PRACTICO 4

1

1

$\exp(2-3\pi i) = e^2(\cos(-3\pi) + i\sin(-3\pi)) = e^2(-1+i.0) = -e^2$

2

$\exp\left(\frac{2+i\pi}{4}\right) = \exp\left(\frac{1}{2}+i\frac{\pi}{4}\right) = \sqrt{e}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right) = \sqrt{e}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)$

$= \sqrt{e}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{e}{2}}(1+i)$

3

$\exp(\dot{z}+\pi i) = \exp(z).\exp(\pi i) = \exp(z).\, e^0(\cos(\pi)+i\sin(\pi)) = \exp(z)(-1+i.0)$

$= -\exp(z)$

2

1 $e^z = e^x e^{iy} = 2e^{i\pi} \iff \begin{cases} e^x = 2 \\ y = \pi + 2k\pi \end{cases}, k \in \mathbb{Z}$

$\iff \begin{cases} x = \ln(2) \\ y = \pi(2k+1) \end{cases}$

$\therefore z = \ln(2) + i\pi(2k+1)$

2 $1+\sqrt{3}i = 2e^{i\frac{\pi}{3}}$

$e^z = e^x e^{iy} = 2e^{i\frac{\pi}{3}} \iff \begin{cases} e^x = 2 \\ y = \frac{\pi}{3} + 2k\pi \end{cases}$

$\therefore z = \ln(2) + i\pi\left(\frac{1}{3}+2k\right)$

3

$$1 = 1e^{i0}$$

$$e^{2z-1} = e^{z}e^{z}\frac{1}{e} = e^{2x}e^{i2y}e^{-1} = e^{2x-1}e^{i2y} = 1e^{i0}$$

$$\iff \begin{cases} e^{2x-1} = 1 \\ 2y = 0 + 2k\pi \end{cases}$$

$$\iff \begin{cases} x = \frac{1}{2} \\ y = k\pi \end{cases}$$

$$\therefore z = \frac{1}{2} + ik\pi$$

4

$$|e^{-2z}| = |e^{-2x}(\cos(-2y) + i\sin(-2y))| = \sqrt{(e^{-2x})^2\cos^2(-2y) + (e^{-2x})^2\sin^2(-2y)}$$

$$= e^{-2x} < 1$$

$$\iff \ln(e^{-2x}) < \ln(1)$$

$$\iff -2x < 0$$

$$\iff x > 0$$

5

$e^{z} = e^{x}\cos(y) + i\,e^{x}\sin(y)$, luego:

$$Re(e^{z}) = e^{x}\cos(y) = 0 \iff \cos(y) = 0$$

$$\iff y = \frac{\pi}{2} + k\pi$$

6

$$Im(e^{z}) = e^{x}\sin(x) = 0 \iff \sin(y) = 0$$

$$\iff y = k\pi$$

Otra forma: f y $\bar{f}$ analíticas $\Rightarrow$ f constante (por teorema)

entonces f no constante $\Longrightarrow$ f no analítica $\vee$ $\bar{f}$ no analítica

como f no es constante, y $f = \bar{f}$ $\Rightarrow$ f no es analítica

3

$$\exp(\bar{z}) = e^x e^{i(-y)} = e^x(\cos(-y) + i\sin(-y)) = e^x(\cos(y) - i\sin(y))$$

$$\overline{\exp(z)} = \overline{e^x e^{iy}} = \overline{e^x \cos(y) + i\sin(y)} = e^x(\cos(y) - i\sin(y))$$

$$\therefore \exp(\bar{z}) = \overline{\exp(z)}$$

$$\begin{cases} u(x,y) = e^x \cos(y) \\ v(x,y) = -e^x \sin(y) \end{cases}$$

$$\frac{\partial u}{\partial x} = e^x \cos(y) \qquad \frac{\partial v}{\partial x} = -e^x \sin(y)$$

$$\frac{\partial u}{\partial y} = -e^x \sin(y) \qquad \frac{\partial v}{\partial y} = -e^x \cos(y)$$

Luego:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Longleftrightarrow \begin{cases} e^x \cos(y) = -e^x \cos(y) \\ -e^x \sin(y) = e^x \sin(y) \end{cases}$$

$$\Longleftrightarrow \begin{cases} \cos(y) = -\cos(y) \\ -\sin(y) = \sin(y) \end{cases}$$

$$\Longleftrightarrow \begin{cases} \cos(y) = 0 \\ \sin(y) = 0 \end{cases}$$

$\rightsquigarrow$ No tiene solución

$\Rightarrow$ No cumplen C-R en ningun punto

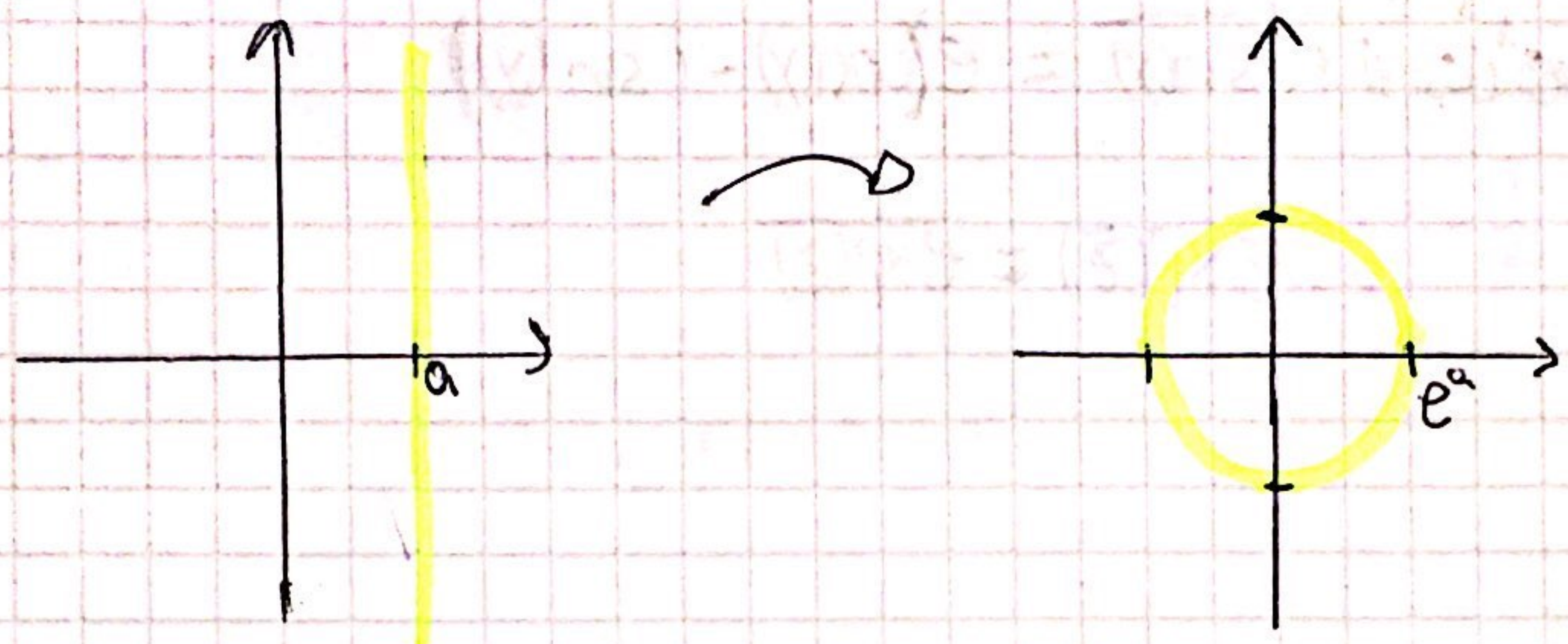
$\Rightarrow$ No son derivables en ningun punto

