Project Summary

June 4, 2025

1 Where I stood

I developed an easy Likelihood Ratio Test (LRT) to test wheter, inside a certain mass bin, the ratio of the number of events $\alpha = \dfrac{N_1}{N_2}$ between two different measurments is different than some threshold α_0, with a certain confidence level c.l. (let's say 95%).

$$N_1, N_2 \longrightarrow N_2, \alpha \longrightarrow \lambda_{\text{LR}} \text{ (Test statistic; asymtotically } \chi^2 \text{ distributed)} \longrightarrow \text{p-value} \longrightarrow \text{is } \alpha \neq \alpha_0?$$

Reverting the test, gives what α has to be measured to conclude that $\alpha \neq \alpha_0$ with a certain confidence level. This implies solving for α the following equation:

$$\xi \alpha_0^{\overline{1+\alpha}} - \alpha_0 - 1 = 0$$

where

$$\xi = \left[\exp\left(\frac{\lambda_{\text{LR}}}{2N_2}\right)\left(1+\frac{1}{\alpha}\right)^{\alpha}(1+\alpha)\right]^{\frac{1}{1+\alpha}}$$

The solutions are the intercepts of the curve with the x axis here: `https://www.desmos.com/calculator/shw88qougk`.

2 Where I am now

I tried to relate the number of events inside a mass bin to the actual merger rate. After some research, I figured out that whatever I was going to write should have included the spacetime volume sensitivity of the detector (Equation 15 here `https://iopscience.iop.org/article/10.3847/2041-8205/833/1/L1/pdf`). That's how reformulated the problem:

- Focus on a certain mass bin Δm_i
- The merger rate per unit volume is $\dfrac{d^2N}{dt_s dV_c}$, where dt_s is the infinitesimal time interval in the source frame, and dV_c and is the infinitesimal volume associated to a redshift interval dz
- Write the merger rate in the mass bin Δm_i as $R(\Delta m_i) = \int_{\Delta m_i} dm \int d\theta \dfrac{d^4N}{dm d\theta dt_s dV_c}$, where θ is the parameter space of an event
- Change variable: $dV_c = \dfrac{dV_c}{dz}dz$; $dt_s = \dfrac{dt_s}{dt_o}dt_o$
- $\dfrac{d^4N}{dm d\theta dt_s dV_c} = \dfrac{d^4N}{dm d\theta dt_o dz}\dfrac{dz}{dV_c}\dfrac{dt_o}{dt_s}$, where $\dfrac{dt_o}{dt_s} = 1+z$
- Introduce the detection probability $p_{\text{det}}(\theta, m, z)$ such that $d^4N_{\text{obs}} = p_{\text{det}}(\theta, m, z)d^4N$
- We have: $R(\Delta m_i) = \int_{\Delta m_i} dm \int d\theta \int_0^{z_{\max,i}} dz \dfrac{d^4N_{\text{obs}}}{dm d\theta dt_o dz}\dfrac{dz}{dV_c}\dfrac{1+z}{p_{\text{det}}(\theta, m, z)}$, where $z_{\max,i}$ is the maximum (observed/observable) redshift in the bin Δm_i
- If we are counting events: $R(\Delta m_i) = \dfrac{1}{T}\sum_{j=1}^{N_i} \dfrac{dz}{dV_c}(z_j)\dfrac{1+z_j}{p_{\text{det}}(\theta_j, m_j, z_j)}$, where N_i is the number of events observed inside the bin Δm_i , and T is the total observation time
- **Define** $C(\theta, m, z) = T\int_0^{z_{\max,i}} dz \dfrac{dV_c}{dz}\dfrac{p_{\text{det}}(\theta, m, z)}{1+z}$
- For the j-th event: $C(\theta_j, m_j, z_j) = T\left[\dfrac{dV_c}{dz}(z_j)\dfrac{p_{\text{det}}(\theta_j, m_j, z_j)}{1+z_j}\right]$
- **Then**: $R(\Delta m_i) = \sum_{j=1}^{N_i} \dfrac{1}{C(\theta_j, m_j, z_j)}$.
- If we have the posterior of the j-th event, we can rewrite: $R(\Delta m_i) \simeq \sum_{j=1}^{N_i} \dfrac{1}{\langle C(\theta, m, z)\rangle_j}$, where $\langle C(\theta, m, z)\rangle_j = \int d\theta C(\theta, m, z)p(\theta|\text{data}_j) \simeq$

$\frac{1}{M_j}\sum_{k=1}^{M_j} C(\theta_{j,k}, m_{j,k}, z_{j,k})$ and M_j is the number of samples drawn from the posterior of the j-th event.

- **Finally**: $R(\Delta m_i) = N_i \left\langle \frac{1}{C(\theta, m, z)} \right\rangle_i$, where $\left\langle \frac{1}{C(\theta, m, z)} \right\rangle_i = \frac{1}{N_i}\sum_{j=1}^{N_i} \frac{1}{\langle C(\theta, m, z)\rangle_j}$

In my reformulation nearly all the physics should be encoded in the detection probability $p_{\rm det}(\theta, m, z)$.

2.1 How to extend the LRT to the merger rate

- Suppose two measurments of the merger rate in a certain mass bin Δm_i: $R_1(\Delta m_i)$ and $R_2(\Delta m_i)$.
- Define $A_i = \frac{R_1(\Delta m_i)}{R_2(\Delta m_i)} = \alpha_i a_i$, where

  $\alpha_i = \frac{N_{1,i}}{N_{2,i}}$ and $a_i = \frac{\left\langle \frac{1}{C(\theta, m, z)} \right\rangle_{1,i}}{\left\langle \frac{1}{C(\theta, m, z)} \right\rangle_{2,i}}$
- I want to test whether $A_i \neq A_0$ with a certain confidence level $\Longrightarrow$ test wheter $\alpha_i \neq \alpha_0 = \frac{A_0}{a_i}$. This approach should work because after the measurments, a_i is just a number.

Under this light, the test functional form is the same but for a change of variable. Once we fix a_i, I can revert the test like before to know what I have to measure to conclude that $A_i \neq A_0$ with a certain confidence level. The result are the intercepts of the red curve with the x axis here: `https://www.desmos.com/calculator/defltq9len` (different link than above).

My general understanding of the problem is that a_i is some proxy of the sensitivity of the detector in the mass bin Δm_i. Therefore the idea is that, for a fixed value of α (the countings rate I'm going to get between two different measurments), I can predict what value of a_i I will need to say that $A \neq A_0$. This information might be translated into some requirment on the detection probability. Following the same logic used for α, the result for a are the intercepts of the green curve with the x axis at the same link above here.

3 Doubts

- Maybe my reformulation is wrong. I hope not that much
- Not 100% sure that treating a like a number works as I think. It depends on the number of events after all. However should be fine if it is obtainable in closed form from the data (I think).

I have some more other plots, but the math is fully covered in the links above.