Number Theory and Mathematics

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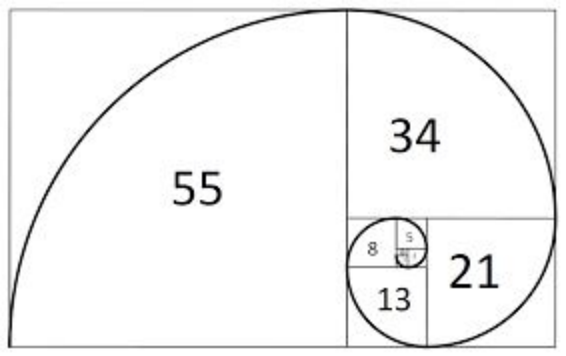
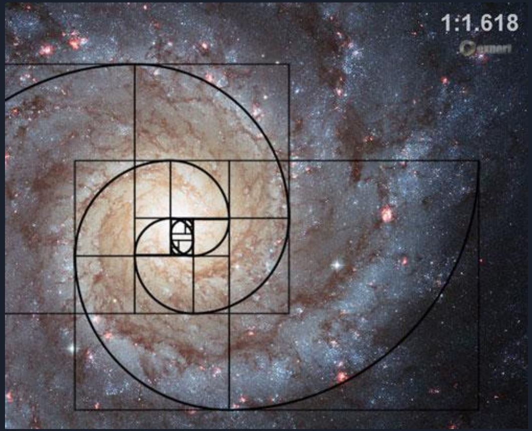
Introduction

- Mathematics is the theoretical backbone of computer science and all the sciences.
- Different branches of mathematics intersect in their applications to Computer Science
- We will be focused on covering sequences, series, summations, as well as number theory along with prime numbers.

Sequences

- Fundamentally, a sequence is just a list of numbers generated based on some rule or pattern.
- The pattern satisfying a sequence can be easy to spot, or it can require great amounts of thinking.
- General types: arithmetic and geometric sequences.
- In an arithmetic sequence the numbers share a common difference
- In a geometric sequence, the numbers share a common ratio
- There are other sequences that may follow different rules, one such example is the fibonacci sequence.
- Fibonacci numbers have been studied extensively and are found in nature.
- Fibonacci sequence is a recursive sequence.
- Spiral Galaxies, the shapes of flowers to name a few have been found to follow the fibonacci sequence.





Series

- Series are basically sequences but instead of just listing the numbers separately, we are summing up all the terms.
- Series can either converge or diverge, in simple terms they either sum up to a particular finite value or don't. If they don't we say the series diverges.
- One example of an interesting series is the harmonic series, which sums up all possible fractions of the form 1/n.
- Harmonic series diverges, but it is related to number theory and the distribution of prime numbers.

The Harmonic Series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\ldots=\infty$$

Interesting example involving a series

- Let's look at an interesting example of a series involving the sum of natural numbers from 1 - 100.
- Which way would you calculate the sum if you were given this problem as an assignment?
- We want to calculate the sum of the numbers from 1-100, that is $1+2+3+4…+100 = ?$.
- One way to do it is to simply manually add every number until 100. This method is the straightforward way to do it but a simpler and more elegant way exists. We cover this in the next slide.

Series Summation continued

- To find the sum of numbers from 1-100, we can pair the integers so that each pair has a sum of 101.
- Since there are 50 such pairs of 101 that can be formed, the sum is simply 101*50 = 5050.
- Legend says that the famous mathematician Gauss solved this problem as a boy in school when he was assigned the problem for busy work. However, using this method he solved it quicker than the teacher anticipated.
- If we had to write a program, it would be optimal to write a program that utilizes the 50 pairs of 101 to compute the sum.
- For such types of series, a general formula exists (Gaussian summation).

$$\underbrace{1+\underbrace{2+\underbrace{3+\cdots+98}_{=101}+99}_{=101}+100}_{=101}$$

