General projection theory

- Types of map projections
- Curvlinear coordinate systems
- Deformation of map projections
- Properties of map projections

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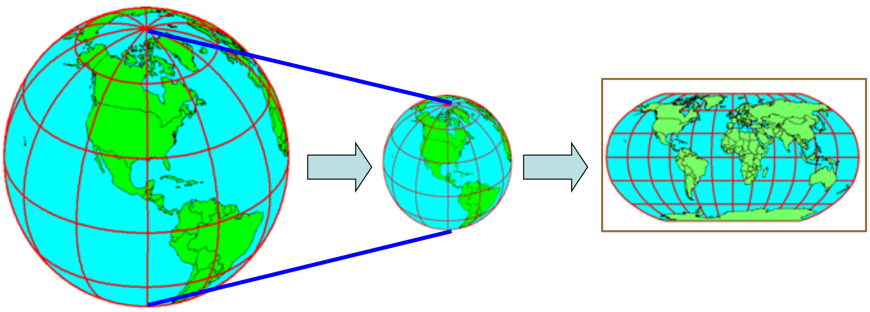
Map projections: basic concepts



Map scale

$= \dfrac{\text{Globe distance}}{\text{Earth distance}}$

(e.g. 1:1000 000)

Projection scale

$= \dfrac{\text{Map distance}}{\text{Globe distance}}$

(e.g. 0.9996)

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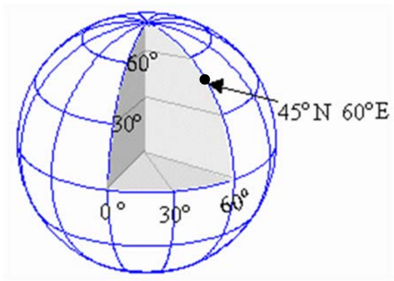
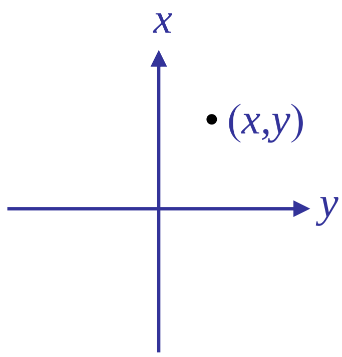
Geodetic vs projected coordinates





(ϕ, λ) ←→ (x, y)

Map Projection

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Classification of map projections

- Projection surfaces
- Relation of projection surfaces to earth surface
- Projection surface's normal/axis in relation to the earth's polar axis

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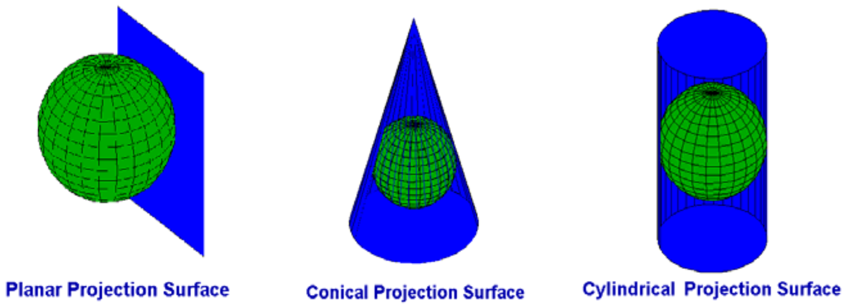
Classification by projection surfaces



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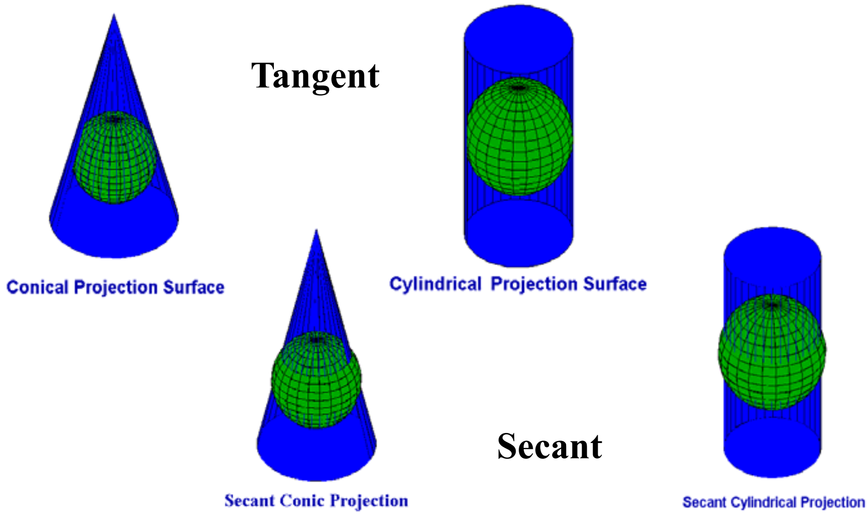
Relation of projection surface to earth surface



