Chapter 3 Counting

§3.1. Basic Principles of Counting

3.1.1. *Addition Principle (for Disjoint Sets)*

If a task T_1 can be performed in n_1 ways and a task T_2 can be performed in n_2 ways, then the number of ways of performing task T_1 or task T_2 is n_1+n_2.

e.g. There are 5 subjects offered in the morning, and there are 4 subjects offered in the afternoon. If a student wants to take only one subject, then he/she has $5+4=9$ choices.

Note Addition Principle is also called Rule of Sum.

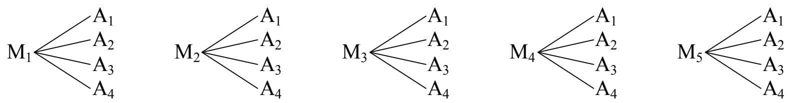
3.1.2. *Multiplication Principle*

Suppose two tasks T_1 and T_2 are to be performed in sequence. If T_1 can be performed in n_1 ways, and for each of these ways T_2 can be performed in n_2 ways, then the sequence T_1T_2 can be performed in n_1n_2 ways.

e.g. There are 5 subjects offered in the morning, and there are 4 subjects offered in the afternoon. If a student wants to take exactly one subject in the morning and exactly one subject in the afternoon, then he/she has $5\cdot 4=20$ choices.

Morning subject	Afternoon subject
M_1	A_1
M_2	A_2
M_3	A_3
M_4	A_4
M_5	

These choices can be represented by the following tree diagram:



Notes

i/. Multiplication Principle is also called Rule of Product.

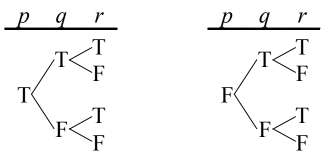
ii/. Multiplication Principle can be generalized for more than two tasks: $T_1 - n_1$ ways, $T_2 - n_2$ ways, …, $T_k - n_k$ ways. Then there are $n_1\cdot n_2\cdot\cdots\cdot n_k$ ways to perform the tasks in sequence.

3.1.3. *More Examples*

1) Let S be a compound proposition involving 3 logical variables p, q, and r. We know that each logical variable has 2 possible truth values. Thus, if we want to setup a truth table to determine the possible truth values of S, according to the (generalized) Multiplication Principle, we have to consider $2\cdot 2\cdot 2=8$ cases.

   These 8 cases are usually listed in the order as shown in the right table. We can also list these 8 cases using the following tree diagram.

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F



   Note that the order of listing the cases in the tree diagram is consistent with those listed in the table.

2) In Macao, each license plate contains 2 letters followed by 4 digits. It follows from the Multiplication Principle that there are $26\cdot 26\cdot 10^4=6{,}760{,}000$ possible different license plates.

3) Prove that if $|A|=n$, then $|\wp(A)|=2^n$, where $n\in\mathbb{Z}_+$.

   Proof: When $n=0$, $A=\varnothing \Rightarrow \wp(A)=\{\varnothing\}$. Therefore, the result is true for $n=0$.

   In what follows, suppose $n\geq 1$.

   To find $|\wp(A)|$, we want to assign to each subset of A an n-digit binary number.

   Let us illustrate this idea by looking at the case when $n=3$ and $A=\{a, b, c\}$. Consider, for instance, the subset $\{a, c\}$. We will assign the 3-digit binary number 101 to this subset. The reason is as follows. The 1st digit (counting from left to right) is 1 $\because$ a is in the subset, the 2nd digit is 0 $\because$ b is not in the subset, and so on.

   Thus there is a one-to-one correspondence between the subsets of A and the n-digit binary numbers.

e.g. $A=\{a, b, c\}$

subset of A	binary representation of the subset abc
$\varnothing$	000
$\{a\}$	100
$\{b\}$	010
$\{c\}$	001
$\{a, b\}$	110
$\{a, c\}$	101
$\{b, c\}$	011
$\{a, b, c\}$	111

Totally, according to the Multiplication Principle, there are 2^n $(=\underbrace{2\cdot 2\cdot\cdots\cdot 2}_{n\text{ factors}})$ n-digit binary numbers because each digit is either 0 or 1.

