6. (Geodetic) Observation equations

- Linearization of nonlinear observation equations
- Adjustment by elements with nonlinear observation equations
- Observation equations of common geodetic measurements
- Numerical example

Nonlinear observation equations

- n measurements,
- m necessary observations,
- m unknown parameters
- each observation is a non-linear function of m unknown parameters

$$
\left.\begin{aligned}
\tilde{\ell}_1 &= f_1(x_1, x_2, \cdots, x_m) \\
\tilde{\ell}_2 &= f_2(x_1, x_2, \cdots, x_m) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\tilde{\ell}_i &= f_i(x_1, x_2, \cdots, x_m) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\tilde{\ell}_n &= f_n(x_1, x_2, \cdots, x_m)
\end{aligned}\right\}
$$

Unknown parameters

Approximate values of unknown parameters:

$$\underset{m\cdot 1}{X}^{0} = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \cdots \\ x_m^0 \end{bmatrix}$$

Corrections to the approximate values:

$$\underset{m\cdot 1}{\delta X} = \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \cdots \\ \delta x_m \end{bmatrix}$$

Correct values of unknown parameters:

$$\underset{m\cdot 1}{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{bmatrix} = \underset{m\cdot 1}{X}^{0} + \underset{m\cdot 1}{\delta X}$$

Linearization of one nonlinear function

$$\widetilde{\ell}_i = \ell_i - \varepsilon_i = f(x_1^0 + \delta x_1, x_2^0 + \delta x_2, \cdots, x_m^0 + \delta x_m)$$

$$= f(x_1^0,\ x_2^0,\ \cdots,\ x_m^0) + \frac{1}{1!}\left[\frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f}{\partial x_m}\delta x_m\right] +$$

$$+\frac{1}{2!}\left[\frac{\partial^2 f}{\partial x_1^2}\delta x_1^2 + \frac{\partial^2 f}{\partial x_1 \partial x_2}\delta x_1 \delta x_2 + \cdots + \frac{\partial^2 f}{\partial x_1 \partial x_m}\delta x_1 \delta x_m\right] + \cdots$$

If corrections δx_i are sufficiently small, terms of order 2 and higher can be neglected. *Keeping linear terms = linearization !*

$$\ell_i - \varepsilon_i = a_{i1}\delta x_1 + a_{i2}\delta x_2 + \cdots + a_{im}\delta x_m + c_i$$

$$a_{ij} = \frac{\partial f_i}{\partial x_j}\ , \quad c_i = f_i(x_1^0,\ x_2^0,\ \cdots,\ x_m^0)\ , \quad i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m$$

Linearization of all n measurements

$$\begin{rcases}
\widetilde{\ell}_1 = f_1(x_1, x_2, \cdots, x_m) \\
\widetilde{\ell}_2 = f_2(x_1, x_2, \cdots, x_m) \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\widetilde{\ell}_i = f_i(x_1, x_2, \cdots, x_m) \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\widetilde{\ell}_n = f_n(x_1, x_2, \cdots, x_m)
\end{rcases}$$

$$\underset{n\cdot 1}{L} - \underset{n\cdot 1}{\varepsilon} = \underset{n\cdot m}{A}\ \underset{m\cdot 1}{\delta X}$$

$$\underset{n\cdot 1}{L} = L' - c, \quad L' = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \cdots \\ \ell_n \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{bmatrix},$$

$$c_i = f_i(x_1^0,\ x_2^0,\ \cdots,\ x_m^0)$$

$$\underset{n\cdot m}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$$a_{ij} = \frac{\partial f_i}{\partial x_j} \quad (1 \le i \le n;\ 1 \le j \le m)$$

Estimate of the unknown parameters

Least squares estimate of parameter correction: $\delta\widehat{X} = (A^\top PA)^{-1}A^\top P\ L$

Least squares estimate of the total parameters: $\widehat{X} = X^0 + \delta\widehat{X}$

Variance-covariance matrix: $C_{\widehat{X}\widehat{X}} = C_{\delta\widehat{X}\delta\widehat{X}} = \sigma_0^2\ (A^\top PA)^{-1}$

Unbiased estimate of the variance factor: $\widehat{\sigma}_0^2 = \dfrac{\widehat{\varepsilon}^\top P\widehat{\varepsilon}}{n-m}$

