2. Error propagation

If the variance-covariance matrix (VCM) of n measurements is known,
how can we calculate the VCM
for a set of functions of these n measurements ?

- one linear function of n measurements
  - n uncorrelated (independent) measurements
- a vector of m linear functions
- one non-linear function
  - n uncorrelated (independent) measurements
- **one vector of m non-linear functions**
- Propagation of co-factor matrices

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Notations

- n measurements (observed values): $x_1, x_2, \cdots, x_n$
- their true errrors: $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$
- … observed values – true values $\varepsilon_i = x_i - \tilde{x}_i \qquad (1 \le i \le n)$

$$\underset{n\cdot 1}{\varepsilon} = \underset{n\cdot 1}{x} - \underset{n\cdot 1}{\tilde{x}}$$

$$\underset{n\cdot 1}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdots \\ \varepsilon_n \end{bmatrix}, \quad \underset{n\cdot 1}{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}, \quad \underset{n\cdot 1}{\tilde{x}} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \cdots \\ \tilde{x}_n \end{bmatrix}$$

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Error properties of n measurements

$$E(\varepsilon) = E\left\{\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdots \\ \varepsilon_n \end{bmatrix}\right\} = \begin{bmatrix} E(\varepsilon_1) \\ E(\varepsilon_2) \\ \cdots \\ E(\varepsilon_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} = \underset{n\cdot 1}{0}$$

$$\underset{n\cdot n}{C_{\varepsilon\varepsilon}} = E\left\{[\varepsilon - E(\varepsilon)]\,[\varepsilon - E(\varepsilon)]^\top\right\} = E\left\{\varepsilon\varepsilon^\top\right\} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix}$$

$$E\{x\} = E\{\widetilde{x} + \varepsilon\} = E\{\widetilde{x}\} = \widetilde{x}$$

$$C_{xx} = E\left\{[x - E(x)][x - E(x)]^\top\right\} = E[\varepsilon\varepsilon^\top] = \underset{n\cdot n}{C_{\varepsilon\varepsilon}}$$

$$\underset{n\cdot n}{C_{xx}} = \underset{n\cdot n}{C_{\varepsilon\varepsilon}} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix}$$

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Error propagation for one linear function

$$y = a_1x_1 + a_2x_2 + \cdots + a_nx_n = a \cdot x$$

$$\underset{1\cdot n}{a} = (\, a_1, a_2, \cdots, a_n)$$

$$\widetilde{y} = a_1\widetilde{x}_1 + a_2\widetilde{x}_2 + \cdots + a_n\widetilde{x}_n = a \cdot \widetilde{x}$$

$$\varepsilon_y = y - \widetilde{y} = ax - a\widetilde{x} = a \cdot \varepsilon$$

$$\underset{n\cdot 1}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdots \\ \varepsilon_n \end{bmatrix}$$

$$\sigma_y^2 = E\left[\varepsilon_y^2\right] = E\left[a\varepsilon \cdot (a\varepsilon)^\top\right] = a \cdot E\left[\varepsilon\varepsilon^\top\right] \cdot a^\top = a\, C_{xx} a^\top$$

$$= (\, a_1, a_2, \cdots, a_n) \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \end{bmatrix}$$

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Error propagation law for uncorrelated observations

$\sigma_{ij} = 0$ for $i \neq j$; C_{xx} becomes a diagonal matrix

$$y = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = a \cdot x$$

$$\boxed{\sigma_y^2 = a_1^2 \cdot \sigma_1^2 + a_2^2 \cdot \sigma_2^2 + \cdots + a_n^2 \cdot \sigma_n^2}$$

Example 1.2

Given: $y = 2\,x_1 - 3\,x_2 - x_3$ and $\sigma_1 = 1\;mm$, $\sigma_2 = 2\;mm$, $\sigma_3 = 3\;mm$; ε_1, ε_2, ε_3 are independent of each other.
Sought: $\sigma_y = ?$

$$\sigma_y^2 = (+2)^2 1^2 + (-3)^2 2^2 + (-1)^2 3^2 = 4 + 36 + 9 = 49\;mm^2\,, \quad \text{or} \quad \sigma_y = \sqrt{49} = 7\;mm$$

