### 4. Variance analysis & regression analysis

- Different distributions of random variables
- Confidence intervals
- Hypothesis tests (statistical tests)
- Variance analysis
- Regression analysis (least squares fitting)

### Statiscal properties of a random variable $\varepsilon$

• Distribution function F(x)  $P(\varepsilon \le x) = F(x) = \int_{-\infty}^{x} f(\varepsilon)d\varepsilon$   $(-\infty < x < +\infty)$  probability that  $\varepsilon < x$ 

• Density function f(x)  $\frac{\partial F(x)}{\partial x} = f(x)$ 

• Expectation (average)  $E(\varepsilon) = \int_{-\infty}^{+\infty} \varepsilon \ f(\varepsilon) d\varepsilon$ 

 $\qquad \qquad var(\varepsilon) = \sigma^2 = E\left\{ [\varepsilon - E(\varepsilon)]^2 \right\} = \int_{-\infty}^{+\infty} [\varepsilon - E(\varepsilon)]^2 \cdot f(\varepsilon) \cdot d\varepsilon$ 

### Several types of distributions

- Binomial distribution
- Uniform distribution
- Normal distribution.
   Standard normal distribution
- $\chi^2$ -distribution
- *t*-distribution
- $\bullet$  F-distribution

#### Binomial distribution

- A random experiment with only two outcomes
- Probabilities: p, q, p+q=1
- Density (frequency) function: Probability that experiment is repeated *n* times and outcome with probability *p* occurs *k* times

$$f(k) = \binom{n}{k} p^k q^{n-k} = \frac{n!}{(n-k)! \ k!} p^k q^{n-k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 3\cdot 2\cdot 1} p^k q^{n-k} \quad (k=0,1,2,\cdots,n)$$

- Distribution function:  $F(x) = \sum_{k=0}^{x} f(k) = \sum_{k=0}^{x} {n \choose k} p^k q^{n-k}$
- Expectation and variance:

$$E(x) = \sum_{k=0}^{x} k f(k) = np$$
,  $Var(x) = \sum_{k=0}^{x} \{(k - np)^2 f(k)\} = npq$ 

#### Example

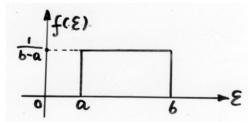
the sign of errors:

$$p = q = \frac{1}{2}$$

### Uniform distribution

 Constant density function inside interval

$$f(arepsilon) = \left\{ egin{array}{ll} rac{1}{b-a} & & ext{for } a \leq arepsilon \leq arepsilon \ & ext{otherwise} \end{array} 
ight.$$



• Distribution function:

$$F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases}$$

### Uniform distribution

• Expectation, Variance:

$$E(\varepsilon) = \int_{a}^{b} \varepsilon \frac{1}{b-a} d\varepsilon = \frac{a+b}{2}$$

$$Var(\varepsilon) = E\left\{ [\varepsilon - E(\varepsilon)]^2 \right\} = \frac{(b-a)^2}{12}$$

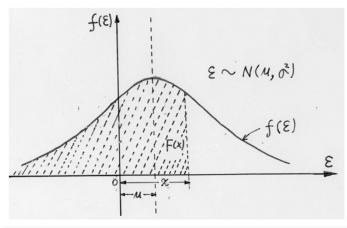
#### **Example**

Rounding errors:  $a = -\frac{1}{2}$ ,  $b = +\frac{1}{2}$ 

Expectation and variance:  $E(\varepsilon) = 0$ ,  $Var(\varepsilon) = \frac{1}{12}$ 

### Density function of a normal distribution

$$f(\varepsilon) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2} (\varepsilon - \mu)^2\right\} \quad (-\infty < \varepsilon < +\infty)$$



$$F(x) = \int_{-\infty}^{x} f(\varepsilon) d\varepsilon = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{1}{2\sigma^{2}} \left(\varepsilon - \mu\right)^{2}\right\} \cdot d\varepsilon \quad (-\infty < x < +\infty)$$

# Normal distribution $\varepsilon \sim N(\mu, \sigma^2)$

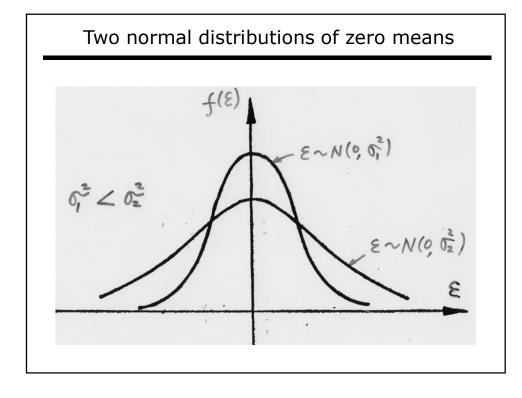
Expectation:

$$E(\varepsilon) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varepsilon \cdot \exp\left\{-\frac{1}{2\sigma^2} \left(\varepsilon - \mu\right)^2\right\} \cdot d\varepsilon = \mu$$

• Variance:

$$Var(\varepsilon) = E\left\{ \left[ \varepsilon - E(\varepsilon) \right]^2 \right\} = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[ \varepsilon - E(\varepsilon) \right]^2 \cdot \exp\left\{ -\frac{1}{2\sigma^2} \left( \varepsilon - \mu \right)^2 \right\} \cdot d\varepsilon = \sigma^2$$

- $\longrightarrow$  Normal distribution is defined by  $(\mu, \sigma^2)$ 
  - Most errors follow normal distributions
  - Central limiting theorem: Sum of n independent random variables of equal expectations/variances converges to a normal distribution



### Standard normal distribution N(0,1)

$$\varepsilon \sim N(\mu, \sigma^2)$$

$$\tau = \frac{\varepsilon - \mu}{\sigma}$$

$$E(\tau) = 0 , \quad Var(\tau) = 1$$

 $\longrightarrow au$  is said to have standard normal distribution

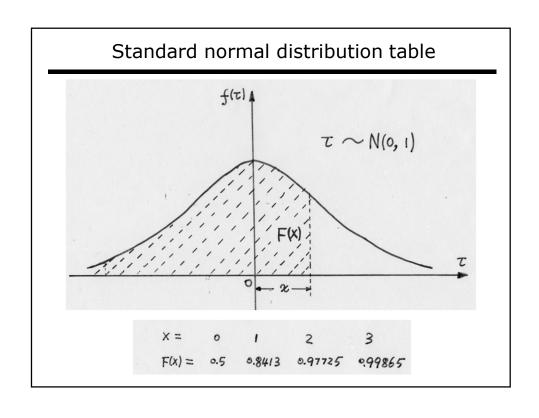
$$au \sim N(0,1)$$

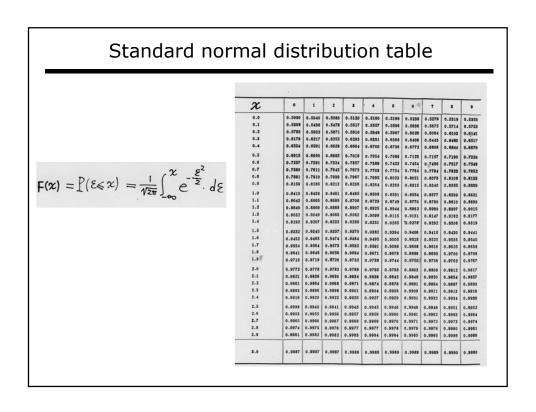
• Density function:

$$f(\tau) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\tau^2\right\} \qquad (-\infty < \tau < +\infty)$$

• Distribution function:

$$F(x) = \int_{-\infty}^{x} f(\tau)d\tau = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{1}{2}\tau^{2}\right\} \cdot d\tau \quad (-\infty < x < +\infty)$$





#### Normal distribution of multiple variables

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$E(\varepsilon) = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \dots \\ \mu_n \end{bmatrix} = \begin{matrix} \mu \\ \dots \\ \mu_n \end{matrix}, \quad \underset{n \cdot n}{C} = E\left\{ \left[\varepsilon - E(\varepsilon)\right] \left[\varepsilon - E(\varepsilon)\right]^\top \right\} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \dots & \sigma_{2n} \\ \dots & \dots & \dots & \dots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_n^2 \end{bmatrix}$$

$$f(\varepsilon) = \frac{1}{\left(2\pi\right)^{\frac{n}{2}} |C|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2} \left(\varepsilon - \mu\right)^{\top} C^{-1} \left(\varepsilon - \mu\right)\right\}$$

