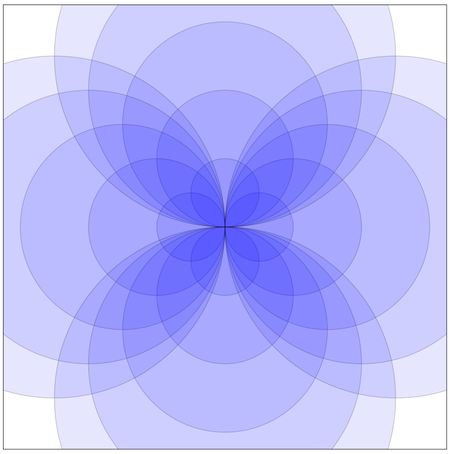
MX4557 Complex Analysis

F. Olukoya



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Module Handbook

Students are asked to make themselves familiar with the information on key institutional policies which have been made available within MyAberdeen or on the university website.

These policies are relevant to all students and will be useful to you throughout your studies. They contain important information and address issues such as what to do if you are absent, how to raise an appeal or a complaint and how seriously the University takes your feedback.

These institutional policies should be read in conjunction with this Course Description Form, in which School-specific policies are detailed. Further information can be found on the University's Infohub webpage or by visiting the Infohub.

Welcome to *MX4557 Complex Analysis*. The aim of the module is to further develop understanding of the concepts, techniques, and tools of calculus. Calculus is the mathematical study of variation. This course emphasises differential and integral calculus in one variable, and sequences and series of functions.

Intended Learning Outcomes (ILO's)

By the end of the course, you should be able

i. to state and illustrate the definitions of the concepts introduced in the course,

ii. to state the theorems of the course, to explain their significance, and to give examples to indicate the role of the hypotheses,

iii. to demonstrate knowledge and understanding of proof techniques used in the course,

iv. to use the methods and results of the course to solve problems at levels similar to those seen in the course.

In particular, you should:

- Be able to perform routine calculations with complex functions.
- Understand the concept of analyticity and be familiar with the Cauchy-Riemann equations.
- Know the necessary background and development of an elementary version of Cauchy's theorem.
- Know the important consequences of Cauchy's theorem: Cauchy's integral formulae, Liouville's Theorem and Taylor Series.
- Know about Laurent Series and the classification of isolated singularities.
- Know the Cauchy residue theorem and some of its applications.

Lecture notes and recommended reading

A full-set of gapped notes is available on MyAberdeen[1]. The gaps will be revealed in due course as the term progress.

Most library books dealing with elementary complex analysis are useful for this course.

[1]These are based on Ben Martin's excellent set of notes

Possible codes in QML are 515.30-01 (Functions of a Complex Variable), 515-01 (Mathematical Analysis) and 620.001 51 (Engineering Mathematics).

For specific recommendations, see the Leganto Reading on MyAberdeen (click Books & Tools). Further references are mentioned in the lecture notes.

Logistics

Lectures

This course is taught in person in the Fraser Noble Building, FN156.
Lectures take place on Tuesdays 14:00–15:00 and Thursdays 13:00–14:00, weeks 27–34, 38-40, with the first lecture being on January 28, 2024.

Tutorials

Tutorials are on Fridays 9:00–10:00 in the Fraser Noble Building, FN156, and will start during the second week of lectures with the first tutorial being on February 4, 2024. Students should work through the current problem sheet (Table Table 1 indicates which problem sheet a tutorial covers.) and attempt as many exercises as possible.

Timetable

Details of all sessions (lectures and tutorials) will be available on your timetable. Please make sure that you have the correct day, time and room for each session. You should check this regularly as there are occasionally changes, particularly in the first couple of weeks of the semester.

Table 1 displays a detailed schedule for the semester.

Week	Day	Chapter	Material
27	T	1	Complex numbers and functions
	Th	1	Complex numbers and functions
	F		
28	T	2	Complex-valued functions
	Th	2	Complex-valued functions
	F		Tutorial 1: Problem Sheet 1
29	T	3	Topology of the complex plane
	Th	3	Topology of the complex plane ***(Assignment)***
	F		Tutorial 2: Problem Sheet 2
30	T	4	Holomorphic functions
	Th	4	Holomorphic functions
	F		Tutorial 3: Problem Sheet 3 ***(Assignment due)***
31	T	5	Power series
	Th	5	Power series
	F		Tutorial 4: Problem Sheet 4
32	T	6	Countour integrals
	Th	6	Countour integrals
	F		Tutorial 5: Problem Sheet 5
33	T	7	Cauchy's Theorem
	Th	7	Cauchy's Theorem
	F		Tutorial 6: Problem Sheet 6
34	T		Applications of Cauchy's Theorem
	Th	8	Applications of Cauchy's Theorem ***(Assignment)***
	F	8	Tutorial 7: Problem Sheet 7
35-37			***Spring Break (Assignment due)***
	T	9	Laurent series and singularities

Week	Day	Chapter	Material
38	Th	9	Laurent series and singularities
	F		Tutorial 8: Problem Sheet 8
	T	10	Residue Theorem
39	Th	10	Residue Theorem
	F		Tutorial 9: Problem Sheet 9
	T	11	Applications to integrals
40	Th	11	Applications to integrals
	F		Tutorial 10: Problem Sheet 10
41			***Exams***

Table 1: Detailed Schedule

Attendance requirements

You should attend classes regularly and do the work of the course. If you fail to do this, you may be asked to discuss your reasons with the Course Organiser and possibly be reported to the Registry as an unauthorised withdrawal from the course.
https://www.abdn.ac.uk/students/academic-life/monitoring-and-student-progress.php

Private Study

In addition to lectures and tutorials, you should aim to spend at least **six** hours a week working on the course. During this time you should

(a) work through the example sheets,
(b) study your lecture notes and related texts,
(c) prepare in-course assignments.

Lecturers

This semester Dr. Feyisayo (Shayo) Olukoya will be the lecturer on the module. As mentioned above you can reach me by email at feyisayo.olukoya1@abdn.ac.uk; you should also feel free to arrange an in-person meeting my office is FN163 in the Fraser Noble; meeting virtually over teams is also an option.

MyAberdeen

All resources for the module (lecture notes, problem sheets, solutions e.t.c) will be made available on MyAberdeen at the appropriate time.

Occasionally, important information will be distributed to your university email account.

Assessment

Continuous Assessment

There are two Continuous Assessments and a final exam for this course.

- Assignment 1 (15%): Released Thursday 13 February. The deadline is Friday 21 February at 3pm.
- Assignment 2(15%): Released Thursday 20 March. The deadline is Friday 28 March at 3pm.

Assignments will be released on MyAberdeen and you should submit your solutions as a pdf file on MyAberdeen by the due date. Only in exceptional circumstances will work handed in after the due date be accepted for formal assessment.

Individual work

Your submission for Continuous Assessment must be your own individual work unless stated otherwise. Further information and a reference to the University Code of Conduct on Student Discipline can be found in the Information Booklet.

You are encouraged to discuss exercises for this course with other students, for example in small groups. You may shop around for ideas but it is expected that you possess and use your own critical faculties, to write up your own solutions. You must not copy sentences from fellow students. If you do not understand the solution of another student, do not attempt to use it as your own solution.

Formative assessment

At the end of each chapter in the notes, you will find problem sheets with questions addressing the content covered in that chapter. Although these sheets do not contribute to the continuous assessment component, you are ***strongly encouraged*** to attempt them as they are designed to consolidate your understanding and enhance your problem-solving skills. Full solutions are provided and will appear after the sheet or sheets have been covered in example classes.

Final Exam

The final exam will be during Exam Week and comprises 70% of the module mark. It is an unseen, closed-book examination. The examination will require you to state definitions, state (and possibly prove) results, and apply these to solving problems. You should be able to state every definition and result in the module unless they are marked in the lecture notes as non-examinable.

Resits

Resit In the summer there will be a resit examination consisting of a single two-hour exam paper. The CGS mark awarded for the resit will be the maximum of

(a) the mark obtained by using the resit examination result combined with the carried-forward CA mark as in the original diet and

(b) the mark based on the resit examination performance alone.

Calculators in exams

Calculators will **not** be allowed in the main examination or the resit examination.

Student Support

For advice on academic and non-academic issue (resonable adjutsments, financial, international, personal or health matters) please contact Student Services.

You can find the University's education policy for students by following this link.

If you have any problems with the course—mathematical or organisational—please contact me, or your class representative, or Mark Grant (who is in charge of undergraduate teaching).

Report a bug

If you encounter any issues with these notes or would like to leave feedback regarding lectures, tutorials, assessments, etc., please feel free to report a bug. I will review submissions periodically, so do share your thoughts.

Part I

Complex numbers and functions

Chapter 1

Geometry of the Complex plane

This section is a brief reminder of Sections 3 and 4 of MA1006 Algebra.

Definition 1.1. The *complex numbers* $\mathbb{C}$ are the set of all pairs $z = (x, y) \in \mathbb{R}^2$ of real numbers with the addition

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) \tag{1.1}$$

and the multiplication

$$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1), \tag{1.2}$$

where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. We call $x = \mathfrak{R}(z)$ the *real part* and $y = \mathfrak{I}(z)$ the *imaginary part* of z.

From Section 3.8 in MA1006 Algebra we recall:

Proposition 1.1. *The complex numbers are a field.*

Remark 1.1.

Of course, $\mathbb{C}$ is a one-dimensional vector space over itself. Restricting the scalar multiplication to $\mathbb{R}$ makes $\mathbb{C}$ into a vector space over $\mathbb{R}$, isomorphic to $\mathbb{R}^2$, of dimension two with standard basis $1, i \in \mathbb{C}$.

Proposition 1.2.

a. *In the standard basis, every* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{M}_{2\times 2}(\mathbb{R})$ *corresponds uniquely to an* $\mathbb{R}$*-linear map* $T_A \colon \mathbb{C} \to \mathbb{C}$, *namely*

$$T_A(x + iy) = (ax + by) + i(cx + dy). \tag{1.3}$$

b. T_A *is* $\mathbb{C}$*-linear* $\iff$ $a = d$ *and* $b = -c$. *In this case,*

$$T_A(z) = \alpha \cdot z, \qquad \alpha = a + ic. \tag{1.4}$$

Proof.

$\square$

Definition 1.2. The *conjugate* of $z \in \mathbb{C}$ is the complex number

$$\overline{z} = (x, -y),$$

and the *modulus* (also called *absolute value*) is

$$|z| = \sqrt{x^2 + y^2} \geqslant 0.$$

Proposition 1.3. *The following formulas hold for* $z, w \in \mathbb{C}$:

$$\overline{z \cdot w} = \overline{z} \cdot \overline{w} \qquad \overline{z + w} = \overline{z} + \overline{w} \tag{1.5}$$

$$\overline{\overline{z}} = z \qquad \bar{i} = -i,\ \overline{1} = 1 \tag{1.6}$$

$$z \cdot \overline{z} = |z|^2 \qquad |z \cdot w| = |z| \cdot |w| \tag{1.7}$$

$$\Re(z) = \frac{z + \overline{z}}{2} \qquad \Im(z) = \frac{z - \overline{z}}{2i} \tag{1.8}$$

$$z^{-1} = \frac{\overline{z}}{|z|^2} \quad \textit{if } z \neq 0 \tag{1.9}$$

Proposition 1.4. *The following inequalities hold for* $z, w \in \mathbb{C}$:

$$|z + w| \leqslant |z| + |w| \qquad |z - w| \geqslant \big||z| - |w|\big| \tag{1.10}$$

$$|\Re(z)| \leqslant |z| \qquad |\Im(z)| \leqslant |z| \tag{1.11}$$

Proposition 1.5. *For every non-zero complex number $z = (x, y)$ there is an* ***argument*** *$\theta \in \mathbb{R}$ and a* ***radius*** *$r = |z| > 0$ such* ***that***

$$x = r\cos(\theta), y = r\sin(\theta). \tag{1.12}$$

This representation is unique up to replacing θ by $\theta + 2\pi k$ for any $k \in \mathbb{Z}$.

Proof. (omitted)

Since $x^2 + y^2 = r^2(\cos(\theta)^2 + \sin(\theta)^2) = r^2$, the radius must be $r = |z|$. Define the complex number $w = r^{-1}z$ and write $w = u + iv$ for its real and imaginary parts.

We will prove the existence of $\theta \in \mathbb{R}$ with $u = \cos(\theta)$, $v = \sin(\theta)$. This also proves the existence of a representation Equation 1.12, by multiplying by r. Since $u^2 + v^2 = |w|^2 = r^{-2}|z|^2 = 1$, we know $|u| \leqslant 1$, $|v| \leqslant 1$. Recall that $\cos\colon [0, \pi] \to [-1, 1]$ and $\sin\colon [-\pi/2, \pi/2] \to [-1, 1]$ are bijections. Hence

$$\begin{aligned} u &= \cos(\alpha) && \text{for some } \alpha \in [0, \pi], \\ v &= \sin(\beta) && \text{for some } \beta \in [-\pi/2, \pi/2]. \end{aligned}$$

As $\sin(\beta)^2 = v^2 = 1 - u^2 = 1 - \cos(\alpha)^2 = \sin(\alpha)^2$, we have $\sin(\alpha) = \pm\sin(\beta) = \sin(\pm\beta)$. To produce the correct θ, we distinguish two cases.

