8

Price Formation and Learning in Equilibrium under Asymmetric Information

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Abstract

This chapter studies the financial equilibrium and its properties among asymmetrically informed market participants starting with the seminal work of Kyle (1985). Using its continuous-time formulation by Back (1992) as the underlying framework, equilibrium strategies of informed traders and market makers will be derived in the original model as well as in a number of key extensions including the models that account for competition among multiple insiders, default risk and dynamic information acquisition. Moreover, the interplay between the batch auction model of Kyle and the sequential arrival model of Glosten and Milgrom (1985) will be discussed. The mathematical analysis will rely on the combination of stochastic filtering and Markovian bridge techniques that are tailored for this equilibrium framework. Finally, by incorporating risk averse market makers to the model we will obtain an equilibrium that simultaneously exhibits price reversal and permanent impact, and thereby bridging the gap between the earlier and more recent market microstructure models.

8.1 Introduction

One of the goals of Market Microstructure (MS) models is to understand the *temporary* and *permanent* impacts of the trades on the asset price and how the price-setting rules evolve in time. In real markets bid and ask prices are announced by *specialists* or *dealers*, whom we will henceforth collectively call *market makers*. The early literature on market microstructure (Garman, 1976; Stoll, 1978; Amihud and Mendelson, 1980; Ho and Stoll, 1981) have started with the simple observation that the trades could involve some implicit costs due to the need for immediate execution, which is provided by the market makers. At the same time, the market makers take into account their inventory level when making pricing decisions. These works have concluded that the market makers adjust the prices in order to keep their inventories around a certain level in the long run: they lower the price when their inventory levels are too high and raise

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the prices when they are short large quantities. As the market makers want to keep their inventories around a fixed level, the impact of trades are *transitory* since the prices are also expected to mean revert.

The MS research have later shifted its focus to models with asymmetric information, which account for permanent changes in the price. The canonical model of markets with asymmetric information is due to Kyle (1985). He studied a market for a single risky asset whose price is determined in equilibrium in discrete time. The key feature of this model is that the market makers cannot distinguish between the informed and uninformed trades and compete to fill the net demand. In this model market makers 'learn' from the net demand by 'filtering' what the informed trader knows, which is 'corrupted' by the demand of the uninformed traders. The market makers learn from the order flow and they update their pricing strategies as a result of this learning mechanism.

In contrast to the batch arrival model of Kyle (1985), Glosten and Milgrom (1985) study the equilibrium pricing in a model where market makers quote bid and ask prices and market orders of unit size arrive sequentially. Nevertheless, the market makers do not know whether the arriving order is informed or not. Thus, a similar learning mechanism has to take place in order to price the risky asset efficiently.

This chapter will give a brief discussion on the fundamentals of the original Kyle model with risk neutral market makers as well as its extensions to include dynamic information flow, multiple informed traders, and defult risk. Moreover, a suitable version of the Glosten–Milgrom model will be presented and its connection to the Kyle model will be discussed.

The empirical studies on the inventories of market makers demonstrate mean reversion, which is a sign of risk aversion. In Section 8.8 we shall study the impact of market makers being risk averse on equilibrium. Consistent with empirical studies such a change will result in mean reverting inventories for market makers. From another perspective having risk averse market makers in the Kyle model bridges the earlier MS literature with that following Kyle's framework.

Surveying, even only listing, all the relevant literature in this limited space is impossible. The last section nevertheless is devoted to brief remarks on some other works that are closely related to the topics discussed in earlier pages.

8.2 The Kyle model

8.2.1 A toy example

To get a flavour of the Kyle model suppose that there is an asset whose value V will be revealed at time 1. Assume further the existence of an insider who knows the value of V at time 0. To simplify the matters the insider will be allowed to trade once at time 0 and liquidate her position at time 1.

At time 0 there are also *noise traders* who are not strategic and their cumulative demand for the asset is given by $\nu \sim N(0, \sigma_\nu^2)$. Consistent with the term 'noise' ν is assumed to be independent of V.

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If the insider trades θ many shares, the market makers observe the net demand $Y := \theta + n$ and take the opposite side to clear the market by setting a price. They know the distribution of V but no other relevant information regarding its value. The market makers are *risk neutral* and compete in a Bertrand fashion to fill the aggregate order Y. That is, the price $h(y)$ chosen by the market makers for $Y = y$ is such that their expected profit is 0. Since they will also liquidate their position at time 1 at price V, this implies

$$h(y) = E[V|Y = y]. \tag{8.1}$$

Given this *pricing rule* of market makers the insider finds her optimal trading amount based on her private information. In this idealisation of the market the market price of the traded asset will be determined in a Bayesian Nash-type equilibrium:

The pair (θ, h) will constitute and equilibrium if

1. Given h, θ maximises the expected profit of the insider;
2. Given θ, h satsifies (8.1).

Suppose further that $V \sim N(0, \sigma^2)$. Let us next observe that a *linear equilibrium* in which $h(y) = a + \lambda y$ and $\theta = \alpha + \beta V$ exists. First of all, if $h(y) = a + \lambda y$, the insider's optimisation problem given $V = v$ is

$$\max_{\alpha,\beta} E[(\alpha + \beta v)(v - a - \lambda(v + \alpha + \beta v)).$$

The profit/loss is quadratic in parameters and the first order condition yields:

$$\alpha + \beta v = \frac{v - a}{2\lambda}. \tag{8.2}$$

On the other hand, (8.1) requires

$$a + \lambda Y = E[V|Y].$$

Now, since (V, v) is a Gaussian vector, the conditional distribution of V given Y is also Gaussian, which can be determined by Bayes' rule. Formally,

$$P(V \in dv|Y = y) \sim \frac{P(Y \in dy|V = v)}{dy} P(V \in dv).$$

Moreover, given $V = v$, $Y := v + \theta \sim N(\alpha + \beta v, \sigma_v^2)$. Thus, $P(Y \in dy \mid V = v)$ is proportional to

$$\exp\left(-\frac{(y - \alpha - \beta v)^2}{2\sigma_v^2}\right).$$

Hence,

$$P(V \in dv|Y = y) \sim \exp\left(-\frac{(v - \hat{\mu})^2}{2\Sigma^2}\right),$$

where

$$\frac{1}{\Sigma^2} = \frac{1}{\sigma^2} + \frac{\beta^2}{\sigma_v^2}, \quad \hat{\mu} = \beta(y - \alpha)\frac{\Sigma^2}{\sigma_v^2}.$$

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That is, V is Gaussian with mean $\hat{\mu}$ and variance Σ^2 given $Y = y$. Thus,

$$a + \lambda y = \beta(y - \alpha)\frac{\Sigma^2}{\sigma_\nu^2} = \beta(y - \alpha)\frac{\sigma^2}{\beta^2\sigma^2 + \sigma_\nu^2},$$

which in turn yields

$$\lambda = \frac{\beta\sigma^2}{\beta^2\sigma^2 + \sigma_\nu^2} \text{ and } a = -\frac{\alpha}{\beta}\lambda.$$

Recall that (8.2) implies $2\lambda\beta = 1$. Therefore,

$$\beta = \frac{\sigma_\nu}{\sigma} \text{ and } \lambda = \frac{\sigma}{2\sigma_\nu}.$$

The remaining two equations for a and α are satisfied only if $a = \alpha = 0$.

