Time series modelling in time domain

- Static, sequential adjustment
  - Adjustment by elements, with pseudo-observations
  - Adjustment by elements in two steps
  - Sequential adjustment by elements
- Dynamic system and Kalman filtering
  - Mathematical model of dynamic systems
  - Discrete Kalman filtering

Adjustment with pseudo-observations

- Observation equations:

$$\underset{n\cdot 1}{L} - \underset{n\cdot 1}{\varepsilon} = \underset{n\cdot m}{A}\ \underset{m\cdot 1}{X}$$

$$E(\varepsilon) = 0\,, \quad E(\varepsilon\varepsilon^\top) = C_{\varepsilon\varepsilon} = \sigma_0^2\ P^{-1}$$

- Stochastic parameters with a priori information:

$$E\,(X) = \mu_x\,,$$

$$E\left[(X-\mu_x)(X-\mu_x)^\top\right] = C_{XX} = \sigma_0^2\ P_x^{-1}$$

- Pseudo-observations:

$$L_x - \varepsilon_x = X \qquad\qquad L_x = \mu_x$$

$$E\left[\varepsilon_x\right] = 0, \quad E\left[\varepsilon_x\varepsilon_x^\top\right] = C_{XX} = \sigma_0^2 \cdot P_x^{-1}$$

$$E\left[\varepsilon\varepsilon_x^\top\right] = C_{\varepsilon X} = 0$$

Adjustment with pseudo-observations

- Joint observation equations: $\begin{bmatrix} \underset{n\cdot 1}{L} \\ \underset{m\cdot 1}{L_x} \end{bmatrix} - \begin{bmatrix} \underset{n\cdot 1}{\varepsilon} \\ \underset{m\cdot 1}{\varepsilon_x} \end{bmatrix} = \begin{bmatrix} \underset{n\cdot m}{A} \\ \underset{m\cdot m}{I} \end{bmatrix} \underset{m\cdot 1}{X}$
- Joint error vector: $V = \begin{bmatrix} \varepsilon \\ \varepsilon_x \end{bmatrix}, \quad E(V) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$
- Joint variance-covariance matrix:

$$C_{VV} = E\left(VV^{\top}\right) = \begin{bmatrix} C_{\varepsilon\varepsilon} & C_{\varepsilon X} \\ C_{X\varepsilon} & C_{XX} \end{bmatrix} = \begin{bmatrix} \sigma_0^2 P^{-1} & 0 \\ 0 & \sigma_0^2 P_x^{-1} \end{bmatrix} = \sigma_0^2\, P_v^{-1}$$

- Joint weight matrix: $P_v = \begin{bmatrix} P & 0 \\ 0 & P_x \end{bmatrix}$

Least squares estimates

$$V^{\top} P_v\, V = \varepsilon^{\top} P \varepsilon + \varepsilon_x^{\top} P_x \varepsilon_x = minimum$$

$$\widehat{X} = \left\{ \begin{bmatrix} A \\ I \end{bmatrix}^{\top} \begin{bmatrix} P & 0 \\ 0 & P_x \end{bmatrix} \begin{bmatrix} A \\ I \end{bmatrix} \right\}^{-1} \begin{bmatrix} A \\ I \end{bmatrix}^{\top} \begin{bmatrix} P & 0 \\ 0 & P_x \end{bmatrix} \begin{bmatrix} L \\ \mu_x \end{bmatrix}$$

$$= \mu_x + (A^{\top} P A + P_x)^{-1} A^{\top} P (L - A\mu_x)$$

$$\begin{bmatrix} \widehat{\varepsilon} \\ \widehat{\varepsilon}_x \end{bmatrix} = \begin{bmatrix} L \\ \mu_x \end{bmatrix} - \begin{bmatrix} A \\ I \end{bmatrix} \widehat{X} = \begin{bmatrix} L - A(A^{\top} P A + P_x)^{-1}(A^{\top} P L + P_x \mu_x) \\ \mu_x - (A^{\top} P A + P_x)^{-1}(A^{\top} P L + P_x \mu_x) \end{bmatrix}$$

$$\widehat{\sigma}_0^2 = \frac{\widehat{\varepsilon}^{\top} P \widehat{\varepsilon} + \widehat{\varepsilon}_x^{\top} P_x \widehat{\varepsilon}_x}{n}$$

