# Numerical Relativity Homework 2

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## 1 Sod Shock Tube Problem

The Sod shock tube problem consists of a one-dimensional Riemann problem with initial discontinuities in density and pressure. The time evolution of the system can be computed by solving the Euler equations. Since the solution to this problem can be computed exactly, it is useful for testing the accuracy of numerical codes. In this exercise, we solve the Sod problem with the Einstein Toolkit (ETK) using different resolutions and compare the results with the exact solution.

The solution to the problem is described by three characteristics, each related to the propagation speed of the fluid in different regions. These can be associated with either a rarefaction wave, a shock wave, or a contact discontinuity. Specifically, the pressure and the velocity of the fluid develop a rarefaction wave and a shock wave, while the density also develops a contact discontinuity. The exact solution along with the initial conditions is shown in Figure 1.

For the numerical solutions, the HLLE Riemann solver has been used, along with the Minmod slope limiter. The domain extends in the range $[-0.5, 0.5]$, and the evolution proceeds up to time $t = 0.4$. The grid spacings used are $\{0.005, 0.0025, 0.00125, 0.000625\}$, corresponding respectively to $\{200, 400, 800, 1600\}$ grid points.

### 1.1 Highest Resolution: 1600 points

{sec:highres1600}

We will use the results obtained with the highest resolution to showcase how the numerical solutions look. Figure 2 shows some snapshots of the profiles of density, pressure, and velocity of the fluid, including the initial and the final ones. As can be seen, S and CD waves travel in opposite directions with respect to the R waves. We point out that the fact that the initial profile in Figure 2 doesn't perfectly resemble a step (as it should) is due to the way the numerical results are interpolated on the chosen uniform grid $[-0.45, 0.45]$. Interpolation effects are also present in the final profile, though they are less evident. Figure 3 compares the interpolated initial data with the raw initial data actually used by the ETK.

The numerical solution captures the discontinuities in the physical variables observed for the Sod problem. Accuracy depends on the number of grid points, but slight smoothing of the discontinuities seems to be a characteristic feature of the HLLE Riemann solver. This also applies to the boundaries of the rarefaction wave, where the solution slope changes rapidly. This behavior isn't related to the interpolation of the data on the plot's grid, as shown in the next section (1.2).

## 1.2 Resolutions Comparison

{sec:rescomp}

Now we examine the results obtained with different resolutions and compare them to the exact solution. To avoid visualization artifacts, we will use the raw data from the ETK and not the interpolated ones. Figures 4, 5, and 6 show the final iteration of the evolution of density, pressure, and velocity respectively.

As mentioned in the previous section (1.1), the smoothing of the discontinuities appears to be a feature of the Riemann solver rather than of the chosen resolution. At the same time, as expected, it is indeed true that a finer spacing of the numerical grid reduces the observed smoothing. Moreover, we point out that there are no evident numerical oscillations near the discontinuities for any of the resolutions used.

Overall, considering its simplicity and low computational cost, the HLLE solver showed good handling of the Sod problem resolution. However, simulations where high shock resolution is needed, might require more advanced and computationally expensive Riemann solvers.