# $\chi^2$ -distribution

$$\chi^2 = \varepsilon_1^2 + \varepsilon_2^2 + \dots + \varepsilon_n^2$$

$$\varepsilon_i \sim N(0,1)$$

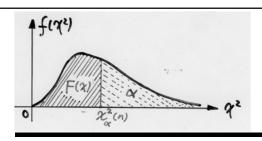
n is called the degree of freedom of  $\chi^2\,$ 

$$f(\chi^2) = \frac{1}{\frac{n}{2}\Gamma(\frac{n}{2})} \, \left(\chi^2\right)^{\frac{n}{2}-1} \exp\left\{-\frac{1}{2}\chi^2\right\} \, , \quad (\chi^2 > 0)$$

$$\Gamma(n) = \int_0^{+\infty} x^{n-1} e^{-x} dx \quad (n > 0)$$

$$F(x) = \int_{-\infty}^{x} f(x) dx$$

$$E\left(\chi^{2}\right)=n\;,\;\;Var\left(\chi^{2}\right)=E\left\{ \left[\chi^{2}-E(\chi^{2})\right]^{2}\right\} =2n$$



| n  | a=0.995 | 0.99    | 0.975  | 0.95   | 0.90     | 0.75   |
|----|---------|---------|--------|--------|----------|--------|
| 1  | - 1     | _       | 0.001  | 0.004  | 0.016    | 0.192  |
| 2  | 0.010   | 0.020   | 0.051  | 0.103  | 0.211    | 0.575  |
| 3  | 0.072   | 0.115   | 0.216  | 0.352  | 0.584    | 1.213  |
| 4  | 0.207   | 0.297   | 0.484  | 0.711  | 1.064    | 1.923  |
| 6  | 0.412   | 0.554   | 0.831  | 1.145  | 1.619    | 2.575  |
| 6  | 0.676   | 0.872   | 1.237  | 1.635  | 2.204    | 3,455  |
| 7  | 0.989   | 1.239   | 1.690  | 2.167  | 2.833    | 4.255  |
| 8  | 1.344   | 1.646   | 2.180  | 2.733  | 3.490    | 5.071  |
| 9  | 1.735   | 2.088   | 2.700  | 3.325  | 4.163    | 5.899  |
| 19 | 2.156   | 2.558   | 3.247  | 3.940  | 4.865    | 6.737  |
| 11 | 2.603   | 3.053   | 3.816  | 4.575  | 5.578    | 7.524  |
| 12 | 3.074   | 3.571 - | 4.404  | 5.226  | * 6.304  | 8.433  |
| 13 | 3.563   | 4.107   | 5.009  | 5.892  | 7.042    | 9.259  |
| 14 | 4.075   | 4.660   | 5.629  | 6.571  | 7,790    | 10.155 |
| 15 | 4.601   | 5.229   | 6.262  | 7.261  | 8.547    | 11.037 |
| 16 | 5.142   | 5.812   | 6.908  | 7.962  | 9.312    | 11.512 |
| 17 | 5.697   | 6.408   | 7.564  | 8.672  | 10.085   | 12.792 |
| 18 | 6.265   | 7.015   | 8.231  | 9.390  | 10.865   | 13.675 |
| 19 | 6.844   | 7.633   | 8.907  | 10.117 | 11.651   | 14.552 |
| 20 | 7.434   | 8.260   | 9.591  | 10.851 | 12.443   | 15.452 |
| 21 | 8.034   | 8.897   | 10.283 | 11.591 | 13.240   | 16.344 |
| 22 | 8.643   | 9.542   | 10.982 | 12.338 | 14.042   | 17.249 |
| 23 | 9.260   | 10.196  | 11.689 | 13.091 | 14.848   | 18,137 |
| 24 | 9.885   | 10.856  | 12.401 | 13.848 | 15.659   | 19.037 |
| 25 | 10.520  | 11.524  | 13.120 | 14.611 | 16.473   | 19.939 |
| 25 | 11.160  | 12.198- | 13.844 | 15.379 | 17.292   | 20.843 |
| 27 | 11.808  | 12.879  | 14.573 | 16.151 | 18.114   | 21.749 |
| 28 | 12.461  | 13.565  | 15.303 | 16.928 | 18.939   | 22.657 |
| 29 | 13.121  | 14,257  | 16.047 | 17.708 | . 19.768 | 23.567 |

Note the risk level  $\boldsymbol{\alpha}$  and the correpsonding critical value

# t-distribution, t(n)

If  $X \sim N(0,1)$  and  $Y \sim \chi^2(n)$  are independent of each other, then

$$t = \frac{X}{\sqrt{Y/n}}$$

follows the t-distribution with degree of freedom n

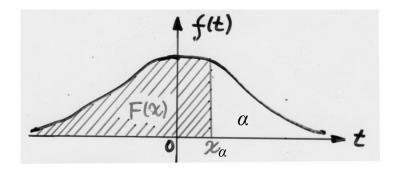
$$f(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi} \,\Gamma\left(\frac{n}{2}\right)} \cdot \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}} \qquad (-\infty < t < +\infty)$$

$$E(t) = 0 \ (n > 1), \ Var(t) = \frac{n}{n-2} \ (n > 2)$$

t(n) approaches N(0,1) when n approaches infinity

# Curve of the t-distribution

$$f(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi} \, \Gamma\left(\frac{n}{2}\right)} \cdot \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}} \qquad (-\infty < t < +\infty)$$



