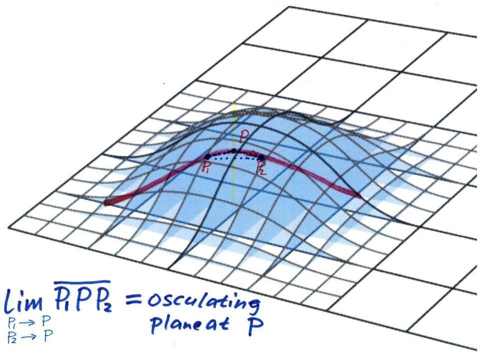
Geodetic lines & basic geodetic problems

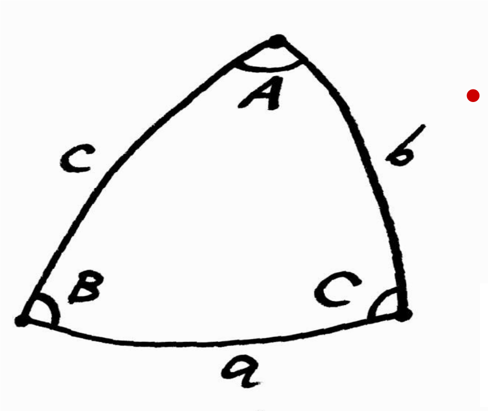
- Definitions: geodetic lines, geodetic triangles, geodetic azimuth, normal sections vs geodetic lines
- Three basic equations of geodetic lines
- Clairaut's equation
- Basic geodetic problems
  on the plane, on the sphere, on the reference ellipsoid
- Helmert-type diffeencial formulas of geodetic lines
- Practical projects related to geodetic lines

Definition of geodetic lines

- Between two points on a curved surface, a curve which has the shortest length is called a **geodesic** or **geodetic line**
- Every point P on a curve ℓ situated on a curved surface has an osculating plane
- If the osculating plane of P always contains the ellipsoidal normal at P, then ℓ is a geodetic line



Geodetic triangle and excess



- 3 geodetic lines form a *geodetic triangle*
- *geodetic excess* :

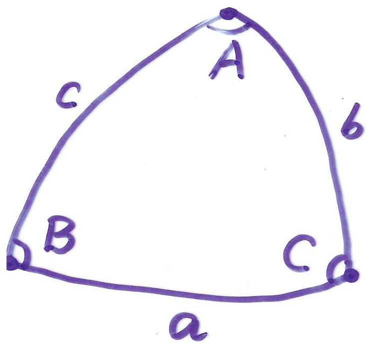
$\varepsilon = A + B + C - 180°$

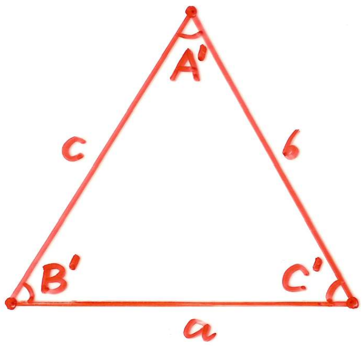
$$\varepsilon = \iint_T \frac{dT}{R_m^2} \approx \frac{T}{R_m^2}$$

$$R_m = \sqrt{MN} = \frac{a\sqrt{1-e^2}}{1-e^2\sin^2\phi}$$

$$\phi_m = \frac{\phi_A + \phi_B + \phi_C}{3}$$

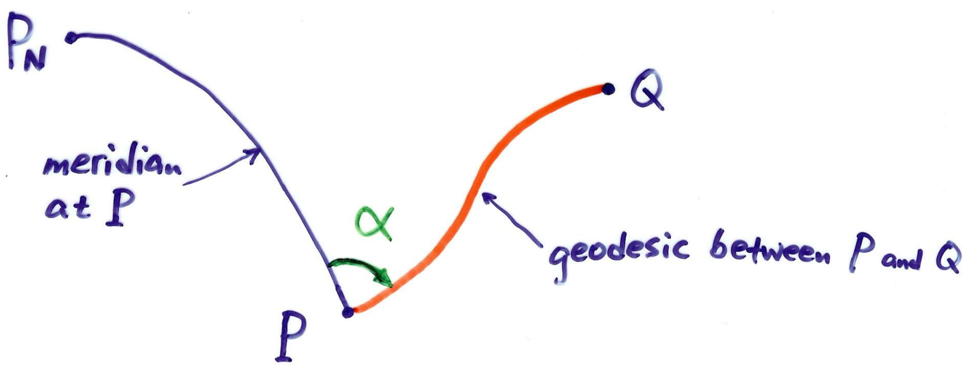
Approximation of a geodetic triangle by a planar triangle



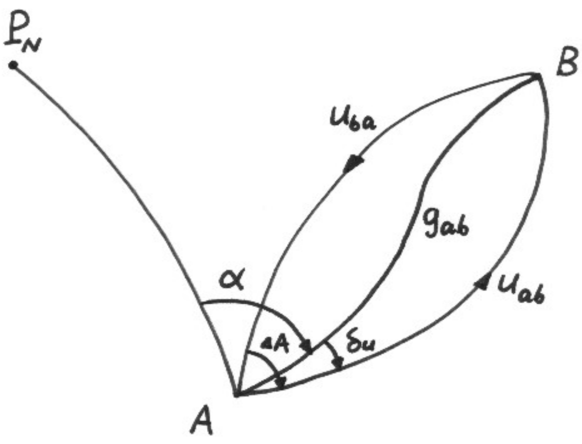


$$\boxed{A' = A - \frac{\varepsilon}{3}, \quad B' = B - \frac{\varepsilon}{3}, \quad C' = C - \frac{\varepsilon}{3}}$$

