PRACTICO 3

(1)

1

$$f'(z_0) = \lim_{h\to 0} \frac{3Re(z_0+h) + 4Im(z_0+h) - 3Re(z_0) - 4Im(z_0)\,i}{h}$$

$$= \lim_{h\to 0} \frac{3Re(z_0) + 4Im(z_0) + 3Re(h) + 4Im(h)\,i - 3Re(z_0) - 4Im(z_0)\,i}{h}$$

$$= \lim_{h\to 0} \frac{3Re(h) + 4Im(h)\,i}{Re(h) + Im(h)\,i}$$

Sea $h = Re(h)$

$$f'(z_0) = \lim_{h\to 0} \frac{3Re(h)}{Re(h)} = 3$$

Sea $h = Im(h)$

$$f'(z_0) = \lim_{h\to 0} \frac{4Im(h)\,i}{Im(h)\,i} = 4$$

$$\Rightarrow \nexists f'(z_0)$$

3

$$f'(z_0) = \lim_{h\to 0} \frac{\overline{z_0+h} - \overline{z_0}}{h} = \lim_{h\to 0} \frac{\overline{z_0} + \overline{h} - \overline{z_0}}{h} = \lim_{h\to 0} \frac{\overline{h}}{h}$$

Sea $h = t \in \mathbb{R}$

$$\lim_{h\to 0} \frac{\overline{h}}{h} = \lim_{h\to 0} 1 = 1$$

Sea $h = i.t$

$$\lim_{h\to 0} \frac{\overline{h}}{h} = \lim_{h\to 0} -1 = -1$$

$$\Rightarrow \nexists f'(z_0)$$

4

$$f'(z_0) = \lim_{h\to 0} \frac{|(z_0+h)|^2 - |z_0|^2}{h} = \lim_{h\to 0} \frac{(z_0+h)(\overline{z_0+h}) - z_0\bar{z}_0}{h}$$

$$= \lim_{h\to 0} \frac{z_0\bar{z}_0 + z_0\bar{h} + h\bar{z}_0 + h\bar{h} - z_0\bar{z}_0}{h} = \lim_{h\to 0} \frac{z_0\bar{h} + h\bar{z}_0 + h\bar{h}}{h}$$

$$= \lim_{h\to 0} z_0\frac{\bar{h}}{h} + \bar{z}_0 + \bar{h}$$

Si $h = t \in \mathbb{R}$

$$\lim_{h\to 0} z_0\frac{\bar{h}}{h} + \bar{z}_0 + \bar{h} = z_0 + \bar{z}_0$$

Si $h = i.t$

$$\lim_{h\to 0} z_0\frac{\bar{h}}{h} + \bar{z}_0 + \bar{h} = -z_0 + \bar{z}_0$$

$\Rightarrow$ Si $z_0 \neq 0$ entonces $\nexists f'(z_0)$

2

Notar que $\quad f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{f(z)}{z} = \lim_{z\to 0} \frac{f'(z)}{?} = f'(0) = 1$

1

$$\lim_{z\to 0} \frac{f(2z)}{z} \underset{\text{lhopital}}{=} \lim_{z\to 0} 2.\frac{f'(2z)}{1} = 2.f'(0) = 2.1 = 2$$

2

$$\lim_{z\to 0} \frac{f(z^2)}{z} \underset{\text{lhopital}}{=} \lim_{z\to 0} \frac{2z f'(z^2)}{1} = \frac{2.0.f'(0)}{1} = 2.0.1 = 0$$

3

$$\lim_{z\to 0} \frac{f(z^2-z)}{z} = \lim_{z\to 0} \frac{f(z^2-z)}{z} \cdot \frac{z-1}{z-1} = \lim_{z\to 0} \frac{f(z^2-z)}{z^2-z} \cdot \lim_{z\to 0} z-1$$

$$= 1 \cdot (-1) = -1$$

4

$$\lim_{z\to i} \frac{f(z^2+1)}{z-i} = \lim_{z\to i} \frac{f(z^2+1)}{z-i} \frac{z+i}{z-i} = \underbrace{\lim_{z\to i} \frac{f(z^2+1)}{z^2+1}}_{= \lim_{t\to 0} \frac{f(t)}{t}} \cdot \lim_{z\to i} z+i = 1 \cdot 2i = 2i$$