Kyle's lambda:

A widely used metric for the amount of liquidity available in a given market is the so-called *Kyle's lambda*. It is a measurement of the sensitivity of prices to the volume and is roughly defined as the inverse of the volume needed to move the prices by one unit. More precisely, it is the derivative of the function h defined above with respect to y, which is given by λ! As such, a low λ is a sign of low liquidity costs. Given the above description of λ a liquid market requires a sufficiently large volume of noise trading in the presence of asymmetric information. This is quite reasonable: the higher the adverse selection faced by the market makers, the higher the level of compensation they require to clear the market.

The value of information:

Information acquisition is costly. Although how the informed trader has obtained her private information is not modelled in the Kyle model, it is possible to compute the value of private information. Given the above explicit characterisation of equilibrium the equilibrium level of wealth of the insider is given by

$$(1 - \lambda\beta)\beta v^2 = \beta\frac{v^2}{2}$$

It is also not difficult to see that an uninformed strategic trader will make 0 expected profit in this model as the prices evolve as a martingale for the uninformed traders. Thus, the value of information equals *ex ante*, i.e. unconditional, profit, which is given by

$$\beta\frac{\sigma^2}{2} = \frac{\sigma\sigma_\nu}{2}. \tag{8.3}$$

8.2.2 The Kyle model in continuous time

If a trader has some private information regarding the future value of the asset, she would like to take advantage of this and trade dynamically, not just once as above. The continuous time version of the Kyle model is formalised in Back

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(1992). Although in the literature it is usually assumed that the informed investor knows the future asset value perfectly, this is not a necessary assumption as we shall soon see.

Let us suppose that the time-1 value of the traded asset is given by some random variable V, which will become public knowledge at $t = 1$ to all market participants.

We shall work on a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,1]}, \mathbb{Q})$.

Three types of agents trade in the market. They differ in their information sets, and objectives, as follows.

- *Noise/liquidity traders* trade for liquidity reasons, and their total demand at time t is given by a standard $(\mathcal{G}_t)$-Brownian motion B independent of V. This normalisation of the variance of the noise trades is without loss of generality as long as the variance process is perfect knowledge among all market participants.
- *Market makers* observe only the total demand

$$Y = \theta + B,$$

where θ is the demand process of the informed trader. The admissibility condition imposed later on θ will entail in particular that Y is a semimartingale.

They set the price of the risky asset via a *Bertrand competition* and clear the market. We assume that the market makers set the price as a function of the total order process at time t, i.e. we consider pricing functionals $S\left(Y_{[0,t]}, t\right)$ of the following form

$$S\left(Y_{[0,t]}, t\right) = H(t, Y_t), \qquad \forall t \in [0,1). \tag{8.4}$$

Moreover, a pricing rule H has to be admissible in the sense of Definition 8.1. In particular, $H \in C^{1,2}$ and, therefore, S will be a semimartingale as well.
- *The informed trader (insider)* observes the price process $S_t = H(t, Y_t)$ and her private signal, Z, which is possibly time varying Markov process and is independent of B. Based on her signal, she makes an educated guess about V. We shall assume a Markovian framework in the sense that

$$E[V|\sigma(Z_t; t \le 1)] = E[V|Z_1].$$

Thus, there exists a measurable function f such that

$$f(Z_1) = E[V|\sigma(Z_t; t \le 1)].$$

We assume that Z_t is a continuous random variable for each $t > 0$ and f is continuous. Moreover, f can be taken strictly increasing. This entails in particular that the larger the signal Z_1 the larger the value of the risky asset for the informed trader.

She is assumed to be risk-neutral, her objective is to maximize the expected

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final wealth.

$$\sup_{\theta\in\mathcal{A}(H)} E^{0,z}\left[W_1^\theta\right], \text{ where}$$

$$W_1^\theta = (V - S_{1-}))\theta_{1-} + \int_0^{1-} \theta_{s-}dS_s.$$

However, using the tower property of conditional expectations, the above problem is equivalent to

$$\sup_{\theta\in\mathcal{A}(H)} E^{0,z}\left[W_1^\theta\right], \text{ where} \tag{8.5}$$

$$W_1^\theta = (f(Z_1) - S_{1-})\theta_{1-} + \int_0^{1-} \theta_{s-}dS_s. \tag{8.6}$$

In above $\mathcal{A}(H)$ is the set of admissible trading strategies for the given pricing rule[1] H, which will be defined in Definition 8.3. Moreover, $E^{0,z}$ is the expectation with respect to $P^{0,z}$, which is the probability measure on $\sigma(Y_s, Z_s; s \leq 1)$ generated by (Y, Z) with $Y_0 = 0$ and $Z = z$.

The informed trader and the market makers not only differ in their information sets but also in their probability measures. To precisely define the probability measure of the market makers consider $\mathcal{F} := \sigma(B_t, Z_t; t \leq 1)$ and let $Q^{0,z}$ be the probability measure on $\mathcal{F}$ generated by (B, Z) with $B_0 = 0$ and $Z_0 = z$. Next introduce the probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ by

$$\mathbb{P}(e) = \int_{\mathbb{R}} Q^{0,z}(e)\mathbb{Q}(Z_0 \in dz), \tag{8.7}$$

for any $e \in \mathcal{F}$. This is the probability measure used by the uninformed market makers in this model. Note that the probability measure of the informed can be *singular* with respect to that of the market makers. Indeed, if Z_0 has a continuous distribution, $P^{0,z}(Z_0 = z) = 1$ while $\mathbb{P}(Z_0 = z) = 0$.

Due to the discrepancies in the null sets of the market makers and those of the informed trader there are also delicate issues regarding the completion of filtration. As such a technical discussion will muddle the presentation and won't have a significant contribution to the understanding of the fundamentals of the model, the interested reader is referred to Section 6.1 in Çetin and Danilova (2018a). What is important to know at this point is that the insider's filtration $\mathcal{F}^I$ is generated by (Z, S) while the market makers' filtration is generated by the observation of Y only.

