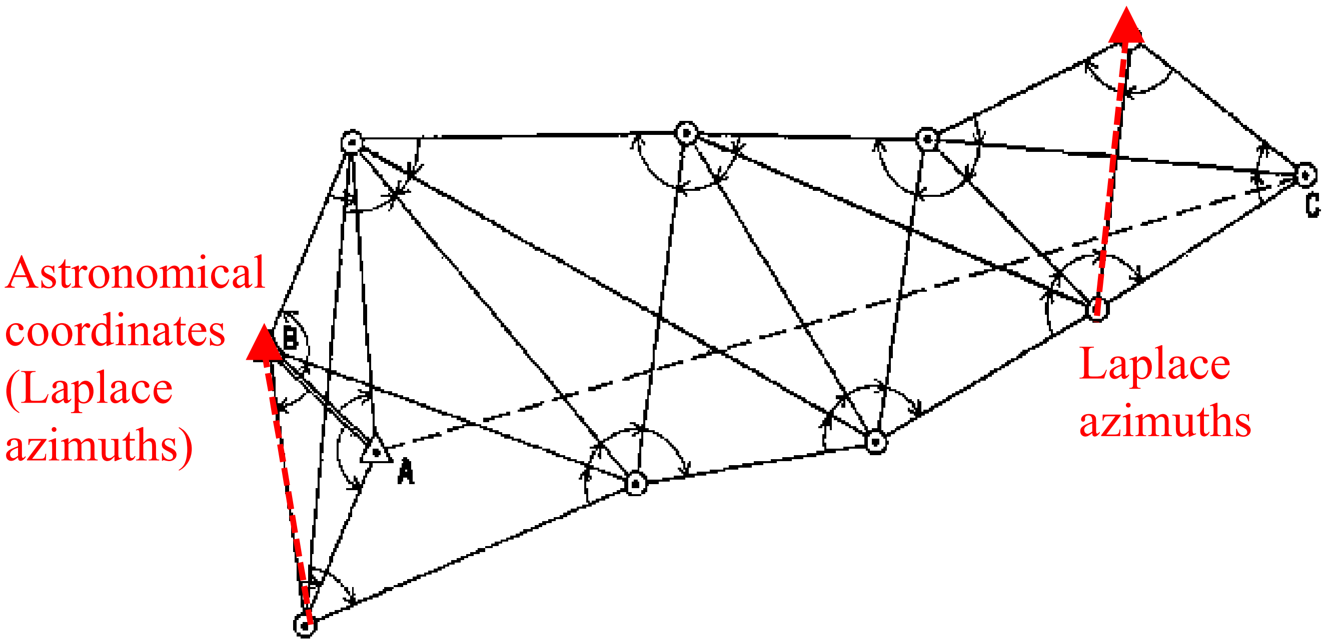
Astrogeodetic triangulation and geodetic datums

- General procedure of astrogeodetic triangulation
- Geodetic datum (Ellipsoidal datum)
- Local triangulation-based coordinate systems versus global, GNSS-based coordinate systems

General procedures

- Select a reference ellipsoid and define a *datum*
- Design a network of triangles
- Astronomical observations (*a fewer*)
- Geodetic measurements (angles, distances etc)
- Reduction of ground meaurements to the ellipsoid
- Least squares adjustment on the ellipsoid
- Define an official 2D coordinate system
- Re-estimation of new ellipsoidal parameters

Principle of Triangulation



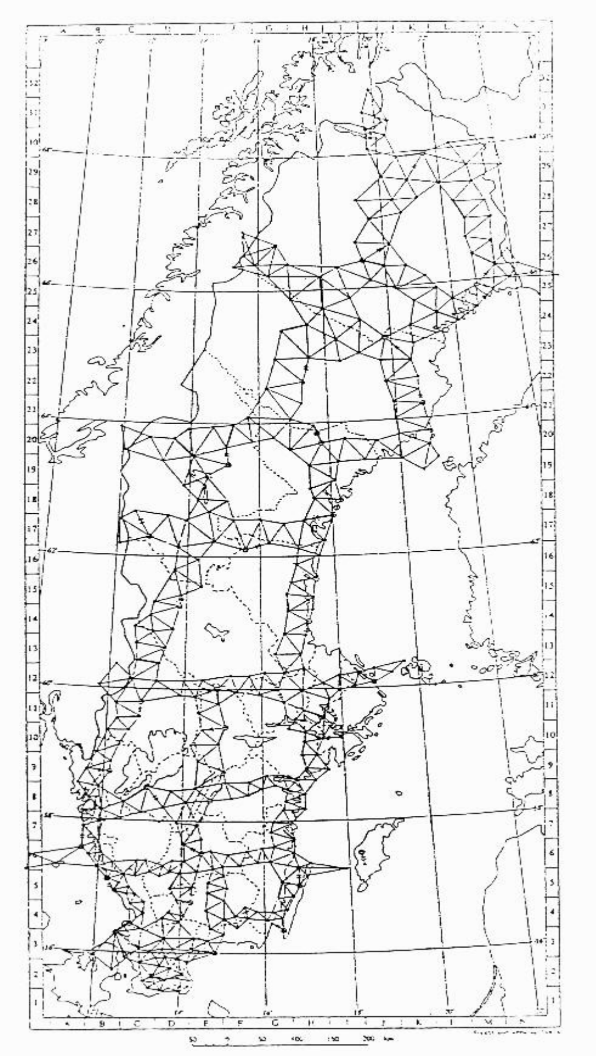
Astrogeodetic networks

- Networks are divided into orders: 1st, 2nd, 3rd
- 1st-order network contains triangle chains
- 2nd-order networks fill in between 1st-order chains
- Triangle chains along meridians/parallels
- Ground points on top of mountains, with towers