$\Rightarrow$ No son analíticas en ningun punto

4

$$\exp(z) = e^x (\cos(y) + i.\sin(y))$$

1



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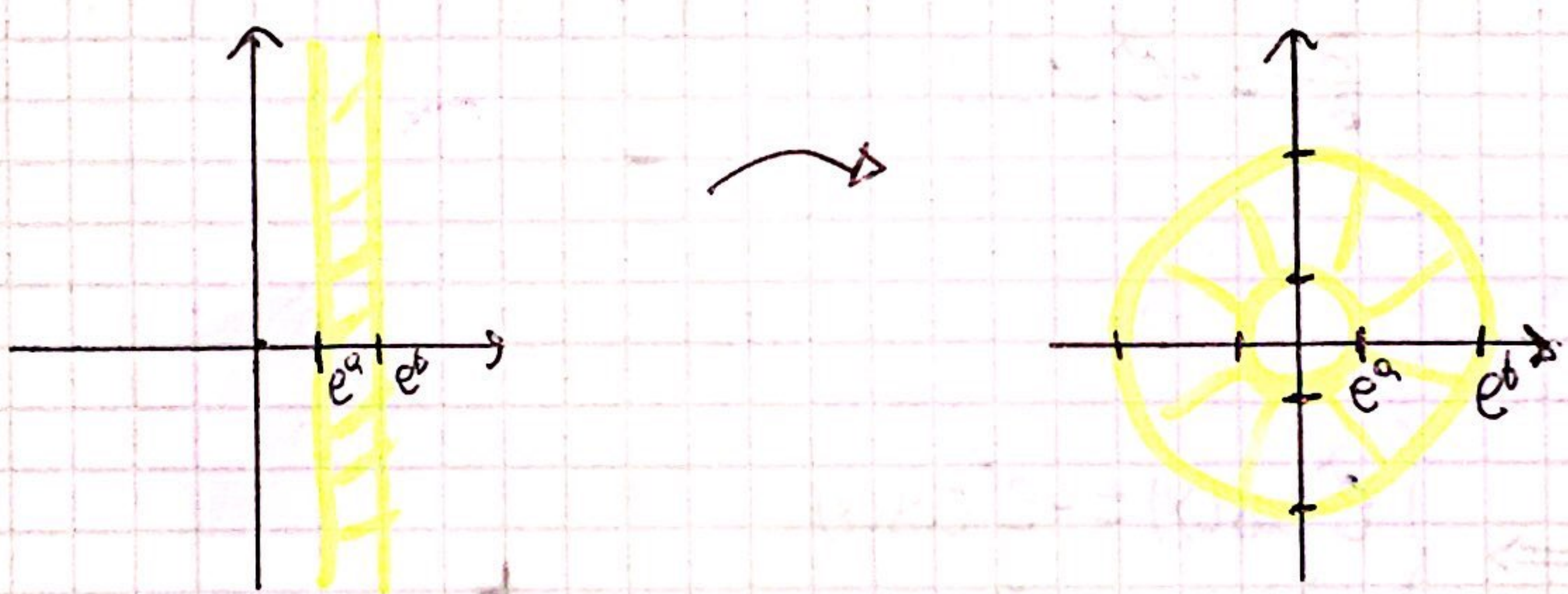
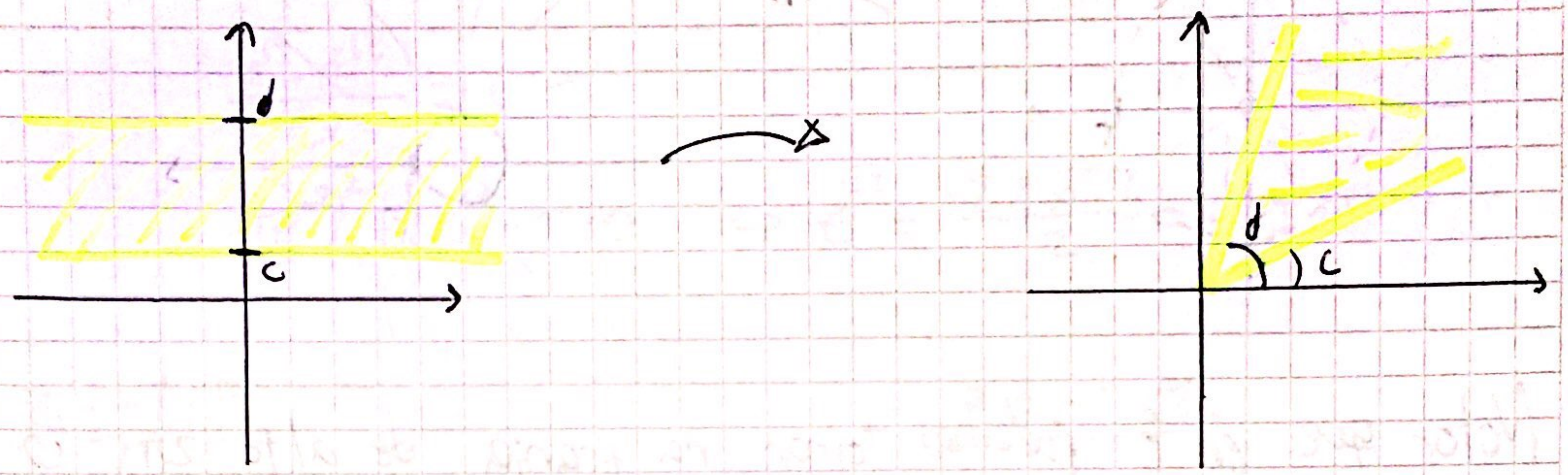


