Modern stellar dynamics, lecture 7: distribution function-based models

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Distribution function in stellar dynamics

Distribution function $f(\mathbf{x}, \mathbf{v})$ is the probability density in the phase space (in this lecture, normalized as $\int f(\mathbf{x}, \mathbf{v})\, \mathrm{d}^3x\, \mathrm{d}^3v = M$).

Jeans theorem tells us that in a steady state, the DF may only depend on the integrals of motion (which themselves depend on the potential): $f(\mathbf{x}, \mathbf{v}) \;=\; f\big(\mathcal{I}(\mathbf{x}, \mathbf{v};\, \Phi)\big)$

One can choose $\mathcal{I}$ to be energy, angular momentum L (in spherical systems) or its z-component (in axisymmetric systems), etc., or alternatively the triplet of actions $\mathbf{J}$.

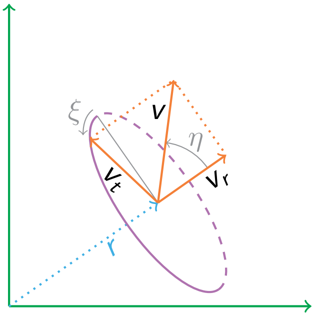
The goal is to find $f(\mathcal{I})$ given $\rho(\mathbf{x})$ and $\Phi(\mathbf{x})$ (*not necessarily* related by the Poisson equation), or vice versa, given $f(\mathcal{I})$, determine ρ and Φ *related* by the Poisson equation (that is, gravitationally self-consistent).

Distribution function in spherical systems

…may depend on E and L (and in principle on L_z if the model is rotating).

So we write $f(\mathbf{x},\mathbf{v}) = f(E,L)$, but the DF is still understood as the probability density in the 6d phase space, not in the E,L space!

To obtain the density generated by the DF in the given potential (and various other quantities of interest, e.g., velocity dispersions), we integrate the DF over the 3d velocity at the given position.



Define the radial and tangential velocity components v_r, v_t in the spherical polar coordinates aligned with the radius vector:

$v_r = v\cos\eta,\;\; v_t = v\sin\eta,\;\; \eta\in[0..\pi]$.

Then $E = \Phi(r) + \frac{1}{2}v^2,\;\; L = r\,v_t,$ and integration over velocity is performed as

$$\int d^3v\; f(\mathbf{x},\mathbf{v}) = \int_0^{v_\mathrm{escape}} dv\; 2\pi\, v^2 \int_0^\pi d\eta\;\sin\eta\; f\Big(\Phi(r)+\tfrac{1}{2}v^2,\; r\,v\,\sin\eta\Big),$$

where $v_\mathrm{escape} \equiv \sqrt{-2\big[\Phi_\infty - \Phi(r)\big]}$ and Φ_∞ is the potential at the edge of the model.

Distribution function in spherical systems

Obviously, one cannot *uniquely* determine a function of two variables $f(E, L)$ from two functions of one variable $\rho(r)$, $\Phi(r)$: same fundamental problem as in the Jeans equations.

Different choices of DF may give the same density profile, but will have different kinematic properties.

On the other hand, we may reasonably expect that *isotropic* DFs $f(E)$ have a 1:1 correspondence to spherical density–potential pairs.

The integration over velocity is reduced to

$$\rho(r) = \int_0^{v_{\mathrm{escape}}(r)} \mathrm{d}v \; 4\pi \, v^2 \; f\left(\Phi(r) + \tfrac{1}{2}v^2\right),$$

and changing the integration variable from v to E, we get

$$\rho(r) = 4\pi \int_{\Phi(r)}^{\Phi_\infty} \mathrm{d}E \; \sqrt{2\left[E - \Phi(r)\right]} \; f(E).$$

Consider first the problem of determining $\rho(r)$ and $\Phi(r)$ from $f(E)$.

