PRACTICO 7

1

1

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

$$\Rightarrow \cos\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{-2n}$$

$$\Rightarrow z \cdot \cos\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} z^{-2n+1} = z^1 \underbrace{-\frac{1}{2}}_{b_1} z^{-1} + \dots$$

$$\therefore \operatorname*{Res}_{z=0} f(z) = -\frac{1}{2}$$

2

$$\frac{z - \sin(z)}{z} \overset{\text{si } z \neq 0}{=} 1 - \frac{\sin(z)}{z} = 1 - \frac{1}{z} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

$$= 1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}$$ $\rightsquigarrow$ No tiene terminos con $z^{-1} \Rightarrow b_1 = 0$

$$\therefore \operatorname*{Res}_{z=0} f(z) = 0$$

3

$$\frac{e^{z+1}}{z^3} = \frac{e}{z^3} \cdot e^z = \frac{e}{z^3} \sum_{n=0}^{\infty} \frac{z^n}{n!} = \frac{e}{z^3}\left(1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \dots\right)$$

$$= \frac{e}{z^3} + \frac{e}{z^2} + \underbrace{\frac{e}{2z}} + \frac{e}{3!} + \frac{e z^4}{4!} + \dots$$

$$\Rightarrow b_1 = \frac{e}{2}$$

$$\therefore \operatorname*{Res}_{z=0} f(z) = \frac{e}{2}$$

2

1

$$\frac{e^{-z}}{z^2} \overset{z\neq 0}{=} \frac{1}{z^2}\cdot\sum_{n=0}^{\infty}\frac{(-z)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}z^{n-2} = \frac{1}{z^2} \underbrace{- \frac{1}{z}}_{b_1=-1} + \cdots$$

$$\Longrightarrow \operatorname*{Res}_{z=0} f(z) = -1$$

$\frac{e^{-z}}{z^2}$ es analítica en todo $z\neq 0$, en particular lo es en C y en su interior excepto en una cantidad finita de singularidad

$$\Longrightarrow \int_C \frac{e^{-z}}{z^2}\,dz = 2\pi i(-1) = -2\pi i$$

2

$$\frac{e^{-z}}{(z-1)^2} = \frac{e^{-z+1}}{e(z-1)^2} = \frac{1}{e(z-1)^2}\sum_{n=0}^{\infty}\frac{(-z+1)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{e\,n!}(z-1)^{n-2}$$

singularidad en $z=1$

$$= \frac{1}{e}(z-1)^{-2} \underbrace{- \frac{1}{e}}_{b_1}(z-1)^{-1} + \cdots$$

$$\Longrightarrow \operatorname*{Res}_{z=1} f(z) = -\frac{1}{e}$$

$$\Longrightarrow \int_C \frac{e^{-z}}{(z-1)^2}\,dz = 2\pi i\cdot -\frac{1}{e} = -\frac{2\pi}{e}i$$

3

$$z^2 e^{\frac{1}{z}} = z^2\sum_{n=0}^{\infty}\frac{1}{n!}z^{-n} = \sum_{n=0}^{\infty}\frac{z^{-n+2}}{n!} = z^2 + z + \frac{1}{2} + \frac{1}{z}\underbrace{\frac{1}{3!}}_{b_1} + \cdots$$

$$\Longrightarrow \operatorname*{Res}_{z=0} f(z) = \frac{1}{6}$$

$$\Longrightarrow \int_C z^2 e^{\frac{1}{z}} dz = 2\pi i \cdot \frac{1}{6} = \frac{\pi}{3} i$$

3

$$z^2 e^{\frac{1}{z}} = z^2 \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{z^{-n+2}}{n!} = z^2 + z + \frac{1}{2!} + \underbrace{\frac{1}{z} \cdot \frac{1}{3!}}_{b_1} + \cdots$$

$$\Longrightarrow \operatorname*{Res}_{z=0} f(z) = \frac{1}{6}$$

$$\Longrightarrow \int_C z^2 e^{\frac{1}{z}} = 2\pi i \cdot \frac{1}{6} = \frac{\pi}{3} i$$

4

$$\frac{z+1}{z^2 - 2z} = \frac{z+1}{z(z-2)}$$ $\rightsquigarrow$ Singularidades en $z=0$ y $z=2$