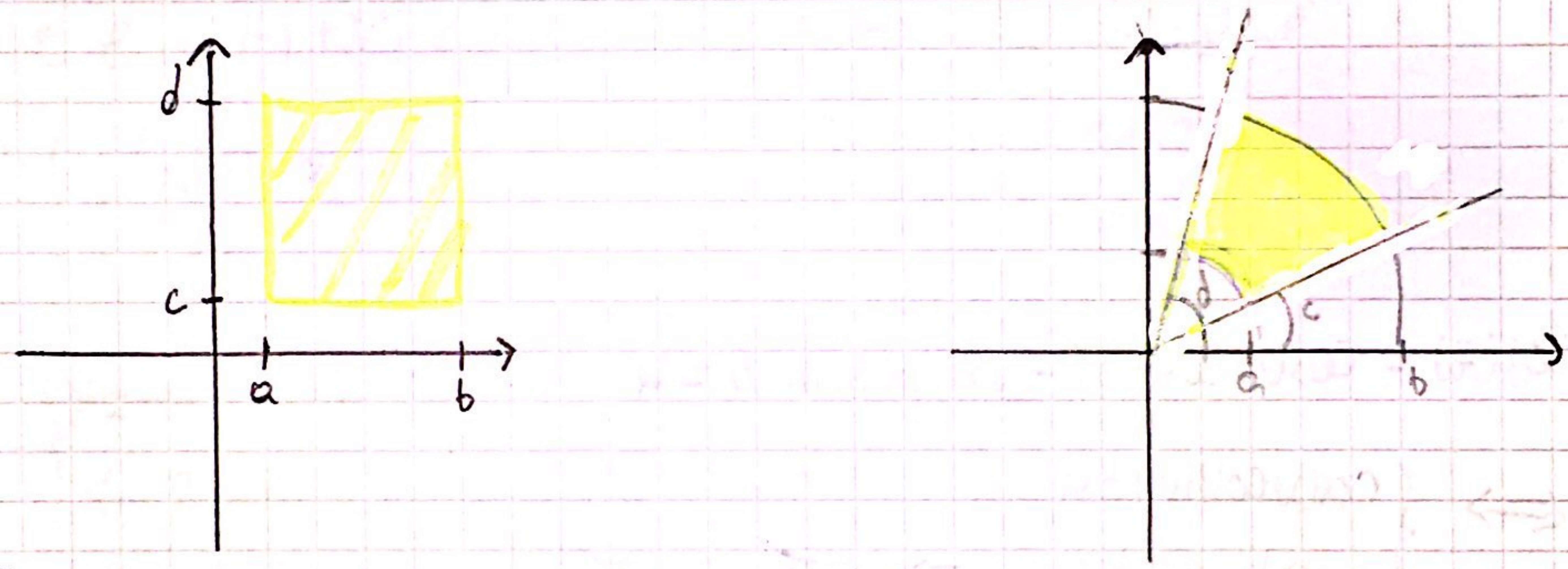
3

$$\exp(z) = e^x \cos(c) + i\sin(c) \Rightarrow \arg(\exp(z)) = c$$

4



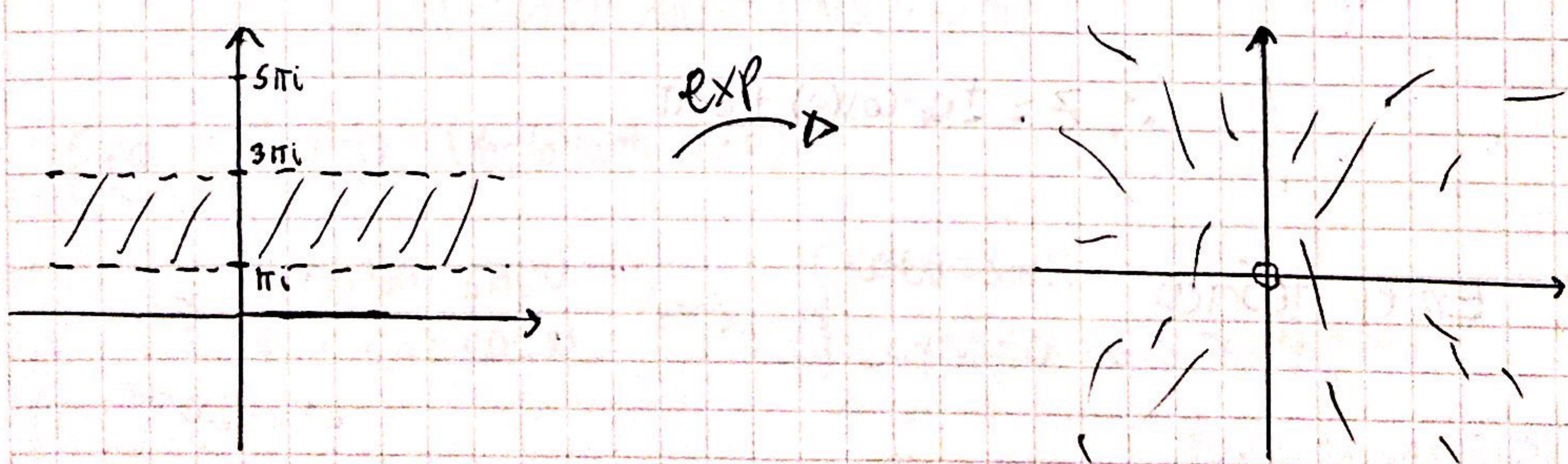
5



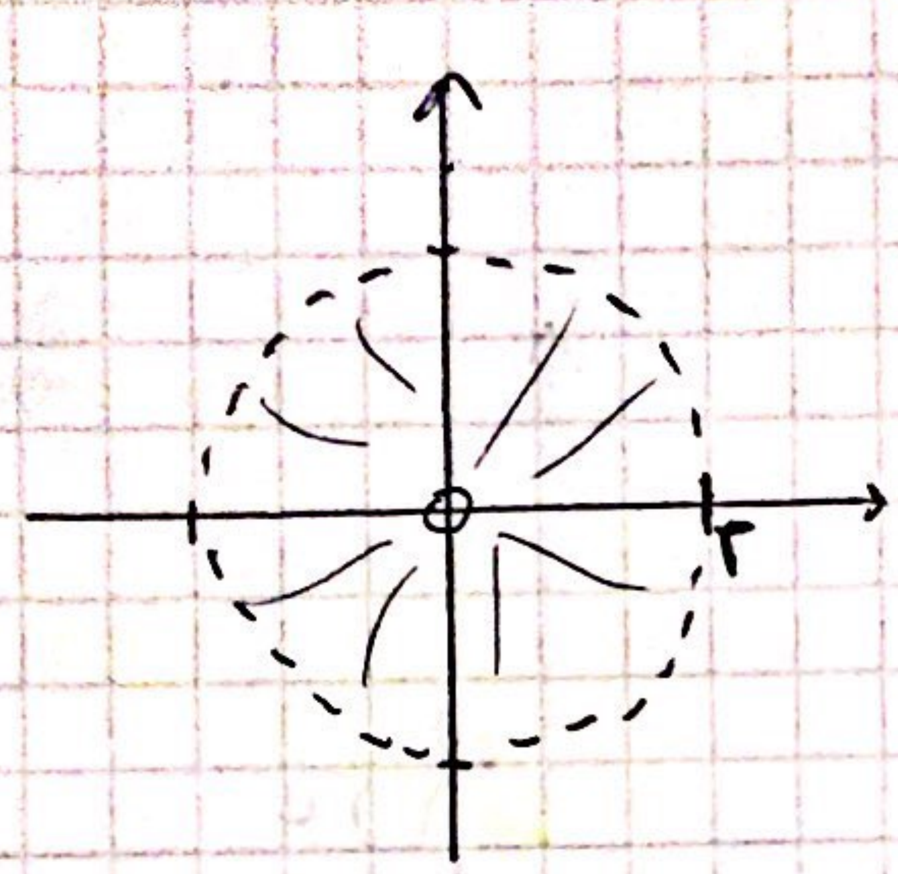
5

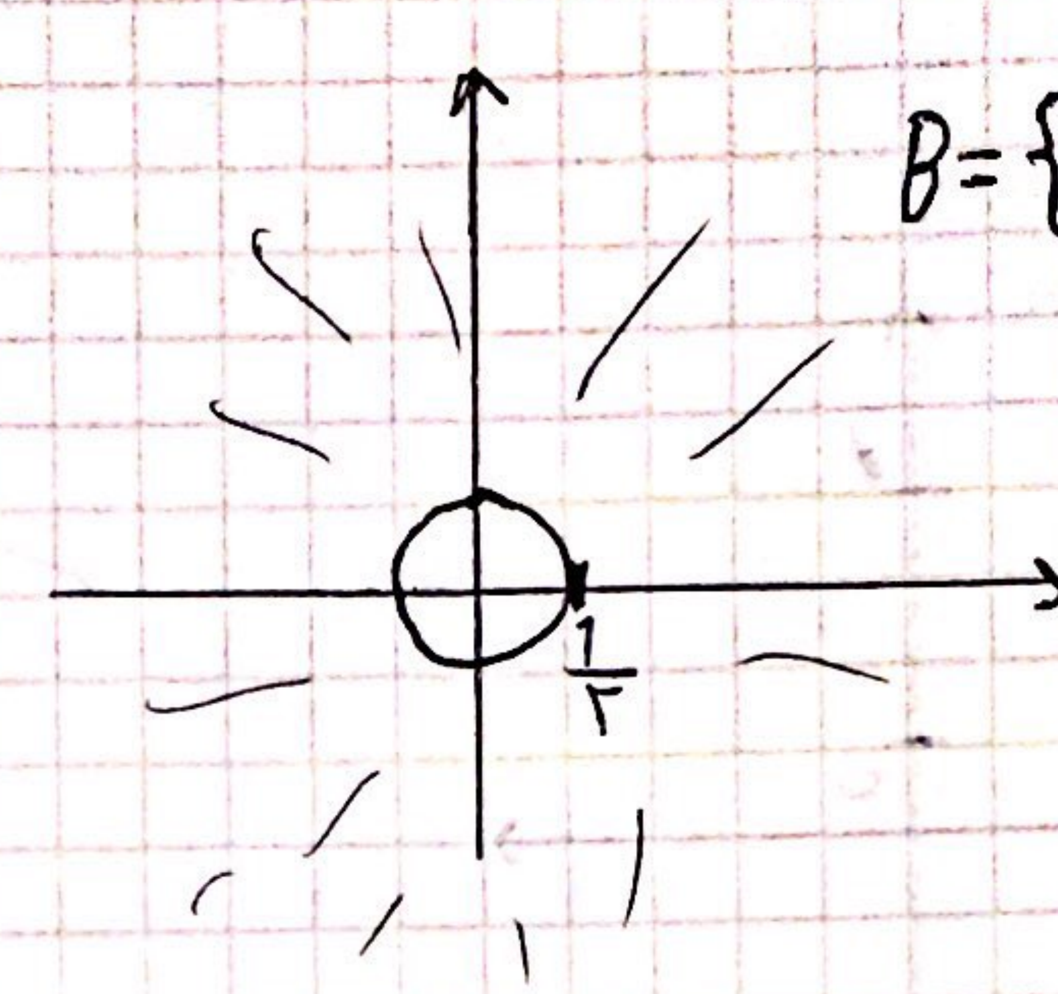
sabemos exp es sobreyectiva sobre $\mathbb{C}-\{0\}$ y además periódica en $2\pi i$. Entonces si el conjunto en el que volvamos exp tiene de alto $2\pi i$ y de largo al eje real entonces exp $\rightsquigarrow \mathbb{C}-\{0\}$

Es decir:



Luego notar que para cualquier $r>0$ el mapeo de $\frac{1}{z}$ es:





$B=\{t \in \mathbb{C} / t=\frac{1}{z}\}$

Notar que en B podemos tomar una franja de alto $2\pi i$ con longitud el eje real $\Rightarrow \exp(B) \rightsquigarrow \mathbb{C}-\{0\}$

6

1

$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y) = a$$

$$\iff \begin{cases} \cos(x)\cosh(y)=a \\ \sin(x)\sinh(y)=0 \end{cases} \Rightarrow \overset{x=k\pi}{\cancel{x=0}} \vee y=0$$

Si $x=k\pi$

$(-1)^k\cosh(y)=a \rightsquigarrow$ absurdo $\forall y\neq 0$ pues $\cosh(y)>a$ $\forall y\neq 0$

Si $y=0$

$\cos(x)=a \Rightarrow x=\arccos(a)+2k\pi$

$$\therefore z=\pm\arccos(a)+2k\pi$$

