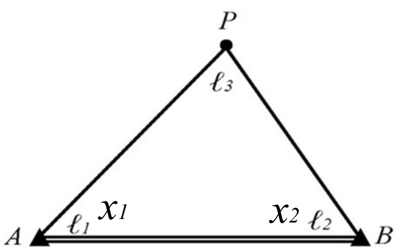
5. Least squares adjustment in linear models

Gauss-Markov model, adjustment by elements

- A simple example
- General formulas for adjustment by elements
  - linear observation equations
  - least squares estimates
  - estimates of variance factor & variance-covariance matrices
- Non-linear adjustment by elements
  - linearization of observation equations
  - computation procedures
  - observation equations for common geodetic measurements
  - datum problem

A simple example



To determine the geometry, two necessary measurements are neeed, e.g. x_1, x_2.

All 3 measurements can be expressed as linear functions of x_1, x_2 :

$$\ell_1 = x_1$$
$$\ell_2 = x_2$$
$$\ell_3 = 180^0 - x_1 - x_2$$

In a planar triangle with 2 fixed points (A,B), three internal angles have been measured.

Observation equations : $\longrightarrow$

$$\begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 180^0 \end{bmatrix}$$

Due to errors, above equations will not hold!

A simple example

Adding errros, we have the following 3 observation equations:

$$\begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} - \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 180^0 \end{bmatrix}$$

$$\underset{3\cdot 1}{L} - \underset{3\cdot 1}{\varepsilon} = \underset{3\cdot 2}{A}\,\underset{2\cdot 1}{X}$$

$$\underset{3\cdot 1}{L} = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 180^0 \end{bmatrix}, \quad \underset{3\cdot 1}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}, \quad \underset{3\cdot 2}{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad \underset{2\cdot 1}{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

3 equations, 5 unknowns **!**

Adjustment by elements: solve for X from L by least squares principle:

$$L - \widehat{\varepsilon} = A\widehat{X}$$

$$\widehat{\varepsilon}^\top P \widehat{\varepsilon} = \text{ minimum}$$

Adjustment by Elements

- *n* measurements have been made
- The problem requires *m* necessary measurements
- These *m* necessary measurements can be represented by *m* unknown parameters (elements)
- Each measurement can be expressed by a linear function of the *m* unknown parameters:

$$\widetilde{\ell}_1 = \ell_1 - \varepsilon_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m + c_1$$

$$\widetilde{\ell}_2 = \ell_2 - \varepsilon_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m + c_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\widetilde{\ell}_n = \ell_n - \varepsilon_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m + c_n$$

Observation Equations

$$\underset{n\cdot 1}{L} - \underset{n\cdot 1}{\varepsilon} = \underset{n\cdot m}{A}\,\underset{m\cdot 1}{X}$$

$$\underset{n\cdot 1}{L} = L' - c\ , \quad L' = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \cdots \\ \ell_n \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{bmatrix}, \quad \underset{n\cdot 1}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdots \\ \varepsilon_n \end{bmatrix}$$

L' = original measurements. L = reduced measurements

$$\underset{n\cdot m}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}, \quad \underset{m\cdot 1}{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{bmatrix}$$

A = design matrix. X = unknown parameters (elements)

Adjustment by elements

A priori information on the measurements:

$$E[\varepsilon] = \begin{bmatrix} E(\varepsilon_1) \\ E(\varepsilon_2) \\ \cdots \\ E(\varepsilon_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} = \underset{n\cdot 1}{0}$$

$$\underset{n\cdot n}{C_{\varepsilon\varepsilon}} = E\left\{[\varepsilon - E(\varepsilon)][\varepsilon - E(\varepsilon)]^\top\right\} = E(\varepsilon\varepsilon^\top) = \sigma_0^2\ P^{-1}$$

Observation equations:

$$\underset{n\cdot 1}{L} - \underset{n\cdot 1}{\varepsilon} = \underset{n\cdot m}{A}\,\underset{m\cdot 1}{X}$$

$$\boxed{L - \widehat{\varepsilon} = A\widehat{X}}$$

n observations,
n+*m* unknowns !!!

