AI1149 Geodata Quality and Time Series Analysis

Exercise Problems

Huaan Fan

KTH Geodesy and Satellite Positioning

October 2023

Contents

1 Errors and Error Propagation

Problem 1

A distance with correct length $\widetilde{\ell} = 53.76$ *metres* has been measured 10 times with 2 different types of instruments. The measurement results and their weights are listed below.

i	ℓ_i	p_i
	(metre)	
1	53.75	2
2	53.72	2
3	53.77	2
4	53.84	2
5	53.74	2
6	53.88	1
7	53.76	1
8	53.69	1
9	53.68	1
10	53.79	1

(a) Find the standard error of unit weight

(b) Does ℓ_7 have higher precision than ℓ_4 and ℓ_6 ? Explain why?

Problem 2

An observation vector x and its variance-covariance matrix are defined as follows:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad C_{xx} = \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} \quad (mm^2)$$

If the variance factor $\sigma_0^2 = 2\ mm^2$, calculate

(a) cofactor matrix Q of x

(b) weight (p_2) of element x_2

(c) weight matrix P of x

(d) compare p_2 computed in (b) with the second diagonal element of P computed in (c)

Problem 3

Between two benchmarks (P_1 ,P_2), the forward height difference h_1 (from P_1 to P_2)and backward height difference h_2 (from P_2 to P_1) have been measured independently, with equal standard error σ. Note that h_1 and h_2 have opposite signs. If the mean value $\overline{h} = \frac{h_1 - h_2}{2}$ of these two height differences has standard error $\sigma_{\overline{h}} = 3\ mm$, what is the standard error of the difference $d = h_1 + h_2$.

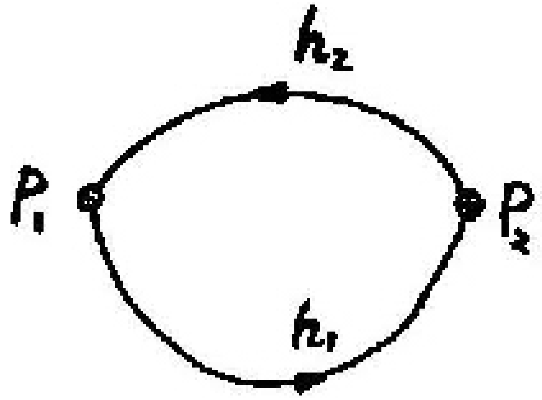
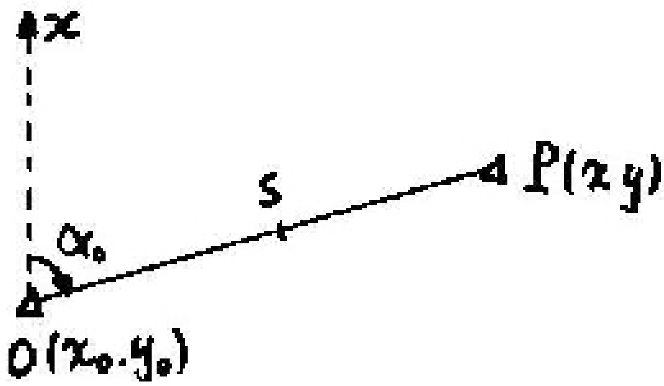


Figure 1: Reciprocal levelling

Problem 4

To determine the coordinates (x, y) of an unknown point P, the azimuth α and distance s from a fixed point O have been measured independently. The emasured values and their standard errors are: $\alpha = 60.0000^o$, $s = 100.000\ metres$, $\sigma_\alpha = 3''$, $\sigma_s = 3\ cm$. calculate the variance-covariance matrix of the coordinate vector $z = \begin{bmatrix} x \\ y \end{bmatrix}$.

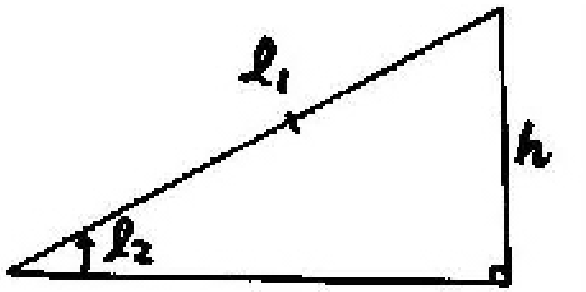


Point positioning by polar method

Problem 5

To determine the height difference h, the slope distance ℓ_1 and its vertical angle ℓ_2 have been measured independently using a totalstation. The measured values are as follows: $\ell_1 = 4125.300\ metres$, $\ell_2 = 20.0000^o$. The angle measurement has a standard error $\sigma_2 = 6''$ while distance measurement ℓ_1 has a constant error of $\pm 4\ mm$ and a scale error $2\ ppm$.

