Chapter 1 Sets

§1.1. Overview

A *set* is a collection of well-defined objects.

e.g. The set of all students in this class
The set of all integers
The set of all rational numbers

Strictly speaking, the above sentence is **not** a definition for the term "set". Actually, "set" should be acce undefined term. There are some terms associated with the concept of sets — "belong", "element" (a "member").

Notations

1) Usually we use capital letters, like A, B, C, … to denote sets, and small letters, like a, b, c, … to de elements.
2) If a is an element of a set A, we denote this by
$$a \in A \text{ (read as "}a\text{ belongs to }A\text{" or "}a\text{ is in }A\text{" or "}A\text{ contains }a\text{").}$$
3) If a is not an element of A, we write $a \notin A$.
4) If, for example, a, b, and c are all the elements of the set A, we denote this by
$$A = \{a, b, c\}.$$

e.g. Let $A = \{1, 3, 5, 7\}$. Then A contains exactly 4 elements, viz. 1, 3, 5, 7. Here $1 \in A$, but $2 \notin A$.

Notes

1) Repeated elements in a set can be ignored. Thus, e.g., the sets $\{a, c, b, c\}$ and $\{a, b, c\}$ are the same.
2) The order in which the elements of a set are listed is not important. Thus, e.g., the sets $\{a, b, c, d\}$ and { are the same.

§1.2. Some Frequently Used Sets

1.2.1. $\mathbb{Z}^+$

This denotes the set of all positive integers, i.e. $\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}$.

1.2.2. $\mathbb{Z}$

This denotes the set of all integers, i.e. $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. Hand-written form of $\mathbb{Z}$: $\mathbb{Z}$.
[Note 0 is neither positive nor negative.]

1.2.3. $\mathbb{Z}^-$

This denotes the set of all negative integers, i.e. $\mathbb{Z}^- = \{-1, -2, -3, -4, \ldots\}$.

1.2.4. $\mathbb{Z}_+$

This denotes the set of all nonnegative integers, i.e. $\mathbb{Z}_+ = \{0, 1, 2, 3, 4, \ldots\}$.

1.2.5. $\mathbb{Z}_-$

This denotes the set of all nonpositive integers, i.e. $\mathbb{Z}_- = \{0, -1, -2, -3, -4, \ldots\}$.

1.2.6. $\mathbb{Q}$

This denotes the set of all rational numbers. A rational number is a number that can be expressed as a **ra integers**. Hand-written form of $\mathbb{Q}$: $\mathbb{Q}$.

Example 2 Prove that $\sqrt{2} \notin \mathbb{Q}$ (i.e. $\sqrt{2}$ is irrational).

Proof: We would use "proof by contradiction".

Suppose $\sqrt{2} \in \mathbb{Q}$. Then by the definition of rational numbers, $\sqrt{2} = \frac{a}{b}$ for some positive integers a and b.

Further, we may suppose the fraction $\frac{a}{b}$ is already in reduced form, that is, we may suppose

$$a \text{ and } b \text{ have no common factor} > 1 \quad \text{--------------------------} \quad (*).$$

Note that

$$\sqrt{2} = \frac{a}{b} \Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2 \quad \text{--------------------------------} \quad (1).$$

(1) tells us that a^2 is even.

Since the square of an odd integer is still odd, we infer that <u>a is even</u>. That is, $a=2c$ for some positive integer c. Put this into (1):

$$(2c)^2 = 2b^2 \Rightarrow 4c^2 = 2b^2 \Rightarrow b^2 = 2c^2 \quad \text{----------------------} \quad (2).$$

Using the same argument as before, we see from (2) that <u>b is also even</u>.

Now a and b are both even. This means that a and b have 2 as a common factor. But this contradicts $(*)$.

This contradiction leads to the conclusion that $\sqrt{2}$ cannot be rational, which is what we want to prove.

Remark i/. Using the method employed in the proof of Example 1, we can show that all recurring decimals are rational.

ii/. Two integers a and b satisfying $(*)$ of Example 2 are said to be <u>*relatively prime*</u> (to each other). For instance, 24 and 25 are relatively prime, while 24 and 141 are not relatively prime [we also say that 24 is relatively prime to 25, but not relatively prime to 141].

iii/. Similar to the proof of Example 2, it can be shown that $\sqrt{3}$ is irrational (exercise).