| n         a ≈ 0.995         0.99         0.975         0.95         0.90         0.75           1         —         —         0.001         0.004         0.016         0.102           2         0.010         0.020         0.051         0.103         0.211         0.573           3         0.072         0.115         0.216         0.352         0.581         1.213           4         0.207         0.287         0.484         0.711         1.064         1.223           6         0.476         0.552         0.331         1.116         1.610         2.573           7         0.999         1.239         1.169         2.167         2.33         3.433           8         1.344         1.466         2.189         2.733         3.403         1.655           9         1.735         2.088         2.700         3.325         4.165         5.899           10         2.136         2.538         3.247         3.940         4.865         6.737           11         2.603         3.053         3.816         4.676         5.578         7.541           12         3.074         3.571         4.404         5.2  |     | The     | e t-di | strib  | ution  | table  | 9      |
|--|-----|---------|--------|--------|--------|--------|--------|
| 1  |     |         |        |        |        |        |        |
| 2 0.010 0.020 0.051 0.103 0.211 0.575 3 0.072 0.115 0.216 0.352 0.584 1.213 4 0.207 0.207 0.207 0.484 0.711 1.061 1.223 6 0.412 0.554 0.831 1.145 1.610 2.575 7 0.989 1.239 1.699 2.167 2.833 4.255 7 0.989 1.239 1.699 2.167 2.833 4.255 8 1.344 1.646 2.180 2.733 3.400 5.071 9 1.735 2.088 2.709 3.325 4.168 5.899 1.735 1.213 2.185 2.204 4.885 6.737 1.213 2.185 2.204 4.885 6.737 1.213 2.185 2.204 2.833 3.400 5.071 1.2399 1.2399 1.2399 1.2399 1.239 1.239 1.239 1.239 1.239 1.2399 1.2399 1.239 1.2399 1.2399 1.239  | n   | a=0.995 | 0.99   | 0.975  | 0.95   | 0.90   | 0.75   |
| 3 0.072 0.115 0.216 0.352 0.584 1.213 4 0.711 1.061 1.292 6 0.412 0.554 0.831 1.145 1.610 2.575 6 0.412 0.554 0.831 1.145 1.610 2.575 7 0.989 1.239 1.090 2.167 2.833 4.255 8 1.344 1.646 2.180 2.733 3.400 5.071 1 2.603 3.455 1 2.264 3.455 1 2.264 3.455 1 2.265 1  | 1   | - 1     |        | 0.001  | 0.004  | 0.016  | 0.102  |
| 3 0.072 0.115 0.216 0.352 0.591 1.213 4 0.7011 1.061 1.933 6 0.412 0.554 0.831 1.145 1.610 2.675 6 0.676 0.472 1.237 1.655 2.204 3.455 7 0.989 1.239 1.239 1.090 2.167 2.833 4.235 9 1.755 2.083 3.227 3.090 2.167 2.833 4.235 1.090 2.167 2.833 4.235 1.090 2.167 2.833 4.235 1.090 2.167 2.833 4.235 1.090 2.167 2.833 4.235 1.090 2.167 2.833 4.235 1.090 2.167 2.833 4.235 1.090 2.167 2.833 4.235 1.090 2.167 2.833 4.235 1.090 2.167 2.833 4.235 1.090 2.167 2.833 4.235 1.090 2.167 2.833 4.235 1.090 2.167 2.833 4.235 1.090 2.167 2.833 4.235 1.090 2.167 2.833 1.090 2.167 2.833 1.227 2.838 3.247 3.940 4.865 6.777 2.131 2.210 2.158 3.247 3.940 4.865 6.777 2.131 2.210 2.158 3.247 3.940 4.865 6.777 2.121 | 2   | 0.010   | 0.020  | 0.051  | 0.103  |        |        |
| 6 0.412 0.554 0.831 1.145 1.610 2.655 6 0.871 1.299 1.690 2.167 2.833 1.255 8 1.344 1.646 2.180 2.733 3.400 5.071 9 1.735 2.088 2.700 3.325 4.165 5.899 10 2.136 2.588 3.247 3.940 4.863 6.737 11 2.603 3.083 3.8184 4.655 6.737 11 2.603 3.083 3.8184 4.655 6.737 11 2.603 3.083 3.8184 4.655 6.737 11 2.603 3.083 3.8184 4.655 6.579 7.543 3.355 4.107 5.600 5.629 6.671 7.790 10.155 13 4.601 5.229 6.202 7.261 8.547 11.037 |     |         | 0.115  | 0.216  |        | 0.584  |        |
| 6 0.076 0.472 1.237 1.035 2.206 2.303 3.55 7 7 0.989 1.759 2.167 2.833 1.355 8 1.344 1.646 2.180 2.790 3.325 4.165 5.071 9 1.755 2.088 2.700 3.325 4.166 5.679 10 2.136 2.588 3.247 3.940 4.865 6.737 11 2.603 3.053 3.816 4.576 5.678 7.341 13 3.565 4.107 8.009 5.892 7.042 0.259 13 3.400 4.865 1.331 3.565 4.107 8.009 5.892 7.042 0.259 13 4.609 6.262 7.261 7.700 10.185 15 4.609 6.262 7.261 7.700 10.185 15 4.609 6.262 7.261 7.700 10.185 15 4.609 6.262 7.261 7.700 10.185 15 4.609 6.262 7.261 7.700 10.185 15 4.609 6.262 7.261 7.700 10.185 15 4.609 6.262 7.261 7.700 10.185 15 4.609 6.262 7.261 7.700 10.185 15 4.609 6.262 7.261 7.700 10.185 15 4.609 6.262 7.261 7.700 10.185 15 4.609 6.262 7.261 7.700 10.185 12.702 18 6.265 7.015 6.231 9.390 10.805 13.673 19 6.844 7.633 8.907 10.117 11.651 14.552 20 7.434 8.260 9.591 10.851 12.443 15.452 12 8.034 8.887 10.283 11.597 13.240 16.344 12.22 8.643 9.502 10.106 11.669 13.1092 13.338 14.642 7.263 12.29 12.20 10.186 13.20 14.643 15.452 12.29 12.20 10.186 13.20 14.643 15.452 12.20 14.643 15.20 14.644 15.20 12.20 14.643 15.20 14.644 15.20 12.20 14.654 13.565 15.308 16.928 18.939 12.657 13.20 14.654 13.20 12.20 14.654 13.20 12.20 14.654 13.20 12.20 13.20 14.654 13.20 12.20 13.20 14.654 13.20 12.20 13.20 13.20 14.654 13.20 12.20 13.20  |     |         |        |        |        | 1.064  | 1.923  |
| 7 0.989 1.239 1.099 2.167 2.833 4.225 8 1.236 9 1.735 2.088 2.700 3.325 4.168 5.899 10 2.136 2.588 3.247 3.940 4.805 6.737 11 2.003 3.053 3.416 4.875 5.879 11 2.003 3.053 3.416 4.875 5.879 11 2.003 3.053 3.416 4.876 5.879 11 2.003 3.055 4.168 5.899 11 2.003 3.055 4.168 5.899 11 2.003 3.055 4.168 5.899 11 2.003 3.055 4.169 5.899 11 2.003 3.055 4.169 5.899 11 2.003 3.055 4.169 5.899 11 2.003 3.055 4.169 5.899 11 2.003 3.055 4.169 5.899 11 2.003 3.055 4.169 5.100 5.1 |     | 0.412   | 0.554  | 0.831  | 1.145  | 1.610  | 2.575  |
| \$ 1.344   |     |         | 0.872  | 1.237  | 1.635  | 2.204  | 3.455  |
| 9 1.735  |     | 0.989   | 1.239  | 1.690  |        | 2.833  | 4.255  |
| 10   |     |         |        |        |        | 3.490  | 5.071  |
| 11   |     |         |        |        |        |        | 5.899  |
| 12 3.074 3.571 4.404 5.226 6.304 8.431 13 3.565 4.107 5.009 5.892 7.042 9.255 14 4.075 4.660 5.629 6.671 7.700 10.155 15 4.601 5.229 6.262 7.261 8.547 11.037 16 5.142 5.812 6.908 7.964 8.672 10.085 12.792 17 5.697 6.408 7.564 8.672 10.085 12.792 18 6.284 7.633 8.007 10.115 10.865 12.792 20 7.434 8.260 9.591 10.851 12.442 21 8.034 8.897 10.283 11.591 10.851 12.442 22 8.643 9.542 10.982 12.338 14.042 17.240 23 9.260 10.196 11.669 13.091 14.646 18.137 24 9.885 10.856 12.401 13.484 15.659 19.037 25 10.500 11.524 13.120 14.611 16.473 19.039 27 11.100 11.100 11.100 11.669 13.091 14.646 18.137 28 11.100 12.198 13.120 14.611 16.473 19.039 29 12.241 13.565 15.008 16.928 13.930 12.338 14.042 17.292 17.033 13.120 14.611 16.473 19.039 17.030 17.0 |     |         |        |        |        | 4.863  | 6.737  |
| 13 3.563 4.107 5.009 5.099 7.042 0.259 14 4.075 4.660 5.629 6.571 7.790 10.155 15 4.691 5.229 6.262 7.261 8.547 11.037 16 5.142 6.812 6.908 7.962 9.312 11.512 17 5.697 6.408 7.564 8.672 10.085 12.792 18 6.265 7.015 8.231 9.390 10.865 12.792 20 7.434 8.260 9.591 10.855 12.443 15.452 21 8.034 8.897 10.283 11.555 12.443 15.452 22 8.034 8.897 10.283 11.595 13.244 15.22 23 8.034 8.897 10.283 11.595 13.244 15.22 24 9.855 10.656 12.401 13.468 15.659 19.037 25 10.520 11.524 13.120 14.611 16.73 19.039 26 11.160 12.198 13.844 15.379 17.292 20.413 27 11.808 12.879 14.573 16.151 18.237 17.292 20.413 27 11.808 12.879 14.573 16.151 18.93 12.247 28 12.461 13.565 15.308 16.928 18.939 22.637 30 13.737 14.954 16.791 18.493 20.599 24.473 31 14.458 15.655 17.539 19.288 18.939 22.637 33 15.134 16.364 16.791 18.493 20.599 24.473 31 14.458 15.655 17.539 19.288 12.241 20.313 14.458 15.659 39.244.473 31 14.458 15.655 17.539 19.288 12.939 22.637 33 15.134 16.362 18.291 20.077 22.271 26.331 14.558 17.798 19.808 20.599 24.473 31 14.458 15.655 17.539 19.288 22.637 33 15.134 16.362 18.291 20.077 22.271 26.331 13.100 14.614 16.51 16.559 19.007 17.798 19.608 20.569 22.465 24.797 29.054 24.473 31 16.586 19.980 20.589 22.465 24.797 29.054 29.973 38 19.289 20.691 22.878 24.4884 24.075 26.492 30.893 38 19.289 20.691 22.878 24.4884 24.075 26.492 30.893  |     |         |        |        |        |        | 7.524  |
| 14 4.075 4.680 5.629 6.871 7.790 10.115 13 4.601 5.229 6.262 7.261 8.547 11.017 16 5.142 5.812 6.908 7.962 9.312 11.512 17 5.997 6.408 7.564 8.672 10.085 12.792 18 6.265 7.015 8.231 9.390 10.805 13.673 19 6.44 7.633 8.007 10.155 11.651 14.512 20 7.644 8.897 10.263 11.651 14.512 21 8.034 8.897 10.263 11.697 13.240 11.615 12.202 22 8.643 0.542 10.923 11.597 13.240 15.412 23 9.260 10.196 11.689 13.091 14.848 15.597 12.240 13.848 15.659 19.37 24 9.855 10.856 12.401 13.848 15.659 19.37 25 11.100 12.158 13.120 14.611 19.473 19.939 12.27 11.698 12.27 11.698 12.27 11.698 12.279 11.698 12.279 13.240 12.27 12.29 13.31 14.651 12.279 13.240 13.618 15.659 19.37 11.698 12.279 14.691 13.691 14.691 12.279 13.240 13.279 14.691 13.691 14.691 12.279 13.290 14.691 13.691 12.279 13.290 14.691 13.691 12.279 13.290 14.691 13.691 15.479 12.290 13.291 14.691 10.473 19.339 13.279 14.2898 13.291 13.291 14.297 13.291 13 |     |         |        |        |        |        | 8.433  |
| 15   |     |         |        |        |        |        |        |
| 16 5.142 5.812 6.008 7.062 9.312 11.02 17 5.507 6.408 7.564 8.672 10.085 12.702 18 6.265 7.015 8.231 9.390 10.865 12.702 19 6.844 7.633 8.907 10.117 11.651 14.592 20 7.434 8.260 9.591 10.851 12.443 15.452 21 8.034 8.897 10.283 11.591 12.246 15.452 22 8.643 9.542 10.923 11.591 12.246 17.240 22 9.260 10.196 11.669 13.091 14.86 15.650 19.307 24 9.265 10.856 12.401 13.848 15.655 19.037 25 10.520 11.524 13.120 14.611 16.473 19.392 26 11.160 12.198 13.844 15.379 17.292 20.813 27 11.808 12.879 14.573 16.151 18.114 17.792 28 12.241 13.255 15.308 18.925 11.939 22.637 29 13.212 14.257 18.040 11.515 18.114 17.79 21 12.461 13.565 15.308 18.925 18.939 22.637 22 13.461 13.565 15.308 18.925 18.939 22.637 23 13.737 14.884 17.579 19.281 20.593 24.473 24 13.65 15.060 17.599 19.281 20.593 24.473 25 13.747 17.759 19.241 20.595 22.657 26 13.757 14.864 17.599 19.281 20.599 24.473 27 18.65 17.767 19.047 20.867 22.110 27.213 28 17.192 18.600 20.569 22.465 24.797 29.054 29 13.659 19.960 22.166 24.075 26.492 30.893 21 15.585 19.960 20.569 24.4075 26.492 30.893 21 12.289 20.691 22.878 24.884  |     |         |        |        |        |        |        |
| 17 5.597 6.408 7.564 8.672 10.085 12.792 18 6.255 7.015 8.231 9.390 10.865 13.675 19 6.844 7.633 8.907 10.117 11.651 14.592 20 7.434 8.260 9.591 10.851 12.443 15.452 21 8.034 8.897 10.283 11.597 13.240 16.314 22 8.643 9.542 10.982 13.338 14.642 7.240 23 9.260 10.166 11.669 13.091 14.868 18.137 24 9.885 10.886 13.300 14.861 16.55 19.007 25 11.100 12.108 13.844 15.379 17.292 20.413 27 11.808 12.108 13.844 15.379 17.292 20.413 27 11.808 12.490 14.257 16.047 17.708 19.768 23.567 28 12.461 13.565 15.308 16.928 18.939 22.637 30 13.737 14.954 16.791 18.493 20.599 24.478 31 14.458 15.655 17.539 19.281 20.402 23.567 30 13.737 14.954 16.791 18.493 20.599 24.478 31 14.458 15.655 17.539 19.281 20.402 22.271 26.331 31 14.458 15.655 17.539 19.281 20.402 22.271 26.331 33 15.615 17.074 18.096 20.072 22.271 26.331 34 17.185 17.074 18.096 20.569 22.465 24.797 29.054 35 17.192 18.500 20.569 22.465 24.797 29.054 36 17.887 19.233 20.599 22.4684 24.797 29.054 37 18.589 19.980 22.106 24.075 26.492 30.893 38 19.289 20.691 22.2878 24.4884 24.075 26.492 30.893   |     |         |        |        |        |        |        |
