Ministry of Education and Science
Federal State Budgetary Educational
Institution of Higher Education
“Bauman Moscow State Technical University”
(BMSTU)

Faculty of “Fundamental Sciences”
Department of “Higher Mathematics”

Coursework Report 1-st semester of the 2020—2021 academic year.

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Contents

1 Goals and objectives of the coursework

1.1 Goals

— development of capabilities necessary for future professional activity and auxiliary for the successful completion of the baccalaureate course.

1.2 Objectives

1. Development of familiarity with the software necessary for future professional activity.
2. Development of the ability to research required information in specialised literature and other sources.
3. Development of the skills needed to formulate reports and present results.

1.3 Individual objectives

1. Examine the methods used to display mathematical information in the LaTeX typesetting system.
2. Examine the possibilities and potential of the Git version control system.
3. Learn to typeset texts with mathematical context, containing formulas and graphs, in the LaTeX system. For this, complete the installation of the open-source distributive TeXLive and compiler TeXStudio.
4. Recreate in the LaTeX system your variant of the sample calculations from the Mathematical Analysis course.
5. Register on the online platform GitHub and upload the initial tex-files and the results of their compilation (in pdf format).

2 Report

The relevance of this topic is dictated by the fundamental necessity of the ability to operate the typesetting systems of LaTeX and TexStudio for the ultimate scope of displaying text, mathematical formulae and graphs. The skills acquired through the completion of this coursework may be applied in such processes as the writing of course projects, thesis work and future professional activity.

Moreover, the LaTeX typesetting system allows for the usage of a large number of external instrumental packages, which greatly facilitate the process of displaying information in various fields of engineering, technical and scientific activity.

3 Individual objectives

3.1 Limits and Continuity.

Question #1.

Conditions: Consider the sequence $a_n = \dfrac{7n-1}{n+1}$ and the value $c = 7$. Prove that $\lim\limits_{n\to\infty} a_n = c$. More precisely, $\forall\varepsilon > 0$ find the minimum value $N = N(\varepsilon)$ such that $|a_n - c| < \varepsilon$, $\forall n > N(\varepsilon)$. Fill in the following table.

ε	0,1	0,01	0,001
$N(\varepsilon)$			

Solution: If c is the limit of a_n, then $\forall\varepsilon > 0$ and $\forall n > N(\varepsilon)$, $|a_n - c| < \varepsilon$ must hold true. By substituting the values for a_n and c into the above inequality we achieve the following:

$$\left|\frac{7n-1}{n+1} - 7\right| < \varepsilon.$$

Next we open the modulus to form a double inequality and rewrite the difference to have a common denominator.

$$-\varepsilon < \frac{-8}{n+1} < \varepsilon.$$

Obviously, the right inequality holds true $\forall n \in \mathbb{N}$, therefore form this point on we will only consider the left inequality:

$$-\varepsilon < \frac{-8}{n+1}.$$

At this point, by performing a series of transformations we can isolate n and formulate a formula for $N(\varepsilon)$.

$$-\varepsilon < \frac{-8}{n+1},$$

$$\varepsilon > \frac{8}{n+1},$$

$$n+1 > \frac{8}{\varepsilon},$$

$$n > \frac{8}{\varepsilon} - 1,$$

$$N(\varepsilon) = \left\lfloor \frac{8}{\varepsilon} - 1 \right\rfloor.$$

Where $\lfloor\ \rfloor$ is the floor function.

The existence of a $N = N(\varepsilon)$ function proves that $|a_n - c| < \varepsilon$ holds true for $\forall\varepsilon > 0$ and $\forall n > N(\varepsilon) \iff \lim\limits_{n\to\infty} a_n = c$. We may fill in the table:

ε	0,1	0,01	0,001
$N(\varepsilon)$	79	799	7999

Verification:

$$|a_{80} - c| = \frac{-8}{81} < 0.1,$$

$$|a_{800} - c| = \frac{-8}{801} < 0.01,$$

$$|a_{8000} - c| = \frac{-8}{8001} < 0.001.$$

Question #2.

