Math 023. HW-2 Fangzheng Yu.

Chap 12.3.

7. $\vec{a} = 2\vec{i} + \vec{j}$, $\vec{b} = \vec{i} - \vec{j} + \vec{k}$

so $\vec{a} \cdot \vec{b} = (2\vec{i} + \vec{j}) \cdot (\vec{i} - \vec{j} + \vec{k})$

$= 2|\vec{i}|^2 - \vec{i} \cdot \vec{j} + 2 \cdot \vec{i} \cdot \vec{k} + \vec{j} \cdot \vec{i} - |\vec{j}|^2 + \vec{j} \cdot \vec{k}$

since. $\vec{i}, \vec{j}, \vec{k}$ are perpendicular with each other.

$\vec{a} \cdot \vec{b} = 2|\vec{i}|^2 - |\vec{j}|^2 = 1$

27. Let this unit vector be $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$

So. $\begin{cases} \vec{u} \cdot (\vec{i} + \vec{j}) = 0 \\ \vec{u} \cdot (\vec{i} + \vec{k}) = 0 \end{cases} \Rightarrow -a = c = b.$

And $|\vec{u}| = 1$.

So $a^2 + b^2 + c^2 = 1 \Rightarrow b = c = \frac{\sqrt{3}}{3}, a = -\frac{\sqrt{3}}{3}$

So $\vec{u} = -\frac{\sqrt{3}}{3}\vec{i} + \frac{\sqrt{3}}{3}\vec{j} + \frac{\sqrt{3}}{3}\vec{k}$

31. $y = x^2$, $y = x^3$

to get the intersection: $\begin{cases} y = x^2 \\ y = x^3 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

so the point A is $(0,0)$

$\tan\alpha = (x^2)'_{x=0} = 0.$

$\tan\beta = (x^3)'_{x=0} = 0.$

so $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = 0.$

so the acute angle is 0.

The other intersection point is $(1,1)$.

$\tan\alpha = (x^2)'_{x=1} = 2$

$\tan\beta = (x^3)'_{x=1} = 3.$

so $\tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\alpha\tan\beta} = \frac{1}{1+6} = \frac{1}{7}$

so the acute angle is $\tan^{-1}(\frac{1}{7})$

39.

$\vec{a} = \langle -5, 12 \rangle$, $\vec{b} = \langle 4, 6 \rangle$.

$Proj_{\vec{a}} \vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\cdot\frac{\vec{a}}{|\vec{a}|} = 4\cdot\frac{1}{13}\langle -5, 12\rangle = \frac{4}{13}\langle -5, 12\rangle = \langle -\frac{20}{13}, \frac{48}{13}\rangle$

$Comp_{\vec{a}} \vec{b} = \cos\theta\cdot|\vec{b}| = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{52}{13} = 4$

47. $\vec{a} = \langle 3, 0, -1 \rangle$.

Let $\vec{b} = \langle a, b, c \rangle$.

So $Comp_{\vec{a}} \vec{b} = \cos\theta|\vec{b}| = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{3a-c}{\sqrt{10}}$.

so $\frac{3a-c}{\sqrt{10}} = 2$.

so we can let $\vec{b} = \langle \frac{2}{3}\sqrt{10}, 0, 0 \rangle$ to adapt to the question.

61. Thm3: $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

proof:

Let two vectors be $\vec{a}, \vec{b}$.

And $\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cos\theta$. (by Thm3.)

so $|\vec{a}\cdot\vec{b}| = |\vec{a}||\vec{b}|\cdot|\cos\theta|$.

since $\cos\theta \in [0, 1]$.

$|\vec{a}\cdot\vec{b}| = |\vec{a}||\vec{b}|\cdot|\cos\theta| \le |\vec{a}|\cdot|\vec{b}|$

so. $|\vec{a}\cdot\vec{b}| \le |\vec{a}|\cdot|\vec{b}|$ (equals when $\vec{a}$ an $\vec{b}$ are perpend

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12.4. Chapter.

2.

$\vec{a} = \langle 4,3,-2\rangle$, $\vec{b} = \langle 2,-1,1\rangle$.

$$\vec{c} = \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 3 & -2 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix}\cdot\vec{i} - \begin{vmatrix} 4 & -2 \\ 2 & 1 \end{vmatrix}\cdot\vec{j} + \begin{vmatrix} 4 & 3 \\ 2 & -1 \end{vmatrix}\vec{k}$$

$$= \vec{i} - 8\vec{j} - 10\vec{k} = \langle 1,-8,-10\rangle.$$

so. $\vec{a}\cdot\vec{c} = \langle 4,3,-2\rangle\cdot\langle 1,-8,-10\rangle$

$= 4-24+20 = 0$

$\vec{b}\cdot\vec{c} = \langle 2,-1,1\rangle\cdot\langle 1,-8,-10\rangle$

$= 2+8-10 = 0$.

so. $\vec{c}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

28.