Spherical polytrope models

Assume $f(E) = A\,|E|^{n-3/2}$, after substituting $E = \Phi\,x$, we have

$$\rho = 4\pi \int_{\Phi}^{0} \mathrm{d}E\, \sqrt{2(E-\Phi)}\, f(E)$$

$$= 4\sqrt{2}\,\pi\, A \int_0^1 \mathrm{d}x\; |\Phi|^{3/2+n-3/2}\, x^{n-3/2}\, (1-x)^{1/2}$$

$$= 4\sqrt{2}\,\pi\, A\, |\Phi|^n\, \mathrm{B}(n-1/2, 3/2) \;=\; 2\sqrt{2}\,\pi^{3/2}\, \tfrac{\Gamma(n-1/2)}{\Gamma(n+1)}\, G\, A\, |\Phi|^n, \qquad n > 1/2.$$

Plugging this into the Poisson equation, we get

$$4\pi\, G\, \rho = \nabla^2\Phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 8\sqrt{2}\,\pi^{5/2}\tfrac{\Gamma(n-1/2)}{\Gamma(n+1)}\, A\, \big[-\Phi(r)\big]^n.$$

This is the Lane–Emden equation, also appearing in the theory of stellar structure.

It admits a power-law solution $\Phi = \Phi_0\, r^{-\alpha}$, and we want $\Phi_0 < 0$ and $0 < \alpha \le 1$:

$$-\alpha(1-\alpha)\,\Phi_0\, r^{-\alpha-2} = 8\sqrt{2}\,\pi^{5/2}\tfrac{\Gamma(n-1/2)}{\Gamma(n+1)}\, A\,[-\Phi_0]^n\, r^{-n\alpha} \quad \Rightarrow\; \alpha = 2/(n-1).$$

These scale-free models exist for $n \ge 3$ and have $\rho \propto r^{-\alpha-2}$ (i.e., ∞ total mass and $\Phi(0) = -\infty$); in the limit $\alpha \to 0$ this is a singular isothermal sphere.

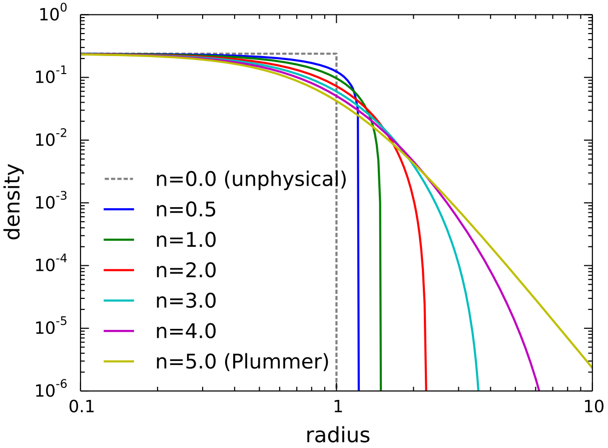
Spherical polytrope models

The Lane–Emden equation also admits more interesting non-power-law solutions that are regular at origin; in the cases $n = 1$ and $n = 5$ the solution is analytic.

Namely, the Plummer model $\Phi(r) = -\dfrac{G\,M}{\sqrt{r^2 + a^2}}$ satisfies the Lane–Emden equation for $n = 5$ and $A = \dfrac{24\sqrt{2}\,a^2}{7\pi^3\,G^5\,M^4}$.

Polytropes with $1/2 < n < 5$ have finite radius and mass; Plummer model has infinite radius but finite mass, and for $n > 5$ both radius and mass are infinite.

The density profile of these models is the same as in a self-gravitating gas sphere with a polytropic equation of state $P \propto \rho^\gamma$, $\gamma = 1 + 1/n$.



Isothermal and lowered isothermal models

The limit of polytropic models as $n \to \infty$ is the isothermal DF $f(E) = A \exp\left(-\frac{E}{\sigma^2}\right)$.

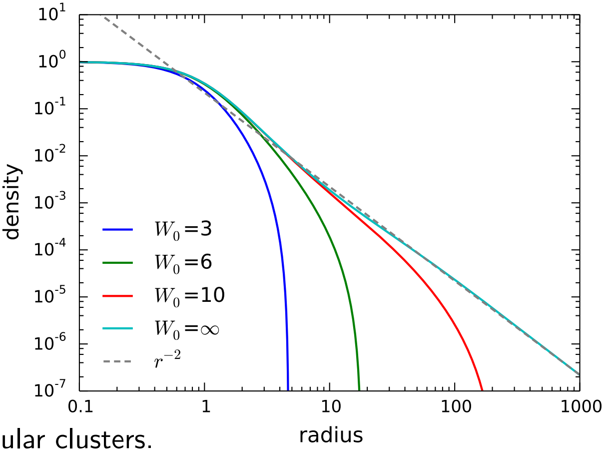
$$\rho = 4\pi A \int_\Phi^\infty \mathrm{d}E \, \sqrt{2(E-\Phi)} \, A \exp(-E/\sigma^2) = 2\sqrt{2}\,\pi^{3/2} A\,\sigma^3 \exp(-\Phi/\sigma^2).$$