| 18 6.255 7.015 8.231 9.390 10.865 13.675 19 6.844 7.633 8.907 10.117 11.651 14.512 20 7.434 8.260 9.591 10.851 13.246 15.452 21 8.643 9.542 10.825 11.591 13.246 15.452 22 9.260 10.196 11.669 13.091 14.846 15.452 22 9.260 10.196 11.669 13.091 14.846 15.452 17.240 12.40 13.856 12.401 13.848 15.655 19.037 12.24 13.120 14.611 16.473 19.037 12.25 10.520 11.524 13.120 14.611 16.473 19.037 12.27 11.808 12.879 14.573 16.151 18.114 12.779 12.27 13.805 13.407 13.848 15.655 12.461 13.848 15.655 19.037 12.29 12.451 13.555 15.308 18.925 13.939 12.537 12.29 13.121 14.257 16.047 17.708 15.057 15.058 13.507 13.507 15.058 13.507 13.50 |     |         |        |        |        |        |        |
| 19   |     |         |        |        |        |        |        |
| 20 7.434 8.260 9.591 10.851 12.443 15.452 21 8.034 8.897 10.285 11.59F 13.240 16.314 22 8.643 9.542 10.982 12.338 14.042 17.240 23 9.260 10.196 11.669 13.091 14.848 15.655 24 9.885 10.886 12.401 13.848 15.655 19.037 25 10.520 11.524 13.120 14.611 16.473 19.039 26 11.160 12.198 13.844 15.379 17.292 20.843 27 11.808 12.879 14.573 16.151 18.114 21.749 28 12.461 13.565 15.308 18.928 11.939 22.657 29 13.121 14.257 16.047 17.708 19.768 23.557 29 13.737 14.984 18.799 17.798 20.599 24.473 31 14.458 15.655 18.291 20.201 18.291 20.599 24.473 32 15.158 17.074 18.291 20.913 34 16.591 17.779 19.047 20.867 23.557 28.359 34.1509 20.551 35.1509 20.551 35.1509    |     |         |        |        |        |        |        |
| 21 8.034 8.897 10.283 11.597 13.240 16.314 22 8.643 0.542 10.982 12.338 14.042 17.240 12.338 14.042 17.240 10.982 12.338 14.042 17.240 12.338 14.042 17.240 12.338 14.042 17.240 12.338 14.042 17.240 12.338 14.042 17.240 12.338 14.042 17.240 12.338 14.042 17.240 12.338 14.042 17.240 12.338 14.041 16.473 19.039 12.340  |     |         |        |        |        |        |        |
| 22 8.643 9.542 10.982 12.338 14.042 17.240 23 9.260 10.196 11.689 13.091 14.646 18.137 24 9.885 10.886 12.401 13.848 15.659 19.037 25 10.520 11.524 13.120 14.611 16.473 19.339 26 11.160 12.198 13.844 15.379 17.292 20.413 27 11.808 12.879 14.673 16.151 18.114 21.749 28 12.461 13.565 15.308 18.928 18.939 22.657 29 13.121 14.257 16.047 17.708 19.768 23.567 30 13.737 14.984 18.791 18.493 20.599 24.473 31 14.458 15.655 17.639 19.220 20.543 32 15.134 10.304 18.791 18.493 20.599 24.473 33 14.458 10.658 17.639 19.262 21.214 25.390 34 16.501 17.789 19.806 21.664 20.867 22.271 26.331 35 17.192 18.609 20.569 22.465 24.797 29.054 36 17.887 19.233 21.336 23.269 25.643 29.973 37 18.586 19.980 22.106 24.075 26.492 30.893 38 19.289 20.691 22.2878 24.884  |     |         |        |        |        |        |        |
| 22 9.260 10.106 11.680 13.091 14.848 18.137 24 9.855 10.856 12.401 13.848 15.659 19.037 25 10.520 11.524 13.120 14.611 19.473 19.039 26 11.160 12.198 13.844 15.779 17.292 20.641 27 11.808 12.879 14.673 16.151 18.114 21.749 28 12.461 13.565 15.308 16.228 18.939 22.657 29 13.121 14.257 16.047 17.708 19.768 23.567 30 13.737 14.954 16.791 18.493 20.599 24.473 31 14.458 15.655 17.539 19.281 21.434 25.399 32 15.134 16.362 18.291 20.072 22.271 26.391 33 15.815 17.074 19.047 20.867 23.110 27.219 34 16.501 17.789 19.866 21.664 23.952 26.136 35 17.192 18.500 20.569 22.465 24.797 29.054 36 17.887 19.233 21.336 23.269 25.643 29.973 37 18.586 19.960 22.166 24.075 26.492 30.893 38 19.288 20.691 22.878 24.884 27.943 31.815  |     |         |        |        |        |        |        |
| 24         9.855         10.856         12.401         13.848         15.659         19.037           25         10.520         11.524         13.120         14.6611         16.473         19.037           26         11.160         12.198         13.844         15.379         17.292         20.413           27         11.808         12.879         14.573         16.151         18.114         21.749           25         12.461         13.565         15.308         16.928         18.939         22.637           30         13.737         14.954         16.791         18.493         20.599         24.473           31         14.455         16.655         17.539         19.281         22.214         26.301           32         15.134         16.302         18.291         20.072         22.271         26.301           33         15.136         16.302         19.802         21.662         21.572         22.272         27.216           34         15.601         17.789         19.806         21.664         21.797         29.054         29.054           35         17.192         18.609         20.569         22.465         24.797  |     |         |        |        |        |        |        |
| 25 10.520 11.524 13.120 14.611 16.473 19.939 26 11.160 12.198 13.844 15.379 17.292 20.613 27 11.808 12.199 14.673 16.161 19.114 21.749 28 12.461 13.565 15.308 16.228 19.399 29 13.121 14.257 16.047 17.708 19.768 22.637 30 13.737 14.954 16.791 18.493 20.599 24.473 31 14.458 15.655 17.539 19.281 21.434 25.399 32 15.134 16.362 18.291 20.072 22.271 26.391 33 15.815 17.074 19.047 20.867 23.110 27.219 34 16.501 17.789 19.666 21.664 23.952 28.136 35 17.192 18.509 20.569 22.465 24.797 29.054 36 17.887 19.233 21.336 23.269 25.643 29.973 37 18.586 19.960 22.166 24.075 26.492 30.893 38 19.289 20.691 22.1878 24.884 27.443 11.815  |     |         |        |        |        |        |        |
| 26 11.100 12.108 13.864 15.379 17.292 20.413 27 11.808 12.079 14.675 16.161 18.114 21.749 27 11.808 12.079 14.675 16.161 18.114 21.749 29 12.461 13.565 15.308 16.928 18.939 22.657 29 13.121 14.257 16.047 17.708 19.768 23.567 30 13.737 14.954 16.791 18.493 20.599 24.473 31 14.455 15.655 17.539 19.281 21.434 20.599 24.473 32 15.134 16.362 18.291 20.072 22.271 26.301 33 15.134 16.362 18.291 20.072 22.271 26.301 33 15.134 16.500 20.500 22.465 24.797 29.054 23.693 35 17.192 18.600 20.569 22.465 24.797 29.054 36 17.887 19.233 21.335 23.269 25.643 29.973 37 16.585 19.980 22.106 24.075 26.492 30.893 38 19.289 20.691 22.878 24.884 27.443 31.815  |     |         |        |        |        |        |        |
| 27 11.808 12.879 14.673 16.161 19.114 21.749 28 12.461 13.565 15.308 16.928 19.399 22.637 29 13.121 14.257 16.047 17.708 19.768 23.567 30 13.737 14.954 16.791 18.493 20.599 24.473 31 14.458 15.655 17.539 19.281 21.434 25.399 32 15.134 16.362 18.291 20.072 22.271 26.391 33 15.815 17.074 19.047 20.867 23.110 27.219 34 16.501 17.789 19.806 21.664 23.952 28.136 35 17.192 18.509 20.569 22.465 24.797 29.054 36 17.887 19.233 21.336 23.269 25.643 29.973 37 18.586 19.960 22.166 24.075 26.492 30.893 38 19.289 20.691 22.1878 24.884 27.343 31.815   |     |         |        |        |        |        |        |
| 28 12.461 13.565 15.308 16.928 18.939 22.637 29 13.121 14.257 16.047 17.708 19.768 23.567 30 13.737 14.964 16.791 18.493 20.599 24.475 31 14.458 15.655 17.539 19.281 21.434 25.399 32 15.134 16.362 18.291 20.072 22.271 26.331 33 15.815 17.074 19.047 20.667 23.110 27.219 34 17.192 18.500 20.569 22.465 24.797 28.103 35 17.192 18.500 20.569 22.465 24.797 28.103 36 17.887 19.233 21.336 23.269 26.443 29.973 37 18.585 19.980 22.106 24.075 26.492 30.893 38 19.289 20.691 22.1878 24.084 27.443 31.815  |     |         |        |        |        |        |        |
| 29 13.121 14.257 16.047 17.708 19.768 23.567 30 13.737 14.954 16.701 18.403 20.599 24.473 31 14.458 15.655 17.539 19.281 21.434 25.399 32 15.134 16.362 18.291 20.072 22.271 26.391 33 15.815 17.074 19.047 20.867 23.110 27.219 34 16.501 17.789 19.806 21.664 23.952 28.136 35 17.192 18.509 20.569 22.465 24.797 29.054 36 17.887 19.233 21.336 23.269 25.643 29.973 37 18.586 19.986 22.166 24.075 26.492 30.893 38 19.289 20.691 22.878 24.884 27.343 31.815  |     |         |        |        |        |        |        |
| 30 13.737 14.954 16.791 18.493 20.599 24.473 31 14.458 15.655 17.539 19.281 21.434 25.390 32 15.134 16.362 18.291 20.072 22.271 26.391 33 15.815 17.074 18.047 20.867 23.110 27.219 34 16.501 17.759 19.866 21.664 23.952 28.136 35 17.192 18.509 20.569 22.465 24.797 29.051 36 17.887 19.233 21.336 23.269 25.643 29.973 37 18.586 19.980 22.106 24.075 26.492 30.893 38 19.289 20.691 22.2878 24.884 27.343 31.815  |     |         |        |        |        |        |        |
| 31 14.458 15.655 17.539 19.281 21.434 25.39) 32 15.134 16.362 18.291 20.072 22.271 26.304 33 15.815 17.074 19.047 20.867 23.110 27.219 34 16.501 17.789 19.806 21.664 23.952 28.136 35 17.192 18.509 20.569 22.465 24.797 29.054 36 17.887 19.233 21.336 23.269 25.643 29.973 37 18.586 19.986 22.106 24.075 26.492 30.893 38 19.289 20.691 22.878 24.884 27.343 31.815  |     |         |        |        |        |        |        |
| 32 15.134 16.302 18.291 20.072 22.271 26.331 3 15.815 17.074 19.047 20.867 21.110 27.219 34 16.501 17.789 19.806 21.664 21.052 28.136 35 17.192 18.609 20.669 22.465 22.797 29.054 36 17.887 19.233 21.336 23.269 25.643 29.973 37 18.589 19.960 22.106 24.075 26.492 30.893 38 19.289 20.691 22.878 24.884 27.343 31.815  |     |         |        | 17.539 | 19.281 |        |        |
| 33 15.815 17.074 19.047 20.867 23.110 27.219 34 16.501 17.789 19.806 21.664 23.952 28.136 35 17.192 18.509 20.569 22.465 24.797 29.054 36 17.887 19.233 21.336 23.269 25.643 29.973 37 18.586 19.960 22.106 24.075 26.492 30.893 38 19.289 20.691 22.878 24.884 27.343 31.815  |     |         |        | 18.291 | 20.072 |        |        |
| 34         16.501         17.789         19.806         21.664         23.952         28.136           35         17.192         18.509         20.569         22.465         24.797         29.054           36         17.887         19.233         21.336         23.269         25.643         29.973           37         18.589         19.960         22.106         24.075         26.492         30.893           38         19.289         20.691         22.878         24.884         27.343         31.815   | 33  | 15.815  | 17.074 | 19.047 | 20.867 | 23.110 |        |
| 35 17.192 18.509 20.569 22.465 24.797 29.054 36 17.887 19.233 21.335 23.269 25.643 29.973 37 18.586 19.960 22.106 24.075 26.492 30.893 38 19.289 20.691 22.878 24.884 27.343 31.815  | 34  |         | 17.789 | 19.806 | 21.664 |        |        |
| 37 18.586 19.960 22.106 24.075 26.492 30.893 38 19.289 20.691 22.878 24.884 27.343 31.815  | 3.5 |         | 18.509 | 20.569 | 22.465 | 24.797 |        |
| 37 18.586 19.960 22.106 24.075 26.492 30.893 38 19.289 20.691 22.878 24.884 27.343 31.815  | 36  | 17.887  | 19.233 |        |        | 25.643 | 29.973 |
| 10.000   | 37  | 18.586  | 19.960 |        |        |        |        |
| 39 19.996 21.426 23.654 25.695 28.196 32.737   |     |         |        |        |        | 27.343 | 31.815 |
|  |     |         |        |        |        | 28.196 | 32.737 |
| 40 20.707 22.164 24.433 26.509 29.051 33.660   | 40  | 20.707  |        |        |        | 29.051 | 33.660 |
| 41 21.421 22.906 25.215 27.326 29.907 34.555   | 41  | 21.421  | 22.906 | 25.215 | 27.326 | 29.907 | 34.555 |
| 42 22.138 23.650 25.999 28.144 30.765 35.510   | 42  | 22.138  | 23.650 | 25.999 |        | 30.765 | 35.510 |
| 43 22.859 24.398 26.785 28.965 31.625 36.436<br>44 23.584 25.148 27.575 29.787 32.487 37.363   |     |         |        |        |        | 31.625 |        |