Luego buscamos series de Laurent centrada en 0 y en 2

$z=0$

$$\frac{z+1}{z(z-2)} = \frac{z+1}{z} \cdot \frac{1}{z-2} = -\frac{z+1}{z} \cdot \frac{1}{2-z} = -\frac{z+1}{2z} \cdot \frac{1}{1-\frac{z}{2}}$$

$$= -\frac{z+1}{2z} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = -(z+1) \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = -z \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

Sii $\left|\frac{z}{2}\right| < 1$

Sii $|z| < 2$

$$= -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty} \frac{z^{n-1}}{2^{n+1}}$$

No tiene termino z^{-1}

$$\hookrightarrow = \underbrace{-\frac{z^{-1}}{2}}_{b_1} - \frac{1}{2} + \cdots$$

$$\therefore \operatorname*{Res}_{z=0} f(z) = -\frac{1}{2}$$

$\underline{z=1}$

$$\frac{z+1}{z(z-2)} = \frac{z+1}{z-2}\cdot\frac{1}{z} = \frac{z-2+3}{z-2}\cdot\frac{1}{z} = \left(\frac{z-2}{z-2}+\frac{3}{z-2}\right)\frac{1}{z}$$

$$= \left(1+\frac{3}{z-2}\right)\frac{1}{z} = \left(1+\frac{3}{z-2}\right)\frac{1}{2+z-2} = \left(1+\frac{3}{z-2}\right)\frac{\frac{1}{2}}{1-\left(-\frac{(z-2)}{2}\right)}$$

$$= \frac{1}{2}\left(1+\frac{3}{z-2}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}(z-2)^n = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}(z-2)^n + \frac{3}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}(z-2)^{n-1}$$

Si $\left|\frac{z-2}{2}\right|<1$

$|z-2|<2$

No tiene término z^{-1}

$= \frac{3}{2}\left((z-2)^{-1}+\cdots\right)$

$b_1 = \frac{3}{2}$

$$\Rightarrow \operatorname*{Res}_{z=1} f(z) = \frac{3}{2}$$

Por teorema de los residuos de Cauchy:

$$\int_C \frac{z+1}{z^2-2z}\,dz = 2\pi i\left(-\frac{1}{2}+\frac{3}{2}\right) = 2\pi i$$

3

$e^{\frac{1}{z}}$ es analítica en $\mathbb{C}-\{0\}$

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!\,z^n} = 1+\underbrace{\frac{1}{z}}_{b_1}+\frac{1}{2}\cdot\frac{1}{z^2}+\cdots$$

$$\therefore \operatorname*{Res}_{z=0} f(z) = 1$$

Suponiendo γ simple y orientada positivamente:

$$\int_\gamma e^{\frac{1}{z}}\,dz = 2\pi i\cdot 1 = 2\pi i$$

Si γ es orientada negativamente: $\int_\gamma e^{\frac{1}{z}}\,dz = -2\pi i$

4

1

$$ze^{\frac{1}{z}} = z\sum_{n=0}^{\infty}\frac{1}{n!}z^{-n} = \sum_{n=0}^{\infty}\frac{z^{-n+1}}{n!}$$

$\to \geq 1 \ \forall n \geq 0 \Rightarrow$ No tiene parte principal

$\Rightarrow b_n = 0$

$\Rightarrow z_0 = 0$ es una singularidad evitable

2

$\frac{z^2}{1+z}$ no singularidad aislada en $z=-1$

Luego:

$$\frac{z^2}{1+z} = \frac{(z-1)(z+1)+1}{z+1} = (z-1) + \frac{1}{z+1}$$

$b_1 = 1$, $b_n = 0 \ \forall n \geq 2$

Dado que $b_n \neq 0$ para una cantidad finita de valores de n (en este caso uno solo) $\Rightarrow z_0 = 1$ es un polo

3

$\frac{\sin}{z}$ tiene singularidad aislada en $z=0$

$$\frac{\sin(z)}{z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n}$$

$\Rightarrow$ No tiene parte principal

$\Rightarrow b_n = 0 \ \forall n$

$\therefore z=0$ es una singularidad evitable

5

llamo $z_0 = w_0$

g es analítica en un abierto que contiene a z_0:

$$\Rightarrow g(z) = \sum_{n=0}^{\infty} {}_g a_n (z-z_0)^n = {}_g a_0 + {}_g a_1 (z-z_0) + \cdots$$

para $|z-z_0| < R$ para algún $R > 0$

f tiene un polo simple en z_0:

$$\Longrightarrow f(z) = \sum_{n=0}^{\infty} {}_f a_n (z-z_0)^n + {}_f b_1 (z-z_0)^{-1}$$

donde $\quad {}_f b_1 = \operatorname*{Res}_{z=z_0} f(z)$

Por lo tanto:

$$f(z) \cdot g(z) = \left(\sum_{n=0}^{\infty} {}_g a_n (z-z_0)^n \right) \cdot \left(\sum_{n=0}^{\infty} {}_f a_n (z-z_0)^n + {}_f b_1 (z-z_0)^{-1} \right)$$

$$= \left({}_g a_0 + {}_g a_1 (z-z_0) + \cdots \right) \left(\sum_{n=0}^{\infty} {}_f a_n (z-z_0)^n + {}_f b_1 (z-z_0)^{-1} \right)$$

$$\Longrightarrow \operatorname*{Res}_{z=z_0} f \cdot g = {}_{fg} b_1 = \underbrace{{}_g a_0}_{= \frac{g^{(0)}(z_0)}{0!} = g(z_0)} \, {}_f b_1 = g(z_0) \cdot \operatorname*{Res}_{z=z_0} f(z)$$

6

1

Si f es analítica en z_0, $f(z_0) \neq 0$ y $g(z) = \frac{f(z)}{z-z_0}$,

por teorema z_0 es un polo de orden 1 de g. Además

$\operatorname*{Res}_{z=z_0} g = f(z_0)$.

2

Si f es analítica en z_0, luego,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

$$\Longrightarrow g(z) = \frac{f(z)}{z-z_0} = \sum_{n=0}^{\infty} C_n (z-z_0)^{n-1}$$

$$= C_0 (z-z_0)^{-1} + \cdots$$

↳ termino de orden $(z-z_0)^m$ con $m \geq 0$

7

1

$\frac{z+1}{z^2+9}$ tiene singularidades en $z = \pm 3i$

$$\frac{z+1}{(z+3i)(z-3i)} = \begin{cases} \dfrac{\frac{z+1}{z-3i}}{z+3i} & A \\[2ex] \dfrac{\frac{z+1}{z+3i}}{z-3i} & B \end{cases}$$

A

Tomando $\varphi(z) = \frac{z+1}{z-3i}$, φ es analítica en $z=-3i$ y

$\varphi(-3i) = \frac{-3i+1}{-6i} = \frac{1}{2} + \frac{1}{6}i \neq 0$

$\Longrightarrow z=-3i$ es un polo de orden 1

$\Longrightarrow \operatorname*{Res}_{z=-3i} f(z) = \varphi(-3) = \frac{1}{2} + \frac{1}{6}i$

B

Tomando $\varphi(z) = \frac{z+1}{z+3i}$, φ es analítica en $z=3i$ y

$\varphi(3i) = \frac{3i+1}{6i} = \frac{1}{2} - \frac{1}{6}i \neq 0$

$\Longrightarrow f$ tiene un polo de orden 1 en $z=3i$

$\Longrightarrow \operatorname*{Res}_{z=3i} f(z) = \varphi(3i) = \frac{1}{2} - \frac{1}{6}i$

3

$$\left(\frac{z}{2z+1}\right)^3 = \frac{z^3}{(2z+1)^3} = \frac{z^3}{8(z+\frac{1}{2})^3}$$ $\rightsquigarrow z=-\frac{1}{2}$ es una singularidad

$$= \frac{\frac{z^3}{8}}{(z+\frac{1}{2})^3}$$

Tomando $\varphi(z) = \frac{z^3}{8}$, φ es analítica en $z=-\frac{1}{2}$ y $\varphi(-\frac{1}{2}) = -\frac{1}{64} \neq 0$