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Error propagation for a vector

$$\underset{m\cdot 1}{y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \underset{m\cdot n}{A}\;\underset{n\cdot 1}{x}$$

$$\underset{m\cdot n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\underset{m\cdot m}{C_{yy}} = E\left\{[y - E(y)][y - E(y)]^\top\right\} = A\;E\left\{[x - E(x)][x - E(x)]^\top\right\} A^\top =$$

$$\boxed{= A\;C_{xx} A^\top}$$

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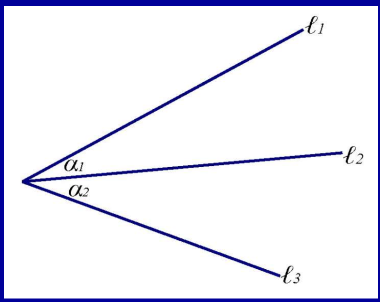
Example: Error propagation



$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -1 & +1 & 0 \\ 0 & -1 & +1 \end{bmatrix} \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} = A \cdot L$$

$$A = \begin{bmatrix} -1 & +1 & 0 \\ 0 & -1 & +1 \end{bmatrix}, \quad L = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix}$$

$$C_{\alpha\alpha} = AC_{LL}A^\top = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$$

$$\boxed{\sigma_1 = \sigma_2 = \sigma_3 = \pm 1.41''}$$

$$C_{LL} = \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \sigma_3^2 \end{bmatrix} \approx \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}$$

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Error propagation for two vectors: covariance matrix

$$\underset{m\cdot 1}{y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \underset{m\cdot n}{A} \; \underset{n\cdot 1}{x}$$

$$\underset{p\cdot 1}{z} = \begin{bmatrix} z_1 \\ z_2 \\ \cdots \\ z_p \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \underset{p\cdot n}{B} \; \underset{n\cdot 1}{x}$$

$$\underset{m\cdot n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix}$$

$$C_{yz} = E\left\{[y - E(y)][z - E(z)]^\top\right\} = A\,E\left\{[x - E(x)][x - E(x)]^\top\right\}B^\top = AC_{xx}B^\top$$

$$\boxed{= AC_{xx}B^\top}$$

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Taylor series expansion of a non-linear function

$$y = f(x_1, x_2, \cdots, x_n)$$

$$y^0 = f(x_1^0, x_2^0, \cdots, x_n^0)$$

$$\left.\begin{array}{l} x_i = x_i^0 + \delta x_i \quad (1 \le i \le n) \\ y = y^0 + \delta y \end{array}\right\}$$

Taylor series expansion

$$y = f(x_1^0 + \delta x_1,\ x_2^0 + \delta x_2,\ \cdots,\ x_n^0 + \delta x_n) =$$

$$= f(x_1^0, x_2^0, \cdots, x_n^0) + \frac{1}{1!}\left[\frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f}{\partial x_n}\delta x_n\right]$$

på grund av små värden används enbart första graden (strunta i raden nedan)

$$+\frac{1}{2!}\left[\frac{\partial^2 f}{\partial x_1^2}\,\delta x_1^2 + \frac{\partial^2 f}{\partial x_1 \partial x_2}\,\delta x_1\,\delta x_2 + \cdots + \frac{\partial^2 f}{\partial x_1 \partial x_n}\,\delta x_1\,\delta x_n\right] + \cdots$$

$$\delta y \approx \frac{\partial f}{\partial x_1}\,\delta x_1 + \frac{\partial f}{\partial x_2}\,\delta x_2 + \cdots + \frac{\partial f}{\partial x_n}\,\delta x_n$$