3 En el teórico

4

$$\sin(z) = i$$

$$\frac{e^{iz}-e^{-iz}}{2i} = i$$

$$(e^{iz})^2 + 2e^{iz} - 1 = 0 \implies e^{iz} = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

$z_0 = -1+\sqrt{2}$

$$e^{iz} = -1+\sqrt{2}$$

$$\iff e^{-y}e^{ix} = |-1+\sqrt{2}|\, e^{i0}$$

$$\iff \begin{cases} e^{-y} = \sqrt{2}-1 \\ x = 2k\pi \end{cases}$$

$$\iff \begin{cases} y = -\ln(\sqrt{2}-1) \\ x = 2k\pi \end{cases}$$

$z_1 = -1-\sqrt{2}$

$$e^{iz} = -1-\sqrt{2}$$

$$\iff e^{-y}e^{ix} = |-1-\sqrt{2}|\, e^{i\pi}$$

$$\iff \begin{cases} e^{-y} = 1+\sqrt{2} \\ x = \pi + 2k\pi \end{cases}$$

$$\iff \begin{cases} y = -\ln(1+\sqrt{2}) \\ x = \pi + 2k\pi \end{cases}$$

$$\therefore\ z = 2k\pi - i.\ln(\sqrt{2}-1) \quad \vee \quad z = \pi(2k+1) - i.\ln(1+\sqrt{2})$$

5

$$\sin(z) = \cos(z)$$

$$\iff \begin{cases} \cos(x)\cosh(y) = \sin(x)\cosh(y) \\ -\sin(x)\sinh(y) = \cos(x)\sinh(y) \end{cases}$$

Sea $\sinh(y) \neq 0$ (por lo tanto $y \neq 0$)

$$\implies \begin{cases} \cos(x) = \sin(x) \\ -\sin(x) = \cos(x) \end{cases} \iff \begin{cases} \cos(x) = \sin(x) \\ 2\cos(x) = 0 \end{cases} \implies x = \frac{\pi}{2} + 2k\pi$$

