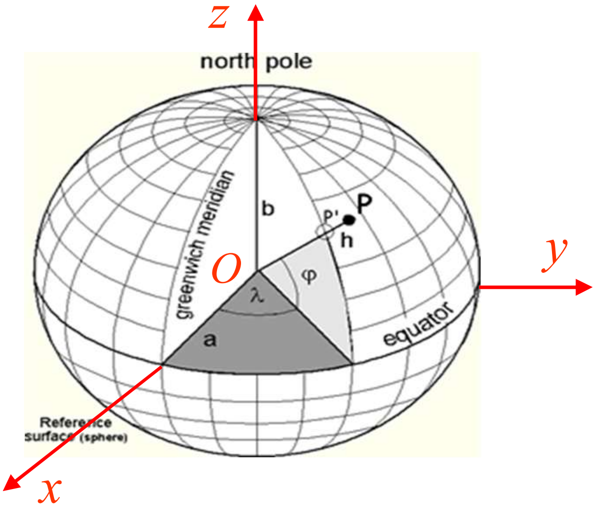
Earth sphere and reference ellipsoid

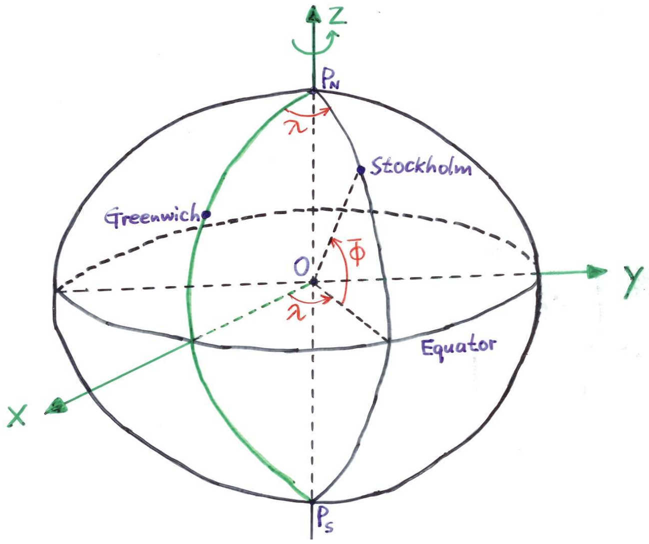
- Earth sphere and main elements
- Spherical coordinates of points on and above the sphere
- Definition of reference ellipsoid
- Ellipsoidal parameters
- Radius of curvature on the reference ellipsoid
- Length of the Meridian and rounding errors in latitude/longitude

Earth sphere & main elements

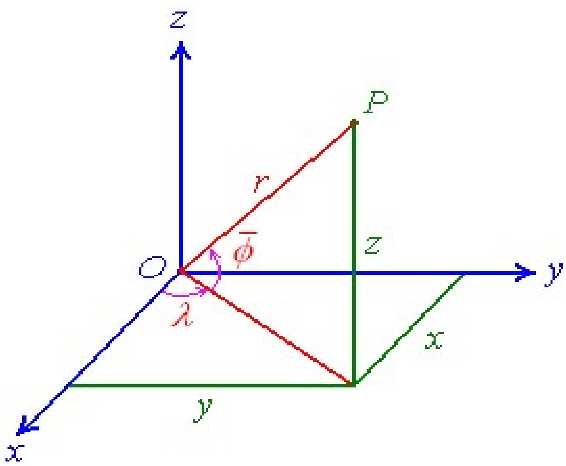


- Geocenter, North Pole and Polar axis
- Meridians and the *Greenwich Meridian*
- Equator and parallel circles
- Spherical coordinates

Spherical coordinates on the earth sphere



Spherical coordinates of point above the sphere



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \ \cos\bar{\phi}\ \cos\lambda \\ r \ \cos\bar{\phi}\ \sin\lambda \\ r \ \sin\bar{\phi} \end{bmatrix}$$

