Math 263: Homework 1

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1. (a) The number is:

$$\frac{9!}{2!3!} = 30240$$

   (b) If "O" is on the start and end, the number of ways:

$$\frac{7!}{3!}$$

   If "E" is on the start and end, the number of ways:

$$\frac{7!}{2!}$$

   Thus, The number is:

$$\frac{7!}{3!} + \frac{7!}{2!} = 3360$$

2. Imagine the three sisters to be a person, so there is "7" persons now.
   Order the "7" group at first (7!), and then order in the "3" group (3!).
   The number of the ways is:

$$7!3! = 30240$$

3. (a) The number of ways is:

$$\binom{2}{1}\binom{5}{1}\binom{6}{1}\binom{9}{3} = 5040$$

   (b) If we exclude the situation in question (a) from the whole situation, we would get the answer.
   The number of ways is:

$$\binom{11}{5}\frac{5!}{3!} - \binom{2}{1}\binom{5}{1}\binom{6}{1}\binom{9}{3} = 9240 - 5040 = 4200$$

4. The number of ways is:

$$\binom{4}{8} - \binom{6}{2} = 70 - 15 = 55$$

5. (a) The number of shortest paths:

$$\binom{6}{3} = 20$$

   (b) The number of shortest paths:

$$\binom{6}{3} - \binom{3}{1}\binom{3}{1} = 20 - 9 = 11$$

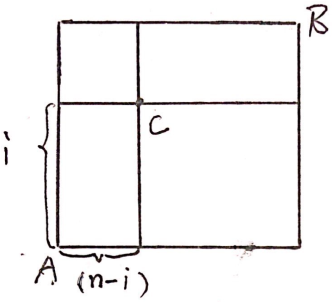

Figure 1: point c and n-by-n grid

(c) *proof1.* Imagaine that we have a n x n grid.
And the left in the equation is the number of the distinct shortest ways that a point moves from A to B.
And the right is the number of the distinct shortest ways that a point moves from A to B and pass through C. C is point whose horizontal distance is $(n-i)$ and veritcal distance is i. And i is an integer ranging from 0 to n, so C can be all the point on the grid's diagonal. (As shown in the figure 1) And it is clear that if the point moves from A to B, it must pass some of the point on the grid's diagonal. Thus, the right equals left. ***i.e.*** $\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \binom{2n}{n}$. □

proof2. Consider we have a polynomial $(x+1)^{2n}$, and we want to get the coefficienct of the item whose power is n. It is clear that the coefficient of x with power of n is $\binom{2n}{n}$.
We can view the equation as $(x+1)^n(x+1)^n$. In this two separate polynomial, let the power of x in the first polynomial is i, so the second is $n-i$. And the coefficienct for the first is $\binom{n}{i}$, and the second is $\binom{n}{n-i}$. So we can get the item $\binom{n}{i}x^i\binom{n}{n-i}x^{n-i}$, which contains x with power of n. And then if sum all this coefficiencts, we get $\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$.
Thus, $\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \binom{2n}{n}$. □