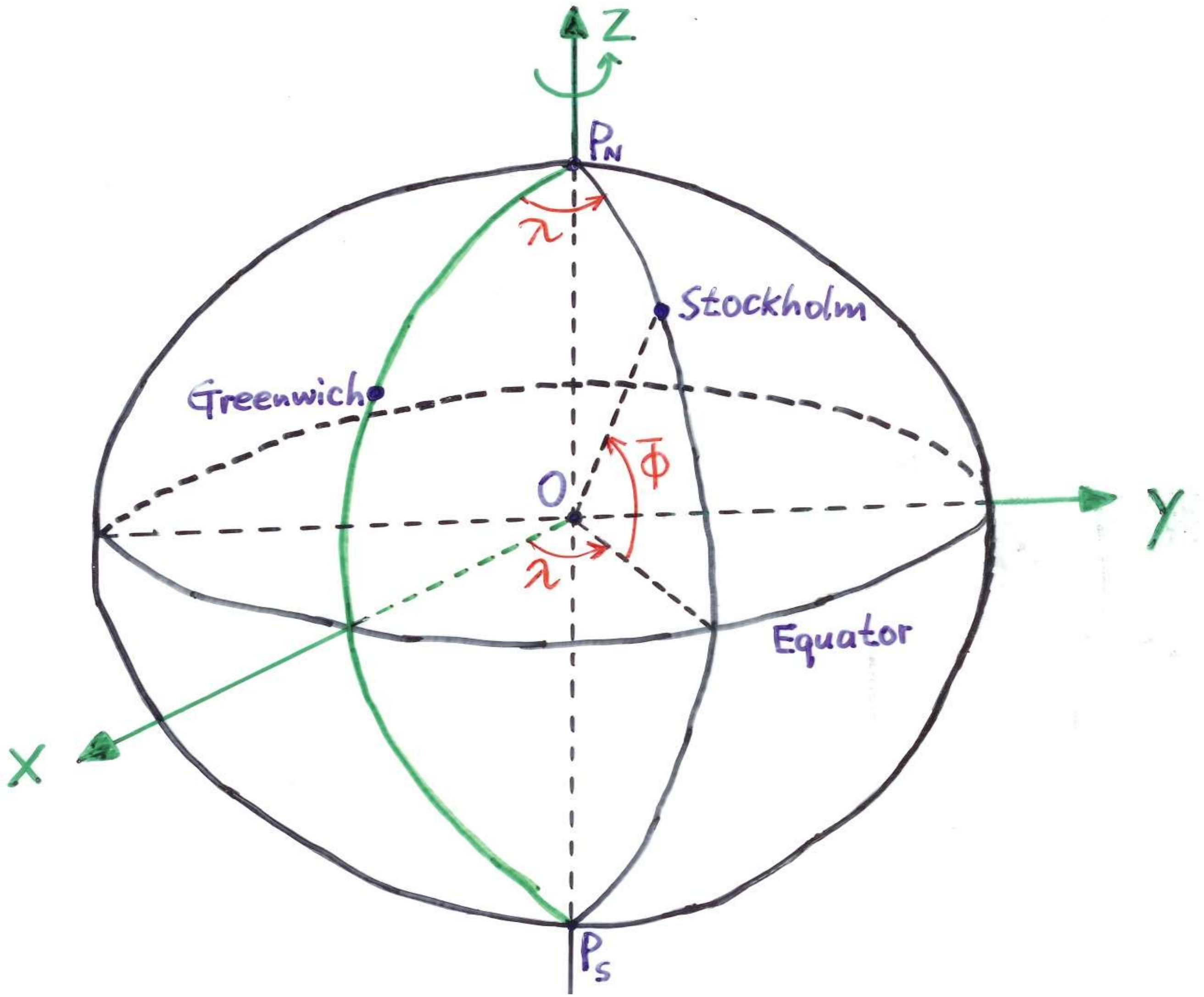
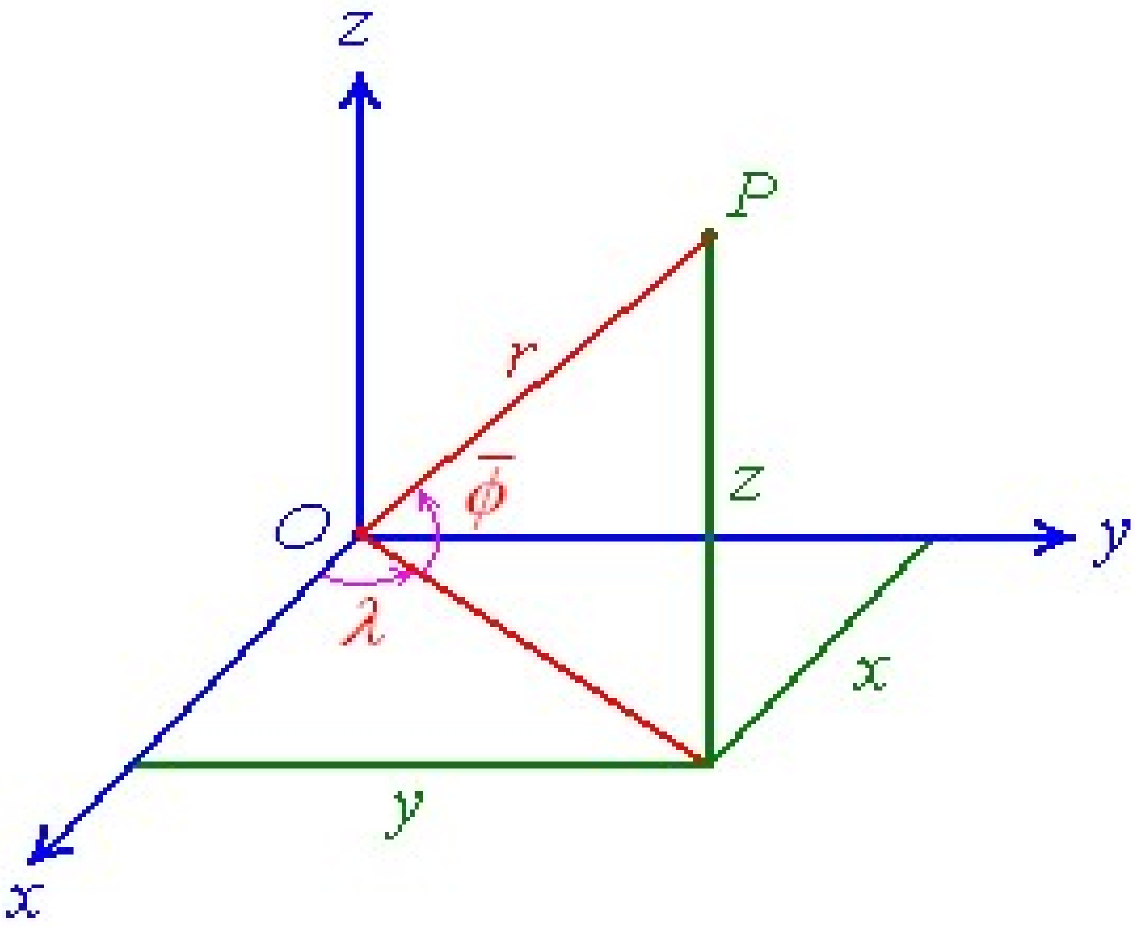
Geodetic coordinates

- Spherical coordinates vs rectangular coordinates (x,y,z)
- **Geodetic coordinates** *(φ, λ, h) vs* rectangular coordinates
- Reduced latitude for a point **on** the reference ellipsoid
- Differential formulas between (x, y, z) and (φ, λ, h)
- Topocentric coordinates

