PRACTICO 5

1

1

$$\int_1^2 \left(\frac{1}{t}-i\right)^2 dt = \int_1^2 \left(\frac{1}{t^2} - \frac{2i}{t} - 1\right) dt = -\frac{1}{t} - 2i\ln(t) - t \Big|_1^2$$

$$= \left(-\frac{1}{2} - 2i\ln(2) - 2\right) - \left(-1 - 2i.0 - 1\right) = -\frac{1}{2} - 2i\ln(2) - 2 + 2$$

$$= -\frac{1}{2} - i.2\ln(2)$$

2

$$\int_0^{\frac{\pi}{6}} e^{2ti}\, dt = \int_{h(0)=0}^{h(\frac{\pi}{6})=\frac{\pi}{3}} e^{ih}\, \frac{dh}{2} = \frac{1}{2}\, \frac{e^{ih}}{i}\Big|_0^{\frac{\pi}{3}} = \frac{1}{2i}\left(e^{i\frac{\pi}{3}} - e^0\right)$$

$h = 2t$

$\frac{dh}{2} = dt$

$$= -\frac{i}{2}\left(e^{i\frac{\pi}{3}} - 1\right) = -\frac{i}{2}\left(\cos(\tfrac{\pi}{3}) + i\sin(\tfrac{\pi}{3}) - 1\right)$$

$$= -\frac{i}{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i - 1\right) = \frac{\sqrt{3}}{4} + i\frac{1}{4}$$

3

$$\int_{-1}^{1} \frac{dt}{t-i} = \int_{-1-i}^{1-i} \frac{dh}{h} = Ln(h)\Big|_{-1-i}^{1-i} = Ln(1-i) - Ln(-1-i)$$

$h = t-i$

$dh = dt$

$h(1) = 1-i$

$h(-1) = -1-i$

$$= \ln(\sqrt{2}) + i\left(-\frac{\pi}{3}\right) - \ln(\sqrt{2}) - i\frac{5\pi}{6} = -i\left(\frac{\pi}{3} + \frac{5\pi}{6}\right) = -i\frac{7\pi}{6}$$

8

$$\int_0^{e^{\pi}=-1} \exp(i\,Ln(t))\, dt = \int_0^{-1} t^i\, dt = \frac{t^{i+1}}{i+1}\Big|_0^{-1} = \frac{(-1)^{i+1}}{i+1}$$

$$= \frac{1}{i+1}\exp((i+1)Ln(-1)) = \frac{\exp((i+1)(i\pi))}{i+1} = \frac{\exp(i\pi - \pi)}{i+1}$$

$$= \frac{e^{i\pi-\pi}}{i+1} = \frac{1}{i+1}\,\frac{e^{i\pi}}{e^{\pi}} = \frac{i}{e^{\pi}(i+1)} = \frac{i}{e^{\pi}}\cdot\frac{(i-1)}{i^2-1} = \frac{i^2-i}{-2e^{\pi}}$$

$$= \frac{-i-1}{-2e^{\pi}} = \frac{1}{2e^{\pi}} + i\,\frac{1}{2e^{\pi}}$$

2

Si $m=n$

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta = \int_0^{2\pi} e^{i(m-n)\theta}\, d\theta = \int_0^{2\pi} d\theta = 2\pi$$

Si $m\neq n$

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta = \int_0^{2\pi} e^{i(m-n)\theta}\, d\theta = \left.\frac{e^{i(m-n)\theta}}{i(m-n)}\right|_0^{2\pi}$$

$$= \frac{1}{i(m-n)}\left(e^{i2\pi(m-n)}-1\right)$$ m,n son enteros $\Rightarrow k=m-n$ es entero

$$= \frac{1}{ik}\left(e^{i2k\pi}-1\right) = \frac{1}{ik}(1-1) = 0$$

3

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt$$

$$f(z) = \frac{z+2}{z}, \quad z'(t) = 2i\,e^{it}, \quad f(z(t)) = \frac{2e^{it}+2}{2e^{it}}$$

4

$$\int_C \frac{z+2}{z}\, dz = 2i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{it}\, dt + 2i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt = 2i\left.\frac{e^{it}}{i}\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 2it$$

$$= 2(e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}}) + 2i\pi = 2(i+i) + 2i\pi = i(4+2\pi)$$