It follows from the correspondence that $|\wp(A)|=2^n$.

§3.2. Permutations

Problem

Suppose we have a set A with n elements ($n\in\mathbb{Z}^+$), and suppose r is a positive integer not exceeding n. How many sequences, each of length r, can be formed using elements from A if all elements in the sequences must be distinct?

e.g. Let $A=\{a, b, c, d\}$. Then

$$abc, acb, bcd, cbd, \ldots$$

are some of the different sequences of length 3.

Solution

Let T_1 be the task of choosing an element from A for the 1st position, T_2 the task of choosing an element (different from the chosen element) from A for the 2nd position, and so on, until T_r is performed for the r^{th} position.

T_1 can be performed in n ways since every element in A can be chosen for the 1st position. Once T_1 is performed, only $n-1$ elements remain, and so T_2 can be performed in $n-1$ ways. Continuing in this way until finally T_r is performed in $n-(r-1)=n-r+1$ ways.

By the Multiplication Principle, the required number of ways of choosing these sequences is

$$\underbrace{n\cdot(n-1)\cdot\cdots\cdot(n-r+1)}_{r\text{ factors}}.$$

The last expression is usually denoted by <u>$P(n, r)$</u> or <u>${}_nP_r$</u> or <u>P_r^n</u>. That is,

$$P(n, r)=n\cdot(n-1)\cdot\cdots\cdot(n-r+1) \qquad (n, r\in\mathbb{Z}^+, r\leq n) \quad \text{--------------} \quad (1)$$

Summing up, $P(n, r)$ represents the number of ways of permuting r distinct objects, which are taken from n distinct objects, and $P(n, r)$ can be calculated by using the above formula.

Factorial

When $r=n$ ($n\in\mathbb{Z}^+$), $P(n, r)=P(n, n)=n\cdot(n-1)\cdot\cdots\cdot 2\cdot 1$. This expression is usually denoted by <u>$n!$</u> (read as "<u>factorial n</u>" or "<u>n factorial</u>"). That is, <u>$n!=n\cdot(n-1)\cdot\cdots\cdot 2\cdot 1$</u> if $n\in\mathbb{Z}^+$. Also, we define <u>$0!=1$</u>.

Factorial Representation of $P(n, r)$

Observe that $n\cdot(n-1)\cdot\cdots\cdot(n-r+1)=\dfrac{n\cdot(n-1)\cdot\cdots\cdot(n-r+1)\cdot(n-r)\cdot\cdots\cdot 1}{(n-r)\cdot(n-r+1)\cdot\cdots\cdot 1}=\dfrac{n!}{(n-r)!}$.

Thus
$$P(n, r)=\frac{n!}{(n-r)!}. \quad \text{--} \quad (2)$$

Notes

i/. From formula (1), we have $P(n, n)=n!$.

From formula (2), we have $P(n, n)=\dfrac{n!}{0!}$.

These two resulting equations would be consistent if $0!=1$. This is why we have defined $0!=1$.

ii/. Formula (2) is useful for studying the properties of $P(n, r)$ only. To evaluate $P(n, r)$, we better use formula (1).

e.g. ${}_6P_3=6\cdot 5\cdot 4=120$; ${}_{1000}P_4=1000\cdot 999\cdot 998\cdot 997=994{,}010{,}994{,}000$.

iii/. $n!$ grows rapidly.

e.g. $6!=120$; $10!=3{,}628{,}800$; $69!=1.711\ldots\cdot 10^{98}$; $1000!=4.02387\ldots\cdot 10^{2567}$.

For large n, we may use Stirling's formula:

$$n!\approx\left(\frac{n}{e}\right)^n\cdot\sqrt{2n\pi}$$

where $e=\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n=2.718281828459\ldots$ is called the <u>base of natural logarithm</u>.

Examples

1) Find the number of permutations of the letters from $\{a, b, c, d\}$ taken 2 at a time.

   Solution: Here n=4, r=2. Required number=${}_4P_2=4\cdot 3=12$.

2) Find the number of permutations of the letters from $\{a, b, c, d\}$ taken all at a time.

