CHAPTER TWO

Mathematical Logic - Part I

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Table of Contents

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Logic:

- defines a formal language for representing knowledge and for making logical inferences
- It helps us to understand how to construct a valid argument

Logic defines:

- Syntax of statements
- The meaning of statements
- The rules of logical inference (manipulation)

Propositions

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Propositions

Definition:

A proposition (also called a statement) is a declarative sentence that is either **true** or **false** but **not both**.

Example

The following are propositions

- The earth is round.
- the earth is moving around the sun
- $2+3=5$
- $\sqrt{3}$ is rational

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Example

Some sentences that are not propositions.

- What time is it?
- Read this carefully
- $x + 1 = 2$
- $x + y = z$

- The first 2 sentences are not proposition because they are not declarative sentences.
- Sentences 3 and 4 are not proposition because they are neither true or false.(depending on the actual value of the unknown variables) Propositions 1 to 3 are true, whereas 4 is false.

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Propositions

Notations

1. Usually we use letters to denote **propositional variables** (or logical variables), that is, variables that represent propositions, just as letters are used to denote numerical variables. The conventional letters used for propositional variable are $p, q, r, s, \ldots$.
2. The truth values "true" and "false" are abbreviated as "T" and "F" respectively.
3. We use following way

$$p\text{: The sun is shinning today}$$
$$q\text{: 24 is even.}$$

4. Propositions that can't be expressed in terms of simpler propositions are called **atomic propositions**

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Compound Propositions

Compound Propositions

Mathematics always needs methods for producing new propositions from those that we already have. Many mathematical statements are constructed by combining one or more propositions.

Definition:

A **compound proposition** is from existing propositions using logical connectives. such as *not*, *and*, *or*, *if ... then*, etc.

Example

The following are compound proposition:

- the earth is not flat
- 2 is prime or 3 is even
- If a^2 is even, then a is even ($a \in \mathbb{Z}$)

The truth value of a compound proposition depends only on the truth values of the propositions being combined and on the type of the connectives used.

Logical Operations (connectives)

Table of Contents

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Negation

Definition: Negation

Let p be a proposition. The **negation** of p, denoted by $\sim p$ (also denoted by $\neg p$), is the statement

"It is not the case that p"

The proposition $\sim p$ is read "not p". The truth value of the negation of p, $\sim p$, is the opposite of the truth value of p.

Remark: The notation for the negation operator is not standardized. Although $\sim p$ and $\neg p$ are the most common notations used in mathematics to express the negation of p, other notations you might see are $\bar{p}$, $-p$, p', Np, and $!p$.

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Negation

p	$\sim p$
T	F
F	T

Table: The truth table for the negation of a proposition

Find the negation of the proposition

p: Jacky's PC runs Linux

Example

The negation is

- $\sim p$: It is not the case that Jacky's PC runs Linux.
- $\sim p$: Jacky's PC does not run Linux.

Negation

Examples

Find the negation of the propositions

1. q: Vandana's smartphone has at least 32 GB of memory

Solutions:

$\sim q$: It is not the case that Vandana's smartphone has at least 32 GB of memory

$\sim q$: Vandana's smartphone has less than 32 GB of memory.

Negation

Examples

Find the negation of the propositions

1. q: Vandana's smartphone has at least 32 GB of memory
2. r: $2+3>1$

Solutions:

$\sim r$: $2+3 \leq 1$

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Negation

Examples

Find the negation of the propositions

1. q: Vandana's smartphone has at least 32 GB of memory
2. r: $2+3>1$
3. s: All student of this class wear glasses.

Solutions:

$\sim p$: There is at least one student of this class who does not ware glasses.

Remark: Let q: All students of this class do not wear glasses.

Note that q is not the negation of s, because s and q could be both false.

Table of Contents

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Conjunction

Definition: conjunction

Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$, is the proposition "p and q". The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table: Truth table for $p \wedge q$

Example

Let p: It is snowing, and q: I am cold.
Then $p \wedge q$: It is snowing and I am cold.

Conjunction

Example

Find the conjunction of the propositions p and q:

- p: Rebeccas PC has more than 200 GB free hard disk space.
- q: The processor in Rebeccas PC runs faster than 2 GHz.

solution

The conjunction of these propositions,$p \wedge q$, is
$p \wedge q$: Rebeccas PC has more than 200 GB free hard disk space, and the processor in Rebeccas PC runs faster than 2 GHz.

Table of Contents

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Disjunction

Definition: disjunction

Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "p or q". The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table: Truth table for $p \vee q$

Example

Let p: 2 is a positive integer, and q: $\sqrt{2}$ is a rational number.

Then $p \vee q$: 2 is a positive integer or $\sqrt{2}$ is a rational number.

Disjunction

Example

Find the disjunction of the propositions p and q:

- p: Rebeccas PC has more than 200 GB free hard disk space.
- q: The processor in Rebeccas PC runs faster than 2 GHz.

solution

The conjunction of these propositions,$p \vee q$, is
$p \vee q$: Rebeccas PC has more than 200 GB free hard disk space, or the processor in Rebeccas PC runs faster than 2 GHz.