$\Longrightarrow z=-\frac{1}{2}$ es un polo de orden 3 de f

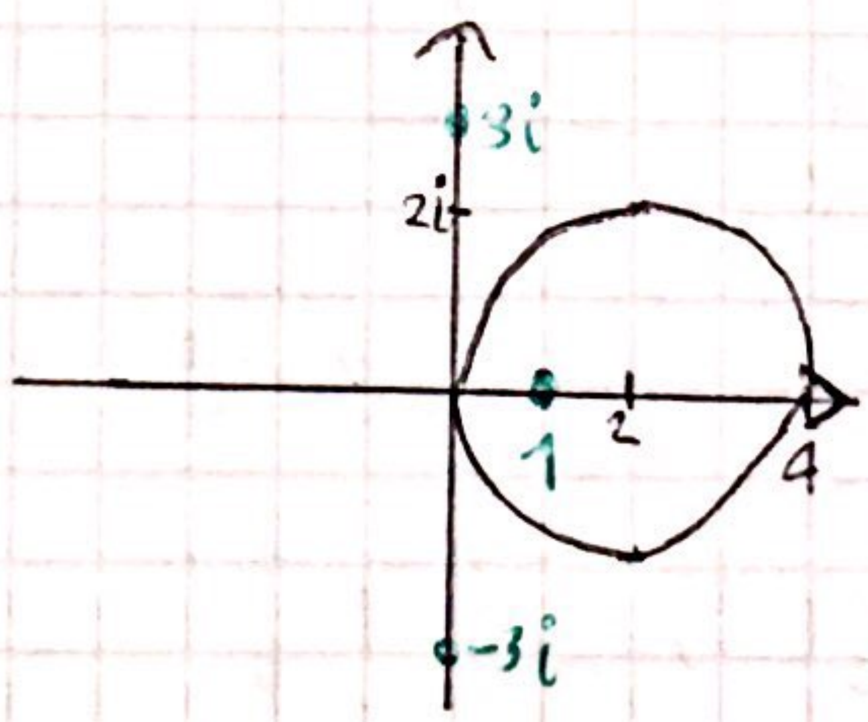
$\Longrightarrow \operatorname{Res}_{z=-\frac{1}{2}} f(z) =$

$\varphi'(z) = \frac{3z^2}{8}$, $\varphi''(z) = \frac{3z}{4} \Longrightarrow \varphi''(-\frac{1}{2}) = -\frac{3}{8}$

$\therefore \operatorname{Res}_{z=-\frac{1}{2}} f(z) = \frac{-\frac{3}{8}}{2} = -\frac{3}{16}$

8

1



• Singularidades

f es analítica en C y en sus puntos interiores excepto en una cantidad finita de singularidades aisladas

Por lo tanto $\int_C f(z)dz = 2\pi i \cdot \operatorname{Res}_{z=1} f(z)$

$$\frac{3z^3+2}{(z-1)(z^2+9)} = \frac{\frac{3z^3+2}{z^2+9}}{z-1}$$

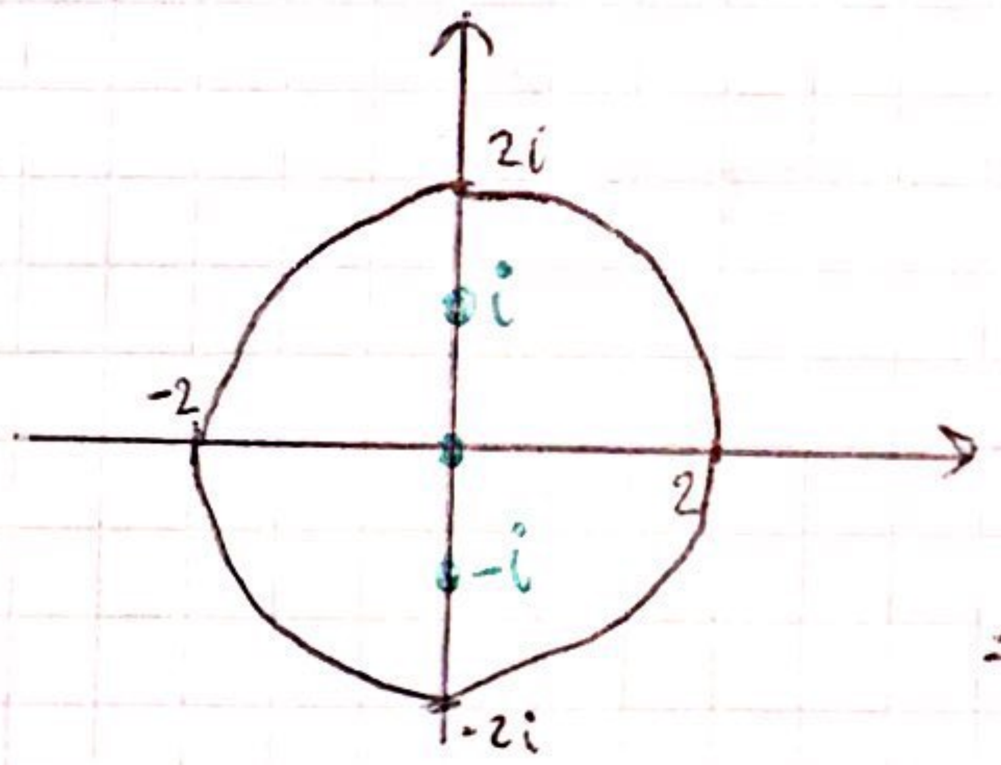
Tomando $\varphi(z) = \frac{3z^2+2}{z^2+9}$, φ es analítica en $z=1$ y $\varphi(1) = \frac{1}{2} \neq 0$