The estimates are to be uniquely determined by least squares principle:

$$\boxed{\widehat{\varepsilon}^\top P \widehat{\varepsilon} = \text{minimum}}$$

Estimate of the unknown parameters

Lagrangian function to solve for a conditional minimization problem:

$$F(X) = \varepsilon^\top P \varepsilon = (L - AX)^\top P (L - AX)$$

$$\frac{\partial F}{\partial X}\Big|_{x=\widehat{X},\ \varepsilon=\widehat{\varepsilon}} = 2\,\widehat{\varepsilon}^\top\ P\,(-A) = 0$$

$$A^\top P \widehat{\varepsilon} = 0$$

Normal equation:

$$A^\top P A \widehat{X} = A^\top P L$$

$$\boxed{L - \widehat{\varepsilon} = A\widehat{X}}$$

Least squares estimate of the unknown parameters:

$$\widehat{X} = (A^\top P A)^{-1} A^\top P L$$

Other least squares estimates

$$\boxed{L - \widehat{\varepsilon} = A\widehat{X}}$$

Least squares estimate of the errors (residuals):

$$\widehat{\varepsilon} = L - A\widehat{X} = \left[I - A(A^\top P A)^{-1} A^\top P\right] L$$

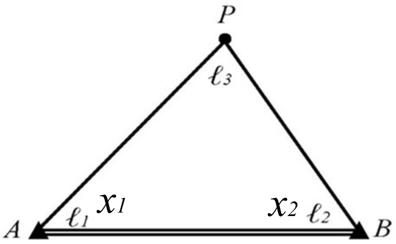
Least squares estimate of the *reduced* observations:

$$\widehat{L} = L - \widehat{\varepsilon} = A\widehat{X} = A(A^\top P A)^{-1} A^\top P L$$

Least squares estimate of the *original* observations:

$$\widehat{L}' = \widehat{L} + c$$

A simple example



To determine the geometry, two necessary measurements are neeed, e.g. x_1, x_2.

All 3 measurements can be expressed as linear functions of x_1, x_2 :

$$\underset{3\cdot 1}{L} = \begin{bmatrix} 60^0\ 00'\ 03'' \\ 60^0\ 00'\ 03'' \\ 59^0\ 59'\ 51'' \end{bmatrix}$$

$$\underset{3\cdot 3}{P} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = \underset{3\cdot 3}{I}$$

$$\begin{aligned} \ell_1 &= x_1 \\ \ell_2 &= x_2 \\ \ell_3 &= 180^0 - x_1 - x_2 \end{aligned}$$

$$\begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 180^0 \end{bmatrix}$$

Observation equations

$$\begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} - \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 180^0 \end{bmatrix}$$

$$\underset{3\cdot 1}{L} - \underset{3\cdot 1}{\varepsilon} = \underset{3\cdot 2}{A}\,\underset{2\cdot 1}{X}$$

$$\underset{3\cdot 1}{L} = L' - c = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 180^0 \end{bmatrix} = \begin{bmatrix} 60^0\ 00'\ 03'' \\ 60^0\ 00'\ 03'' \\ -120^0\ 00'\ 09'' \end{bmatrix}$$

$$\underset{3\cdot 1}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}$$

$$\underset{3\cdot 2}{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad \underset{2\cdot 1}{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Adjustment computations

$$A^\top PA = A^\top A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad (A^\top PA)^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\widehat{X} = (A^\top A)^{-1}A^\top L = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} L$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 60^0\ 00'\ 03'' \\ 60^0\ 00'\ 03'' \\ -120^0\ 00'\ 09'' \end{bmatrix} = \begin{bmatrix} 60^0\ 00'\ 04'' \\ 60^0\ 00'\ 04'' \end{bmatrix}$$

$$\widehat{L} = A\widehat{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 60^0\ 00'\ 04'' \\ 60^0\ 00'\ 04'' \end{bmatrix} = \begin{bmatrix} 60^0\ 00'\ 04'' \\ 60^0\ 00'\ 04'' \\ -120^0\ 00'\ 08'' \end{bmatrix}$$

$$\widehat{L'} = \widehat{L} + c = \begin{bmatrix} 60^0\ 00'\ 04'' \\ 60^0\ 00'\ 04'' \\ -120^0\ 00'\ 08'' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 180^0 \end{bmatrix} = \begin{bmatrix} 60^0\ 00'\ 04'' \\ 60^0\ 00'\ 04'' \\ 59^0\ 59'\ 52'' \end{bmatrix}$$

Variance factor & variance-covariance matrices

Unbiased estimate of the variance factor:

$$\widehat{\sigma}_0^2 = \frac{\widehat{\varepsilon}^\top P\widehat{\varepsilon}}{n-m}$$