Linearization ! 9

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Error propagation for a non-linear function

$$\varepsilon_y \approx \frac{\partial f}{\partial x_1}\varepsilon_1 + \frac{\partial f}{\partial x_2}\varepsilon_2 + \cdots + \frac{\partial f}{\partial x_n}\varepsilon_n = a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_n\varepsilon_n = a \cdot \varepsilon$$

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdots \\ \varepsilon_n \end{bmatrix}, \quad a = (a_1,\ a_2,\ \cdots,\ a_n)\ , \quad a_i = \frac{\partial f}{\partial x_i}\ , \quad i = 1, 2, \cdots, n$$

$$\sigma_y^2 = E\left\{[y - E(y)]^2\right\} = a\ C_{xx} a^\top$$

$$\underset{n\cdot n}{C_{xx}} = \underset{n\cdot n}{C_{\varepsilon\varepsilon}} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix}$$

For uncorrelated observations ($\sigma_{ij} = 0$, $i \neq j$):

$$\sigma_y^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_1^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_2^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_n^2$$

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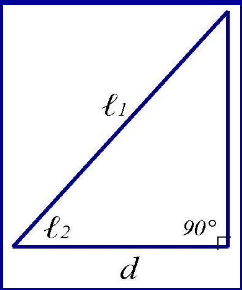
Error propagation for a non-linear function

Example 1.3

A slope distance has been measured to $\ell_1 = 103.132$ metres with standard error $\sigma_1 = \pm 3\ mm$. The corresponding vertical angle has been measured to $\ell_2 = 45^0$ with standard error $\sigma_2 = \pm 6''$ (Cf Figure 1.1). What is the standard error (σ_d) of horizontal distance $d = \ell_1 \cos \ell_2$?



$\ell_1 = 103.132$ metres $\quad \sigma_1 = \pm 3\ mm$

$\ell_2 = 45^0 \quad \sigma_2 = \pm 6''$

$d = \ell_1 \cos \ell_2 \quad \sigma_d =$?

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Error propagation for a non-linear function

$d = \ell_1 \cos \ell_2$

$$\varepsilon_d = \frac{\partial f}{\partial \ell_1}\ \varepsilon_1 + \frac{\partial f}{\partial \ell_2}\ \varepsilon_2$$

$$\frac{\partial f}{\partial \ell_1} = \cos \ell_2 = \cos 45^0 = \frac{\sqrt{2}}{2} \quad \text{(no unit)}$$

$$\frac{\partial f}{\partial \ell_2} = -\ell_1\ \sin \ell_2 = 103.132\ \frac{\sqrt{2}}{2} \quad \text{(metres)}$$

$$\sigma_d^2 = \left(\frac{\partial f}{\partial \ell_1}\right)^2 \sigma_1^2 + \left(\frac{\partial f}{\partial \ell_2}\right)^2 \sigma_2^2 = \left(\frac{\sqrt{2}}{2}\right)^2 \sigma_1^2 + \left(103.132^m\ \frac{\sqrt{2}}{2}\right)^2 \sigma_2^2$$

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Error propagation for a non-linear function

$$\sigma_d^2 = \left(\frac{\partial f}{\partial \ell_1}\right)^2 \sigma_1^2 + \left(\frac{\partial f}{\partial \ell_2}\right)^2 \sigma_2^2 = \left(\frac{\sqrt{2}}{2}\right)^2 \sigma_1^2 + \left(103.132^m \frac{\sqrt{2}}{2}\right)^2 \sigma_2^2$$

$$1 \ \ radian = \rho'' = \frac{360 \ \cdot 3600''}{2\pi} \approx \ 206\ 265''$$

$$\sigma_d^2 = \frac{1}{2}\ (3^{mm})^2 + \left(103\ 132^{mm} \frac{\sqrt{2}}{2}\right)^2 \ \left(\frac{6''}{\rho''}\right)^2 \approx \frac{9}{2} + \left(\frac{1}{2}\right)^2 \ \frac{1}{2} \cdot 36 = 9\ mm^2$$

$$\sigma_d \ = \sqrt{9} = 3 \ \ mm$$

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Error propagation for m non-linear functions

