Chapter 2 Logic

§2.1. Propositions

Definition: A proposition (also called statement) is a declarative sentence that is either true or false.

Examples

1) The following are propositions:

"The earth is round"
"The earth is moving around the sun"
"2+3=5"
"$\sqrt{3}$ is rational"

The above propositions, except the last one, are true.

2) The following are not propositions:

"Do you speak English?"
"Take two aspirins"
"$3-x=5$"

The last sentence is not a proposition because it has no definite truth value, i.e., it is true or false depending on the actual value of the unknown x.

Notations

1) Usually we use p, q, r, … to represent propositions. They are also called logical variables.
2) The truth values "true" and "false" are abbreviated as "T" and "F" respectively.
3) We use the following way

p: The sun is shinning today
q: 24 is even

to mean "p denotes the proposition 'The sun is shinning today'" and "q denotes the proposition '24 is even'".

§2.2. Compound Propositions

Definition: A compound proposition is formed from existing propositions using logical connectives, such as *not*, *and*, *or*, *if ... then*, etc.

e.g. The following are compound propositions:

"The earth is not flat"
"2 is prime or 3 is even"
"If a^2 is even, then a is even" ($a \in \mathbb{Z}$)

The truth value of a compound proposition depends only on the truth values of the propositions being combined and on the type of the connectives used. Details are given in the next two sections.

§2.3. Logical Operations

2.3.1. *Negation*

Definition: Negation of p is a proposition which is true when p is false and is false when p is true. Negation of p is denoted by $\sim p$ (read as "not p").

The above definition can be summarized in the following truth table:

p	$\sim p$
T	F
F	T

e.g. Let p: All students of this class wear glasses.
Then $\sim p$: There is at least one student of this class who does not wear glasses.
Remark Let q: All students of this class do not wear glasses.
Note that q is not the negation of p, because p and q could be both false.

2.3.2. *Conjunction*

Definition: Conjunction of p and q is a proposition which is true when both p and q are true, and false otherwise. Conjunction of p and q is denoted by $p\wedge q$ (read as "p and q").

p	q	$p\wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

e.g. Let p: It is snowing, and q: I am cold.
Then $p\wedge q$: It is snowing and I am cold.

2.3.3. *Disjunction*

Definition: Disjunction of p and q is a proposition which is false when both p and q are false, and true otherwise. Disjunction of p and q is denoted by $p\vee q$ (read as "p or q").

p	q	$p\vee q$
T	T	T
T	F	T
F	T	T
F	F	F

e.g. Let p: 2 is a positive integer, and q: $\sqrt{2}$ is a rational number.
Then $p\vee q$: 2 is a positive integer or $\sqrt{2}$ is a rational number.

2.3.4. *Conditional (Implication)*

Definition: Implication of q by p is a proposition which is false when p is true and q is false, and true otherwise. Implication of q by p is denoted by $p\to q$.

p	q	$p\to q$
T	T	T
T	F	F
F	T	T
F	F	T

"$p\to q$" is read as

"If p then q"
or "p implies q"
or "p is a sufficient condition of q"
or "q is a necessary condition of p".

Examples

1) Let p: a is even, and q: a^2 is even ($a\in\mathbb{Z}$).
Then $p\to q$: If a is even, then a^2 is even,
and $q\to p$: If a^2 is even, then a is even.

Remark i/. In this example, both $p\to q$ and $q\to p$ are proved to be true.
ii/. Generally speaking, it is possible that $p\to q$ is true, but $q\to p$ is false (consider, e.g. p: x=2 and q: x^2=4; here the universal set is $\mathbb{R}$).

2) Let p: a^2 is a multiple of 3, and q: a is a multiple of 3.
Then $p\to q$: If a^2 is a multiple of 3, then a is a multiple of 3.

Remark The above conditional can be proved to be true, and it plays a very important role in the proof of "$\sqrt{3}\notin\mathbb{Q}$".

2.3.5. *Biconditional (Equivalence)*

Definition: Equivalence of p and q is a proposition which is true if both p and q are true or both p and q are false, and false otherwise. Equivalence of p and q is denoted by $p\leftrightarrow q$ (read as "p if and only if q" or "p iff q").

p	q	$p\leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

e.g. Let p: a^2 is a multiple of 3, and q: a is a multiple of 3.
Then $p\leftrightarrow q$: a^2 is a multiple of 3 iff a is a multiple of 3.