Estimates of other quantities

Least squares estimates of
- the erorrs (residuals),
- the reduced measurments and
- the original measurements:

$$\widehat{\varepsilon} = L - A\delta\widehat{X} = \left[I - A(A^\top PA)^{-1}A^\top P\right]\ L$$

$$\widehat{L} = L - \widehat{\varepsilon} = A\ \delta\widehat{X} = A(A^\top PA)^{-1}A^\top PL$$

$$\widehat{L}' = \ \widehat{L} + c$$

…… and their variance-covariance matrices:

$$C_{\widehat{\varepsilon}\widehat{\varepsilon}} = \sigma_0^2\left[P^{-1} - A(A^\top PA)^{-1}A^\top\right]$$

$$C_{\widehat{L}\widehat{L}} = C_{\widehat{L}'\widehat{L}'} = \sigma_0^2\ A(A^\top PA)^{-1}A^\top$$

Selection of unknown parameters

- Most often, coordinates of unknown points are chosen as unknown parameters
- Sufficient initial data (datum) must exist.

  Otherwise the network has datum defect.
  Then the normal equation matrix will be singular.
- A network without any initial data is called a *free network*.
- Adjustment of networks with datum defect requires the use of generalized matrix inverses. See Chapter 4 !

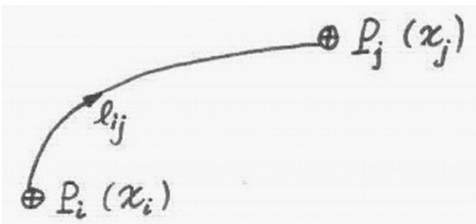
Observation equations of common geodetic measurements

- height differences from levelling
- 3D or 2D distances
- horizontal angles
- vertical angles
- GPS measurements – code pseudo-range
- GPS measurements – phase pseudo-range

Most often, coordinates of points are chosen as unknown parameters

A height difference from levelling

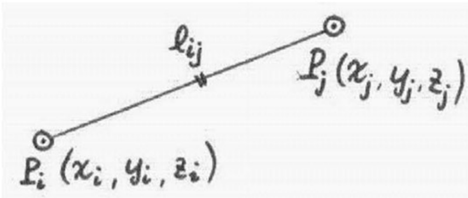
Heights of unknown benchmarks are chosen as unknown parameters



$$\ell_{ij} - \varepsilon_{ij} = x_j - x_i$$

Linear observation equation !

3D distance measurement



Coordinates are chosen as the unknown parameters

$$\ell_{ij} - \varepsilon_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$$

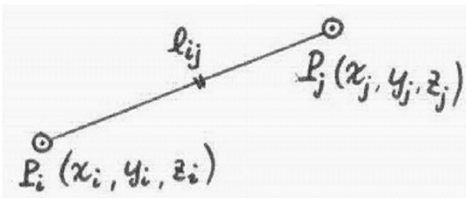
Nonlinear equation !

To linearize the above equation, approximate coordinates are needed.

From approximate coordinates, approximate distance can be derived:

$$s_{ij}^0 = \sqrt{(x_j^0 - x_i^0)^2 + (y_j^0 - y_i^0)^2 + (z_j^0 - z_i^0)^2}$$

Linearization of a 3D distance



$$(\ell_{ij} - s_{ij}^0) - \varepsilon_{ij} = a \cdot \delta x_i + b \cdot \delta y_i + c \cdot \delta z_i + d \cdot \delta x_j + e \cdot \delta y_j + f \cdot \delta z_j$$

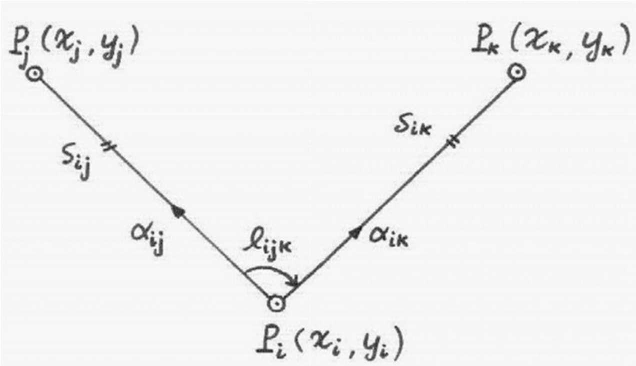
$$a = \frac{\partial \ell_{ij}}{\partial x_i} = -\frac{x_j^0 - x_i^0}{s_{ij}^0} \qquad d = \frac{\partial \ell_{ij}}{\partial x_j} = \frac{x_j^0 - x_i^0}{s_{ij}^0} = -a$$

$$b = \frac{\partial \ell_{ij}}{\partial y_i} = -\frac{y_j^0 - y_i^0}{s_{ij}^0} \qquad e = \frac{\partial \ell_{ij}}{\partial y_j} = \frac{y_j^0 - y_i^0}{s_{ij}^0} = -b$$

$$c = \frac{\partial \ell_{ij}}{\partial z_i} = -\frac{z_j^0 - z_i^0}{s_{ij}^0} \qquad f = \frac{\partial \ell_{ij}}{\partial z_j} = \frac{z_j^0 - z_i^0}{s_{ij}^0} = -c$$

For 2D horizontal distances, terms related to z are to be omitted.