Geodetic azimuth

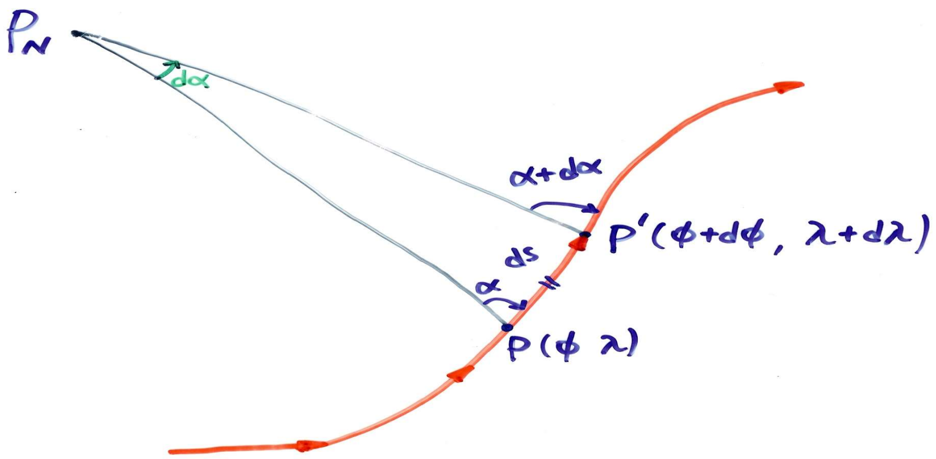


Geodetic line & normal sections

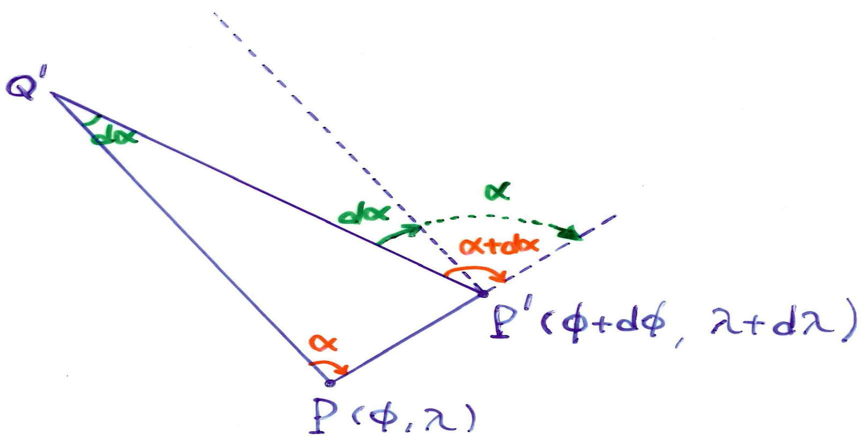


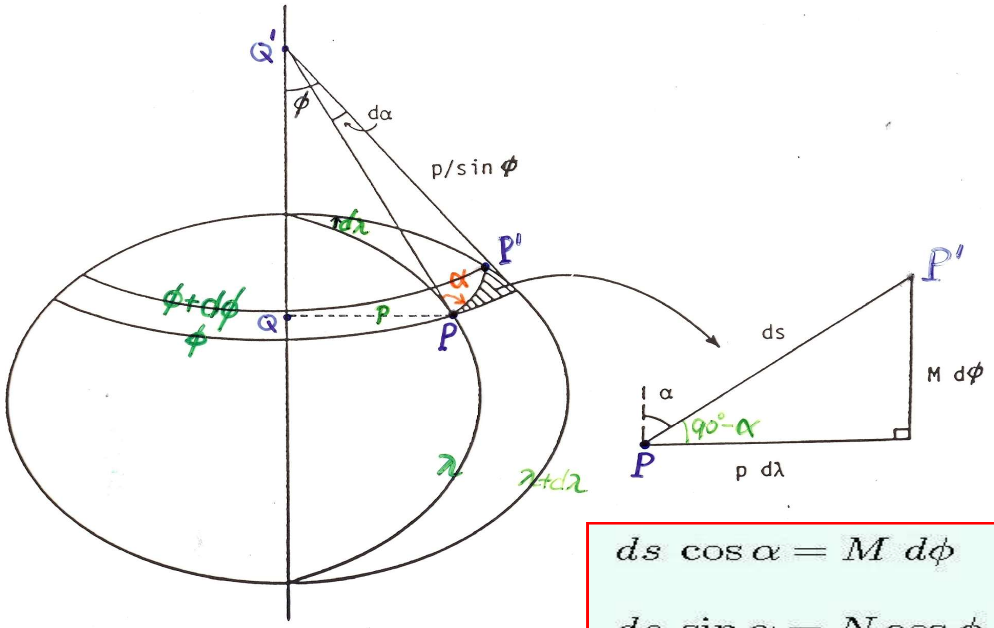
- Ellipsoidal normal,
  Normal plan, normal section
- Normal sections: u_{ab} u_{ba}
- Geodetic line: g_{ab}
- Geodetic line g_{ab} does not coincide with normal sections (u_{ab}, u_{ba})