$\Rightarrow$ f tiene un polo simple de orden 1 en $z=1$

$\Rightarrow$ $\underset{z=1}{\text{Res}}\, f(z) = \varphi(1) = \frac{1}{2}$

$\therefore \int_C f(z)dz = 2\pi i \cdot \frac{1}{2} = \pi i$

5



f tiene ~~tiene~~ singularidades aisladas en $z=0, i, -i$

$\Rightarrow \int_C f(z)dz = 2\pi i \left(\underset{z=0}{\text{Res}}\, f(z) + \underset{z=i}{\text{Res}}\, f(z) + \underset{z=-i}{\text{Res}}\, f(z) \right)$

$z=0$

$$\frac{\cosh(\pi z)}{z(z^2+1)} = \frac{\frac{\cosh(\pi z)}{z^2+1}}{z}$$

Tomando $\varphi(z) = \frac{\cosh(\pi z)}{z^2+1}$, φ es analítica en $z=0$ y $\varphi(0)=1 \neq 0$

$\Rightarrow$ $z=0$ es un polo de orden 1 de f

$\Rightarrow$ $\underset{z=0}{\text{Res}}\, f(z) = \varphi(0) = 1$

$z=i$

$$\frac{\cosh(\pi z)}{z(z^2+1)} = \frac{\frac{\cos(\pi z)}{z(z+i)}}{z-i}$$

Tomando $\varphi(z)=\frac{\cosh(\pi z)}{z(z+i)}$, φ es analítica en i y

$$\varphi(i) = \frac{\cosh(\pi i)}{-2} = \frac{e^{\pi i}+e^{-\pi i}}{-4} = \frac{-1-1}{-4} = \frac{1}{2} \neq 0$$

$$\Longrightarrow \operatorname*{Res}_{z=i} f(z) = \varphi(i) = \frac{1}{2}$$

$z=-i$

$$\frac{\cosh(\pi z)}{z(z^2+1)} = \frac{\frac{\cosh(\pi z)}{z(z-i)}}{z+i}$$

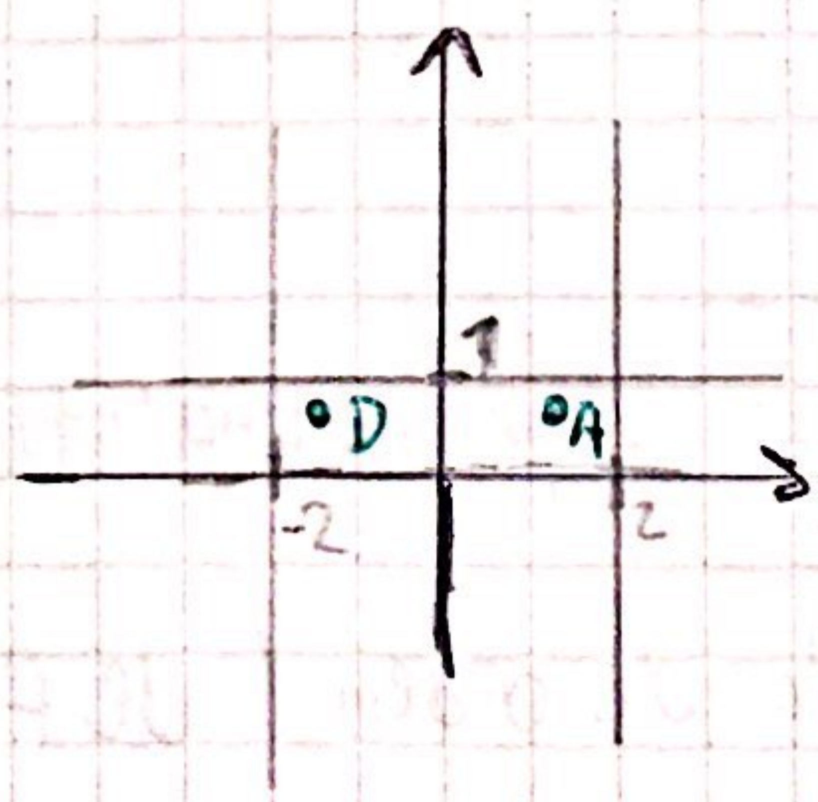
Tomando $\varphi(z)=\frac{\cosh(\pi z)}{z(z-i)}$, φ es analítica en $-i$ y