$\cosh(y) \neq 0$

Pero es tal caso $\cos(x) \neq \sin(x)$

Sea $y=0$

$$\cos(x)=\sin(x)$$
$$\Rightarrow \cos^2(x)=\sin^2(x)$$
$$1-\sin^2(x)=\sin^2(x)$$
$$1=2\sin^2(x)$$
$$\frac{1}{\sqrt{2}}=|\sin(x)| \Rightarrow \sin(x)=\sqrt{\tfrac{1}{2}} \lor \sin(x)=-\sqrt{\tfrac{1}{2}}$$
$$\Rightarrow \left(x=\frac{\pi}{4}+2k\pi \lor x=\frac{3\pi}{4}+2k\pi\right) \lor \left(x=\frac{5\pi}{4}+2k\pi \lor x=\frac{7\pi}{4}+2k\pi\right)$$

Pero $\sin(x)\neq\cos(x)$ si $x=\frac{3\pi}{4}+2k\pi \lor x=\frac{7\pi}{4}+2k\pi$. Finalmente:

$$x=\frac{\pi}{4}+2k\pi \lor x=\frac{5\pi}{4}+2k\pi \Rightarrow x=\frac{\pi}{4}+k\pi$$

$$\therefore z=\frac{\pi}{4}+k\pi$$

9

$$\frac{e^z+e^{-z}}{2}=i$$
$$e^z+e^{-z}=2i$$
$$(e^z)^2+1=2ie^z$$
$$(e^z)^2-2ie^z+1=0$$
$$\Rightarrow e^z=\frac{2i\pm\sqrt{-4-4}}{2}=\frac{2i\pm2i\sqrt{2}}{2}=i\pm i\sqrt{2}=i(1\pm\sqrt{2})$$
$$\Rightarrow e^z=i(1+\sqrt{2}) \quad \lor \quad e^z=i(1-\sqrt{2})$$
$$\Rightarrow e^xe^{iy}=(1+\sqrt{2})e^{i\frac{\pi}{2}} \quad \lor \quad e^xe^{iy}=(1-\sqrt{2})e^{i\frac{\pi}{2}}$$

$$\Rightarrow \begin{cases} e^x = 1+\sqrt{2} \\ y = \frac{\pi}{2} + 2k\pi \end{cases} \quad \vee \quad \begin{cases} e^x = 1-\sqrt{2} \\ y = \frac{\pi}{2} + 2k\pi \end{cases}$$ $\rightsquigarrow$ No tiene solución

$$\therefore\ z = \ln(1+\sqrt{2}) + i\left(\frac{\pi}{2} + 2k\pi\right)$$

10

$$\frac{e^z + e^{-z}}{2} = \frac{1}{2}$$

$$e^z + e^{-z} = 1$$

$$(e^z)^2 + 1 = e^z$$

$$(e^z)^2 - e^z + 1 = 0$$

$$\Rightarrow e^z = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\Rightarrow e^z = \frac{1}{2} + \frac{\sqrt{3}}{2} i \quad \vee \quad e^z = \frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$$\Rightarrow e^x e^{iy} = e^{i\frac{\pi}{3}} \quad \vee \quad e^x e^{iy} = e^{i\frac{5\pi}{3}}$$

$$\Rightarrow \begin{cases} e^x = 1 \\ y = \frac{\pi}{3} + 2k\pi \end{cases} \quad \vee \quad \begin{cases} e^x = 1 \\ y = \frac{5\pi}{3} + 2k\pi \end{cases}$$

$$\therefore\ z = i\left(\frac{\pi}{3} + 2k\pi\right) \quad \vee \quad z = i\left(\frac{5\pi}{3} + 2k\pi\right)$$

7

1

$$\cos(z)+i\sin(z)=\frac{e^{iz}+e^{-iz}}{2}+i\,\frac{e^{iz}-e^{-iz}}{2i}=\frac{e^{iz}+e^{-iz}+e^{iz}-e^{-iz}}{2}=e^{iz}$$

2

$$\overline{\sin(z)}=\overline{\frac{e^{iz}-e^{-iz}}{2i}}=\frac{\overline{e^{iz}}-\overline{e^{-iz}}}{\overline{2i}}=\frac{e^{-i\bar{z}}-e^{i\bar{z}}}{-2i}=\frac{e^{i\bar{z}}-e^{-i\bar{z}}}{2i}=\sin(\bar{z})$$

3

$$\sin\left(\frac{\pi}{2}-z\right)=\frac{e^{i(\frac{\pi}{2}-z)}-e^{-i(\frac{\pi}{2}-z)}}{2i}=\frac{e^{i\frac{\pi}{2}}e^{-iz}-e^{iz}e^{-i\frac{\pi}{2}}}{2i}$$

$$=\frac{i\,e^{-iz}-(-i)e^{iz}}{2i}=\frac{e^{iz}+e^{-iz}}{2}=\cos(z)$$

4

$$\sin^2(z)+\cos^2(z)=\left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2+\left(\frac{e^{iz}+e^{-iz}}{2}\right)^2$$

$$=\frac{(e^{iz})^2-2+(e^{-iz})^2}{(2i)^2}+\frac{(e^{iz})^2+2+(e^{-iz})^2}{2^2}$$

$$=\frac{(e^{iz})^2+(e^{-iz})^2-2}{-4}+\frac{(e^{iz})^2+(e^{-iz})^2+2}{4}$$

$$=-\frac{(e^{iz})^2}{4}-\frac{(e^{-iz})^2}{4}+\frac{1}{2}+\frac{(e^{iz})^2}{4}+\frac{(e^{-iz})^2}{4}+\frac{1}{2}$$

$$=1$$

8

$$i\sin(z) = i\,\frac{e^{iz}-e^{-iz}}{2i} = \frac{e^{iz}-e^{-iz}}{2} = \sinh(iz)$$