Spherical coordinates on a spherical earth



Spherical vs rectangular coordinates



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \ \cos\bar{\phi}\ \cos\lambda \\ r \ \cos\bar{\phi}\ \sin\lambda \\ r \ \sin\bar{\phi} \end{bmatrix}$$

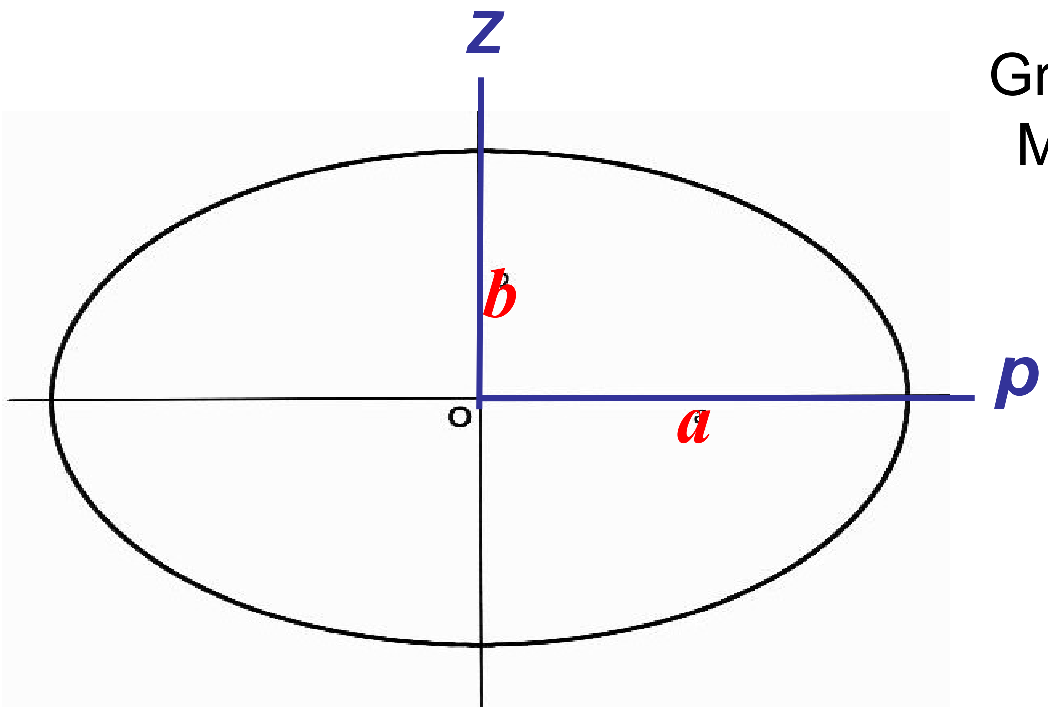
$$r = \sqrt{x^2 + y^2 + z^2}$$

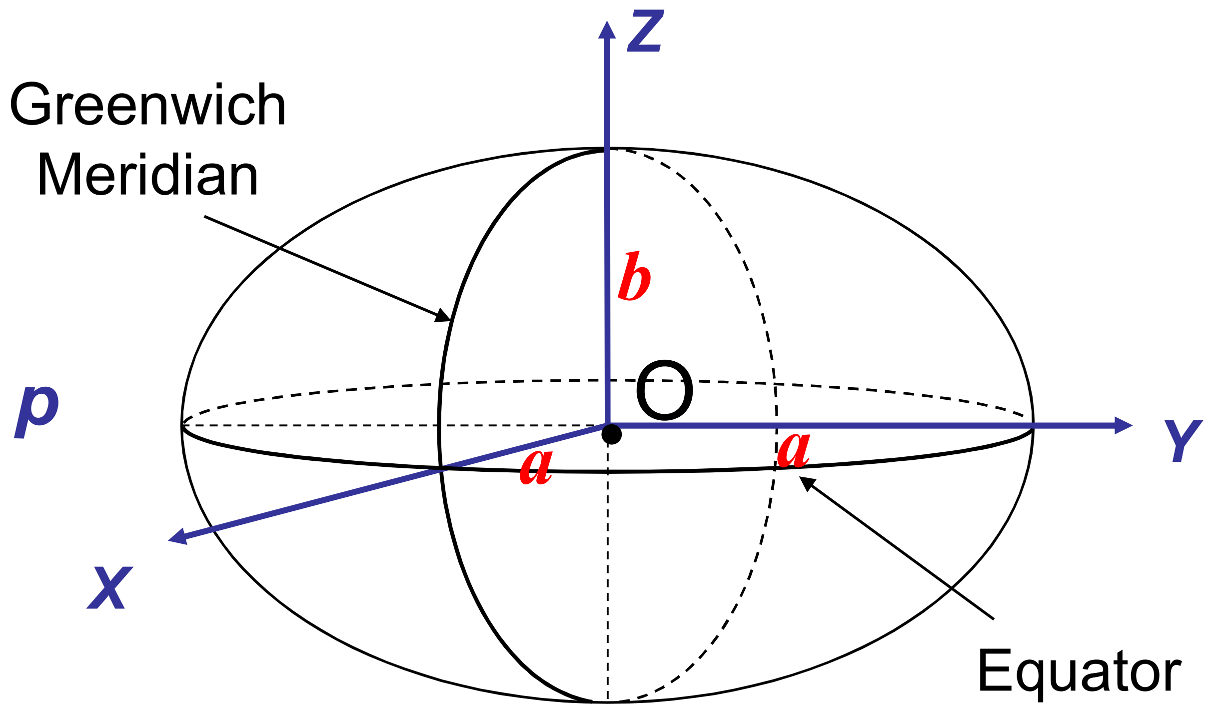
$$\tan\bar{\phi} = \frac{z}{\sqrt{x^2 + y^2}}$$

$$\tan\lambda = \frac{y}{x}$$

Pay attention to quadrants !

