ROYAL INSTITUTE
OF TECHNOLOGY

Exercise Problems in Map Projections and Reference Systems

Huaan Fan

Division of Surveying and Geodesy
Royal Institute of Technology (KTH)
100 44 Stockholm

October 2023

Contents

1 Spherical trigonometry and reference ellipsoids

Problem 1.1

In a spherical triangle on a sphere of radius $R = 6371\ km$, two sides $a,\ b$ and one angle C are given :

$$a = 30.000\,000\,000^0, \quad b = 60.000\,000\,000^0, \quad C = 60.000\,000\,000^0$$

a. Find the third side c and the other two angles $A,\ B$ of the triangle ;

b. Find the spherical excess ε of the triangle ;

c. Find the area T of the triangle .

Problem 1.2

A spherical triangle has two angles (A, B) and the common side between them (c) given as follows :

$$A = 60.000\,000\,000^0, \quad B = 60.000\,000\,000^0, \quad c = 30.000\,000\,000^0$$

Find the third angle (C) as well as the other two sides (a, b).

Problem 1.3

On the GRS 80 reference ellipsoid ($a = 6378137.000\ metres$, $e^2 = 0.0066943800229$), calculate:

a. the ellipsoidal parameters: b,

b. $1/f$

c. e'^2

d. radius (R) of the mean earth sphere, i.e. a sphere that has the same volume as the reference ellipsoid. $R = \sqrt[3]{a^2 b}$

f. total length of the equator

Problem 1.4

For point P at latitude $\phi = 60.000\,000\,000^0$ on the GRS 80 reference ellipsoid ($a = 6378137.000\ metres$, $e^2 = 0.0066943800229$), compute:

a. the radius (M) of curvature of the meridian at P;

b. the radius (N) of curvature in the prime vertical at P;

c. the mean radius (R_m) of curvature at P;

d. the radius (R_α) of curvature of a normal section at P of azimuth $\alpha = 45^0$

e. radius $(p = N\cos\phi)$ and perimeter $(\ell = 2\pi p)$ of the complete parallel circle through P .

2 Geodetic coordinates

Problem 2.1

On the GRS 80 reference ellipsoid ($a = 6378137.000\ metres\ ,\ \ e^2 = 0.0066943800229$), there is a point P with geodetic coordinates: $\phi = 60^0$, $\lambda = 30^0$, $h = 0$. Calculate:

a. the rectangular coordinates (x, y, z) of P.

b. the spherical coordinates $(r, \bar{\phi}, \lambda)$ of P ;

c. the reduced latitude (β) of P;

d. calculate the latitude differences: $\phi - \overline{\phi},\ \phi - \beta$.

Problem 2.2

Let point P in space have the following geocentric rectangular coordinates:

$$\begin{aligned} x &= 3042053.7355\ m \\ y &= 988423.1757\ m \\ z &= 5503075.2100\ m \end{aligned}$$

Use the *closed formulas* to transform (x, y, z) into geodetic coordinates (ϕ, λ, h) with respect to the GRS 80 reference ellipsoid ($a = 6378137.000\ metres\ ,\ \ e^2 = 0.0066943800229$).

Problem 2.3

Use the iterative formulas to transform (x, y, z) given in **Problem 2.2** into geodetic coordinates (ϕ, λ, h) with respect to the GRS 80 reference ellipsoid ($a = 6378137.000\ metres\ ,\ \ e^2 = 0.0066943800229$).

The iteration can stop if

$$|\Delta h| < 0.1\ mm \qquad and \qquad |\Delta\phi| < 10^{-9} \qquad degrees$$

where Δh, $\Delta\phi$ are height and latitude changes between two successive iterations, respectively. For each iteration i, provide the intermediate results for $\phi_i,\ |\Delta\phi_i|\ ,\ N_i,\ h_i,\ |\Delta h_i|$.