8mm



Trigonometric height determination

(a) Find the linear relation between the true errors of ℓ_1, ℓ_2 and h .

(b) Calculate the standard error of h .

(c) In order to significantly improve the precision of h derived from ℓ_1 and ℓ_2, which measurement (ℓ_1 or ℓ_2) should be determined more precisely ?

2 Statistical Analysis

Problem 1

If ε is a normally distributed variable, $\varepsilon \sim N(\mu,\ \sigma^2)$. Find $P\{\mu - 3\sigma \leq \varepsilon \leq \mu + 2\sigma\}$, i.e. the probability that ε belongs to the confidence interval $[\mu - 3\sigma,\ \mu + 2\sigma]$.

Problem 2

A distance has been measured 10 times with a tape of standard error $\sigma = 4\ mm$. The measured values are as follows:

i	Measurements (metre)
1	1234.103
2	1234.110
3	1234.105
4	1234.108
5	1234.111
6	1234.113
7	1234.104
8	1234.107
9	1234.109
10	1234.110

Find out if this distance is equal to 1234.110 *metres* at 5% risk level.

Problem 3

Make the same test in **Problem 2** when the theoretical standard error σ is not known.

Problem 4

In order to study the difference of angle measurements during day time and night time, an angle α has been observed independently 8 times using the same theodolite but at different times. The angle is roughly $55^o\ 23'$ and the second ($''$) part of the measurements are listed in the following table.

i		1	2	3	4	5	6	7	8
Day time		18.8	19.8	20.9	21.5	19.5	21.0	21.2	20.5
Night time		20.3	20.0	18.8	19.0	20.1	20.2	19.1	17.7

Use $F - test$ to find out if daytime and night time emasurment have the same accuracy at 5% risk level.

Problem 5

A baseline has been measured using three different types of totalstations. The results (in metre) are listed below.

Measurement No (j)	Instrument $i = 1$	Instrument $i = 2$	Instrument $i = 3$
1	476.266	476.276	476.312
2	476.286	476.290	476.284
3	476.279	476.304	476.296
4		476.283	
$n_i \rightarrow$	3	4	3

If all measurements have the same theoretical variance σ^2, test the hypothesis that all instruments give the same baseline length at 5% risk level.

Problem 6

To study the possible correlation between baseline length x and *DGPS* baseline precision y, baseline errors for 10 different baselines have been determined in an experiment:

i	x_i	y_i
1	0.5	3.40
2	0.8	3.60
3	1.0	3.98
4	1.2	4.22
5	1.5	4.61
6	2.0	5.12
7	2.5	5.40
8	3.0	6.21
9	3.5	6.48
10	3.8	7.10

(a) Plot the data points in a graph to see if there is a linear trend between between x_i and y_i

(b) Estimate the least squares coefficients $(\alpha,\ \beta)$ of a linear regression model between x_i and y_i :

$$y_i = \alpha + \beta \cdot x_i$$

(c) Estimate the *aposteriori* unit-weight standard error $\widehat{\sigma}_0$, and standard errors of $\widehat{\alpha}$ and $\widehat{\beta}$

(d) If the measurements y_i have known standard error $\sigma_0 = 0.15$, use χ^2-test to check if the computed regression line is significant at 5% risk level

3 Adjustment by Elements in Linear Models

(Problems should be solved manually, without uisng electronic calculators or computers)

Problem 1

Repeat **Example 3.1** in **Chapter 3,** assuming now that the three angle measurements have different weights: $p_1 = p_2 = 2,\ p_3 = 1$.

(a) Find the observation equations

(b) Calculate the least squares estimates of the parameters, residuals and original/reduced measurements

(c) Find the unit-weight standard error

(d) Compute the cofactor matrix and variance-covariance matrix of $\widehat{X}$, $\widehat{\varepsilon}$ and adjusted observations $\widehat{L}$, respectively.