6

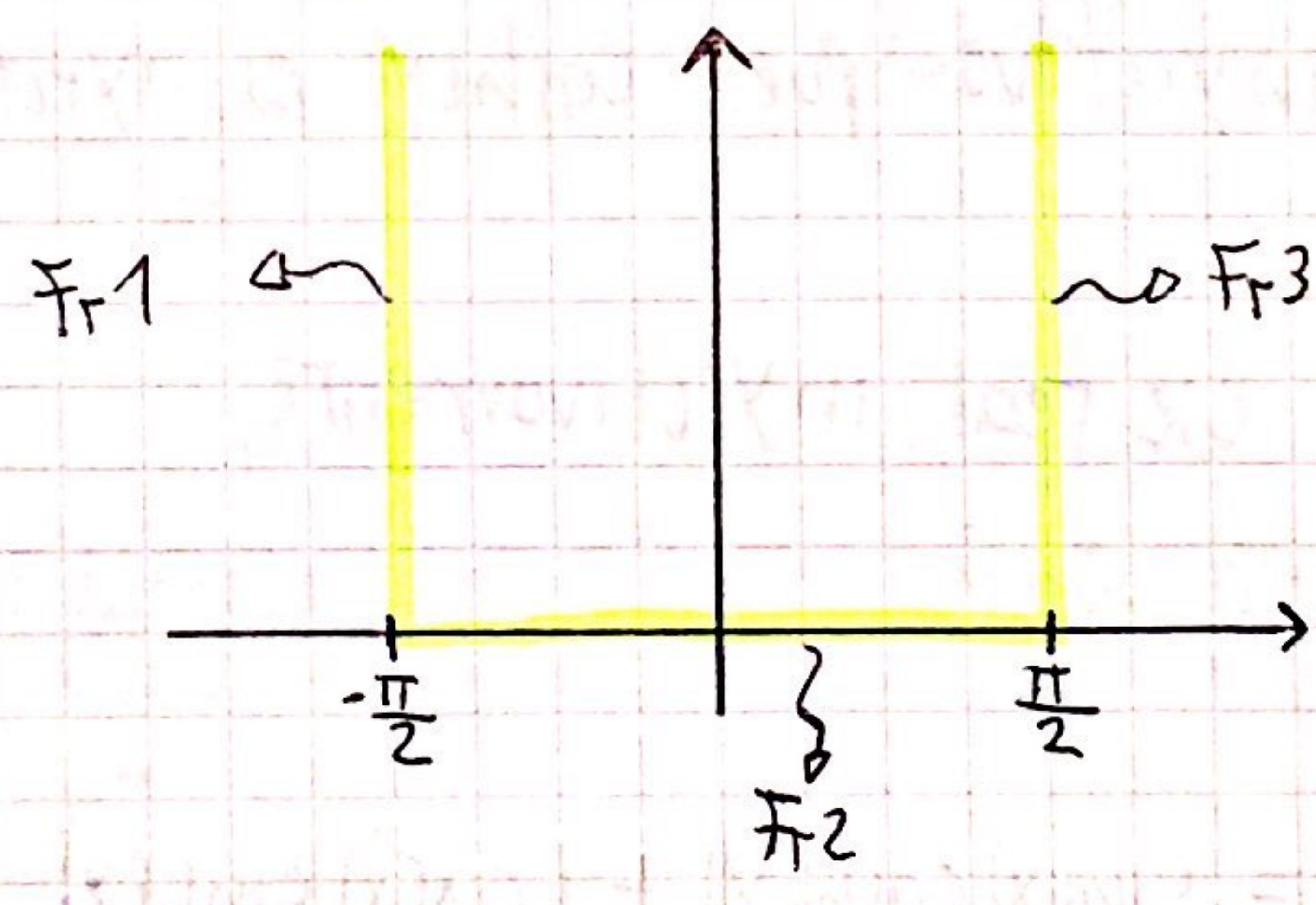
$$\cos(z)\cos(w) - \sin(z)\sin(w) = \frac{e^{iz}+e^{-iz}}{2}\cdot\frac{e^{iw}+e^{-iw}}{2} - \frac{e^{iz}-e^{-iz}}{2i}\cdot\frac{e^{iw}-e^{-iw}}{2i}$$

$$= \frac{e^{i(z+w)}+e^{i(w-z)}+e^{i(z-w)}+e^{-i(z+w)}}{4} + \frac{e^{i(z+w)}-e^{i(w-z)}-e^{i(z-w)}+e^{-i(z+w)}}{4}$$

$$= \frac{e^{i(z+w)}+e^{-i(z+w)}}{2} = \cos(z+w)$$

8

1

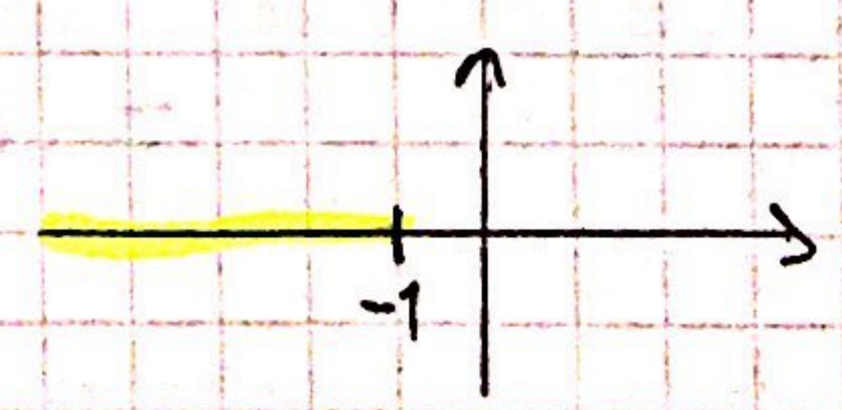


$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

Fr1

Tenemos que $x = -\frac{\pi}{2}$, $y \geq 0$. $\cos(-\frac{\pi}{2}) = 0$, $\sin(-\frac{\pi}{2}) = -1$

$\Longrightarrow \sin(z) = -\cosh(y)$ $\rightsquigarrow$ mapea:

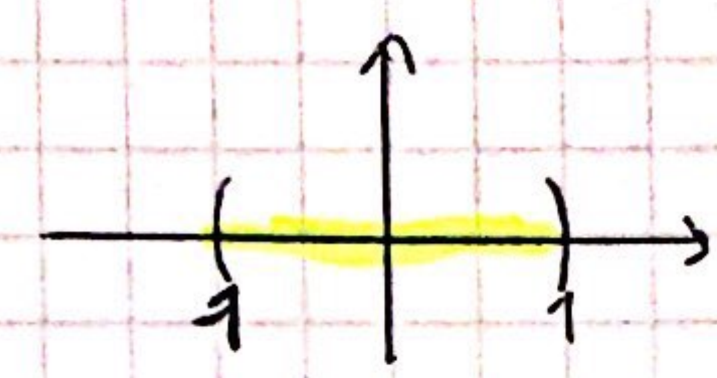


Y ese mapeo es inyectivo pues $\cosh(y) > 1$, y es inyectiva

Fr 2

Tenemos $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $y = 0$. $\sinh(0) = 0$ y $\cosh(0) = 1$

$\Rightarrow \sin(z) = \sin(x)$ $\rightsquigarrow$ mapea:

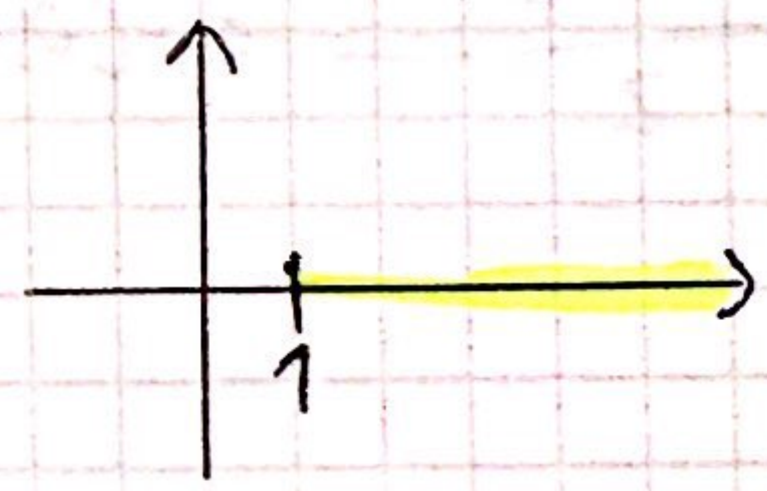


Y el mapeo es inyectivo pues sin es inyectivo en $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Fr 3

Tenemos $x = \frac{\pi}{2}$, $y \geq 0$. Luego $\cos\left(\frac{\pi}{2}\right) = 0$ y $\sin\left(\frac{\pi}{2}\right) = 1$

$\Rightarrow \sin(z) = \cosh(y)$ $\rightsquigarrow$ mapea:



Y el mapeo es inyectivo pues $\cosh(y)$ es inyectivo $\forall y \geq 0$

$\therefore$ Cubrimos todo el eje real inyectivamente.

