7. Gross errors and local redundancies

- Direct statistical tests
- Data snooping
  - ✓ make a *preliminary* least squares adjustment
  - ✓ check the estimated errors *one by one*
- Local redundancy, errros of hypothesis tests, minimum detectable gross errors and reliablility

Classification of measurement errors

- Systematic errors
  - influences of instruments, environment or surveyors etc
  - eliminate the causes, reduce/eliminate the effect
  - apply theoretical correction <u>or</u> automatic detection
- Random errors

  $$\boxed{\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \varepsilon_i}{n} = 0}$$

  - arithmatic mean approaches zero
  - equal chance for positive and negative errors
  - more small errors than larger errors
  - magnitude of errors is limited
  - use statistical methods: normal distribution
- Gross errors
  - mistakes which should be avoided. Automatic detection

Direct tests

Main ideas

- We start from the assumption that the measurement errors are random errors with given distributions. This is actually our null hypothesis (H_0). The alternative hypothesis (H_1) is simply that the measurement errors are not from the assumed distribution;
- We then construct some quantities which can be computed from observation data and which have well-defined statistical distribution when the above assumption holds;
- Now we test the computed sample values of the above statistics against the theoretical critical values at certain risk level. If the test is passed, we then accept H_0. Otherwise, we may suspect there might exist systematic or gross errors in the measurement results.

Direct tests

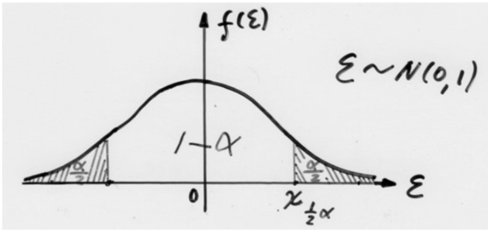
- Testing the maximum value of the errors
- Testing the sum of the errors
- Testing the number of positive errors versus number of negative errors
- Testing the order of the signs of errors
- Testing the sum of positive errors squared versus negative errors squared

Testing maximum absolute value of errors

Statistics : $\varepsilon_m = \max_{1 \le i \le n} (|\varepsilon_i|)$

If : $\varepsilon_i \sim N(0, \sigma^2)$

Then : $P\left\{\frac{\varepsilon_m}{\sigma} < c_{\frac{1}{2}\alpha}\right\} = 1 - \alpha$



Conclusion : if $|\varepsilon_m| < \sigma\ c_{\frac{1}{2}\alpha}$, we accept $\varepsilon_i \sim N(0, \sigma^2)$

Testing the sum of errors

Statistics : $s = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n$

If : $\varepsilon_i \sim N(0, \sigma^2)$ are uncorrelated with each other,

$\longrightarrow$ s will have normal distribution $N(0, n\sigma^2)$

Then : $P\left\{\left|\frac{s}{\sqrt{n}\sigma}\right| < c_{\frac{1}{2}\alpha}\right\} = 1 - \alpha$

Conclusion : If $|s| < \sqrt{n} \cdot \sigma \cdot c_{\frac{1}{2}\alpha}$, we accept $\varepsilon_i \sim N(0, \sigma^2)$

Number of positive vs negative errors

Statistics : s_+ *is number of positive errors* ε_i $(1 \le i \le n)$ *and*
s_- *is number of negative errors :*

s_+ *follows binomial distribution with expectation and variance:* $\mu = \frac{1}{2}n, \quad \sigma^2 = \frac{n}{4}$

