3. Statistical concepts

- Different distributions of random variables
- Confidence intervals
- Hypothesis tests (statistical tests)

Statiscal properties of a random variable ε

- Distribution function F(x) $P(\varepsilon \le x) = F(x) = \int_{-\infty}^{x} f(\varepsilon)d\varepsilon$ $(-\infty < x < +\infty)$ probability that $\varepsilon < x$
- Density function f(x) $\frac{\partial F(x)}{\partial x} = f(x)$
- Expectation (average) $E(\varepsilon) = \int_{-\infty}^{+\infty} \varepsilon \ f(\varepsilon) d\varepsilon$

Several types of distributions

- Binomial distribution
- Uniform distribution
- Normal distribution.
 Standard normal distribution
- χ^2 -distribution
- *t*-distribution
- \bullet F-distribution

Binomial distribution

- A random experiment with only two outcomes
- Probabilities: p, q, p+q=1
- Density (frequency) function: Probability that experiment is repeated *n* times and outcome with probability *p* occurs *k* times

$$f(k) = \binom{n}{k} p^k q^{n-k} = \frac{n!}{(n-k)! \ k!} p^k q^{n-k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 3\cdot 2\cdot 1} p^k q^{n-k} \quad (k=0,1,2,\cdots,n)$$

- Distribution function: $F(x) = \sum_{k=0}^{x} f(k) = \sum_{k=0}^{x} {n \choose k} p^k q^{n-k}$
- Expectation and variance:

$$E(x) = \sum_{k=0}^{x} k f(k) = np$$
, $Var(x) = \sum_{k=0}^{x} \{(k - np)^2 f(k)\} = npq$

Example

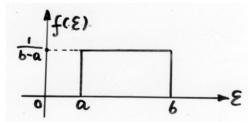
the sign of errors:

$$p = q = \frac{1}{2}$$

Uniform distribution

 Constant density function inside interval

$$f(arepsilon) = \left\{ egin{array}{ll} rac{1}{b-a} & & ext{for } a \leq arepsilon \leq arepsilon \ & ext{otherwise} \end{array}
ight.$$



• Distribution function:

$$F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases}$$

Uniform distribution

• Expectation, Variance:

$$E(\varepsilon) = \int_{a}^{b} \varepsilon \frac{1}{b-a} d\varepsilon = \frac{a+b}{2}$$

$$Var(\varepsilon) = E\left\{ [\varepsilon - E(\varepsilon)]^2 \right\} = \frac{(b-a)^2}{12}$$

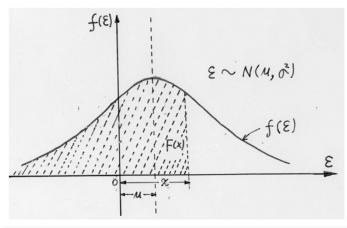
Example

Rounding errors: $a = -\frac{1}{2}$, $b = +\frac{1}{2}$

Expectation and variance: $E(\varepsilon) = 0$, $Var(\varepsilon) = \frac{1}{12}$

Density function of a normal distribution

$$f(\varepsilon) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2} (\varepsilon - \mu)^2\right\} \quad (-\infty < \varepsilon < +\infty)$$



$$F(x) = \int_{-\infty}^{x} f(\varepsilon) d\varepsilon = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{1}{2\sigma^{2}} \left(\varepsilon - \mu\right)^{2}\right\} \cdot d\varepsilon \quad (-\infty < x < +\infty)$$

Normal distribution $\varepsilon \sim N(\mu, \sigma^2)$

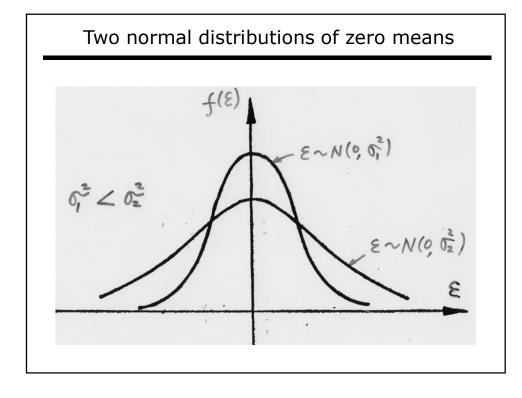
Expectation:

$$E(\varepsilon) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varepsilon \cdot \exp\left\{-\frac{1}{2\sigma^2} \left(\varepsilon - \mu\right)^2\right\} \cdot d\varepsilon = \mu$$

• Variance:

$$Var(\varepsilon) = E\left\{ \left[\varepsilon - E(\varepsilon) \right]^2 \right\} = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[\varepsilon - E(\varepsilon) \right]^2 \cdot \exp\left\{ -\frac{1}{2\sigma^2} \left(\varepsilon - \mu \right)^2 \right\} \cdot d\varepsilon = \sigma^2$$

- \longrightarrow Normal distribution is defined by (μ, σ^2)
 - Most errors follow normal distributions
 - Central limiting theorem: Sum of n independent random variables of equal expectations/variances converges to a normal distribution