### F-distribution

If X,Y are two independent  $\chi^2-$ distributed random variables :

$$X \sim \chi^2(n)$$
,  $Y \sim \chi^2(m)$ 

$$\longrightarrow$$
  $Z = \frac{X/n}{Y/m}$ 

is called a F-distribution with first degree of freedom n, and 2nd degree of freedom m

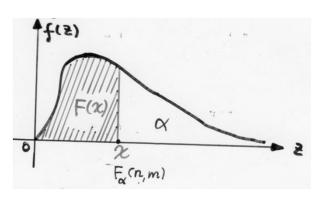
$$f(z) = \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{n}{m}\right) \left(\frac{n}{m}z\right)^{\frac{n}{2}-1} \left(1 + \frac{n}{m}z\right)^{-\frac{n+m}{2}} \qquad (z \ge 0)$$

$$E(z)=\frac{n}{m-2} \quad \ (m>2)$$

$$Var(z) = rac{2m^2(n+m-2)}{n(m-2)^2(m-4)} ~~(m>4)$$

#### F-distribution

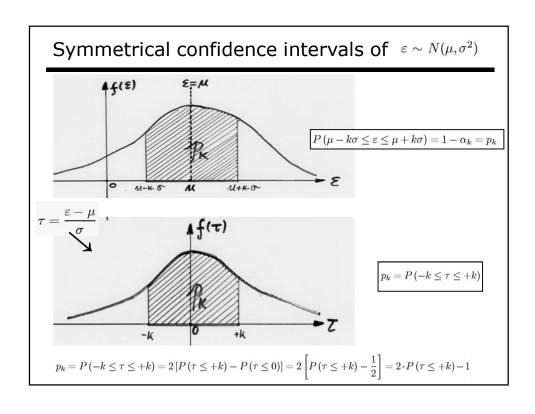
$$f(z) = \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{n}{m}\right) \left(\frac{n}{m}z\right)^{\frac{n}{2}-1} \left(1 + \frac{n}{m}z\right)^{-\frac{n+m}{2}} \qquad (z \ge 0)$$