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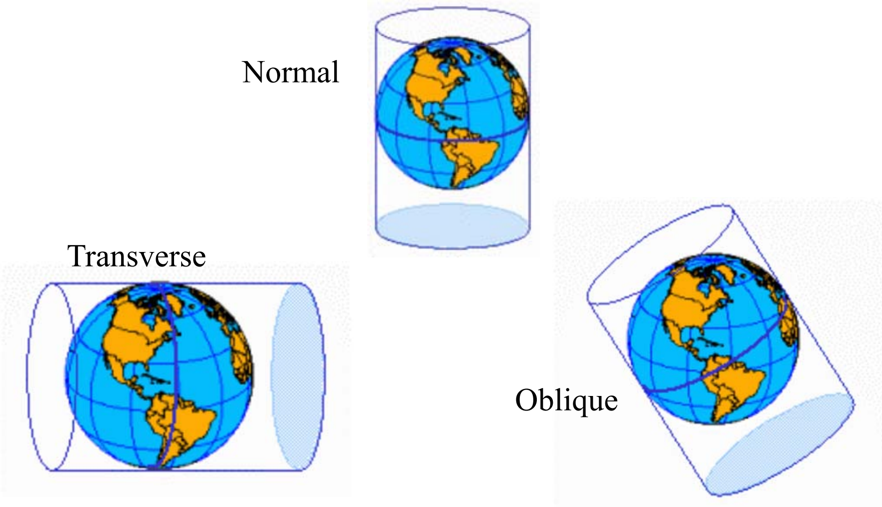
Relation to the polar axis



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Properties of map projections

- Conformal projections (*vinkelriktig*)
  - Angles not changed. Shape of small figures not changed
  - Distances changed
  - Same scale in all directions at the ***same*** point

  (but: different scales at ***different*** points!)
- Equal-area projections (*ytriktig*)
  - Area of small figures unchanged => angles/distances changed
- Equidistant projections (*avståndsriktig*)
  - Some distances not changed (meridians or parallels)
  - For a fewer distances, not all distances unchanged

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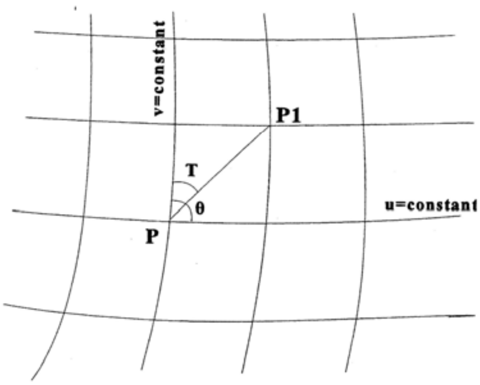
Curvlinear coordinate system



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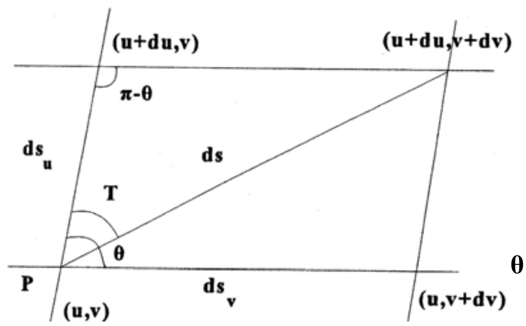
Distance differential (*metric*)



$$ds^2 = ds_u^2 + ds_v^2 - 2\,ds_u\,ds_v\cos(\pi-\theta) =$$

$$= ds_u^2 + 2\,ds_u\,ds_v\cos\theta + ds_v^2$$

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Distance differential in 3D

$$X = X(u,v)$$
$$Y = Y(u,v)$$
$$Z = Z(u,v)$$

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$dx = \frac{\partial x}{\partial u} \cdot du + \frac{\partial x}{\partial v} \cdot dv$$

$$dy = \frac{\partial y}{\partial u} \cdot du + \frac{\partial y}{\partial v} \cdot dv$$

$$dz = \frac{\partial z}{\partial u} \cdot du + \frac{\partial z}{\partial v} \cdot dv$$

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Distance differential

$$ds^2 = \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2\right] du^2 +$$
$$+ 2\left[\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}\right] du \cdot dv +$$
$$+ \left[\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] dv^2$$