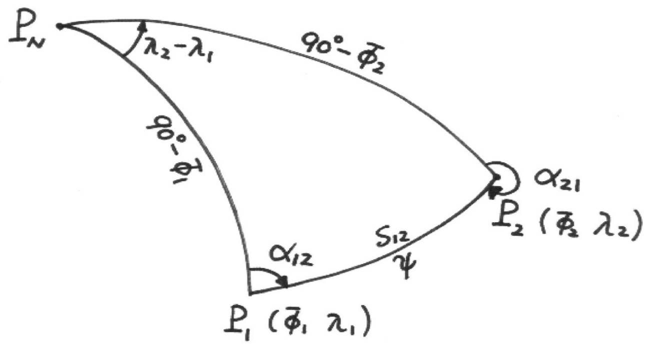
$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\tan\bar{\phi} = \frac{z}{\sqrt{x^2 + y^2}}$$

$$\tan\lambda = \frac{y}{x}$$

Pay attention to quadrants !

Basic geodetic problems on the sphere

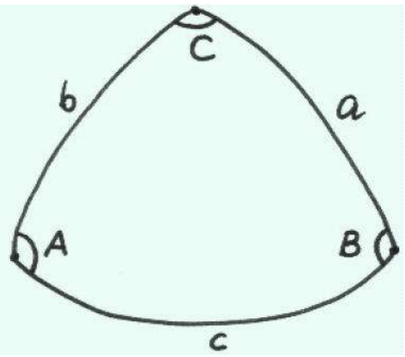


- Direct problem: Find P_2 when P_1 and α_{12}, s_{12} are known
- Inverse problem: Find α_{12}, α_{21}, s_{12} when P_1 , P_2 are known

Concepts of spherical trigonometry

- A plane through the spherical centre cuts the sphere along a great circle. The shortest distance between two points on the sphere is along the great circle through the two points.
- The length of a part of a great circle can be expressed by its geocentric angle → angular distance, spherical distance. Distances/lengths on the sphere have angular units (*radian, deg/min/sec*)
- Three great circles form a spherical triangle
- On a spherical earth, positions can be defined by spherical coordinates (*geocentric latitude, longitude*)

Spherical excess ε



- Sum of 3 internal angles in a spherical triangle is >180°:

  A+B+C **>** *180° (π)*

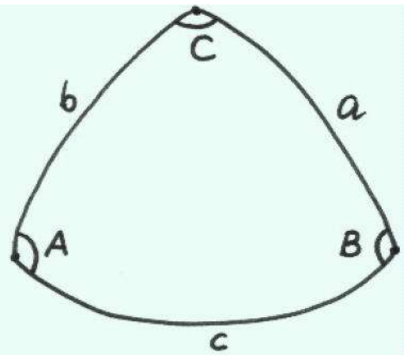
- Spherical excess ε :

  $\varepsilon = A+B+C - 180°$

- $\varepsilon = T/R^2$, T=area, R=radius

- $0 \leq \varepsilon \leq 360°\ (2\pi)$

Spherical trigonometry



Sine theorem

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

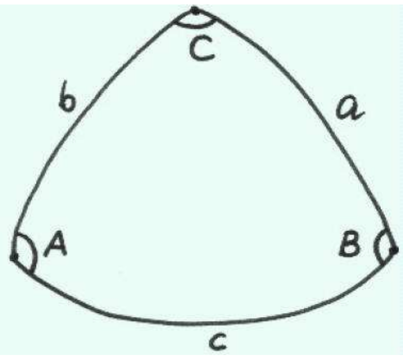
Cosine theorem for sides

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

Cosine theorem for angles

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

Spherical trigonometry



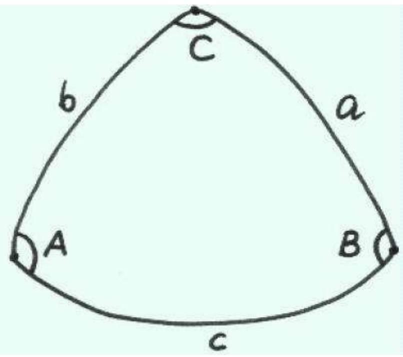
Third theorem for sides

$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A$$
$$\sin a \cos C = \cos c \sin b - \sin c \cos b \cos A$$