Problem 2.4

A point P within the Onsala Space Observatory has the following rectangular coordinates in $ITRF97$:

$$\begin{pmatrix} X_p \\ Y_p \\ Z_p \end{pmatrix} = \begin{pmatrix} 3\ 370\ 641.970 \\ 711\ 866.128 \\ 5\ 349\ 796.160 \end{pmatrix} metre$$

With respect to GRS 80 reference ellipsoid ($a = 6\ 378\ 137\ metre,\ e^2 = 0.006\ 694\ 380\ 0229$), the geodetic coordinates of P are:

$$\begin{pmatrix} \phi_p \\ \lambda_p \\ h_p \end{pmatrix} = \begin{pmatrix} 57.395483300^0 \\ 11.925395149^0 \\ 43.372\ metre \end{pmatrix}$$

a. compute the rotation matrix $Q(\phi_p, \lambda_p)$ from the global geocentric coordinate system to the topocentric coordinate system $P - NEU$ at P with respect to the ellipsoidal normal at P

b. If point A has the following geocentric coordinates in $ITRF97$, compute the topocentric coordinates (N, E, U) of point A in the topocentric cooridnate system $P - NEU$.

$$\begin{pmatrix} X_A \\ Y_A \\ Z_A \end{pmatrix} = \begin{pmatrix} 3098889.388 \\ 1011032.696 \\ 5463980.133 \end{pmatrix} \; metre$$

c. If point P is moving with the following velocity vector in the $ITRF97$,

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} -1.36 \\ +1.47 \\ +0.84 \end{pmatrix} \; cm/year$$

what is the velocity vecotor of P in the topocentric coordinate system $P - NEU$:

$$\begin{pmatrix} \dot{N} \\ \dot{E} \\ \dot{U} \end{pmatrix} =?$$

d. How fast and in which direction is point P moving in the horizontal plane of P ?

3 Basic geodetic problems and geodetic lines

Problem 3.1

On the $GRS80$ reference ellipsoid, a man starts from point P ($\phi_p = 60^0\ N$, $\lambda_p = 18^0\ E$) and walks 100 metres in three different directions (see below). Use the differential equations of the geodesics to **estimate** the positions of the man after 10 m walking.

a. toward the exact north direction

b. toward the exact east direction

c. in the direction of azimuth $\alpha = 60^0$.

Problem 3.2

On the spherical Earth of radius $R = 6371\ 000.000\ m$, the distance s_{12} and azimuth α_{12} from Stockholm ($\overline{\phi}_1 = 59^0\ 20'\ N$, $\lambda_1 = \ 18^0\ 05'\ E$) to the Swedish scientific research station WASA are:

$$s_{12} = 14\ 723\ 000.000\ m, \quad \alpha_{12} = 181^0\ 50'\ 00.00000''$$

Calculate the geocentric latitude ($\overline{\phi}_2$) and longitude (λ_2) of the station WASA.

Problem 3.3

Use the GRS 80 ellipsoidal parameters and Gauss' Mean Arguments Method to calculate the distance (s_{12}) and azimuths (α_{12}, α_{21}) from Uppsala to Stockholm, based on their geodetic coordinates given below:

Uppsala :	$\phi_1 = 59^0\ 55'\ 00.00000''\ N,$	$\lambda_1 = 17^0\ 38'\ 00.00000''\ E$
Stockholm :	$\phi_2 = 59^0\ 20'\ 00.00000''\ N,$	$\lambda_2 = 18^0\ 05'\ 00.00000''\ E$

4 Deformation parameters for simple projections

Map projection coordinates (x, y) of azimuthal projections can be expressed as functions of the spherical coordinates (ϕ, λ):

$$\begin{cases} x = f_x(\phi, \Delta\lambda) \\ y = f_y(\phi, \Delta\lambda) \\ \Delta\lambda = \lambda - \lambda_0 \end{cases}$$

where λ_0 is the longitude of the reference meridian (e.g. at Greenwich $\lambda_0 = 0$).