Variance-covariance matrices of the estimated quantities:

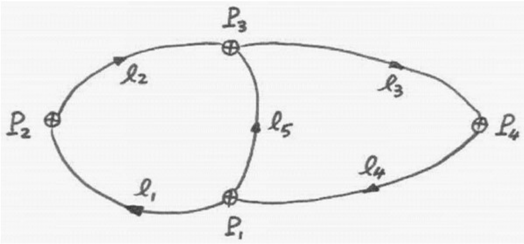
$$C_{\widehat{X}\widehat{X}} = \widehat{\sigma}_0^2\ (A^\top PA)^{-1}$$

$$C_{\widehat{\varepsilon}\widehat{\varepsilon}} = \widehat{\sigma}_0^2\ [P^{-1} - A(A^\top PA)^{-1}A^\top]$$

$$C_{\widehat{L}\widehat{L}} = \widehat{\sigma}_0^2\ A(A^\top PA)^{-1}A^\top$$

$$C_{\widehat{L'}\widehat{L'}} = C_{\widehat{L}\widehat{L}}$$

Example 2.2



- A network of benchmarks
- P_4 has fixed height H_4=10.000 m
- 5 un-correlated height difference measurements
- empirical weights are inversely proportional to levelling distances

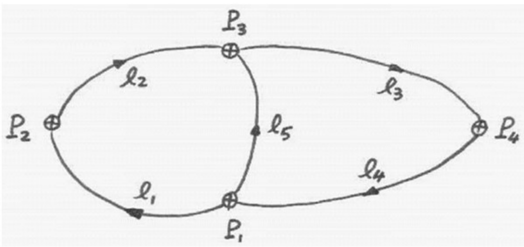
$$\begin{aligned}
\ell_1 &= +1.002\ metres, & s_1 &\approx 1\ km \\
\ell_2 &= +2.004\ metres, & s_2 &\approx 1\ km \\
\ell_3 &= -2.001\ metres, & s_3 &\approx 2\ km \\
\ell_4 &= -1.002\ metres, & s_4 &\approx 2\ km \\
\ell_5 &= +3.012\ metres, & s_5 &\approx 1\ km
\end{aligned}$$

$$p_i = \frac{2^{\ km}}{s_i^{\ km}}$$

$$\underset{5\cdot 1}{L'} = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ \ell_5 \end{bmatrix} = \begin{bmatrix} +1.002 \\ +2.004 \\ -2.001 \\ -1.002 \\ +3.012 \end{bmatrix} \ (m)$$

$$\underset{5\cdot 5}{P} = \begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 2 \end{bmatrix}$$

Observation equations



- To determine the heights of all benchmarks, 3 necessary measurements are needed
- Choose the heights of P_1 , P_2 and P_3 as unknown parameters: x_1, x_2, x_3 :

$$\begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ \ell_5 \end{bmatrix} - \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ H_4 \\ -H_4 \\ 0 \end{bmatrix}$$

Observation equations

$$\begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ \ell_5 \end{bmatrix} - \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ H_4 \\ -H_4 \\ 0 \end{bmatrix}$$

$$\longrightarrow \quad L' - \varepsilon = AX + c$$

$$\longrightarrow \quad L - \varepsilon = AX$$

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad L = L' - c = \begin{bmatrix} +1.002 \\ +2.004 \\ -12.001 \\ +8.998 \\ +3.012 \end{bmatrix} (m)$$

Adjustment computations

$$A^\top P = \begin{bmatrix} -1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 1 & -2 \\ 2 & -2 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 2 \end{bmatrix}$$

$$A^\top PA = \begin{bmatrix} -2 & 0 & 0 & 1 & -2 \\ 2 & -2 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 5 \end{bmatrix}$$

$$(A^\top PA)^{-1} = \tfrac{1}{28} \begin{bmatrix} 16 & 14 & 12 \\ 14 & 21 & 14 \\ 12 & 14 & 16 \end{bmatrix}$$

$$(A^\top PA)^{-1} A^\top P = \tfrac{1}{14} \begin{bmatrix} -2 & -2 & -6 & 8 & -4 \\ 7 & -7 & -7 & 7 & 0 \\ 2 & 2 & -8 & 6 & 4 \end{bmatrix}$$