Third theorem for angles

$$\sin A \cos b = \cos B \sin C + \sin B \cos C \cos a$$
$$\sin A \cos c = \cos C \sin B + \sin C \cos B \cos a$$

Spherical trigonometry



Napier's formulas

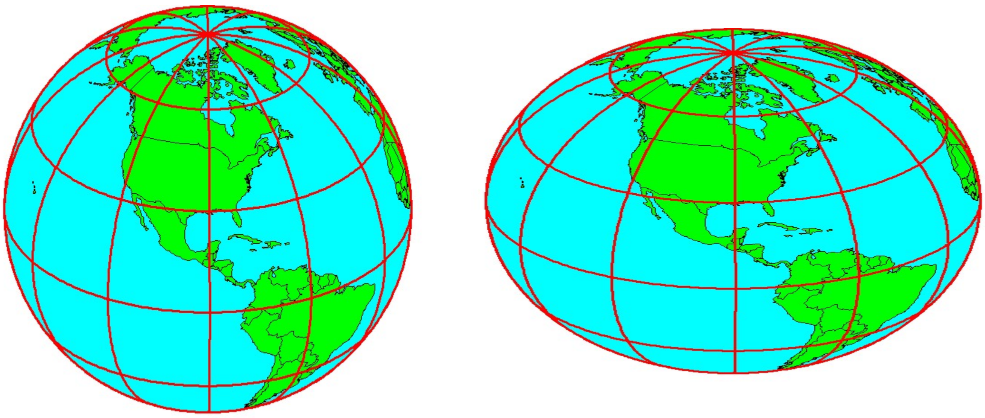
$$\tan \frac{1}{2}(b+c) = \frac{\cos \frac{1}{2}(B-C)}{\cos \frac{1}{2}(B+C)} \tan \frac{1}{2} a$$

$$\tan \frac{1}{2}(b-c) = \frac{\sin \frac{1}{2}(B-C)}{\sin \frac{1}{2}(B+C)} \tan \frac{1}{2} a$$

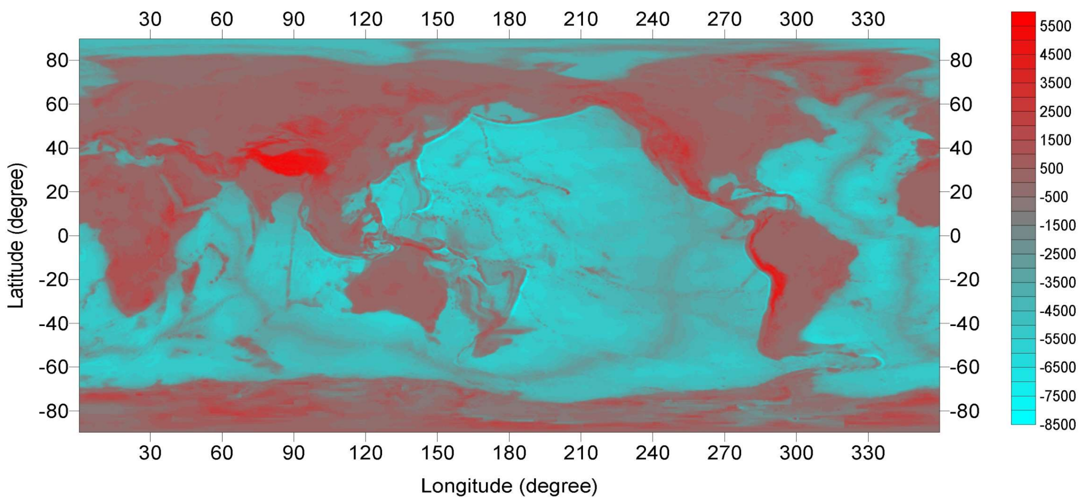
$$\tan \frac{1}{2}(B+C) = \frac{\cos \frac{1}{2}(b-c)}{\cos \frac{1}{2}(b+c)} \cot \frac{1}{2} A$$