A 2D horizontal angle

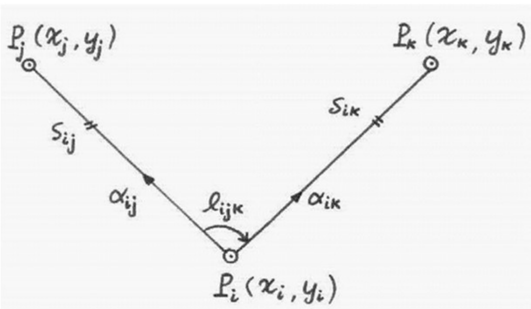


$$\ell_{ijk} - \varepsilon_{ijk} = \arctan\left(\frac{y_k - y_i}{x_k - x_i}\right) - \arctan\left(\frac{y_j - y_i}{x_j - x_i}\right)$$

Nonlinear equation !

Quantities derived from approximate coordinates

To linearize the nonlinear observation equation, approximate coordinates are introduced.



From the approximate coordinates, approximate values of angles, azimuths and distances can be derived.

$$\beta^0_{ijk} = \alpha^0_{ik} - \alpha^0_{ij}$$

$$\alpha^0_{ik} = \arctan\left(\frac{y^0_k - y^0_i}{x^0_k - x^0_i}\right)$$

$$\alpha^0_{ij} = \arctan\left(\frac{y^0_j - y^0_i}{x^0_j - x^0_i}\right)$$

$$s^0_{ij} = \sqrt{(x^0_j - x^0_i)^2 + (y^0_j - y^0_i)^2}$$

$$s^0_{ik} = \sqrt{(x^0_k - x^0_i)^2 + (y^0_k - y^0_i)^2}\ .$$

Linearization of a horizontal angle

$$\boxed{(\ell_{ijk} - \beta_{ijk}^0) - \varepsilon_{ijk} = a \cdot \delta x_i + b \cdot \delta y_i + c \cdot \delta x_j + d \cdot \delta y_j + e \cdot \delta x_k + f \cdot \delta y_k}$$

$$a = \frac{\partial \ell_{ijk}}{\partial x_i} = \frac{y_k^0 - y_i^0}{\left(s_{ik}^0\right)^2} - \frac{y_j^0 - y_i^0}{\left(s_{ij}^0\right)^2} = \frac{\sin \alpha_{ik}^0}{s_{ik}^0} - \frac{\sin \alpha_{ij}^0}{s_{ij}^0}$$

$$b = \frac{\partial \ell_{ijk}}{\partial y_i} = -\frac{x_k^0 - x_i^0}{\left(s_{ik}^0\right)^2} + \frac{x_j^0 - x_i^0}{\left(s_{ij}^0\right)^2} = -\frac{\cos \alpha_{ik}^0}{s_{ik}^0} + \frac{\cos \alpha_{ij}^0}{s_{ij}^0}$$

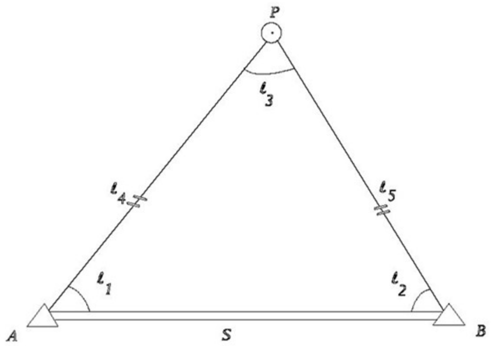
$$c = \frac{\partial \ell_{ijk}}{\partial x_j} = \frac{y_j^0 - y_i^0}{\left(s_{ij}^0\right)^2} = \frac{\sin \alpha_{ij}^0}{s_{ij}^0}$$

$$d = \frac{\partial \ell_{ijk}}{\partial y_j} = -\frac{x_j^0 - x_i^0}{\left(s_{ij}^0\right)^2} = -\frac{\cos \alpha_{ij}^0}{s_{ij}^0}$$

$$e = \frac{\partial \ell_{ijk}}{\partial x_k} = -\frac{y_k^0 - y_i^0}{\left(s_{ik}^0\right)^2} = -\frac{\sin \alpha_{ik}^0}{s_{ik}^0}$$

$$f = \frac{\partial \ell_{ijk}}{\partial y_k} = \frac{x_k^0 - x_i^0}{\left(s_{ik}^0\right)^2} = \frac{\cos \alpha_{ik}^0}{s_{ik}^0}$$

