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MT4508: Dynamical Systems

University of
St Andrews

Contents

Module Handbook

Welcome to MT4508: Dynamical Systems. In this module you will study *dynamical systems*. A dynamical system consists of a set of possible states and a rule that determines the present state based on past states. The module is split into two parts based on the two types of dynamical systems we will encounter: *discrete* and *continuous* dynamical systems. The aim is largely applied: we will look at techniques that help us understand long term behaviours of the dynamical system, particularly stability and chaos. Our discussion will mainly be motivated by a selection of hands-on examples which model dynamics of interest. In this sense the module is more practical than theoretical— the emphasis will be on applying techniques to examples with brief indications of why the techniques should be expected to work more generally.

Structure

The module follows closely the book *Chaos - An introduction to Dynamical systems* (Alligood et al., 2000) as such we have indicated below the Chapters/Sections of the book that will be covered in each week of the semester. The module is comprised of the following parts:

1. Introduction to dynamical systems (covers Chapter 1 from Section 1.1 to Section 1.3).
2. Periodic points and stability (covers Section 1.4 of Chapter 1 up to Section 2.5 Definition 2.12 of Chapter 2).
3. Higher-dimensional dynamics (covers Chapter 2 Section 2.5 Example 2.13 to Section 2.6 Example 2.20 of Chapter 2).
4. Higher-dimensional stability (covers Chapter 3 Sections 3.1 and 3.2 and Chapter 4 Sections 4.1 to 4.4).
5. Fractals (covers Chapter 4 from Section 4.5 to Definition 5.1 of Chapter 5).
6. Chaos in Higher Dimensions (covers Chapter 5 Sections 5.1 and 5.6 up; Chapter 6 Sections 6.1 and 6.2; Chapter 7 Section 7.1).
7. Equilibria and stability in continuous time dynamical systems (covers Chapter 7 Sections 7.2 to 7.5).

8. Lyapunov Functions (covers Chapter 7 Section 7.6 to Chapter 8 Section 8.1 (Theorem 8.3)).
9. Limit sets in 2-dimensions (covers Chapter 8 Section 8.1 and 8.2 and Sections 9.1 and 9.2 of Chapter 9).
10. Lorenz chaos and bifurcation (Covers Chapter 11 Sections 11.1 and 11.2).

Prerequisites

- MT3504

Lecture notes and bibliography

The lecture note are based on the following text (Alligood et al., 2000)

- Alligood, Sauer, Yorke, *Chaos - An introduction to Dynamical systems*, Springer 200.

There are hard and electronic copies available from the University library;

Other recommended texts:

- Ott, *Chaos in Dynamical SYstems*, Cambridge UP, 1993 (2nd Edition 2002)
- Strogatz, *Nonlinear Dynamics and Chaos*, Westview Press, 2000
- Baker and Gollub, *Chaotic Dynamics*- An Introduction, Cambridge UP, 1990 (2nd edition 1996)
- Drazin, *Nonlinear Systems*, Cambridge UP, 1992
- Guckenheimer and Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vetcor Fields*, Springer 1983.

Logistics

Lectures

The module runs in Semester 2 with 2.5 lectures each week: 3 in even weeks (Monday, Tuesday and Thursday) and 2 in odd weeks (Tuesday and Thursday). In weeks 1-5, Lectures will be delivered live online at 10am on Monday (even weeks), Tuesday and Thursday; in weeks 6-11 Lectures will be delivered in person in Maths Lecture Theatre C and livestreamed over teams. A recording of each lecture with be made available on Panopto after the class. Table 0.2 sets out a timetable for the semester. If you have any questions while going through the content then *do* email me fo55@st-andrews.ac.uk. I am more than happy to arrange a meeting over teams or in person.

Lecturers

This semester Dr. Feyisayo (Shayo) Olukoya will be the sole lecturer on the module. As mentioned above you can reach me by **email**; you should also feel free to arrange an in-person meeting my office is **Room 217** in the mathematics building; meeting virtually over teams is also an option.

Tutorials

Starting in Week 2 there will be one tutorial per week. There are 2 groups; you will need to sign up.

Group 1	Tuesday 13:00	Maths Lecture Theatre B
Group 2	Thursday 13:00	Maths Lecture Theatre B (Livestreamed & recorded)

TABLE 0.1: Tutorial groups

Tutorial sheets will be uploaded on Moodle the week before the relevant tutorial and solutions will appear on Moodle on Friday at 17:00 the week after. If you are unable to make it in-person to a tutorial then do join in the Thursday tutorial which will be live-streamed on Teams (the Thursday tutorial will also be recorded with the recording made available via Moodle after the class). The Tuesday tutorial is an in-person only class, it will **not** be recorded or livestreamed.

Moodle

All resources for the module (lecture notes, problem sheets, solutions e.t.c) will be made available on moodle at the appropriate time.

Scheduling

Table 0.2 displays the schedule for the semester. The key below explains what all the symbols mean.

- L = Lecture (10am on Teams & in-person in Weeks 6 - 11 in Maths Lecture Theatre C)
- Tut = Tutorial (In-person)
- Reading party: Honours reading party; Thursday lecture and tutorial cancelled.

Assessment

The module is assessed entirely by a final exam.

Week Beginning	Week	Mon	Tue	Wed	Thu	Fri
17 Jan	1		L		L	
24	2 Tut	L	L		L	
31	3 Tut		L		L	
7 Feb	4 Tut	L	L		L	
14	5 Tut		L		L	
21	**Vacation**					
28	6 Tut	L	L		L	
7 Mar	7 Tut		L	**Reading Party**		
14	8 Tut	L	L		L	
21	9 Tut		L		L	
28	10 Tut	L	L		L	
4 Apr	11 Tut		L			
11	**Revision**					

TABLE 0.2: Timetable

90% Rule

You must complete 90% of assessment in a module to have the opportunity to gain credit in the module. If you do not complete at least 90% of assessment, you will be given a **FINAL** academic alert in the module and receive a grade 0X. In MT4508, this only applies if you fail to sit the final exam.

Student Support

For advice on any non-academic issue (including financial, international, personal or health matters) or if you are unsure who to go to for help, please contact the Advice and Support Centre. You can drop-in on 79 North Street, phone 01334 462020 or email theasc@st-andrews.ac.uk.

You can also contact the school's well-being officer by email at tdhc@st-andrews.ac.uk.

Acknowledgements

Part I

Discrete time dynamical systems

1

Introduction

Covers up to and including Section 1.3 of Chapter 1 (Alligood et al., 2000).

1.1 What is a "Dynamical System"?

Definition 1.1. A *dynamical system* consists of a set of possible states, together with a rule that determines the present state in terms of past states.

Example 1.1. Let us assume that t denotes time.

    1. The set of states is the set of real numbers $\mathbb{R}$. The rule is given by a differential equation. For example, something of the form:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F(u)$$

where $F : \mathbb{R} \to \mathbb{R}$ is a function. Time t is called a *continuous parameter* in this system and such a system is called a *continuous time* dynamical system.

    2. Our set of states is again the set of real numbers $\mathbb{R}$. Our rule is as follows:

$$x_n = f(x_{n-1})$$

where $f : \mathbb{R} \to \mathbb{R}$ is a function and $n \in \mathbb{N}$. The index n takes the role of t in 1. above. Note that this is a discrete parameter. Such a dynamical system is therefore called a *discrete time* dynamical system.
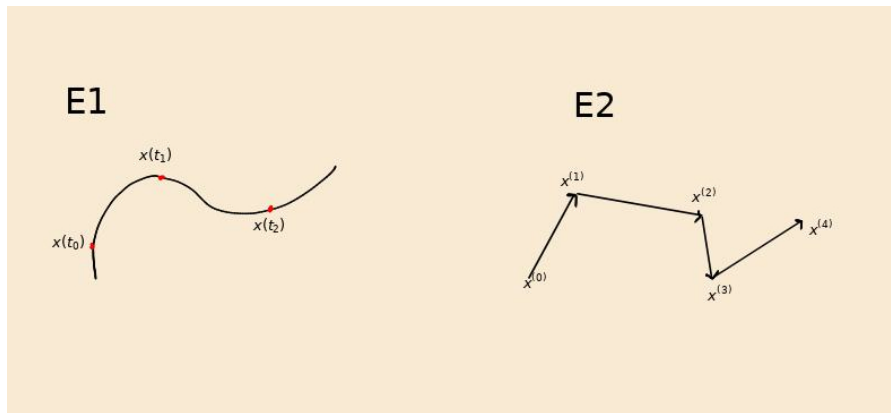
FIGURE 1.1: Continuous and Discrete dynamical systems

Initially, we will consider discrete time dynamical systems as they are simpler to study. We will then consider continuous time dynamical system.

All dynamical systems in the course will have the following property:

> the present state is uniquely determined by past states.

Dynamical systems with this property are called *deterministic*.

More generally, we have *random* or *stochastic processes* where the current state is stochastically determined from past states e.g. by flipping a coin or rolling a die. We will not be studying these.

We begin out study of discrete time dynamical systems with *one-dimensional maps*.

1.1.1 One-dimensional (1D) maps

First, the definition of a map.

Definition 1.2. A function whose domain (input) space and range (output) space are the same, is called a *map*.

A 1D-map is a map with domain space (and so range space) equal to a subset of $\mathbb{R}$.

> **Note**
> In this module, unless explicitly stated otherwise, **all** functions will be differentiable infinitely often (so-called *smooth* functions).

Example 1.2. Consider the 1D-map

$$f : \mathbb{R} \to \mathbb{R}, \ x \mapsto 2x.$$

This gives rise to a discrete time dynamical system:

$$x_n = 2x_{n-1}.$$

We can imagine that this models how a population of bacteria grows with time. At some initial time, 1, we have an initial population x_1 of bacteria; at each time step (say hourly intervals), the population of bacteria is twice the population at the previous time step.

In general, so long as we know the value of the initial population, x_1, we can calculate the population at all later times by successively applying the map f. For instance:

$$x_2 = f(x_1) = 2x_1,$$
$$x_3 = f(x_2) = f(f(x_1)) = f^2(x_1) = 4x_1$$
$$\vdots$$
$$x_n = f(x_{n-1}) = f^2(x_{n-2}) = f^{n-1}(x_1) = 2^{n-1}x_1$$

As can be seen, in this discerete time dynamical system, the population will grow without bound. This type of growth, in which the population is *multiplied by a constant factor at each time step* (2 in this example) is called *exponential growth*.

Example 1.2 is rather unphysical since in reality population growth is limited (e.g by a finite food supply). Perhaps we can find a better model by introducing to a term to account for increased competition for resources.

Example 1.3 (The logistic map). Define a map $g : [0, 1] \to [0, 1]$ by $g(x) = 2x(1 - x)$. We have a corresponding discrete time dynamical system

$$x_n = g(x_{n-1}) = 2x_{n-1}(1 - x_{n-1}).$$

We consider the behaviour of g in two different regimes:

- for x very small, $g(x) = 2x(1-x) \simeq 2x$
- for x very close to but strictly less than 1, $g(x) = 2x(1-x) \simeq 2(1-x) < 1$.

We see that for small x we have a similar doubling behaviour as in Example 1.2, however, for large x the population begins to decrease.

This new behaviour is due to the non-linear dependence of g in x. It is an extremely important ingredient for the rich behaviour of **all** dynamical systems.

1.1.2   Orbits and fixed points

We introduce some properties of maps.

Definition 1.3. Let f be a map and x a point in the domain space of f. The *orbit* of x under f is the set

$$\{x, f(x), f^2(x), f^3(x), ...\}$$

of iterates of x under f. The starting point x of the orbit is called the *initial value* of the orbit.

A point p in the domain space of f is called a *fixed point* of f if its orbit has size 1, that is $f(p) = p$.

Let us compute the orbit of a generic point $x_0 \in [0,1]$ under the logistic map of Example 1.3:

$$
\begin{aligned}
n = 1, \quad & x_1 = 2(1-x_0)x_0 \\
n = 2, \quad & x_2 = 2(1-x_1)x_1 = 2(1-2(1-x_0)x_0) \cdot 2(1-x_0)x_0 \\
n = 3, \quad & x_3 = 2(1-x_2)x_2 = ... \\
\vdots \quad &
\end{aligned}
$$

One might hope that such an orbit will approach a specific value. We can perform some computational experiments.

FIGURE 1.2: Computational experiments

Code Challenge

The code for the plot is shown below. Do experiment with the code by changing the parameter and initial value. Are you able to write a more efficient code?

Show Python Code on P138

R variant

The same graph can be plotted in R, although we will mainly use python for our computational experiments in this module.

Show R code on P140

Figures 1.2 above shows the orbits under the logistic map with initial values 0.1 and 0.9 respectively. We see that the orbits converge towards the value $x = 1/2$, that is $\lim_{n \to \infty} f^n(x_0) = 1/2$. The point $1/2$ is in fact a fixed point of the discrete time dynamical system corresponding to the logistic map.

We can explicitly compute the fixed points of the logistic map by solving the equation $g(p) = 2(1-p)p = p$. This gives rise to the quadratic

$$2p^2 - p = 0$$

which has solutions $p = 0$ and $p = 1/2$.

It is in fact the case that for all initial values (except 0 and 1 which converge to 0), the system converges to the fixed point at $x = 1/2$. (The reader is encouraged to perform their own experiments with different initial values either in R or in Python in order to confirm this.) Thus, the system settles to a steady state (a point from which it never changes) for all initial values.

1.2 Cobweb Plots

Cobweb plots are a tool that enable us to graphically visualise the orbit of a point under a one-dimensional map. They can be useful in investigating the qualitative behaviour of the related dynamical system. For instance, in subsection 1.1.2 we investigated orbits of the the logistic map with parameter 2 with different initial values. We saw that for all initial values in the open interval $(0, 1)$, the orbits converged to the fixed point at $1/2$. Cobweb plots enable us to infer such long-term status of an initial condition under repeated application of a map.

1.2.1 Constructing cobweb plots

We construct a cobweb plot for a generic 1D map $f : U \to U$ (where U is a subset of R) with an initial value x_0 as follows:

> 0. First we plot the the curve representing $y = f(x)$ and the line $y = x$.
> 1. Draw a vertical from the point $(x_0, 0)$ on the x-axis until we hit the curve $y = f(x)$ at the point $(x_0, f(x_0))$
> 2. Draw a horizontal line from the point $(x_0, f(x_0))$ to the line $y = x$ (this of course hits the line at coordinate $(f(x_0), f(x_0))$).
> 3. Draw a vertical line from the point $(f(x_0), f(x_0))$ on the line $y = x$ to the point $(f(x_0), f(f(x_0)))$ on the curve $y = f(x)$.
> 4. Repeat steps 2-4 with $(f(x_0), f(f(x_0)))$ in place of $(x_0, f(x_0))$.

Notice that points on the intersection of the line $y = x$ with the curve $y = f(x)$ are fixed points. We characterise two types of fixed points based on the behaviour of the cobweb plot near the fixed point. Fixed points which have an "inward spiral" (that is if we construct the cobweb plot with an initial value closed to the fixed point, we spiral in towards the fixed point) are called *stable/attracting fixed points*. Fixed points with an "outward spiral" (cobweb plots with an initial value closed to the fixed point, spiral out and away from the fixed point) are called *unstable/repelling fixed points*.

Let us construct the cobweb plot for the logistic map $g(x) = 2x(1-x)$.
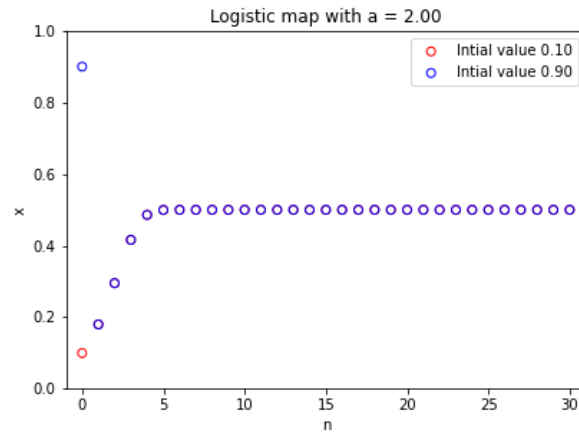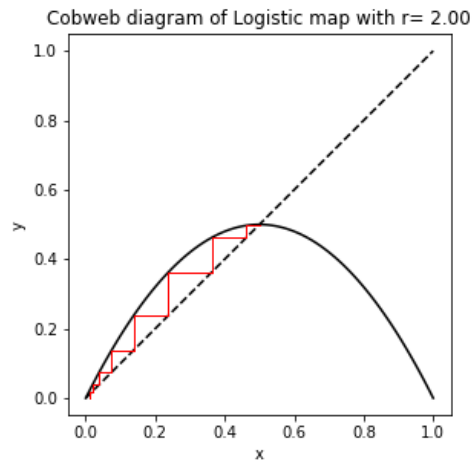

FIGURE 1.3: Cobweb plot of the logistic map with parameter 2

Code Challenge

The code for the plot is shown below. Do experiment with the code by changing the parameter and initial value.

Show Code on P142

We can experiment with different parameter values (and feel free to make your own experiments with the code).
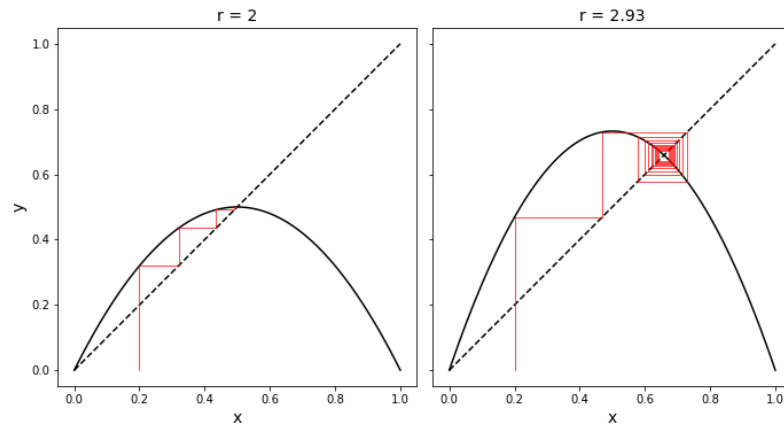
FIGURE 1.4: Cobweb comparison for different parameter values

Figure 1.4 shows cobweb plots of the logistic map with parameters 2 and 2.93. We see that for parameter 2.93 convergence to the fixed point at 0.6587 is slower than for parameter $r = 2$ with the same initial starting point of 0.2.

We can fix the logistic parameter and vary the starting points instead as in Figure 1.5.
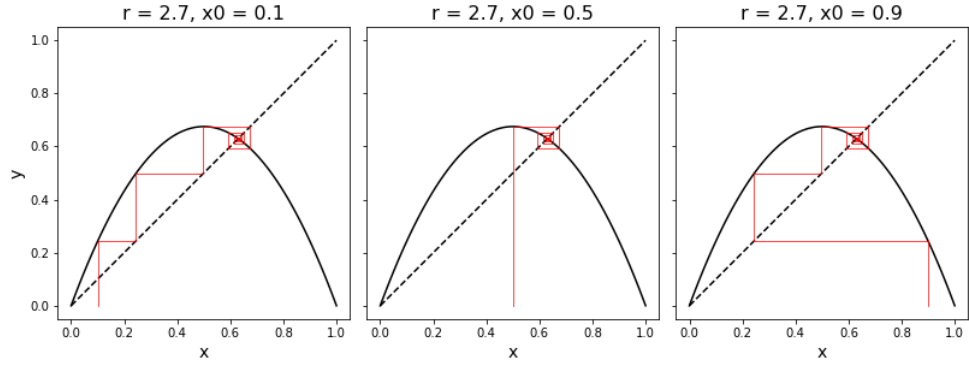
FIGURE 1.5: Cobweb comparison for different initial values

1.2.2   Stability of fixed points

We characterised fixed points by investigating the properties of the cobweb dia-
gram 'near' the fixed points. Let us make this a little more precise.

For $\epsilon > 0$, write $N_\epsilon(p)$ for the set of points which are a distance of at most ϵ
from p. That is

$$N_\epsilon(p) = \{x \in \mathbb{R} : |x - p| < \epsilon\}.$$

We call $N_\epsilon(p)$ the ϵ-*neighbourhood at* p.

We now give a formal definition of attracting and repelling fixed points using this
notation.

Definition 1.4. Let $U \subseteq \mathbb{R}$ and $f : U \to U$ be a map with a fixed point p.
Then p is a called a *sink/attracting fixed point* if there is an $\epsilon > 0$ such that
$\lim_{k \to \infty} f^k(x) = p$ for all x in the domain of f which are in $N_\epsilon(p)$. If there is an
$\epsilon > 0$ such that for all $x \neq p$ in the domain of f which also lie in $N_\epsilon(p)$, there is
a $k \in \mathbb{N}$ with $f^k(x) \notin N_\epsilon(p)$, then p is called a *source/repelling fixed point*.

Thus points near to attracting fixed points draw nearer and nearer to the fixed

points, while points close to repelling fixed points are repelled further and further from the fixed point.

Suppose we have a fixed point p and an initial value $x_{n-1} = p + \epsilon$ close to the fixed point ($\epsilon \ll 1$). We calculate $x_n = f(x_{n-1}) = f(p + \epsilon)$. Taking a Taylor expansion for f about p we estimate

$$x_n = f(p) + f'(p)\epsilon + f''(p)\frac{\epsilon^2}{2} + ...$$

. Using the fact that p is a fixed point, we have $x_n \simeq p + f'(p)\epsilon$ and so

$$|x_n - p| \simeq |f'(p)||\epsilon|.$$

Thus if the derivative of f at p is strictly greater than 1, we expect the distance $|x_n - p|$ to grow and p to be a source; if the derivative of f at p is strictly less than 1 on the other hand, we expect the distance $|x_n - p|$ to decrease and p to be a sink. This is exactly what Theorem 1.1 says.

Theorem 1.1 (Stability of fixed points). *Let $U \subseteq \mathbb{R}$ and $f : U \to U$ be a map with a fixed point p.*

1. *If $|f'(p)| < 1$ then p is a sink.*
2. *If $|f'(p)| > 1$ then p is a source.*

NOTE

It is the derivative of f *at* the fixed point p that determines the stability of p.

Theorem 1.1 is a one way implication. That is, the theorem *does not* imply that if p is a sink then $|f'(p)| <$ or if p is a source then $|f'(p)| > 1$. Although the definition of p simply requires finding an ϵ-neighbourhood at p for which points are attracted to p under iterations of f, this does not rule out the possibility that some points *outside* the ϵ-neighbourhood are attracted to p under iterations of the map f.

Definition 1.5. Let f be a 1D map and p an attracting fixed point of f. The set of *all* points which are attracted to p under iterations of f is called the *basin of attraction of p*. That is, the basin of attraction of a sink p is the set of all points x for which $\lim_{k \to \infty} f^k(x) = p$.

The basin of attraction can exhibit many interesting properties especially at its boundary and we will investigate these in subsequent lectures.

1.3 Problem Sheet 1

For Week 2.

Question 1

Consider the 1D map

$$f(x) = ax + x^3$$

where a is a real parameter.

- i. Find the fixed points of f.
- ii. Investigate how the stability of the fixed points depends on a.
- iii. For $a = 1$ there is exactly one fixed point. Is this fixed point stable or unstable?

Question 2

Find the fixed points of the map

$$f(x) = 2x^2 - 5x$$

and check their stability.

Verify that $\{1-\sqrt{2}, 1+\sqrt{2}\}$ is a period-2 orbit of the map. Is this period-2 orbit stable.

Question 3

Assume the growth of a population of bacteria is described by the discrete time dynamical system

$$x_n = \frac{2}{1 + \frac{x_{n-1}}{k}} x_{n-1},$$

where x_n is the population of bacteria at time n and k is a positive constant.

i. How does the system evolve in the two limiting cases

   a. x_n small ($x_n/k \ll 1$),
   b. x_n large ($x_n/k \gg 1$).

ii. For an arbitrary initial population x_0, find the value of x_n for every n.

   [Hint: Use $y_n = 1/x_n$ to rewrite the equation of the system.]

iii. What is the limiting value of x_n as $n \to \infty$?

iv. Determine the fixed points of the system and their stability. Discuss the connection between the exact solution and the fixed points.

2

Periodic points and stability

Covers Section 1.4 of Chapter 1 to Section 2.5 Definition 2.12 of Chapter 2 (Alligood et al., 2000).

In Chapter 1 we investigated the logistic map mainly for parameters in the open interval $(2, 3)$. We saw largely uniform behaviour for parameters in this interval, namely that all values in the interval $(0, 1)$ converged to a single attracting fixed point. However, we get quite different behaviour when we consider logistic parameter bigger than 3. Indeed applying Theorem 1.1 to the logistic map $g_r(x) = rx(1-x)$ for an arbitrary value of r we see that the stability of the fixed points depends on the parameter r.

The fixed points of $g_r(x)$ are obtained by solving $p = rp(1 - p)$. Rearranging yields:
$$p(r(1 - p) - 1) = 0.$$

This has solutions $p = 0$ and $p = 0$ and $p = \frac{r-1}{r}$ (of course r is non-zero). Computing the derivative of g_r at these fixed points, we obtain:

$$g'(0) = r \qquad \text{and} \qquad g'\left(\frac{r-1}{r}\right) = 2 - r.$$

Therefore in order to have a stable fixed point at zero we require $|r| < 1$; the non-zero fixed point is stable/attracting when $1 < r < 3$. We expect therefore to have different behaviour for $r > 3$. Let us investigate what happens for r just bigger than 3.
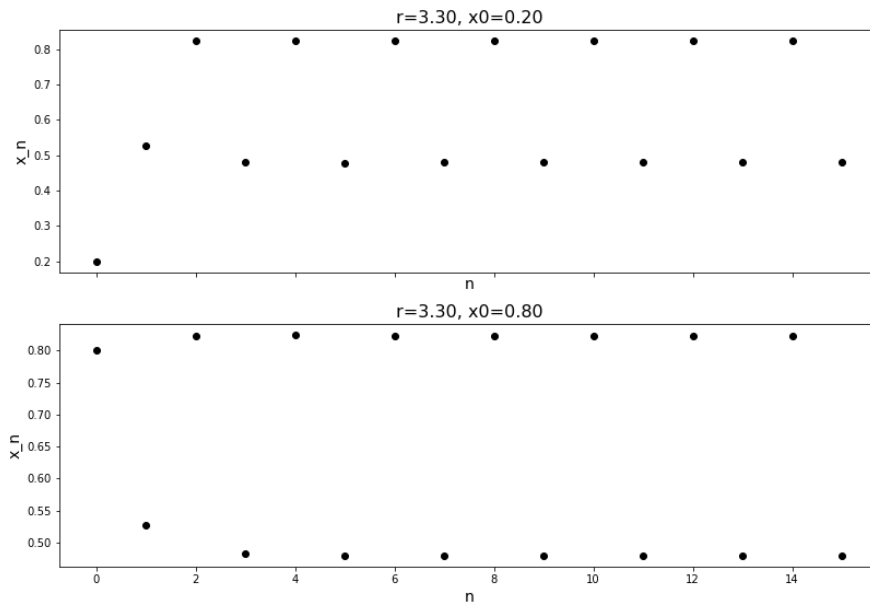
FIGURE 2.1: Orbits of logistic map for r =3.3

The plots will seem to indicate that for initial values in $(0, 1)$ orbits eventually oscillate between two values. This is indeed what happens.

2.1 Stability of periodic points

We introduce terminology to characterise this new type of orbit.

Definition 2.1. Let f be a map. We call a point p in the domain of p a *periodic point of period k/period-k point* if

$$f^k(p) = p$$

and k is the minimal positive integer for which this happens. The orbit with initial point p is called a *periodic orbit of period k/period-k orbit.*

In other words, a periodic point is a point which will repeat itself under iterations of the dynamical system. A dynamical system which exhibits a stable periodic orbit is often called an *oscillator*. Notice that a periodic point of period k is simply a fixed point of f^k.

Definition 2.2. Let f be a map and p a period-k point. Then the period-k orbit of p is a *sink/periodic sink/stable* if p is a sink of f^k. Likewise, the period-k orbit of p is a *source/periodic source/unstable* if p is a source of f^k.

Definition 2.2 is tricky to apply in practise as it requires computing f^k. At this stage it is useful to recall Theorem 1.1 which gives a test of stability of a map f at a point p depending on the size of the derivative of the map at the fixed point p. We can apply this test to determine stability of a periodic orbit so long as we are able to compute $(f^k)'$. Thankfully, this is can be done by repeated application of the chain rule.

Recall that the chain rule states that $(f \circ g)'(x) = f'(g(x))g'(x)$. Applying the chain rule to f^2 gives:

$$(f^2)'(x) = f'(f(x))f'(x). \tag{2.1}$$

Consider two points $p := p_1$ and p_2 in a period-2 orbit of a 1D map f. Equation (2.1) implies

$$(f^2)'(p) = f'(p_2)f'(p_1).$$

Thus if

$$|f'(p_1)f'(p_2)| < 1$$

then the periodic orbit $(p = p_1, p_2)$ of period 2 is stable.

The above can be generalised by induction to an orbit of period k.

> ### Stability test for periodic orbits
> Given a one-dimensional map f with a periodic orbit $(p_1, p_2, \dots, p_k)$ of period k, then
>
> $$(f^k)'(p_1) = f'(p_k)f'(p_{k-1}) \dots f'(p_1).$$
>
> Consequently, if
>
> $$|f'(p_k)f'(p_{k-1}) \dots f'(p_1)| < 1$$
>
> the periodic orbit $(p_1, p_2, \dots, p_k)$ is stable/a sink; the period-k orbit $(p_1, p_2, \dots, p_k)$ is unstable/a source if
>
> $$|f'(p_k)f'(p_{k-1}) \dots f'(p_1)| > 1.$$

Example 2.1 (Periodic 2 orbits of the logistic map)**.** Let us compute the period 2 orbits of the logistic map $g(x) = rx(1-x)$ for $r > 0$. These of course will be the fixed points of g^2 except those which are also fixed points of g (0 and $\frac{r-1}{r}$). We compute

$$g^2(x) = g(g(x)) = g(rx(1-x)) = r^2 x(1-x)(1 - rx(1-x)).$$

Thus we want to find values p such that $p = r^2 p(1-p)(1 - rp(1-p))$.

Rearranging we have

$$p\left(r^2(1-p)(1 - rp(1-p)) - 1\right) = 0.$$

We now make use of the fact that both p and $r(1-p) - 1$ are roots of $r^2 p(1-p)(1 - rp(1-p))$ (since $p = 0$ and $p = \frac{r-1}{r}$ are both fixed points of g^2) to deduce that $r(1-p) - 1$ is a factor of $r^2(1-p)(1 - rp(1-p)) - 1$. Thus there are $a, b, c \in \mathbb{R}$ such that

$$(r(1-p) - 1)(ap^2 + bp + c) = r^2(1-p)(1 - rp(1-p)) - 1.$$

We compare coefficients to figure out what a, b, c are.

$$p^3 : -ar = -r^3$$
$$p^2 : ar - br - a = 2r^3$$
$$p^1 : -rc - b + br = -r^2 - r^3$$
$$p^0 : rc - c = r^2 - 1$$

Looking at the p^3 and p^0 coefficients we see that $a = r^2$ and $c = r + 1$. From either the p coefficients or p^2 coefficients we see that $b = -r(r+1)$.