Differential change along a geodetic line



Azimuth change along a geodetic line





$$ds \ \cos\alpha = M \ d\phi$$

$$ds \ \sin\alpha = N\cos\phi \ d\lambda$$

Basic equations of Geodetic lines

$$ds \ \cos\alpha = M \ d\phi$$

$$ds \ \sin\alpha = N\cos\phi \ d\lambda$$

$$d\alpha = \sin\phi \ d\lambda$$

$$M = \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}} = \frac{a(1-e^2)}{W^3}$$

$$N = \frac{a}{W} = \frac{a}{\sqrt{1-e^2\sin^2\phi}}$$

$$W = \sqrt{1-e^2\sin^2\phi}$$

$$p = N \cdot \cos\phi$$

$$\frac{dp}{d\phi} = -M \cdot \sin\phi$$

Clairaut's equation of geodetic lines

$$p \bullet \sin\alpha = \text{constant} \qquad (p = N\cos\varphi)$$

- Clairaut's equation can be derived from the three basic differantial equations of geodetic lines
- Clairaut's equation shows how a geodetic line goes
- On a spherical surface, Clairaut's equation reduces to the sine theorem:

$$\cos\varphi \bullet \sin\alpha = \text{constant}$$

Example of *one* geodesic

givet: $\varphi_1 = 59° \; 20' \; N$ $\qquad \varphi_2 = 39° \; 55'$

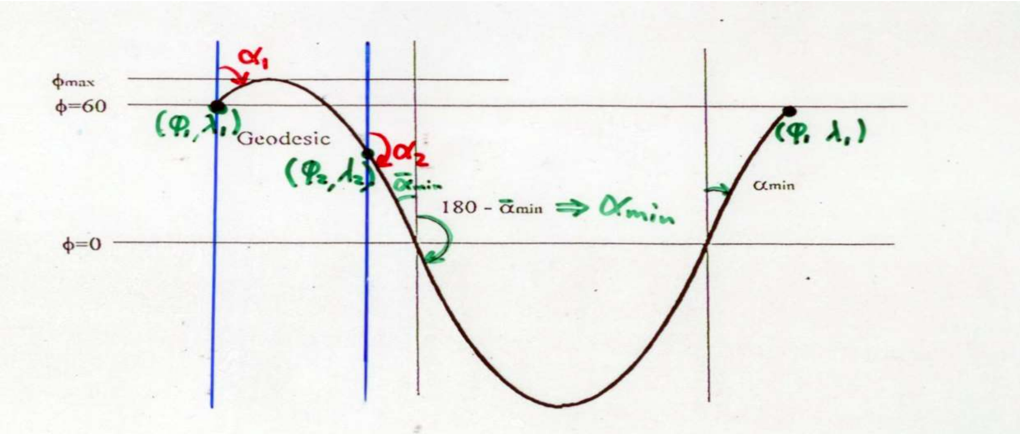
$\lambda_1 = 18° \; 5' \; E$

$\alpha_1 = 60° \; 51'$

sökt: $\alpha_2 = ?$

$\alpha_{min} = ?$

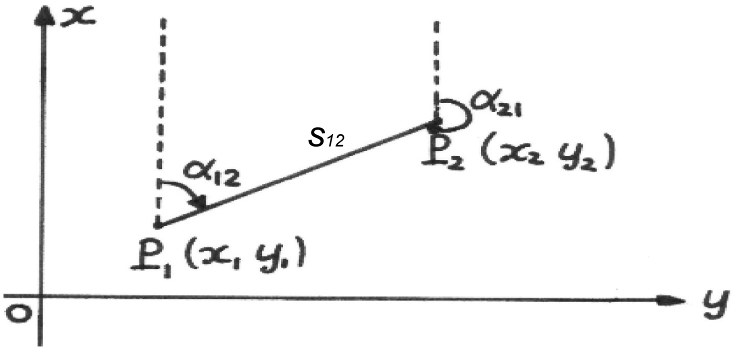
Example of *one* geodesic



Basic geodetic problems

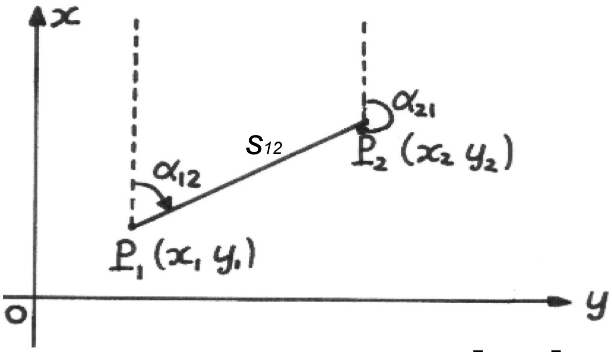
- Basic problems on the plane
- Basic problems on the sphere
- Basic problems on the reference ellipsoid
  - Outline of solutions
  - Gauss method of mean arguments
  - Bessl's method for long distances
  - Example: maritime boundary delimitation
  - Example: global triangulation on the ellipsoid

Basic problems on the plane



- Direct problem: Find P_2 when P_1 and α_{12}, s_{12} are known
- Inverse problem: Find α_{12}, α_{21}, s_{12} when P_1 , P_2 are known

