PRÁCTICO 6

1

Tomemos como válido el teorema de series reales, es decir:

$$\sum_{n=1}^{\infty} |a_n| \text{ converge} \Longrightarrow \sum_{n=1}^{\infty} a_n \text{ converge} \qquad (\alpha)$$

Sea $z_n = x_n + i y_n$. Es claro que $0 \leq |x_i| \leq |z_i|$
$0 \leq |y_i| \leq |z_i|$

Como $\sum_{n=1}^{\infty} |z_n|$ es convergente, por criterio de comparación de series reales: $\sum_{i=1}^{\infty} |x_i|$, $\sum_{i=1}^{\infty} |y_i|$ convergen

$$\Longrightarrow \sum_{i=1}^{\infty} x_i, \sum_{i=1}^{\infty} y_i \text{ convergen}$$

por (α)

$$\Longrightarrow \sum_{i=1}^{\infty} z_n \text{ convergen}$$

2

Sea $\sum_{n=0}^{k} z^n = 1 + z^1 + z^2 + \cdots + z^k = S_k$

$$\Longrightarrow z \cdot S_k = z + z^2 + \cdots + z^{k+1}$$

$$\Longrightarrow S_k - zS_k = 1 - z^{k+1}$$

$$S_k(1-z) = 1 - z^{k+1}$$

$$S_k = \frac{1-z^{k+1}}{1-z}$$

Sea $|z|<1$

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1}{1-z} - \lim_{k\to\infty} \frac{z^{k+1}}{1-z} = \frac{1}{1-z} - 0 = \frac{1}{1-z}$$

Sea $|z|\geq 1$

$$\lim_{k\to\infty} S_n = \lim_{k\to\infty} \frac{1}{1-z} - \lim_{k\to\infty} \frac{z^{k+1}}{1-z} = \infty \text{ no diverge}$$

$$\therefore \sum_{n=0}^{\infty} z^n = \lim_{k\to\infty} \sum_{n=0}^{k} z^n = \lim_{k\to\infty} S_n = \frac{1}{1-z} \iff |z|<1$$

3

Sea $z_n = x_n + iy_n$, $S = a+ib$

Luego $$\sum_{n=1}^{\infty} z_n = S \iff \sum_{n=1}^{\infty} x_n + i\sum_{n=1}^{\infty} y_n = S = a+ib$$

$$\iff \sum_{n=1}^{\infty} x_n = a \wedge \sum_{n=1}^{\infty} y_n = b$$

Por lo tanto:

$$\sum_{n=1}^{\infty} \bar{z}_n = \sum_{n=1}^{\infty} x_n - i\sum_{n=1}^{\infty} y_n = a - ib = \bar{S}$$

Sea $c = k + pi$. $c.z_n = kx_n + iky_n + ipx_n - py_n$

$$= (kx_n - py_n) + i(ky_n + px_n)$$

$$\Rightarrow \sum_{n=1}^{\infty} c.z_n = k\sum_{n=1}^{\infty} x_n - p\sum_{n=1}^{\infty} y_n + i\left(k\sum_{n=1}^{\infty} y_n + p\sum_{n=1}^{\infty} x_n\right)$$

$$= ka - pb + i(kb + pa)$$

$C.S = (k+pi)(a+bi) = ka + kbi + api - pb = ka - pb + i(kb+ap)$

$$\therefore \sum_{n=1}^{\infty} cz_n = c.S$$

4 $e^z = e^{z-1} \cdot e$. Sea $f(z) = e^z$, luego $f^{(n)}(z) = e^z \Longrightarrow f^{(n)}(0) = 1$

Por serie de macleurin:

$$e^z = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=1}^{\infty} \frac{z^n}{n!}$$

$$\Longrightarrow e^{z-1} = \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!}$$

$$\Longrightarrow e^z = e^{z-1} e = e \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!}$$

5

$$f(z) = \frac{z}{9} \cdot \frac{1}{1 - \left(-\frac{z^4}{9}\right)} \underset{\substack{\text{Si } \left|-\frac{z^4}{9}\right| < 1 \\ \text{Si } |z| < \sqrt[4]{9}}}{=} \frac{z}{9} \cdot \sum_{n=0}^{\infty} \left(-\frac{z^4}{9}\right)^n = \frac{z}{9} \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n}}{9^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+1}}{9^{n+1}}$$

