Time series modelling in time domain

- Moving average
- Exponential smoothing
- Autocorrelation plot
- Autoregression model

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Time series analysis

- Characteristics of time series:
  - Trends or other systematic variations
  - Periodic oscillations: with fixed or varing patterns
  - Irregularity: unexpected situations or events or scenarios, spikes in short time span
- Stationary time series (*stationary stochastic process*):
  - Mean, variance, covariances will not change with time
  - White noise: uniform uncorrelated normally distributed variations
- Modelling of time series
  - Reveal the characteristics or structure of the series
  - Facilitate prediction
  - Handle accompanying noises

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Moving average

Original time series: $x_1,\ x_2,\ x_3,\ \ldots,\ x_t,\ \ldots,\ x_N$

Smoothed time series: $y_2,\ y_3,\ \ldots,\ y_t,\ \ldots,\ y_N$

Simple moving average:

$$y_t = \frac{1}{M}\sum_{i=1}^{M} x_{t-i}$$

Centered moving average:
(M is odd)

$$y_t = \frac{1}{M}\sum_{i=-M2}^{M2} x_{t-i} \qquad M2 = \frac{M-1}{2}$$

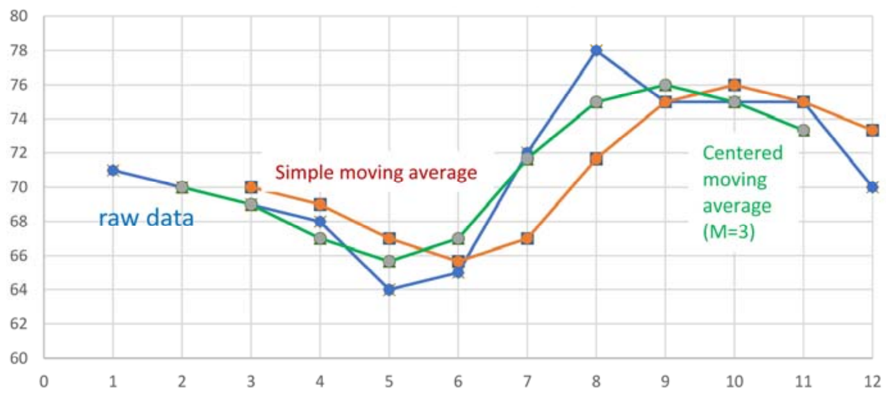
Centered moving average:
(M is even)

$$y_t = \frac{1}{M}\sum_{i=-M2}^{M2} x_{t-i}\ , \quad M2 = \frac{M}{2}$$

Example of moving average

t	x(t)	Simple averaging			Simple moving averaging			Centered moving averaging (M=3)		
		smoothed	error	error squared	smoothed	error	error squared	smoothed	error	error squared
1	**71**	71	0	0						
2	**70**	71	-1	1				70,00	0,00	0,00
3	**69**	71	-2	4	70,00	-1,00	1,00	69,00	0,00	0,00
4	**68**	71	-3	9	69,00	-1,00	1,00	67,00	1,00	1,00
5	**64**	71	-7	49	67,00	-3,00	9,00	65,67	-1,67	2,78
6	**65**	71	-6	36	65,67	-0,67	0,44	67,00	-2,00	4,00
7	**72**	71	1	1	67,00	5,00	25,00	71,67	0,33	0,11
8	**78**	71	7	49	71,67	6,33	40,11	75,00	3,00	9,00
9	**75**	71	4	16	75,00	0,00	0,00	76,00	-1,00	1,00
10	**75**	71	4	16	76,00	-1,00	1,00	75,00	0,00	0,00
11	**75**	71	4	16	75,00	0,00	0,00	73,33	1,67	2,78
12	**70**	71	-1	1	73,33	-3,33	11,11			
SUM=	852			198	709,67		88,67	709,67		20,67
Average=	71	Root of mean square error =		**4,06**	70,967	MSE=	**2,72**	70,967	MSE=	**1,31**

Example of moving average



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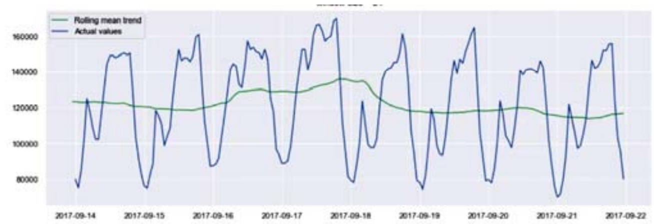
Example of simple moving average