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

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Distance differential

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$

$$F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$$

$$G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$$

→ ds^2 is a *quadratic form* of du, dv

$$\boxed{\begin{array}{l} E > 0 \\ G > 0 \\ EG\text{-}F^2 > 0 \end{array}}$$

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Distance differential in 2 forms

$$ds^2 = ds_u^2 + ds_v^2 - 2\,ds_u\,ds_v \cos(\pi - \theta) =$$

$$= ds_u^2 + 2\,ds_u\,ds_v \cos\theta + ds_v^2$$

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

$$ds_u = \sqrt{E} \cdot du \qquad \cos\theta = \frac{F}{+\sqrt{EG}}$$

$$ds_v = \sqrt{G} \cdot dv$$

on ellipsoid ⟶ $ds_\varphi = M\,d\varphi$

$ds_{\Delta\lambda} = P\,d\Delta\lambda$

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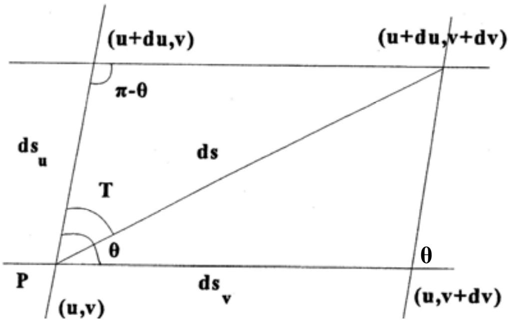
Area of the parallelogram



$$d\sigma = ds_u \, ds_v \sin\theta$$

$$= \sqrt{EG - F^2}\, du\, dv$$

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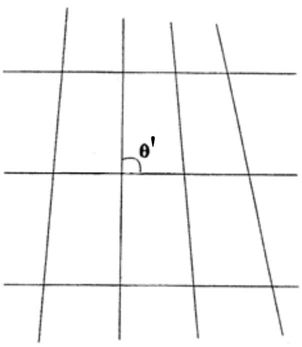
Differentials on projection plane



$$\Delta\lambda = \lambda - \lambda_0$$

$$X = f_X(\varphi, \Delta\lambda)$$

$$Y = f_Y(\varphi, \Delta\lambda)$$

$$ds'^2 = e\,d\varphi^2 + 2f\,d\varphi\,d\Delta\lambda + g\,d\Delta\lambda^2$$

$$e = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2$$

$$f = \frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \Delta\lambda} + \frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial \Delta\lambda}$$

$$g = \left(\frac{\partial x}{\partial \Delta\lambda}\right)^2 + \left(\frac{\partial y}{\partial \Delta\lambda}\right)^2$$

$$H = \sqrt{eg - f^2} = \frac{\partial x}{\partial \varphi}\frac{\partial y}{\partial \Delta\lambda} - \frac{\partial x}{\partial \Delta\lambda}\frac{\partial y}{\partial \varphi}$$

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Linear scales of map projections

$$\mu = \frac{ds'}{ds}$$

$$h = \frac{ds_\varphi{}'}{ds_\varphi} \qquad k = \frac{ds_\lambda{}'}{ds_\lambda}$$

$$ds'_\psi = \sqrt{e}\cdot d\varphi \qquad ds_\varphi = M\,d\varphi$$

$$ds'_{\Delta\lambda} = \sqrt{g}\cdot d\Delta\lambda \qquad ds_{\Delta\lambda} = P\,d\Delta\lambda$$

$$h = \frac{ds'_\varphi}{ds_\varphi} = \frac{\sqrt{e}\,d\varphi}{M\,d\varphi} = \frac{\sqrt{e}}{M}$$

$$k = \frac{ds'_{\Delta\lambda}}{ds_{\Delta\lambda}} = \frac{\sqrt{g}\,d\Delta\lambda}{P\,d\Delta\lambda} = \frac{\sqrt{g}}{P}$$

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Area scale of map projections

$$\xi = \frac{d\sigma'}{d\sigma} = \frac{H\,d\varphi\,d\Delta\lambda}{MP\,d\varphi\,d\Delta\lambda} = \frac{H}{MP}$$

$$H = \sqrt{eg - f^2}$$

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Scale of map projections

Projections for a **spherical** earth:

$$\eta = \frac{\sqrt{e}\, d\bar{\phi}}{R\, d\,\bar{\phi}} = \frac{\sqrt{e}}{R}, \quad \kappa = \frac{\sqrt{g}\, d\lambda}{R\ \cos\bar{\phi}\ d\lambda} = \frac{\sqrt{g}}{R\ \cos\bar{\phi}}$$

$$\xi = \frac{\sqrt{eg - f^2}\ d\bar{\phi}\ d\lambda}{R^2\ \cos\bar{\phi}\ d\bar{\phi}\ d\lambda} = \frac{\sqrt{eg - f^2}}{R^2\ \cos\bar{\phi}}$$