Standard normal distribution N(0,1)

$$\varepsilon \sim N(\mu, \sigma^2)$$

$$\tau = \frac{\varepsilon - \mu}{\sigma}$$

$$E(\tau) = 0 , \quad Var(\tau) = 1$$

 $\longrightarrow au$ is said to have standard normal distribution

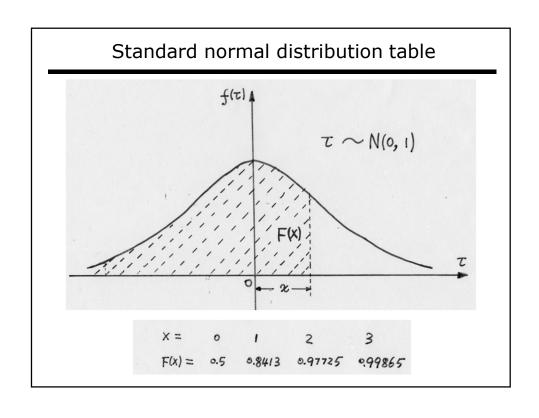
$$au \sim N(0,1)$$

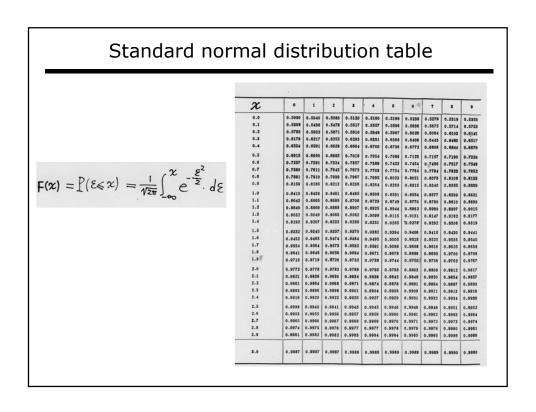
• Density function:

$$f(\tau) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\tau^2\right\} \qquad (-\infty < \tau < +\infty)$$

• Distribution function:

$$F(x) = \int_{-\infty}^{x} f(\tau)d\tau = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{1}{2}\tau^{2}\right\} \cdot d\tau \quad (-\infty < x < +\infty)$$





Normal distribution of multiple variables

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$E(\varepsilon) = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \dots \\ \mu_n \end{bmatrix} = \begin{matrix} \mu \\ \dots \\ \mu_n \end{matrix}, \quad \underset{n \cdot n}{C} = E\left\{ \left[\varepsilon - E(\varepsilon)\right] \left[\varepsilon - E(\varepsilon)\right]^\top \right\} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \dots & \sigma_{2n} \\ \dots & \dots & \dots & \dots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_n^2 \end{bmatrix}$$

$$f(\varepsilon) = \frac{1}{\left(2\pi\right)^{\frac{n}{2}} |C|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2} \left(\varepsilon - \mu\right)^{\top} C^{-1} \left(\varepsilon - \mu\right)\right\}$$

χ^2 -distribution

$$\chi^2 = \varepsilon_1^2 + \varepsilon_2^2 + \dots + \varepsilon_n^2$$

$$\varepsilon_i \sim N(0,1)$$

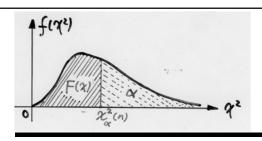
n is called the degree of freedom of $\chi^2\,$

$$f(\chi^2) = \frac{1}{\frac{n}{2}\Gamma(\frac{n}{2})} \, \left(\chi^2\right)^{\frac{n}{2}-1} \exp\left\{-\frac{1}{2}\chi^2\right\} \, , \quad (\chi^2 > 0)$$

$$\Gamma(n) = \int_0^{+\infty} x^{n-1} e^{-x} dx \quad (n > 0)$$

$$F(x) = \int_{-\infty}^{x} f(x) dx$$

$$E\left(\chi^{2}\right)=n\;,\;\;Var\left(\chi^{2}\right)=E\left\{ \left[\chi^{2}-E(\chi^{2})\right]^{2}\right\} =2n$$



n	a=0.995	0.99	0.975	0.95	0.90	0.75
1	- 1	_	0.001	0.004	0.016	0.192
2	0.010	0.020	0.051	0.103	0.211	0.575
3	0.072	0.115	0.216	0.352	0.584	1.213
4	0.207	0.297	0.484	0.711	1.064	1.923
6	0.412	0.554	0.831	1.145	1.619	2.575
6	0.676	0.872	1.237	1.635	2.204	3,455
7	0.989	1.239	1.690	2.167	2.833	4.255
8	1.344	1.646	2.180	2.733	3.490	5.071
9	1.735	2.088	2.700	3.325	4.163	5.899
19	2.156	2.558	3.247	3.940	4.865	6.737
11	2.603	3.053	3.816	4.575	5.578	7.524
12	3.074	3.571 -	4.404	5.226	* 6.304	8.433
13	3.563	4.107	5.009	5.892	7.042	9.259
14	4.075	4.660	5.629	6.571	7,790	10.155
15	4.601	5.229	6.262	7.261	8.547	11.037
16	5.142	5.812	6.908	7.962	9.312	11.512
17	5.697	6.408	7.564	8.672	10.085	12.792
18	6.265	7.015	8.231	9.390	10.865	13.675
19	6.844	7.633	8.907	10.117	11.651	14.552
20	7.434	8.260	9.591	10.851	12.443	15.452
21	8.034	8.897	10.283	11.591	13.240	16.344
22	8.643	9.542	10.982	12.338	14.042	17.249
23	9.260	10.196	11.689	13.091	14.848	18,137
24	9.885	10.856	12.401	13.848	15.659	19.037
25	10.520	11.524	13.120	14.611	16.473	19.939
25	11.160	12.198-	13.844	15.379	17.292	20.843
27	11.808	12.879	14.573	16.151	18.114	21.749
28	12.461	13.565	15.303	16.928	18.939	22.657
29	13.121	14,257	16.047	17.708	. 19.768	23.567

Note the risk level $\boldsymbol{\alpha}$ and the correpsonding critical value

t-distribution, t(n)