$$\varphi(-i) = \frac{\cos(-\pi i)}{-2} = \frac{e^{-\pi i}+e^{\pi i}}{-4} = \frac{-2}{-4} = \frac{1}{2} \neq 0$$

$$\Longrightarrow \operatorname*{Res}_{z=-i} f(z) = \varphi(-i) = \frac{1}{2}$$

$$\therefore \int_C f(z)\,dz = 2\pi i\left(1+\frac{1}{2}+\frac{1}{2}\right) = 4\pi i$$

9



A, D son las únicas singularidades de f en el interior de la curva

$(z^2-1)^2+3=0 \rightsquigarrow$ singularidades de f

$(z^2-1)^2=-3$

$z^2-1=\pm\sqrt{3}\,i$

$z^2-1=\sqrt{3}\,i$

$z^2=\sqrt{3}\,i+1$

$\Longrightarrow z=\begin{cases}\sqrt{\frac{3}{2}}+\frac{\sqrt{2}}{2}i & A\\ -\sqrt{\frac{3}{2}}-\frac{\sqrt{2}}{2}i & B\end{cases}$

$z^2-1=-\sqrt{3}\,i$

$z^2=1-\sqrt{3}\,i$

$\Longrightarrow z=\begin{cases}\sqrt{\frac{3}{2}}-\frac{\sqrt{2}}{2}i & C\\ -\sqrt{\frac{3}{2}}+\frac{\sqrt{2}}{2}i & D\end{cases}$

$$\int_\gamma f(z)\,dz=2\pi i\left(\operatorname*{Res}_A f+\operatorname*{Res}_D f\right)$$

A

Dado que $(z^2-1)^2+3=(x-A)(x-B)(x-C)(x-D)$

$f(z)=\dfrac{\frac{1}{(x-B)(x-C)(x-D)}}{x-A}$, tomando $\varphi(z)=\dfrac{1}{(x-B)(x-C)(x-D)}$

, φ es analítica en A y $\varphi(A)\neq 0$

$\Longrightarrow A$ es un polo de orden 1 de f

$\Longrightarrow \operatorname*{Res}_A f=\varphi(A)=\dfrac{1}{(2A)(\sqrt{2}\,i)\left(\sqrt{\frac{3}{2}}\right)}=\dfrac{1}{(\sqrt{6}+\sqrt{2}\,i)(\sqrt{2}\,i)\left(\sqrt{\frac{3}{2}}\right)}$

$=\dfrac{1}{-\frac{3}{\sqrt{2}}+\sqrt{\frac{3}{2}}\,i}=\dfrac{\frac{3}{\sqrt{2}}-\frac{3}{2}i}{\frac{27}{4}}=\dfrac{2\sqrt{2}}{9}-\dfrac{2}{9}i$

D

$f(z)=\dfrac{\frac{1}{(x-A)(x-B)(x-C)}}{x-D}$, Tomando $\varphi(z)=\dfrac{1}{(x-B)(x-C)(x-D)}$

φ es analítica en D y $\varphi(D)\neq 0$

$\Longrightarrow A$ es un polo de orden 1 de f

$$\Longrightarrow \operatorname{Res}_{D} f = \varphi(D) = \frac{1}{\left(-\frac{\sqrt{3}}{2}\right)(\sqrt{2}\,i)(2D)} = \frac{1}{\left(-\sqrt{\frac{3}{2}}\,i\right)(\sqrt{6}-\sqrt{2}\,i)} = \frac{1}{-\sqrt{3}-3i}$$