For each of the two azimuhal projections given in **Problem 4.1** and **Problem 4.2**, perform the following tasks:

a) calculate the *first foundamental coefficients* (FFC): $e,\ f,\ g,\ H$

b) calculate the general deformation parameters, $\eta,\ k$ and ξ

c) comment on the characteristics of the projection (is it conformal, equivalent or equidistant?)

d) are the projections of the meridians and parallel circles vertical to each other ? Are there curves which are equidistant?

Problem 4.1

Eratostenes, 200 B.C., created a simple cylinder projection for the sphere:

$$\begin{cases} x = R\ \cdot \overline{\phi} \\ y = R \cdot \Delta\lambda \end{cases}$$

Problem 4.2

Lambert constructed in the middle of the 18th century the following map projection for the sphere:

$$\begin{cases} x = R\ \cdot \sin\overline{\phi} \\ y = R \cdot \Delta\lambda \end{cases}$$

Problem 4.3

Assume that the earth is a sphere of radius $R = 6371\ 000\ m$. A point P_0 on the roof of KTH L-Buildinghas the following geocentric coordinates:

$$P_0 : \quad \overline{\phi}_0 = 59^0\ 21'\ 00'\ N, \quad \lambda_0 = 18^0\ 04'\ 09''\ E$$

The following geocentric coordinates refer to the Southern Tower of Charles Bridge in Prague (which has been damaged in 1648 by the cannon fire of the Swedish army during the Thirty-Years´ War):

$$P : \qquad \overline{\phi} = 50^0\ 05'\ 10''\ N, \quad \lambda = 14^0\ 24'\ 50''\ E$$

Calculate the oblique azimuthal projection coordinates (x, y) for point P, with P_0 as the map centre (projection centre), using the method of:

a. orthographic azimuthal projection

b. Lambert equal-area azimuthal projection

c. equidistant azimuthal projection

d. stereographic azimthal projection

e. Gnonomic azimuthal projection

For each of the five tasks above, calculate also the distance $r = \sqrt{x^2 + y^2}$ between P_0 (Stockholm, the map center) and P (the Southern Tower, the projection point) on the map projection plane.

Problem 4.4 *(conical projections)*

Assume that the earth is a sphere of radius $R = 6371\ 000\ m$. We use *conical projection* to calculate the map cooridnates (x, y) for point P with point P_0 as the map centre. The geocentric coordinates of P and P_0 are as given in **Problem 4.3**. The two standard parallel circles of the conical projections have the following geocentric latitudes:

$$\overline{\phi}_1 = 45^0, \quad \overline{\phi}_2 = 65^0$$

The two used projection methods are:

a. Abers equal-area projection

b. Lambert conformal conical projection

For each of the two tasks above, calculate also the distance $r = \sqrt{x^2 + y^2}$ between P_0 and P on the map projection plane.

5 Cylindrical projections

Problem 5.1 *(normal Mercator projection for a sphere)*

Compute the map projection coordinates $x,\ y$ of the *normal Mercator projection for a sphere* of radius $R = 6371\ 000$ *meter* for a point P at KTH with spherical coordinates:

$$\overline{\phi} = 59^0\ 21'\ \ 00''\ N\ , \quad \lambda = 18^0\ 04'\ 09''\ E$$

Use the Equator as the reference parallel (i.e. $\overline{\phi}_0 = 0$). The reference meridian (central meridian) has longitude: $\lambda_0 = 15^0$.

Problem 5.2 *(normal Mercator projection for an ellipsoid)*

Compute the planar coordinates $x,\ y$ of the *normal Mercator projection* for a point P on the GRS 80 reference ellipsoid ($a = 6378137\ m,\ 1/f = 298.257\ 222\ 101$). Point P has the following geodetic coordinates:

$$\phi = 59^0\ 21'\ \ 00''\ N\ , \quad \lambda = 18^0\ 04'\ 09''\ E$$

Use the Equator as the reference parallel (i.e. $\phi_0 = 0$). The reference meridian (central meridian) has longitude: $\lambda_0 = 15^0$.