An example of adjustment by elements

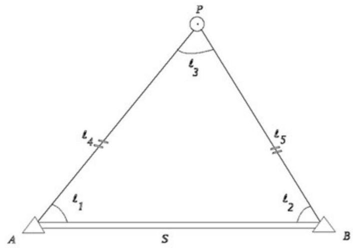


In a 2D triangulation network, P is an unknown point and A, B are fixed points with given coordinates :

$$x_A = 6\ 500\ 000.000 \ \ m\ , \quad x_B = 6\ 500\ 060.000 \ \ m$$
$$y_A = 1\ 500\ 000.000 \ \ m\ , \quad y_B = 1\ 500\ 080.000 \ \ m$$

The meaurements

- 3 angles and 2 distances
- all uncorrelated.



$$L' = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ \ell_5 \end{bmatrix} = \begin{bmatrix} 60^0 \;\; 00' \; 05'' \\ 60^0 \;\; 00' \; 05'' \\ 59^0 \;\; 59' \; 58'' \\ 100.008 \; m \\ 99.997 \; m \end{bmatrix} \quad , \quad \sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{bmatrix} = \begin{bmatrix} \pm 6'' \\ \pm 6'' \\ \pm 6'' \\ \pm 3 \; mm \\ \pm 3 \; mm \end{bmatrix}$$

If we chose variance factor as $\sigma_0 = \pm$ 3 mm, then the weight matrix becomes:

$$p_i = \frac{\sigma_0^2}{\sigma_i^2} \longrightarrow \underset{5\cdot 5}{P} = \begin{bmatrix} \frac{1}{4} & & & & \\ & \frac{1}{4} & & & \\ & & \frac{1}{4} & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

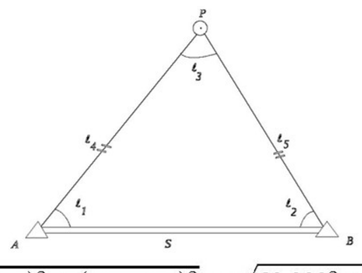
Analysis of the network

- n=5 measurements, $\longrightarrow$ n=5 observation equations
- To determine the position of P, m=2 necessary measurements are needed
- $\longrightarrow$ m=2 unknown parameters in adjustment by elements
- We choose the coordiantes of P, (x_p, y_p), as parameters
- There are n-m=3 *over-determinations* (*redundancies*)

Procedures to compute linearized observation equations

- Find approximate coordinates for the unknown point P
- Calculate approximate distances and angles
- Calculate linearized observation equations for angles
- Calculate linearized observation equations for distances

Find approximate coordinates of P



$$S = s_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} = \sqrt{60.000^2 + 80.000^2} = 100.000\ (m)$$

$$\alpha_{AB} = \arctan\left(\frac{y_B - y_A}{x_B - x_A}\right) = \arctan\left(\frac{80.000}{60.000}\right) = 53.1301024^0 = 53^0\ 7'\ 48.37''$$

$$\alpha_{BA} = \alpha_{AB} + 180^0 = 233.1301024^0 = 233^0\ 7'\ 48.37''$$

$$\boxed{\alpha^0_{AP} = \alpha_{AB} - \ell_1 = 353^0\ 7'\ 43.35''}$$

$$x^0_P = x_A + \ell_4 \cos\alpha^0_{AP} = x_A + 99.2897\ m = 6\ 500\ 099.2897\ m$$
$$y^0_p = y_A + \ell_4 \sin\alpha^0_{AP} = y_A - 11.9649\ m = 1\ 499\ 988.0351\ m$$

Calculate approximate angles & distances

From	*To*	*Distance* s^0 *(m)*	*Azimuth* α^0 $(^{0\,\prime\,\prime\prime})$	*Remark*
A	B	100.0000	$53^0\ 07'\ 48.37''$	fixed
A	P	100.0080	$353^0\ 07'\ 43.35''$	approximate
B	A	100.0000	$233^0\ 07'\ 48.37''$	fixed
B	P	100.0061	$293^0\ 08'\ 00.18''$	approximate
P	A	100.0080	$173^0\ 07'\ 43.35''$	***Approximate***
P	B	100.0061	$113^0\ 08'\ 00.18''$	approximate