We can now define the rational expectations equilibrium of this market, i.e. a pair consisting of an *admissible* pricing rule and an *admissible* trading strategy such that: *a)* given the pricing rule the trading strategy is optimal, *b)* given the trading strategy, the pricing rule is *rational* in the following sense:

$$H(t, Y_t) = S_t = \mathbb{E}\left[V|\mathcal{F}_t^M\right] = \mathbb{E}\left[f(Z_1)|\mathcal{F}_t^M\right], \tag{8.8}$$

[1] Note that this implies the insider's optimal trading strategy takes into account the *feedback effect*, i.e. that prices react to her trading strategy.

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where $\mathbb{E}$ corresponds to the expectation operator under $\mathbb{P}$. Note that the last equality follows from the tower property of conditional expectations and the independence of B from V and Z as

$$\mathbb{E}\left[V|\mathcal{F}_t^M\right] = \mathbb{E}\left[\mathbb{E}\left[V|\sigma(B_s, Z_s; s \leq t)\right]\middle|\mathcal{F}_t^M\right] = \mathbb{E}\left[\mathbb{E}\left[V|\sigma(Z_s; s \leq t)\right]\middle|\mathcal{F}_t^M\right].$$

Observe that in view of (8.8) what is important is not the exact value of V but its valuation by the informed trader. That is, the informed trader does not have to be an insider.

To formalize the above notion of equilibrium, we first define the sets of admissible pricing rules and trading strategies.

Definition 8.1 An *admissible pricing rule* is any function H fulfilling the following conditions:

1. $H \in C^{1,2}([0,1) \times \mathbb{R})$.
2. $x \mapsto H(t,x)$ is strictly increasing for every $t \in [0,1)$;

Remark 8.2 The strict monotonicity of H in the space variable implies H is invertible prior to time 1, thus, the filtration of the insider is generated by Y and Z. This in turn implies that $(\mathcal{F}_t^{S,Z}) = (\mathcal{F}_t^{B,Z})$, i.e. the insider has full information about the market.

In view of the above one can take $\mathcal{F}_t^I = \mathcal{F}_t^{B,Z}$ for all $t \in [0,1]$.

Definition 8.3 An $\mathcal{F}^{B,Z}$-adapted θ is said to be an admissible trading strategy for a given pricing rule H if

1. θ is adapted and absolutely continuous on $(\Omega, \mathcal{F}, (\mathcal{F}_t^{B,Z}), Q^{0,z})$; that is, $d\theta_t = \alpha_t dt$ for some adapted and integrable α.
2. and no doubling strategies are allowed, i.e. for all $z \in \mathbb{R}$

$$E^{0,z}\left[\int_0^1 H^2(t, X_t)\, dt\right] < \infty. \tag{8.9}$$

The set of admissible trading strategies for the given H is denoted with $\mathcal{A}(H)$.

The hypothesis of absolutely continuity is standard in the literature. It was proved by Back (1992) that this restriction was without loss of generality when the insider's signal is static, i.e. $Z_t = Z_0, t \leq 1$. That it suffices to consider only the absolutely continuous strategies in the dynamic case has been recently proved in Çetin and Danilova (2018b).

Definition 8.4 A couple $(H^*\theta^*)$ is said to form an equilibrium if H^* is an admissible pricing rule, $\theta^* \in \mathcal{A}(H^*)$, and the following conditions are satisfied:

1. *Market efficiency condition:* given θ^*, H^* is a rational pricing rule, i.e. it satisfies (8.8).
2. *Insider optimality condition:* given H^*, θ^* solves the insider optimization problem:

$$\mathbb{E}[W_1^{\theta^*}] = \sup_{\theta \in \mathcal{A}(H^*)} \mathbb{E}[W_1^{\theta}].$$

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8.3 The static Kyle equilibrium

In this section we consider the case when the private signal of the informed trader is unchanged during the trading period, i.e. $Z_t = Z_1, t \leq 1$. That is, we are considering the extension of the toy example to the case of continuous trading. We shall also assume without loss of generality that Z_1 is standard normal. Before finding the optimal strategy of the insider let us formally deduce the Hamilton-Jacobi-Bellmann (HJB) equation associated to the value function of the insider.

Let H be any rational pricing rule and suppose that $d\theta_t = \alpha_t dt$. First, notice that a standard application of integration-by-parts formula applied to W_1^θ gives

$$W_1^\theta = \int_0^1 (f(Z_1) - S_s)\alpha_s \, ds. \tag{8.10}$$

Furthermore,

$$E^{0,z}\left[\int_0^1 (f(Z_1) - S_s)\alpha_s ds\right] = E^{0,z}\left[\int_0^1 (f(z) - S_s)\alpha_s ds\right]. \tag{8.11}$$

In view of (8.10) and (8.11), insider's optimization problem becomes

$$\sup_\theta E^{0,z}[W_1^\theta] = \sup_\theta E^{0,z}\left[\int_0^1 (f(z) - H(s, Y_s))\alpha_s ds\right]. \tag{8.12}$$

Let us now introduce the value function of the insider:

$$\phi(t, y, z) := \operatorname{ess\,sup}_\alpha E^{0,z}\left[\int_t^1 (f(z) - H(s, Y_s))\alpha_s ds | Y_t = y, Z = z\right], \quad t \in [0, 1].$$

Applying formally the dynamic programming principle, we get the following HJB equation:

$$0 = \sup_\alpha \left(\left[\phi_y + f(z) - H(t, y)\right] \alpha\right) + \phi_t + \frac{1}{2}\phi_{yy}. \tag{8.13}$$

Thus, for the finiteness of the value function and the existence of an optimal α we need

$$\phi_y + f(z) - H(t, y) = 0 \tag{8.14}$$

$$\phi_t + \frac{1}{2}\phi_{yy} = 0. \tag{8.15}$$

Differentiating (8.14) with respect to y and since from (8.14) it follows that $\phi_y = H(t, y) - f(z)$, we get

$$\phi_{yy} = H_y(t, y), \ \phi_{yyy} = H_{yy}. \tag{8.16}$$

Since differentiation (8.14) with respect to t gives

$$\phi_{yt} = H_t(t, y),$$

(8.16) implies after differentiating (8.15) with respect to y

$$H_t(t, y) + \frac{1}{2}H_{yy}(t, y) = 0. \tag{8.17}$$

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Thus, the equations (8.15) and (8.17)seem to be necessary to have a finite solution to the insider's problem.