1.2.7. $\mathbb{R}$

This denotes the set of all real numbers. This set contains all rational and irrational numbers.

Note The symbols $\mathbb{Q}^+$, $\mathbb{Q}^-$, $\mathbb{Q}_+$, $\mathbb{Q}_-$, $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}_+$, and $\mathbb{R}_-$ can be similarly interpreted. Hand-written form of $\mathbb{R}$: .

§1.3. Set Descriptions

The following are two common ways of describing sets:

1.3.1. *Enumeration Method*

To describe a set by enumeration method, we simply list the elements of the set, e.g. $\{1, 2, 3\}$.

1.3.2. *Descriptive Property Method*

To describe a set by descriptive property method, we specify the property that the elements of the set in common using the notation $\{x \mid P(x)\}$ (read as "<u>The set of all those x such that $P(x)$</u>").

Examples

1) Using the descriptive property method, the set $\{1, 2, 3\}$ might be expressed as

$$\{x \mid x \text{ is a positive integer less than } 4\}.$$

[Read as "<u>The set of all those objects x such that x is a positive integer less than 4</u>"]

2) The set consisting of all the letters in the word "byte" might be described respectively by the enumeration method and the descriptive property method as follows:

$$\{\text{b, e, t, y}\}$$
$$\{x \mid x \text{ is a letter in the word "byte"}\}$$

§1.4. Equality of Sets

Definition: Two sets A and B are said to be <u>equal</u> if they have the same elements. This is denoted by

$A=B$ (read as "<u>A is equal to B</u>" or "<u>A equals B</u>")

e.g. Let $A=\{1, 2, 3\}$, $B=\{x \mid x$ is a positive integer less than $4\}$, and $C=\{1, 2\}$. Then $A=B$, and $A\neq C$.

§1.5. Empty Set

Definition: The set contains no elements.

Notation: { } or $\varnothing$.

Note: An empty set is also called a *null set*.

Examples

1) $\{x \mid x \text{ is a real number and } x^2=-1\}=\varnothing$ $\because$ the square of any real number must be nonnegative.
   Remark The left-hand side of the above equality can be concisely written as
   $$\{x\in\mathbb{R} \mid x^2=-1\}$$
   [Read as "The set of all those real numbers x such that $x^2=-1$"]
2) $\{x\in\mathbb{Q} \mid x^2=2\}=\varnothing$ $\because$ it is proved that $\sqrt{2}\notin\mathbb{Q}$.
   However, $\{x\in\mathbb{R} \mid x^2=2\}\neq\varnothing$ $\because$ $\sqrt{2}$ belongs to this set.
3) Let S be the set of all those students in this class who failed in *Discrete Mathematics*. If all students passed, then $S=\varnothing$ (and vice versa).

§1.6. Miscellaneous Examples

1) Let $A=\{1, 2, 4, a, b, c\}$ (a, b, and c are interpreted as ordinary letters). Determine true or false:
   (a) $\varnothing\in A$ (b) $\{\ \}\notin A$ (c) $A\in A$
   Ans.: (a) F, (b) T, (c) F ('T' stands for 'true', and 'F' stands for 'false').
   [Please provide explanations yourself.]
2) List the elements in the set $A=\{n\in\mathbb{Z} \mid n^2=2\}$.
   Ans.: $A=\{\ \}$ or $A=\varnothing$.
3) Write the following set in the form $\{x \mid P(x)\}$: $\{-2, -1, 0, 1, 2\}$ (i.e. rewrite this set using the descriptive property method).
   Ans.: $\{x\in\mathbb{Z} \mid |x|<3\}$ or $\{x\in\mathbb{Z} \mid |x|\le 2\}$.
   Remark i/. There are other possible answers.
   ii/. Recall that $|x|$ is called the absolute value of x and is defined as
   $$|x|=\begin{cases} x & \text{if } x\ge 0 \\ -x & \text{if } x<0 \end{cases}$$

§1.7. Subsets

Definition: Let A and B be two given sets. If every element of A is also an element of B, then A is said to be a subset of B. In that case, we write $A\subseteq B$.

We write $A\nsubseteq B$ to mean "A is not a subset of B".