Case 1 $\alpha \in [0, \pi/2]$. Then $\alpha = \pm\beta$ by the injectivity of the sine function on the interval $[-\pi/2, \pi/2]$. Setting $\theta = \pm\alpha = \beta$, we find that $u = \cos(\theta)$ and $v = \sin(\theta)$, as required.

Case 2 $\alpha \in [\pi/2, \pi]$. Then $\pi - \alpha, \beta \in [-\pi/2, \pi/2]$ and $\sin(\pi - \alpha) = \sin(\alpha) = \pm\sin(\beta)$, so $\pi - \alpha = \pm\beta$ by injectivity. Setting $\theta = \pm\alpha = \pm\pi - \beta$, we find $u = \cos(\theta), v = \sin(\theta)$, using trigonometric identities.

This completes the existence part of the proof. For uniqueness, we already know that $r = |z| > 0$ is unique, so it remains to consider

$$x = r\cos(\theta_1) = r\cos(\theta_2), \qquad y = r\sin(\theta_1) = r\sin(\theta_2).$$

To translate the situation into an interval that we understand, pick $k_1, k_2 \in \mathbb{Z}$ so that $\theta_1 + 2\pi k_1, \theta_2 + 2\pi k_2 \in [-\pi, \pi)$. Then

$$\cos(\theta_1 + 2\pi k_1) = \cos(\theta_1) = \cos(\theta_2) = \cos(\theta_2 + 2\pi k_2).$$

Using the injectivity of the cosine function and considering cases as above, we find that $\theta_1 + 2\pi k_1 = \pm(\theta_2 + 2\pi k_2)$. If the sign is '$+$' we get $\theta_1 - \theta_2 = 2\pi(k_2 - k_1)$ and we are done, so suppose $\theta_1 + 2\pi k_1 = -(\theta_2 + 2\pi k_2)$. Then

$$\sin(\theta_1) = \sin(\theta_2) = \sin(\theta_2 + 2\pi k_1) = -\sin(\theta_1 + 2\pi k_1) = -\sin(\theta_1)$$

implies $\sin(\theta_1) = 0$. Therefore θ_1 is a multiple of 2π, which implies that $\theta_1 + 2\pi k_1 = -(\theta_2 + 2\pi k_2) = 0$ since these numbers were chosen in $[-\pi, \pi)$ and we have $2\pi\mathbb{Z} \cap [-\pi, \pi) = \{0\}$. Hence $\theta_1 - \theta_2 = 2\pi(k_2 - k_1)$. $\square$

To get around the non-uniqueness of the argument in polar coordinates, we restrict θ to lie in a half-open interval of length 2π. Here is the most common convention.

Definition 1.3. The **principal argument** of a non-zero $z \in \mathbb{C}$ is the unique $\theta \in (-\pi, \pi]$ such that Equation 1.12 holds, and we write $\arg(z) = \theta$.

Definition 1.4. The value of the **exponential function** at the complex number

$z = x + i\theta$, where $x, \theta \in \mathbb{R}$, is defined as

$$e^{x+i\theta} = e^x\big(\cos(\theta) + i\sin(\theta)\big). \tag{1.13}$$

Proposition 1.6. $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$ *for all* $z_1, z_2 \in \mathbb{C}$. *Moreover, we have* $(e^z)^n = e^{nz}$ *for all* $z \in \mathbb{C}$, $n \in \mathbb{Z}$.

Proof.

□

The polar form can be applied to the construction of n^{th} roots.

Proposition 1.7. *Every complex number $z \neq 0$ has an n^{th} root w satisfying $w^n = z$. If w is an n^{th} root of z, the set of all n^{th} roots of z is*

$$\{w, \zeta_n \cdot w, \zeta_n^2 \cdot w, \dots, \zeta_n^{n-1} \cdot w\}.$$

Proof.

Questions for further discussion

- The complex numbers are obtained by 'adjoining' a symbol i with $i^2 = -1$. If instead we would have adjoined a different symbol ϵ with $\epsilon^2 = -1$, would the set of elements $x + \epsilon y$ still define a field?
- The real numbers have a total order '$\leqslant$'. Why doesn't it make sense to extend this definition to the complex numbers?
- Describe geometrically the set $R_n = \{1, \zeta_n, \dots, (\zeta_n)^{n-1}\}$ of n^{th} roots of unity. Find a connection between R_n and the cyclic group $C_n = \{\overline{0}, \dots, \overline{n-1}\}$ of integers modulo n from MX3020 Group Theory.

1.1 Exercises

Note

This problem sheet is intended as a recap and contains more problems than can be discussed during the tutorials.

Exercise 1.1

Verify

$$z = x + iy$$

and

$$i^2 = -1$$

straight from the definition Equation 1.2.

Exercise 1.2

How many real solutions x does $x^2 + 1 = 0$ have? Show that the polynomial equation $z^2 + 1 = 0$ has exactly two solutions $z \in \mathbb{C}$.

Exercise 1.3

Give examples of complex numbers $z, w \neq 0$ such that $z^2 + w^2 = 0$.

Exercise 1.4

Sketch the position of the complex numbers $i, 1+i, \frac{3+2i}{4}$ in the plane.

Exercise 1.5

Express the following complex numbers z in the form $x + iy$ with $x, y \in \mathbb{R}$.

$$(1+i)^{20},\ (5+3i)(1+2i),\ (1-i)(2+3i),\ (1-i)i(1+i),\ \frac{2+i}{1-i}$$

Exercise 1.6

Express the following complex numbers z in the form $x + iy$ with $x, y \in \mathbb{R}$.

$$1/i,\ \frac{1}{1+i},\ \frac{3+i}{3-i}$$

Exercise 1.7

Find the modulus and the conjugate of the following complex numbers.

$$2+i,\ i,\ 5-3i,\ \frac{1+i}{2+i}$$

Exercise 1.8

Describe the sets $A = \{z \in \mathbb{C} \mid \Im(z) > 0\}$, $B = \{z \in \mathbb{C} \mid \Re(z) \leqslant 1\}$, $C = \{z \in \mathbb{C} \mid \Re((1+i)z) = 0\}$, and $A \cap B$ geometrically.

Exercise 1.9

Describe the set $D = \{z \in \mathbb{C} \mid z \cdot \overline{z} = 1\}$ geometrically.

Hint: Write $z = re^{i\theta}$ in polar form.

Exercise 1.10

Draw all nine sets described by the following conditions on the complex number z.

$$\begin{array}{ccc} |z| = 1, & |z| < 1, & 1 < |z| < 2, \\ |1+z| > 1, & |2-z| < 2, & 3 < |z+i| < 4, \\ |z-1| < |z+1|, & |z| = |z+1|, & |z-1| = |z+i|. \end{array}$$

Exercise 1.11

Let $S = \{x + iy \in \mathbb{C} \mid 0 \leqslant x, y \leqslant 1\}$. Draw S and the sets

$$\begin{array}{cc} A = \{2z \mid z \in S\}, & B = \{\overline{z} \mid z \in S\}, \\ C = \{-z \mid z \in S\}, & D = \{z^2 \mid z \in S\}. \end{array}$$

Exercise 1.12

Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disk. Draw the sets

$$A = \{2z \mid z \in D\}, \quad B = \{z^2 \mid z \in D\}, \quad C = \{|z| \mid z \in D\}.$$

Exercise 1.13

Show that $i = e^{i\pi/2}$ and $-1 = e^{i\pi}$.

Exercise 1.14

Express the following complex numbers z in the form $x + iy$ with $x, y \in \mathbb{R}$.

$$e^{i\pi/4},\ e^{i\pi},\ e^{i\frac{2\pi}{3}}$$

Exercise 1.15

Write each of the following complex numbers in polar form $re^{i\theta}$ with $r > 0$ and $-\pi < \theta \leqslant \pi$.

$$i, \quad -1, \quad -i, \quad 1+i, \quad 1-i, \quad i-1, \quad \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

Draw each of these numbers in the complex plane.

Exercise 1.16

Calculate i^{2021} and $(1+i)^{20}$.

Exercise 1.17

Solve the equation $(1-i)^n - 2075 = 2021$ and find $n \in \mathbb{N}$.

Exercise 1.18

Prove that for $z \in \mathbb{R}$ we have

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Use these equations to extend the definition of the functions $\cos(z), \sin(z)$ to complex arguments $z \in \mathbb{C}$. Find $z \in \mathbb{C}$ with $\sin(z) = 2$.

Hint: Put $w = e^{iz}$ and reduce to a quadratic equation.

Exercise 1.19

Prove the following statements.

a. $\overline{zw} = \overline{z} \cdot \overline{w}$ for all $z, w \in \mathbb{C}$

b. $\overline{z_1 z_2 \cdots z_n} = \overline{z}_1 \overline{z}_2 \cdots \overline{z}_n$ for all $z_1, z_2, \ldots, z_n \in \mathbb{C}$ (use induction)

c. $\overline{(z^n)} = (\overline{z})^n$ for all $z \in \mathbb{C}$

d. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial with real coefficients $a_0, \ldots, a_n \in \mathbb{R}$. Prove that $\overline{p(z)} = p(\overline{z})$. Deduce that all roots of $p(z)$ occur in complex conjugate pairs.

Exercise 1.20

Show that $|z| = |-z|$ and $|\overline{z}| = |z|$. Prove also that $|\lambda z| = \lambda |z|$ for all $\lambda \geqslant 0$.

Exercise 1.21

Prove that $\mathfrak{R}(z) = \frac{1}{2}(z + \overline{z})$, $\mathfrak{I}(z) = \frac{1}{2i}(z - \overline{z})$.

Exercise 1.22

Prove that $\overline{e^z} = e^{\overline{z}}$. Deduce that $|e^z| = e^{\Re(z)}$.

Exercise 1.23

Prove that $|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$.

Exercise 1.24

Show that $|z+w|^2 = |z|^2 + 2\Re(z\overline{w}) + |w|^2$. Use this to determine the conditions on z, w for $|z+w| = |z| + |w|$ to hold.

Exercise 1.25

Assuming we know the triangle inequality $|z+w| \leqslant |z| + |w|$ for all $z, w \in \mathbb{C}$, prove the reverse triangle inequality

$$|z-w| \geqslant \big||z| - |w|\big|.$$

Exercise 1.26

Let K be a field with $\mathbb{R} \subset K \subset \mathbb{C}$. Prove that $K = \mathbb{R}$ or $K = \mathbb{C}$.

Further resources

- Freitag–Busam (Freitag and Busam 2009, chap. 1) for additional exercises and historical background.
- https://en.wikipedia.org/wiki/Complex_number for overview and history
- https://youtu.be/T647CGsuOVU for some visualization

Chapter 2

Complex-valued functions

Definition 2.1. A *complex function* $f\colon D \to \mathbb{C}$ is a map with *domain of definition* $D \subset \mathbb{C}$ and codomain the complex plane. Thus, f assigns to each $z = x + iy \in D$ in the domain a complex number

$$f(z) = u(z) + iv(z).$$

We call $u\colon D \to \mathbb{R}$ the *real part* and $v\colon D \to \mathbb{R}$ the *imaginary part* of the complex function f.

Example 2.1.

Example 2.2.

We have already met several complex functions in the previous section.

Example 2.3.

Example 2.4.

Definition 2.2. We define the following subsets of the complex plane:

$$\begin{aligned}
\mathbb{C}^\times &= \{z \in \mathbb{C} \mid z \neq 0\} && \textbf{punctured plane} \\
\mathbb{C}^- &= \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leqslant 0\} && \textbf{slit plane} \\
S &= \{z = x + iy \mid y \in (-\pi, \pi)\} && \textbf{principal strip} \\
&= \{z = x + iy \mid y > 0\} && \textbf{upper half-plane}
\end{aligned}$$

Example 2.5.

Proposition 2.1.

a. *The exponential function is surjective onto* $\mathbb{C}^{\times}$.

b. We have

$$\mathsf{exp}(\mathsf{log}(w)) = w \; (\forall w \in \mathbb{C}^-) \tag{2.1}$$

$$\mathsf{log}(\mathsf{exp}(z)) = z \;\; (\forall z \in S). \tag{2.2}$$

Hence the restriction $\mathsf{exp}\,|_S$ *of the exponential to the principal strip is a bijection* $\mathsf{exp}\,|_S \colon S \to \mathbb{C}^-$ *onto the slit plane, with inverse* $\mathsf{log}(w)$.

Proof.

$\square$

Since the *graph* $\Gamma(f) = \{(z,w) \in D \times \mathbb{C} \mid f(z) = w\}$ of a complex function is a subset of four-dimensional space, we cannot visualize complex functions as easily as real functions. We will now discuss some alternatives.