3

$$\int_C f(z)dz = \int_0^{\pi} \frac{2e^{it}+2}{2} \cdot 2e^{it}dt = 2i\int_0^{\pi}(e^{it}+1)\,dt$$

$$= 2i\int_0^{\pi} e^{it}dt + 2i\int_0^{\pi} dt = 2i\,\frac{e^{it}}{i}\Big|_0^{\pi} + 2i\pi$$

$$= 2(e^{i\pi} - 1) + 2i\pi = -4 + 2i\pi$$

2

$$\int_C f(z)\,dz = \int_C \frac{z+2}{z}\,dz = 2i\int_{\pi}^{2\pi} e^{it}dt + 2i\int_{\pi}^{2\pi} dt = 2i.\,\frac{e^{it}}{i}\Big|_{\pi}^{2\pi} + 2i\pi$$

$$= 2(e^{i2\pi} - e^{i\pi}) + 2i\pi = 4 + i2\pi$$

1

$$\int_C f(z)dz = \int_C \frac{z+2}{z}\,dz = 2i\int_0^{\pi} e^{it}dt + 2i\int_0^{\pi} dt = 2i\,\frac{e^{it}}{i}\Big|_0^{\pi} + 2i\pi$$

$$= 2(e^{i\pi} - 1) + 2i\pi = -4 + i2\pi$$

5

1 $\quad \gamma(t) = z_0 + Re^{it} \qquad \gamma'(t) = iRe^{it}$

$$\int_C \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{1}{z_0 + Re^{it} - z_0} \cdot iRe^{it}dt = i\int_0^{2\pi} dt = 2\pi i$$

2

$$\int_C (z-z_0)^{n-1} dz = \int_0^{2\pi} (z_0 + Re^{it} - z_0)^{n-1} iRe^{it} dt$$

$$= \int_0^{2\pi} R^{n-1} e^{it(n-1)} Re^{it} i dt$$

$$= i \int_0^{2\pi} R^n e^{itn} dt = iR^n \int_0^{2\pi} e^{itn} dt$$

$$= iR^n \left. \frac{e^{inT}}{in} \right|_0^{2\pi} = R^n (e^{i2n\pi} - e^0) = R^n (1-1) = 0.$$

7

1

$$|z^4+1| \geq ||z^4| - |1|| = |16-1| = 15$$

$$|z-2| \leq |z| + |-2| = 2+2 = 4$$

$$f(z) = \frac{z-2}{z^4+1} \Longrightarrow |f(z)| \leq \frac{4}{15} = M$$

$$long(C) = \frac{1}{2}\pi \cdot radio = \pi$$

$$\Longrightarrow \left| \int_C \frac{z-2}{z^4+1} dz \right| \leq M . long(C) = \frac{4\pi}{15}$$

2

$$|z^2-1| \geq ||z|^2 - |1|| = |4-1| = 3$$

$$f(z) = \frac{1}{z^2-1} \Longrightarrow |f(z)| \leq \frac{1}{3} = M$$

$$\Longrightarrow \left|\int_C f(z)\,dz\right| \leq M\cdot \operatorname{long}(C) = \frac{\pi}{3}$$

(8)

$$\int_\gamma e^{iz^2}\,dz = \int_0^{\frac{\pi}{4}} e^{i(re^{it})^2}\, i r e^{it}\,dz$$

$$\Longrightarrow \left|\int_\gamma e^{iz^2}\,dz\right| \leq \int_0^{\frac{\pi}{4}} \left|e^{i(re^{it})^2}\right|\cdot\left|ire^{it}\right| dt = \int_0^{\frac{\pi}{4}} r\cdot\left|e^{ir^2e^{i2t}}\right| dt$$

Notar que:

$$ir^2e^{i2t} = ir^2(\cos(2t) + i\sin(2t)) = ir^2\cos(2t) - r^2\sin(2t)$$

$$\Longrightarrow \left|e^{ir^2e^{i2t}}\right| = \left|e^{ir^2\cos(2t)}\cdot e^{-r^2\sin(2t)}\right| = \left|e^{ir^2\cos(2t)}\right|\cdot\left|e^{-r^2\sin(2t)}\right| = e^{-r^2\sin(2t)}$$

Por lo tanto

$$\left|\int_\gamma e^{iz^2}\,dz\right| \leq r\int_0^{\frac{\pi}{4}} \left|e^{ir^2e^{i2t}}\right| dt = r\int_0^{\pi} e^{-r^2\sin(2t)}\,dt$$