Examples

1) Let $A=\{1, 2, 3, 4, 5, 6\}$, $B=\{2, 4, 5\}$, and $C=\{1, 2, 3, 4, 5\}$.
   We have $B\subseteq A$, $B\subseteq C$, and $C\subseteq A$;
   but $A\nsubseteq B$ ($\because$ $1\in A$ & $1\notin B$) and $C\nsubseteq B$ ($\because$ $3\in C$ & $3\notin B$).
2) $\mathbb{Z}^+\subseteq\mathbb{Z}$, $\mathbb{Z}\subseteq\mathbb{Q}$ (i.e. every integer is rational $\because$ for any integer x, $x=\frac{x}{1}$);
   $\mathbb{Q}\subseteq\mathbb{R}$, but $\mathbb{R}\nsubseteq\mathbb{Q}$ ($\because$ $\sqrt{2}\in\mathbb{R}$ & $\sqrt{2}\notin\mathbb{Q}$).

Notes

i/. If A is a set, then it is clear from the definition that $A\subseteq A$. That is, every set is a subset of itself.

ii/. If A is a set, then $\varnothing\subseteq A$. That is, empty set is a subset of any set.
Proof: Suppose, to the contrary, that $\varnothing\nsubseteq A$.
Then, according to the definition of subset, there exists at least one element $x\in\varnothing$ such that $x\notin A$.
But $\varnothing$ has no elements. $\therefore$ $\varnothing\subseteq A$.

§1.8. Proper Subsets

Definition: If A is a subset of B, $A \neq B$ and $A \neq \varnothing$, then A is said to be a proper subset of B. In that case, we write $A \subset B$.

Examples

1) $\{a, b\}$ and $\{x, y\}$ are proper subsets of $\{x, y, c, b, a\}$.
2) $\mathbb{Z}^+ \subset \mathbb{Z}$, $\mathbb{Z} \subset \mathbb{Q}$, and $\mathbb{Q} \subset \mathbb{R}$.

§1.9. Embedded Sets

Definition: A set that contains sets is called an embedded set.

Examples

1) $\{\{a, b\}, d\}$, $\{\{a, b, c\}, \{a, d, e, f\}, \{b, e, g\}\}$, and $\{a, \{a\}, \{\{a\}\}\}$are all embedded sets.
2) Let $S_1 = \{\text{John, Mary}\}$, $S_2 = \{\{\text{John, Mary}\}\}$, and $S_3 = \{\{\{\text{John, Mary}\}\}\}$.
   S_2 and S_3 are different embedded sets. Note that John $\in S_1$, but John $\notin S_2$.
   Note also that $S_2 = \{S_1\}$ and $S_3 = \{\{S_1\}\} = \{S_2\}$. Here $S_1 \in S_2$ and $S_2 \in S_3$, but $S_1 \notin S_3$.

§1.10. More Miscellaneous Examples

1) Determine true or false:
   (a) $\varnothing \subseteq \varnothing$ (b) $\varnothing \in \varnothing$ (c) $\varnothing \in \{\varnothing\}$ (d) $\{a, b\} \in \{a, b, c, \{a, b, c\}\}$
   (e) $\{a, b\} \subseteq \{b, a, c, \{a, b\}\}$ (f) $\{a, b\} \in \{b, a, c, \{a, b\}\}$ (g) $\{a, \varnothing\} \subseteq \{a, \{a, \varnothing\}\}$
   Ans.: (a) T, (b) F, (c) T, (d) F, (e) T, (f) T, (g) F.
   [Please provide explanations yourself.]
2) Give an example of sets A, B, and C such that $A \in B$, $B \in C$, and $A \notin C$.
   Ans.: Consider the sets $A = \varnothing$, $B = \{\varnothing\}$, and $C = \{\{\varnothing\}\}$.
   Note that C contains exactly one element, viz. $\{\varnothing\}$, which is not the same as $\varnothing$. $\therefore A \notin C$.
3) Let $A = \{1, 2, 5, 8, 11\}$. Determine true or false:
   (a) $\{1, 6\} \nsubseteq A$ (b) $\{8, 1\} \in A$ (c) $\{2\} \subset A$ (d) $A \subseteq \{11, 2, 5, 1, 8, 4\}$
   Ans.: (a) T $\because 6 \notin A$
   (b) F $\because$ A is not an embedded set
   (c) T $\because 2 \in A$, $\{2\} \neq A$, and $\{2\} \neq \varnothing$
   (d) T $\because$ every element of A is also in $\{11, 2, 5, 1, 8, 4\}$

§1.11. Power Set

Definition: Power set of a set A is the set which contains exactly all the subsets of A. This set is denoted by $\wp(A)$.