Basic problems on the plane



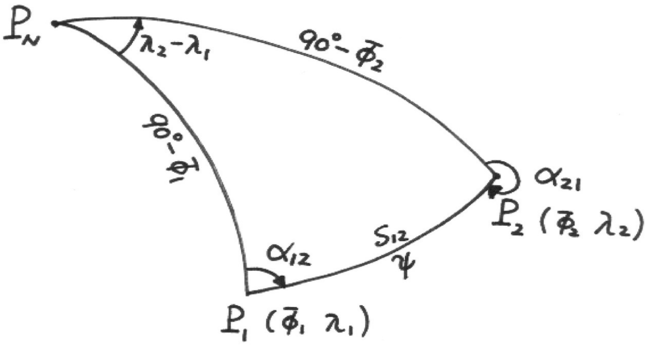
- Direct problem:

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} s_{12} \ \cos\alpha_{12} \\ s_{12} \ \sin\alpha_{12} \end{bmatrix}$$

- Inverse problem:

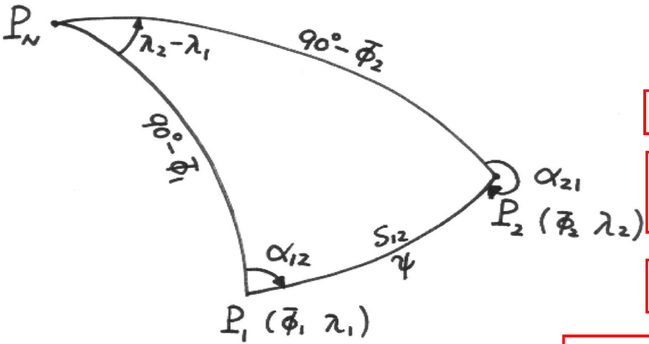
$$\begin{bmatrix} s_{12} \\ \alpha_{12} \\ \alpha_{21} \end{bmatrix} = \begin{bmatrix} \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ \arctan\{(y_2-y_1)/(x_2-x_1)\} \\ \alpha_{12}+180^0 \end{bmatrix}$$

Basic problems on the sphere



- Direct problem: Find P_2 when P_1 and α_{12}, s_{12} are known
- Inverse problem: Find α_{12}, α_{21}, s_{12} when P_1 , P_2 are known

Direct problem on the sphere



$$\psi = \frac{s_{12}}{R}$$

$$\boxed{sin\overline{\phi}_2 = \cos\psi\ \sin\overline{\phi}_1 + \sin\psi\ \cos\overline{\phi}_1\ \cos\alpha_{12}}$$

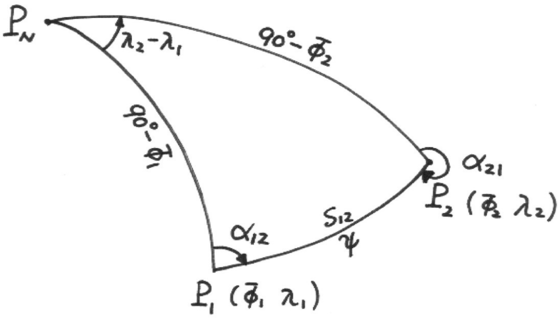
$$\boxed{\cos\Delta\lambda = \frac{\cos\overline{\phi}_1\ \sin\overline{\phi}_2 - \sin\psi\ \cos\alpha_{12}}{\sin\overline{\phi}_1\ \cos\overline{\phi}_2}}$$

$$\boxed{\lambda_2 = \lambda_1 \pm \Delta\lambda}$$

$$\boxed{\cos\alpha_{21} = \frac{\sin\overline{\phi}_1\ \cos\overline{\phi}_2 - \cos\overline{\phi}_1\ \sin\overline{\phi}_2\ \cos(\lambda_2-\lambda_1)}{\sin\psi}}$$

- If $\alpha_{12} < 180°$, then take the *plus* sign
- If $\alpha_{12} > 180°$, then take the *minus* sign

Inverse problem on the sphere



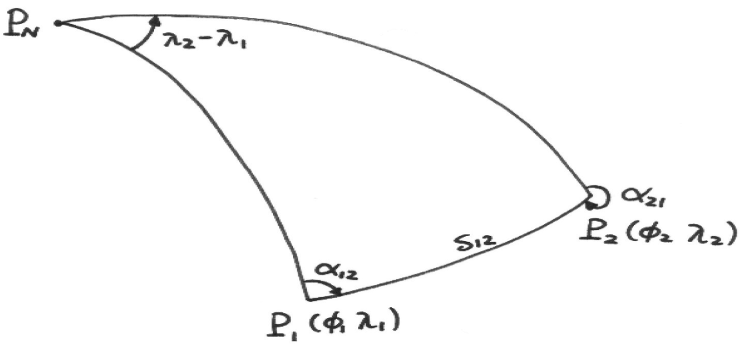
$$\cos\psi = \sin\overline{\phi}_1 \ \sin\overline{\phi}_2 + \cos\overline{\phi}_1 \ \cos\overline{\phi}_2 \ \cos(\lambda_2 - \lambda_1)$$

$$\cos\alpha_{12} = \frac{\cos\overline{\phi}_1 \ \sin\overline{\phi}_2 - \sin\overline{\phi}_1 \ \cos\overline{\phi}_2 \ \cos(\lambda_2 - \lambda_1)}{\sin\psi}$$

$$\cos\alpha_{21} = \frac{\sin\overline{\phi}_1 \ \cos\overline{\phi}_2 - \cos\overline{\phi}_1 \ \sin\overline{\phi}_2 \ \cos(\lambda_2 - \lambda_1)}{\sin\psi}$$