Linearization of a horizontal angle

$$\boxed{(\ell_{ijk} - \beta^0_{ijk}) - \varepsilon_{ijk} = a \cdot \delta x_i + b \cdot \delta y_i + c \cdot \delta x_j + d \cdot \delta y_j + e \cdot \delta x_k + f \cdot \delta y_k}$$

$$a = \frac{\partial \ell_{ijk}}{\partial x_i} = \frac{y_k^0 - y_i^0}{\left(s_{ik}^0\right)^2} - \frac{y_j^0 - y_i^0}{\left(s_{ij}^0\right)^2} = \frac{\sin \alpha_{ik}^0}{s_{ik}^0} - \frac{\sin \alpha_{ij}^0}{s_{ij}^0}$$

$$b = \frac{\partial \ell_{ijk}}{\partial y_i} = -\frac{x_k^0 - x_i^0}{\left(s_{ik}^0\right)^2} + \frac{x_j^0 - x_i^0}{\left(s_{ij}^0\right)^2} = -\frac{\cos \alpha_{ik}^0}{s_{ik}^0} + \frac{\cos \alpha_{ij}^0}{s_{ij}^0}$$

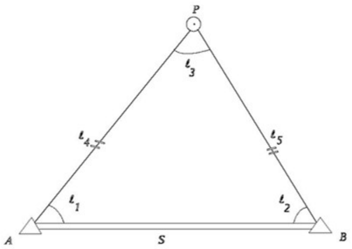
$$c = \frac{\partial \ell_{ijk}}{\partial x_j} = \frac{y_j^0 - y_i^0}{\left(s_{ij}^0\right)^2} = \frac{\sin \alpha_{ij}^0}{s_{ij}^0}$$

$$d = \frac{\partial \ell_{ijk}}{\partial y_j} = -\frac{x_j^0 - x_i^0}{\left(s_{ij}^0\right)^2} = -\frac{\cos \alpha_{ij}^0}{s_{ij}^0}$$

$$e = \frac{\partial \ell_{ijk}}{\partial x_k} = -\frac{y_k^0 - y_i^0}{\left(s_{ik}^0\right)^2} = -\frac{\sin \alpha_{ik}^0}{s_{ik}^0}$$

$$f = \frac{\partial \ell_{ijk}}{\partial y_k} = \frac{x_k^0 - x_i^0}{\left(s_{ik}^0\right)^2} = \frac{\cos \alpha_{ik}^0}{s_{ik}^0}$$

Linearized equations for angles



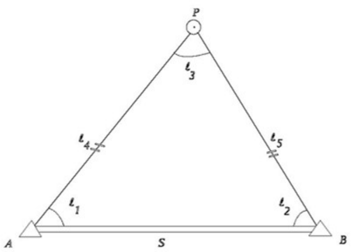
If one of the 2 points is fixed, terms related to that point do *not* exist.

$$\ell_1 - \varepsilon_1 = \left(\alpha_{AB} - \alpha^0_{AP}\right) + \frac{\rho''}{1000}\frac{\sin\alpha^0_{AP}}{s^0_{AP}}\delta x_p - \frac{\rho''}{1000}\frac{\cos\alpha^0_{AP}}{s^0_{AP}}\delta y_p$$

$$\ell_2 - \varepsilon_2 = \left(\alpha^0_{BP} - \alpha_{BA}\right) - \frac{\rho''}{1000}\frac{\sin\alpha^0_{BP}}{s^0_{BP}}\delta x_p + \frac{\rho''}{1000}\frac{\cos\alpha^0_{BP}}{s^0_{BP}}\delta y_p$$

$$\ell_3 - \varepsilon_3 = \left(\alpha^0_{PA} - \alpha^0_{PB}\right) + \frac{\rho''}{1000}\left(\frac{\sin\alpha^0_{PA}}{s^0_{PA}} - \frac{\sin^0_{PB}}{s^0_{PB}}\right)\delta x_p - \frac{\rho''}{1000}\left(\frac{\cos\alpha^0_{PA}}{s^0_{PA}} - \frac{\cos\alpha^0_{PB}}{s^0_{PB}}\right)\delta y_p$$

Linearized equations for angles



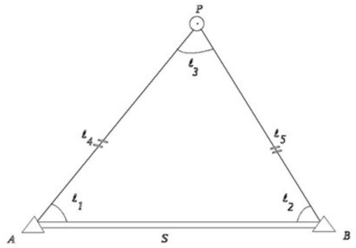
$\boxed{\frac{\rho''}{1000}} \longrightarrow$ coordinate corrections in mm and residuals in ”

$$\ell_1 - \varepsilon_1 = \left(\alpha_{AB} - \alpha^0_{AP}\right) + \boxed{\frac{\rho''}{1000}}\frac{\sin\alpha^0_{AP}}{s^0_{AP}}\delta x_p - \boxed{\frac{\rho''}{1000}}\frac{\cos\alpha^0_{AP}}{s^0_{AP}}\delta y_p$$