9

1

$$|\cos(z)|^2 = \cos^2(x) + \sinh^2(y) \geq \sinh^2(y)$$

$$\Rightarrow |\cos(z)| \geq |\sinh(y)|$$

$$|\sin(z)|^2 = \sin^2(x) + \sinh^2(y) = \sin^2(x) + \cosh^2(y) - 1$$

$$= \cosh^2(y) - \underbrace{\cos^2(x)}_{\geq 0}$$

$$\Longrightarrow |\sin(z)|^2 \leq \cosh^2(y)$$

$$\Longrightarrow |\sin(z)| \leq |\cosh(y)| = \cosh(y)$$

2

$$|\sin(z)|^2 + |\cos(z)|^2 = \sin^2(x) + \sinh^2(y) + \cos^2(x) + \sinh^2(y)$$

$$= 1 + 2\underbrace{\sinh^2(y)}_{\geq 0} \geq 1$$

12

Sea $\quad \mu(x,y) = \mu(r\cos(\theta), r\sin(\theta))$
$\qquad\quad V(x,y) = V(r\cos(\theta), r\sin(\theta))$

C-R en polares $\begin{cases} \dfrac{\partial \mu}{\partial r} = \dfrac{1}{r}\dfrac{\partial V}{\partial \theta} \\ \dfrac{\partial V}{\partial r} = -\dfrac{1}{r}\dfrac{\partial \mu}{\partial \theta} \end{cases}$ $\qquad \ell = \{ z \in \mathbb{C} / \arg(z) = \pi \}$

$\operatorname{Log}(z) = \underbrace{\ln(r)}_{\mu} + i.\underbrace{\theta}_{V}$ $\qquad$ donde $\pi < \theta < 3\pi$

$$\frac{\partial \mu}{\partial r} = \frac{1}{r}, \quad \frac{\partial \mu}{\partial \theta} = 0, \quad \frac{\partial V}{\partial r} = 0, \quad \frac{\partial V}{\partial \theta} = 1$$

Luego:

$$\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{1}{r}\cdot 1 = \frac{1}{r} = \frac{\partial \mu}{\partial r}$$

$$-\frac{1}{r}\frac{\partial \mu}{\partial \theta} = -\frac{1}{r}\cdot 0 = 0 = \frac{\partial V}{\partial r}$$

$\therefore$ Cumplen C-R $\forall r \neq 0 \Rightarrow$ Cumple C-R $\forall\, \mathbb{C} - \ell$ (α)

Por otro lado:

$$\frac{\partial u}{\partial r}(r,\theta) = \frac{1}{r} \quad \text{continua } \forall r \neq 0 \implies \text{continua en } \mathbb{C}-\ell$$

$$\frac{\partial u}{\partial \theta}(r,\theta) = 0 \quad \text{continua en } \mathbb{C} \implies \text{continua en } \mathbb{C}-\ell$$

$$\frac{\partial v}{\partial r}(r,\theta) = 0 \quad \text{continua en } \mathbb{C} \implies \text{continua en } \mathbb{C}-\ell$$

$$\frac{\partial v}{\partial \theta}(r,\theta) = 1 \quad \text{continua en } \mathbb{C} \implies \text{continua en } \mathbb{C}-\ell$$

(β)

$(\alpha) \wedge (\beta) \implies$ es analítica en $\mathbb{C}-\ell$

13

$$\overline{Log(z)} = \overline{\ln(|z|) + i\theta} = \ln(|z|) + i(-\theta) = Log(\bar{z}) \quad \forall z \in \mathbb{C}-\ell$$

(Notar que si $z = re^{i\theta}$ y $\theta = Arg(z) \implies \bar{z} \in \mathbb{C}-\ell$)

$$\overline{z^{\lambda}} = \overline{\exp(\lambda . Ln(z))} \underset{\text{ej 3}}{=} \exp(\overline{\lambda . Ln(z)}) = \exp(\bar{\lambda} . \overline{Ln(z)})$$

$$= \exp(\bar{\lambda} \, Ln(\bar{z})) = \bar{z}^{\bar{\lambda}}$$

14

f es inyectiva si $f(z_1) = f(z_2) \implies z_1 = z_2$

Sean $z_1, z_2 \in A$:

$$Ln(z_1^2+1) = Ln(z_2^2+1)$$

$$\exp(Ln(z_1^2+1)) = \exp(Ln(z_2^2+1))$$

$$z_1^2+1 = z_2^2+1 \quad \rightsquigarrow \text{ pues } \begin{matrix} z_1^2+1 \neq 0 \\ z_2^2+1 \neq 0 \end{matrix} \quad \forall z_1, z_2 \in A$$

$$z_1^2 = z_2^2$$

$$r_1^2 e^{2i\theta_1} = r_2^2 e^{2i\theta_2} \qquad \text{con } \theta_1, \theta_2 \in [-\pi, \pi]$$

$$\Longleftrightarrow \begin{cases} r_1^2 = r_2^2 \\ 2\theta_1 = 2\theta_2 + 2k\pi \end{cases}$$

$$\Longleftrightarrow \begin{cases} r_1 = r_2 \\ \theta_1 = \theta_2 + k\pi \end{cases}$$

$$\Longleftrightarrow \quad z_1 = z_2 \ \vee \ z_1 = -z_2$$

$\rightsquigarrow$ pero si $z_1 \in A \Rightarrow -z_2 \notin A$

$\therefore$ Es inyectiva en A

17

Sea $Ln(z_1 z_2) = w_1$

$$\Longrightarrow \exp(w_1) = \exp(Ln(z_1 z_2)) = z_1 z_2$$

Sea $w_2 = Ln(z_1) + Ln(z_2)$

$$\Longrightarrow \exp(w_2) = \exp(Ln(z_1) + Ln(z_2)) = \exp(Ln(z_1)) . \exp(Ln(z_2))$$
$$= z_1 . z_2$$

$$\Longrightarrow \exp(w_1) = \exp(w_2)$$

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exp es Periódica

$\Longrightarrow w_1 = w_2 + 2k\pi i$

$\Longrightarrow Log(z_1 z_2) = Log(z_1) + Log(z_2) + 2k\pi i$

$\Longleftrightarrow \underbrace{Arg(z_1 z_2)}_{\alpha} = \underbrace{Arg(z_1)}_{\theta_1} + \underbrace{Arg(z_2)}_{\theta_2} + 2k\pi$

Sabemos que $-\pi < \alpha < \pi$, $-\pi < \theta_1 < \pi$, $-\pi < \theta_2 < \pi$.