If n is large : $\frac{s_+ - \frac{1}{2}n}{\frac{1}{2}\sqrt{n}} \sim N(0,1)$

Then :

$$P\left\{\left|\frac{s_+ - \frac{1}{2}n}{\frac{1}{2}\sqrt{n}}\right| < c_{\frac{1}{2}\alpha}\right\} = P\left\{\left|\frac{s_+ - s_-}{\sqrt{n}}\right| < c_{\frac{1}{2}\alpha}\right\} = 1 - \alpha$$

$$P\left\{|s_+ - s_-| < \sqrt{n}c_{\frac{1}{2}\alpha}\right\} = 1 - \alpha$$

Conclusion : if $|s_+ - s_-| < \sqrt{n}c_{\frac{1}{2}\alpha}$ we accept H_0

$$\boxed{H_0 :\ s_+ = s_- \quad \text{and} \quad H_1 :\ s_+ \neq s_-}$$

Testing the randomness of signs

Statistics : s_1 *is number of error pairs with the same sign*
s_0 *is number of error pairs with different signs*

$$s_1 = x_1 + x_2 + \cdots + x_{n-1}$$

$$x_i = \begin{cases} 1 & \text{if } \varepsilon_i \text{ and } \varepsilon_{i+1} \text{ have the same sign} \\ 0 & \text{otherwise} \end{cases}$$

Then : s_1 *follows binomial distribution and approaches standard normal distribution when n is large:*

$$\frac{s_1 - \frac{1}{2}(n-1)}{\frac{1}{2}\sqrt{n-1}} = \frac{s_1 - s_0}{\sqrt{n-1}} \sim N(0,1)$$

Conclusion : if $|s_1 - s_0| < \sqrt{n-1}c_{\frac{1}{2}\alpha}$, we accept H_0

$$\boxed{H_0 :\ s_1 = s_0 \quad \text{and} \quad H_1 :\ s_1 \neq s_0}$$

Difference of positive/negative errors squared

Statistics : $s^2 = \lambda_1\varepsilon_1^2 + \lambda_2\varepsilon_2^2 + \cdots + \lambda_n\varepsilon_n^2 \qquad \lambda_i = \begin{cases} +1 & \text{if } \varepsilon_i > 0 \\ -1 & \text{if } \varepsilon_i < 0 \end{cases}$

$$P(\lambda_i = +1) = P(\lambda_i = -1) = \frac{1}{2} \qquad (i = 1, 2, \cdots, n)$$

$$\left.\begin{aligned} E(\lambda_i) &= (+1)\cdot\tfrac{1}{2} + (-1)\cdot\tfrac{1}{2} = 0 \\ E(\lambda_i^2) &= (+1)^2\cdot\tfrac{1}{2} + (-1)^2\cdot\tfrac{1}{2} = 1 \end{aligned}\right\}$$

$$\left.\begin{aligned} E(\lambda_i\varepsilon_i^2) &= E(\lambda_i)\cdot E(\varepsilon_i^2) = 0 \\ E\left[(\lambda_i\varepsilon_i^2)^2\right] &= E(\lambda_i^2)\cdot E(\varepsilon_i^4) = E(\varepsilon_i^4) = 3\sigma^4 \end{aligned}\right\}$$

$$\left.\begin{aligned} E(s^2) &= \textstyle\sum_{i=1}^n \left[E(\lambda_i\varepsilon_i^2)\right] = 0 \\ E\left[(s^2)^2\right] &= \textstyle\sum_{i=1}^n \left[E(\lambda_i^2\varepsilon_i^4)\right] = \sum_{i=1}^n \left[E(\lambda_i^2)\cdot E(\varepsilon_i^4)\right] = 3n\sigma^4 \end{aligned}\right\}$$

Difference of positive/negative errors squared

When n is large, s^2 *approaches normal distribution*

$$\longrightarrow \qquad P\left\{\left|\frac{s^2}{\sqrt{3n}\sigma^2}\right| < c_{\frac{1}{2}\alpha}\right\} = 1 - \alpha$$