Problem 2

In the following levelling network, $A,\ B,\ C$ are three fixed benchmarks with known heights. To determine the height of an unknown benchmark P, three height differences ($\ell_1,\ \ell_2,\ \ell_3$) have been measured indepedently.

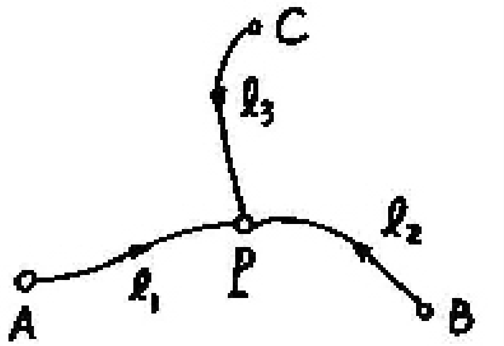


Figure 2: Levelling with one unknown benchmark

$$\begin{cases} H_A = 10.421\ m \\ H_B = 12.261\ m \\ H_C = 11.559\ m \end{cases},\quad \begin{cases} \ell_1 = +1.680\ m \\ \ell_2 = -0.148\ m \\ \ell_3 = +0.557\ m \end{cases}$$

The lengths of all three levelling lines are roughly the same and thus all height differences are regarded to have equal weights. Adjust the three obervations using the method of adjustment by elements.

(a) Find the observation equations

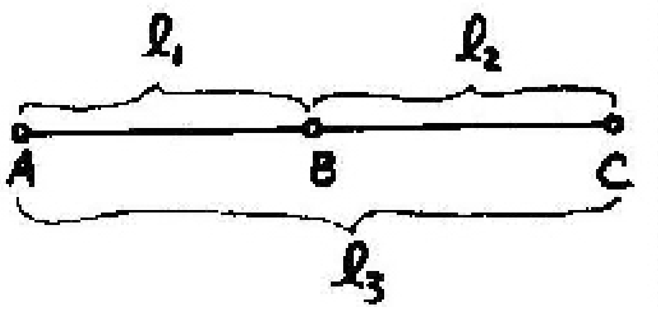
(b) Calculate the least squares estimates of the height of P, residuals and height differences

(c) Find the unit-weight standard error

(d) Compute the cofactor matrix and variance-covariance matrix of $\widehat{X}$, $\widehat{\varepsilon}$ and adjusted observations $\widehat{L}$, respectively.

Problem 3

Between three points $(A,\ B,\ C)$ on a straight line, three distances have been measured independently: $\ell_1 = 100.006\ m,\ \ell_2 = 99.999\ m,\ \ell_3 = 200.002\ m.$



3 distances on a straight line

The weight p_i of ℓ_i is assumed to be inversely proportional to the distance (in metre) squared: $p_i = \frac{40\ 000\ m^2}{\ell_i^2}$. Adjust the observations using the method of adjustment by elements.

(a) Find the weight matrix

(b) Find the observation equations

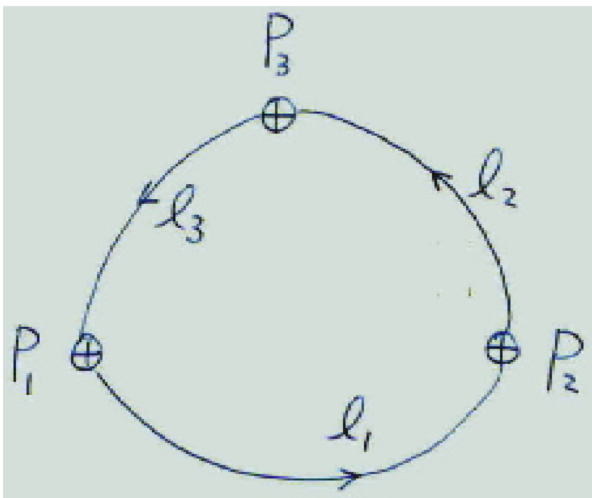
(c) Calculate the least squares estimates of the unknowns, residuals and distances

(d) Find the unit-weight standard error

(e) Compute the cofactor matrix and variance-covariance matrix of the unknowns, $\widehat{\varepsilon}$ and adjusted observations $\widehat{L}$, respectively.