Conditions: Evaluate the limits of the following functions:

$$\textbf{(a)} \qquad \lim_{x\to -1} \frac{x^3 - 3x - 2}{x^2 - x - 2},$$

$$\textbf{(b)} \quad \lim_{x\to +\infty} \frac{1 - \sqrt[3]{4x^4 - x^7\sqrt{x}}}{2x^2 - 3x + 5},$$

$$\textbf{(c)} \quad \lim_{x\to 3} \frac{\sqrt{x+13} - 2\sqrt{x+1}}{x^2 - 9},$$

$$\textbf{(d)} \quad \lim_{x\to 0} (2 - \cos 3x)^{\frac{1}{\ln(1+x^2)}},$$

$$\textbf{(e)} \quad \lim_{x\to 0} \left(\frac{\sin 2x}{\arcsin 3x}\right)^{\operatorname{arccot} x},$$

$$\textbf{(f)} \qquad \lim_{x\to 2} \frac{\lg(5 - 2x)}{\sqrt{10 - 3x} - 2}.$$

Solutions:

(a)

$$\lim_{x\to -1} \frac{x^3 - 3x - 2}{x^2 - x - 2} = \left[\frac{0}{0}\right] = \lim_{x\to -1} \frac{(x+1)(x^2 - x - 2)}{x^2 - x - 2} = \lim_{x\to -1} (x+1) = 0.$$

(b)

$$\lim_{x\to +\infty} \frac{1 - \sqrt[3]{4x^4 - x^7\sqrt{x}}}{2x^2 - 3x + 5} = \left[\frac{1 - \sqrt[3]{\infty - \infty}}{\infty - \infty}\right] = \lim_{x\to +\infty} \frac{1 - x^{5/2} \cdot \sqrt[3]{4x^{-7/2} - 1}}{x^2 \cdot (2 - 3x^{-1} + 5x^{-2})} =$$

$$\lim_{x\to +\infty} \frac{x^{5/2} \cdot (x^{-5/2} - \sqrt[3]{4x^{-7/2} - 1})}{x^2 \cdot (2 - 3x^{-1} + 5x^{-2})} = \lim_{x\to +\infty} \sqrt{x} \cdot \frac{x^{-5/2} - \sqrt[3]{4x^{-7/2} - 1}}{2 - 3x^{-1} + 5x^{-2}} = +\infty.$$

(c)

$$\lim_{x\to 3} \frac{\sqrt{x+13} - 2\sqrt{x+1}}{x^2 - 9} = \left[\frac{0}{0}\right] = \lim_{x\to 3} \frac{(x+13) - 4(x+1)}{(x^2 - 9)(\sqrt{x+13} + 2\sqrt{x+1})} =$$

$$\lim_{x\to 3} \frac{-3x + 9}{(x-3)(x+3)(\sqrt{x+13} + 2\sqrt{x+1})} = \lim_{x\to 3} \frac{-3}{(x+3)(\sqrt{x+13} + 2\sqrt{x+1})} = -\frac{1}{16}.$$

(d)

$$\lim_{x\to0}(2-\cos 3x)^{\frac{1}{\ln(1+x^2)}}=[1^\infty]=\lim_{x\to0}\exp\left(\frac{\ln(2-\cos 3x)}{\ln(1+x^2)}\right)=\left|\begin{array}{l}\ln(2-\cos 3x)\sim 1-\cos 3x\\ \ln(1+x^2)\sim x^2\end{array}\right|=$$
$$\exp\left(\lim_{x\to0}\frac{1-\cos 3x}{x^2}\right)=\left|\,1-\cos 3x\sim\frac{(3x)^2}{2}\,\right|=\exp\left(\lim_{x\to0}\frac{9x^2}{2x^2}\right)=e^{\frac{9}{2}}=(\sqrt{e})^9.$$

(e)

$$\lim_{x\to0}\left(\frac{\sin 2x}{\arcsin 3x}\right)^{\operatorname{arccot} x}=\left[\left(\frac{0}{0}\right)^{\frac{\pi}{2}}\right]=\left|\begin{array}{l}\sin 2x\sim 2x\\ \arcsin 3x\sim 3x\end{array}\right|=\left(\frac{2}{3}\right)^{\lim\limits_{x\to0}\operatorname{arccot} x}=\left(\frac{2}{3}\right)^{\pi/2}.$$

(f)

$$\lim_{x\to2}\frac{\lg(5-2x)}{\sqrt{10-3x}-2}=\left[\frac{0}{0}\right]=\left|\begin{array}{l}t=x-2\\ t\to0\end{array}\right|=\lim_{t\to0}\frac{\lg(1-2t)}{\sqrt{4-3t}-2}=\lim_{t\to0}\frac{\lg(1-2t)}{2\cdot\sqrt{1-\frac{3t}{4}}-2}=$$
$$\left|\begin{array}{l}\lg(1-2t)\sim\dfrac{-2t}{\ln 10}\\ \sqrt{1-\frac{3t}{4}}-1\sim\dfrac{1}{2}\cdot\dfrac{-3t}{4}\end{array}\right|=\frac{1}{2\ln 10}\cdot\lim_{t\to0}\frac{16t}{3t}=\frac{8}{3\ln 10}.$$

Question #3.