From **ellipse** to **ellipsoid of revolution**

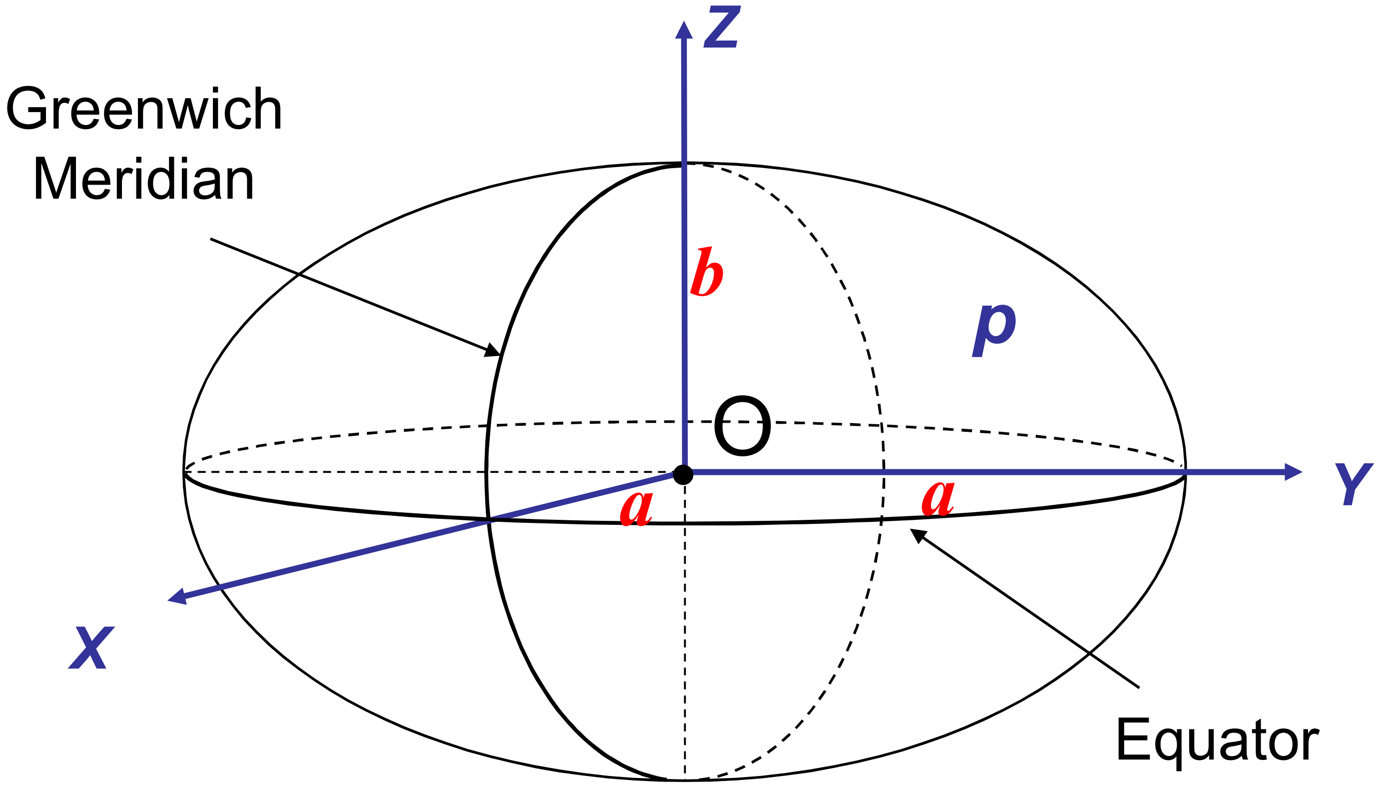




$$\frac{p^2}{a^2} + \frac{z^2}{b^2} = 1$$

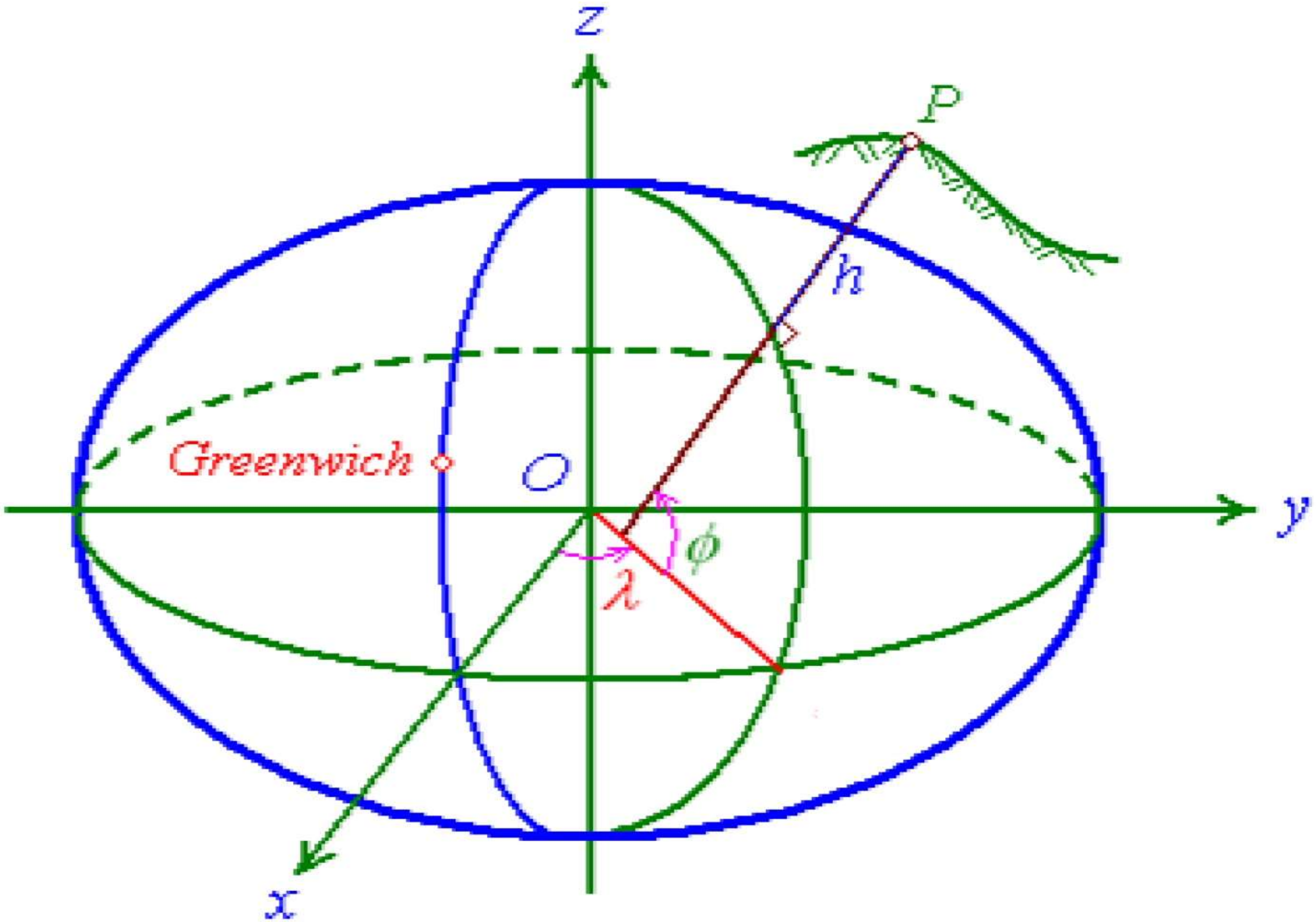
$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1$$