Problem 4

In the levelling network illustrated below, P_3 is a fixed benchmark with height $H_3 = +1.000\ m$ and $P_1,\ P_2$ are unknown benchmarks.



The measured height differences (uncorrelated) and their weights are listed below:

$$\begin{array}{ll} \ell_1 = +1.054\,m & p_1 = 1 \\ \ell_2 = -0.700\,m & p_2 = 1 \\ \ell_3 = -0.351\,m & p_3 = 1 \end{array}$$

Adjust the three obervations using the method of adjustment by elements.

(a) Find the observation equations

(b) Calculate the least squares estimates of the unknowns, the residuals and height differences

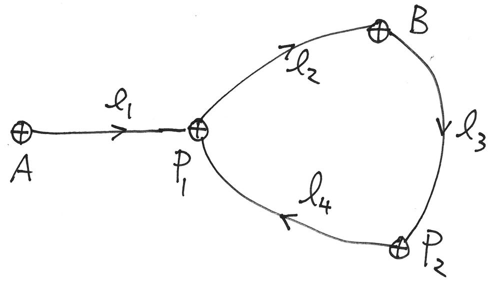
(c) Find the unit-weight standard error

(d) Compute the cofactor matrix and variance-covariance matrix of the unknowns, the residuals and adjusted observations $\widehat{L}$, respectively.

Problem 5

In the following levelling network, benchmarks A and B are fixed with known heights: $H_A = 0\ m,\ H_B = 2.000\ m$. Benchmarks P_1 and P_2 are unknown points. 4 height differences have been measured with the following values:

$$L' = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \end{bmatrix} = \begin{bmatrix} 1.001 \\ 1.000 \\ 1.004 \\ -1.996 \end{bmatrix} \ (m)$$



All measurements are un-correlated with each other and have equal weights. Adjustment by elements will be carried out using the heights of $P_1,\ P_2$ as unknown parameters x_1, x_2 :

(a) Find the observation equations

(b) Calculate the least squares estimates of the parameters, residuals and adjusted observations.

(c) Calculate the *a posteriori* estimate of the variance factor σ_0^2 .

(d) Find the variance-covariance matrices of the estimated parameters $\widehat{X}$, residuals $\widehat{\varepsilon}$ and adjusted observations $\widehat{L}$.

4 Adjustment by Elements in Non-Linear Models

Adjust the network in **Problem 2 of Exercise 5**. To determine the 2D coordinates (x, y) of an unknown point P, 4 angles have been measured independently with the same precision $\sigma_i = 3.0''$. In addition, one distance (ℓ_5) has also been measured with standard error $\sigma_5 = 1.5\ cm$.

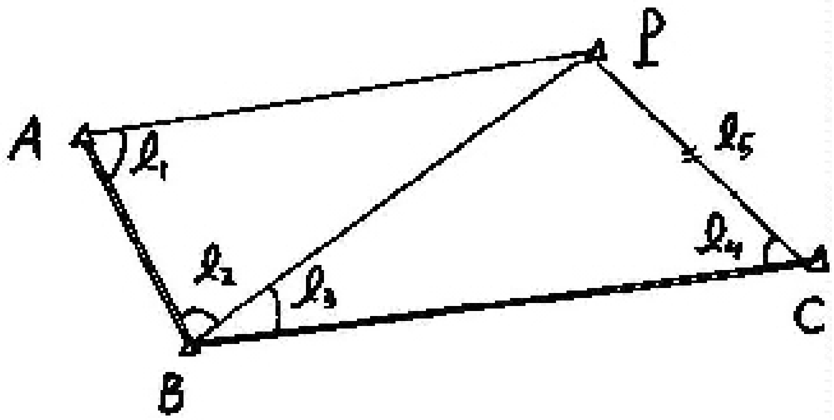


Figure 3: Point positioning from 3 fixed points

The measurements and the given coordinates are listed below:

$$\begin{cases} \ell_1 = 60^o\ 00'\ 00.2'' \\ \ell_2 = 70^o\ 53'\ 38.2'' \\ \ell_3 = 49^o\ 06'\ 25.0'' \\ \ell_4 = 29^o\ 59'\ 57.7'' \\ \ell_5 = 3464.121\ m \end{cases}, \quad \begin{cases} x_A = 2732.651\ m,\ y_A = 1000.314\ m \\ x_B = 1000.600\ m,\ y_B = 2000.314\ m \\ x_C = 1000.600\ m,\ y_C = 6500.314 \end{cases}$$

(a) Find the number of independent unknowns

(b) Find the weight matrix P using the *a priori* unit-weight standard error $\sigma_0 = 3.0''$

(c) Calculate the approximate coordinates $(x_p^o,\ y_p^o\ )$ of P, (e.g. using the given coordinates of the fixed point C and observed values $\ell_4,\ \ell_5$)

(d) Find all observation equations. If they are not linear, linearize them! Angular residuals shall be in $''$ (arcsecond) and distance residual in *cm*.