Luego como $\sin(2x) \geq \frac{4}{\pi}x \quad \forall x \in [0, \frac{\pi}{4}]$

$$\Longrightarrow -r^2\sin(2x) \leq -r^2\frac{4}{\pi}x$$

$$e^{-r^2\sin(2x)} \leq e^{-r^2\frac{4}{\pi}x}$$

Luego:

$$\left|\int_\gamma e^{iz^2}dz\right| \leq r\int_0^{\frac{\pi}{4}} e^{-r^2\sin(2t)}dt \leq r\int_0^{\frac{\pi}{4}} e^{-r^2\frac{4}{\pi}t}dt = r\cdot\frac{-1}{r^2\frac{4}{\pi}}\cdot e^{-r^2\frac{4}{\pi}t}\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{-1}{r\cdot\frac{4}{\pi}}\cdot\left(e^{-r^2}-1\right) = \frac{\pi(1-e^{-r^2})}{4r}$$

9

$$\gamma'(t) = ie^{1+it}$$

$$\int_\gamma \frac{1}{\operatorname{Log}(z)}dz = \int_0^\pi \frac{1}{\operatorname{Log}(e^{1+it})}\cdot i.e^{1+it}dt$$

$$\longrightarrow \left|\int_\gamma \frac{1}{\operatorname{Log}(z)}dz\right| \leq \int_0^\pi \frac{|e^{1+it}|}{|\operatorname{Log}(e^{1+it})|}dt = \int_0^\pi \frac{|e.e^{it}|}{|1+it|}dt$$

$$= e\int_0^\pi \frac{dt}{|1+it|} = e\int_0^\pi \frac{dt}{\sqrt{1+t^2}}$$

Sea $F(z) = \operatorname{Ln}(t+\sqrt{t^2+1})$:

$$F'(z) = \frac{1}{t+\sqrt{t^2+1}}\cdot\left(1+\frac{2t}{2\sqrt{t^2+1}}\right) = \frac{1}{t+\sqrt{t^2+1}}\cdot\frac{2\sqrt{t^2+1}+2t}{2\sqrt{t^2+1}}$$

$$= \frac{1}{t+\sqrt{t^2+1}}\cdot\frac{\sqrt{t^2+1}+t}{\sqrt{t^2+1}} = \frac{1}{\sqrt{t^2+1}}$$

$\therefore \operatorname{Ln}(t+\sqrt{t^2+1})$ es primitiva de $\frac{1}{\sqrt{t^2+1}}$.

Por lo tanto:

$$e\int_0^\pi \frac{dt}{\sqrt{1+t^2}} = e.\operatorname{Ln}(t+\sqrt{t^2+1})\Big|_0^\pi = e\operatorname{Ln}(\pi+\sqrt{\pi^2+1})$$

(10)

Sea $\xi:[a,b]\to\mathbb{C}$ una parametrización de C. Sea $f(z)=z^n$.

$$\int_C z^n dz = \int_a^b f(\xi(t))\,\xi'(t)\,dt = \int_a^b \xi^n(t)\,\xi'(t)\,dt$$

Sea $F(t)=\dfrac{\xi^{n+1}(t)}{n+1}$. Luego:

$$F'(t)=\frac{(n+1)\cdot\xi^n(t)\,\xi'(t)}{n+1} = \xi^n(t)\,\xi'(t)$$

Por lo tanto F es primitiva de $\xi^n(t)\cdot\xi'(t)$.

$$\int_C z^n dz = \int_a^b \xi^n(t)\,\xi'(t)\,dt = \frac{\xi^{n+1}(t)}{n+1}\Big|_a^b = \frac{1}{n+1}\left(\xi^{n+1}(b)-\xi^{n+1}(a)\right) = \frac{1}{n+1}\left(z_2^{n+1}-z_1^{n+1}\right)$$

(12)

a) Notar que $z_0 = i\cdot\frac{\pi}{2} \in \operatorname{Int}(C)$

$f(z)=e^{-z}$ es analítica en z_0

$$\Longrightarrow f\left(i\tfrac{\pi}{2}\right) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-i\frac{\pi}{2}}\,dz$$

formula integral de Cauchy

$$\Longrightarrow \int_C \frac{e^{-z}}{z-i\frac{\pi}{2}} = 2\pi i\cdot e^{-i\frac{\pi}{2}} = 2\pi i(-i) = 2\pi$$

b) $f(z)=\dfrac{\cos(z)}{z^2+8}$ tiene singularidades en $\pm\sqrt{-8}$ y no pertenecen al interior de C ni a su frontera

$$\Longrightarrow \frac{1}{2\pi i}\int_C \frac{f(z)}{z} = f(0)$$

$$\Longrightarrow \int_C \frac{\cos(z)}{z(z^2+8)}\,dz = 2\pi i \cdot \frac{1}{8} = i\frac{\pi}{4}$$

c) $\dfrac{z}{2z+1}$ tiene singularidades en $z_0 = -\frac{1}{2}$

$$\frac{z}{2z+1} = \frac{1}{2}\,\frac{z}{z-(-\frac{1}{2})} = \frac{\frac{z}{2}}{z-(-\frac{1}{2})}$$