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Table of Contents

Definition: conditional (Implication)

Conditional statement or implication of q by p is a proposition which is false when p is true and q is false, and true otherwise. Implication of q by p is denoted by $p \to q$
$p \to q$ is read as "if p, then q" and has this truth table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Table: Truth table of $p \to q$

Example

If p denotes "I am at home." and q denote "It is raining" then $p \to q$ denotes "If I am at home then it is raining."
In $p \to q$, p is the *hypothesis* (*antecedent* or *premise*) and q is the *conclusion* (or *consequence*)

Examples

1. Let p: a is even, and q: a^2 is even ($a \in \mathbb{Z}$).
   Then $p \to q$: If a is even, then a^2 is even.
   and $q \to p$: If a^2 is even, then a is even
   Remark:
   i. In this example, both $p \to q$ and $q \to p$ are proved to be true.
   ii. Generally speaking, it is possible that $p \to q$ is true, but $q \to p$ is false (consider, e.g. $p: \ x = 2$ and $q := x^2 = 4$; here the universal set is $\mathbb{R}$
2. Let p: a^2 is a multiple of 3, and q : a is a multiple of 3.
   Then $p \to q$: If a^2 is a multiple of 3, then a is a multiple of 3.
   Remark: The above conditional can be proved to be true, and it plays a very important role in the proof of "$\sqrt{3} \notin \mathbb{Q}$"

Understanding Implication

- In $p \to q$ there does not need to be any connection between the antecedent or the consequent. The "meaning" of $p \to q$ depends only on the truth values of p and q.
- These implications are perfectly fine, but would not be used in ordinary English.
  - "If the moon is made of green cheese, then I have more money than Bill Gates."
  - "If the moon is made of green cheese then I'm on welfare."
  - "If $1 + 1 = 3$, then your grandma wears combat boots."
- One way to view the logical conditional is to think of an obligation or contract.
  - "If I am elected, then I will lower taxes."
  - "If you get 100% on the final, then you will get an A."
- If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge. Something similar holds for the professor. This corresponds to the case where p is true and q is false.

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Different Ways of Expressing $p \to q$

- if p, then q
- if p, q
- p implies q
- p only if q
- q unless $\sim p$
- q if p
- a necessary condition for p is q
- q when p
- q whenever p
- p is sufficient for q
- q is necessary for p
- q follows from p
- q provided that p
- a sufficient condition for q is p

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Table of Contents

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Definition: Biconditional (Equivalence)

If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$ read as "p if and only if q (or p iff q)". The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table: Truth table for $p \leftrightarrow q$

Example

Let p: a^2 is a multiple of 3, and q: a is a multiple of 3. Then $p \leftrightarrow q$: a^2 is a multiple of 3 iff a is a multiple of 3.

Remark: The above biconditional can be proved to be true.

Evaluate Compound Propositions

Evaluate Compound Propositions

A compound proposition may have many logical variables.
For example, the compound proposition $s : p \to (q \wedge (p \to r))$ involves three logical variables (namely p, q, and r) and two logical operations (namely $\to$ and $\wedge$).
Usually, the truth value of a compound proposition is given in the from of a table, as illustrated by the following examples.

Examples

1. Determine the truth table for $(p \to q) \leftrightarrow (\sim q \to \sim p)$
2. Determine the truth table for $((p \to q) \wedge (q \to r)) \to (p \to r)$

Order of Operations for Logical Operators

1. $\sim$ Evaluate negations first.
2. $\wedge, \vee$ Evaluate $\wedge$ and $\vee$ second. When both are present, parentheses may be needed.
3. $\to, \leftrightarrow$ Evaluate $\to$ and $\leftrightarrow$ third. When both are present, parentheses may be needed.

Evaluate Compound Propositions

Examples

1. **Determine the truth table for** $(p \to q) \leftrightarrow (\sim q \to \sim p)$
2. Determine the truth table for $((p \to q) \wedge (q \to r)) \to (p \to r)$

Solution: Question 1

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

From the table, we see that $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is always true.

Evaluate Compound Propositions

Remarks:

i. A proposition that is always true is called a *tautology*. A tautology is denoted by **T**(boldface T).

ii. A proposition that is always false is called a *contradiction*. A contradiction is denoted by **F** (boldface F).

iii. We write "$p \Rightarrow q$" to mean "$p \rightarrow q$" is a tautology".

iv. We write "$p \Leftrightarrow q$" (read as "p is equivalent to q") to mean "$p \leftrightarrow q$" is a tautology". The above example shows that

$$p \rightarrow q \Leftrightarrow \sim q \rightarrow \sim p$$

v. We do not use the equal sign for equivalent propositions. For instance, **we do not write** $p \rightarrow q = \sim q \rightarrow \sim p$. the notation $p \equiv q$ also denotes that p and q are logically equivalent.

Examples

2. **Determine the truth table for** $((p \to q) \wedge (q \to r)) \to (p \to r)$

Solution: Question 2

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$((p\to q)\wedge(q\to r))\to(p\to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

$\therefore$ $((p \to q) \wedge (q \to r)) \to (p \to r)$ is a tautology, i.e. $((p \to q) \wedge (q \to r)) \Rightarrow (p \to r)$

Remark If we take a closer look at the above table, we **do not** have $((p \to q) \wedge (q \to r)) \Leftrightarrow (p \to r)$