Problem 5.3 *(transverse Mercator projection for a sphere)*

Compute the planar coordinates $x,\ y$ of the *transverse Mercator projection for a sphere* of radius $R = 6371\ 000$ *meter* for a point P with spherical coordinates:

$$\overline{\phi} = 59^0\ 21'\ \ 00''\ N\ , \quad \lambda = 18^0\ 04'\ 09''\ E$$

Use the Equator as the reference parallel (i.e. $\overline{\phi}_0 = 0$). The reference meridian (central meridian) has longitude: $\lambda_0 = 15^0$.

Problem 5.4 *(transverse Mercator projection for an ellipsoid)*

Compute the planar coordinates $x,\ y$ of the *Universal Transverse Mercator (UTM) projection* in UTM zone 33 for a point P on the GRS 80 reference ellipsoid ($a = 6378137\ m,\ 1/f = 298.257\ 222\ 101$). Point P has the following geodetic coordinates:

$$\phi = 59^0\ 21'\ \ 00''\ N\ , \quad \lambda = 18^0\ 04'\ 09''\ E$$

6 Geodetic astronomy

Problem 6.1

On March 1, 1981, $UT1 = 0^h$ the Equation of the Equinox ($Eq.E$) is -0.8^s. compute for this epoch:

a. JD

b. T_0

c. $(GMST)_0$

d. $GAST$ when $SNT =$ $22^h\ 50^m\ 42.0^s$ at the same day (SNT = Swedish standard time)

Problem 6.2

At the official Swedish time $SNT = 22^h\ 50^m\ 42.0^s$ on March 1, 1981, a star was observed in the South of the local meridian from a ground station. The equatorial coordinates of the star are as follows:

$$\delta = 51^0\ 1'\ 14'' \ , \quad \alpha = 9^h\ 38^m\ 11.1^s$$

The height angle of the star has been measued to $h = 80^0\ 25'\ 34''$. Find out the astronomical latitude (Φ) and longitude (Λ) of this observation station. (Hints: use results from **Problem 7.1**)

Problem 6.3

Assume that at a point A, the geocentric coordinates referred to the instantaneous pole of the earth has been determined to (unit: metres) :

$$\begin{aligned} X &= 3\ 042\ 053.735 \\ Y &= \ \ 988\ 423.176 \\ Z &= 5\ 503\ 075.210 \end{aligned}$$

The polar position at the time of coordinates determination is :

$$\begin{aligned} x_p &= +0.12'' \\ y_p &= +0.15'' \end{aligned}$$

The gravity value at A is $g = 9.78\ m/s^2$ and GRS-80 reference ellipsoid will be used .

a. compute the geocentric coordinates of A referred to the CIO pole ;

b. estimate the changes in the latitude and longitude of A when the CIO pole is used ;

c. compute the geocentric latitude $\overline{\phi}$ at A and thereafter the potential change due to the polar motion ;

d. find out the surface gravity change at A due to the polar motion ;

e. how much will the sea level near A change due to the polar motion ?

Problem 6.4

On February 20, 2001, at $t = 15^h\ 15^m$ (Swedish standard time), the Earth orientation parameters have the following values:

$$\begin{cases} \Delta\varepsilon = -1.707'' \\ \Delta\psi = -15.447'' \\ x_p = -0.0466'' \\ y_p = +0.4391 \\ dUT1 = UT1 - UTC = 0.0471^s \end{cases}$$

In the International Celestial Reference Frame referred to the mean equator and equinox at $J2000.0$, an artificial satellite has the following geocentric coordinates:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{ICRF} = \begin{pmatrix} 12\,791\,872.0418 \\ 4\,173\,410.3284 \\ 22\,669\,864.5407 \end{pmatrix} metre$$