$f(z) = \frac{z}{2}$ es analítica en $int(C)$ y en su frontera

$$\Longrightarrow \frac{1}{2\pi i}\int_C \frac{f(z)}{z-(-\frac{1}{2})}\,dz = f(-\tfrac{1}{2})$$

$$\Longrightarrow \int_C \frac{z}{2z+1}\,dz = 2\pi i \cdot f(-\tfrac{1}{2}) = 2\pi i \cdot -\frac{1}{4} = -i\frac{\pi}{2}$$

d) $z_0 = 0$ pertenece al interior de C. Luego siendo $f(z) = \cosh(z)$, esta es analítica en el interior de C, por lo tanto:

$$\Longrightarrow f'''(0) = \frac{3!}{2\pi i}\int_C \frac{\cosh(z)}{z^4}\,dz$$

form int de Cauchy con derivadas

$$\Longrightarrow \int_C \frac{\cosh(z)}{z^4}\,dz = \frac{2\pi i}{6}\cdot \sinh(0) = 0$$

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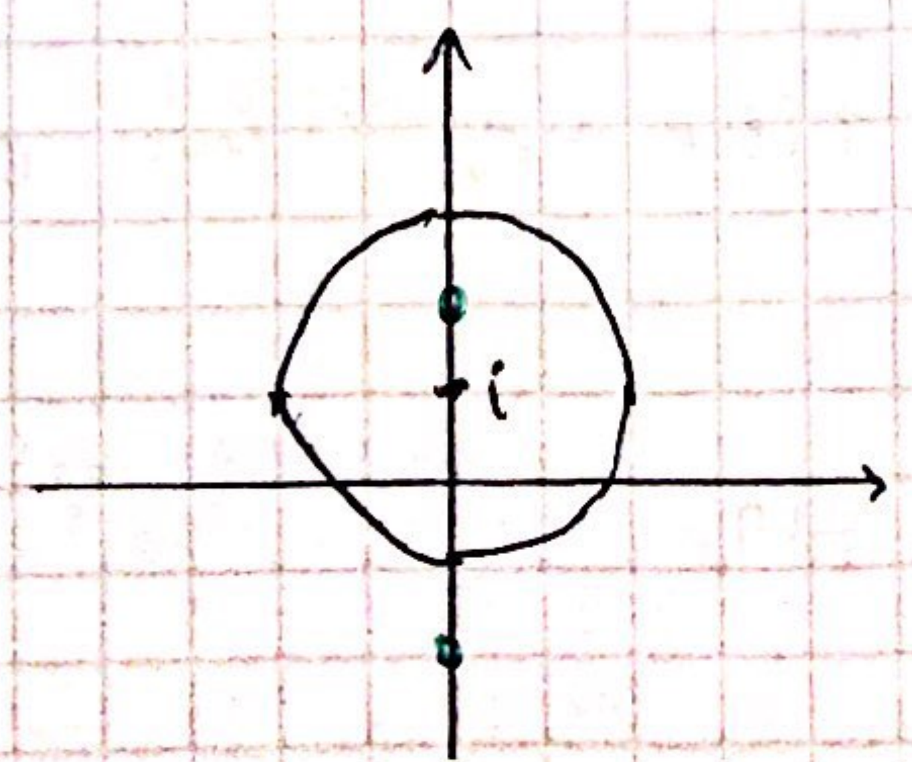
a)



$g(z)$ tiene singularidades en $\pm 2i$.