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, \cdots, x_n) \\ f_2(x_1, x_2, \cdots, x_n) \\ \cdots \\ f_m(x_1, x_2, \cdots, x_n) \end{bmatrix}$$

$$\varepsilon_y = \begin{bmatrix} \varepsilon_{y_1} \\ \varepsilon_{y_2} \\ \cdots \\ \varepsilon_{y_m} \end{bmatrix} = \underset{m\cdot n}{A} \ \underset{n\cdot 1}{\varepsilon}$$

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdots \\ \varepsilon_n \end{bmatrix},$$

$$\underset{m\cdot n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$a_{ij} = \frac{\partial f_i}{\partial x_j} \ . \quad (i = 1, 2, \cdots, m; \ j = 1, 2, \cdots, n)$$

$$\boxed{C_{yy} = E\left\{[y - E(y)][y - E(y)]^\top\right\} = A\ C_{xx} A^\top}$$

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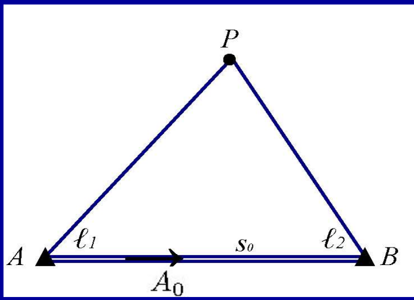
Example: Error propagation for 2 non-linear functions



Relations between the two measurements and the derived coordinates:

$A_0 = 120^0$

$s_0 = 750.000$ metres

$\ell_1 = \ell_2 = 60^0$

$\sigma_1 = \sigma_2 = \sigma =$ 2 ”

$$x = x_a + s_0 \frac{\sin \ell_2 \ \cos(A_0 - \ell_1)}{\sin(\ell_1 + \ell_2)}$$

$$y = y_a + s_0 \frac{\sin \ell_2 \ \sin(A_0 - \ell_1)}{\sin(\ell_1 + \ell_2)}$$

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Example: Error propagation for 2 non-linear functions

Relations between true errors of derived coordinates and true errors of the measurements

$$x = x_a + s_0 \frac{\sin \ell_2 \ \cos(A_0 - \ell_1)}{\sin(\ell_1 + \ell_2)}$$

$$y = y_a + s_0 \frac{\sin \ell_2 \ \sin(A_0 - \ell_1)}{\sin(\ell_1 + \ell_2)}$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \end{bmatrix} = A \cdot \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

$$\underset{2\cdot 2}{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$a_{11} = \frac{\partial x}{\partial \ell_1} = -\frac{s_0}{\rho''} \frac{\sin \ell_2 \cos(A_0 + \ell_2)}{\sin^2(\ell_1 + \ell_2)}$$

$$a_{12} = \frac{\partial x}{\partial \ell_2} = +\frac{s_0}{\rho''} \frac{\sin \ell_1 \cos(A_0 - \ell_1)}{\sin^2(\ell_1 + \ell_2)}$$

$$a_{21} = \frac{\partial y}{\partial \ell_1} = -\frac{s_0}{\rho''} \frac{\sin \ell_2 \sin(A_0 + \ell_2)}{\sin^2(\ell_1 + \ell_2)}$$

$$a_{22} = \frac{\partial y}{\partial \ell_2} = +\frac{s_0}{\rho''} \frac{\sin \ell_1 \sin(A_0 - \ell_1)}{\sin^2(\ell_1 + \ell_2)}$$

$$A = \frac{s_0}{0.75 \cdot \rho''} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{4} \\ 0 & \frac{3}{4} \end{bmatrix}$$

$\ell_1 = \ell_2 = 60°$

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Example: Error propagation for 2 non-linear functions

Relation between variance-covariane matrices

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \end{bmatrix} = A \cdot \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \qquad A = \frac{s_0}{0.75 \cdot \rho''} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{4} \\ 0 & \frac{3}{4} \end{bmatrix}$$