Image grid

A useful way to picture a complex function is to sketch its values on a grid G. The image grid $f(G)$ is a distorted version of the original grid which can be navigated easily using the grid lines. For example, to determine $f(1+2i)$, take one step in x-direction and two steps in y-direction on the distorted grid. Formally, let $G = \{z = x + iy \in \mathbb{C} \mid x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$ be the *unit grid* and define the *image grid* as (see Figure 2.1)

$$f(G) = \{w = u + iv \in \mathbb{C} \mid \exists z \in G : f(z) = w\}.$$

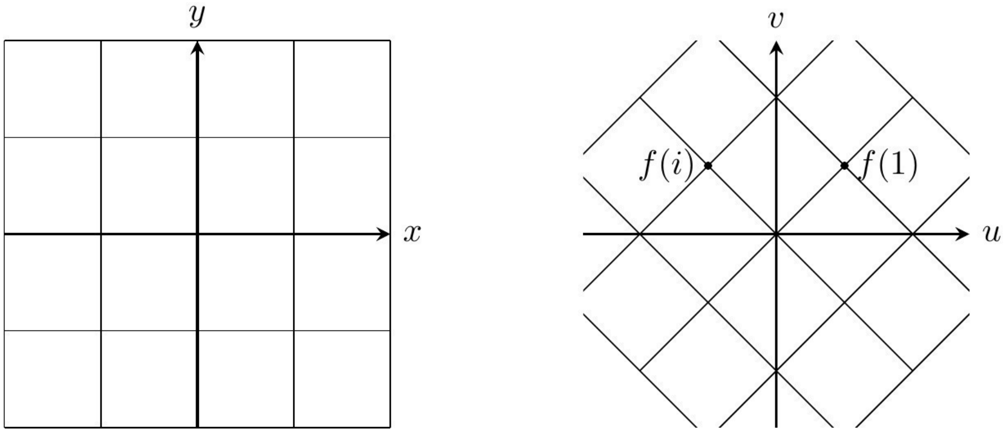


Figure 2.1: The unit grid G and the image grid $f(G)$ for $f(z) = \frac{1+i}{\sqrt{2}}z$

In practice, the image grid can often be described by finding a familiar equation that all of its members $u + iv$ satisfy. When this is not possible, a computer will help sketching an approximate image grid.

Example 2.6.

Example 2.7.

The previous example can be generalized.

Example 2.8.

Domain colouring

We represent each unit complex number $e^{i\theta}$ by a color on the color wheel. The modulus r of an arbitrary complex number $re^{i\theta}$ will be represented by the lightness of the color. This assigns a unique color to each complex number, see Figure 2.2. Pure white is never used and would correspond to infinity. Pure black corresponds to the origin.

This is less useful for calculations by hand but generates artistic images using a computer.

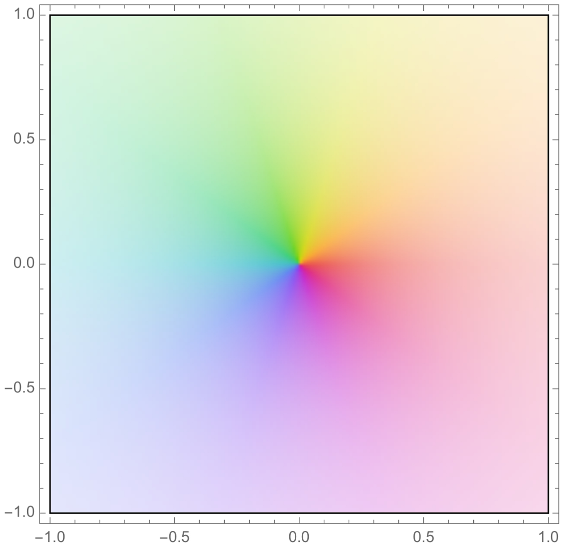


Figure 2.2: Representing complex numbers z by color

This can be used for visualizing complex functions. Draw each point z in the domain of $w = f(z)$ using the color for w.

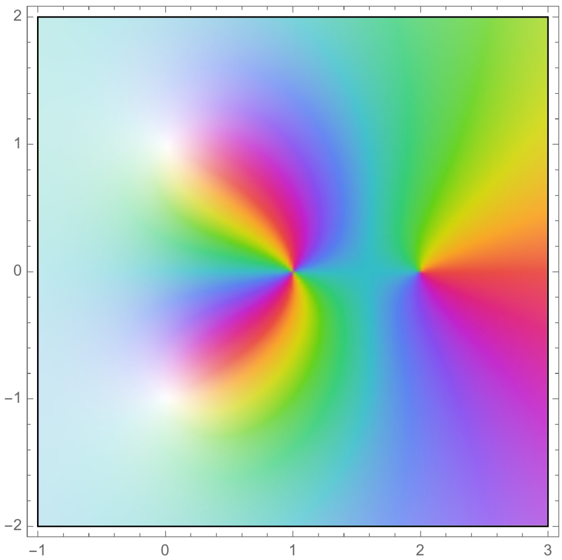


Figure 2.3: Domain colouring of $f(z) = \frac{(z-2)(z-1)^2}{z^2+1}$. Notice that near $z = \pm i$, where f is undefined, the function tends to infinity (pure white). The zeros of f at $z = 1, 2$ can also be seen.

3-dimensional graphs

Another approach is to plot the 3-dimensional graph of any of the following real-valued functions $D \to \mathbb{R}$

$$u, v, |f| = \sqrt{u^2 + v^2}.$$

Again, the missing information can be color-coded (see Figure 2.4).

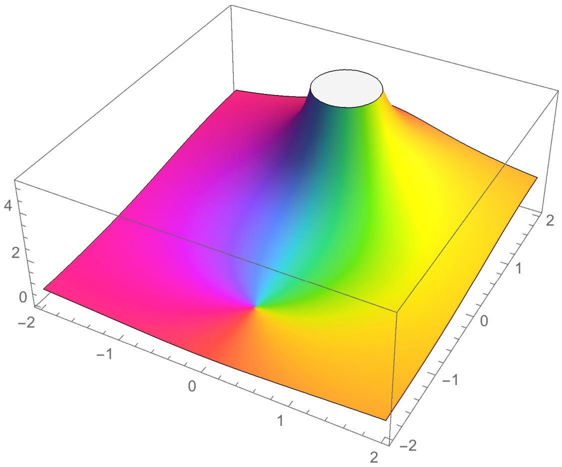


Figure 2.4: The graph of $|f(z)|$ colored by arg(z)

Questions for further discussion

- What are the functions in Example 2.1 geometrically?
- Does Equation 2.2 remain valid for all $z \in \mathbb{C}$? What is the correct modification?
- The two zeros in Figure 2.3 have a slightly different character. What is the difference between the zeros that might account for this?
- In (**?**) and (**?**)} almost all the image gridlines meet at a right angle. The only exception is at $f(0)$ in Example 2.6. Use polar coordinates to explain this behavior of $f(z) = z^2$ at $z = 0$.
- Is it always possible to extend the domain of definition of a complex function?

Is this always sensible?

- Can you think of other branches of the logarithm?

2.1 Exercises

Exercise 2.1

Describe the image set of the complex function $f(z) = \frac{1+z}{1-z}$ with domain $D = \mathbb{C} \setminus \{1\}$. In other words, determine the set of all $w \in \mathbb{C}$ for which $w = \frac{1+z}{1-z}$ has a solution $z \in D$.

Exercise 2.2

Sketch the following curves $z(t)$ in the complex plane, where t is a real parameter.

a. $z(t) = t(1+i)$ for $0 \leqslant t \leqslant 1$
b. $z(t) = \cos(t) + i\sin(t)$ for $0 \leqslant t \leqslant \pi$
c. $z(t) = \cos(t) - i\sin(t)$ for $0 \leqslant t \leqslant \pi/2$
d. $z(t) = \frac{1}{1+it}$ for $t \in \mathbb{R}$

Exercise 2.3

Let $\log(z)$ be the principal branch of the logarithm. Compute

$$\log(2i), \log(1+i), \log(-3i), \log(5).$$

Exercise 2.4

Describe the following complex functions geometrically.

$$f(z) = 3z, f(z) = iz, f(z) = \frac{(1+i)}{\sqrt{2}}z$$

Exercise 2.5

Determine a domain of definition for the following complex functions.

$$f(z) = \frac{1}{z}, f(z) = \frac{1+z}{z-1}, f(z) = \frac{z^2-4}{z^2+2z}, f(z) = \frac{1}{\exp(z)}.$$

Exercise 2.6

Determine the domain of definition for $f(z) = \frac{1}{\sin(z)}$.

Exercise 2.7

Find all solutions to the following equations:

a. $e^z = -1$,

b. $\sin(z) = -i$

Exercise 2.8

Prove that the composition $f \circ f'$ of two Möbius transformations is again a Möbius transformation. If we associate to f, f' the matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, A' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathsf{M}_{2\times 2}(\mathbb{C}),$$

show that $f \circ f'$ is associated to the matrix product AA'.

Exercise 2.9

Draw the image grid for $\exp\colon \mathbb{C} \to \mathbb{C}^\times$.

Further resources

- https://youtu.be/NtoIXhUgqSk 5 ways to visualize a complex function
- Animation of Möbius transformations: https://youtu.be/0z1fIsUNhO4 and https://www-users.cse.umn.edu/~arnold/moebius/

Chapter 3

Topology of the complex plane

In this section we explain that most of the facts about limits, series, and continuity carry over from real analysis essentially without change.

The modulus of complex numbers defines a *distance* $d(z,w) = |z-w|$ on the plane (this is the usual Euclidean distance), which determines the following standard terminology for metric spaces.

Definition 3.1. The *open disc* of radius $0 \leqslant r \leqslant +\infty$ centered at $z_0 \in \mathbb{C}$ is

$$D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

The *closed disc* $\overline{D}_r(z_0)$ is the set of all $z \in \mathbb{C}$ with $|z - z_0| \leqslant r$.

Definition 3.2. A subset $O \subset \mathbb{C}$ is *open* if for every point $z_0 \in O$ there exists $r > 0$ such that $D_r(z_0) \subset O$ (see Figure 3.1). A subset $C \subset \mathbb{C}$ is called *closed* if the complement $O = \mathbb{C} \setminus C$ is an open subset.

A subset $B \subset \mathbb{C}$ is *bounded* if $B \subset D_r(0)$ for some $0 < r < \infty$.

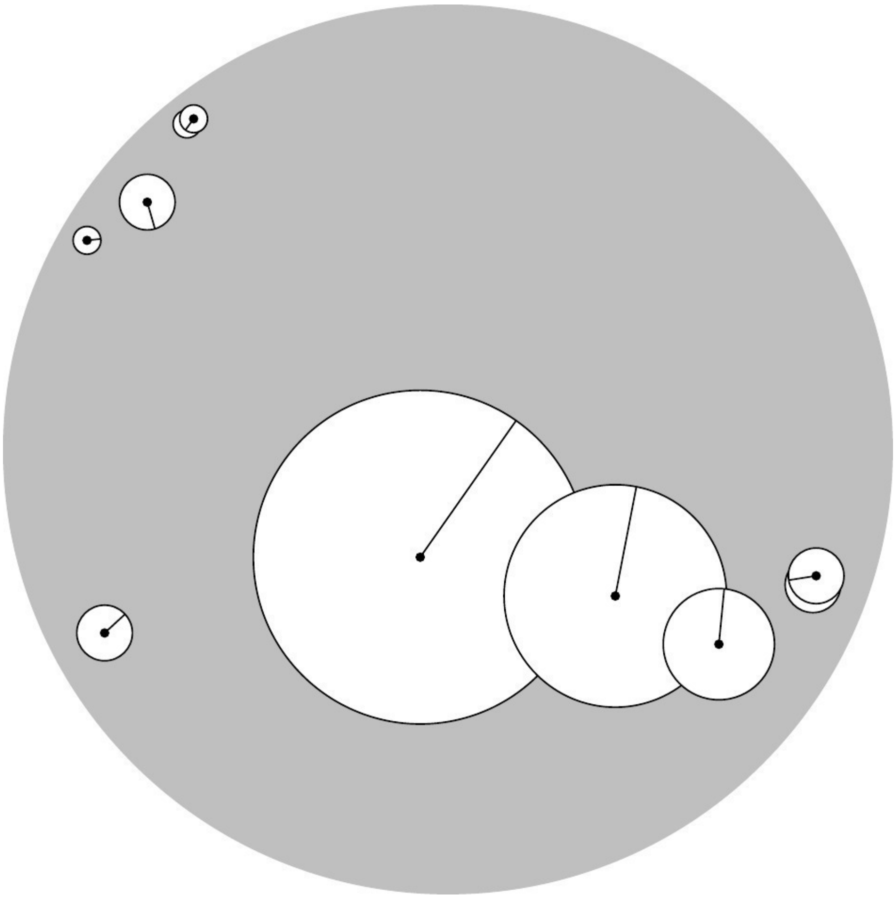


Figure 3.1: The gray open disc is an open subset of $\mathbb{C}$.

Remark 3.1. The above notion of open set determines a topology on $\mathbb{C}$.

Definition 3.3. A sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ has the *limit* $\zeta \in \mathbb{C}$ (or is *convergent* to ζ), written $\lim_{n\to\infty} z_n = \zeta$, if

$$\forall \epsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geqslant n_0 : |z_n - \zeta| < \epsilon.$$

Equivalently, $\lim_{n\to\infty} |z_n - \zeta| = 0$. We call $(z_n)_{n\in\mathbb{N}}$ a *Cauchy sequence* if

$$\forall \epsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n, m \geqslant n_0 : |z_n - z_m| < \epsilon.$$

The same argument as in real analysis shows that the limit w is unique and that every convergent sequence is a Cauchy sequence. The converse is also true by the completeness of real numbers.