- If $\lambda_2 > \lambda_1$, α_{21} should be replaced by $360° - \alpha_{21}$
- If $\lambda_2 < \lambda_1$, α_{12} should be replaced by $360° - \alpha_{12}$

Basic problems on the ellipsoid



- Direct problem: Find P_2 when P_1 and α_{12}, s_{12} are known
- Inverse problem: Find α_{12}, α_{21}, s_{12} when P_1 , P_2 are known

Methods for BGPs on the ellipsoid

- Series expansions

$$\phi_2 = \phi_1 + \frac{1}{1!}\frac{d\phi}{ds}s + \frac{1}{2!}\frac{d^2\phi}{ds^2}s^2 + \frac{1}{3!}\frac{d^3\phi}{ds^3}s^3 + \cdots\cdots$$

$$\lambda_2 = \lambda_1 + \frac{1}{1!}\frac{d\lambda}{ds}s + \frac{1}{2!}\frac{d^2\lambda}{ds^2}s^2 + \frac{1}{3!}\frac{d^3\lambda}{ds^3}s^3 + \cdots\cdots$$

$$\alpha_{21} = \alpha_{12} + \frac{1}{1!}\frac{d\alpha}{ds}s + \frac{1}{2!}\frac{d^2\alpha}{ds^2}s^2 + \frac{1}{3!}\frac{d^3\alpha}{ds^3}s^3 + \cdots\cdots$$

$$\boxed{\frac{d\phi}{ds} = \frac{\cos\alpha}{M}\,, \quad \frac{d\lambda}{ds} = \frac{\sin\alpha}{N\cos\phi}\,, \quad \frac{d\alpha}{ds} = \frac{\sin\alpha}{N\cot\phi}}$$

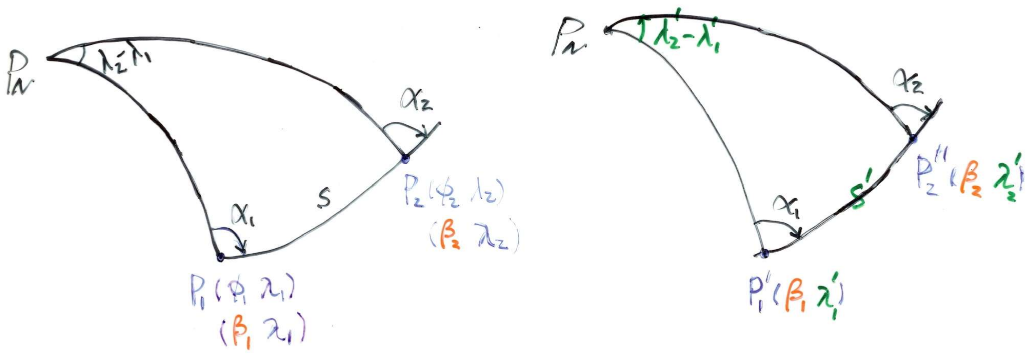
$$\boxed{\frac{d^n\phi}{ds^n} = \frac{\partial\left(\frac{d^{n-1}\phi}{ds^{n-1}}\right)}{\partial\phi}\frac{d\phi}{ds} + \frac{\partial\left(\frac{d^{n-1}\phi}{ds^{n-1}}\right)}{\partial\alpha}\frac{d\alpha}{ds}}$$

- Numerical integration

$$\frac{d\lambda}{ds} \to \infty\,, \quad \frac{d\alpha}{ds} \to \infty\,, \quad as\ \phi \to 90^0$$

- Spherical projection

Bessel’s spherical projection



Gauss' method of mean arguments for medium distances <600km

- Series expansion around the average latitude
- Series expansion up to 4th terms
- Accurate for medium distances up to 600 km
- Solution to the inverse problem is straight-forward
- Solution to the direct problem is evaluated iteratively.
  - *Spherical solution can be the initial values*

Gauss' method of mean arguments

Solution to the inverse problem

$$\phi_m = \frac{\phi_1 + \phi_2}{2}\ , \quad \lambda_m = \frac{\lambda_1 + \lambda_2}{2}\ , \quad \alpha_m = \frac{\alpha_1 + \alpha_2 \pm 180^0}{2}$$

$$\begin{cases} M = \frac{a(1-e^2)}{W^3}\ , \quad N = \frac{a}{W}\ , \quad W = \sqrt{1 - e^2 \sin^2 \phi_m} \\ \eta_m = e' \cos\phi_m\ , \quad t_m = \tan\phi_m\ , \quad {V_m}^2 = 1 + {\eta_m}^2 \end{cases}$$