Adjustment with pseudo-observations

$$C_{\hat{X}\hat{X}} = (A^\top PA + P_x)^{-1} \begin{bmatrix} A^\top P & P_x \end{bmatrix} \hat{\sigma}_0^2 \begin{bmatrix} P^{-1} & 0 \\ 0 & P_x^{-1} \end{bmatrix} \begin{bmatrix} PA \\ P_x \end{bmatrix} (A^\top PA + P_x)^{-1}$$

$$= \hat{\sigma}_0^2 \, (A^\top PA + P_x)^{-1} = \hat{\sigma}_0^2 \cdot P_{\hat{X}}^{-1}$$

$$P_{\hat{X}} = A^\top PA + P_x$$

$$\underset{n\cdot 1}{L} - \underset{n\cdot 1}{\varepsilon} = \underset{n\cdot m}{A} \; \underset{m\cdot 1}{X}$$

$$\hat{X} = \mu_x + (A^\top PA + P_x)^{-1} A^\top P (L - A\mu_x)$$

$$P_{\hat{X}} = A^\top PA + P_x$$

Sequential adjustment in two groups

$$\begin{bmatrix} \underset{r\cdot 1}{L_1} \\ \underset{s\cdot 1}{L_2} \end{bmatrix} - \begin{bmatrix} \underset{r\cdot 1}{\varepsilon_1} \\ \underset{s\cdot 1}{\varepsilon_2} \end{bmatrix} = \begin{bmatrix} \underset{r\cdot m}{A_1} \\ \underset{s\cdot m}{A_2} \end{bmatrix} \underset{m\cdot 1}{X}$$

$$\underset{n\cdot 1}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}, \quad E(\varepsilon) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C_{\varepsilon\varepsilon} = E(\varepsilon\varepsilon^\top) = \sigma_0^2 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}^{-1}$$

- Least squares estimate using one the first group:

$$\hat{X}^{(1)} = (A_1^\top P_1 A_1)^{-1} A_1^\top P_1 L_1$$

$$C_{\hat{X}\hat{X}}^{(1)} = \sigma_0^2 \, (A_1^\top P_1 A_1)^{-1} = \sigma_0^2 \left(P_x^{(1)}\right)^{-1}$$

Adjustment in two groups

$$\widehat{X}^{(2)} = \widehat{X}^{(1)} + \left[A_2^\top P_2 A_2 + P_x^{(1)}\right]^{-1} A_2^\top P_2 \left[L_2 - A_2 \widehat{X}^{(1)}\right]$$

$$= \widehat{X}^{(1)} + K_2 \left[L_2 - A_2 \widehat{X}^{(1)}\right]$$

$$K_2 = \left[A_2^\top P_2 A_2 + P_x^{(1)}\right]^{-1} A_2^\top P_2$$ (Kalman gain)

$$P_x^{(2)} = A_2^\top P_2 A_2 + P_x^{(1)}$$

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Proof of equivalence

$$\begin{bmatrix} \underset{r\cdot 1}{L_1} \\ \underset{s\cdot 1}{L_2} \end{bmatrix} - \begin{bmatrix} \underset{r\cdot 1}{\varepsilon_1} \\ \underset{s\cdot 1}{\varepsilon_2} \end{bmatrix} = \begin{bmatrix} \underset{r\cdot m}{A_1} \\ \underset{s\cdot m}{A_2} \end{bmatrix} \underset{m\cdot 1}{X}$$

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^\top \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \widehat{X} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^\top \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} \underset{r\cdot 1}{L_1} \\ \underset{s\cdot 1}{L_2} \end{bmatrix}$$

$$\left(A_1^\top P_1 A_1 + A_2^\top P_2 A_2\right) \widehat{X} = A_1^\top P_1 L_1 + A_2^\top P_2 L_2$$

$$\widehat{X} = (A_1^\top P_1 A_1 + A_2^\top P_2 A_2)^{-1} \left[A_1^\top P_1 L_1 + A_2^\top P_2 L_2\right]$$

$$= (A_1^\top P_1 A_1 + A_2^\top P_2 A_2)^{-1} \left[(A_1^\top P_1 A_1)\,(A_1^\top P_1 A_1)^{-1} A_1^\top P_1 L_1 + A_2^\top P_2 L_2\right]$$

$$= (A_1^\top P_1 A_1 + A_2^\top P_2 A_2)^{-1} \left[(A_1^\top P_1 A_1 + A_2^\top P_2 A_2 - A_2^\top P_2 A_2)\,\widehat{X}^{(1)} + A_2^\top P_2 L_2\right]$$

$$= \widehat{X}^{(1)} + \left[A_2^\top P_2 A_2 + P_x^{(1)}\right]^{-1} A_2^\top P_2 \left[L_2 - A_2 \widehat{X}^{(1)}\right] = \widehat{X}^{(1)} + K_2 \left[L_2 - A_2 \widehat{X}^{(1)}\right]$$