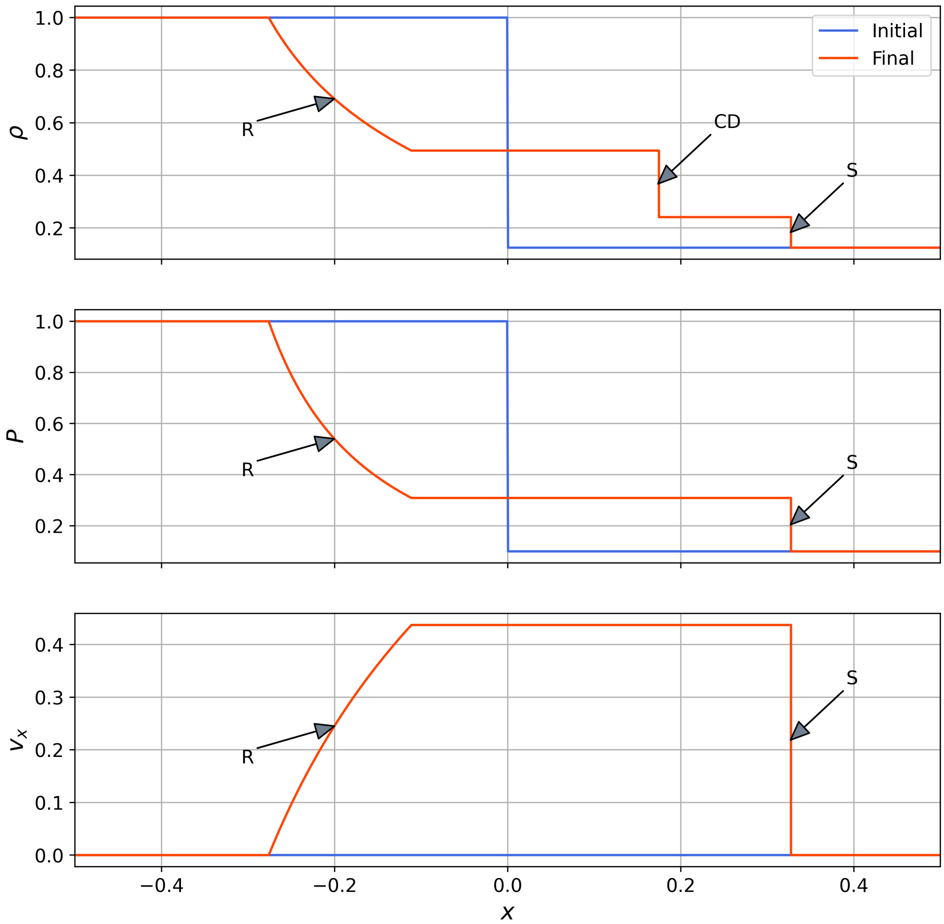


Figure 1: Exact Solution; Initial and final conditions; The arrows points at the different types of waves: R (Rarefaction), S (Shock) and CD (Contact Discontinuity).

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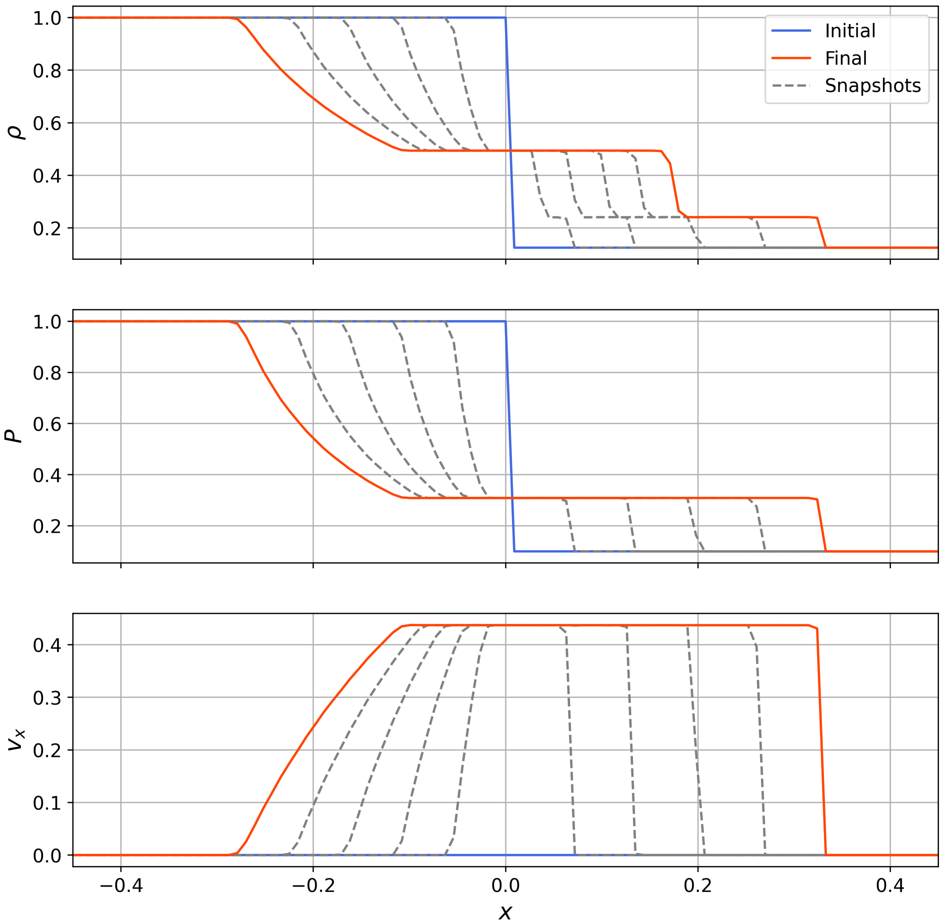


Figure 2: Numerical Solution; 1600 points; Snapshots.