$$\left.\begin{array}{l}
C_1 = 1/M \\
C_2 = 1/N \\
C_3 = 1/24 \\
C_4 = (1 + {\eta_m}^2 - 9\ {\eta_m}^2\ {t_m}^2)/(24\ {V_m}^4) \\
C_5 = (1 - 2\ {\eta_m}^2)/24 \\
C_6 = {\eta_m}^2({t_m}^2 - 1 - {\eta_m}^2 - 4{\eta_m}^2{t_m}^2)/(8{V_m}^4) \\
C_7 = {V_m}^2/12 \\
C_8 = (3 + 5{\eta_m}^2)/(24\ {V_m}^2) \\
C_9 = 1/2880 \\
C_{10} = (4 + 15\ {t_m}^2)\ \cos^2\phi_m\ /1440 \\
C_{11} = (12\ {t_m}^2 + {t_m}^4)\ \cos^4\phi_m\ /2880 \\
C_{12} = (14 + 40\ {t_m}^2 + 15\ {t_m}^4)\ \cos^4\phi_m\ /2880 \\
C_{13} = 1/192 \\
C_{14} = \sin^2\phi_m\ /48 \\
C_{15} = (7 - 6\ {t_m}^2)\ \cos^4\phi_m\ /1440
\end{array}\right\}$$

Gauss' method of mean arguments

Solution to the inverse problem

$$
\begin{cases}
\delta_1 = \frac{\Delta\phi \cos\frac{1}{2}\Delta\lambda}{C_1} \cdot \left[1 + C_5 \ \cos^2\phi_m \ \Delta\lambda^2 - C_6 \ \Delta\phi^2 - (C_{10} \ \Delta\phi^2 \ \Delta\lambda^2 + C_{12}\Delta\lambda^4)\right] \\
\delta_2 = \frac{\Delta\lambda \cos\phi_m}{C_2} \cdot \left[1 - C_3 \ \sin^2\phi_m \ \Delta\lambda^2 + C_4 \ \Delta\phi^2 + (C_9\Delta\phi^4 - C_{10} \ \Delta\phi^2 \ \Delta\lambda^2 - C_{11}\Delta\lambda^4)\right] \\
\Delta\alpha = \sin\phi_m \ \Delta\lambda \cdot \left[1 + C_7 \ \cos^2\phi_m \ \Delta\lambda^2 + C_8 \ \Delta\phi^2 + (C_{13} \ \Delta\phi^4 - C_{14} \ \Delta\phi^2 \ \Delta\lambda^2 + C_{15} \ \Delta\lambda^4)\right]
\end{cases}
$$

$$
\begin{cases}
s_{12} = \sqrt{{\delta_1}^2 + {\delta_2}^2} \\
\alpha_m = \arctan(\delta_2/\delta_1) \\
\alpha_{12} = \alpha_m - \frac{1}{2}\Delta\alpha \\
\alpha_{21} = \alpha_m + \frac{1}{2}\Delta\alpha + 180^0
\end{cases}
$$

Attention: Δφ, Δλ must be in radian !

Gauss' method of mean arguments

Solution to the direct problem

- Find initial values of $\phi_2^0, \ \lambda_2^0$ e.g. from spherical solution
- Compute mean values and *C*-coefficients

$$
\phi_m = \frac{\phi_1 + \phi_2^0}{2} \ , \quad \lambda_m = \frac{\lambda_1 + \lambda_2^0}{2} \ , \quad \alpha_m^0 = \frac{\alpha_1^0 + \alpha_2^0 \pm 180^0}{2}
$$

$$
\left.
\begin{array}{l}
C_1 = 1/M \\
C_2 = 1/N \\
C_3 = 1/24 \\
C_4 = (1 + {\eta_m}^2 - 9 \ {\eta_m}^2 \ {t_m}^2)/(24 \ {V_m}^4) \\
C_5 = (1 - 2 \ {\eta_m}^2)/24 \\
C_6 = {\eta_m}^2({t_m}^2 - 1 - {\eta_m}^2 - 4{\eta_m}^2{t_m}^2)/(8{V_m}^4) \\
C_7 = {V_m}^2/12 \\
C_8 = (3 + 5{\eta_m}^2)/(24 \ {V_m}^2) \\
C_9 = 1/2880 \\
C_{10} = (4 + 15 \ {t_m}^2) \ \cos^2\phi_m \ /1440 \\
C_{11} = (12 \ {t_m}^2 + {t_m}^4) \ \cos^4\phi_m \ /2880 \\
C_{12} = (14 + 40 \ {t_m}^2 + 15 \ {t_m}^4) \ \cos^4\phi_m \ /2880 \\
C_{13} = 1/192 \\
C_{14} = \sin^2\phi_m \ /48 \\
C_{15} = (7 - 6 \ {t_m}^2) \ \cos^4\phi_m \ /1440
\end{array}
\right\}
\quad
\begin{array}{l}
M = \frac{a(1-e^2)}{W^3} \ , \quad N = \frac{a}{W} \ , \quad W = \sqrt{1 - e^2\sin^2\phi_m} \\
\eta_m = e'\cos\phi_m \ , \quad t_m = \tan\phi_m \ , \quad {V_m}^2 = 1 + {\eta_m}^2
\end{array}
$$

