Notes on generating initial conditions using Eddington's inversion

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1 The Eddington Inversion

We start with an equilibrium, spherical model described by $\rho(r)$.

So far, we have postulated the distribution function and derived the profile. What about the other way around? In general, this is quite hard but there is direct method for isotropic distributions, $f = f(E)$. For this case, we know:

$$\rho(r) = \int_0^\infty d\mathbf{v} f(E) = 4\pi \int_0^{v_{max}(r)} dv v^2 f(E)$$

$v_{max}(r) = \sqrt{2(E_T - V(r))}$ where $v_{max}(r) < \infty$ for finite models where $v_{max}(r)$ is, essentially, the escape velocity from the potential at radius r. Since $E = v^2/2 + V(r)$ we may change variable from E to v by noting that $v = \sqrt{2(E - V(r))}$ and $dv = dE/\sqrt{2(E-V(r))}$ which gives red

$$\rho(r) = 4\pi \int_{V(r)}^{E_T} dE \sqrt{2(E - V(r))} f(E).$$

Now the gravitational potential is a monotonic function of r so we may change variables from r to V. The equation now becomes

$$\rho(V) = 4\sqrt{2}\pi \int_V^{E_T} dE \sqrt{E - V} f(E)$$

and may be differentiated in V to give:

$$\frac{d\rho(V)}{dV} = -\sqrt{8}\pi \int_V^{E_T} dE \frac{1}{\sqrt{E-V}} f(E).$$

This has the form of a well-known integral equation called a *Volterra integral equation of the second kind* and more specifically called the Abel integral equation[1]. This equation

[1] Ref: Tricomi, "Integral Equations," Chapter 1, Dover

has the solution:

$$f(E) = \frac{1}{\sqrt{8}\pi^2} \frac{d}{dE} \int_E^{E_T} dV \frac{1}{\sqrt{V-E}} \frac{d\rho(V)}{dV}. \tag{1}$$

This form the distribution function is called *Eddington's formula*. Although the solution is guaranteed to be unique is is *not* guaranteed to be physical; there may be value of E for which $f(E) < 0$. In this case, one requires a more general distribution function (e.g. $f = f(E,L)$) to find a solution.

2 Notes on the numerical computation of $f(E)$

1. Tabulate $d\rho/dV$. If the density profile is steep at small radius (e.g. NFW profile), I recommend tabulating $V(r)$ vs $\log(\rho(r)+\delta)$ where the radial grid r is logarithmically spaced. The quantity δ is some small density offset to prevent taking the logarithm of zero.

2. Use standard finite-difference derivatives to compute

$$\frac{d\log(\rho(r)+\delta)}{dV(r)}$$

and then get $d\rho/dV$ using

$$\frac{d\rho(r)}{dV(r)} = [\rho(r)+\delta] \frac{d\log(\rho(r)+\delta)}{dV(r)}.$$

3. Change variables in the integral in equation (1) to remove the square-root divergence. E.g. choose $Q = \sqrt{V-E}$ which implies that $V = E + Q^2$.

4. Tabulate the integral as a function of E.

5. Finally, perform the finite-difference derivative of the integral as a function of E to get $f(E)$.

6. The run of $f(E)$ will also be steep with decreasing values of energy, E. This means that you will need a denser grid at small values of energy. I recommend tabulating $\log(f(E))$ as a function of $log(E - E_{min})$ where $E_{min} < V(r_{min})$.

This is only one suggestion for tabulating these functions to make the numerics better behaved. The goal is choose a spacing and transformation so that the functions, ρ or $f(E)$, do not vary by more than order unity over a few grid points.

3 Notes on generating phase-space points

1. The function $M(r)/M(\infty) \in [0,1]$ is the cumulative probability distribution for mass. Therefore, we can choose the random variate r_i according to this probability function by choosing a unit random variate $\mathscr{R}_i$ and solving $\mathscr{R}_i = M(r)/M(\infty)$ to get r_i.