$$\ell_2 - \varepsilon_2 = \left(\alpha^0_{BP} - \alpha_{BA}\right) - \boxed{\frac{\rho''}{1000}}\frac{\sin\alpha^0_{BP}}{s^0_{BP}}\delta x_p + \boxed{\frac{\rho''}{1000}}\frac{\cos\alpha^0_{BP}}{s^0_{BP}}\delta y_p$$

$$\ell_3 - \varepsilon_3 = \left(\alpha^0_{PA} - \alpha^0_{PB}\right) + \boxed{\frac{\rho''}{1000}}\left(\frac{\sin\alpha^0_{PA}}{s^0_{PA}} - \frac{\sin^0_{PB}}{s^0_{PB}}\right)\delta x_p - \boxed{\frac{\rho''}{1000}}\left(\frac{\cos\alpha^0_{PA}}{s^0_{PA}} - \frac{\cos\alpha^0_{PB}}{s^0_{PB}}\right)\delta y_p$$

Linearized equations for distances



$$\ell_4 - \varepsilon_4 = s^0_{AP} + \cos\alpha^0_{AP}\delta x_p + \sin\alpha^0_{AP}\delta y_p$$

$$\ell_5 - \varepsilon_5 = s^0_{PB} + \cos\alpha^0_{PB}\delta x_p + \sin\alpha^0_{PB}\delta y_p$$

Numerical values of observation equations

$$\begin{aligned}
0\,'' - \varepsilon_1 &= -0.246\,754\,45\;\delta x_p - 2.047\,670\,68\;\delta y_p \\
-8.81\,'' - \varepsilon_2 &= +1.896\,681\,98\;\delta x_p + 0.810\,309\,87\;\delta y_p \\
14.83\,'' - \varepsilon_3 &= -1.649\,927\,53\;\delta x_p + 1.237\,360\,81\;\delta y_p \\
0\,^{mm} - \varepsilon_4 &= +0.992\,817\,41\;\delta x_p - 0.119\,639\,41\;\delta y_p \\
-9.12\,^{mm} - \varepsilon_5 &= +0.392\,872\,97\;\delta x_p - 0.919\,592\,75\;\delta y_p
\end{aligned}$$

$$\downarrow$$

$$\underset{5\cdot 1}{L} - \underset{5\cdot 1}{\varepsilon} = \underset{5\cdot 2}{A}\,\underset{2\cdot 1}{X}$$

$$\uparrow$$

$$L = \begin{bmatrix} 0 \\ -8.81 \\ +14.83 \\ 0 \\ -9.12 \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix}, \quad A = \begin{bmatrix} -0.246\,754\,45 & -2.047\,670\,68 \\ +1.896\,681\,98 & +0.810\,309\,87 \\ -1.649\,927\,53 & +1.237\,360\,81 \\ +0.992\,817\,41 & -0.119\,639\,41 \\ +0.392\,872\,97 & -0.919\,592\,75 \end{bmatrix}, \quad \delta X = \begin{bmatrix} \delta x_P \\ \delta y_P \end{bmatrix}$$

Calculation of least squares estimates

$$A^\top PA = \begin{bmatrix} +2.735\,173 & -0.479\,909 \\ -0.479\,909 & +2.455\,119 \end{bmatrix}, \quad |A^\top PA| = 6.484\,863$$

$$(A^\top PA)^{-1} = \frac{1}{6.484\,863}\begin{bmatrix} +2.455\,119 & +0.479\,909 \\ +0.479\,909 & +2.735\,173 \end{bmatrix} = \begin{bmatrix} +0.378\,592 & +0.074\,005 \\ +0.074\,005 & +0.421\,778 \end{bmatrix}$$

$$A^\top PL = \begin{bmatrix} -13.890\,199 \\ +11.199\,127 \end{bmatrix}$$

$$\delta\widehat{X} = (A^\top PA)^{-1}A^\top PL = \begin{bmatrix} -4.43 \\ +3.70 \end{bmatrix} \;(mm)$$

$$\widehat{X} = X^0 + \delta\widehat{X} = \begin{bmatrix} 6\,500\,099.2897 \\ 1\,499\,988.0351 \end{bmatrix} + \begin{bmatrix} -0.00443 \\ +0.00370 \end{bmatrix} = \begin{bmatrix} 6500\,099.2853 \\ 1499\,988.0388 \end{bmatrix} \;(m)$$