Example of a moving average on a 12h window



Example of a moving average on a 24h window

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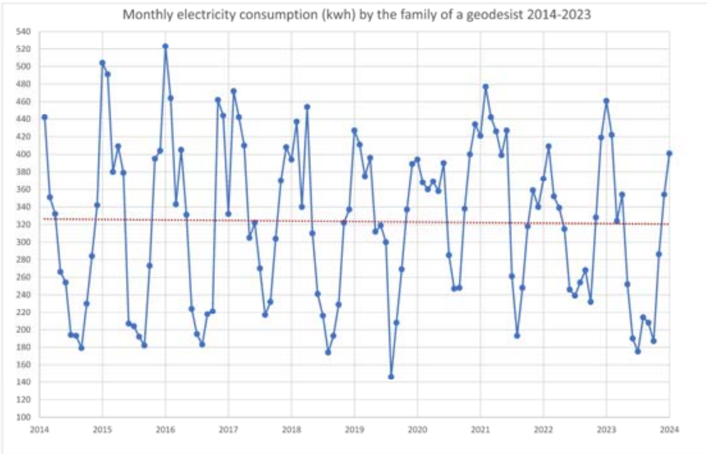
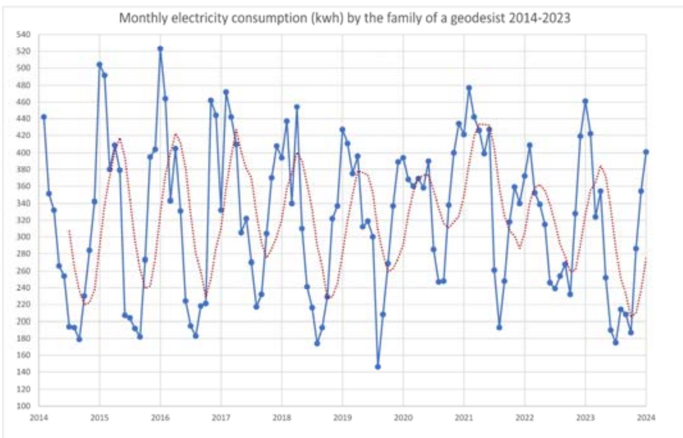
Example of simple moving average



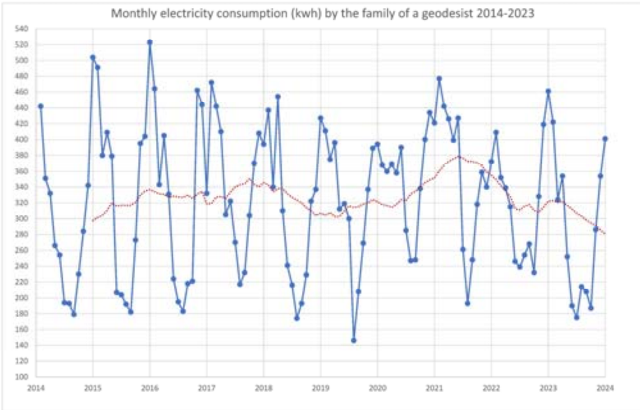
Example of simple moving average

M = 6 months



Example of simple moving average

M = 12 months



Exponential smoothing

Original time series $x_t, \quad t = 1,\ 2,\ 3,\ ...,\ N$

New smoothed time series: $y_t, \quad t = 2,\ 3,\ 4,\ ...,\ N$

$$y_2 = x_1$$

$$\boxed{y_t = \alpha\ x_{t-1} + (1-\alpha)\ y_{t-1} \qquad 0 < \alpha < 1 \quad t \geq 3}$$

$$\begin{aligned} y_3 &= \alpha\ x_2 + (1-\alpha)\ y_2 \\ y_4 &= \alpha\ x_3 + (1-\alpha)\ y_3 \\ y_5 &= \alpha\ x_4 + (1-\alpha)\ x_4 \\ &\ldots\ldots \end{aligned}$$

Exponential smoothing

$$y_t = \alpha \sum_{i=1}^{t-2} \left[(1-\alpha)^{i-1} x_{t-i} \right] + (1-\alpha)^{t-2} x_1 \,, \qquad t \geq 2$$