If $X \sim N(0,1)$ and $Y \sim \chi^2(n)$ are independent of each other, then

$$t = \frac{X}{\sqrt{Y/n}}$$

follows the t-distribution with degree of freedom n

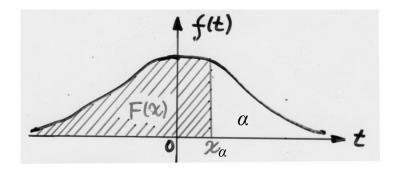
$$f(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi} \,\Gamma\left(\frac{n}{2}\right)} \cdot \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}} \qquad (-\infty < t < +\infty)$$

$$E(t) = 0 \ (n > 1), \ Var(t) = \frac{n}{n-2} \ (n > 2)$$

t(n) approaches N(0,1) when n approaches infinity

Curve of the t-distribution

$$f(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi} \, \Gamma\left(\frac{n}{2}\right)} \cdot \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}} \qquad (-\infty < t < +\infty)$$



n a ≈ 0.995 0.99 0.975 0.95 0.90 0.75 1 — — 0.001 0.004 0.016 0.102 2 0.010 0.020 0.051 0.103 0.211 0.573 3 0.072 0.115 0.216 0.352 0.581 1.213 4 0.207 0.287 0.484 0.711 1.064 1.223 6 0.476 0.552 0.331 1.116 1.610 2.573 7 0.999 1.239 1.169 2.167 2.33 3.433 8 1.344 1.466 2.189 2.733 3.403 1.655 9 1.735 2.088 2.700 3.325 4.165 5.899 10 2.136 2.538 3.247 3.940 4.865 6.737 11 2.603 3.053 3.816 4.676 5.578 7.541 12 3.074 3.571 4.404 5.2		The	e t-di	strib	ution	table	9
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11							5.899
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37 18.586 19.960 22.106 24.075 26.492 30.893 38 19.289 20.691 22.878 24.884 27.343 31.815	3.5		18.509	20.569	22.465	24.797	
37 18.586 19.960 22.106 24.075 26.492 30.893 38 19.289 20.691 22.878 24.884 27.343 31.815	36	17.887	19.233			25.643	29.973
10.000	37	18.586	19.960				
39 19.996 21.426 23.654 25.695 28.196 32.737						27.343	31.815
						28.196	32.737
40 20.707 22.164 24.433 26.509 29.051 33.660	40	20.707				29.051	33.660
41 21.421 22.906 25.215 27.326 29.907 34.555	41	21.421	22.906	25.215	27.326	29.907	34.555
42 22.138 23.650 25.999 28.144 30.765 35.510	42	22.138	23.650	25.999		30.765	35.510
43 22.859 24.398 26.785 28.965 31.625 36.436 44 23.584 25.148 27.575 29.787 32.487 37.363						31.625	

F-distribution

If X,Y are two independent χ^2- distributed random variables :

$$X \sim \chi^2(n)$$
, $Y \sim \chi^2(m)$

$$\longrightarrow$$
 $Z = \frac{X/n}{Y/m}$

is called a F-distribution with first degree of freedom n, and 2nd degree of freedom m

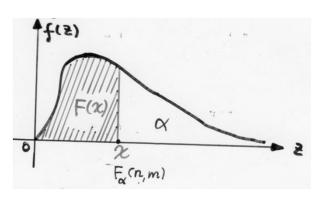
$$f(z) = \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{n}{m}\right) \left(\frac{n}{m}z\right)^{\frac{n}{2}-1} \left(1 + \frac{n}{m}z\right)^{-\frac{n+m}{2}} \qquad (z \ge 0)$$

$$E(z)=\frac{n}{m-2} \quad \ (m>2)$$

$$Var(z) = rac{2m^2(n+m-2)}{n(m-2)^2(m-4)} ~~(m>4)$$

F-distribution

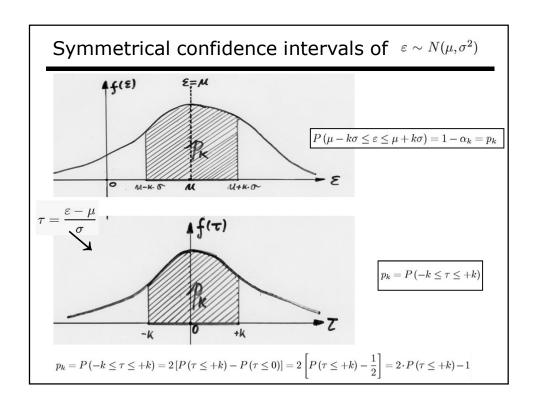
$$f(z) = \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{n}{m}\right) \left(\frac{n}{m}z\right)^{\frac{n}{2}-1} \left(1 + \frac{n}{m}z\right)^{-\frac{n+m}{2}} \qquad (z \ge 0)$$