$\overrightarrow{PQ} = \langle 2,3,1\rangle$

$\overrightarrow{PR} = \langle 6,5,6\rangle$

$\overrightarrow{PS} = \langle 4,2,5\rangle$

$$\overrightarrow{PQ}\times\overrightarrow{PS} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ 4 & 2 & 5 \end{vmatrix} = 13\vec{i} - 6\vec{j} - 8\vec{k}$$

so

$$\text{Area} = |\overrightarrow{PQ}\times\overrightarrow{PS}| = \sqrt{13^2+6^2+8^2} = \sqrt{269}$$

31. $P(0,-2,0)$, $Q(4,1,-2)$, $R(5,3,1)$

so. $\overrightarrow{PQ} = \langle 4,3,-2\rangle$, $\overrightarrow{PR} = \langle 5,5,1\rangle$

so the orthogonal vector

$$\vec{c} = \overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 3 & -2 \\ 5 & 5 & 1 \end{vmatrix} = 13\vec{i} - 14\vec{j} + 5\vec{k} = \langle 13,-14,5\rangle$$

$$\text{Area}_{\triangle PQR} = \frac{1}{2}|\vec{c}| = \frac{1}{2}\sqrt{13^2+5^2+14^2} = \frac{1}{2}\sqrt{390}$$

38. $A(1,3,2)$

$B(3,-1,6)$

$C(5,2,0)$

$D(3,6,-4)$

$\overrightarrow{AB} = \langle 2,-4,4 \rangle$

$\overrightarrow{AC} = \langle 4,-1,-2 \rangle$

$\overrightarrow{AD} = \langle 2,3,-6 \rangle$

So $\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{vmatrix} 2 & -4 & 4 \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix} = 2\cdot 12-(-4)\cdot(-20)+4\cdot 14$

$= 24-80+56 = 80-80 = 0$

So A,B,C,D are both in a same plane.

49. $(\vec{a}-\vec{b}) \times (\vec{a}+\vec{b}) = 2(\vec{a} \times \vec{b})$

proof: $(\vec{a}-\vec{b}) \times (\vec{a}+\vec{b}) = \vec{a} \times (\vec{a}+\vec{b}) - \vec{b} \times (\vec{a}+\vec{b})$

$= \vec{a}\times\vec{a} + \vec{a}\times\vec{b} - \vec{b}\times\vec{a} - \vec{b}\times\vec{b}$

Since $\vec{a}\times\vec{a} = -\vec{a}\times\vec{a}$,

$\vec{a}\times\vec{a} = 0$.

so, $\vec{b}\times\vec{b} = 0$.

So $(\vec{a}-\vec{b}) \times (\vec{a}+\vec{b}) = 0+0+\vec{a}\times\vec{b} - \vec{b}\times\vec{a}$

$= 0+0+\vec{a}\times\vec{b} + \vec{a}\times\vec{b} = 2(\vec{a}\times\vec{b})$.

Thus. $(\vec{a}-\vec{b}) \times (\vec{a}+\vec{b}) = 2(\vec{a}\times\vec{b})$

2. vector $\vec{v} = \langle 1, 3, -\frac{2}{3} \rangle$, point $(6, -5, 2)$

so we have the line's vector

$$\vec{r} = \vec{r_0} + t\vec{v}$$

$$= \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

$$= \langle 6 + t, -5 + 3t, 2 - \frac{2}{3}t \rangle$$

So we have the line:

$$\frac{x-6}{1} = \frac{y+5}{3} = \frac{z-2}{-\frac{2}{3}}$$

13. the line A's direction: $\vec{r_A} = \langle 2, 6, -4 \rangle = 2 \cdot \langle 1, 3, -2 \rangle$

the line B's direction: $\vec{r_B} = \langle 5, 15, -10 \rangle = 5 \cdot \langle 1, 3, -2 \rangle$

so two line parallel.

17. The vector equation of this line's segment is:

①: $\vec{r}(t) = (t)\langle 6, -1, 9 \rangle + (1-t)\langle 7, 6, 0 \rangle$ $t \in [0, 1]$.

$$= \langle 6t + 7 - 7t, -t + 6 - 6t, 9t \rangle$$

$$= \langle -t + 7, -7t + 6, 9t \rangle \quad ①$$

so, $\vec{r}(t) = (-t+7)\vec{i} + (-7t+6)\vec{j} + 9t\vec{k}$, $t \in [0, 1]$

② or, $\vec{r}(t) = (1-t)\langle 6, -1, 9 \rangle + t\langle 7, 6, 0 \rangle$, $t \in [0, 1]$

$$= \langle t + 6, 7t - 1, 9 - 9t \rangle$$

so $\vec{r}(t) = (t+6)\vec{i} + (7t-1)\vec{j} + (9-9t)\vec{k}$, $t \in [0, 1]$

18. The vector equation of this line:

$\vec{r}(t) = (1-t)\langle -2, 18, 31\rangle + t\langle 11, -4, 48\rangle, t \in [0,1]$

$= \langle 13t-2, 18-22t, 31+17t\rangle$

So the parametric equation is:

$$\frac{x+2}{-13} = \frac{y-18}{-22} = \frac{z-31}{17}$$

23. The normal vector of the plane is: $\langle 1, -2, 5\rangle$

And the point (x_0, y_0, z_0) is $(0,0,0)$

so the plane's equation is: $(x-x_0)-2(y-y_0)+5(z-z_0)=0$

$\Rightarrow x-2y+5z=0.$

31. Let A be point $(0,1,1)$

B be point $(1,0,1)$

C be point $(1,1,0)$

so $\vec{AB}=(1,-1,0)$

$\vec{AC}=(1,0,-1)$

one of the normal vector $\vec{n} = \vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \vec{i}+\vec{j}+\vec{k}$

$= \langle 1, 1, 1\rangle$

so the plane's equation is:

$(x-0)+(y-1)+(z-1)=0$

$x+y+z=2.$