Putting these together we get:

$$r^2(1-p)(1 - rp(1-p)) - 1 = (r(1-p) - 1)(r^2 p^2 - r(r+1)p + r + 1).$$

We find the period 2 points by factorising the term $r^2 p^2 - r(r+1)p + r + 1$. We can do so using the quadratic formula to get:

$$\begin{aligned}
p_{1,2} &= \frac{r(r+1) \pm \sqrt{r^2(r+1)^2 - 4r^2(r+1)}}{2r^2} \\
&= \frac{r + 1 \pm \sqrt{(r+1)^2 - 4(r+1)}}{2r} \\
&= \frac{r + 1 \pm \sqrt{(r+1)(r-3)}}{2r}.
\end{aligned}$$

Observe that if $r < 3$ then the above has no real solutions.

Let us construct a cobweb plot to investigate the stability of the period two points for a parameter value just bigger than 3.
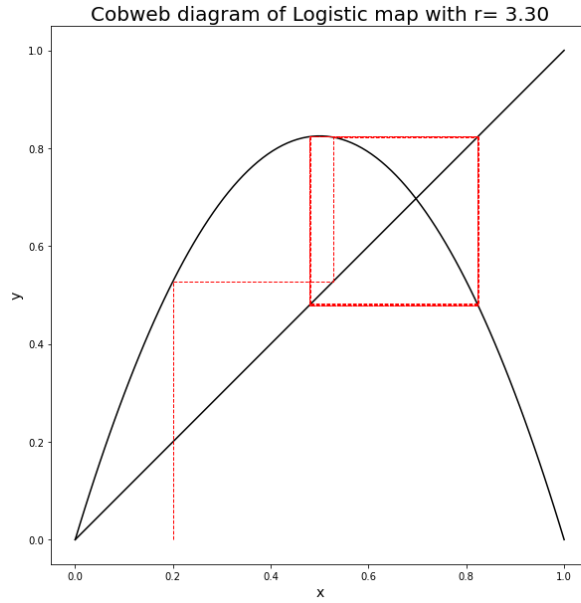
FIGURE 2.2: Cobweb diagram illustrating stable period 2 orbit.

Figure 2.2 demonstrates that for $r = 3.3$ the period 2 points are stable.

For the generic logistic map, stability of the period 2 orbits depends on the parameter r. First we compute $g'(x) = r(1 - 2x)$.

Thus,

$$
\begin{aligned}
|g'(p_1)g'(p_2)| &= |g'(p_1)||g'(p_2)| \\
&= r^2(1 - 2p_1 - 2p_2 + 4p_1p_2) \\
&= r^2 - 2r(r + 1) + 4(r + 1) \\
&= |-r^2 + 2r + 4|
\end{aligned}
$$

Thus, for stability of the period two points, we require $|-r^2 + 2r + 4| < 1$ which gives $3 < r < 1 + \sqrt{6}$. We note that for these values of r the period 1 points are unstable fixed points.

2.2 The family of logistic maps

Let us summarise what we have learnt about the logistic map $g(x) = rx(1-x)$ so far. We have shown that for values of r between $0 \leq r < 1$ there is a single sink at $x = 0$. For $1 < r < 3$, we have seen that there is a single sink at $x = \dfrac{r-1}{r}$. Lastly in Subsection 2.1 we saw that period 2 orbits emerge for $r > 3$ and these are stable when $3 < r < 1 + \sqrt{6}$. We now investigate what happens when $r \geq 1 + \sqrt{6}$. Of course we know that the period 2 orbits become unstable in this regime. Overall, the behaviour of the dynamical system changes significantly. We get what is called *bifurcation*, more precisely *period-doubling bifurcation/ period-doubling cascade*.

Let us introduce some terminology for this new dynamic.

Definition 2.3 (Period-doubling bifurcation)**.** A change from a stable periodic orbit of size N to a stable periodic orbit of size $2N$ which occurs when the parameter r is increased.

Definition 2.4 (Birfucation diagram)**.** A diagram that summarises the succession of period-doubling produced as r is increased. The diagram is constructed using the following steps:

0. Pick an initial value of r.
1. Choose a random $x \in (0,1)$.
2. Calculate the orbit of x under the logistic map $g(x)$.
3. Ignore the firs 100 orbits (to give time to settle to a periodic sink) and plot the the orbit beginning with iterate 101.
4. Increment r and repeats Steps 1 - 5.

In Figure 2.3 we plot the bifurcation diagram of the logistic map. Feel free to copy the code in order to perform your own experimentation.
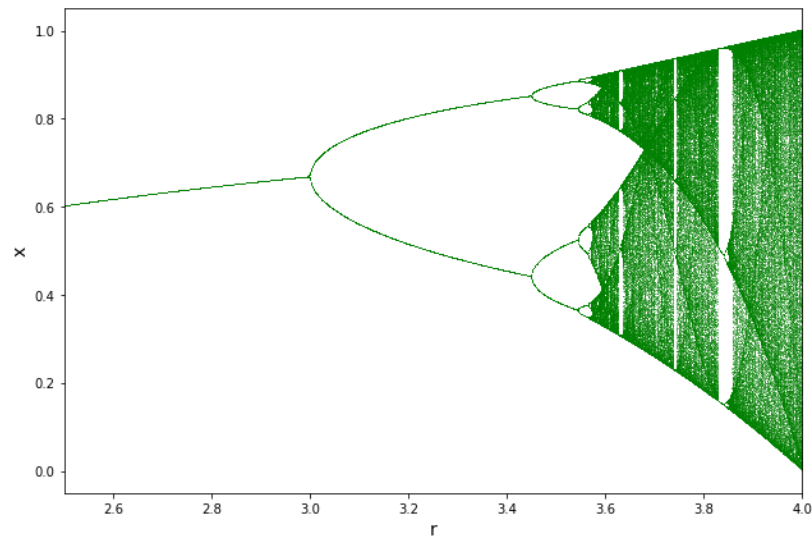
FIGURE 2.3: Bifurcation diagram of the logistic map

Code Challenge

The code for the plot is shown below. Do experiment with the code by changing the parameter range and initial values. Are you able to write a more efficient code/produce a more artistic diagram?

Show Code on P144
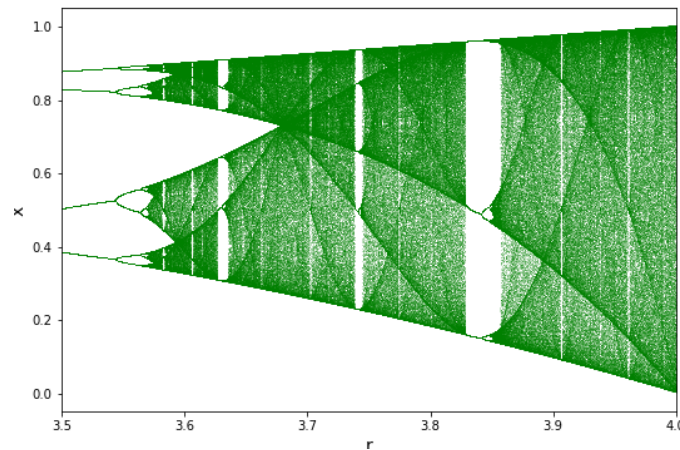
FIGURE 2.4: Bifurcation diagram of the logistic map zooming in on the range (3.5,4)

We point out some key features of the bifurcation diagram 2.3

- For $r < 1$, there is one attractor namely the sink at 0.
- For $1 < r < 3$ there is one attractor $\frac{r-1}{r}$.
- The first bifurcation occurs at r=3 and subsequently at r= 3.45, 3.54, 3.564, 3.569 and so on until just beyond $r \simeq 3.57$. That is at $r = 3$ we have the first period doubling from a period one sink to a period two sink, then at the next bifurcation an period 4 sink appears, then a period 8 sink at the next and so on illustrating the period doubling cascade mentioned above.
- For $r \gtrsim 3.57$ the diagram becomes very complicated. The system becomes *chaotic*. A key feature of this chaotic behaviour is that a small change in a state of the system (e.g. a small change in initial condition) can result in large differences in the next state. These changes are such that the system's behaviour is unpredictable in general.
- At several values of r just beyond 3.57 a small number of x values are produced — these are the *white spaces* in the diagram. In these regions no fixed points or attracting orbits seem to exist. On the other hand, in the same r range, there are also *black spaces* filed in with attracting points/orbits these are called *attracting sets/ chaotic attractors*.
- A close-up of the bifurcation diagram shows that at $r = 3.83$, a period 3 attractor emerges which also exhibits a period-doubling cascade to period 6 orbits, then period 12 and so on.
- Generally, in the range $(3.57, 4)$ there is a rich interleaving of "chaos" and "order". A small change in r can make a stable system chaotic and vice versa.
- The diagram stops at $r = 4$. After this point, no attracting sets exist.

2.2.1 Initial conditions and Chaos

What do we mean by "chaotic behaviour"? There are two key features:

1. Non-periodicity.
2. Sensitive dependence on initial conditions.

Let us say a little more about the 2nd condition. Suppose we start and orbit from an initial point $x_0^{(1)}$ and another orbit from an initial point very close to the first $x_0^{(2)} = x_0^{(1)} + \epsilon$ where $0 < |\epsilon| \ll 1$. If the orbits move strongly away from each other with n, that is, $|x_n^{(1)} - x_n^{(2)}|$ "becomes big", then the system is sensitive to initial conditions. In such situations making long-term predictions of the behaviour of the dynamical system becomes very difficult, as slightly different initial conditions can result vastly different long-term behaviour.

More precisely, we have the following definition.

Definition 2.5 (Sensitive dependence on initial conditions)**.** Let f be a 1D map. A point x_0 has *sensitive dependence on initial conditions* if there is a $d > 0$ such that for any $\epsilon > 0$ the ϵ-neighbourhood $N_\epsilon(x_0)$ of x_0 contains a point x (in the domain of f) such that $|f^k(x) - f^k(x_0)| \geq d$ for some non-negative integer k. That is, there are points arbitrarily close to x_0 which are eventually mapped at least d units from the corresponding image of x_0. Note that generally the closer x is to x_0 the larger k will need to be. Such a point x_0 is sometimes called a *sensitive point*.

Figure 2.5 shows two orbits with initial values $x_0^{(1)} = 0.3$ and $x_0^{(2)} = 0.3 + 0.001$. We can see that there are several points at which corresponding images of both orbits are a distance 0.5 (for instance) from one another.
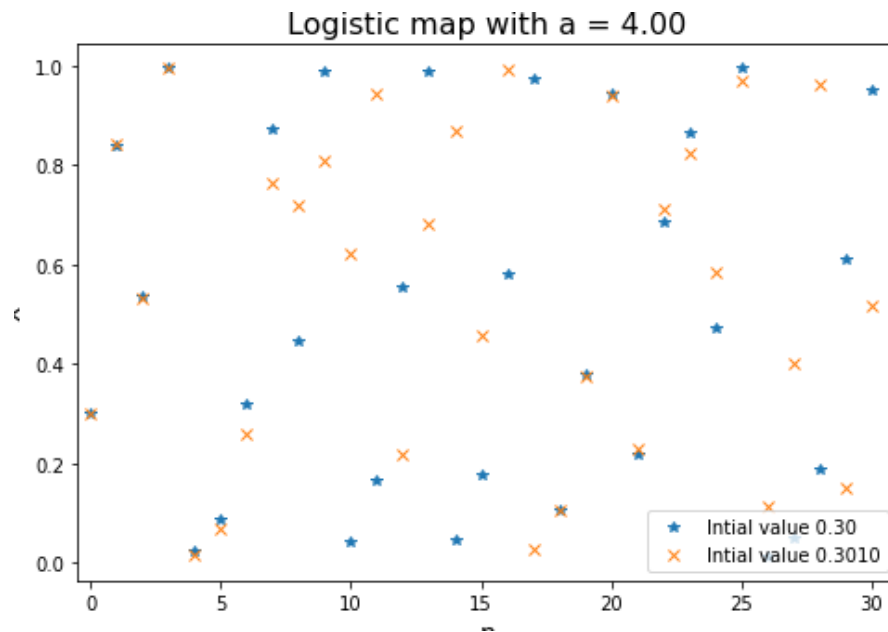
FIGURE 2.5: Two orbits of the logistic map with parameter 4 with close initial points.

Perhaps the most interesting chaotic dynamical systems are those that exhibit attracting behaviour (for example the logistic map with parameter values close (and equal) to 4). That is, systems with orbits whose trajectories converge to an attractor.

2.3 Higher-dimensional maps

Now that we have established some key features of dynamical systems associated to one-dimensional maps, we begin to think about higher-dimensional maps. The features introduced for 1D maps are instances of more general phenomena in higher dimensions.

Below is an example of the dynamical system associated to a 2-dimensional map.

Example 2.2 (Hènon's map)**.** Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (a - x^2 + by, x)$. This gives rise to the dynamical system:

$$x_{n+1} = a - x_n^2 + by_n$$
$$y_{n+1} = x_n$$

2.3.1 Sinks, sources and saddles

Let us define the notion of an ϵ-neighbourhood in higher dimensions. Let $\epsilon > 0$, then we write:

$$N_\epsilon(\boldsymbol{p}) := \{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x} - \boldsymbol{p}| < \epsilon\}$$

that is $N_\epsilon(\boldsymbol{p})$ is the set of points in $\mathbb{R}^n$ which are a distance at most ϵ from p.

We can now define sources, and sinks as before.

Definition 2.6. Let $\boldsymbol{f}$ be a map defined on a subset of $\mathbb{R}^n$ and let $\boldsymbol{p}$ be a fixed point of $\boldsymbol{f}$. If there is an ϵ-neighbourhood $N_\epsilon(\boldsymbol{p})$ of p such that for all $\boldsymbol{v}$ in the domain of $\boldsymbol{f}$ with $\boldsymbol{v} \in N_\epsilon(\boldsymbol{p})$,

$$\lim_{k \to \infty} \boldsymbol{f}^k(\boldsymbol{v}) = \boldsymbol{p},$$

then $\boldsymbol{v}$ is called a *sink/attractor/stable fixed point*. If, on the other hand, there is an ϵ-neighbourhood $N_\epsilon(\boldsymbol{p})$ of $\boldsymbol{p}$ such that for all points $\boldsymbol{v} \neq \boldsymbol{p}$ in the domain of $\boldsymbol{f}$ with $\boldsymbol{v} \in N_\epsilon(\boldsymbol{p})$, there is a $k \in \mathbb{N}$ such that $f^k(\boldsymbol{v}) \notin N_\epsilon(\boldsymbol{p})$, then $\boldsymbol{p}$ is called a *source/repeller*.

Note that fixed points of a map $\boldsymbol{f}$ defined on a subset of $\mathbb{R}^n$ are exactly what we expect them to be, that is they are points $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$.

In higher dimensions a new dynamic emerges around fixed points giving rise to a new type of fixed point — a so called *saddle point*. Saddle points have at least one attracting direction and at least one repelling direction. Saddle points are *unstable fixed points* and are very important for chaotic behaviour.
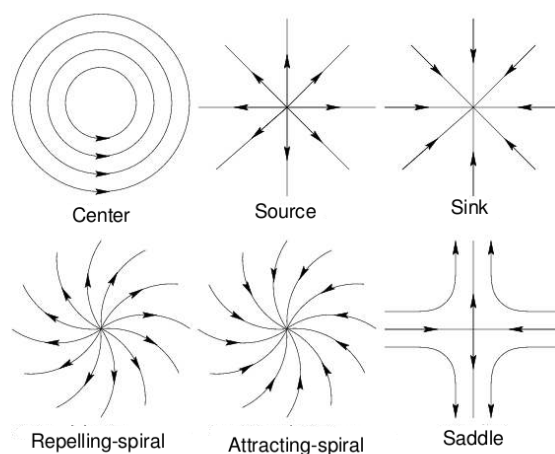
FIGURE 2.6: Higher-dimensional dynamics

Let $\boldsymbol{f} = (f_1, f_2, \dots, f_n)$ be a map on a subset of $\mathbb{R}^n$ and consider the associated dynamical system $\boldsymbol{x}_{n+1} = \boldsymbol{f}(\boldsymbol{x}_n)$. Suppose $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$ is a fixed point of $\boldsymbol{f}$. How do we determine the stability of this fixed point? Let us repeat a strategy we applied fruitfully in the 1D case but now generalised suitably for higher dimensional maps. Consider an initial value $\boldsymbol{x}_0 = \boldsymbol{p} + \boldsymbol{\epsilon}$ close to the fixed point $\boldsymbol{p}$ where $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \dots, \epsilon_n) \in \mathbb{R}^n$ and $|\boldsymbol{\epsilon}| \ll 1$. We can approximate the next point $\boldsymbol{x}_1 = \boldsymbol{f}(\boldsymbol{x}_0)$ in the orbit of $\boldsymbol{x}_0$ by taking a multivariate Taylor expansion of $\boldsymbol{f}$ about the point $\boldsymbol{p}$. That is we have:

$$\boldsymbol{x}_1 = \boldsymbol{f}(\boldsymbol{p}) + \sum_{i=1}^{n} \frac{\partial \boldsymbol{f}(\boldsymbol{p})}{\partial x_i} \epsilon_i + \ldots$$

$$= \boldsymbol{f}(\boldsymbol{p}) + \begin{pmatrix} \sum_{i=1}^{n} \dfrac{\partial f_1}{\partial x_i}(\boldsymbol{p})\epsilon_i \\ \sum_{i=1}^{n} \dfrac{\partial f_2}{\partial x_i}(\boldsymbol{p})\epsilon_i \\ \vdots \\ \sum_{i=1}^{n} \dfrac{\partial f_n}{\partial x_i}(\boldsymbol{p})\epsilon_i \end{pmatrix} + \ldots$$

$$\simeq \boldsymbol{f}(\boldsymbol{p}) + \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(\boldsymbol{p})\epsilon_1 + \dfrac{\partial f_1}{\partial x_2}(\boldsymbol{p})\epsilon_2 + \ldots + \dfrac{\partial f_1}{\partial x_n}(\boldsymbol{p})\epsilon_n \\ \dfrac{\partial f_2}{\partial x_1}(\boldsymbol{p})\epsilon_1 + \dfrac{\partial f_2}{\partial x_2}(\boldsymbol{p})\epsilon_2 + \ldots + \dfrac{\partial f_2}{\partial x_n}(\boldsymbol{p})\epsilon_n \\ \vdots \\ \dfrac{\partial f_n}{\partial x_1}(\boldsymbol{p})\epsilon_1 + \dfrac{\partial f_n}{\partial x_2}(\boldsymbol{p})\epsilon_2 + \ldots + \dfrac{\partial f_n}{\partial x_n}(\boldsymbol{p})\epsilon_n \end{pmatrix}$$

$$= \boldsymbol{p} + \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(\boldsymbol{p}) & \dfrac{\partial f_1}{\partial x_2}(\boldsymbol{p}) & \ldots & \dfrac{\partial f_1}{\partial x_n}(\boldsymbol{p}) \\ \dfrac{\partial f_2}{\partial x_1}(\boldsymbol{p}) & \dfrac{\partial f_2}{\partial x_2}(\boldsymbol{p}) & \ldots & \dfrac{\partial f_2}{\partial x_n}(\boldsymbol{p}) \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1}(\boldsymbol{p}) & \dfrac{\partial f_n}{\partial x_2}(\boldsymbol{p}) & \ldots & \dfrac{\partial f_n}{\partial x_n}(\boldsymbol{p}) \end{pmatrix} \cdot \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix}$$

The matrix

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(\boldsymbol{p}) & \dfrac{\partial f_1}{\partial x_2}(\boldsymbol{p}) & \ldots & \dfrac{\partial f_1}{\partial x_n}(\boldsymbol{p}) \\ \dfrac{\partial f_2}{\partial x_1}(\boldsymbol{p}) & \dfrac{\partial f_2}{\partial x_2}(\boldsymbol{p}) & \ldots & \dfrac{\partial f_2}{\partial x_n}(\boldsymbol{p}) \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1}(\boldsymbol{p}) & \dfrac{\partial f_n}{\partial x_2}(\boldsymbol{p}) & \ldots & \dfrac{\partial f_n}{\partial x_n}(\boldsymbol{p}) \end{pmatrix}$$

is called the *Jacobian (matrix) of* $\boldsymbol{f}$ *at* $\boldsymbol{p}$ and we write $\mathbf{D}\boldsymbol{f}(p)$ for this matrix. (The matrix is named after the mathematician Carl Gustav Jacob Jacobi). The matrix

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \ldots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \ldots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \ldots & \dfrac{\partial f_n}{\partial x_n} \end{pmatrix}$$

is the *Jacobian matrix of $\boldsymbol{f}$*. Write A for the matrix $\mathbf{D}\boldsymbol{f}(p)$, and recall (from MT3501 or MT2501) that the map $A : \mathbb{R}^n \to \mathbb{R}^n$ by $\boldsymbol{x} \mapsto A \cdot \boldsymbol{x}$ is a *linear map* that is, it satisfies the condition $A(a\boldsymbol{x} + b\boldsymbol{y}) = aA\boldsymbol{x} + bA\boldsymbol{y}$ for $a, b \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

Now if $A\epsilon$ remains small enough, we can repeat the process to approximate $\boldsymbol{x}_2 = \boldsymbol{f}(x_1)$ as $\boldsymbol{x}_2 \simeq \boldsymbol{p} + A^2\epsilon$. In particular, so long as $A^n\epsilon$ is small, $x_{n+1} \simeq A^{n+1}\epsilon$. Thus to understand when the distance $|\boldsymbol{x}_n - \boldsymbol{p}|$ grows or shrinks, we need to understand how $A = \mathbf{D}\boldsymbol{f}(p)$ acts on $\mathbb{R}^n$.

The effect of A on a small disc about centred at the origin can be determined by the *eigenvalues* of A. Recall that an eigenvector $\boldsymbol{v}$ of A with *eigenvalue* $\lambda \in \mathbb{R}$ is a vector satisfying the equation $A\boldsymbol{v} = \lambda\boldsymbol{v}$. Thus if $\boldsymbol{v}$ is an eigenvector of A with eigenvalue λ where $|\lambda| > 1$, then $|A\boldsymbol{v}| > |\boldsymbol{v}|$. In particular if we chose ϵ to be a multiple of an eigenvector $\boldsymbol{v}$ of A with eigenvalue λ, then

$$|\boldsymbol{x}_1 - \boldsymbol{p}| \simeq |A\epsilon| = |\lambda||\epsilon|.$$

More generally,

$$|\boldsymbol{x}_n - \boldsymbol{p}| \simeq |A^n\epsilon| = |\lambda|^n|\epsilon|.$$

Therefore, we get growth away from the fixed point if $|\lambda| > 1$ and if $|\lambda| < 1$ the orbit moves closer to the fixed point.

We can now state the following analogue of Theorem 1.1 which gives a stability test for higher dimensional maps.

Theorem 2.1. *Let $\boldsymbol{f}$ be a map on a subset of $\mathbb{R}^n$ and $\boldsymbol{p}$ a fixed point of $\boldsymbol{f}$.*

>   *1. If the magnitude of each eigenvalue of $\mathbf{D}\boldsymbol{f}(\boldsymbol{p})$ is strictly less than 1, then $\boldsymbol{p}$ is a sink.*
>   *2. If the magnitude of each eigenvalue of $\mathbf{D}\boldsymbol{f}(\boldsymbol{p})$ is strictly greater than 1, then $\boldsymbol{p}$ is a source.*
>   *3. If at least on eigenvalue of eigenvalue of $\mathbf{D}\boldsymbol{f}(\boldsymbol{p})$ has magnitude strictly bigger than 1, then $\boldsymbol{p}$ is unstable.*

We can further characterise fixed points based on the eigenvalues of the Jacobian.

Definition 2.7 (Classification of fixed points). Let $\boldsymbol{f}$ be a map on a subset of $\mathbb{R}^n$ and $\boldsymbol{p}$ be a fixed point of f. Then

>   1. $\boldsymbol{p}$ is *hyperbolic* if none of the eigenvalues of $\mathbf{D}\boldsymbol{f}(\boldsymbol{p})$ has magnitude 1;
>   2. if $\boldsymbol{p}$ is hyperbolic and $\mathbf{D}\boldsymbol{f}(\boldsymbol{p})$ has at least one eigenvalue with magnitude strictly bigger than 1 and at least one eigenvalue with magnitude strictly less than 1, then a $\boldsymbol{p}$ is called a *saddle (point)*.

As in Definition 2.2 we can extend the above definition to period-k orbits by replacing f with f^k. For instance a period k orbit with initial vector p is a saddle if p is a saddle of f^k.

2.4 Problem Sheet 2

For Week 3.

Question 1

Consider the 1D map f defined on the closed unit interval as

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2(1-x) & \text{if } 1/2 \le x \le 1. \end{cases}$$

a. Carefully sketch $y = f(x)$ for $0 \le x \le 1$ and the diagonal $y = x$. On the basis of your diagram how many fixed points do you expect the map to have. [Hint: A fairly accurate diagram is useful here. This question might be easier done via python.]

b. For each of the suspected fixed points of $f(x)$ chose an initial condition $x_1 \in [0, 1]$ close to the fixed point and sketch the first 3 steps of a cobweb plot (you can use the diagram sketched in part a.). Use your cobweb plots to comment on the expected stability properties of each fixed point.

c. Calculate all fixed points of the map and determine their stability. Briefly compare your result with what you expected from parts a. and b.

d. Find the period-2 orbit of this map and determine its stability.

Question 2

Given $x_n = ax_{n-1} - x_{n-1}^3$, $n = 1, 2, 3 \dots$ and $a \in \mathbb{R}$

i. Verify that $\{\sqrt{1+a}, -\sqrt{1+a}\}$ for $a > -1$ is a period-2 orbit.

ii. Investigate the stability of this orbit.

Question 3

Determine the eigenvalues and eigenvectors of the following matrices:

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \qquad B = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \qquad C = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}.$$

3

Higher-dimensional dynamics

Covers Chapter 2 Section 2.5 Example 2.13 to Section 2.6 Example 2.20 of Chapter 2 (Alligood et al., 2000).

3.1 Example

Let us apply Theorem 2.1 to an example we encountered earlier on: the Hènon map.

Example 3.1 (The Hènon map II). The Hènon map is the map $\boldsymbol{f}_{a,b}(x,y) = (a - x^2 + by, x)$. For this example we take $a = 0$ and $b = 0.4$ to get:

$$\boldsymbol{f}(x,y) = \boldsymbol{f}_{0,0.4}(x,y) = (-x^2 + by, x).$$

We compute the fixed points by solving the equation

$$\boldsymbol{f}(x,y) = (x,y).$$

From the second coordinates we find that $x = y$; the first coordinate then gives $x^2 - bx + x = 0$.

Solving this gives two solutions, $x = y = 0$ and $x = y = b - 1$.

To investigate the stability of the fixed point we need to compute the Jacobian of $\boldsymbol{f}$ at each of the two fixed points and compute the eigenvalues of the resulting two matrices. We write $\boldsymbol{f} = (f_1(x,y), f_2(x,y))$ where $f_1(x,y) = -x^2 + by$ and $f_2(x,y) = x$.

Thus,

$$\mathbf{D}\boldsymbol{f}(x,y) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix}$$

For $x = y = 0$**:**

We have

$$\mathbf{D}f(0,0) = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$

We find the eigenvalues by solving the characteristic equation: $\det(\lambda I - \mathbf{D}f(0,0)) = 0$. This gives:

$$\begin{vmatrix} \lambda & -b \\ -1 & \lambda \end{vmatrix} = \lambda^2 - b = 0$$

This has solutions $\lambda = \pm\sqrt{b}$. Clearly the magnitude of the eigenvalues, $\sqrt{0.4}$, is strictly less than 1 and so we conclude that $(0,0)$ is a sink.

For $x = y = b - 1$**:**

We have

$$\mathbf{D}f(-3/5, -3/5) = \begin{pmatrix} 2(1-b) & -b \\ -1 & 0 \end{pmatrix}$$

We solve the characteristic equation as before.

$$\det(\lambda I - \mathbf{D}f(b-1, b-1)) = \begin{vmatrix} \lambda + 2(b-1) & b \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 2(b-1)\lambda - b = 0.$$

This gives

$$\lambda = (1-b) \pm \sqrt{(b-1)^2 + b} = (1-b) \pm \sqrt{b^2 - b + 1}.$$

For $b = 0.4$, we get eigenvalues $\lambda_1 = (3 + \sqrt{19})/5$ and $\lambda_2 = (3 - \sqrt{19})/5$. Thus by Theorem 2.1 $(-0.6, -0.6)$ is a saddle point.

To get an idea of the dynamics of the Hènon map, we plot below a bifurcation diagram where we keep b fixed at 0.4 and allow a to vary between 0 and 1.25. On the y axis we plot only the x values after sufficiently many iterates have been obtained. Feel free to experiment with the code.
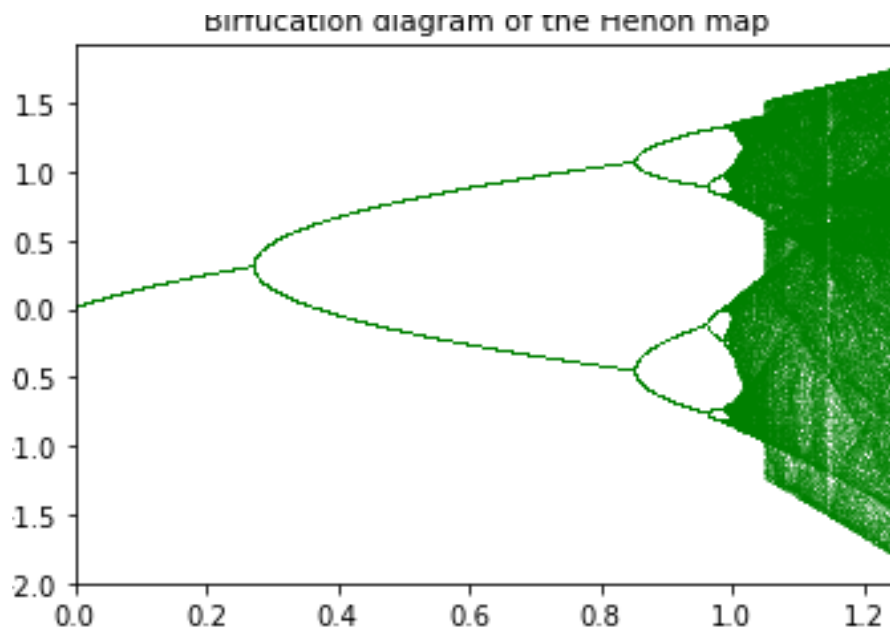
FIGURE 3.1: Bifurcation diagram of Hènon's map

> **Code Challenge**
>
> The code for the plot is shown below. Do experiment with the code by changing the parameter range and initial value. Are you able to write a more efficient code? How about randomising the initial x values?