		F-distribution table																
						Ρ{	F(n,	,m) >	F _a (n	,m) }	= α	= 0.	05					
								n										
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
m / 2	161	200	216	225	230	234	237 19.4	239	241 19.4	242 19.4	243 19.4	244 19.4	245 19.4	245 19.4	246 19.4	246 19.4	247 19.4	247 19.4
3	18.5	19.0	19.2 9.28	19.2 9.12	19.3	19.3	8.89	8.85	8.81	8.79	8.76	8.74	8.73	8.71	8.70	8.69	8.68	8.67
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94	5.91	5.89	5.87	5.86	5.84	5.83	5.82
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74 .	4.70	4.68	4.66	4.64	4.62	4.60	4.59	4.58
6 7	5.99	5.14 4.74	4.76	4.53	4.39 3.97	4.28 3.87	4.21 3.79	4.15	4.10	4.06 3.64	4.03 3.60	4.00 3.57	3.98	3.96	3.94	3.92	3.91	3.90
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28	3.26	3.24	3.22	3.20	3.19	3.17
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07	3.05	3.03	3.01	2.99	2.97	2.96
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94	2.91	2.89	2.86	2.85	2.83	2.81	2.80
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82	2.79	2.76	2.74	2.72	2.70	2.69	2.67
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63	2.60	2.58	2.55	2.53	2.51	2.50	2.48
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57	2.53	2.51	2.48	2.46	2.44	2.43	2.41
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.51	2.48	2.45	2.42	2.40	2.38	2.37	2.35
16	4.49	3.63	3.24	3.01 2.96	2.85	2.74	2.66	2.59	2.54	2.49	2.46	2.42	2.40	2.37	2.35	2.33	2.32	2.30
18	4.41	3.59 3.55	3.16	2.96	2.77	2.66	2.58	2.55	2.49	2.43	2.37	2.34	2.33	2.29	2.27	2.25	2.23	2.20
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.34	2.31	2.28	2.26	2.23	2.21	2.20	2.18
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.31	2.28	2.25	2.22	2.20	2.18	2.17	2.15
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.28	2.25	2.22	2.20	2.18	2.16	2.14	2.12
23	4.28	3.44	3.03	2.82	2.64	2.53	2.44	2.37	2.34	2.30	2.23	2.23	2.18	2.15	2.13	2.13	2.09	2.07
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.21	2.18	2.15	2.13	2.11	2.09	2.07	2.05
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.20	2.16	2.14	2.11	2.09	2.07	2.05	2.04
M 26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15	2.12	2.09	2.07	2.05	2.03	2.00
28	4.20		2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.15	2.13	2.09	2.06	2.04	2.04	2.02	1.9
29	4.18		2.93	2.70	2.55	2.43		2.28	2.22	2.18	2.14	2.10	2.08	2.05	2.03	2.01	1.99	1.9
30	4.17		2.92	2.69	2.53	2.42		2.27	2.21	2.16	2.13	2.09	2.06	2.04	2.01	1.99	1.98	1.9
32	4.15	3.29	2.90	2.67	2.51	2.40		2.24	2.19	2.14	2.10	2.07	2.04	2.01	1.99	1.97	1.95	1.9
34	4.13	3.28 3.26	2.88	2.65	2.49	2.38	2.29	12.23 2.21	2.17	2.12	2.08	2.05	2.02	1.99	1.97	1.95	1.93	1.9
38	4.10	3.24	2.85	2.62	2.46	2.35	2.26	2.19	2.14	2.09	2.05	2.02	1.99	1.96	1.94	1.92	1.90	1.8

Confidence intervals

- A confidence interval of $\, \varepsilon \,$ is an interval which contains the expectation $\mu \,$
- The probability α that ε lies *outside* the confidence interval is called the *risk level*
- The probability that $\pmb{\varepsilon}$ lies *inside* the confidence interval is $1-\alpha$, called the *confidence level*
- \bullet Confidence intervals are often taken as symmetrical intervals around the expectation μ



Probability inside confidence intervals of $\varepsilon \sim N(\mu, \sigma^2)$

$$p_k = P(-k \le \tau \le +k) = 2[P(\tau \le +k) - P(\tau \le 0)] = 2\left[P(\tau \le +k) - \frac{1}{2}\right] = 2 \cdot P(\tau \le +k) - 1$$
$$p_1 = P(-1 \le \tau \le +1) = 2 \cdot 0.84134 - 1 = 68.27\%$$

$$p_2 = P(-2 \le \tau \le +2) = 2 \cdot 0.97725 - 1 = 95.45\%$$

$$p_3 = P(-3 \le \tau \le +3) = 2 \cdot 0.99865 - 1 = 99.73\%$$

Table 1.5: Confidence Intervals of Normal Distributions

k	interval for ε	interval for $ au$	confidence level $(p_k = 1 - \alpha_k)$	$risk\ level\ (\alpha_k)$
1	$\mu \pm 1\sigma$	±1	68.27 %	31.73 %
2	$\mu \pm 2\sigma$	±2	95.45 %	4.55 %
3	$\mu \pm 3\sigma$	±3	99.73 %	0.27 %
4	$\mu \pm 4\sigma$	±4	99.99 %	0.01 %

 \rightarrow 2 σ or 3 σ can be the maximum error tolerance!

Statistical Analysis (2/2)

- Different distributions of random variables
- Confidence intervals
- Hypothesis tests (statistical tests)
- Variance analysis
- Regression analysis (least squares fitting)

Sample mean and sample variance

Let $x_i \sim N(\mu, \sigma^2)$ $(i=1,2,\cdots,n)$ be n independent normally distributed variables with equal expectation μ and equal variance σ^2 . A sample mean \overline{x} and an estimated sample variance $\hat{\sigma}^2$ can be calculated:

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \; , \quad \widehat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2$$

$$\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$$
 \longrightarrow $u = \frac{\overline{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$

$$\frac{(n-1)\widehat{\sigma}^2}{\sigma^2} = \sum_{i=1}^n \left(\frac{\overline{x} - x_i}{\sigma}\right)^2 \sim \chi^2(n-1) \longrightarrow t' = \frac{\frac{\overline{x} - \mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)\widehat{\sigma}^2}{\sigma^2} \cdot \frac{1}{n-1}}} = \frac{\overline{x} - \mu}{\widehat{\sigma}/\sqrt{n}} \sim t(n-1)$$

$$Var\left(\widehat{\sigma}^{2}\right) = E\left\{\left(\widehat{\sigma}^{2} - \sigma^{2}\right)^{2}\right\} = \frac{2\sigma^{4}}{n-1}$$