Before presenting a solution of the equilibrium let's briefly observe one immediate consequence of (8.17). First recall that

$$dY_t = dB_t + \alpha_t dt,$$

where α_t is the rate of trade of the informed trader. Since the market makers only observe the batched order and cannot differentiate between the informed and the uninformed, the decomposition of the total order into a martingale component and a drift component will be different for market makers. The theory of non-linear filtering comes to the rescue here and one can write

$$dY_t = dB_t^Y + \hat{\alpha}_t dt,$$

where B^Y is the so-called innovation process, i.e. a Brownian motion with respect to the filtration of the market makers, and $\hat{\alpha}_t = \mathbb{E}[\alpha_t | \mathcal{F}_t^M]$.

Next set $S_t = H(t, Y_t)$ and observe in view of (8.17) that

$$dS_t = H_y(t, Y_t) dB_t^Y + H_y(t, Y_t)\hat{\alpha}_t dt.$$

Since S must be a martingale in equilibrium and $H_y > 0$, we expect to have $\hat{\alpha} \equiv 0$ in equilibrium. That is, the informed trader should hide her trades among the noise traders so that she gives the impression that she does not trade (well, locally)! We shall see that this is indeed the case in equilibrium.

Theorem 8.5 *Let H be an admissible pricing rule satisfying* (8.17) *and assume that $Z_t = Z_1$, $t \leq 1$, where Z_1 is a standard normal random variable. Then $\theta \in \mathcal{A}(H)$ is an optimal strategy if $H(1-, Y_{1-}) = f(Z_1)$, $P^{0,z}$-a.s..*

Proof Using Itô's formula we obtain

$$\begin{aligned} dH(t, Y_t) &= H_t(t, Y_t)dt + H_y(t, Y_t)dY_t + \frac{1}{2}H_{yy}(t, Y_t)d[Y, Y]_t \\ &= H_y(t, Y_t)dY_t. \end{aligned}$$

Also recall that

$$W_1^\theta = f(Z_1)\theta_1 - \int_0^1 H(t, Y_t))d\theta_t. \tag{8.18}$$

Consider the function

$$\Psi^a(t, x) := \int_{\xi(t,a)}^x (H(t, u) - a)du + \frac{1}{2}\int_t^1 H_y(s, \xi(s, a))ds \tag{8.19}$$

where $\xi(t, a)$ is the unique solution of $H(t, \xi(t, a)) = a$. Direct differentiation with respect to x gives that

$$\Psi_x^a(t, x) = H(t, x) - a. \tag{8.20}$$

Differentiating above with respect to x gives

$$\Psi_{xx}^a(t, x) = H_x(t, x). \tag{8.21}$$

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Direct differentiation of $\Psi^a(t,x)$ with respect to t gives

$$\Psi^a_t(t,x) = \int_{\xi(t,a)}^{x} H_t(t,u)du - \frac{1}{2}H_x(t,\xi(t,a))$$
$$= -\frac{1}{2}H_x(t,x).$$

Combining the above with (8.21) gives

$$\Psi^a_t + \frac{1}{2}\Psi^a_{xx} = 0. \tag{8.22}$$

Applying Ito's formula we thereby deduce

$$d\Psi^a(t,Y_t) = (H(t,Y_t) - a)\,dY_t.$$

The above implies

$$\Psi^a(1-,Y_{1-}) = \Psi^a(0,0) + \int_0^{1-} H(t,Y_t)(dB_t + d\theta_t) - a(B_1 + \theta_1).$$

Combining the above and (8.18) yields

$$E^{0,z}\left[W_1^\theta\right] = E^{0,z}\left[\Psi^{f(Z_1)}(0,0) - \Psi^{f(Z_1)}(1-,Y_1) - f(Z_1)B_1 + \int_0^{1-} H(t,Y_t)dB_t\right]$$
$$= E^{0,z}\left[\Psi^{f(Z_1)}(0,0) - \Psi^{f(Z_1)}(1-,Y_{1-})\right].$$

Moreover, $\Psi^{f(Z_1)}(1-,Y_{1-}) \geq 0$ with an equality if and only if $H(1-,Y_{1-}) = f(Z_1)$. Therefore, $E^{0,z}\left[W_1^\theta\right] \leq E^{0,z}\left[\Psi^{f(Z)}(0,0)\right]$ for all admissible θs, and equality is reached if and only if $H(1-,Y_{1-}) = f(Z_1)$, $P^{0,z}$-a.s.. □

The above result shows that the insider will drive the market prices to her own valuation at time 1. We will see that this will be the case in many other extensions.

Let us now compute the equilibrium in the case of bounded asset value.

Theorem 8.6 *Suppose f is bounded. Define θ by setting $\theta_0 = 0$ and*

$$d\theta_t = \frac{Z_1 - Y_t}{1-t}dt.$$

Let H be the unique solution of

$$H_t + \frac{1}{2}H_{yy} = 0, \quad H(1,y) = f(y).$$

Then, (H,θ) is an equilibrium. In particular, Y is a Brownian motion in its own filtration and $Y_1 = Z_1$.

Proof First note that since f is bounded, H is bounded by the same constant due to its Feynman–Kac representation. Thus, to show that θ is admissible it suffices to show that it is a semimartingale. Indeed, given $Z_1 = z$

$$Y_t = B_t + \theta$$

is a Brownian bridge converging to z. Thus, Y is a $P^{0,z}$-semimartingale for each

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z. Consequently, θ is a $P^{0,z}$-semimartingale for each z. Moreover, $H(1, Y_1) = f(Z_1)$, $P^{0,z}$-a.s.. Thus, θ is optimal given H.

Therefore, it remains to show that H is a rational pricing rule. Note that if Y is a Brownian motion in its own filtration,

$$H(t, Y_t) = \mathbb{E}[f(Y_1) \mid \mathcal{F}_t^Y]$$

due to the Feynman–Kac representation of H, which in turn implies H is a rational pricing rule.

Let us next show that Y is a Brownian motion in its own filtration. This requires finding the conditional distribution of Z given Y. This is a classical Kalman–Bucy filtering problem on $[0, T]$ for any $T < 1$. It is well-known (see, e.g., Theorem 3.4 in Çetin and Danilova, 2018a) the conditional distribution of Z given $\mathcal{F}_t^Y$ is Gaussian with mean $\widehat{X}_t := \mathbb{E}[Z|\mathcal{F}_t^Y]$ and variance $v(t)$, where

$$(1-t)^2 v'(t) + v^2(t) = 0,$$

and

$$\widehat{X}_t = \int_0^t \frac{v(s)}{1-s} dN_s,$$

where N is the innovation process.