If: $\left|s^2\right| < \sqrt{3n}\sigma^2\, c_{\frac{1}{2}\alpha}$

Then: s^2 *is statistically equal to zero*

Example of direct tests

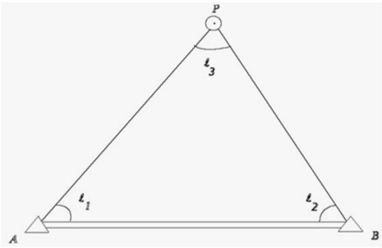


Table 1.1: List of Triangular Misclosures

i	w_i ($''$)	i	w_i ($''$)	i	w_i ($''$)
1	+1.5	11	-2.0	21	-1.1
2	+1.0	12	-0.7	22	-0.4
3	+0.8	13	-0.8	23	-1.0
4	-1.1	14	-1.2	24	-0.5
5	+0.6	15	+0.8	25	+0.2
6	+1.1	16	-0.3	26	+0.3
7	+0.2	17	+0.6	27	+1.8
8	-0.3	18	+0.8	28	+0.6
9	-0.5	19	-0.3	29	-1.1
10	+0.6	20	-0.9	30	-1.3

The standard error of w_i is assumed to be $\sigma = \pm 0.93''$. We want to test the randomness of this group of triangular misclosures using the five different tests described above. The risk level is chosen to be: $\alpha = 4.55\%$, for which the critical value of the standard normal distribution is $c_{\frac{1}{2}\alpha} = 2$.

5 statistical tests

(i) Testing the maximum absolute value of errors

$|w_m| = |w_{11}| = |-2.0| = 2.0'' > \sigma \cdot c_{\frac{1}{2}\alpha} = 1.86''$

$\Longrightarrow$ test is not passed !

(ii) Testing the sum of errors

$|s| = |2.6| = 2.6 < \sqrt{n}\sigma \cdot c_{\frac{1}{2}\alpha} \approx 10.2$

$\Longrightarrow$ test is passed !

(iii) Testing the number of positive versus negative errors

$|s_+ - s_-| = |14 - 16| = 2 < \sqrt{n} \cdot c_{\frac{1}{2}\alpha} \approx 11$

$\Longrightarrow$ test is passed !

(iv) Testing the order of positive versus negative errors

$|s_1 - s_0|\,| = |18 - 11| = 7 < \sqrt{n-1} \cdot c_{\frac{1}{2}\alpha} \approx 11$

$\Longrightarrow$ test is passed !

(v) Testing the sum of positive versus negative errors squared

$|s^2| = |3.40| = 3.4 < \sqrt{3n}\sigma^2 \cdot c_{\frac{1}{2}\alpha} \approx 16.41$

$\Longrightarrow$ test is passed !

Data Snooping

- *Based on the results of a least squares adjustment.*
- *Data snooping aims to check if one of the measurements contains a very big gross error*
- *In order to find that big gross error, data snooping makes statistical test of each measurement, one by one.*

$$H_0 : \begin{cases} \underset{n\cdot 1}{L} - \underset{n\cdot 1}{\varepsilon} = \underset{n\cdot m}{A}\ \underset{m\cdot 1}{X} \\ E\{\varepsilon\} = 0 \\ E\left\{\varepsilon\varepsilon^\top\right\} = \sigma_0^2 Q = \sigma_0^2 P^{-1} \end{cases}$$

$$H_1 : \begin{cases} \underset{n\cdot 1}{L} - \underset{n\cdot 1}{\varepsilon} = \underset{n\cdot m}{A}\ \underset{m\cdot 1}{X} + \underset{n\cdot 1}{e_i} \cdot \Delta_i \\ E\{\varepsilon\} = 0 \\ E\left\{\varepsilon\varepsilon^\top\right\} = \sigma_0^2 Q = \sigma_0^2 P^{-1} \end{cases} \qquad \underset{n\cdot 1}{e_i} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