Averaging/smoothing: sum of all coeeficients is equal to **1**

$$\alpha \sum_{i=1}^{t-2} (1-\alpha)^{i-1} + (1-\alpha)^{t-2}$$

$$= \alpha \sum_{i=0}^{t-3} (1-\alpha)^{i-1} + (1-\alpha)^{t-2}$$

$$\boxed{\sum_{i=0}^{n} z^i = \frac{1-z^{n+1}}{1-z}}$$

$$= \alpha \frac{1-(1-\alpha)^{t-3+1}}{1-(1-\alpha)} + (1-\alpha)^{t-2} = 1$$

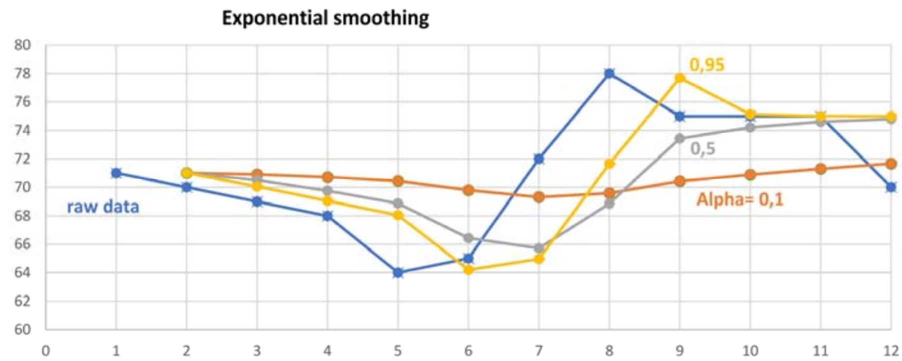
Examples of exponential smoothing

t	x(t)	Alpha = 0,1			Alpha=0,3			Alpha=0,5		
		y(t)	error	error squared	y(t)	error	error squared	y(t)	error	error squared
1	**71**									
2	**70**	71,00	-1,00	1,00	71,00	-1,00	1,00	71,00	-1,00	1,00
3	**69**	70,90	-1,90	3,61	70,70	-1,70	2,89	70,50	-1,50	2,25
4	**68**	70,71	-2,71	7,34	70,19	-2,19	4,80	69,75	-1,75	3,06
5	**64**	70,44	-6,44	41,46	69,53	-5,53	30,61	68,88	-4,88	23,77
6	**65**	69,80	-4,80	22,99	67,87	-2,87	8,25	66,44	-1,44	2,07
7	**72**	69,32	2,68	7,21	67,01	4,99	24,89	65,72	6,28	39,45
8	**78**	69,58	8,42	70,83	68,51	9,49	90,10	68,86	9,14	83,55
9	**75**	70,43	4,57	20,92	71,36	3,64	13,28	73,43	1,57	2,47
10	**75**	70,88	4,12	16,95	72,45	2,55	6,51	74,21	0,79	0,62
11	**75**	71,29	3,71	13,73	73,21	1,79	3,19	74,61	0,39	0,15
12	**70**	71,67	-1,67	2,77	73,75	-3,75	14,06	74,80	-4,80	23,08
SUM=	852			208,82			199,59			181,46
Average=	71	Root of mean square error =		**4,36**			**4,26**		MSE=	**4,06**