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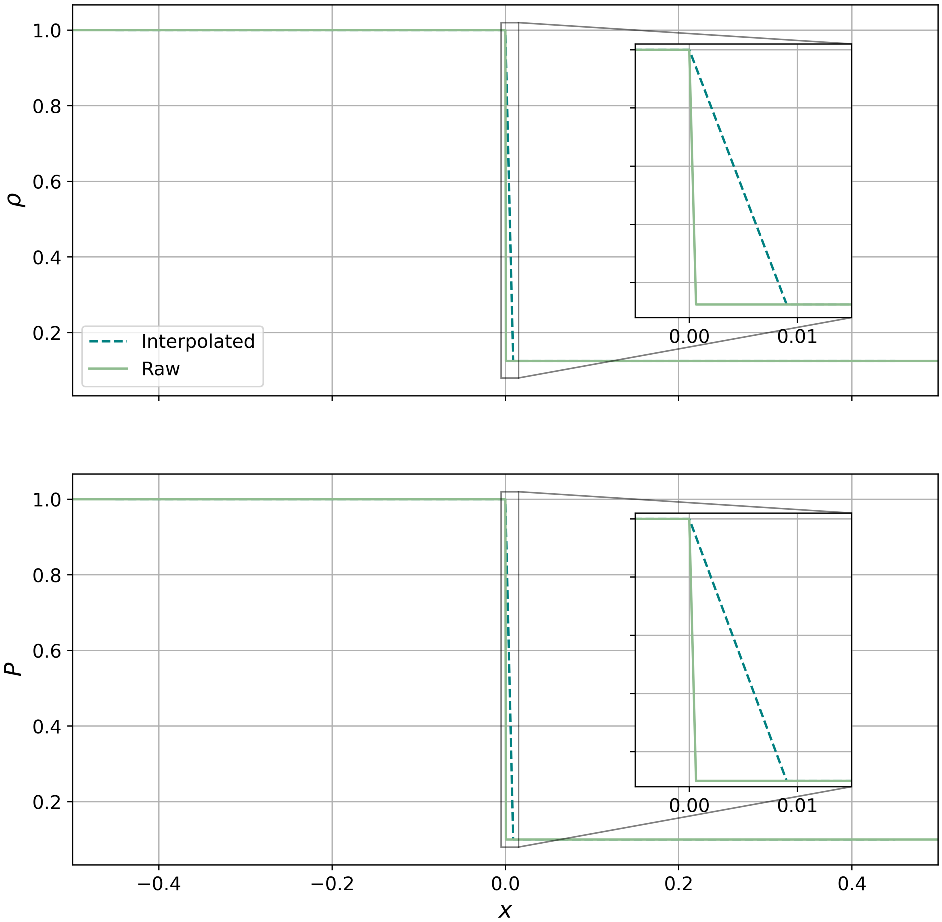


Figure 3: Initial data; 1600 points; Raw and interpolated.