|       |      | F-distribution table |              |              |              |              |              |               |                   |              |              |              |             |             |             |             |             |             |
|-------|------|----------------------|--------------|--------------|--------------|--------------|--------------|---------------|-------------------|--------------|--------------|--------------|-------------|-------------|-------------|-------------|-------------|-------------|
|       |      |                      |              |              |              | Ρ{           | F(n,         | ,m) >         | F <sub>a</sub> (n | ,m) }        | = α          | = 0.         | 05          |             |             |             |             |             |
|       |      |                      |              |              |              |              |              | n             |                   |              |              |              |             |             |             |             |             |             |
|       | 1    | 2                    | 3            | 4            | 5            | 6            | 7            | 8             | 9                 | 10           | 11           | 12           | 13          | 14          | 15          | 16          | 17          | 18          |
| m / 2 | 161  | 200                  | 216          | 225          | 230          | 234          | 237<br>19.4  | 239           | 241<br>19.4       | 242<br>19.4  | 243<br>19.4  | 244<br>19.4  | 245<br>19.4 | 245<br>19.4 | 246<br>19.4 | 246<br>19.4 | 247<br>19.4 | 247<br>19.4 |
| 3     | 18.5 | 19.0                 | 19.2<br>9.28 | 19.2<br>9.12 | 19.3         | 19.3         | 8.89         | 8.85          | 8.81              | 8.79         | 8.76         | 8.74         | 8.73        | 8.71        | 8.70        | 8.69        | 8.68        | 8.67        |
| 4     | 7.71 | 6.94                 | 6.59         | 6.39         | 6.26         | 6.16         | 6.09         | 6.04          | 6.00              | 5.96         | 5.94         | 5.91         | 5.89        | 5.87        | 5.86        | 5.84        | 5.83        | 5.82        |
| 5     | 6.61 | 5.79                 | 5.41         | 5.19         | 5.05         | 4.95         | 4.88         | 4.82          | 4.77              | 4.74 .       | 4.70         | 4.68         | 4.66        | 4.64        | 4.62        | 4.60        | 4.59        | 4.58        |
| 6 7   | 5.99 | 5.14<br>4.74         | 4.76         | 4.53         | 4.39<br>3.97 | 4.28<br>3.87 | 4.21<br>3.79 | 4.15          | 4.10              | 4.06<br>3.64 | 4.03<br>3.60 | 4.00<br>3.57 | 3.98        | 3.96        | 3.94        | 3.92        | 3.91        | 3.90        |
| 8     | 5.32 | 4.46                 | 4.07         | 3.84         | 3.69         | 3.58         | 3.50         | 3.44          | 3.39              | 3.35         | 3.31         | 3.28         | 3.26        | 3.24        | 3.22        | 3.20        | 3.19        | 3.17        |
| 9     | 5.12 | 4.26                 | 3.86         | 3.63         | 3.48         | 3.37         | 3.29         | 3.23          | 3.18              | 3.14         | 3.10         | 3.07         | 3.05        | 3.03        | 3.01        | 2.99        | 2.97        | 2.96        |
| 10    | 4.96 | 4.10                 | 3.71         | 3.48         | 3.33         | 3.22         | 3.14         | 3.07          | 3.02              | 2.98         | 2.94         | 2.91         | 2.89        | 2.86        | 2.85        | 2.83        | 2.81        | 2.80        |
| 11    | 4.84 | 3.98                 | 3.59         | 3.36         | 3.20         | 3.09         | 3.01         | 2.95          | 2.90              | 2.85         | 2.82         | 2.79         | 2.76        | 2.74        | 2.72        | 2.70        | 2.69        | 2.67        |
| 13    | 4.67 | 3.81                 | 3.41         | 3.18         | 3.03         | 2.92         | 2.83         | 2.77          | 2.71              | 2.67         | 2.63         | 2.60         | 2.58        | 2.55        | 2.53        | 2.51        | 2.50        | 2.48        |
| 14    | 4.60 | 3.74                 | 3.34         | 3.11         | 2.96         | 2.85         | 2.76         | 2.70          | 2.65              | 2.60         | 2.57         | 2.53         | 2.51        | 2.48        | 2.46        | 2.44        | 2.43        | 2.41        |
| 15    | 4.54 | 3.68                 | 3.29         | 3.06         | 2.90         | 2.79         | 2.71         | 2.64          | 2.59              | 2.54         | 2.51         | 2.48         | 2.45        | 2.42        | 2.40        | 2.38        | 2.37        | 2.35        |
| 16    | 4.49 | 3.63                 | 3.24         | 3.01<br>2.96 | 2.85         | 2.74         | 2.66         | 2.59          | 2.54              | 2.49         | 2.46         | 2.42         | 2.40        | 2.37        | 2.35        | 2.33        | 2.32        | 2.30        |
| 18    | 4.41 | 3.59<br>3.55         | 3.16         | 2.96         | 2.77         | 2.66         | 2.58         | 2.55          | 2.49              | 2.43         | 2.37         | 2.34         | 2.33        | 2.29        | 2.27        | 2.25        | 2.23        | 2.20        |
| 19    | 4.38 | 3.52                 | 3.13         | 2.90         | 2.74         | 2.63         | 2.54         | 2.48          | 2.42              | 2.38         | 2.34         | 2.31         | 2.28        | 2.26        | 2.23        | 2.21        | 2.20        | 2.18        |
| 20    | 4.35 | 3.49                 | 3.10         | 2.87         | 2.71         | 2.60         | 2.51         | 2.45          | 2.39              | 2.35         | 2.31         | 2.28         | 2.25        | 2.22        | 2.20        | 2.18        | 2.17        | 2.15        |
| 21    | 4.32 | 3.47                 | 3.07         | 2.84         | 2.68         | 2.57         | 2.49         | 2.42          | 2.37              | 2.32         | 2.28         | 2.25         | 2.22        | 2.20        | 2.18        | 2.16        | 2.14        | 2.12        |
| 23    | 4.28 | 3.44                 | 3.03         | 2.82         | 2.64         | 2.53         | 2.44         | 2.37          | 2.34              | 2.30         | 2.23         | 2.23         | 2.18        | 2.15        | 2.13        | 2.13        | 2.09        | 2.07        |
| 24    | 4.26 | 3.40                 | 3.01         | 2.78         | 2.62         | 2.51         | 2.42         | 2.36          | 2.30              | 2.25         | 2.21         | 2.18         | 2.15        | 2.13        | 2.11        | 2.09        | 2.07        | 2.05        |
| 25    | 4.24 | 3.39                 | 2.99         | 2.76         | 2.60         | 2.49         | 2.40         | 2.34          | 2.28              | 2.24         | 2.20         | 2.16         | 2.14        | 2.11        | 2.09        | 2.07        | 2.05        | 2.04        |
| M 26  | 4.23 | 3.37                 | 2.98         | 2.74         | 2.59         | 2.47         | 2.39         | 2.32          | 2.27              | 2.22         | 2.18         | 2.15         | 2.12        | 2.09        | 2.07        | 2.05        | 2.03        | 2.00        |
| 28    | 4.20 |                      | 2.95         | 2.71         | 2.56         | 2.45         | 2.36         | 2.29          | 2.24              | 2.19         | 2.15         | 2.13         | 2.09        | 2.06        | 2.04        | 2.04        | 2.02        | 1.9         |
| 29    | 4.18 |                      | 2.93         | 2.70         | 2.55         | 2.43         |              | 2.28          | 2.22              | 2.18         | 2.14         | 2.10         | 2.08        | 2.05        | 2.03        | 2.01        | 1.99        | 1.9         |
| 30    | 4.17 |                      | 2.92         | 2.69         | 2.53         | 2.42         |              | 2.27          | 2.21              | 2.16         | 2.13         | 2.09         | 2.06        | 2.04        | 2.01        | 1.99        | 1.98        | 1.9         |
| 32    | 4.15 | 3.29                 | 2.90         | 2.67         | 2.51         | 2.40         |              | 2.24          | 2.19              | 2.14         | 2.10         | 2.07         | 2.04        | 2.01        | 1.99        | 1.97        | 1.95        | 1.9         |
| 34    | 4.13 | 3.28<br>3.26         | 2.88         | 2.65         | 2.49         | 2.38         | 2.29         | 12.23<br>2.21 | 2.17              | 2.12         | 2.08         | 2.05         | 2.02        | 1.99        | 1.97        | 1.95        | 1.93        | 1.9         |
| 38    | 4.10 | 3.24                 | 2.85         | 2.62         | 2.46         | 2.35         | 2.26         | 2.19          | 2.14              | 2.09         | 2.05         | 2.02         | 1.99        | 1.96        | 1.94        | 1.92        | 1.90        | 1.8         |

# Confidence intervals

- A confidence interval of  $\, \varepsilon \,$  is an interval which contains the expectation  $\mu \,$
- The probability  $\alpha$  that  $\varepsilon$  lies *outside* the confidence interval is called the *risk level*
- The probability that  $\pmb{\varepsilon}$  lies *inside* the confidence interval is  $1-\alpha$  , called the *confidence level*
- $\bullet$  Confidence intervals are often taken as symmetrical intervals around the expectation  $\mu$



#### Probability inside confidence intervals of $\varepsilon \sim N(\mu, \sigma^2)$

$$p_{k} = P\left(-k \le \tau \le +k\right) = 2\left[P\left(\tau \le +k\right) - P\left(\tau \le 0\right)\right] = 2\left[P\left(\tau \le +k\right) - \frac{1}{2}\right] = 2 \cdot P\left(\tau \le +k\right) - 1$$

$$p_1 = P(-1 \le \tau \le +1) = 2 \cdot 0.84134 - 1 = 68.27\%$$

$$p_2 = P(-2 \le \tau \le +2) = 2 \cdot 0.97725 - 1 = 95.45\%$$

$$p_3 = P(-3 \le \tau \le +3) = 2 \cdot 0.99865 - 1 = 99.73\%$$

Table 1.5: Confidence Intervals of Normal Distributions

| k | interval for $\varepsilon$ | interval for $	au$ | confidence level $(p_k = 1 - \alpha_k)$ | $risk\ level\ (\alpha_k)$ |
|---|----------------------------|--------------------|---|---------------------------|
| 1 | $\mu \pm 1\sigma$          | ±1                 | 68.27~%                                 | 31.73 %                   |
| 2 | $\mu \pm 2\sigma$          | ±2                 | 95.45 %                                 | 4.55 %                    |
| 3 | $\mu \pm 3\sigma$          | ±3                 | 99.73 %                                 | 0.27 %                    |
| 4 | $\mu \pm 4\sigma$          | ±4                 | 99.99~%                                 | 0.01 %                    |

 $\rightarrow$  2 $\sigma$  or 3 $\sigma$  can be the maximum error tolerance!

### Statistical Analysis (2/2)

- Different distributions of random variables
- Confidence intervals
- Hypothesis tests (statistical tests)
- Variance analysis
- Regression analysis (least squares fitting)

### Sample mean and sample variance

Let  $x_i \sim N(\mu, \sigma^2)$   $(i=1,2,\cdots,n)$  be n independent normally distributed variables with equal expectation  $\mu$  and equal variance  $\sigma^2$ . A sample mean  $\overline{x}$  and an estimated sample variance  $\hat{\sigma}^2$  can be calculated:

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \; , \quad \widehat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2$$

$$\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$$
  $\longrightarrow$   $u = \frac{\overline{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$ 

$$\frac{(n-1)\widehat{\sigma}^2}{\sigma^2} = \sum_{i=1}^n \left(\frac{\overline{x} - x_i}{\sigma}\right)^2 \sim \chi^2(n-1) \longrightarrow t' = \frac{\frac{\overline{x} - \mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)\widehat{\sigma}^2}{\sigma^2} \cdot \frac{1}{n-1}}} = \frac{\overline{x} - \mu}{\widehat{\sigma}/\sqrt{n}} \sim t(n-1)$$

$$Var\left(\widehat{\sigma}^{2}\right) = E\left\{\left(\widehat{\sigma}^{2} - \sigma^{2}\right)^{2}\right\} = \frac{2\sigma^{4}}{n-1}$$

#### 2 measurement samples

$$x_{i} \sim N(\underline{\mu}_{1}, \sigma_{1}^{2}) \ (i = 1, 2, \cdots, n) \qquad y_{i} \sim N(\underline{\mu}_{2}, \sigma_{2}^{2}) \ (i = 1, 2, \cdots, m)$$

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_{i} , \quad \widehat{\sigma}_{1}^{2} = \frac{1}{n-1} \sum_{i=1}^{n} (x_{i} - \overline{x})^{2} \qquad \overline{y} = \frac{1}{m} \sum_{i=1}^{m} x_{i} , \quad \widehat{\sigma}_{2}^{2} = \frac{1}{m-1} \sum_{i=1}^{m} (y_{i} - \overline{y})^{2}$$

$$\frac{(m-1)\widehat{\sigma}_{1}^{2}}{\sigma_{1}^{2}} \sim \chi^{2}(m-1)$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$F' = \frac{\frac{(n-1)\widehat{\sigma}_{1}^{2}}{\sigma_{1}^{2}} \cdot \frac{1}{n-1}}{\frac{(m-1)\widehat{\sigma}_{2}^{2}}{\sigma_{2}^{2}} \cdot \frac{1}{m-1}} = \frac{\sigma_{2}^{2} \cdot \widehat{\sigma}_{1}^{2}}{\sigma_{1}^{2} \cdot \widehat{\sigma}_{2}^{2}} \sim F(n-1, m-1)$$