Examples of exponential smoothing

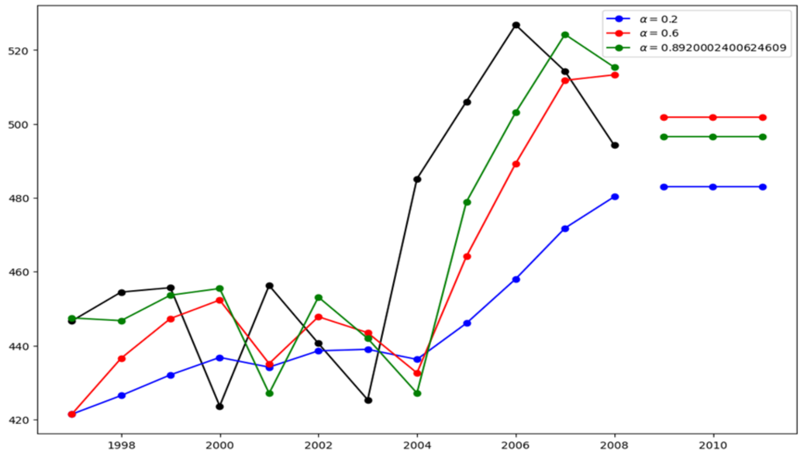
t	x(t)	Alpha = 0,75 y(t)	error	error squared	Alpha=0,95 y(t)	error	error squared	Alpha=0,99 y(t)	error	error squared
1	**71**									
2	**70**	71,00	-1,00	1,00	71,00	-1,00	1,00	71,00	-1,00	1,00
3	**69**	70,25	-1,25	1,56	70,05	-1,05	1,10	70,01	-1,01	1,02
4	**68**	69,31	-1,31	1,72	69,05	-1,05	1,11	69,01	-1,01	1,02
5	**64**	68,33	-4,33	18,73	68,05	-4,05	16,42	68,01	-4,01	16,08
6	**65**	65,08	-0,08	0,01	64,20	0,80	0,64	64,04	0,96	0,92
7	**72**	65,02	6,98	48,71	64,96	7,04	49,56	64,99	7,01	49,13
8	**78**	70,26	7,74	59,98	71,65	6,35	40,35	71,93	6,07	36,85
9	**75**	76,06	-1,06	1,13	77,68	-2,68	7,20	77,94	-2,94	8,64
10	**75**	75,27	-0,27	0,07	75,13	-0,13	0,02	75,03	-0,03	0,00
11	**75**	75,07	-0,07	0,00	75,01	-0,01	0,00	75,00	0,00	0,00
12	**70**	75,02	-5,02	25,17	75,00	-5,00	25,00	75,00	-5,00	25,00
SUM=	852			158,09			142,39			139,66
Average=	71	Root of mean square error =		**3,79**			**3,60**			**3,56**

Examples of exponential smoothing

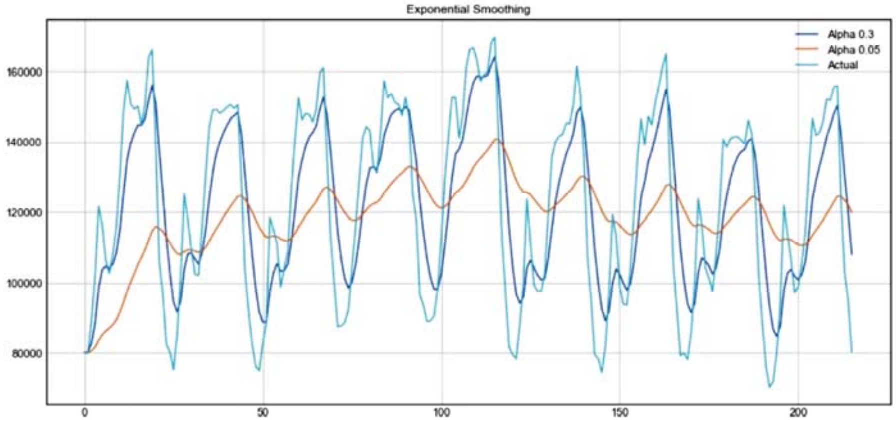


Examples of exponential smoothing

Saudi Arabian oil production in million tones



Examples of exponential smoothing



Autocorrelation

Data points at different time epochs are correlated

$x_t, \quad t = 1,\ 2,\ 3,\ \ldots,\ N$

Mean of a time series: $\overline{x} = \frac{1}{N}\sum_{t=1}^{N} x_i$

Variance of a time series: $\gamma_0 = \frac{1}{N}\sum_{t=1}^{N}(x_t - \overline{x})^2$

Autocovariance function for time lag τ :

$$\boxed{\gamma_\tau = \frac{1}{N}\sum_{t=1}^{N-\tau}\left[(x_t - \overline{x})\ (x_{t+\tau} - \overline{x})\right], \quad \tau = 1, 2, 3, \ \ldots\ldots}$$