$$4\pi\, G\,\rho = \nabla^2\Phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 8\sqrt{2}\,\pi^{5/2}\, G\, A\,\sigma^3 \exp\big(-\Phi(r)/\sigma^2\big).$$

Again it's easy to find a power-law solution – singular isothermal sphere $\Phi = 2\sigma^2 \ln r, \quad \rho = \sigma^2/(2\pi\, G\, r^2)$.

A non-singular numerically computed solution has a finite-density core and tends to $\rho \propto r^{-2}$ at large r.

A modification of the DF to $f(E) = A\big[\exp(-E/\sigma^2) - 1\big]$ produces truncated isothermal (King) models parametrized by the relative depth of the potential well $W_0 \equiv [\Phi(r_{\mathrm{trunc}}) - \Phi(0)]/\sigma^2$, which are often used to describe globular clusters.



Eddington inversion formula

A general expression for the density produced by an *isotropic* DF $f(E)$ is

$$\rho(r) = \int_0^{\sqrt{2\left[\Phi_\infty - \Phi(r)\right]}} \mathrm{d}v\; 4\pi\, v^2\, f\left(\Phi(r) + \frac{v^2}{2}\right) = 4\sqrt{2}\,\pi \int_{\Phi(r)}^{\Phi_\infty} \mathrm{d}E\; \sqrt{E - \Phi(r)}\; f(E).$$

usually 0

We want to determine the DF $f(E)$ that produces the given density $\rho(r)$ in the given potential $\Phi(r)$ (not necessarily related by the Poisson equation).

This integral expression looks quite like the fractional integral from Lecture 2, so can be inverted using the Abel formula, but with two modifications: first, we express ρ as a function of Φ rather than r (this is possible since the potential is monotonic with radius); and second, we need to move $\sqrt{E - \Phi}$ into the denominator by differentiating both sides w.r.t. Φ:

$$\frac{\mathrm{d}\rho(\Phi)}{\mathrm{d}\Phi} = 4\sqrt{2}\pi \frac{\mathrm{d}}{\mathrm{d}\Phi} \int_{\Phi}^{\Phi_\infty} \mathrm{d}E\; \sqrt{E - \Phi}\; f(E) = -2\sqrt{2}\pi \int_{\Phi}^{\Phi_\infty} \mathrm{d}E\; \frac{f(E)}{\sqrt{E - \Phi}}.$$

The Abel inversion formula then gives the Eddington DF:

$$f(E) = \frac{1}{\sqrt{8}\,\pi^2} \frac{\mathrm{d}}{\mathrm{d}E} \int_E^{\Phi_\infty} \frac{\mathrm{d}\Phi}{\sqrt{\Phi - E}} \frac{\mathrm{d}\rho}{\mathrm{d}\Phi} = \frac{1}{\sqrt{8}\,\pi^2} \left[\int_E^{\Phi_\infty} \frac{\mathrm{d}\Phi}{\sqrt{\Phi - E}} \frac{\mathrm{d}^2\rho}{\mathrm{d}\Phi^2} - \frac{1}{\sqrt{\Phi_\infty - E}} \cancel{\left.\frac{\mathrm{d}\rho}{\mathrm{d}\Phi}\right|_{\Phi = \Phi_\infty}} \right]$$

usually 0, but see Lacroix+ 2018

Eddington inversion formula

$$f(E) = \frac{1}{\sqrt{8}\,\pi^2}\,\frac{\mathrm{d}}{\mathrm{d}E}\int_E^{\Phi_\infty}\frac{\mathrm{d}\Phi}{\sqrt{\Phi-E}}\,\frac{\mathrm{d}\rho}{\mathrm{d}\Phi}.$$

The DF obtained by the Eddington inversion is not guaranteed to be non-negative; however, unlike Jeans equations, we may explicitly check this condition.