Examples

1) $A = \varnothing$
   $\wp(A) = \{\varnothing\}$
2) $A = \{1\}$
   $\wp(A) = \{\varnothing, \{1\}\}$
3) $A = \{1, 2\}$
   $\wp(A) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$
4) $A = \{1, 2, 3\}$
   $\wp(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

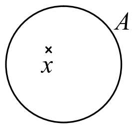
Notes

i/. For any set A, $\varnothing \in \wp(A)$ $\because$ $\varnothing$ is a subset of any set.

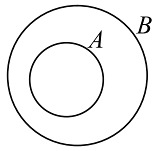
ii/. If A has n elements, then $\wp(A)$ has 2^n elements (this will be proved later).

§1.12. Venn Diagrams

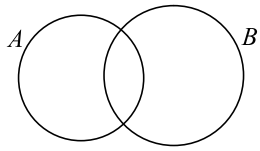
Venn diagrams are used to show relationship between sets. Some examples are as shown below.



$x \in A$



$A \subseteq B$



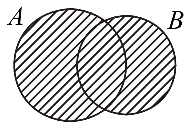
$A \nsubseteq B$ and $B \nsubseteq A$

§1.13. Set Operations

1.13.1. ***Union***

Definition: The union of two sets A and B is the set that consists of all elements that belong to A or B (or both). This set is denoted by $A \cup B$, i.e.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$



$A \cup B$

Note: In mathematics, the word "or" is used in the inclusive sense. That is, when we say "p or q", we already mean "p or q or both".

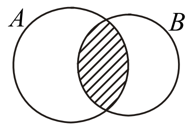
Examples

1) $\{a, b\} \cup \{c, d\} = \{a, b, c, d\}$
2) $\{a, b\} \cup \{a, c\} = \{a, b, c\}$
3) $\{a, b\} \cup \varnothing = \{a, b\}$ [In general, $A \cup \varnothing = A$ for any set A.]
4) $\{a, b\} \cup \{\{a, b\}\} = \{a, b, \{a, b\}\}$

1.13.2. ***Intersection***

Definition: The intersection of two sets A and B, denoted by $A \cap B$, is the set that consists of all elements that belong to both A and B, i.e.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$



$A \cap B$

Examples

1) $\{a, b\} \cap \{c, d\} = \varnothing$
2) $\{a, b\} \cap \{a, c\} = \{a\}$
3) $\{a, b\} \cap \varnothing = \varnothing$ [In general, $A \cap \varnothing = \varnothing$ for any set A.]
4) $\{a, b\} \cap \{\{a, b\}\} = \varnothing$

1.13.3. ***Notes on Union and Intersection***

i/. Two sets A and B that have no common elements are said to be disjoint. That is, A and B are disjoint if and only if $A \cap B = \varnothing$.

ii/. According to the definitions,

$$A \cup B = B \cup A \quad \text{and} \quad A \cap B = B \cap A$$

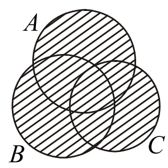
for any sets A and B (see also §1.14).

iii/. If $A \subseteq B$, then $A \cup B = B$ and $A \cap B = A$.

iv/. The operations union and intersection can be defined for three or more sets in the same way. Thus

$$A \cup B \cup C = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in C\}.$$

$$A \cap B \cap C = \{x \mid x \in A \text{ and } x \in B \text{ and } x \in C\}.$$



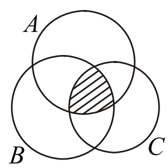
$A \cup B \cup C$

e.g. Let $A = \{1, 2, 3, 4, 5, 7\}$, $B = \{1, 3, 8, 9\}$, and $C = \{1, 3, 6, 8\}$. Then $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A \cap B \cap C = \{1, 3\}$.

Notation for more sets:

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{k=1}^{n} A_k$$

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{k=1}^{n} A_k$$

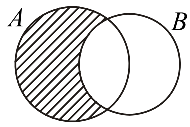


$A \cap B \cap C$

1.13.4. ***Difference***

Definition: The difference of two sets A and B, denoted by $A-B$, is the set that consists of all those elements of A which do not belong to B, i.e.

$$A-B=\{x \mid x\in A \text{ and } x\notin B\}.$$



$A-B$

Examples

1) $\{a, b, c\}-\{a\}=\{b, c\}$, $\{a, b, c\}-\{d, e\}=\{a, b, c\}$.
2) $\{a, b, c\}-\{a, d\}=\{b, c\}$, $\{a, d\}-\{a, b, c\}=\{d\}$.