Least squares estimates

$$\widehat{X} = (A^\top PA)^{-1}A^\top PL = \frac{1}{14}\begin{bmatrix} -2 & -2 & -6 & 8 & -4 \\ 7 & -7 & -7 & 7 & 0 \\ 2 & 2 & -8 & 6 & 4 \end{bmatrix}\begin{bmatrix} +1.002 \\ +2.004 \\ -12.001 \\ +8.998 \\ +3.012 \end{bmatrix} = \begin{bmatrix} 8.9950 \\ 9.9985 \\ 12.0040 \end{bmatrix} \ (m)$$

$$\widehat{L} = A\widehat{X} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 8.9950 \\ 9.9985 \\ 12.0040 \end{bmatrix} = \begin{bmatrix} 1.0035 \\ 2.0055 \\ -12.0040 \\ 8.9950 \\ 3.0090 \end{bmatrix} \ (m)$$

$$\widehat{L}' = \widehat{L} + c = \begin{bmatrix} 1.0035 \\ 2.0055 \\ -12.0040 \\ 8.9950 \\ 3.0090 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10.0000 \\ -10.000 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.0035 \\ 2.0055 \\ -2.0040 \\ -1.0050 \\ 3.0090 \end{bmatrix} \ (m)$$

$$\widehat{\varepsilon} = L - A\widehat{X} = L - \widehat{L} = \begin{bmatrix} +1.002 \\ +2.004 \\ -12.001 \\ +8.998 \\ +3.012 \end{bmatrix} - \begin{bmatrix} 1.0035 \\ 2.0055 \\ -12.0040 \\ 8.9950 \\ 3.0090 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -1.5 \\ +3.0 \\ +3.0 \\ +3.0 \end{bmatrix} \ (mm)$$

Variance factor and co-factor matrices

Variance factor:

$$\widehat{\sigma}_0^2 = \frac{\widehat{\varepsilon}^\top P\widehat{\varepsilon}}{n-m} = \frac{1}{5-3}{}_i \sum_1^5 (p_i\widehat{\varepsilon}_i\widehat{\varepsilon}_i) = 22.50 \ mm^2$$

$$\widehat{\sigma}_0 = \sqrt{22.50} \approx \pm 4.74 \ mm$$

Co-factor matrices:

$$\sigma_{\widehat{x}_1} = \widehat{\sigma}_0\sqrt{16/28} \approx \pm 3.59 \quad mm$$
$$\sigma_{\widehat{x}_2} = \widehat{\sigma}_0\sqrt{21/28} \approx \pm 4.11 \quad mm$$
$$\sigma_{\widehat{x}_3} = \widehat{\sigma}_0\sqrt{16/28} \approx \pm 3.59 \quad mm$$

$$Q_{\widehat{X}\widehat{X}} = (A^\top PA)^{-1} = \frac{1}{28}\begin{bmatrix} 16 & 14 & 12 \\ 14 & 21 & 14 \\ 12 & 14 & 16 \end{bmatrix}$$

$$Q_{\widehat{L}\widehat{L}} = A(A^\top PA)^{-1}A^\top = \frac{1}{28}\begin{bmatrix} 9 & -5 & -2 & -2 & 4 \\ -5 & 9 & -2 & -2 & 4 \\ -2 & -2 & 16 & -12 & -4 \\ -2 & -2 & -12 & 16 & -4 \\ 4 & 4 & -4 & -4 & 8 \end{bmatrix}$$

$$Q_{\widehat{\varepsilon}\widehat{\varepsilon}} = P^{-1} - Q_{\widehat{L}\widehat{L}} = \frac{1}{28}\begin{bmatrix} 5 & 5 & 2 & 2 & -4 \\ 5 & 5 & 2 & 2 & -4 \\ 2 & 2 & 12 & 12 & 4 \\ 2 & 2 & 12 & 12 & 4 \\ -4 & -4 & 4 & 4 & 6 \end{bmatrix}$$