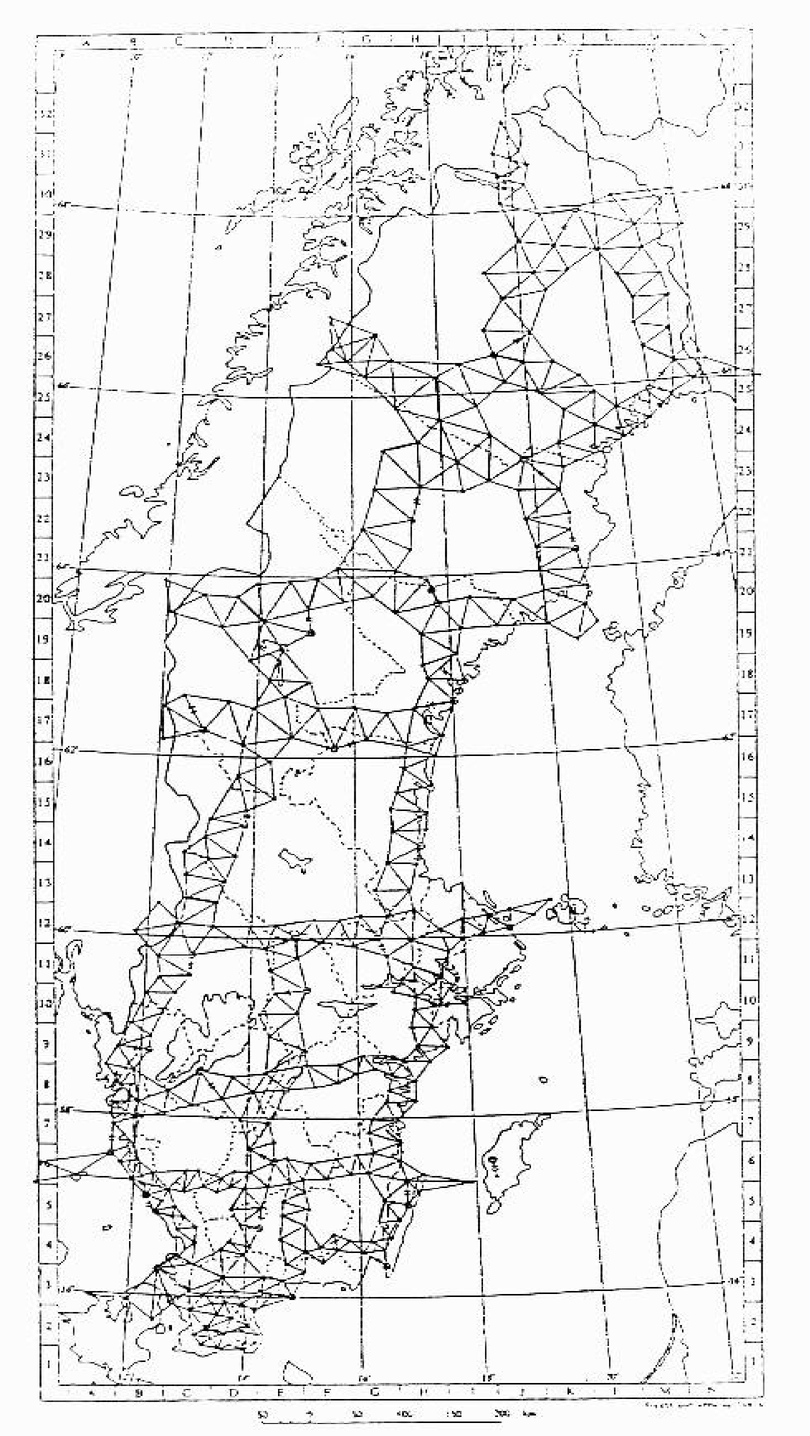
Astrogeodetic measurements

- Absolute position (Φ, Λ) of a few points determined by astronomical observations. (0.5" – 1")
- Astronomical azimuths provide orientation and control of error culmulations in angle measurements
- Only a fewer distances are measured using invar wires due to distance measurement difficulties (in the past) (1:300 000)
- Large number of angles are measured by theodolites (0.5" – 1")

1st Swedish national triangulation



- Divided into orders: 1st, 2nd, 3rd
- 1st-order network in triangle chains (sides up to 30 km)
- Chains along meridians/parallels
- 2nd-order networks as fill in between
- Towers on mountain tops

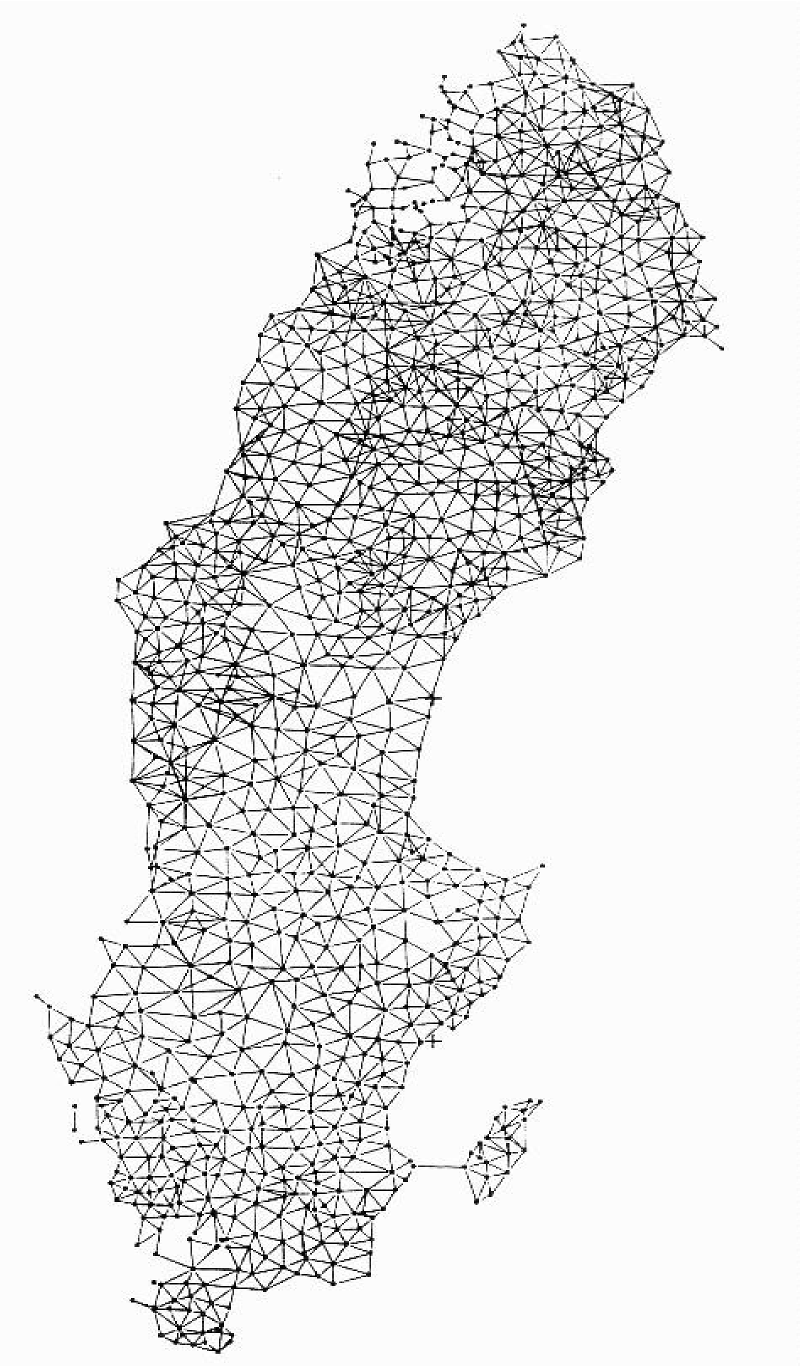


2nd Swedish National Triangulation Network (RT38)

- Field work during 1905-1950
- Triangle chains (EW, NS) at about 200 km distances
- 366 points with sides of ~ 30 km
- Astronomical coordinates determined at 30 junction points
- 11 baselines meaurements by invar wire (accuracy: 1:300 000, 3 ppm)
- Large number of angle/direction measurements (accuracy: ~ 1”)
- **Bessel’s ellipsoid** was used: $a=6377397.155$, $1/f=299.1528128$
- Gauss-Krüger Projection, y+1500km, $\lambda_0=15.808\ 277\ 777^{o}$
- Coordinate system named as **RT 38**


