \mathbb{C}$ with $\Re(z) > 1$ and uniformly on every subset $S_\delta = \{z \in \mathbb{C} \mid \Re(z) > 1 + \delta\}$ with $\delta > 0$.

Show Solution 5.8 on P201

Chapter 6

Contour integrals

In this section, we will generalize the integral $\int_a^b f(x)dx$ from calculus in two ways. Firstly, we allow $f = u + iv$ to be a complex function. Secondly, we will define the integral over more general curves γ than intervals $[a,b] \subset \mathbb{R}$.

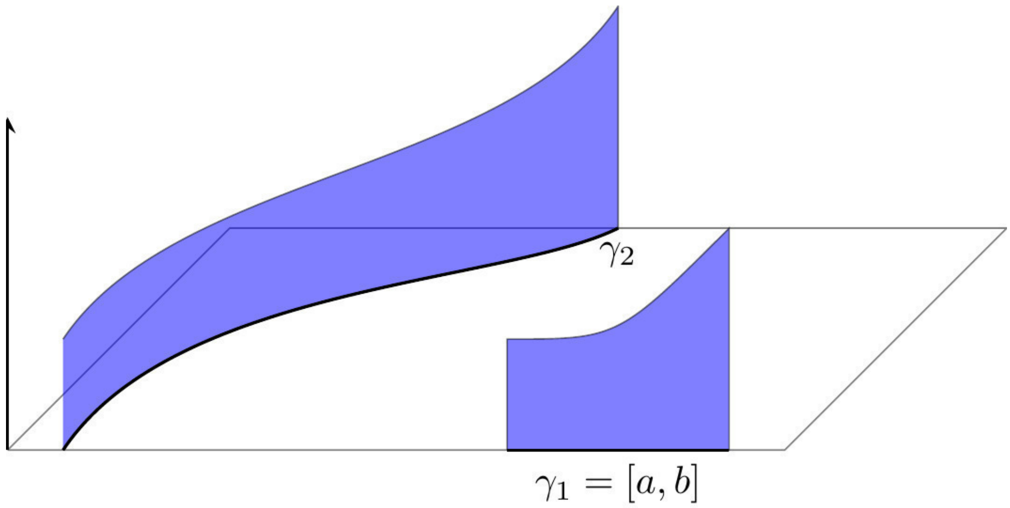


Figure 6.1: Generalizing the integral (the area, up to sign) of a real-valued function over an interval γ_1 to a general differentiable curve γ_2 by $\int_a^b f(\gamma(t))|\gamma'(t)|dt$. We will use a *different* generalization where we omit the modulus.

Definition 6.1. Let $f\colon [a,b] \to \mathbb{C}$ be a continuous complex-valued function on an interval with real and imaginary parts $f = u + iv$, $u, v\colon [a,b] \to \mathbb{R}$. Define

$$\int_a^b f(x)dx = \int_a^b u(x)dx + i\int_a^b v(x)dx. \tag{6.1}$$

Familiar rules for integrals (linearity, substitution) carry over to complex-valued functions, but the following estimate is more tricky to prove.

Proposition 6.1. $\left|\int_a^b f(x)dx\right| \leqslant \int_a^b |f(x)|dx$

Proof.

Pick a complex number $0 \neq c \in \mathbb{C}$ such that $c\int_a^b f(x)dx \in \mathbb{R}$. Then

$$\begin{aligned}
|c|\left|\int_a^b f(x)dx\right| &= \left|c\int_a^b f(x)dx\right| \\
&= \left|\Re\int_a^b cf(x)dx\right| && \text{as } c\int_a^b f(x)dx \in \mathbb{R} \\
&= \left|\int_a^b \Re(cf(x))dx\right| \\
&\leqslant \int_a^b |\Re(cf(x))|dx && \text{from calculus} \\
&\leqslant \int_a^b |cf(x)|dx \\
&= |c|\int_a^b |f(x)|dx,
\end{aligned}$$

where the third line follows from Equation 6.1 and the fifth from Equation 1.13. The result follows by dividing by $|c| \neq 0$.

□

Definition 6.2. A **curve** (or **path**) in the plane is a continuous map

$$[a,b] \xrightarrow{\gamma} \mathbb{C}$$

on a closed interval. Decompose $\gamma(t) = u(t) + iv(t)$ into real and imaginary parts. The curve γ is **differentiable** if u, v are differentiable on $[a,b]$ (including one-sided derivatives at the endpoints), and the curve γ is **continuously differentiable** (or C^1) if the derivatives $u'(t), v'(t)$ are continuous on $[a,b]$. The curve γ is **piecewise $\mathbf{C^1}$** if there exists a subdivision of the interval

$$a = t_0 < t_1 < \cdots < t_n = b \tag{6.2}$$

such that each of the restrictions $\gamma|_{[t_{k-1},t_k]}$, $k = 1, \dots, n$, is a continuously differentiable curve. In this case we call the subdivision **admissible**. For any admissible subdivision, the **length** of the curve γ is

$$L(\gamma) = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} |\gamma'(t)| dt. \tag{6.3}$$

We call $\gamma([a,b]) \subset \mathbb{C}$ the **image** of the curve. When the image is contained in a subset $D \subset \mathbb{C}$, we say that γ is a **curve in *D***.

A curve is **closed** if $\gamma(a) = \gamma(b)$, and then we call $\gamma(a)$ the **base** of the **loop** (or **contour**) γ.

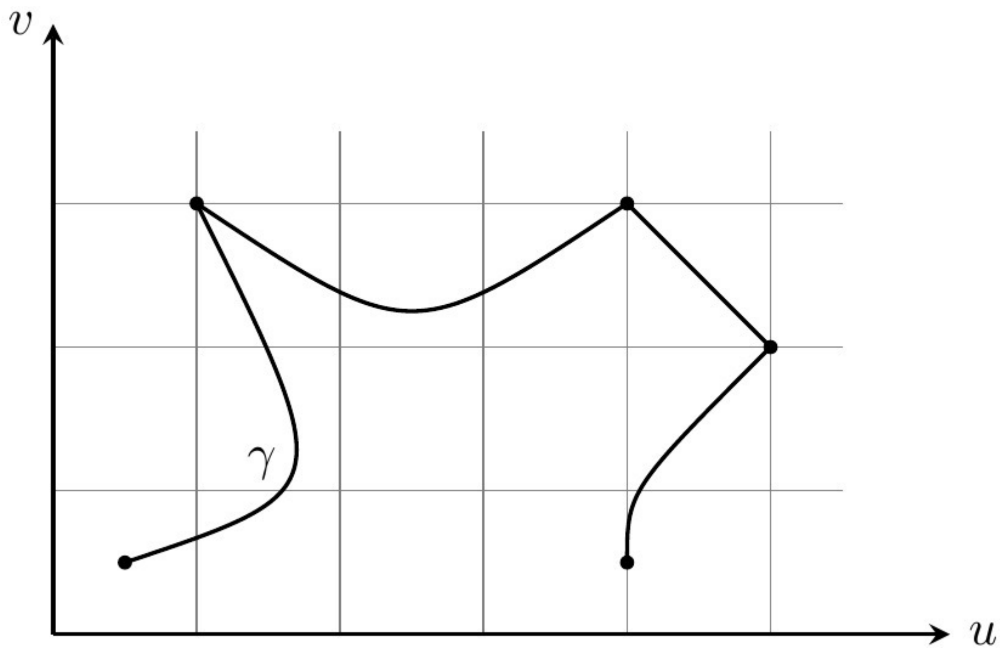


Figure 6.2: A piecewise C^1 curve

Definition 6.3. Let $f\colon D \to \mathbb{C}$ be a continuous complex function. Let γ be a piecewise C^1 curve in D. Pick an admissible subdivision Equation 6.2. The **path integral** (or **contour integral** if γ is closed) of f over the curve γ is

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(\gamma(t))\gamma'(t)dt. \tag{6.4}$$

Example 6.1.

The curve $\gamma(t) = t$ parameterizes the interval $[a, b]$. In this case, Equation 6.4 reduces to the integral $\int_a^b f(x)dx$ from Equation 6.1.

Example 6.2.

Let $\gamma(t) = p$ be constant. Then $\int_\gamma f(z)dz = 0$ for all f.

Example 6.3 (important)**.**

Let $f(z) = z^n$, $n \in \mathbb{Z}$, and $\gamma(t) = e^{it}$, $t \in [a, b]$.

$$\int_\gamma f(z)dz = \int_a^b e^{int} ie^{it} dt = \begin{cases} i(b-a) & \text{if } n = -1, \\ \frac{e^{i(n+1)b} - e^{i(n+1)a}}{n+1} & \text{if } n \neq -1. \end{cases}$$

In particular, for the **boundary curve** of the disk $\overline{D}_r(z_0)$ defined by

$$[0, 2\pi] \xrightarrow{\gamma_{\partial D_r(z_0)}} \overline{D}_r(z_0), \gamma_{\partial D_r(z_0)}(t) = z_0 + re^{it}, \tag{6.5}$$

which we denote by $\partial D_r(z_0)$, we have

$$\int_{\partial D_r(z_0)} (z - z_0)^n = \begin{cases} 2\pi i & \text{if } n = -1, \\ 0 & \text{if } n \neq -1. \end{cases} \tag{6.6}$$

Lemma 6.1.

a. *The integral Equation 6.4 and the length of a curve Equation 6.3 are independent of the choice of admissible subdivision.*

b. *Let $\varphi \colon [c, d] \to [a, b]$ be a strictly monotone increasing, continuously differentiable bijection. Let γ be a piecewise C^1 curve. Then $\gamma \circ \varphi$ is also a piecewise C^1 curve and*

$$\int_{\gamma\circ\varphi} f(z)dz = \int_\gamma f(z)dz.$$

Hence the curve integral is independent of the parameterization of γ. Similarly, $L(\gamma \circ \varphi) = L(\gamma)$ for the lengths.

Proof.

a. Since every two admissible subdivisions have a common admissible subdivision, it suffices to prove that Equation 6.4 remains unchanged upon passing to a subdivision. Every subdivision is obtained by inductively inserting points, so it suffices to consider the case of a single insertion, say $t_{k-1} < t_* < t_k$. Then only one summand in Equation 6.4 changes, and our claim follows from

$$\int_{t_{k-1}}^{t_k} f(\gamma(t))\gamma'(t)dt = \int_{t_{k-1}}^{t_*} f(\gamma(t))\gamma'(t)dt + \int_{t_*}^{t_k} f(\gamma(t))\gamma'(t)dt.$$

The same argument applies to the length of a curve.

b. As φ is bijective, there are unique $s_k \in [c, d]$ with $\varphi(s_k) = t_k$ for $k = 0, \dots, n$. Of course, $s_0 = c$ and $s_n = d$. The monotonicity implies that $s_{k-1} < s_k$ for all k and that φ maps $[s_{k-1}, s_k]$ bijectively onto $[t_{k-1}, t_k]$. The subdivision $c = s_0 < s_1 < \cdots < s_n = d$ is again admissible, since $\gamma \circ \varphi|_{[s_{k-1}, s_k]}$ is continuously differentiable for all k by the chain rule.

   Using the change of variables $s = \varphi(t)$, we compute

$$\begin{aligned}\int_{\gamma\circ\varphi} f(z)dz &= \sum_{k=1}^{n} \int_{s_{k-1}}^{s_k} f(\gamma(\varphi(t)))\gamma'(\varphi(t))\varphi'(t)dt \\ &= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(\gamma(s))\gamma'(s)ds = \int_\gamma f(z)dz.\end{aligned}$$

□

Due to Lemma 6.1(b) we view curves differing only by a parametrization as **equivalent**. In particular, we may translate and scale the domain $[a, b]$.

Definition 6.4.

a. Let $[a, b] \xrightarrow{\gamma_1} \mathbb{C}$, $[b, c] \xrightarrow{\gamma_2} \mathbb{C}$. Assume $\gamma_1(b) = \gamma_2(b)$. Then the **concatenation** of γ_1 with γ_2 is the curve

$$[a,c] \xrightarrow{\gamma_1 * \gamma_2} \mathbb{C}, (\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [a,b], \\ \gamma_2(t) & \text{if } t \in [b,c]. \end{cases}$$

b. The **opposite** of a curve $[a,b] \xrightarrow{\gamma} \mathbb{C}$ is the curve

$$[-b,-a] \xrightarrow{-\gamma} \mathbb{C}, (-\gamma)(t) = \gamma(-t).$$

Proposition 6.2. *Let $f, g\colon D \to \mathbb{C}$ be continuous complex functions and let $[a,b] \xrightarrow{\gamma} D$ be a piecewise C^1 curve.*

a. The integral over a curve is linear: for all $\lambda \in \mathbb{C}$ we have

$$\int_\gamma \lambda f(z) + g(z) dz = \lambda \int_\gamma f(z) dz + \int_\gamma g(z) dz.$$

b. For the opposite curve,

$$\int_{-\gamma} f(z) dz = -\int_\gamma f(z) dz.$$

c. For the concatenation of curves,

$$\int_{\gamma_1 * \gamma_2} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz.$$

d. Suppose that $|f(z)| \leqslant M$ for all $z \in \gamma([a,b])$. Then

$$\left| \int_\gamma f(z) dz \right| \leqslant M \cdot L(\gamma). \tag{6.7}$$

Proof.

a. follows from the linearity of the ordinary integral and b. is simply the change of variables $s = -t$ in the integral. The argument for c. is the same as for Lemma 6.1(a). We prove d. for a trivial subdivision ($n = 1$):

$$\begin{aligned}\left|\int_\gamma f(z)dz\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \\ &\leqslant \int_a^b |f(\gamma(t))||\gamma'(t)|dt \\ &\leqslant M\int_a^b |\gamma'(t)|dt.\end{aligned}$$

where the second line follows from Proposition 6.1.

The general case is obtained by applying this to each subinterval $[t_{k-1}, t_k]$ and then summing the resulting estimates over k.

□

Remark 6.1. There is a generalization of the path integral to curves γ of bounded variation called the Riemann–Stieltjes integral. Proposition 6.2 continuous to hold with the length $L(\gamma)$ replaced by the total variation. For example, Lipschitz continuous maps are of bounded variation.

From Equation 6.7 we obtain:

Corollary 6.1. *Let $(f_n)_{n\in\mathbb{N}}$ be a uniformly convergent sequence of complex functions on D. For every curve $\gamma\colon [a,b] \to D$ we have*

$$\lim_{n\to\infty}\int_\gamma f_n(z)dz = \int_\gamma \lim_{n\to\infty} f_n(z)dz.$$

Proposition 6.3 (Complex FTC). *Let f be a continuous complex function on an open set U and let $[a,b] \xrightarrow{\gamma} U$ be a piecewise C^1 curve. Suppose F is a holomorphic function on U with $F' = f$ (we call F a* ***primitive*** *of f). Then*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)). \tag{6.8}$$

In particular, for every closed curve

$$\int_\gamma f(z)dz = 0. \tag{6.9}$$

Proof.

By the chain rule,

$$\frac{d}{dt}F(\gamma(t)) = f(\gamma(t))\gamma'(t).$$

Using the (ordinary) fundamental theorem of calculus to evaluate Equation 6.4 gives

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} F(\gamma(t_k)) - F(\gamma(t_{k-1})) = F(\gamma(t_n)) - F(\gamma(t_0)).$$

□

If F is holomorphic with $F' = 0$ and U is path-connected, see Definition 8.1(a) below, we recover from Equation 6.8 the familiar fact that F is a constant. Of course, this holds more generally for differentiable functions F.

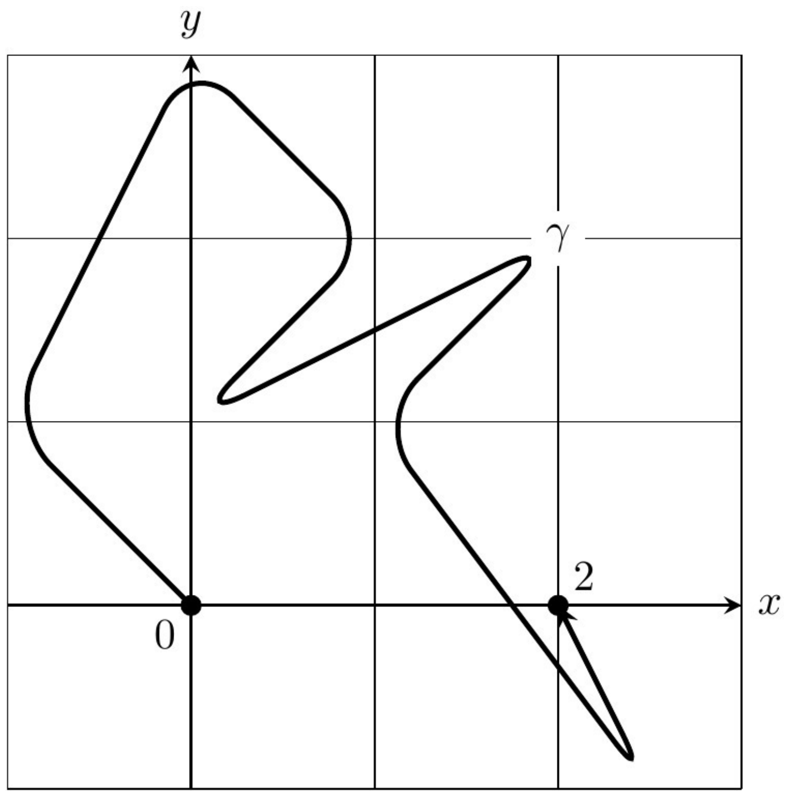


Figure 6.3: Seemingly complicated path integrals are easy to evaluate using the complex FTC. For the depicted contour, $\int_\gamma e^z dz = e^2 - 1$.

Example 6.4.

Since the entire function $f(z) = az + b$ has the primitive $\frac{1}{2}az^2 + bz$, we have $\int_\gamma (az + b)dz = 0$ for every closed curve. More generally, Theorem 5.2 can be used to construct primitives for power series.

Remark 6.2. Conversely, every holomorphic function has a holomorphic primitive on a *simply connected* domain U (for example, a disk). This will be proven in Theorem 8.5.

Questions for further discussion

- Is there an analogue of Proposition 6.3 where F is only assumed to be real differentiable?
- Find the integral of a power series $P = \sum_{n=0}^{\infty} a_n z^n$ with positive radius of convergence $\rho > 0$ along an arbitrary curve γ in $D_\rho(0)$?

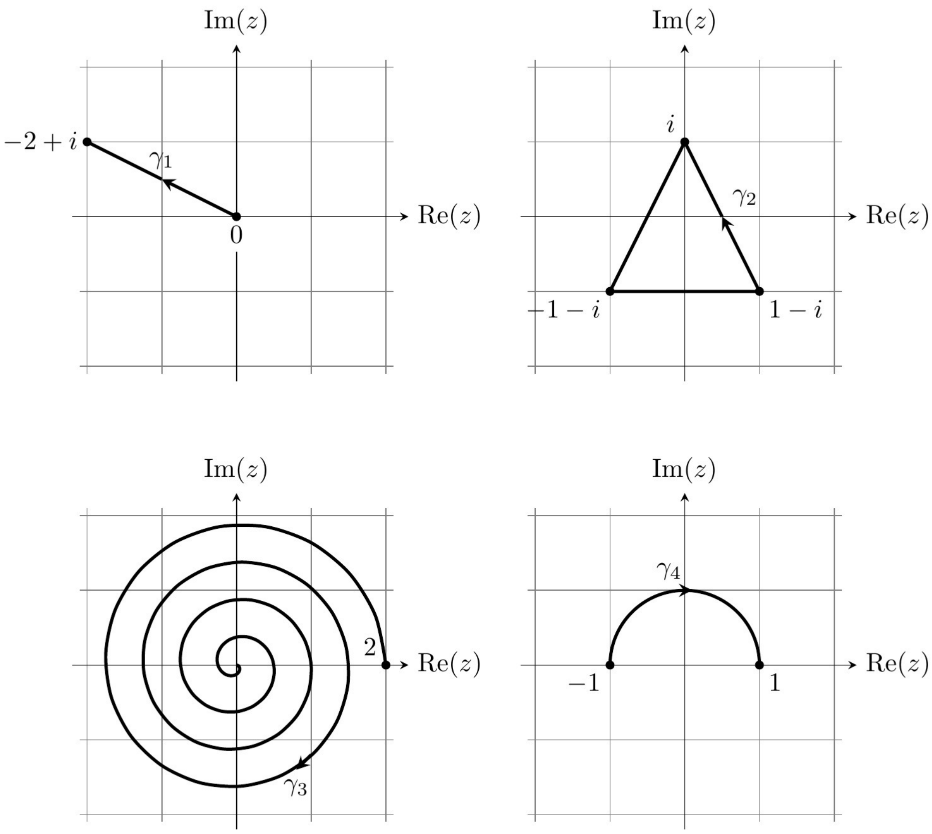
6.1 Exercises

Exercise 6.1

Recall from Example 4.6 that $\log\colon \mathbb{C}^- \to \mathbb{C}$ is a holomorphic primitive of $1/z$ on the slit plane $\mathbb{C}^-$. Combine Equation 6.9 and Example 6.3 to show that there is *no* holomorphic primitive of $1/z$ on the punctured plane $\mathbb{C}^\times$.

Exercise 6.1

For the curves γ_k, $k = 1, 2, 3, 4$, as in the following sketch



a. find piecewise C^1 parameterisations $\gamma_k \colon [a, b] \to \mathbb{C}$,

b. compute the length $L(\gamma_k)$ (γ_3 may be omitted)

c. evaluate $\int_{\gamma_k} z^2 dz$.

Exercise 6.2

Define $\gamma_\epsilon \colon [-\pi+\epsilon, \pi-\epsilon] \to \mathbb{C}$, $\gamma_\epsilon(t) = e^{it}$ for $0 < \epsilon < \pi$. Compute

$$\lim_{\epsilon \to 0} \int_{\gamma_\epsilon} z^{1/2} dz,$$

where the square root $z^{1/2} = \exp(\log(z)/2)$ is defined as on Sheet 5 using the principal branch of the logarithm.

(*Aside:* $z^{1/2}$ is integrable in the sense of measure theory and using the more general integral defined there we can rewrite the above integral as $\int_{\partial D_1(0)} z^{1/2} dz$.)

Exercise 6.3

Let γ be the straight line from $z = 1$ to $z = i$. Let $f(z) = 1/z^4$. Determine the maximum of $|f(z)|$ over all $z \in \mathfrak{I}(\gamma)$. Use this to estimate

$$\left| \int_\gamma f(z) dz \right| \leqslant 4\sqrt{2}.$$

Then compute $\int_\gamma f(z)dz$ directly and compare this to the estimate.

Exercise 6.4

Let $f(z) \colon \partial D_r(0) \to \mathbb{C}$ be a continuous function. Show that

$$\overline{\int_{\partial D_r(z_0)} f(z) dz} = -r^2 \int_{\partial D_r(z_0)} \overline{f(z)} (z - z_0)^{-2} dz.$$

Exercise 6.5

Let $D_1, D_2 \subset \mathbb{C}$ be closed subsets and $f\colon D_1 \cup D_2 \to \mathbb{C}$ be continuous. Suppose that $\int_{\gamma_1} f(z)dz = 0$ for every closed curve γ_1 in D_1 and that $\int_{\gamma_2} f(z)dz = 0$ for every closed curve γ_2 in D_2. Show that if $D_1 \cap D_2$ is path-connected, then $\int_\gamma f(z)dz = 0$ for every closed curved $\gamma\colon [a,b] \to D_1 \cup D_2$.

Hint: Subdivide $[a,b]$ into subintervals $[t_{k-1}, t_k]$ which map under γ entirely to D_1 or entirely to D_2.

Part III

Cauchy’s theorem and residues

Chapter 7

Cauchy's theorem

Definition 7.1. Let $D \subset \mathbb{C}$ and $\gamma_0, \gamma_1 \colon [a,b] \to D$ curves in D.

a. A **homotopy**[1] in D between paths γ_0, γ_1 is a continuous map

$$\Gamma \colon [0,1] \times [a,b] \longrightarrow D, (s,t) \longmapsto \Gamma_s(t),$$

such that $\Gamma_0(t) = \gamma_0(t)$ and $\Gamma_1(t) = \gamma_1(t)$ for all $t \in [a,b]$. Then γ_0, γ_1 are called (freely) **homotopic paths** in D.

If, additionally, $\Gamma_s(a) = p$ and $\Gamma_s(b) = q$ are constant in $s \in [0,1]$, we call Γ a **path homotopy** in D and γ_0, γ_1 **path-homotopic** in D.

b. If γ_0, γ_1 are loops, a **homotopy of loops** in D is a homotopy Γ in D with the additional property that Γ_s is a loop for each $s \in [0,1]$. Then γ_0, γ_1 are called (freely) **homotopic loops** in D.

A loop is **null-homotopic** in D if there is a homotopy of loops in D to the constant loop.

[1] From Greek *homos* 'one and the same' and *topos* 'place, region, space'

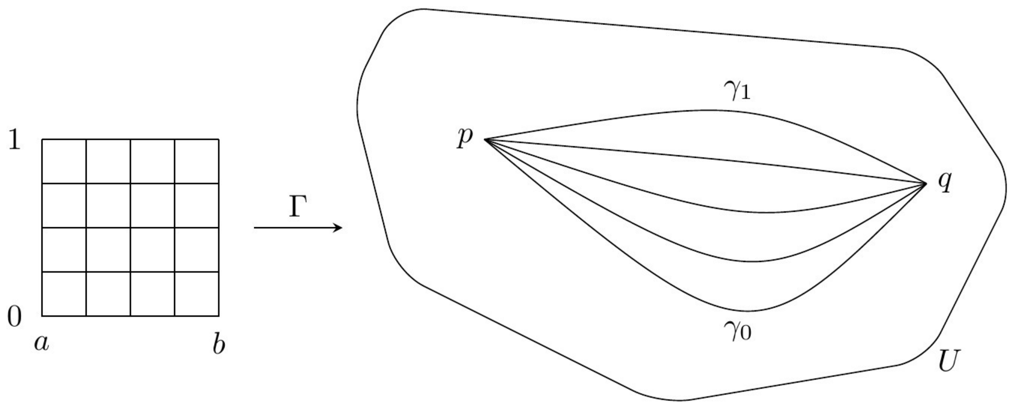


Figure 7.1: A path homotopy is a video of paths with fixed endpoints.

Remark 7.1. In the following we will also suppose that Γ is piecewise C^1, meaning there exist subdivisions

$$0 = s_0 < s_1 < \cdots < s_m = 1, \qquad a = t_0 < t_1 < \cdots < t_n = b$$

such that each restriction $\Gamma|_{[s_{j-1},s_j]\times[t_{k-1},t_k]}$ is continuously differentiable.

Example 7.1.

A set D is **star-shaped** if there exists a **focal point** $z_0 \in D$ such that for each $z \in D$ the straight line segment $tz + (1-t)z_0$, $t \in [0,1]$, is contained in D. Disks, the complex plane, and rectangles are star-shaped. Every loop γ in a star-shaped domain is null-homotopic. If the focal point agrees with the base of the loop, the path homotopy is

$$\Gamma_s(t) = (1-s)\gamma(t) + sz_0. \tag{7.1}$$

In general, the path homotopy is more complicated to write down. We omit it, since we will not need this fact below.