Remark The above biconditional can be proved to be true.

§2.4. Evaluate Compound Propositions

A compound proposition may have many logical variables.

For example, the compound proposition s: $p\to(q\wedge(p\to r))$ involves three logical variables (namely p, q, and r) and two logical operations (namely $\to$ and $\wedge$).

Usually, the truth value of a compound proposition is given in the form of a table, as illustrated by the following examples.

Examples

1) Determine the truth table for $(p\to q)\leftrightarrow(\sim q\to\sim p)$.

Solution:

p	q	$p\to q$	$\leftrightarrow$	$\sim q$	$\to$	$\sim p$
T	T	T	T	F	T	F
T	F	F	T	T	F	F
F	T	T	T	F	T	T
F	F	T	T	T	T	T

From the table, we see that $(p\to q)\leftrightarrow(\sim q\to\sim p)$ is always true.

Remarks

i/. A proposition that is always true is called a tautology. A tautology is denoted by **T** (boldface T).

ii/. A proposition that is always false is called a contradiction. A contradiction is denoted by **F** (boldface F).

iii/. We write "$p\Rightarrow q$" to mean "$p\to q$ is a tautology".

iv/. We write "$p\Leftrightarrow q$" (read as "p is equivalent to q") to mean "$p\leftrightarrow q$ is a tautology". The above example shows that

$$p\to q\Leftrightarrow\sim q\to\sim p.$$

v/. We do not use the equal sign for equivalent propositions. For instance, we **do not write** "$p\to q=\sim q\to\sim p$".

2) Determine the truth table for $((p\to q)\wedge(q\to r))\to(p\to r)$.

Solution:

p	q	r	$(p\to q)$	$\wedge$	$(q\to r)$	$\to$	$(p\to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	F
T	F	T	F	F	T	T	T
T	F	F	F	F	T	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

$\therefore$ $((p\to q)\wedge(q\to r))\to(p\to r)$ is a tautology, i.e. $((p\to q)\wedge(q\to r))\Rightarrow(p\to r)$.

Remark If we take a closer look at the above table, we **do not** have $(p\to q)\wedge(q\to r)\Leftrightarrow(p\to r)$.

§2.5. Properties of Logical Operations

In what follows, p, q, and r denote arbitrary propositions; **T** denotes a tautology, and **F** denotes a contradiction.

1. $p\wedge p\Leftrightarrow p$, $p\vee p\Leftrightarrow p$ (Idempotent Law))
2. $p\wedge q\Leftrightarrow q\wedge p$, $p\vee q\Leftrightarrow q\vee p$ (Commutative Law)
3. $(p\wedge q)\wedge r\Leftrightarrow p\wedge(q\wedge r)$, $(p\vee q)\vee r\Leftrightarrow p\vee(q\vee r)$ (Associative Law)
4. $p\wedge(q\vee r)\Leftrightarrow(p\wedge q)\vee(p\wedge r)$, $p\vee(q\wedge r)\Leftrightarrow(p\vee q)\wedge(p\vee r)$ (Distributive Law)
5. $\left.\begin{array}{l} p\wedge\mathbf{T}\Leftrightarrow p \\ p\vee\mathbf{F}\Leftrightarrow p \end{array}\right\}$ Identity Law $\quad$ $\begin{array}{l} p\vee\mathbf{T}\Leftrightarrow\mathbf{T} \\ p\wedge\mathbf{F}\Leftrightarrow\mathbf{F} \end{array}$

   Remark a/. **T** (tautology) plays the role of "identity" for conjunction.

   b/. **F** (contradiction) plays the role of "identity" for disjunction.
6. $\sim(\sim p)\Leftrightarrow p$, $p\wedge(\sim p)\Leftrightarrow\mathbf{F}$, $p\vee(\sim p)\Leftrightarrow\mathbf{T}$, $\sim\mathbf{T}\Leftrightarrow\mathbf{F}$, $\sim\mathbf{F}\Leftrightarrow\mathbf{T}$ (Negation Law)
7. $\sim(p\wedge q)\Leftrightarrow(\sim p)\vee(\sim q)$, $\sim(p\vee q)\Leftrightarrow(\sim p)\wedge(\sim q)$ (De Morgan's Law)