To use the Eddington inversion formula, one needs to express ρ as a function of Φ, which is not always an easy task; the integral itself can be taken analytically only in some simple cases, and even then the resulting expressions may be unwieldy:

for example, the $\gamma = 1$ Dehnen model: $\Phi(r) = -\dfrac{G\,M}{r+a}$, $\rho(r) = \dfrac{M\,a}{2\pi\,r\,(r+a)^3}$ $\Rightarrow$

$$f(E) = \frac{1}{2^{7/2}\pi^3\,(G\,M\,a)^{3/2}}\left[\frac{\sqrt{x}\,(1-2x)(8x^2-8x-3)}{(1-x)^2} + \frac{3\arcsin\sqrt{x}}{(1-x)^{5/2}}\right],\quad x \equiv -\frac{E\,a}{G\,M}.$$

On the other hand, this integral can be computed numerically for any given combination of ρ and Φ.

Anisotropic spherical systems

There are infinitely many $f(E, L)$ that generate the given $\rho(r)$ in the given $\Phi(r)$.

One possible choice is to consider a factorized DF $f(E, L) = f_E(E)\, f_L(L)$.
For example, $f_L = L^{-2\beta}$ produces models with a constant velocity anisotropy $\beta = 1 - \sigma_t^2/(2\sigma_r^2)$.

$$\rho(r) = \int_0^{\sqrt{2\left[\Phi_\infty - \Phi(r)\right]}} \mathrm{d}v\; 2\pi\, v^2\; f_E\left(\Phi(r) + \frac{v^2}{2}\right) \int_0^{\pi} \mathrm{d}\eta\; \sin\eta\, (r\, v\, \sin\eta)^{-2\beta}$$
$$= 2\pi \int_{\Phi(r)}^{\Phi_\infty} \mathrm{d}E\; \left\{2[E - \Phi(r)]\right\}^{1/2-\beta}\; r^{-2\beta}\; f_E(E)\, \mathrm{B}(1/2, 1-\beta).$$

This is very similar to the expression in the isotropic case, and we proceed in the same way: first express $\rho(r)\, r^{2\beta}$ as a function of Φ, then take as many derivatives w.r.t. Φ as needed to put $[E - \Phi]^a$ in the denominator with $0 \le a < 1$, and then applying the Abel integral inversion.
For half-integer values of β, there is no fractional integration at all:
e.g., $\beta = 1/2 \;\Rightarrow\; f_E(E) = -\dfrac{1}{2\pi^2}\dfrac{\mathrm{d}(\rho\, r)}{\mathrm{d}\Phi}$, $\quad \beta = -1/2 \;\Rightarrow\; f_E(E) = \dfrac{1}{2\pi^2}\dfrac{\mathrm{d}^2(\rho/r)}{\mathrm{d}\Phi^2}$.

Anisotropic spherical systems

Another common approach [Osipkov 1979; Merritt 1985] is to consider DFs of the form $f(E,L) = f_Q(Q),\;\; Q \equiv E + L^2/(2r_a^2),$ where r_a is the "anisotropy radius".

$$\rho(r) = \int_0^\pi \mathrm{d}\eta\;\sin\eta \int_0^{\sqrt{2\left[\Phi_\infty - \Phi(r)\right]}} \mathrm{d}v\; 2\pi\, v^2\; f_Q\left(\Phi(r) + \frac{v^2}{2}\left[1 + \frac{r^2}{r_a^2}\sin^2\eta\right]\right).$$

Changing the integration variable from v to Q in the inner integral, we get

$$\rho(r) = 2\pi \int_0^\pi \mathrm{d}\eta\;\sin\eta \int_{\Phi(r)}^{\Phi_\infty} \mathrm{d}Q\; f_Q(Q) \frac{\sqrt{2(Q-\Phi)}}{\left[1 + (r/r_a)^2\,\sin^2\eta\right]^{3/2}}.$$

After exchanging the order of integration, the inner integral is taken analytically,

$$\rho(r) = \frac{4\sqrt{2}\,\pi}{1 + (r/r_a)^2} \int_{\Phi(r)}^{\Phi_\infty} \mathrm{d}Q\; \sqrt{Q - \Phi}\; f(Q).$$

This is again very similar to the isotropic case – we need to express $\left[1+(r/r_a)^2\right]\rho(r)$ as a function of Φ, then differentiate both sides w.r.t. Φ, and apply the Abel inversion formula.