Luego:

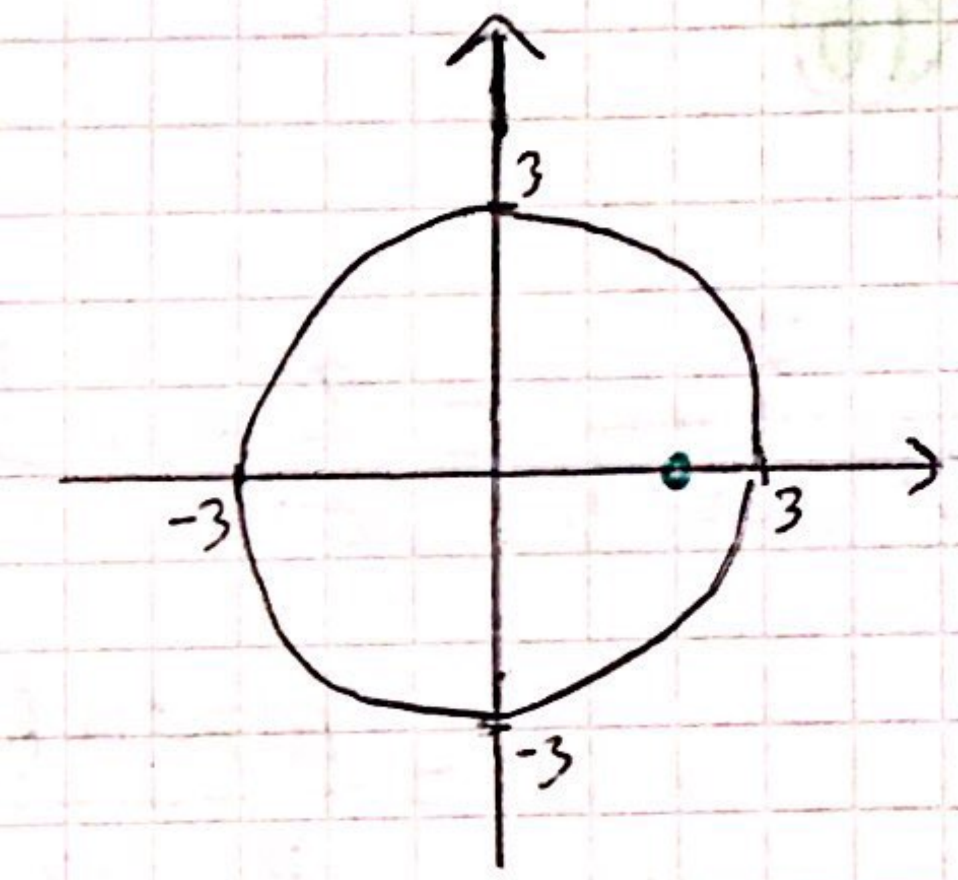
$$\frac{1}{z^2+4} = \frac{1}{(z-2i)(z+2i)} = \frac{\frac{1}{z+2i}}{z-2i}$$

Sea $f(z)=\frac{1}{z+2i}$, f es analítica en C y su interior

$$\Longrightarrow \int_C g(z)\,dz = 2\pi i \cdot f(2i) = 2\pi i \,\frac{1}{4i} = \frac{\pi}{2}$$

14

$$g(z) = \int_C \underbrace{\frac{2z^2-z-2}{z-2}}_{\text{tiene singularidades en } z_0=2} dz$$



Sea $f(z)=2z^2-z-2$ es entera y por lo tanto analítica en C y en su interior

$$\Longrightarrow f(2)=\frac{1}{2\pi i}\int_C \frac{2z^2-z-2}{z-2}\,dz$$

$$\Longrightarrow \int_C \frac{2z^2-z-2}{z-2}\,dz = 2\pi i\, f(2) = 2\pi i . 4 = 8\pi i$$

$$\therefore g(z) = 8\pi i$$

Además si $|w|>3 \Longrightarrow \frac{2z^2-z-2}{z-w}$ no tiene singularidades ni en C ni en su interior, por lo tanto es analítica en $D=C\cup B_3(0)$, el cual es simplemente conexo

$$\Longrightarrow \int_C \frac{2z^2-z-2}{z-w}\,dz = 0$$

15

1

$f(z) = e^{az}$ es entera, por lo tanto analítica en C y su interior.

$\frac{e^{az}}{z}$ tiene singularidad en $z_0 = 0$

$$\Longrightarrow f(0) = \frac{1}{2\pi i}\int_C \frac{e^{az}}{z}\,dz \Longrightarrow \int_C \frac{e^{az}}{z}\,dz = 2\pi i\, f(0) = 2\pi i$$