   Solution: Required number=${}_4P_4=4!=4\cdot 3\cdot 2\cdot 1=24$.

   To list these 24 permutations, we may use a tree diagram:

   - a
     - b < c—d, d—c
     - c < b—d, d—b
     - d < b—c, c—b
   - b
     - a < c—d, d—c
     - c < a—d, d—a
     - d < a—c, c—a
   - c
     - a < b—d, d—b
     - b < a—d, d—a
     - d < a—b, b—a
   - d
     - a < b—c, c—b
     - b < a—c, c—a
     - c < a—b, b—a

3) Determine the number of ways in which we can make up strings of four distinct letters followed by three distinct digits.

   Solution: This involves two different permutations: permutation of 4 distinct letters from the 26 letters (a to z), and permutation of 3 distinct digits from the 10 digits (0 to 9).

   By the Multiplication Principle,

   $$\text{required number}={}_{26}P_4\cdot {}_{10}P_3\,(=26\cdot 25\cdot 24\cdot 23\cdot 10\cdot 9\cdot 8=258{,}336{,}000).$$

§3.3. Permutations with Repetition

Problem

In how many ways can we permute the letters of the word "DADDY"?

Solution

Firstly tag the D's in order to distinguish among them temporarily. That is, instead of considering the permutations of "DADDY", we consider the permutations of "$D_1AD_2D_3Y$". According to formula (1) of the previous section, there are $5!=120$ permutations of the 5 objects D_1, D_2, D_3, A, and Y.

But actually the D's are the same here, so some of these 120 permutations are the same (referring to the original problem) except for the order in which the D's appear. For example, $D_1AD_2D_3Y$, $D_1AD_3D_2Y$, $D_2AD_1D_3Y$, and so on, are the same (referring to the original problem), i.e. DADDY. In fact, there are $3!=6$ permutations in this example.

Suppose there are x permutations of the letters of the word "DADDY" (i.e. x is the required number). Each of these permutations (like the above example) generates $3!$ permutations regarding the D's different. Thus, if the D's are tagged, then there are $x\cdot 3!$ permutations, which is the same as the number of permutations of 5 distinct objects. Hence

$$x\cdot 3!=5!$$

$$\Rightarrow \qquad x=\frac{5!}{3!}=5\cdot 4=20.$$

Using the same argument, we can prove the following

Theorem

The number of permutations that can be formed from a collection of n objects, in which one of them appears k_1 times, another one appears k_2 times, and so on, is given by

$$\frac{n!}{k_1!\,k_2!\cdots k_r!} \qquad (k_1+k_2+\cdots+k_r=n).$$

Examples

1) Find the number of ways to permute 4 balls – 2 red, 1 blue, 1 white.

   Solution: Here $k_1=2$, $k_2=1$, $k_3=1$, and $n=4$. $\therefore$ Required number$=\frac{4!}{2!\,1!\,1!}=4\cdot 3=12$.

2) Find the number of ways to paint 12 offices so that 3 of them will be green, 2 of them pink, 2 of them yellow, and the remaining ones white.

   Solution: Required number$=\frac{12!}{3!\,2!\,2!\,5!}=\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6}{6\cdot 2\cdot 2}=166{,}320$.

§3.4. Combinations

Problem

Let A be a set with n elements, and let $r \in \mathbb{Z}$ such that $1 \le r \le n$. How many different subsets of A are there, exactly r elements?

Solution

Suppose there are x such subsets. For each of these subsets, there are $r!$ ways to permute the elements of t Therefore, totally there are $x \cdot r!$ ways to permute r distinct objects which are taken from n distinct objects. T

$$x \cdot r! = {}_nP_r \quad \Rightarrow \quad x = \frac{{}_nP_r}{r!}$$

The last expression is usually denoted by $\underline{C(n, r)}$ or $\underline{{}_nC_r}$ or $\underline{C_r^n}$. Thus,

$${}_nC_r = \frac{{}_nP_r}{r!} \qquad\qquad (n, r \in \mathbb{Z}^+, r \le n)$$

Summing up, $C(n, r)$ represents the number of combinations of choosing r distinct objects from n distinct ob $C(n, r)$ can be calculated by using the above formula.