LS adjustment under *H0*

$$H_0 : \begin{cases} \underset{n\cdot 1}{L} - \underset{n\cdot 1}{\varepsilon} = \underset{n\cdot m}{A}\ \underset{m\cdot 1}{X} \\ E\{\varepsilon\} = 0 \\ E\left\{\varepsilon\varepsilon^\top\right\} = \sigma_0^2 Q = \sigma_0^2 P^{-1} \end{cases}$$

$$\widehat{X} = \left(A^\top P A\right)^{-1} A^\top P L$$

$$\widehat{\varepsilon} = \left[I - A(A^\top P A)^{-1} A^\top P\right] L$$

$$C_{\widehat{\varepsilon\varepsilon}} = \sigma_0^2 \cdot Q_{\widehat{\varepsilon\varepsilon}}$$

$$Q_{\widehat{\varepsilon\varepsilon}} = P^{-1} - A(A^\top P A)^{-1} A^\top = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix}$$

$$\Omega = \widehat{\varepsilon}^\top P\, \widehat{\varepsilon} = L^\top \left[P - PA(A^\top P A)^{-1} A^\top P\right] L$$

LS adjutsment under *H*1

$$H_1 : \begin{cases} \underset{n\cdot 1}{L} - \underset{n\cdot 1}{\varepsilon} = \underset{n\cdot m}{A}\ \underset{m\cdot 1}{X} + \underset{n\cdot 1}{e_i} \cdot \Delta_i \\ E\{\varepsilon\} = 0 \\ E\left\{\varepsilon\varepsilon^\top\right\} = \sigma_0^2 Q = \sigma_0^2 P^{-1} \end{cases} \qquad \underset{n\cdot 1}{e_i} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

$$\widehat{X}_\Delta = \widehat{X} - \left(A^\top P A\right)^{-1} A^\top P e_i \cdot \widehat{\Delta}_i$$

$$\widehat{\Delta}_i = \left(e_i^\top P Q_{\widehat{\varepsilon}\widehat{\varepsilon}} P e_i\right)^{-1} e_i^\top P \widehat{\varepsilon}$$

$$\Omega_\Delta = \widehat{\varepsilon}_\Delta^\top P \widehat{\varepsilon}_\Delta = \widehat{\varepsilon}^\top P \widehat{\varepsilon} - \Delta\Omega = \Omega - \Delta\Omega$$

$$\Delta\Omega = \widehat{\varepsilon}^\top P e_i \left(e_i^\top P Q_{\widehat{\varepsilon}\widehat{\varepsilon}} P e_i\right)^{-1} e_i^\top P \widehat{\varepsilon}$$

For a diagonal weight matrix

$$\underset{n\cdot n}{P} = \begin{bmatrix} p_1 & & & \\ & p_2 & & \\ & & \cdots & \\ & & & p_n \end{bmatrix}$$

$$\Delta\Omega = \frac{\widehat{\varepsilon}_i^2}{q_{ii}}$$

$$\widehat{\Delta}_i = \frac{\widehat{\varepsilon}_i}{p_i \cdot q_{ii}}$$

$$\boxed{\frac{\Omega}{\sigma_0^2} \sim \chi^2(n-m)} \qquad \frac{\Delta\Omega}{\sigma_0^2} \sim \chi^2(1) \quad \text{or equivalently:} \quad \frac{\sqrt{\Delta\Omega}}{\sigma_0} \sim N(0,1)$$

$$\boxed{u_i = \frac{\sqrt{\Delta\Omega}}{\sigma_0} = \frac{\widehat{\varepsilon}_i}{\sigma_0\sqrt{q_{ii}}} = \frac{\widehat{\varepsilon}_i}{\sigma_{\widehat{\varepsilon}_i}} \sim N(0,1)}$$

Procedure of data snooping

- *Make a least squares adjustment*
- *Overall test of Ω : χ^2 -test*

If $\Omega < \sigma_0^2 \cdot \chi_{n-m,\,\alpha}$, we accept H_0 (i.e. there is no gross error) and reject H_0 otherwise. $\chi_{n-m,\,\alpha}$ is the critical value of $\chi^2(n-m)$ at risk level α.