4

1

$$\lim_{z\to 0} \frac{4z^2+9z}{5z^2+8z} \overset{\text{lhopital}}{=} \lim_{z\to 0} \frac{8z+9}{10z+8} \overset{\text{lhopital}}{=} \lim_{z\to 0} \frac{8}{10} = \frac{4}{5}$$

4

$$\lim_{z\to i} \frac{z^{10}+1}{z^6+1} \overset{\text{lhopital}}{=} \lim_{z\to i} \frac{z^9}{z^5} = \lim_{z\to i} z^4 = i^4 = 1$$

5

$$\lim_{z\to 2} \frac{z^2+3z-10}{z^2-z-2} = \lim_{z\to 2} \frac{2z+3}{2z-1} = \frac{4+3}{4-1} = \frac{7}{3}$$

3

Notar que:

$$f'(w_0) = \lim_{z\to w_0} \frac{f(z)-f(w_0)}{z-w_0} = \lim_{z\to w_0} \frac{f(z)}{z-w_0}$$

$$g'(w_0) = \lim_{z\to w_0} \frac{g(z)-g(w_0)}{z-w_0} = \lim_{z\to w_0} \frac{g(z)}{z-w_0}$$

Luego:

$$\frac{f'(w_0)}{g'(w_0)} = \frac{\lim_{z\to w_0} \frac{f(z)}{z-w_0}}{\lim_{z\to w_0} \frac{g(z)}{z-w_0}} = \lim_{z\to w_0} \frac{f(z)}{z-w_0} : \frac{g(z)}{z-w_0} = \lim_{z\to w_0} \frac{f(z)}{g(z)}$$

6

1

$f(z) = i(x+iy) - 2 = \underbrace{2-y}_{\mu(x,y)} + i\underbrace{x}_{v(x,y)}$

$\frac{\partial \mu}{\partial x} = 0 \qquad \frac{\partial \mu}{\partial y} = -1$

$\frac{\partial v}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = 0$

$\therefore \mu, v$ son C^1 en $\mathbb{C}$

además: $\begin{cases} \frac{\partial \mu}{\partial x} = 0 = \frac{\partial v}{\partial y} \\ \frac{\partial \mu}{\partial y} = -1 = -\frac{\partial v}{\partial x} \end{cases} \Rightarrow$ Cumplen C-R

$\therefore \exists f'(z_0) \; \forall z_0 \in \mathbb{C}$ y $f'(z) = i$

3

$\mu(x,y) = \sqrt{xy}$

$v(x,y) = 0$

$\frac{\partial \mu}{\partial x} = \frac{1}{2}\sqrt{\frac{y}{x}} \qquad \frac{\partial v}{\partial x} = 0$

$\frac{\partial \mu}{\partial y} = \frac{1}{2}\sqrt{\frac{x}{y}} \qquad \frac{\partial v}{\partial y} = 0$

Notar que

$\left.\begin{array}{l} \frac{\partial \mu}{\partial x} = \frac{\partial v}{\partial y} \iff y = 0 \wedge x \neq 0 \\ \frac{\partial \mu}{\partial y} = -\frac{\partial v}{\partial x} \iff x = 0 \wedge y \neq 0 \end{array}\right\}$ No cumplen C-R para ningun (x,y)

$\therefore \nexists f'(z)$ para ningun $z \in \mathbb{C}$

5

$\mu(x,y) = 2xy$

$v(x,y) = x^2 + y^2$

$\frac{\partial \mu}{\partial x} = 2y \qquad \frac{\partial v}{\partial y} = 2y$

$\frac{\partial \mu}{\partial y} = 2x \qquad \frac{\partial v}{\partial x} = 2x \qquad \therefore \mu, v$ son C^1

$$\begin{cases} \frac{\partial \mu}{\partial x} = 2y = \frac{\partial v}{\partial x} \\ \frac{\partial \mu}{\partial y} = -\frac{\partial v}{\partial x} \iff x = 0 \end{cases}$$

$\therefore \exists f'(z_0) \ \forall z_0 \in \{ z = x+iy \in \mathbb{C} / x = 0 \}$ y además $f'(z) = 2y + i2x$

8

i $\quad \mu(x,y) = xy(x-y)$

$\frac{\partial \mu}{\partial x} = 2xy - y^2 = \frac{\partial v}{\partial y} \quad (\alpha)$

$\frac{\partial \mu}{\partial y} = x^2 - 2xy = -\frac{\partial v}{\partial x} \quad (\beta)$