Notes

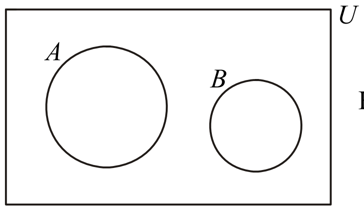
i/. In general, $A-B\neq B-A$.

ii/. The set $A-B$ is also called the complement of B with respect to A ["with respect to" is usually abbreviated as "w.r.t."] or complement of B in A.

1.13.5. ***Complement***

Universal Set

Universal set is a set including all the sets under discussion. This set is usually denoted by U. In a Venn diagram, U is represented by a rectangle.

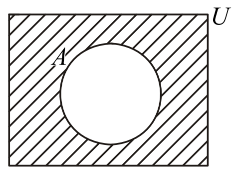


Here, U includes A, B, …

Complement of A

Definition: If U is the universal set, then $U-A$ is called the complement of A, and is denoted by A^c (or $\overline{A}$). That is,

$$A^c=U-A=\{x\in U \mid x\notin A\}.$$



A^c

Examples

1) Let $U=\{x\in\mathbb{Z}^+ \mid x\leq 10\}$ and $A=\{2, 3, 5, 7\}$. Then $A^c=\{1, 4, 6, 8, 9, 10\}$.
2) Let $U=\mathbb{Z}$. Let A and B denote respectively the set of all odd integers and the set of all even integers. Then $A^c=B$ and $B^c=A$.

Notes

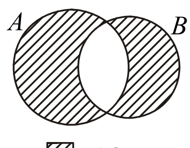
i/. When we talk about the complement of a set, we must have a universal set lying at the back in the first place.

ii/. Using the terminology of the previous subsection, A^c is the complement of A w.r.t. U.

iii/. There is no "real universal set". That is, there does not exist a set that contains everything. A famous paradox, called "Russell's Paradox", is related to this.

1.13.6. ***Symmetric Difference***

Definition: The symmetric difference of two sets A and B is the set that contains all the elements that belong to A or B, but not to both A and B. This set is denoted by $A\oplus B$.



$A\oplus B$

It is readily seen from the Venn diagram that

$$\boxed{A\oplus B=(A\cup B)-(A\cap B)=(A-B)\cup(B-A)}.$$

Examples

1) $\{a, b\}\oplus\{a, c\}=\{b, c\}$
2) $\{a, b\}\oplus\varnothing=\{a, b\}$. In fact, $A\oplus\varnothing=A$ for any set A.
3) $\{a, b\}\oplus\{a, b\}=\varnothing$. In fact, $A\oplus A=\varnothing$ for any set A.
4) Let $A=\{2, 3, 5, 7, 11\}$ and $B=\{2, 3, 6, 12\}$.
   Then $A\oplus B=(A\cup B)-(A\cap B)=\{2, 3, 5, 6, 7, 11, 12\}-\{2, 3\}=\{5, 6, 7, 11, 12\}$.

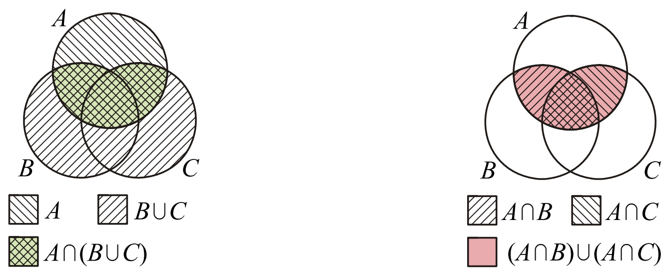
§1.14. Properties of Set Operations

In what follows, A, B, and C denote arbitrary sets, and U denotes the universal set.