2003 10 11


2003 10 11



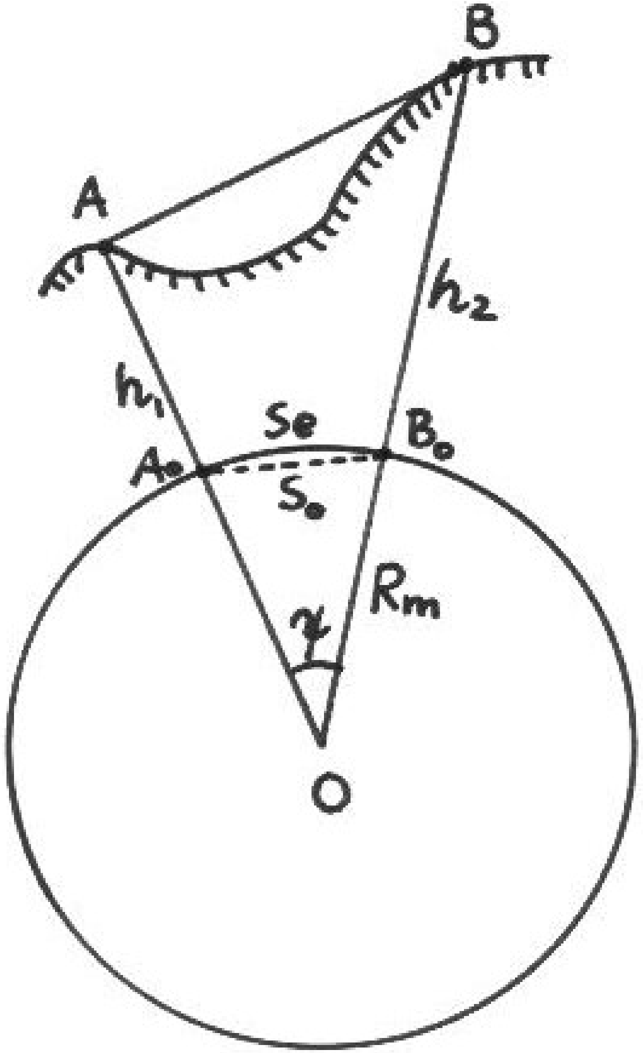
The 3rd Swedish National Triangulation Network (RT 90)

- Field work during 1967-1982
- Complete network with 15295 distance measurements using geodimeters
- Including 366 first-order points of RT 38
- Including angles and baselines from RT 38
- 5424 angle (direction) measurements
- Observations are adjusted in ED 87 using Hayford ellipsoid (a=63781388m, 1/f=297)
- Coordinates at 366 common points in ED87 are fitted to coordinates in RT 38
- Final coordinates are referred to a non-geocentric **Bessel's ellipsoid**
- Gauss-Kruger map projection with mid-meridian $\lambda_0 = 15.808\ 277\ 777^\circ$
- Coordinate system is designated as **RT 90**
- RT 90 = improved coordinates in RT 38

Reduction of ground measurements down to the reference ellipsoid

- Reduce slope **distances** on the ground to distances of the geodetic line on the ellipsoid
- Reduce **astronomical coordinates** to obtain geodetic coordinates (*deflection of the vertical is needed*)
- Reduce surface **angles** to angles of geodetic triangles
- Reduce astronomical **zenith distances** to otain geodetic zenith distances

Reduction of slope distances



$$s_0 = \sqrt{\frac{s^2 - (h_2 - h_1)^2}{\left(1 + \frac{h_1}{R_m}\right)\left(1 + \frac{h_2}{R_m}\right)}}$$

$$R_m = \frac{a\sqrt{1-e^2}}{W^2} = \frac{a\sqrt{1-e^2}}{1 - e^2 \sin^2 \phi_m}$$

$$s_e = 2R_m \cdot \arcsin\left(\frac{s_0}{2R_m}\right)$$

Reduction of astronomical coordinates

$$\phi = \Phi - \xi$$
$$\lambda = \Lambda - \eta / \cos \phi$$

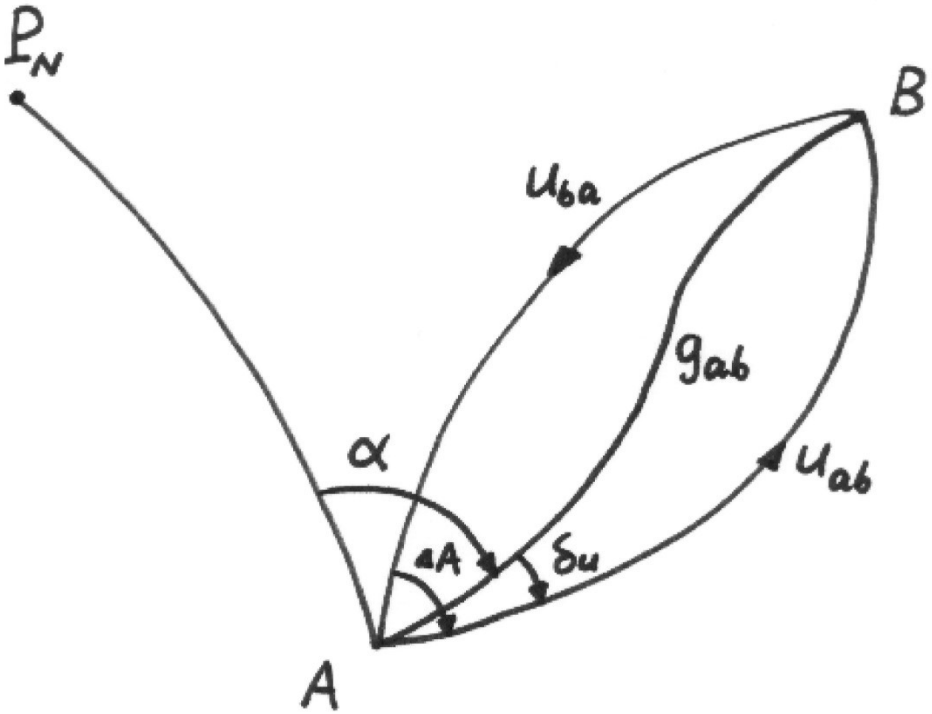
$$\alpha_{ab} = A_{ab} - \Delta A_1 - \Delta A_2$$

$$\left.\begin{array}{l} \Delta A_1 = \eta \; tg\phi = (\Lambda - \lambda) \sin \phi \\ \Delta A_2 = (\xi \sin \alpha_{ab} - \eta \cos \alpha_{ab}) \cot z_{ab} \end{array}\right\}$$

When z_{ab} ~90° , we get the **Laplace condition**:

$$\boxed{A_{ab} - \alpha_{ab} = (\Lambda - \lambda) \sin \phi}$$

Reduction of ground directions (angles)



- Reduction due to deflection of the vertical

$$A_{ab} - \alpha_{ab} = (\Lambda - \lambda) \sin \phi$$

- Reduction due to discrepancy between normal section and geodesic

$$\delta_u \approx \Delta A/3 \approx \frac{e^2 \cdot s^2}{12\ N^2} \cos^2 \phi\ \ \sin 2\alpha$$