2 measurement samples

$$x_{i} \sim N(\underline{\mu}_{1}, \sigma_{1}^{2}) \ (i = 1, 2, \cdots, n) \qquad y_{i} \sim N(\underline{\mu}_{2}, \sigma_{2}^{2}) \ (i = 1, 2, \cdots, m)$$

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_{i} , \quad \widehat{\sigma}_{1}^{2} = \frac{1}{n-1} \sum_{i=1}^{n} (x_{i} - \overline{x})^{2} \qquad \overline{y} = \frac{1}{m} \sum_{i=1}^{m} x_{i} , \quad \widehat{\sigma}_{2}^{2} = \frac{1}{m-1} \sum_{i=1}^{m} (y_{i} - \overline{y})^{2}$$

$$\frac{(m-1)\widehat{\sigma}_{1}^{2}}{\sigma_{1}^{2}} \sim \chi^{2}(m-1)$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$F' = \frac{\frac{(n-1)\widehat{\sigma}_{1}^{2}}{\sigma_{1}^{2}} \cdot \frac{1}{n-1}}{\frac{(m-1)\widehat{\sigma}_{2}^{2}}{\sigma_{2}^{2}} \cdot \frac{1}{m-1}} = \frac{\sigma_{2}^{2} \cdot \widehat{\sigma}_{1}^{2}}{\sigma_{1}^{2} \cdot \widehat{\sigma}_{2}^{2}} \sim F(n-1, m-1)$$

Hypothesis tests (statistical tests)

- Make measurements / collect samples
- Compute sample statistics (mean, variance, etc)
- Define the null-hypothesis H₀ and the alternative hypothesis H₁
- Choose risk level α
- Find critical value from the distribution table
- Compare computed statistics against the critical value to decide whether to accept H₀ or H₁

Test expectation when variance is known

u-test

 x_1, x_2, \dots, x_n are n independent measurements of the same normal distribution $N(\mu, \sigma^2)$

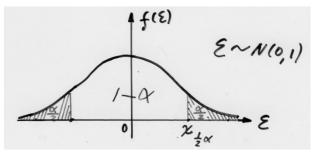
$$\overline{x} = rac{1}{n} \sum_{i=1}^n x_i \qquad \qquad ar{x} \sim N(\mu, rac{\sigma^2}{n}) \quad ext{and} \quad rac{ar{x} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1)$$

Null hypothesis H_0 :	$\overline{x}=\mu$
Alternative hypothesis H_1 :	$\overline{x} \neq \mu$

Choose risk level $\alpha = 5\%$

Test expectation when variance is known

Find the cirical value $x_{\frac{1}{2}\alpha}$ of N(0,1)



$$P\left\{\left|\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right| \leq x_{\frac{1}{2}\alpha}\right\} = 1 - \alpha \quad or: P\left\{\left|\bar{x}-\mu\right| \leq c_1\right\} = 1 - \alpha$$

Compare critical value $x_{\frac{1}{2}\alpha}$ with computed $\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ Decide whether to accept H₀ or H₁

Test expectation when variance is unknown

t-test

 x_1, x_2, \dots, x_n are n independent measurements of the same normal distribution $N(\mu, \sigma^2)$

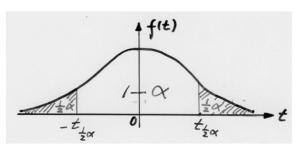
$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$
 $\widehat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2$ $\frac{\overline{x} - \mu}{\widehat{\sigma}/\sqrt{n}} \sim t(n-1)$

Null hypothesis H_0 :	$\overline{x} = \mu$
Alternative hypothesis H_1 :	$\overline{x} \neq \mu$

Choose risk level $\alpha = 5\%$

Test expectation when variance is unknown

Find the cirical value $t_{\frac{1}{2}\alpha}(n-1)$ of t-distribution



$$P\left\{\frac{\overline{x}-\mu}{\widehat{\sigma}/\sqrt{n}} \leq t_{\frac{1}{2}\alpha}(n-1)\right\} = 1-\alpha \quad or: \ P\left\{|\bar{x}-\mu| \leq c_2\right\} = 1-\alpha$$

Compare $\frac{\overline{x}-\mu}{\widehat{\sigma}/\sqrt{n}}$ with $t_{\frac{1}{2}\alpha}(n-1)$

Decide whether to accept Ho or H1

Test whether two samples have the same variance

F-test

$$x_i \sim N(\mu_1, \sigma_1^2) \; (i=1,2,\cdots,n) \qquad \quad y_i \sim N(\mu_2, \sigma_2^2) \; (i=1,2,\cdots,m)$$

$$\overline{x} = rac{1}{n} \sum_{i=1}^n x_i \; , \quad \widehat{\sigma}_1^2 = rac{1}{n-1} \sum_{i=1}^n (x_i - \overline{x})^2 \qquad \overline{y} = rac{1}{m} \sum_{i=1}^m x_i \; , \quad \widehat{\sigma}_2^2 = rac{1}{m-1} \sum_{i=1}^m (y_i - \overline{y})^2$$

$$\frac{(n-1)\widehat{\sigma}_1^2}{\sigma_1^2} \sim \chi^2(n-1) \qquad \frac{(m-1)\widehat{\sigma}_2^2}{\sigma_2^2} \sim \chi^2(m-1)$$

$$F' = \frac{\frac{(n-1)\widehat{\sigma}_1^2}{\sigma_1^2} \cdot \frac{1}{n-1}}{\frac{(m-1)\widehat{\sigma}_2^2}{\sigma_2^2} \cdot \frac{1}{m-1}} = \frac{\sigma_2^2 \cdot \widehat{\sigma}_1^2}{\sigma_1^2 \cdot \widehat{\sigma}_2^2} \sim F(n-1, m-1)$$