1. $A\cup A=A$, $A\cap A=A$ (Idempotent Law)
2. $A\cup B=B\cup A$, $A\cap B=B\cap A$ (Commutative Law)
3. $(A\cup B)\cup C=A\cup(B\cup C)$, $(A\cap B)\cap C=A\cap(B\cap C)$ (Associative Law)
4. $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$, $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$ (Distributive Law)
   Remark a/. The first equality can be described as "$\cap$ is distributive over $\cup$".
   b/. The second equality can be similarly described as "$\cup$ is distributive over $\cap$".
5. $\left.\begin{array}{l} A\cup\varnothing=A \\ A\cap U=A \end{array}\right\}$ Identity Law $\quad\begin{array}{l} A\cap\varnothing=\varnothing \\ A\cup U=U \end{array}$
   Remark a/. The "Identity Law" for usual addition is $x+0=x$ for all $x\in\mathbb{R}$.
   b/. The "Identity Law" for usual multiplication is $x\cdot 1=x$ for all $x\in\mathbb{R}$.
6. $(A^c)^c=A$, $A\cup A^c=U$, $A\cap A^c=\varnothing$, $\varnothing^c=U$, $U^c=\varnothing$ (Complement Law)
7. $(A\cup B)^c=A^c\cap B^c$, $(A\cap B)^c=A^c\cup B^c$ (De Morgan's Law)

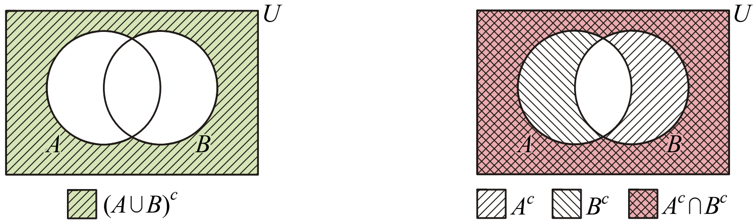
Notes

i/. Verification of $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$ by Venn diagrams:



These diagrams show that $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$.

ii/. Verification of $(A\cup B)^c=A^c\cap B^c$ by Venn diagrams:



These diagrams show that $(A\cup B)^c=A^c\cap B^c$.

Exercise Verify by Venn diagrams that $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$ and $(A\cap B)^c=A^c\cup B^c$.

§1.15. Cardinality

Definitions and Notations

a) The cardinality of a finite set is the number of elements in the set. The cardinality of a set A is denoted by $|A|$.
   The concept of cardinality can be extended to infinite sets. However, in this course, we will only consider cardinalities for finite sets.
b) Let x be a real number. The integer part of x, denoted by $[x]$, is the greatest integer not exceeding x.
   e.g. $[\frac{1}{3}]=0$, $[-3]=-3$, $[\pi]=3$, & $[-\pi]=-4$.
c) Two positive integers a and b are said to be relatively prime to each other if they have no positive common factor other than 1.
   e.g. i/. 24 is relatively prime to 35
   ii/. 1 is relatively prime to any positive integer n
   iii/. 133 is not relatively prime to 98

Examples on Cardinality

1) If $A=\{2, 3, 5, 7\}$, then $|A|=4$.

2) $|\varnothing|=0$

3) Let n denote a nonnegative integer. If $|A|=n$, then $|\wp(A)|=2^n$ (will be proved later).

4) Let $A=\{n\in\mathbb{Z}^+ \,|\, n\le 100 \text{ and } n \text{ is divisible by } 6\}$. Then $|A|=16$.

   Proof: Let $n\in A$. By the definition of divisibility, $n=6k$ for some positive integer k, where $6k\le 100$.
   Note that $6k\le 100 \Rightarrow k\le \frac{100}{6}=16.666\ldots$ This implies that $A=\{6\cdot 1, 6\cdot 2, 6\cdot 3, \cdots, 6\cdot 16\}$.
   $\therefore\ |A|=16$ $\left(\text{note that } 16=\left[\frac{100}{6}\right]\right)$.

5) The above example can be generalized as follows.
   Let $m, d\in\mathbb{Z}^+$. Then

$$\boxed{|\{n\in\mathbb{Z}^+ \,|\, n\le m \text{ and } n \text{ is divisible by } d\}|=\left[\frac{m}{d}\right]}$$

6) Let $A=\{n\in\mathbb{Z}^+ \,|\, n\le 20 \text{ and } n \text{ is relatively prime to } 6\}$. Find $|A|$.

   Solution: We do this by using the enumeration method:

$$A=\{1, 5, 7, 11, 13, 17, 19\}.$$

   $\therefore\ |A|=7$.