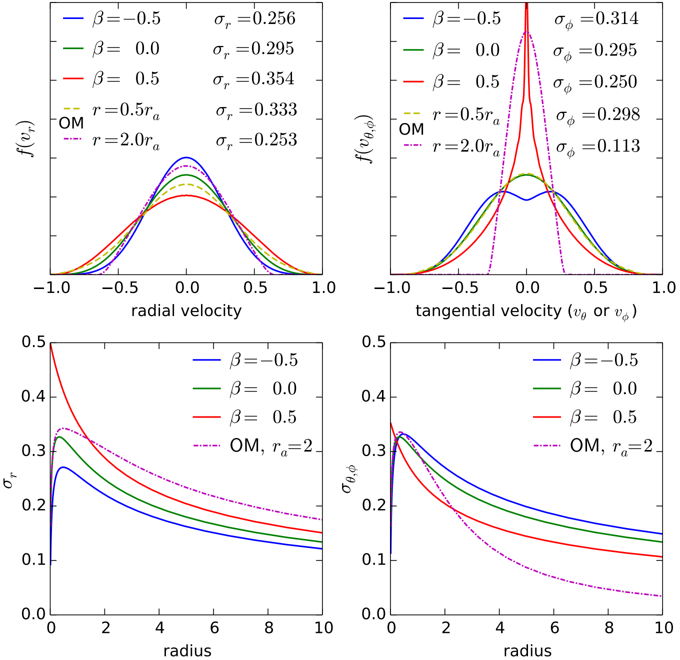
These models have radially varying velocity anisotropy profiles $\beta(r) = r^2/(r^2 + r_a^2)$, which change from isotropic (0) at $r \ll r_a$ to completely radial (1) at $r \gg r_a$.

Anisotropic spherical systems

Velocity distributions of anisotropic DF models may be quite strongly non-gaussian!

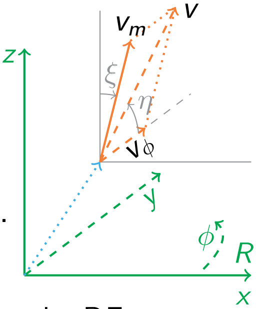
In addition to constant-β and Osipkov–Merritt models (or a combination of both using the same Abel inversion technique [Cuddeford 1991]), there are various other DF forms that permit an inversion $\rho, \Phi \;\Rightarrow\; f(E, L)$ [Gerhard 1991; Saha 1992; Wojtak+2008].

The opposite approach $f(E, L) \Rightarrow \rho, \Phi$ also produces anisotropic generalizations, e.g., of King models [Michie 1963; Gieles&Zocchi 2015].



Distribution function in axisymmetric systems

With two classical integrals of motion, we can construct DFs of the form $f(E, L_z)$.
Define the velocity components in the cylindrical coordinates as follows: azimuthal $v_\phi = v\,\cos\eta$ and meridional $v_m = v\,\sin\eta$, $\eta \in [0..\pi]$, with v_m further split into $v_R = v_m\,\sin\xi$ and $v_z = v_m\,\cos\xi$, $\xi \in [0..2\pi)$.
Then the density is computed from the DF as



$$\rho(R,z) = \int_0^{v_\mathrm{escape}} \mathrm{d}v\; 2\pi\, v^2 \int_0^\pi \mathrm{d}\eta\;\sin\eta\; f\left(\Phi(R,z) + \tfrac{1}{2}v^2,\; R\,v\cos\eta\right).$$

Since ρ and Φ are also functions of two variables R, z, we might expect that there is a unique correspondence between the DF and the potential–density pair. This is indeed the case, but the derivation of $f(E, L_z)$ from ρ, Φ (analogue of the Eddington inversion formula) is a rather cumbersome task involving contour integrals in the complex plane [Hunter&Qian 1993].

Moreover, these two-integral models necessarily have equal velocity dispersions in the two meridional plane components σ_R, σ_z, which is rather unrealistic (in the Milky Way disc, $\sigma_R/\sigma_z \simeq 1.5-2$). Thus we need three-integral models $f(E, L_z, I_3)$.