Integro (α) respecto a y:

$v(x,y) = xy^2 - \frac{y^3}{3} + C$ → Cte respecto a $y \Rightarrow C = g(x)$

derivando lo anterior:

$$\frac{\partial v}{\partial x} = y^2 + g'(x) = -x^2 + 2xy \implies g'(x) = -x^2 + 2xy - y^2$$

$\llcorner$ por (β)

cte respecto a x

$$\implies g(x) = -x^2 y + xy^2 - \frac{y^3}{3} + K$$

$\implies$ g depende de y $\implies$ g no es constante respecto a y

$\implies$ No existe $v(x,y)$ tal que cumple C-R

$\implies$ No puede ser la parte real de una función analítica

ii

$$u(x,y) = xy(x-2y)$$

$$\frac{\partial u}{\partial x} = 2xy - 2y^2 = \frac{\partial v}{\partial y} \quad (\alpha)$$

$$\frac{\partial u}{\partial y} = x^2 - 4xy = -\frac{\partial v}{\partial x} \quad (\beta)$$

Integro (α) respecto a y:

$$v(x,y) = xy^2 - \frac{y^3}{3} + c$$

$\llcorner$ constante respecto a y, es decir $C = g(x)$

derivando lo anterior:

$$\frac{\partial v}{\partial x} = y^2 + g'(x) = 4xy - x^2 \implies g'(x) = 4xy - x^2 - y^2$$

$\llcorner$ por (β)

cte respecto a x

$$\implies g(x) = 2x^2 y - \frac{x^3}{3} - y^2 x + K$$

$\implies$ g depende de y $\implies$ g no es constante con respecto a y

$\implies$ No existe $v(x,y)$ tal que cumple C-R

$\implies$ No puede ser la parte real de una función analítica

10

1

$Dom(\mu) = \mathbb{R}^2$

$\frac{\partial \mu}{\partial x} = 2(1-y)$ $\frac{\partial^2 \mu}{\partial x^2} = 0$

$\frac{\partial \mu}{\partial y} = -2x$ $\frac{\partial^2 \mu}{\partial y^2} = 0$

$\Rightarrow \frac{\partial^2 \mu}{\partial x^2} + \frac{\partial^2 \mu}{\partial y^2} = 0$ $\therefore \mu$ es armónica

Por otro lado:

$$\begin{cases} \frac{\partial \mu}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial \mu}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Rightarrow \begin{cases} \frac{\partial v}{\partial y} = 2-2y & (1°) \\ \frac{\partial v}{\partial x} = 2x & (2°) \end{cases}$$

Integro (1°) respecto a y:

$$v(x,y) = 2y - y^2 + g(x) \Rightarrow \frac{\partial v}{\partial x} = g'(x) \overset{\text{por } (2°)}{=} 2x$$

$$\Rightarrow g(x) = x^2 + \underbrace{C}_{\text{constante respecto a } x}$$

$$\Rightarrow v(x,y) = 2y - y^2 + x^2 + C$$

v cumple C-R con μ.

$\frac{\partial^2 v}{\partial x^2} = 0$, $\frac{\partial^2 v}{\partial y^2} = 0$ $\Rightarrow$ v es armónica

$\therefore$ v es conjugado armónico de μ.

3

$Dom(u) = \mathbb{R}^2$

$\frac{\partial u}{\partial x} = \sin(y)\cosh(x)$ $\frac{\partial^2 u}{\partial x^2} = \sin(y)\sinh(x)$

$\frac{\partial u}{\partial y} = \cos(y)\sinh(x)$ $\frac{\partial^2 u}{\partial y^2} = -\sin(y)\sinh(x)$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \sin(y)\sinh(x) - \sin(y)\sinh(x) = 0$

$\therefore u$ es armónica

Por otro lado:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Rightarrow \begin{cases} \frac{\partial v}{\partial y} = \sin(y)\cosh(x) & (1°) \\ \frac{\partial v}{\partial x} = -\cos(y)\sinh(x) & (2°) \end{cases}$$

Integro (1°) respecto a y:

$v(x,y) = -\cos(y)\cosh(x) + g(x) \Rightarrow \frac{\partial v}{\partial x} = -\cos(y)\sinh(x) + g'(x)$

$= -\cos(y)\sinh(x)$

$\hookrightarrow$ por (2°)