- *Individual test: u-test (snooping)*

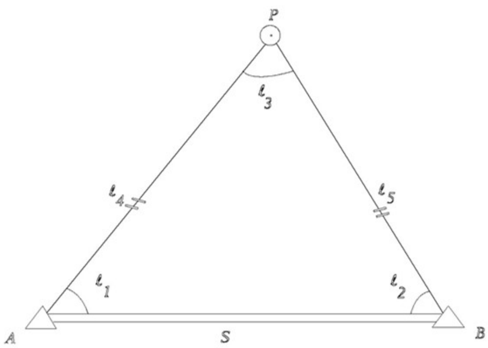
$$u_i = \frac{\sqrt{\Delta\Omega}}{\sigma_0} = \frac{\widehat{\varepsilon}_i}{\sigma_0\sqrt{q_{ii}}} = \frac{\widehat{\varepsilon}_i}{\sigma_{\widehat{\varepsilon}_i}} \sim N(0,1)$$

If $|u_i| < c_{\frac{1}{2}\alpha}$, we accept H_0 and otherwise reject H_0. Here $c_{\frac{1}{2}\alpha}$ denotes the critical value of $N(0,1)$ at risk level α.

- *If estimated variance factor is used, individual test by t-test:*

$$\widehat{\sigma}_0^2 = \frac{\Omega}{n-m} = \frac{\widehat{\varepsilon}^\top P \widehat{\varepsilon}}{n-m} \qquad\qquad w_i = \frac{\widehat{\varepsilon}_i}{\widehat{\sigma}_0\sqrt{q_{ii}}} \sim t(n-m-1)$$

Data snooping: Example 2.4, 3.3



$$\underset{5\cdot 5}{P} = \begin{bmatrix} \frac{1}{4} & & & & \\ & \frac{1}{4} & & & \\ & & \frac{1}{4} & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

Least squares adjustment: $\longrightarrow$ $$\widehat{\varepsilon} = \begin{bmatrix} +6.45\,'' \\ -3.40\,'' \\ +2.95\,'' \\ +4.82\,^{mm} \\ -3.98\,^{mm} \end{bmatrix}$$

Data snooping: Example 2.4, 3.3

$$Q_{\widehat{L}\widehat{L}} = \begin{bmatrix} +1.8664 & -1.1792 & -0.6872 & -0.1377 & +0.7148 \\ -1.1792 & +1.8664 & -0.6872 & +0.7148 & -0.1377 \\ -0.6872 & -0.6872 & +1.3744 & -0.5771 & -0.5771 \\ -0.1377 & +0.7148 & -0.5771 & +0.3616 & +0.1230 \\ +0.7148 & -0.1377 & -0.5771 & +0.1230 & +0.3616 \end{bmatrix}$$

$$Q_{\widehat{\varepsilon}\widehat{\varepsilon}} = P^{-1} - Q_{\widehat{L}\widehat{L}} = \begin{bmatrix} 4 & & & & \\ & 4 & & & \\ & & 4 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} - Q_{\widehat{L}\widehat{L}}$$

$$Q_{\widehat{\varepsilon}\widehat{\varepsilon}} = \begin{bmatrix} +2.1336 & +1.1792 & +0.6872 & +0.1377 & -0.7148 \\ +1.1792 & +2.1336 & +0.6872 & -0.7148 & +0.1377 \\ +0.6872 & +0.6872 & +2.6256 & +0.5771 & +0.5771 \\ +0.1377 & -0.7148 & +0.5771 & +0.6384 & -0.1230 \\ -0.7148 & +0.1377 & +0.5771 & -0.1230 & +0.6384 \end{bmatrix}$$