Distribution functions for axisymmetric discs

Most stars in disc galaxies move on close-to-circular orbits, which can be considered in the epicyclic approximation, i.e., separable motion in $\Phi_{\rm eff}(R)$ and $\Phi_z(z)$.

Consider first an infinitely cold system with all stars on *exactly* circular orbits in the $z=0$ plane. Then the DF for the 2d planar motion is

$f(E, L_z) = S(L_z)\,\delta\big[E - E_{\rm circ}(L_z)\big]$,

where $E_{\rm circ}(L_z) \equiv \Phi(R_{\rm circ}) + \dfrac{R_{\rm circ}}{2}\dfrac{\partial\Phi}{\partial R}\Big|_{R=R_{\rm circ}}$ and $R_{\rm circ}(L_z)$ is the root of $R^3\,\dfrac{\partial\Phi}{\partial R} = L_z$.

The corresponding surface density is $\Sigma(R) = \dfrac{\pi\,\kappa}{\Omega}\,S(\Omega^2\,R)$, where κ and Ω are the radial and azimuthal epicyclic frequencies evaluated at the radius R.

To produce "warm" discs, we replace δ-function by a Gaussian and add a vertical dimension with a third [approximate] integral of motion $E_z \equiv \Phi(R,z) - \Phi(R,0) + \frac{1}{2}v_z^2$:

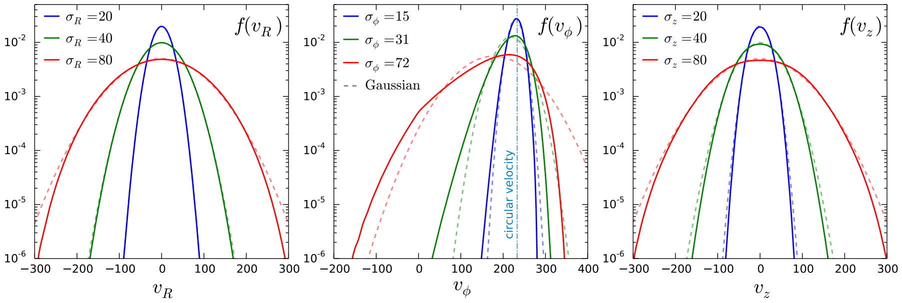
$$f(E_R, L_z, E_z) = S(L_z)\,\exp\left[-\frac{E_R - E_{\rm circ}(L_z)}{\sigma_R^2(L_z)} - \frac{E_z}{\sigma_z^2(L_z)}\right].$$

[Shu 1969; see also Dehnen 1999 for a similar recipe]

The three adjustable functions S, σ_R and σ_z control the surface density and radial and vertical velocity dispersions (and match them quite well when $\sigma \ll v_{\rm circ}$).

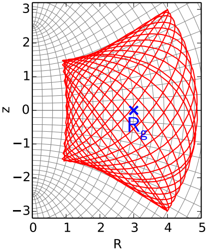
Distribution functions for warm axisymmetric discs

Increasing the velocity dispersion in the disc DF keeps $f(v_R)$ and $f(v_z)$ quite close to Gaussian distributions, but produces increasingly asymmetrically shaped $f(v_\phi)$, which are peaked at a different v_ϕ than the local circular velocity $v_{\rm circ}$ – a phenomenon known as "asymmetric drift". The high-velocity wing $v_\phi > v_{\rm circ}$ is produced by stars near their pericentres (i.e., coming from larger radii), and conversely, $v_\phi < v_{\rm circ}$ represent stars coming from smaller radii near their apocentres. The latter population is typically more numerous since both surface density and velocity dispersion decrease with radius, thus normally $\overline{v_\phi} < v_{\rm circ}$; however, for some populations these trends may be reversed.



Action-based distribution functions

The epicyclic approximation is only valid for cold enough orbits (low eccentricity and inclination). For larger excursions in R and z the motion is not quite separable in the cylindrical coordinates, but is still regular for most orbits, hence conserves 3 integrals, which may be chosen to be actions (usually computed in the Stäckel approximation).