(e) Form the normal equation and calculate the inverse of the normal equation matrix $N = A^\top PA$

(f) Compute the least squares estimates of the unknowns and the adjusted coordinates coordinates $(\widehat{x}_p,\ \widehat{y}_p\ )$ of P

(g) Compute the least squares residuals and the adjusted observations $\widehat{L}$

(h) Estimate the *aposteriori* unit-weight standard errror $\widehat{\sigma}_0$

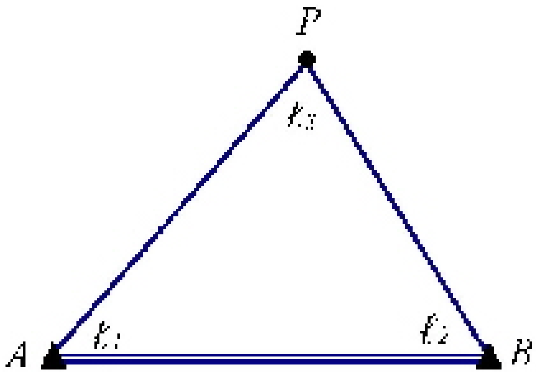
(i) Compute the cofactor matrices and variance-covariance matrices of the unknowns, the residuals $\widehat{\varepsilon}$ and the adjusted observations $\widehat{L}$

(j) Compute the error ellipse elements of point P .

5 Local redundancy, data snooping and reliability

Problem 1

Example 3.1 and *Problem 1* in *Exercise 6* deal with the same geodetic network but with different weights.



For *Example 3.1*:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad Q_{\widehat{\varepsilon}\widehat{\varepsilon}} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

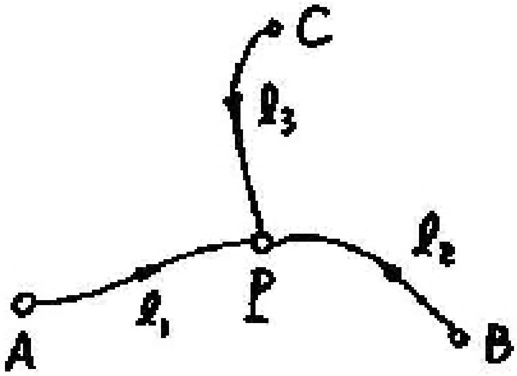
For **Problem 1** in *Exercise 6*:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix}, \quad Q_{\widehat{\varepsilon}\widehat{\varepsilon}} = \frac{1}{8}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

For each of the above 2 situations,
(a) calculate the redundancy matrix R and local redundancies r_i.
(b) verify that the sum ($\sum r_i$) of all local redundancies is equal to the total redundancy $(n-m)$ of the nwtwork
(c) comment on the local redundancies r_i: if measurements have different r_i,speculate why. Does a measurement with higher weight also have higher local redundancy and consequently higher reliability ?

Problem 2

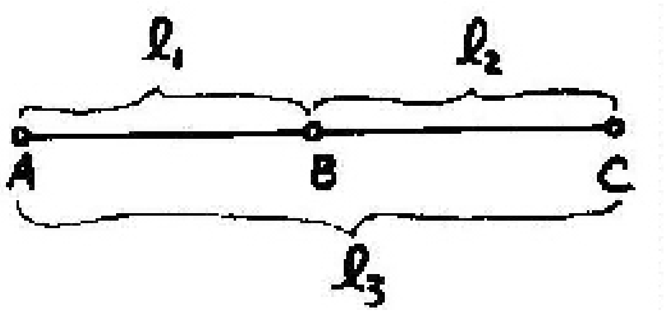
Consider the geodetic network in *Problem 2, Exercise 6.*



(a) calculate the redundancy matrix R and local redundancies r_i.
(b) verify that the sum ($\sum r_i$) of all local redundancies is equal to the total redundancy $(n-m)$ of the nwtwork
(c) comment on the local redundancies r_i: speculate why we have such r_i.