Sequential adjustment in q groups

$$\begin{bmatrix} L_1 \\ L_2 \\ \cdots \\ L_q \end{bmatrix} - \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdots \\ \varepsilon_q \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ \cdots \\ A_q \end{bmatrix} X$$

$$\underset{n_k\cdot 1}{L_k} - \underset{n_k\cdot 1}{\varepsilon_k} = \underset{n_k\cdot m}{A_k} \underset{m\cdot 1}{X}, \quad \left(1 \le k \le q, \sum_{k=1}^{q} n_k = n\right)$$

$\widehat{X}^{(k)}$: least squares estimate of X based on the first k groups of observations $L_1, L_2, \cdots, L_k$

$C_{\widehat{X}\widehat{X}}^{(k)}$: variance-covariance matrix of $\widehat{X}^{(k)}$

$P_x^{(k)}$: weight matrix of $\widehat{X}^{(k)}$.

Sequential adjustment

The adjustment of $L_1, L_2, \cdots, L_{q-1}, L_q$ can be made sequentially at q steps. At the first step ($k = 1$), the observations in L_1 are adjusted in the usual way, leading to the first least squares estimate $\widehat{X}^{(1)}$, $P_x^{(1)}$. At step k ($k = 2, 3, 4, \cdots, q$), L_k is adjusted using the previous least squares estimate $\widehat{X}^{(k-1)}$ and its weight matrix $P_x^{(k-1)}$ (estimated at step $k-1$) as the a priori information for parameters X to produce the least squares estimate $\widehat{X}^{(k)}$, $P_x^{(k)}$ (based on observations $L_1, L_2, \cdots, L_k$).

$$\widehat{X}^{(1)} = (A_1^\top P_1 A_1)^{-1} A_1^\top P_1 L_1$$

$$C_{\widehat{X}\widehat{X}}^{(1)} = \sigma_0^2 \, (A_1^\top P_1 A_1)^{-1} = \sigma_0^2 \left(P_x^{(1)}\right)^{-1}$$

$$P_x^{(1)} = A_1^\top P_1 A_1$$

$$\widehat{X}^{(k)} = \widehat{X}^{(k-1)} + K_k \left(L_k - A_k \widehat{X}^{(k-1)}\right)$$

$$P_x^{(k)} = A_k^\top P_k A_k + P_x^{(k-1)}$$

$$K_k = \left(A_k^\top P_k A_k + P_x^{(k-1)}\right)^{-1} A_k^\top P_k$$

$$= \left(P_x^{(k-1)}\right)^{-1} A_k^\top \left[(P_k)^{-1} + A_k \left(P_x^{(k-1)}\right)^{-1} A_k^\top\right]^{-1}$$

Mathematical Model of a dynamic system

$$\underset{m\cdot 1}{\dot{X}}(t) = \underset{m\cdot m}{G}(t)\ \underset{m\cdot 1}{X}(t) + \underset{m\cdot q}{H}(t)\ \underset{q\cdot 1}{v}(t)$$

$X(t)$ = state vector of the dynamical system

$\dot{X}(t)$ = time derivative of the state vector
$G(t)$ = transition matrix of the state vector
$H(t)$ = transition matrix of the system noise vector
$v(t)$ = system noise vector

$$E\{v(t)\} = 0, \quad E\{v(t_1)v(t_2)^\top\} = \underset{q\cdot q}{D}(t)\cdot\delta(t_2 - t_1)$$

$$\delta(\tau) = 0 \quad \text{for } \tau \neq 0, \quad \int_{-\infty}^{+\infty} \delta(\tau)d\tau = 1$$

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Observation equations

$$\underset{n\cdot 1}{L}(t) = \underset{n\cdot m}{A}(t)\ \underset{m\cdot 1}{X}(t) + \underset{n\cdot 1}{\varepsilon}(t)$$