Action-based disc DFs are constructed by replacing $E_R = \Phi(R,0) - \Phi(R_\mathrm{g}) + \frac{1}{2}v_R^2$ with $\kappa\, J_R$ and $E_z \equiv \Phi(R_\mathrm{g}, z) - \Phi(R_\mathrm{g}, 0) + \frac{1}{2}v_z^2$ with $\nu\, J_z$:

$$f(\mathbf{J}) = \frac{\Sigma\,\Omega}{2\pi^2\,\kappa^2} \times \frac{\kappa}{\sigma_r^2}\exp\left[-\frac{\kappa\,J_r}{\sigma_r^2}\right] \times \frac{\nu}{\sigma_z^2}\exp\left[-\frac{\nu\,J_z}{\sigma_z^2}\right] \times \begin{cases} 1 & \text{if } J_\phi \ge 0, \\ \exp\left(\dfrac{2\Omega\,J_\phi}{\sigma_r^2}\right) & \text{if } J_\phi < 0, \end{cases}$$

where Σ and $\sigma_{r,z}$ are adjustable functions of R_g controlling the surface density and velocity dispersion profiles, κ, ν, Ω are epicyclic frequencies (also functions of R_g), and the guiding-centre radius R_g is a function of [mainly] J_ϕ [Binney&McMillan 2012].

Spheroidal models can be constructed with “double-power-law” action-based DFs

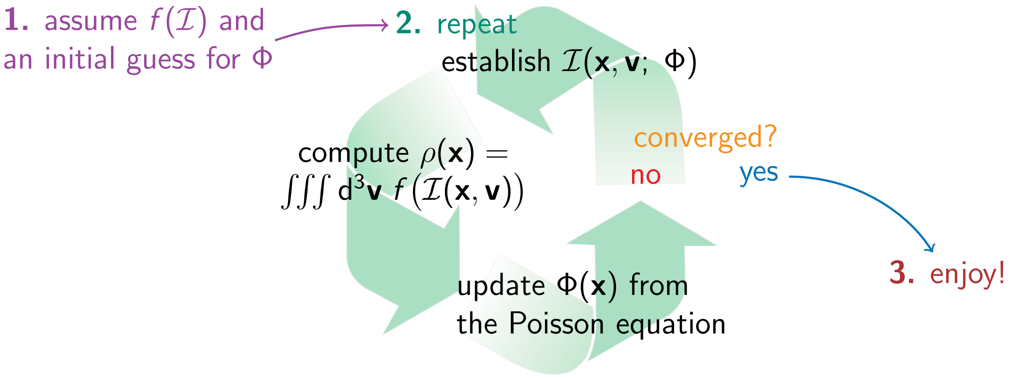
$$f(\mathbf{J}) = A\left[1 + \left(J_0/h(\mathbf{J})\right)^\eta\right]^{\Gamma/\eta}\left[1 + \left(g(\mathbf{J})/J_0\right)^\eta\right]^{(\Gamma-\mathrm{B})/\eta}$$ [Posti+ 2015],

where $h(\mathbf{J})$, $g(\mathbf{J})$ are some linear combinations of actions at small and large radii.

Self-consistent distribution function-based models

A general DF $f(\mathcal{I})$ is specified in terms of integrals of motion in the given potential $\mathcal{I}(\mathbf{x}, \mathbf{v};\ \Phi)$. To compute the density $\rho(\mathbf{x})$ generated by this DF, one needs to know $\Phi(\mathbf{x})$, but in the gravitationally self-consistent case, Φ is determined by ρ via the Poisson equation – thus we have a circular dependency.

Such models are constructed by the iterative approach [Prendergast & Tomer 1975; Rowley 1988; Kuijken & Dubinski 1995; Widrow+ 2005], which works best for action-based DFs [Binney 2014; Piffl+ 2015; Sanders & Evans 2016; Vasiliev 2019]:



Summary

- Steady-state distribution functions depend only on the integrals of motion in the given potential.
- At least for spherical and axisymmetric systems, there is a considerable freedom in constructing a DF that produces a given density profile.
- Therefore, the path from ρ, Φ to f is not unique – one must choose a functional form of the DF first, and then use some sort of "inversion formula" (e.g., Eddington).
- The path from f to a *self-consistent* ρ, Φ pair isn't easy either; in spherical systems it reduces to solving a 2nd order ODE, and more generally, requires an iterative solution.
- Bottom line: it's complicated but doable.