Data snooping: Example 2.4, 3.3

Risk level: 5%

$$c_{\frac{1}{2}\alpha} = 1.96, \quad t_{\frac{1}{2}\alpha}(n-m-1) = t_{\frac{1}{2}\alpha}(2) = 4.3$$

i	$\widehat{\varepsilon}_i$	q_{ii}	$\sqrt{q_{ii}}$	$u_i = \frac{\widehat{\varepsilon}_i}{\sigma_0 \times \sqrt{q_{ii}}}$ $(\sigma_0 = 3\ ^{mm})$	$w_i = \frac{\widehat{\varepsilon}_i}{\widehat{\sigma}_0 \times \sqrt{q_{ii}}}$ $(\widehat{\sigma}_0 = 4.26\ ^{mm})$
1	$+6.45''$	2.1336	1.46	$+1.47$	$+1.04$
2	$-3.40''$	2.1336	1.46	-0.78	-0.55
3	$+2.95''$	2.6256	1.62	$+0.61$	$+0.43$
4	$+4.82^{mm}$	0.6384	0.80	$+2.01$	$+1.41$
5	-3.98^{mm}	0.6384	0.80	-1.66	-1.17
				$\alpha = 5\%,\ c_{\frac{1}{2}\alpha} = 1.96$	$\alpha = 5\%,\ t_{\frac{1}{2}\alpha}(2) = 4.3$

Local redundancies

$$\underset{n\cdot n}{R} = Q_{\tilde{\varepsilon}\tilde{\varepsilon}} \cdot P = \underset{n\cdot n}{I} - A\left(A^\top P A\right)^{-1} A^\top P = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix}$$

$$tr(R) = tr\left[\underset{n\cdot n}{I}\right] - tr\left[A\left(A^\top P A\right)^{-1} A^\top P\right] =$$

$$= n - tr\left[\left(A^\top P A\right)^{-1} A^\top P A\right] = n - m = r$$

$$r_i = (R)_{ii} = r_{ii} \qquad (i = 1, 2, \cdots, n)$$

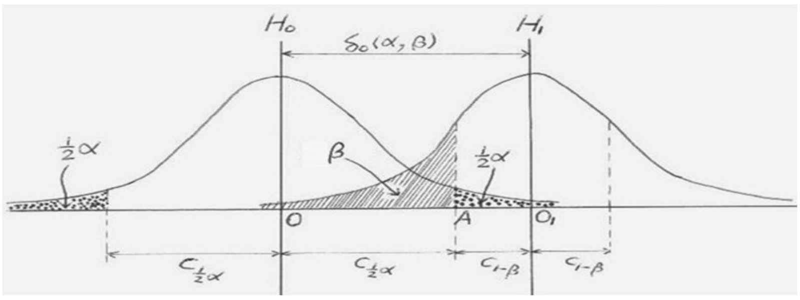
$$r_1 + r_2 + \cdots + r_n = r$$

$$0 \le r_i \le 1$$

If weight matrix P is diagonal ! $\longrightarrow$ $r_i = q_{ii} \cdot p_i$

Errors of hypothesis tests

$$\boxed{H_0 \;:\; u_i \sim N(0,1); \qquad H_1 \;:\; u_i \sim N(\delta_0, 1)}$$

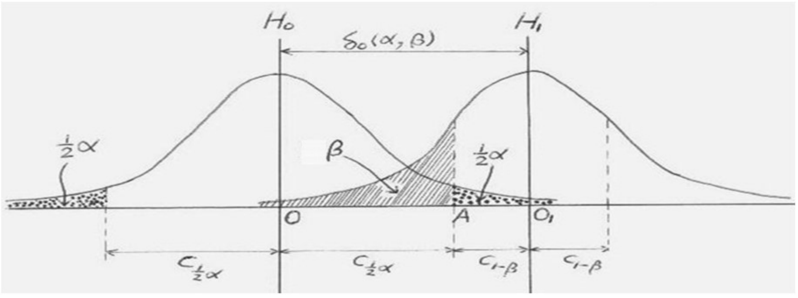


Type I error:
H_o is true but rejected incorrectly when $|u_i|$ *is larger than* $c_{\frac{1}{2}\alpha}$

Type II error:
H_o is false but accepted incorrectly when $|u_i|$ *is smaller than* $c_{\frac{1}{2}\alpha}$
Probability of type II error is β. 1 - β is called the power of the test.