Notes

i/. The properties listed here are the counterparts of the set operations, and they can be proved by using truth tables.

ii/. $\wedge$ is the counterpart of $\cap$, and $\vee$ is the counterpart of $\cup$.

iii/. **T** is the counterpart of U (universal set), and **F** is the counterpart of $\varnothing$ (empty set).

iv/. $\sim$ (negation) is the counterpart of ^{c} (complement).

§2.6. Quantifiers

2.6.1. *Predicate*

Definition: A predicate is a sentence which involves variables and that this sentence becomes a proposition when specific values are assigned to the variables.

e.g. "$x<8$", "$x^2+2x-3=0$", and "$x-4y=5$" are predicates.

Usually we use $P(x)$, $Q(x)$, and $R(x, y)$, etc. to represent predicates.

Note A predicate is also called a propositional function. As a practice in mathematics, sometimes a predicate is also referred to as a "proposition".

2.6.2. *Universal Quantifier* ($\forall$)

A proposition involving the universal quantifier is usually written in the form $\forall x\ P(x)$, and is read as "For all x, $P(x)$". "$\forall x\ P(x)$" is referred to as a proposition quantified by the universal quantifier.

When we talk about a quantified proposition, we must have a universal set lying at the back in the first place.

Examples

1) The quantified proposition "$\forall n\in\mathbb{Z}^+,\ 1+2+\cdots+n=\frac{1}{2}n(n+1)$" can be translated as "For all positive integers n, $1+2+\cdots+n$ is equal to $\frac{1}{2}n(n+1)$".

   Here, $U=\mathbb{Z}^+$, and $P(n)$: $1+2+\cdots+n=\frac{1}{2}n(n+1)$.

2) "$\forall n\in\mathbb{Z}^+,\ 1^2+2^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$" is another similar example.

   This example and the previous one can be proved to be true quantified propositions by using a method called *Mathematical Induction* (see §2.8).

3) Let U=the set of all students of this class, and let $P(x)$: x is a boy.

   Then "$\forall x\ P(x)$" means "For all students x of this class, x is a boy", which can be translated further as (in the language of ordinary English) "All students of this class are boys".

2.6.3. *Existential Quantifier* ($\exists$)

A proposition involving the existential quantifier is usually written in the form $\exists x\ P(x)$, and is read as "There exists an x such that $P(x)$". "$\exists x\ P(x)$" is referred to as a proposition quantified by the existential quantifier.

Examples

1) "$\exists x\in\mathbb{R}\ x^2+2x-3=0$" means "There exists a real number x such that $x^2+2x-3=0$". This proposition is true $\because$ $x^2+2x-3=0$ is satisfied when $x=1$.

2) Let p: "$\exists x\in\mathbb{R}\ x^2=2$" and q: "$\exists x\in\mathbb{Q}\ x^2=2$". Then p is true and q is false $\because$ $\sqrt{2}$ is a real number but $\sqrt{2}$ is irrational.

3) Let U=the set of all students of this class, and let $P(x)$: x is a girl.

   Then "$\exists x\ P(x)$" means "There exists a student x of this class such that x is a girl", which can be translated further as "There is a girl in this class".

2.6.4. *Negation of a Quantified Proposition*

(a) $\sim(\forall x\ P(x))\Leftrightarrow\exists x\ (\sim P(x))$

e.g. Let U=the set of all prime numbers, and let $P(x)$: x is odd.

Then $\sim(\forall x\ P(x))\Leftrightarrow\exists x\ (\sim P(x))$. That is, the negation of "All prime numbers are odd" is "There is at least one prime number x such that x is not odd", which is equivalent to "There is at least one even prime number".

Note that this negation is **not** equivalent to "All prime numbers are even".

(b) $\sim(\exists x\ P(x))\Leftrightarrow\forall x\ (\sim P(x))$

e.g. Negation of "$\exists x\in\mathbb{Q}\ x^2=3$" is "$\forall x\in\mathbb{Q}\ x^2\neq 3$".