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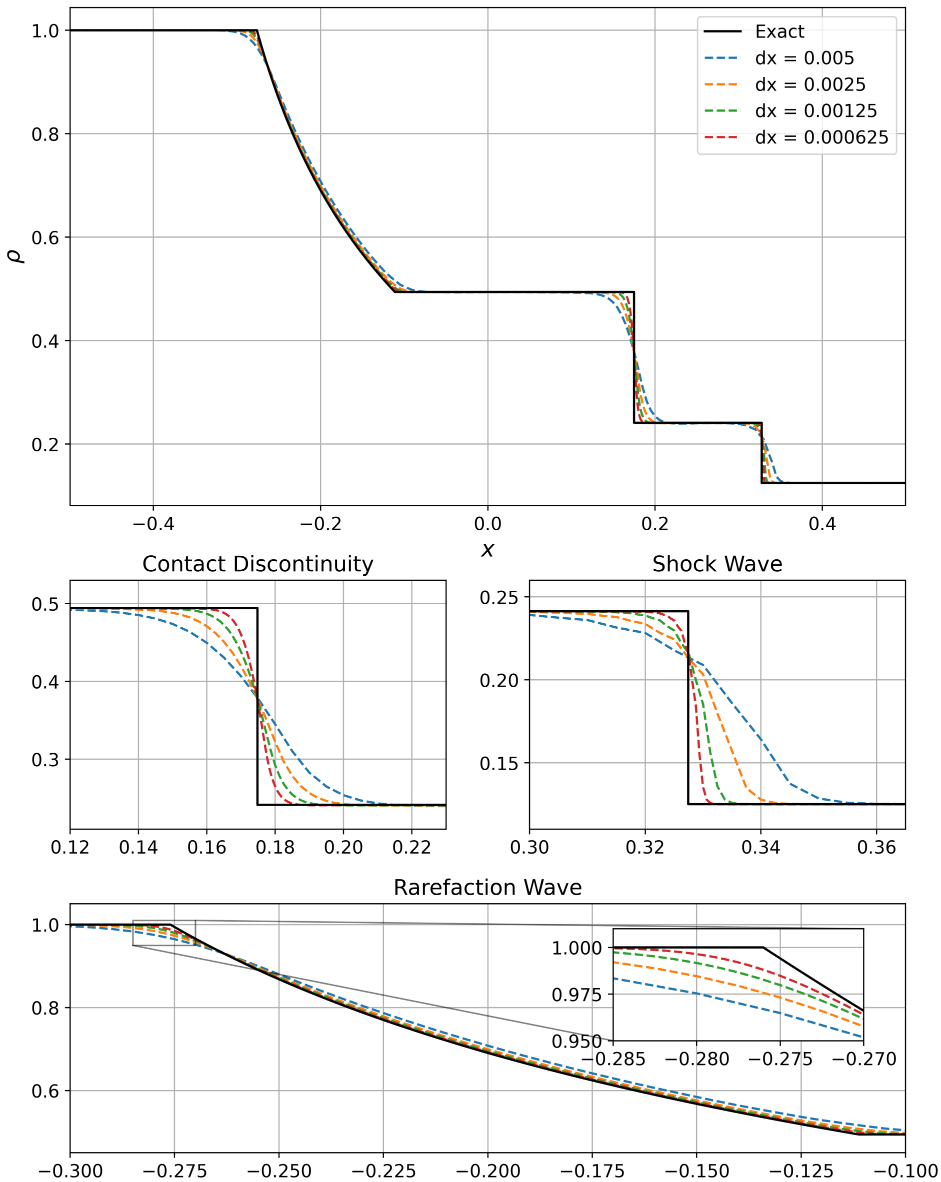