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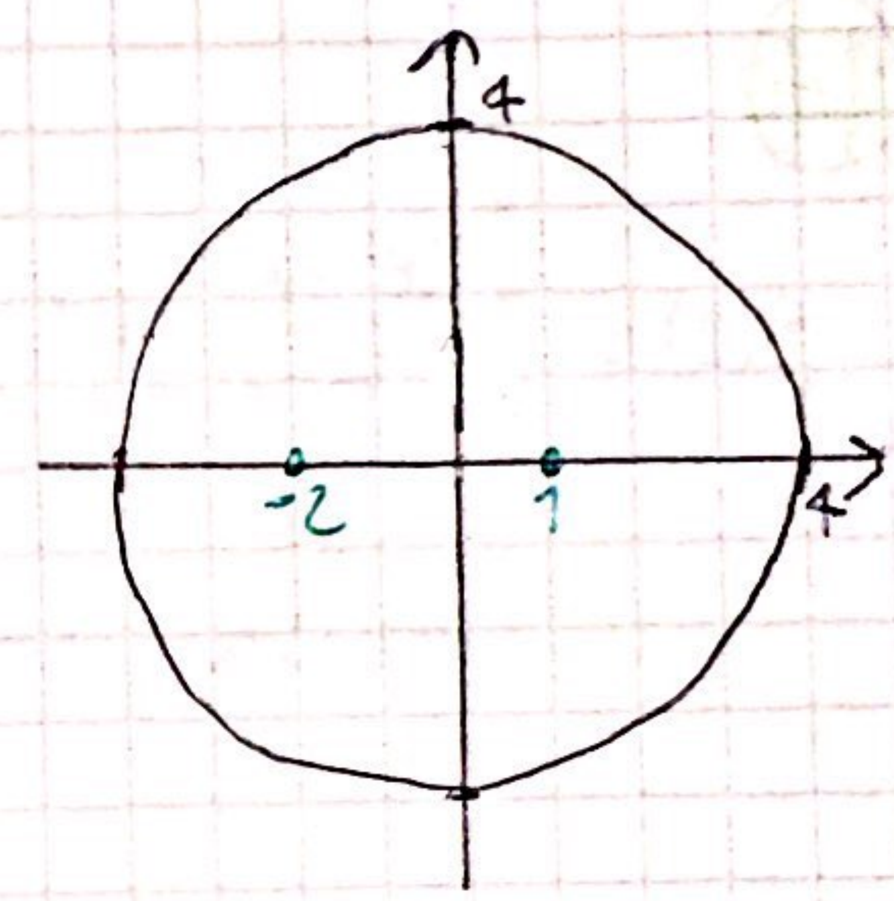

$$\frac{z}{(z+2)(z-1)} = \frac{A}{z+2} + \frac{B}{z-1}$$

$$= \frac{A(z-1)+B(z+2)}{(z+2)(z-1)}$$

$$= \frac{z(A+B)+(2B-A)}{(z+2)(z-1)}$$

$$\Longrightarrow \begin{cases} A+B=1 & \Longrightarrow 3B=1 \Longrightarrow B=\frac{1}{3} \\ 2B-A=0 & \Longrightarrow A=2B \Longrightarrow A=\frac{2}{3} \end{cases}$$

$$\Longrightarrow \frac{z}{(z+2)(z-1)} = \frac{2}{3}\frac{1}{z+2} + \frac{1}{3}\frac{1}{z-1}$$

$$\Longrightarrow \int_C \frac{z}{(z+2)(z-1)}\,dz = \frac{2}{3}\underbrace{\int_C \frac{dz}{z+2}}_{A} + \frac{1}{3}\underbrace{\int_C \frac{dz}{z-1}}_{B}$$

A

$\frac{1}{z+2}$ tiene singularidad en $z_0=-2$

$f(z)=1$ es analítica en C y su interior

$$\Longrightarrow \int_C \frac{dz}{z+2} = (-2)\,2\pi i = 2\pi i$$

B

$\frac{1}{z-1}$ tiene singularidad en $z_0=1$

$f(z)=1$ es analítica en C y su interior

$$\Longrightarrow \int_C \frac{dz}{z-i} = f(-1)\,2\pi i = 2\pi i$$

Finalmente

$$\int_C \frac{1}{(z+2)(z-1)}\,dz = \frac{2}{3}2\pi i + \frac{1}{3}2\pi i = 2\pi i$$

18, 19 No hacer

20 Por hipótesis $|u(x,y)| \leq M \quad \forall (x,y) \in \mathbb{C}$. Sea $g(z)=\exp(f(z))$

$|g(z)| = |e^{u(x,y)+iv(x,y)}| = e^{u(x,y)} \leq e^{M} \Longrightarrow g$ es acotada (α)

f entera $\Longrightarrow \exp(f(z)) = g(z)$ es entera (β)

$\alpha \wedge \beta \underset{\text{Liouville}}{\Longrightarrow} g$ es constante $\Longrightarrow f$ es constante.