§2.7. Miscellaneous Examples

1) Given that $p \rightarrow q$ is false, determine the value of $(\sim p) \wedge q$.

   Solution: Since $p \rightarrow q$ is false, we need to consider only one case:

p	q	$\sim p$	$\wedge$	q
T	F	F	$\underline{\underline{\text{F}}}$	F

2) Simplify $(p \wedge q) \vee p$ (i.e., replace the given one by a logically equivalent simpler proposition).

   Solution:
   $$\begin{aligned} (p\wedge q)\vee p &\Leftrightarrow (p\wedge q)\vee(p\wedge \mathbf{T}) && \text{(Identity Law)}\\ &\Leftrightarrow p\wedge(q\vee \mathbf{T}) && \text{(Distributive Law)}\\ &\Leftrightarrow p\wedge \mathbf{T} && \text{(Property 5)}\\ &\Leftrightarrow p && \text{(Identity Law)} \end{aligned}$$

   Remark i/. This can also be done by setting up a truth table.
   ii/. Do not use the Distributive Law in the first step; otherwise we will go into a cycle.

3) Let $P(x)$: x is even, and $Q(x, y)$: $3x-5y=1$. The variables x and y represent integers (i.e. $U=\mathbb{Z}$ here). Write an English sentence that corresponds to $\forall x\ (P(x)\vee(\exists y\ Q(x, y)))$.

   Solution: For all integers x, x is even or there exists an integer y such that $3x-5y=1$.

4) Write the negation of the proposition in Example 3 symbolically.

   Solution:
   $$\begin{aligned} \sim\forall x\ (P(x)\vee(\exists y\ Q(x,y))) &\Leftrightarrow \exists x \sim (P(x)\vee(\exists y\ Q(x,y))) && \text{(By 2.6.4 (a))}\\ &\Leftrightarrow \exists x\ ((\sim P(x))\wedge\sim(\exists y\ Q(x,y))) && \text{(By De Morgan's Law)}\\ &\Leftrightarrow \exists x\ ((\sim P(x))\wedge(\forall y \sim Q(x,y))) && \text{(By 2.6.4 (b))} \end{aligned}$$

§2.8. Mathematical Induction

This is a very powerful proof technique, which applies to propositions (i.e. predicates) involving a positive integral variable n. "$1+2+\cdots+n=\frac{1}{2}n(n+1)$" is a typical example of such a proposition.

Principle of Mathematical Induction

For a given proposition $P(n)$ involving a positive integer n, if we can show that

(B) $P(n)$ is true for $n=n_0$, where n_0 is a certain specific number; and

(I) $P(k+1)$ is true under the assumption that $P(k)$ is true for some integer $k\geq n_0$.

Then $P(n)$ is true for all $n\geq n_0$.

Notes

i/. Condition (B) is called the basis, and (I) the induction step.

ii/. The assumption mentioned in condition (I) is called the induction assumption or induction hypothesis.

iii/. When we apply the principle, we need only to verify conditions (B) and (I). Whenever (B) and (I) are met, the conclusion is true.

iv/. The value of n_0 depends on the proposition. Usually, $n_0=1$.

v/. This principle will be abbreviated as PMI.

Examples

1) Prove that $1^2+2^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)\ \forall n\in\mathbb{Z}^+$.

   Proof: Let $P(n)$: $1^2+2^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$.

   When $n=1$, LHS$=1^2=1$, and RHS$=\frac{1}{6}\cdot 1\cdot(1+1)\cdot(2\cdot 1+1)=\frac{6}{6}=1$.

   $\therefore$ LHS=RHS, i.e. $P(1)$ is true.