Show code on P146

We make some observations on the bifurcation diagram 3.1.

- The bifurcation diagram shows *striking similarities* with the bifurcation diagram of the logistic map (Figure 2.3).
- At a=0.27, a *period-doubling* bifurcation occurs: the fixed point loses stability and a period 2 orbit (which is a sink) emerges.
- The period 2 orbit becomes unstable at 0.85, at which point a period 4 orbit, an attracting orbit, emerges and so on.
- In general, for $b = 0.4$ and $a > 0.85$, the attractors of the Hènon map become more complex.

3.1.1 Stable and Unstable Manifolds

In order to explore the dynamics of dynamical systems corresponding to higher dimensional maps, we require some more terminology – the notion of a *manifold*.

Loosely speaking an *n-dimensional manifold* is a set of point whose geometry/topology "locally resembles" $\mathbb{R}^n$. That is a small segment of the manifold looks like a small segment of $\mathbb{R}^n$. For instance a 1-dimensional manifold "locally resembles" a curve.

Examples of one dimensional manifolds are a circle "o" or the disjoint union of two circles. The letters "A" and "X" are not manifolds, for instance, a small neighbourhood at the centre of "X" where different line segments meet does not resemble a line segment. The letter S is not a manifold as zooming in near either of its end points, we resemble a half line instead of a line. If we remove the two end points of S, then it becomes a manifold. The letter S with its end points is a so-called *manifold with boundary*.

Examples of 2-dimensional manifolds are spheres, tori since locally they look like a *plane*. Note that the unions of two manifolds is again a manifold.

Let us now apply these definitions to dynamics of higher-dimensional maps. Recall that a fixed/periodic point of a map f is called a saddle/periodic saddle point if it has at least one attracting direction and one repelling direction. We can consider the set of all points which are attracted to the saddle point under iterations of the map, and likewise the set of all points which are repelled from the saddle under iterations of the map. These are respectively termed the *stable manifold* and *unstable manifold* of the saddle point p. The terminology is not accidental and we shall later see why stable and unstable manifolds of a fixed point are indeed manifolds.

For the moment let us illustrate a saddle point with an example.

Example 3.2 (Movement towards and away from a saddle point). Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(x, y) \mapsto (2x, y/2)$. This map has one fixed point at $x = y = 0$.

The Jacobian matrix of f at the point $(0,0)$ is the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

which has eigenvalues $(2, 1/2)$ (eigenvectors corresponding to the x and y axis respectively). Thus we see that along the x axis we have movement away from the saddle point at the origin and along the y axis we have movement towards the origin.
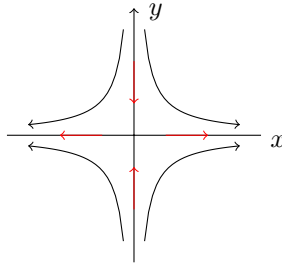
3.2   Stability of manifolds associated with a saddle

In this module most higher dimensional maps we consider will be *one-to-one/injective*. This property means that for most functions we consider, their *inverses* exist.

Definition 3.1. Let $\boldsymbol{f}$ be a map on a subset of $\mathbb{R}^n$. Then $\boldsymbol{f}$ is called *one-to-one/injective* if for $\boldsymbol{v}_1, \boldsymbol{v}_2$ in the domain of $\boldsymbol{f}$

$$\boldsymbol{f}(\boldsymbol{v}_2) = \boldsymbol{f}(\boldsymbol{v}_2) \implies \boldsymbol{v}_1 = \boldsymbol{v}_2.$$

That is, $\boldsymbol{f}$ is injective if and only if *distinct points* in its domain have *distinct* images.

The one-to-one property means that we can "reverse" the map in order to find the inverse of $\boldsymbol{f}$. That is if $\boldsymbol{f}(\boldsymbol{v}_1) = \boldsymbol{v}_2$, then $\boldsymbol{f}^{-1}(\boldsymbol{v}_2) = \boldsymbol{v}_1$. Being one-to-one guarantees that there is no ambiguity in this definition; there is one and only one element in the domain of $\boldsymbol{f}$ whose image under $\boldsymbol{f}$ is $\boldsymbol{v}_2$.

Let us now formally define stable and unstable manifolds of a saddle fixed/periodic point.

Definition 3.2 (Stable and unstable manifolds). Let $\boldsymbol{f}$ be a (smooth) one-to-one map on a subset of $\mathbb{R}^n$ and let $\boldsymbol{p}$ be a saddle fixed or saddle periodic point of $\boldsymbol{f}$.

The *stable manifold* of $\boldsymbol{p}$ denoted $\mathcal{S}(\boldsymbol{p})$ is the set of points $\boldsymbol{v}$ in the domain of $\boldsymbol{p}$ such that

$$\lim_{k \to \infty} |\boldsymbol{f}^k(\boldsymbol{v}) - \boldsymbol{f}^k(\boldsymbol{p})| = 0.$$

The *unstable manifold* of $\boldsymbol{p}$ denoted $\mathcal{U}(\boldsymbol{p})$ is the set of points $\boldsymbol{v}$ in the domain of $\boldsymbol{p}$ such that

$$\lim_{k \to \infty} |\boldsymbol{f}^{-k}(\boldsymbol{v}) - \boldsymbol{f}^{-k}(\boldsymbol{p})| = 0.$$

Notice that the unstable manifold of $\boldsymbol{p}$ is simply the stable manifold of $\boldsymbol{p}$ under the inverse map. Also observe that $\mathcal{S}(\boldsymbol{P})$ and $\mathcal{U}(\boldsymbol{p})$ both contain $\boldsymbol{p}$.

3.2.1   Inverse Maps – a reminder

As discussed earlier on, for a one-to-one map $\boldsymbol{f}$ on a subset of $\mathbb{R}^n$ its inverse map $\boldsymbol{f}^{-1}$ exists. In particular such a map $\boldsymbol{f}$ is called *invertible*.

Example 3.3. Consider the map $\boldsymbol{f}_1 : \mathbb{R}^2 \to \mathbb{R}^2$ by $(x, y) \mapsto (x^2, y^2)$. This is not a one-to-one map on $\mathbb{R}^2$, for instance $\boldsymbol{f}_1(-1, -1) = \boldsymbol{f}_1(1, -1) = \boldsymbol{f}_1(-1, 1)$ and so on. If, however, we restrict the domain to the set $\mathbb{R}^{2+}$ of pairs of positive real numbers, then $\boldsymbol{f}$ is indeed one-to-one (no two distinct positive real numbers square to the same value). The inverse of $\boldsymbol{f}_1$ as a map on $\mathbb{R}^{2+}$ is simply $\boldsymbol{f}_1^{-1}(x, y) = (\sqrt{x}, \sqrt{y})$.

The map $\boldsymbol{f}_2 : \mathbb{R}^2 \to \mathbb{R}^2$ by $(x, y) \mapsto (x^3, y^3)$ is a one-to-one map. Now two distinct real numbers cube to the same value. The inverse map of $\boldsymbol{f}_2$ is given by taking cube roots. That is $\boldsymbol{f}_2^{-1}(x, y) = (\sqrt[3]{x}, \sqrt[3]{y})$.

How do we compute the inverse map in general? In general we make use of the one-to-one property as follows. Suppose $\boldsymbol{f}^{-1}(\boldsymbol{v}_1) = \boldsymbol{v}_2$, then $\boldsymbol{f}(\boldsymbol{v}_2) = \boldsymbol{v}_1$. We then solve the latter equation to obtain the coordinates of $\boldsymbol{v}_2$ in terms of the coordinates of $\boldsymbol{v}_1$. Let us illustrate this with an example.

Example 3.4. Let $\boldsymbol{f} = (x + 2y, x^3)$ be a map on $\mathbb{R}^2$. Let (x_1, y_1) be such that $f(x_1, y_1) = (x, y)$ (note that $\boldsymbol{f}^{-1}(x, y) = (x_1, y_1)$ necessarily as $\boldsymbol{f}$ is one-to=one). Then

$$x = x_1 + 2y_1$$
$$y = x_1^3$$

Thus we get

$$x_1 = y^{1/3}$$
$$y_1 = (x - x_1)/2 = (x - y^{1/3})/2$$

Therefore $\boldsymbol{f}^{-1}(x, y) = (x_1, y_1) = (y^{1/3}, (x - y^{1/3})/2)$. The reader can verify that $\boldsymbol{f}(\boldsymbol{f}^{-1}(x, y)) = (x, y)$.

3.2.2 How do we characterise the stable and unstable manifolds of a saddle?

Identifying the stable and unstable manifolds of a saddle fixed point using Definition 3.2, in general, is quite tricky especially when we have no nice closed form expression for $\boldsymbol{f}^{k}(\boldsymbol{x})$ and $\boldsymbol{f}^{-k}(\boldsymbol{x})$. However, Definition 2.7 gives us a clue. The stable manifold must be tangent, at the saddle point, to the line that passes through the saddle in the direction of the eigenvector corresponding to the (contracting) eigenvalue which is strictly less than 1; the unstable manifold must be tangent at the saddle point to the line that passes through the saddle in the direction of the (expanding) eigenvector corresponding to the eigenvalue which is strictly larger than 1. This together with the fact that that stable and unstable manifolds of saddles are indeed manifolds can be enough. For example if we know that the stable and unstable manifolds must be 1-dimensional manifolds, then all we need to do is find a curve which contains the saddle, is tangent to the relevant line and satisfies the relevant condition in Definition 3.2. In general though, we will need to estimate the manifolds of the saddle via some computational means.

If we restrict simply to the plane, then with the aid of the Stable Manifold Theorem[1], we can say quite a bit.

---

[1] We unfortunately do not cover the Stable Manifold Theorem in this module however a proof can be found in Chapter 10 of (Alligood et al., 2000).

Plane Maps

For a map $\boldsymbol{f}$ of the plane, as a consequence of the Stable Manifold Theorem, stable and unstable manifolds of a saddle fixed point $\boldsymbol{p}$ *are always one-dimensional manifolds i.e. either straight lines or curves.* Therefore, in the plane, in order to identify the stable and unstable manifold of a saddle we need to find a one-dimensional manifold C satisfying the following conditions:

1. C passes though the saddle,
2. C is *invariant* under $\boldsymbol{f}$ and $\boldsymbol{f}^{-1}$, that is

$$C = \boldsymbol{f}(C) = \boldsymbol{f}^{-1}(C),$$

3. C is tangent at the saddle either
   i. to the line through the saddle in the direction of the contracting eigenvector (if C is the stable manifold), or,
   ii. to the line through the saddle in the direction of the expanding eigenvector (if C is the unstable manifold),
4. For all $\boldsymbol{v} \in C$ either

$$\lim_{k \to \infty} |\boldsymbol{f}^k(\boldsymbol{v}) - \boldsymbol{f}^k(\boldsymbol{p})| = 0$$

(if C is the stable manifold) or

$$\lim_{k \to \infty} |\boldsymbol{f}^{-k}(\boldsymbol{v}) - \boldsymbol{f}^{-k}(\boldsymbol{p})| = 0$$

(when C is the unstable manifold.)

For *linear maps* (only linear terms in the definition of the map; alternatively the usual definition of linear maps from MT2501/MT3501), Definition 2.7 gives the whole answer in n-dimensions and not only in the plane.

Linear Maps

For a linear map, in 2-dimensions the eigenvector corresponding to the contracting eigenvalue determine the stable manifold — *the line through the saddle point in the direction of the eigenvector*; the eigenvector corresponding to the expanding eigenvalue determines the unstable manifold — *the line through the fixed point in the direction of the eigenvector*. In higher-dimensions the stable manifold is determined by the subspace spanned by the contracting eigenvectors (we translate so that the origin is moved to the saddle fixed point) and the unstable manifold is determined by the subspace spanned by the contracting eigenvectors (translate such that the origin is moved to the fixed point).

Let us do some examples to illustrate this.

Example 3.5. Consider the linear map $\boldsymbol{f}(x, y) = (2x, y/2)$ from Example 3.2. We showed that $(0, 0)$ is a saddle point of $\boldsymbol{f}$. In particular, we showed that the Jacobian of $\boldsymbol{f}$ at $(0, 0)$ has eigenvalues $2, 1/2$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We concluded that points lying on the x-axis move away from the origin under iterations of the map $\boldsymbol{f}$ while points lying on the y-axis move towards the orgin under applications of the map $\boldsymbol{f}$. From this it follows that the x-axis is the unstable manifold/*outgoing direction* of the saddle point $(0, 0)$ and the y-axis is the stable manifold/*incoming direction* of $(0, 0)$.

Example 3.6. Let us again consider a linear map $\boldsymbol{f}(x, y) = (-2x + 5y/2, -5x + 11y/2)$. The fixed points of $\boldsymbol{f}$ are found by solving $\boldsymbol{f}(x, y) = (x, y)$ which gives $(0, 0)$ as the unique fixed point. The Jacobian of $\boldsymbol{f}$ is simply the matrix of coefficients of $\boldsymbol{f}$ since $\boldsymbol{f}$ is a linear map. That is

$$\mathbf{D}\boldsymbol{f}(0, 0) = \begin{pmatrix} -2 & 5/2 \\ -5 & 11/2 \end{pmatrix}$$

We find the eigenvalues in the usual way to be 0.5 and 3. The eigenvector corresponding to eigenvalue $\lambda \in \{0.5, 3\}$ can then be found by solving the equation $(A - \lambda I)\boldsymbol{v} = 0$ where $A = \mathbf{D}\boldsymbol{f}(0, 0)$. This gives eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Thus, we see that points on the line $y = x$ corresponding to the first eigenvalue move closer to the origin under iterations of the map $\boldsymbol{f}$ since such vectors are scaled by a factor of 0.5. Points on the line $y = 2x$ (that is the set of points $\left\{ t \begin{pmatrix} 1 \\ 2 \end{pmatrix} : t \in \mathbb{R} \right\}$) move away from the orgin under applications of f as they are scaled by a factor of 3.

Therefore the line $y = x$ is the stable manifold of the saddle $(0, 0)$ and the line $y = 2x$ is the unstable manifold of the saddle $(0, 0)$.
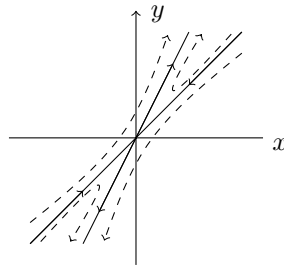

FIGURE 3.2: Stable and unstable manifolds of a regular saddle

Example 3.7. We consider again a linear map $\boldsymbol{f}(x, y) = (2x + 5y, -0.5y)$. This has a fixed point at $(0, 0)$. Once more we compute the eigenvalues and eigenvectors of the Jacobian of $\boldsymbol{f}$ at the fixed points (just the matrix of coefficients for a linear map) using the usual methods of linear algebra. We get eigenvalues $2, -0.5$ and corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

We see that the line $y = -x/2$ in the direction of the vector $(2, -1)$ is the stable manifold of the saddle $(0, 0)$ — points on this line move towards the origin under iterations of $\boldsymbol{f}$. The x-axis (the line $y = 0$) is the unstable saddle points on this line move away from the origin under iterations of the map.

Notice the new interesting behaviour that occurs on the stable manifold of $(0, 0)$. On each iteration of the map, a point on the saddle flips from one side of the origin to another. Generally this behaviour occurs happens for all fixed points for which the Jacobian has negative eigenvalues.

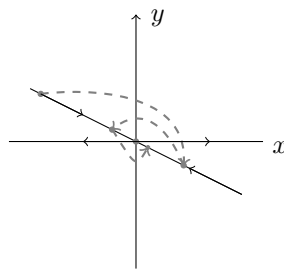A saddle with at least one negative eigenvalue is sometimes called a *flip saddle*.

FIGURE 3.3: Stable and unstable manifold of a flip saddle

3.3   Problem Sheet 3

For week 4.

Question 1

The Hènon map is defined by

$$\boldsymbol{f}(x, y) = (a - x^2 + by, x)$$

with real parameters a and b.

a. Show that the two fixed points of the Hènon map are given by

$$\boldsymbol{p}_+ = (x_+, x_+), \quad \boldsymbol{p}_- = (x_-, x_-)$$

with

$$x_\pm = \frac{1}{2}\left(b - 1 \pm \sqrt{(b-1)^2 + 4a}\right), \quad (b-1)^2 + 4a \geq 0.$$

b. Investigate the stability of these fixed points for $b = 1$ and $a > 0$ and classify them as sinks, sources or saddles.

Question 2

Let

$$\boldsymbol{f}(x, y, z) = (x^2 y, y^2, xz + y)$$

be a map on $\mathbb{R}^3$. Find the fixed points of $\boldsymbol{f}$ and investigate their stability.

Question 3

Determine the fixed points of the map

$$\boldsymbol{f}(x, y) = (x + y, y + a\sin(bx)),$$

where a and b are real and positive and investigate the stability of the fixed points.

Question 4

Consider the Hènon map

$$\boldsymbol{f}(x, y) = (a - x^2 + by, x)$$

for $a > 0$ and $b = 1 - \sqrt{a}$. Verify that $\{(\sqrt{a}, 0), (0, \sqrt{a})\}$ is a period 2-orbit of the Hènon map. Determine the stability of the period-2 orbit.

Question 5

Find the fixed points of the map

$$\boldsymbol{f}(x, y) = \left(\frac{4}{\pi} \tan^{-1}(x), \frac{y}{2} \right)$$

where $\tan^{-1}(x) : \mathbb{R} \to (-\pi/2, \pi/2)$ is the inverse tangent function. Determine the stability of the fixed points.

4

Higher-dimensional stability

Covers Chapter 3 Sections 3.1 and 3.2 and Chapter 4 Sections 4.1 to 4.4 (Alligood et al., 2000)

We concluded Chapter 3 by characterising the stable and unstable manifolds of linear maps. We now turn to non-linear maps. We begin with a map we have already encountered (Examples 2.2 3.1).

4.1 Non-linear examples

Example 4.1 (Hènon map III)**.** Recall that the Hènon map is the map $\boldsymbol{f}_{a,b}(x,y) = (a - x^2 + by, x)$. This has an associated discrete time dynamical system:

$$x_{n+1} = a - x_n^2 + by_n$$
$$y_{n+1} = x_n$$

Notice that the new x terms depends in a non-linear way on the preceding x and y terms. In Example 3.1 we plotted a bifurcation diagram for the Hènon map and saw that it had similar dynamics to the logistic map (2.3).

Let us investigate the dynamics of the orbits further. Figure 4.1 show orbits for the Hernon map for b fixed at -0.3 and, $a = 1.28$ for the left figure and 1.4 for the right figure. The figure is obtained by iterating the map for each initial value until the orbits either converge to the period-2 sink (plotted as crosses in both figures) or diverge towards infinity. The initial values that converge to the period 2 sink are coloured white and initial values that diverge are coloured black.

Thus the black points form the *basin of infinity* of the period 2-sink while the white points are the *basin of attraction* of the period-2 sink.
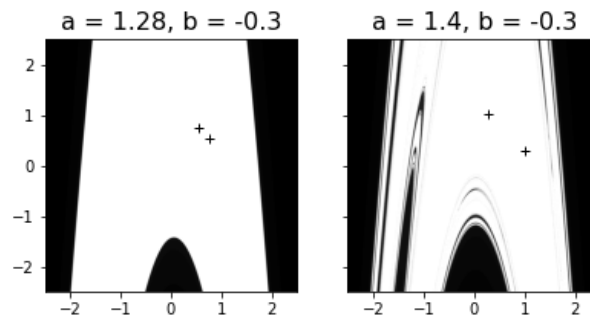
FIGURE 4.1: A square of initial conditions for the Hènon map with b=-0.3

A similar picture is seen when we fix b at -0.3 and increase a to 2: orbits in the black region diverge to infinity, and orbits in the white region are attracted to the "2-piece" attractor in the white region.
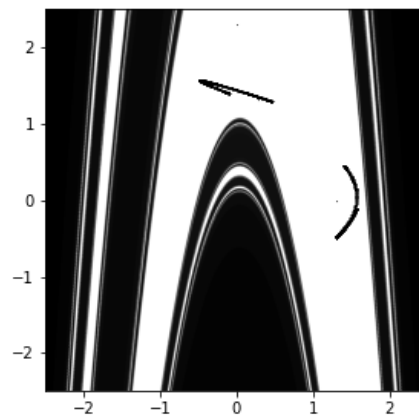

FIGURE 4.2: Attractors for the Hènon map with a=2, b=-0.3

Figure 4.3 zooms in on the white region. We choose a value for a (keeping b fixed as 0.4), we iterate the Hènon map for different start values in the white region

until they settle down to an attracting orbit. Note that as we increase a the length of the attracting orbits increase and then decrease again.
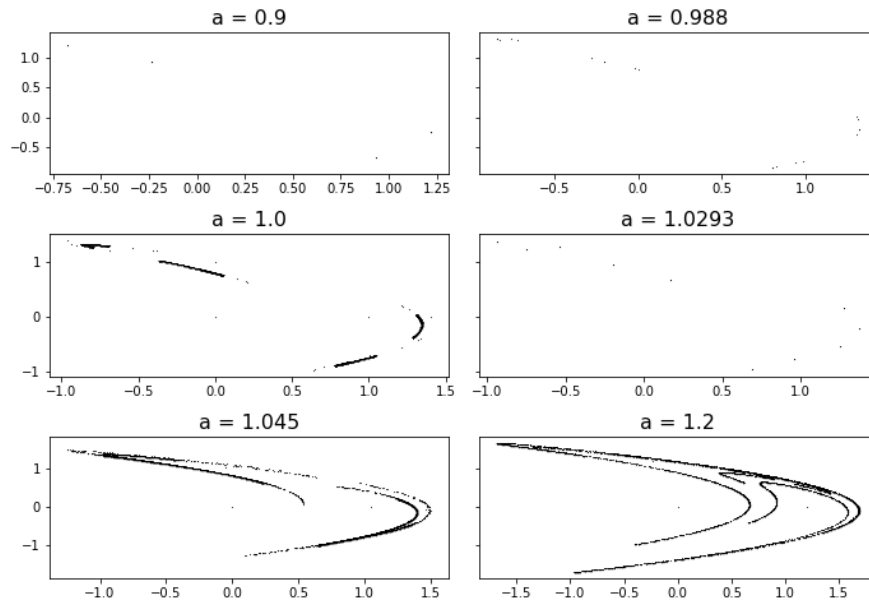
FIGURE 4.3: Attractors for the Hernon map with b=0.4

4.1.1 Chaos in 1-dimensional maps

We return to the notion of chaos for 1-dimensional maps which we have so far been a little vague about. Recall that in Section 2.2 we said that chaos has at least 2 distinct characteristics

1. Non-periodicity
2. Sensitive dependence on initial conditions.

We now introduce a third component in characterising chaotic behaviour. This third characteristic is determined by the *Lyapunov exponent* (named after the mathematician Aleksandr Mikhailovich Lyapunov). The Lyapunov exponent estimates the *rate of separation of trajectories* which are infinitesimally close to each other and gives us a way of making the notion of "sensitive dependence on initial conditions" precise.

Let us give a precise definition.

Definition 4.1 (Lyapunov exponent)**.** Let f be a (smooth) map defined on a

subset of $\mathbb{R}$. The *Lyapunov number* $L(x_1)$ of the orbit $(x_1, x_2, ...)$ is defined by the following limit (if it exists):

$$L(x_1) = \lim_{n\to\infty} (|f'(x_1)||f'(x_2)| ... |f'(x_n)|)^{\frac{1}{n}}.$$

The *Lyapunov exponent* $h(x_1)$ is defined by the following limit (if it exists):

$$h(x_1) = \lim_{n\to\infty} \frac{\ln|f'(x_1)| + \ln|f'(x_2)| + ... + \ln|f'(x_n)|}{n}.$$

Note that the Lyapunov exponent h exists/is well-defined if and only if the Lyapunov number L exists and is non-zero in which case $\ln L = h$. The requirement that the limit exists is necessary since the limit might not exist for some orbits. For instance if an orbit with initial point x has $f'(x) = 0$, then automatically the Lyapunov exponent of the orbit will be undefined.

Example 4.2. Let f be a 1D map with a fixed point p. Then the Lyapunov number of the orbit p is computed as follows:

$$L(p) = \lim_{n\to\infty} (|f'(p)||f'(p)| ... |f'(p)|)^{\frac{1}{n}} = \lim_{n\to\infty} (|f'(p)|^n)^{\frac{1}{n}} = |f'(p)|.$$

Therefore the Lyapunov exponent of the orbit of p is $\ln(|f'(p)|)$.

For a period k orbit $(x_1, x_2, ..., x_k)$, the Lyapunov exponent is given by

$$h(x_1) = \frac{\ln|f'(x_1)| + \ln|f'(x_2)| + ... + \ln|f'(x_k)|}{k}$$

and the Lyapunov number is

$$(|f'(x_1)||f'(x_2)| ... |f'(x_k)|)^{\frac{1}{k}}.$$

The other notion we have been rather vague about so far is what it means to be "non-periodic." Before we say exactly what we mean by this, we introduce the notion of being "close to" a periodic orbit.

Definition 4.2. Let f be a 1D map. An orbit $(x_1, x_2, ...)$ is called *asymptotically periodic* if it converges (pointwise) to a periodic orbit. That is, $(x_1, x_2, ...)$ if there exists a periodic orbit $(y_1, y_2, ...)$ (of period k say) such that

$$\lim_{n\to\infty} |x_n - y_n| = 0.$$

The orbit $(x_1, x_2, ...)$ is called *eventually periodic* if there is some $N \in \mathbb{N}$ such that $|x_m - y_m| = 0$ for all $m > N$.

Note that the orbit of any point in the basin of attraction of a periodic point is asymptotically periodic.

When we introduced the Lyapunov exponent, we said it measures how quickly trajectories which are infinitesimally close to one another drift apart. In particular we expect two points whose orbits "eventually coincide" to have the same Lyapunov exponent (if it exists). Thus, the following result which says that the Lyapunov exponent of an orbit asymptotically periodic orbit is equal to the Lyapunov exponent of the periodic orbit it converges to is perhaps not surprising.

Theorem 4.1. *Let f be a 1D map. Let $(x_1, x_2, ...)$ be an orbit which is asymptotically periodic and suppose that it converges to the periodic orbit $(y_1, y_2, ...)$. Suppose further that $f'(x_i) \neq 0$ for all i. Then the Lyapunov exponent of the orbits $(x_1, x_2, ...)$ and $(y_1, y_2, ...)$ are identical (assuming they both exist).*

4.2   Chaotic Orbits

We can now define formally what it means for an orbit of a 1D map to be chaotic in a special case – *bounded orbits*. An orbit $(x_1, x_2, ...)$ of a 1D map f is said to be *bounded* if there is an $M \in \mathbb{R}$ such that $|x_i| < M$ for all i.

Definition 4.3 (Chaotic orbits)**.** Let f be a 1D map and $(x_1, x_2, ...)$ be a bounded orbit. Then the orbit is *chaotic* if

1. it is *not* asymptotically periodic (Non-periodicity);
2. the Lyapunov exponent $h(x_1)$ is greater than zero (Sensitivity to initial conditions).

Example 4.3 (Doubling map)**.** Define the map $f : [0, 1) \to [0, 1)$ as follows:

$$f(x) = \begin{cases} 2x & 0 \leq x < 1/2 \\ 2x - 1 & 1/2 \leq x < 1 \end{cases}$$

Note that this map can equivalently be written as $f(x) = 2x \mod 1$.

Figure 4.4 gives a sketch of the function.
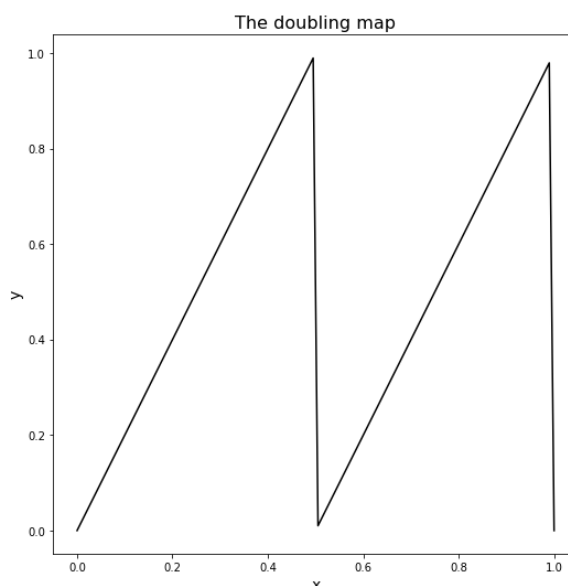
FIGURE 4.4: Doubling map

Note that f is not a smooth map, it is not continuous or differentiable at a $1/2$ (although it is a piecewise smooth function and so differentiable at all other points).

Let us investigate the nature of orbits of points not equal to $1/2$.

We begin by computing the Lyapunov exponent. Let $(x_1, x_2, ...)$ be an orbit such that $x_i \neq 1/2$ for all $i \in \mathbb{N}$, then:

$$h(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\infty} \ln |f'(x_i)|$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\infty} \ln 2 = \ln 2 > 0$$

Therefore an orbit which is not asymptotically periodic (and so which avoids the problematic point at $1/2$) is a chaotic orbit.

Let us see if we can characterise the asymptotically periodic orbits. The nature of the map means that it makes sense to consider binary expansion. That is for each $x \in [0, 1)$ we write x as $0.b_1 b_2 ...$ corresponding to the sum:

$$x = \sum_{n=1}^{\infty} \frac{b_n}{2^n}.$$

Now the effect of multiply by 2 is simply to shift the decimal point rightward, working modulo 1 means we get rid of the b_1 term. That is $f(x) = 0.b_2 b_3 \dots$. It therefore follows that a periodic orbit corresponds to an element of $[0,1)$ of the form $0.(b_1 b_2 \dots b_k)(b_1 b_2 \dots b_k)\dots = 0.\overline{(b_1 b_2 \dots b_k)}$. It therefore follows that an eventually periodic orbit corresponds to a rational numbers (rational numbers are characterised by the property that they have eventually periodic base n expansions). Now, it is not hard to see that an asymptotically periodic orbit of the map f must in fact be eventually periodic. Thus the asymptotically periodic orbits of the map $f(x)$ are precisely the rational numbers in $[0,1)$. Consequently, *all* irrational points give rise to chaotic orbits of f.