$$\tan \frac{1}{2}(B-C) = \frac{\sin \frac{1}{2}(b-c)}{\sin \frac{1}{2}(b+c)} \cot \frac{1}{2} A$$

Spherical vs ellipsoidal earth



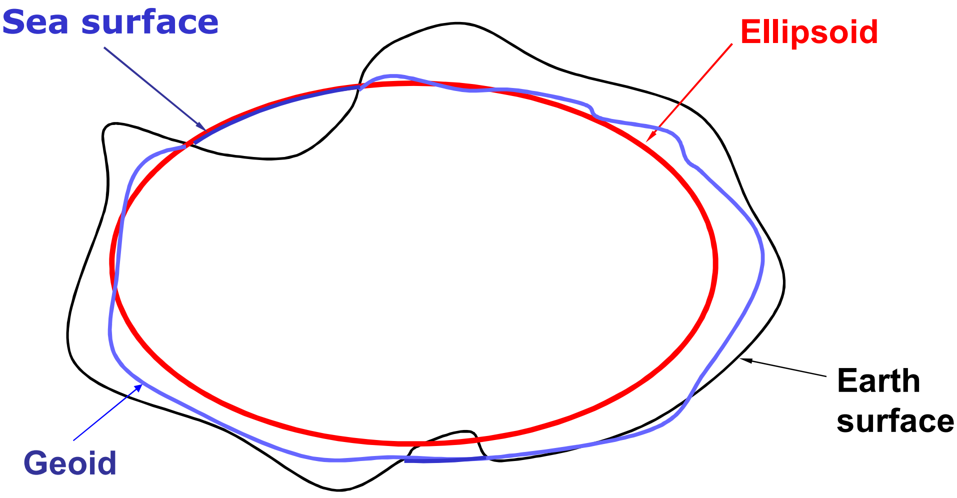
Earth's physical surface



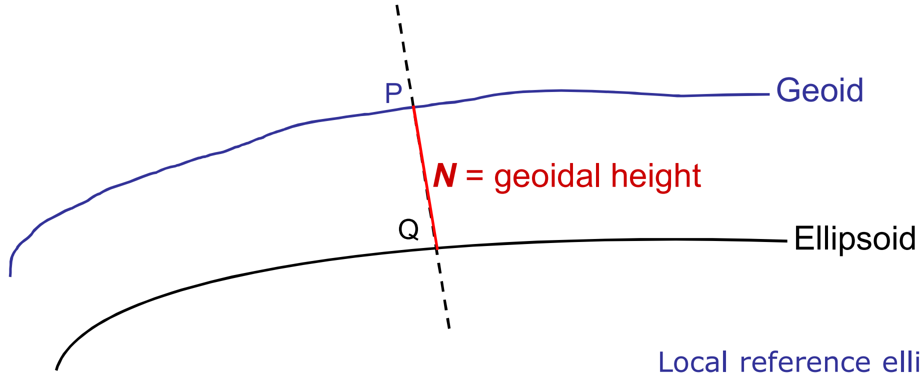
Reference ellipsoid fitted to the geoid

- The continental area of the earth surface is irregular
  - Mt Everest at 8848m above the sea level
- 67% of earth surface is covered by the sea. The sea level is much smoother than the natural surface of the earth
- Global Mean Sea Level (MSL) is a part of a special equi-potential surface in the earth’s gravity field, called the **geoid**
- The geoid coincides with the global mean sea level **and** extends below the continent
- Reference ellipsoid is most close to the geoid in the least squares sense