$\Rightarrow g'(x) = 0$

$\Rightarrow g(x) = c \sim$ cte respecto a x

$\Rightarrow v(x,y) = -\cos(y)\cosh(x) + c$ cumple C-R con u

$\frac{\partial^2 v}{\partial x^2} = -\cos(y)\cosh(x)$

$\frac{\partial^2 v}{\partial y^2} = \cos(y)\cosh(x)$

$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \Rightarrow v$ es armónica

5 $Dom(u) = \mathbb{R}^2$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial u}{\partial y} = -6xy \qquad \frac{\partial^2 u}{\partial y^2} = -6x$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u \text{ es armónica}$$

Por otro lado:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Rightarrow \begin{cases} \frac{\partial v}{\partial y} = 3x^2 - 3y^2 & (1^o) \\ \frac{\partial v}{\partial x} = 6xy & (2^o) \end{cases}$$

Integro (1°) respecto a y:

$$v(x,y) = 3x^2y - y^3 + g(x) \Rightarrow \frac{\partial v}{\partial x} = 6xy + g'(x) \overset{\text{Por } (2^o)}{=} 6xy$$

$$\Rightarrow g'(x) = 0 \Rightarrow g(x) = C \text{ cte respecto a } x$$

$$\Rightarrow v(x,y) = 3x^2y - y^3 + C$$

$$\frac{\partial^2 v}{\partial x^2} = 6y \qquad \frac{\partial^2 v}{\partial y^2} = -6y \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \Rightarrow v \text{ es armónica}$$

11

Sea $q(z) = f(z) - g(z)$, luego $q'(z) = f'(z) - g'(z) \overset{\text{por hipótesis}}{=} 0$

f, g son analíticas $\Rightarrow f - g$ son analíticas $\Rightarrow q$ es analítica

q analítica y $q'(z) = 0 \Rightarrow q$ es constante

$\Rightarrow q(z) = c$

$\Rightarrow f(z) - g(z) = c$

14

1 Sea $f(z) = \mu(x,y) + i\,v(x,y)$

Dado que $f(z) \in \mathbb{R} \Rightarrow v(x,y) = 0 \Rightarrow \begin{cases} \frac{\partial v}{\partial x} = 0 & (\alpha) \\ \frac{\partial v}{\partial y} = 0 & (\beta) \end{cases}$

Dado que f es analítica, cumple C-R:

$$\begin{cases} \frac{\partial \mu}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial \mu}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Rightarrow \begin{cases} \frac{\partial \mu}{\partial x} = 0 \\ \frac{\partial \mu}{\partial y} = 0 \end{cases} \Rightarrow \mu \text{ es constante en } D$$

μ, v constante en $D \Rightarrow f$ constante en D

2

$$\mu(x,y) = c \Rightarrow \begin{cases} \frac{\partial \mu}{\partial x} = 0 \\ \frac{\partial \mu}{\partial y} = 0 \end{cases}$$

Dado que f es analítica, cumple C-R:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Rightarrow \begin{cases} \frac{\partial v}{\partial x} = 0 \\ \frac{\partial v}{\partial y} = 0 \end{cases} \Rightarrow v \text{ es constante en } D$$

u, v constante en $D \Rightarrow f$ es constante en D

3

- Si $u = 0 = v$, cumple que $|f| = cte$ y además u, v cte $\Rightarrow f$ cte
- Si $u = 0$ y $v \neq 0$ para que $|f| = cte$ v debería ser cte, por lo tanto u, v serían constantes $\Rightarrow f$ es constante
- Si $u \neq 0$ y $v = 0$ para que $|f| = cte$ u debería ser constante, por lo tanto u, v serían constantes $\Rightarrow f$ es constantes

Sean u, v no nulas:

Sea $g(x,y) = |f(x+iy)|^2 = cte^2$ y $f(x+iy) = u(x,y) + iv(x,y)$. Luego $g(x,y) = u(x,y)^2 + v(x,y)^2$ y tenemos que:

$$\frac{\partial g}{\partial x} = 2u(x,y)\frac{\partial u}{\partial x} + 2v(x,y)\frac{\partial v}{\partial x} = 0 \iff u(x,y)\cdot\frac{\partial u}{\partial x} + v(x,y)\frac{\partial v}{\partial x} = 0$$

pues $g = cte^2$

$$\frac{\partial g}{\partial y} = 2u(x,y)\frac{\partial u}{\partial y} + 2v(x,y)\frac{\partial v}{\partial y} = 0 \iff u(x,y)\cdot\frac{\partial u}{\partial y} + v(x,y)\frac{\partial v}{\partial y} = 0$$