- Reduction due to height of the object aove the ellipsoid

$$\delta_h \approx -\frac{h}{2N}\ e'^2\ \ \cos^2 \phi\ \ \sin 2\alpha$$

Reduction of astronomical zenith distance

- Astronomical zenith distances refer to the plumb line (vertical line=gravity vector)

- Geodetic zenith distances refer to the ellipsoidal normal

- Reduction depends on the deflection of the vertical and the azimuth:

$$z_e = z_s + \xi \ \cos\alpha + \eta \ \sin\alpha$$

Least squares adjustment on the reference ellipsoid

- **Unknown parameters to be determined:** geodetic latitudes and longitudes of unknown triangulation points

- **Observations:** distances, azimuths, angles/directions

- **Observation equation:** an explicit function of the unknown parameters for each observation

- **If an observation equation is non-linear:** the observation equation must be linearized using approximate geodetic latitudes/longitudes

Observation equation of a distance measurement

$$s_{ij} - \varepsilon_{ij} = s(\phi_i^0 + \delta\phi_i,\ \lambda_i^0 + \delta\lambda_i,\ \phi_j^0 + \delta\phi_j,\ \lambda_j^0 + \delta\lambda_j)$$

$$\approx s_{ij}^0 + \frac{\partial s}{\partial \phi_i} \cdot \delta\phi_i + \frac{\partial s}{\partial \lambda_i} \cdot \delta\lambda_i + \frac{\partial s}{\partial \phi_j} \cdot \delta\phi_j + \frac{\partial s}{\partial \lambda_j} \cdot \delta\lambda_j$$

$$\ell_{ij} - \varepsilon_{ij} = a_i \cdot \delta\phi_i + b_i \cdot \delta\lambda_i + a_j \cdot \delta\phi_j + b_j \cdot \delta\lambda_j$$

$$\begin{aligned}
a_i &= -M_i^0 \cos\alpha_{ij}^0 \\
b_i &= N_j^0 \cos\phi_j^0 \sin\alpha_{ji}^0 \\
a_j &= -M_j^0 \cos\alpha_{ji}^0 \\
b_j &= -N_j^0 \cos\phi_j^0 \sin\alpha_{ji}^0
\end{aligned}$$

$$\ell_{ij} = s_{ij} - s_{ij}^0$$

For linearization:

$$\begin{aligned}
\phi_i &= \phi_i^0 + \delta\phi_i \\
\lambda_i &= \lambda_i^0 + \delta\lambda_i \\
\phi_j &= \phi_j^0 + \delta\phi_j \\
\lambda_j &= \lambda_j^0 + \delta\lambda_j
\end{aligned}$$

$$\begin{aligned}
s_{ij}^0 &= s(\phi_i^0,\ \lambda_i^0,\ \phi_j^0,\ \lambda_j^0) \\
\alpha_{ij}^0 &= \alpha(\phi_i^0,\ \lambda_i^0,\ \phi_j^0,\ \lambda_j^0)
\end{aligned}$$

Observation equation of an azimuth

$$\alpha_{ij} - \varepsilon_{ij} = \alpha(\phi_i^0 + \delta\phi_i,\ \lambda_i^0 + \delta\lambda_i,\ \phi_j^0 + \delta\phi_j,\ \lambda_j^0 + \delta\lambda_j)$$

$$\approx \alpha_{ij}^0 + \frac{\partial\alpha}{\partial\phi_i}\cdot\delta\phi_i + \frac{\partial\alpha}{\partial\lambda_i}\cdot\delta\lambda_i + \frac{\partial\alpha}{\partial\phi_j}\cdot\delta\phi_j + \frac{\partial\alpha}{\partial\lambda_j}\cdot\delta\lambda_j$$

$$\ell_{ij} - \varepsilon_{ij} = c_i\cdot\delta\phi_i + d_i\cdot\delta\lambda_i + c_j\cdot\delta\phi_j + d_j\cdot\delta\lambda_j$$

$$\left.\begin{aligned}
c_i &= M_i^0 \sin\alpha_{ij}^0 / s_{ij}^0 \\
d_i &= N_j^0 \cos\phi_j^0 \cos\alpha_{ji}^0 / s_{ij}^0 \\
c_j &= M_j^0 \sin\alpha_{ji}^0 / s_{ij}^0 \\
d_j &= -N_j^0 \cos\phi_j^0 \cos\alpha_{ji}^0 / s_{ij}^0 \\
\\
\ell_{ij} &= \alpha_{ij} - \alpha_{ij}^0
\end{aligned}\right\}$$

Least squares adjustment on the reference ellipsoid

Final observation equations for all measurements

$$L - \varepsilon = A\,X$$

$L:$ (reduced) measurements (azimuths, angles, distances etc)
$\varepsilon:$ residuals (errors)
$X:$ unknowns (coordinate corrections, other parameters)
$A:$ design matrix

Least squares solution:

$$\hat{X} = \left(A^\top P A\right)^{-1} A^\top P L$$

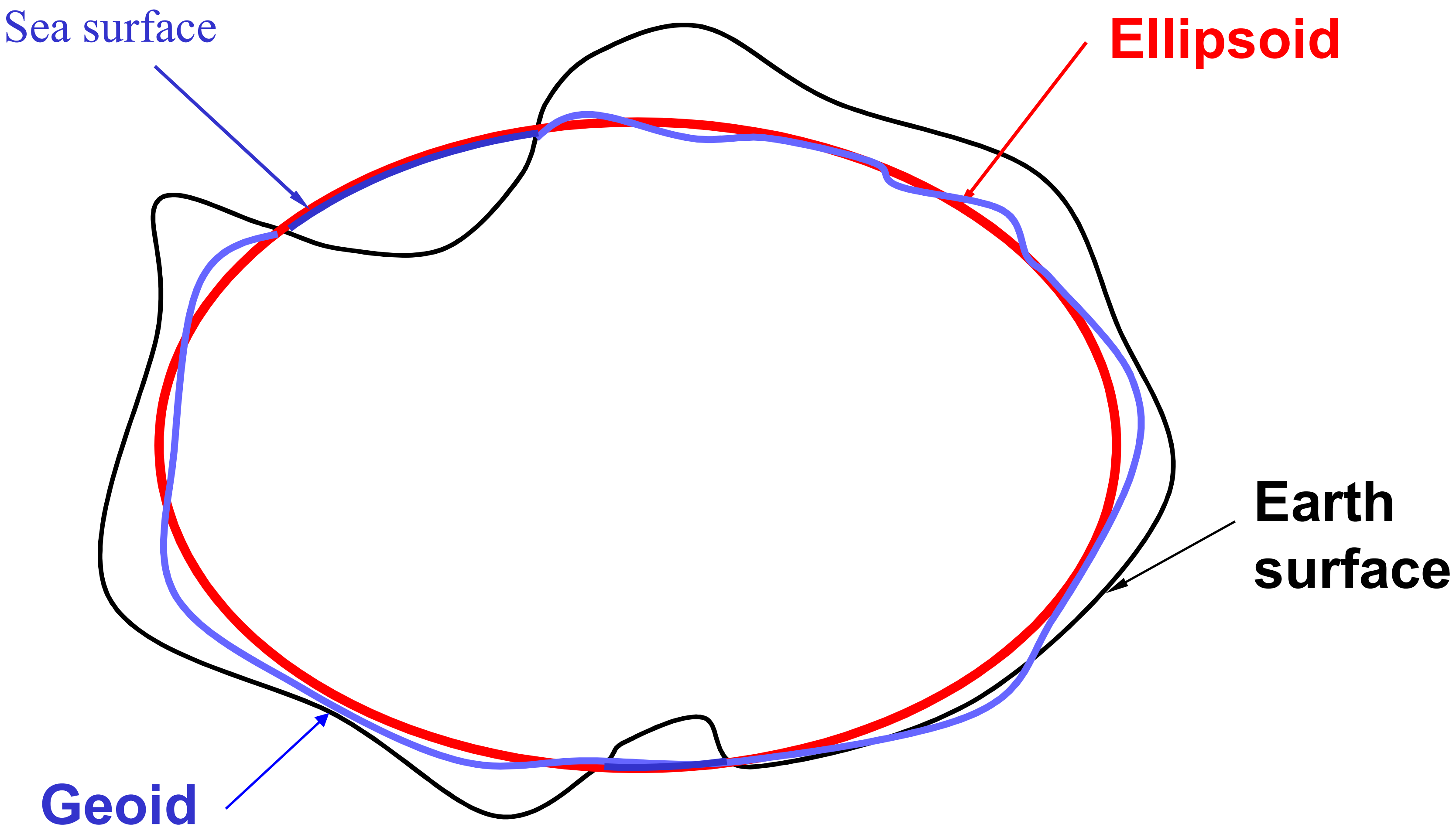
Geodetic datum (*ellipsoidal datum*)