The unique solution of the ODE with $v(0) = 1$ is given by $v(t) = 1 - t$. Consequently, $\widehat{X} = N$, i.e. $\widehat{X}$ is an $\mathcal{F}^Y$-Brownian motion. Let us now see that $\widehat{X} = Y$.

Indeed,

$$d\widehat{X}_t = dN_t = dY_t - \frac{\widehat{X}_t - Y_t}{1-t} dt.$$

In other words, $\widehat{X}$ solves an SDE given Y. Since this is a linear SDE, it has a unique solution, which is given by Y itself. Hence Y is an $\mathcal{F}^Y$-Brownian motion. □

Some remarks are in order. First of all, the boundedness assumption is only imposed for the brevity of the proof of admissibility and is easily satisfied for many natural boundary conditions.

The total order process Y is a Brownian motion in its own filtration. Thus, the distribution of Y is the same as that of noise trades. That is, the insider hides her orders among the noise traders and, thereby, the *inconspicuous trade theorem* holds.

The *Kyle's lambda* or the market impact of trades is given by

$$\lambda(t, y) := H_y(t, y).$$

Thus, the flatter f the more liquid is the market. Also note that the insider is indifferent among all bridge strategies that bring the market price to $f(Z_1)$ at time 1. One such bridge is when

$$dY_t = dB_t + k \frac{Z_1 - Y_t}{1-t} dt$$

for some k while H is still as in Theorem 8.6. Although this is optimal for the

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insider, it cannot make an equilibrium when combined with H since $H(t, Y_t)$ will not be a martingale when Y is as above.

8.4 The static Kyle model with multiple insiders

The equilibrium in the previous section shows that the informed trader trades moderately in the sense that she reveals her private information slowly. In fact she only reveals her hand fully at the end of the trading period. This is crucially dependent on the fact that the informed trader has a monopoly over the meaningful information on the future asset price. Indeed, Holden and Subrahmanyam (1992) conjectured by taking the continuous-time limit of their discrete-time model that the insiders reveal their information immediately in case of two or more insiders possessing the same information.

This conjecture was later proven by Back et al. (2000) in the setting of the previous section under the assumption that V is normally distributed with mean 0. Moreover, they have also considered the case of multiple insiders when their private information are not perfectly correlated and have established the existence of equilibrium in a special case.

To present their results let's denote the number of insiders by $N \geq 2$ and assume that

$$V = \sum_{i=1}^{N} Z^i,$$

where Z_i is the private signal of insider i. It is assumed that the private information is symmetric; that is, the joint distribution of Z^is is invariant to permutations.

They also limit themselves to linear equilibria given the Gaussian structure, where the rate of trade of insider i at time t is of the form

$$\alpha_i(t)S_t + \beta_i(t)Z^i$$

and α and β are deterministic functions, and the price changes is given by

$$dS_t = \lambda(t)\left\{dB_t + \sum_{i=1}^{N}\left(\alpha_i(t)S_t + \beta_i(t)Z^i\right) dt\right\}$$

with λ a deterministic function.

Note that the equilibrium rate of trade for the insider and the equilibrium price process obtained in Section 8.3 when f is affine is of this form.

Despite all these simplifying assumptions the solution of the individual insider's optimisation problem is still a difficult task. However, Back et al. obtain a clever resolution of this stochastic control problem. Their main result is the following theorem that describes the equilibrium in this setting.

Theorem 8.7 *Let*

$$\phi := \frac{\mathrm{Var}(V)}{\mathrm{Var}(NZ^i)}$$

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and consider the constant

$$\kappa = \int_1^\infty x^{2(N-2)/N} e^{-2x(1-\phi)/N\phi} dx.$$

If $N > 1$ and the Z^i are perfectly correlated, i.e. $\phi = 1$, there is no equilibrium. Otherwise, there is a unique linear equilibrium. Set $\Sigma(0) = \mathrm{Var}(V)$ and define $\Sigma(t)$ for each $t < 1$ by

$$\int_1^{\frac{\Sigma(0)}{\Sigma(t)}} x^{2(N-2)/N} e^{-2x(1-\phi)/N\phi} dx = \kappa t.$$

An equilibrium is

$$\begin{aligned}
\beta(t) &= \left(\frac{\kappa}{\Sigma(0)}\right)^{\frac{1}{2}} \left(\frac{\Sigma(t)}{\Sigma(0)}\right)^{\frac{N-2}{N}} \exp\left(\frac{1}{N}\frac{1-\phi}{\phi}\frac{\Sigma(0)}{\Sigma(t)}\right),\\
\alpha(t) &= -\frac{\beta(t)}{N},\\
\lambda(t) &= \beta(t)\Sigma(t).
\end{aligned}$$

Furthermore, $\Sigma(t)$ is the conditional variance of V given market makers' information at time t.

It is easy to see that $\Sigma(t) \to 0$ and, thus, $\beta(t) \to \infty$ as $t \to 1$. This implies,

$$S_1 = \sum_{i=1}^N Z^i = V,$$

establishing that the prices converge to the true value at the end of trading.

In case $N = 1$ the above characterisation yields $\phi = \kappa = 1$. Thus, $\Sigma(t) = (1-t)\Sigma(0)$. Therefore,

$$\beta(t) = \frac{1}{(1-t)\sqrt{\Sigma(0)}}, \quad \alpha(t) = -\frac{1}{(1-t)\sqrt{\Sigma(0)}}, \text{ and } \lambda(t) = \sqrt{\Sigma(0)},$$

coinciding with the findings of Theorem 8.6.

Given the competition among insiders one naturally wonders whether they trade more or less aggressively compared to the monopolist insider of Kyle. Fortunately, using the explicit form of equilibrium it is easy to analyse the impact of competition among the informed traders. Back et al. measures the intensity of informed trading by the coefficient of $V^i - S$ in the rate of trade for insider i, where V^i is the private valuation of the traded asset by insider i, and is shown to be a linear combination of market price and initial signal as follows:

$$V^i = (1-\delta(t))S_t + \delta(t)NZ^i,$$

where

$$\delta(t) = \frac{\phi\Sigma(t)}{(1-\phi)\Sigma(0) + \phi\Sigma(t)}.$$

If $N = 2$ and the signals are not perfectly correlated, trade intensity is easily

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less than or equal to $\frac{1}{1-t}$, which is the corresponding intensity for the monopolist insider. Thus, the insiders reveals less in the presence of competition.