#### Hypothesis tests (statistical tests)

- Make measurements / collect samples
- Compute sample statistics (mean, variance, etc)
- Define the null-hypothesis H<sub>0</sub> and the alternative hypothesis H<sub>1</sub>
- Choose risk level  $\alpha$
- Find critical value from the distribution table
- Compare computed statistics against the critical value to decide whether to accept H<sub>0</sub> or H<sub>1</sub>

### Test expectation when variance is known

u-test

 $x_1, x_2, \dots, x_n$  are n independent measurements of the same normal distribution  $N(\mu, \sigma^2)$ 

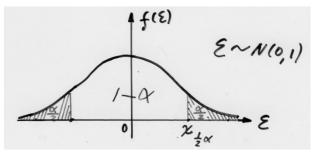
$$\overline{x} = rac{1}{n} \sum_{i=1}^n x_i \qquad \qquad ar{x} \sim N(\mu, rac{\sigma^2}{n}) \quad ext{and} \quad rac{ar{x} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1)$$

| Null hypothesis $H_0$ :        | $\overline{x}=\mu$      |
|--------------------------------|-------------------------|
| Alternative hypothesis $H_1$ : | $\overline{x} \neq \mu$ |

Choose risk level  $\alpha = 5\%$ 

# Test expectation when variance is known

Find the cirical value  $x_{\frac{1}{2}\alpha}$  of N(0,1)



$$P\left\{\left|\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right| \leq x_{\frac{1}{2}\alpha}\right\} = 1 - \alpha \quad or: P\left\{\left|\bar{x}-\mu\right| \leq c_1\right\} = 1 - \alpha$$

Compare critical value  $x_{\frac{1}{2}\alpha}$  with computed  $\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$  Decide whether to accept H<sub>0</sub> or H<sub>1</sub>

#### Test expectation when variance is unknown

#### t-test

 $x_1, x_2, \dots, x_n$  are n independent measurements of the same normal distribution  $N(\mu, \sigma^2)$ 

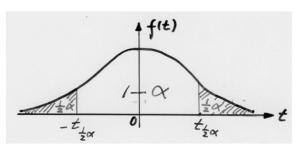
$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$
  $\widehat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2$   $\frac{\overline{x} - \mu}{\widehat{\sigma}/\sqrt{n}} \sim t(n-1)$ 

| Null hypothesis $H_0$ :        | $\overline{x} = \mu$    |
|--------------------------------|-------------------------|
| Alternative hypothesis $H_1$ : | $\overline{x} \neq \mu$ |

Choose risk level  $\alpha = 5\%$ 

#### Test expectation when variance is unknown

Find the cirical value  $t_{\frac{1}{2}\alpha}(n-1)$  of t-distribution



$$P\left\{\frac{\overline{x}-\mu}{\widehat{\sigma}/\sqrt{n}} \leq t_{\frac{1}{2}\alpha}(n-1)\right\} = 1-\alpha \quad or: \ P\left\{|\bar{x}-\mu| \leq c_2\right\} = 1-\alpha$$

Compare  $\frac{\overline{x}-\mu}{\widehat{\sigma}/\sqrt{n}}$  with  $t_{\frac{1}{2}\alpha}(n-1)$ 

Decide whether to accept Ho or H1

#### Test whether two samples have the same variance

#### F-test

$$x_i \sim N(\mu_1, \sigma_1^2) \; (i=1,2,\cdots,n) \qquad \quad y_i \sim N(\mu_2, \sigma_2^2) \; (i=1,2,\cdots,m)$$

$$\overline{x} = rac{1}{n} \sum_{i=1}^n x_i \; , \quad \widehat{\sigma}_1^2 = rac{1}{n-1} \sum_{i=1}^n (x_i - \overline{x})^2 \qquad \overline{y} = rac{1}{m} \sum_{i=1}^m x_i \; , \quad \widehat{\sigma}_2^2 = rac{1}{m-1} \sum_{i=1}^m (y_i - \overline{y})^2$$

$$\frac{(n-1)\widehat{\sigma}_1^2}{\sigma_1^2} \sim \chi^2(n-1) \qquad \frac{(m-1)\widehat{\sigma}_2^2}{\sigma_2^2} \sim \chi^2(m-1)$$

$$F' = \frac{\frac{(n-1)\widehat{\sigma}_1^2}{\sigma_1^2} \cdot \frac{1}{n-1}}{\frac{(m-1)\widehat{\sigma}_2^2}{\sigma_2^2} \cdot \frac{1}{m-1}} = \frac{\sigma_2^2 \cdot \widehat{\sigma}_1^2}{\sigma_1^2 \cdot \widehat{\sigma}_2^2} \sim F(n-1, m-1)$$

? 
$$H_0: \ \sigma_1^2 = \sigma_2^2 \ ; \quad H_1: \sigma_1^2 
eq \sigma_2^2$$

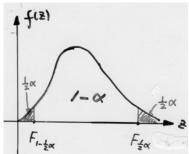
#### Test whether two samples have the same variance

Under Ho,

$$\frac{\widehat{\sigma}_1^2}{\widehat{\sigma}_2^2} \sim F(n-1, m-1)$$

Choose risk level  $\alpha$ 

$$P\left\{F_{1-\frac{1}{2}\alpha}(n-1,m-1)<\frac{\widehat{\sigma}_1^2}{\widehat{\sigma}_2^2}< F_{\frac{1}{2}\alpha}(n-1,m-1)\right\}=1-\alpha$$



Order so that 
$$\frac{\widehat{\sigma}_1^2}{\widehat{\sigma}_2^2} > 1$$

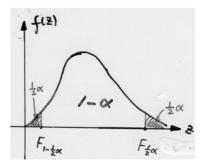
 $F_{1-\frac{1}{2}\alpha}(n-1,m-1)$  is always smaller than 1 for small  $\alpha.$ 

$$P\left\{\frac{\widehat{\sigma}_{1}^{2}}{\widehat{\sigma}_{2}^{2}} < F_{\frac{1}{2}\alpha}(n-1,m-1)\right\} = 1 - \alpha$$

Test whether two samples have the same variance

**Example.** n=41, m=61,  $\alpha=5\%$ 

$$F_{rac{1}{2}lpha}(n-1,m-1) = 1.74\;,\;\;\; rac{\widehat{\sigma}_1^2}{\widehat{\sigma}_2^2} = (4.2/3.0)^2 = 1.96 > F_{rac{1}{2}lpha}(n-1,m-1)$$



#### **Conclusion:**

the two samples have significantly different variances at risk level  $\alpha$ =5%

# Variance Analysis

| roups)   | Mea             | surem | ents            |     |               | No.            | Distributions                              |
|----------|-----------------|-------|-----------------|-----|---------------|----------------|--|
| 1        | l <sub>11</sub> | l,2   | L13             |     | lin,          | n,             | N(M,, 02)                                  |
| 2        | l21             | L22   | l <sub>23</sub> |     | $l_{2n_2}$    | n2             | $N(M_1, \sigma^2)$<br>$N(M_2, \sigma^2)$   |
| i        | $l_{i_1}$       | liz   | li3             | ••• | $\ell_{in_i}$ | n,             | $\mathcal{N}(\mathcal{U}_i, \sigma^2)$     |
| m        | Lmi             | lm2   | L <sub>m3</sub> |     | Lmnm          | n <sub>m</sub> | $\mathcal{N}(\mathcal{M}_{m}, \sigma^{2})$ |
| Hypothes | ж. Ц            | . 11  | = 1/-           | =   | = Mn          |                | 2  |

#### Internal variation

• Within each group, compute mean and variance

$$\widehat{\mu}_i = rac{1}{n_i} \sum_{j=1}^{n_i} \ell_{ij} \quad (i=1,2,3,\cdots,m) \quad \widehat{\sigma}_i^2 = rac{1}{n_i-1} \sum_{j=1}^{n_i} (\ell_{ij} - \widehat{\mu}_i)^2 \quad (i=1,2,3,\cdots,m)$$

• Internal variation

$$S_I^2 = \sum_{i=1}^m \sum_{i=1}^{n_i} (\ell_{ij} - \widehat{\mu}_i)^2 = \sum_{i=1}^m \left\{ (n_i - 1) \cdot \widehat{\sigma}_i^2 \right\}$$

• Number of measurements:  $n = \sum_{i=1}^{m} n_i = n_1 + n_2 + \cdots + n_m$ 

• Number of group averages: *m* 

• Internal variance:  $\widehat{\sigma}_I^2 = \frac{S_I^2}{n-m} = \frac{1}{n-m} \sum_{i=1}^m \sum_{j=1}^{n_i} (\ell_{ij} - \widehat{\mu}_i)^2$ 

$$\frac{S_I^2}{\sigma^2} \sim \chi^2(n-m)$$

#### External variation

- Overall mean  $\widehat{\mu} = \frac{1}{n} \sum_{i=1}^{m} \sum_{j=1}^{n_i} \ell_{ij} = \frac{1}{n} \sum_{i=1}^{m} (n_i \cdot \widehat{\mu}_i)$
- External variation  $S_E^2 = \sum_{i=1}^m \sum_{j=1}^{n_i} (\widehat{\mu}_i \widehat{\mu})^2 = \sum_{i=1}^m n_i (\widehat{\mu}_i \widehat{\mu})^2$

$$\frac{S_E^2}{\sigma^2} \sim \chi^2(m-1)$$

• External variance  $\widehat{\sigma}_E^2 = \frac{1}{m-1} \sum_{i=1}^m \left\{ n_i \, (\widehat{\mu}_i - \widehat{\mu})^2 \right\} = \frac{S_E^2}{m-1}$ 