- A *geodetic datum* consists of :
  - reference ellipsoid (size, shape and position defined) **and**
  - a set of ground triangulation points whose 2D geodetic coordinates (φ,λ) are computed with respect to this reference ellipsoid

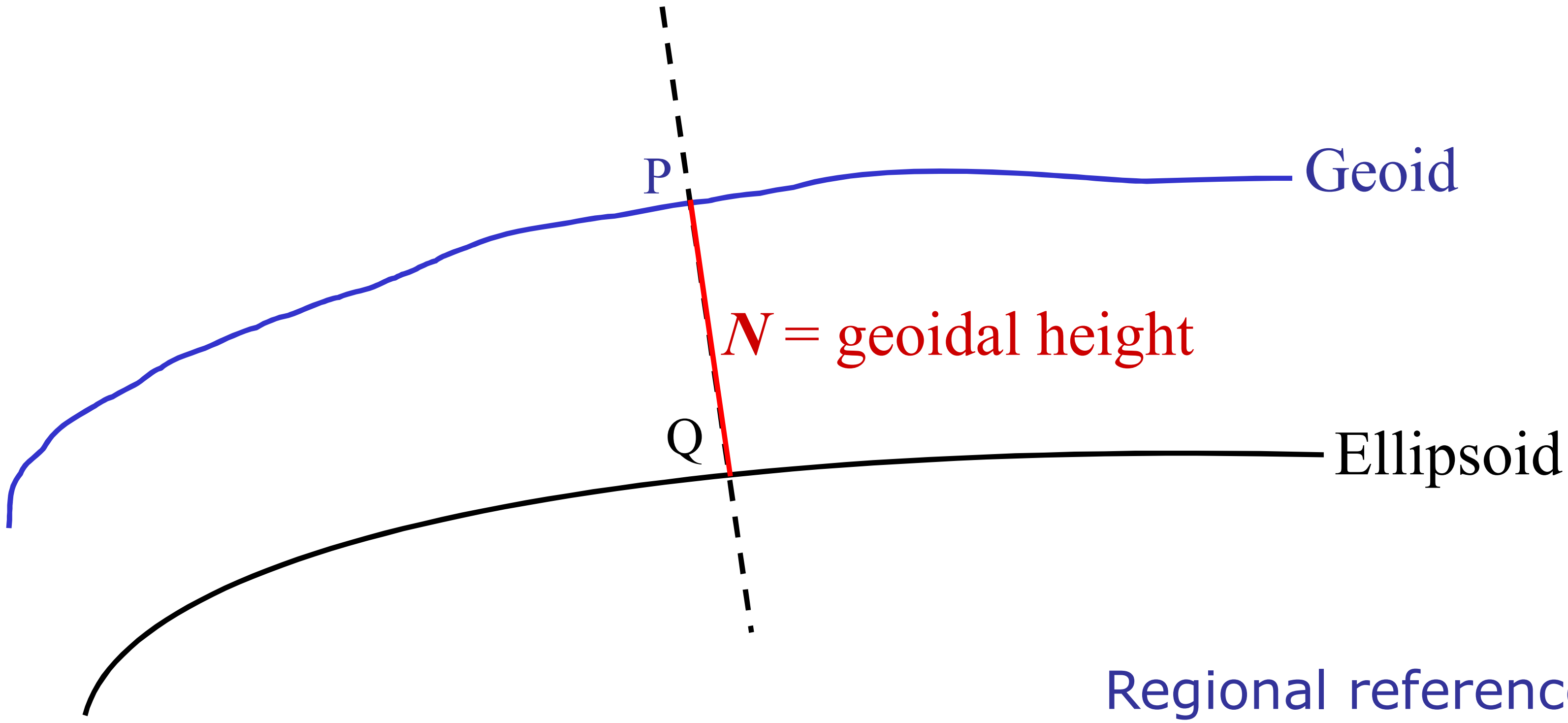
- Assumptions:
  - The minor axis of ellipsoid is parallel to earth rotation axis;
  - The initial median plane of the ellipsoid is parallel to the *astronomical* Greenwich meridian plane

- The position of the ellipsoid is often/best defined by the geocentric coordinates of the ellipsoidal centre (x_0, y_0, z_0)