6 cos

$f'(z) = -\sin(z)$

$f''(z) = -\cos(z)$

$f'''(z) = \sin(z)$

$f''''(z) = \cos(z)$

$\Longrightarrow$

$f(\frac{\pi}{2}) = 0$

$f'(\frac{\pi}{2}) = -1$

$f''(\frac{\pi}{2}) = 0$

$f'''(\frac{\pi}{2}) = 1$

$f''''(\frac{\pi}{2}) = 0$

$$\Rightarrow f^{(n)}(z_0) = \begin{cases} (-1)^n & \text{si n es impar} \\ 0 & \text{si n es par} \end{cases}$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{2})}{n!} (z-\tfrac{\pi}{2})^n = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} (z-\tfrac{\pi}{2})^{2m+1}$$

Sinh

$$f'(z) = \cosh(z) \qquad \Rightarrow \qquad f'(\pi i) = \frac{e^{\pi i}+e^{-\pi i}}{2} = \frac{-1-1}{2} = -1$$

$$f''(z) = \sinh(z) \qquad\qquad f''(\pi i) = \frac{e^{\pi i}-e^{-\pi i}}{2} = \frac{-1-(-1)}{2} = 0$$

$$\Rightarrow f^{(n)}(\pi i) = \begin{cases} 0 & n \text{ par} \\ -1 & n \text{ impar} \end{cases}$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(2n+1)!} (z-i\pi)^{2n+1}$$

7

Sabemos que $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$

$$\Rightarrow \sin(z^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n+2}$$

$$\Rightarrow f^{(4n)}(0) = f^{(4n+1)}(0) = f^{(4n+3)}(0) = 0 \qquad \forall n \in \mathbb{N}$$

Además

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n+2} \overset{k=2n+1}{=} \sum_{k=1}^{\infty} \frac{(-1)^{\frac{k-1}{2}}}{k!} z^{2k}$$

$\Rightarrow f^{(2k+1)}(0) = 0 \quad \forall k \geq 1$ (para $k=0$ está incluido arriba en la $4n+1$)

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(8)

$$f(z) = (1+z)^{\lambda}$$

$$f'(z) = \lambda(1+z)^{\lambda-1}$$

$$f''(z) = \lambda(\lambda-1)(1+z)^{\lambda-2}$$

$$\vdots$$

$$f^{(n)}(z) = \lambda(\lambda-1)\cdots(\lambda-n+1)(1+z)^{\lambda-n}$$

$$\Longrightarrow f^{n}(0) = \lambda(\lambda-1)\cdot\cdots\cdot(\lambda-n+1)$$

$$\Longrightarrow \quad f(z) = \sum_{n=0}^{\infty} \frac{\lambda(\lambda-1)\cdot\cdots\cdot(\lambda-n+1)}{n!} z^n$$

$$= \sum_{n=0}^{\infty} \binom{\lambda}{n} z^n \quad \text{si } |z|<R \text{ para algun } R$$

Sabemos que converge si $\lim_{n\to\infty} \left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| < 1$:

$$\lim_{n\to\infty} \left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| = \lim_{n\to\infty} \left|\frac{\binom{\lambda}{n+1}}{\binom{\lambda}{n}}\right| \cdot \left|\frac{z^{n+1}}{z^n}\right| = \lim_{n\to\infty} \left|\frac{\frac{\lambda-n+2}{(n+1)!}}{\frac{1}{n}}\right| \cdot |z|$$

$$= \lim_{n\to\infty} \left|\frac{\lambda-n+2}{n+1}\right| \cdot |z| = \lim_{n\to\infty} \left|\underbrace{\frac{\lambda+2}{n+1}}_{\to 0} - \underbrace{\frac{n}{n+1}}_{\to 1}\right| |z| = |z|$$

Por lo tanto converge si $|z|<1$

(9) 1

$$g(z) = \sin(\pi z) \qquad g(1)=0$$

$$g'(z) = \pi\cos(\pi z) \qquad g'(1) = -\pi$$

$$g''(z) = -\pi^2\sin(\pi z) \qquad g''(1) = 0$$

$$g'''(z) = -\pi^3 \cos(\pi z) \qquad g'''(1) = \pi^3$$

$$g''''(z) = \pi^4 \sin(\pi z) \qquad g''''(1) = 0$$

$$\Rightarrow g^{(n)}(1) = \begin{cases} 0 & \text{si } n \text{ es par} \\ (-1)^{3\frac{(n+1)}{2}} \pi^n & \text{si } n \text{ es impar} \end{cases}$$

$$\Rightarrow g(z) = \sum_{n=0}^{\infty} \frac{(-1)^{3n} \pi^{2n-1}}{(2n-1)!} (z-1)^{2n-1}$$

$$\Rightarrow \frac{\sin(\pi z)}{(z-1)^3} = \frac{g(z)}{(z-1)^3} = \sum_{n=0}^{\infty} \frac{(-1)^{3n} \pi^{2n-1}}{(2n-1)!} (z-1)^{2n-4}$$