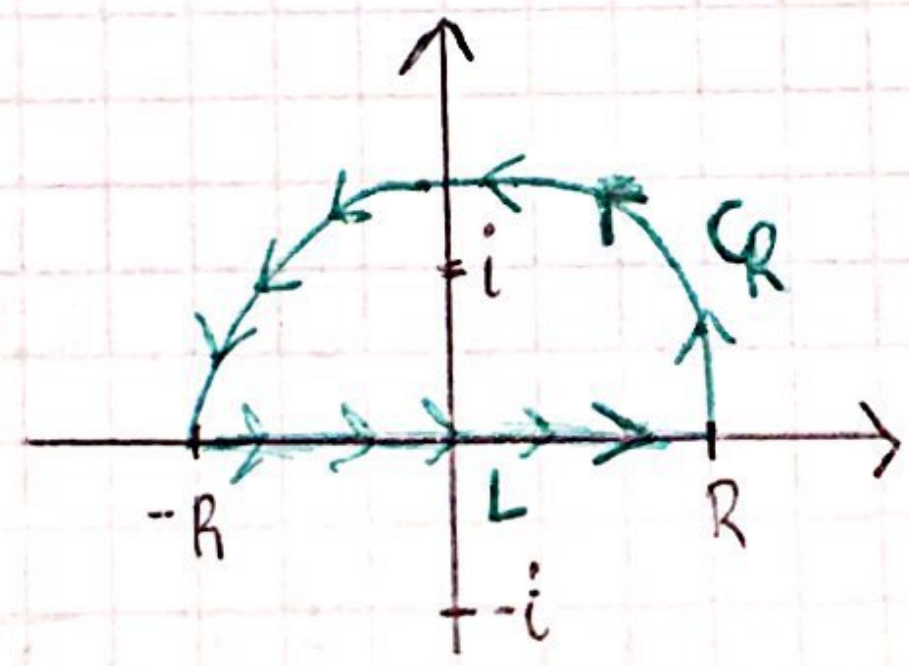
$$= \frac{-\sqrt{3}+3i}{\sqrt{12}} = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

10

1

Sea $f(z)=\frac{1}{z^2+1}$. f tiene singularidades en $z=\pm i$.

Luego tomamos algun $R>1$ y definimos $C=\underbrace{[-R,R]}_{L}+C_R$ → de circunferencia de radio R



Sabemos que:

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz \quad \text{y además} \quad \int_C f(z)\,dz = 2\pi i \operatorname{Res}_{z=i} f(z)$$

Dado que $f(z)=\frac{1}{z^2+1}=\frac{\frac{1}{z+i}}{z-i}$, tomando $\varphi(z)=\frac{1}{z+i}$, φ es analítica y no nula en $z=i$

$\Longrightarrow z=i$ es un polo de orden 1 de f

$$\Longrightarrow \operatorname{Res}_{z=i} f(z) = \varphi(i) = \frac{1}{2i} = -\frac{1}{2}i$$

Luego siendo $z\in C_R$:

$$|z^2+1| \geq \left||z|^2-1\right| = |R^2-1| = R^2-1$$

$$\therefore |f(z)| \leq \frac{1}{R^2-1} \quad \forall z\in C_R$$

$$\Longrightarrow \left| \int_{C_R} f(z)dz \right| \leq \frac{1}{R^2-1} \cdot \overbrace{\pi R}^{long(C_R)} = \frac{\pi R}{R^2-1}$$

Cuando $R \to \infty$ tenemos que $\frac{\pi R}{R^2-1} \to 0 \Longrightarrow \lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$

Por lo tanto:

$$\lim_{R\to\infty} \left(\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz \right) = 2\pi i \left(-\tfrac{1}{2} i\right) = \pi$$