Fitting ellipsoids to the geoid



The geoid-ellipsoid separation



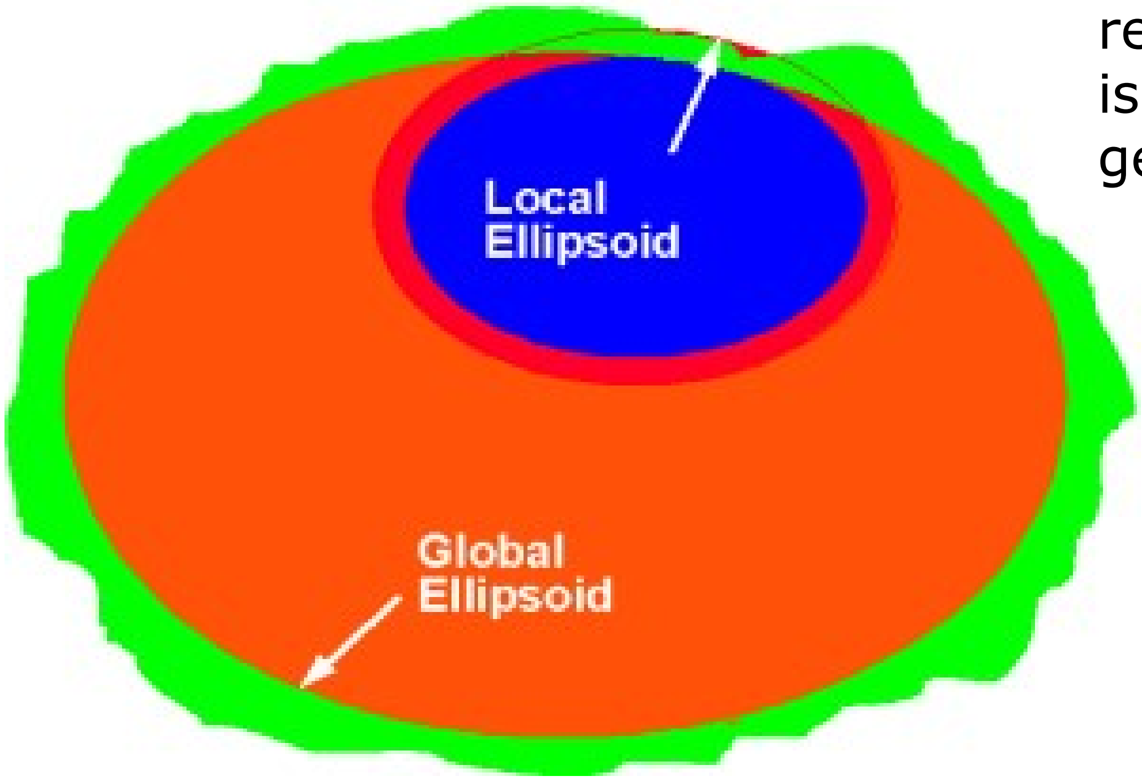
Regional reference ellipsoid

Mean Earth Ellipsoid (σ=whole earth) (σ_1 = local area, a country)

$$\iint_{\sigma} N^2 d\sigma = \text{minimum}$$

$$\iint_{\sigma_1} N^2 d\sigma = \text{minimum}$$

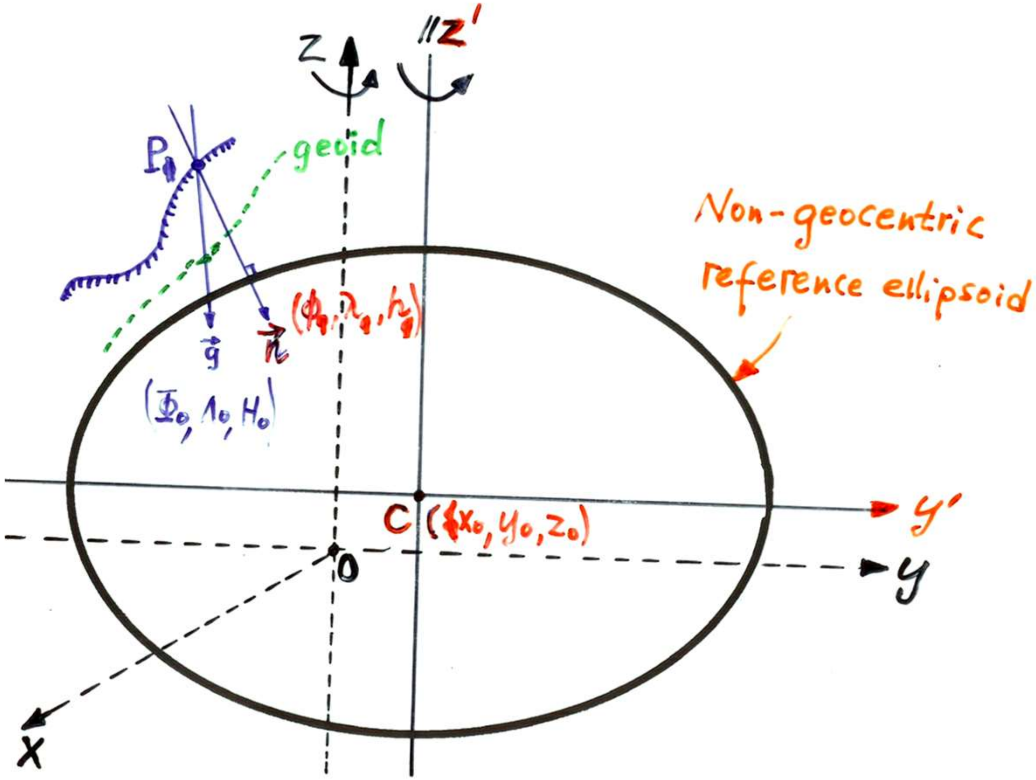
Local vs global ellipsoids



A locally fitted reference ellipsoid is most often not geocentric

A globally fitted reference ellipsoid (*Mean Earth Ellipsoid*) is geocentric

Positioning of the ellipsoid



O: geocentre
C: centre of the reference ellipsoid

Assumption: ellipsoid's minor axis ***Z'*** is parallel to the Earth rotation axis *Z*

→ the equator of the ellipsoid is parallel to the Earth's equator

→ position of the ellipsoid in relation to the earth can be defined by the geocentric coordinates (x_0, y_0, z_0) of the ellipsoidal center *C*

3 (or 2) ways to position the ellipsoid

- by geocentric coordinate (x_0, y_0, z_0) of ellipsoidal center C
- by geodetic coordinate ($\varphi_1, \lambda_1, h_1$) of the initial point P_1
- by geoid height N_1 and deflection of the vertical (ξ_1, η_1) of P_1

Geodetic datum parametres

Name	**a (m)**	**1/f**	**x_0 (m)**	**y_0 (m)**	**z_0 (m)**
WGS 84	6 378 137	298.257223563	0	0	0
RT 90 (*Bessel*)	6 377 397.155	299.1528128	414.098	41.338	603.063
Pulkovo 1942 (*Krassovski*)	6 378 245	298.3	-25.0	141.0	78.5
Adindan 4, Ethiopia (Clarke 1880)	6 378 249.145	293.465	+162	+12	-206

Coordinate change due to geodetic datum change

If geodetic datum parameters (a, f, x_0, y_0, z_0) have slight changes (δa, δf, δx_0, δy_0, δz_0)´, correspondingy the geodetic coordinates (φ, λ, h) of a ground point P will have slight changes by ($\delta\varphi$, $\delta\lambda$, δh)

$$\begin{bmatrix} a \cdot \delta\phi \\ a\cos\phi \cdot \delta\lambda \\ dh \end{bmatrix} = \Omega(\phi,\lambda) \cdot \begin{bmatrix} \delta x_0 \\ \delta y_0 \\ \delta z_0 \end{bmatrix} + \Theta(\phi,\lambda) \cdot \begin{bmatrix} \delta a \\ a \cdot \delta f \end{bmatrix}$$

$$\Omega(\phi,\lambda) = \begin{bmatrix} \sin\phi\ \cos\lambda & \sin\phi\ \sin\lambda & -\cos\phi \\ \sin\lambda & -\cos\lambda & 0 \\ -\cos\phi\ \cos\lambda & -\cos\phi\ \sin\lambda & -\sin\phi \end{bmatrix}, \quad \Theta(\phi,\lambda) = \begin{bmatrix} 0 & \sin 2\phi \\ 0 & 0 \\ -1 & \sin^2\phi \end{bmatrix}$$