?
$$H_0: \ \sigma_1^2 = \sigma_2^2 \ ; \quad H_1: \sigma_1^2
eq \sigma_2^2$$

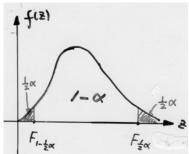
Test whether two samples have the same variance

Under Ho,

$$\frac{\widehat{\sigma}_1^2}{\widehat{\sigma}_2^2} \sim F(n-1, m-1)$$

Choose risk level α

$$P\left\{F_{1-\frac{1}{2}\alpha}(n-1,m-1)<\frac{\widehat{\sigma}_1^2}{\widehat{\sigma}_2^2}< F_{\frac{1}{2}\alpha}(n-1,m-1)\right\}=1-\alpha$$



Order so that
$$\frac{\widehat{\sigma}_1^2}{\widehat{\sigma}_2^2} > 1$$

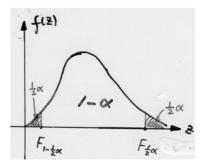
 $F_{1-\frac{1}{2}\alpha}(n-1,m-1)$ is always smaller than 1 for small $\alpha.$

$$P\left\{\frac{\widehat{\sigma}_{1}^{2}}{\widehat{\sigma}_{2}^{2}} < F_{\frac{1}{2}\alpha}(n-1,m-1)\right\} = 1 - \alpha$$

Test whether two samples have the same variance

Example. n=41, m=61, $\alpha=5\%$

$$F_{rac{1}{2}lpha}(n-1,m-1) = 1.74\;,\;\;\; rac{\widehat{\sigma}_1^2}{\widehat{\sigma}_2^2} = (4.2/3.0)^2 = 1.96 > F_{rac{1}{2}lpha}(n-1,m-1)$$



Conclusion:

the two samples have significantly different variances at risk level α =5%

Variance Analysis

roups)	Mea	surem	ents			No.	Distributions
1	l ₁₁	l,2	L13		lin,	n,	N(M,, 02)
2	l21	L22	l ₂₃		l_{2n_2}	n2	$N(M_1, \sigma^2)$ $N(M_2, \sigma^2)$
i	l_{i_1}	liz	li3	•••	ℓ_{in_i}	n,	$\mathcal{N}(\mathcal{U}_i, \sigma^2)$
m	Lmi	lm2	L _{m3}		Lmnm	n _m	$\mathcal{N}(\mathcal{M}_{m}, \sigma^{2})$
Hypothes	ж. Ц	. 11	= 1/-	=	= Mn		2

Internal variation

• Within each group, compute mean and variance

$$\widehat{\mu}_i = rac{1}{n_i} \sum_{j=1}^{n_i} \ell_{ij} \quad (i=1,2,3,\cdots,m) \quad \widehat{\sigma}_i^2 = rac{1}{n_i-1} \sum_{j=1}^{n_i} (\ell_{ij} - \widehat{\mu}_i)^2 \quad (i=1,2,3,\cdots,m)$$

• Internal variation

$$S_I^2 = \sum_{i=1}^m \sum_{i=1}^{n_i} (\ell_{ij} - \widehat{\mu}_i)^2 = \sum_{i=1}^m \left\{ (n_i - 1) \cdot \widehat{\sigma}_i^2 \right\}$$

• Number of measurements: $n = \sum_{i=1}^{m} n_i = n_1 + n_2 + \cdots + n_m$

• Number of group averages: *m*

• Internal variance: $\widehat{\sigma}_I^2 = \frac{S_I^2}{n-m} = \frac{1}{n-m} \sum_{i=1}^m \sum_{j=1}^{n_i} (\ell_{ij} - \widehat{\mu}_i)^2$

$$\frac{S_I^2}{\sigma^2} \sim \chi^2(n-m)$$

External variation

- Overall mean $\widehat{\mu} = \frac{1}{n} \sum_{i=1}^{m} \sum_{j=1}^{n_i} \ell_{ij} = \frac{1}{n} \sum_{i=1}^{m} (n_i \cdot \widehat{\mu}_i)$
- External variation $S_E^2 = \sum_{i=1}^m \sum_{j=1}^{n_i} (\widehat{\mu}_i \widehat{\mu})^2 = \sum_{i=1}^m n_i (\widehat{\mu}_i \widehat{\mu})^2$

$$\frac{S_E^2}{\sigma^2} \sim \chi^2(m-1)$$

• External variance $\widehat{\sigma}_E^2 = \frac{1}{m-1} \sum_{i=1}^m \left\{ n_i \, (\widehat{\mu}_i - \widehat{\mu})^2 \right\} = \frac{S_E^2}{m-1}$

Total variation

Total variation

$$\begin{split} S_T^2 &= S_T^2(\widehat{\mu}) = \sum_{i=1}^m \sum_{j=1}^{n_i} \left(\ell_{ij} - \widehat{\mu}\right)^2 \\ &= \sum_{i=1}^m \sum_{j=1}^{n_i} \left(\ell_{ij} - \widehat{\mu}_i\right)^2 + \sum_{i=1}^m n_i \left(\widehat{\mu}_i - \widehat{\mu}\right)^2 \\ S_T^2 &= S_I^2 + S_E^2 \\ \\ \frac{S_T^2}{\sigma^2} &= \frac{S_I^2}{\sigma^2} + \frac{S_E^2}{\sigma^2} \\ &\sim \chi^2(n-1) \end{split}$$