Transform the above coordinates in the $ICRF$ to coordinates in the $ITRF$ by the following steps:

a. computer the Julian date (JD_0) and Julian Century (T_0) for $UT1 = 0$

b. compute $UT1$, the Julian date (JD) and Julian Century (T) for epoch $t = 15^h\ 15^m$

c. compute the precession parameters $\xi,\ z,\ \vartheta$ and the precession matrix P (Check if P is orthogonal: $PP^\top = P^\top P = I$)

d. compute the mean obliquity of the ecliptic (ε_0), the true obliquity of the ecliptic ($\varepsilon = \varepsilon_0 + \Delta\varepsilon$) and the nutation matrix $N(\varepsilon_0, \Delta\varepsilon, \Delta\psi)$.((Check if N is orthogonal)

e. compute $(GMST)_0$, $GMST$, $GAST$ for epoch t,as well as . (**Hints**: to compute $GMST$, one need to use the Julian century T_0 computed in (**a**)

f. compute the rotation matrix $R_3(GAST)$, the polar motion matrix $R_p(x_p, y_p)$/ and the joint rotation matrix $R = R_p \cdot R_3 \cdot N \cdot P$ used for transformation from $ICRS$ to $ITRF$. Check if $R_2,\ R_p$ and R are orthogonal).

g. compute the coordinates of the satellite in $ITRF$, $(X, Y, Z)_{ITRS}$

h. compare the geocentric distance to satellite, $r = \sqrt{X^2 + Y^2 + Z^2}$, computed in $ICRF$ and $ITRF$ respectively.

7 Geodetic triangulation datums and height systems

Problem 7.1

Assume that the centre of a local Hayford ellipsoid (a = 6378388 meter; $1/f$ = 297) has geocentric coordinates :

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} -100 \\ -100 \\ -100 \end{pmatrix} \text{ (metres)}$$

With respect to this non-geocentric Hayford ellipsoid, point P has the following geodetic coordinates:

$$\begin{pmatrix} \phi \\ \lambda \\ h \end{pmatrix} = \begin{pmatrix} 60^0 \\ 18^0 \\ 100\ m \end{pmatrix}$$

With respect to the **geocentric** GRS 80 ellipsoid (a' = 6378137 meter; f' = 1/298.257 222 101), point P willl have different coordinates $(\phi + \delta\phi, \lambda + \delta\lambda, h + \delta h)$.

Consider changes in the coordinates of P from System 1 (associated with a non-geocentric Hayford ellipsoid) to System 2 (associated with a geocentric GRS 80 ellipsoid). Use the differential formulas to estimate the coordinate changes ($\delta\phi$, $\delta\lambda$, δh) caused by:

a. (x_0, y_0, z_0)

b. $\delta a = a' - a$

c. $\delta f = f' - f$

d. sum of the above three factors together

Problem 7.2

Through GPS observations at a point P, the following geodetic coordinates in $SWEREF$ 99 have been obtained:

$$\phi = 59^0\ 21'\ 0.04297''$$
$$\lambda = 18^0\ \ 4'\ 9.30051''$$
$$h = 60.000\ metre$$

The geoid height of P in $SWEN01L$ has been found to be: $N_{SWEN01L} = 23.193\ metre$. Compute:

a. the rectangular coordinates (X, Y, Z) of P in $SWEREF$ 99

b. the rectangular coordinates (X, Y, Z) of P in RT 90

c. the geodetic coordinates (ϕ, λ) of P in RT 90

d. the planar coordinates (x, y) on the Gauss-Kruger projection plane RT 90 2.5 $gon\ V$ (λ_0 = 15.808 277 777^0, $k_0 = 1$, y_0 = 1 500 km)

e. height above the mean sea level for point P in RH 70.

Problem 7.3

A point P lies on the equator with dynamic height $H_d = 100.000\ metre$. The mean normal gravity at the equator and at latitude 45^0 are: $\gamma_e = 978.0\ Gal$, $\gamma_{45} = 980.6\ Gal$. How much will the dynamic height at P differ from its normal height ?