This should lead one to conjecture that the markets are informationally less efficient when there is a competition among insiders. Indeed, the residual uncertainty at time t as measured by $\Sigma(t)$ is greater than $1-t$ when $N = 2$ and V is standard nnormal. It is a straightforward exercise in Gaussian filtering to conclude from Theorem 8.6 that in case of a monopolist insider the conditional variance of V given market makers' information at time t equals $1-t$.

Another important metric is, of course, the market depth as measured by Kyle's lambda. In case of monopolistic insider $\lambda = 1$ once we assume that V is normally distributed. Back et al. show that

$$\lim_{t\to 1} \frac{1}{\lambda(t)} = 0.$$

In other words, the market approaches to complete illiquidity as the date of public announcement of V approaches.

In summary, the competition leads to relatively low informed trading intensity, lower level of informational efficiency, and lower liquidity.

8.5 Dynamic Kyle equilibrium

Section 8.3 assumes that the informed trader receives a private information only at the beginning of the trading period. In this section we shall relax this assumption by considering the case of a single informed trader receiving a continuous signal converging to Z_1 as time approaches to the public announcement date of the value of the traded asset.

Following Back and Pedersen (1998) we assume that the private signal of the insider is the following Gaussian process:

$$Z_t = Z_0 + \int_0^t \sigma(s)dW_s,$$

where Z_0 is a mean-zero Normal random variable, W is a Brownian motion independent of B, and $\mathrm{Var}(Z_1) = 1$. This normalisation is for the sake of easy comparison with the static equilibrium from Section 8.3. Back and Pedersen placed a certain restriction on the mapping $t \mapsto \mathrm{Var}(Z_t)$, which have been relaxed by Danilova (2010). The following assumption on σ follows Danilova (2010) (see also Section 5.1 in Çetin and Danilova, 2018a).

Assumption 8.8 Let $c = \mathrm{Var}(Z_0)$ and define $\Sigma(t) = c + \int_0^t \sigma^2(s)ds$. Then Σ satisfies the following conditions:

1. $\Sigma(t) > t$ for every $t \in (0,1)$, and $\Sigma(1) = 1$.
2. $\int_0^t \frac{1}{(\Sigma(s)-s)^2} ds < \infty$ for all $t \in [0,1)$.
3. $\lim_{t\to 1} s^2(t)S(t)\log S(t) = 0$, where $s(t) = \exp\left(-\int_0^t \frac{1}{\Sigma(s)-s} ds\right)$ and $S(t) = \int_0^t \frac{1+\sigma^2(r)}{s^2(r)} dr$.

4. σ is bounded.

Although the third condition above seems involved, it is satisfied in practical situations. For instance, it is always satisfied if $S(1) < \infty$ since $s(1) = 0$ under the first condition. Also, an application of L'Hôpital rule shows its validity when σ is constant.

In this case the optimal strategy of the insider is still to bring the market price to her own time-1 valuation gradually. More precisely, the equilibrium total order process is given by

$$Y_t = B_t + \int_0^t \frac{Z_s - Y_s}{\Sigma(s) - s} ds.$$

As in static case, the informed traders' trades are inconspicuous, i.e. Y is a Brownian motion in its own filtration.

There is no change in the equilibrium pricing rule. Indeed, it is given by the solution to the same boundary value problem:

$$H_t + \frac{1}{2} H_{yy} = 0, \qquad H(1, y) = f(y). \tag{8.23}$$

This in particular implies the Kyle's lambda, i.e. $H_y(t, Y_t)$, have the same properties, too.

Thus, whether the information flow is dynamic or static does not have any impact on the qualitative properties of the equilibrium when there is a single informed trader.

One can also consider more general Markovian information flows. The reader is referred to Campi et al. (2011) for the details and, in particular, the concept of general dynamic Markovian bridges (see also Çetin and Danilova, 2018a).

8.6 The Kyle model and default risk

In earlier sections we have considered the pricing of a default-free risky asset. However, it is also possible to use a similar framework when the risky asset is also subject to default. This was analysed by Campi and Çetin (2007) in a static setting and by Campi et al. (2013) in a dynamic one.

Suppose for simplicity that the normalised cash balances of the firm is modelled by $1 + \beta_t$, where β is a standard Brownian motion, and the default occurs at time T_0, where

$$T_0 = \inf\{t > 0 : 1 + \beta_t = 0\}.$$

If the insider has perfect knowledge of T_0, analogous to the case studied in Section 8.3, the problem can be treated within the paradigm of static information flow as done in Campi and Çetin (2007). The other possibility is that the insider receives a dynamic information flow that gradually reveals the default time. What is assumed in Campi et al. (2013) is that the insider's signal is given by

$$Z_t = 1 + \beta_{\Sigma(t)},$$

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where Σ is a continuously differentiable function with $\Sigma(0) = 0$, $\Sigma(1) = 1$ and $\Sigma(t) > t$ for $t \in (0, 1)$. This in particular implies that

$$Z_t = 1 + \int_0^t \sigma(s) dW_s,$$

where $\sigma(t) = \sqrt{\Sigma'(t)}$, as well as $T_0 = \Sigma(\tau)$ with

$$\tau = \inf\{t > 0 : Z_t = 0\}.$$

Now let us consider the pricing of defaultable asset whose value at time-1 is given by $1_{[T_0>1]} f(Z_1)$ for some continuous and strictly increasing f. The information structure is almost identical to the previous cases except that the market makers not only observe the total order process Y but also whether the default has occurred or not, i.e the default indicator process $D_t := 1_{[T_0>t]}$.

In earlier default-free models the insider's goal was to bring the market valuation of the risky asset to her own valuation using a Brownian bridge strategy. A similar phenomenon occurs here as well. However, the insider now must convey relevant information not only about the value of Z_1 but also regarding the default time. And the analogue of the Brownian bridge in this setting is the Bessel bridge.

Following Chapter 8 of Çetin and Danilova (2018a) we shall consider two cases: (1) the static case that comprises the insider knowing τ and Z_1 in advance; and (2) the dynamic case in which the insider observes the process Z only, where Σ is satisfying Assumption 8.8 with $c = 0$.