Geometry of an ellipsoid of revolution



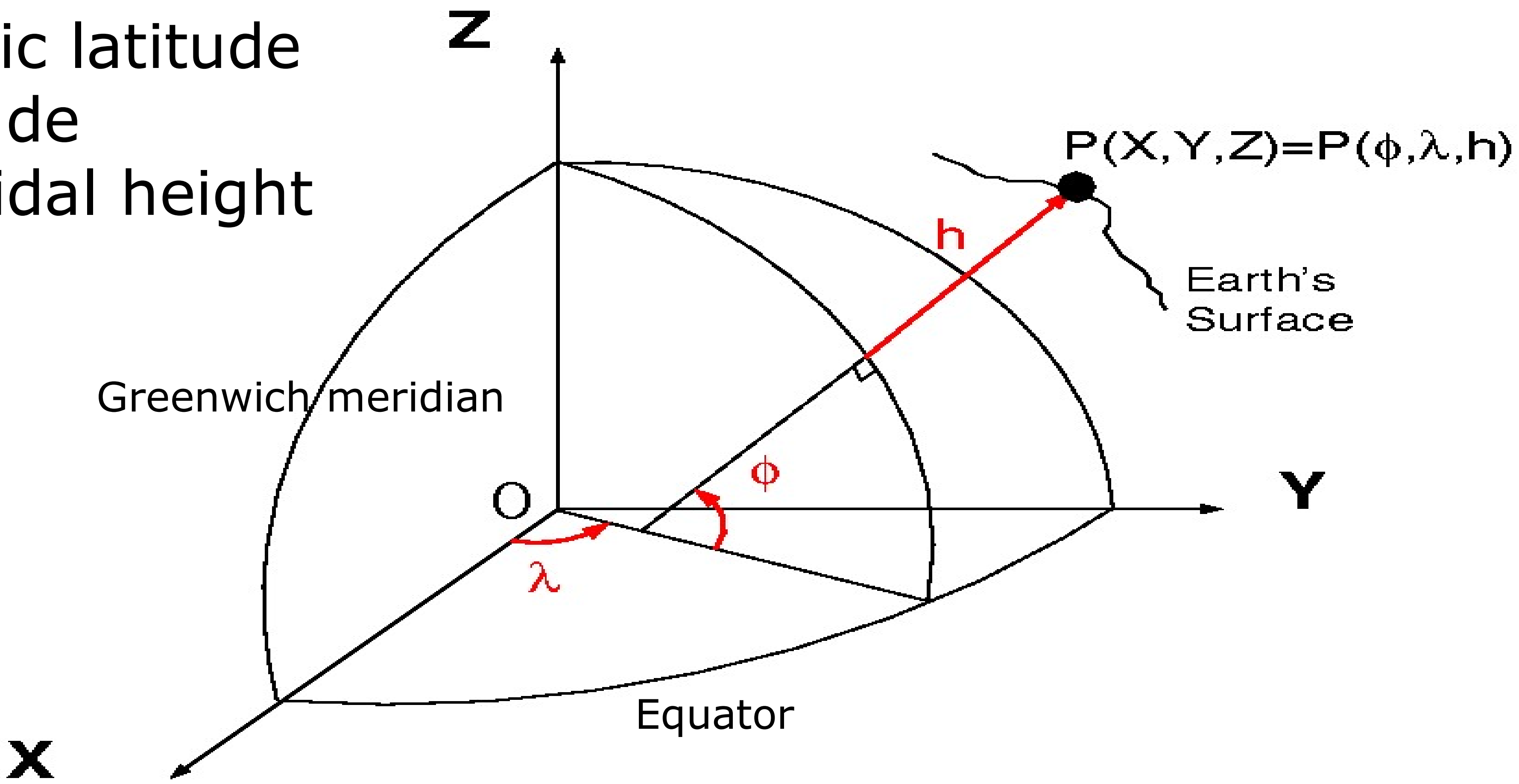
- ✓ rotation axis
- ✓ equatorial plane and equator
- ✓ parallel planes and parallel circles
- ✓ meridian planes and meridians
- ✓ Greenwich meridian (Prime Meridian)

Definition of Geodetic Coordinates



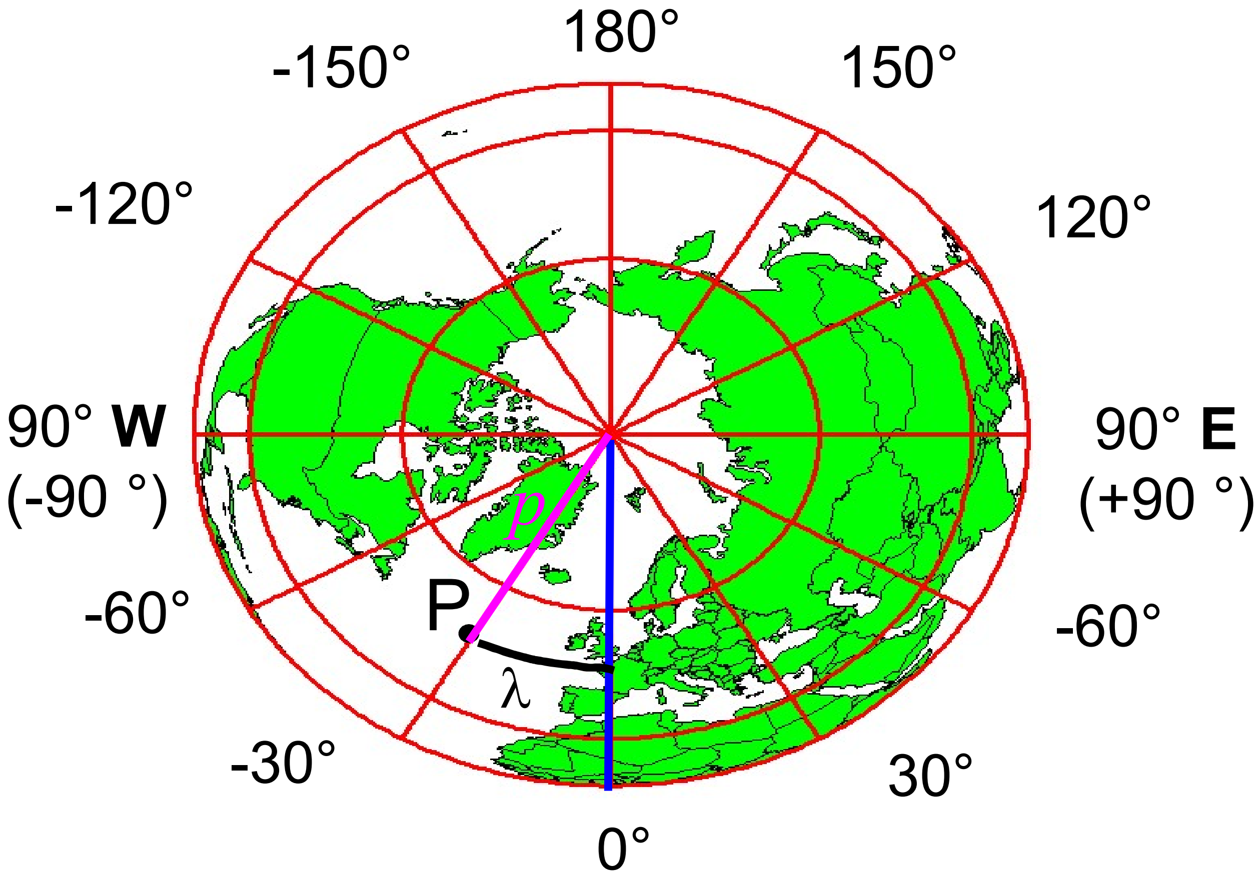
Definition of Geodetic Coordinates

φ: geodetic latitude
λ: longitude
h: ellipsoidal height

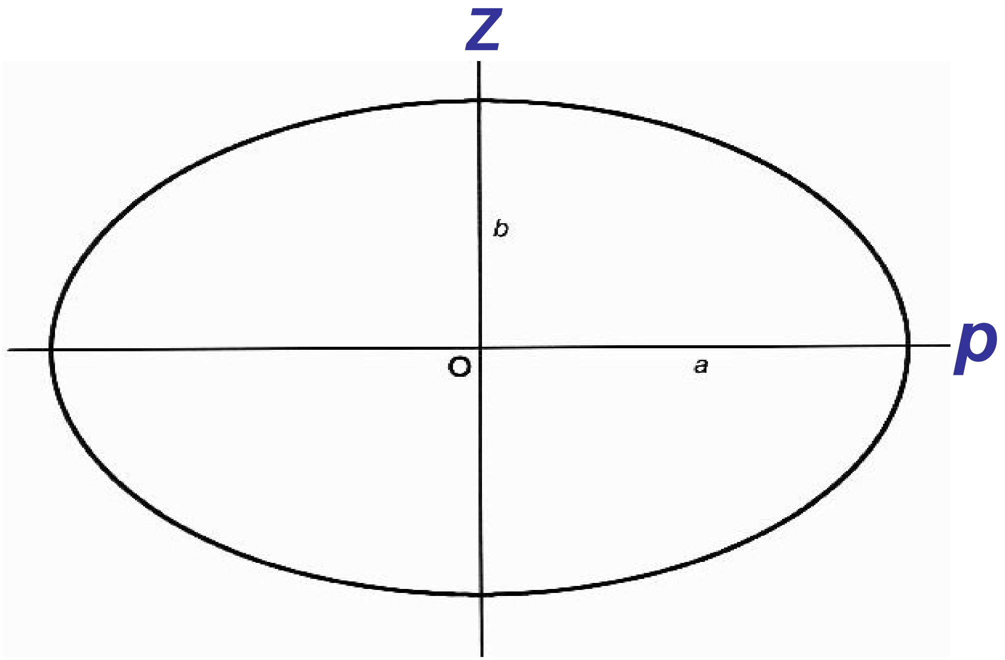


Definition of Longitude, λ

λ = the angle between a cutting plane on the prime meridian and the cutting plane on the meridian through point P



