A zero mark will be given if you copy someone else's work or you let someone copy your work.

COMP122/22-12 Data Structures and Algorithms	**0 – 50 points**
Trees and Mathematical Induction	2022-03-07 *Due Date* — 2022-03-14
Class Code	
Student No.	DO **NOT** WRITE YOUR NAME

1. For integer $n \geqslant 0$, suppose the coefficients of a polynomial

$$a_0x^0 + a_1x^1 + \ldots a_{n-1}x^{n-1} = \sum_{i=0}^{n-1} a_i x^i$$

are stored in a Python list a of length n, from $a[0]$ down to $a[n-1]$, respectively. According to the Horner's rule, the polynomial can be evaluated by the following tail-recursive function.

$$f(x,a,n,s) = \begin{cases} s & \text{if } n = 0, \\ f(x,a,n-1,a[n-1]+sx) & \text{if } n \geqslant 1. \end{cases}$$

Prove by mathematical induction that, for integer $n \geqslant 0$,

$$f(x,a,n,s) = sx^n + \sum_{i=0}^{n-1} a_i x^i.$$

Note, $\left(\sum_{i=k}^{l} \cdots\right) = 0$, if $k > l$. (9 points)

Base case: for $n = 0$,

$$\begin{aligned} f(x,a,n,s) &= s && \text{[by } f\text{]} \\ &= s + 0 = sx^0 + \sum_{i=0}^{0-1} a_i x^i. && \text{[by arithmetic and } \textstyle\sum\text{]} \end{aligned}$$

Induction step: for $n \geqslant 1$,

$$\begin{aligned} f(x,a,n,s) &= f(x,a,n-1,sx+a[n-1]) && \text{[by } f\text{]} \\ &= (sx + a_{n-1})x^{n-1} + \sum_{i=0}^{n-2} a_i x^i && \text{[by induction hypothesis]} \\ &= sx^n + a_{n-1}x^{n-1} + \sum_{i=0}^{n-2} a_i x^i && \text{[by arithmetic]} \\ &= sx^n + \sum_{i=0}^{n-1} a_i x^i. && \text{[by } \textstyle\sum\text{]} \end{aligned}$$

(1)

2. Let n and s be integers, and $f(n,s) = \begin{cases} s & \text{if } n = 0, \\ 1 - s + 2f(n-1, f(n-1,s)) & \text{if } n \geqslant 1. \end{cases}$

Prove by mathematical induction that $f(n,s) = s + \frac{4^n - 1}{3}$ for all $n \geqslant 0$. (10 points)

Base case: when $n = 0$, we have $f(0,s) = s = s + \frac{4^0 - 1}{3}$.
Induction step: when $n \geqslant 1$,

$$\begin{aligned} f(n,s) &= 1 - s + 2f(n-1, f(n-1,s)) && \text{[by } f\text{]} \\ &= 1 - s + 2f\left(n-1, s + \frac{4^{n-1}-1}{3}\right) && \text{[by induction hypothesis]} \\ &= 1 - s + 2\left(\left(s + \frac{4^{n-1}-1}{3}\right) + \frac{4^{n-1}-1}{3}\right) && \text{[by induction hypothesis]} \\ &= 1 - s + 2\left(s + 2\frac{4^{n-1}-1}{3}\right) \\ &= 1 - s + 2s + 4\frac{4^{n-1}-1}{3} \\ &= s + \frac{4^n - 1}{3}. && \text{[by arithmetic]} \end{aligned}$$

______ (2).

3. For the tree shown in Figure 1:

a) Which node is the root? S (3). (1 point)

b) What is the depth of the tree? 3 (4). (2 points)

c) List all the leaf nodes? $R, D, C, L, B, I, H, G, A, E$ (5). (2 points)

d) List the pre-order traversal sequence:

$S, R, Q, M, D, C, P, L, K, B, O, I, H, G, N, F, A, E$ (6). (4 points)

e) List the post-order traversal sequence:

$R, D, C, M, Q, L, B, K, P, I, H, G, O, A, F, E, N, S$ (7). (4 points)

f) For each node in the tree:

- name the parent node,
- list the children,
- list the siblings,
- compute the depth, and
- compute the height.