In the static case the insider's strategy in the equilibrium is given by

$$d\theta_t = \left(\frac{q_x(1-t, Y_t, Z_1)}{q(1-t, Y_t, Z_1)} 1_{[T_0>1]} + \frac{\ell_a(T_0 - t, Y_t)}{\ell(T_0 - t, Y_t)} 1_{[\tau \le 1]} \right) dt$$

where

$$q(t, x, z) = \frac{1}{\sqrt{2\pi t}} \left(\exp\left(-\frac{(x-z)^2}{2t} \right) - \exp\left(-\frac{(x+z)^2}{2t} \right) \right) \text{ and}$$
$$\ell(t, a) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t} \right).$$

Therefore, if the insider knows that the default will not happen before time 1, she will bring the total demand to the same level as Z_1 as she did in earlier models. However, if the default is going to take place before time 1, she will drive the total demand to 0 at the time of default.

A similar but more complicated trading strategy is employed in the dynamic case. Note that since $\Sigma(t) > t$ for $t \in (0, 1)$, if $T_0 < 1$, the insider will receive the news of default a bit earlier, more precisely, at time τ, than T_0 since $\tau = \Sigma^{-1}(T_0) < T_0$. How much in advance depends on the structure of Σ and how significantly it differs from the identity function. Such difference can indeed happen. It is documented that there is a difference between the recorded default time and the economic default time (see Guo et al., 2014).

In the dynamic case the trading strategy of the informed trader in equilibrium is given by

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$$d\theta_t = \left(\frac{q_x(\Sigma(t) - t, Y_t, Z_t)}{q(\Sigma(t) - t, Y_t, Z_t)} 1_{[t \le \tau \wedge 1]} + \frac{\ell_a(T_0 - t, Y_t)}{\ell(T_0 - t, Y_t)} 1_{[\tau \wedge 1 < t \le T_0 \wedge 1]} \right) dt.$$

Again, the total order process is driven to Z_1 in case of no-default whereas it converges to 0 when default happens before time 1.

In both cases the pricing rule is given by the solution of a boundary value akin to the one given by (8.23) with the extra side condition that H vanishes at 0. The solution to this boundary value problem is given by a Feynman–Kac representation in terms of a *killed* Brownian motion (see Campi et al., 2013, for details).

As in the default-free case, there is no qualitative difference between the equilibrium with a dynamical signal and the one with a static signal.

8.7 Glosten–Milgrom model

Glosten and Milgrom (1985) study a model in which competitive risk-neutral market makers quote bid and ask prices to trade a single unit of an asset with a trader who submits a market order. The market order can be informed or coming from a *noise* trader who trade for liquidity reasons endogenous to the model. We shall be using the version of the Glosten–Milgrom model studied in Çetin and Xing (2013) that is a formalisation of the version considered by Back and Baruch (2004).

In this model the cumulative demand of the noise traders is given by the difference of two jump process X^{B} (representing buy orders) and X^{S} (representing sell orders). Each order is of fixed magnitude of δ and the arrival times of buy and sell orders are following two independent Poisson processes of constant intensity β.

The value of the risky asset V is either 0 or 1. This value will become public information at time 1 but is already known to the insider at time 0.

As usual, the market makers only observe the total order flow, and the insider is assumed to observe the noise orders as well. Note that in this model the insider will never trade of size different than δ or trade at the same time in the same direction since such actions will immediately reveal the presence of the insider and whether she is buying or selling.

Çetin and Xing show that, differently from the Kyle model, the insider uses a mixed strategy. That is, the trades of the insider not only depend on the total order and her private information but also on an extra randomisation. She achieves this by randomly meeting the orders of the noise traders by submitting a market order in the opposite direction and, thus, in a way acting like a market maker.

The techniques for establishing the equilibrium in this model is quite different than the ones discussed in earlier sections and relies on enlargement of filtration arguments for point process (see Çetin and Xing, 2013, for details). However, the equilibrium strategy of the insider is still a bridge strategy: The equilibrium price converges to V at the end of the trading horizon.

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Çetin and Xing also study the asymptotics of Glosten–Milgrom equilibria by setting $\beta = (2\delta^2)^{-1}$ and letting $\delta \to 0$. It is shown therein that the limiting equilibrium is that of a Kyle model where V is Bernoulli random variable taking values in $\{0, 1\}$. Thus, the continuous-time Kyle model can be viewed as an idealisation of a Glosten–Milgrom model with high trading activity.

8.8 Risk aversion of market makers

Whereas the risk-neutrality of the market makers makes the model tractable, it is not consistent with the observed market behaviour. Indeed, there is vast empirical evidence that the market makers are risk averse and quote prices in a way to ensure their inventories mean revert around a target level at a speed determined by their risk aversion (see Huang and Stoll, 1997, and Madhavan and Smidt, 1993, for New York Stock Exchange, Hansch et al., 1998, for London Stock Exchange, Bjønnes and Rime, 2005, for Foreign Exchange; for a survey of related literature and results, see Sections 1.2 and 1.3 in Biais et al., 2005).

Although relaxing the assumption of market makers' risk neutrality is natural and has been prompted by empirical evidence, there have been limited attempts in the literature for a theoretical investigation of its impact due to the technical complexity of the model. Subrahmanyam (1991) considered a one-period Kyle model where market makers with identical exponential utilities set the price assuming autarky utility, i.e. their mark-to-market utilities are martingales.

Çetin and Danilova (2016) developed and solved a continuous-time version of the problem introduced by Subrahmanyam. The model assumes N identical market makers quoting prices assuming autarky utilities. The market makers are risk averse and have exponential utility with risk aversion coefficient γ. Moreover, the total demand is assumed to be split equally in case of draw, which will be the case in equilibrium as the market makers are identical. The information flow of the insider is static, i.e. the insider knows V from the beginning. Note that the model presented below is for an insider, and not for an informed trader with an unbiased estimator for V. This is due to the fact that the market makers are risk-averse, thus the argument leading to (8.8) no longer holds as one needs to work with certainty equivalents due to risk aversion.

Given this warning the characterisation of the equilibrium in this market is as follows: The market makers choose the pricing rule H so that

$$H_t + \frac{\sigma^2}{2} H_{yy} = 0$$

and the total demand for the asset in its own filtration has the decomposition

$$dY_t = \sigma\, dB_t^Y - \frac{\sigma^2\gamma}{2N} Y_t H_y(t, Y_t)\, dt,$$

where B^Y is an $\mathcal{F}^Y$-Brownian motion. On the other hand the optimality condition of the informed trader requires that $H(1, Y_1) = f(V)$. It turns out that the

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equilibrium dynamics in this framework is given by the forward-backward system

$$\begin{aligned} H_t + \frac{\sigma^2}{2} H_{yy} &= 0; \\ dY_t &= \sigma d\beta_t - \frac{\sigma^2\gamma}{2N} Y_t H_y(t, Y_t)\, dt; \\ H(1, Y_1) &\stackrel{d}{=} f(V), \end{aligned} \tag{8.24}$$

where the last equality is equality in distribution, β is a given Brownian motion, and the solution of the SDE is required to be strong. Note that the terminal condition of the backward PDE is determined by the time-1 distribution of the solution of the forward SDE, which itself depends on the solution of the PDE.