Autocorrelation coefficient: $\rho(\tau) = \frac{\gamma_\tau}{\gamma_0}$

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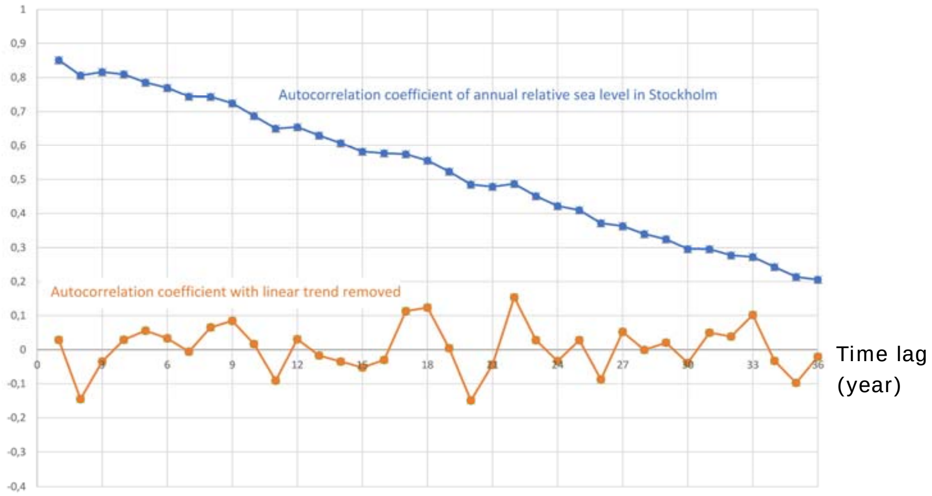
Autocorrelation coefficients of relative sea level in Stockholm



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Autoregression (AR) model

$x_t, \quad t = 1,\ 2,\ 3,\ \ldots,\ N$

$E(x_t) = \mu, \quad Var(x_t) = \sigma^2,$

$E(\varepsilon_t) = 0, \quad Var(\varepsilon_t) = \sigma_\varepsilon^2$

$$\boxed{x_t = \sum_{i=1}^{p} \alpha_i x_{t-i} + \varepsilon_t}$$

Coefficients α_i can be determined via least squares adjustment,

or via Yue-Walker equation:

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \cdots \\ \gamma_p \end{bmatrix} = \begin{bmatrix} \gamma_0 & \gamma_{-1} & \gamma_{-2} & \cdots \\ \gamma_1 & \gamma_0 & \gamma_{-1} & \cdots \\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots \\ \cdots\cdot & \cdots & \cdots & \cdots \\ \gamma_{p-1} & \gamma_{p-2} & \gamma_{p-3} & \cdots \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_p \end{bmatrix}$$

AR model for $p = 1, 2$

$p = 1$

$$x_t = \alpha\ x_{t-1} + \varepsilon_t$$

$$\gamma_1 = \gamma_0\ \alpha_1 \quad \rightarrow \quad \alpha_1 = \frac{\gamma_1}{\gamma_0} = \rho(1)$$

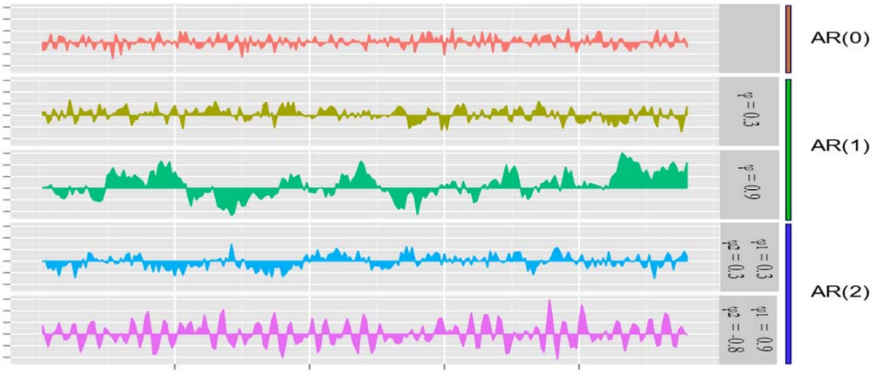
$$\sigma^2 = \alpha^2\ \sigma^2 + \sigma_\varepsilon^2 \quad \rightarrow \quad \sigma^2 = \frac{\sigma_\varepsilon^2}{1-\alpha^2}$$

$p = 2$

$$x_t = \alpha_1\ x_{t-1} + \alpha_2\ x_{t-2} + \varepsilon_t$$

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} \gamma_0 & \gamma_{-1} \\ \gamma_1 & \gamma_0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

Examples of AR for 0, 1, 2



p=0, only white noise

p=1, if α is close to 1, mainly white noise, if α is close to 1 it is smoothing

p=2, if two αs are positive, high frequency noise is reduced; if two αs have opposite signs, output oscillates and change directions

Other types of time series models

- Autoregression moving average model (ARMA)

$$x_t = \sum_{i=1}^{p} \alpha_i x_{t-i} + \sum_{i=1}^{q} \beta_i \, \varepsilon_{t-i}$$

- Autoregression integrated moving average model (ARIMA)
- Double (triple) exponential smoothing
- Box-jenkins model