Fill your answers in Table 1. (6 points)

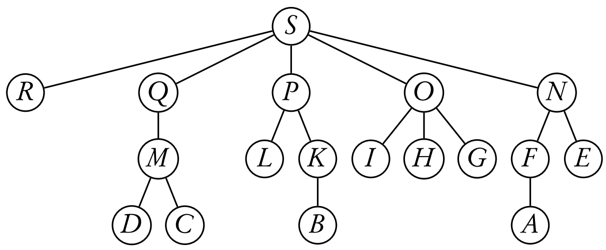

Figure 1: A Tree

	Parent	Children	Siblings	Depth	Height
A	F	--	--	3	0
B	K	--	--	3	0
C	M	--	D	3	0
D	M	--	C	3	0
E	N	--	F	2	0
F	N	A	E	2	1
G	O	--	H,I	2	0
H	O	--	G,I	2	0
I	O	--	G,H	2	0
K	P	B	L	2	1
L	P	--	K	2	0
M	Q	C,D	--	2	1
N	S	E,F	O,P,Q,R	1	2
O	S	G,H,I	N,P,Q,R	1	1
P	S	K,L	N,O,Q,R	1	2
Q	S	M	N,O,P,R	1	2
R	S	--	N,O,P,Q	1	0
S	--	N,O,P,Q,R	--	0	3

Table 1: Tree Nodes

4. For a perfectly balanced binary tree of size n ($n \geqslant 1$) and height h ($h \geqslant 0$), prove by mathematical induction on n that

$$2^h \leqslant n \leqslant 2^{h+1} - 1.$$

Try to reduce to the subtrees and apply the properties:

$$\text{(perfectly balanced tree) } n_{s-} \leqslant n_{s+} \leqslant n_{s-} + 1 \quad \text{and} \quad \text{(binary tree) } h = \max(h_{s+}, h_{s-}) + 1,$$

where $s+$ and $s-$ are the subtrees, and $s+$ is the possibly larger one. **12 points**

Base case:
1) for $n = 1$, we have $h = 0$, the tree is obviously perfectly balanced and

$$2^0 \leqslant 1 \leqslant 2^1 - 1,$$

2) for $n = 2$, we have $h = 1$, the tree is still perfectly balanced and

$$2^1 \leqslant 2 \leqslant 2^2 - 1.$$

Induction step: for $n \geqslant 3$, suppose $s+$ is the larger subtree and $s-$ is the smaller subtree. Clearly, both $s+$ and $s-$ have at least one node. We have

$$
\begin{aligned}
2^h &= 2^{\max(h_{s+}, h_{s-})+1} && \text{[by binary tree]} \\
&= \max(2 \times 2^{h_{s+}}, 2 \times 2^{h_{s-}}) && \text{[by inequality, arithmetic]} \\
&\leqslant \max(2n_{s+}, 2n_{s-}) && \text{[by induction hypothesis]} \\
&= 2n_{s+} && \text{[by perfectly balanced tree } (n_{s-} \leqslant n_{s+})\text{, max]} \\
&\leqslant n_{s+} + (n_{s-} + 1) && \text{[by perfectly balanced tree } (n_{s+} \leqslant n_{s-} + 1)\text{]} \\
&= n && {}^{*}\ \text{[by binary tree]} \\
&\leqslant (2^{h_{s+}+1} - 1) + ((2^{h_{s-}+1} - 1) + 1) && \text{[by induction hypothesis]} \\
&\leqslant (2^h - 1) + (2^h - 1) + 1 && \text{[by binary tree, max]} \\
&= 2^{h+1} - 1. && \text{[by arithmetic]}
\end{aligned}
$$

(8)