Relation between *x, y* and *p, λ*

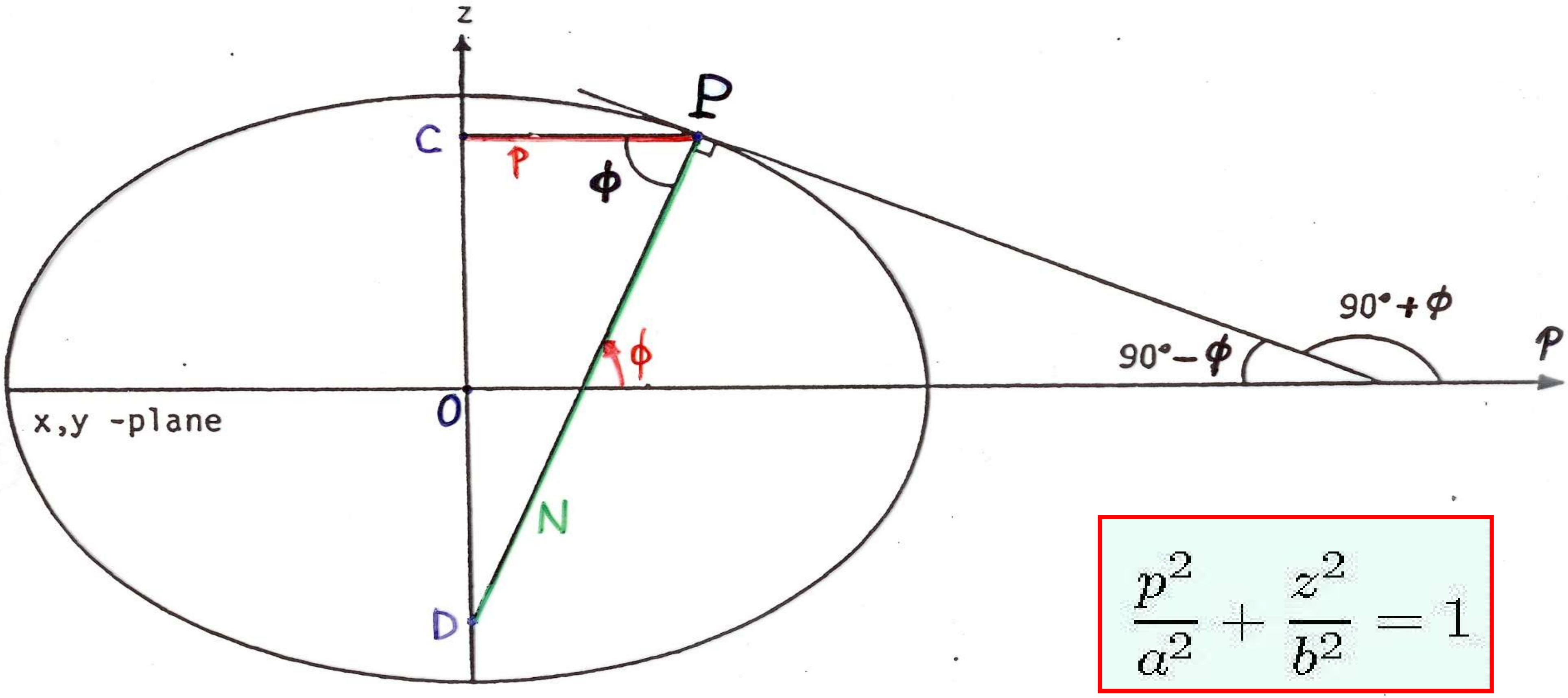


$$x = p \cdot \cos \lambda$$
$$y = p \cdot \sin \lambda$$

$$p = \sqrt{x^2 + y^2}$$

$$\frac{p^2}{a^2} + \frac{z^2}{b^2} = 1$$

Meridian ellipse and latitude

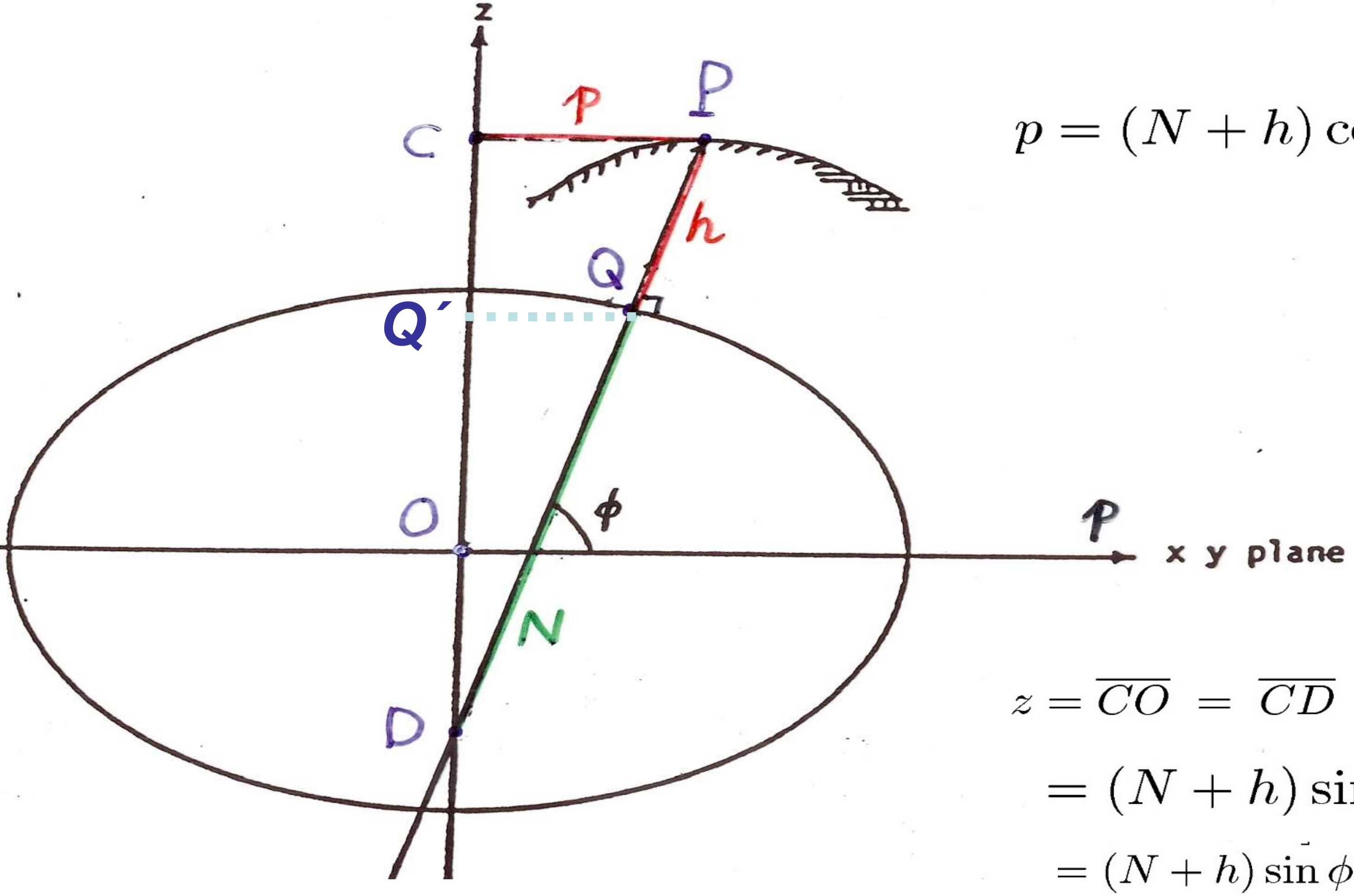


$$\frac{p^2}{a^2} + \frac{z^2}{b^2} = 1$$

Point *P* on the reference ellipsoid

$$x = p \cdot \cos\lambda = N\cos\phi \cdot \cos\lambda$$
$$y = p \cdot \sin\lambda = N\cos\phi \cdot \sin\lambda$$
$$z = (1 - e^2)\ N\ \sin\phi$$

Point *P* above the reference ellipsoid



$$p = (N + h)\cos\phi$$

$$z = \overline{CO} = \overline{CD} - \overline{DO} = \overline{CD} - \left[\overline{DQ'} - \overline{OQ'}\right]$$

$$= (N + h)\sin\phi - \left[N\sin\phi - z_{h=0}\right] =$$

$$= (N + h)\sin\phi - \left[N\sin\phi - \left(1 - e^2\right)N\sin\phi\right]$$

$$= \left[N\left(1 - e^2\right) + h\right]\sin\phi$$

From geodetic to Cartesian coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (N+h)\ \cos\phi\ \cos\lambda \\ (N+h)\ \cos\phi\ \sin\lambda \\ \left[N(1-e^2)+h\right]\ \sin\phi \end{pmatrix}$$

Remember these formulas!

$$N = \frac{a}{W} = \frac{a}{\sqrt{1-e^2\sin^2\phi}}$$

Transformation from Cartesian coordinates to geodetic coordinates

Approximate and sufficiently accurate, closed formulas

$$\boxed{\tan\lambda = \frac{y}{x}}$$

Pay attention to quadrants !

$$p = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{z}{p\sqrt{1-e^2}}$$

$$\boxed{\tan\phi = \frac{z + \frac{a\cdot e^2}{\sqrt{1-e^2}}\sin^3\theta}{p - a\cdot e^2\cdot\cos^3\theta}}$$