Using the factorial notation, we have

$${}_nC_r = \frac{n!}{r!\,(n-r)!}$$

Note that this formula implies

$${}_nC_r = {}_nC_{n-r}$$

For example, $${}_{20}C_{18} = {}_{20}C_2 = \frac{20 \cdot 19}{2 \cdot 1} = 190.$$

Examples

1) How many committees of 4 members can be formed from 10 people?

   Solution: Required number $= {}_{10}C_4 = \dfrac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$.

2) A farmer buys 3 cows, 2 pigs and 4 hens from a man who has 6 cows, 5 pigs and 8 hens. How mar does the farmer have?

   Solution: The farmer can choose the cows in ${}_6C_3$ ways, the pigs in ${}_5C_2$ ways, and the hens in ${}_8C_4$ ways According to the Multiplication Principle, altogether the farmer can choose the animals in

   $${}_6C_3 \cdot {}_5C_2 \cdot {}_8C_4 = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} \cdot \frac{5 \cdot 4}{2 \cdot 1} \cdot \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 20 \cdot 10 \cdot 70 = 14{,}000$$

   ways.

§3.5. Miscellaneous Examples

1) In how many ways can three men and three women be seated in a row if men and women must occup seats?

   Solution: The figure on the right shows 2 of the required ways. The 3 women (or men) can permute among themselves.

W_1	M_1	W_2	M_2
M_1	W_1	M_2	W_2

   $\therefore$ Required number $= {}_3P_3 \cdot {}_3P_3 \cdot {}_2P_2 = 3! \cdot 3! \cdot 2! = 6 \cdot 6 \cdot 2 = 72$.

2) In how many ways can we arrange the letters of the word "COMMITTEE"?

   Solution: This is a problem of permutation with repetition.

   $\therefore$ Required number $= \dfrac{9!}{2! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1!} = 45{,}360$.

4) How many 4-member committees can be formed among 12 students if student A or student B must be included?

Solution: *Method* I

Number of committees that includes A but not $B={}_{10}C_3=120$.

Number of committees that includes B but not $A={}_{10}C_3=120$.

Number of committees that includes both A and $B={}_{10}C_2=45$.

$\therefore$ Required number$=2\cdot{}_{10}C_3+{}_{10}C_2=285$.

Method II

The number of required committees is equal to "number of all possible 4-member committees" minus "number of 4-member committees excluding A and B".

$\therefore$ Required number$={}_{12}C_4-{}_{10}C_4=495-210=285$.

Remark i/. This problem can also be solved by applying PIE. Using this third method, the required number is given by ${}_{11}C_3+{}_{11}C_3-{}_{10}C_2=285$. Details are left as an exercise.

ii/. Among the above methods, the second one is the best because we do not have to consider too many cases.

5) In how many ways can 4 people be divided into 2 pairs?

Solution: Suppose the 4 people are A, B, C, and D.

To form the 1st pair, there are ${}_4C_2=6$ ways (viz. $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C\}$, $\{B, D\}$, and $\{C, D\}$). Once the 1st pair is chosen, there is ${}_2C_2=1$ way to form the 2nd pair (see the figure).

① $\{\{A, B\}, \{C, D\}\}$　④ $\{\{B, C\}, \{A, D\}\}$
② $\{\{A, C\}, \{B, D\}\}$　⑤ $\{\{B, D\}, \{A, C\}\}$
③ $\{\{A, D\}, \{B, C\}\}$　⑥ $\{\{C, D\}, \{A, B\}\}$

However there are repetitions in the above consideration (e.g. ① is the same as ⑥, ② is the same as ⑤, and ③ is the same as ④), because the order of forming the pairs is taken into consideration.

$\therefore$ Required number of ways$=\dfrac{{}_4C_2\cdot{}_2C_2}{2!}=3$.

§3.6. Binomial Theorem

We have

$$(a+b)^0=1$$
$$(a+b)^1=a+b$$
$$(a+b)^2=a^2+2ab+b^2$$
$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$$
$$\cdots$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
....