Total variance

$$\widehat{\sigma}_T^2 = \frac{S_T^2}{n-1} = \frac{S_I^2 + S_E^2}{n-1} = \frac{(n-m) \cdot \widehat{\sigma}_I^2 + (m-1) \cdot \widehat{\sigma}_E^2}{(n-m) + (m-1)}$$

Summary of variations

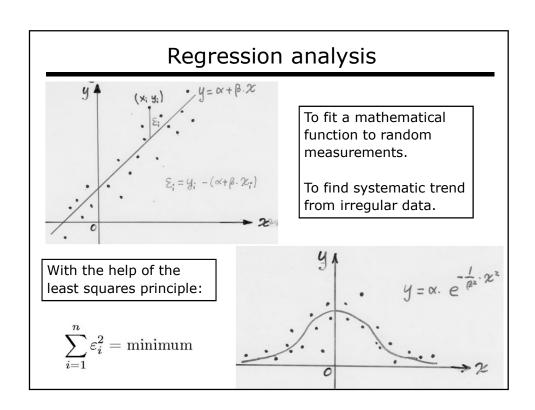
Type	Variations	Degree of Freedom	Distributions	Variances
Internal	$S_{i}^{2} = \sum_{i=1}^{m} \sum_{j=1}^{n_{i}} (\ell_{ij} - \hat{\mu}_{i})^{2}$	n-m	$\frac{S_1^2}{\sigma^2} \sim \chi^2(n-m)$	$\hat{C}_{\mathbf{I}}^2 = \frac{S_{\mathbf{I}}^2}{n-m}$
External	$S_E^2 = \sum_{i=1}^m n_i \left(\hat{u}_i - \hat{u} \right)^2$		$\frac{S_E^2}{\sigma^2} \sim \chi^2 (m-1)$	
Total	$S_T^2 = \sum_{i=1}^{m} \sum_{j=1}^{n_i} (l_{ij} - \hat{\mu})^2$ $= S_I^2 + S_E^2$	n-1	$\frac{S_{T}^{2}}{\sigma^{2}} \sim \gamma^{2}(n-1)$ $= \frac{S_{z}^{2}}{\sigma^{2}} + \frac{S_{\varepsilon}^{2}}{\sigma^{2}}$	$\hat{O}_{T}^{2} = \frac{S_{T}^{2}}{n-1}$
	$= S_{I} + S_{E}$. 1 2		
	$\Gamma_{o} = \frac{S_{E}}{m-1}$	$=\frac{0}{\hat{k}^2}$	~ F(m-1	, n-m)

Variance analysis

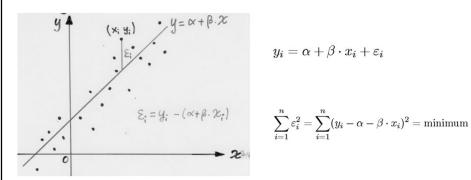
 Comparison of the external and the internal variances:

$$F_0 = rac{S_E^2/(m-1)}{S_I^2/(n-m)} = rac{\widehat{\sigma}_E^2}{\widehat{\sigma}_I^2} \sim F(m-1, n-m)$$

- For chosen risk level α , find the critical value
- If $F_0 > F_{\alpha}(m-1,n-m)$, reject H_0 at risk level α



Linear regression of one variable



$$y_i = \alpha + \beta \cdot x_i + \varepsilon_i$$

$$\sum_{i=1}^n \varepsilon_i^2 = \sum_{i=1}^n (y_i - \alpha - \beta \cdot x_i)^2 = \text{minimum}$$

$$\left.\frac{\partial}{\partial\alpha}\left\{\sum_{i=1}^{n}(y_{i}-\alpha-\beta\cdot x_{i})^{2}\right\}_{\alpha=\widehat{\alpha},\;\beta=\widehat{\beta}}\right.=\sum_{i=1}^{n}2(y_{i}-\widehat{\alpha}-\widehat{\beta}\cdot x_{i})\left(-1\right)=0$$

$$\frac{\partial}{\partial \beta} \left\{ \sum_{i=1}^n (y_i - \alpha - \beta \cdot x_i)^2 \right\}_{\alpha = \widehat{\alpha}, \; \beta = \widehat{\beta}} = \sum_{i=1}^n 2(y_i - \widehat{\alpha} - \widehat{\beta} \cdot x_i) \left(-x_i \right) = 0$$

Computation of regression coefficients

$$egin{array}{ll} q_{11} &= n & q_{12} & \sum_{i=1}^n x_i \ q_{22} &= \sum_{i=1}^n x_i^2 \ q_{12} &= q_{12} & \sum_{i=1}^n x_i^2 \ q_{22} &= \sum_{i=1}^n x_i^2 \ w_1 &= \sum_{i=1}^n y_i \ w_2 &= \sum_{i=1}^n x_i y_i \end{array}$$

$$q_{11} = n$$
 $q_{12} = \sum_{i=1}^{n} x_i$
 $q_{22} = \sum_{i=1}^{n} x_i^2$
 $w_1 = \sum_{i=1}^{n} y_i$

$$q_{22} = \sum_{i=1}^{n} x_i^2$$
 $w_1 = \sum_{i=1}^{n} y_i$
 $w_2 = \sum_{i=1}^{n} x_i y_i$

$$\left[\begin{array}{c} \widehat{\alpha} \\ \widehat{\beta} \end{array} \right] = \frac{1}{q_{11}q_{22} - q_{12}^2} \left[\begin{array}{cc} q_{22} & -q_{12} \\ -q_{12} & q_{11} \end{array} \right] \left[\begin{array}{c} w_1 \\ w_2 \end{array} \right] = \frac{1}{q_{11}q_{22} - q_{12}^2} \left[\begin{array}{c} q_{22}w_1 - q_{12}w_2 \\ -q_{12}w_1 + q_{11}w_2 \end{array} \right]$$