4

$$\frac{1}{z(1+z^2)} = \frac{1}{z} \cdot \frac{1}{1+z^2} = \frac{1}{z} \cdot \frac{1}{1-(-z^2)} \overset{\text{si } |z^2|<1}{=} \frac{1}{z} \sum_{n=0}^{\infty} (-z^2)^n$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n z^{2n} = \sum_{n=0}^{\infty} (-1)^n z^{2n-1}$$

5

Sea $u = \frac{1}{z}$, luego $|z|>1 \iff |u|<1$. Luego:

$$\frac{1}{z(1+z^2)} = \frac{1}{z+z^3} = \frac{1}{\frac{1}{u}+\frac{1}{u^3}} = \frac{u^3}{u^2+1} = u^3 \frac{1}{1-(-u^2)} = u^3 \sum_{n=0}^{\infty} (-1)^n u^{2n}$$

Sii $|u|^2<1$ sii $|u|<1$ sii $|z|>1$

$$= \sum_{n=0}^{\infty} (-1)^n u^{2n+3} = \sum_{n=0}^{\infty} (-1)^n z^{-2n-3}$$

6

$$\frac{z}{(z+1)(z+2)} = \frac{1}{z+2} \cdot \frac{z}{z+1}$$

Es de notar que:

$$\frac{1}{z+1} = \frac{1}{z+2-1} = -\frac{1}{1-(z+2)} \underset{\text{Si } |z+2|<1}{=} -\sum_{n=0}^{\infty} (z+2)^n$$

Luego:

$$\frac{z}{(z+1)(z+2)} = \frac{z+2-2}{(z+1)(z+2)} = \frac{z+2}{(z+1)(z+2)} - \frac{2}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{2}{z+2}\cdot\frac{1}{z+1}$$

$$= -\sum_{n=0}^{\infty} (z+2)^n + \frac{2}{z+2}\sum_{n=0}^{\infty}(z+2)^n = 2\sum_{n=0}^{\infty}(z+2)^{n-1} - \sum_{n=0}^{\infty}(z+2)^n$$

$$= 2\sum_{m=-1}^{\infty}(z+2)^m - \sum_{n=0}^{\infty}(z+2)^n = 2(z+2)^{-1} + 2\sum_{m=0}^{\infty}(z+2)^m - \sum_{n=0}^{\infty}(z+2)^n$$

$$= 2(z+2)^{-1} + 2\sum_{m=0}^{\infty}(z+2)^m$$

3

Sea $U = \frac{1}{z-i}$, luego $|U|<1 \iff |z-i|>1$

$$\frac{1}{z} = \frac{1}{i+\frac{1}{U}} = \frac{1}{\frac{iU+1}{U}} = \frac{U}{iU+1} = U\cdot\frac{1}{1-(-iU)} = U\cdot\sum_{n=0}^{\infty}(-iU)^n$$

Sii $|iU| = |U| < 1$ Sii $|z-i|>1$

$$= \sum_{n=0}^{\infty}(-1)^n i^n U^{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n i^n}{(z-i)^n}$$

7

$$\frac{w}{z-w} = \frac{\frac{w}{z}}{1-\frac{w}{z}} = \frac{w}{z}\cdot\frac{1}{1-\frac{w}{z}} = \frac{w}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n = \sum_{n=0}^{\infty}\frac{w^{n+1}}{z^{n+1}}$$

Sii $\left|\frac{w}{z}\right|<1$ Sii $|w|<|z|$

10

1

$$\frac{d\left(\frac{1}{1-z}\right)}{dz} = \frac{1}{(1-z)^2}$$

$$\Rightarrow \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} \frac{d(z^n)}{dz} = \sum_{n=1}^{\infty} n z^{n-1} \underset{m=n-1}{=} \sum_{m=0}^{\infty} (m+1) z^m$$

2

$$\frac{d^2\left(\frac{1}{1-z}\right)}{dz} = \frac{2}{(1-z)^3}$$

$$\Rightarrow \frac{2}{(1-z)^3} = \sum_{n=1}^{\infty} (n+1) \frac{d(z^n)}{dz} = \sum_{n=1}^{\infty} (n+1)\, n\, z^{n-1} = \sum_{m=0}^{\infty} (m+2)(m+1) z^m$$