Figure 4.5 shows a cobweb plot of the doubling map with irrational initial point.
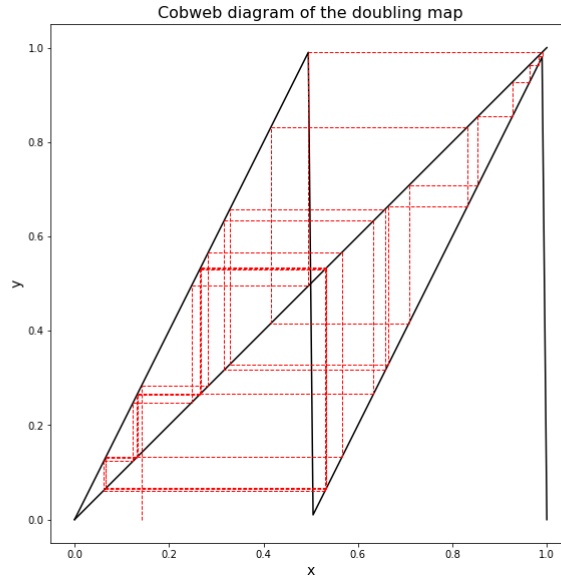

FIGURE 4.5: Cobweb plot of doubling map with irrational initial point

4.2.1 Fractals

Although we have thus far not explicitly drawn it out, *fractals* have been hiding in our discussions of dynamical systems so far. For instance attracting sets dynamical systems with chaotic behaviour are usually fractal sets and the boundaries of basins of attractions can a have a fractal structure.

What exactly is a fractal? This is not a course in fractal geometry so we are going to be a little vague. There are two main characteristics of a fractal:

- complicated structure over a wide range of length scales,
- self-similarity — repetition of structures at different length scales.

A key example of a fractal is the Cantor set. We can construct this set using an iterative process as follows. Consider the unit interval $k_0 := [0,1]$. In the first pass we delete the middle third from the interval so we obtain the set $k_1 := [0,1] \backslash (1/3, 2/3) = [0, 1/3] \sqcup [2/3, 1]$. We repeat the process on the two subintervals created, we delete the middle third of each $((1/9, 2/9), (7/9, 8/9))$ to obtain 4 intervals making up a set

$$k_2 := [0, 1/9] \sqcup [2/9, 3/9] \sqcup [6/9, 7/9] \sqcup [8/9, 1].$$

We repeat the process on each of the remaining subintervals. Continuing on in this way indefinitely gives rise to a set $k_\infty = \mathfrak{C}$ of points on the unit interval. Figure 4.6 gives a graphical depiction of this construction. The resulting set $\mathfrak{C}$ is call the *middle-third Cantor set* after the mathematician Georg Cantor who first studied such sets.
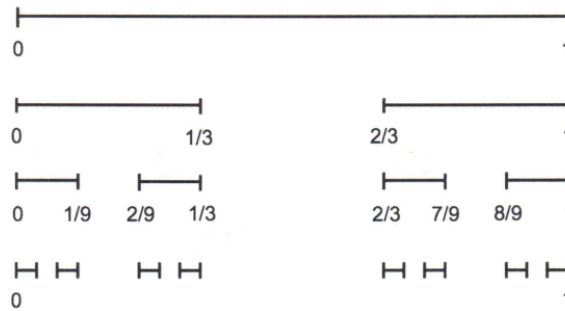
FIGURE 4.6: Construction of the middle-third Cantor set

We observe an interesting property of the middle-third Cantor set when trying to answer the following question, what is the length of the set $\mathfrak{C}$? Notice that there are infinitely many points in the set $\mathfrak{C}$.

After the first pass, the set k_1 has length $2/3$ (2 subintervals of length $1/3$). After the second pass we are left with a set k_2 of length $4/9$ (4 subintervals of length $1/9$). In particular after the nth pass we are left with a set k_n of length $(2/3)^n$ (2^n intervals of length 3^n). In the limit we see that $\mathfrak{C}$ has length 0 or *measure* zero. Thus we have the seemingly contradictory proposition of a set which contains infinitely many points but has no length.

Generally a set $X \subset \mathbb{R}$ is said to have *measure zero* if for any ϵ greater than zero, there is a choice of intervals of total length strictly less than ϵ whose union contains X.

One can generalise the Cantor set construction from $\mathbb{R}$ to $\mathbb{R}^2$ to create interesting fractals. For example Figure 4.7 shows the Sierpinski gasket. Here we begin with an equilateral triangle and at each pass remove the "middle triangle" of the equilateral triangles left over. The resulting set of points in the plane is the Sierpinski gasket.
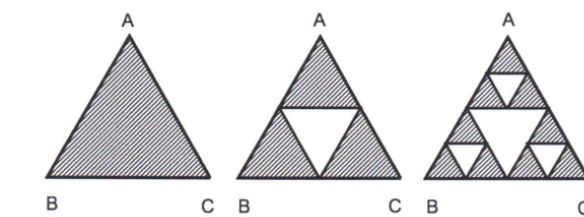
FIGURE 4.7: 3 steps of the Sierpinski gasket

4.3 Fractals and deterministic dynamical systems

At the end of Section 4.2 we introduced a key player in the module who thus far had been lurking in the background of our discussions — *fractals*. In this section we explore some of the instances in which fractals occur in dynamical system.

4.3.1 Fractals from deterministic dynamical systems

Consider the 1D map

$$f_a(x) = \begin{cases} ax & x \leq 1/2 \\ a(1-x) & x > 1/2 \end{cases}$$

for $a > 0$. This map is often called the *tent map* as its graph looks like a tent. As with the doubling map this is a piecewise-smooth and not a smooth map. For differentiable values in $[0, 1]$, the Lyapunov exponent is equal to $\ln(a)$. Therefore all non-asymptotically periodic orbits of the tent map are chaotic when $\ln(a) > 0$.

Figure 4.8 compares the tent map $f = f_a$ $(a = 2)$ against the logistic map $g = g_a$ $(a = 4)$.
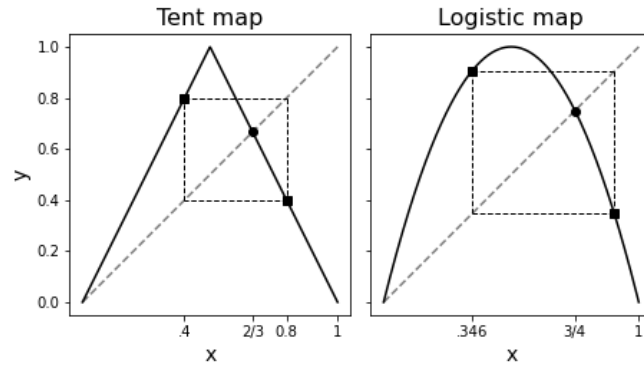
FIGURE 4.8: Comparison of the tent and logistic maps

- The point $1/2$ is a critical point of both maps: it is the local maximum of the logistic map and the joining point of tent map.
- For both maps $1/2$ is mapped first to 1 and then to 0 which is a fixed point of both maps.
- The other fixed point of the tent map is at $2/3$; for the logistic map it is $3/4$.
- Both maps have a period two orbit: $(0.4, 0.8)$ for the tent map and $\left(\frac{5-\sqrt{5}}{8}, \frac{5+\sqrt{5}}{8}\right)$ for the logistic map.
- The fixed points are sources.
- Their period 2 orbits are also sources: $|f'(0.4)||f'(0.8)| = 4$ and $\left|g'\left(\frac{5-\sqrt{5}}{8}\right)\right|\left|g'\left(\frac{5+\sqrt{5}}{8}\right)\right| = 4$.

In some ways one can think of the tent map as a non-smooth version of the logistic map.

Now fix $a = 3$ and consider the tent map $f = f_3$ (shown in Figure 4.9).

We are going think about the set of points in the unit interval whose orbits under f remain inside the unit square. As with the doubling map, the slope of 3 means that considering a ternary expansion might prove fruitful. Another way of visualising this is to consider breaking up the unit interval $[0, 1]$ into blocks of $1/3$, $1/9$ and so on and thinking about which blocks are eventually mapped outside of the square (see Figure 4.9)
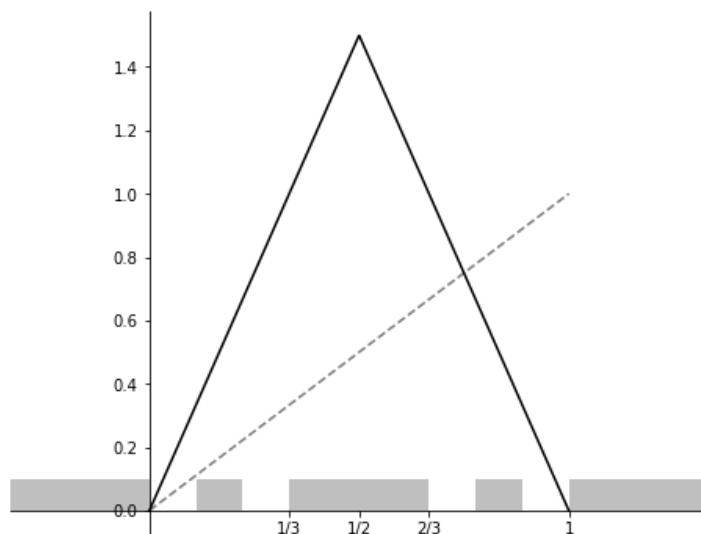
FIGURE 4.9: Tent map with a=3 and escaping orbits highlighted

Observe that points in the interval $(1/3, 2/3)$ are mapped in one iteration of the map f outside of the unit square. Points mapped into the interval $(1/3, 2/3)$ in one iteration, are mapped outside of the square after 2 iterations.

Let us delete the interval $(1/3, 2/3)$ which leaves us with two subintervals $[0, 1/3]$, $[2/3, 1]$. The middle third of these intervals $(1/9, 2/9)$ and $(7, 9, 8/9)$. Are mapped under one iteration to the middle third of the interval $[0, 1]$ that is $(1/3, 2/3)$. In particular, after two iterations points in $(1/9, 2/9)$ and $(7, 9, 8/9)$ are mapped out of the unit interval. This pattern repeats. If we delete the intervals $(1/9, 2/9)$ and $(7, 9, 8/9)$, the middle third of the intervals that are left will map outside the unit square in 3 iterations of f.

Therefore in the limit, the set of points in the unit interval whose orbits under f remain in the unit square is precisely points in the middle third Cantor set $\mathfrak{C}$ — a fractal set.

What about points outside the unit interval? All such points have orbits converging to $-\infty$ (in the first iteration they are mapped to the negative real line and from then on they remain on the negative real line and triple in size with each application of f).

Therefore the basin of infinity (the set of points whose orbits diverge to $-\infty$) is the set

$$(-\infty, 0) \sqcup (1, \infty) \sqcup (1/3, 2/3) \sqcup (1/9, 2/9) \sqcup (7/8, 8/9) \ldots .$$

4.3.2   Fractal basin boundaries

Let us now consider higher dimensional maps to see how such maps might give rise to fractal structures.

Let f be a map on a subset of $\mathbb{R}^n$ and suppose that p is an attracting fixed/periodic point of f. Recall that the basin of attraction of p is the set of points v such that

$$\lim_{k\to\infty} |f^k(v) - f^k(p)| = 0.$$

Figure 4.10 shows self-similarity in an attracting orbit of the Hènon map. The sub-figures are successive magnifications on a portion of the orbit in the top left.
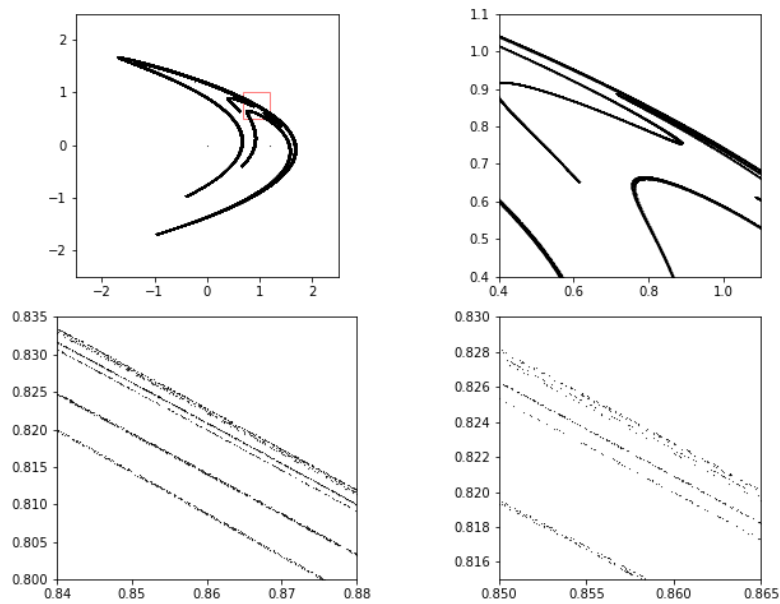
FIGURE 4.10: Zooming in on an orbit of the Hènon map

Similarly Figure 4.12 reveals fractal structure in the basin of infinity of the Hènon map. The sub-figures are magnifications of the left portion of figure 4.11
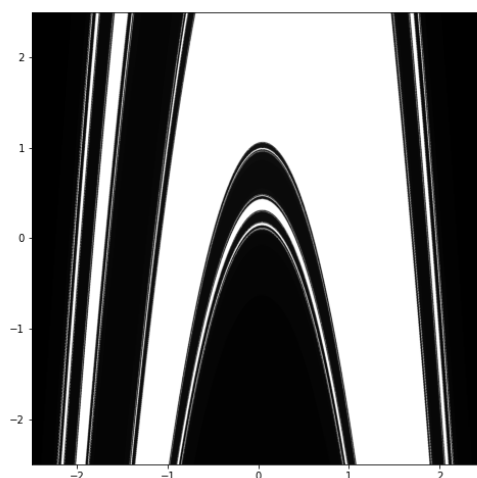
FIGURE 4.11: Attracting basin of the Hènon map
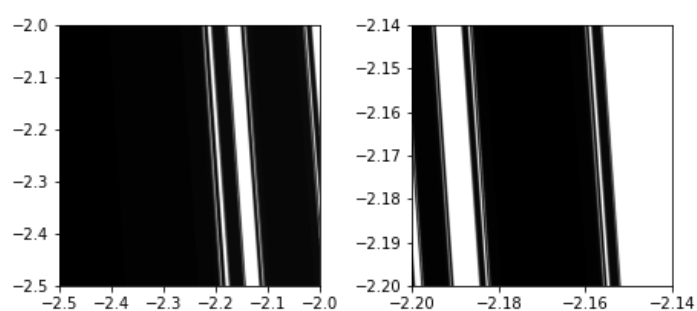
FIGURE 4.12: Zooming in on the attracting basin of the Hènon map

How do fractal basin boundaries arise? Figure 4.13 gives a schematic for how fractal basins might arise.
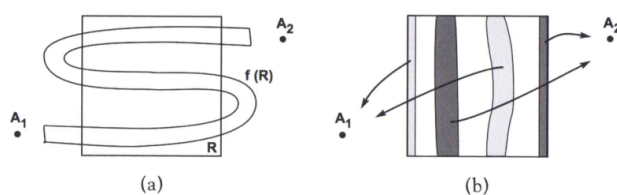
FIGURE 4.13: Fractal basin schematic.

We consider a map f of the plane and a square region R. The map f has two attracting fixed points A_1 and A_2 to the left and right (respectively) of the square R. The image of the region R under the map f is the S shaped strip. The sub-figure on the right of Figure 4.13 shows which portions are mapped out of the R to the left and which are mapped (in one iteration) out of R to the right. In particular, the light-grey shaded regions are mapped under f to the out of R to the left and the dark shaded regions are mapped out of R to the right. In the next iteration the three white strips remaining will each develop 4 shaded strips (2 light-grey and 2-dark grey) corresponding to points which are mapped under two iterates of f out of R to the left and under 2 iterates of f out of R to the right. This process then repeats giving rise to a Cantor-like construction. The set of points in R which remain in R under repeated applications of f is a fractal set.

4.4   Problem Sheet 4

For week 5.

Question 1

a. Show that
$$f(x, y) = (ax, bx^3 + cy)$$
has a saddle fixed point at $(0,0)$ if either $|a| < 1$ and $|c| > 1$ or $|a| > 1$ and $|c| < 1$.

b. Find the inverse map f^{-1}.

[Hint: Set $f_1(x, y) = ax$ and $f_2(x, y) = bx^3 + cy$ and solve
$$x = f_1(u, v), \qquad y = f_2(u, v)$$
for $u(x, y)$ and $v(x, y)$. The inverse map is given by $f^{-1}(x, y) = (u(x, y), v(x, y))$]

c. Set $q = \frac{b}{a^3 - c}$. Show that
$$S = \{(t, qt^3) : t \in \mathbb{R}\}$$
is invariant under f. That is for $(x, y) \in S$, $f(x, y)$ **and** $f^{-1}(x, y)$ are also in S.

d. What are the stable and unstable manifold of $(0,0)$ in the regimes
   i. $|a| < 1$ and $|c| > 1$;
   ii. $|a| > 1$ and $|c| < 1$?

Question 2

Consider the map
$$f(x, y, z) = \left(\frac{x}{2}, \frac{y}{2}, 2z - (x^2 + y^2) \right).$$

a. Show that $(0,0,0)$ is the only fixed point of the map and that it is a saddle point. What are the directions at $(0,0,0)$ along which contraction and expansion occur?

b. Show that the unstable manifold $\mathcal{U}((0,0,0))$ is the z-axis.

c. Show that the stable manifold of $\mathcal{S}((0,0,0))$ is the paraboloid
$$z = \frac{4(x^2 + y^2)}{7}.$$

Question 3

The tent map $T_2(x) : [0, 1] \to [0, 1]$ is defined by

$$
T_2(x) = \begin{cases} 2x & \text{for } x \leq 1/2 \\ 2(1 - x) & \text{for } x > 1/2 \end{cases}.
$$

a. Sketch the map $T_2(x)$.

b. For every orbit $(x_1, x_2, ...)$ of $T_2(x)$ with $x_i \neq 1/2$ for all $i \in \mathbb{N}$, calculate the Lyapunov number $L(x_1)$ and the Lyapunov exponent $h(x_1)$.

c. Show that $T_2(x)$ does not have any stable period-k orbits or fixed points. Argue that any asymptotically periodic orbit of $T_2(x)$ must be eventually periodic.

d. Write L for the set of all points in $[0, 1]$ strictly less than $1/2$ and R for the set of all points strictly greater than $1/2$. Let $(x_1, x_2, x_3, ...)$ be an orbit with $x_i \neq 1/2$ for all $i \in \mathbb{N}$. We assign an infinite sequence, an *itinerary*, $l_1 l_2 l_3 ... \in \{L, R\}^{\mathbb{N}}$ to the orbit $(x_1, x_2, x_3, ...)$ in the following way: for $i \in \mathbb{N}$ $l_i = L$ if $x_i \in L$ otherwise $l_i = R$.

Graphically or otherwise show that each itinerary $l_1 l_2 ... l_k \in \{L, R\}^k$ consisting of k symbols corresponds to an interval of length 2^{-k}.

Explain why each infinite itinerary corresponds to the orbit of exactly one point in $[0, 1]$.

e. Using points a. to d. show that $T_2(x)$ has infinitely many chaotic orbits.

5

Fractals

Covers Chapter 4 from Section 4.5 to Definition 5.1 of Chapter 5 (Alligood et al., 2000).

5.1 Complex maps and Julia sets

As another example of how fractals might arise from dynamical systems, consider a complex map $P_c : \mathbb{C} \to \mathbb{C}$ defined by $P_c(z) = z^2 + c$ for a constant $c \in \mathbb{C}$. This defines a complex discrete dynamical system:

$$z_{n+1} = z_n^2 + c.$$

We can also think of this as a map on $\mathbb{R}^2$ as $\boldsymbol{f}(x, y) = (x^2 - y^2 + a, 2xy + b)$ where $c = a + ib \in \mathbb{C}$ and so as a discrete dynamical system on $\mathbb{R}^2$.

For $c = 0$, $P_0 = z^2$ and we have a fixed point at 0 which is in fact an attracting fixed point. All points $z \in \mathbb{C}$ with $|z| < 1$ are attracted to 0 under iterations of the map P_0. Therefore the interior of the unit disk is the basin of attraction of 0.

Notice that points on the boundary $S = \{z \in \mathbb{C} : |z| = 1\}$ of the unit disk are mapped under P_0 to other points on the unit disk. While the set $K = \{z \in \mathbb{C} : |z| > 1\}$ of points outside the closed unit disk map to infinity under iterations of P_0. Therefore S forms the boundary between the basin of 0 and the basin of infinity.

The dynamics differ if we change the value of c, indeed 0 is no longer a fixed point once c is non-zero. However the orbit of 0 play an important role in the dynamics of P_c. Therefore one might wonder for which values of c does the orbit with initial point 0 not diverge to infinity? The set of points $c \in \mathbb{Z}$ for which the orbit of 0 is bounded is called the *Mandelbrot set* (after the mathematician Benoit Mandelbrot) and is denoted M . That is,

$$M = \{c \in \mathbb{C} : \exists M \in \mathbb{R}, |P_c^n(0)| < M \forall n \in \mathbb{N}\}.$$
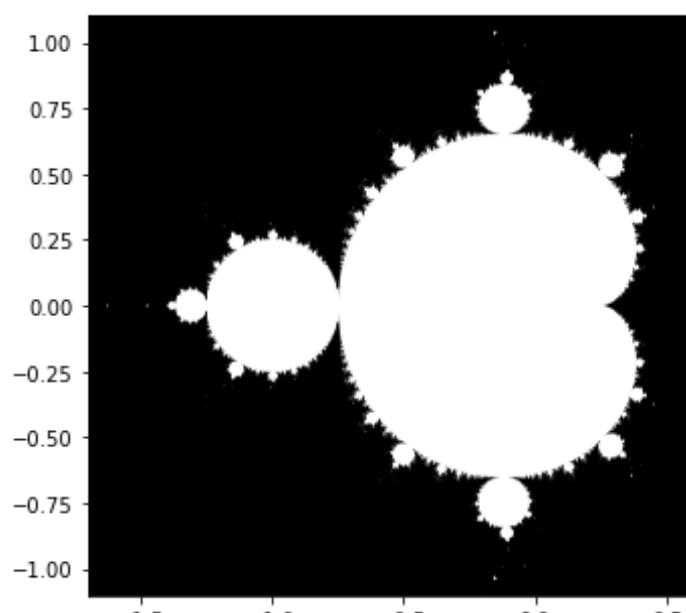
FIGURE 5.1: A picture of the Mandelbrot set.

For a fixed $c \in \mathbb{C}$, the boundary between orbits which diverge to infinity under P_c and orbits which are bounded is an important set. It is the *Julia set of $P_c(z)$* (after the mathematician Gaston Maurice Julia). The Julia sets are good examples of fractal sets.

Figure 5.2 shows examples of Julia sets for values of $\mathbb{C}$ inside the Mandelbrot set.
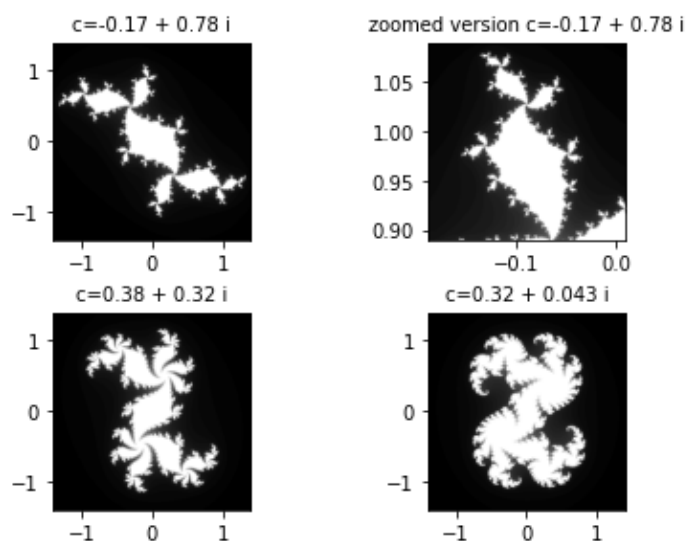
FIGURE 5.2: Julia sets for values of c in the Mandelbrot set

5.1.1 Fractal dimension

When we constructed the middle third Cantor set, we observed that although the resulting set $\mathfrak{C}$ contained infinitely many points it had zero length. But perhaps we were using the wrong measure. A *fractal dimension* is a way of measuring the complexity of a fractal set. Of course we expect such a measure to generalise the normal notion of "dimension" that we are used to. That is we expect a fractal dimension to give the value 1 for the dimension of an interval, 2 for a square 3 for a cube and so on.

There are many different ways to measure the dimension of a fractal set and in some instances they coincide. We only introduce a couple in this module. The first is the *box counting* dimension.

Let us first consider the definition of the box-countingdimension for standard subsets of $\mathbb{R}$ and $\mathbb{R}^2$ before generalising to subsets of $\mathbb{R}^n$.

Consider the unit interval $[0, 1]$. Given $\epsilon > 0$, what is the minimum number intervals of size ϵ that we need to cover $[0, 1]$? That is we want a minimal collection X of intervals $U = [a, b]$ such that $|a - b| = \epsilon$ and $\bigcup_{U \in X} U \supseteq [0, 1]$? Clearly we need $1/\epsilon$ intervals of size ϵ. If we had instead considered the interval $[0, 42]$, then the minimal number of intervals of size has size $42/\epsilon$. In particular the number of intervals is *linear* in $1/\epsilon$ (i.e of the form $c1/\epsilon$ for a constant c depending on the length of the interval) representing in some sense the 1-dimensionality of the interval.

What about a square D in $\mathbb{R}^2$? Given $\epsilon > 0$, what is the minimum number of squares of size ϵ^2 that we need to cover the square D? We find in this case that we need $c(1/\epsilon^2)$ squares of size ϵ^2 for a constant c depending on the area of the square. In particular the number of squares is *quadratic* in the $1/\epsilon$ representing the two dimensionality of the square.

In the general case, given a cube $D \in \mathbb{R}^d$ and writing $N(\epsilon)$ for the minimum number of *boxes* of side length ϵ in $\mathbb{R}^d$ required to cover the D. We find $N(\epsilon) = c(1/\epsilon^d)$ where c is a constant depending on the size of the cube D. Thus the exponent give the dimension. We can rearrange this expression by taking logarithms to obtain an equation for the exponent in terms of $N(\epsilon)$ and c.

$$d = \frac{\ln(N(\epsilon)) - \ln(c)}{\ln(1/\epsilon^d)}.$$

If c roughly constant for all small ϵ, then the contribution of $\ln(c)$ is negligible. Thus we have the following definition.

Definition 5.1 (Box-counting dimension). A bounded set $S \subseteq \mathbb{R}^n$ has *box-countingdimension*

$$\mathrm{boxdim}(s) = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)}$$

when the limit exists.

See Figure 5.3 for an illustration of applying this in practice.
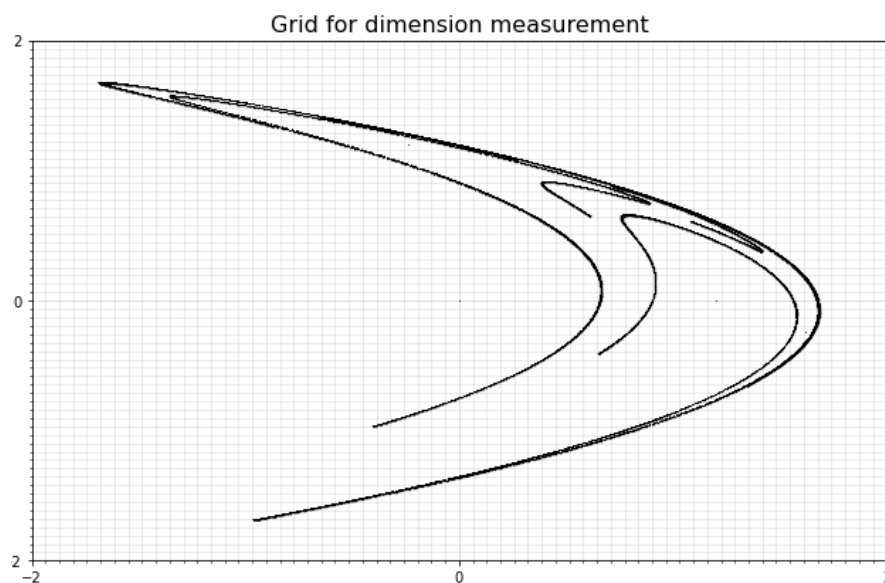
FIGURE 5.3: Boxcounting dimension for an orbit of the Hènon map

Example 5.1. Let us compute the box-counting dimension of the middle-third Cantor set $\mathfrak{C}$. Now we expect a number between 0 and 1 since $\mathfrak{C}$ is a subset of $\mathbb{R}$.

Essentially the minimal length boxes are given to us by the construction of the middle third Cantor set (Subsection 4.2.1). Taking $\epsilon = (1/3)^n$, from the construction of the middle third Cantor set we deduce that $N(\epsilon) = 2^n$ (at the nth step of the construction, we have 2^n intervals of size $1/3^n$).

Therefore

$$\text{boxdim}(\mathfrak{C}) = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)} = \lim_{n \to \infty} \frac{\ln 2^n}{\ln 3^n} = \ln 2 / \ln 3.$$

5.2 Fractal dimensions

5.2.1 Correlation dimension

Another notion of dimension of a fractal set is the so called *correlation dimension*. As opposed to the box-counting dimension which applies to any bounded subset of $\mathbb{R}^n$ the correlation dimension is **only defined** for the orbit of a dynamical system.

Definition 5.2 (Correlation dimension). Let $S = (v_0, v_1, ...)$ be an orbit of a map $f : \mathbb{R}^n \to \mathbb{R}^n$. Let $r \in \mathbb{R}$ be a positive number and set $S_N = (v_0, v_1, ..., v_n)$ to be the first N points in the orbit.

The *correlation function* $C(r)$ is defined as:

$$C(r) = \lim_{N \to \infty} \frac{|\{(i, j) : i \neq j, 0 \leq i, j \leq N, |v_i - v_j| < r\}|}{N^2}.$$

The *correlation dimension* of the orbit S is defined as

$$\mathrm{cordim}(S) = \lim_{r \to 0} \frac{\ln C(r)}{\ln r}$$

when the limit exists.

> ### Note
> For an orbit which is *not* asymptotically periodic:
> - when r tends to 0 the correlation function tends to 0 (since the size of the numerator tends to 0);
> - when r approaches infinity then $C(r)$ tends to 1 since the size of the numerator approaches N^2.
>
> If for small r the correlation function is $C(r)$ is of the order r^d, that is $C(r) \simeq r^d$ then d is the correlation dimension of S.

Let us compare the box-counting and correlation dimensions of the Hènon attractor from Figure 5.3.