$$N = \frac{a}{\sqrt{1 - e^2\sin^2\phi}}$$

$$\boxed{h = \frac{p}{\cos\phi} - N}$$

Transformation from Cartesian coordinates to geodetic coordinates

When the point P is right **on** the reference ellipsoid (h=0)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (N+h)\ \cos\phi\ \cos\lambda \\ (N+h)\ \cos\phi\ \sin\lambda \\ \left[N(1-e^2)+h\right]\ \sin\phi \end{pmatrix}$$

$$\tan\phi = \frac{1}{1-e^2}\frac{z}{\sqrt{x^2+y^2}}, \qquad \tan\lambda = \frac{y}{x}$$

Pay attention to quadrants !

Iterative method to compute *(φ, λ, h)*

When the point P is **not on** the reference ellipsoid (h≠0)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (N+h)\ \cos\phi\ \cos\lambda \\ (N+h)\ \cos\phi\ \sin\lambda \\ \left[N(1-e^2)+h\right]\ \sin\phi \end{pmatrix}$$

$$\tan\lambda = \frac{y}{x}$$

$\boxed{h_0 = 0}$

$$\tan\phi_n = \frac{z}{p\left(1 - e^2 \frac{N_{n-1}}{N_{n-1}+h_{n-1}}\right)} \quad (n = 1, 2, 3, \cdots)$$

$$N_n = \frac{a}{\sqrt{1 - e^2 sin^2\phi_n}} \quad (n = 1, 2, 3, \cdots)$$

$$h_n = \frac{p}{cos\phi_n} - N_n$$

Transformation from Cartesian coordinates to geodetic coordinates

When the point P is not on the reference ellipsoid $(h \neq 0)$

Iterative computation of latitude and height:

$n=0$: assume $h_0 = 0$,

$n=1$: $\rightarrow \varphi_1 \rightarrow N_1 \rightarrow h_1$

$n=2$: $\rightarrow \varphi_2 \rightarrow N_2 \rightarrow h_2$

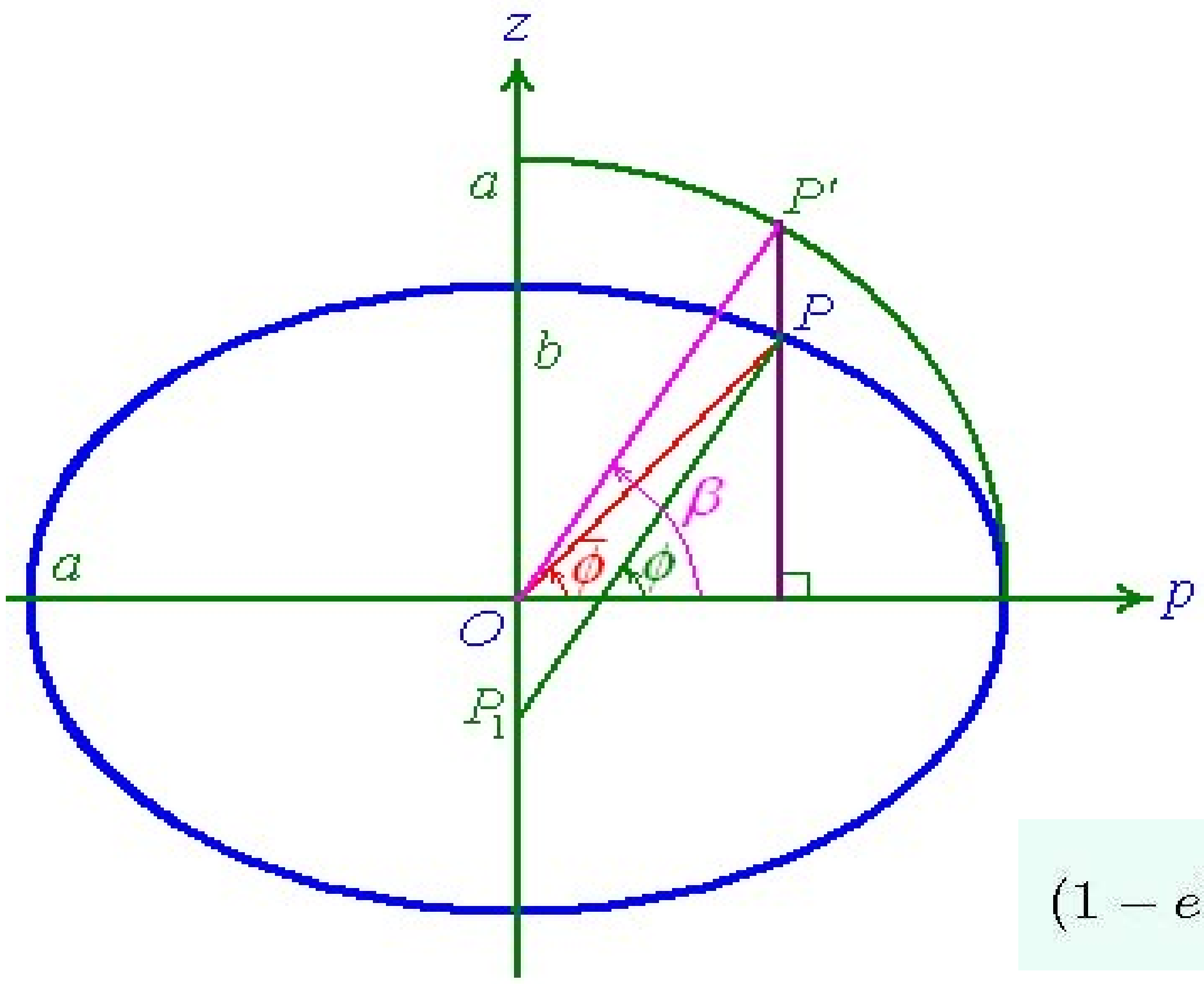
$n=3$: $\rightarrow \varphi_3 \rightarrow N_3 \rightarrow h_3$