Gauss' method of mean arguments

Solution to the direct problem

- Compute coordinate differences:

$$\begin{cases} \Delta\phi = C_1 \frac{s_{12}\cos\alpha_m}{\cos\frac{1}{2}\Delta\lambda} \cdot \left[1 - C_5 \ \cos^2\phi_m \ \Delta\lambda^2 + C_6 \ \Delta\phi^2 + (C_{10} \ \Delta\phi^2 \ \Delta\lambda^2 + C_{12}\Delta\lambda^4)\right] \\ \Delta\lambda = C_2 \frac{s_{12}\sin\alpha_m}{\cos\phi_m} \cdot \left[1 + C_3 \ \sin^2\phi_m \ \Delta\lambda^2 - C_4 \ \Delta\phi^2 - (C_9\Delta\phi^4 - C_{10} \ \Delta\phi^2 \ \Delta\lambda^2 - C_{11}\Delta\lambda^4)\right] \end{cases}$$

- Compute coordinates of P_2 :

$$\begin{cases} \phi_2 = \phi_1 + \Delta\phi \\ \lambda_2 = \lambda_1 + \Delta\lambda \end{cases}$$

- Iterate using computed coordinates of P_2 as new initial values

Helmert's differential formulas

$$\begin{aligned} \phi_2 &= \phi_2(\phi_1, \ \lambda_1, \ s, \ \alpha_{12}, \ a, \ f) \\ \lambda_2 &= \lambda_2(\phi_1, \ \lambda_1, \ s, \ \alpha_{12}, \ a, \ f) \\ \alpha_{21} &= \alpha_{21}(\phi_1, \ \lambda_1, \ s, \ \alpha_{12}, \ a, \ f) \end{aligned}$$

$$\begin{cases} d\phi_2 = \frac{\partial\phi_2}{\partial\phi_1} \cdot d\phi_1 + \frac{\partial\phi_2}{\partial s} \cdot ds + \frac{\partial\phi_2}{\partial\alpha_{12}} \cdot d\alpha_{12} + \frac{\partial\phi_2}{\partial a} \cdot da + \frac{\partial\phi_2}{\partial f} \cdot df \\ d\lambda_2 = \frac{\partial\lambda_2}{\partial\phi_1} \cdot d\phi_1 + d\lambda_1 + \frac{\partial\lambda_2}{\partial s} \cdot ds + \frac{\partial\lambda_2}{\partial\alpha_{12}} \cdot d\alpha_{12} + \frac{\partial\lambda_2}{\partial a} \cdot da + \frac{\partial\lambda_2}{\partial f} \cdot df \\ d\alpha_{21} = \frac{\partial\alpha_2}{\partial\phi_1} \cdot d\phi_1 + \frac{\partial\alpha_2}{\partial s} \cdot ds + \frac{\partial\alpha_2}{\partial\alpha_{12}} \cdot d\alpha_{12} + \frac{\partial\alpha_2}{\partial a} \cdot da + \frac{\partial\alpha_2}{\partial f} \cdot df \end{cases}$$

Helmert's differential formulas

$$\boxed{\begin{aligned} d\phi_2 &= p_1 \cdot d\phi_1 + p_3 \cdot ds + p_4 \cdot d\alpha_{12} + p_5 \cdot \tfrac{da}{a} + p_6 \cdot df \\ \cos\phi_2 \cdot d\lambda_2 &= q_1 \cdot d\phi_1 + d\lambda_1 + q_3 \cdot ds + q_4 \cdot d\alpha_{12} + q_5 \cdot \tfrac{da}{a} + q_6 \cdot df \\ \cot\phi_2 \cdot d\alpha_{21} &= r_1 \cdot d\phi_1 + r_3 \cdot ds + r_4 \cdot d\alpha_{12} + r_5 \cdot \tfrac{da}{a} + r_6 \cdot df \end{aligned}}$$

$$\left.\begin{aligned}
p_1 &= -\tfrac{M_1}{M_2}\left[\cos\alpha_{12}\cos\alpha_{21} - \left(\tfrac{dm}{ds}\right)_2 \sin\alpha_{12}\sin\alpha_{21}\right] \\
p_3 &= +\tfrac{\cos\alpha_{21}}{M_2} \\
p_4 &= -\tfrac{m}{M_2} \cdot \sin\alpha_{21} \\
p_5 &= -\tfrac{s}{M_2} \cdot \cos\alpha_{21} \\
p_6 &= 2 \cdot \Delta\phi - \left(3\Delta\phi - \tfrac{1}{2}\Delta\lambda \sin\phi_1 \cos\phi_1\right)\sin^2\phi_1 \\
\\
q_1 &= \tfrac{M_1}{N_2} \cdot \sin\Delta\lambda \cdot \sin\phi_2 \\
q_3 &= \tfrac{\sin\alpha_{21}}{N_2} \\
q_4 &= W_2 \cdot \sin\Delta\lambda \cdot \cos\phi_1 \cdot \cot\alpha_{21} \\
q_5 &= -\tfrac{s}{a} W_2 \sin\alpha_{21} \\
q_6 &= -\Delta\lambda\ \sin^2\phi_1\ \cos\phi_1 \\
\\
r_1 &= \tfrac{\sin\Delta\lambda}{\sin\phi_2} \cdot \left(1 - \frac{1}{4}e^2 \sin^2 2\phi_1\right) \\
r_3 &= q_3 \\
r_4 &= q_4 + \cos\tfrac{s}{a} \cdot \cot\phi_2 \\
r_5 &= -\tfrac{s}{N_2}\sin\alpha_{21} \\
r_6 &= q_6 + \Delta\phi \cdot \Delta\lambda \tfrac{\cos^4\phi_1}{\sin\phi_1} \\
\\
\Delta\phi &= \phi_2 - \phi_1 \\
\Delta\lambda &= \lambda_2 - \lambda_1
\end{aligned}\right\} \quad (s < 50\ km)$$