Figure 4: Density; Final data; Resolution comparison. {fig:rho_final_rescom

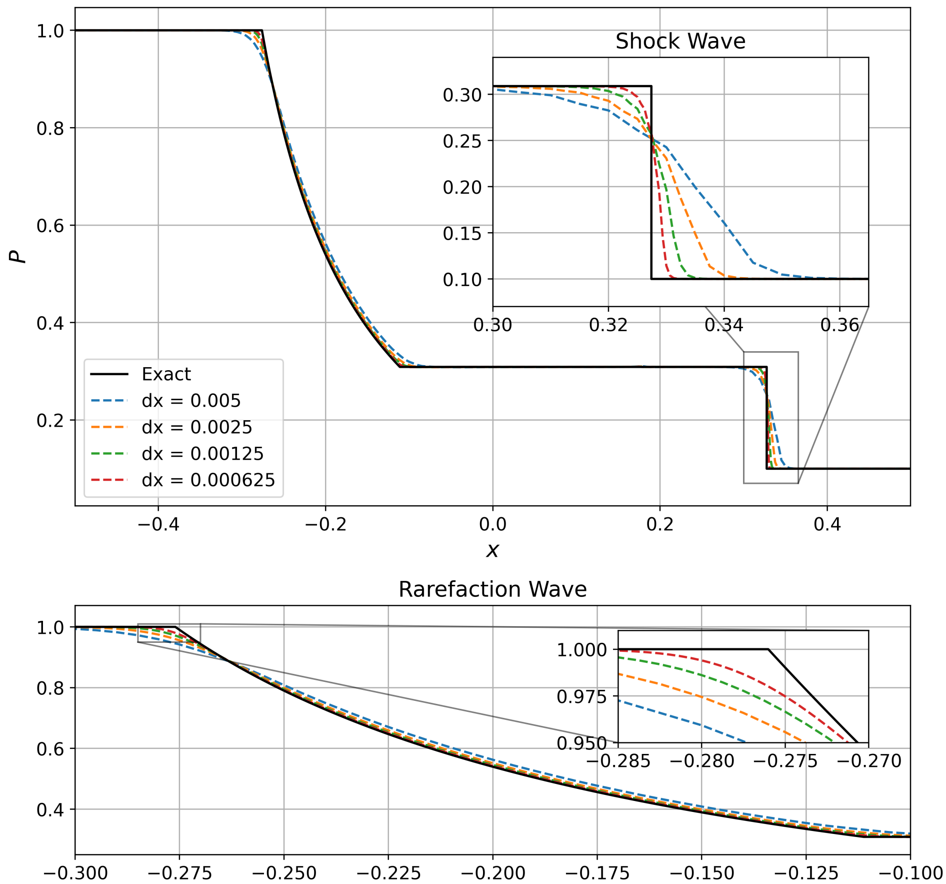


Figure 5: Pressure; Final data; Resolution comparison.

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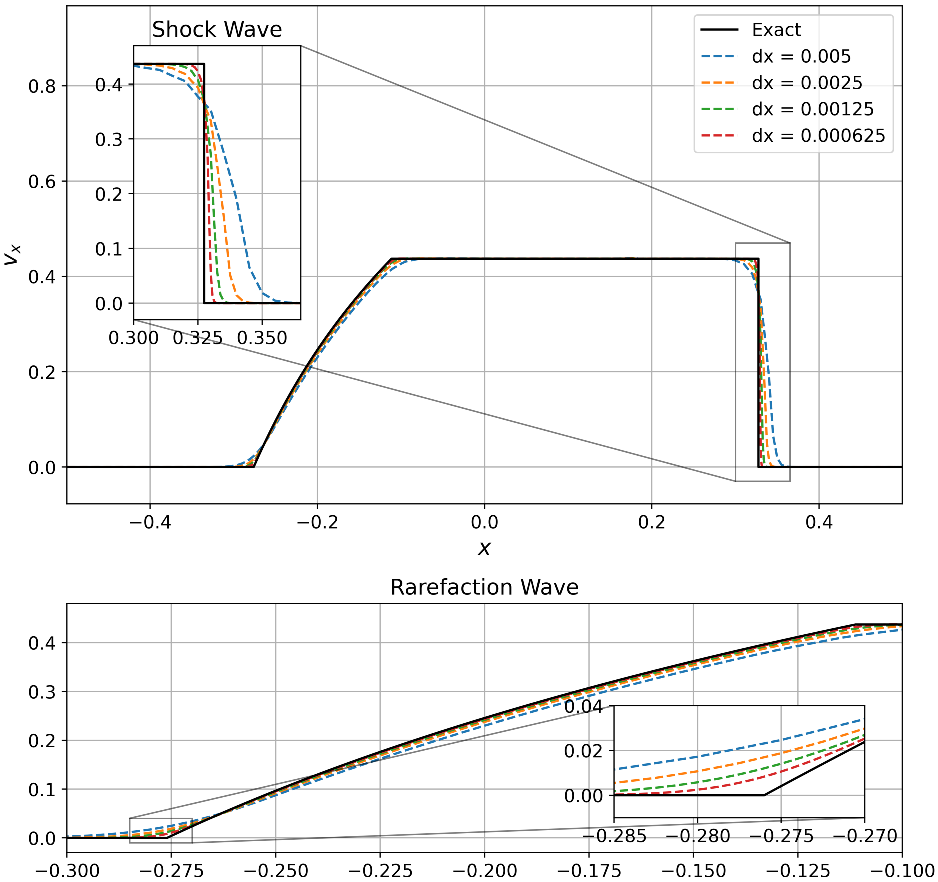


Figure 6: Velocity; Final data; Resolution comparison.

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# 2 TOV Evolution