Mean Variance Theory

Thursday, 21 September 2023 8:23 PM

3 portfolio derivation

$E[r_i] = z_i$ for $i = 1, 2, 3$

portfolio

Assets	1	2	& 3
$E[\cdot]$	z_1	z_2	z_3
Weights:	w_1	w_2	w_3

Let $Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ $\quad W = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ $\quad \mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\mu_p = W^T Z \text{ or } Z^T W = w_1 z_1 + w_2 z_2 + w_3 z_3$$

"Sum of weights are equal to 1" $\rightarrow$ $\mathbb{1}^T W = W^T \mathbb{1} = 1$

Let $\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{bmatrix}$, $\quad \sigma_p^2 = W^T \Sigma W$

We are trying to solve the following optimisation problem.

$$\min_{w_1, w_2, w_3} \{\sigma_p^2\} = W^T \Sigma W \quad \text{such that} \quad W^T \mathbb{1} = 1 \quad \text{and} \quad \mu_p = W^T Z$$

Lagrange: let $L(W, \lambda, \gamma) = \frac{1}{2} W^T \Sigma W + \lambda(1 - W^T \mathbb{1}) + \gamma(\mu_p - W^T Z)$

First order conditions:

$$\frac{dL}{dW} = 0, \quad \frac{dL}{d\lambda} = 0 \quad \& \quad \frac{dL}{d\gamma} = 0$$

matrix calculus

$$* \ \frac{dL}{dW} = \frac{1}{2}(2\Sigma W) - \lambda \mathbb{1} - \gamma Z$$

$$\frac{dL}{dW} = \Sigma W - \lambda \mathbb{1} - \gamma Z = 0 \qquad (1)$$

$$* \ \frac{dL}{d\lambda} = 1 - W^T \mathbb{1} = 0 \qquad (2)$$

$$* \ \frac{dL}{d\gamma} = \mu_p - W^T Z = 0 \qquad (3)$$

From (1), $\Sigma W = \lambda \mathbb{1} + \gamma Z$

since we want to solve for W that minimises σ_p^2, multiply (1) by Σ^{-1}:

$$W = \lambda \Sigma^{-1} \mathbb{1} + \gamma \Sigma^{-1} Z \qquad (4)$$

$$\bar{w} = \lambda \Sigma^{-1} \mathbb{1} + \gamma \Sigma^{-1} \bar{z} \quad (4)$$

Aside: Helpful Scalars [Memorise]

$$A = \mathbb{1}^T \Sigma^{-1} \mathbb{1}$$

$$B = \mathbb{1}^T \Sigma^{-1} \bar{z} = \bar{z}^T \Sigma^{-1} \mathbb{1}$$

$$C = \bar{z}^T \Sigma^{-1} \bar{z}$$

$$\Delta = AC - B^2$$

Note : $\bar{w} = \lambda \Sigma^{-1} \mathbb{1} + \gamma \Sigma^{-1} \bar{z}$ (4)

to solve for $\bar{w}$, we need λ and γ. therefore we aim to create a system of equations so that we can solve them simultaneously.

From (2), $1 = \bar{w}^T \mathbb{1} = \mathbb{1}^T \bar{w}$

Multiplying (4) by $\mathbb{1}^T$, $\mathbb{1}^T \bar{w} = \lambda \mathbb{1}^T \Sigma^{-1} \mathbb{1} + \gamma \mathbb{1}^T \Sigma^{-1} \bar{z}$

$$\Rightarrow \quad 1 = A\lambda + B\gamma \quad (5)$$

From (3), $\mu_p = \bar{w}^T \bar{z} = \bar{z}^T \bar{w}$

Multiplying (4) by $\bar{z}^T$, $\bar{z}^T \bar{w} = \lambda \bar{z}^T \Sigma^{-1} \mathbb{1} + \gamma \bar{z}^T \Sigma^{-1} \bar{z}$

$$\Rightarrow \quad \mu_p = B\lambda + C\gamma \quad (6)$$

We don't know what Mu is since Mu requires the weights of the portfolio which we are trying to find.

Using (5) and (6), we can now solve for λ and γ.

Solving (5) and (6) simultaneously

$$1 = A\lambda + B\gamma \quad (5)$$

$$\mu_p = B\lambda + C\gamma \quad (6)$$

$\lambda = \dfrac{1 - B\gamma}{A}$ from (5),

We now have $\mu_p = \dfrac{B}{A}(1 - B\gamma) + C\gamma$ from (6)

$$A\mu_p = B - B^2\gamma + AC\gamma$$

$$A\mu_p - B = \gamma(AC - B^2)$$

$$A\mu_p - B = \gamma(AC - B^2)$$

$$\therefore \gamma = \frac{A\mu_p - B}{AC - B^2} = \frac{A\mu_p - B}{\Delta}$$

which means that $$\lambda = \frac{1 - \frac{B}{\Delta}(A\mu_p - B)}{A}$$

$$\lambda = \frac{\Delta - AB\mu_p + B^2}{A\Delta} = \frac{AC - B^2 - AB\mu_p + B^2}{A\Delta} = \frac{A(C - B\mu_p)}{A\Delta}$$

$$\therefore \lambda = \frac{C - B\mu_p}{\Delta}$$

Now we can sub λ and γ to solve for W

$$W = \lambda\Sigma^{-1}\mathbb{1} + \gamma\Sigma^{-1}Z \qquad (4)$$

Here, we are trying to solve for Mu so that we can solve for W, this is done by creating a function of Mu that equates to variance of portfolio. We then find Mu that minimizes the variance. We use this Mu to find W.