Inverse formulas

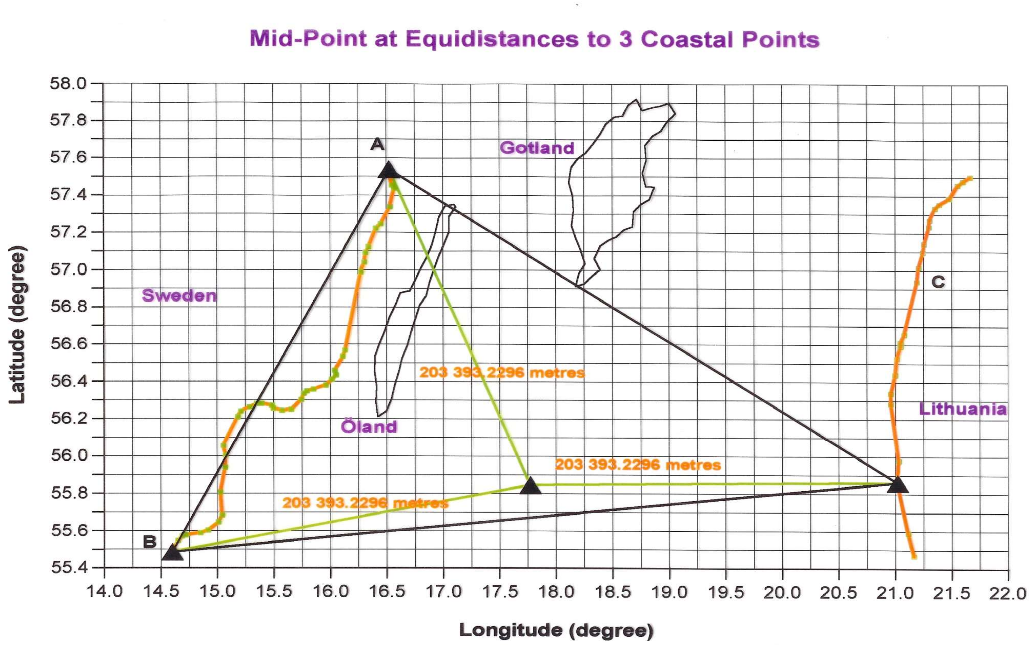
$$\left.\begin{aligned} s &= s(\phi_1,\ \lambda_1,\ \phi_2,\ \lambda_2) \\ \alpha_{12} &= \alpha_1(\phi_1,\ \lambda_1,\ \phi_2,\ \lambda_2) \end{aligned}\right\}$$

$$\boxed{\begin{aligned} ds &= \tfrac{\partial s}{\partial\phi_1}\, d\phi_1 + \tfrac{\partial s}{\partial\lambda_1}\, d\lambda_1 + \tfrac{\partial s}{\partial\phi_2}\, d\phi_2 + \tfrac{\partial s}{\partial\lambda_2}\, d\lambda_2 = a_1\, d\phi_1 + b_1\, d\lambda_1 + a_2\, d\phi_2 + b_2\, d\lambda_2 \\ d\alpha_{12} &= \tfrac{\partial\alpha_{12}}{\partial\phi_1}\, d\phi_1 + \tfrac{\partial\alpha_{12}}{\partial\lambda_1}\, d\lambda_1 + \tfrac{\partial\alpha_{12}}{\partial\phi_2}\, d\phi_2 + \tfrac{\partial\alpha_{12}}{\partial\lambda_2}\, d\lambda_2 = c_1\, d\phi_1 + d_1\, d\lambda_1 + c_2\, d\phi_2 + d_2\, d\lambda_2 \end{aligned}}$$

$$\begin{aligned}
a_1 &= -M_1\cos\alpha_{12} & c_1 &= \tfrac{1}{s}M_1\sin\alpha_{12} \\
b_1 &= N_2\cos\phi_2\sin\alpha_{21} & d_1 &= \tfrac{1}{s}N_2\cos\phi_2\cos\alpha_{21} \\
a_2 &= -M_2\cos\alpha_{21} & c_2 &= \tfrac{1}{s}M_2\sin\alpha_{21} \\
b_2 &= -N_2\cos\phi_2\sin\alpha_{21} & d_2 &= -\tfrac{1}{s}N_2\cos\phi_2\cos\alpha_{21}
\end{aligned}$$

Practical project 1: maritime boundary delimitation in the Baltic Sea

- GALOS: Geodetic Aspects of the Law of the Sea
- Find mid-points between two coastal lines
- All points are assumed to be on the ellipsoid
- Use Gauss' method of mean arguments to calculate the lengths of geodetic lines
- Differential formulas used to converge from approximate solutions to final solutions



Practical project 2: *Numerical* bisection on the reference ellipsoid

- Global traingulation with geodetic lines
- Two points are fixed with given coordinates and azimuths to a 3rd point are also given
- How to find the geodetic/latitude/longitude of the 3rd point ?
- Spherical solution is used as a approximate solution
- Differential formulas used to correct the approximate solutions
- Numerical differentiation used (*finite differencing*)
- Iteration is used.