### Total variation

Total variation

$$\begin{split} S_T^2 &= S_T^2(\widehat{\mu}) = \sum_{i=1}^m \sum_{j=1}^{n_i} \left(\ell_{ij} - \widehat{\mu}\right)^2 \\ &= \sum_{i=1}^m \sum_{j=1}^{n_i} \left(\ell_{ij} - \widehat{\mu}_i\right)^2 + \sum_{i=1}^m n_i \left(\widehat{\mu}_i - \widehat{\mu}\right)^2 \\ S_T^2 &= S_I^2 + S_E^2 \\ \\ \frac{S_T^2}{\sigma^2} &= \frac{S_I^2}{\sigma^2} + \frac{S_E^2}{\sigma^2} \\ &\sim \chi^2(n-1) \end{split}$$

Total variance

$$\widehat{\sigma}_T^2 = \frac{S_T^2}{n-1} = \frac{S_I^2 + S_E^2}{n-1} = \frac{(n-m) \cdot \widehat{\sigma}_I^2 + (m-1) \cdot \widehat{\sigma}_E^2}{(n-m) + (m-1)}$$

# Summary of variations

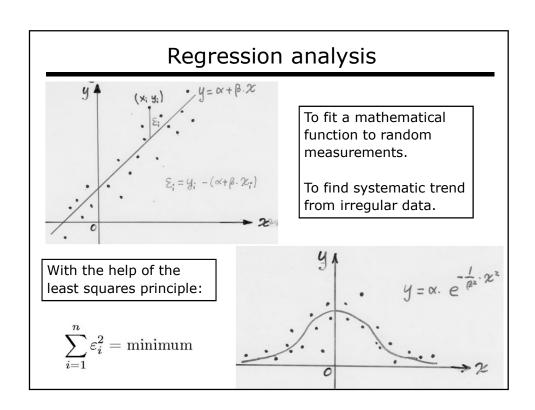
| Type     | Variations   | Degree of Freedom      | Distributions   | Variances   |
|----------|--|------------------------|---|---|
| Internal | $S_{i}^{2} = \sum_{i=1}^{m} \sum_{j=1}^{n_{i}} (\ell_{ij} - \hat{\mu}_{i})^{2}$    | n-m                    | $\frac{S_1^2}{\sigma^2} \sim \chi^2(n-m)$   | $\hat{C}_{\mathbf{I}}^2 = \frac{S_{\mathbf{I}}^2}{n-m}$ |
| External | $S_E^2 = \sum_{i=1}^m n_i \left( \hat{u}_i - \hat{u} \right)^2$                    |                        | $\frac{S_E^2}{\sigma^2} \sim \chi^2 (m-1)$  |   |
| Total    | $S_T^2 = \sum_{i=1}^{m} \sum_{j=1}^{n_i} (l_{ij} - \hat{\mu})^2$ $= S_I^2 + S_E^2$ | n-1                    | $\frac{S_{T}^{2}}{\sigma^{2}} \sim \gamma^{2}(n-1)$ $= \frac{S_{z}^{2}}{\sigma^{2}} + \frac{S_{\varepsilon}^{2}}{\sigma^{2}}$ | $\hat{O}_{T}^{2} = \frac{S_{T}^{2}}{n-1}$               |
|          | $= S_{I} + S_{E}$  | . 1 2                  |   |   |
|          | $\Gamma_{o} = \frac{S_{E}}{m-1}$   | $=\frac{0}{\hat{k}^2}$ | ~ F(m-1   | , n-m)  |

### Variance analysis

 Comparison of the external and the internal variances:

$$F_0 = rac{S_E^2/(m-1)}{S_I^2/(n-m)} = rac{\widehat{\sigma}_E^2}{\widehat{\sigma}_I^2} \sim F(m-1, n-m)$$

- For chosen risk level  $\alpha$ , find the critical value
- If  $F_0 > F_{\alpha}(m-1,n-m)$  , reject  $H_0$  at risk level  $\alpha$



Linear regression of one variable 
$$y_i = \alpha + \beta \cdot x_i + \varepsilon_i$$

$$\sum_{i=1}^n \varepsilon_i^2 = \sum_{i=1}^n (y_i - \alpha - \beta \cdot x_i)^2 = \min$$

$$\frac{\partial}{\partial \alpha} \left\{ \sum_{i=1}^n (y_i - \alpha - \beta \cdot x_i)^2 \right\}_{\alpha = \widehat{\alpha}, \ \beta = \widehat{\beta}} = \sum_{i=1}^n 2(y_i - \widehat{\alpha} - \widehat{\beta} \cdot x_i) (-1) = 0$$

$$\frac{\partial}{\partial \beta} \left\{ \sum_{i=1}^n (y_i - \alpha - \beta \cdot x_i)^2 \right\}_{\alpha = \widehat{\alpha}, \ \beta = \widehat{\beta}} = \sum_{i=1}^n 2(y_i - \widehat{\alpha} - \widehat{\beta} \cdot x_i) (-x_i) = 0$$

# Standardfel bör vara minst 3x större än beräknat fel

Reduce non-linear relations into linear regression model

$$y_i = \frac{x_i}{\alpha \cdot x_i + \beta} \longrightarrow Y_i = \frac{1}{y_i}, \quad X_i = \frac{1}{x_i} \longrightarrow Y_i = \alpha + \beta \cdot X_i$$

$$y_i = \alpha \cdot e^{-\frac{x_i^2}{\beta^2}} \longrightarrow \ln y_i = \ln \alpha - \frac{1}{\beta^2} \cdot x_i^2 \longrightarrow Y_i = a + b \cdot X_i$$

$$X_i = x_i^2, \quad Y_i = \ln y_i, \quad a = \ln \alpha, \quad b = -\frac{1}{\beta^2}$$

$$\widehat{\alpha} = e^{\widehat{\alpha}}, \quad \widehat{\beta} = \sqrt{-\frac{1}{\widehat{b}}}$$

$$\sigma_i^2=a^2+b^2\cdot s_i^2$$
  $y_i=lpha+eta\cdot x_i$   $y_i=lpha+eta\cdot x_i$   $y_i=lpha+eta\cdot x_i$   $y_i=lpha+eta\cdot x_i$ 

#### Constant and scale errors of EDM instruments

$$\sigma_i^2 = a^2 + b^2 \cdot s_i^2$$

| i  | $s_i (km)$ | $\sigma_i (cm)$ |
|----|------------|-----------------|
| 1  | 0.5        | 2.9             |
| 2  | 1.2        | 3.1             |
| 3  | 1.9        | 3.2             |
| 4  | 3.0        | 3.4             |
| 5  | 3.7        | 3.5             |
| 6  | 4.4        | 3.7             |
| 7  | 4.9        | 3.8             |
| 8  | 5.1        | 4.1             |
| 9  | 5.7        | 4.2             |
| 10 | 6.0        | 4.4             |

#### Constant and scale errors of EDM instruments

$$y_i = \sigma_i^2, \quad x_i = s_i^2, \quad lpha = a^2, \quad eta = b^2$$
  $y_i = lpha + eta \cdot x_i$ 

| i      | $x_i = s_i^2 \ (km^2)$ | $y_i = \sigma_i^2 \; (cm^2)$ | $x_i^2$   | $x_i \cdot y_i$ |
|--------|------------------------|------------------------------|-----------|-----------------|
| 1      | 0.25                   | 8.41                         | 0.0625    | 2.1025          |
| 2      | 1.44                   | 9.61                         | 2.0736    | 13.8384         |
| 3      | 3.61                   | 10.24                        | 13.0321   | 36.9664         |
| 4      | 9.00                   | 11.56                        | 81.0000   | 104.0400        |
| 5      | 13.69                  | 12.25                        | 187.4161  | 167.7025        |
| 6      | 19.36                  | 13.69                        | 374.8096  | 265.0384        |
| 7      | 24.01                  | 14.44                        | 576.4801  | 346.7044        |
| 8      | 26.01                  | 16.81                        | 676.5201  | 437.2281        |
| 9      | 32.49                  | 17.64                        | 1055.6001 | 573.1236        |
| 10     | 36.00                  | 19.36                        | 1296.000  | 696.9600        |
| $\sum$ | 165.86                 | 134.01                       | 4262.9942 | 2643.7043       |

#### Constant and scale errors of EDM instruments

$$\left[\begin{array}{cc} 10 & 165.86 \\ 165.86 & 4262.9942 \end{array}\right] \left[\begin{array}{c} \widehat{\alpha} \\ \widehat{\beta} \end{array}\right] = \left[\begin{array}{c} 134.01 \\ 2643.7043 \end{array}\right]$$

$$\left[\begin{array}{c} \widehat{\alpha} \\ \widehat{\beta} \end{array}\right] = \frac{1}{15120.403} \, \left[\begin{array}{cc} 4262.9942 & -165.86 \\ -165.86 & 10 \end{array}\right] \left[\begin{array}{c} 134.01 \\ 2643.7043 \end{array}\right] = \left[\begin{array}{c} 8.783 \\ 0.278 \end{array}\right]$$

$$\widehat{\sigma}^2 = rac{1}{10-2} \sum_{i=1}^n \left( y_i - \widehat{lpha} - \widehat{eta} \cdot x_i 
ight)^2 = 0.3782 \quad 
ightarrow \quad \widehat{\sigma} = \pm 0.615 \; (cm^2)$$

$$\sigma_{\widehat{\alpha}} = \widehat{\sigma} \cdot \sqrt{\frac{4262.9942}{15120.403}} = 0.326 \ (cm^2), \quad \sigma_{\widehat{\beta}} = \widehat{\sigma} \cdot \sqrt{\frac{10}{15120.403}} = 0.015 \ \ (cm/km)^2$$

$$\widehat{a} = \sqrt{\widehat{lpha}} = \sqrt{8.783} = 2.96 \; cm, \quad \widehat{b} = \sqrt{\widehat{eta}} = \sqrt{0.278} = 0.53 \; (cm/km) = 5.3 \; ppm$$