Projections for a **ellipsoidal** earth:

$$\eta = \frac{\sqrt{e}}{M}, \quad \kappa = \frac{\sqrt{g}}{N\ \cos\phi}, \quad \xi = \frac{\sqrt{eg - f^2}}{M\ N\ \cos\phi}$$

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Projection properties

- Conformal projections: $h = k$
- Equal-area projections: $h \bullet k = 1$
- Equidistant projections:
  - $h = 1$ for meridians
  - $k = 1$ for parallel circles
  - $\mu = 1$ for some curves (e.g. great circles)

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Example: Analysis of a certain map projection

The planar coordinates of a certain map projection are computed as:

$$x = \rho_0 - \rho \cos n(\lambda - \lambda_0)$$
$$y = \rho \sin n(\lambda - \lambda_0)$$

where ρ_0, λ_0 and n are constants. ρ is a function of latitude ϕ. ρ and its derivative are :

$$\rho = \rho(\phi) = \frac{R}{n}\sqrt{C - 2n\sin\phi}, \qquad \frac{d\rho}{d\phi} = -R\frac{\cos\phi}{\sqrt{C - 2n\sin\phi}}$$

where R is the mean radius of the earth sphere and C is another constant.

For this projection, find out :

1. a) the first fundamental coefficients e, f, g
   b) the scale factor of the meridian
   c) the scale factor of the parallel circle
   d) the angle θ' between the projections of the meridians and parallel circls
   e) the area scale factor ξ
   f) Is this projection conformal or equivalent ? Why?
   g) If a parallel circle is equidistant, what is the latitude ϕ_0 of this parallel circle ?

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a)

$$\frac{\partial x}{\partial \phi} = -\frac{\partial \rho}{\partial \phi}\cos n(\lambda - \lambda_0) = R\frac{\cos\phi}{\sqrt{C-2n\sin\phi}}\cos n(\lambda - \lambda_0)$$

$$\frac{\partial y}{\partial \phi} = \frac{\partial \rho}{\partial \phi}\sin n(\lambda - \lambda_0) = -R\frac{\cos\phi}{\sqrt{C-2n\sin\phi}}\sin n(\lambda - \lambda_0)$$

$$\frac{\partial x}{\partial \lambda} = n\,\rho\,\sin n(\lambda - \lambda_0)$$

$$\frac{\partial y}{\partial \lambda} = n\,\rho\,\cos n(\lambda - \lambda_0)$$

$$e = (\tfrac{\partial x}{\partial \phi})^2 + (\tfrac{\partial y}{\partial \phi})^2 = \left[R\frac{\cos\phi}{\sqrt{C-2n\sin\phi}}\right]^2, \qquad \sqrt{e} = R\frac{\cos\phi}{\sqrt{C-2n\sin\phi}}$$

$$f = \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \lambda} = 0$$

$$g = (\tfrac{\partial x}{\partial \lambda})^2 + (\tfrac{\partial y}{\partial \lambda})^2 = (n\rho)^2 = \left[R\sqrt{C-2n\sin\phi}\right]^2, \qquad \sqrt{g} = R\sqrt{C-2n\sin\phi}$$

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b)

$$h = \frac{\sqrt{e}}{R} = \frac{\cos\phi}{\sqrt{C-2n\ \sin\phi}}$$

c)

$$k = \frac{\sqrt{g}}{R\cos\phi} = \frac{\sqrt{C-2n\ \sin\phi}}{\cos\phi}$$

d)

$$\cos\theta' = \frac{f}{\sqrt{eg}} = 0 \quad \rightarrow \quad \cos\theta' = 0 \quad \rightarrow \quad \theta' = 90^0$$

e)

$$\sqrt{eg} = R\ \frac{\cos\phi}{\sqrt{C-2n\ \sin\phi}}\ R\sqrt{C-2n\ \sin\phi} = R^2\cos\phi$$

$$\xi = \frac{\sqrt{eg}}{R^2\cos\phi} = 1$$

f)

Because $\xi = 1$, this projection is an equivalent projection.

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g)

A parallel circle at latitude ϕ_0 is equidistant if k is equal to 1. Then we have:

$$k = \frac{\sqrt{C-2n\ \sin\phi_0}}{\cos\phi_0} = 1 \quad \text{or:} \quad C - 2n\sin\phi_0 = \cos^2\phi_0 = 1 - \sin^2\phi_0$$

$$\text{or:} \quad t^2 - 2n\,t + (C-1) = 0 \qquad (t = \sin\phi_0)$$

$$t = n \pm \sqrt{n^2+1-C} \quad \rightarrow \quad \phi_0 = \arcsin\left[n \pm \sqrt{n^2+1-C}\right]$$

Abers equal-area projection

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