Recall that the box-counting dimension of a set S is defined as $\lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)}$. Choose box-sizes of the from $\epsilon = 2^{-2}, 2^{-3}, ...$. Plotting $\log_2 N(\epsilon)$ against $\log_2(1/\epsilon)$, the box-counting dimension of S can be estimated as the slope of the resulting graph. The result is approximately 1.27
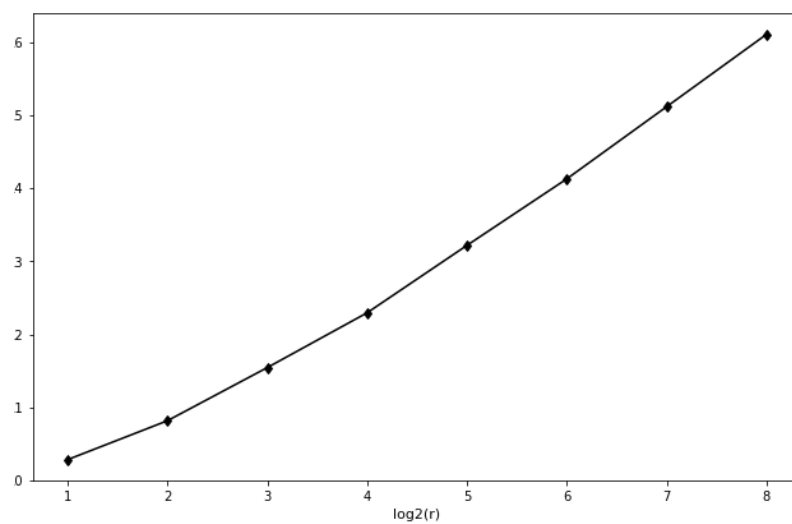
FIGURE 5.4: Box-counting dimension of the Hènon attractor

For the correlation dimension we can take a similar approach. We plot $\log_2(C(r))$ against $\log_2(r)$ where we consider the first 1000 points of the orbit. We take values of $r = 2^{-2}, 2^{-3}, \dots$. The correlation dimension is then the slope of the resulting graph as r tends to 0. The result is approximately 1.23
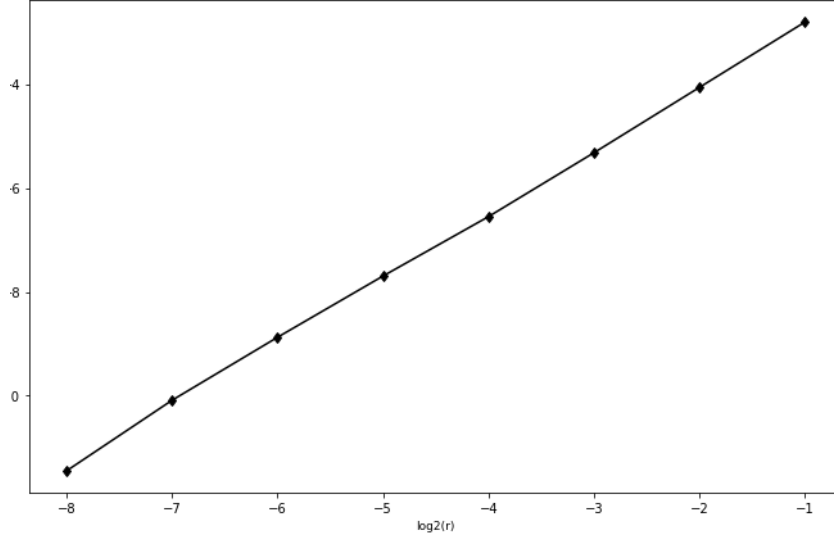
FIGURE 5.5: Correlation dimension of the Hènon attractor

5.2.2 Chaos in higher-dimensions

In Subsection 4.1.1 we gave a precise definition of what is meant by a chaotic orbit for a bounded orbit of a one-dimensional map. We were able to say exactly what is meant by non-periodicity (not asymptotically periodic) and we characterised sensitivity to initial conditions in terms of the Lyapunov exponent of the orbit. We now generalise this to higher-dimensional maps. As we have seen (2.3) a new dynamic emerges in dimensions greater than 1 — saddle fixed/saddle periodic points — these have have expansion/stretching in some directions and contraction in others. The expansion and compression factors are determined by the eigenvalues of the Jacobian of the map at the fixed point. The Lyapunov number for higher-dimensional maps ought to give the stretching/attracting factor on orbits near the repelling/attracting directions of the saddle point. The Lyapunov number should quantify the stretching/shrinking per iteration of the map on orbits originating near the saddle point.

Definition 5.3 (Lyapunov exponent). Let $\boldsymbol{f}$ be a smooth map defined on a subset of $\mathbb{R}^n$ and let $J_m = \mathbf{D}\boldsymbol{f}^m(\boldsymbol{v}_0)$ (that is J_m is the Jacobian of $\boldsymbol{f}^m$ at v_0). The matrix J_m induces a linear map on $\mathbb{R}^n$ and maps the n-dimensional unit sphere to an ellipsoid (recall that our maps are invertible). Let r_k^m be the length of the kth longest orthogonal axis of the ellipsoid (that J_m maps the unit sphere to). Then r_k^m measures the contraction and expansion near the orbit of $\boldsymbol{v}_0$ during the first n iterations of $\boldsymbol{f}$. The kth *Lyapunov number* of $\boldsymbol{v}_0$ is defined as

$$L_k = \lim_{m \to \infty} (r_k^m)^{1/m}$$

when the limit exists.

The kth *Lyapunov exponent* of $\mathbf{v}_0$ is $h_k = \ln L_k$.

Notice that by choice of the r_k's it follows that $L_1 \geq L_2 \geq ... \geq L_n$ and $h_1 \geq h_2 \geq ... \geq h_n$.

Now we have the ingredients required to define chaotic orbits in higher dimensions.

5.3   Problem Sheet 5

For week 6.

Question 1

Consider the map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathbf{f}(x, y) = (1 - ax^2 + y, bx).$$

a. Verify that a period-2 orbit of $\mathbf{f}(x, y)$ is given by

$$\mathbf{p}_0 = (x_+, y_+), \quad \mathbf{p}_1 = (x_-, y_-)$$

where

$$x_+ = \frac{1}{2a} \left[1 - b + \sqrt{4a - 3(1 - b)^2} \right] \quad (5.1)$$

$$y_+ = \frac{b}{2a} \left[1 - b - \sqrt{4a - 3(1 - b)^2} \right] \quad (5.2)$$

$$x_- = \frac{1}{2a} \left[1 - b - \sqrt{4a - 3(1 - b)^2} \right] \quad (5.3)$$

$$y_- = \frac{b}{2a} \left[1 - b + \sqrt{4a - 3(1 - b)^2} \right] \quad (5.4)$$

where $a > 3(1 - b)^2/4$.

b. Let $b = 0$. Show that the period 2-orbit is stable for $3/4 < a < 5/4$ and unstable for $a > 5/4$.

Question 2

A map f of the unit interval is defined as follows

$$f(x) = 10x \mod 1 = \begin{cases} 10x, & 0 \le x < \frac{1}{10} \\ 10x - 1, & \frac{1}{10} \le x < \frac{2}{10} \\ 10x - 2, & \frac{2}{10} \le x < \frac{3}{10} \\ \vdots & \vdots \\ 10x - 9, & \frac{9}{10} \le x < 1 \\ 0, & x = 1 \end{cases}$$

a. Sketch the map and the straight line $y = x$. On the basis of your skecth, how many fixed points do you expect this map to have?

b. State the definition of the Lyapunov number and Lyapunov exponent for one-dimensional maps.

c. Calculate the Lyapunov number and Lyapunov exponent for all orbits of the map $f(x)$ that avoid

$$x_i = \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}, \text{ and } \frac{9}{10}.$$

Can this map have any stable fixed points or periodic points?

d. State the definition of chaotic orbits for one-dimensional maps.

e. Briefly explain why you expect the map $f(x)$ defined above to have chaotic orbits.

[Hint: It might help to use the decimal representation of numbers in the interval $[0, 1]$.]

Question 3

A one-dimensional map on the real numbers is defined by

$$f(x) = rx(1 - x), \quad 0 \le r < \infty.$$

Let $r > 2 + \sqrt{5}$. For any orbit of $f(x)$ that remains in $[0, 1]$ find a lower bound for $|f'(x)|$. Hence show that the Lyapunov exponents of such orbits is greater than 0 if it exists. When would such an orbit be chaotic?

6

Chaos in Higher Dimensions

Covers Chapter 5 Sections 5.1 and 5.6; Chapter 6 Sections 6.1 and 6.2; Chapter 7 Section 7.1 (Alligood et al., 2000).

6.1 Chaotic Orbits in higher dimensions

Definition 6.1. Let f be an n-dimensional map $n \geq 1$. Let $(x_0, x_1, ...)$ be a bounded orbit of f. The orbit is called *chaotic* if:

1. it is not asymptotically periodic;
2. no Lyapunov number is exactly one;
3. $L_1(x_0) > 1$ (alternatively $h_1(x_0) > 0$).

This definition should be compared with Definition 4.3: the first condition is not surprising; while the second 2 characterise sensitivity to initial conditions suitable generalised for higher dimensions.

6.1.1 The horeshoe map

Let us consider an example with interesting dynamics. There are two key features of the example which follows:

- *Stretching*: nearby points are being pulled further away from one another.
- *folding* distant points are being mixed together

The aim is to see how chaotic orbits can arise from when stretching and folding are present are present.

We schematically describe the mapping (See Figure 6.1). The process is as follows:

1. we have a square W with vertices A, B, C, D embedded in the plane $\mathbb{R}^2$,

2. a continuous, injective map $\boldsymbol{h} : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\boldsymbol{h}(A) = A*, \ \boldsymbol{h}(B) = B*, \ \boldsymbol{h}(C) = C*, \ \boldsymbol{h}(D) = D*$$

3. We assume that $\boldsymbol{h}$ uniformly contracts distances horizontally and expands distances vertically.
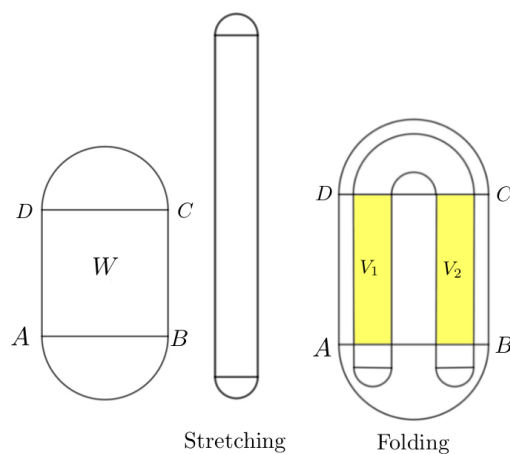
FIGURE 6.1: Stretching and folding of the horseshoe map

Stretching and folding in the horseshoe map
- The fold (upper annulus) is outside the initial square W.
- The intersection $W \cap h(W)$ consists if two strips V_1 and V_2.
- On the 2$^{\text{nd}}$ iteration of h, we apply h to the horseshoe. This results in a "4-stranded" horseshoe sitting inside the simple "2-stranded" horseshoe.
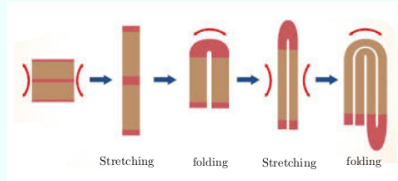
FIGURE 6.2: Two iterations of the horseshoe map
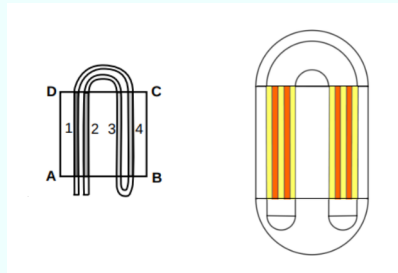
FIGURE 6.3: Four-stranded orange horseshoe inside original yellow horseshoe.

Forward and backward orbits

Let x_0 be a point in the square W. Under h, x_0 maps to x_1 and x_0 maps under h^2 to a point x_2. This gives us a forward orbit $(x_0, x_1, x_2, ...)$. Note that the orbits might escape the square at some point, for instance if x_0 lies near the bottom of the square then it is stretched out of the square in the first iteration (as h).

We can also consider a backward orbit of x_0 as our map h is injective. We think of the point $x_{-1} \in W$ (if it exists) such that $h(x_{-1}) = x_0$ and a point x_{-2} (if it exists) such that $h(x_{-2}) = x_{-1}$ and so on. This gives rise to an orbit $(x_0, x_{-1} x_{-2} ...)$

Forward and backward orbits that always remain in the square

Let us try to characterise the points whose forwards and backwards orbits never escape the square W.

Let us first think about the points with images inside of the horseshoe. That is points in W with images lying the two vertical strips V_1 and V_2 (see figure 6.1).

We can figure this out either by unfolding the horseshoe or by applying the inverse of h to the region $h(W) \cap W$.

Such a point x_0 whose image $\boldsymbol{h}(\boldsymbol{x}_0)$ lies in one of the vertical strips must originate from one of the two vertical strips H_0 or H_1 (Figure 6.4)
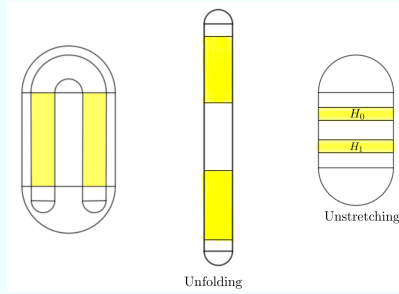
FIGURE 6.4: Unfolding to find pre-images of vertical strips.

Points $\boldsymbol{x}_0$ in W whose pre-images also lie in W lie on the vertical strips. Therefore point in W with image and pre-image in W lie on the intersection of the horizontal and vertical strips (see Figure 6.5).
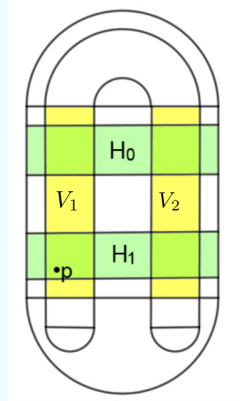
FIGURE 6.5: Points whose image and pre-image lie in the square W.

We can see that with successive iterations we obtain a fractal type construction (See Figure 4.13 in Subsection 4.3.2). The horizontal strips double with each step on the bottom and on the top (symmetrically) and become thinner, similarly for the vertical strips. The points in the intersections of the corresponding horizontal and vertical strips which remain in the limit correspond to the set

$$\Lambda = \{\boldsymbol{x} \in W : \boldsymbol{h}^k(\boldsymbol{x}) \in W \,\forall\; k \in \mathbb{Z}\}.$$

6.2 Chaotic attractors

Thus far we have seen that the orbits of maps can be attracted to stable fixed points, stable periodic points and saddle fixed and saddle periodic points (along attracting axes). A question one can as is whether or not chaotic orbits can be "attracting"? What do we even mean by "attracting" in the context of chaotic orbits? Roughly speaking, we mean that a significant portion of initial conditions are attracted to the chaotic orbit.

Consider Figures 6.6 and 6.7.
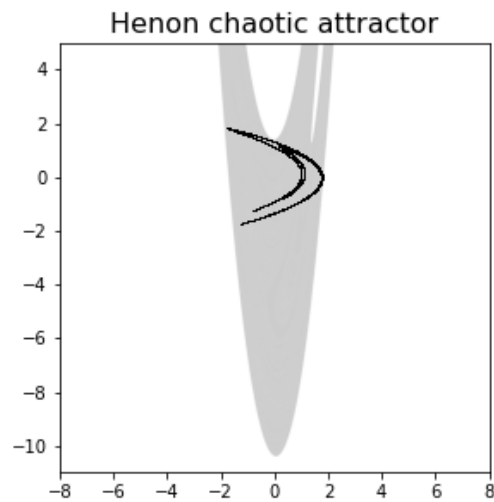
FIGURE 6.6: Basin of chaotic attractor of the Hènon map
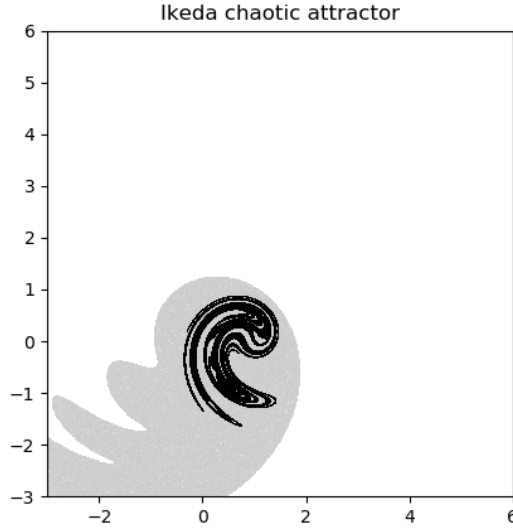
FIGURE 6.7: Basin of chaotic attractor of the Ikeda map

Figure 6.6 shows a chaotic orbit (non-periodic with Lyapunov exponent bigger than 0) of the Hènon map for $a = 1.4$ and $b = 0.3$. Figure 6.7 shows a chaotic orbit of the Ikeda map. The Ikeda map is the map $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\boldsymbol{f}(x, y) = (R + C_2(x \cos \tau - y \sin \tau), C_2(x \sin \tau + y \cos \tau))$$

where $\tau = C1 - C3/(1 + x^2 + y^2)$ and R, C_1, C_2, and C_3 are fixed real valued parameters. In Figure 6.7 $R = 0.9, C_1, C_3 = 0.4, 6$. The grey area containing the black chaotic orbits represent points whose orbits converge to the chaotic orbit. The white region represents points which diverge to infinity or converge to a different attracting orbit.

These two examples illustrate two key characteristics of a "chaotic attractor"

1) It contains a contains a chaotic orbit;
2) Attracts a set of initial values which has a non-zero area in the plane.

6.2.1 Forward Limit Set

A consequence of the second attribute of a chaotic attractor given above is it is what is a called a *forward limit set*. We shall say precisely what we mean by this below. For the moment we roughly mean that a chaotic attractor is what is left over after throwing a away a (very) large number of initial points (say 1 million)

of the orbit. The attracting nature of the chaotic attractor means we return to the vicinity of set of points that remain again and again.

Consider Figure 6.8
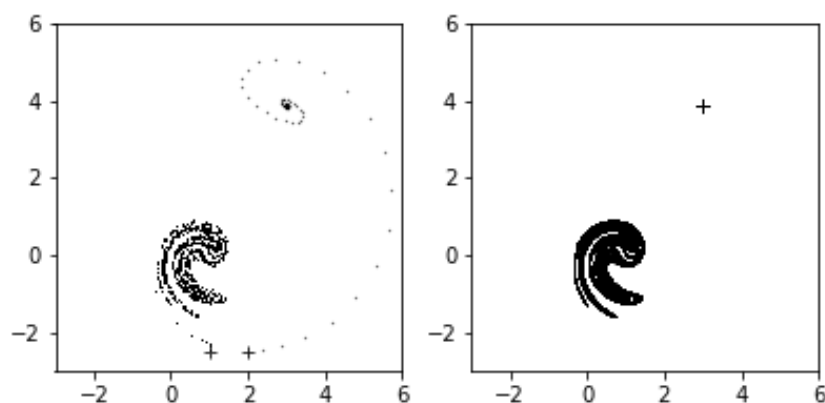
FIGURE 6.8: Two attractors of the Ikeda map

The first subplot shows the first 1000 points of two different orbits of the Ikeda map. The second subplot shows iterates 1,000,001 to 2,000,000 of the same two orbits. We can see that one orbit approaches the chaotic attractor and the other approaches a sink.

Let us now give a precise definition of the forward limit set.

Definition 6.2 (Forward limit set). Let f be a map (1D or ND) and x_0 a point in the domain of f. The *forward limit set* of the orbit $\{f^n(x_0) : n \in \mathbb{N}\}$ is the set

$$\omega(x_0) = \{x : \forall\, N \in \mathbb{N} \text{ and } \epsilon > 0, \exists\, n > N \text{ such that } |f^n(x_0) - x| < \epsilon\}$$

The forward limit set is sometimes called the ω-limit set ("omega limit set").

Given points x_0, x_1 in the domain of a map f, We say that the orbit of x_1 is *attracted to* the forward limit set of x_0 if $\omega(x_1)$ is contained in $\omega(x_0)$.

For the examples in Figure 6.8 the ω-limit set of the second initial condition, the one attracted to the sink, is the sink. Indeed any fixed point is its own forward limit set.

Let us now formally define we mean by a chaotic attractor.

Definition 6.3 (Chaotic attractor). Let f be a map and x_0 a point in the domain of f. Suppose that the orbit of x_0 is chaotic. If x_0 is in its ω-limit set $\omega(x_0)$, then $\omega(x_0)$ is called a *chaotic set*.

An *attractor* is any forward limit set which attracts a set of initial values with a non-zero measure (i.e a measure appropriate to the dimension of the domain of the map: length in $\mathbb{R}$, area in $\mathbb{R}^2$, volume in higher dimensions). The set of initial conditions attracted to the attractor is called the *basin of attraction of the attractor*.

A *chaotic attractor* is a chaotic set that is also an attractor.

We conclude our studies of discrete-time dynamical systems. We have characterised key features of the dynamics of discrete dynamical systems in one and higher dimensions and have explored the concept of chaos. We have seen how in the midst of/ out of chaos order can arise.

6.3 Continuous-time dynamical systems

As promised, the remainder of the module will be focused on continuous time dynamical systems. Whereas discrete time dynamical systems arose from maps on $\mathbb{R}^n$ (for some n), continuous time dynamical systems are defined by *differential equations*. In this module we only consider Ordinary Differential Equations (ODE's) as opposed to partial or stochastic differential equations. ODE's are differential equations whose solutions are functions of one independent variable t (which in this module represents time). Solutions we seek are of the form $x(t)$ where x typically represents a physical quantity that depends on time; x is a dependent variable.

There are two main types of ODE's

- Autonomous ODE's: the independent variable t does not explicitly appear in the ODE e.g.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax$$

- Non-autonomous ODE's: the variable t appears explicitly in the ODE. e.g.

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -c\frac{\mathrm{d}x}{\mathrm{d}t} - \sin x + \rho \sin t.$$

We only focus on autonomous ODE's; any non-autonomous dynamical system can be recast as an autonomous one.

For instance, for the system

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -c\frac{\mathrm{d}x}{\mathrm{d}t} - \sin x + \rho \sin t.$$

we can introduce a new variable y satisfying $\frac{\mathrm{d}y}{\mathrm{d}t} = 1$. This gives a 2-dimensional continuous time autonomous system governed by the rules

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -c\frac{\mathrm{d}x}{\mathrm{d}t} = \sin x + \rho \sin y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1$$

Let $\boldsymbol{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ describe the state of a continuous time dynamical system $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(\boldsymbol{x})$ on $\mathbb{R}^n$ where $\boldsymbol{R} : \mathbb{R}^n \to \mathbb{R}^n$ is a map. The evolution of the system is governed by the rule

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(\boldsymbol{x}).$$

Example 6.1. The ODE $\frac{\mathrm{d}x}{\mathrm{d}t} = ax(1 - x)$ describes dynamical system. As the variable x approaches 1, $\frac{\mathrm{d}x}{\mathrm{d}t}$ approaches 0 and the population x approaches a steady state. This a 1D-non-linear ODE and gives rise to a 1D continuous time dynamical systemm on $\mathbb{R}$.

What about a $2D$ example? Let $\boldsymbol{R} = (R_x(x, y), R_y(x, y))$ be a map on $\mathbb{R}^2$. The pair of ODES:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = R_x(x, y) := ax + by^2$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = R_y(x, y) := cy + dy^2$$

Gives rise to a continuous time dynamical system on $\mathbb{R}^2$.

The technique we applied to recast a non-autonomous ODE as an autonomous one by introducing an extra variable can be repeatedly applied to recast a high-order ODE as a system of first order linear ODE's.

Example 6.2. Consider the non-linear damped oscillator governed by the rule

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -c\frac{\mathrm{d}x}{\mathrm{d}t} - \sin x \tag{6.1}$$

A second order ODE. Introduce a new variable y satisfying $y = \dfrac{\mathrm{d}x}{\mathrm{d}t}$ so that $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\mathrm{d}^2 x}{\mathrm{d}t^2}$. Then we can rewrite (6.1) as the system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -cy - \sin x.$$

To solve a system of linear ODE's we need to have two conditions satisfied:

- Initial conditions for each variable x_i of the ODE. That is for a system of n linear ODE's with variables $x_1, x_2, \ldots, x_n$ we need initial conditions

$$x_1(0) = x_{1,0}, x_2(0) = x_{2,0}, \ldots, x_n(0) = x_{n,0}$$

- For a given set of initial conditions, the ODE must have a *unique* solution. Moreover, this solution should be continuous (no singularities).

The set of all solutions of an ODES is called the *flow* of the ODE. To be more precise

Definition 6.4. The *flow* of an autonomous ODE is the function $\boldsymbol{F}$ of time t and the initial value $\boldsymbol{x}_0$ which represents the set of solutions. So $\boldsymbol{F}(t, \boldsymbol{x}_0)$ (alternatively $\boldsymbol{F}_t(\boldsymbol{x}_0)$) is the value at time t of the solution with initial value $\boldsymbol{x}_0$.

The reason for the alternative notation is that in some situations we might want to think of the evolution of the dynamical system at a fixed time $t = T$ units later (the time-T-map), the notation $\boldsymbol{F}_T(\boldsymbol{x}_0)$ makes it apparent that we are fixing $t = T$ and looking at the solution T time units later of the initial condition $\boldsymbol{x}_0$. By considering the time T-map we discretise our dynamical system: $\boldsymbol{x}_{n+1} = \boldsymbol{F}_T(\boldsymbol{x}_n)$. The resulting discrete time dynamical system gives the evolution of the system at time units of T.

Example 6.3 (Example of a flow). Consider the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax$$

for a a real constant. This is a separable ODE and has solutions $x(t) = Ce^{at}$ where $C = x(0)$.

Given an initial value $x(0) = x_0$, the solution of the *initial value problem*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax, \ x(0) = x_0$$

is $x(t) = x_0 e^{at}$.

The flow of the ODE is the function $F(t, x) = xe^{at}$, for initial condition $x = x_0$, we have $F(t, x_0) = x_0 e^{at}$.

Figure 6.9 show solutions for various values of x_0 in the regimes $a > 0$ and $a < 0$.
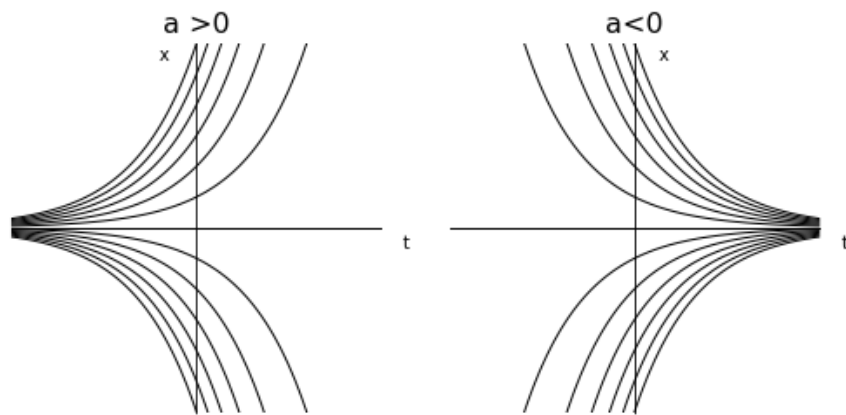
FIGURE 6.9: Flow of linear ODE dx/dt = ax

6.3.1 Equilibria

When we considered discrete time dynamical systems we found it useful to study fixed points and their stability. For continuous time dynamical systems, as these

are governed by ODE's, the equivalent of a fixed point is called an *equilibrium* these are solutions that do not change in time.

Definition 6.5 (Equilibrium)**.** A constant solution of the autonomous differential equation $\dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x)$ is called an *equilibrium* of the equation.

Example 6.4. Consider the ODE $\dfrac{\mathrm{d}x}{\mathrm{d}t} = ax$. The solution $x = 0$ is an equilibrium of the equation.

In general equilibria of ODE's $\dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(\boldsymbol{x})$ are constant solutions.

6.4 Problem Sheet 6

For week 7.

Question 1

Let $\mathfrak{C}_5$ denote the middle-1/5 Cantor set. This set is defined similarly to the middle-third Cantor set only at each step we remove the middle 5th part of the intervals remaining. Determine the box-counting dimension of $\mathfrak{C}_5$.

Question 2

The skinny baker's map is a map from $[0,1] \times [0,1]$ to itself defined as follows:

$$
\boldsymbol{B}(x,y) = \begin{cases} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } 0 \le y \le \frac{1}{2} \\[2ex] \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ -1 \end{pmatrix} & \text{if } \frac{1}{2} < y \le 1 \end{cases}
$$

a. Sketch the image of the unit square under the action of the map. What image do you expect after two and three iterates? Can you describe the set of points the original unit square is mapped onto if the number of iterations goes to infinity?

b. Calculate the Lyapunov numbers and exponents of orbits of the skinny baker map which never map onto the line $y = 1/2$.

[Hint: use the fact that as the skinny baker's map is a linear map, up to translation, it is equal to the map defined by its Jacobian matrix. You can then use the fact that the expansion/contraction factors r_k^n are given by the square roots of the eigenvalues of $\mathbf{D}\boldsymbol{f}^n \cdot (\mathbf{D}\boldsymbol{f}^n)^t$. Here, for a matrix A, we write A^t for the transpose of A.]

c. What can you conclude about the possibility of chaotic orbits?

Question 3

Consider Smale's Horseshoe map h (Subsection 6.1.1 in the notes). Recall that Λ is defined to be the set of points whose forwards and backwards orbits always remain in the square W. We can associate to each point of Λ a sequence of 0's and 1's as follows

- The intersection of the image of the unit square with the unit square defines two vertical strips V_1 and V_2.
- $h^n(x)$ must be in V_1 or V_2 if $x \in \Lambda$ since Λ is a subset of $V_1 \cup V_2$.
- If $x \in V_1$ the sequence begin with 0, otherwise $x \in V_2$ and the sequence begins with 1.
- If $h(x) \in V_1$ the sequence continues with 0 otherwise $h(x) \in V_2$ and the sequence continues with 1.
- If $h^n(x) \in V_1$ then the $(n+1)^{\text{st}}$ symbol in the sequence is 0, otherwise it is 1 and so on.

One application of the horseshoe map can be thought of as shifting our position in the sequence one symbol to the right. Using this analogy show that Smale's horseshoe map has two fixed points. To what symbols do the fixed points correspond?

Part II

Continuous time dynamical systems

7

Equilibria and stability in continuous time dynamical systems

Covers Chapter 7 Sections 7.2 to 7.5 (Alligood et al., 2000).

7.1 Equilibria

We concluded the last chapter by stating that the analogue of fixed points in continuous time dynamical system is an *equilibrium*. To reiterate, equilibria occur as solutions to the equation

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(\boldsymbol{x}) = 0.$$

Example 7.1. 1. The ODE $\frac{\mathrm{d}x}{\mathrm{d}t} = ax(1-x)$ has equilibria at $x = 0$ and $x = 1$.

2. Writing $\boldsymbol{x} = (x, y)$ for an arbitrary point in $\mathbb{R}^2$, the ODE $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(\boldsymbol{x}) = (y, -ay - \sin x)$. Has equilibria at points in the set $\{(n\pi, 0) : n \in \mathbb{Z}\}$. Since these are the points at which $\boldsymbol{R}(\boldsymbol{x}) = 0$.