Error estimates of parameters (coordinates)

$$\widehat{\sigma}_0^2 = \frac{\widehat{\varepsilon}^\top P\widehat{\varepsilon}}{n-m} = \frac{1}{5-2}\sum_{i=1}^{n}\{p_i\widehat{\varepsilon}_i^2\} \approx 18.1885 \;\; mm^2$$

$$\widehat{\sigma}_0 \approx \pm 4.26\; mm$$

$$C_{\widehat{X}\widehat{X}} = \widehat{\sigma}_0^2(A^\top PA)^{-1} = \widehat{\sigma}_0^2 \begin{bmatrix} +0.378\,592 & +0.074\,005 \\ +0.074\,005 & +0.421\,778 \end{bmatrix}$$

$$\sigma_{\widehat{x}_P} = \widehat{\sigma}_0\sqrt{0.378572} \approx \pm 2.62\; mm;$$

$$\sigma_{\widehat{y}_P} = \widehat{\sigma}_0\sqrt{0.421778} \approx \pm 2.77\; mm.$$

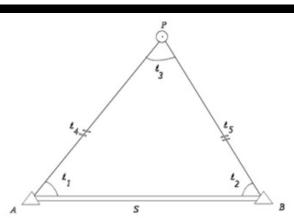
$$\sigma_P = \sqrt{\sigma^2_{\widehat{x}_P} + \sigma^2_{\widehat{y}_P}} = \widehat{\sigma}_0\sqrt{0.378572 + 0.421778} \approx \pm 3.82\; mm$$

Calculation of least squares estimates

$$\hat{\varepsilon} = L - A\,\delta\hat{X} = \begin{bmatrix} +6.45\,'' \\ -3.40\,'' \\ +2.95\,'' \\ +4.82\,^{mm} \\ -3.98\,^{mm} \end{bmatrix}$$

$$\hat{L}' = L' - \hat{\varepsilon} = \begin{bmatrix} 60^0 \;\; 00' \; 05'' \\ 60^0 \;\; 00' \; 03'' \\ 59^0 \;\; 59' \; 58'' \\ 100.008\; m \\ 99.997\; m \end{bmatrix} - \begin{bmatrix} +6.45\,'' \\ -3.40\,'' \\ +2.95\,'' \\ +4.82\,^{mm} \\ -3.98\,^{mm} \end{bmatrix} = \begin{bmatrix} 59^0 \;\; 59' \; 58.55'' \\ 60^0 \;\; 00' \; 06.40'' \\ 59^0 \;\; 59' \; 55.05'' \\ 100.0032\; m \\ 100.0010\; m \end{bmatrix}$$

Error estimates of adjusted measurements



$$C_{\hat{L}\hat{L}} = \hat{\sigma}_0^2 \; A(A^\top P A)^{-1} A^\top =$$

$$= \hat{\sigma}_0^2 \begin{bmatrix} +1.866\,333 & -1.179\,235 & -0.687\,098 & -0.137\,684 & +0.714\,774 \\ -1.179\,235 & +1.866\,364 & -0.687\,129 & +0.714\,765 & -0.137\,698 \\ -0.687\,098 & -0.687\,129 & +1.374\,227 & -0.577\,081 & -0.577\,076 \\ -0.137\,684 & +0.714\,765 & -0.577\,081 & +0.361\,630 & +0.123\,031 \\ +0.714\,774 & -0.137\,698 & -0.577\,076 & +0.123\,031 & +0.361\,639 \end{bmatrix}$$

$$\begin{aligned} \hat{\sigma}_1 &= \hat{\sigma}_0\sqrt{1.866333} \approx \pm 5.83'' \\ \hat{\sigma}_2 &= \hat{\sigma}_0\sqrt{1.866364} \approx \pm 5.83'' \\ \hat{\sigma}_3 &= \hat{\sigma}_0\sqrt{1.374227} \approx \pm 5.00'' \\ \hat{\sigma}_4 &= \hat{\sigma}_0\sqrt{0.361630} \approx \pm 2.56\; mm \\ \hat{\sigma}_5 &= \hat{\sigma}_0\sqrt{0.361639} \approx \pm 2.56\; mm \end{aligned} \qquad \begin{matrix} \longrightarrow \\ \longleftarrow \end{matrix} \qquad \sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{bmatrix} = \begin{bmatrix} \pm 6'' \\ \pm 6'' \\ \pm 6'' \\ \pm 3\; mm \\ \pm 3\; mm \end{bmatrix}$$

Observation equation of the vertical angle

Local 3D measurements:

- ✓ slope distance (3D)
- ✓ horizontal angle (2D)
- ✓ vertical angle

$$\ell_{ij} - \varepsilon_{ij} = arctan\left(\frac{z_j - z_i}{\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}}\right)$$

approximate coordinates $(x_i^0,\ y_i^0,\ z_i^0)$, $(x_j^0,\ y_j^0,\ z_j^0)$ of P_i and P_j

$$\ell_{ij}^0 = arctan\left(\frac{z_j^0 - z_i^0}{\sqrt{(x_j^0 - x_i^0)^2 + \left(y_j^0 - y_i^0\right)^2}}\right)$$

$$(\ell_{ij} - \ell_{ij}^0) - \varepsilon_{ij} = a\ \delta x_i + b\ \delta y_i + c\ \delta z_i + d\ \delta x_j + e\ \delta y_j + f\ \delta z_j$$

Coefficients for vertical angle measurement

$$a = \frac{1}{s_{ij}^{02}}\ \frac{(x_j^0 - x_i^0)\ (z_j^0 - z_i^0)}{\sqrt{(x_j^0 - x_i^0)^2 + \left(y_j^0 - y_i^0\right)^2}}$$

$$b = \frac{1}{s_{ij}^{02}}\ \frac{(y_j^0 - y_i^0)\ (z_j^0 - z_i^0)}{\sqrt{(x_j^0 - x_i^0)^2 + \left(y_j^0 - y_i^0\right)^2}}$$

$$\begin{aligned} c &= \frac{1}{s_{ij}^{02}}\ \sqrt{(x_j^0 - x_i^0)^2 + \left(y_j^0 - y_i^0\right)^2} \\ d &= -a\ , \quad e = -b\ , \quad f = -c \\ s_{ij}^0 &= \sqrt{(x_j^0 - x_i^0)^2 + \left(y_j^0 - y_i^0\right)^2 + (z_j^0 - z_i^0)^2} \end{aligned}$$

GPS code (pseudo-range) measurement

$$s_{ik} = c_0 \cdot \Delta t_{ik} = \sqrt{(x_k - x_i)^2 + (y_k - y_i)^2 + (z_k - z_i)^2}$$

approximate coordinates $(x_i^0,\ y_i^0,\ z_i^0)$ of the receiver P_i

$$s_{ik}^0 = \sqrt{(x_k - x_i^0)^2 + (y_k - y_i^0)^2 + (z_k - z_i^0)^2}$$

$$(\ell_{ik} - s_{ik}^0) - \varepsilon_{ik} = a\ \delta x_i + b\ \delta y_i + c\ \delta z_i + d\ \delta t_{ik}$$

$a = \frac{\partial \ell_{ik}}{\partial x_i} = -\frac{x_k^0 - x_i^0}{s_{ik}^0}$

$b = \frac{\partial \ell_{ik}}{\partial y_i} = -\frac{y_k^0 - y_i^0}{s_{ik}^0}$

$c = \frac{\partial \ell_{ik}}{\partial z_i} = -\frac{z_k^0 - z_i^0}{s_{ik}^0}$

$d = c_0$

GPS phase measurement

$$s_{ik} = \lambda \cdot (\phi_{ik} + N_{ik})$$

$$\phi_{ik} - \varepsilon_{ik} = \frac{1}{\lambda}(s_{ik} + c_0\ \delta t_{ik}) - N_{ik} = \frac{1}{\lambda}\sqrt{(x_k - x_i)^2 + (y_k - y_i)^2 + (z_k - z_i)^2} + \frac{c_0}{\lambda}\ \delta t_{ik} - N_{ik}$$

approximate coordinates $(x_i^0,\ y_i^0,\ z_i^0)$ of the receiver P_i

$$\phi_{ik}^0 = \tfrac{1}{\lambda} s_{ik}^0 = \tfrac{1}{\lambda}\sqrt{(x_k - x_i^0)^2 + (y_k - y_i^0)^2 + (z_k - z_i^0)^2}$$

$$(\phi_{ik} - \phi_{ik}^0) - \varepsilon_{ik} = a\ \delta x_i + b\ \delta y_i + c\ \delta z_i + d\ \delta t_{ik} + e\ N_{ik}$$

$a = \frac{\partial \ell_{ik}}{\partial x_i} = -\frac{x_k^0 - x_i^0}{s_{ik}^0}\ \frac{1}{\lambda}$

$b = \frac{\partial \ell_{ik}}{\partial y_i} = -\frac{y_k^0 - y_i^0}{s_{ik}^0}\ \frac{1}{\lambda}$

$c = \frac{\partial \ell_{ik}}{\partial z_i} = -\frac{z_k^0 - z_i^0}{s_{ik}^0}\ \frac{1}{\lambda}$

$d = \frac{1}{\lambda} c_0$

$e = -1$