Proposition 3.1. *For a sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$, the following are equivalent:*

a. *There exists $\zeta \in \mathbb{C}$ such that $\zeta = \lim_{n\to\infty} z_n$.*

b. *$(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.*

Proof.

$\square$

The advantage of Cauchy sequences is that one does not need to know the value of the limit in advance.

Definition 3.4. A *series* of complex numbers $(z_k)_{k\in\mathbb{N}}$ converges to the *limit* ζ, written $\zeta = \sum_{k=0}^{\infty} z_k$, if the sequence of partials sums $\left(w_n = \sum_{k=0}^{n} z_k\right)_{n\in\mathbb{N}}$ converges to ζ.

We call a series *absolutely convergent* if the series $\sum_{k=0}^{\infty} |z_k|$ is convergent.

As in real analysis, the Cauchy criterion implies that every absolutely convergent series is convergent. Absolutely convergent series may be rearranged and orders of summation may be exchanged.

Although $\infty \notin \mathbb{C}$, it will be convenient to define $\lim_{n\to\infty} z_n = \infty$ to mean that the sequence $(z_n)_{n\in\mathbb{N}}$ eventually leaves every disk. Symbolically,

$$\forall r > 0 \; \exists n_0 \in \mathbb{N} \; \forall n \geqslant n_0 : |z_n| > r.$$

We call $\mathbb{C} \cup \{\infty\}$ the *extended complex plane*.

Definition 3.5. A point $\zeta \in \mathbb{C} \cup \{\infty\}$ is in the *closure* of $D \subset \mathbb{C}$ if there exists a sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in D$ and $\lim_{n\to\infty} z_n = \zeta$.

Let $f\colon D \to \mathbb{C}$ be a complex function and let $\zeta \in \mathbb{C}\cup\{\infty\}$ be in the closure of D. The function $f(z)$ has the *limit* $w \in \mathbb{C} \cup \{\infty\}$ as $z \to \zeta$, written $\lim_{z\to\zeta} f(z) = w$, if for *every* sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in D$ and $\lim_{n\to\infty} z_n = \zeta$ we have $\lim_{n\to\infty} f(z_n) = w$. An equivalent ε-δ-definition is

$$\forall \varepsilon > 0 \; \exists \delta > 0 : 0 < |z - \zeta| < \delta, z \in D \implies |f(z) - w| < \varepsilon.$$

A complex function $f\colon D \to \mathbb{C}$ is *continuous at* $\zeta \in D$ if $\lim_{z\to\zeta} f(z) = f(\zeta)$. We call f *continuous on* D if f is continuous at every $\zeta \in D$.

Questions for further discussion

- Explain the difference between the notions 'limit of a function', 'limit of a sequence', and 'limit of a series'.

3.1 Exercises

Exercise 3.1

Recall the ratio and root test for series of real numbers.

Exercise 3.2

For which $z \in \mathbb{C}$ do the following limits exist?

$$\lim_{n\to\infty} n^{1/n} z, \quad \lim_{n\to\infty} z^n, \quad \lim_{n\to\infty} \frac{z^n}{n}, \quad \lim_{n\to\infty} \frac{z^n}{n!}, \quad \lim_{n\to\infty} \frac{z^n}{n^n}, \quad \lim_{n\to\infty} n! z.$$

Exercise 3.3

Show that every convergent sequence $(z_n)_{n\in\mathbb{N}}$ of complex numbers is bounded.

Exercise 3.4

Let $\sum_{k=0}^{\infty} z_k$ be a convergent series of complex numbers. Show that $\lim_{k\to\infty} z_k = 0$.

Exercise 3.5

For which $z \in \mathbb{C}$ do the following series converge?

$$\sum_{k=0}^{\infty} k z^k, \quad \sum_{k=0}^{\infty} (kz)^k$$

Exercise 3.6

Let $f\colon \mathbb{C} \to \mathbb{C}$ be a complex function. Show that

$$\lim_{z\to\infty} f(z) = w \iff \lim_{z\to 0} f\left(\frac{1}{z}\right) = w.$$

Exercise 3.7

The Riemann sphere is $\mathbb{S} = \{(a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 1\}$. Show that the stereographic projection

$$F\colon \mathbb{S} \longrightarrow \mathbb{C} \cup \{\infty\}, F(a, b, c) = \begin{cases} \frac{a+ib}{1-c} & \text{if } c \neq 1, \\ \infty & \text{if } c = 1. \end{cases}$$

is a bijection between the Riemann sphere and the extended complex plane. Find a formula for the inverse function.

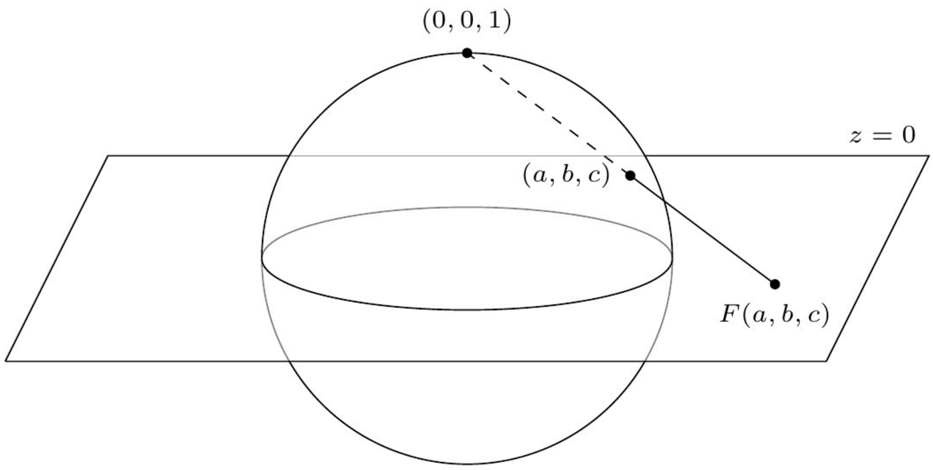


Figure 3.2: Stereographic projection from the north pole

Further resources

Part II

Differentiation and contour integrals

Chapter 4

Holomorphic functions

The following definition is a key concept for this course.

Definition 4.1. A complex function $f\colon U \to \mathbb{C}$ with domain an open set $U \subset \mathbb{C}$ is *complex differentiable* at a point $z_0 \in U$ if the limit

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{4.1}$$

exists. We call f *holomorphic* on U if f is complex differentiable at every $z_0 \in U$. The set of all holomorphic functions on U is denoted $\mathcal{O}(U)$. A function that is holomorphic[1] on $U = \mathbb{C}$ is *entire*.

[1]From Greek *holos* 'whole, complete' and *morphē* 'form, shape

Proposition 4.1. *Let $f, g\colon U \to \mathbb{C}$ be complex differentiable at z_0. Then:*

a. $f + g$ is complex differentiable at z_0 with

$$(f+g)'(z_0) = f'(z_0) + g'(z_0).$$

b. fg is complex differentiable at z_0 with

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0).$$

c. If $g(z_0) \neq 0$, then f/g is complex differentiable at z_0 with

$$(f/g)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}.$$

Proof.

□

Example 4.1.

Proposition 4.2. *Let $f\colon U \to \mathbb{C}$, $g\colon V \to \mathbb{C}$ be complex functions with $f(U) \subset V$, and U, V open. Suppose that f is complex differentiable at $z_0 \in U$ and that g is complex differentiable at $w_0 = f(z_0) \in V$.*

Then $g \circ f$ is complex differentiable at z_0 and

$$(g \circ f)'(z_0) = g'(w_0)f'(z_0). \tag{4.2}$$

Proof.

Proposition 4.3. *For a complex function $f = u + iv$ the following are equivalent:*

a. *f is complex differentiable at z_0*

b. *f is real differentiable at z_0 and the* Cauchy–Riemann equations

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0), \qquad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0), \tag{4.3}$$

hold.

In this case, $f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

Proof.

$\square$

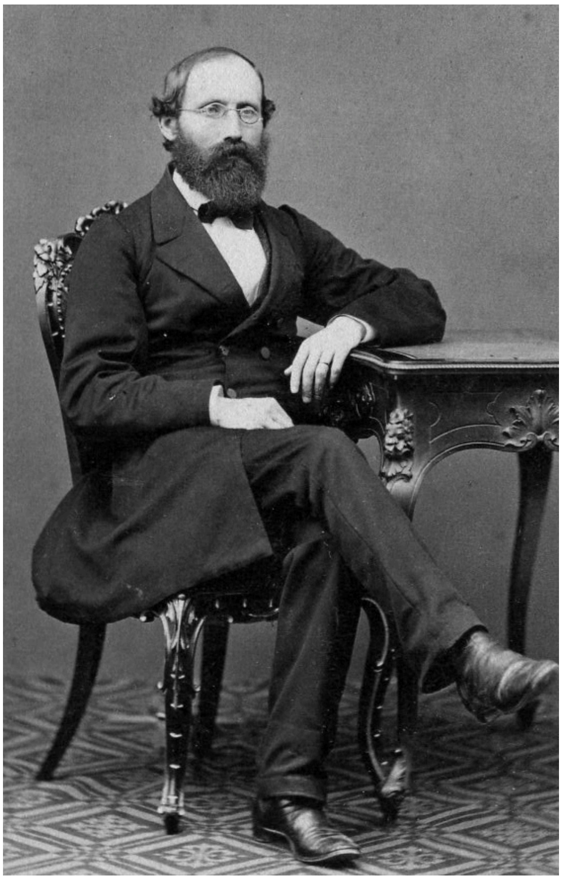


Figure 4.1: Bernhard Riemann. Familienarchiv Thomas Schilling, ca. 1862. WikiMedia Commons. Public Domain

Example 4.2.

Example 4.3.

Example 4.4.

Example 4.5.

Theorem 4.1. *Let* $f\colon U \to \mathbb{C}$ *be a holomorphic function and* $z_0 \in U$ *with* $f'(z_0) \neq 0$. *Then there exist open sets* $V, W \subset \mathbb{C}$ *with* $z_0 \in V \subset U$ *and* $w_0 = f(z_0) \in W$ *with the property that the restriction* $f|_V$ *becomes a bijection* $V \to W$ *with holomorphic inverse function* $(f|_V)^{-1}\colon W \to V$ *and*

$$\frac{\partial (f|_V)^{-1}}{\partial w}(w) = \frac{1}{f'\big((f|_V)^{-1}(w)\big)}. \tag{4.4}$$

Proof.

Remark 4.1. Our proof is a simple application of the ordinary inverse function theorem. We will see two different proofs later in the course.

Example 4.6.

Questions for further discussion

- $x \mapsto x^3$ is a real differentiable bijection whose inverse function is not differentiable at $x = 0$. What about the complex analogue $z \mapsto z^{1/3}$?

4.1 Exercises

Exercise 4.1

Write the following complex functions in the form $f = u + iv$:

$$f_1(z) = \sin(z), \qquad f_2(z) = e^{z^2}, \qquad f_3(z) = \cosh(z)$$

Exercise 4.2

At which points $z \in \mathbb{C}$ are the following functions complex differentiable? At which points are they real differentiable?

$$f_1(z) = z, \qquad f_2(z) = \overline{z}, \qquad f_3(z) = z^3 + z,$$
$$f_4(z) = \frac{1}{2iz}, \qquad f_5(z) = |z|^2, \qquad f_6(z) = \frac{|z|^2}{\overline{z}}$$

Exercise 4.3

Let $f\colon V \to W$ be a bijection of open sets $V, W \subset \mathbb{C}$. Assume that f is complex differentiable at z_0 and that f^{-1} is complex differentiable at $w_0 = f(z_0)$. Show that $f'(z_0) \neq 0$. (The same result holds for real differentiable maps to show $\det J_f(z_0) \neq 0$.)

Exercise 4.4

Let $f\colon U \to \mathbb{C}$ be a holomorphic function with domain an open set $U \subset \mathbb{C}$. Suppose also that $f'(z)$ is holomorphic. Write $f(z) = u(x,y)+iv(x,y)$, where $z = x+iy$. As we will see, a consequence of f being holomorphic, is that u and v have continuous second order derivatives. Show that:

a. $|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$

b. Both u, v satisfy the Laplace equation $\Delta(u) = \Delta(v) = 0$, where the Laplace operator is defined as $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

c. Fix $n \in \mathbb{N}$. Use b. to find a solution to $\Delta(u) = 0$ that satisfies $u(z) = \cos(n\theta)$ for all $|z| = 1$, $z = e^{i\theta}$ on the unit circle.

d. Does the converse to b. hold?

Exercise 4.5

Find subdomains where sin, cos, sinh, cosh, tan, tanh are injective. Compute the derivative of an inverse using the chain rule. Deduce an explicit formula for the derivative of an inverse.

Exercise 4.6

Show that the set $\mathcal{O}(U)$ of holomorphic functions on U becomes a ring under the operations of point-wise addition and multiplication:

$$(f+g)(z) = f(z)+g(z), \qquad (fg)(z) = f(z)g(z).$$

Exercise 4.7

Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function with image $f(\mathbb{C}) \subset \mathbb{R}$. Prove that f is constant.

