Numerical Relativity 2023-2024

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Homework 1

1 Advection Equation [max 2 pages]

Given the advection equation in 1D $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$ build a numerical code to solve it on a grid with extent $x \in [0, 10]$ and with initial conditions given by

$$u(x, t = 0) = \exp[-(x - x_0)^2] \ , \tag{1}$$

with $x_0 = 5$. Solve the equation using the following schemes:

1. FTCS
2. Lax-Friedrichs
3. Leapfrog
4. Lax-Wendroff

Use Courant factor $c_f = 0.5$ and compare the results obtained with the different methods, paying attention to their stability and dissipation properties. Plot $u(x, t)$ at different times (including $t = 0$ and $t = 20$) and the evolution of the L2-norm of $u(x, t)$. Use at least $J = 101$ points in the x direction, so that the spacing Δx is at least $0.1 = 10/(J - 1)$, and terminate your simulation at $t = 20$. Use periodic boundary conditions. Modify the number of points and/or the Courant factor c_f to check how your results change.

1.1 Optional

Replace the periodic boundary condition with outflow boundary conditions and run using two stable methods selected from the ones used before. Plot the L2-norm for the two methods using a logarithmic scale in the y axis.

2 Step Function [max 2 pages]

Solve the advection equation, but using as initial data a step function instead of a Gaussian profile: $u(x, t = 0) = 1$ for $x \in [4, 6]$ and $u(x, t = 0) = 0$ in the rest of the domain. Compare the results obtained when using the Lax-Friedrichs and the Lax-Wendroff schemes. Use $c_f = 0.5$, $J = 101$, and terminate the evolution at $t = 20$. Plot $u(x, t)$ at different times and the evolution of the L2-norm of $u(x, t)$. Check what happens when changing the number of points and/or the Courant factor.

3 Burgers' Equation [max 2 pages]

Given the Burgers' equation in 1D $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ build a numerical code to solve it on a grid with extent $x \in [0, 10]$ and with initial conditions given by

$$u(x, t = 0) = 10\,\mathrm{e}^{-(x-x_0)^2}\ , \tag{2}$$

with $x_0 = 5$. Compute the solution using both the flux-conservative and the non flux-conservative versions of the upwind scheme. Use Courant factor $c_f = 0.5$, a grid with at least $J = 101$ points with periodic boundary conditions, and terminate the evolution at $t = 0.5$. Compare the solutions computed with the two different methods by plotting $u(x, t)$ at different times (including $t = 0.5$). What happens when you increase the resolution?

Figures do not count toward the maximum number of page limit. Use an A4 page format and a font size of at least 11.

Note: in order to get admitted to the oral exam you are requested to submit the answers to all these questions as a single pdf document via email at least two weeks before the oral exam. Include the source codes used to solve the exercises at the end of the pdf document or provide a link to a Google drive folder containing them.