Figure 7.2: Augustin Louis, Baron Cauchy. Lithograph by J. Boilly, 1821. Wellcome Collection. Public Domain Mark

Theorem 7.1 (Cauchy's theorem)**.** *Let $f: U \to \mathbb{C}$ be a holomorphic function on an open set. Let γ_0, γ_1 be piecewise C^1 curves in U that are path-homotopic in U. Then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Proof.

Pick a path homotopy $\Gamma: [0,1] \times [a,b] \to U$. By reparameterizing we may assume $[a,b] = [0,1]$. Write $R^{(0)} = [0,1] \times [0,1]$ for the domain of Γ and let $\partial R^{(0)}$ be its piecewise linear boundary path, with the obvious counterclockwise parameterization (see Figure 7.3).

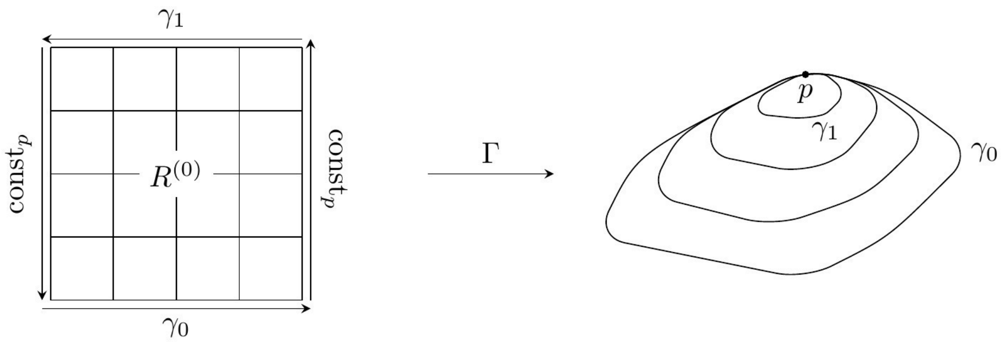


Figure 7.3: Path homotopy when γ_0, γ_1 are loops.

The boundary path has four segments and the path integral along each constant path vanishes. Using Proposition 6.2(b),(c), we then find that

$$\int_{\gamma_0} f(z)dz - \int_{\gamma_1} f(z)dz = \int_{\Gamma(\partial R^{(0)})} f(z)dz.$$

Subdivide $R^{(0)}$ into four congruent rectangles $R_i^{(0)}$ as in Figure 7.4. Using the fact that the path integrals in opposite directions cancel, we find

$$\left|\int_{\Gamma(\partial R^{(0)})} f(z)dz\right| = \left|\sum_{i=1}^{4} \int_{\Gamma(\partial R_i^{(0)})} f(z)dz\right| \leqslant \sum_{i=1}^{4} \left|\int_{\Gamma(\partial R_i^{(0)})} f(z)dz\right|.$$

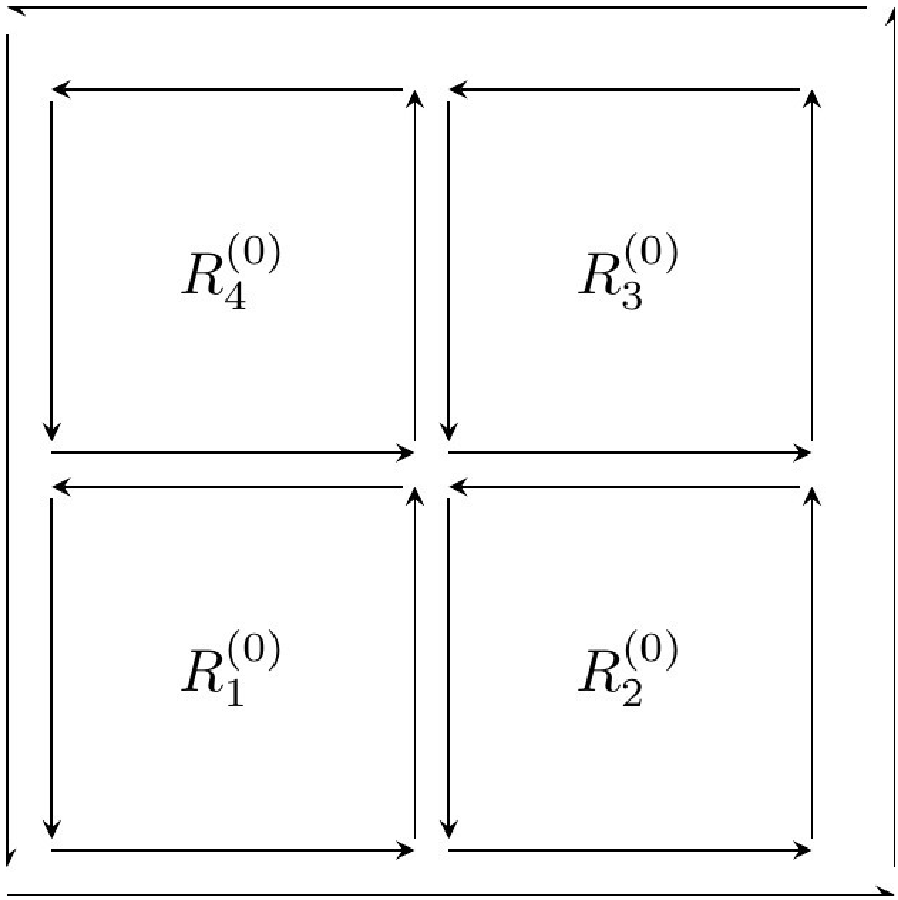


Figure 7.4: Subdivision and cancellation of opposite paths

Let $R^{(1)}$ be the rectangle $R_i^{(0)}$ for which $\left|\int_{\Gamma(\partial R_i^{(0)})} f(z)dz\right|$ is maximal. Then

$$\left|\int_{\Gamma(\partial R^{(0)})} f(z)dz\right| \leqslant 4\left|\int_{\Gamma(\partial R^{(1)})} f(z)dz\right|.$$

Now repeat this process with $R^{(1)}$ to obtain $R^{(2)}$ and so forth. This yields a sequence of rectangles $R^{(n)}$ as in Figure 7.5 with sides of length 2^{-n} and

$$\left|\int_{\Gamma(\partial R^{(0)})} f(z)dz\right| \leqslant 4^n\left|\int_{\Gamma(\partial R^{(n)})} f(z)dz\right|. \tag{7.2}$$

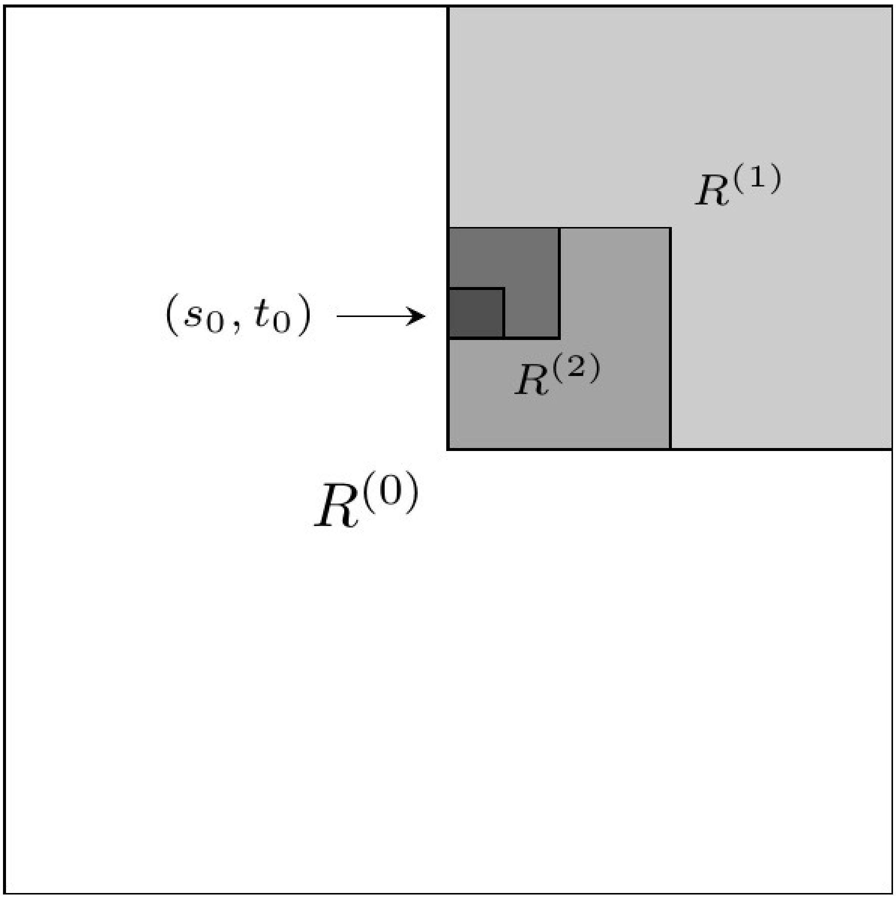


Figure 7.5: Convergent sequence of rectangles $\bigcap R^{(n)} = \{(s_0, t_0)\}$

As the side lengths tend to zero, the midpoints $(s^{(n)}, t^{(n)})$ of the rectangles $R^{(n)}$ are a Cauchy sequence, so converge to some limit (s_0, t_0). Let $z_0 = \Gamma_{s_0}(t_0)$. Since f is complex differentiable at z_0, we may write

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + (z - z_0)\rho(z), \qquad \lim_{z \to z_0} \rho(z) = 0. \tag{7.3}$$

By Example 6.4,

$$\int_{\Gamma(\partial R^{(n)})} (f(z_0) + (z - z_0)f'(z_0))\, dz = 0. \tag{7.4}$$

As Γ is piecewise C[1], its derivative is bounded in norm by some $C > 0$.

We may also estimate Γ in a neighbourhood of (s_0, t_0) as

$$\begin{aligned} \Gamma(s,t) &= \Gamma(s_0,t_0) + ((s,t)-(s_0,t_0))\,\Gamma'(s_0,t_0) + ((s,t)-(s_0,t_0))\,\phi(s,t), \\ & \lim_{(s,t)\to(s_0,t_0)} \phi(s,t) = 0. \end{aligned}$$

Let $\epsilon > 0$. Pick $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, $z \in \Gamma(R^{(n)})$, $(s,t) \in R^{(n)}$, we have $|\rho(z)|, |\phi(s,t)| < \epsilon$. As the side lengths of $R^{(n)}$ are 2^{-n} and $z_0 \in \Gamma(R^{(n)})$, for all $z \in \Gamma(\partial R^{(n)})$, we have

$$\begin{aligned} |z - z_0| &= |\Gamma(s,t) - \Gamma(s_0,t_0)| \\ &\leqslant |(s,t)-(s_0,t_0)||\Gamma'(s_0,t_0)| + |(s,t)-(s_0,t_0)||\phi(s,t)| \\ &\leqslant 2^{-n}\sqrt{2}C + 2^{-n}\sqrt{2}\epsilon \\ &\leqslant \sqrt{2}2^{-n}(C+\epsilon) \end{aligned}$$

where $(s,t) \in \partial(R^{(n)}$ is any point in $\Gamma^{-1}(z)$

Combining this with Equation 7.3 and Equation 7.4, we can estimate

$$\begin{aligned} \left|\int_{\Gamma(\partial R^{(n)})} f(z)dz\right| &= \left|\int_{\Gamma(\partial R^{(n)})} (z-z_0)\rho(z)dz\right| \\ &\leqslant L(\Gamma(\partial R^{(n)}))\sqrt{2}2^{-n}(C+\epsilon)\epsilon = 4C \cdot 2^{-n}\sqrt{2}2^{-n}(C+\epsilon)\epsilon \end{aligned}$$

The last inequality follows by Equation 6.7.

Hence

$$\left|\int_\gamma f(z)dz\right| = \left|\int_{\Gamma(\partial R^{(0)})} f(z)dz\right| \leqslant 4C(C+\epsilon)\sqrt{2}\epsilon.$$

The final inequality follows from Equation 7.2.

As $\epsilon > 0$ is arbitrary, the left hand side must be zero.

$\square$

Theorem 7.2. *Let γ_0, γ_1 be piecewise C^1 loops in U that are (freely) homotopic in U. Suppose $f\colon U \to \mathbb{C}$ is a holomorphic function. Then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

In particular, if γ is a loop that is homotopic in U to the constant loop, then

$$\int_\gamma f(z)dz = 0. \tag{7.5}$$

Proof.

Let Γ be the homotopy between γ_0 and γ_1. Let $\eta(s) = \Gamma_s(0)$. Figure 7.6 describes the construction of a path homotopy between $\eta * \gamma_0 * (-\eta)$ and γ_1.

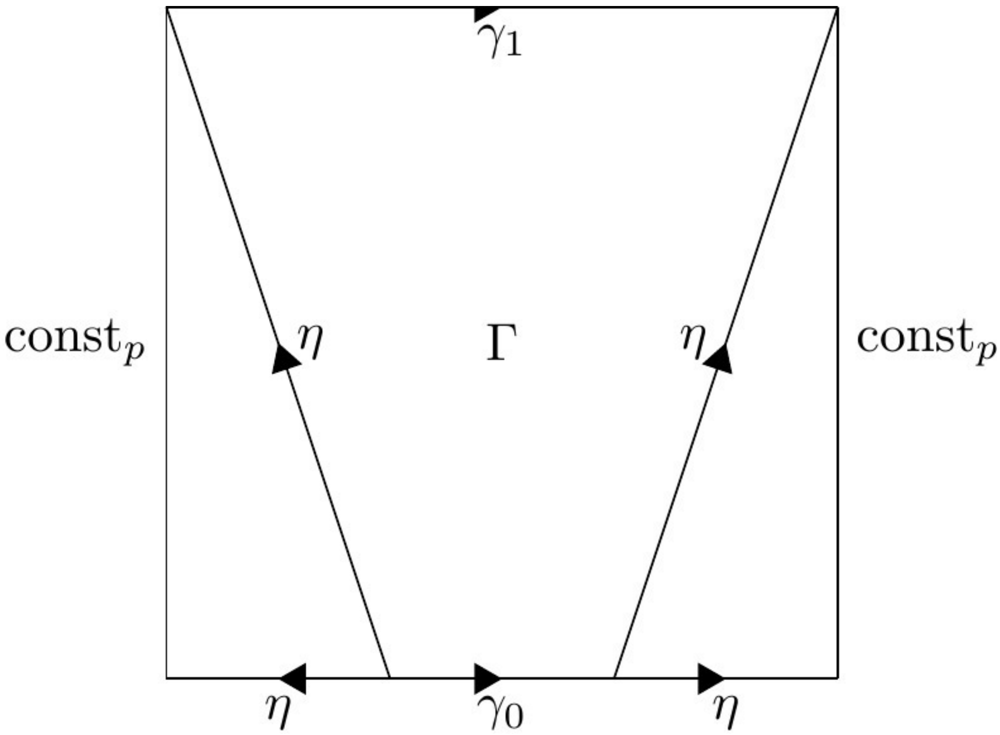


Figure 7.6: Construction of path homotopy

Hence Theorem 7.1 implies our claim, using the fact that the path integrals over η and $-\eta$ cancel each other.

□

Questions for further discussion

- How should Figure 7.6 be interpreted? Can you give a formula for the path homotopy (consider cases)?
- Give a counterexample to Cauchy's theorem when (i) γ is not null-homotopic (ii) $f(z)$ is not holomorphic
- Does Cauchy's theorem hold for $f(z) = \overline{z}$?

7.1 Exercises

Exercise 7.1

(a) Sketch the curves

$$\gamma_0\colon [-1,1] \longrightarrow \mathbb{C}, \qquad \gamma_0(t) = t,$$
$$\gamma_1\colon [-1,1] \longrightarrow \mathbb{C}, \qquad \gamma_1(t) = e^{i\pi\frac{1-t}{2}}$$

and show that they are path-homotopic.

(b) Let $0 < r_0 < r_1$ and $z_0 \in \mathbb{C}$. Prove that the loops $\partial D_{r_0}(z_0)$ and $\partial D_{r_1}(z_0)$ are freely homotopic in the closed annulus $\overline{A}_{r_0,r_1}(z_0)$.

Exercise 7.2

Let $\alpha + i\beta \in \mathbb{C}$. Determine

$$\int_a^b e^{(\alpha+i\beta)t} dt$$

to compute

$$\int_a^b e^{\alpha t} \cos(\beta t) dt.$$

Exercise 7.3

Let $\mathbb{C}^- = \mathbb{C} \setminus (-\infty, 0]$ be the slit plane.

a. Show that any two points in $\mathbb{C}^-$ may be connected by a path in $\mathbb{C}^-$. Hence $\mathbb{C}^-$ is *path-connected*.

b. Show that every closed curve $\gamma\colon [a, b] \to \mathbb{C}^-$ is null-homotopic in $\mathbb{C}^-$ by finding a homotopy

$$\Gamma\colon [0, 1] \times [a, b] \longrightarrow \mathbb{C}^-, (s, t) \longmapsto \Gamma_s(t)$$

satisfying $\Gamma_0(t) = \gamma(t)$ and $\Gamma_1(t) = 1$ for all $t \in [a, b]$. Hence $\mathbb{C}^-$ is *simply-connected*.

c. Prove the analogues of a. and b. for a disk $D_r(z_0)$.

d. Use Cauchy's Theorem to prove that the punctured plane $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ is not simply-connected. That is, there exists a closed curve in $\mathbb{C}^\times$ that is not null-homotopic in $\mathbb{C}^\times$.

Exercise 7.4

Let $0 < b < 1$.

a. Using the geometric series, find the power series expansion

$$\frac{1}{z - 1/b} = \sum_{n=0}^{\infty} a_n (z-b)^n$$

with center $z_0 = b$ and determine the radius of convergence ρ.

b. Use a. to show that for all $0 < r < \rho$

$$\int_{\partial D_r(b)} \frac{dz}{(z-b)(z-1/b)} = \frac{2\pi i}{b - 1/b}.$$

c. Use c. to compute

$$\int_0^{2\pi} \frac{dt}{1 - 2b\cos(t) + b^2}.$$

Exercise 7.5

Let $\gamma\colon [0,1] \to D$ be a curve in $D \subset \mathbb{C}$ and let $-\gamma\colon [0,1] \to D$, $(-\gamma)(t) = \gamma(1-t)$ be the opposite curve. Prove that $\gamma * (-\gamma)$ is path-homotopic to a constant loop.

Chapter 8

Applications of Cauchy's theorem

Theorem 8.1 (Cauchy's integral formula)**.** *Let $f\colon U \to \mathbb{C}$ be a holomorphic function and assume that $\overline{D}_r(z_0) \subset U$ with boundary curve*

$$[0, 2\pi] \xrightarrow{\gamma_{\partial D_r(z_0)}} \overline{D}_r(z_0), \gamma_{\partial D_r(z_0)}(t) = z_0 + re^{it}.$$

Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta, \qquad \forall z \in D_r(z_0). \tag{8.1}$$

Proof.

Fix $z \in D_r(z_0)$. Since f is complex differentiable at z, we have

$$f(\zeta) = f(z) + (\zeta - z)f'(z) + (\zeta - z)\rho(\zeta - z), \lim_{\zeta \to z} \rho(\zeta - z) = 0. \tag{8.2}$$

There is a homotopy in $U \setminus \{z\}$ between the loops $\partial D_r(z_0)$ and $\partial D_s(z)$, for all $s > 0$ with $D_s(z) \subset D_r(z_0)$, see Figure 8.1.

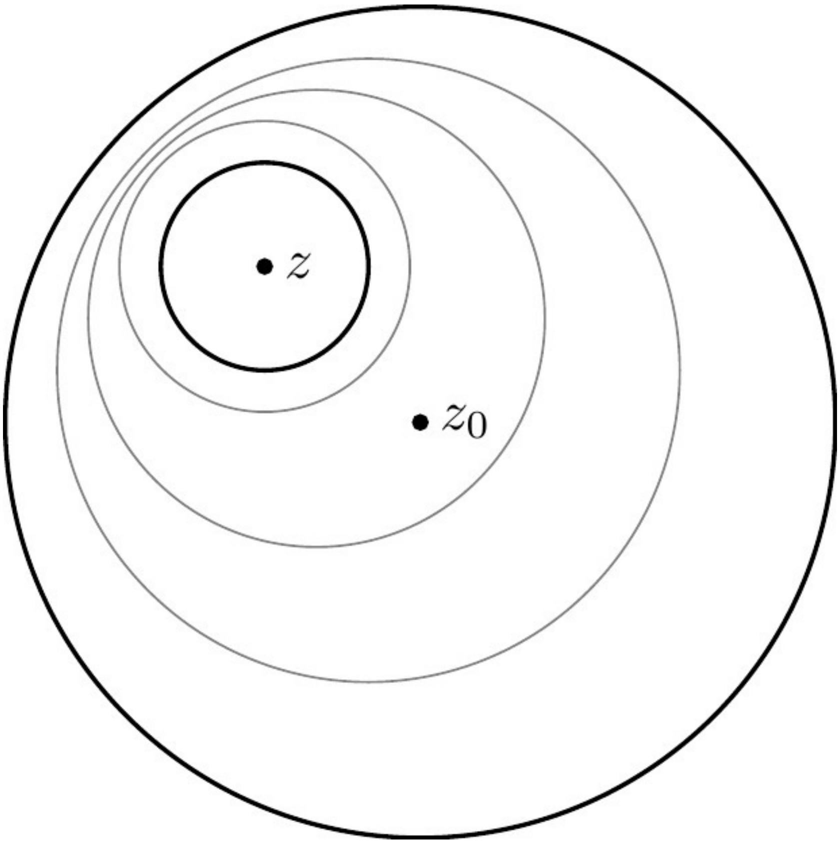


Figure 8.1: Homotopy between $\partial D_r(z_0)$ and $\partial D_s(z)$

Moreover, $\zeta \mapsto \frac{f(\zeta)}{\zeta - z}$ is a holomorphic function on $U \setminus \{z\}$.

By Theorem 7.1 we have

$$\int_{\partial D_r(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{\partial D_s(z)} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

applying Equation 8.2,

$$\int_{\partial D_s(z)} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{\partial D_s(z)} \frac{f(z)}{\zeta - z} d\zeta + \int_{\partial D_s(z)} f'(z) d\zeta + \int_{\partial D_s(z)} \rho(\zeta - z) d\zeta$$

applying Equation 6.6 first with $n = -1$ and then $n = 0$, we have

$$\int_{\partial D_s(z)} \frac{f(z)}{\zeta - z} d\zeta + \int_{\partial D_s(z)} f'(z) d\zeta + \int_{\partial D_s(z)} \rho(\zeta - z) d\zeta = 2\pi i f(z) \tag{8.3}$$

$$+ 0 \tag{8.4}$$

$$+ \int_{\partial D_s(z)} \rho(\zeta - z) d\zeta. \tag{8.5}$$

Consequently,

$$\int_{\partial D_r(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta = 2\pi i f(z) + 0 + \int_{\partial D_s(z)} \rho(\zeta - z) d\zeta.$$

Since $\lim_{\zeta \to z} \rho(\zeta - z) = 0$ there exists a bound $|\rho(\zeta - z)| \leqslant M$. Hence the final term can be estimated using Equation 6.7 as

$$\left| \int_{\partial D_s(z)} \rho(\zeta - z) d\zeta \right| \leqslant 2\pi s M.$$

This tends to zero as $s \to 0$, and the left hand side is independent of s.

□

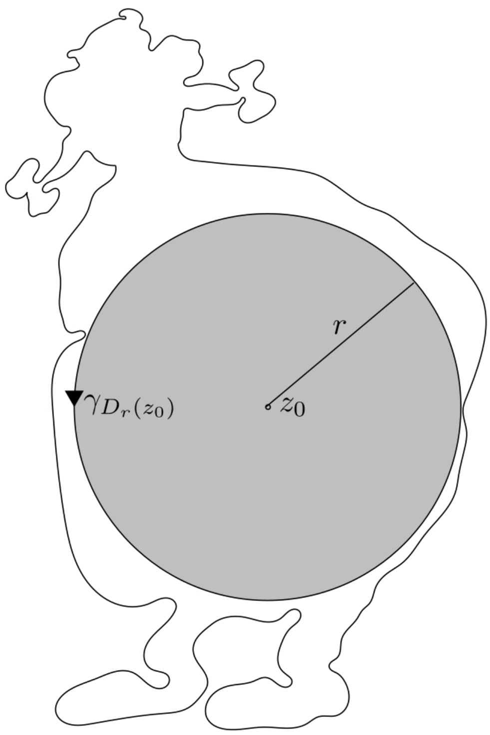


Figure 8.2: Boundary curve of largest disk fitting inside U

Theorem 8.2 (Higher Cauchy integral formula)**.** *Let $f\colon U \to \mathbb{C}$ be a holomorphic function and assume that $\overline{D}_r(z_0) \subset U$. Then f is equal on $D_r(z_0)$ to its Taylor power series*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \tag{8.6}$$

which has radius of convergence $\rho \geqslant r$. In particular, f is infinitely complex differentiable on the open set U and has a primitive on $D_r(z_0)$. Moreover,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\partial D_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta. \tag{8.7}$$

Proof.

Let $z \in D_r(z_0)$. The integrand in Equation 8.1 can be rewritten as

$$\frac{f(\zeta)}{\zeta - z} = \frac{\frac{f(\zeta)}{\zeta - z_0}}{1 - \frac{z-z_0}{\zeta - z_0}} = \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n, \tag{8.8}$$

where the geometric series converges uniformly for all ζ with $|z - z_0| < |\zeta - z_0|$.

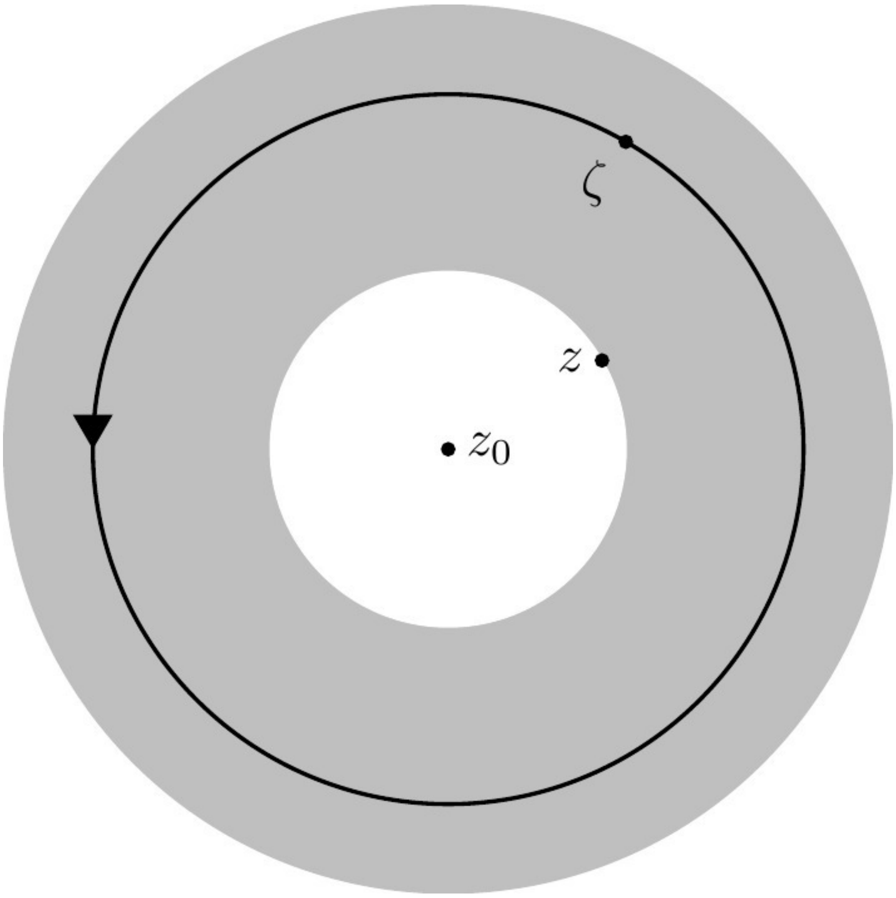


Figure 8.3: The region $|z - z_0| < |\zeta - z_0|$ and the contour of integration

In particular, this holds for all $\zeta \in \partial D_r(z_0)$ and using Corollary 6.1 we may exchange the limit and the integral. Hence for all $|z - z_0| < r$ we have a convergent series

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n. \tag{8.9}$$

In particular, the series Equation 8.9 has radius of convergence $\rho \geqslant r$.