Variance-covariance matrices

$$\sigma_{\widehat{x}_1} = \widehat{\sigma}_0 \sqrt{16/28} \approx \pm 3.59 \ \ mm$$
$$\sigma_{\widehat{x}_2} = \widehat{\sigma}_0 \sqrt{21/28} \approx \pm 4.11 \ \ mm$$
$$\sigma_{\widehat{x}_3} = \widehat{\sigma}_0 \sqrt{16/28} \approx \pm 3.59 \ \ mm$$

$$C_{\widehat{X}\widehat{X}} = \widehat{\sigma}_0^2 \, Q_{\widehat{X}\widehat{X}} = \tfrac{22.50}{28} \begin{bmatrix} 16 & 14 & 12 \\ 14 & 21 & 14 \\ 12 & 14 & 16 \end{bmatrix}$$

$$C_{\widehat{L}\widehat{L}} = \widehat{\sigma}_0^2 \, Q_{\widehat{L}\widehat{L}} = \tfrac{22.50}{28} \begin{bmatrix} 9 & -5 & -2 & -2 & 4 \\ -5 & 9 & -2 & -2 & 4 \\ -2 & -2 & 16 & -12 & -4 \\ -2 & -2 & -12 & 16 & -4 \\ 4 & 4 & -4 & -4 & 8 \end{bmatrix}$$

$$C_{\widehat{\varepsilon}\widehat{\varepsilon}} = \widehat{\sigma}_0^2 \, Q_{\widehat{\varepsilon}\widehat{\varepsilon}} = \tfrac{22.50}{28} \begin{bmatrix} 5 & 5 & 2 & 2 & -4 \\ 5 & 5 & 2 & 2 & -4 \\ 2 & 2 & 12 & 12 & 4 \\ 2 & 2 & 12 & 12 & 4 \\ -4 & -4 & 4 & 4 & 6 \end{bmatrix}$$

Direct Adjustment

$$\underset{n\cdot 1}{L} = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \cdots \\ \ell_n \end{bmatrix}$$

$$\underset{n\cdot n}{P} = \begin{bmatrix} p_1 & & & \\ & p_2 & & \\ & & \cdots & \\ & & & p_n \end{bmatrix}$$

$$C_{\varepsilon\varepsilon} = \sigma_0^2 \cdot P^{-1}$$

$$\underset{n\cdot 1}{L} - \underset{n\cdot 1}{\varepsilon} = \underset{n\cdot m}{A} \cdot \underset{m\cdot 1}{X} \quad (m = 1)$$

$$\underset{n\cdot 1}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdots \\ \varepsilon_n \end{bmatrix}, \quad \underset{n\cdot m}{A} = \underset{n\cdot 1}{A} = \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix}, \quad \underset{m\cdot 1}{X} = \underset{1\cdot 1}{X} = [\, x \,]$$

Direct Adjustment

$$A^\top PA = \sum_{i=1}^{n} p_i, \quad A^\top PL = \sum_{i=1}^{n} p_i \ell_i$$

$$\widehat{X} = \widehat{x} = \left(A^\top PA\right)^{-1} A^\top PL = \frac{\sum_{i=1}^{n} p_i \ell_i}{\sum_{i=1}^{n} p_i} = \underbrace{\frac{p_1\ell_1 + p_2\ell_2 + \cdots + p_n\ell_n}{p_1 + p_2 + \cdots + p_n}}_{\text{Weighted mean}}$$

$$\widehat{\varepsilon} = L - AX = L - A \cdot \widehat{x} = \begin{bmatrix} \ell_1 - \widehat{x} \\ \ell_2 - \widehat{x} \\ \cdots \\ \ell_n - \widehat{x} \end{bmatrix}$$

$$\widehat{\sigma}_0^2 = \frac{\widehat{\varepsilon}^\top P \widehat{\varepsilon}}{n - m} = \frac{1}{n-1} \sum_{i=1}^{n} p_i \widehat{\varepsilon}_i^2 = \frac{1}{n-1} \sum_{i=1}^{n} p_i \left(\ell_i - \widehat{x}\right)^2$$

Direct Adjustment

$$\widehat{x} = \frac{p_1\ell_1 + p_2\ell_2 + \cdots + p_n\ell_n}{p_1 + p_2 + \cdots + p_n}$$

$$\sigma_{\widehat{x}}^2 = \widehat{\sigma}_0^2 \cdot \left(A^\top PA\right)^{-1} = \widehat{\sigma}_0^2 \cdot \frac{1}{\sum_{i=1}^{n} p_i} \qquad \longrightarrow \qquad p_x = \sum_{i=1}^{n} p_i$$

When all direct measurements have equal weights $p_i = 1$

$$\widehat{x} = \frac{\sum_{i=1}^{n} \ell_i}{n} = \frac{\ell_1 + \ell_2 + \cdots + \ell_n}{n}$$

$$\widehat{\sigma}_0^2 = \frac{1}{n-1} \sum (\ell_i - \widehat{x})^2, \quad \sigma_{\widehat{x}}^2 = \widehat{\sigma}_0^2 \cdot \frac{1}{n}, \quad p_x = n$$