Now that we have analogues of fixed point for continuous time dynamical systems, we introduce the notion of stability in these systems.

Definition 7.1 (Continuous stability)**.** Consider an autonomous differential equation with flow $\boldsymbol{F}(t, x)$. An equilibrium of $\boldsymbol{v}$ of the system is called

- *stable* if all nearby solutions stay close to the equilibrium for all future times. Formally for any neighbourhood N of $\boldsymbol{v}$ there is a neighbourhood $N_1 \subseteq N$ of $\boldsymbol{v}$ such that for all points $\boldsymbol{x} \in N_1$, $\boldsymbol{F}(t, \boldsymbol{x})$ remains in N_1 for all $t \geq 0$.
- *unstable* if it is not stable.
- *asymptotically stable* if it is both stable and attracting (nearby solutions converge to the equilibrium)

- *globally asymptotically stable* if it is asymptotically stable and **all** initial values converge to the equilibrium.

The example below generalises a scheme we applied when investigating stability of fixed points for discrete time dynamical systems (see Subsections 2.1 and 2.3.1).

Example 7.2. Consider an ODE

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(\boldsymbol{x})$$

and suppose $\boldsymbol{x}_e$ is an equilibrium of the ODE. Let $\epsilon > 0$ be very small so that $\boldsymbol{x} = \boldsymbol{x}_e + \epsilon \boldsymbol{u}$ is an initial condition very close to the equilibrium $\boldsymbol{x}_e$. Then

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(\boldsymbol{x}_e + \epsilon \boldsymbol{u}) = \frac{\mathrm{d}\boldsymbol{x}_e}{\mathrm{d}t} + \epsilon \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} \tag{7.1}$$

Now using the fact that $\boldsymbol{x}_e$ is an equilibrium, and taking a Taylor series approximation of $\boldsymbol{R}$ at $\boldsymbol{x}_e$ Equation (7.1) becomes:

$$\epsilon \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} \simeq \epsilon \boldsymbol{DR}(\boldsymbol{x}_e) \cdot \boldsymbol{u}$$

Cancelling out ϵ (which is non-zero by assumption) we have:

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} \simeq \boldsymbol{DR}(\boldsymbol{x}_e) \cdot \boldsymbol{u}$$

A linear ODE with constant coefficients for u.

Our analysis now follows as in the discrete time case.

Let λ_i, $i = 1, 2, \dots, n$ be the eigenvalues of $\boldsymbol{DR}(\boldsymbol{x}_e)$ and $\boldsymbol{v}_i$ the corresponding eigenvectors. The general solution for $\boldsymbol{u}$ is given by

$$\boldsymbol{u}(t) = \sum_{i=1}^{n} u_i \boldsymbol{v}_i e^{\lambda_i t}$$

where the u_i's are constants and $\boldsymbol{u}_0 = \sum_{i=1}^{n} u_i \boldsymbol{v}_i$.

Thus we expect that if the real part of all eigenvalues of $\boldsymbol{DR}(\boldsymbol{x}_e)$ are strictly less than 0 then the equilibrium is stable; if the real part of one of the eigenvalues is positive, the equilibrium is unstable.

The following result sums up this conclusion.

Theorem 7.1. *Let x_e be an equilibrium of $\dfrac{\mathrm{d}x}{\mathrm{d}t} = R(x)$. If the real part of each eigenvalue of $DR(x_e)$ is strictly less than zero, then x_e is asymptotically stable. If the real part of at least one eigenvalue is strictly positive, then x_e is unstable.*

Example 7.3. Let us apply Theorem 7.1 to classify the equilibria of the system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 - y^2$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = xy - 4$$

There equilibria occur at $(-2, -2)$ and $(2, 2)$. Write $f(x, y)$ for the function on $\mathbb{R}^2$ given by $(x, y) \mapsto (x^2 - y^2, xy - 4)$. The Jacobian of f is:

$$\begin{pmatrix} 2x & -2y \\ y & x \end{pmatrix}$$

At the point $(x, y) = (2, 2)$ we have:

$$Df(2, 2) = \begin{pmatrix} 4 & -4 \\ 2 & 2 \end{pmatrix}$$

The eigenvalues of this matrix are the roots of the equation $x^2 - 6x + 16 = 0$. This has complex solutions $\lambda_1, \lambda_2 = 3 \pm i\sqrt{7}$. Therefore the point $(2, 2)$ is an unstable equilibrium of the system.

At the point $(x, y) = (-2, -2)$, the Jacobian matrix is:

$$Df(-2, -2) = \begin{pmatrix} -4 & 4 \\ -2 & -2 \end{pmatrix}$$

This matrix has eigenvalues $\lambda_1, \lambda_2 = -3 \pm i\sqrt{7}$. Therefore the point $(-2, -2)$ is an asymptotically stable fixed point.

As with higher-dimensional discrete time dynamical systems, it is possible that an equilibrium of a continuous time system, has at least one attracting direction and at least one repelling direction. In this case we again apply the term *saddle* to describe such an equilibrium. The following definition should be compared with 2.7.

Definition 7.2. An equilibrium x_e of a system $\dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x)$ is called *hyperbolic* if none of the eigenvalues of $Df(x_e)$ has real part equal to 0.

If an equilibrium $\boldsymbol{x}_e$ of $\dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x})$ is hyperbolic and has at least one eigenvalue with a positive real part and at least one eigenvalue with a negative real part, then $\boldsymbol{x}_e$ is called a *saddle*.

We can similarly extend Definition 3.2 to continuous time dynamical systems using the flow $\boldsymbol{F}(t, \boldsymbol{x})$ of the differential equation.

Definition 7.3. Let $\boldsymbol{x}_e$ be a saddle of a system $\dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x})$. Let $\boldsymbol{F}(t, \boldsymbol{x})$ be the flow of the system.

The *stable manifold* of $\boldsymbol{x}_e$ is the set $\boldsymbol{x}$ of points in $\mathbb{R}^n$ such that

$$\lim_{t \to \infty} |\boldsymbol{F}(t, \boldsymbol{x}) - \boldsymbol{F}(t, \boldsymbol{x}_e)| = 0.$$

The *unstable manifold* of $\boldsymbol{x}_e$ is the set $\boldsymbol{x}$ of points in $\mathbb{R}^n$ such that

$$\lim_{t \to \infty} |\boldsymbol{F}(-t, \boldsymbol{x}) - \boldsymbol{F}(-t, \boldsymbol{x}_e)| = 0.$$

7.2 Lyapunov Functions I

Looking over the argument in Example 7.2, we see that it is only valid for deducing stability in a small neighbourhood of the equilibrium point i.e for small perturbations from the equilibrium point. This is a consequence of approximating the function $\mathbb{R}(x)$ with a linear function via its Taylor expansion (this technique is commonly referred to as *linearisation*).

Figure 7.1 shows an equilibrium point which is stable against small perturbations to the left and to the right, but is unstable for large perturbations.
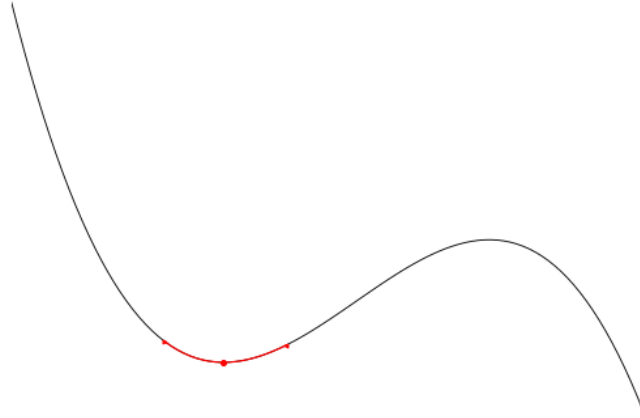
FIGURE 7.1: Ball in a potential well

We need to take a different approach then when investigating stability against large perturbations which occurs for example for globally asymptotically stable equilibria. The approach which is of use in this situation is the method of *Lyapunov functions*.

Example 7.4. Perhaps the best way of motivating Lyapunov functions is to consider the motion of a dynamical system through a potential field. That is, we have a potential field $V = V(x(t))$ (which tells us the potential energy of at a point x), and a dynamical system governed by the rules

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \tag{7.2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{\partial V}{\partial x} \tag{7.3}$$

One way of thinking about this is as follows. If x represents position, then $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ is velocity (change of position with time) and $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ is acceleration. Now acceleration is proportional to the force at the point x and the force is equal to minus the gradient of the velocity $\left(-\nabla V = -\dfrac{\partial V}{\partial x}\right.$ in this example). Thus the dynamical system moving through a potential can be thought of in terms of Newton's 2nd Law of motion.

From equation (7.3) we have

$$\frac{d^2x}{dt^2} + \frac{\partial V}{\partial x} = 0 \tag{7.4}$$

Multiplying by equation (7.4) by $\frac{dx}{dt}$ and using $\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dV}{dt}$ we get:

$$\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dV}{dt} = \frac{d}{dt}\left(1/2\left(\frac{dx}{dt}\right)^2 + V(x)\right) = 0. \tag{7.5}$$

Lastly observe that the total energy is the sum of the potential and kinetic energy and so the term $1/2\left(\frac{dx}{dt}\right)^2 + V(x)$ is just the total energy $E(x,y)$. Equation (7.5) can now be interpreted as saying that the total energy in the system is conserved along an orbit — it does not change with time; in symbols for an initial value (x,y) and $F(t,(x,y))$ the flow of the system of differential equations, $\frac{dE}{dt}(F(t,(x,y))) = 0$ for all t.

Note

The inclusion of the phrase "along a trajectory" is essential, since we considered the motion of a particle (i.e the orbit of an initial value) in the vector field. In particular for *different orbits* the total energy E along each orbit might be different.

For a dynamical system moving in a potential field as above, we have two ways of visualising the orbits of the dynamical system.

  i. By sketching the potential field $V(x)$
  ii. By plotting phase plane trajectories. That is at each point $\left(x, y = \frac{dx}{dt}\right)$ we plot the gradient vector $\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$ at the point.

Let us illustrate approach i. using Example 7.4

Example 7.5. Suppose we are in the context of Example 7.4 and the potential $V(x)$ is as in Figure 7.2
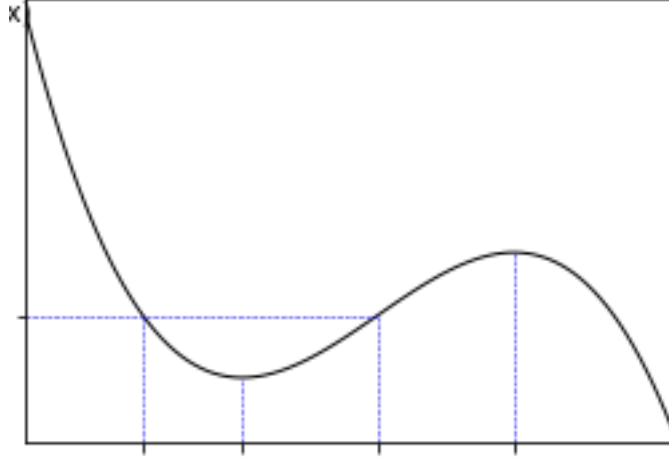
FIGURE 7.2: Sketch of potential field

Thus the potential has a local minimum at x_{min} and a local maximum at x_{max}, and $E_1 = V(x_1)$ is the potential energy at x_1. We suppose the trajectory begins at rest from the point x_1, so that initially there is no kinetic energy and the total energy is equal to the potential energy (note that this implies that $y_1 = 0$ since the kinetic energy at (x_1, y_1) is equal to $y_1^2/2$). This means that the energy along the trajectory/orbit is constantly equal to E_1. Thus, to figure out the subsequent behaviour of the trajectory the key value to consider is $E_1 - V(x, y) = 1/2y^2 = E_{kin}$ where E_{kin} is the kinetic energy.

Now at points where $V(x) = E_1$, there is no kinetic energy and so $y = \dfrac{\mathrm{d}x}{\mathrm{d}t} = 0$ for such points. Points at which $V(x) = E_1$ are called *turning points* — the velocity changes direction around these points.

If $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0$ and $\dfrac{\partial V}{\partial x} = 0$, then x is an equilibrium point: the particle is at rest and has no forces acting on it. For this example, x_{min} is an equilibrium point.

If orbits originating near an equilibrium point oscillate around it, then in particular they never move far away from it, this means that the equilibrium is stable. For instance, the equilibrium at x_{min} is stable, since orbits originating near x_{min} remain stuck in the *potential well* between the points x_1 and x_2.

For orbits originating at rest from the local maximum x_{max}, they either get stuck in a potential well to the left of x_{max} or move away from x_{max} towards $+\infty$ — x_{max} is an unstable equilibrium.

7.3   Problem Sheet 7

For week 8.

Question 1

Write the following equations describing continuous-time dynamical systems as a set of first order autonomous ODEs. What is the dimension of the phase space of these dynamical systems?

a. $\dfrac{d^2 x}{dt^2} + (x^2 - 1)\dfrac{dx}{dt} + x = 0,$

b. $\dfrac{d^2 x}{dt^2} - x\sin^2 t = 0$

Question 2

Solve the ODE

$$\frac{dx}{dt} = x^2$$

with initial condition $x(t_0) = x_0 > 0$. Show that the solution goes to infinity (blows up) at a finite time.

Question 3

Consider the 2D dynamical system give in polar coordinates r and θ by

$$\frac{dr}{dt} = r(1 - r), \quad \frac{d\theta}{dt} = 1.$$

a. Solve the differential equations. Sketch in the (x, y)-plane, indicating typical behaviour, solutions corresponding to initial conditions $r_0 < 1$, $r_0 = 1$ and $r_0 > 1$ where $r_0 = r(0)$.
b. Determine the time-2π map of the system.
c. Consider only the r-part of the time-2π map. Determine the fixed points and their stability. To which solutions of the differential equations do the fixed points correspond?

8

Lyapunov Functions

Covers Chapter 7 Section 7.6 to Chapter 8 Section 8.1 (Theorem 8.3) (Alligood et al., 2000).

8.1 Lyapunov Functions II

We continue with Example 7.4 and illustrate method ii. (plotting phase plane trajectories) for visualising the orbit of a dynamical system moving in a potential field.

Example 8.1. The context is as follows. We have a dynamical system governed by the rule:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \tag{8.1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{\partial V}{\partial x} \tag{8.2}$$

where $V(x)$ is a potential field (see 8.2).

We examine how $y = \dfrac{\mathrm{d}x}{\mathrm{d}t}$ changes as x increases and $E_1 - V(x)$ changes.

As x increases from x_1 to x_2, the difference $E_1 - V(x)$ representing the kinetic energy, increases from 0 to a maximum value at x_{min} and then decreases to 0 again. Consequently the velocity $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ also increases from 0 to some maximum value at x_{min} and then decreases to 0 again.

Now recollecting that x_1 and x_2 are turning points, the velocity changes direction and reverses the process describes above: it moves from x_2 to x_1, with the velocity (now negative) decreasing from 0 to a minimum at x_{min} and then increasing back to 0 at x_2 at which point the whole process repeats. Therefore we have a *closed*

orbit between x_1 and x_2 and initial values in this region gives rise to *periodic orbits*: (as noted before) the system oscillates between the values x_1 and x_2.

Figure 8.1 illustrates the trajectory of orbits originating nearby x_{min},

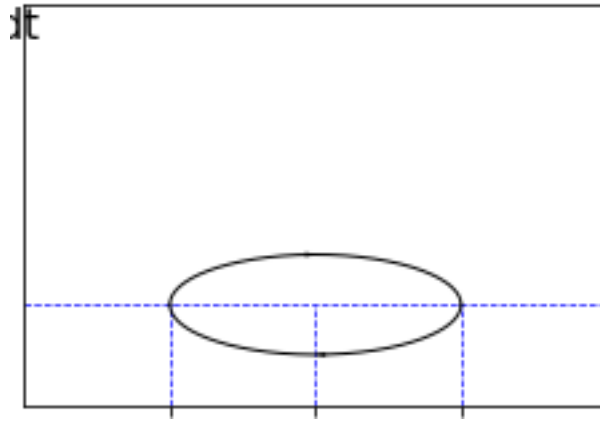FIGURE 8.1: Dynamical system in a potential well: higher energy trajectory.

For a trajectory originating near x_{min} with a higher total energy $E_2 > E_1$ but less than the value at the local maximum x_{max}, a similar behaviour occurs. We have oscillation about x_{min} between two new values x_3 and x_4. See Figure 8.2.
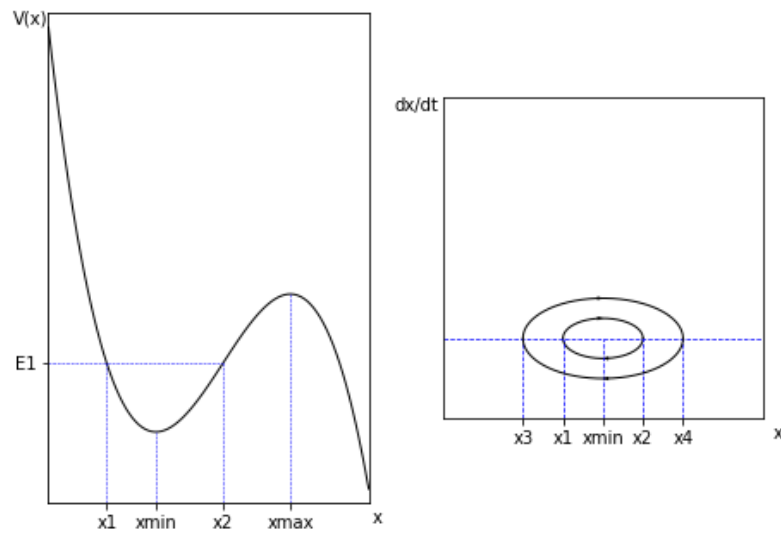

FIGURE 8.2: Dynamical system in a potential well: higher energy trajectory.

What if we introduce a damping term $c\dfrac{\mathrm{d}^2 x}{\mathrm{d}t^2}$ for $c > 0$? In this situation the dynamical system is now governed by the rules:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \tag{8.3}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\left(cy + \frac{\partial V}{\partial x}\right) \tag{8.4}$$

Thus we have:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + c\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial V}{\partial x} = 0.$$

Multiplying through by $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ and using the product rule gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(1/2\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \boldsymbol{V}(x)\right) + c\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = 0.$$

Recognising the term in the first bracket as the energy at the point (x, y), we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -c\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2.$$

Since c is positive, we conclude that the energy decreases with time. The system eventually settle to an equilibrium and so we will need energy to perturb the equilibrium.

Let us now return to our key aim — understanding stability under large perturbations. The idea from the example we have considered is to make use of functions which behave like an energy function for the system. These are *Lyapunov functions*.

What behaviour do we expect such a function to have?

Consider a dynamical system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(\boldsymbol{x}) = (R_1(\boldsymbol{x}), R_2(\boldsymbol{x}), \dots, R_n(\boldsymbol{x}))$$

and an initial point $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$. Suppose we have a function $E(\boldsymbol{x})$, a real-valued function which acts like an energy function for the system. Now

observing that E is a function of $\boldsymbol{x}$ which is an implicit function of t, by the chain rule we have:

$$
\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}x_1}{\mathrm{d}t}\frac{\partial E}{\partial x_1} + \frac{\partial x_2}{\partial t}\frac{\partial E}{\partial x_2} + ... + \frac{\mathrm{d}x_n}{\mathrm{d}t}\frac{\partial E}{\partial x_n} \\
&= R_1(\boldsymbol{x})\frac{\partial E}{\partial x_1} + R_2(\boldsymbol{x})\frac{\partial E}{\partial x_2} + ... + R_n(\boldsymbol{x})\frac{\partial E}{\partial x_n}
\end{aligned}
$$

Since $\boldsymbol{R}(x) = 0$ when x is an equilibrium point, then E satisfies $\dfrac{\mathrm{d}E}{\mathrm{d}t} = 0$ at equilibrium points. As E acts like an energy function, then it *increases away from a stable equilibrium*. That is as a trajectory moves towards a stable steady point, E decreases $\left(\dfrac{\mathrm{d}E}{\mathrm{d}t} \leq 0\right)$.

Using this key observation we can formally defined Lyapunov functions.

Definition 8.1 (Lyapunov Functions). Consider a dynamical system

$$
\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(\boldsymbol{x})
$$

with an equilibrium at a point $\boldsymbol{x}_e$. A function $E : \mathbb{R}^n \to \mathbb{R}$ is called a *Lyapunov function* for x_e, if for some neighbourhood N of $\boldsymbol{x}_e$, the following things hold:

1) $E(\boldsymbol{x}_e) = 0$ and $E(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in N\backslash\{\boldsymbol{x}_e\}$;
2) $\dfrac{\mathrm{d}E}{\mathrm{d}t}(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x} \in N$.

If, in place of 2 E satisfies the stronger condition

$$
2'. \quad \frac{\mathrm{d}E}{\mathrm{d}t}(\boldsymbol{x}) < 0 \text{ for all } \boldsymbol{x} \in N\backslash\{\boldsymbol{x}_e\},
$$

then it is called a *strict Lyapunov function*.

8.2 Equilibira and Lyapunov functions

The following theorem characterises when an equilibrium point of a continuous dynamical system is stable using Lyapunov functions.

Theorem 8.1. *Let*

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(\boldsymbol{x})$$

be the governing rule of a continuous time dynamical system. Suppose that $\boldsymbol{x}_e$ is an equilibrium point of the system.

If there exists a Lyapunov function for $\boldsymbol{x}_e$, then $\boldsymbol{x}_e$ is stable.

If there is a strict Lyapunov function for $\boldsymbol{x}_e$ then $\boldsymbol{x}_e$ is asymptotically stable.

Note

- If $\boldsymbol{x}_e$ is an equilibrium point of a dynamical system which has a strict Lyapunov function, E then any neighbourhood N on which $\dfrac{\mathrm{d}E}{\mathrm{d}t}(\boldsymbol{x}) < 0$ for all $\boldsymbol{x} \in N$ is contained in the basin of attraction of $\boldsymbol{x}_e$.
- The derivative of a Lyapunov function $E = E(\boldsymbol{x})$ is computed using the chain rule as follows:

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \cdot \nabla E \\
&= \frac{\mathrm{d}x_1}{\mathrm{d}t}\frac{\partial E}{\partial x_1} + \frac{\mathrm{d}x_2}{\mathrm{d}t}\frac{\partial E}{\partial x_2} + ... + \frac{\mathrm{d}x_n}{\mathrm{d}t}\frac{\partial E}{\partial x_n} \\
&= \boldsymbol{R}(\boldsymbol{x}) \cdot \nabla E
\end{aligned}$$

The following is a strengthening of theorem 8.1 where we have written $\dfrac{\mathrm{d}E}{\mathrm{d}t}$ as $\boldsymbol{R}(\boldsymbol{x}) \cdot \nabla E$. Note the stronger assumption that E is smooth — all derivatives exist and are continuous. This will mainly be a formality in this module as we mainly on deal with smooth functions in this module.

Theorem 8.2. *Let*

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(\boldsymbol{x})$$

be the governing rule of a continuous time dynamical system with $\boldsymbol{x}_e$ an equilibrium point of the system. Suppose there is a function $E : \mathbb{R}^n \to \mathbb{R}$ such that

- E *and its derivatives are continuous;*
- *there is a neighbourhood N of $\boldsymbol{x}_e$ on which $E(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in N\backslash\{\boldsymbol{x}_e\}$ and $E(\boldsymbol{x}_e) = 0$.*

  i. If $\boldsymbol{R} \cdot \nabla E \leq 0$ for all $\boldsymbol{x} \in N$, then $\boldsymbol{x}_e$ is stable.
  ii. If $\boldsymbol{R} \cdot \nabla E < 0$ for all $\boldsymbol{x} \in N\backslash\{\boldsymbol{x}_e\}$, then $\boldsymbol{x}_e$ is asymptotically stable.

     iii. if $\boldsymbol{R} \cdot \nabla E > 0$ for all $\boldsymbol{x} \in N\backslash\{\boldsymbol{x}_e\}$, $\boldsymbol{x}_e$ is unstable.

Let us do some examples.

Example 8.2. Consider the dynamical system governed by the rules:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -(cy + \sin x)$$

for $c \geq 0$. The trajectories of this dynamical system can model, for instance, a damped pendulum (when $c > 0$).

For $c = 0$ the system neither gains nor looses energy. In this case, the energy in the system is given by:

$$E = y^2/2 + V(x).$$

We can find $V(x)$, $\sin x$ (the negative of the force). This gives $V(x) = -\cos x + b$ for some constant b. Therefore

$$E = y^2/2 - \cos x + b$$

for

(In the specific instance of an undamped pendulum, it is natural assume that at angle 0 the potential energy of the particle is 0, and so $V(x) = -\cos x + 1$.)

In order to have a physical analogy in mind, we take $b = 1$.

The equilibria occur at $y = 0$ and $x = n\pi$ for $n \in \mathbb{Z}$.

Note that $E(2n\pi, 0) = 0$ for all $n \in \mathbb{Z}$. Whilst for $x \neq 2n\pi$, $\cos(x) < 1$ and so $E(x, y) > 0$.

By construction we know that $\dfrac{\mathrm{d}E}{\mathrm{d}t} = 0$ since in the undamped context we have conservation of energy. (One can also verify this fact by direct computation.)

It therefore follows that $E = y^2/2 - \cos x + 1$ satisfies the two conditions (Definition 8.1) required to be a Lyapunov function at $(2n\pi, 0)$ for a suitably chosen neighbourhood about $(2n\pi, 0)$ (e.g $N_\pi((2n\pi, 0))$).

Notice that E always satisfies condition 1 of Definition 8.1 at $(2n\pi, 0)$ regardless of the value of c. To see whether E remains a Lyapunov function for these equilibria even when $c \neq 0$ we compute $\dfrac{\mathrm{d}E}{\mathrm{d}t}$.

$$\frac{\mathrm{d}E}{\mathrm{d}t} = y\frac{\mathrm{d}y}{\mathrm{d}t} + \sin x \frac{\mathrm{d}x}{\mathrm{d}t} = -cy^2 - y\sin x + y\sin x = -cy^2 < 0$$

Notice that $-cy^2 \leq 0$ and is equal to 0 only when $y = 0$. However, any neighbourhood of $(2n\pi, 0)$ contains arbitrarily many points where $y = 0$. Therefore $E(x, y)$ is again a non-strict Lyapunov function for $(2n\pi, 0)$ when $c > 0$. However, we know, using the method of linearisation, that $(2n\pi, 0)$ is asymptotically stable.

Example 8.3. Consider the system of ODE's

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + y - x(x^2 + y^2)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x + ay - y(x^2 + y^2)$$

Defining $\boldsymbol{R} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\boldsymbol{R}(x, y) = (ax + y - x(x^2 + y^2), -x + ay - y(x^2 + y^2))$$

we can write the ODE as $\dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(x, y)$. Clearly $(x, y) = (0, 0)$ is an equilibrium of the system. What about its stability? We apply Theorem 8.2. We need to find a (smooth) function $E : \mathbb{R}^2 \to \mathbb{R}$ such that $E((0,0)) = 0$ and $E(x, y) > 0$ for all $(x, y) \neq (0, 0)$ in a neighbourhood around $(0, 0)$. Clearly $E(x, y) = 1/2(x^2 + y^2)$ satisfies both conditions. Thus we can determine stability at $(0, 0)$ by computing $\boldsymbol{R} \cdot \nabla E$:

$$\begin{aligned}
\boldsymbol{R} \cdot \nabla E &= x(ax + y - x(x^2 + y^2)) + y(-x + ay - y(x^2 + y^2)) \\
&= ax^2 - x^2(x^2 + y^2) + ay^2 - y^2(x^2 + y^2) \\
&= a(x^2 + y^2) - (x^2 + y^2)^2 \\
&= (x^2 + y^2)(a - (x^2 + y^2))
\end{aligned}$$

For $a \leq 0$ and all $(x, y) \neq (0, 0)$, $\boldsymbol{R} \cdot \nabla E < 0$ and so $(0, 0)$ is asymptotically stable. For $(x^2 + y^2) < a$, $\boldsymbol{R} \cdot \nabla E > 0$ and $(0, 0)$ is unstable.

8.3 Periodic orbits and limit sets

We have encountered periodicity in continuous time dynamical system (Example 8.1), as with discrete time dynamical systems we now investigate stability of period orbits and orbits more generally.

For autonomous differential equations, bounded trajectories may converge to closed curves called *periodic orbits/cycles*.

Now when we studied stability of orbits for discrete dynamical systems, we found the notion of a forward limit set helpful (Definition 6.2) in characterising the dynamics around the orbit. The definition that follows extends this definition to continuous time dynamical system, and essentially defines the forward limit set as the set of points towards which all initial orbits converge to as t gets larger and larger.

Definition 8.2 (omega-limit set). Let

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(\boldsymbol{x})$$

be a differential equation and let $\boldsymbol{F}(t, \boldsymbol{x})$ be its flow.

A point $\boldsymbol{z} \in \mathbb{R}^n$ is in the *ω-limit set/forward limit set* $\omega(\boldsymbol{x}_0)$ of the trajectory $\boldsymbol{F}(t, \boldsymbol{x}_0)$ if there is a sequence of points "far out" along the orbit which converges to $\boldsymbol{z}$. That is $\boldsymbol{z} \in \omega(\boldsymbol{x}_0)$ if there is an increasing sequence $(t_1 \leq t_2 \leq t_3 \dots)$ with $\lim_{n\to\infty} t_n = \infty$ and

$$\lim_{n\to\infty} \boldsymbol{F}(t_n, \boldsymbol{x}_0) = \boldsymbol{z}.$$

A point $\boldsymbol{z} \in \mathbb{R}^n$ is in the *α-limit set/backward limit set* $\alpha(\boldsymbol{x}_0)$ if there is a decreasing sequence $(t_1 \geq t_2 \geq t_3 \geq \dots)$ with $\lim_{n\to\infty} t_n = -\infty$ such that

$$\lim_{n\to\infty} \boldsymbol{F}(t_n, \boldsymbol{x}_0) = \boldsymbol{z}.$$

> **Note**
> - The ω-limit $\omega(\boldsymbol{x}_0)$ set of a point $\boldsymbol{x}_0$ for the ODE $\dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(\boldsymbol{x})$ is equal to the α limit set $\alpha(\boldsymbol{x}_0)$ of $\boldsymbol{x}_0$ for the ODE $\dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = -\boldsymbol{R}(\boldsymbol{x})$.
> - If $\boldsymbol{x}_0$ is an equilibrium point, then $\omega(x_0) = \alpha(x_0)$.

Example 8.4. Let r and θ be polar coordinates (so $r > 0$, $0 \leq \theta \leq 2\pi$, $x, y =$

$r \cos \theta, r \sin \theta)$ in the plan and consider the differential equation below for $a, b > 0$.