Finally, Equation 8.7 (and therefore Equation 8.6) follow from the fact that the power series Equation 8.9 is differentiated termwise, see Theorem 5.2.

□

Theorem 8.3 (Liouville). *Every bounded entire function is constant.*

Proof.

Suppose $|f(z)| < C$ for all $z \in \mathbb{C}$. Then by Equation 8.7 with $z_0 = 0$ we have

$$|f^{(n)}(0)| = \left| \frac{n!}{2\pi i} \int_{\partial D_r(0)} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta \right|,$$

by Equation 6.7,

$$\left| \frac{n!}{2\pi i} \int_{\partial D_r(0)} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta \right| \leqslant \frac{n!}{2\pi} L(\partial D_r(0)) \frac{C}{r^{n+1}} = \frac{n! C}{r^n}$$

and so

$$|f^{(n)}(0)| \leqslant \frac{n! C}{r^n}$$

for all $r > 0$. Letting $r \to \infty$, this implies that $f^{(n)}(0) = 0$ for all $n > 0$. Now Equation 8.6 proves that $f(z)$ is constant.

□

Figure 8.4: Carl Friedrich Gauß, 1777-1855, Österreichische Nationalbibliothek. Public Domain

Theorem 8.4 (Fundamental theorem of algebra)**.** *Every non-constant polynomial* $P(z) = a_n z^n + ... + a_1 z + a_0$ *with* $a_i \in \mathbb{C}$ *has a complex root.*

Proof.

Assume by contradiction that $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then $1/P(z)$ is holomorphic and bounded. Indeed, assuming $a_n \neq 0$, we find

$$|P(z)| \geqslant |a_n||z|^n - |a_{n-1}||z|^{n-1} - ... - |a_0| \to +\infty \text{as } z \to \infty.$$

Hence $1/|P(z)| \to 0$ as $z \to \infty$. In particular, $|1/P(z)|$ is bounded.

$\square$

Definition 8.1.

a. A subset $D \subset \mathbb{C}$ is **path-connected** if for all $z_0, z_1 \in D$ there exists a (piecewise C^1) curve γ in D with $\gamma(0) = z_0$, $\gamma(1) = z_1$.
b. A path-connected subset D is **simply connected** if every loop in D is (freely) homotopic in D to a constant loop.

Example 8.1.

Every star-shaped set D is simply connected. Firstly, D is path-connected since any $z \in D$ can be connected by a straight line $tz + (1-t)z_0$ to the focal point z_0. Secondly, if $\gamma\colon [a,b] \to D$ is a loop, then $\Gamma_s(t) = sz_0 + (1-s)\gamma(t)$ is a homotopy in D from γ to the constant loop.

Theorem 8.5. *Let $f\colon U \to \mathbb{C}$ be a holomorphic function. Suppose that U is simply connected and let $z_0 \in U$. Define $F(z)$ for each $z \in U$ by choosing a piecewise C^1 curve γ with $\gamma(0) = z_0$, $\gamma(1) = z$ and defining*

$$F(z) = \int_\gamma f(\zeta) d\zeta. \tag{8.10}$$

Then F is well-defined and is the unique holomorphic function on U such that $F' = f$ and $F(z_0) = 0$.

Proof.

For any two curves γ_0, γ_1 as in the statement of the theorem, the concatenation $\gamma_0 * (-\gamma_1)$ is a loop which is null-homotopic by assumption. By Theorem 7.2 we have

$$\int_{\gamma_0} f(\zeta) d\zeta - \int_{\gamma_1} f(\zeta) d\zeta = 0,$$

hence Equation 8.10 is well-defined. It remains to check that F is complex differentiable at every point $z_* \in U$ for which it suffices to restrict attention to the disk $\overline{D}_r(z_*) \subset U$.

Fix a path γ_* in U from z_0 to z_*. For each $z \in D_r(z_*)$, let γ be the concatenation of γ_* with a path η from z_* to z in the disk $D_r(z_*)$. Then

$$F(z) = \int_\gamma f(\zeta) d\zeta = \int_{\gamma_*} f(\zeta) d\zeta + \int_\eta f(z) d\zeta.$$

The first integral is a constant independent of z. By Theorem 8.2, f has a holomorphic primitive g on $D_r(z_*)$ and η is a path in that disk, so we can apply the complex FTC Equation 6.8 to the second integral and write

$$F(z) = \int_{\gamma_*} f(\zeta) d\zeta + g(z) - g(z_*).$$

This shows that F is a holomorphic function of z with $F' = g' = f$.

To prove uniqueness, suppose that $\tilde{F}$ is another holomorphic function with $\tilde{F}' = f$ and $\tilde{F}(z_0) = w_0$. Then $F - \tilde{F} = c$ is constant on U and by evaluating at z_0 we find $c = F(z_0) - \tilde{F}(z_0) = w_0 - w_0 = 0$.

□

Example 8.2.

Let $U \subset \mathbb{C}$ be a simply connected subset with $0 \notin U$ and $f(z) = 1/z$. Fix $w_0 \in \mathbb{C}$ such that $z_0 = e^{w_0} \in U$. Define the holomorphic function $F(z)$ on U by Equation 8.10. Then $\ell(z) = F(z) + w_0$ is the unique **branch of the logarithm** on U such that $\ell(z_0) = w_0$.

For $z_0 = 1$, $w_0 = 0$, and $U = \mathbb{C}^-$, we recover the principal branch of the logarithm, since $\log'(z) = 1/z$ by Equation 4.9 and $\log(1) = 0$.

Questions for further discussion

- How should $\int_{-i}^{1+i} z^2 dz$ be interpreted? What is the result?
- Why is $\mathbb{C}^\times$ path-connected but not simply connected?

Hint: consider $\frac{1}{2\pi i} \int_\gamma \frac{1}{z} dz$ for a loop γ in $\mathbb{C}^\times$. +

It is a mysterious calculus fact that $\arctan(x) = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{2n+1}$ diverges for $|x| > 1$ although $\arctan\colon \mathbb{R} \to \mathbb{R}$ is smooth. In terms of the principal logarithm, $\arctan(z) = \frac{i}{2} \log\left(\frac{i+z}{i-z}\right)$. Apply this to give a geometric explanation of the divergence using Theorem 8.2.

8.1 Exercises

Exercise 8.1

Compute the following integrals:

i. $\int_{\partial D_1(0)} \frac{\sin(z)}{z^2} dz$,

ii. $\int_{\partial D_{1/2}(0)} \frac{\cos(z)}{z-1} dz$,

iii. $\int_{\partial D_2(1)} \frac{\sin(\cos(z))}{z-1} dz$.

Exercise 8.2

Let $f\colon \mathbb{C} \to \mathbb{C}$ be entire. Suppose there is a constant $C > 0$ such that $|f(z)| \leqslant C|z|^d$ for all $z \in \mathbb{C}$. Prove that $f(z)$ is a polynomial of degree $\leqslant d$.

Hint: Generalize the proof of Liouville's theorem.

Exercise 8.3

Let $f\colon U \to \mathbb{C}$ be holomorphic and $\overline{D}_r(z_0) \subset U$. Show that

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\varphi}) d\varphi$$

and interpret this equation geometrically.

Exercise 8.4

Let $f\colon U \to \mathbb{C}$ be holomorphic and $\overline{D}_r(z_0) \subset U$.

a. Prove that $g(x) = f(z_0 + re^{ix})$ is a 2π-periodic function $\mathbb{R} \to \mathbb{C}$.

b. Show that the Cauchy integral formula implies an absolutely and uniformly convergent *Fourier expansion*

$$g(x) = \sum_{n=0}^{\infty} \gamma_n e^{inx}, \qquad \forall x \in \mathbb{R},$$

with only non-negative Fourier modes. Moreover, show that $\gamma_n = \frac{1}{2\pi} \int_0^{2\pi} g(x) e^{-inx} dx.\$$

Note. More generally, a bi-infinite Fourier series $\sum_{n=-\infty}^{\infty} \gamma_n e^{inx}$ can be obtained as a superposition $g_1(x) + g_2(-x)$.

Exercise 8.5

Let $f\colon U \to \mathbb{C}$ be holomorphic and $\overline{D}_r(z_0) \subset U$. Using polar coordinates show that for the (surface) integral over the unit disk $\overline{D}_r(x_0) = \{(x, y) \in \mathbb{R}^2 \mid (x - x_0)^2 + (y - y_0)^2 \leqslant r\}$ we have

$$f(z_0) = \frac{1}{\pi r^2} \int_{\overline{D}_r(z_0)} f(x + iy)dxdy$$

and interpret this equation geometrically.

Chapter 9

Laurent series and singularities

Definition 9.1. A **formal bilateral series** is an expression of the form

$$P = \sum_{n=-\infty}^{\infty} a_n T^n \tag{9.1}$$

with **coefficients** $a_n \in \mathbb{C}$. This is just a bi-infinite sequence $(a_n)_{n\in\mathbb{Z}}$ of complex numbers. Equivalently, we have a pair of formal power series

$$P_+ = \sum_{n=0}^{\infty} a_n T^n, \qquad\qquad P_- = \sum_{n=1}^{\infty} a_{-n} T^n, \tag{9.2}$$

such that

$$P = P_+(T) + P_-(T^{-1}).$$

P_- is called the **principal part**. The **residue** of P is the coefficient a_{-1}.

Definition 9.2. Let P be a bilateral series Equation 9.1. The **domain** $D(P)$ is the set of all $z \in \mathbb{C} \setminus \{0\}$ such that both series of complex numbers

$$P_+(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad P_-(z^{-1}) = \sum_{n=1}^{\infty} a_{-n} z^{-n}$$

converge. We obtain a complex function

$$D(P) \longrightarrow \mathbb{C}, P(z) := \sum_{n=-\infty}^{\infty} a_n z^n := P_+(z) + P_-(z^{-1}). \tag{9.3}$$

More generally, fix a **center** $z_0 \in \mathbb{C}$. We then have a complex function

$$P(z - z_0) := \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n := P_+(z - z_0) + P_-((z - z_0)^{-1}).$$

with domain $D(P, z_0) = z_0 + D(P)$ differing from $P(z)$ only by a translation.

Remark 9.1. Bi-infinite series of complex numbers $\sum_{n \in \mathbb{Z}} a_n$ must be treated with care. We have avoided these issues by viewing them as pairs of ordinary series. Another approach would be to pick a bijection $\nu \colon \mathbb{N} \to \mathbb{Z}$ and consider the ordinary series $\sum_{k=0}^{\infty} a_{\nu(k)}$. However, the limit depends on the choice of ν unless the series is absolutely convergent.

Definition 9.3. The **open annulus** $A_{r,R}(z_0)$ centered at $z_0 \in \mathbb{C}$ with radii $0 \leqslant r, R \leqslant +\infty$ is the (possibly empty) subset

$$A_{r,R}(z_0) = \{z \in \mathbb{C} \mid r < |z - z_0| < R\} = D_R(z_0) \cap (\mathbb{C} \setminus \overline{D}_r(z_0)).$$

The **punctured open disk** centered at $z_0 \in \mathbb{C}$ with radius $0 < R \leqslant +\infty$ is

$$D_R^{\times}(z_0) = D_R(z_0) \setminus \{z_0\} = A_{0,R}(z_0).$$

The **closed annulus** is $\overline{A}_{r,R}(z_0) = \overline{D}_R(z_0) \cap (\mathbb{C} \setminus D_r(z_0))$.

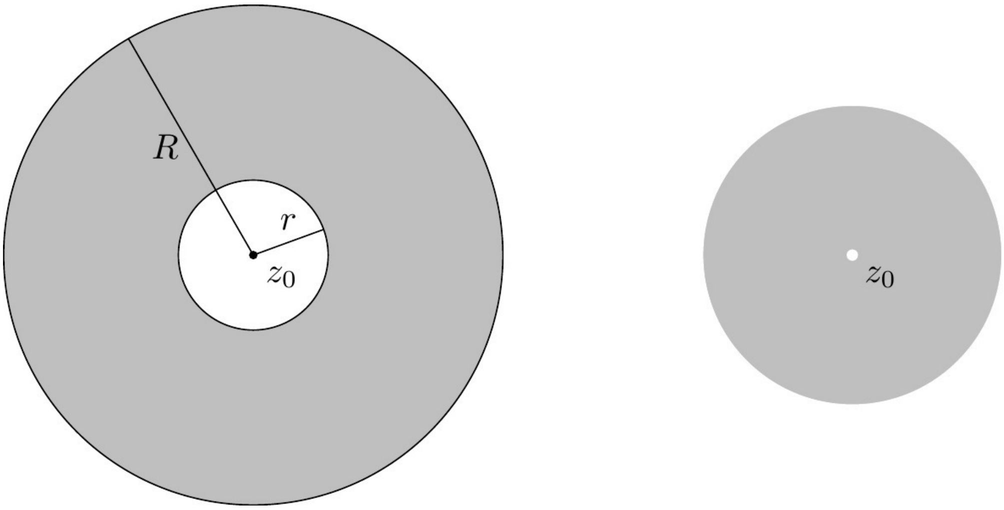


Figure 9.1: A closed annulus and a punctured open disc centered at z_0

Using the inversion

$$i \colon \mathbb{C}^\times \longrightarrow \mathbb{C}^\times, z \longmapsto w = z^{-1}, \tag{9.4}$$

we can rewrite Equation 9.3 as $P(z) = P_+(z) + P_-(w)$. Therefore we can treat the principal part as a power series as well, but on a second complex w-plane corresponding to the original complex z-plane via Equation 9.4.

Lemma 9.1. *Let* $0 \leqslant r \leqslant +\infty$. *Then*

$$i^{-1}(D_r(0)) = \mathbb{C}^\times \setminus \overline{D}_{1/r}(0), \qquad i^{-1}(\overline{D}_r(0)) = \mathbb{C}^\times \setminus D_{1/r}(0), \tag{9.5}$$

using the conventions $1/0 = +\infty$ *and* $1/+\infty = 0$.

Proposition 9.1. *Let* P *be a bilateral series. If* $D(P, z_0) \neq \emptyset$, *there are unique* ***radii of convergence*** $0 \leqslant r, R \leqslant +\infty$ *such that*

$$A_{r,R}(z_0) \subset D(P) \subset \overline{A}_{r,R}(z_0). \tag{9.6}$$

Here $r = 1/\rho_-$, $R = \rho_+$ for the radii of convergence $\rho_\pm$ of the power series $P_\pm$ as defined in Theorem 5.1.

Moreover, the series $P_+(z)$, $P_-(1/z)$ converge absolutely and uniformly on every annulus $A_{r',R'}(z_0)$ with $r < r' < R' < R$.

Proof.

For simplicity, put $z_0 = 0$. According to Theorem 5.1 the power series P_+, P_- have radii of convergence $\rho_\pm \geqslant 0$ and disklike domains

$$D_{\rho_\pm}(0) \subset D(P_\pm) \subset \overline{D}_{\rho_\pm}(0). \tag{9.7}$$

As $P(z) = P_+(z) + P_-(w)$ with $i(z) = w$, we can rewrite Definition 9.2 in terms of the preimage operation $i^{-1}(\sqcup)$ as

$$D(P) = D(P_+) \cap i^{-1}(D(P_-)).$$

Applying i^{-1} to Equation 9.7 for P_- and using Equation 9.5 gives

$$\mathbb{C}^\times \setminus \overline{D}_{1/\rho_-}(0) \subset i^{-1}(D(P_-)) \subset \mathbb{C}^\times \setminus D_{1/\rho_-}(0).$$

Combining this with Equation 9.7 for P_+ implies

$$\underbrace{(\mathbb{C}^\times \setminus \overline{D}_{1/\rho_-}(0)) \cap D_{\rho_+}(0)}_{A_{r,R}(0)=} \subset D(P) \subset \underbrace{(\mathbb{C}^\times \setminus D_{1/\rho_-}(0)) \cap \overline{D}_{\rho_+}(0)}_{\subset \overline{A}_{r,R}(0)}$$

where we have set $r = 1/\rho_-$, $R = \rho_+$ (see Figure 9.2).

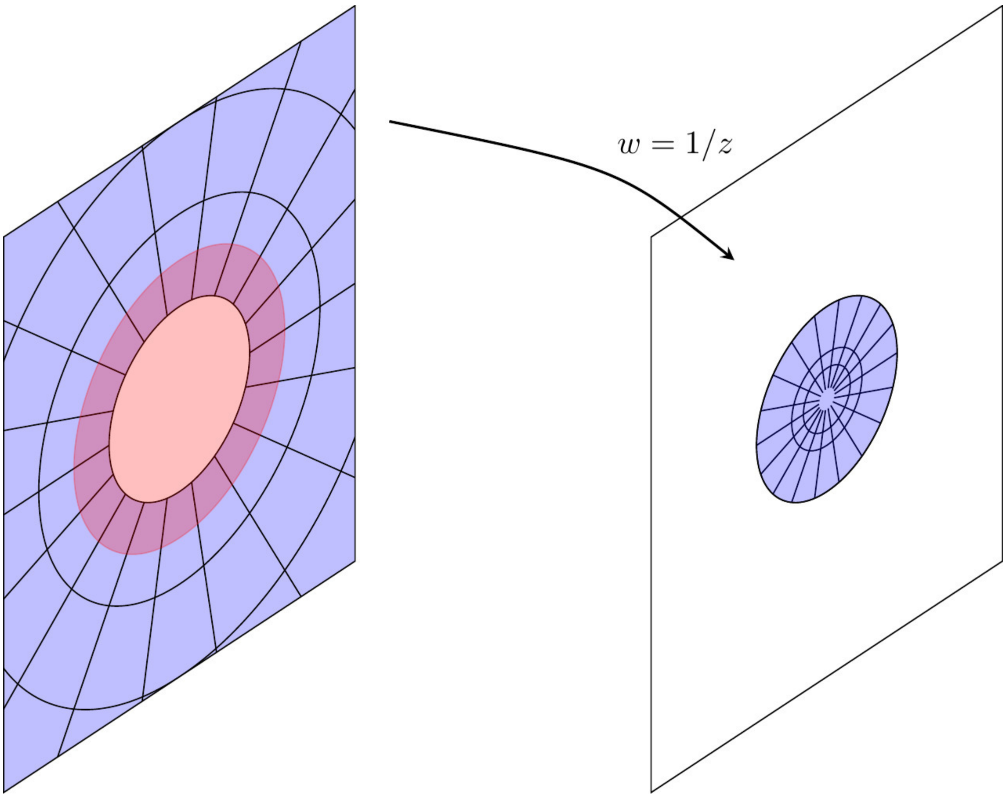


Figure 9.2: Annulus of convergence for a bilateral series. The series $P_+(z)$ converges on the red disk in the z-plane and the series $P_-(w)$ on the blue disk in the w-plane, corresponding to the complement of a disk on the z-plane. $P(z) = P_+(z) + P_-(w)$ converges on the overlap, an annulus in the z-plane.

The absolute and uniform convergence on smaller annuli follows from the corresponding properties for power series stated in Theorem 5.1.

□

If the domain of a bilateral series P with center z_0 contains an non-empty open annulus $A_{r,R}(z_0) \subset D(P, z_0)$, then P is called a (convergent) **Laurent series** at z_0.

Figure 9.3: Pierre Laurent, 1813-1854, Wikipedia. Public Domain

Theorem 9.1 (Laurent expansion)**.** *Let $f\colon U \to \mathbb{C}$ be a holomorphic function on an open set containing a closed annulus $\overline{A}_{r,R}(z_0)$. Then f is equal on $A_{r,R}(z_0)$ to the convergent Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Moreover,

$$a_n = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \tag{9.8}$$

Proof.

Let $z \in A_{r,R}(z_0)$. Let γ be the contour shown in Figure 9.4 that runs around and connects the two boundary components of the annulus. Pick $s > 0$ such that $D_s(z) \subset A_{r,R}(z_0)$. Then γ is freely homotopic to $\partial D_s(z)$.

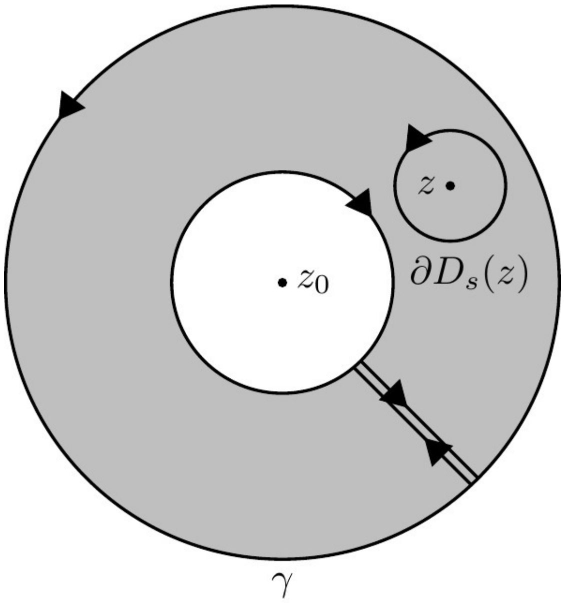


Figure 9.4: Contour of integration for Laurent expansion

In the path integral over γ, the pieces connecting the two boundary components cancel and the inner circle is negatively oriented. Hence, by Equation 8.1

$$f(z) = \frac{1}{2\pi i} \int_{\partial D_s(z)} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

applying Theorem 7.2

$$\frac{1}{2\pi i} \int_{\partial D_s(z)} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\partial A_{r,R}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

thus,

$$f(z) = \frac{1}{2\pi i} \int_{\partial D_R(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta$$

Rewrite the first integral as in Equation 8.8,

$$\frac{f(\zeta)}{\zeta - z} = \frac{\frac{f(\zeta)}{\zeta - z_0}}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n.$$

The geometric series converges uniformly because $|z - z_0| < |\zeta - z_0| = R$. We then apply Corollary 6.1 to exchange the limit and the integral. We similarly rewrite the second integral using

$$\frac{f(\zeta)}{z - \zeta} = \frac{\frac{f(\zeta)}{z - z_0}}{1 - \frac{\zeta - z_0}{z - z_0}} = \frac{f(\zeta)}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{\zeta - z_0}{z - z_0} \right)^n$$

and again apply Corollary 6.1, now using $r = |\zeta - z_0| < |z - z_0|$.

□

Definition 9.4. Let $f\colon U \to \mathbb{C}$ be a holomorphic function. A point $z_0 \in \mathbb{C} \setminus U$ is an **isolated singularity** of f if there exists $R > 0$ such that $D_R^\times(z_0) \subset U$.

By applying Theorem 9.1 to $A_{r,R}(z_0)$ for all $0 < r < R$ we obtain a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{9.9}$$

that converges on $D_R^\times(z_0)$.

Definition 9.5. An isolated singularity of f is called

a. **removable**,

b. a **pole** of order $m \geqslant 1$,

c. **essential**,

if the principal part $\sum_{n=-\infty}^{-1} a_n(z-z_0)^n$ of the Laurent expansion Equation 9.9

a. is zero,
b. has $a_{-m} \neq 0$ and $a_n = 0$ for all $n < -m$,
c. has infinitely many non-zero terms.

We can restate Definition 9.5 without reference to Laurent expansions. If z_0 is a removable singularity, then Equation 9.9 is a power series which by Theorem 5.1 defines a holomorphic $D_r(z_0)$. In other words,

$$z_0 \text{ removable singularity} \iff f \text{ extends holomorphically to } U \cup \{z_0\}.$$

If z_0 is a pole of order m, then $(z-z_0)^m f(z)$ has a removable singularity. So $g(z) = (z-z_0)^m f(z) = g(z)$ is a holomorphic function on $U \cup \{z_0\}$ and

$$z_0 \text{ pole of order } m \iff f(z) = \frac{g(z)}{(z-z_0)^m},\ g \text{ holomorphic on } U \cup \{z_0\}$$
$$\text{with } g(z_0) \neq 0.$$

Theorem 9.2 (Riemann removable singularities)**.** *Let z_0 be an isolated singularity of a holomorphic function $f\colon U \to \mathbb{C}$. Suppose there exist $R, C > 0$ such that $|f(z)| < C$ for all $z \in D_R^\times(z_0)$. Then z_0 is a removable singularity.*

Proof.

For all $n < 0$ we can estimate Equation 9.8 using Equation 6.7 and get

$$|a_n| \leqslant \frac{1}{2\pi} L(\partial D_R(z_0)) \frac{C}{R^{n+1}} = CR^{-n} \xrightarrow{R\to 0} 0.$$

□

Questions for further discussion

- Define the sum of formal Laurent series in the same way as Equation 5.2. Why can't we use Equation 5.3 to define the product of formal Laurent series?
- In Definition 9.2, why did we exclude $z = z_0$?
- For an isolated singularity, why is $U \cup \{z_0\}$ always an open set?
- Does the converse of the Riemann removable singularities theorem hold?

9.1 Exercises

Exercise 9.1

Determine whether the singularities of the following holomorphic functions $f(z)$ are isolated. If so, compute the residue. When applicable, find also the order of the pole or a holomorphic extension.

i. $\frac{z+i}{z^2+1}$,
ii. $\frac{1-\cos(z)}{z^2}$,
iii. $\frac{1}{\cos(z)-1}$,
iv. $e^{1/z}$

Exercise 9.2

Determine whether the domain $D(P)$ of the following formal bilateral series P is non-empty and, if so, find the radii of convergence. i. $\sum_{n=-\infty}^{\infty} \frac{z^n}{|n|!}$,

ii. $\sum_{n=-\infty}^{\infty} z^n$,

iii. $\sum_{n=-\infty}^{\infty} \frac{|n|^{|n|}}{|n|!} z^n$.

Exercise 9.3

Use the geometric series to find the Laurent expansion of

$$f(z) = \frac{1}{z(1-z)}$$

on the domains

i. $0 < |z| < 1$,

ii. $0 < |z-1| < 1$,

iii. $1 < |z| < +\infty$.