Linear Regression

$$y_i = \alpha + \beta \cdot x_i + \varepsilon_i \qquad (1 \le i \le n)$$

$$\underset{n\cdot 1}{L} - \underset{n\cdot 1}{\varepsilon} = \underset{n\cdot m}{A} \cdot \underset{m\cdot 1}{X} \qquad (m = 2)$$

$$\underset{n\cdot 1}{L} = \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix}, \quad \underset{n\cdot 1}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdots \\ \varepsilon_n \end{bmatrix} \qquad \underset{n\cdot n}{P} = \begin{bmatrix} p_1 & & & \\ & p_2 & & \\ & & \cdots & \\ & & & p_n \end{bmatrix}$$

$$\underset{n\cdot m}{A} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \cdots & \cdots \\ 1 & x_n \end{bmatrix}, \quad \underset{m\cdot 1}{X} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Linear Regression

$$A^\top PA = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} p_1 & & & \\ & p_2 & & \\ & & \cdots & \\ & & & p_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \cdots & \cdots \\ 1 & x_n \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^n (p_i) & \sum_{i=1}^n (p_i x_i) \\ \sum_{i=1}^n (p_i x_i) & \sum_{i=1}^n \left(p_i x_i^2\right) \end{bmatrix}$$

$$A^\top PL = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} p_1 & & & \\ & p_2 & & \\ & & \cdots & \\ & & & p_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^n (p_i y_i) \\ \sum_{i=1}^n (p_i x_i y_i) \end{bmatrix}$$

Direct Adjustment

$$\widehat{X} = \begin{bmatrix} \widehat{\alpha} \\ \widehat{\beta} \end{bmatrix} = (A^\top PA)^{-1}A^\top PL$$

$$= \frac{1}{n\sum_{i=1}^n (p_i x_i^2) - \left[\sum_{i=1}^n (p_i x_i)\right]^2} \begin{bmatrix} \sum_{i=1}^n (p_i x_i^2) & -\sum_{i=1}^n (p_i x_i) \\ -\sum_{i=1}^n (p_i x_i) & \sum_{i=1}^n (p_i) \end{bmatrix} \begin{bmatrix} \sum_{i=1}^n (p_i y_i) \\ \sum_{i=1}^n (p_i x_i y_i) \end{bmatrix}$$

$$\widehat{\varepsilon} = L - A\widehat{X} = \begin{bmatrix} y_1 - \widehat{\alpha} - \widehat{\beta} \cdot x_1 \\ y_2 - \widehat{\alpha} - \widehat{\beta} \cdot x_2 \\ \cdots\cdots\cdots\cdots\cdots \\ y_n - \widehat{\alpha} - \widehat{\beta} \cdot x_n \end{bmatrix}$$

$$\widehat{\sigma}^2 = \frac{\widehat{\varepsilon}^\top P \widehat{\varepsilon}}{n-m} = \frac{1}{n-2}\sum_{i=1}^n p_i \widehat{\varepsilon}_i^2 = \frac{1}{n-2}\sum_{i=1}^n \left\{ p_i \left(y_i - \widehat{\alpha} - \widehat{\beta} \cdot x_i \right)^2 \right\}$$

$$C_{\widehat{X}\widehat{X}} = \begin{bmatrix} \sigma_{\widehat{\alpha}}^2 & \sigma_{\widehat{\alpha}\widehat{\beta}} \\ \sigma_{\widehat{\alpha}\widehat{\beta}} & \sigma_{\widehat{\beta}}^2 \end{bmatrix} = \widehat{\sigma}_0^2 \cdot \left(A^\top PA\right)^{-1}$$

$$= \frac{\widehat{\sigma}_0^2}{n\sum_{i=1}^n (p_i x_i^2) - \left[\sum_{i=1}^n (p_i x_i)\right]^2} \begin{bmatrix} \sum_{i=1}^n (p_i x_i^2) & -\sum_{i=1}^n (p_i x_i) \\ -\sum_{i=1}^n (p_i x_i) & \sum_{i=1}^n (p_i) \end{bmatrix}$$

Summary of adjustment procedures

Adjustment by elements

- Analyze the problem (network of measurements) to decide m
- Select m parameters (most often coordinates)
- Form observation equations (one by one first)
- Specify weight matrix
- Calculate least squares estimates
- Calculate variance factor and VC-matrices