Exercise 4.8

Let U be an open set and assume that U is *path-connected*, meaning that for all $z_0, z_1 \in U$ there exists a continuously differentiable map $\gamma\colon [0,1] \to U$ such that $\gamma(0) = z_0$, $\gamma(1) = z_1$.

a. Let $f\colon U \to \mathbb{C}$ be a holomorphic function such that $f'(z) = 0$ for all $z \in U$. Prove that $f(z)$ is a constant function.

   Hint: Consider the real and imaginary parts of $f \circ \gamma$.

b. Show that the only entire functions $f(z)$ satisfying $f'(z) = Cf(z)$ for a constant $C \in \mathbb{C}$ are the functions $f(z) = De^{Cz}$, $D \in \mathbb{C}$.

c. Find all entire solutions $f(z)$ of $f''(z) = f(z)$.

   Hint: Consider $g = f + f'$, $h = f - f'$

Chapter 5

Power Series

Definition 5.1. A *formal power series* is an expression of the form

$$P = \sum_{n=0}^{\infty} a_n T^n \tag{5.1}$$

with *coefficents* $a_n \in \mathbb{C}$. As no convergence is required, this is really just a sequence $(a_n)_{n\in\mathbb{N}}$ of complex numbers.

We call a_0 the *constant term* of P. The *order* of P is the smallest $n \in \mathbb{N}$ such that $a_n \neq 0$.

Example 5.1.

Definition 5.2. Let $\mathbb{C}[\![T]\!]$ be the set of all formal power series. Define the addition and multiplication of $P = \sum_{n=0}^{\infty} a_n T^n$, $Q = \sum_{n=0}^{\infty} b_n T^n \in \mathbb{C}[\![T]\!]$ by

$$P + Q = \sum_{n=0}^{\infty} (a_n + b_n) T^n, \tag{5.2}$$

$$PQ = \sum_{n=0}^{\infty} c_n T^n, c_n = \sum_{i+j=n} a_i b_j. \tag{5.3}$$

Proposition 5.1.

a. *These operations make $\mathbb{C}[\![T]\!]$ a commutative ring with unit $1,$ the series with constant term 1 and all higher coefficients zero.*
b. *$P \in \mathbb{C}[\![T]\!]$ has a multiplicative inverse $\iff$ the constant term of P is non-zero.*

Proof.

Example 5.2.

Definition 5.3. The *domain* $D(P)$ of a formal power series Equation 5.1 is the set of all $z \in \mathbb{C}$ such that the series of complex numbers

$$P(z) = \sum_{n=0}^{\infty} a_n z^n,$$

obtained by substituting T by z, converges. We obtain a complex function

$$D(P) \longrightarrow \mathbb{C}, z \longmapsto P(z).$$

More generally, fix a *center* $z_0 \in \mathbb{C}$. We then have a complex function

$$D(P, z_0) := z_0 + D(P) \longrightarrow \mathbb{C}, z \longmapsto P(z - z_0) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{5.4}$$

which differs from $P(z)$ only by a translation.

Example 5.3.

Recall the following concept from analysis.

Definition 5.4. For a complex function $f\colon D \to \mathbb{C}$, the *uniform norm* is

$$\|f\|_{\infty,D} = \sup_{z\in D} |f(z)|. \tag{5.5}$$

A sequence of functions $(f_n)_{n\in\mathbb{N}}$ is *uniformly convergent* to f on D if

$$\|f - f_n\|_{\infty,D} \to 0 \text{ as } n \to \infty.$$

A series $\sum_{n=0}^{\infty} f_n(z)$ is a *uniformly convergent series* on D if the sequence of *partial sums* $P_n(z) = \sum_{k=0}^{n} f_k(z)$ converges uniformly.

Uniform convergence on a subset $C \subset D$ refers to the uniform convergence of the functions restricted to C.

Theorem 5.1. *Every formal power series P has a unique* radius of convergence $0 \leqslant \rho \leqslant +\infty$ *such that for every center z_0*

$$D_\rho(z_0) \subset D(P, z_0) \subset \overline{D}_\rho(z_0). \tag{5.6}$$

In fact,

$$\rho = \sup \left\{ r \geqslant 0 \;\middle|\; (|a_k| r^k)_{k\in\mathbb{N}} \text{ is a bounded sequence} \right\}. \tag{5.7}$$

Moreover, $P(z-z_0)$ converges absolutely and uniformly on every smaller disk $D_r(z_0)$ with $0 < r < \rho$ and diverges at every point $z \in \mathbb{C}$ with $|z - z_0| > \rho$.

Proof.

$\square$

Remark 5.1. The domain $D(P)$ of a formal power series is roughly a disk of radius ρ, with uncertain behavior on the boundary circle $|z| = \rho$. Determining the behavior on the boundary can be very difficult. On the other hand, the radius of convergence is easy to compute. If you find $z \in \mathbb{C}$ for which $P(z)$ converges (for example, using the ratio or the root test), you can deduce $|z| < \rho$. If $P(z)$ diverges at $z \in \mathbb{C}$, you know $|z| \geqslant \rho$.

Example 5.4 (optional).

Corollary 5.1. *The restriction of the complex function $f(z) = P(z - z_0)$ defined in Equation 5.4 to $D_\rho(z_0) \subset D(P, z_0)$ is continuous.*

Proof.

$\square$

Remark 5.2. Continuity need not hold on $D(P, z_0)$ (Sierpinski, 1916).

Example 5.5.

When the radius of convergence is positive, we drop the word ‘formal’ and simply speak of a (convergent) *power series*.

Example 5.6.

Example 5.7 (optional).

Theorem 5.2. *Let P be a formal power series. Assume the radius of convergence $\rho > 0$ is positive. Then for each center $z_0 \in \mathbb{C}$ the function*

$$P\colon D_\rho(z_0) \longrightarrow \mathbb{C}, z \longmapsto P(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{5.8}$$

is holomorphic with derivative given by termwise differentiation,

$$P'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}. \tag{5.9}$$

Hence P is infinitely complex differentiable, by induction.

Proof.

□

The *identity theorem* for power series is the following result.

Corollary 5.2. *Let $P = \sum_{n=0}^{\infty} a_n z^n$, $Q = \sum_{n=0}^{\infty} b_n z^n$ be formal power series with positive radius of convergence. Let $0 \neq z_\ell \in D(P) \cap D(Q)$ be a null sequence, $\lim_{\ell\to\infty} z_\ell = 0$, of non-zero complex numbers in the common domain. If $P(z_\ell) =$*

$Q(z_\ell)$ for all ℓ, then all coefficients $a_n = b_n$ agree.

Proof.

$\square$

Questions for further discussion

- Give precise statements of the comparison test, the ratio test and root test. Recall how the ratio test is proven by comparison with the geometric series.
- Find a power series P with $D(P) = \overline{D}_1(0) \setminus \{\pm 1, \pm i\}$.
- Explain why 'is a bounded sequence' in Equation 5.7 can be replaced by 'is a null sequence'.

5.1 Exercises

Exercise 5.1

Find the radius of convergence for the following power series centered at the origin.

i. $\sum_{n=0}^{\infty} \frac{z^n}{n^3}$,
ii. $\sum_{n=0}^{\infty} z^{3n}$,
iii. $\sum_{n=0}^{\infty} \frac{z^n}{n^n}$.

Hint: Recall the ratio and root tests for series of complex numbers

Exercise 5.2

Treating e^z, $\sin(z)$, $\cos(z)$ as formal power series with their usual Taylor expansion, find the terms of order ≤ 3 of the following power series:

i. $e^z \sin(z)$,
ii. $\sin(z)\cos(z)$,
iii. $1/\cos(z)$

Exercise 5.3

Let $P = \sum_{n=0}^{\infty} a_n z^n$, $Q = \sum_{n=0}^{\infty} b_n z^n$ be power series with positive radii of convergence $\rho_P, \rho_Q > 0$. Show that:

a. $P+Q = \sum_{n=0}^{\infty}(a_n+b_n)z^n$ has radius of convergence $\rho \geqslant \min(\rho_P, \rho_Q)$.
b. $PQ = \sum_{n=0}^{\infty}\left(\sum_{i+j=n} a_i b_j\right) z^n$ has radius of convergence $\rho \geqslant \min(\rho_P, \rho_Q)$.

Exercise 5.4

Find a solution to the non-linear differential equation

$$f'(z) + f(z)^2 = 0, \qquad f(0) = 1$$

on a disk centered at $z_0 = 0$ by making the ansatz $f(z) = \sum_{n=0}^{\infty} a_n z^n$, inductively determining the coefficients a_n, and finding the radius of convergence.

Exercise 5.5

Let $P(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series centered at $z_0 = 0$ and assume that the radius of convergence $\rho > 0$ is positive. Suppose that $P(z) \in \mathbb{R}$ for all $z \in D_\rho(0) \cap \mathbb{R}$. Prove that all coefficients a_n, $n \in \mathbb{N}$, must be real numbers. Deduce that $\overline{P(z)} = P(\overline{z})$ for all $z \in D_\rho(0)$.

Exercise 5.6

Prove that the ring of formal power series $\mathbb{C}[\![T]\!]$ is an *integral domain*. In other words, show that $PQ = 0 \implies P = 0$ or $Q = 0$.

Exercise 5.7

The *binomial coefficient* of a complex number $\alpha \neq 0$ and $k \in \mathbb{N}$ is defined as

$$\binom{\alpha}{k} = \begin{cases} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

The *binomial series* is the formal power series

$$B_\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} T^k.$$

a. Determine the radius of convergence of B_α.

b. Prove that if $\alpha \in \mathbb{N}$, then $B_\alpha(z) = (z+1)^\alpha$.

c. Show the *generalized Vandermonde* identity for $\alpha, \beta, \alpha + \beta \in \mathbb{C}^\times$,

$$\sum_{k=0}^{n} \binom{\alpha}{k} \binom{\beta}{n-k} = \binom{\alpha+\beta}{n}. \tag{$\star$}$$

d. Prove that for $\alpha = 1/k$ the complex function $B_\alpha(z)$ satisfies $B_\alpha(z)^k = z + 1$ on its domain. Hence $B_\alpha(z)$ is a k-th root of the function $z + 1$.

Exercise 5.8

For $a > 0$ and $z \in \mathbb{C}$ define $a^z = \exp(\log(a)z)$. Show that:

a. $a^z b^z = (ab)^z$ for all $a, b > 0,\ z \in \mathbb{C}$

b. $a^z a^w = a^{z+w}$ for all $a > 0,\ z, w \in \mathbb{C}$

c. $|a^z| = a^{\Re(z)}$ for all $a > 0,\ z \in \mathbb{C}$

The *Riemann ζ-function* is defined as

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad \text{for } \Re(z) > 1. \tag{$\star$}$$

d. Prove that the series $(\star)$ converges absolutely for all $z \in \mathbb{C}$ with $\Re(z) > 1$ and uniformly on every subset $S_\delta = \{z \in \mathbb{C} \mid \Re(z) > 1 + \delta\}$ with $\delta > 0$.

Chapter 6

Contour integrals

In this section, we will generalize the integral $\int_a^b f(x)dx$ from calculus in two ways. Firstly, we allow $f = u + iv$ to be a complex function. Secondly, we will define the integral over more general curves γ than intervals $[a, b] \subset \mathbb{R}$.

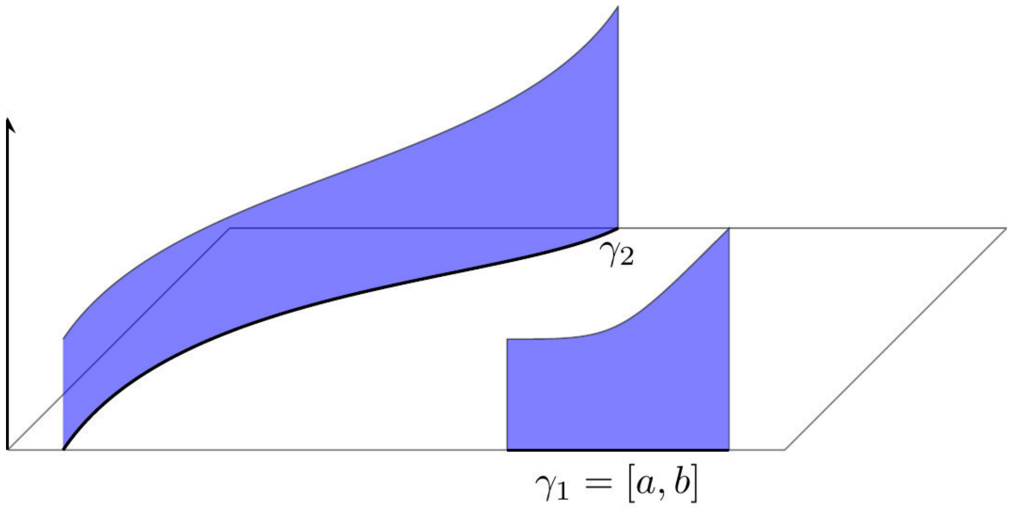


Figure 6.1: Generalizing the integral (the area, up to sign) of a real-valued function over an interval γ_1 to a general differentiable curve γ_2 by $\int_a^b f(\gamma(t))|\gamma'(t)|dt$. We will use a *different* generalization where we omit the modulus.