Coordinate change due to geodetic datum change

$$\begin{bmatrix} a\cdot\delta\phi \\ a\cos\phi\cdot\delta\lambda \\ \delta h \end{bmatrix} = \sum(\phi,\lambda)\cdot\begin{bmatrix} a\cdot\delta\phi_1 \\ a\cos\phi_1\cdot\delta\lambda_1 \\ \delta h_1 \end{bmatrix} + \Pi(\phi,\lambda)\cdot\begin{bmatrix} \delta a \\ a\cdot\delta f \end{bmatrix}$$

$$\sum_{3\times 3}(\phi,\lambda) = \Omega(\phi,\lambda)\cdot[\Omega(\phi_1,\lambda_1)]^{-1} =$$

$$= \begin{bmatrix} \cos\phi_1\cos\phi+\sin\phi_1\sin\phi\cos(\lambda-\lambda_1) & -\sin\phi\sin(\lambda-\lambda_1) & \sin\phi_1\cos\phi-\cos\phi_1\sin\phi\cos(\lambda-\lambda_1) \\ \sin\phi_1\sin(\lambda-\lambda_1) & \cos(\lambda-\lambda_1) & -\cos\phi_1\sin(\lambda-\lambda_1) \\ \cos\phi_1\sin\phi-\sin\phi_1\cos\phi\cos(\lambda-\lambda_1) & \cos\phi\sin(\lambda-\lambda_1) & \sin\phi_1\sin\phi+\cos\phi_1\cos\phi\cos(\lambda-\lambda_1) \end{bmatrix} \tag{2.128}$$

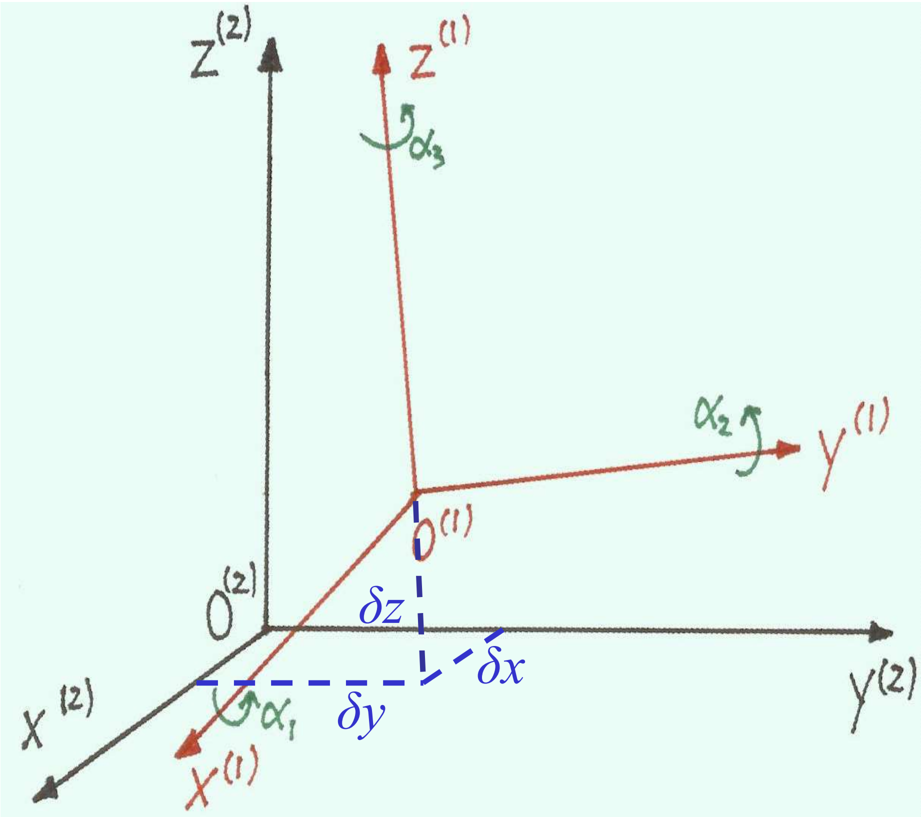
$$\prod_{3\times 2}(\phi,\lambda) = \Theta(\phi,\lambda) - \Omega(\phi,\lambda)\cdot[\Omega(\phi_1,\lambda_1)]^{-1}\Theta(\phi_1,\lambda_1) = \left[\begin{matrix} \sin\phi_1\cos\phi-\cos\phi_1\sin\phi\cos(\lambda-\lambda_1) \\ -\cos\phi_1\sin(\lambda-\lambda_1) \\ \sin\phi_1\sin\phi+\cos\phi_1\cos\phi\cos(\lambda-\lambda_1)-1 \end{matrix}\right.$$

$$\left.\begin{matrix} [\sin\phi_1\cos\phi-\cos\phi_1\sin\phi\cos(\lambda-\lambda_1)]\sin^2\phi_1+2\cos\phi(\sin\phi-\sin\phi_1) \\ -\cos\phi_1\sin(\lambda-\lambda_1)\sin^2\phi_1 \\ [\sin\phi_1\sin\phi+\cos\phi_1\cos\phi\cos(\lambda-\lambda_1)]\sin^2\phi_1+(\sin^2\phi-2\sin\phi_1\sin\phi) \end{matrix}\right] \tag{2.129}$$

Local triangulation vs global systems: Reference ellipsoids

- Theoretically, coordinate system or positions can be uniquely defined using Cartesian cooridnates (x,y,z)
- In principle, transformation of two Cartesian coordinate systems are independent of reference ellipsoids
- Reference ellipsoids are needed to define (geographic) geodetic coordinates (φ,λ,h)
- Local/global systems *often* use different reference ellipsoids
- Reference ellipsoids affect also map projections

Local triangulation system vs global geocentric systems



- Global system (e.g. GPS) is geocentric while local systems are most often non-geocentric: $\delta x \neq 0$, $\delta y \neq 0$, $\delta z \neq 0$
- The axes of local and global systems are often not parallel, rather with small angles ($\alpha_1, \alpha_2, \alpha_3$) between them
- Linear scales can be significantly different due to difficulties of distance measurements in triangulation
- Difference between local and global systems can be modelled by 3 translations, 3 rotations, 1 scale change