Conditions:

(a) Show that the given functions $f(x)=x^2+\sqrt{x}+\sin x$ and $g(x)=\ln\cos\sqrt{x}$ are infinitely-large or infinitely-small functions as $x\to0+$;

(b) For both functions find their main part (equivalent function in the form $C(x-x_0)^\alpha$ if $x\to x_0$ or $C(x)^\alpha$ if $x\to\infty$), state their asymptotic orders of growth;

(c) Compare the functions $f(x)$ and $g(x)$ as $x\to0+$.

Solutions:

(a) To show that the given functions are infinitely-small as $x\to0+$ we should evaluate their limits with the given tendency of the argument x:

$$\lim_{x\to0+}f(x)=\lim_{x\to0+}(x^2+\sqrt{x}+\sin x)=[0+0+0]=0.$$

$$\lim_{x\to0+}g(x)=\lim_{x\to0+}\ln\cos\sqrt{x}=[\ln\cos 0]=[\ln 1]=0.$$

(b) The functions $f(x)$ and $g(x)$ are infinitely-small as $x \to 0+$, therefore their main parts have the form of $C(x)^{\alpha}$, and may be found using asymptotically equivalent functions.

$$\sin x \sim x, \text{ as } x \to 0+ \implies f(x) = x^2 + \sqrt{x} + \sin x \;\sim\; x^2 + \sqrt{x} + x,$$

$\because$ an infinitely-small polynomial is equivalent to its monomial of the lowest power:

$$x^2 + \sqrt{x} + x \;\sim\; \sqrt{x} = x^{(1/2)},$$
$$\therefore\; f(x) = x^2 + \sqrt{x} + \sin x \;\sim\; x^{(1/2)}.$$

$$1 - \cos x \sim \frac{x^2}{2}, \text{ as } x \to 0+ \implies g(x) = \ln\cos\sqrt{x} = \ln(1 + (\cos\sqrt{x} - 1)) \;\sim\; \ln(1 - \frac{x}{2}),$$
$$\ln(1+x) \sim x, \text{ as } x \to 0+ \implies \ln(1 - \frac{x}{2}) \sim -\frac{x}{2},$$
$$\therefore\; g(x) = \ln\cos\sqrt{x} \;\sim\; -\frac{x}{2}.$$

Consequently, as $x \to 0+$, the main part of $f(x)$ is $1(x)^{1/2}$ and the main part of $g(x)$ is $-\frac{1}{2}(x)^1$.

(c) To compare $f(x)$ and $g(x)$ as $x \to 0+$, we should consider the limit of the relation between the main parts of $f(x)$ and $g(x)$:

$$\lim_{x\to 0+} \frac{g(x)}{f(x)} = \lim_{x\to 0+} \frac{-(1/2)x}{\sqrt{x}} = -\frac{1}{2} \cdot \lim_{x\to 0+} x^{\frac{1}{2}} = 0.$$

$\therefore$ $g(x)$ is infinitely-small relative to $f(x)$, as $x \to 0+$. In Bachmann-Landau notation this may be represented as $g(x) = o(f(x))$.

Moreover, it is possible to calculate the exact asymptotic order of growth of one function against the other as $x \to 0+$. Let us consider the following relation

$$\lim_{x\to 0+} \frac{g(x)}{(f(x))^{\alpha}} = C,$$

where α is the asymptotic order of growth of g(x) relative to f(x) and C is a constant.

$$\lim_{x\to 0+} \frac{g(x)}{(f(x))^{\alpha}} = C,$$
$$\lim_{x\to 0+} \frac{-(1/2)x}{(\sqrt{x})^{\alpha}} = C,$$
$$-\frac{1}{2} \cdot \lim_{x\to 0+} \frac{x}{x^{\frac{\alpha}{2}}} = C,$$
$$\therefore\; \alpha = 2.$$

Question #4.