   Assume $P(k)$ is true for some $k\in\mathbb{Z}^+$.
   That is, for this k, we have
   $$1^2+2^2+\cdots+k^2=\frac{1}{6}k(k+1)(2k+1).$$

   Then

$$\begin{aligned}
1^2+2^2+\cdots+k^2+(k+1)^2 &= \tfrac{1}{6}k(k+1)(2k+1)+(k+1)^2 && \text{(By induction assumption)}\\
&= \tfrac{1}{6}(k+1)[k(2k+1)+6(k+1)]\\
&= \tfrac{1}{6}(k+1)(2k^2+7k+6))\\
&= \tfrac{1}{6}(k+1)(k+2)(2k+3)\\
&= \tfrac{1}{6}(k+1)[(k+1)+1][2(k+1)+1]
\end{aligned}$$

$\therefore P(k+1)$ is also true.

$\therefore$ By PMI, $P(n)$ is true $\forall n\in\mathbb{Z}^+$.

Remark For the first line in the proof, **do not** write "Let $P(n)$: $1^2+2^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$ $\forall n\in\mathbb{Z}^+$". Note that "$1^2+2^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$ $\forall n\in\mathbb{Z}^+$" is the same as "$1^2+2^2+\cdots+m^2=\frac{1}{6}m(m+1)(2m+1)$ $\forall m\in\mathbb{Z}^+$". Hence n plays no role in "$1^2+2^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$ $\forall n\in\mathbb{Z}^+$".

2) Prove that $\forall n\geq 4$ $(n\in\mathbb{Z})$, $n!\geq 2^n$.

Proof: Let $P(n)$: $n!\geq 2^n$.

When $n=4$, LHS$=4!=24$, and RHS$=2^4=16$. $\therefore$ LHS$>$RHS, i.e. $P(4)$ is true.

Assume $P(k)$ is true for some $k\geq 4$. That is, $k!\geq 2^k$.

Then
$$\begin{aligned}
(k+1)! &= (k+1)\cdot k!\\
&> (k+1)\cdot 2^k && \text{(By induction assumption)}\\
&> 2\cdot 2^k && (k+1>2 \because k\geq 4)\\
&= 2^{k+1}
\end{aligned}$$

$\therefore P(k+1)$ is also true. Hence by PMI, $P(n)$ is true $\forall n\geq 4$.

Remark $n!$ is called factorial n (or n factorial), and is defined as $n!=1\cdot 2\cdot\cdots\cdot n$ if $n\in\mathbb{Z}^+$.

3) Prove that 8^n-2^n is divisible by 6 $\forall n\in\mathbb{Z}^+$.

Proof: Let $P(n)$: 8^n-2^n is divisible by 6.

$8^1-2^1=6$ which is divisible by 6, i.e. $P(1)$ is true.

Assume $P(k)$ is true for some $k\in\mathbb{Z}^+$. That is, $8^k-2^k=6a$ for some $a\in\mathbb{Z}$.

Then
$$\begin{aligned}
8^{k+1}-2^{k+1} &= (6+2)\cdot 8^k-2\cdot 2^k\\
&= 6\cdot 8^k+2(8^k-2^k)\\
&= 6\cdot 8^k+2\cdot 6a && \text{(By induction hypothesis)}\\
&= 6(8^k+2a)
\end{aligned}$$

$\therefore P(k+1)$ is also true $\because 8^k+2a\in\mathbb{Z}$.

$\therefore$ By PMI, $P(n)$ is true $\forall n\in\mathbb{Z}^+$.

4) Prove that n^5-n is divisible by 5 $\forall n\in\mathbb{Z}^+$.

Proof: Let $P(n)$: n^5-n is divisible by 5.

$1^5-1=0=5\cdot 0\Rightarrow P(1)$ is true.

Assume $P(k)$ is true for some $k\in\mathbb{Z}^+$. That is, $k^5-k=5a$ for some $a\in\mathbb{Z}$.

Then
$$\begin{aligned}
(k+1)^5-(k+1) &= k^5+5k^4+10k^3+10k^2+5k+1-k-1\\
&= k^5-k+5(k^4+2k^3+2k^2+k)\\
&= 5a+5(k^4+2k^3+2k^2+k) && \text{(By induction hypothesis)}\\
&= 5(a+k^4+2k^3+2k^2+k)
\end{aligned}$$

$\therefore P(k+1)$ is also true $\because a+k^4+2k^3+2k^2+k\in\mathbb{Z}$.

$\therefore$ By PMI, $P(n)$ is true $\forall n\in\mathbb{Z}^+$.