Fitting an ellipsoid to the geoid



Fitting *an* ellipsoid to *the* geoid



Mean Earth Ellipsoid (σ=whole earth)

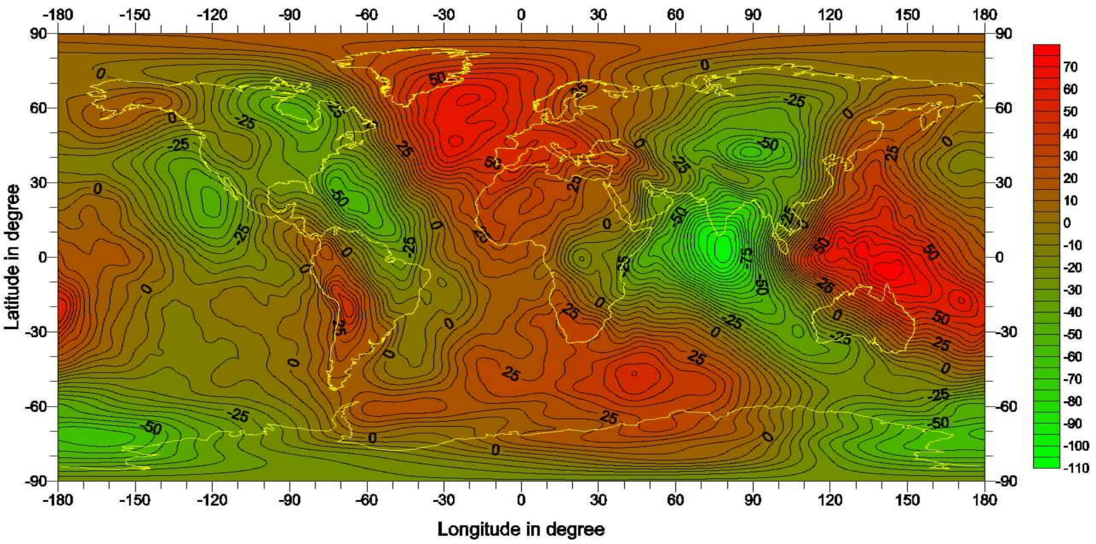
$$\iint_{\sigma} N^2 d\sigma = \text{minimum}$$

Local reference ellipsoid

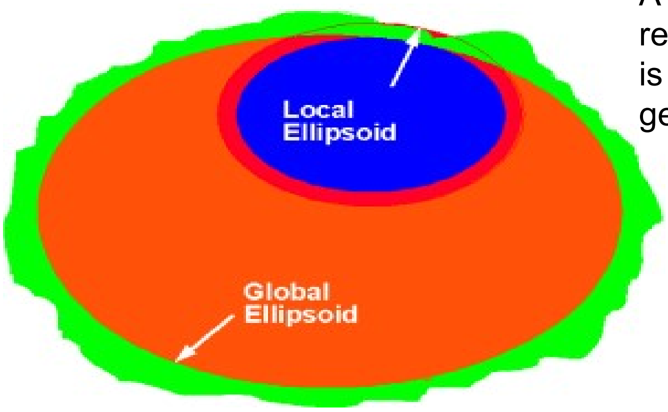
(σ_1 = local area, a country)

$$\iint_{\sigma_1} N^2 d\sigma = \text{minimum}$$

World Geoid JGM2S (Ellipsoid: GRS80)