Definition 6.1. Let $f\colon [a,b] \to \mathbb{C}$ be a continuous complex-valued function on an interval with real and imaginary parts $f = u + iv$, $u, v\colon [a,b] \to \mathbb{R}$. Define

$$\int_a^b f(x)dx = \int_a^b u(x)dx + i\int_a^b v(x)dx. \tag{6.1}$$

Familiar rules for integrals (linearity, substitution) carry over to complex-valued functions, but the following estimate is more tricky to prove.

Proposition 6.1. $\left|\int_a^b f(x)dx\right| \leqslant \int_a^b |f(x)|dx$

Proof.

$\square$

Definition 6.2. A **curve** (or **path**) in the plane is a continuous map

$$[a,b] \xrightarrow{\gamma} \mathbb{C}$$

on a closed interval. Decompose $\gamma(t) = u(t) + iv(t)$ into real and imaginary parts. The curve γ is **differentiable** if u, v are differentiable on $[a,b]$ (including one-sided derivatives at the endpoints), and the curve γ is **continuously differentiable** (or C^1) if the derivatives $u'(t), v'(t)$ are continuous on $[a,b]$. The curve γ is **piecewise $\mathbf{C}^1$** if there exists a subdivision of the interval

$$a = t_0 < t_1 < \cdots < t_n = b \tag{6.2}$$

such that each of the restrictions $\gamma|_{[t_{k-1},t_k]}$, $k = 1, \dots, n$, is a continuously differentiable curve. In this case we call the subdivision **admissible**. For any admissible subdivision, the **length** of the curve γ is

$$L(\gamma) = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} |\gamma'(t)| dt. \tag{6.3}$$

We call $\gamma([a,b]) \subset \mathbb{C}$ the **image** of the curve. When the image is contained in a subset $D \subset \mathbb{C}$, we say that γ is a **curve in *D***.

A curve is **closed** if $\gamma(a) = \gamma(b)$, and then we call $\gamma(a)$ the **base** of the **loop** (or **contour**) γ.

Definition 6.3. Let $f\colon D \to \mathbb{C}$ be a continuous complex function. Let γ be a piecewise C^1 curve in D. Pick an admissible subdivision Equation 6.2. The **path integral** (or **contour integral** if γ is closed) of f over the curve γ is

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(\gamma(t))\gamma'(t)dt. \tag{6.4}$$

Example 6.1.

Example 6.2.

Example 6.3 (important)**.**

Lemma 6.1.

a. *The integral Equation 6.4 and the length of a curve Equation 6.3 are independent of the choice of admissible subdivision.*

b. *Let $\varphi\colon [c,d] \to [a,b]$ be a strictly monotone increasing, continuously differentiable bijection. Let γ be a piecewise C¹ curve. Then $\gamma \circ \varphi$ is also a piecewise C¹ curve and*

$$\int_{\gamma\circ\varphi} f(z)dz = \int_{\gamma} f(z)dz.$$

Hence the curve integral is independent of the parameterization of γ. Similarly, $L(\gamma \circ \varphi) = L(\gamma)$ for the lengths.

Proof.

$\square$

Due to Lemma 6.1(b) we view curves differing only by a parametrization as **equivalent**. In particular, we may translate and scale the domain $[a, b]$.

Definition 6.4.

a. Let $[a, b] \xrightarrow{\gamma_1} \mathbb{C}$, $[b, c] \xrightarrow{\gamma_2} \mathbb{C}$. Assume $\gamma_1(b) = \gamma_2(b)$. Then the **concatenation** of γ_1 with γ_2 is the curve

$$[a,c] \xrightarrow{\gamma_1 * \gamma_2} \mathbb{C}, (\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [a,b], \\ \gamma_2(t) & \text{if } t \in [b,c]. \end{cases}$$

b. The **opposite** of a curve $[a,b] \xrightarrow{\gamma} \mathbb{C}$ is the curve

$$[-b,-a] \xrightarrow{-\gamma} \mathbb{C}, (-\gamma)(t) = \gamma(-t).$$

Proposition 6.2. *Let $f, g \colon D \to \mathbb{C}$ be continuous complex functions and let $[a,b] \xrightarrow{\gamma} D$ be a piecewise C¹ curve.*

a. The integral over a curve is linear: for all $\lambda \in \mathbb{C}$ we have

$$\int_\gamma \lambda f(z) + g(z) dz = \lambda \int_\gamma f(z) dz + \int_\gamma g(z) dz.$$

b. For the opposite curve,

$$\int_{-\gamma} f(z) dz = -\int_\gamma f(z) dz.$$

c. For the concatenation of curves,

$$\int_{\gamma_1 * \gamma_2} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz.$$

d. Suppose that $|f(z)| \leqslant M$ for all $z \in \gamma([a,b])$. Then

$$\left| \int_\gamma f(z) dz \right| \leqslant M \cdot L(\gamma). \tag{6.5}$$

Proof.

$\square$

Remark 6.1. There is a generalization of the path integral to curves γ of bounded variation called the Riemann–Stieltjes integral. Proposition 6.2 continuous to hold with the length $L(\gamma)$ replaced by the total variation. For example, Lipschitz continuous maps are of bounded variation.

From Equation 6.5 we obtain:

Corollary 6.1. *Let $(f_n)_{n\in\mathbb{N}}$ be a uniformly convergent sequence of complex functions on D. For every curve $\gamma\colon [a,b] \to D$ we have*

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma \lim_{n\to\infty} f_n(z)dz.$$

Proposition 6.3 (Complex FTC)**.** *Let f be a continuous complex function on an open set U and let $[a,b] \xrightarrow{\gamma} U$ be a piecewise C¹ curve. Suppose F is a holomorphic function on U with $F' = f$ (we call F a **primitive** of f). Then*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)). \tag{6.6}$$

In particular, for every closed curve

$$\int_\gamma f(z)dz = 0. \tag{6.7}$$

Proof.

$\square$

If F is holomorphic with $F' = 0$ and U is path-connected, see Definition 8.1(a) below, we recover from Equation 6.6 the familiar fact that F is a constant. Of course, this holds more generally for differentiable functions F.

Example 6.4.

Remark 6.2. Conversely, every holomorphic function has a holomorphic primitive on a *simply connected* domain U (for example, a disk). This will be proven in Theorem 8.5.

Questions for further discussion

- Is there an analogue of Proposition 6.3 where F is only assumed to be real differentiable?
- Find the integral of a power series $P = \sum_{n=0}^{\infty} a_n z^n$ with positive radius of convergence $\rho > 0$ along an arbitrary curve γ in $D_\rho(0)$?

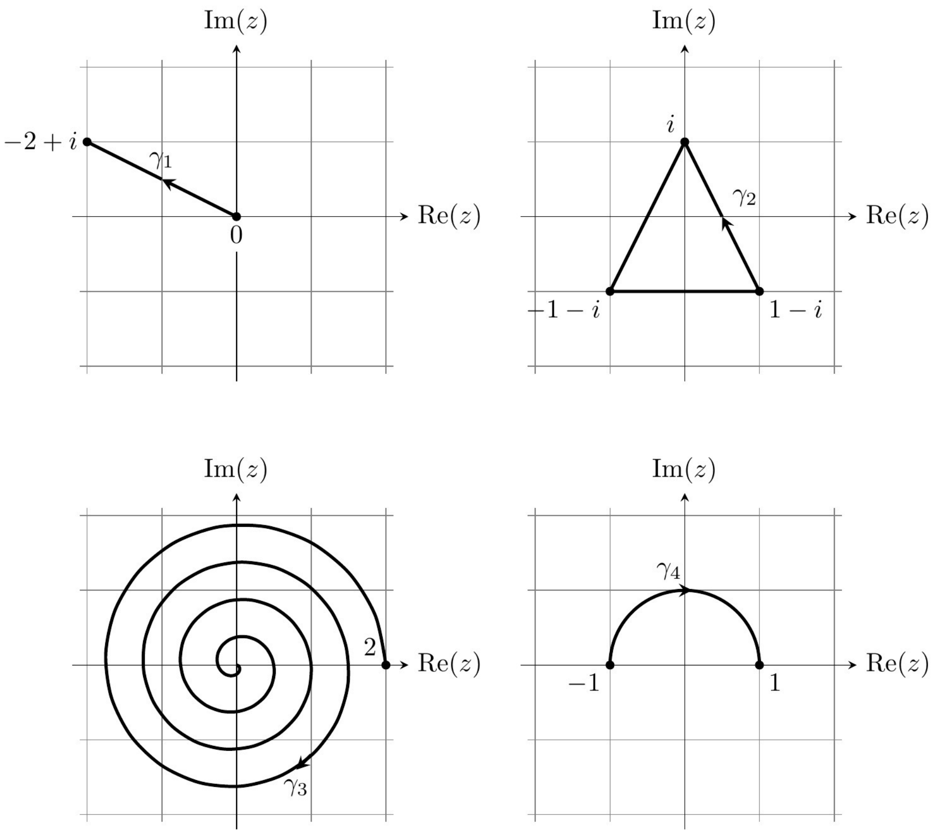
6.1 Exercises

Exercise 6.1

Recall from Example 4.6 that $\log\colon \mathbb{C}^- \to \mathbb{C}$ is a holomorphic primitive of $1/z$ on the slit plane $\mathbb{C}^-$. Combine Equation 6.7 and Example 6.3 to show that there is *no* holomorphic primitive of $1/z$ on the punctured plane $\mathbb{C}^\times$.

Exercise 6.1

For the curves γ_k, $k = 1, 2, 3, 4$, as in the following sketch



a. find piecewise C^1 parameterisations $\gamma_k \colon [a, b] \to \mathbb{C}$,

b. compute the length $L(\gamma_k)$ (γ_3 may be omitted)

c. evaluate $\int_{\gamma_k} z^2 dz$.

Exercise 6.2

Define $\gamma_\epsilon\colon [-\pi+\epsilon, \pi-\epsilon] \to \mathbb{C}$, $\gamma_\epsilon(t) = e^{it}$ for $0 < \epsilon < \pi$. Compute

$$\lim_{\epsilon\to 0} \int_{\gamma_\epsilon} z^{1/2} dz,$$

where the square root $z^{1/2} = \exp(\log(z)/2)$ is defined as on Sheet 5 using the principal branch of the logarithm.

(*Aside:* $z^{1/2}$ is integrable in the sense of measure theory and using the more general integral defined there we can rewrite the above integral as $\int_{\partial D_1(0)} z^{1/2} dz$.)

Exercise 6.3

Let γ be the straight line from $z = 1$ to $z = i$. Let $f(z) = 1/z^4$. Determine the maximum of $|f(z)|$ over all $z \in \mathfrak{I}(\gamma)$. Use this to estimate

$$\left| \int_\gamma f(z) dz \right| \leqslant 4\sqrt{2}.$$

Then compute $\int_\gamma f(z) dz$ directly and compare this to the estimate.

Exercise 6.4

Let $f(z)\colon \partial D_r(0) \to \mathbb{C}$ be a continuous function. Show that

$$\overline{\int_{\partial D_r(z_0)} f(z) dz} = -r^2 \int_{\partial D_r(z_0)} \overline{f(z)} (z - z_0)^{-2} dz.$$

Exercise 6.5

Let $D_1, D_2 \subset \mathbb{C}$ be closed subsets and $f\colon D_1 \cup D_2 \to \mathbb{C}$ be continuous. Suppose that $\int_{\gamma_1} f(z)dz = 0$ for every closed curve γ_1 in D_1 and that $\int_{\gamma_2} f(z)dz = 0$ for every closed curve γ_2 in D_2. Show that if $D_1 \cap D_2$ is path-connected, then $\int_\gamma f(z)dz = 0$ for every closed curved $\gamma\colon [a,b] \to D_1 \cup D_2$.

Hint: Subdivide $[a,b]$ into subintervals $[t_{k-1}, t_k]$ which map under γ entirely to D_1 or entirely to D_2.

Part III

Cauchy's theorem and residues

Chapter 7

Cauchy's theorem

Definition 7.1. Let $D \subset \mathbb{C}$ and $\gamma_0, \gamma_1 \colon [a, b] \to D$ curves in D.

a. A **homotopy**[1] in D between paths γ_0, γ_1 is a continuous map

$$\Gamma \colon [0,1] \times [a,b] \longrightarrow D, (s,t) \longmapsto \Gamma_s(t),$$

such that $\Gamma_0(t) = \gamma_0(t)$ and $\Gamma_1(t) = \gamma_1(t)$ for all $t \in [a, b]$. Then γ_0, γ_1 are called (freely) **homotopic paths** in D.

If, additionally, $\Gamma_s(a) = p$ and $\Gamma_s(b) = q$ are constant in $s \in [0, 1]$, we call Γ a **path homotopy** in D and γ_0, γ_1 **path-homotopic** in D.

b. If γ_0, γ_1 are loops, a **homotopy of loops** in D is a homotopy Γ in D with the additional property that Γ_s is a loop for each $s \in [0, 1]$. Then γ_0, γ_1 are called (freely) **homotopic loops** in D.

A loop is **null-homotopic** in D if there is a homotopy of loops in D to the constant loop.