Problem 3

For the geodetic network described in *Problem 3, Exercise 6,* consider two situations: (i) all measurements have unit weights, (ii) all measurements have different weights: $p_1 = p_2 = 4,\ p_2 = 1$.



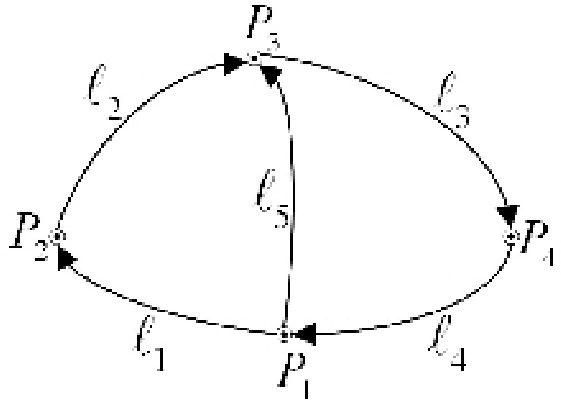
3 distances on a straight line

For each situation,
(a) calculate the redundancy matrix R and local redundancies r_i.
(b) verify that the sum ($\sum r_i$) of all local redundancies is equal to the total redundancy $(n-m)$ of the nwtwork
(c) comment on r_i: if different measurements have different r_i,speculate why. Does a measurement with higher weight also have higher local redundancy and consequently higher reliability ?

•

Problem 4

This problem concerns a levelling network treated in **Example 2.2** and **Example 3.2** in the compendium. The levelling network has one fixed benchmark P_4 with given height $H_4 = 10.000$ metres, 3 unknown benchmarks P_1, P_2 , P_3 and five height difference meaurements ℓ_i $(1 \le i \le 5)$.



The original observations and their weight matrix are:

$$\underset{5\cdot 1}{L'} = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ \ell_5 \end{bmatrix} = \begin{bmatrix} +1.002 \\ +2.004 \\ -2.001 \\ -1.002 \\ +3.012 \end{bmatrix} (m), \quad \underset{5\cdot 5}{P} = \begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 2 \end{bmatrix}$$

Choosing the heights of benchmarks P_1, P_2 and P_3 as the unknown parameters x_1, x_2 and x_3, respectively, the observation equations become :

$$L - \varepsilon = AX$$

with :

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} +1.002 \\ +2.004 \\ -12.001 \\ +8.998 \\ +3.012 \end{bmatrix} (m)$$

Through least squares adjustment, the residuals ε and its cofactor matrix are found:

$$\widehat{\varepsilon} = \begin{bmatrix} -1.5 \\ -1.5 \\ +3.0 \\ +3.0 \\ +3.0 \end{bmatrix} (mm); \quad Q_{\widehat{\varepsilon}\widehat{\varepsilon}} = \frac{1}{28} \begin{bmatrix} 5 & 5 & 2 & 2 & -4 \\ 5 & 5 & 2 & 2 & -4 \\ 2 & 2 & 12 & 12 & 4 \\ 2 & 2 & 12 & 12 & 4 \\ -4 & -4 & 4 & 4 & 6 \end{bmatrix}$$

The theoretical (true) variance factor and the estimated *a posteriori* variance factor $\widehat{\sigma}_0^2$ are as follows:

$$\sigma_0 = \pm 9.0 \; mm^2 \,, \qquad \widehat{\sigma}_0 = 22.50 \; mm^2$$

1. When σ_0 is available, use data snooping (overall test and individual $u-$tests) to test if the observations contain any gross errors
2. Use $\widehat{\sigma}_0$ and $t-$test to test if the observations contain any gross errors

Problem 5

This problem concerns the same levelling network as in **Problem 1**. Assume that the theoretical variance factor is: $\sigma_0^2 = 22.50 \; mm^2$, the risk level of statistical tests is: $\alpha = 5\%$ and the power of tests is 80%. Calculate:

1. the redundancy matrix R and the local redundancies for each observation
2. the *internal* reliability for each observation
3. the *external* reliability for each observation