$\ell_1 = \ell_2 = 60°$

$$C_{\varepsilon\varepsilon} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (\sigma^2 = 4\,''^2)$$

$$C_p = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{yx} & \sigma_y^2 \end{bmatrix} = A\, C_{\varepsilon\varepsilon}\, A^\top = \sigma^2\, AA^\top = \left(\frac{\sigma\ s_0}{0.75 \cdot \rho''} \right)^2 \frac{3}{16} \begin{bmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix}$$

For s0=750 m, σ=2"

$$C_p = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{yx} & \sigma_y^2 \end{bmatrix} = \begin{bmatrix} 88.14 & 30.53 \\ 30.53 & 52.88 \end{bmatrix} \quad (mm^2)$$

$$\sigma_x = \sqrt{88.14} = \pm 9.4\ mm\ , \quad \sigma_y = \sqrt{52.88} = \pm 7.3\ mm.$$

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Propagation of the co-factor matrix

$$C_{xx} = \sigma_0^2 \cdot Q_{xx} = \sigma_0^2 \cdot P_{xx}^{-1}$$

Linear functions

$$\underset{m \cdot 1}{y} = \underset{m \cdot n}{A}\ \underset{n \cdot 1}{x}$$

$$\underset{m \cdot n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$C_{yy} = A\, C_{xx} A^\top = A \left[\sigma_0^2 \cdot Q_{xx} \right] A^\top$$

Non-linear functions

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, \cdots, x_n) \\ f_2(x_1, x_2, \cdots, x_n) \\ \cdots \\ f_m(x_1, x_2, \cdots, x_n) \end{bmatrix}$$

$$a_{ij} = \frac{\partial f_i}{\partial x_j}\ . \quad (i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n)$$

$$C_{yy} = \sigma_0^2 \cdot Q_{yy} = \sigma_0^2 \cdot P_{yy}^{-1}$$

$$Q_{yy} = P_{yy}^{-1} = A Q_{xx} A^\top$$

→ Propagation law for co-factor matrices (weight matrices)

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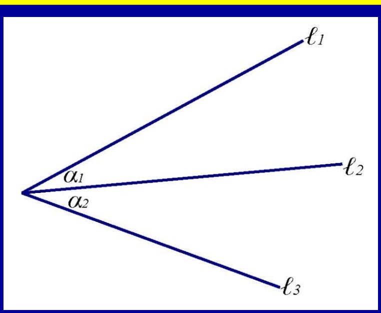
Example: Error propagation



$\sigma_1 = \sigma_2 = \sigma_3 = \pm 1.41''$

$$C_{LL} = \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \sigma_3^2 \end{bmatrix} \approx \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}$$

σ0 = 2 ”

$$Q_{LL} = \frac{C_{LL}}{\sigma_0^2} = \begin{bmatrix} \frac{1}{2} & & \\ & \frac{1}{2} & \\ & & \frac{1}{2} \end{bmatrix}$$

$$P_{LL} = (Q_{LL})^{-1} = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}$$

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -1 & +1 & 0 \\ 0 & -1 & +1 \end{bmatrix} \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} = A \cdot L$$

$$C_{\alpha\alpha} = A C_{LL} A^{\top} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$$

α is correlated !

$$Q_{\alpha\alpha} = \frac{C_{\alpha\alpha}}{\sigma_0^2} = \frac{1}{2} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

σ0 = 2 ”

$$Q_{aa} = A\ Q_{LL} A^{\top} = \frac{1}{2} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$P_{\alpha\alpha} = Q_{\alpha\alpha}^{-1} = \frac{2}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$p_{\alpha_1} = 1$$

Weight matrix does not contain weights when components are correlated !

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Summary: Errors and Error Propagation

- Error propagation for one linear function
- Error propagation for one non-linear function
- Error propagation for m linear/non-linear functions
- Error propagation for co-factor matrices

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