$L(t)$ = observation vector (time dependent in general)
$A(t)$ = design matrix
$\varepsilon(t)$ = observation error vector

$$E\{\varepsilon(t)\} = 0, \quad E\{\varepsilon(t_1)\varepsilon(t_2)^\top\} = \underset{n\cdot n}{C}(t)\cdot\delta(t_2 - t_1)$$

$$E\{v(t_1)\varepsilon(t_2)^\top\} = \underset{q\cdot n}{0}$$

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Discrete models

$$\underset{m\cdot 1}{X_k} = \underset{m\cdot m}{G_{k,k-1}} \cdot \underset{m\cdot 1}{X_{k-1}} + \underset{m\cdot q}{H_k} \cdot \underset{q\cdot 1}{v_k} \qquad (k = 1, 2, 3, \cdots)$$

$$\underset{n\cdot 1}{L_k} = \underset{n\cdot m}{A_k} \cdot \underset{m\cdot 1}{X_k} + \underset{n\cdot 1}{\varepsilon_k} \qquad (k = 1, 2, 3, \cdots)$$

$$E(v_k) = 0$$

$$E(\varepsilon_k) = 0$$

$$E\left(v_k v_j^\top\right) = \underset{q\cdot q}{D_k} \cdot \delta_{kj}$$

$$E\left(\varepsilon_k \varepsilon_j^\top\right) = \underset{n\cdot n}{C_k} \cdot \delta_{kj}$$

$$E\left(v_k \varepsilon_j^\top\right) = 0$$

$$\delta_{kj} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

Kalman filtering: step 1 ($k=1$)

- Prediction of state vectors from initial values:

$$\widehat{X}_{1,0} = G_{1,0} \cdot \widehat{X}_{0,0}$$

$$P^x_{1,0} = \left(G_{1,0} Q_{0,0} G_{1,0}^\top + H_1 D_1 H_1^\top\right)^{-1}$$

- Filtering using observations L_1

$$\widehat{X}_{1,1} = \widehat{X}_{1,0} + \left(A_1^\top P_1 A_1 + P^x_{1,0}\right)^{-1} A_1^\top P_1 \left(L_1 - A_1 \widehat{X}_{1,0}\right)$$

$$= \widehat{X}_{1,0} + \left[A_1^\top C_1^{-1} A_1 + (Q_{1,0})^{-1}\right]^{-1} A_1^\top (Q_{1,0})^{-1} \left(L_1 - A_1 \widehat{X}_{1,0}\right)$$

$$P^x_{1,1} = A_1^\top P_1 A_1 + P^x_{1,0}$$

Kalman filtering: step 2 ($k=2$)

- Prediction from previous values:

$$X_1 = G_{1,0} \cdot X_0 + H_1 \cdot v_1$$

$$\widehat{X}_{2,1} = G_{2,1}\widehat{X}_{1,1}$$

$$P^x_{2,1} = \left(G_{2,1}Q_{1,1}G^\top_{2,1} + H_2 D_2 H^\top_2\right)^{-1}$$

- Filtering using observations L_2:

$$L_2 - \varepsilon_2 = A_2 X_2$$
$$C_2 = \sigma_0^2 \cdot P_2^{-1}$$

$$\widehat{X}_{2,2} = \widehat{X}_{2,1} + \left(A_2^\top P_2 A_2 + P^x_{2,1}\right)^{-1} A_2^\top P_2 \left(L_2 - A_2\widehat{X}_{2,1}\right)$$

$$P^x_{2,2} = A_2^\top P_2 A_2 + P^x_{2,1}$$

Kalman filtering: step k ($k=3,4,5\ldots$)

- Prediction from previous values:

$$X_k = G_{k,k-1} \cdot X_{k-1} + H_k \cdot v_k$$

$$\widehat{X}_{k,k-1} = G_{k,k-1} \cdot \widehat{X}_{k-1,k-1}$$

$$P^x_{k,k-1} = \left(G_{k,k-1}Q_{k,k-1}\, G^\top_{k,k-1} + H_k D_k H^\top_k\right)^{-1}$$

- Filtering using observations L_k

$$L_k = A_k \cdot X_k + \varepsilon_k$$

$$\widehat{X}_{k,k} = \widehat{X}_{k,k-1} + K_k\left[L_k - A_k\widehat{X}_{k,k-1}\right]$$

$$K_k = \left[A_k^\top C_k^{-1} A_k + (Q_{k,k-1})^{-1}\right]^{-1} A_k^\top (Q_{k,k-1})^{-1} = \left(A_k^\top P_k A_k + P^x_{k,k-1}\right)^{-1} A_k^\top P_k$$

Sequential adjustment vs Kalman filtering

$$X_k = G_{k,k-1} \cdot X_{k-1} + H_k \cdot v_k$$
$$L_k = A_k \cdot X_k + \varepsilon_k$$

$$G_{k,k-1} = I \quad \text{and} \quad H_k = 0$$

$$L_k = A_k \cdot X_k + \varepsilon_k$$