11

1

$$f(z) = \frac{1}{z^2}$$

$$f'(z) = -2 \cdot \frac{1}{z^3}$$

$$f''(z) = 6\,\frac{1}{z^4} \qquad \Rightarrow f^{(n)}(z) = (n+1)!\, z^{-(n+2)} (-1)^{n+3}$$

$$f'''(z) = -24\,\frac{1}{z^5} \qquad \Rightarrow f^{(n)}(-1) = (n+1)!\,(-1)^{-n}(-1)^{-2}(-1)^{n+3}$$

$$f''''(z) = 120\,\frac{1}{z^6} \qquad = (n+1)!$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!}(z+1)^n = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!}(z+1)^n = \sum_{n=0}^{\infty} (n+1)(z+1)^n$$

si $|z-z_0| < R$

$$\lim_{n\to\infty} \left| \frac{(n+2)}{(n+1)} \right| = 1$$

$\therefore R = 1$

2

$f(z) = Ln(z)$ $\quad f(i) = i\frac{\pi}{2}$

$f'(z) = \frac{1}{z}$ $\quad f'(i) = -i$

$f''(z) = -\frac{1}{z^2} \Rightarrow f''(i) = 1$

$f'''(z) = 2 \cdot \frac{1}{z^3}$ $\quad f'''(i) = -2i$

$f''''(z) = -6\frac{1}{z^4}$ $\quad f''''(i) = -6$

$$\Rightarrow f^{n}(z) = \begin{cases} Ln(z) & \text{si } n=0 \\ \frac{(n-1)!(-1)^{n+1}}{z^n} & \text{si } n \geq 1 \end{cases}$$

$$f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{(n-1)!(-1)^{n+1}}{i^n n!}(z-i)^n = i\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\, i^n}(z-i)^n$$

Luego:

$$\lim_{n\to\infty} \left| \frac{n \cdot i^n}{(n+1)\, i^{n+1}} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

$$\therefore R = 1$$

3

$$\frac{1}{(1+i)-\sqrt{2}z} = \frac{1}{1+i} \cdot \frac{1}{1-\frac{\sqrt{2}}{1+i}z} = \frac{1}{1+i}\sum_{n=0}^{\infty}\left(\frac{\sqrt{2}}{1+i}\right)^n z^n = \sum_{n=0}^{\infty}\frac{(\sqrt{2})^n}{(1+i)^{n+1}} z^n$$

Sii $\left|\frac{\sqrt{2}}{1+i} z\right| < 1$

Sii $|z| < 1 \Rightarrow R = 1$

4

$$\frac{d(\arctan(z))}{dz} = \frac{1}{1+z^2} = \sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n}$$

Si $|z|<1 \Rightarrow R=1$

$$\Rightarrow \arctan(z) = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{2n+1} + C \Rightarrow \arctan(0) = C \Rightarrow C=0$$

$$\therefore \arctan(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} z^{2n+1}$$

8

$$e^{2z} = \frac{e^{2z-2}}{e^2} = \frac{1}{e^2}\sum_{n=0}^{\infty}\frac{(2z-2)^n}{n!} = \frac{1}{e^2}\sum_{n=0}^{\infty}\frac{2^n}{n!}(z-1)^n$$

$$\Rightarrow z e^{2z} = (z-1)e^{2z} + e^{2z} = \frac{z-1}{e^2}\sum_{n=0}^{\infty}\frac{2^n(z-1)^n}{n!} + \frac{1}{e^2}\sum_{n=0}^{\infty}\frac{2^n(z-1)^n}{n!}$$

$$= \frac{1}{e^2}\sum_{n=0}^{\infty}\frac{2^n}{n!}(z-1)^{n+1} + \frac{1}{e^2}\sum_{n=0}^{\infty}\frac{2^n(z-1)^n}{n!}$$ ¿?

12

Taylor

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n}$$

Sii $|-z^2|<1$ sii $|z|<1$

Laurent

Sea $\frac{1}{z^2} = \mu$, luego $|\mu|<1 \Leftrightarrow |z|>1$

Luego:

$$\frac{1}{1+z^2} = \frac{\frac{1}{z^2}}{\frac{1}{z^2}+1} = \frac{1}{z^2}\cdot\frac{1}{1-(-\frac{1}{z^2})} = \mu\cdot\frac{1}{1-(-\mu)}$$

$$= \mu\sum_{n=0}^{\infty}(-\mu)^n = \mu\sum_{n=0}^{\infty}(-1)^n\mu^n = \sum_{n=0}^{\infty}(-1)^n\mu^{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+1}}$$

14

1

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(n+1)!} : \frac{(-1)^n}{n!}\right| = \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n+1} = 0$$

$\therefore R = \infty$

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2

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(1-\frac{1}{n}\right)^n\right|} = \lim_{n\to\infty} 1-\frac{1}{n} = 1$$

$\therefore R = 1$