Exercise 9.4

Let $P = \sum_{n=-\infty}^{\infty} a_n T^n$, $Q = \sum_{n=-\infty}^{\infty} b_n T^n$ be formal bilateral series. Suppose that both domains $D(P, z_0)$, $D(Q, z_0)$ contain a non-empty annulus $A_{r,R}(z_0)$. Assume that for some radius s with $r < s < R$ we have

$$P(z) = Q(z), \qquad \text{for all } |z - z_0| = s.$$

Prove the *identity theorem*, namely that $a_n = b_n$ for all $n \in \mathbb{Z}$.

Exercise 9.5

Let U be open and $z_0 \in U$. Let $f\colon U \to \mathbb{C}$ and $g\colon U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic functions.

i. If $f(z_0) \neq 0$ and g has a pole of order one at z_0, prove that fg has a pole of order one at z_0 with

$$\mathrm{Res}_{z_0}(fg) = f(z_0)\,\mathrm{Res}_{z_0}(g).$$

ii. If $f(z_0) = 0$ and $f'(z_0) \neq 0$, prove that $1/f(z)$ has a pole of order one at z_0 with $\mathrm{Res}_{z_0}(1/f) = 1/f'(z_0)$.

Chapter 10

Residue theorem

Definition 10.1. Let z_0 be an isolated singularity of a holomorphic function $f\colon U \to \mathbb{C}$. The **residue** $\operatorname{Res}_{z_0}(f)$ is the residue of the Laurent series Equation 9.9. In other words,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \implies \operatorname{Res}_{z_0}(f) = a_{-1}.$$

Proposition 10.1. *Let z_0 be an isolated singularity of both the holomorphic functions $f, g\colon U \to \mathbb{C}$.*

a. $\operatorname{Res}_{z_0}(f+g) = \operatorname{Res}_{z_0}(f) + \operatorname{Res}_{z_0}(g)$

b. $\operatorname{Res}_{z_0}(\lambda f) = \lambda \operatorname{Res}_{z_0}(f)$ *for all* $\lambda \in \mathbb{C}$

c. If z_0 is a pole of order m, then $\operatorname{Res}_{z_0}(f) = \frac{1}{(m-1)!} \lim_{z\to z_0} h^{(m-1)}(z)$ *for* $h(z) = (z-z_0)^m f(z)$.

Proof.

a. and b.are left as an Exercise.

b. If $f = \sum_{n=-m}^{\infty} a_n (z-z_0)^n$, then $h(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^{n+m}$ is a power series which we can differentiate termwise by Theorem 5.2 to get

$$h^{(m-1)}(z) = \sum_{n=-m}^{\infty} (n+m)(n+m-1)\cdots(n+2)a_n(z-z_0)^{n+1}.$$

The limit $z \to z_0$ of this power series is its constant term, $(m-1)!a_{-1}$.

□

Example 10.1.

$$\mathrm{Res}_0(z^n) = \begin{cases} 0 & \text{if } n \neq -1, \\ 1 & \text{if } n = -1. \end{cases}$$

Example 10.2.

The best way to compute a residue is often to work out the Laurent expansion using the calculus of formal power series from Chapter 5.

(a) Consider $f(z) = \frac{\sin(z^2)}{z^7}$. As $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n+2-7},$$

hence $\mathrm{Res}_0\, f(z) = -1/6$ is the coefficient of the term with $n = 1$.

(b) Let $g(z) = \frac{\cos(z)-1}{(\exp(z)-1)^3}$. It suffices to work with finite portions of the involved power series. For this, introduce the notation $O(z^m)$ to mean 'terms involving z^n of order $m \leqslant n$'. We will use this in an informal way, which can be made precise in algebra by saying that $O(z^n)$ is the ideal in the ring of formal power series generated by z^n. Notice the properties

$$O(z^m)O(z^n) = O(z^{m+n}), \tag{10.1}$$

$$O(z^m)^n = O(z^{mn}), \tag{10.2}$$

$$O(z^m) + O(z^n) = O(z^{\min(m,n)}), \tag{10.3}$$

$$1/O(z^n) = O(z^{-n}). \tag{10.4}$$

Now compute

$$\begin{aligned} \exp(z) - 1 &= z + z^2/2 + O(z^3), \\ \implies (\exp(z)-1)^3 &= \frac{z^6}{8} + \frac{3z^5}{4} + \frac{3z^4}{2} + z^3 + O(z^5) \\ &= z^3 + \frac{3z^4}{2} + O(z^5). \end{aligned}$$

To find the expansion of the inverse $1/(\exp(z)-1)^3$, we apply the proof strategy of Proposition 5.1(b) to $z^{-3}(\exp(z)-1)^3 = 1 + \frac{3z}{2} + O(z^2)$. In other words, we compare coefficients to solve

$$\left(1 + \frac{3z}{2} + ...\right)(a_0 + a_1 z + a_2 z^2 + ...) = 1$$

recursively for a_0, a_1, a_2. This gives

$$1/(\exp(z)-1)^3 = z^{-3}\left(1 - \frac{3z}{2} + \frac{9z^2}{4} + O(z^3)\right) = z^{-3} - \frac{3z^{-2}}{2} + \frac{9z^{-1}}{4} + O(z^0).$$

Finally, multiply this by $\cos(z) - 1 = -\frac{z^2}{2} + \frac{z^4}{24} + O(z^5)$ to get

$$g(z) = -\frac{1}{2z} + \frac{3}{4} - \frac{13z}{12} - \frac{z^2}{16} + O(z^2)$$

from which we read off $\mathrm{Res}_0\, g(z) = -1/2$.

Definition 10.2. Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed piecewise C^1 curve and let $z_0 \in \mathbb{C} \setminus \gamma([a,b])$. The **winding number** of γ around z_0 is

$$W_{z_0}(\gamma) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}.$$

Example 10.3.

Let $\gamma\colon [0,2\pi] \to \mathbb{C}$, $\gamma(t) = z_0 + re^{it}$, be the boundary curve of the disk $\overline{D}_r(z_0)$. Then Equation 6.6 implies $W_{z_0}(\gamma) = 1$. More generally, for every $w_0 \in D_r(z_0)$ the Cauchy integral formula Equation 8.1 gives

$$W_{w_0}(\gamma) = 1.$$

Pick a subdivision $a = t_0 < t_1 < \cdots < t_n = b$ such that $\gamma|_{[t_{k-1},t_k]}$ is continuously differentiable. Using branches of the logarithm, we can write $\gamma(t) = z_0 + r(t)e^{i\theta(t)}$ for $r(t) = |\gamma(t) - z_0|$ and a continuous function $\theta\colon [a,b] \to \mathbb{R}$ with $\theta|_{[t_{k-1},t_k]}$ continuously differentiable. With this notation, the following holds.

Proposition 10.2.

(a) We have $W_{z_0}(\gamma) = \frac{\theta(b)-\theta(a)}{2\pi}$, which is an integer.

(b) For closed curves γ_0, γ_1 in $\mathbb{C} \setminus \{z_0\}$ we have

$$W_{z_0}(\gamma_0) = W_{z_0}(\gamma_1) \iff \gamma_0, \gamma_1 \text{ are homotopic in } \mathbb{C} \setminus \{z_0\}.$$

Proof.

(a) Over $[t_{k-1}, t_k]$ we have $\gamma'(t) = r'(t)e^{i\theta(t)} + r(t)\theta'(t)e^{i\theta(t)}$. Using

$$\frac{\gamma'(t)}{\gamma(t) - z_0} = \frac{r'(t)}{r(t)} + i\theta'(t) \tag{10.5}$$

we compute

$$2\pi i W_{z_0}(\gamma) = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{\gamma'(t)}{\gamma(t)} dt$$

by Equation 10.5 we have

$$\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{\gamma'(t)}{\gamma(t)} dt = \sum_{k=1}^{n} \left(\int_{t_{k-1}}^{t_k} \frac{d}{dt} \log r(t) dt + i \int_{t_{k-1}}^{t_k} \theta'(t) dt \right)$$

using FTC, the right hand side becomes

$$\sum_{k=1}^{n} \Big(\log r(t_k) - \log r(t_{k-1}) + i\theta(t_k) - i\theta(t_{k-1}) \Big)$$

observing that these are telescoping sums, we conclude that

$$2\pi i W_{z_0}(\gamma) = i(\theta(t_n) - \theta(t_0)) = i(\theta(b) - \theta(a))$$

and the claimed formula follows. From $\gamma(a) = \gamma(b)$ we have $e^{i\theta(a)} = e^{i\theta(b)}$, so Proposition 1.5 implies $\theta(b) - \theta(a) = 2\pi k$ for some $k \in \mathbb{Z}$.

(b) By Theorem 7.2 applied to $U = \mathbb{C} \setminus \{z_0\}$ and $f(z) = 1/(z - z_0)$, the implication '$\Longleftarrow$' holds.

Conversely, assume $W_{z_0}(\gamma_0) = W_{z_0}(\gamma_1)$. Write $\gamma_k(t) = z_0 + r_k(t)e^{i\theta_k(t)}$ for $k = 0, 1$. As $\gamma_k(a) = \gamma_k(b)$, we have $r_k(a) = r_k(b)$ and $\theta_k(a) = \theta_k(b) - 2\pi i n_k$ for some $n_k \in \mathbb{Z}$. By assumption, $\frac{\theta_0(b) - \theta_0(a)}{2\pi} = \frac{\theta_1(b) - \theta_1(a)}{2\pi}$, hence $n_0 = n_1$.

Define the homotopy by

$$\Gamma_s(t) = z_0 + [(1-s)r_0(t) + sr_1(t)] \operatorname{exp}(i(1-s)\theta_0(t) + is\theta_1(t)).$$

Clearly, $\Gamma_k = \gamma_k$ for $k = 0, 1$. The only thing to check is that Γ_s is a closed curve for each $s \in [0,1]$, that is, to show $\Gamma_s(b) = \Gamma_s(a)$. As $r_k(a) = r_k(b)$, this is equivalent to

$$\operatorname{exp}(i(1-s)(\theta_1(b) - \theta_1(a)) + is(\theta_0(b) - \theta_0(a))) = 1.$$

This holds since $\theta_1(b) - \theta_1(a) = \theta_0(b) - \theta_0(a) = 2\pi n_0$ and

$$\operatorname{exp}(i(1-s)2\pi n_0 + is2\pi n_0) = \operatorname{exp}(2\pi i n_0) = 1.$$

□

Theorem 10.1 (Residue theorem). *Let $U \subset \mathbb{C}$ be an open set, $\gamma\colon [a,b] \to U$ a null-homotopic piecewise C^1 loop in U, and $z_1, \dots, z_n \in U \setminus \gamma([a,b])$ points not on the loop γ. If f is a holomorphic function on $U \setminus \{z_1, \dots, z_n\}$, then*

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{k=1}^{n} W_{z_k}(\gamma) \operatorname{Res}_{z_k}(f).$$

Proof.

For each $k = 1, \dots, n$ we find $r_k > 0$ and Laurent expansions

$$f(z) = \sum_{n=-\infty}^{\infty} a_n^{(k)}(z - z_k)^n \qquad \text{for all } z \in D^\times_{r_k}(z_k).$$

Let $P_k(z) = \sum_{n=-\infty}^{-1} a_n^{(k)}(z-z_k)^n$ be the corresponding principal part. Hence $a_{-1}^{(k)} = \mathrm{Res}_{z_k}(f)$. By Proposition 9.1, $P_k(z)$ is holomorphic on $D_{+\infty}^{\times}(z_0)$. Also, $f(z) - \sum_{k=1}^{n} P_k(z)$ defines a holomorphic function on U since its singularities at $\{z_1, \dots, z_n\}$ are removable. By Cauchy's Theorem 7.1,

$$\int_\gamma f(z)dz = \sum_{k=1}^{b} \int_\gamma P_k(z)dz.$$

By Proposition 10.2(b), the loop γ is homotopic in $\mathbb{C} \setminus \{z_k\}$ to $\partial D_{r_k}(z_k) * \cdots * \partial D_{r_k}(z_k)$ (the same loop traversed $W_{z_k}(\gamma)$-times). Combined with the fact that P_k is holomorphic on $\mathbb{C} \setminus \{z_k\}$, this implies by Theorem 7.2 that

$$\begin{aligned}\int_\gamma P_k(z)dz &= W_{z_k}(\gamma) \int_{\partial D_{r_k}(z_k)} P_k(z)dz \\ &= W_{z_k}(\gamma) \int_{\partial D_{r_k}(z_k)} \sum_{n=-\infty}^{-1} a_n^{(k)}(z-z_k)^n dz\end{aligned}$$

by uniform convergence,

$$W_{z_k}(\gamma) \int_{\partial D_{r_k}(z_k)} \sum_{n=-\infty}^{-1} a_n^{(k)}(z-z_k)^n dz = W_{z_k}(\gamma) \sum_{n=-\infty}^{-1} a_n^{(k)} \int_{\partial D_{r_k}(z_k)} (z-z_k)^n dz$$

using Equation 6.6 we have

$$W_{z_k}(\gamma) \sum_{n=-\infty}^{-1} a_n^{(k)} \int_{\partial D_{r_k}(z_k)} (z-z_k)^n dz = W_{z_k}(\gamma) a_{-1}^{(k)} 2\pi i$$

as required.

□

10.1 Exercises

Exercise 10.1

Determine the winding numbers $W_{z_0}(\gamma)$ in the following cases.

a. $\gamma\colon [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = e^{it}$, for $z_0 = 0, 2$.

b. $\gamma\colon [2\pi, 10\pi] \to \mathbb{C}$, $\gamma(t) = \begin{cases} te^{it} & \text{if } t \in [2\pi, 6\pi], \\ 12\pi - t & \text{if } t \in [6\pi, 10\pi], \end{cases}$ for $z_0 = 0, 20$.

c. $\gamma\colon [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = (2 + \cos(t))e^{2it}$, for $z_0 = 0, 4$.

Exercise 10.2

Use the residue theorem to compute

$$\int_{\partial D_r(0)} \frac{(z+2)^2}{z(z-1)^2} dz$$

for all $0 < r \neq 1$.

Hint: Consider the cases $r > 1$ and $r < 1$ separately.

Exercise 10.3

Apply the residue theorem to the contour $\partial D_1(0)$ and a suitable holomorphic function $f(z)$ to compute the integral

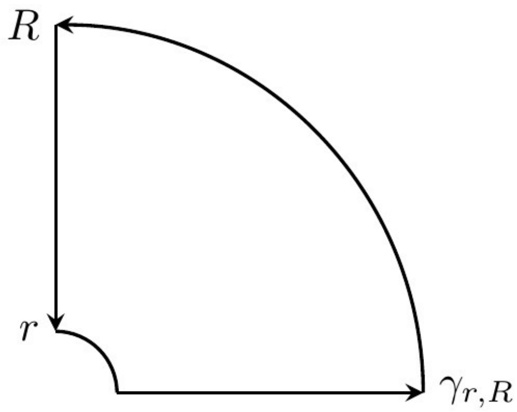
$$\int_0^{2\pi} \frac{dx}{2 + \cos(x)}.$$

Exercise 10.4

Compute the integral

$$\int_0^{+\infty} \frac{\cos(x)}{\sqrt{x}} dx$$

by considering $f(z) = \frac{e^{iz}}{\sqrt{z}}$ and the following contour $\gamma_{r,R}$ for $r \to 0$, $R \to +\infty$.



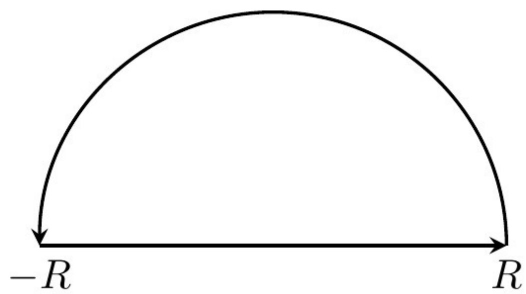
Hint: Recall that $\int_0^{\infty} e^{-x^2} dx = \sqrt{\pi}/2$.

Exercise 10.5

Compute the integral

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^6}$$

by applying the residue theorem to $f(z) = \frac{1}{1+z^6}$ and the following upper semi-circular contour γ_R for $R \to +\infty$.



Chapter 11

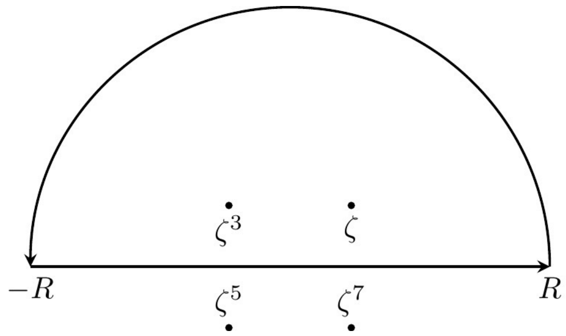
Applications to integrals

The residue theorem can be used to compute integrals of functions that do not have an obvious primitive. For this, we first need to find a suitable holomorphic function and contour to which we can apply the residue theorem, which is guesswork. The original integral is often a part of this contour and one shows that the rest of contour integral tends to zero.

We give three examples that illustrate the general technique.

Example 11.1.

Let $f(z) = \frac{1}{1+z^4}$, which is holomorphic except for poles of order one at all points ζ^{1+2k}, $k = 0, 1, 2, 3$, where $\zeta = e^{i\pi/4}$. For $R > 0$ introduce the semi-circular contour $\eta_R \colon [0, \pi] \to \mathbb{C}$, $\eta_R(t) = Re^{it}$ and let γ_R be the concatenation of η_R with the real interval $[-R, R]$, parametrized as in Example 6.1.



The residues are

$$\mathsf{Res}_{\zeta^{1+2k}}(f) = \frac{-\zeta^{1+2k}}{4}.$$

When $R > 1$, the only isolated singularities inside the contour are ζ, ζ^3, hence $W_\zeta(\gamma_R) = W_{\zeta^3}(\gamma_R) = 1$ (more formally, this holds by Example 10.3 and Proposition 10.2) and $W_{\zeta^5}(\gamma_R) = W_{\zeta^7}(\gamma_R) = 0$ (more formally, by Cauchy's Theorem 7.1). By the residue Theorem 10.1,

$$\int_{\gamma_R} f(z)dz = 2\pi i[W_\zeta(\gamma_R)\,\mathsf{Res}_\zeta(f) + W_{\zeta^3}(\gamma_R)\,\mathsf{Res}_{\zeta^3}(f)] = \frac{\pi}{\sqrt{2}}$$

The contour integral decomposes as

$$\frac{\pi}{\sqrt{2}} = \int_{\gamma_R} f(z)dz = \int_{\eta_R} f(z)dz + \int_{[-R,R]} f(z)dz$$

and we have

$$\int_{-\infty}^{+\infty} f(z)dz = \lim_{R\to+\infty} \int_{[-R,R]} f(z)dz.$$

We claim that $\int_{\eta_R} f(z)dz \to 0$ as $R \to +\infty$. We have $L(\eta_R) = \pi R$ and

$$|f(z)| = \frac{1}{|z^4+1|} \leqslant \frac{1}{R^4-1}$$

where the inequality follows by Equation 1.12.

Hence Equation 6.7 implies $\left|\int_{\eta_R} f(z)dz\right| \leqslant \frac{\pi R}{R^4-1}$, which indeed tends to zero as $R \to +\infty$. Putting things together, we get

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}.$$

More generally, we can use the approach above to **compute all integrals** $\int_{-\infty}^{+\infty} f(z)dz$ **of rational functions** $f(z) = \frac{p(z)}{q(z)}$ with $\deg(q) \geqslant \deg(p) + 2$ and $q(x) \neq 0$ for all $x \in \mathbb{R}$.

Integrals made only of trigonometric functions can often be evaluated using the residue theorem and the following method.

Example 11.2.

If $x \in \mathbb{R}$, then $\cos(x) = \frac{z+1/z}{2}$, $\sin(x) = \frac{z-1/z}{2i}$ for $z = e^{ix}$. Using this, we can change variables and use $dz = izdx$ to rewrite

$$\int_0^{2\pi} \frac{\cos(x)^2}{\sin(x)+2}dx = \int_{\partial D_1(0)} \frac{(1+z^2)^2}{2z^2(z^2+4iz-1)}dz$$

as a contour integral, which is evaluated using the residue theorem. The singularities of $f(z) = \frac{(1+z^2)^2}{2z^2(z^2+4iz-1)} = \frac{(1+z^2)^2}{2z^2(z-z_+)(z-z_-)}$ are at $z = 0$ (pole of order two) and $z_\pm = -i(2 \pm \sqrt{3})$ (poles of order one). Using Proposition 10.1(c), we find that the residues are

$$\begin{aligned}
\mathrm{Res}_0(f) &= \lim_{z\to 0} \frac{d}{dz} \frac{(1+z^2)^2}{2(z^2+4iz-1)} = -2i, \\
\mathrm{Res}_{z_-}(f) &= \lim_{z\to z_-} \frac{(1+z^2)^2}{2z^2(z-z_+)} = \frac{(1+z_-^2)^2}{2z_-^2(z_- - z_+)} = i\sqrt{3}.
\end{aligned}$$

Now the residue theorem implies

$$\int_{\partial D_1(0)} \frac{(1+z^2)^2}{2z^2(z^2+4iz-1)} dz = 2\pi i(-2i+i\sqrt{3}) = 2\pi(\sqrt{3}-2).$$

This example can be generalized to compute integrals over $[0, 2\pi]$ of rational algebraic functions of $\sin(x)$ and $\cos(x)$.

The following integral was already computed in calculus using polar coordinates in the plane.

Example 11.3.

$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$

Let $a = (1+i)\sqrt{\frac{\pi}{2}}$. Then $a^2 = i\pi$. The function $f(z) = \frac{e^{-z^2}}{1+e^{-2az}}$ is holomorphic except for poles of order one at all points $a\left(k+\frac{1}{2}\right)$, $k \in \mathbb{Z}$ (simple zeros of the denominator). Using the power series expansion of the exponential, we find

$$\begin{aligned} 1+e^{-2az} &= 1+e^{-2a(z-\frac{a}{2})}e^{a^2} = 1-e^{-2a(z-\frac{a}{2})} \\ &= 2a\left(z-\frac{a}{2}\right) - \frac{1}{2!}4a^2\left(z-\frac{a}{2}\right)^2 + \ldots \end{aligned}$$

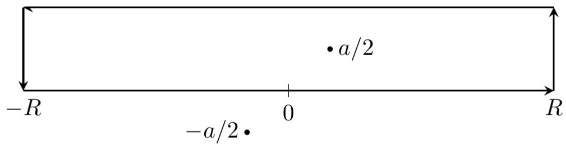
and this gives the residue

$$\mathrm{Res}_{a/2}(f) = \lim_{z\to a/2} \frac{e^{-z^2}(z-\frac{a}{2})}{1+e^{-2az}} = \frac{e^{-a^2/4}}{2a} = \frac{-i}{2\sqrt{\pi}}.$$

From $a^2 = i\pi$ it is not hard to check

$$f(z) - f(z+a) = e^{-z^2}.$$

Hence the integral over $z \in [-\infty, +\infty]$ can be computed using the two horizontal parts of the following contour γ_R as $R \to +\infty$.



The only singularity inside the contour is $a/2$, so the residue theorem gives

$$\int_{\gamma_R} f(z)dz = 2\pi i \frac{-i}{2\sqrt{\pi}} = \sqrt{\pi}.$$

The contour integral over γ_R splits into four parts. We estimate the vertical parts $\eta_{\pm R}$ of the integral as follows. Let $z = x + iy$ with $x = \pm R$ and $y \in [0, \sqrt{\pi/2}]$. Then

$$|f(z)| \leqslant \frac{e^{-\Re(z^2)}}{1 - e^{-\Re(2az)}} = \frac{e^{y^2 - x^2}}{1 - e^{\sqrt{2\pi}(y-x)}} \leqslant \frac{e^{\pi/2 - x^2}}{1 - e^{-\sqrt{2\pi}x}}$$

tends to zero as $|x| \to +\infty$, uniformly in $y \in [0, \sqrt{\pi/2}]$. Hence Equation 6.7 implies

$$\int_{\eta_{\pm R}} f(z)dz \xrightarrow{R \to \infty} 0$$

and we conclude

$$\begin{aligned}
\sqrt{\pi} &= \int_{\gamma_R} f(z)dz \\
&= \int_{-R}^{+R} f(z)dz + \int_{R}^{-R} f(z+a)dz + \int_{\eta_R} f(z)dz + \int_{\eta_{-R}} f(z)dz \\
&\xrightarrow{R \to +\infty} \int_{-\infty}^{+\infty} f(z)dz - \int_{-\infty}^{+\infty} f(z+a)dz = \int_{-\infty}^{+\infty} e^{-x^2} dx.
\end{aligned}$$

Appendix A

All Solutions

A.1 Chapter 1 solutions

Solution 1.1

Verify

$$z = x + iy$$

and

$$i^2 = -1$$

straight from the definition Equation 1.2.

Let $z = (x, y) \in \mathbb{C}$. Identifying x with $(x, 0)$ and y with $(y, 0)$. We calculate

$$iy = (0, 1) \cdot (y, 0) = (0 - 0, 0 + y) = (0, y).$$

Consequently,

$$x + iy = (x, 0) + (0, y) = (x, y) = z.$$

To see that $i^2 = -1$, we calculate

$$(0, 1)(0, 1) = (0 - 1, 0 + 0) = (-1, 0).$$

This yields the desired conclusion under the identification $-1 = (-1, 0)$.

Return to Exercise 1.1 on P23

Solution 1.2

How many real solutions x does $x^2 + 1 = 0$ have? Show that the polynomial equation $z^2 + 1 = 0$ has exactly two solutions $z \in \mathbb{C}$.

This equation has no reals solutions. To find all solutions in $\mathbb{C}$, we first rearrange to get $z^2 = -1$. This means that z must be one of the square roots of -1. There are at least two ways to find all possible values of z.

The first approach uses Proposition 1.7.

Expressing -1 in polar form, we have $-1 = e^{i\pi}$.

By Proposition 1.7 the square roots of -1, are

$$e^{\frac{i(\pi+2k\pi)}{2}}$$

for $k = 0, 1$. This gives exactly two solutions:

$$i = e^{\frac{i(\pi)}{2}}, \text{ and } -i = e^{\frac{i(\pi+2\pi)}{2}}.$$

The other approach is as follows. Write $z = x + iy$. Then,

$$z^2 = (x + iy)(x + iy) = (x^2 - y^2) + i(2xy) = -1.$$

This gives us two equations

$$x^2 - y^2 = -1 \tag{A.1}$$

$$2xy = 0. \tag{A.2}$$

The second equation imples one of x or y is 0.

If x is 0, then $-y^2 = -1$ and so $y = \pm 1$.

If y is 0, then $x^2 = -1$ and has no solutions (since x is a real number).

We conclude that $x = 0$ and $y = \pm 1$, which gives exactly two values for z, i and $-i$.