Problem 7.4

Between a fixed benchmark A and an unknown point B, levelling and gravity measurements have been made with the following results for height differences Δh_i and gravity g_i $(i = 1, 2, 3, 4)$:

i	$\Delta h_i\ (metre)$	$g_i\ \ (m/s^2)$
1	-1.010	9.80372
2	-1.511	9.80312
3	$+0.307$	9.80320
4	$+2.032$	9.80067

Benchmark A has normal height $H_A^* = 20.000\ metre$. The mean gravity at A, B are: $\overline{\gamma}_A = 9.80000\ m/s^2$, $\overline{\gamma}_B = 9.79820\ m/s^2$. Calculate:

a. geopotential number (C_A) at A;

b. geometrical height difference (Δh_{AB}) from A to B ;

c. difference (ΔC_{AB}) between geopotential numbers at A and B ;

d. geopotential number (C_B) at B;

e. normal height (H_B^*) at B ;

f. normal height difference (ΔH_{AB}^*) from A to B ;

g. difference between the geometrical and the normal height differences between A and B.

8 Estimation of Helmert transformation parameters

Rectangular coordinates (x, y, z) at 20 ground points defined in two different coordinate systems are listed below. The two systems are quite close to each other, i.e. the scale change and rotations are very small.

System 1	x	y	z
point 1	2441775.419	799268.100	5818729.162
point 2	3464655.838	845749.989	5270271.528
point 3	3309991.828	828932.118	5370882.280
point 4	3160763.338	759160.187	5469345.504
point 5	2248123.493	865686.595	5886425.596
point 6	3022573.157	802945.690	5540683.951
point 7	3104219.427	998384.028	5463290.505
point 8	2998189.685	931451.634	5533398.462
point 9	3199093.294	932231.327	5420322.483
point 10	3370658.823	711876.990	5349786.786
point 11	3341340.173	957912.343	5330003.236
point 12	2534031.166	975174.455	5752078.309
point 13	2838909.903	903822.098	5620660.184
point 14	2902495.079	761455.843	5609859.672
point 15	2682407.890	950395.934	5688993.082
point 16	2620258.868	779138.041	5743799.267
point 17	3246470.535	1077900.355	5365277.896
point 18	3249408.275	692757.965	5426396.948
point 19	2763885.496	733247.387	5682653.347
point 20	2368885.005	994492.233	5818478.154
System 2	x	y	z
Point 1	2441276.712	799286.666	5818162.025
Point 2	3464161.275	845805.461	5269712.429
Point 3	3309496.800	828981.942	5370322.060
Point 4	3160269.913	759204.574	5468784.081
Point 5	2247621.426	865698.413	5885856.498
Point 6	3022077.340	802985.055	5540121.276
Point 7	3103716.966	998426.412	5462727.814
Point 8	2997689.029	931490.201	5532835.154
Point 9	3198593.776	932277.179	5419760.966
Point 10	3370168.626	711928.884	5349227.574
Point 11	3340840.578	957963.383	5329442.724
Point 12	2533526.497	975196.347	5751510.935
Point 13	2838409.359	903854.897	5620095.593
Point 14	2902000.172	761490.908	5609296.343
Point 15	2681904.794	950423.098	5688426.909
Point 16	2619761.810	779162.964	5743233.630
Point 17	3245966.134	1077947.976	5364716.214
Point 18	3248918.041	692805.543	5425836.841
Point 19	2763390.878	733277.458	5682089.111
Point 20	2368378.937	994508.273	5817909.286

All coordinates are assumed to be uncorrelated with each other and have equal weights.

Assume that all coordinates are uncorrelated with equal weights.

Use the linearized Helmert transformation model to estimate the 7 transformation parameters δx, δy, δz, δs, α_1, α_2 and α_3 , as well as the standrad errors of the 7 parameters.

The required unit is *metre* for the translation parameters, *ppm* for the scale change and $''$ (arcsecond) for the rotational angles.