Under the assumption that f is bounded with a continuous derivative, it is shown in Çetin and Danilova (2016) via a Schauder's fixed point argument the existence of a solution to the above system. Moreover, the solution to the SDE given in (8.24) has a smooth transition density $q(s, y; t, z)$ implying that the equilibrium level of demand in this economy is given by

$$Y_t = \sigma B_t - \int_0^t \frac{\sigma^2\gamma}{2N} Y_s H_y(s, Y_s)\, ds + \sigma^2 \int_0^t \frac{q_y}{q}(s, Y_s; 1, H^{-1}(1, f(V)))\, ds, \tag{8.25}$$

while the price is similarly given by $H(t, Y_t)$. Since the price is always a strictly increasing function of the demand, the solution to (8.24) is mean reverting as predicted by the empirical studies.

Interestingly the price chosen by the market makers in the above model is a solution to a particular backward stochastic differential equation (BSDE). If one denotes by S the price set by the market makers, then given any (not necessarily the optimal) trading strategy of the informed trader, S satisfies

$$dS_t = -\frac{\sigma^2\gamma}{2N} Y_t \lambda_t^2\, dt + \sigma\lambda_t dB_t^Y, \tag{8.26}$$

where B^Y is a Brownian motion in the natural filtration of Y, and S_1 is determined via the terminal condition

$$\exp(\gamma Y_1 S_1) = \mathbb{E}\left[\exp(\gamma Y_1 V) \middle| \mathcal{F}_1^Y\right]. \tag{8.27}$$

If one can find a solution (S, λ) to the above BSDE, then S determines the price in this market while λ can be considered as the *price impact* of the trades given that the martingale part of Y is σB^Y extending the notion of price impact in the previous Markovian equilibria where it is given by $H_y(t, Y_t)$.

Although at the first sight the above BSDE seems to be quadratic in λ, its coefficient is proportional to Y, which is in general unbounded being a Brownian motion. In the Markovian setting Y will be the solution to a forward SDE

$$dY_t = \sigma dB_t^Y + \hat{\alpha}(t, Y_t, S_t, Z_t)dt. \tag{8.28}$$

Even if one tries to handle this difficulty via localisation, the terminal condition (8.27) is highly non-standard and depends possibly on the whole history of Y.

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The equilibrium found in Çetin and Danilova (2016) in fact shows that when the informed trader is behaving optimally, there is a Markovian solution to the above BSDE when Y is defined by the equilibrium demand process.

In the Kyle model and its extensions discussed in earlier sections, the Kyle's lambda is a martingale. In fact, it was conjectured in Kyle (1985) that:

> [. . .] neither increasing nor decreasing depth is consistent with behavior by the informed trader which is "stable" enough to sustain an equilibrium. If depth ever increases, the insider wants to destabilize prices (before the increase in depth) to generate unbounded profits. If depth ever decreases, the insider wants to incorporate all of his private information into the price immediately.

However, when the market makers are risk averse, the Kyle's lambda is no longer a martingale, while the insider still has bounded profits. This is due to the risk sharing between the market makers and the insider. Indeed, if the trader attempts to follow the strategy outlined by Kyle, she would be moving the total order away from its mean, leaving the market makers exposed to the risk of large orders. This would violate the risk sharing mechanism in equilibrium and cause the market makers to adjust the prices eliminating favourable gains for the insider.

It is also shown in Çetin and Danilova (2016) that the sensitivity of prices to the total order can be a submartingale for certain model parameters. This implies that the execution costs are, on average, increasing toward the end of a trading period, which is consistent with the empirical results obtained in Madhavan et al. (1997).

8.9 Conclusion and further remarks

In all the models discussed so far the informed traders reveal their private information slowly and make sure that the market prices converge to their own valuation by the end of trading period. However, all models considered assumed a single traded asset. Two notable extensions of the single-period Kyle model to multiple assets are Caballé and Krishnan (1994) and Garcia del Molino et al. (2020). A more recent paper (Cocquemas et al., 2020) uses methods from optimal transport to study equilibrium with multiple assets. Back studies in a continuous-time setting an extension of the Kyle model to allow trading in an option on the stock and shows that possibility of trading in the option introduces a stochastic volatility component to the stock (Back, 1993). Stochastic volatility of equilibrium prices is also obtained in the setting considered by Collin-Dufresne and Fos (2016).

Choi et al. (2019) study a dynamic Kyle model in discrete time in which there are two strategic traders: one is the informed trader as above and the other is an uninformed trader with a target amount in the traded stock to liquidate, which is unknown to the others. The model therefore combines the informed trading model of Kyle with the literature on optimal execution for uninformed traders with liquidity motives. Although it cannot be solved in closed form, the equilibrium can be computed numerically and the model has testable implications.

Risk aversion of the insider is studied in a one-period setting by Subrahmanyam

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(1991) and in continuous-time but restrictive assumptions by Cho (2003). In an unpublished manuscript that constitutes a part of the dissertation of P. Shi, Danilova and Shi established the equilibrium in a fairly general setting for a risk-averse insider with static information flow (Danilova and Shi, 2014). More recently, Bose and Ekren (2020) studied the equilibrium with a risk-averse insider receiving a static signal using methods of optimal transport.

Risk aversion of the market makers and its impact on market risk premium is also the subject of Ying (2020).

Back et al. (2018a) propose a model of informed trading in Kyle's framework that allows for the detection of information events based on market data.

In all these models the terminal value of the asset V is exogenous. Back et al. study activist trading in Kyle's model in which the terminal value of the traded asset depends on the trades of the activist (Back et al., 2018b), and, thus, is endogenously determined in equilibrium.

An important aspect of the research literature that has not been touched upon so far in this chapter is that on limit order markets and the equilibria therein. This is mostly due to the technical difficulties involved in modelling and the scarcity of solvable models in contrast to the Kyle model. Moreover, the competition among market makers submitting limit orders are fundamentally different. Bernhardt and Hughson (1997) show that the limit order traders makes positive expected gains if there are only finitely many of them, while only two market makers are enough to drive the profits to zero in the Kyle model. Glosten (1994) assumes infinitely many limit order traders to get around this issue. Çetin and Waelbroeck (2020) propose a setting to combine the Glosten model of limit order trading and the Kyle model in a single equilibrium framework, albeit in a single-period model!

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