Smallest detectable shift

$$\boxed{H_0 \ : \quad u_i \sim N(0,1); \qquad H_1 \ : \quad u_i \sim N(\delta_0,1)}$$



$$\delta_0\left(\alpha,\beta\right) = c_{\frac{1}{2}\alpha} + c_{1-\beta}$$

$\delta_0\left(\alpha,\beta\right)$ *is the smallest shift in a standard normal distriution u_i that can be detected by a statistical test for the chosen α and β .*

Normally one chooses $\alpha = 5\%$ and $\beta = 20\%$, which gives $\delta_0 \approx 2.80$.

Internal reliability

$$u_i = \frac{\sqrt{\Delta\Omega}}{\sigma_0} = \frac{\widehat{\varepsilon}_i}{\sigma_0\sqrt{q_{ii}}} = \frac{\widehat{\varepsilon}_i}{\sigma_{\widehat{\varepsilon}_i}} \sim N\left(0,1\right)$$

$$\widehat{\Delta}_i = \frac{\widehat{\varepsilon}_i}{p_i \cdot q_{ii}}$$

$$r_i = q_{ii} \cdot p_i \qquad \sigma_i = \frac{\sigma_0}{\sqrt{p_i}}$$

$$\widehat{\Delta}_i = \frac{\sigma_0 \cdot \sqrt{q_{ii}}}{p_i \cdot q_{ii}} \cdot u_i = \frac{\sigma_0/\sqrt{p_i}}{\sqrt{p_i \cdot q_{ii}}} = \frac{\sigma_i}{\sqrt{r_i}} \cdot u_i$$

$\longrightarrow$ $\boxed{\theta_i = |\Delta_i|_{\min} = \frac{\sigma_i}{\sqrt{r_i}} \cdot \delta_0(\alpha,\beta)}$

$\longrightarrow$ *Smallest gross error in measurement No.* ***i*** *that is possible to detect through statistical test*

$\longrightarrow$ *Internal reliability of measurement No.* ***i*** *(Baarda, 1967)*

External reliability

$$\Delta_i^{(X)} = \widehat{X}_\Delta - \widehat{X} = -\left(A^\top PA\right)^{-1} A^\top Pe_i \cdot \widehat{\Delta}_i$$

$$\Delta_i^{(\widehat{L})} = A\left(\widehat{X}_\Delta - \widehat{X}\right) = -A\left(A^\top PA\right)^{-1} A^\top Pe_i\widehat{\Delta}_i$$

$$= (I - R) \cdot e_i\widehat{\Delta}_i$$

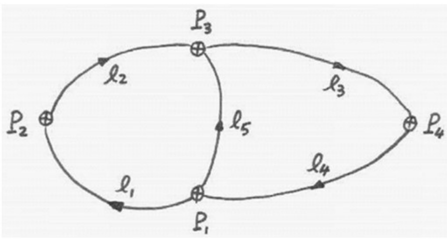
$$\Delta_i^{(\widehat{\ell}_i)} = (1 - r_i) \cdot \widehat{\Delta}_i$$

$$\varphi_i = \left|\Delta_i^{(\widehat{\ell}_i)}\right|_{\min} = (1 - r_i) \cdot \frac{\sigma_i}{\sqrt{r_i}}\delta_0(\alpha, \beta) = (1 - r_i) \cdot \theta_i$$

$\longrightarrow$ *External reliability of measurement No.* i *(Baarda, 1967)*

Example 6.3 (3.2)

$\ell_1 = +1.002\ metres,\quad s_1 \approx 1\ km$
$\ell_2 = +2.004\ metres,\quad s_2 \approx 1\ km$
$\ell_3 = -2.001\ metres,\quad s_3 \approx 2\ km$
$\ell_4 = -1.002\ metres,\quad s_4 \approx 2\ km$
$\ell_5 = +3.012\ metres,\quad s_5 \approx 1\ km$



$$p_i = \frac{2^{\ km}}{s_i^{\ km}}$$

$$\underset{5\cdot 5}{P} = \begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 2 \end{bmatrix}$$