Pascal Δ
(or Yanghui Δ)

The above coefficients form the Pascal Triangle (also called Yanghui Triangle). In fact, these coefficients are the combination numbers, e.g.

$$(a+b)^4={}_4C_0a^4+{}_4C_1a^3b+{}_4C_2a^2b^2+{}_4C_3ab^3+{}_4C_4b^4$$

More generally, we have

$$(a+b)^n={}_nC_0a^n+{}_nC_1a^{n-1}b+\cdots+{}_nC_{n-1}ab^{n-1}+{}_nC_nb^n=\sum_{r=0}^{n}{}_nC_r\,a^{n-r}b^r \qquad (\forall n\in\mathbb{Z}^+)$$

This is called the Binomial Theorem for positive integral exponent. It could be proved by using PMI (problem 9 of Assignment 3 could be used in the induction step; details are left as an exercise).

Examples

1) Prove that for any $n\in\mathbb{Z}^+$, ${}_nC_0+{}_nC_1+\cdots+{}_nC_{n-1}+{}_nC_n=2^n$.

Proof: Simply take $a=b=1$ in the Binomial Theorem.

2) Prove that for any $n\in\mathbb{Z}^+$, if $|A|=n$, then $|\wp(A)|=2^n$.

Proof: We have already proved this in 3.1.3. Here we give a second proof.

Recall that for any $r\in\mathbb{Z}_+$ with $r\le n$, ${}_nC_r$ represents the number of those subsets S such that $|S|=r$. Thus the total number of subsets (i.e. $|\wp(A)|$) is given by ${}_nC_0+{}_nC_1+\cdots+{}_nC_{n-1}+{}_nC_n$. It follows from Example 1 that $|\wp(A)|={}_nC_0+{}_nC_1+\cdots+{}_nC_{n-1}+{}_nC_n=2^n$.

§3.7. Pigeonhole Principle

Pigeonhole Principle (also called Drawer Principle)

If there are n pigeons and m pigeonholes, and if $n>m$ (i.e. there are more pigeons than pigeonholes), then there is at least one pigeonhole which contains two or more pigeons.

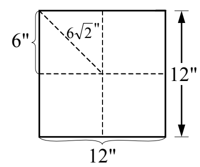
Examples

1) Among any 13 students, there are at least 2 students have the same month of birth because there are only 12 months.

2) Among any 367 people, there are at least 2 people have the same birthday because there are only 366 possible birthdays (note that 29th of February is a possible birthday).

3) Five pins are randomly distributed on a 1'×1' square board (1'=one foot=12 inches=12"). Show that there are always at least two pins such that the distance between them is less than 8.5". Can you improve this upper bound?

   Proof: Divide the board into 4 identical small squares as shown in the picture. These small squares play the role of pigeonholes, and the pins play the role of pigeons.

   Observe that for each of these small squares, the largest distance between any two points of the square $=6\sqrt{2}$" (=length of the diagonal).

   According to the Pigeonhole Principle, there are at least 2 pins that belong to the same small square. It follows from the above observation that the distance between these 2 points does not exceed $6\sqrt{2}$" $=8.485...$"<8.5".

   The above argument shows that $6\sqrt{2}$" is a better upper bound.

4) Show that among any 5 distinct numbers chosen from the set $A=\{1, 2, 3, 4, 5, 6, 7, 8\}$, there are at least 2 of them will add up to 9.

   Proof: Divide the given set A into 4 smaller sets:

   $$A_1=\{1, 8\}, A_2=\{2, 7\}, A_3=\{3, 6\}, A_4=\{4, 5\}.$$

   Observe that (i) $A_1 \cup A_2 \cup A_3 \cup A_4=A$, and the A_i's are pairwise disjoint (i.e. $A_i \cap A_j=\varnothing$ whenever $i \neq j$), and (ii) for $k=1, 2, 3$, or 4, the 2 elements of A_k will add up to 9.

   Observation (i) means that the A_i's can play the role of pigeonholes. Here, the 5 chosen numbers play the role of pigeons.

   By the Pigeonhole Principle, among the 5 chosen numbers, there are at least 2 of them belong to the same A_k for some $1 \leq k \leq 4$. By observation (ii), these 2 numbers will add up to 9.