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r(1 - r)$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 8$$

There is an equilibrium when $r = 0$ (i.e at the origin). Therefore $\omega(0, 0) = \alpha(0, 0) = \{(0, 0)\}$.

When $r = 1$ we have a periodic orbit/cycle. Solutions spiral in towards the periodic solution at $r = 1$. The point p on the unit circle is in the ω-limit set of the orbit depicted in Figure 8.3 since there are points $\boldsymbol{F}(t_1, \boldsymbol{x}_0), \boldsymbol{F}(t_2, \boldsymbol{x}_0), \boldsymbol{F}(t_3, \boldsymbol{x}_0) \ldots$ which converge to p. The entire unit circle is the ω-limit set of the trajectory of any point in $\mathbb{R}^2 \backslash \{(0, 0)\}$ (see Figure 8.3).
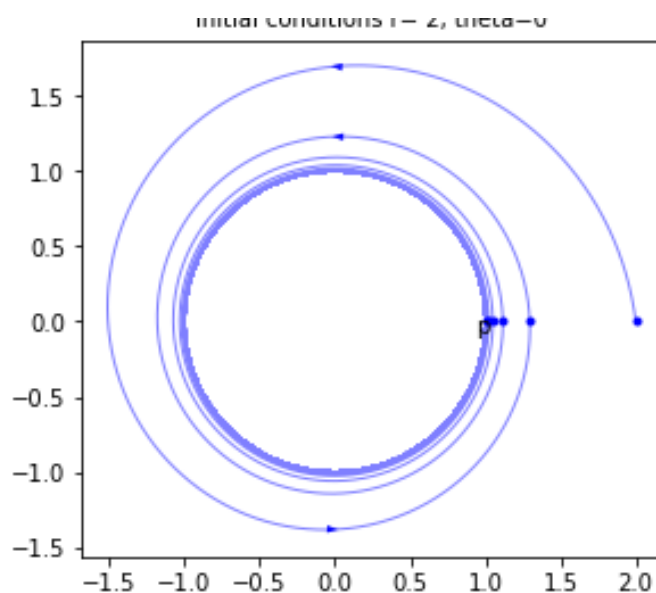

FIGURE 8.3: omega-limit set

Example 8.5. Consider the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = x(a - x)$ for $a > 0$. Equilibria occur at the points $x = 0, x = a$. All trajectories with $x_0 > 0$ converge to the point $x = a$ and so a is an asymptotically stable equilibrium and $\omega(x_0) = \{a\}$ for all $x_0 > 0$.

All trajectories with $x_0 < 0$ diverge to -infinity. In this case $\omega(x_0) = \{\}$ when $x_0 < 0$.

Note

The above examples are characteristic of ω-limit sets in one-dimension. In particular, for one-dimensional differential equations, ω-limit sets are either singletons sets or they are empty. Thus a *trajectory either diverges to infinite or converges to a single equilibrium*. This is precisely the content of the theorem that follows.

Theorem 8.3. *Let* $\dfrac{\mathrm{d}x}{\mathrm{d}t} = R(x)$ *be a differential equation.* **All** *solutions to the differential equation are either monotonically increasing or monotonically decreasing as a function of* t. *For* $x_0 \in \mathbb{R}$ *if the orbit* $\boldsymbol{F}(t, x_0)$, $t \geq 0$, *is bounded then* $\omega(x_0)$ *consists solely of an equilibrium.*

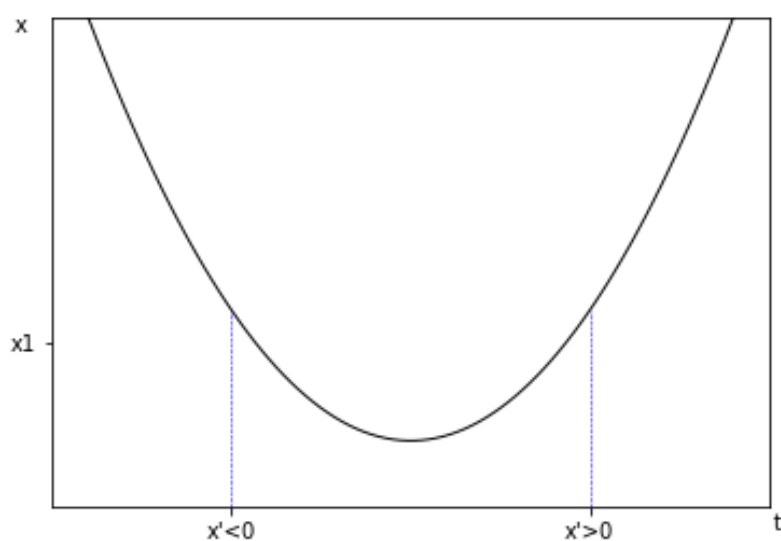**Example 8.6.** Consider the solution curve shown in Figure 8.4

FIGURE 8.4: Non-monotonic curve

The curve is monotonically decreasing $\left(\dfrac{\mathrm{d}x}{\mathrm{d}t} < 0\right)$ until the turning point at which it begins to increase monotonically $\left(\dfrac{\mathrm{d}x}{\mathrm{d}t} > 0\right)$. In particular, this curve cannot be a solution to **any** differential equation of the form $\dfrac{\mathrm{d}x}{\mathrm{d}t} = R(x)$.
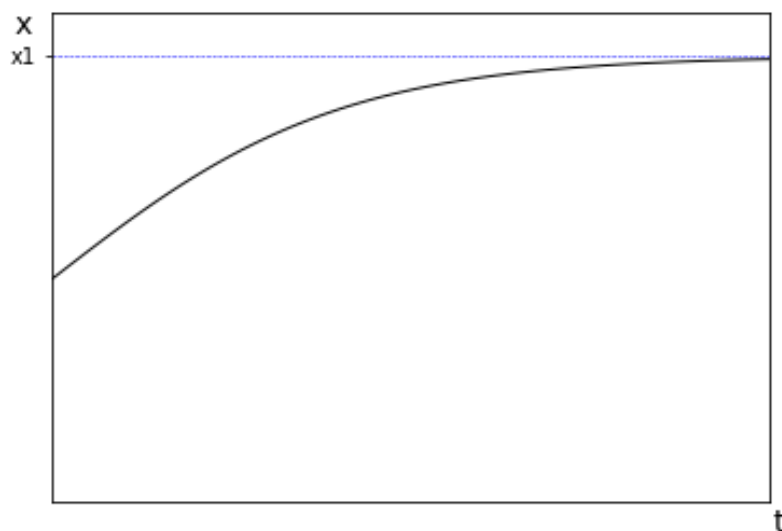
What about the solution curve in Figure 8.5

FIGURE 8.5: Monotonic curve

For all initial conditions below the equilibrium value at x_1, the solution curve increases monotonically to the equilibrium value x_1. The ω-limit set is just the singleton consisting of the equilibrium value.

We have already seen that 2D systems can have more complicated behaviour: there are periodic orbits/cycles which can attract solutions and are themselves ω-limit sets.

8.4 Problem Sheet 8

For week 9.

Question 1

Consider the nonlinear system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + y - x\sqrt{x^2 + y^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x + ay - y\sqrt{x^2 + y^2}$$

where a is a real parameter.

Show that $(x, y) = (0, 0)$ is the only equilibrium point. Investigate the linear stability of this equilibrium and how it depends on the parameter a.

Question 2

The Lorenz equations are given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x) \tag{8.5}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = rx - y - xz \tag{8.6}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -bz + xy \tag{8.7}$$

For positive real parameters σ, r and b.

a. Determine the equilibria of the Lorenz equations. Assume that $b = 1$. Sketch the bifurcation diagram by plotting the x-value of each equilibrium as a function of r.

b. Investigate how the stability of the equilibrium $x = y = z = 0$ depends on r when σ and b are fixed.

c. Show that for the Lorenz equations

$$\nabla \cdot \boldsymbol{R} = \frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y} + \frac{\partial R_z}{\partial z} < 0.$$

[Note that dynamical systems with this property are called *dissipative*.]

9

Limit sets in 2-dimensions

Covers Chapter 8 Section 8.1 and 8.2 and Sections 9.1 and 9.2 of Chapter 9 (Alligood et al., 2000).

9.1 Limit cycles

Let us formally define a cycle/periodic orbit of a continuous time dynamical system.

Definition 9.1. Consider the differential equation

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(\boldsymbol{x})$$

with flow $\boldsymbol{F}(t, \boldsymbol{x})$. Let $\boldsymbol{x}_0$ be a point and suppose that $\boldsymbol{x}_0$ is **not** an equilibrium point. If there is a $T > 0$ such that

$$\boldsymbol{F}(t + T, \boldsymbol{x}_0) = \boldsymbol{F}(t, \boldsymbol{x}_0) \tag{9.1}$$

for all t, then the orbit $\boldsymbol{F}(t, \boldsymbol{x}_0)$ is called a *periodic orbit/cycle*. The smallest such T such that Equation (9.1) holds is called the *period* of the orbit.

> **Note**
>
> A period orbit/cycle is a *simple closed* curve in $\mathbb{R}^n$. Simple means that the curve does not repeat points apart from the end points (this follows from the minimality of the period) and closed means that the curve ends where it begins (a consequence of Equation (9.1)).

As with discrete time dynamical systems, periodic orbits of continuous time dynamical system can also be attracting.

Definition 9.2. A *limit cycle* is a periodic orbit which attracts nearby solutions.

Example 9.1. Consider the differential equation $\dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(x)$ given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - by - x\sqrt{x^2 + y^2} \tag{9.2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = bx + ay - y\sqrt{x^2 + y^2} \tag{9.3}$$

The point $(x, y) = (0, 0)$ is an equilibrium point. Consider the function $E : \mathbb{R}^2 \to \mathbb{R}$ given by $E(x, y) = (x^2 + y^2)/2$. Clearly $E(0, 0) = 0$ and E is always strictly bigger than zero at non-zero coordinates. Computing $\boldsymbol{R} \cdot \nabla E$ we have:

$$\boldsymbol{R} \cdot \nabla E = (x^2 + y^2)\left(a - \sqrt{x^2 + y^2}\right).$$

If $a \leq 0$ then $\boldsymbol{R} \cdot \nabla E$ is strictly less than 0 whenever $(x, y) \in \mathbb{R}^2$ does not lie on the origin and $(0, 0)$ is a globally asymptotically stable equilibrium. If $a > 0$, then there is a neighbourhood of $(0, 0)$ on which $a - \sqrt{x^2 + y^2} > 0$ and so $\boldsymbol{R} \cdot \nabla E > 0$ at all points $(x, y) \neq (0, 0)$ in some neighbourhood of a, we conclude that $(0, 0)$ is an unstable equilibrium. We can confirm this by computing the eigenvalues of the Jacobian of $\boldsymbol{R}$ at $(0, 0)$.

$$\boldsymbol{DR}(0, 0) = \begin{pmatrix} a - \left(\sqrt{x^2 + y^2} + \frac{x^2}{\sqrt{x^2+y^2}}\right) & -b - \frac{xy}{\sqrt{x^2+y^2}} \\ b - \frac{xy}{\sqrt{x^2+y^2}} & a - \left(\sqrt{x^2 + y^2} + \frac{y^2}{\sqrt{x^2+y^2}}\right) \end{pmatrix}.$$

Now the terms with denominator equal to $\sqrt{x^2 + y^2}$ cause problems. To get around this, observe that

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} \frac{y^2}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} = \frac{xy}{\sqrt{x^2 + y^2}} = 0.$$

This can be seen by noting that $0 \leq \frac{x^2}{\sqrt{x^2+y^2}} \leq \sqrt{x^2 + y^2}$ and $\lim_{(x,y)\to(0,0)} \lim_{(x,y)\to(0,0)} \sqrt{x^2 + y^2} = 0$. Therefore

$$\boldsymbol{DR}(0, 0) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

The eigenvalues of this matrix are $a \pm ib$. Therefore Theorem 7.1 now implies that $(0, 0)$ is stable for $a < 0$ and unstable for $a > 0$.

We can verify this by solving the differential equation. While it looks a little imposing in Cartesian coordinates, it becomes rather tame in polar coordinates. Recall that for polar coordinates we take $x = r\cos\theta$ and $y = r\sin\theta$ where, plotting the point (x, y) in the plane, θ is the angle, moving clockwise, from the positive real axis to the line from the origin to the point (x, y). Thus $r > 0$ and $0 \le \theta \le 2\pi$. Notice that $r^2 = x^2 + y^2$ and $\tan\theta = y/x$. We therefore compute $\dfrac{\mathrm{d}r}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$ as follows.

$$2r\frac{\mathrm{d}r}{\mathrm{d}t} = 2\left(x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t}\right) = 2\left(a(x^2 + y^2) - (x^2 + y^2)\sqrt{x^2 + y^2}\right)$$

$$\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{x\dfrac{\mathrm{d}y}{\mathrm{d}t} - y\dfrac{\mathrm{d}x}{\mathrm{d}t}}{x^2} = \frac{b(x^2 + y^2)}{x^2}$$

Substituting $x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = x^2 + y^2$ we get:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r(a - r)$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = b$$

The differential equation for r can be solved by separating the variables and using partial fractions. With some algebraic manipulations we get:

$$r = \frac{ar_0}{r_0 + (a - r_0)e^{-at}}.$$

The differential equation for θ gives $\theta = bt + \theta_0$. Thus, solutions are given by:

$$x(t) = r(t)\cos(bt + \theta_0)$$
$$y(t) = r(t)\sin(bt + \theta_0)$$

In order to understand the behaviour of the system as t grows towards infinity, r is the crucial variable (since θ determines the angle in the plane).

If $a < 0$ then as t tends to positive infinity, e^{-at} tends to positive infinity (a is negative) and r tends to 0: all initial conditions tend to the stable equilibrium at $(0, 0)$. See figure 9.1.
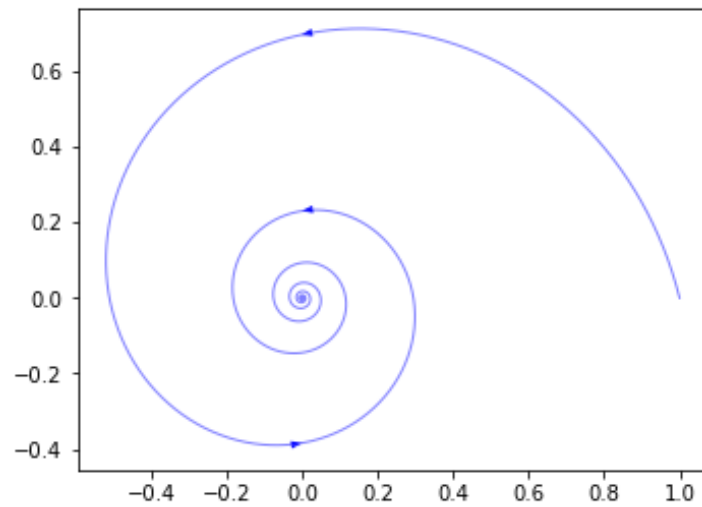
FIGURE 9.1: Origin is stable equilibrium for a<0

If $a > 0$, then $\lim_{t\to\infty} e^{-at}$ is 0 and so $\lim_{t\to\infty} r(t) = a$. All solutions tend to the limit cycle — the circle of radius a centred at the origin (see Figure 9.2)
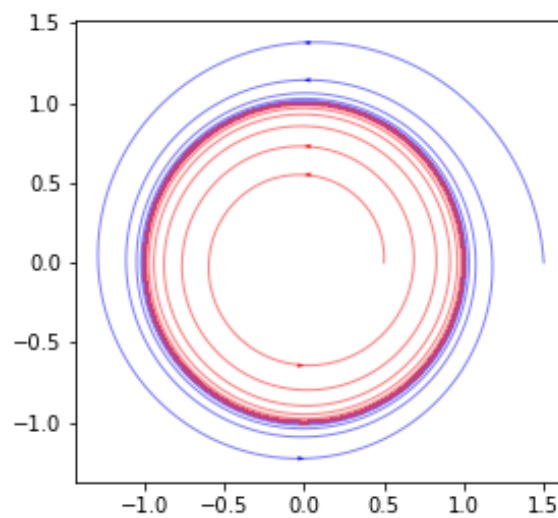
FIGURE 9.2: Limit cycle for a>0

Example 9.1 shows a periodic orbit which is stable i.e attracting. We determined this by solving the differential equation. But how might one determine the stability

of a general cycle? We use the same strategy we successfully applied in previous contexts (see Example 7.2 and Subsections 2.1 and 2.3.1). We linearise by considering a small perturbation from the limit cycle. This gives rise to a system of 1st order linear ODES. Alternatively, if we are able to compute the flow of the ODE, then we can use the stability of fixed point of the flow.

Let us consider Example 9.1. We computed the flow of the ODE:

$$\boldsymbol{F}(t, r, \theta) = \left(\frac{ar}{r + (a - r)e^{(-at)}}, bt + \theta \right).$$

To examine the stability of periodic solutions with period $t = 2\pi$, say. We can examine the "r-part" of the time-2π map of the flow:

$$f(r) = \frac{ar}{r + (a - r)e^{(-2\pi a)}}$$

noting that $r = a$ is a fixed point of this map since $r = a$ corresponds to the initial condition $(r(0), \theta(0)) = (a, \theta_0)$ which is on the limit cycle. we compute $\left| \dfrac{\mathrm{d}f}{\mathrm{d}r}(a) \right|$ to investigate the stability of the fixed point a of f.

$$
\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}r} &= \frac{a(r + (a - r)e^{-2\pi a}) - ar(1 - e^{-2\pi a})}{(r + (a - r)e^{-2\pi a})^2} \\
&= \frac{a^2 e^{-2\pi a}}{(r + (a - r)e^{-2\pi a})^2}
\end{aligned}
$$

Therefore $\left| \dfrac{\mathrm{d}f}{\mathrm{d}r}(a) \right| = e^{-2\pi a}$. Now $e^{-2\pi a} > 1$ when $a \leq 0$ otherwise $e^{-2\pi a} < 1$ in which case $a > 0$. Therefore the periodic orbit is unstable for $a \leq 0$ and stable for $a > 0$.

9.2 Planar dynamics of continuous time dynamical systems

For 1 dimensional ODE's we were able to complete characterise ω-limit sets — they are either the $\emptyset$ (in which case the corresponding trajectory diverges to 0) or they consist of a single equilibrium point (see Theorem 8.3). Can we similarly characterise ω-limit sets for 2 dimensional ODE's? The answer is yes.

> **Note**
>
> The ω-limit set of a bounded trajectory $\boldsymbol{F}(t, \boldsymbol{v}_0)$ in 2D is of one of the following forms:
>
> i. an equilibrium;
> ii. a limit cycle;
> iii. for all $\boldsymbol{u} \in \omega(\boldsymbol{v}_0)$, $\omega(\boldsymbol{u})$ is an equilibrium.
>
> Said differently, the chaotic phenomenon that we encountered in discrete time dynamical systems (see Chapter 6) does **not** occur in 2D continuous time dynamical systems.

Once we get to 3-dimensions, chaotic behaviour emerges.

Example 9.2 (The Lorenz equations). Consider the Lorenz equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sigma x + \sigma y \tag{9.4}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -xz + rx - y \tag{9.5}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - bz \tag{9.6}$$

for positive real parameters σ, r and b.

The Lorenz equations were derived to model convection in a fluid heated from below. We assume there are two parallel plates T_l and T_u with some fluid between the two plates which is at rest at time $t = 0$. (See Figure 9.3)
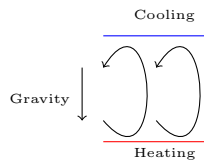

FIGURE 9.3: Two plates with fluid heated from below

If the difference $T_l - T_u$ in temperature between the bottom and top plates is small, then heat is transported upwards via thermal conduction. If $T_l - T_u$ exceeds a critical value, an instability arises and the fluid itself moves in convection rolls transporting heat upwards.

The Lorenz equations describe how the warm fluid below rises and the cool fluid above sinks setting up a clockwise or counter-clockwise current.

Lorenz showed that the equations display *chaotic* (sensitivity to initial conditions & non-periodicity) solutions and for certain values of σ, r and b *chaotic attractors* emerge.

The Lorenz equations roughly say that:

- x is proportional to the circulatory fluid flow velocity;
- y is proportional to the temperature difference between ascending and descending fluid elements;
- z is proportional to the distortion of the vertical temperature profile from its equilibrium;
- the width of the flow rolls is proportional to the parameter b;
- σ and r are dimensionless numbers describing properties of the fluid; σ is the Prandtl number and r is the Rayleigh number.

Lorenz found that for parameter values $\sigma = 10, b = 8.3$ and r just greater than 24.74 chaotic behaviour emerges.

9.2.1 Properties of the Lorenz dynamical system

The Lorenz system is *dissipative*

This roughly means that the systems is *volume decreasing* — the volume of a given phase space decreases with time.

Definition 9.3. A dynamical system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(\boldsymbol{x}) = (R_1(\boldsymbol{x}), R_2(\boldsymbol{x}), ..., R_n(\boldsymbol{x}))$$

is called *dissipative* if

$$\nabla \cdot \boldsymbol{R} = \frac{\partial R_1}{\partial x_1} + \frac{\partial R_2}{\partial x_2} + ... + \frac{\partial R_n}{\partial x_n} < 0$$

Dissipative systems are examples of *thermodynamically open systems*. That is a system which is far from thermodynamic equilibrium and which can exchange energy and matter with its environment.
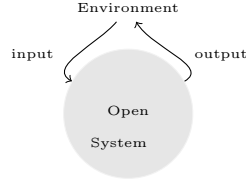
FIGURE 9.4: A thermodynamically open system

For the Lorenz equations:

$$\nabla \cdot \boldsymbol{R} = \frac{\partial}{\partial x}(-\sigma x + \sigma y) + \frac{\partial}{\partial y}(-xz + rx - y) + \frac{\partial}{\partial z}(xy - bz) = -\sigma - 1 - b < 0$$

Equilibria of the Lorenz system.

suppose σ and b are fixed at 10 and $8/3$ respectively. The point $(x, y, z) = (0, 0, 0)$ is an equilibrium for all r. When $r < 1$, $(0, 0, 0)$ is a stable attractor and for $r \geq 1$ $(0, 0, 0)$ is unstable (compute the Jacobian at $(0, 0, 0)$ for instance).

The orbit with initial conditions $(x, y, z) = (0, 0, 0)$ at $t = 0$ corresponds to the fluid at rest with a linear temperature profile — hot at the bottom and cool at the top.

For $r \geq 1$, two new equilibria exist:

$$C_+ = (x_+, y_+, z_+) = \left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1 \right)$$
$$C_- = (x_-, y_-, z_-) = \left(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1 \right)$$

these two equilibria $C_\pm$ correspond to convective rolls; they branch off from the origin at $r = 1$ and move away as r increases. At $r = 1$, $C_\pm$ are stable and they remain stable for $r < r_u \simeq 24.74$. For $r > r_u$ all three equilibria are unstable and chaotic behaviour emerges. Below (Fig 9.5) are some images of the chaotic attractor for fixed $r = 28$.
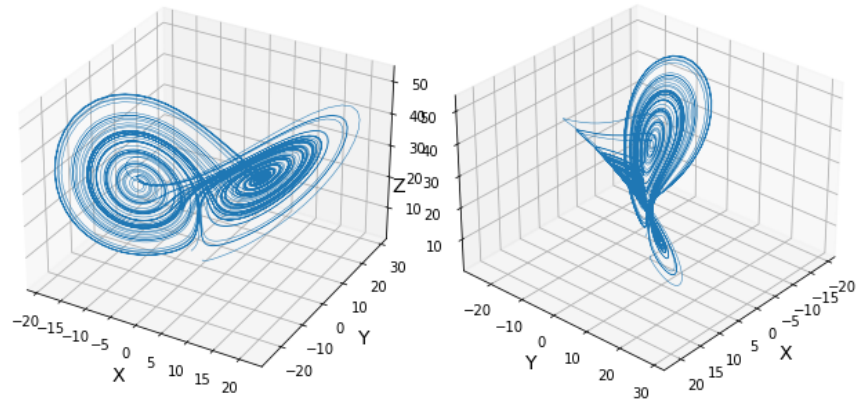
FIGURE 9.5: Rotated views of the Lorenz chaotic attractor r=28, sigma=10, b=8/3

9.3   Problem Sheet 9

For week 10.

Question 1

Consider the nonlinear system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + y - x\sqrt{x^2 + y^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x + ay - y\sqrt{x^2 + y^2}$$

where a is a real parameter. In Tutorial 8 we showed that $(0,0)$ is the only equilibrium point of the system and that it is asymptotically stable when $a < 0$ and unstable when $a > 0$.

a. Use the Lyapunov function

$$E(x,y) = \frac{x^2 + y^2}{2}$$

to show that the origin is a stable equilibrium for $a \leq 0$ and unstable for $a > 0$. For which values of a is the origin asymptotically stable?

b. Show that the system can be written using polar coordinates as:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = (a - r)r, \tag{9.7}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -1. \tag{9.8}$$

c. Verify that the solution of the transformed system with initial conditions $r(0) = r_0$ and $\theta(0) = \theta_0$ is given by:

$$r = \frac{ar_0}{r_0 + (a - r_0)e^{-at}},$$

$$\theta = -t + \theta_0.$$

Show, provided $a > 0$, that all solutions tend to a limit cycle as $t \to \infty$.

d. Use the solution in c. to show that for all $a > 0$ one can define a Poincaré map $P : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$r_n = P(r_{n-1}) = \frac{ar}{r + (a - r)e^{-2\pi a}}.$$

[Note a Poincaré map is similar to a time-T map, however instead of orbits at fixed time intervals, we consider the intersection point of the orbit with an $(n - 1)$-dimensional disk transverse to the orbit. In two dimensions, we're looking at a line transverse to the orbit. See Section 11.7 of (Alligood et al., 2000) for more detail.]

e. Using the Poincaré map, or otherwise, show that the limit cycle is stable for $a > 0$.

Question 2

Consider the Lorenz equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x) \tag{9.9}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = rx - y - xz \tag{9.10}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -bz + xy \tag{9.11}$$

For positive real parameters σ, r and b.

Use the Lyapunov function

$$E(x, y, z) = \frac{x^2 + \sigma y^2 + \sigma z^2}{2}$$

to show that the equilibrium $x = y = z = 0$ is stable for $r \leq 1$.

10

Lorenz chaos and bifurcation

Covers Chapter 11 Sections 11.1 and 11.2 of (Alligood et al., 2000).

10.1   Properties of the Lorenz dynamical system II

We continue our study of the Lorenz dynamical system. Recall that this dynamical system is governed by the Lorenz equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sigma x + \sigma y \tag{10.1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -xz + rx - y \tag{10.2}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - bz. \tag{10.3}$$

We have seen that the system is dissipative and characterises the behaviour of some of its equilibria. The next observation we make about the system is on a so-called *trapping region*.

Trapping region

In the Lorenz dynamical system **all** orbits eventually enter and never leave a neighbourhood (sphere) around the origin.

Let us give an argument that sketches out how this can occur.

Define $\boldsymbol{E}(x, yz) = x^2/2 + y^2/2 + (z - (\sigma + r))^2/2$ (the $-(\sigma + r)$ is added in so we have nice cancellations). Notice that $\boldsymbol{E}$ is 0 at the point $(0, 0(\sigma + r))$ and strictly greater than 0 otherwise. We apply a similar strategy as with Lyapunov functions. We compute $\dfrac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t}$.

$$\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} = x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t} + (z - (\sigma + r))\frac{\mathrm{d}z}{\mathrm{d}t}$$
$$= -\sigma x^2 + \sigma xy - xyz + rxy - y^2 + xyz - bz2 - (\sigma + r)xy + bz(\sigma + r)$$
$$= -\sigma x^2 - y^2 - bz^2 + bz(\sigma + r)$$

Notice that $-\sigma x^2 - y^2 - bz^2$ is negative. Since we are thinking about convergence towards a sphere, our analysis is helped by considering spherical coordinates: $R^2 = x^2 + y^2 + z^2$, $x = R\sin\theta\cos\phi$, $y = R\sin\theta\sin\phi$ and $z = R\cos\theta$ (R is the line from the origin to the point (x, y, z), θ is the angle from the positive z axis to the line from the origin to the point (x, y, z), $R\sin\theta$ is the the projection of the line in the x, y plane, ϕ is the angle (clockwise) from the positive x-axis to the projected line). Set $d := \min\{\sigma, 1, b\}$. Then

$$\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} \leq -d(x^2 + y^2 + z^2) + bz(\sigma + r) = -dR^2 + b\cos\theta R(\sigma + r)$$
$$\leq -R(dR - b(\sigma + r))$$

Therefore $\dfrac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} < 0$ when $dR - b(\sigma + r) > 0$. Hence, for $R > \frac{-b(\sigma+r)}{d}$, $\dfrac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} < 0$. This means that all solutions outside the sphere of radius $R = \frac{-b(\sigma+r)}{d}$ move towards the sphere and eventually enter the sphere. All solutions inside the sphere cannot leave it (the sphere is in a potential well).

10.1.1   Lyapunov Exponents in Flows

Recall the definition of the Lyapunov exponent (5.3). We can extend this definition to orbits of continuous time dynamical system using the flow $\boldsymbol{F}(t, \boldsymbol{x})$ of the differential equation governing the system. We define the Lyapunov number and exponent of an orbit as the Lyapunov number of the time-1 flow $\boldsymbol{F}_1(\boldsymbol{x})$. Recall that the time-1 map of the flow gives rise to a dynamical system defined by the rule:

$$\boldsymbol{x}_{n+1} = \boldsymbol{F}_1(\boldsymbol{x}_n) = \boldsymbol{F}(1, \boldsymbol{x}_n).$$

We can therefore apply Definition 5.3 using $\boldsymbol{F}_1$ to define Lyapunov numbers and exponents of an orbit of a continuous time dynamical system.

For the the Lorenz dynamical system, one can show that a typical orbit has one positive Lyapunov exponent which is approximate 0.905.

Now that we have the definition of Lyapunov numbers for continuous time dynamical system, then we can give a precise definition of a chaotic orbit.

Definition 10.1. Consider a dynamical system governed by the rule $\dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(\boldsymbol{x})$ and with flow $\boldsymbol{F}_t(\boldsymbol{x})$. Let $\boldsymbol{x}_0 \in \mathbb{R}^n$. The orbit $\boldsymbol{F}(t, \boldsymbol{x}_0)$ is called *chaotic* if the following things hold:

1. The orbit $(\boldsymbol{F}(t, \boldsymbol{x}_0))_{t \geq 0}$ is bounded (it is not in the basin of infinity.)
2. The orbit $\boldsymbol{F}(t, \boldsymbol{x}_0)$ has one positive Lyapunov exponent. (Sensitivity to initial conditions)
3. The ω-limit set $\omega(\boldsymbol{x}_0)$ of $\boldsymbol{x}_0$ is not periodic and does not consist only of equilibria (or equilibria and connecting arcs). (non-periodicity).