Conditions: Find the points of discontinuity of the function $y = f(x)$ and state their class. Plot the graph of $f(x)$ in the neighborhood of each point of discontinuity.

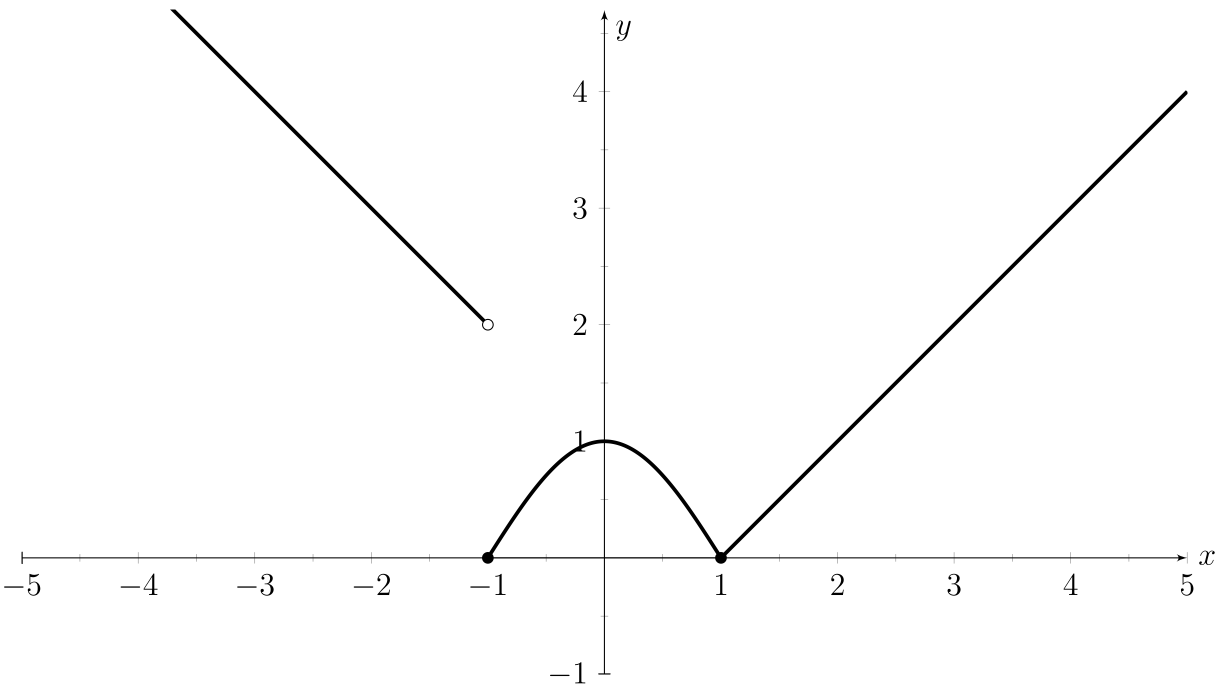
$$f(x) \equiv \begin{cases} \cos\left(\dfrac{\pi x}{2}\right), & |x| \leqslant 1, \\ |x-1|, & |x| > 1. \end{cases}$$

Solution: Both $\cos\left(\dfrac{\pi x}{2}\right)$ and $|x-1|$ are continuous functions on $\mathbb{R}$, therefore a discontinuity may only occur in the points $x_1 = -1$ and $x_2 = 1$. As such, we should evaluate the one-sided limits of $f(x)$ in these two points:

$$\lim_{x\to(-1)-} f(x) = \lim_{x\to(-1)-} |x-1| = 2, \qquad \lim_{x\to(-1)+} f(x) = \lim_{x\to(-1)+} \cos\left(\frac{\pi x}{2}\right) = 0,$$
$$\therefore \text{ Both limits exist and are finite, yet } \lim_{x\to(-1)-} f(x) \neq \lim_{x\to(-1)+} f(x),$$
$$\implies x_1 = -1, \text{ is a point of jump discontinuity.}$$

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} \cos\left(\frac{\pi x}{2}\right) = 0, \qquad \lim_{x\to 1+} f(x) = \lim_{x\to 1+} |x-1| = 0,$$
$$\therefore \text{ Both limits exist, are finite and are equal to 0,}$$
$$\implies x_2 = 1, \text{ is not a point of discontinuity.}$$

Knowing the points of discontinuity and their classification, it is possible to plot the graph of $f(x)$ in the neighborhood of the points of discontinuity:



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