…………………………………………

$n-1$: $\rightarrow \varphi_{n-1} \rightarrow N_{n-1} \rightarrow h_{n-1}$

n : $\rightarrow \varphi_n \rightarrow N_n \rightarrow h_n$

3 types of latitutude on the ellipsoid



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \ \cos\beta \ \cos\lambda \\ a \ \cos\beta \ \sin\lambda \\ a\sqrt{1-e^2} \ \sin\beta \end{pmatrix}$$

$$(1-e^2)\tan\phi = \tan\overline{\phi} = \sqrt{1-e^2}\tan\beta$$

Geodetic versus Cartesian coordinates

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (N+h)\ \cos\phi\ \cos\lambda \\ (N+h)\ \cos\phi\ \sin\lambda \\ \left[N(1-e^2)+h\right]\ \sin\phi \end{bmatrix}$$

$$\boxed{\begin{bmatrix} dx \\ \\ dy \\ \\ dz \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \phi}\cdot d\phi + \frac{\partial x}{\partial \lambda}\cdot d\lambda + \frac{\partial x}{\partial h}\cdot dh \\ \\ \frac{\partial y}{\partial \phi}\cdot d\phi + \frac{\partial y}{\partial \lambda}\cdot d\lambda + \frac{\partial y}{\partial h}\cdot dh \\ \\ \frac{\partial z}{\partial \phi}\cdot d\phi + \frac{\partial z}{\partial \lambda}\cdot d\lambda + \frac{\partial z}{\partial h}\cdot dh \end{bmatrix} = Q^\top \cdot \kappa^{-1} \cdot \begin{bmatrix} M\cdot d\phi \\ N\cos\phi\cdot d\lambda \\ dh \end{bmatrix}}$$

$$\kappa = \begin{bmatrix} \frac{M}{M+h} & & \\ & \frac{N}{N+h} & \\ & & 1 \end{bmatrix} \qquad Q(\phi,\lambda) = \begin{bmatrix} -\sin\phi\ \cos\lambda & -\sin\phi\ \sin\lambda & \cos\phi \\ -\sin\lambda & \cos\lambda & 0 \\ \cos\phi\ \cos\lambda & \cos\phi\ \sin\lambda & \sin\phi \end{bmatrix}$$

Geodetic versus Cartesian coordinates

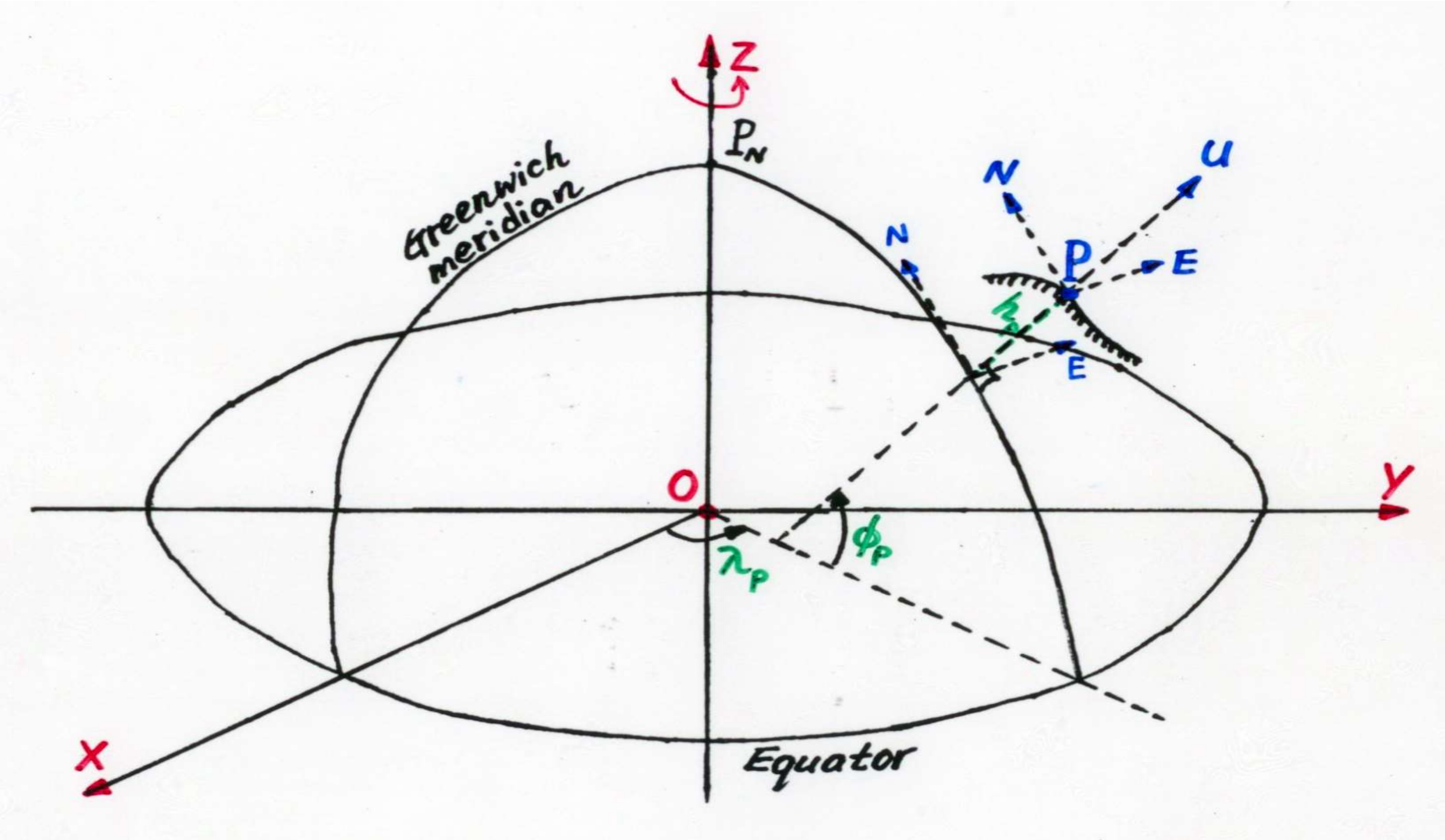
$$\begin{bmatrix} dx \\ \\ dy \\ \\ dz \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \phi} \cdot d\phi + \frac{\partial x}{\partial \lambda} \cdot d\lambda + \frac{\partial x}{\partial h} \cdot dh \\ \\ \frac{\partial y}{\partial \phi} \cdot d\phi + \frac{\partial y}{\partial \lambda} \cdot d\lambda + \frac{\partial y}{\partial h} \cdot dh \\ \\ \frac{\partial z}{\partial \phi} \cdot d\phi + \frac{\partial z}{\partial \lambda} \cdot d\lambda + \frac{\partial z}{\partial h} \cdot dh \end{bmatrix} = Q^\top \cdot \kappa^{-1} \cdot \begin{bmatrix} M \cdot d\phi \\ N \cos\phi \cdot d\lambda \\ dh \end{bmatrix}$$

