Math 163: Homework 2

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February 10, 2022

1. (a)

Theorem. *The product of an even number and an integer is an even number.*

Proof. Let x be an even integer and y be an integer.
Then by definition of even, there is an integer a such that $x = 2a$.
Then,

$$xy = (2a)(y)$$

$$xy = 2a \cdot y$$

By commutative property:

$$xy = 2 \cdot ay$$

By associative property:

$$xy = 2(ay)$$

By closure under production of the integers, since a and y are integers, ay is also an integer.
Thus, by definition of even, this theorem is true. $\square$

(b)

Theorem. *The product of two odd numbers is odd.*

Proof. Let x and y be odd integers
Then by definition of odd, there is an integer a such that $x = 2a + 1$, and there is an integer b such that $y = 2b + 1$. Then,

$$xy = (2a + 1)(2b + 1)$$

By distributive property:

$$xy = 2a(2b + 1) + (2b + 1)$$

By distributive property:

$$xy = 2a \cdot 2b + 2a + (2b + 1)$$

By commutative property:

$$xy = 2 \cdot 2ab + 2a + 2b + 1$$

By distributive property:

$$xy = 2(2ab + a + b) + 1$$

By closure under multiplication of the integers, since a, b and 2 are integers, $2ab$ is also an integer.
By closure under addition of the integers, since a, b, and $2ab$ are integers, $2ab + a + b$ is also an integer.
Thus, by definition of odd, $xy = 2(2ab + a + b) + 1$ is an odd. $\square$

2.

Theorem. *If $a|b$ and $a|c$, then $a|(2b + 3c)$.*

Proof. Let a, b, and c be integers. Let a divides b and a divides c.
By definition of divides, there exists integers k_1 and k_2 such that $b = k_1 a$ and $c = k_2 a$.
Thus,

$$2b + 3c = 2(ak_1) + 3(ak_2)$$

By associative property:

$$2b + 3c = a(2k_1) + a(3k_2)$$

By distributive property:

$$2b + 3c = a(2k_1 + 3k_2)$$

By closure under multiplication and addition of the integer, since k_1, k_2 are integers, $(2k_1 + 3k_2)$ is an integer.
Thus, by definition of divides, a divides $(2b + 3c)$. □

3. (a)

Theorem. *If a, b are integers, then $(ab)^2 = a^2b^2$.*

Proof. Let a and b be integers.
Then,
By the definition of "square":

$$(ab)^2 = (ab)(ab)$$

By associative property:

$$(ab)^2 = a(ba)b$$

By commutative property:

$$(ab)^2 = a(ab)b$$

By associative property:

$$(ab)^2 = (aa)(bb)$$

Thus, by the definition of "square":

$$(ab)^2 = a^2b^2$$

□

(b)

Theorem. *The expansion of $(a + b)^2$ is $a^2 + 2ab + b^2$.*

Proof. Let a and b be integers.
Then,
By the definition of "square":

$$(a + b)^2 = (a + b)(a + b)$$

By distributive property:

$$(a + b)^2 = a(a + b) + b(a + b)$$

By distributive property:

$$(a + b)^2 = aa + ab + ba + bb$$

By commutative property:

$$(a + b)^2 = aa + ab + ab + bb$$

By the definition of "square":

$$(a + b)^2 = a^2 + ab + ab + b^2$$

Thus,

$$(a + b)^2 = a^2 + 2ab + b^2$$

□

(c)

Theorem 1. *The expansion of* $(ab)^2$ *is* $abab$.

Proof. By the definition of "square":

$$(ab)^2 = (ab)(ab)$$

$$(ab)^2 = abab$$

Because there is no commutative law for multiplication, this is done in this step. □

Theorem 2. *The expansion of* $(a+b)^2$ *is* $a^2 + b^2 + ab + ba$.

Proof. By the definition of "square":

$$(a+b)^2 = (a+b)(a+b)$$

By the right distributive law:

$$(a+b)^2 = a(a+b) + b(a+b)$$

By the left distributive law:

$$(a+b)^2 = aa + ab + ba + bb$$

By the definition of "square":

$$(a+b)^2 = a^2 + ab + ba + b^2$$

By the commutativity of addition:

$$(a+b)^2 = a^2 + b^2 + ab + ba$$

Because there is no commutative law for multiplication, this is done in this step. □

The reason why the alien's number system has two distributive laws rather than one is that there is no commutative laws for multiplication. For example, $ab \neq ba$. this two number are totally different numbers, and if this system has just one distributive laws, they may can not figure out the expansion of $(a+b)^2$.

4. (a) This two expressions are not logically equivalent.

Proof.

$P\ Q\ R$	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
$T\ T\ T$	T	T
$T\ T\ F$	F	F
$T\ F\ T$	T	T
$T\ F\ F$	T	T
$F\ T\ T$	T	T
$F\ T\ F$	F	T
$F\ F\ T$	T	T
$F\ F\ F$	F	T

So from this truth table, we can conclude that these two expressions are not logically equivalent. □

(b) This two expressions are not logically equivalent.

Proof.

P	Q	R	$(P \vee Q) \Rightarrow R$	$(P \Rightarrow R) \vee (Q \Rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

So from this truth table, we can conclude that these two expressions are not logically equivalent. □

(c) This two expressions are not logically equivalent.

Proof.

P	Q	R	$(P \wedge Q) \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	T

So from this truth table, we can conclude that these two expressions are not logically equivalent. □

(d) This two expressions are logically equivalent.

Proof.

P	Q	R	$P \Rightarrow (Q \wedge R)$	$(P \Rightarrow Q) \vee (P \Rightarrow R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

So from this truth table, we can conclude that these two expressions are logically equivalent. □

(e) This two expressions are logically equivalent.

Proof.

P	Q	R	$P \Rightarrow (Q \wedge R)$	$(P \Rightarrow Q) \wedge (P \Rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

So from this truth table, we can conclude that these two expressions are logically equivalent. □

5. (a) This statement is a tautology.

Proof.

P	$P \Leftrightarrow \neg(\neg P)$
T	T
F	T

So from this truth table, we can conclude that this statement is a tautology. ☐

(b) This statement is a contradiction.

Proof.

P	$P \wedge \neg P$
T	F
F	F

So from this truth table, we can conclude that this statement is a contradiction.. ☐

(c) This statement is a tautology.

Proof.

P	$P \vee \neg P$
T	T
F	T

So from this truth table, we can conclude that This statement is a tautology. . ☐

(d) This statement is neither tautology or contradiction.

Proof.

P Q	$(P \wedge Q) \vee (\neg P \wedge \neg Q)$
T T	T
T F	F
F T	F
F F	F

So from this truth table, we can conclude that this statement is neither tautology or contradiction. ☐

(e) This statement is a tautology.

Proof.

P Q R	$P \Rightarrow (Q \Rightarrow P)$
T T	T
T T	T
T T	T
T T	T

So from this truth table, we can conclude that this statement is a tautology. ☐