A typical chaotic attractor of the Lorenz system is shown in Figure 10.1. It has positive Lyapunov exponent and is not asymptotically periodic.
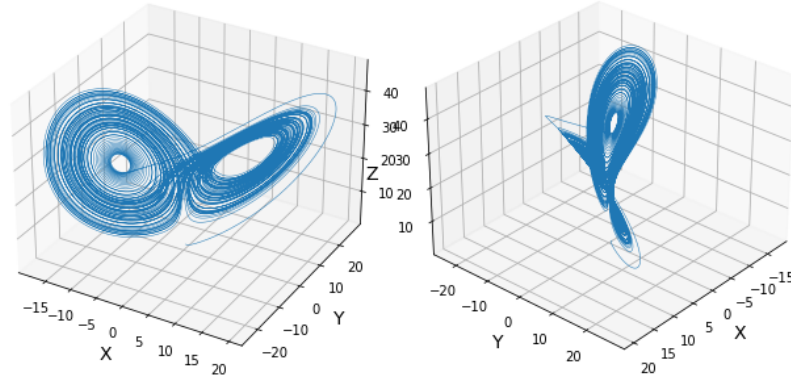

FIGURE 10.1: Rotated views of the Lorenz chaotic attractor r=28, sigma=10, b=8/3

Computation estimates give the Lyapunov numbers L_i and Lyapunov exponents h_i of the above attractor as:

$$\begin{aligned}
h_1 &\simeq 0.905 \pm 0.005; & L1 &\simeq 2.47 \\
h_2 &\simeq 0.0; & L2 &\simeq 1 \\
h_3 &\simeq -14.57 \pm 0.01; & L_3 &\simeq 0.00000047
\end{aligned}$$

The distance between orbits of initial values that start of quite close together on the chaotic attractor increases by a factor of $\simeq 2.47$ per time unit.

Let us examine Figure 10.1. The existence of a chaotic attractor in the orbit

displayed is numerically implied by the fact that almost any initial condition in a close vicinity of the set displayed gives rise to a similar plot. Which suggests *robustness*.

- A trajectory appears to initially to spiral around one equilibrium C_+ or C_- with increasing amplitude.
- Once the distance of the trajectory from the equilibrium exceeds a critical distance, the trajectory spirals about the other equilibrium with increasing amplitude until the critical distance is again exceeded and the process repeats.

10.1.2 Bifurcation (Chapter 11 of (Alligood et al., 2000))

In the last few sections we return to the idea of bifurcation. We encountered the phenomenon of bifurcation early on in the module (see Subsection 2.2). Qualitatively, for a given dynamical system *bifurcation points* are points in the *parameter space* at which qualitative change in the dynamics occur.

Consider a system of ODE's $\dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{R}(\boldsymbol{x}, a)$ where a is a parameter. Let $\boldsymbol{x}_e(a)$ be an equilibrium of the system (typically this will depend on the parameter value a). Stability of $\boldsymbol{x}_e(a)$ depends on the real part of the eigenvalues of the Jacobian of $\boldsymbol{R}$ at the $\boldsymbol{x}_e$. Again these typically depend on the parameter a. For values of a for which the real part of the eigenvalues are negative, the equilibrium $\boldsymbol{x}_e(a)$ is stable; for values of a for which the real part of at least one of the eigenvalues is positive, $\boldsymbol{x}_e(a)$ is unstable. Therefore we have a bifurcation occurring at the parameter point at which the equilibrium moves from stability to instability (or vice versa).

10.2 Bifurcation

The aim of this section is to try to understand the ways in which bifurcation can occur. Therefore as opposed to studying a fixed dynamical system we shall study how a parametrised dynamical system changes as its parameter values change. We will see that there are a limited number of ways the dynamics can change. Let us give a few more definitions to enrich our discussion.

Definition 10.2. A *bifurcation value* is a parameter value at which the number or stability of fixed/periodic points/orbits changes. The orbit/fixed point itself is called a *bifurcation orbit/bifurcation point*.

There are two basic types of bifurcation (we discuss a few more in Subsection 10.3):

1. *Saddle-node* : fixed points emerge
2. *Period doubling* : a fixed point loses stability and a new orbit of double period is born.

Both of these behaviours occur in discrete dynamical systems corresponding to 1D maps. (For instance we encountered period-doubling bifurcation in the logistic map in Section 2.2.)

10.2.1 Saddle-node bifurcation

Let $f_a(x) = a - x^2$ be a 1D map with a single (real) parameter a. The map has no fixed points for $a < -1/4$. When $a = -1/4$, there is a fixed point at $x = -1/2$ and when $a > -1/4$, there are two fixed points:

$$x_\pm = \frac{-1 \pm \sqrt{1 + 4a}}{2}.$$
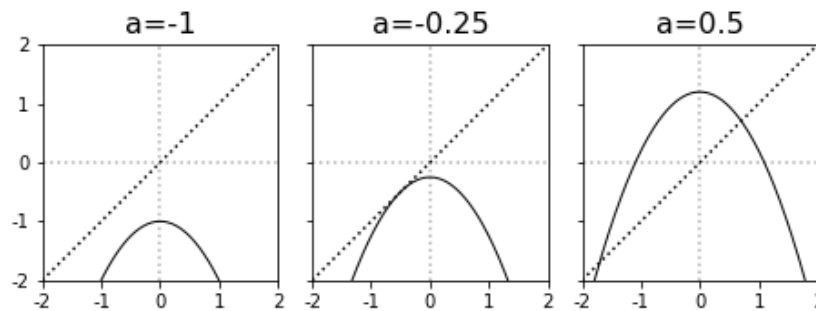
FIGURE 10.2: A saddle-node bifurcation

The point $a = -1/4$ is a *bifurcation value* and $-1/2$ is a bifurcation point. The case in which a **pair** of fixed points emerge in a region where there were none, as a is varied, is called a *saddle-node* bifurcation. At a saddle-node bifurcation, 2 fixed points emerge − one stable and one unstable − as a is increased. The unstable point is a *saddle* and the stable fixed point is an attractor or 'node', hence the terminology "saddle-node".
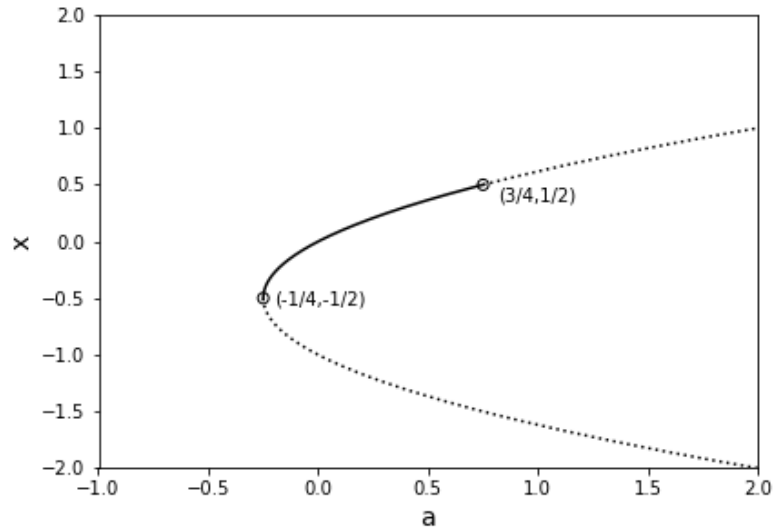
FIGURE 10.3: Bifurcation diagram of a saddle-node bifurcation

In Figure 10.3 we have plotted the fixed points of the map f_a as a function of a. The solid curve represents attracting fixed points (sinks) and the dotted regions are repelling fixed points (sources). The highlighted points $(-1/4, -1/2)$ and $(3/4, 1/2)$ are respectively the location of the saddle-node and period doubling bifurcation. At $a = 3/4$, the fixed point at $1/2$ becomes unstable and a period 2 orbit emerges. For a just above $3/4$ one point in the period 2 orbit is larger than $1/2$ and the other point is less than $1/2$ — a *period doubling bifurcation*.

Of course, period-2 points are fixed points of f_a^2 (Definition 2.1) and so we gain more insight about the period 2 orbits by plotting the graph f_a^2 (see Figure 10.4).
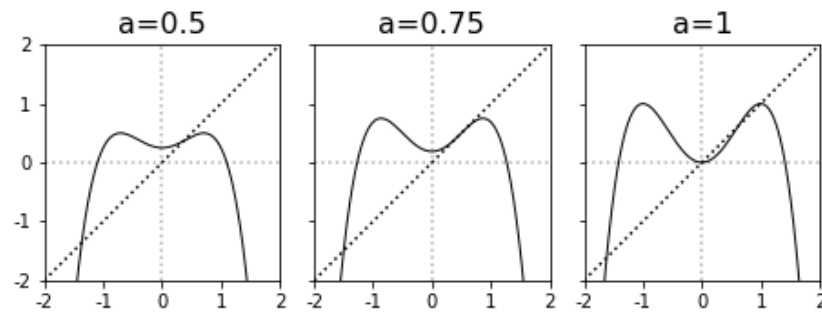
FIGURE 10.4: A saddle-node bifurcation

Figure 10.4 shows plots of f_a^2 for $a = 0.5, 0.75$ and $a - 1$. The point $a = 0.5$ is just before bifurcation occurs. At $a = 0.75$, there is a fixed point (of f_a^2; a period-2 point of f_a) at the tangent. For $a > 0.75$, a new dynamic emerges — there are now 3 non-negative fixed points of f_a: a flip repeller of f_a followed surrounded by a period-2 attractor of f_a.
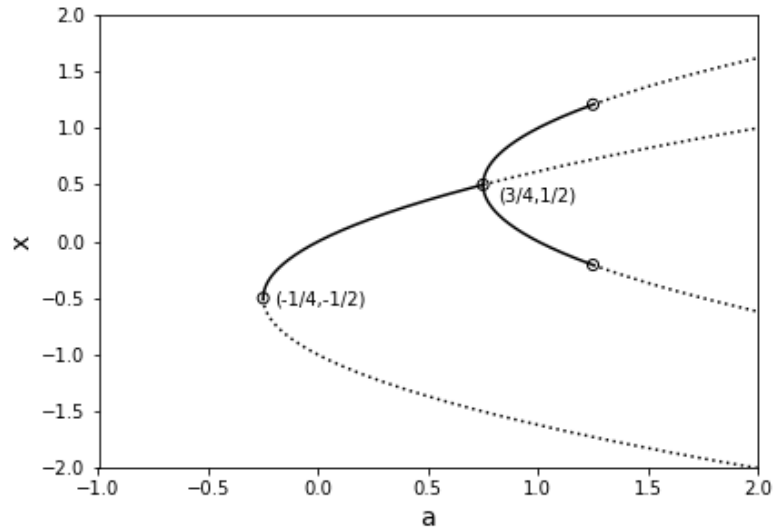
FIGURE 10.5: Bifurcation diagram of a saddle-node bifurcation

In Figure 10.5 we have plotted the period-2 and fixed points of f_a as a function of a. The solid curves are attracting fixed/period-2 points; the dotted curves are repelling fixed/period-2 points. The circled points are the regions where bifurcation occurs.

At $a = 5/4$ a further period-doubling bifurcation occurs and a period-4 sink emerges.

10.3   More types of Bifurcations

In the previous Subsection we introduced two basic types of bifurcations — saddle-node bifurcations (a pair of fixed points, one stable and one unstable, emerge) and period doubling bifurcations (a new orbit of double period emerges).

We continue our discussion of bifurcations in this lecture by considering more ways in which they can arise.

### Transcritical bifurcations

This type of bifurcation is a basic mechanism by which fixed points change stability as the parameter is varied.

For example consider the map $f_r = rx - x^2$ and the continuous time dynamical system $\dfrac{\mathrm{d}x}{\mathrm{d}t} = f_r(x)$. There are equilibria at $x = 0$ and $x = r$. As r is varied the stability of these equilibrium points change. See Figure 10.6
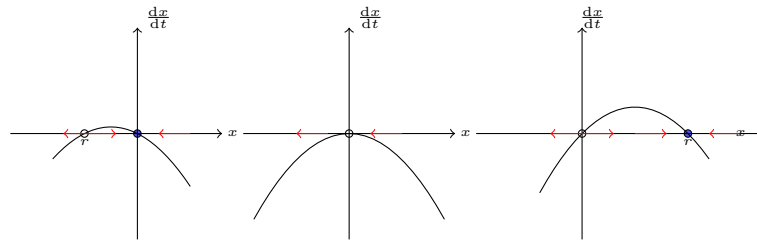
FIGURE 10.6: Plot of fr for varying values of r

When $r < 0$, the non-zero equilibrium is unstable while the 0 equilibrium is stable. When $r = 0$, the two equilibria coincide to a single equilibrium at $x = 0$ which is unstable. When $r > 0$, the equilibrium at 0 becomes unstable and the non-zero equilibrium is stable.

Figure 10.7 plots the equilibrium points of $\dfrac{\mathrm{d}x}{\mathrm{d}t} = f_r(x)$ as a function of r. The dotted lines are unstable equilibria and the solid lines are the stable equilibria. This is an example of a *transcritical bifurcation* diagram.
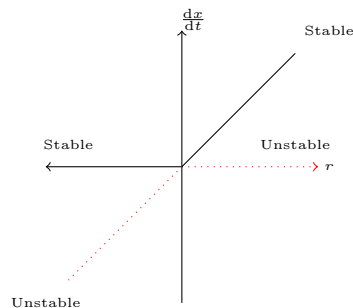
FIGURE 10.7: Plot of fr for varying values of r

We use the stability test for continuous time dynamical systems to determine the stability of the respective fixed points. That is, we compute the Jacobian of f_r at the fixed points — this is simply the derivative of f_r at $x = r$ and $x = 0$. we have

$$f'(0) = r \qquad \text{and} \qquad f'(r) = -r.$$

For $r < 0$, 0 is unstable (the real part of the eigenvalue of the Jacobian of f

at 0 is simply r which is negative); 0 is stable for $r > 0$. The opposite is true of the equilibrium point at $x = r$ — it is unstable when $r > 0$ and stable when $r < 0$. Therefore, as r is varied, the equilibrium points x and r simply interchange stability properties. Note that unlike the saddle-node case, the equilibrium points do not disappear.

Pitch-fork bifurcation

This type of bifurcation generally occurs in systems with symmetry and involves fixed/equilibrium points that emerge and vanish in symmetrical pairs.

Consider a continuous time dynamical system governed by the rule:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx - x^3 = f_r(x).$$

The equilibrium points of the system occur when $x = 0$ and $x_\pm = \pm\sqrt{r}$; note that for $r \leq 0$ only one equilibrium point exists. For $r > 0$ two additional fixed points emerge and so $r = 0$ is a bifurcation value and 0 is a bifurcation point. We investigate stability of the equilibria by computing the derivative at the equilibria.

For $x = 0$, $f_r'(0) = r$. therefore 0 is stable when $r < 0$ and unstable when $r > 0$.

For $x = \pm\sqrt{r}$; $f_r'(x_\pm) = -2r$. Therefore the equilibria $x_\pm$ are stable when $r > 0$ and unstable when $r < 0$.
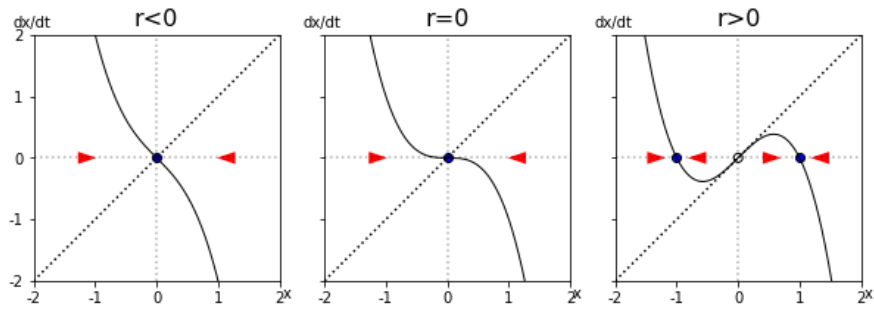
FIGURE 10.8: Plot of f for different parameter values.

The bifurcation diagram is as in Figure 10.9. We continue with the convention that solid curves represent stable equilibria and dotted curves unstable equilibria. Note that the diagram resembles a pitch-fork. The "branches" of the pitch-fork are to the right of the bifurcation/critical point — this is called a *super-critical bifurcation*.
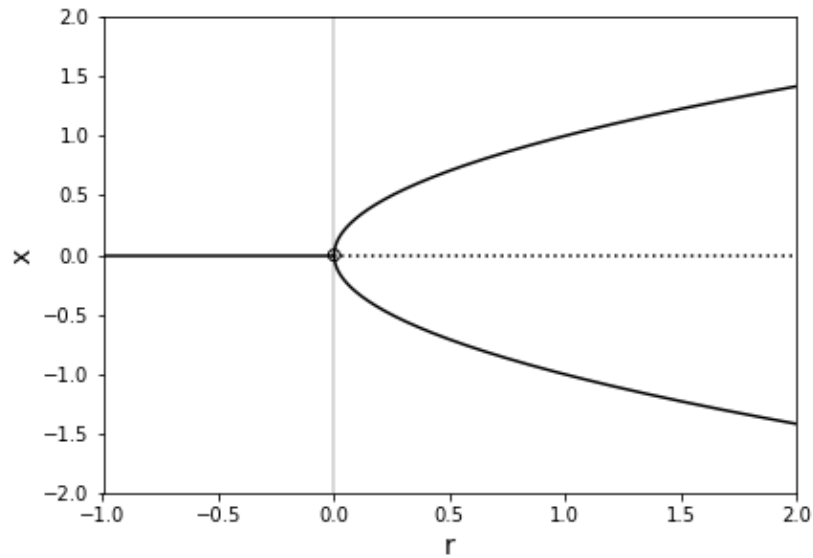
FIGURE 10.9: Bifurcation diagram of a pitchfork bifurcation

The other situation that can arise is when the branches are to the left of the critical point (see Figure 10.10). Notice that the stability behaviour is the inverse of the super-critical pitchfork bifurcation in Figure 10.9. This type of bifurcation, where the branches are to the left of the critical point is called a *sub-critical bifurcation*.
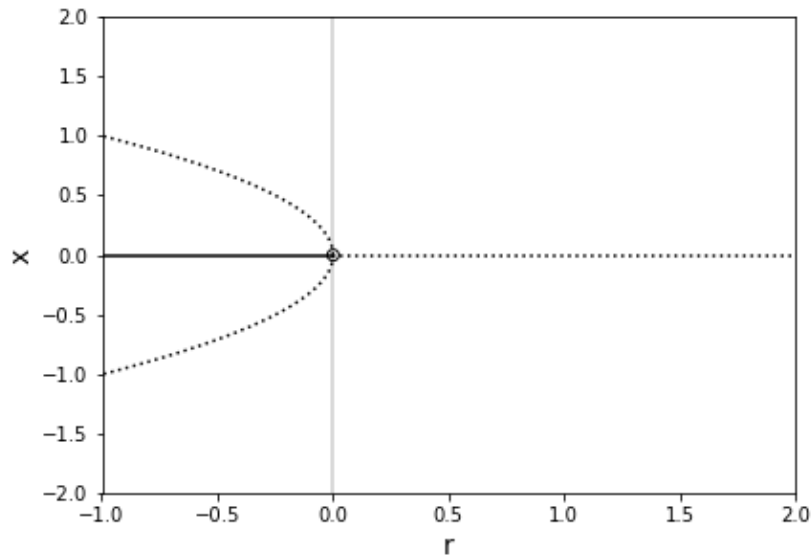
FIGURE 10.10: Bifurcation diagram of a pitchfork bifurcation

An example of a sub-critical bifurcation is the dynamical system governed by the ODE $\dfrac{\mathrm{d}x}{\mathrm{d}t} = ax + x^3$. Further examples can be found in Chapter 11 of (Alligood et al., 2000): a transcritical example is given in Example 11.4 and a pitchfork example in Example 11.5. Note that these examples are for discrete dynamical systems associated to 1D maps and so the stability of the fixed points are determined by the modulus of the derivative at the fixed point (see Subsection 2.1).

10.4 Problem Sheet 10

For Week 11.

Question 1

Consider the Lorenz equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x) \tag{10.4}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = rx - y - xz \tag{10.5}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -bz + xy \tag{10.6}$$

with r, b, σ positive real parameters. We have seen (Sheet 8) that the equilibria of the Lorenz equations are:

$$(0, 0, 0)$$

and

$$x_\pm = y_\pm = \pm\sqrt{b(r - 1)}, z_\pm = r - 1.$$

We have seen (Sheet 8, Sheet 9) that the equilibrium $(0, 0, 0)$ is stable when $r \leq 1$ and unstable for $r > 1$.

a. Show that for $(x\pm, y_\pm, z_\pm)$ the characteristic equation for the eigenvalues λ of the Jacobian matrix $\mathbf{D}\mathbf{R}(x_\pm, y_\pm, z_\pm)$ at the equilibrium point is

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2\sigma b(r - 1) = 0.$$

b. The non-zero equilibria becomes unstable at the value of r at which the real part of at least one eigenvalue vanishes. By seeking solutions of the form $\lambda = iw$ (w a real variable), show that there is a pair of purely imaginary eigenvalues when

$$r = r_u = \sigma\left(\frac{\sigma + b + 3}{\sigma - b - 1}\right).$$

Why do we need to assume that $\sigma > b + 1$ for this result to be meaningful.

Question 2

For each of the systems below sketch the bifurcation diagrams of the equilibrium points x_e as functions of r and classify the bifurcations you find.

i. $\dfrac{\mathrm{d}x}{\mathrm{d}t} = rx + x^2$,

ii. $\dfrac{\mathrm{d}x}{\mathrm{d}t} = rx - \sinh x$,

iii. $\dfrac{\mathrm{d}x}{\mathrm{d}t} = r - \cosh x$.

The End

A

Code

Python Code 1

```python
# Logistic map x_(n+1) = a*x_n*(1.0-x_n)
# simple orbit plotting
# Change parameter 0< a <= 4.0 to get
# different types of orbits

from numpy import *
from matplotlib.pyplot import *

a = 2
# set logistic map parameter (between 0.0 and 4.0)
# single non-zero stable fixed point for 1.0 <a < 3.0
# stable period-2 orbit for 3.0 < a < 1 + sqrt(6) = 3.45
#*************************************************
# define a function for the RHS of the logistic map
def log_map(x):
    return a*x*(1.0-x)
#*************************************************

x01 = 0.1 # set first initial condition (between 0 and 1)
x02 = 0.9 # set second initial condition (between 0 and 1)

nmax = 30 # set maximum number of iterations

# generate the lists etc for plotting

n = linspace(0,nmax,nmax+1)

x1 = zeros(len(n))
x2 = zeros(len(n))
# assign initial conditions to first
#coordinates
x1[0] = x01
x2[0] = x02
# calculate next i points in orbit
for i in range(1,nmax+1,1):
    x1[i] = log_map(x1[i-1])
    x2[i] = log_map(x2[i-1])

# the plot commands next, with axes labels etc
# plot first orbit with red circles
scatter(n,x1, facecolors="none",edgecolors='r',\
label = 'Intial value %.2f' % (x01))
# overplot a second orbit in blue
scatter(n,x2,facecolors="none",edgecolors='b',\
label='Intial value %.2f' % (x02))
# title of plot
title('Logistic map with a = %.2f' % (a))
#label axes
xlabel('n')
ylabel('x')
# x-y range of plot
axis([-1, nmax+1, -0.0, 1.0])
# show legend
```

```
legend()
# show the plot
show()
```

Return to task on P??

R code 2

```r
logistic.map<- function(r, x, N, M){
  # r: logistic map parameter
  # x: initial value
  # N: number of iterations
  # M: number of iteration points to be returned

  z <- 1:N
  z[1] <- x
  for (i in c(1:(N-1))){
    z[i+1] <- r*z[i] * (1 - z[i])
  }

  # Return last M iterations

  z[c((N-M):N)]

}

my.r<- 2      #value of logistic parameter

x_1<- 0.1    #first initial value
x_2<- 0.9    #second initial value

x.axis<- seq(1, 30) #points on x axis
#compute orbits of first initial value
Orbit_1 <- logistic.map(my.r, x_1, N=30, M=30)
#compute orbits of second intial value
Orbit_2 <- logistic.map(my.r, x_2, N=30, M=30)
plot(Orbit_1~x.axis, pch=1, col="red", ylab="x", xlab="n",
ylim = c(0,1.0), main=paste("Logistic  map with a = ",
 my.r, sep=""))
#plot second orbit
points(Orbit_2~x.axis, col="blue")
#add first orbit to plot
legend("topright", c(paste("x_1=",x_1, sep=""),
paste("x_1=",x_2, sep="")), fill=c("blue","red"))
```
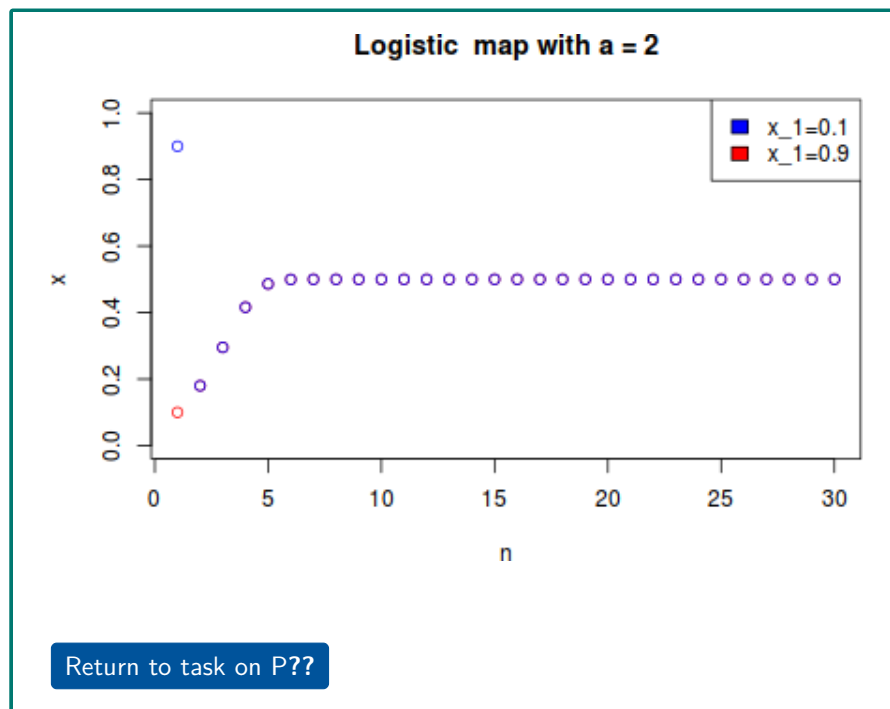
Logistic map with a = 2

Return to task on P??

Code 3

```python
from numpy import *
from matplotlib.pyplot import *

r = 2
# set logistic map parameter (between 0.0 and 4.0)
# single non-zero stable fixed point for 1.0 <a < 3.0
# stable period-2 orbit for 3.0 < a < 1 + sqrt(6) = 3.45
#

#************************************************************

def log_map(x):
    return r*x*(1.0-x)
# define a function for the RHS of the logistic map


def cobweb(f, x0, N, a, b):
# f is function; x0 is starting point,
#N is number of iterated
# a and b are range of domain; typically 0,1
# plot the function being iterated
  t = linspace(a,b, 100)
##array of 100 evenly spaced numbers between a and b
##need sufficently many steps for good graph
  plot(t, f(t), 'k')
  # plot dashed line y=x
  plot(t,t, 'k--')
  #plot the iterates x,y, f(x0)
  x,y = x0,f(x0) # initiating x= x0 and y = f(x0)
  for _ in range(N):
    fy = f(y)
    plot([x,y], [y,y], 'r', linewidth=1)
    plot([y,y], [y,fy], 'r', linewidth=1)
    x ,y= y, fy #iterating x= f(x0) = y, f = f^2*(x0)#
  axes().set_aspect(1)
  # title of plot
  title('Cobweb diagram of Logistic map with r= %.2f' % (r))
  # label of x-axis
  xlabel('x')
  # label of y-axis
  ylabel('y')
  show()
```

Return to task on P**??**

Code 4

```python
from numpy import *
from matplotlib.pyplot import *

def log_map(r,x):
        return r*x*(1.0-x)
# define a function for the RHS of the logistic map


def bifurc(f, x0, a, N, M):
#f is the map, x0 the initial value,
#a is the range from which to begin incrementing r
#N the number of iterations and
# M the number of iterations to plot
        #incrementing r values between a and N
        r= np.linspace(a,4,N)
        # initiating initial values
        x= x0 * np.ones(N)
        #iterating the logmap on the initial values
        for i in range(N):
            x= log_map(r,x)
            if i >= (N - M):
                # plotting the last M orbits;
                # green pixels with transparency .25
                plot(r,x, 'g,', alpha=.25)

        axes().set_xlim(a,4) # setting x axis limit
        # title of plot
        title('Birfucation diagram of the Logistic map')
        # label of x-axis
        xlabel('r')
        # label of y-axis
        ylabel('x')
        # rescaling plot if necessary
        figure_size=gcf().get_size_inches()
        factor= 1
        gcf().set_size_inches(factor * figure_size)
        show()
#bifurcation diagram of the logistic map beginning at r=2.5
bifurc(log_map, 0.1, 2.5, 1500, 500)
```

Return to task on P??

code 5

```python
from numpy import *
from matplotlib.pyplot import *
#define a function for the RHS of the Hènon map
def hern_map(a,b,x,y):
        return (a- x**2 + b*y, x)

def bifurc(f, x0, y0, r, b0, N, M):
  #f is the map,
  #r is the range from which to begin incrementing a
  #(x0,y0) is the coordinate of initial point
  #b0 is the fixed value of b
  #N the number of iterations and
  # M the number of iterations to plot
    #incrementing a values between r and 1.25
    a= np.linspace(r,1.25,N)
    #initiating x and y values
    x= x0*np.ones(N)
    y= y0*np.ones(N)
    # fixed b
    b= b0*np.ones(N)
    # iterating the Henon map on the initial values
    for i in range(N):
      x,y=hern_map(a,b,x,y)
      if i >= (N - M):
        # plotting the last M orbits;
        #green pixels with transparency .25
        plot(a,x, 'g,', alpha=.25)
    axes().set_xlim(r,1.25) # setting x axis limit
    # title of plot
    title('Birfucation diagram of the Hènon map')
    # label of axes
    xlabel('a')
    ylabel('x')
    # rescaling plot if necessary
    figure_size=gcf().get_size_inches()
    factor= 0.65
    gcf().set_size_inches(factor * figure_size)
    show()

bifurc(hern_map,0.1,0.0,0,0.4,1500,300)
```

Return to task on P**??**

B

Solutions

References

Alligood, K., Sauer, T., and Yorke, J. (2000). *Chaos: An Introduction to Dynamical Sytems*. Textbooks in Mathematical Sciences. Springer New York.