$$\boxed{\begin{bmatrix} M \cdot d\phi \\ N \cos\phi \cdot d\lambda \\ dh \end{bmatrix} = \kappa(\phi, \lambda) \cdot Q \cdot \begin{bmatrix} dx \\ \\ dy \\ \\ dz \end{bmatrix}}$$

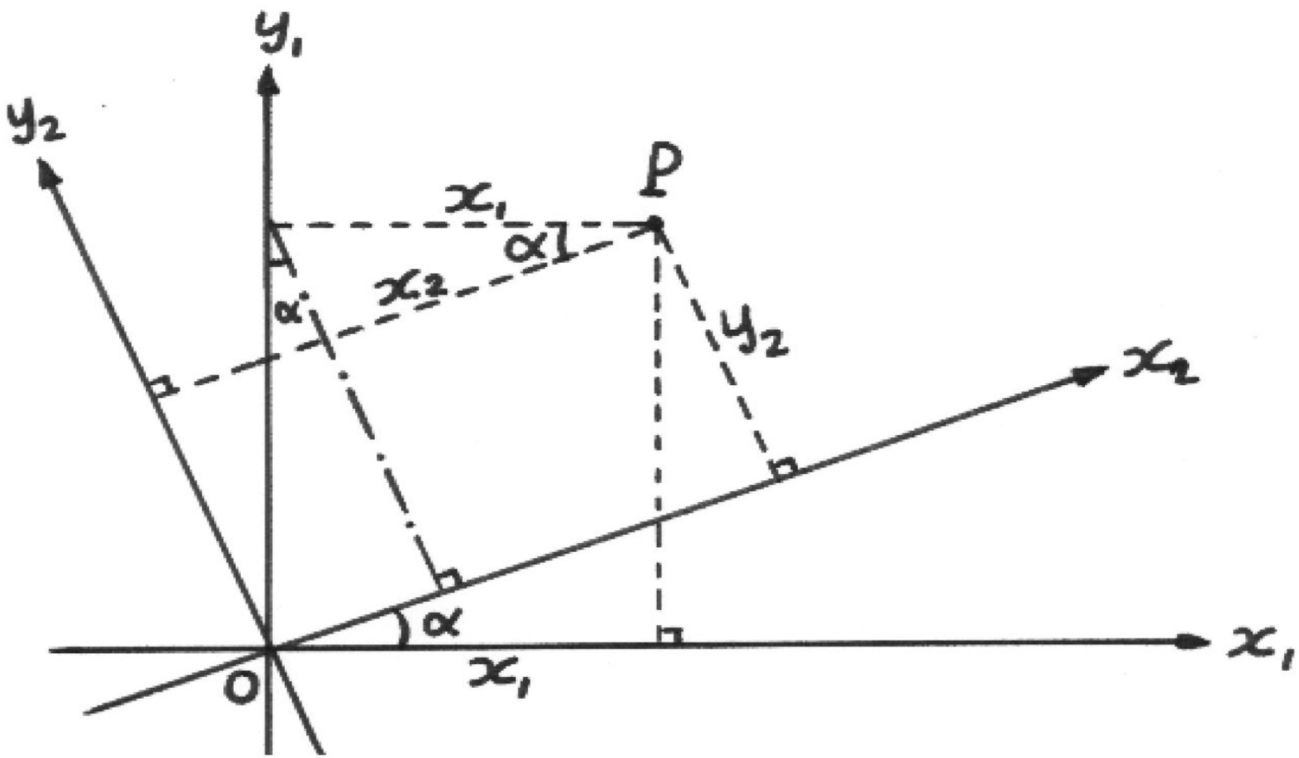
Topocentric coordinates

- Origin at a ground point $P\ (X_p,\ Y_p,\ Z_p)$
- Horizontal axes toward the North (N) and East (E)
- Vertical axis (U) upward along the plumb line
- P-NEU forms a ***left-handed system***
- The plumb line can be approximated by the ellipsoidal normal

Topocentric coordinates (N,E,U)



Rotation around the z-axis



$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = R \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$$R = R_3(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation Matrices

$$R = R_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = R \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$$R = R_2(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$R = R_3(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Topocentric coordinates

$$\begin{bmatrix} -N \\ E \\ U \end{bmatrix} = R_2(90^0 - \Phi_p)\ R_3(\Lambda_p) \begin{bmatrix} X - X_p \\ Y - Y_p \\ Z - Z_p \end{bmatrix}$$

$$\boxed{\begin{bmatrix} N \\ E \\ U \end{bmatrix} = Q(\Phi_p, \Lambda_p) \begin{bmatrix} X - X_p \\ Y - Y_p \\ Z - Z_p \end{bmatrix}}$$

$$Q(\Phi_p, \Lambda_p) = \begin{bmatrix} -\sin\Phi_p \cos\Lambda_p & -\sin\Phi_p \sin\Lambda_p & \cos\Phi_p \\ -\sin\Lambda_p & \cos\Lambda_p & 0 \\ \cos\Phi_p \cos\Lambda_p & \cos\Phi_p \sin\Lambda_p & \sin\Phi_p \end{bmatrix} \approx \begin{bmatrix} -\sin\phi_p \cos\lambda_p & -\sin\phi_p \sin\lambda_p & \cos\phi_p \\ -\sin\lambda_p & \cos\lambda_p & 0 \\ \cos\phi_p \cos\lambda_p & \cos\phi_p \sin\lambda_p & \sin\phi_p \end{bmatrix} \quad \begin{pmatrix} \Phi_p \approx \phi_p \\ \Lambda_p \approx \lambda_p \end{pmatrix}$$

Differential topocentric coordinates

$$\begin{bmatrix} N \\ E \\ U \end{bmatrix} = Q\left(\phi_p, \lambda_p\right) \cdot \begin{bmatrix} X - X_p \\ Y - Y_p \\ Z - Z_p \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} \Delta N \\ \Delta E \\ \Delta U \end{bmatrix} = Q\left(\phi_p, \lambda_p\right) \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}$$

$$\begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} = Q^{-1} \begin{bmatrix} \Delta N \\ \Delta E \\ \Delta U \end{bmatrix} = \begin{bmatrix} -\sin\phi_p \cos\lambda_p & -\sin\lambda_p & \cos\phi_p \cos\lambda_p \\ -\sin\phi_p \sin\lambda_p & \cos\lambda_p & \cos\phi_p \sin\lambda_p \\ \cos\phi_p & 0 & \sin\phi_p \end{bmatrix} \begin{bmatrix} \Delta N \\ \Delta E \\ \Delta U \end{bmatrix}$$