Return to Exercise 1.2 on P23

Exercise 1.3

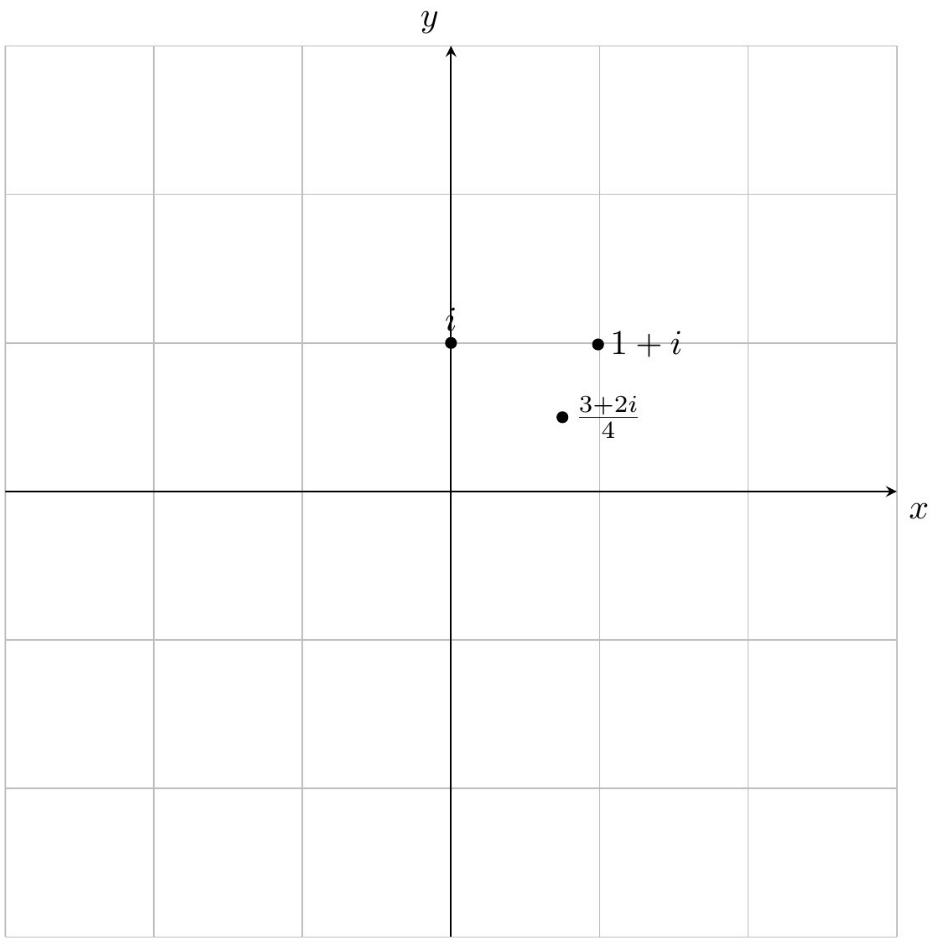
Give examples of complex numbers $z, w \neq 0$ ***such that*** $z^2 + w^2 = 0$.

Take $z = 1$, $w = i$.

Return to Exercise 1.3 on P23

Solution 1.4

Sketch the position of the complex numbers $i, 1+i, \frac{3+2i}{4}$ ***in the plane.***



Return to Exercise 1.4 on P24

Solution 1.5

Express the following complex numbers z ***in the form*** $x+iy$ ***with*** $x, y \in \mathbb{R}$.

$$(1+i)^{20}, \ (5+3i)(1+2i), \ (1-i)(2+3i), \ (1-i)i(1+i), \ \frac{2+i}{1-i}$$

$$(2i)^{10} = -1024,\ -1 + 13i,\ 5 + i,\ 2i,\ \frac{1 + 3i}{2}$$

Return to Exercise 1.5 on P24

Solution 1.6

Express the following complex numbers z in the form $x + iy$ with $x, y \in \mathbb{R}$.

$$1/i,\ \frac{1}{1+i},\ \frac{3+i}{3-i}$$

$$-i,\ \frac{1-i}{2},\ \frac{8+6i}{10}$$

Return to Exercise 1.6 on P24

Solution 1.7

Find the modulus and the conjugate of the following complex numbers.

$$2 + i,\ i,\ 5 - 3i,\ \frac{1+i}{2+i}$$

$$|2 + i| = \sqrt{5}, \qquad \overline{2+i} = 2 - i \tag{A.3}$$

$$|5 - 3i| = \sqrt{34}, \qquad \overline{5-3i} = 5 + 3i \tag{A.4}$$

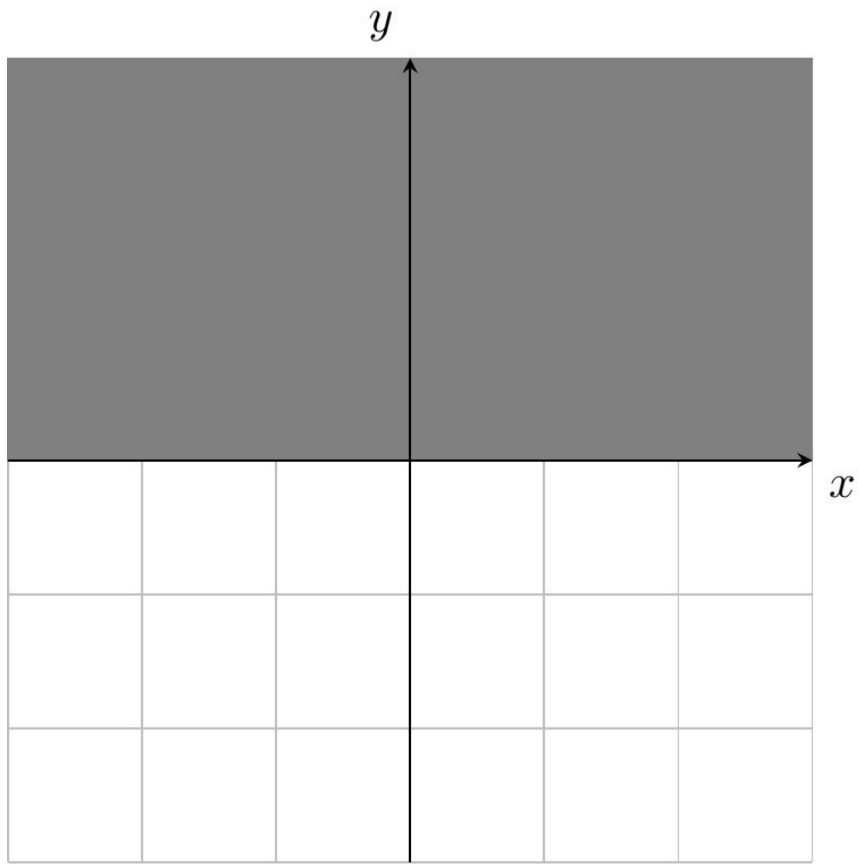
$$|\frac{1+i}{2+i}| = \sqrt{\frac{2}{5}} = \frac{\sqrt{10}}{5}, \qquad \overline{\frac{1+i}{2+i}} = \frac{1-i}{2-i} \tag{A.5}$$

Return to Exercise 1.7 on P24

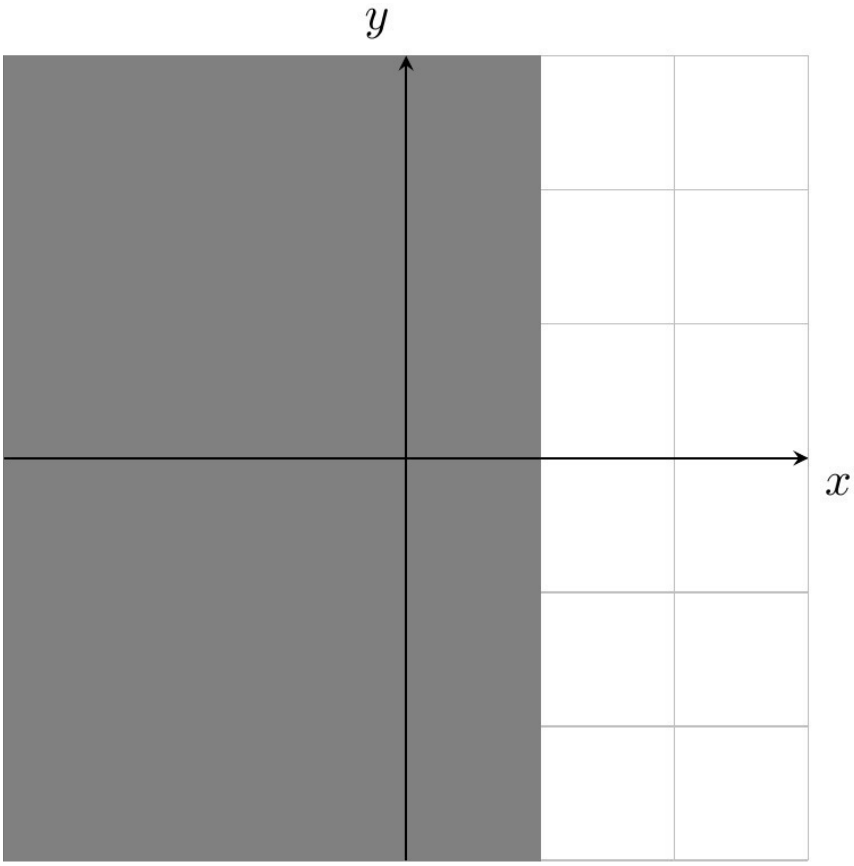
Solution 1.8

Describe the sets $A = \{z \in \mathbb{C} \mid \Im(z) > 0\}$, $B = \{z \in \mathbb{C} \mid \Re(z) \leqslant 1\}$, $C = \{z \in \mathbb{C} \mid \Re((1+i)z) = 0\}$, ***and*** $A \cap B$ ***geometrically.***

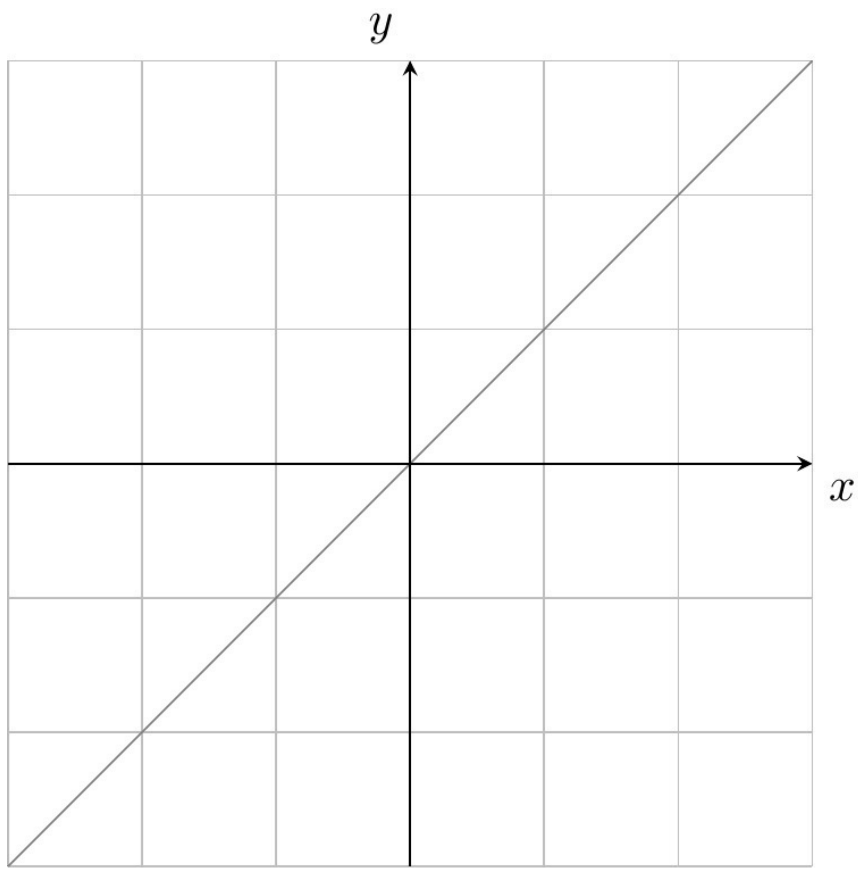
- The set A is the 'upper half-plane'



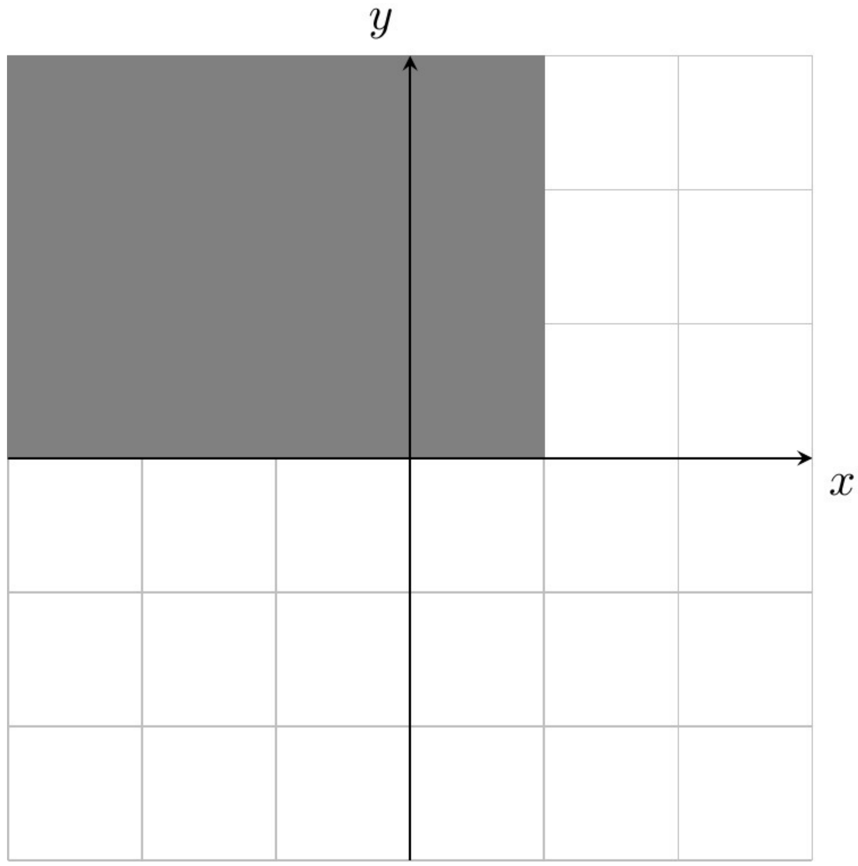
- The set B is the "left half-plane" shifted to the right



- The set C is the straight line $x = y$ since $\Re((1+i)z) = x - y$.



- The intersection of A and B is pictured below



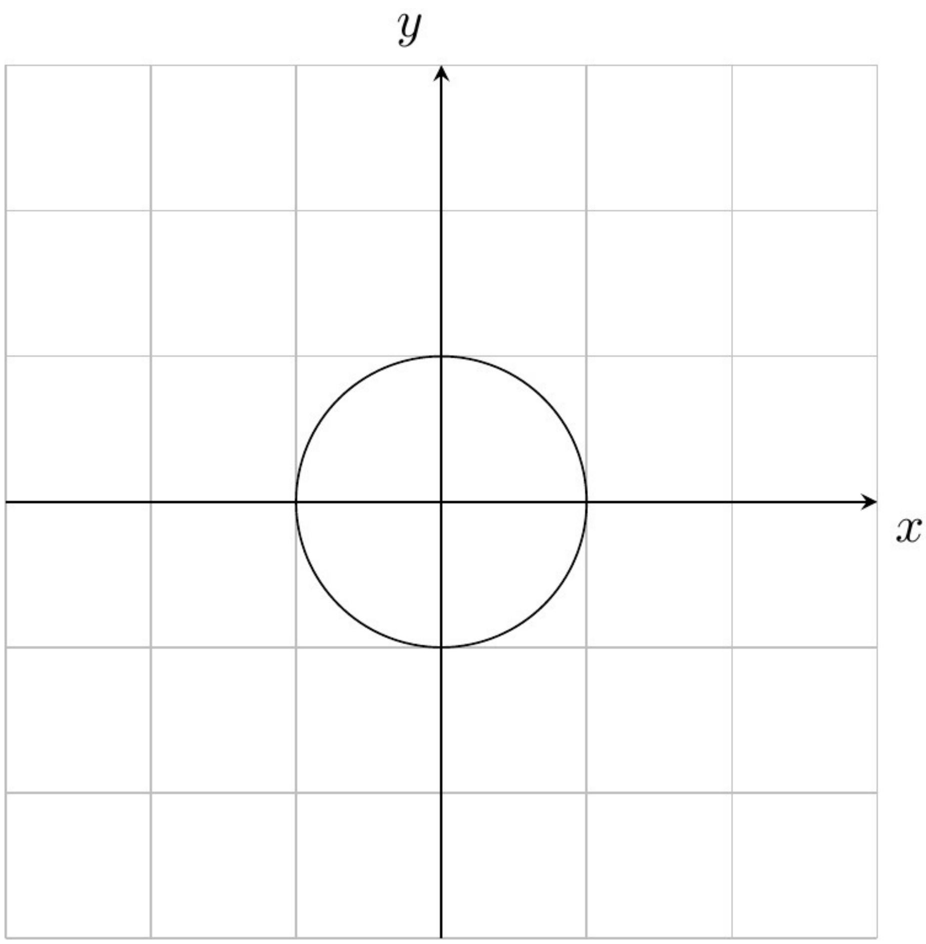
Return to Exercise 1.8 on P24

Solution 1.9

Describe the set $D = \{z \in \mathbb{C} \mid z \cdot \overline{z} = 1\}$ geometrically.

Hint: ***Write $z = re^{i\theta}$ in polar form.***

Note that $z \cdot \overline{z} = |z|^2$. Hence points in D satisfy the equation $|z|^2 = 1$. This is the circle of radius one centred at the origin.



Return to Exercise 1.9 on P25

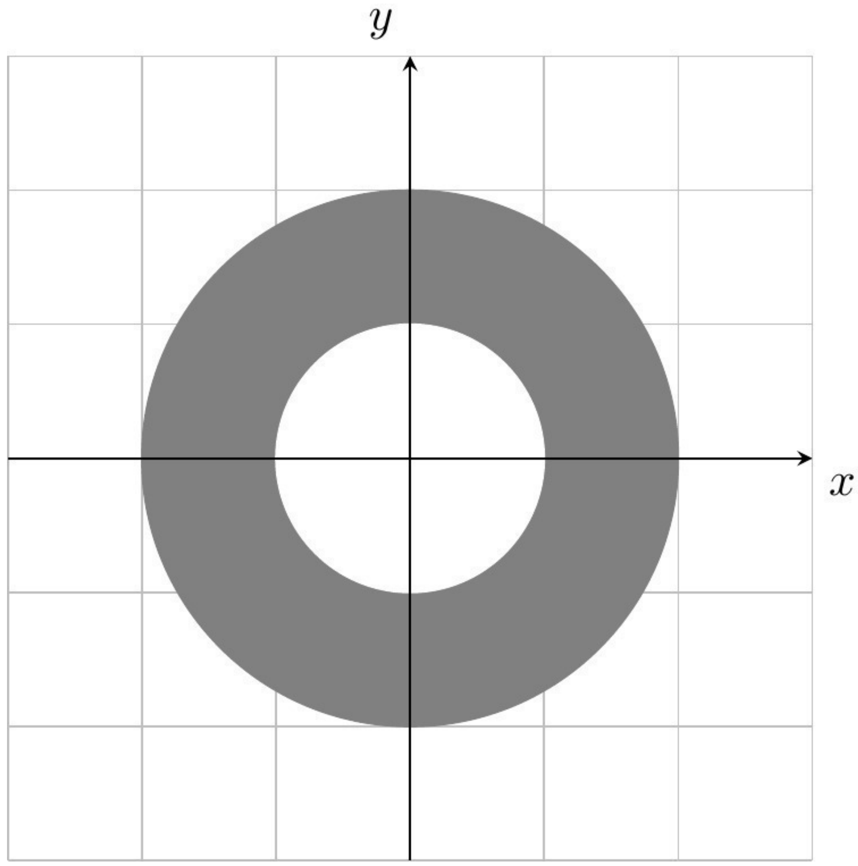
Solution 1.10

Draw all nine sets described by the following conditions on the complex number z.

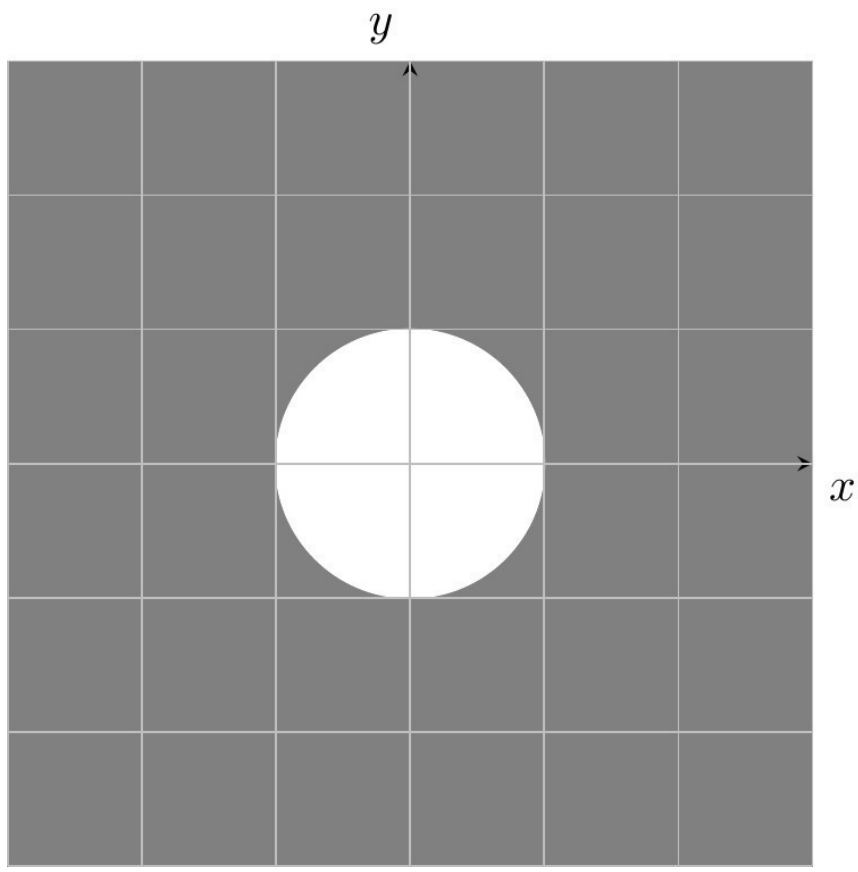
$$|z| = 1, \qquad |z| < 1, \qquad 1 < |z| < 2,$$
$$|1+z| > 1, \qquad |2-z| < 2, \qquad 3 < |z+i| < 4,$$
$$|z-1| < |z+1|, \qquad |z| = |z+1|, \qquad |z-1| = |z+i|.$$

- $|z| = 1$ is the unit circle.

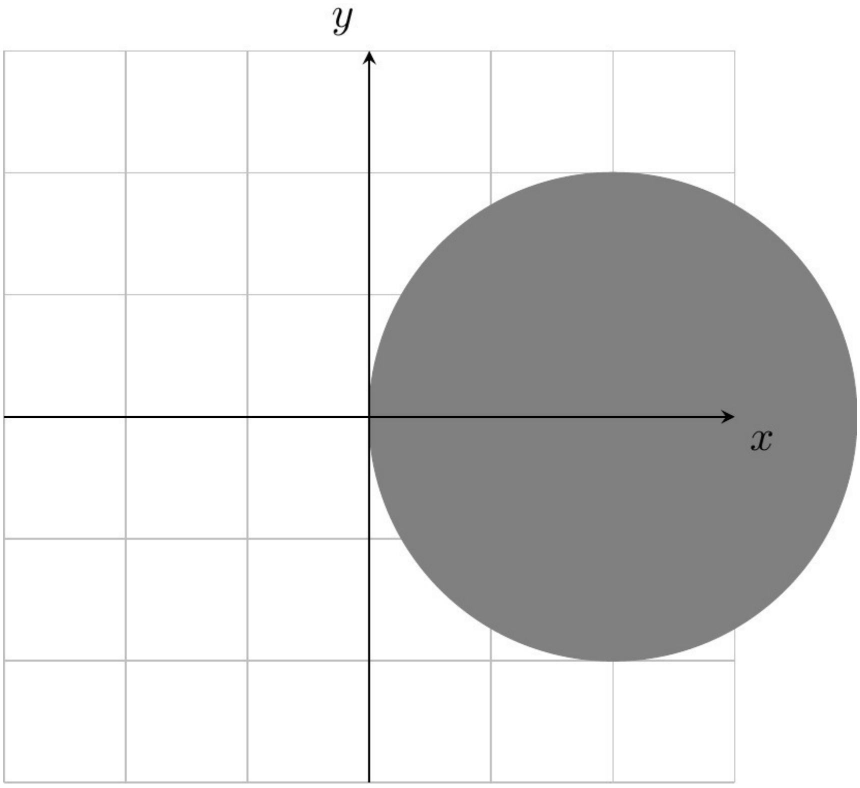
- $|z| < 1$ is the open disk of radius 1 with center the origin.
- $1 < |z| < 2$ is the open annulus centered at the origin with radii 1 and 2.



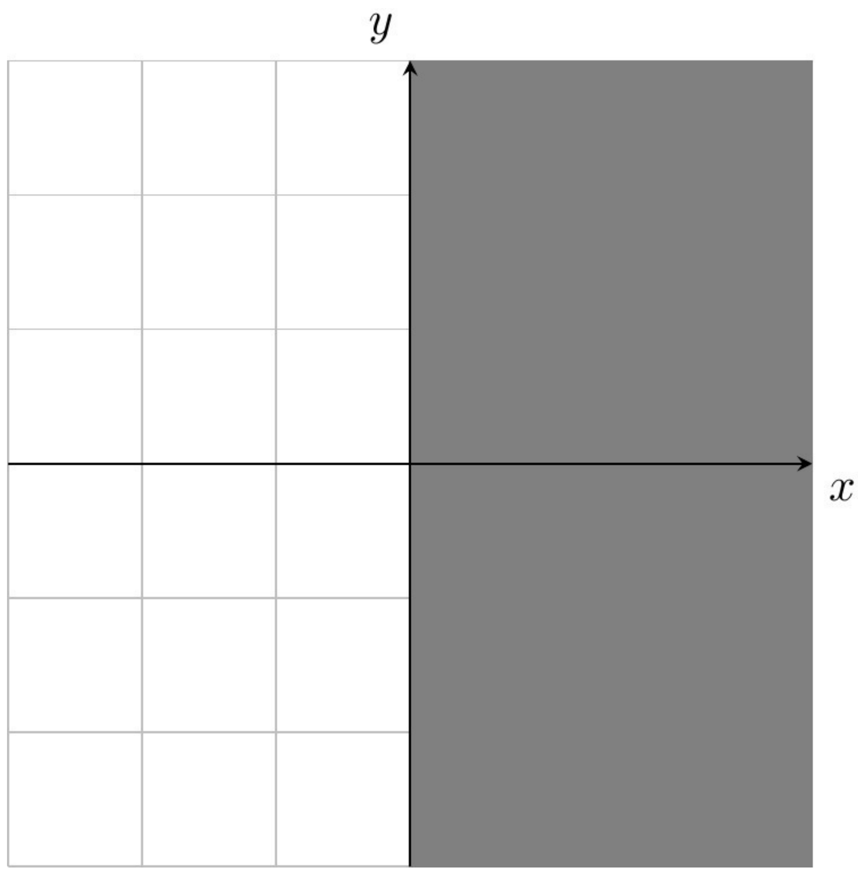
- The complement of the set defined by $|1+z| > 1$ is the set of all $|1+z| \leqslant 1$, the closed disk of radius 1 centered at -1. Hence $\{z \in \mathbb{C} \mid |1+z| > 1\}$ is the complement of this closed disk.