[1] From Greek *homos* 'one and the same' and *topos* 'place, region, space'

Remark 7.1. In the following we will also suppose that Γ is piecewise C^1, meaning there exist subdivisions

$$0 = s_0 < s_1 < \cdots < s_m = 1, \qquad a = t_0 < t_1 < \cdots < t_n = b$$

such that each restriction $\Gamma|_{[s_{j-1},s_j]\times[t_{k-1},t_k]}$ is continuously differentiable.

Example 7.1.

Figure 7.1: Augustin Louis, Baron Cauchy. Lithograph by J. Boilly, 1821. Wellcome Collection. Public Domain Mark

Theorem 7.1 (Cauchy's theorem)**.** *Let $f\colon U \to \mathbb{C}$ be a holomorphic function on an open set. Let γ_0, γ_1 be piecewise C^1 curves in U that are path-homotopic in U. Then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Proof.

$\square$

Theorem 7.2. *Let γ_0, γ_1 be piecewise C^1 loops in U that are (freely) homotopic in U. Suppose $f\colon U \to \mathbb{C}$ is a holomorphic function. Then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

In particular, if γ is a loop that is homotopic in U to the constant loop, then

$$\int_\gamma f(z)dz = 0. \tag{7.1}$$

Proof.

□

Questions for further discussion

- Give a counterexample to Cauchy's theorem when (i) γ is not null-homotopic (ii) $f(z)$ is not holomorphic
- Does Cauchy's theorem hold for $f(z) = \overline{z}$?

7.1 Exercises

Exercise 7.1

(a) Sketch the curves

$$\gamma_0 \colon [-1,1] \longrightarrow \mathbb{C}, \qquad \gamma_0(t) = t,$$
$$\gamma_1 \colon [-1,1] \longrightarrow \mathbb{C}, \qquad \gamma_1(t) = e^{i\pi\frac{1-t}{2}}$$

and show that they are path-homotopic.

(b) Let $0 < r_0 < r_1$ and $z_0 \in \mathbb{C}$. Prove that the loops $\partial D_{r_0}(z_0)$ and $\partial D_{r_1}(z_0)$ are freely homotopic in the closed annulus $\overline{A}_{r_0,r_1}(z_0)$.

Exercise 7.2

Let $\alpha + i\beta \in \mathbb{C}$. Determine

$$\int_a^b e^{(\alpha+i\beta)t} dt$$

to compute

$$\int_a^b e^{\alpha t} \cos(\beta t) dt.$$

Exercise 7.3

Let $\mathbb{C}^- = \mathbb{C} \setminus (-\infty, 0]$ be the slit plane.

a. Show that any two points in $\mathbb{C}^-$ may be connected by a path in $\mathbb{C}^-$. Hence $\mathbb{C}^-$ is *path-connected*.

b. Show that every closed curve $\gamma\colon [a,b] \to \mathbb{C}^-$ is null-homotopic in $\mathbb{C}^-$ by finding a homotopy

$$\Gamma\colon [0,1] \times [a,b] \longrightarrow \mathbb{C}^-, (s,t) \longmapsto \Gamma_s(t)$$

satisfying $\Gamma_0(t) = \gamma(t)$ and $\Gamma_1(t) = 1$ for all $t \in [a,b]$. Hence $\mathbb{C}^-$ is *simply-connected*.

c. Prove the analogues of a. and b. for a disk $D_r(z_0)$.

d. Use Cauchy's Theorem to prove that the punctured plane $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ is not simply-connected. That is, there exists a closed curve in $\mathbb{C}^\times$ that is not null-homotopic in $\mathbb{C}^\times$.

Exercise 7.4

Let $0 < b < 1$.

a. Using the geometric series, find the power series expansion

$$\frac{1}{z - 1/b} = \sum_{n=0}^{\infty} a_n (z-b)^n$$

with center $z_0 = b$ and determine the radius of convergence ρ.

b. Use a. to show that for all $0 < r < \rho$

$$\int_{\partial D_r(b)} \frac{dz}{(z-b)(z-1/b)} = \frac{2\pi i}{b - 1/b}.$$

c. Use c. to compute

$$\int_0^{2\pi} \frac{dt}{1 - 2b\cos(t) + b^2}.$$

Exercise 7.5

Let $\gamma\colon [0,1] \to D$ be a curve in $D \subset \mathbb{C}$ and let $-\gamma\colon [0,1] \to D$, $(-\gamma)(t) = \gamma(1-t)$ be the opposite curve. Prove that $\gamma * (-\gamma)$ is path-homotopic to a constant loop.

Chapter 8

Applications of Cauchy's theorem

Theorem 8.1 (Cauchy's integral formula)**.** *Let $f\colon U \to \mathbb{C}$ be a holomorphic function and assume that $\overline{D}_r(z_0) \subset U$ with boundary curve*

$$[0, 2\pi] \xrightarrow{\gamma_{\partial D_r(z_0)}} \overline{D}_r(z_0), \gamma_{\partial D_r(z_0)}(t) = z_0 + re^{it}.$$

Then

$$f(z) = \frac{1}{2\pi i}\int_{\partial D_r(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta, \qquad \forall z \in D_r(z_0). \tag{8.1}$$

Proof.

□

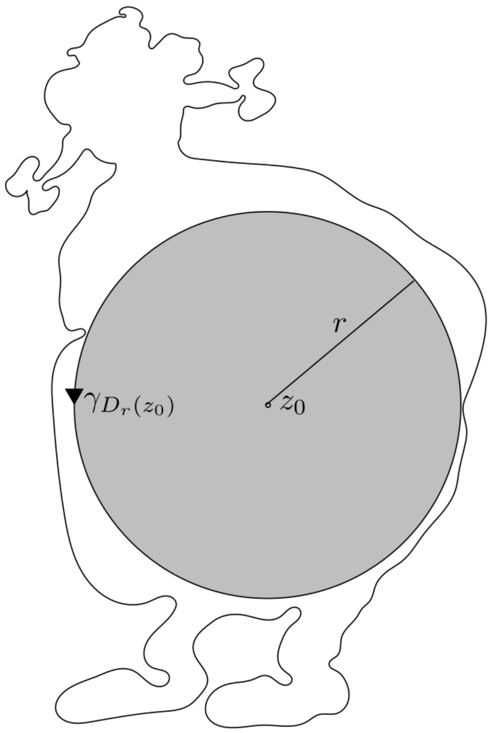


Figure 8.1: Boundary curve of largest disk fitting inside U

Theorem 8.2 (Higher Cauchy integral formula)**.** *Let $f\colon U \to \mathbb{C}$ be a holomorphic function and assume that $\overline{D}_r(z_0) \subset U$. Then f is equal on $D_r(z_0)$ to its Taylor power series*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \tag{8.2}$$

which has radius of convergence $\rho \geqslant r$. In particular, f is infinitely complex differentiable on the open set U and has a primitive on $D_r(z_0)$. Moreover,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta. \tag{8.3}$$

Proof.

□

Theorem 8.3 (Liouville)**.** *Every bounded entire function is constant.*

Proof.

$\square$

Figure 8.2: Carl Friedrich Gauß, 1777-1855, Österreichische Nationalbibliothek. Public Domain

Theorem 8.4 (Fundamental theorem of algebra)**.** *Every non-constant polynomial*

$P(z) = a_n z^n + ... + a_1 z + a_0$ *with* $a_i \in \mathbb{C}$ *has a complex root.*

Proof.

$\square$

Definition 8.1.

a. A subset $D \subset \mathbb{C}$ is **path-connected** if for all $z_0, z_1 \in D$ there exists a (piecewise C^1) curve γ in D with $\gamma(0) = z_0$, $\gamma(1) = z_1$.
b. A path-connected subset D is **simply connected** if every loop in D is (freely) homotopic in D to a constant loop.

Example 8.1.

Theorem 8.5. *Let* $f\colon U \to \mathbb{C}$ *be a holomorphic function. Suppose that* U *is simply* *connected and let $z_0 \in U$. Define $F(z)$ for each $z \in U$ by choosing a piecewise C^1 curve γ with $\gamma(0) = z_0$, $\gamma(1) = z$ and defining*

$$F(z) = \int_\gamma f(\zeta) d\zeta. \tag{8.4}$$

Then F is well-defined and is the unique holomorphic function on U such that $F' = f$ and $F(z_0) = 0$.

Proof.

$\square$

Example 8.2.

Questions for further discussion

- How should $\int_{-i}^{1+i} z^2 dz$ be interpreted? What is the result?
- Why is $\mathbb{C}^\times$ path-connected but not simply connected?

Hint: consider $\frac{1}{2\pi i}\int_\gamma \frac{1}{z}dz$ for a loop γ in $\mathbb{C}^\times$. +

It is a mysterious calculus fact that $\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ diverges for $|x| > 1$ although $\arctan\colon \mathbb{R} \to \mathbb{R}$ is smooth. In terms of the principal logarithm, $\arctan(z) = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)$. Apply this to give a geometric explanation of the divergence using Theorem 8.2.

8.1 Exercises

Exercise 8.1

Compute the following integrals:

i. $\int_{\partial D_1(0)} \frac{\sin(z)}{z^2} dz$,

ii. $\int_{\partial D_{1/2}(0)} \frac{\cos(z)}{z-1} dz$,

iii. $\int_{\partial D_2(1)} \frac{\sin(\cos(z))}{z-1} dz$.

Exercise 8.2

Let $f\colon \mathbb{C} \to \mathbb{C}$ be entire. Suppose there is a constant $C > 0$ such that $|f(z)| \leqslant C|z|^d$ for all $z \in \mathbb{C}$. Prove that $f(z)$ is a polynomial of degree $\leqslant d$.

Hint: Generalize the proof of Liouville's theorem.

Exercise 8.3

Let $f\colon U \to \mathbb{C}$ be holomorphic and $\overline{D}_r(z_0) \subset U$. Show that

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\varphi}) d\varphi$$

and interpret this equation geometrically.

Exercise 8.4

Let $f\colon U \to \mathbb{C}$ be holomorphic and $\overline{D}_r(z_0) \subset U$.

a. Prove that $g(x) = f(z_0 + re^{ix})$ is a 2π-periodic function $\mathbb{R} \to \mathbb{C}$.
b. Show that the Cauchy integral formula implies an absolutely and uniformly convergent *Fourier expansion*

$$g(x) = \sum_{n=0}^{\infty} \gamma_n e^{inx}, \qquad \forall x \in \mathbb{R},$$

with only non-negative Fourier modes. Moreover, show that $\gamma_n = \frac{1}{2\pi} \int_0^{2\pi} g(x) e^{-inx} dx.\$$

Note. More generally, a bi-infinite Fourier series $\sum_{n=-\infty}^{\infty} \gamma_n e^{inx}$ can be obtained as a superposition $g_1(x) + g_2(-x)$.

Exercise 8.5

Let $f\colon U \to \mathbb{C}$ be holomorphic and $\overline{D}_r(z_0) \subset U$. Using polar coordinates show that for the (surface) integral over the unit disk $\overline{D}_r(x_0) = \{(x, y) \in \mathbb{R}^2 \mid (x - x_0)^2 + (y - y_0)^2 \leqslant r\}$ we have

$$f(z_0) = \frac{1}{\pi r^2} \int_{\overline{D}_r(z_0)} f(x + iy) dx dy$$

and interpret this equation geometrically.

Chapter 9

Laurent series and singularities

Definition 9.1. A **formal bilateral series** is an expression of the form

$$P = \sum_{n=-\infty}^{\infty} a_n T^n \tag{9.1}$$

with **coefficients** $a_n \in \mathbb{C}$. This is just a bi-infinite sequence $(a_n)_{n\in\mathbb{Z}}$ of complex numbers. Equivalently, we have a pair of formal power series

$$P_+ = \sum_{n=0}^{\infty} a_n T^n, \qquad P_- = \sum_{n=1}^{\infty} a_{-n} T^n, \tag{9.2}$$

such that

$$P = P_+(T) + P_-(T^{-1}).$$

P_- is called the **principal part**. The **residue** of P is the coefficient a_{-1}.

Definition 9.2. Let P be a bilateral series Equation 9.1. The **domain** $D(P)$ is the set of all $z \in \mathbb{C} \setminus \{0\}$ such that both series of complex numbers

$$P_+(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad P_-(z^{-1}) = \sum_{n=1}^{\infty} a_{-n} z^{-n}$$

converge. We obtain a complex function

$$D(P) \longrightarrow \mathbb{C}, P(z) := \sum_{n=-\infty}^{\infty} a_n z^n := P_+(z) + P_-(z^{-1}). \tag{9.3}$$

More generally, fix a **center** $z_0 \in \mathbb{C}$. We then have a complex function

$$P(z - z_0) := \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n := P_+(z - z_0) + P_-((z - z_0)^{-1}).$$

with domain $D(P, z_0) = z_0 + D(P)$ differing from $P(z)$ only by a translation.

Remark 9.1. Bi-infinite series of complex numbers $\sum_{n \in \mathbb{Z}} a_n$ must be treated with care. We have avoided these issues by viewing them as pairs of ordinary series. Another approach would be to pick a bijection $\nu\colon \mathbb{N} \to \mathbb{Z}$ and consider the ordinary series $\sum_{k=0}^{\infty} a_{\nu(k)}$. However, the limit depends on the choice of ν unless the series is absolutely convergent.