Helmert transformation between 2 Cartesian coordinate systems

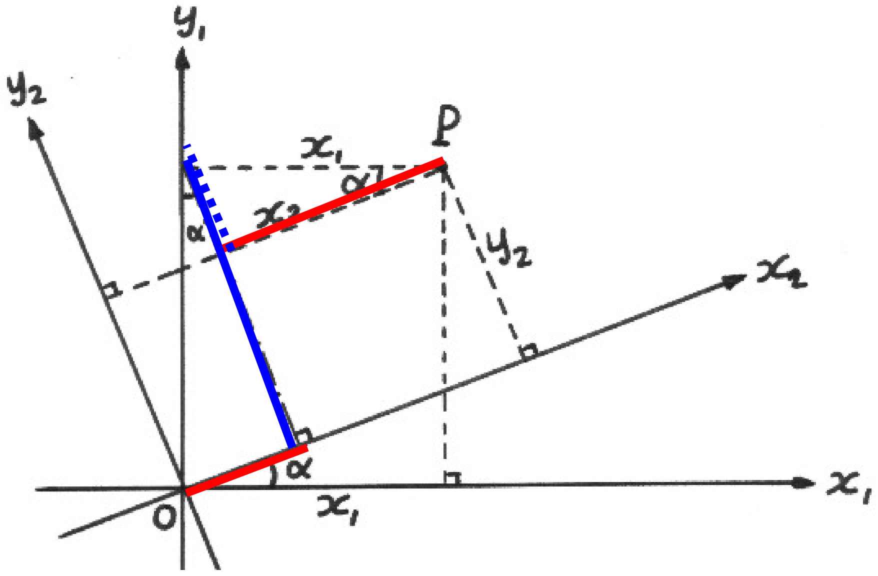
$$\begin{bmatrix} X_i^{(2)} \\ Y_i^{(2)} \\ Z_i^{(2)} \end{bmatrix} = \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} + s \cdot R(\alpha_1, \alpha_2, \alpha_3) \cdot \begin{bmatrix} X_i^{(1)} \\ Y_i^{(1)} \\ Z_i^{(1)} \end{bmatrix}$$

$$s = 1 + \delta s$$

$$\underset{3\times 3}{R} = R(\alpha_1, \alpha_2, \alpha_3) = R_3(\alpha_3) \cdot R_2(\alpha_2) \cdot R_1(\alpha_1)$$

$$= \begin{bmatrix} \cos\alpha_2 \cos\alpha_3 & \cos\alpha_1 \sin\alpha_3 + \sin\alpha_1 \sin\alpha_2 \cos\alpha_3 & \sin\alpha_1 \sin\alpha_3 - \cos\alpha_1 \sin\alpha_2 \cos\alpha_3 \\ -\cos\alpha_2 \sin\alpha_3 & \cos\alpha_1 \cos\alpha_3 - \sin\alpha_1 \sin\alpha_2 \sin\alpha_3 & \sin\alpha_1 \cos\alpha_3 + \cos\alpha_1 \sin\alpha_2 \sin\alpha_3 \\ \sin\alpha_2 & -\sin\alpha_1 \cos\alpha_2 & \cos\alpha_1 \cos\alpha_2 \end{bmatrix}$$

Rotation Matrices



$$R = R_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$R = R_2(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

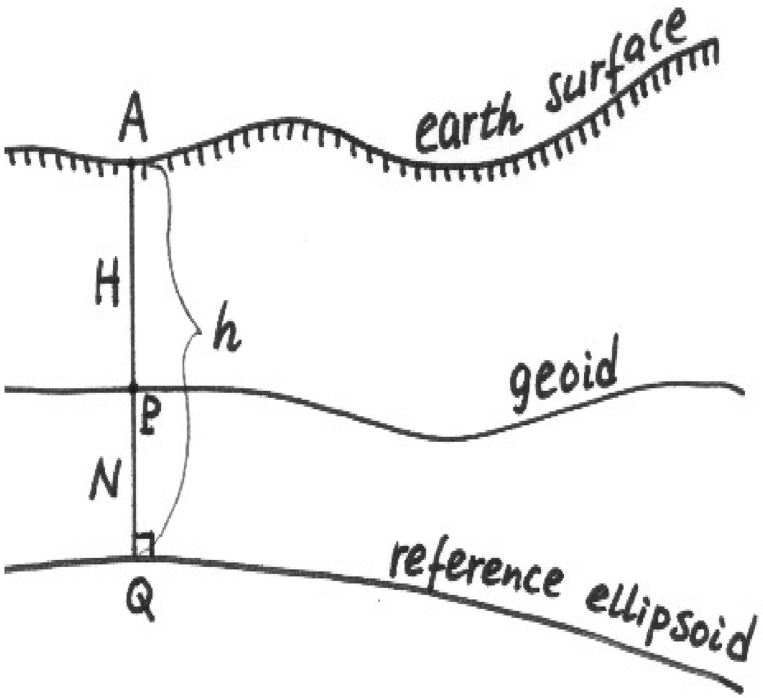
$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = R \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$$R = R_3(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Helmert transformation parameters

From System 1:	**SWEREF 99** (GPS-based)
To System 2:	**RT 90** (Swedish triangulation)
δx (metre):	- 414.0979
δy (metre):	- 41.3381
δz (metre):	- 603.0627
δs *(ppm):*	+ 0.000 000 000 0
α_1 (arcsecond):	- 0.855 043 431 4
α_2 (arcsecond):	+ 2.141 346 518 5
α_3 (arcsecond):	- 7.022 720 951 6

Local triangulation system Vs gloal coordinate systems



- Triangulation gives us basically 2D coordinates (φ,λ)
- GNSS provides ellipsoidal heights h
- Height above MSL (H) from precise levelling refers to the geoid
- h and H are different by definition and related to each other by:

$$h = H + N$$

- h, N must refer to the same ellipsoid
- N depends on gravity field of the earth

$\boxed{2 + 1 \neq 3!}$

- Triangulation + levelling is not equivalent to 3D positioning using e.g. GNSS

Steps from SWEREF 99 to RT 90

- (φ,λ,h) SWEREF 99 $\rightarrow$ (x,y,z) SWEREF 99 ← GRS 80 ellipsoid
- (x,y,z) SWEREF 99 $\rightarrow$ (x,y,z) RT 90 ← Helmert model (7 parameters)
- (x,y,z) RT 90 $\rightarrow$ (φ,λ,h) RT 90 ← Bessel's ellipsoid
- (φ,λ) RT 90 $\rightarrow$ $(X,\ Y)$ UTM / RT 90 ← Bessel's ellipsoid
- $(\ h\ )$ SWEREF 99 $\rightarrow$ (H) ← Geoid model, GRS 80 ellipsoid

Steps from RT90 to SWEREF 99TM

- $(x,y)_{\text{RT 90}} \rightarrow (\varphi,\ \lambda)_{\text{RT 90}} \leftarrow$ Bessel's ellipsoid
- $(\varphi,\lambda,h{=}0/H)_{\text{RT 90}} \rightarrow (x,y,z)_{\text{RT 90}} \leftarrow$ Bessel's ellipsoid
- $(x,y,z)_{\text{RT 90}} \rightarrow (x,y,z)_{\text{SWEREF 99}} \leftarrow$ Helmert model (7 parameters)
- $(x,y,z)_{\text{SWEREF 99}} \rightarrow (\varphi,\lambda,h)_{\text{SWEREF 99}} \leftarrow$ GRS 80 ellipsoid
- $(\varphi,\lambda)_{\text{SWEREF 99}} \rightarrow (x,y)_{\text{SWEREF 99TM}} \leftarrow$ GRS 80 ellipsoid