Multiplying (4) by Σ, $\Sigma W = \lambda\mathbb{1} + \gamma Z$

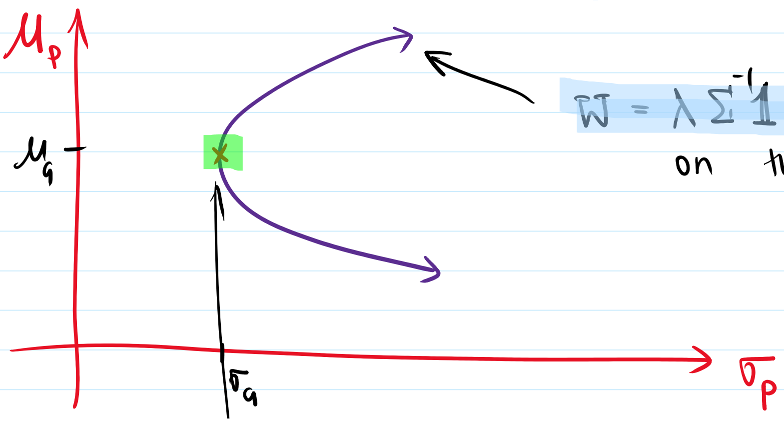
we know that $\sigma_p^2 = W^T\Sigma W$

$\therefore \sigma_{p_{min}}^2 = W^T[\lambda\mathbb{1} + \gamma Z]$ on the minimum variance set

$$\sigma_{p_{min}}^2 = \lambda\overbrace{W^T\mathbb{1}}^{1} + \gamma\underbrace{W^TZ}_{\mu_p}$$

$$\therefore \sigma_{p_{min}}^2 = \lambda + \gamma\mu_p \Rightarrow \sigma_{p_{min}}^2 = \frac{C - B\mu_p}{\Delta} + \left[\frac{A\mu_p - B}{\Delta}\right]\mu_p.$$

$$\therefore \sigma_p^2 = \frac{A\mu_p^2 - 2B\mu_p + C}{\Delta}$$



$W = \lambda\Sigma^{-1}\mathbb{1} + \gamma\Sigma^{-1}Z$ is on this curve

We are interested in the weights that result in the green x i.e. $\bar{w}$ such that (μ_g, σ_g)

$$\therefore \quad \frac{d\sigma_p^2}{d\mu_p} = 0 \implies \frac{2A\mu_p - 2B}{\Delta} = 0$$

$$\therefore \quad \mu_g = \frac{B}{A}$$

$$\sigma_g^2 = \frac{A\left(\frac{B}{A}\right)^2 - 2B\left(\frac{B}{A}\right) + C}{\Delta} = \frac{\frac{B^2}{A} - 2\frac{B^2}{A} + C}{\Delta}$$

$$= \frac{C - \frac{B^2}{A}}{AC - B^2} = \frac{AC - B^2}{A} \times \frac{1}{AC - B^2}$$

$$\sigma_g^2 = \frac{1}{A}$$

$\therefore$ Green X on graph is $\left(\frac{B}{A}, \sqrt{\frac{1}{A}}\right)$ i.e. $\mu_p = \frac{B}{A}$ minimises σ_p^2.

Now we can solve for $\bar{w}$.

$$\bar{w} = \lambda \Sigma^{-1} \mathbb{1} + \gamma \Sigma^{-1} \bar{z} \qquad (4)$$

$$\therefore \gamma = \frac{A\mu_p - B}{AC - B^2} = \frac{A\mu_p - B}{\Delta}$$

$$\therefore \lambda = \frac{C - B\mu_p}{\Delta}$$

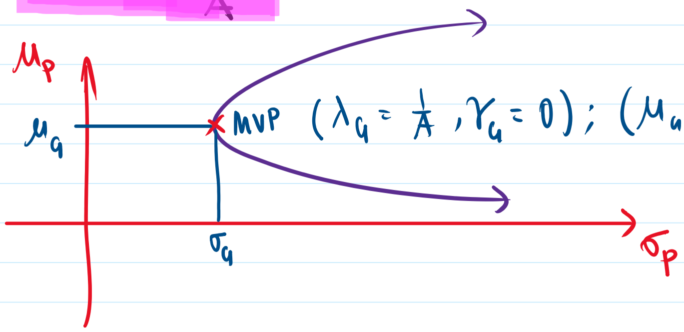
$$\therefore \lambda_g = \frac{C - B\left(\frac{B}{A}\right)}{\Delta} = \frac{C - \frac{B^2}{A}}{\Delta} = \frac{AC - B^2}{A} \times \frac{1}{AC - B^2}$$

$$\lambda_g = \frac{1}{A}$$

$$\therefore \gamma_g = \frac{A\left(\frac{B}{A}\right) - B}{\Delta} = 0$$

$\therefore \quad W_g = \dfrac{\Sigma^{-1}\mathbb{1}}{A}.$

W All possible W give the purple curve such that $W = \lambda \Sigma^{-1}\mathbb{1} + \gamma \Sigma^{-1} Z$



MVP $(\lambda_g = \frac{1}{A}, \gamma_g = 0)$; (μ_g, σ_g) $\therefore$ $\boxed{W_g = \dfrac{\Sigma^{-1}\mathbb{1}}{A}}$