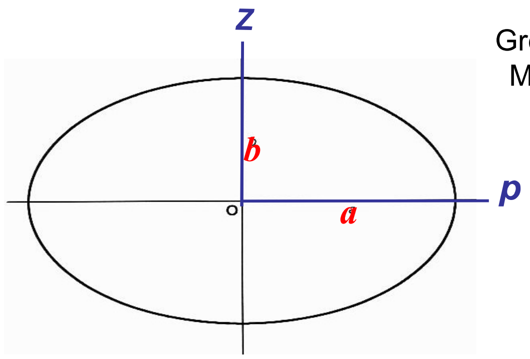
Local vs global reference ellipsoids

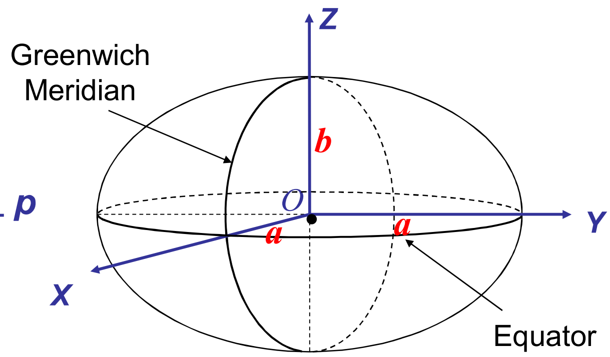


A locally fitted reference ellipsoid is *most often* not geocentric

A globally fitted reference ellipsoid (*Mean Earth Ellpsoid*) is *always* geocentric

From **ellipse** to **ellipsoid of revolution**

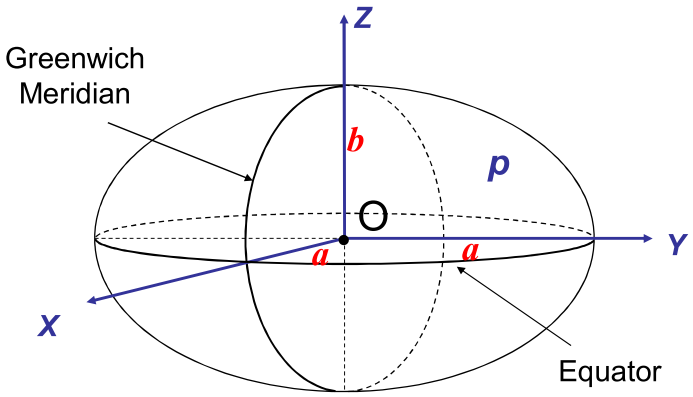




$$\frac{p^2}{a^2} + \frac{z^2}{b^2} = 1$$

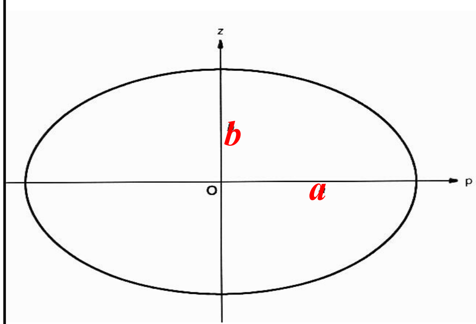
$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1$$

Geometry of an ellipsoid of revolution



- ✓ rotation axis
- ✓ equatorial plane and equator
- ✓ parallel planes and parallel circles
- ✓ meridian planes and meridians
- ✓ Greenwich meridian (Prime Meridian)

Ellipsoidal Parameters



$$f = \frac{a-b}{a}$$

$$e = \frac{\sqrt{a^2-b^2}}{a}$$

$$e' = \frac{\sqrt{a^2-b^2}}{b}$$

Size of the ellipsoid: *a* (or *b*)

Shape of the ellipsoid: *f* (*e*, *e′*)