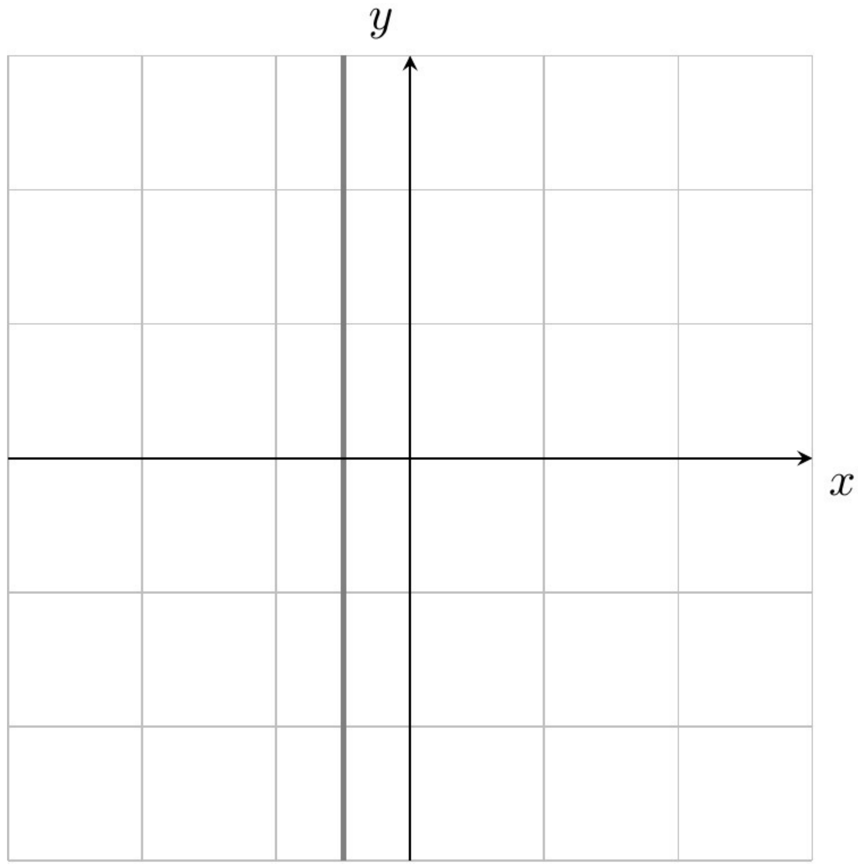
- $|2 - z| < 2$ is the open disk of radius 2 centered at 2.



- $3 < |z+i| < 4$ is an annulus centered at $-i$ with radii 3 and 4.
- $|z-1| < |z+1|$ is the set of points that are closer to 1 than to -1. This is the 'right half-plane' $\{x+iy \mid x > 0\}$. In more detail, write $z = x+iy$. Then the inequality is equivalent to $(x-1)^2 + y^2 < (x+1)^2 + y^2$, so to $-2x < 2x$. This just means $x > 0$.



- $|z| = |z+1|$ is the set of points equidistant from 0 and -1. Write $z = x + iy$. Then the condition is equivalent to $x^2 + y^2 = (x+1)^2 + 2(x+1)y + y^2$, so equivalent to $2x + 1 + y(2x+1) = 0$. This implies $x = -1/2$ and therefore $\{z \in \mathbb{C} \mid |z| = |z+1|\}$ is the straight line $\{x + iy \mid x = -1/2\}$.



- $|z-1| = |z+i|$ is the set of points equidistant from 1 and $-i$. This is again a straight line, namely $\{x - ix \mid x \in \mathbb{R}\}$. \end{itemize}

Return to Exercise 1.10 on P25

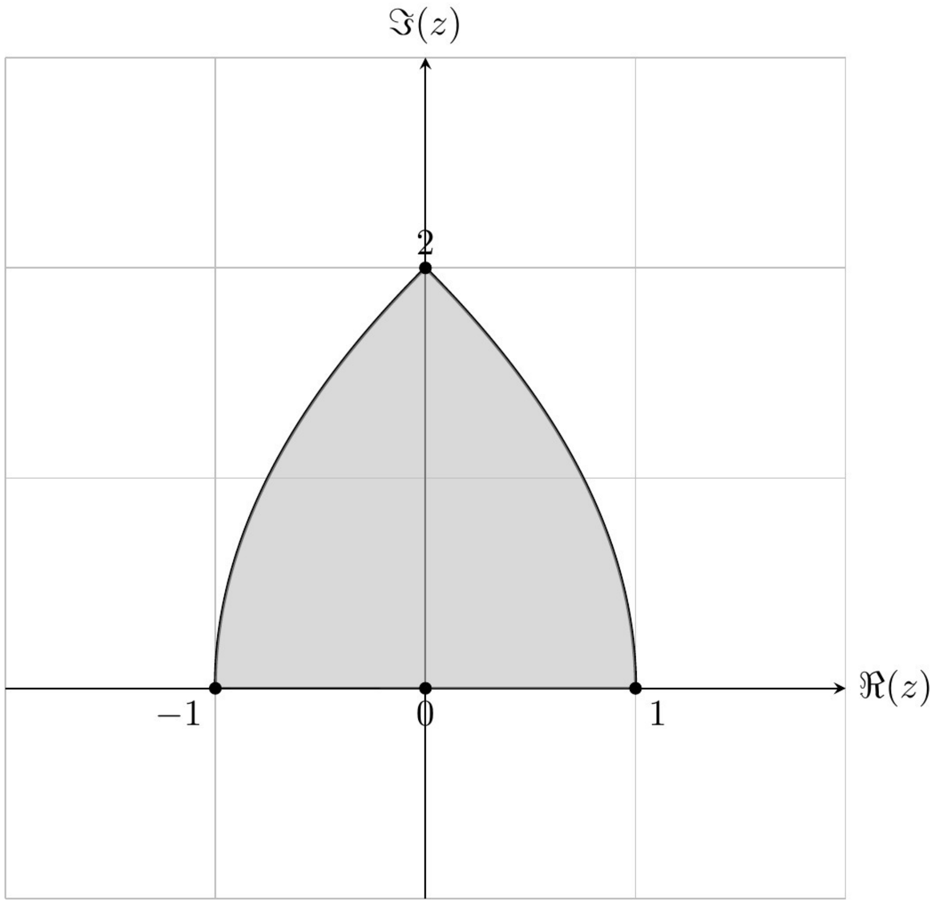
Solution 1.11

Let $S = \{x + iy \in \mathbb{C} \mid 0 \leqslant x, y \leqslant 1\}$***. Draw*** S ***and the sets***

$$A = \{2z \mid z \in S\}, \qquad B = \{\overline{z} \mid z \in S\},$$
$$C = \{-z \mid z \in S\}, \qquad D = \{z^2 \mid z \in S\}.$$

A is the square with corners $\{0, 2, 2+2i, 2i\}$. B is the square with corners $\{0, 1, 1-i, -i\}$. C is the square with corners $\{0, -i, -1-i, -1\}$.

D is much harder to work out. It is shown in the picture. The edge joining 0 to 1 remains as it was. The edge joining 1 to $1+i$ can be parametrised as $\{1+it \mid 0 \leqslant t \leqslant 1\}$ and, on squaring, becomes the curve parametrised as $\{(1-t^2, 2t) \mid 0 \leqslant t \leqslant 1\}$ and this is the right-hand curved edge. In the same way the line joining $1+i$ to i becomes the left-hand curved edge. Finally, the line joining i to 0 becomes the line joining -1 to 0.



Return to Exercise 1.11 on P25

Solution 1.12

Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ ***be the unit disk. Draw the sets***

$$A = \{2z \mid z \in D\}, \quad B = \{z^2 \mid z \in D\}, \quad C = \{|z| \mid z \in D\}.$$

- Since $|2z| = 2|z| < 2 \iff |z| < 1$ the set A is the open disk of radius 2 at the origin.
- As $|z^2| < |z|^2 < 1$ for $z \in D$ we have $B \subset D$. Conversely, if $w \in D$, we know there exists a square root z with $z^2 = w$. As $|z| = |z^2|^{1/2} < |w|^{1/2} < 1$ it follows that $w \in B$. Hence $B = D$.
- C is the real interval $\{x + i0 \mid 0 \leqslant x < 1\} = [0, 1)$.

Return to Exercise 1.12 on P26

Solution 1.13

Show that $i = e^{i\pi/2}$ ***and*** $-1 = e^{i\pi}$.

We have

$$e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = 0 + i = i$$

and

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + 0 = -1.$$

Return to Exercise 1.13 on P26

Solution 1.14

Express the following complex numbers z ***in the form*** $x + iy$ ***with*** $x, y \in \mathbb{R}$.

$$e^{i\pi/4}, \; e^{i\pi}, \; e^{i\frac{2\pi}{3}}$$

$$\frac{\sqrt{2}}{2},\ -1,\ \frac{-1+i\sqrt{3}}{2}$$

Return to Exercise 1.14 on P26

Solution 1.15

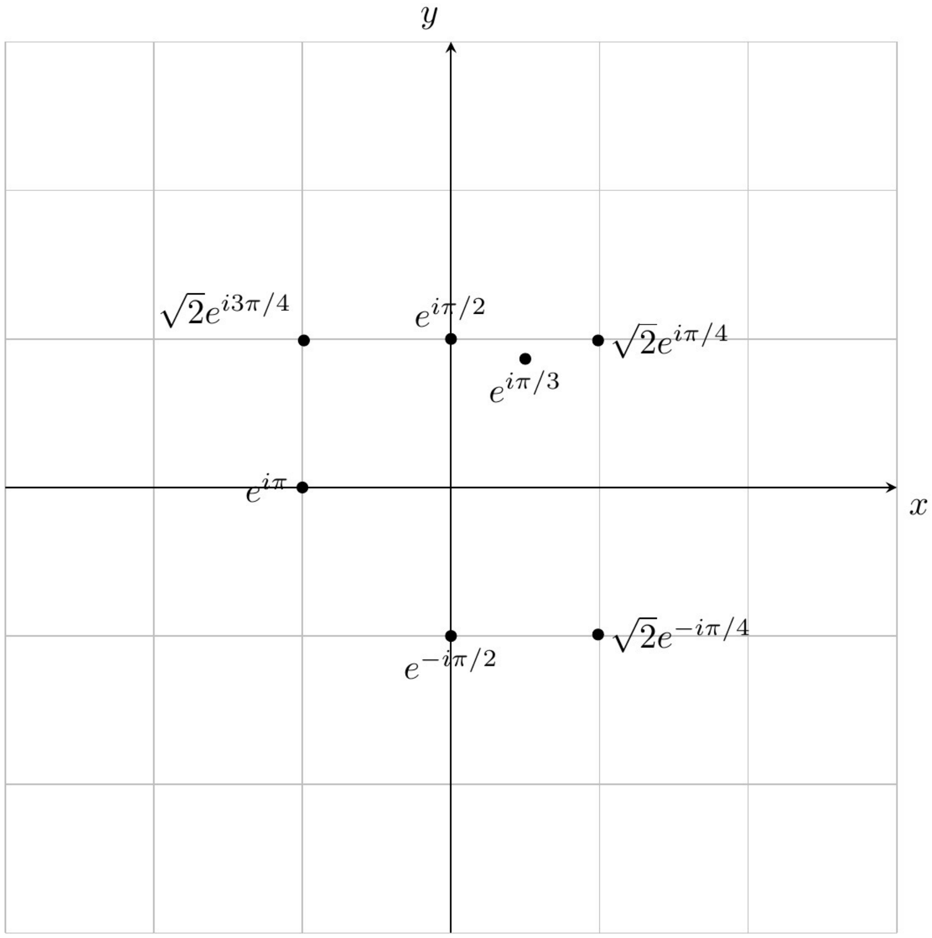
Write each of the following complex numbers in polar form $re^{i\theta}$ with $r > 0$ and $-\pi < \theta \leqslant \pi$.

$$i, \quad -1, \quad -i, \quad 1+i, \quad 1-i, \quad i-1, \quad \frac{1}{2}+i\frac{\sqrt{3}}{2}$$

Draw each of these numbers in the complex plane.

$$e^{i\pi/2}, \quad e^{i\pi}, \quad e^{-i\pi/2}, \quad \sqrt{2}e^{i\pi/4}, \quad \sqrt{2}e^{-i\pi/4}, \quad \sqrt{2}e^{3i\pi/4}, \quad e^{i\pi/3}.$$

Be careful to take the angle in the correct range $-\pi < \theta \leqslant \pi$.



Return to Exercise 1.15 on P26

Solution 1.16

Calculate i^{2021} ***and*** $(1+i)^{20}$.

We use polar coordinates. For the first expression,

$$i^{2021} = (e^{i\pi/2})^{2021} = (e^{i\pi/2})^{2020}(e^{i\pi/2}) = (e^{i\pi/2})^{2020}i.$$

Now we express $(e^{i\pi/2})^{2020}$ as

$$(e^{i\pi})^{1010} = (-1)^{1010} = 1.$$

We conclude that $i^{2021} = i$.

For the second expression, we write

$$(i+1)^{20} = \left(\sqrt{2}e^{i\pi/4}\right)^{20} = 2^{10}e^{i5\pi} = -1024.$$

Return to Exercise 1.16 on P26

Solution 1.17

Solve the equation $(1-i)^n - 2075 = 2021$ ***and find*** $n \in \mathbb{N}$.

Writing $1 - i = \sqrt{2}e^{-i\pi/4}$, the equation is equivalent to $2^{n/2}e^{-in\pi/4} = 4096$. Taking the modulus gives $2^{n/2} = 4096$ so necessarily $n = 24$ and indeed we have $(1-i)^{24} = 4096$.

Return to Exercise 1.17 on P27

Solution 1.18

Prove that for $z \in \mathbb{R}$ ***we have***

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Use these equations to extend the definition of the functions $\cos(z), \sin(z)$ ***to complex arguments*** $z \in \mathbb{C}$. ***Find*** $z \in \mathbb{C}$ ***with*** $\sin(z) = 2$.

Hint: ***Put $w = e^{iz}$ and reduce to a quadratic equation.***

Expanding we have

$$\frac{e^{iz}+e^{-iz}}{2} = \frac{\cos z + i\sin z + \cos z - i\sin z}{2} = \frac{2\cos z}{2} = \cos z \quad \text{(A.6)}$$

$$\frac{e^{iz}-e^{-iz}}{2i} = \frac{\cos z + i\sin z - \cos z + i\sin z}{2i} = \frac{2i\sin z}{2i} = \sin z. \quad \text{(A.7)}$$

We solve

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i} = 2$$

for $z \in \mathbb{C}$. Rearranging, we have

$$e^{iz} - e^{-iz} = 4i.$$

Multiplying through by e^{iz} gives,

$$e^{2iz} - 4ie^{iz} - 1 = 0$$

. Setting $t = e^{iz}$ gives the quadratic euqation:

$$t^2 - 4it - 1 = 0$$

. Solving, we get

$$t = e^{i(x+iy)} = i(2 \pm \sqrt{3}).$$

Expanding out the term $e^{i(x+iy)}$, gives

$$e^{-y}e^{ix} = i(2 \pm \sqrt{3}).$$

We conclude that $x = \pi/2 + 2k\pi$ for $k \in \mathbb{Z}$ (since $e^{ix} = i$) and $y = -\ln(2 \pm \sqrt{3})$ (since $e^{-y} = 2 \pm \sqrt{3}$).

Therefore $z = (\pi/2 + 2k\pi) - \ln(2 \pm \sqrt{3})$ for $k \in \mathbb{Z}$.

Return to Exercise 1.18 on P27

Solution 1.19

Prove the following statements.

- ***a.*** $\overline{zw} = \overline{z} \cdot \overline{w}$ ***for all*** $z, w \in \mathbb{C}$
- ***b.*** $\overline{z_1 z_2 \cdots z_n} = \overline{z}_1 \overline{z}_2 \cdots \overline{z}_n$ ***for all*** $z_1, z_2, \ldots, z_n \in \mathbb{C}$ ***(use induction)***
- ***c.*** $\overline{(z^n)} = (\overline{z})^n$ ***for all*** $z \in \mathbb{C}$
- ***d.*** ***Let*** $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ ***be a polynomial with real coefficients*** $a_0, \ldots, a_n \in \mathbb{R}$. ***Prove that*** $\overline{p(z)} = p(\overline{z})$. ***Deduce that all roots of*** $p(z)$ ***occur in complex conjugate pairs.***

a. Let $z = x + iy$ and $w = u + iv$. Then

$$zw = (xu - yv) + i(xv + yu) \implies \overline{zw} = (xu - yv) - i(xv + yu)$$
$$\overline{z} \cdot \overline{w} = (x - iy)(u - iv) = xu - yv + (-ivx - iyu).$$

b. The base case $n = 1$ is trivial. For the inductive step, we have

$$\begin{aligned} \overline{z_1 \cdots z_n z_{n+1}} &= \overline{z_1 \cdots z_n} \cdot \overline{z_{n+1}} && \text{by (a)} \\ &= (\overline{z_1} \cdots \overline{z_n})\overline{z_{n+1}} && \text{by induction.} \end{aligned}$$

c. follows from (b) by setting $z_1 = \cdots = z_n = z$.

d. We have

$$
\begin{aligned}
\overline{p(z)} &= \overline{a_n z^n + \ldots + a_1 z + a_0} \\
&= \overline{a_n z^n} + \ldots + \overline{a_1 z} + \overline{a_0} && \text{since } \overline{\cdots} \text{ is additive} \\
&= \overline{a_n}\,\overline{z^n} + \ldots + \overline{a_1} \cdot \overline{z} + \overline{a_0} && \text{by a.} \\
&= a_n \overline{z}^n + \ldots + a_1 \overline{z} + a_0 && \text{by (c) and since } a_k \in \mathbb{R} \\
&= p(\overline{z}).
\end{aligned}
$$

Finally, suppose $p(\zeta) = 0$. Then $p(\overline{\zeta}) = \overline{p(\zeta)}$.

Return to Exercise 1.19 on P27

Solution 1.20

Show that $|z| = |-z|$ ***and*** $|\overline{z}| = |z|$. ***Prove also that*** $|\lambda z| = \lambda |z|$ ***for all*** $\lambda \geqslant 0$.

Write $z = x + iy$, then for $\lambda \geq 0$,

$$
\begin{aligned}
|z| &= \sqrt{x^2 + y^2} = \sqrt{(-x)^2 + (-y)^2} = |-z| \\
\lambda|z| &= \lambda\sqrt{x^2 + y^2} = \sqrt{\lambda^2(x^2 + y^2)} = \sqrt{(\lambda x)^2 + (\lambda y)^2} = |\lambda z|
\end{aligned}
$$

Return to Exercise 1.20 on P28

Solution 1.21

Prove that $\Re(z) = \frac{1}{2}(z + \bar{z})$, $\Im(z) = \frac{1}{2i}(z - \bar{z})$.

Let $z = x + iy$. Then

$$z + \overline{z} = x + iy + x - iy = 2x = 2\Re(z)$$

and

$$z - \overline{z} = x + iy - (x - iy) = 2iy = 2i\Im(z).$$

Return to Exercise 1.21 on P28

Solution 1.22

Prove that $\overline{e^z} = e^{\overline{z}}$. ***Deduce that*** $|e^z| = e^{\Re(z)}$.

Let $z = x + iy$, then

$$\begin{aligned}\overline{e^z} &= \overline{e^{x+iy}} = \overline{e^x(\cos(y) + i\sin(y))}\\ &= e^x(\cos(y) - i\sin(y)) = e^x(cos(-y) + i\sin(-y)) = e^{\overline{z}}.\end{aligned}$$

For the second part,

$$|e^z| = \sqrt{e^z \cdot e^{\overline{z}}} = \sqrt{e^{z+\overline{z}}} = \sqrt{e^{2\Re(z)}} = e^{\Re(z)}.$$

Return to Exercise 1.22 on P28

Solution 1.23

Prove that $|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2$.

Write $z = x + iy$ and $w = u + iv$, then

$$|z+w|^2+|z-w|^2 = (x+u)^2+(y+v)^2+(x-u)^2+(y-v)^2$$
$$= 2(x^2+u^2)+2(y^2+v^2) = 2|z|^2+2|w|^2.$$

Return to Exercise 1.23 on P28

Solution 1.24

Show that** $|z+w|^2 = |z|^2+2\Re(z\overline{w})+|w|^2$. **Use this to determine the conditions on** z, w **for** $|z+w| = |z|+|w|$ **to hold.

Write $z = x+iy$ and $w = u+iv$, then

$$|z+w|^2 = (x+u)^2+(y+v)^2 = (x^2+y^2)+2xu+2yv+(y^2+v^2).$$

Noting that $\Re(z\overline{w}) = xu+yv$, we have

$$|z+w|^2 = |z|^2+2\Re(z\overline{w})+|w|^2$$

as required.

For the second part write $w = \lambda z$ for some $\lambda \in \mathbb{C}$. We show that λ is a positive real number. Suppose $|z+w| = |z|+|w|$ then

$$|z+\lambda z| = |z|+|\lambda z|.$$

It follows, assuming $z \neq 0$ that

$$|1 + \lambda| = |1| + |\lambda|.$$

Using the preceding part $2\Re(\lambda) = 2|\lambda|$ and we conclude that $|\lambda| = \Re(\lambda)$ and so λ is a positive real number.

If $z = 0$, then we may take $\lambda = 0$.

Return to Exercise 1.24 on P29

Solution 1.25

Assuming we know the triangle inequality $|z + w| \leqslant |z| + |w|$ for all $z, w \in \mathbb{C}$, prove the reverse triangle inequality

$$|z - w| \geqslant ||z| - |w||.$$

Applying the triangle inequality to z and $w - z$ gives

$$|w| = |z + (w - z)| \leqslant |z| + |w - z|,$$

which we rearrange to

$$|w| - |z| \leqslant |w - z|. \qquad \text{(A)}$$

By exchanging the roles of z and w we get

$$|z| - |w| \leqslant |z - w|. \qquad \text{(B)}$$

The two inequalities (A) and (B) are equivalent to the single inequality $||z| - |w|| \leqslant |z - w|$.

Return to Exercise 1.25 on P29

Solution 1.26

Let K be a field with $\mathbb{R} \subset K \subset \mathbb{C}$. Prove that $K = \mathbb{R}$ or $K = \mathbb{C}$.

Suppose that $K \neq \mathbb{R}$. Then there exists $z = x + iy \in K$ with $y \neq 0$. We know that $x \in K$ since x is real. So, by the basic properties of a field, $iy = z - x \in K$. Now y is real (so in K) and non-zero. So $i = (iy)/y \in K$. That's all we need, since we can now deduce that $x + iy \in K$ for all $x, y \in \mathbb{R}$.

Return to Exercise 1.26 on P29

A.2 Chapter 2 solutions

Solution 2.1

Describe the image set of the complex function $f(z) = \frac{1+z}{1-z}$ with domain $D = \mathbb{C} \setminus \{1\}$. In other words, determine the set of all $w \in \mathbb{C}$ for which $w = \frac{1+z}{1-z}$ has a solution $z \in D$.

If $w \neq -1$, the equation $w = \frac{1+z}{1-z}$ can be rearranged as

$$z = \frac{w-1}{w+1}.$$

Hence $\mathbb{C} \setminus \{-1\}$ is contained in the image of f. Also, -1 is not in the image of f since $-1 = \frac{1+z}{1-z}$ would imply $z - 1 = 1 + z$, so $-1 = 1$, which is impossible.

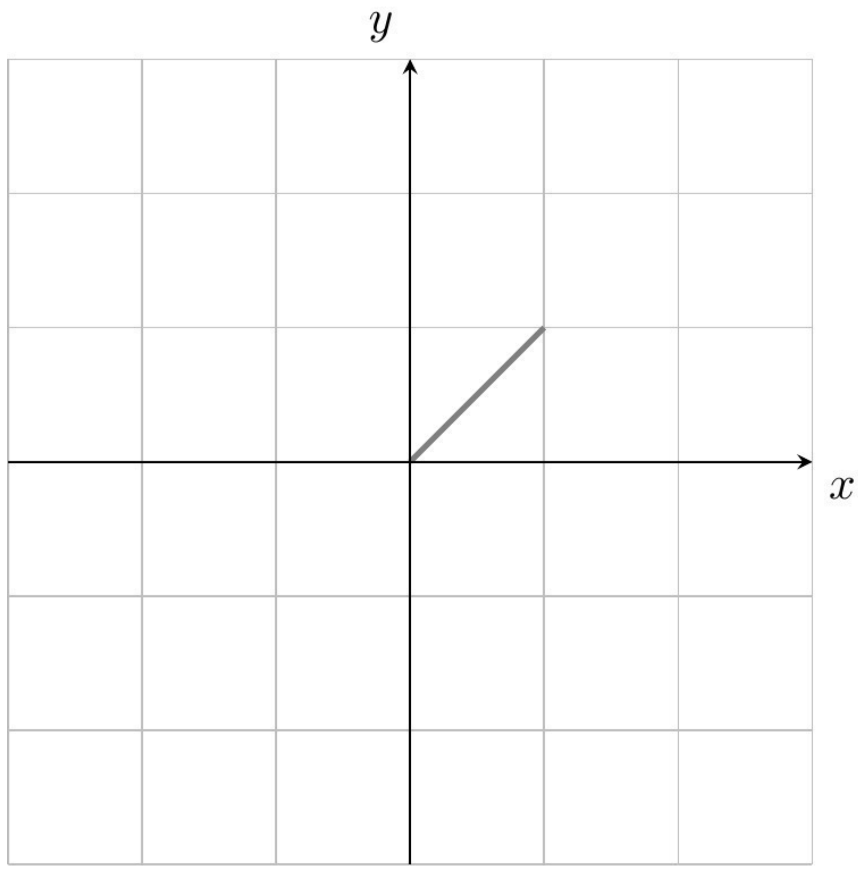
Return to Exercise 2.1 on P40

Solution 2.2

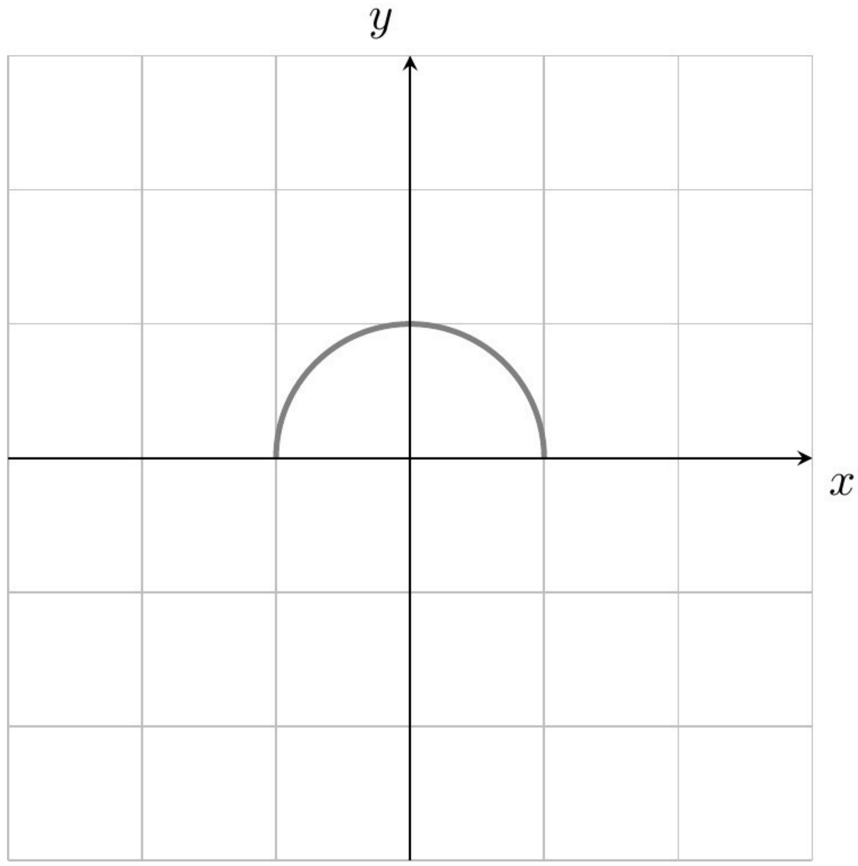
Sketch the following curves $z(t)$ in the complex plane, where t is a real parameter.