Definition 9.3. The **open annulus** $A_{r,R}(z_0)$ centered at $z_0 \in \mathbb{C}$ with radii $0 \leqslant r, R \leqslant +\infty$ is the (possibly empty) subset

$$A_{r,R}(z_0) = \{z \in \mathbb{C} \mid r < |z - z_0| < R\} = D_R(z_0) \cap (\mathbb{C} \setminus \overline{D}_r(z_0)).$$

The **punctured open disk** centered at $z_0 \in \mathbb{C}$ with radius $0 < R \leqslant +\infty$ is

$$D_R^\times(z_0) = D_R(z_0) \setminus \{z_0\} = A_{0,R}(z_0).$$

The **closed annulus** is $\overline{A}_{r,R}(z_0) = \overline{D}_R(z_0) \cap (\mathbb{C} \setminus D_r(z_0))$.

Using the inversion

$$i\colon \mathbb{C}^\times \longrightarrow \mathbb{C}^\times, z \longmapsto w = z^{-1}, \tag{9.4}$$

we can rewrite Equation 9.3 as $P(z) = P_+(z) + P_-(w)$. Therefore we can treat the principal part as a power series as well, but on a second complex w-plane corresponding to the original complex z-plane via Equation 9.4.

Lemma 9.1. *Let* $0 \leqslant r \leqslant +\infty$. *Then*

$$i^{-1}(D_r(0)) = \mathbb{C}^\times \setminus \overline{D}_{1/r}(0), \qquad i^{-1}(\overline{D}_r(0)) = \mathbb{C}^\times \setminus D_{1/r}(0), \tag{9.5}$$

using the conventions $1/0 = +\infty$ *and* $1/+\infty = 0$.

Proposition 9.1. *Let* P *be a bilateral series. If* $D(P, z_0) \neq \emptyset$, *there are unique* ***radii of convergence*** $0 \leqslant r, R \leqslant +\infty$ *such that*

$$A_{r,R}(z_0) \subset D(P) \subset \overline{A}_{r,R}(z_0). \tag{9.6}$$

Here $r = 1/\rho_-$, $R = \rho_+$ *for the radii of convergence* $\rho_\pm$ *of the power series* $P_\pm$ *as defined in Theorem 5.1.*

Moreover, the series $P_+(z)$, $P_-(1/z)$ *converge absolutely and uniformly on every annulus* $A_{r',R'}(z_0)$ *with* $r < r' < R' < R$.

Proof.

□

If the domain of a bilateral series P with center z_0 contains an non-empty open annulus

$A_{r,R}(z_0) \subset D(P, z_0)$, then P is called a (convergent) **Laurent series** at z_0.

Figure 9.1: Pierre Laurent, 1813-1854, Wikipedia. Public Domain

Theorem 9.1 (Laurent expansion)**.** *Let $f\colon U \to \mathbb{C}$ be a holomorphic function on an open set containing a closed annulus $\overline{A}_{r,R}(z_0)$. Then f is equal on $A_{r,R}(z_0)$ to the convergent Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Moreover,

$$a_n = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \tag{9.7}$$

Proof.

$\square$

Definition 9.4. Let $f\colon U \to \mathbb{C}$ be a holomorphic function. A point $z_0 \in \mathbb{C} \setminus U$ is an **isolated singularity** of f if there exists $R > 0$ such that $D_R^{\times}(z_0) \subset U$.

By applying Theorem 9.1 to $A_{r,R}(z_0)$ for all $0 < r < R$ we obtain a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{9.8}$$

that converges on $D_R^{\times}(z_0)$.

Definition 9.5. An isolated singularity of f is called

a. **removable**,

b. a **pole** of order $m \geqslant 1$,

c. **essential**,

if the principal part $\sum_{n=-\infty}^{-1} a_n(z-z_0)^n$ of the Laurent expansion Equation 9.8

a. is zero,

b. has $a_{-m} \neq 0$ and $a_n = 0$ for all $n < -m$,

c. has infinitely many non-zero terms.

Theorem 9.2 (Riemann removable singularities). *Let z_0 be an isolated singularity of a holomorphic function $f\colon U \to \mathbb{C}$. Suppose there exist $R, C > 0$ such that $|f(z)| < C$ for all $z \in D_R^\times(z_0)$. Then z_0 is a removable singularity.*

Proof.

$\square$

Questions for further discussion

- Define the sum of formal Laurent series in the same way as Equation 5.2. Why can't we use Equation 5.3 to define the product of formal Laurent series?
- In Definition 9.2, why did we exclude $z = z_0$?
- For an isolated singularity, why is $U \cup \{z_0\}$ always an open set?
- Does the converse of the Riemann removable singularities theorem hold?

9.1 Exercises

Exercise 9.1

Determine whether the singularities of the following holomorphic functions $f(z)$ are isolated. If so, compute the residue. When applicable, find also the order of the pole or a holomorphic extension.

i. $\frac{z+i}{z^2+1}$,
ii. $\frac{1-\cos(z)}{z^2}$,
iii. $\frac{1}{\cos(z)-1}$,
iv. $e^{1/z}$

Exercise 9.2

Determine whether the domain $D(P)$ of the following formal bilateral series P is non-empty and, if so, find the radii of convergence. i. $\sum_{n=-\infty}^{\infty} \frac{z^n}{|n|!}$,

ii. $\sum_{n=-\infty}^{\infty} z^n$,

iii. $\sum_{n=-\infty}^{\infty} \frac{|n|^{|n|}}{|n|!} z^n$.

Exercise 9.3

Use the geometric series to find the Laurent expansion of

$$f(z) = \frac{1}{z(1-z)}$$

on the domains

i. $0 < |z| < 1$,

ii. $0 < |z-1| < 1$,

iii. $1 < |z| < +\infty$.

Exercise 9.4

Let $P = \sum_{n=-\infty}^{\infty} a_n T^n$, $Q = \sum_{n=-\infty}^{\infty} b_n T^n$ be formal bilateral series. Suppose that both domains $D(P, z_0)$, $D(Q, z_0)$ contain a non-empty annulus $A_{r,R}(z_0)$. Assume that for some radius s with $r < s < R$ we have

$$P(z) = Q(z), \qquad \text{for all } |z - z_0| = s.$$

Prove the *identity theorem*, namely that $a_n = b_n$ for all $n \in \mathbb{Z}$.

Exercise 9.5

Let U be open and $z_0 \in U$. Let $f\colon U \to \mathbb{C}$ and $g\colon U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic functions.

i. If $f(z_0) \neq 0$ and g has a pole of order one at z_0, prove that fg has a pole of order one at z_0 with

$$\mathrm{Res}_{z_0}(fg) = f(z_0)\,\mathrm{Res}_{z_0}(g).$$

ii. If $f(z_0) = 0$ and $f'(z_0) \neq 0$, prove that $1/f(z)$ has a pole of order one at z_0 with $\mathrm{Res}_{z_0}(1/f) = 1/f'(z_0)$.

Chapter 10

Residue theorem

Definition 10.1. Let z_0 be an isolated singularity of a holomorphic function $f\colon U \to \mathbb{C}$. The **residue** $\mathrm{Res}_{z_0}(f)$ is the residue of the Laurent series Equation 9.8. In other words,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \implies \mathrm{Res}_{z_0}(f) = a_{-1}.$$

Proposition 10.1. *Let z_0 be an isolated singularity of both the holomorphic functions $f, g\colon U \to \mathbb{C}$.*

a. $\mathrm{Res}_{z_0}(f + g) = \mathrm{Res}_{z_0}(f) + \mathrm{Res}_{z_0}(g)$

b. $\mathrm{Res}_{z_0}(\lambda f) = \lambda \,\mathrm{Res}_{z_0}(f)$ *for all* $\lambda \in \mathbb{C}$

c. If z_0 is a pole of order m, then $\mathrm{Res}_{z_0}(f) = \frac{1}{(m-1)!} \lim_{z \to z_0} h^{(m-1)}(z)$ *for* $h(z) = (z - z_0)^m f(z)$.

Proof.

□

Example 10.1.

Example 10.2.

Definition 10.2. Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed piecewise C^1 curve and let $z_0 \in \mathbb{C} \setminus \gamma([a,b])$. The **winding number** of γ around z_0 is

$$W_{z_0}(\gamma) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}.$$

Example 10.3.

Pick a subdivision $a = t_0 < t_1 < \cdots < t_n = b$ such that $\gamma|_{[t_{k-1},t_k]}$ is continuously differentiable. Using branches of the logarithm, we can write $\gamma(t) = z_0 + r(t)e^{i\theta(t)}$ for $r(t) = |\gamma(t) - z_0|$ and a continuous function $\theta\colon [a,b] \to \mathbb{R}$ with $\theta|_{[t_{k-1},t_k]}$ continuously differentiable. With this notation, the following holds.

Proposition 10.2.

(a) We have $W_{z_0}(\gamma) = \frac{\theta(b)-\theta(a)}{2\pi}$, *which is an integer.*

(b) For closed curves γ_0, γ_1 *in* $\mathbb{C} \setminus \{z_0\}$ *we have*

$$W_{z_0}(\gamma_0) = W_{z_0}(\gamma_1) \iff \gamma_0, \gamma_1 \text{ are homotopic in } \mathbb{C} \setminus \{z_0\}.$$

Proof.

$\square$

Theorem 10.1 (Residue theorem). *Let $U \subset \mathbb{C}$ be an open set, $\gamma\colon [a,b] \to U$ a null-homotopic piecewise C^1 loop in U, and $z_1, \dots, z_n \in U \setminus \gamma([a,b])$ points not on the loop γ. If f is a holomorphic function on $U \setminus \{z_1, \dots, z_n\}$, then*

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{k=1}^{n} W_{z_k}(\gamma)\,\mathsf{Res}_{z_k}(f).$$

Proof.

$\square$

10.1 Exercises

Exercise 10.1

Determine the winding numbers $W_{z_0}(\gamma)$ in the following cases.

a. $\gamma\colon [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = e^{it}$, for $z_0 = 0, 2$.

b. $\gamma\colon [2\pi, 10\pi] \to \mathbb{C}$, $\gamma(t) = \begin{cases} te^{it} & \text{if } t \in [2\pi, 6\pi], \\ 12\pi - t & \text{if } t \in [6\pi, 10\pi], \end{cases}$ for $z_0 = 0, 20$.

c. $\gamma\colon [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = (2 + \cos(t))e^{2it}$, for $z_0 = 0, 4$.

Exercise 10.2

Use the residue theorem to compute

$$\int_{\partial D_r(0)} \frac{(z+2)^2}{z(z-1)^2} dz$$

for all $0 < r \neq 1$.

Hint: Consider the cases $r > 1$ and $r < 1$ separately.

Exercise 10.3

Apply the residue theorem to the contour $\partial D_1(0)$ and a suitable holomorphic function $f(z)$ to compute the integral

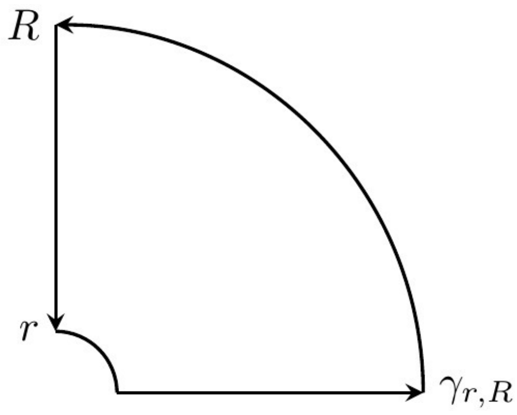
$$\int_0^{2\pi} \frac{dx}{2 + \cos(x)}.$$

Exercise 10.4

Compute the integral

$$\int_0^{+\infty} \frac{\cos(x)}{\sqrt{x}} dx$$

by considering $f(z) = \frac{e^{iz}}{\sqrt{z}}$ and the following contour $\gamma_{r,R}$ for $r \to 0$, $R \to +\infty$.



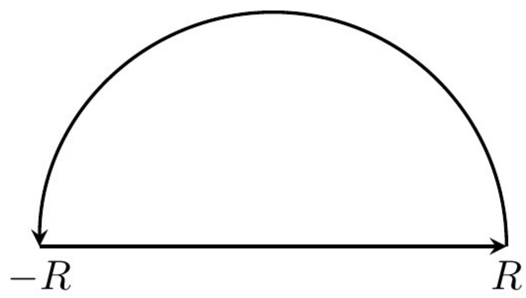
Hint: Recall that $\int_0^{\infty} e^{-x^2} dx = \sqrt{\pi}/2$.

Exercise 10.5

Compute the integral

$$\int_{-\infty}^{+\infty} \frac{dx}{1 + x^6}$$

by applying the residue theorem to $f(z) = \frac{1}{1+z^6}$ and the following upper semi-circular contour γ_R for $R \to +\infty$.



Chapter 11

Applications to integrals

The residue theorem can be used to compute integrals of functions that do not have an obvious primitive. For this, we first need to find a suitable holomorphic function and contour to which we can apply the residue theorem, which is guesswork. The original integral is often a part of this contour and one shows that the rest of contour integral tends to zero.

We give three examples that illustrate the general technique.

Example 11.1.

Integrals made only of trigonometric functions can often be evaluated using the residue theorem and the following method.

Example 11.2.

The following integral was already computed in calculus using polar coordinates in the plane.

Example 11.3.

Appendix A

All Solutions

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