$\Longrightarrow -\pi < \theta_1 + \theta_2 + 2k\pi < \pi$

Veamos la máxima cota: Si $\theta_1 = \pi$, $\theta_2 = \pi \Longrightarrow \theta_1 + \theta_2 = 2\pi \Longrightarrow k = -1$

Veamos la minima cota: Si $\theta_1 = -\pi$, $\theta_2 = -\pi \Longrightarrow \theta_1 + \theta_2 = -2\pi \Longrightarrow k = 1$

$\therefore k \in \{-1, 0, 1\}$

16

Sea $z = re^{i\theta}$, $\theta = Arg(z)$.

$Log(z^\lambda) = Log(r^\lambda e^{i\lambda\theta}) = \ln(r^\lambda) + i\lambda\theta \quad (-\pi < \lambda\theta < \pi)$

$\lambda Log(z) = \lambda(\ln(r) + i\theta) \quad (-\pi < \theta < \pi)$

Luego

$Log(z^\lambda) = \lambda Log(z)$

$\Longleftrightarrow \ln(r^\lambda) + i\lambda\theta = \lambda\ln(r) + i\lambda\theta$

$\Longleftrightarrow \lambda\ln(r) + i\lambda\theta = \lambda\ln(r) + i\lambda\theta$ ✓

$-\pi < \lambda\theta < \pi \;\wedge\; -\pi < \theta < \pi$

$\Longrightarrow -\frac{\pi}{\lambda} < \theta < \frac{\pi}{\lambda}$

$\therefore$ Sea $z = re^{i\theta}$, entonces $Log(z^\lambda) = \lambda \cdot Log(z) \Longleftrightarrow -\frac{\pi}{\lambda} < Arg(z) < \frac{\pi}{\lambda}$

18

$$z^{\lambda} z^{\nu} = \exp(\lambda . Ln(z)) . \exp(\nu Ln(z))$$

$$z^{\lambda+\nu} = \exp((\lambda+\nu) Ln(z)) = \exp(\lambda . Ln(z) + \nu Ln(z))$$

$$= \exp(\lambda . Ln(z)) . \exp(\nu Ln(z))$$

$$\therefore z^{\lambda} z^{\nu} = z^{\lambda+\nu}$$

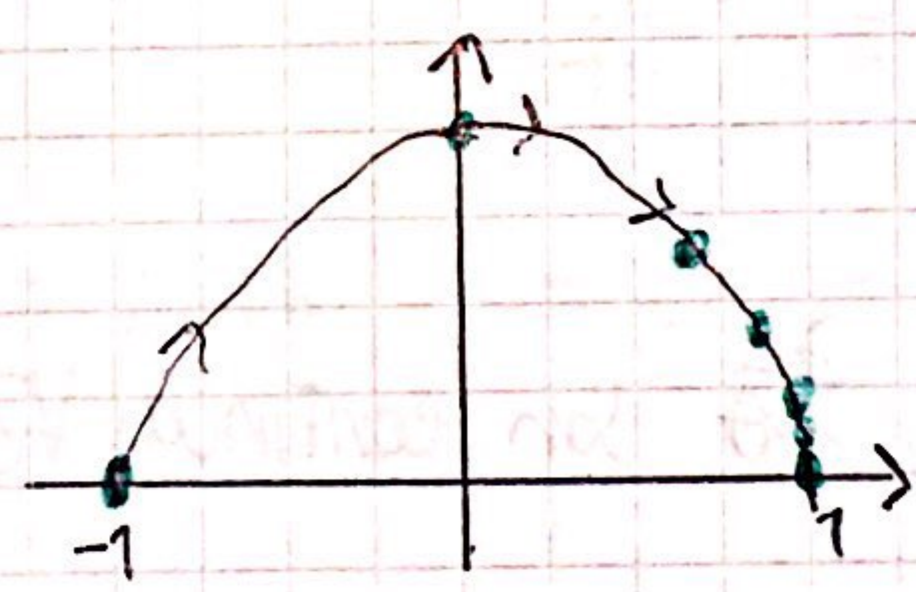
19

i $1^{1-i} = \exp((1-i). Ln(1)) = \exp((1-i).0) = \exp(0) = 1$

vi

$$(-1)^{\frac{1}{n}} = \exp\left(\tfrac{1}{n}(Ln(-1)\right) = \exp\left(\tfrac{1}{n}(\ln(1) + i\pi)\right) = \exp\left(i\tfrac{\pi}{n}\right) = e^{i\frac{\pi}{n}}$$

graficamente:



vii $i^{-i} = i^{i^{i}}$

↳ lo hago abajo:

viii

$$i^{i} = \exp(i Ln(i)) = \exp(i(\ln(1) + i\tfrac{\pi}{2})) = \exp(i^2 \tfrac{\pi}{2}) = e^{-\frac{\pi}{2}} = -i$$

Por lo tanto $i^{i^{i}} = i^{-i}$. Luego:

$$i^{-i} = \exp(-i Ln(i)) = \exp(-i(\ln(1) + i\tfrac{\pi}{2})) = \exp(-i^2 \tfrac{\pi}{2}) = \exp(\tfrac{\pi}{2}) = e^{\frac{\pi}{2}} = i$$

20

Para $z=0$, $\sqrt{z}=0$ y dado que cos es analítica en $z_0=0$, $\cos(\sqrt{z})$ es analítica en $z_0=0$.

Sea $z\neq 0$:

$$\sqrt{z} = r^{\frac{1}{n}} e^{i\overbrace{\left(\frac{\theta+2k\pi}{n}\right)}^{=\alpha}} = r^{\frac{1}{n}}\cos(\alpha) + i r^{\frac{1}{n}}\sin(\alpha)$$

$$\frac{\partial u}{\partial r} = \frac{r^{\frac{1-n}{n}}}{n}\cdot\cos(\alpha) \qquad \frac{\partial v}{\partial r} = \frac{r^{\frac{1-n}{n}}}{n}\sin(\alpha)$$

$$\frac{\partial u}{\partial \theta} = -\frac{r^{\frac{1}{n}}}{n}\sin(\alpha) \qquad \frac{\partial v}{\partial \theta} = \frac{r^{\frac{1}{n}}}{n}\cos(\alpha)$$

Por lo tanto

$$\begin{cases} r\frac{\partial u}{\partial r} = \frac{r^{\frac{1}{n}}}{n}\cos(\alpha) = \frac{\partial v}{\partial \theta} \\ \frac{\partial u}{\partial \theta} = -\frac{r^{\frac{1}{n}}}{n}\sin(\alpha) = -r\frac{\partial v}{\partial r} \end{cases} \Longrightarrow \text{Cumple C-R } \forall z\in\mathbb{C}-\{0\}$$

$\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ son continuas $\forall z\in\mathbb{C}-\{0\}$

$\therefore \sqrt{z}$ es analítica $\forall\ \mathbb{C}-\{0\}$

Y como cos es entera $\Longrightarrow \cos(\sqrt{z})$ es analítica en $\mathbb{C}-\{0\}$

$\therefore \cos(\sqrt{z})$ es entera.