- ***a.*** $z(t) = t(1+i)$ ***for*** $0 \leqslant t \leqslant 1$
- ***b.*** $z(t) = \cos(t) + i\sin(t)$ ***for*** $0 \leqslant t \leqslant \pi$
- ***c.*** $z(t) = \cos(t) - i\sin(t)$ ***for*** $0 \leqslant t \leqslant \pi/2$
- ***d.*** $z(t) = \frac{1}{1+it}$ ***for*** $t \in \mathbb{R}$

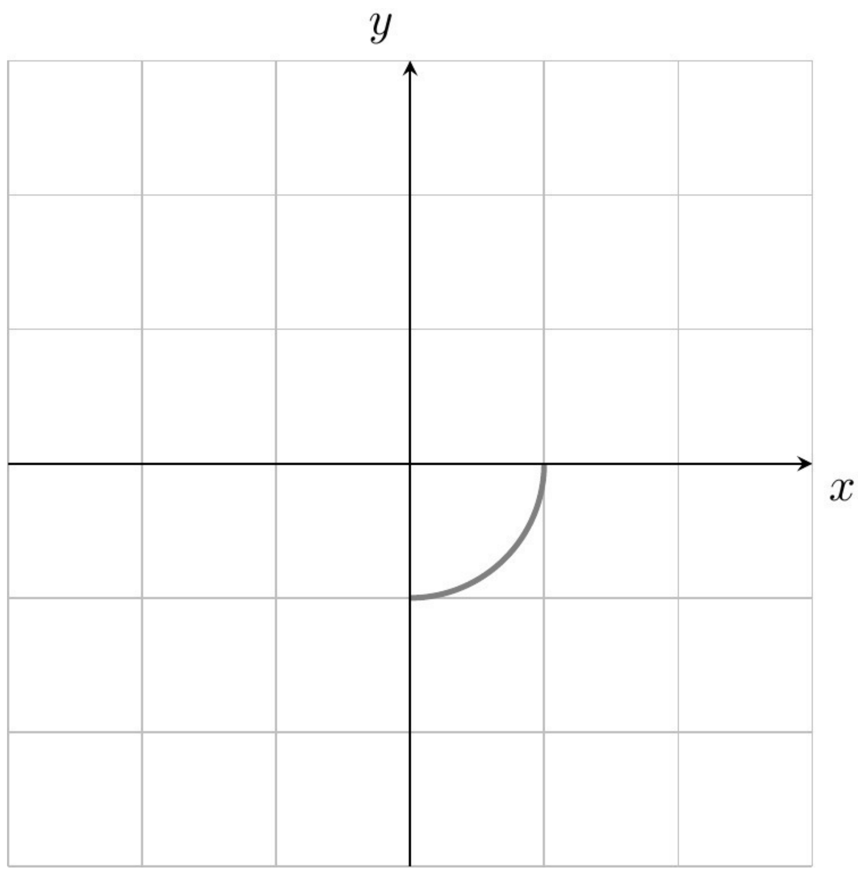
a. is a straight line from the origin to $1+i$.



b. Thinking in polar coordinates, this the upper semicircle from $+1$ to -1.



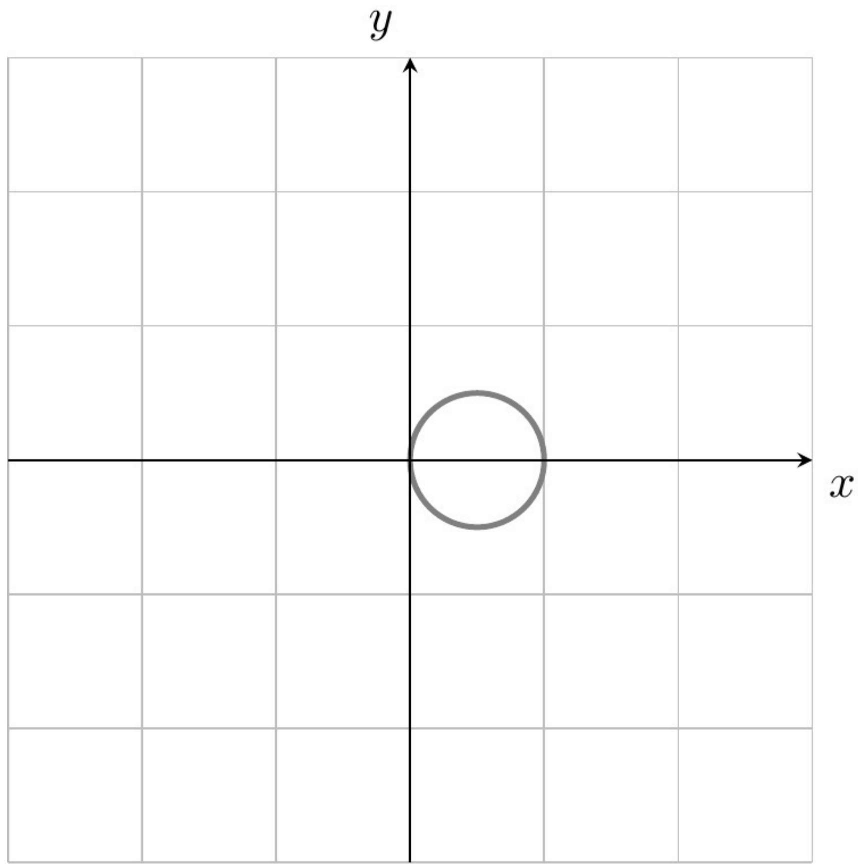
c. We have $z(t) = \cos(-t) + i\sin(-t) = \cos(s) + i\sin(s)$ where $s = -t \in [-\pi/2, 0]$. This is the quarter circle from $+1$ to $-i$.



d. By plotting the points for $t = 0, \pm 1$ and $t \to \pm\infty$, we suspect that $z(t)$ lies entirely on the circle centered at $1/2$. We verify this as follows.

$$\begin{aligned}\left|z(t) - \frac{1}{2}\right| &= \left|\frac{1 - it}{t^2 + 1} - \frac{1 + t^2}{2 + 2t^2}\right| = \left|\frac{1 + (it)^2 - 2it}{2 + 2t^2}\right| \\ &= \left|\frac{(1 + it)^2}{2 + 2t^2}\right| = \frac{|1 + it|^2}{|2 + 2t^2|} = \frac{1 + t^2}{2 + 2t^2} = \frac{1}{2}.\end{aligned}$$

We know that $z(0) = 1$, $\lim_{t \to \pm\infty} z(t) = 0$, $z(\pm 1) = \frac{1 \mp i}{2}$ and that the image of z is a connected subset. Moreover, $z(t) \neq 0$ for all t. It follows that the image of $z(t)$ is a circle of radius $1/2$ centered at $1/2$ with the origin removed. Note, however, that this parameterization of the circle has an unusual speed.



Return to Exercise 2.2 on P40

Solution 2.3

Let $\log(z)$ ***be the principal branch of the logarithm. Compute***

$$\mathbf{log}(2i), \mathbf{log}(1+i), \mathbf{log}(-3i), \mathbf{log}(5).$$

Recall that the principal branch is defined by $\log(z) = \log(r) + i\theta$ if $z = re^{i\theta}$, where $r > 0$, $\theta \in (-\pi, \pi)$. We obtain the values

$$\log(2) + \frac{i\pi}{2}, \quad \log(\sqrt{2}) + \frac{i\pi}{4}, \quad \log(3) - \frac{i\pi}{2}, \log(5).$$

Return to Exercise 2.3 on P40

Solution 2.4

Describe the following complex functions geometrically.

$$f(z) = 3z, f(z) = iz, f(z) = \frac{(1+i)}{\sqrt{2}}z$$

$f(z) = 3z$ scales every complex number (though of as a vector in $\mathbb{R}^2$) by the factor 3.

$f(z) = iz$ rotates counterclockwise by 90 degrees.

As $\frac{1+i}{\sqrt{2}} = e^{i\pi/4}$, the map $f(z) = \frac{(1+i)}{\sqrt{2}}z$ is a counterclockwise rotation by 45 degrees.

Return to Exercise 2.4 on P40

Solution 2.5

Determine a domain of definition for the following complex functions.

$$f(z) = \frac{1}{z}, f(z) = \frac{1+z}{z-1}, f(z) = \frac{z^2-4}{z^2+2z}, f(z) = \frac{1}{\mathbf{exp}(z)}.$$

For $f(z) = 1/z$, $D = \mathbb{C} \setminus \{0\}$.

For $f(z) = \frac{1+z}{z-1}$, $D = \mathbb{C} \setminus \{1\}$.

For $f(z) = \frac{z^2-4}{z^2+2z}$, it seems at first that both $z = 0, -2$ are problematic, but we can rewrite $\frac{z^2-4}{z^2+2z} = \frac{z-2}{z}$ since $z^2 - 4 = (z-2)(z+2)$. We can therefore extend the definition of f to the domain $D = \mathbb{C} \setminus \{0\}$.

For $f(z) = \frac{1}{\exp(z)}$, $D = \mathbb{C}$ as $\exp(z) \neq 0$ for all z. This last fact holds since $|\exp(z)| = e^x \neq 0$ for all $z = x + iy$.

Return to Exercise 2.5 on P41

Solution 2.6

Determine the domain of definition for $f(z) = \frac{1}{\sin(z)}$.

We need to find the points at which $\sin(z) = 0$. Using the equality

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i},$$

we need to find the points $z \in \mathbb{C}$ where $e^{iz} = e^{-iz}$, equivalently, where $e^{2iz} = 1$. Writing $z = x + iy$, we find that $y = 0$, since

$$|e^{2iz}| = e^{-2y} = 1$$

and $x = k\pi$ since $2x = 0 + 2k\pi$, $k \in \mathbb{Z}$.

The domain of $1/\sin(z)$ is therefore $D = \mathbb{C} \setminus \{k\pi : i \in \mathbb{Z}\}$.

Return to Exercise 2.6 on P41

Solution 2.7

Find all solutions to the following equations:

a. $e^z = -1$,

b. $\sin(z) = -i$

a. As $-1 = e^{\pi i}$, the equation $e^z = -1$ implies $e^{z-\pi i} = 1$, hence the set of solutions is $\pi i + 2\pi i\mathbb{Z}$.

b. Recall that $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$. Writing $q = e^{iz}$, the equation $\sin(z) = -i$ becomes $q - q^{-1} = 2$. This leads to the quadratic equation $q^2 - 2q - 1$ which has the roots $q_\pm = 1 \pm \sqrt{2}$. Hence $\sin(z) = -i \iff e^{iz} = q_+$ or $e^{iz} = q_-$. The first equation is equivalent to $iz \in \log(1+\sqrt{2}) + 2\pi i\mathbb{Z}$ and by writing $q_- = (\sqrt{2}-1)e^{\pi i}$ we find that the second equation is equivalent to $iz \in \log(\sqrt{2}-1) + \pi i + 2\pi i\mathbb{Z}$. Summing up, the set of solutions is

$$\{-i\log(1+\sqrt{2}) + 2\pi k \mid k \in \mathbb{Z}\} \cup \{-i\log(\sqrt{2}-1) + \pi + 2\pi k \mid k \in \mathbb{Z}\}.$$

Return to Exercise 2.7 on P41

Solution 2.8

Prove that the composition $f \circ f^{'}$ of two Möbius transformations is again a Möbius transformation. If we associate to $f, f^{'}$ the matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, A^{'} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbf{M}_{2\times 2}(\mathbb{C}),$$

show that $f \circ f^{'}$ is associated to the matrix product $AA^{'}$.

Let $f(z) = \frac{az+b}{cz+d}$ and $g(z) = \frac{\alpha z+\beta}{\gamma+\delta}$ be Möbius transformations. Then,

$$(f \circ g)(z) = \frac{ag(z)+b}{cg(z)+d}$$

. Using the definition of $g(z)$, this expands to

$$\frac{a\alpha z + a\beta + b\gamma z + b\delta}{\gamma z + \delta} \cdot \frac{\gamma z + \delta}{c\alpha z + c\beta + d\gamma z + d\delta}.$$

We consequently have,

$$f(g(z)) = \frac{(a\alpha + b\gamma)z + (a\beta + b\delta)}{(c\alpha + d\gamma)z + (c\beta + d\delta)}.$$

One can verify explicitly that $((a\alpha+b\gamma)(c\beta+d\delta))-((a\beta+b\delta)(c\alpha+d\delta)) \neq 0$, however, this is neatly tackled by the second part of the question, since the product of two invertible matrices is again an invertible matrix.

Notice that,

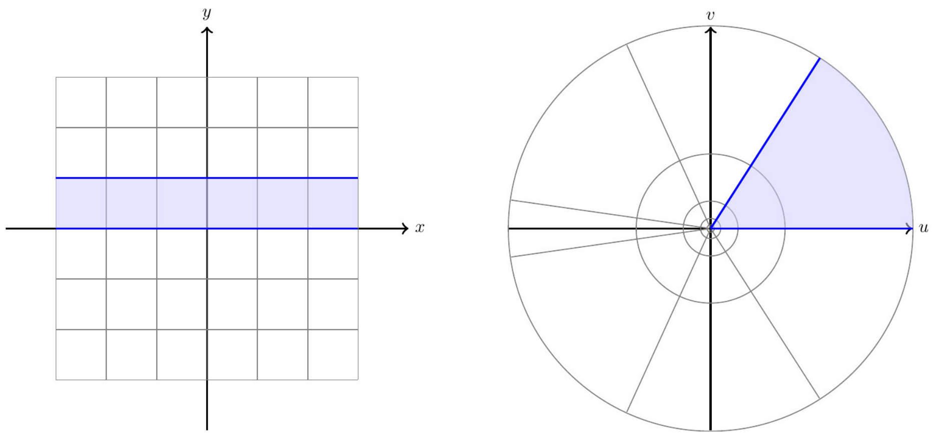
$$AA' = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}$$

is precisley the matrix associated to $f \circ g$.

Return to Exercise 2.8 on P41

Solution 2.9

Draw the image grid for $\exp\colon \mathbb{C} \to \mathbb{C}^\times$.



Return to Exercise 2.9 on P42

A.3 Chapter 3 solutions

Solution 3.1

Recall the ratio and root test for series of real numbers.

Let $\sum_{k=0}^{\infty} a_n$ be a real series.

Ratio test Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

- If $L > 1$, the the series is divergent;
- If $L < 1$ then the series is (absolutely) convergent;
- If $L = 1$ or does not exist, then the test is inconclusive.

Root test Let $C = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$

- if $C > 1$, then the series is divergent;
- if $C < 1$, then the series is (absolutely) convergent;
- if $C = 1$ and the limit approaches strictly from above then the series diverges;

- otherwise the test is inconclusive.

Return to Exercise 3.1 on P48

Solution 3.2

For which $z \in \mathbb{C}$ do the following limits exist?

$$\lim_{n\to\infty} n^{1/n}z, \lim_{n\to\infty} z^n, \lim_{n\to\infty} \frac{z^n}{n}, \lim_{n\to\infty} \frac{z^n}{n!}, \lim_{n\to\infty} \frac{z^n}{n^n}, \lim_{n\to\infty} n!z.$$

The sequence $n^{1/n}$ is convergent with limit equal to 1 and so the sequence $n^{1/n}|z|$ is convergent for all complex numbers z with limit equal to z.

If $|z| > 1$, then z^n is unbounded, hence does not converge. If $|z| < 1$, then $\lim_{n\to\infty} z^n = 0$.

Suppose $|z| = 1$ and that $\lim_{n\to\infty} z^n$ exists. As z^n is a Cauchy sequence, we have in particular that $|z-1| = |z|^n|z-1| = |z^{n+1} - z^n| \to 0$ for $n \to \infty$. Hence we must have $z = 1$ and in this case the sequence indeed converges.

As $\frac{z^{n+1}/(n+1)}{z^n/n} = \frac{n}{n+1}z \to z$ for $n \to \infty$, the series $\sum_{n=0}^{\infty} \frac{z^n}{n}$ converges for all $|z| < 1$ by the ratio test. Hence $\lim \frac{z^n}{n} = 0$ for all $|z| < 1$. Alternatively, apply by l'Hôpital's rule to the real limit $\frac{x^n}{n}$, where $x = |z|$. If $|z| = 1$, then $|z^n/n| = 1/n \to 0$ as $n \to \infty$, so $\lim \frac{z^n}{n} = 0$ is this case also. If $|z| > 1$, then $\left|\frac{z^n}{n}\right| = |z|^n/n$ is unbounded (for example, by l'Hôpital's rule) and hence the sequence is divergent.

As $\frac{z^{n+1}/(n+1)!}{z^n/n!} = \frac{z}{n+1} \to 0$ for $n \to \infty$ the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all z by the ratio test. Hence $\lim_{n\to\infty} \frac{z^n}{n!} = 0$ for all $z \in \mathbb{C}$.

Notice that $n^n > n!$ and so $|z^n|/n! > |z^n|/n^n$ for all n. Since $z^n/n!$ converges to 0, it follows that z^n/n^n converges to 0 for all z as well.

The final limit is divergent as the sequence $n!|z|$ is unbounded.

Return to Exercise 3.2 on P48

Solution 3.3

Show that every convergent sequence $(z_n)_{n\in\mathbb{N}}$ of complex numbers is bounded.

Take $\varepsilon = 1$. By assumption, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ we have $|z - z_n| < \varepsilon = 1$. Apply the reverse triangle inequality from Sheet 1 to obtain from this

$$|z_n| - |z| \leqslant \big|z_n - z\big| < 1.$$

Hence $|z_n| \leqslant C = \max\{|z| + 1, |z_0|, \dots, |z_{n_0}|\}$.

Return to Exercise 3.3 on P48

Solution 3.4

Let $\sum_{k=0}^{\infty} z_k$ be a convergent series of complex numbers. Show that $\lim_{k\to\infty} z_k = 0$.

By definition, the sequence of partial sums $S_n = \sum_{k=0}^{n} z_k$ convergent, hence a Cauchy sequence. In particular, taking $m = n+1$ in the definition of a Cauchy sequence, we have $|z_n| = |S_{n+1} - S_n| \to 0$ as $n \to \infty$.

Return to Exercise 3.4 on P48

Solution 3.5

For which $z \in \mathbb{C}$ do the following series converge?

$$\sum_{k=0}^{\infty} kz^k, \quad \sum_{k=0}^{\infty} (kz)^k$$

Suppose the series $\sum_{k=0}^{\infty} kz^k$ converges for z. Then by (a) and (b), $k|z|^k \leqslant C$ is bounded. Recall that $\lim_{k\to\infty} \sqrt[k]{x} = 1$ for all $x > 0$ and that $\lim_{k\to\infty} \sqrt[k]{k} = 1$. Hence

$$|z| \leqslant \frac{\sqrt[k]{C}}{\sqrt[k]{k}} \longrightarrow 1 \quad \text{as } k \to \infty.$$

It follows that the series can only possibly converge for $|z| \leqslant 1$. We can alternatively use the ratio test to see that convergence fails for $|z| > 1$. If $|z| = 1$, then kz^k is not a null sequence (the modulus is always 1), hence the series does not converge by (b). If $|z| < 1$, the series converges by the ratio test:

$$\left|\frac{(k+1)z^{k+1}}{kz^k}\right| = \left(1 + \frac{1}{k}\right)|z| \longrightarrow |z| < 1, \quad \text{as } k \to \infty.$$

For the next part, suppose that the series $\sum_{k=0}^{\infty} (kz)^k$ converges for z. Then by (a) and (b), $|kz|^k \leqslant C$ is bounded. Increasing C, we may suppose that $C \geqslant 1$ and then $\sqrt[k]{C} \leqslant C$. Now

$$|z| \leqslant \frac{\sqrt[k]{C}}{k} \longrightarrow 0 \quad \text{as } k \to \infty.$$

This implies $z = 0$ and the series indeed converges in this case.

Return to Exercise 3.5 on P48

Solution 3.6

Let $f\colon \mathbb{C} \to \mathbb{C}$ be a complex function. Show that

$$\lim_{z\to\infty} f(z) = w \iff \lim_{z\to 0} f\left(\frac{1}{z}\right) = w.$$

The left hand side means that for all $\varepsilon > 0$ there exists $M > 0$ such that $|f(z) - w| < \varepsilon$ for all $+\infty > |z| > M$. The right hand side means that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(1/u) - w| < \varepsilon$ for all $0 < |u| < \delta$.

Upon setting $z = 1/u$ the inequalities $+\infty > |z| > M$ and $0 < |u| < \delta$ are equivalent for $M = 1/\delta$.

Return to Exercise 3.6 on P49

Solution 3.7

The Riemann sphere is $\mathbb{S} = \{(a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 1\}$. Show that the stereographic projection

$$F\colon \mathbb{S} \longrightarrow \mathbb{C} \cup \{\infty\}, F(a, b, c) = \begin{cases} \frac{a+ib}{1-c} & \textit{if } c \neq 1, \\ \infty & \textit{if } c = 1. \end{cases}$$

is a bijection between the Riemann sphere and the extended complex plane. Find a formula for the inverse function.

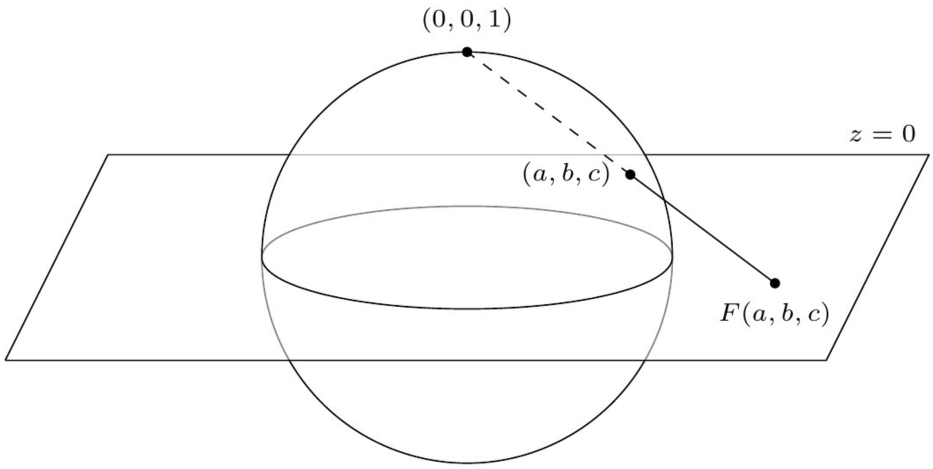


Figure A.1: Stereographic projection from the north pole

We use the sketch to find a putative inverse $G\colon \mathbb{C} \cup \{\infty\} \to \mathbb{S}$. The line from $(0, 0, 1)$ to $z = x + iy \in \mathbb{C}$ is parameterized as

$$\ell(t) = (1-t)(0, 0, 1) + t(x, y, 0) = (tx, tx, 1-t), \qquad t \in [0, 1].$$

To find the intersection points with $\mathbb{S}$, we need to solve

$$1 = (tx)^2 + (ty)^2 + (1-t)^2 = t^2(|z|^2 + 1) - 2t + 1$$

for t. The solutions to this quadratic equation are $t = 0$ and $t = \frac{2}{1+|z|^2}$. Hence

$$G(z) = \ell\left(\frac{2}{1+|z|^2}\right) = \left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{|z|^2-1}{1+|z|^2}\right)$$

for all $z \in \mathbb{C}$. Moreover, according to the sketch we should define $G(\infty) = (0, 0, 1)$. The verifications $F \circ G = \mathrm{id}_{\mathbb{S}}$ and $G \circ F = \mathrm{id}_{\mathbb{C}\cup\{\infty\}}$ are left to the reader.

Return to Exercise 3.7 on P49

A.4 Chapter 4 solutions

Solution 4.1

Write the following complex functions in the form $f = u + iv$:

$$f_1(z) = \mathbf{sin}(z), \qquad f_2(z) = e^{z^2}, \qquad f_3(z) = \mathbf{cosh}(z)$$

By definition, $f_1(z) = \frac{e^{iz} - e^{-iz}}{2i}$. Let $z = x + iy$. Then

$$\begin{aligned} u_1 = \Re(f_1(z)) &= \frac{f_1(z) + \overline{f_1(z)}}{2} = \frac{1}{2}\left(\frac{e^{iz} - e^{-iz}}{2i} - \frac{e^{-i\bar{z}} - e^{i\bar{z}}}{2i}\right) \\ &= \frac{e^{-y}e^{ix} - e^{y}e^{-ix} - e^{-y}e^{-ix} + e^{y}e^{ix}}{4i} \\ &= \frac{e^{ix} - e^{-ix}}{2i} \cdot \frac{e^{y} + e^{-y}}{2} = \sin(x)\cosh(y). \end{aligned}$$

Similarly,

$$v_1 = \cos(x)\sinh(y).$$

Next,

$$f_2(z) = e^{z^2} = e^{x^2 - y^2 + 2ixy} = \underbrace{\cos(x^2 - y^2)}_{=u_2} + i\,\underbrace{\sin(2xy)}_{=v_2}.$$

The function f_3 is treated in the same way as f_1. The result is

$$u_3 = \cosh(x)\cos(y), \quad v_3 = \sinh(x)\sin(y).$$

Return to Exercise 4.1 on P61

Solution 4.2

At which points $z \in \mathbb{C}$ are the following functions complex differentiable? At which points are they real differentiable?

$$f_1(z) = z, \qquad f_2(z) = \bar{z}, \qquad f_3(z) = z^3 + z,$$
$$f_4(z) = \frac{1}{2iz}, \qquad f_5(z) = |z|^2, \qquad f_6(z) = \frac{|z|^2}{\bar{z}}$$

f_1, f_3 and $f_6(z) = z$ are complex differentiable at every $z \in \mathbb{C}$, which implies real differentiability at these points.

f_4 is holomorphic on $\mathbb{C} \setminus \{0\}$.