Basel Problem (Example in summation)

- Famous problem that was attempted by many mathematicians, but was eventually solved by a famous mathematician by the name of Leonhard Euler.
- The series itself is quite simple as seen on the right, involving a sum of the inverse squares of all the natural numbers.
- This is a really interesting problem with an even more interesting result as the sum, seen on the right.
- The sum converges to pi squared over 6
- Like the harmonic series this series is also related to number theory and primes, as we will see in the upcoming slides.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

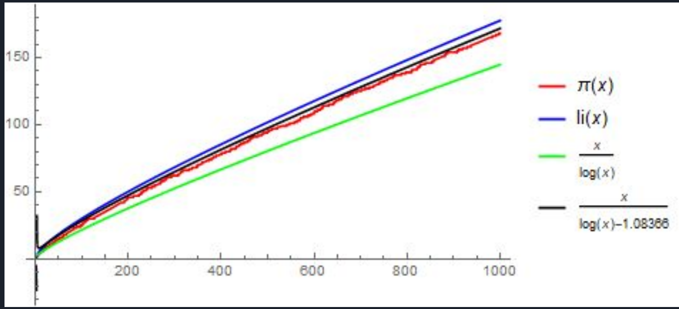
$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots\ldots = \frac{\pi^2}{6}$$

Riemann Zeta Function

- The discussion of the Basel Problem leads us to the Riemann Zeta Function.
- It is named after the mathematician Bernhard Riemann who studied it extensively. It was also known by Euler.
- The function is written as $\zeta(x)$, where x=1 gives the harmonic series, x=2 gives the series from the basel problem. x can be any value where, where the result is a series of the form $\zeta(s) = 1 + 2^{-s} + 3^{-s} + 4^{-s} + \cdots,$
- The Basel problem is simply the riemann zeta function when zeta = 2.
- Riemann zeta function has big implications for prime numbers.
- It has been postulated that the Riemann Zeta function is related to the distribution of prime numbers, that is if it is proved it could have enormous impact on computer science, specifically cryptography.

Prime Numbers

- A prime number is a number p greater than 1 such that p’s only factors are 1 and p.
- Distribution of prime numbers is an ongoing challenge mathematicians continue to tackle. Prime numbers have been studied for centuries.
- Modern encryption is based on large prime numbers and their properties.
- RSA encryption system utilizes two large prime numbers to form an encryption key.
- Prime number theorem: states that $\pi(x)$ is approximately equal to $x/\ln(x)$ for large enough x.
- Simply put $\pi(x)$ is the number of primes less than or equal to x.



Modular Arithmetic and Congruences

- Conceptualized and formulated by Gauss, who made various contributions in mathematics, and physics.
- Modular arithmetic basically just turns the remainder that results from division into an operation.
- Modular arithmetic is utilized in simple shift-ciphers, as well as hashing.
- Check digits also utilize congruences to check if a certain message is valid.
- Another application involves parity bits for when digital information is represented by a bit string.
- The parity bit is simply the sum of the bits mod 2. It follows that the parity bit is 0 if there are an even number of 1's, 1 otherwise.
- It is important to note that even if a parity bit is correct there can still be an error, but if the parity bit is wrong there definitely is an error in the bit string.
- Likewise, check digits also apply to UPCs, and ISBNs with the same concept.

Example involving Congruences (Bit Strings)

- Say we are given a bit string of length 8 with a parity bit that is as follows: 101101101
- We want to find if the bit string is valid. To do this we must check if the parity bit is correct.
- We can utilize the fact that the parity bit is simply: $x_{n+1} = x_1 + x_2 + \cdots + x_n \mathbf{\ mod\ } 2.$
- Likewise: 1 + 0 + 1 + 1 + 0 + 1 + 1 + 0 = 5.
- Since 5 mod 2 is equal 1, and the parity bit is also equal to 1 we can accept the bit string as being valid for our purposes.

References

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