Only 2 independent parameters:

$$b = a(1-f) = a\sqrt{1-e^2} = \frac{a}{\sqrt{1+e'^2}}$$

$$f = 1 - \sqrt{1-e^2} = 1 - \frac{1}{\sqrt{1+e'^2}}$$

$$e^2 = 2f - f^2 = \frac{e'^2}{1+e'^2}$$

$$e'^2 = \frac{2f-f^2}{(1-f)^2} = \frac{e^2}{1-e^2}$$

Widely used reference ellipsoids

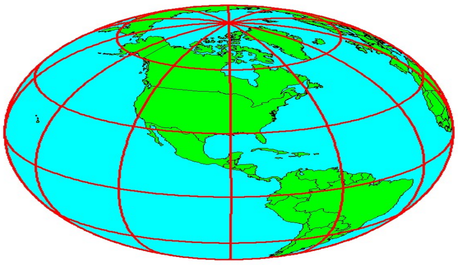
Name	Year	a (m)	1/f	Remarks
Bessel	1841	6 377 397.155	299.152 812 8	Sweden, RT 90
Hayford	1910	6 378 388	297	N America, ED 87
GRS 80	1980	6 378 137	298.257 222 101	Recommended by IAG
WGS 84	1984	6 378 137	298.257 223 563	Used in GPS
Krasovski	1940	6 378 245	298.3	Pulkovo 1942
Clarke	1880	6 378 249.145	293.465	Ethiopia

Some concepts on the ellipsoid

- Normal plane: a plane contains the ellipsoidal normal at a point on the ellipsoid
- Normal section: intersection of a normal plane on the surface of the ellipsoid
- At a point on the surface of the ellipsoid, there are many normal sections in different directions
- One way to under the geomterical property of the ellipsoid is to study the *radius of curvature* of normal sections at a point on the ellipsoid

Radius of curvature on the ellipsoid

- Radius (M) of curvature of the meridian (N-S)



$$M = \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}} = \frac{a(1-e^2)}{W^3}$$

$$W = \sqrt{1-e^2\sin^2\phi}$$

- Radius (N) of curvature in the prime vertical (E-W)

$$N = \frac{a}{W} = \frac{a}{\sqrt{1-e^2\sin^2\phi}}$$

Radius of curvature on the ellipsoid

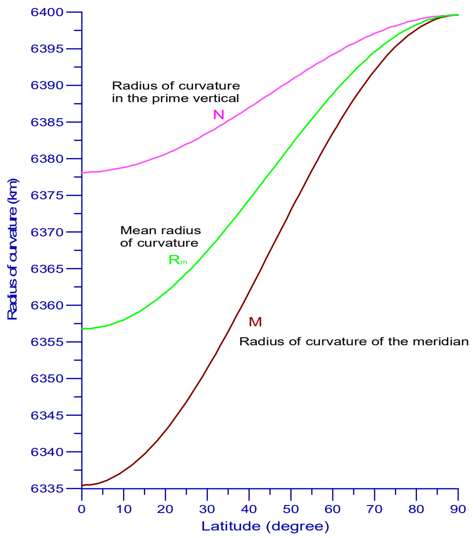
- Radius (Rα) of curvature in direction α

$$\frac{1}{R_\alpha} = \frac{\cos^2\alpha}{M} + \frac{\sin^2\alpha}{N}$$

- Mean radius (Rm) of curvature (Rα averaged over all directions)

$$R_m = \frac{1}{2\pi}\int_0^{2\pi} R_\alpha \, d\alpha = \sqrt{MN} = \frac{a\sqrt{1-e^2}}{W^2}$$

Radius of curvature on the ellipsoid

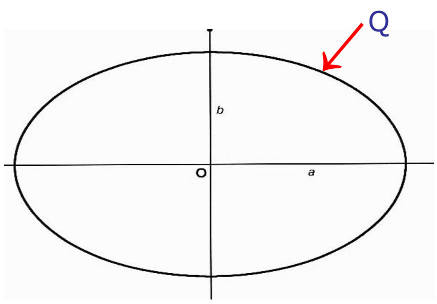


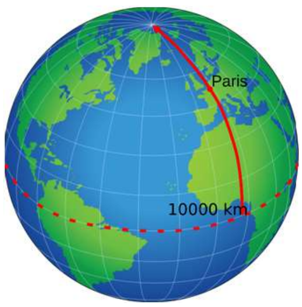
$$M = \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}}$$