$f_2 = u_2 + iv_2$ is not complex differentiable at any $z \in \mathbb{C}$ since the Cauchy–Riemann equations ($\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$) are violated at every $(x, y) \in \mathbb{C}$. Indeed, $u_2 = x$, $v_2 = -y$ and $\frac{\partial u_2}{\partial x} = 1 \neq -1 = \frac{\partial v_2}{\partial y}$. Of course, f_2 is real differentiable on all of $\mathbb{R}^2$ since both component functions $u_2(x, y) = x$, $v_2(x, y) = -y$ are evidently differentiable.

$f_5 = u_5 + iv_5$ is complex differentiable at the origin but not anywhere else. This follows by the Cauchy–Riemann equations, as $u_5 = x^2$, $v_5 = y^2$ and

$$\frac{\partial u_5}{\partial x} = 2x, \quad \frac{\partial u_5}{\partial y} = 0, \quad \frac{\partial v_5}{\partial x} = 0, \quad \frac{\partial v_5}{\partial y} = 2y.$$

f_5 is real differentiable on all of $\mathbb{R}^2$ since both component functions $u_5(x, y) = x^2$, $v_2(x, y) = y^2$ are evidently differentiable.

Return to Exercise 4.2 on P61

Solution 4.3

Let $f: V \to W$ be a bijection of open sets $V, W \subset \mathbb{C}$. Assume that f is complex differentiable at z_0 and that f^{-1} is complex differentiable at $w_0 = f(z_0)$. Show that $f^{'}(z_0) \neq 0$. (The same result holds for real differentiable maps to show $\det J_f(z_0) \neq 0$.)

Let $g = f^{-1}$ so that $g(f(z)) = z$ for all $z \in V$. By the chain rule, $g \circ f$ is complex differentiable at z_0 with derivative

$$g'(f(z_0))f'(z_0) = (gf)'(z_0) = 1.$$

Hence $f'(z_0) \neq 0$.

Return to Exercise 4.3 on P61

Solution 4.4

Let $f: U \to \mathbb{C}$ be a holomorphic function with domain an open set $U \subset \mathbb{C}$. Suppose also that $f^{'}(z)$ is holomorphic. Write $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$. As we will see, a consequence of f being holomorphic, is that u and v have continuous second order derivatives. Show that:

- ***a.*** $|f^{'}(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$
- ***b. Both u, v satisfy the Laplace equation $\Delta(u) = \Delta(v) = 0$, where the Laplace operator is defined as*** $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.
- ***c. Fix $n \in \mathbb{N}$. Use b. to find a solution to $\Delta(u) = 0$ that satisfies $u(z) = \cos(n\theta)$ for all $|z| = 1$, $z = e^{i\theta}$ on the unit circle.***
- ***d. Does the converse to b. hold?***

Since f is holomorphic, the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{$\star$}$$

hold at every $z \in U$. For a., compute

$$
\begin{aligned}
|f'(z)|^2 &= \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 && \text{by Prop. 4.5} \\
&= \left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial v}{\partial x}\right|^2 \\
&= \left|\frac{\partial v}{\partial y}\right|^2 + \left|\frac{-\partial u}{\partial y}\right|^2 && \text{by } (\star) \\
&= \left|\frac{\partial u}{\partial y}\right|^2 + \left|\frac{\partial v}{\partial y}\right|^2.
\end{aligned}
$$

For b., we take the partial derivative $\frac{\partial}{\partial x}$ of the first equation in $(\star)$ to get

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 v}{\partial x \partial y} \\
&= \frac{\partial}{\partial y}\frac{\partial v}{\partial x} && \text{symmetry of 2nd derivatives} \\
&= -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} && \text{by second equation in } (\star) \\
&= -\frac{\partial^2 u}{\partial y^2}.
\end{aligned}
$$

Similarly, take the partial derivative $\frac{\partial}{\partial x}$ of the second equation in $(\star)$ to get

$$
\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial y^2}.
$$

Note that the symmetry of the 2$^{\text{nd}}$ derivatives is a consequence of continuity of the second order derivatives.

For c., let $f(z) = z^n$. Then $u = \Re(z^n) = \frac{z^n + \overline{z}^n}{2} = r^n \cos(n\theta)$ solves Laplace's equation by part (b). Also, $r = |z| = 1$ implies $u(x, y) = \cos(n\theta)$, as required.

For d., No, the converse fails. Take for example $f(z) = \overline{z}$. Then $f = u + iv$ u

and v satisfy Laplace's equation but f is not holomorphic.

Return to Exercise 4.4 on P61

Solution 4.5

Find subdomains where $\sin, \cos, \sinh, \cosh, \tan, \tanh$ ***are injective. Compute the derivative of an inverse using the chain rule. Deduce an explicit formula for the derivative of an inverse.***

$\sin z = \sin w$ implies $e^{iz} - e^{-iz} = e^{iw} - e^{-iw} = a$.

Set $t = e^{iz}$, and $s = e^{iw}$ we find that t and s are roots of the polynomial:

$$x^2 - xa - 1 = 0.$$

This polynomial has two roots, $x = \{r, \frac{1}{r}\}$.

We conclude that either $s = t$ or $st = 1$.

If $s = t$, then $e^{-y}e^{ix} = e^{-v}e^{iu}$. We conclude that $y = v$ and $x - u = 2k\pi$.

If $st = 1$ then, $e^{-y}e^{ix} = e^{v}e^{-iu}$. We conclude that $v = -y$ and $x + u = (2k+1)\pi$. Where in both cases $k \in \mathbb{Z}$. In particular, $\sin z$ is injective on the domain $\{x + iy : x \in (0, \pi)\}$.

In a similar way $\cos(z)$ is also injective on the domain $\{x + iy : x \in (0, \pi)\}$

We can then deduce that $\sinh(z)$ and $\cosh(z)$ are injective on the domain $\{x + iy : y \in (0, \pi)\}$.

For tanh: $\tanh z = \tanh w$ implies $e^{2(z-w)} = 1$. Writing $z = x + iy$ and $w = u + iv$, we have $2(x - u) = 0$ and $2(y - v) = 0 + 2k\pi$. We conclude that tanh is injective on the interval $\{x + iy : y \in (0, \pi)\}\backslash\{i(\pi/2 + k\pi) : k \in \mathbb{Z}\}$.

We can then deduce that $\tan(z) = \tanh(iz)$ is injective on $\{x+iy : x \in (0,\pi)\}\backslash\{(\pi/2+k\pi) : k \in \mathbb{Z}\}$.

Using the chain rule, and the identities $\cos^2+\sin^2 = 1$, $\cosh^2-\sinh^2 = 1$, we have

$$\arcsin'(z) = \frac{1}{\cos(\arcsin(z))} = \frac{1}{\sqrt{1-z^2}}$$

$$\arccos'(z) = \frac{-1}{\sin(\arccos(z))} = \frac{-1}{\sqrt{1-z^2}}$$

$$\operatorname{arcsinh}'(z) = \frac{1}{\cosh(\operatorname{arcsinh}(z))} = \frac{1}{\sqrt{1+z^2}}$$

$$\operatorname{arccosh}'(z) = \frac{1}{\sinh(\operatorname{arccosh}(z))} = \frac{1}{\sqrt{z^2-1}}$$

Return to Exercise 4.5 on P62

Solution 4.6

Show that the set $\mathcal{O}(U)$ of holomorphic functions on U becomes a ring under the operations of point-wise addition and multiplication:

$$(f+g)(z) = f(z)+g(z), \qquad (fg)(z) = f(z)g(z).$$

Since we already know that the set of all functions $U \to \mathbb{C}$ is a ring, it suffices to show that $\mathcal{O}(U)$ is a subring. As constant functions are entire, $\pm 1 \in \mathcal{O}(U)$. If

f, g are holomorphic U, then by the lecture notes $f + g$ and fg are holomorphic on U as well. Finally, $-f = -1 \cdot f$ is holomorphic on U.

Return to Exercise 4.6 on P62

Solution 4.7

Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function with image $f(\mathbb{C}) \subset \mathbb{R}$. Prove that f is constant.

Writing $f = u + iv$, we have have $v = 0$. Hence by the Cauchy–Riemann equations it follows that $\partial u/\partial x = 0$ and $\partial u/\partial y = 0$. This implies that u is constant: for every fixed $y_0 \in \mathbb{R}$ the real-valued differentiable function $\mathbb{R} \to \mathbb{R}$, $x \mapsto u(x, y_0)$ has an everywhere vanishing derivative and is therefore a constant $u(x, y_0) = c(y_0)$, $\forall x \in \mathbb{R}$, with the constant $c(y_0)$ possibly depending on $y_0 \in \mathbb{R}$. For every fixed $x_0 \in \mathbb{R}$ we similarly have that $u(x_0, y) = d(x_0)$ is a constant of y. Fix $(x_0, y_0) \in \mathbb{C}$. Then

$$u(x, y) = d(x) = u(x, y_0) = c(y_0) = u(x_0, y_0)$$

for all $(x, y) \in \mathbb{C}$, so $u(x, y)$ is constant.

Return to Exercise 4.7 on P62

Solution 4.8

Let U be an open set and assume that U is* path-connected, *meaning that for all $z_0, z_1 \in U$ there exists a continuously differentiable map $\gamma\colon [0, 1] \to U$ such that $\gamma(0) = z_0$, $\gamma(1) = z_1$.

a. ***Let $f\colon U \to \mathbb{C}$ be a holomorphic function such that $f^{'}(z) = 0$ for all $z \in U$. Prove that $f(z)$ is a constant function.***

Hint: ***Consider the real and imaginary parts of $f \circ \gamma$.***

b. Show that the only entire functions $f(z)$ ***satisfying*** $f'(z) = Cf(z)$ ***for a constant*** $C \in \mathbb{C}$ ***are the functions*** $f(z) = De^{Cz}$, $D \in \mathbb{C}$.

c. Find all entire solutions $f(z)$ ***of*** $f''(z) = f(z)$.

Hint: ***Consider*** $g = f + f'$, $h = f - f'$

a. Let $f = u + iv$ and fix $z_0 \in U$. We will prove that $f(z) = f(z_0)$ for all $z \in U$. Pick a continuously differentiable map $\gamma\colon [0,1] \to U$ with $\gamma(0) = z_0$, $\gamma(1) = z$. By Proposition 4.5 (the Cauchy–Riemann equations), u, v are real differentiable on U with vanishing derivatives. Hence $u\circ\gamma$ and $v\circ\gamma$ are real functions $[0,1] \to \mathbb{R}$ with vanishing derivatives which we know are constant from calculus. Hence

$$f(z_0) = u(\gamma(0)) + iv(\gamma(0)) = u(\gamma(1)) + iv(\gamma(1)) = f(z).$$

b. Let $g(z) = f(z)e^{-Cz}$. Then $g'(z) = f'(z)e^{-Cz} + f(z)(-Ce^{-Cz}) = 0$. Hence $f(z)e^{-Cz} = g(z) = D$ is constant.

c. Since $g' = g$, $h' = -h$ we know from (b) that $g(z) = Ce^z$, $h(z) = De^{-z}$. Hence $f(z) = \frac{1}{2}(g(z) + h(z)) = \frac{Ce^z + De^{-z}}{2}$ is the general solution. Taking $(C, D) \in \{(1,1), (1,-1)\}$ gives $\cosh(z)$, $\sinh(z)$ and we could alternatively say that the general solution is $A\cosh(z) + B\sinh(z)$.

Return to Exercise 4.8 on P63

A.5 Chapter 5 solutions

Solution 5.1

Find the radius of convergence for the following power series centered at the origin.

i. $\sum_{n=0}^{\infty} \frac{z^n}{n^3}$,
ii. $\sum_{n=0}^{\infty} z^{3n}$,
iii. $\sum_{n=0}^{\infty} \frac{z^n}{n^n}$.

Hint: ***Recall the ratio and root tests for series of complex numbers***

i. We can apply the *ratio test* to determine the convergence of a series of complex numbers $\sum_{n=0}^{\infty} a_n$ provided $a_n \neq 0$ for n sufficiently large. Let $q = \limsup \left|\frac{a_{n+1}}{a_n}\right|$. If $q < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely. If $q > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges. Apply this to $a_n = \frac{z^n}{n^3}$:

$$\left|\frac{z^{n+1}/(n+1)^3}{z^n/n^3}\right| = |z|\left(\frac{n}{n+1}\right)^3 \longrightarrow |z| \quad \text{as } n \longrightarrow \infty.$$

Hence, $q = |z|$ and (i) converges for $|z| < 1$ and diverges for $|z| > 1$, so $\rho = 1$.

ii. By definition, the radius of convergence ρ is the supremum of all real numbers $r \geqslant 0$ such that the sequence $(a_n r^n)_{n\in\mathbb{N}}$ is bounded. Here $a_n = 1$ if 3 divides n and $a_n = 0$ otherwise. This is equivalent to the boundedness of the sequence $(r^{3n})_{n\in\mathbb{N}}$ which holds precisely for $r \leqslant 1$. Thus $\rho = 1$.

iii. Recall the *root test* to determine the convergence of a series of complex numbers $\sum_{n=0}^{\infty} a_n$. Let $q = \limsup \sqrt[n]{|a_n|}$. If $q < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely. If $q > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges. Apply this to $a_n = \frac{z^n}{n^n}$:

$$\sqrt[n]{|a_n|} = |z|/n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Hence iii. converges for all $z \in \mathbb{C}$, and so $\rho = +\infty$.

Return to Exercise 5.1 on P75

Solution 5.2

Treating e^z, $\sin(z)$, $\cos(z)$ as formal power series with their usual Taylor expansion, find the terms of order ≤ 3 of the following power series:

i. $e^z \sin(z)$,

ii. $\sin(z)\cos(z)$,

iii. $1/\cos(z)$

Using the definition of the product of formal power series,

$$\begin{aligned} e^z \sin(z) &= z + z^2 + \frac{z^3}{3} + O(z^4) \\ \sin(z)\cos(z) &= z - \frac{2z^3}{3} + O(z^4). \end{aligned}$$

To verify

$$1/\cos(z) = 1 + \frac{z^2}{2} + O(z^4)$$

we begin with

$$\cos(z) = 1 - z^2/2 + O(z^4)$$

and then using the geometric series,

$$1/\cos(z) = \frac{1}{1 - z^2/2} + O(z^4) = 1 + z^2/2 + O(z^4).$$

Return to Exercise 5.2 on P75

Solution 5.3

Let $P = \sum_{n=0}^{\infty} a_n z^n$, $Q = \sum_{n=0}^{\infty} b_n z^n$ ***be power series with positive radii of convergence*** $\rho_P, \rho_Q > 0$. ***Show that:***

a. $P + Q = \sum_{n=0}^{\infty}(a_n + b_n)z^n$ ***has radius of convergence*** $\rho \geqslant \min(\rho_P, \rho_Q)$.
b. $PQ = \sum_{n=0}^{\infty} \left(\sum_{i+j=n} a_i b_j\right) z^n$ ***has radius of convergence*** $\rho \geqslant \min(\rho_P, \rho_Q)$.

a. Let $z \in D_{\min(\rho_P,\rho_Q)}(0) = D_{\rho_P}(0) \cap D_{\rho_Q}(0)$. Then the series of complex numbers $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ converge, hence the series $\sum_{n=0}^{\infty}(a_n + b_n)z^n$ converges as well. It follows that $D_{\min(\rho_P,\rho_Q)}(0) \subset D(P+Q, 0)$ is in the domain of $P+Q$, hence $\min(\rho_P, \rho_Q) \leqslant \rho$.

b. Recall that the Cauchy product of series of non-negative numbers converges: If $a_n, b_n \geqslant 0$ and $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ converge, then $\sum_{n=0}^{\infty} \left(\sum_{i+j=n} a_i b_j\right)$ converges. To see this, let $A_N = \sum_{n=0}^{N} a_n$, $B_N = \sum_{n=0}^{N} b_n$ be the partial sums. Then the partial sums of the Cauchy product
$$\sum_{n=0}^{N} \sum_{i+j=n} a_i b_j \leqslant A_N B_N$$
are a bounded sequence (since $A_N B_N$ converges) that is increasing, hence convergent by the monotone convergence theorem of calculus.

Given this preliminary, the same argument as for a. proves $\min(\rho_P, \rho_Q) \leqslant \rho$.

Return to Exercise 5.3 on P75

Solution 5.4

Find a solution to the non-linear differential equation

$$f'(z) + f(z)^2 = 0, \qquad f(0) = 1$$

on a disk centered at $z_0 = 0$ by making the ansatz $f(z) = \sum_{n=0}^{\infty} a_n z^n$, inductively determining the coefficients a_n, and finding the radius of convergence.

We substitute the power series into the differential equation. By Theorem 5.2, we have

$$f'(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

and for the product of power series (see Equation 5.3) we have

$$f(z)^2 = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i a_{n-i} \right) z^n.$$

Hence

$$\sum_{n=0}^{\infty} \left[(n+1) a_{n+1} + \sum_{i=0}^{n} a_i a_{n-i} \right] z^n = 0.$$

The identity theorem (Corollary 5.2) now implies

$$a_{n+1} = \frac{-1}{n+1} \sum_{i=0}^{n} a_i a_{n-i}, \qquad \forall n \geqslant 0.$$

By assumption, $a_0 = f(0) = 1$. We solve by recursion,

$$\begin{aligned} a_0 &= 1 \\ a_1 &= -a_0^2 = -1 \\ a_2 &= \frac{-1}{2}(a_0 a_1 + a_1 a_0) = 1 \\ a_3 &= \frac{-1}{3}(a_0 a_2 + a_1^2 + a_2 a_0) = -1 \\ &\vdots \end{aligned}$$

This leads us to the conjecture $a_n = (-1)^n$ for all $n \in \mathbb{N}$. This holds in the base case $n = 0$ and, assuming this result for all integers up to n, we have

$$a_{n+1} = \frac{-1}{n+1} \sum_{i=0}^{n} (-1)^i (-1)^{n-i} = \frac{-1}{n+1} \sum_{i=0}^{n} (-1)^n = (-1)^{n+1},$$

which proves the inductive step. In summary,

$$f(z) = \sum_{n=0}^{\infty} (-1)^n z^n,$$

which we recognize as a geometric series with $f(z) = 1/(1+z)$, whose radius of convergence we know is $\rho = 1$.

Return to Exercise 5.4 on P76

Solution 5.5

Let $P(z) = \sum_{n=0}^{\infty} a_n z^n$ ***be a power series centered at*** $z_0 = 0$ ***and assume that the radius of convergence*** $\rho > 0$ ***is positive. Suppose that*** $P(z) \in \mathbb{R}$ ***for all*** $z \in D_\rho(0) \cap \mathbb{R}$***. Prove that all coefficients*** a_n, $n \in \mathbb{N}$, ***must be real numbers. Deduce that*** $\overline{P(z)} = P(\overline{z})$ ***for all*** $z \in D_\rho(0)$.

For all $z \in D_\rho(0) \cap \mathbb{R}$ we have by assumption the middle equality in

$$\sum_{n=0}^{\infty} a_n z^n = P(z) = \frac{P(z) + \overline{P(z)}}{2} = \sum_{n=0}^{\infty} \frac{a_n + \overline{a_n}}{2} z^n.$$

Pick $n_0 \in \mathbb{N}$ with $1/n_0 < \rho$. We can apply the identity theorem (Corollary 5.2) with $w_n = 1/(n + n_0)$, P as above, and

$$Q = \sum_{n=0}^{\infty} \frac{a_n + \overline{a_n}}{2} z^n$$

to conclude that $a_n = \frac{a_n + \overline{a_n}}{2}$ for all n. In other words, $a_n \in \mathbb{R}$.

We may then deduce that

$$\overline{P(z)} = \overline{\sum_{n=0}^{\infty} a_n z^n} \overset{(\S)}{=} \sum_{n=0}^{\infty} \overline{a_n z^n} = \sum_{n=0}^{\infty} a_n \overline{z}^n = P(\overline{z}).$$

Here $(\S)$ is easy to check from the definition of convergence of partial sums, as

$$\overline{\sum_{n=0}^{N} a_n z^n} = \sum_{n=0}^{N} \overline{a_n z^n}$$

holds for finite sums.

Return to Exercise 5.5 on P76

Solution 5.6

Prove that the ring of formal power series $\mathbb{C}[\![T]\!]$ ***is an* integral domain. *In other words, show that*** $PQ = 0 \implies P = 0$ ***or*** $Q = 0$.

Suppose that $Q \neq 0$ and $PQ = 0$. We must show that $P = 0$. Write $Q = z^n R$ where R has non-vanishing constant term. According to Proposition 5.4(b), R is invertible, that is, we can find $S \in \mathbb{C}[\![T]\!]$ such that

$$RS = 1, \qquad SR = 1.$$

Then

$$0 = PQS = Pz^n RS = Pz^n.$$

This implies $P = 0$ (the coefficients of P and Pz^n are just shifted by n).

Return to Exercise 5.6 on P76

Solution 5.7

The* binomial coefficient *of a complex number* $\alpha \neq 0$ *and* $k \in \mathbb{N}$ *is defined as

$$\binom{\alpha}{k} = \begin{cases} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \textit{if } k > 0, \\ 1 & \textit{if } k = 0. \end{cases}$$

The* binomial series *is the formal power series

$$B_\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} T^k.$$

- ***a.*** ***Determine the radius of convergence of*** B_α.
- ***b.*** ***Prove that if*** $\alpha \in \mathbb{N}$, ***then*** $B_\alpha(z) = (z+1)^\alpha$.
- ***c.*** ***Show the* generalized Vandermonde *identity for*** $\alpha, \beta, \alpha+\beta \in \mathbb{C}^\times$,

$$\sum_{k=0}^{n} \binom{\alpha}{k}\binom{\beta}{n-k} = \binom{\alpha+\beta}{n}. \tag{$\star$}$$

- ***d.*** ***Prove that for*** $\alpha = 1/k$ ***the complex function*** $B_\alpha(z)$ ***satisfies*** $B_\alpha(z)^k = z+1$ ***on its domain. Hence*** $B_\alpha(z)$ ***is a*** k***-th root of the function*** $z+1$.

a. For $\alpha \in \mathbb{C} \setminus \mathbb{N}$ we have

$$\left| \frac{\binom{\alpha}{k+1} z^{k+1}}{\binom{\alpha}{k} z^k} \right| = \left| z \frac{\alpha - k}{k+1} \right| \longrightarrow |z| \quad \text{as} \quad k \to \infty.$$

Therefore, the ratio test implies $\rho = 1$ when $\alpha \notin \mathbb{N}$. When $\alpha \in \mathbb{N}$ only finitely many binomial coefficients $\binom{\alpha}{k}$ are non-zero, so B_α is a polynomial and $\rho = +\infty$ and indeed by the binomial theorem we have

$$B_\alpha(z) = (z+1)^\alpha, \qquad \forall \alpha \in \mathbb{N}.$$

b. As $\binom{\alpha}{k} = 0$ if $k > \alpha$, this is just the binomial theorem:

$$\sum_{k=0}^{\alpha} \binom{\alpha}{k} z^k = (z+1)^\alpha.$$

c. We can deduce from b. that ($\star$) holds for all $\alpha, \beta \in \mathbb{N}$, since by the definition of the product of formal power series, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{\alpha}{k} \binom{\beta}{n-k} \right) z^n &= \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k \cdot \sum_{\ell=0}^{\infty} \binom{\beta}{\ell} z^\ell \\ &= (1+z)^\alpha (1+z)^\beta = (1+z)^{\alpha+\beta} \\ &= \sum_{n=0}^{\infty} \binom{\alpha+\beta}{n} z^n. \end{aligned}$$

Fix $\beta \in \mathbb{N}$. Then both sides of ($\star$) are polynomials in α, which agree on $\mathbb{N}$. Since a non-zero polynomial has only finitely many roots, this implies that both sides of ($\star$) agree for all $\alpha \in \mathbb{C}$. Now fix $\alpha \in \mathbb{C}$. Then both sides of ($\star$) are polynomials in β, which have just shown agree for all $\beta \in \mathbb{N}$. As before, this implies that ($\star$) holds for all $\beta \in \mathbb{C}$. This proves the general case.

d. From c. and the definition of the product of power series, we have

$$B_\alpha(z) B_\beta(z) = B_{\alpha+\beta}(z).$$

By induction,

$$B_\alpha(z)^k = B_{k\alpha}(z) = B_1(z) = z+1.$$

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Solution 5.8

For $a > 0$ and $z \in \mathbb{C}$ define $a^z = \exp(\log(a)z)$. Show that:

- ***a.*** $a^z b^z = (ab)^z$ ***for all*** $a, b > 0,\ z \in \mathbb{C}$
- ***b.*** $a^z a^w = a^{z+w}$ ***for all*** $a > 0,\ z, w \in \mathbb{C}$
- ***c.*** $|a^z| = a^{\Re(z)}$ ***for all*** $a > 0,\ z \in \mathbb{C}$

The **Riemann ζ-function** ***is defined as***

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad \textit{for } \Re(z) > 1. \tag{$\star$}$$

- ***d. Prove that the series ($\star$) converges absolutely for all $z \in \mathbb{C}$ with $\Re(z) > 1$ and uniformly on every subset $S_\delta = \{z \in \mathbb{C} \mid \Re(z) > 1 + \delta\}$ with $\delta > 0$.***

a. We have:

$$\begin{aligned} a^z b^z &= \exp(\log(a)z)\exp(\log(b)z) \\ &= \exp([\log(a) + \log(b)]z) \\ &= \exp(\log(ab)z) = (ab)^z \end{aligned}$$

b.

$$a^z a^w = \exp(\log(a)z)\exp(\log(a)w) = \exp(\log(a)(z+w)) = a^{z+w}$$

c. Notice first that $|\exp(z)|^2 = \exp(z)\overline{\exp(z)} = \exp(z)\exp(\overline{z}) = \exp(z + \overline{z}) = \exp(2\Re z)$. Hence $|\exp(z)| = \exp(2\Re z)^{1/2} = \exp(\Re z)$. Using this,

$$|a^z| = |\exp(\log(a)z)| = \exp(\Re(\log(a)z)) = \exp(\log(a)\Re(z)) = a^{\Re z}.$$

d. The series of absolute values $\sum_{n=1}^{\infty} |1/n^z| = \sum_{n=1}^{\infty} 1/n^x$ converges for $x > 1 + \delta$ by the comparison test: We have $1/n^x \leqslant 1/n^{1+\delta}$ and the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} \leqslant \int_1^{\infty} \frac{dt}{t^{1+\delta}} = \lim_{R\to\infty} \int_1^{R} \frac{dt}{t^{1+\delta}} = \lim_{R\to\infty} \left(1 - \frac{1}{R^{\delta}}\right) = 1 \quad (\dagger)$$

converges by the integral comparison test. It follows that $(\star)$ converges uniformly. Moreover, for fixed $\delta > 0$ the series $(\dagger)$ that we compare with is independent of $z \in S_\delta$. Thus the comparison test for series of functions applies and implies uniform convergence on each S_δ, $\delta > 0$.

Return to Exercise 5.8 on P77

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