Dado que f es analítica, μ, v cumplen C-R, por lo tanto tenemos:

$$\begin{cases} \mu(x,y)\frac{\partial \mu}{\partial x} - v(x,y)\frac{\partial \mu}{\partial y} = 0 \\ \mu(x,y)\frac{\partial \mu}{\partial y} + v(x,y)\frac{\partial \mu}{\partial x} = 0 \end{cases}$$

$$\equiv \underbrace{\begin{pmatrix} \mu(x,y) & -v(x,y) \\ v(x,y) & \mu(x,y) \end{pmatrix}}_{=A} \cdot \begin{pmatrix} \frac{\partial \mu}{\partial x} \\ \frac{\partial \mu}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\det(A) = \mu(x,y)^2 + v(x,y)^2 \neq 0$

$\Rightarrow$ el sistema tiene única solución, la trivial

$\Rightarrow \frac{\partial \mu}{\partial x} = 0 \wedge \frac{\partial \mu}{\partial y} = 0$

$\Rightarrow \mu$ es constante

$$\begin{cases} \mu(x,y)\frac{\partial v}{\partial y} + v(x,y)\frac{\partial v}{\partial x} = 0 \\ -\mu(x,y)\frac{\partial v}{\partial x} + v(x,y)\frac{\partial v}{\partial y} = 0 \end{cases}$$

$$\equiv \underbrace{\begin{pmatrix} v(x,y) & \mu(x,y) \\ -\mu(x,y) & v(x,y) \end{pmatrix}}_{=B} \cdot \begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\det(B) = v(x,y)^2 + \mu(x,y)^2 \neq 0$

$\Rightarrow$ el sistema tiene única solución, la trivial

$\Rightarrow \frac{\partial v}{\partial x} = 0 \wedge \frac{\partial v}{\partial y} = 0$

$\Rightarrow v$ es constante

$\therefore \mu, v$ ctes en $D \Rightarrow f$ es cte.

4

$$v(x,y) = \mu(x,y)^2$$

$v = 0 \Leftrightarrow \mu = 0$ y en tal caso ambas son constante

Sea $\mu \neq 0 \wedge v \neq 0$:

$$\begin{cases} \dfrac{\partial v}{\partial x} = 2\mu(x,y) \cdot \dfrac{\partial \mu}{\partial x} \\[2ex] \dfrac{\partial v}{\partial y} = 2\mu(x,y) \dfrac{\partial \mu}{\partial y} \end{cases}$$

Por C-R: $\begin{cases} \dfrac{\partial \mu}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial \mu}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \iff \begin{cases} \dfrac{\partial \mu}{\partial x} = 2\mu(x,y) \dfrac{\partial \mu}{\partial y} \\[2ex] \dfrac{\partial \mu}{\partial y} = -2\mu(x,y) \dfrac{\partial \mu}{\partial x} \end{cases}$

$$\iff \begin{cases} \dfrac{\partial \mu}{\partial x} - 2\mu(x,y) \dfrac{\partial \mu}{\partial y} = 0 \\[2ex] 2\mu(x,y) \dfrac{\partial \mu}{\partial x} + \dfrac{\partial \mu}{\partial y} = 0 \end{cases} \iff \begin{pmatrix} 1 & 2\mu(x,y) \\ 2\mu(x,y) & 1 \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial \mu}{\partial x} \\[1ex] \dfrac{\partial \mu}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Luego:

$$\det(A) = 1 + 4\mu(x,y) \begin{cases} \text{Si } \det(A) = 0 \Rightarrow \mu(x,y) = -\frac{1}{4} \quad \therefore \mu \text{ es cte} \\[2ex] \text{Si } \det(A) \neq 0 \Rightarrow \text{El sistema tiene solución trivial} \end{cases}$$

$$\Rightarrow \frac{\partial \mu}{\partial x} = 0 \ \wedge \ \frac{\partial \mu}{\partial y} = 0$$

$\therefore \mu$ es cte

Luego $v(x,y) = Cte^2 \Rightarrow v$ es constante

μ, v constantes en $D \Rightarrow f$ es constante

NOTA