2. We can get x_i, y_i, z_i by choosing a random direction on the unit sphere. I described 3 possible algorithms for this in class.

3. We now need to choose our velocity point at radius r. Since

$$\rho(r) = \int d^3v f(E,L) = 2\pi \int_{-v_m}^{v_m} dv_r \int_0^{\sqrt{v_m^2 - v_r^2}} dv_t v_t f(E,L)$$

where v_m is the maximum velocity at radius r: $v_m = \sqrt{2(E_T - V(r))}$. Changing variables $v_t = \sqrt{v_m^2 - v_r^2}x$, we may write

$$\begin{aligned} \rho(r) &= \int_{-v_m}^{v_m} dv_r (v_m^2 - v_r^2) \int_0^1 dxxf(E,L) \\ &= \int_{-v_m}^{v_m} d(v_m^2 v_r - v_r^3/3) \int_0^1 d(x^2/2) f(E,L) \end{aligned}$$

The whole point of this manipulation is transform the velocity variables to that they can be chosen uniformly in some range. This is standard practice when generating random variates.

We can now generate a velocity point by choosing two unit random variates $\mathscr{R}_1, \mathscr{R}_2$ and setting:

$$\mathscr{R}_1 = \frac{3}{4}\left[\frac{v_r}{v_m} - \frac{1}{3}\left(\frac{v_r}{v_m}\right)^3\right] + \frac{1}{2}$$

and solving for v_r, and

$$\mathscr{R}_2 = x^2$$

and solving for x to get $v_t = \sqrt{v_m^2 - v_r^2}x$. The solution of the equation for v_r in terms of v_m and $\mathscr{R}_1$ is readily done in trigonometric form.

4. Given these trial points, we need to decide whether to accept them based on the value of $f(E,L)$. Let's assume that $f = f(E)$ for simplicity. We compute the maximum value of $f(E)$ in the allowed range of $E \in [V(r), E_T)$. Since $f(E)$ generally decreases with increasing E, we can compare the ratio of to a random variate $\mathscr{R}_3$. If

$$f(v^2/2 + V(r))/f(V(r)) > \mathscr{R}_3$$

we accept the velocity point v_r and v_t. Otherwise, we generate a new pair of velocities and try again.

5. One we have our pair of velocities, we need generate the radial unit vector, and the two unit vectors perpendicular to the radial unit vector,

$$\begin{aligned} \mathbf{u}_r &= \frac{(x_i, y_i, z_i)}{r_i} \\ \mathbf{u}_{\perp,1} &= \frac{(-y_i, x_i, 0)}{R_i} \\ \mathbf{u}_{\perp,2} &= \frac{(-x_i z_i, -y_i z_i, R^2)}{R_i r_i} \end{aligned}$$

where $R_i = \sqrt{x_i^2 + y_i^2}$ is the cylindrical radius.

Finally, the radial velocity is

$$\mathbf{v}_r = \mathbf{u}_r v_r.$$

We choose a final random angle to distribute v_t between $\mathbf{u}_{\perp,1}, \mathbf{u}_{\perp,2}$:

$$\mathbf{v}_t = \mathbf{u}_{\perp,1} v_t \cos(2\pi \mathscr{R}_4) + \mathbf{u}_{\perp,2} v_t \sin(2\pi \mathscr{R}_4).$$

6. The velocity vector u_i, v_i, w_i follows by summing up the components of $\mathbf{v}_r + \mathbf{v}_t$.

Note: there is an easier way to do Step (3), if we assume that $f = f(E)$ to start. Rather than break d^3v into $2\pi dv_r dv_t v_t$, we can use $4\pi dv v^2 = 4\pi d(v^3/3)$ where v is the total velocity. We see that the cubic equation in Step(3) simplifies to a cube root. Then, one chooses the direction for $\mathbf{v}$ on the unit sphere, and projects that to Cartesian coordinates. I nearly always use the method in Step (3) above, because it is more general and can be used with anisotropic distributions, $f = f(E, L)$, too.