7) (Exercise) Let $A=\{n\in\mathbb{Z}^+ \,|\, n\le 30 \text{ and } n \text{ is relatively prime to } 15\}$. Find $|A|$.

8) Let $U=\{n\in\mathbb{Z}^+ \,|\, n\le 3000\}$, $S=\{n\in U \,|\, n \text{ is relatively prime to } 12\}$, $A=\{n\in U \,|\, n \text{ is divisible by } 2\}$, and $B=\{n\in U \,|\, n \text{ is divisible by } 3\}$. Express S in terms of A and B (or their complements).

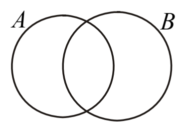
   Solution: Firstly note that 12 has exactly two prime factors, viz. 2 and 3.
   Secondly observe that n ($\in\mathbb{Z}^+$) is not relatively prime to 12 if and only if 2 or 3 is a factor of n. We might express this in terms of sets:

$$\begin{aligned} S^c &= \{n\in U \,|\, n \text{ is not relatively prime to } 12\} \\ &= \{n\in U \,|\, n \text{ is divisible by 2 or by 3}\} \\ &= A\cup B. \end{aligned}$$

   $\therefore$ By De Morgan's Law, $S=(A\cup B)^c=A^c\cap B^c$.

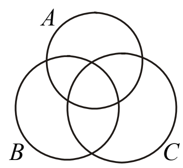
   Remark What can you say about $|S|$?

§1.16. Principle of Inclusion and Exclusion



For two finite sets A and B, their union $A\cup B$ is also finite, and

$$|A\cup B|=|A|+|B|-|A\cap B| \quad \text{--------------------------------} \quad (1).$$



Identity (1) can be readily seen by using a Venn diagram. This is the simplest form of the Principle of Inclusion and Exclusion (this principle will be abbreviated as PIE). PIE is also known as the Addition Principle.

Let A, B, C be finite sets. (1) can be extended to three finite sets as follows:

$$\begin{aligned} |A\cup(B\cup C)| &= |A|+|B\cup C|-|A\cap(B\cup C)| && \text{(By (1))} \\ &= |A|+|B\cup C|-|(A\cap B)\cup(A\cap C)| && \text{(By Distributive Law)} \\ &= |A|+(|B|+|C|-|B\cap C|)-(|A\cap B|+|A\cap C|-|(A\cap B)\cap(A\cap C)|) && \text{(By (1) twice)} \\ &= \underline{|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|} \end{aligned}$$

Similarly, for four finite sets A_1, A_2, A_3, and A_4, we have

$$\begin{aligned} |A_1\cup A_2\cup A_3\cup A_4| = {} & |A_1|+|A_2|+|A_3|+|A_4|-|A_1\cap A_2|-|A_1\cap A_3|-|A_1\cap A_4|-|A_2\cap A_3|-|A_2\cap A_4|-|A_3\cap A_4| \\ & +|A_1\cap A_2\cap A_3|+|A_1\cap A_2\cap A_4|+|A_1\cap A_3\cap A_4|+|A_2\cap A_3\cap A_4|-|A_1\cap A_2\cap A_3\cap A_4| \end{aligned}$$

Examples

1) A computer company must hire 25 programmers to handle system programming jobs, and 40 programmers for application programming. For those hired, 10 will be expected to perform jobs of both types. How many programmers must be hired?

   Solution: Let A, B be the sets of system programmers and application programmers hired respectively. Then, according to the given,
   $$|A|=25, \qquad |B|=40, \qquad |A\cap B|=10.$$
   By PIE, $|A\cup B|=|A|+|B|-|A\cap B|=25+40-10=55$.
   That is, 55 programmers must be hired.

2) A survey has been taken on methods of commuter travel. Each respondent was asked to check BUS, TRAIN, or AUTOMOBILE as a major method of traveling to work. More than one answer was permitted. The results reported were as follows:

   BUS, 30 people; TRAIN, 35 people; AUTOMOBILE, 100 people; BUS & TRAIN, 15 people; BUS & AUTOMOBILE, 15 people; TRAIN & AUTOMOBILE, 20 people; all three methods, 5 people.

   How many people completed a survey form?