Assuming a theoretical variance factor $\sigma_0^2 = 22.5\ mm^2$

$$\begin{cases} \sigma_1 = \sigma_0/\sqrt{p_1} = 3.35\ ^{mm} \\ \sigma_2 = \sigma_0/\sqrt{p_2} = 3.35\ ^{mm} \\ \sigma_3 = \sigma_0/\sqrt{p_3} = 4.74\ ^{mm} \\ \sigma_4 = \sigma_0/\sqrt{p_4} = 4.74\ ^{mm} \\ \sigma_5 = \sigma_0/\sqrt{p_5} = 3.35\ ^{mm} \end{cases}$$

Compute local redundancies

$$Q_{\tilde{\varepsilon}\tilde{\varepsilon}} = \frac{1}{28}\begin{bmatrix} 5 & 5 & 2 & 2 & -4 \\ 5 & 5 & 2 & 2 & -4 \\ 2 & 2 & 12 & 12 & 4 \\ 2 & 2 & 12 & 12 & 4 \\ -4 & -4 & 4 & 4 & 6 \end{bmatrix}$$

$$R = Q_{\tilde{\varepsilon}\tilde{\varepsilon}}P = \frac{1}{28}\begin{bmatrix} 5 & 5 & 2 & 2 & -4 \\ 5 & 5 & 2 & 2 & -4 \\ 2 & 2 & 12 & 12 & 4 \\ 2 & 2 & 12 & 12 & 4 \\ -4 & -4 & 4 & 4 & 6 \end{bmatrix}\begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 2 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 5 & 5 & 1 & 1 & -4 \\ 5 & 5 & 1 & 1 & -4 \\ 2 & 2 & 6 & 6 & 4 \\ 2 & 2 & 6 & 6 & 4 \\ -4 & -4 & 2 & 2 & 6 \end{bmatrix}$$

$$\begin{cases} r_1 = \frac{5}{14} \\ r_2 = \frac{5}{14} \\ r_3 = \frac{6}{14} \\ r_4 = \frac{6}{14} \\ r_5 = \frac{6}{14} \end{cases} \qquad \left(check : \sum_{i=1}^{5} r_i = 2 = n - m \right)$$

Compute internal/external reliabilities

Normally one chooses $\alpha = 5\%$ and $\beta = 20\%$, which gives $\delta_0 \approx 2.80$.

$$\begin{cases} \theta_1 = 2.8 \times \sigma_1/\sqrt{r_1} = 4.7\sigma_1 = 15.7\ ^{mm} \\ \theta_2 = 2.8 \times \sigma_2/\sqrt{r_2} = 4.7\sigma_2 = 15.7\ ^{mm} \\ \theta_3 = 2.8 \times \sigma_3/\sqrt{r_3} = 4.3\sigma_3 = 20.3\ ^{mm} \\ \theta_4 = 2.8 \times \sigma_4/\sqrt{r_4} = 4.3\sigma_4 = 20.3\ ^{mm} \\ \theta_5 = 2.8 \times \sigma_5/\sqrt{r_5} = 4.3\sigma_5 = 14.3\ ^{mm} \end{cases}$$

$$\begin{cases} \varphi_1 = (1 - r_1)\theta_1 = 10.1\ ^{mm} \\ \varphi_2 = (1 - r_2)\theta_2 = 10.1\ ^{mm} \\ \varphi_3 = (1 - r_3)\theta_3 = 11.6\ ^{mm} \\ \varphi_4 = (1 - r_4)\theta_4 = 11.6\ ^{mm} \\ \varphi_5 = (1 - r_5)\theta_5 = 8.2\ ^{mm} \end{cases}$$