$$\widehat{arepsilon}_i = y_i - \widehat{lpha} - \widehat{eta} \cdot x_i$$
 $\widehat{\sigma}^2 = \frac{1}{n-2} \sum_{i=1}^n \widehat{arepsilon}_i^2 = \frac{1}{n-2} \sum_{i=1}^n \left(y_i - \widehat{lpha} - \widehat{eta} \cdot x_i
ight)^2$

$$\sigma_{\widehat{\alpha}} = \widehat{\sigma} \cdot \sqrt{\frac{q_{22}}{q_{11}q_{22} - q_{12}^2}}, \quad \sigma_{\widehat{\beta}} = \widehat{\sigma} \cdot \sqrt{\frac{q_{11}}{q_{11}q_{22} - q_{12}^2}}$$

$$\frac{\widehat{\alpha}}{\sigma_{\widehat{\alpha}}} \sim t(n-2), \quad \frac{\widehat{\beta}}{\sigma_{\widehat{\beta}}} \sim t(n-2)$$

 $\frac{\widehat{\alpha}}{\sigma_{\widehat{\alpha}}} \sim t(n-2), \quad \frac{\widehat{\beta}}{\sigma_{\widehat{\beta}}} \sim t(n-2)$ Are the computed coefficients statistically significant?

Reduce non-linear relations into linear regression model

$$y_i = \frac{x_i}{\alpha \cdot x_i + \beta} \longrightarrow Y_i = \frac{1}{y_i}, \quad X_i = \frac{1}{x_i} \longrightarrow Y_i = \alpha + \beta \cdot X_i$$

$$y_i = \alpha \cdot e^{-rac{x_i^2}{eta^2}} \longrightarrow \ln y_i = \ln \alpha - rac{1}{eta^2} \cdot x_i^2 \longrightarrow Y_i = a + b \cdot X_i$$
 $X_i = x_i^2, \quad Y_i = \ln y_i, \quad a = \ln \alpha, \quad b = -rac{1}{eta^2}$
 $\widehat{\alpha} = e^{\widehat{a}}, \quad \widehat{\beta} = \sqrt{-rac{1}{\widehat{b}}}$

$$\sigma_i^2=a^2+b^2\cdot s_i^2$$
 $y_i=lpha+eta\cdot x_i$ $y_i=lpha+eta\cdot x_i$ $y_i=lpha+eta\cdot x_i$ $y_i=lpha+eta\cdot x_i$

Constant and scale errors of EDM instruments

$$\sigma_i^2 = a^2 + b^2 \cdot s_i^2$$

i	$s_i (km)$	$\sigma_i (cm)$
1	0.5	2.9
2	1.2	3.1
3	1.9	3.2
4	3.0	3.4
5	3.7	3.5
6	4.4	3.7
7	4.9	3.8
8	5.1	4.1
9	5.7	4.2
10	6.0	4.4

Constant and scale errors of EDM instruments

$$y_i = \sigma_i^2, \quad x_i = s_i^2, \quad lpha = a^2, \quad eta = b^2$$
 $y_i = lpha + eta \cdot x_i$

i	$x_i = s_i^2 \ (km^2)$	$y_i = \sigma_i^2 \; (cm^2)$	x_i^2	$x_i \cdot y_i$
1	0.25	8.41	0.0625	2.1025
2	1.44	9.61	2.0736	13.8384
3	3.61	10.24	13.0321	36.9664
4	9.00	11.56	81.0000	104.0400
5	13.69	12.25	187.4161	167.7025
6	19.36	13.69	374.8096	265.0384
7	24.01	14.44	576.4801	346.7044
8	26.01	16.81	676.5201	437.2281
9	32.49	17.64	1055.6001	573.1236
10	36.00	19.36	1296.000	696.9600
\sum	165.86	134.01	4262.9942	2643.7043

Constant and scale errors of EDM instruments

$$\left[\begin{array}{cc} 10 & 165.86 \\ 165.86 & 4262.9942 \end{array}\right] \left[\begin{array}{c} \widehat{\alpha} \\ \widehat{\beta} \end{array}\right] = \left[\begin{array}{c} 134.01 \\ 2643.7043 \end{array}\right]$$

$$\left[\begin{array}{c} \widehat{\alpha} \\ \widehat{\beta} \end{array}\right] = \frac{1}{15120.403} \, \left[\begin{array}{cc} 4262.9942 & -165.86 \\ -165.86 & 10 \end{array}\right] \left[\begin{array}{c} 134.01 \\ 2643.7043 \end{array}\right] = \left[\begin{array}{c} 8.783 \\ 0.278 \end{array}\right]$$

$$\widehat{\sigma}^2 = rac{1}{10-2} \sum_{i=1}^n \left(y_i - \widehat{lpha} - \widehat{eta} \cdot x_i
ight)^2 = 0.3782 \quad
ightarrow \quad \widehat{\sigma} = \pm 0.615 \; (cm^2)$$

$$\sigma_{\widehat{\alpha}} = \widehat{\sigma} \cdot \sqrt{\frac{4262.9942}{15120.403}} = 0.326 \ (cm^2), \quad \sigma_{\widehat{\beta}} = \widehat{\sigma} \cdot \sqrt{\frac{10}{15120.403}} = 0.015 \ \ (cm/km)^2$$

$$\widehat{a} = \sqrt{\widehat{lpha}} = \sqrt{8.783} = 2.96 \; cm, \quad \widehat{b} = \sqrt{\widehat{eta}} = \sqrt{0.278} = 0.53 \; (cm/km) = 5.3 \; ppm$$