   Solution: Let B, T, A be the sets of people traveled by BUS, TRAIN, AUTOMOBILE respectively.
   Then the required number is represented by $|B\cup T\cup A|$.
   According to the given,
   $$|B|=30,\ |T|=35,\ |A|=100,\ |B\cap T|=15,\ |B\cap A|=15,\ |T\cap A|=20,\ |B\cap T\cap A|=5.$$
   By PIE, we have
   $$\begin{aligned}|B\cup T\cup A|&=|B|+|T|+|A|-|B\cap T|-|B\cap A|-|T\cap A|+|B\cap T\cap A|\\&=(30+35+100)-(15+15+20)+5\\&=120.\end{aligned}$$
   That is, 120 people completed the survey form.

3) How many integers in $U=\{1, 2, 3, \ldots, 1000\}$ are divisible by 3 or 5?

   Solution: Let $A=\{n\in U\,|\,n \text{ is divisible by } 3\}$ and $B=\{n\in U\,|\,n \text{ is divisible by } 5\}$.
   Then
   $$\begin{aligned}A\cap B&=\{n\in U\,|\,n \text{ is divisible by 3 and 5}\}\\&=\{n\in U\,|\,n \text{ is divisible by } 15\}. \qquad (\because\ 15=\text{LCM}(3, 5))\end{aligned}$$
   It follows from Example 5 of §1.15 that for any positive integer d,
   $$|\{n\in U\,|\,n \text{ is divisible by } d\}|=\left\lfloor\frac{1000}{d}\right\rfloor.$$
   $\therefore\ |A|=\left\lfloor\frac{1000}{3}\right\rfloor=333,\ |B|=\left\lfloor\frac{1000}{5}\right\rfloor=200,\ |A\cap B|=\left\lfloor\frac{1000}{15}\right\rfloor=66.$

   $\therefore$ The required number$=|A\cup B|=|A|+|B|-|A\cap B|=533-66=467$.
   That is, 467 integers in U are divisible by 3 or 5.

4) Among the integers 1, 2, 3, …, 1000, how many of them are divisible by 4, or by 5, or by 6?

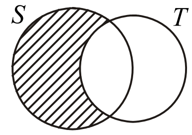
   Solution: Let $U=\{1, 2, 3, \ldots, 1000\}$, $A=\{n\in U\,|\,n \text{ is divisible by } 4\}$, $B=\{n\in U\,|\,n \text{ is divisible by } 5\}$, and $C=\{n\in U\,|\,n \text{ is divisible by } 6\}$.
   Similar to the previous example, if n_1 denotes the required number, then $n_1=|A\cup B\cup C|$.
   Note that LCM(4, 5)$=$20, LCM(4, 6)$=$12, LCM(5, 6)$=$30, and LCM(4, 5, 6)$=$60. Thus
   $$|A|=\left\lfloor\frac{1000}{4}\right\rfloor=250,\ |B|=\left\lfloor\frac{1000}{5}\right\rfloor=200,\ |C|=\left\lfloor\frac{1000}{6}\right\rfloor=166,\ |A\cap B|=\left\lfloor\frac{1000}{20}\right\rfloor=50,$$
   $$|A\cap C|=\left\lfloor\frac{1000}{12}\right\rfloor=83,\ |B\cap C|=\left\lfloor\frac{1000}{30}\right\rfloor=33,\ |A\cap B\cap C|=\left\lfloor\frac{1000}{60}\right\rfloor=16.$$
   By PIE, we have
   $$\begin{aligned}n_1&=|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|\\&=616-166+16\\&=466.\end{aligned}$$
   That is, among the integers 1 to 1000, 466 of them are divisible by 4 or by 5 or by 6.

5) Among the integers 1, 2, 3, …, 1000, how many of them are divisible by 4 or by 6, but not by 5?

Solution: Let U, A, B, and C denote the same sets as in the solution of the previous example.

Let $n_1=|A\cup B\cup C|$, and let n_2 denote the required number here. Then $n_2=|(A\cup C)\cap B^c|$.

It can be seen from a Venn diagram that for any two finite sets S and T, we have



$$|S\cap T^c|=|S\cup T|-|T|. \quad \text{----------} \quad (*)$$

It follows from $(*)$ and Example 4 that

$$\begin{aligned} n_2 &= |(A\cup C)\cup B|-|B| \\ &= n_1-|B| \\ &= 466-200 \\ &= 266. \end{aligned}$$

That is, among the integers 1 to 1000, 266 of them are divisible by 4 or by 6, but not by 5.