$$N = \frac{a}{W} = \frac{a}{\sqrt{1-e^2\sin^2\phi}}$$

$$R_m = \frac{a\sqrt{1-e^2}}{W^2}$$

Length of the Meridian Arc





$$S_m = \int_0^{\phi} ds_m = \int_0^{\phi} M \cdot d\phi = a(1-e^2)\int_0^{\phi} \frac{d\phi}{(1-e^2\sin^2\phi)^{\frac{3}{2}}}$$

$$S_m = a(1-e^2)\left(A_0\phi - \frac{1}{2}A_2\sin 2\phi + \frac{1}{4}A_4\sin 4\phi - \frac{1}{6}A_6\sin 6\phi + \frac{1}{8}A_8\sin 8\phi - \frac{1}{10}A_{10}\sin 10\phi\right)$$

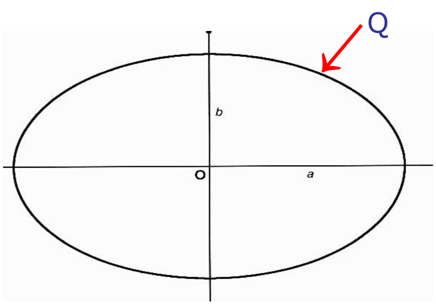
$$Q = S_m(\phi = 90^0) = \frac{\pi}{2}\cdot a(1-e^2)\cdot A_0$$

Length of the Meridian Arc

$$
\begin{aligned}
A_0 &= +1 + \tfrac{3}{4}e^2 + \tfrac{45}{64}e^4 + \tfrac{175}{256}e^6 + \tfrac{11025}{16384}e^8 + \tfrac{43659}{65536}e^{10} + \cdots \\
A_2 &= + \tfrac{3}{4}e^2 + \tfrac{15}{16}e^4 + \tfrac{525}{512}e^6 + \tfrac{2205}{2048}e^8 + \tfrac{72765}{65536}e^{10} + \cdots \\
A_4 &= + \tfrac{15}{64}e^4 + \tfrac{105}{256}e^6 + \tfrac{2205}{4096}e^8 + \tfrac{10395}{16384}e^{10} + \cdots \\
A_6 &= + \tfrac{35}{512}e^6 + \tfrac{315}{2048}e^8 + \tfrac{31185}{131072}e^{10} + \cdots \\
A_8 &= + \tfrac{315}{16384}e^8 + \tfrac{3465}{65536}e^{10} + \cdots \\
A_{10} &= + \tfrac{693}{131072}e^{10} + \cdots
\end{aligned}
$$

Length of the Meridian Arc

Length (Q) of the meridian arc from the Equator to the Pole



$$Q = \frac{\pi}{2} \cdot a(1 - e^2) \cdot A_0$$

GRS 80 ellipsoid

$$Q = 10\,001\,965.7293 \;\; metres$$

$$1^o \sim \frac{Q}{90^o} \approx 111\,132.9525 \; metres$$

$$1' \sim \frac{Q}{90 \times 60'} \approx 1\,852.2159 \; metres$$

$$1'' \sim \frac{Q}{90 \times 60 \times 60''} \approx 30.8703 \; metres$$

How to keep rounding errors < 0.*1 mm*

- 1 degree Meridian arc ~ 111 km ~ 111 000 000 mm

  → 1 mm ~ 0.000 000 01 degree

- If we want to keep rounding errors smaller than ***0.1 mm***, then *one should keep 9 digits when writing latitudes or longitudes in decimal degrees*:

  φ = 59.123 456 789°

- 1″ Meridian arc ~ 31 metres ~ 31 000 mm

  → 1 mm ~ 0.000 03″ (arcsecond)

- To keep rounding errors smaller than **0.1 mm**, *one should keep 6 digits in arcseconds for* **φ** *and* ***λ*** *:*

  φ = 12° 13' 00.123 456 "