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2+1} = \pi$$

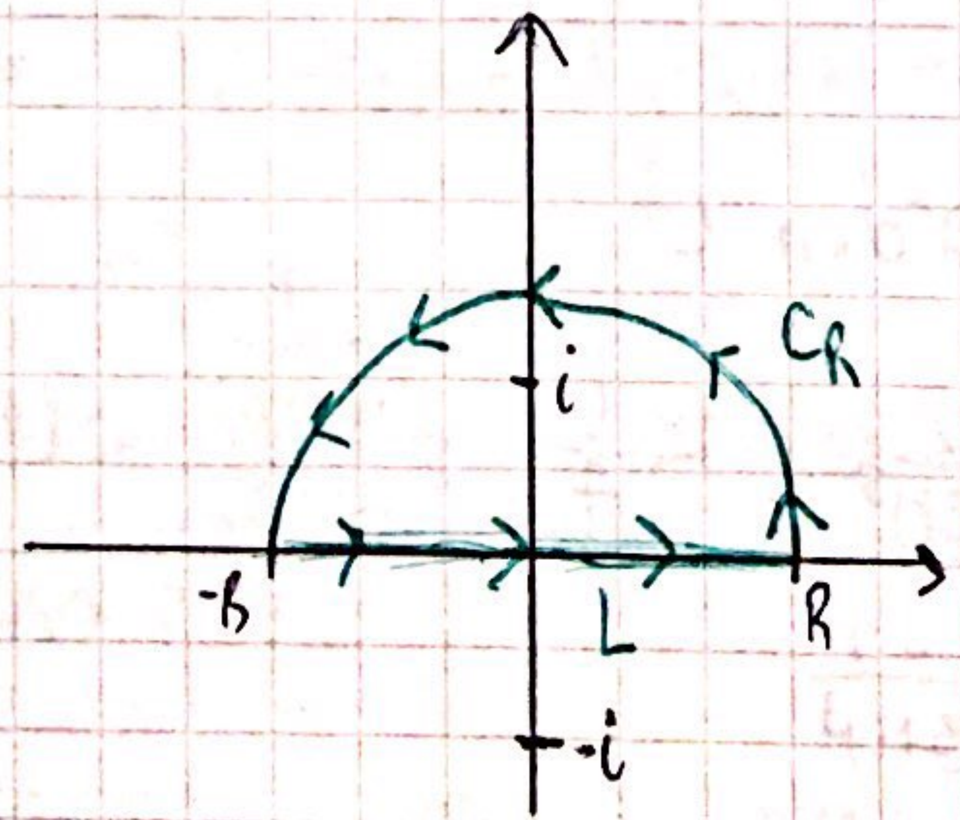
$$\Longrightarrow VP \int_{-\infty}^{\infty} \frac{1}{x^2+1} = \pi$$

$$\underset{f \text{ es par}}{\Longrightarrow} 2 \int_{0}^{\infty} \frac{1}{x^2+1} = \pi \Longrightarrow \int_{0}^{\infty} \frac{1}{x^2+1} = \frac{\pi}{2}$$

2

Sea $f(z) = \frac{1}{(z^2+1)^2}$. f tiene singularidades en $z = \pm 1$

Luego tomamos $R > 1$ y definimos $C = [-R, R] + C_R$



Sabemos que:

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z)$$

Siendo $z \in C_R$:

$$(z^2+1)^2 \geq \left|\, |z|^2-1 \right|^2 = |R^2-1|^2 = (R^2-1)^2$$

$$\therefore |f(z)| \leq \frac{1}{(R^2-1)^2} \quad \forall z \in C_R$$

$$\Longrightarrow \left| \int_{C_R} f(z)\,dz \right| \leq \frac{\pi R}{(R^2-1)^2}$$

Y dado que $\dfrac{\pi R}{(R^2-2)^2} \xrightarrow[R\to\infty]{} 0 \Longrightarrow \lim\limits_{R\to\infty} \int_{C_R} f(z)\,dz = 0$

Luego:

$$\lim_{R\to\infty} \left(\int_{-R}^{R} \frac{1}{(x^2+1)^2}\,dx + \overbrace{\int_{C_R} f(z)\,dz}^{\to 0} \right) = 2\pi i \cdot \operatorname*{Res}_{z=i} f(z)$$

Dado que $f(z) = \dfrac{\frac{1}{(z+i)^2}}{(z-i)^2}$, tomando $\varphi(z) = \dfrac{1}{(z+i)^2}$, φ es analítica

Y $\varphi(z) \neq 0$

$\Longrightarrow z = i$ es un polo de orden 2

$$\Longrightarrow \operatorname*{Res}_{z=i} f(z) = \varphi'(i) = -\frac{2}{(2i)^3} = -\frac{i}{4}$$

$$\varphi'(z) = \frac{-2}{(z+i)^3}$$

Por lo tanto

$$\lim_{R\to\infty}\left(\int_{-R}^{R}\frac{1}{(x^2+1)^2}dx+\int_{C_R}f(z)dz\right)=2\pi i\cdot\left(-\frac{i}{4}\right)=\frac{\pi}{2}$$

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{1}{(x^2+1)^2}dx=\frac{\pi}{2}$$

$$\Longrightarrow VP\int_{-\infty}^{\infty}\frac{1}{(x^2+1)^2}dx=\frac{\pi}{2}\Longrightarrow 2\int_{0}^{\infty}\frac{1}{(x^2+1)^2}dx=\frac{\pi}{2}$$

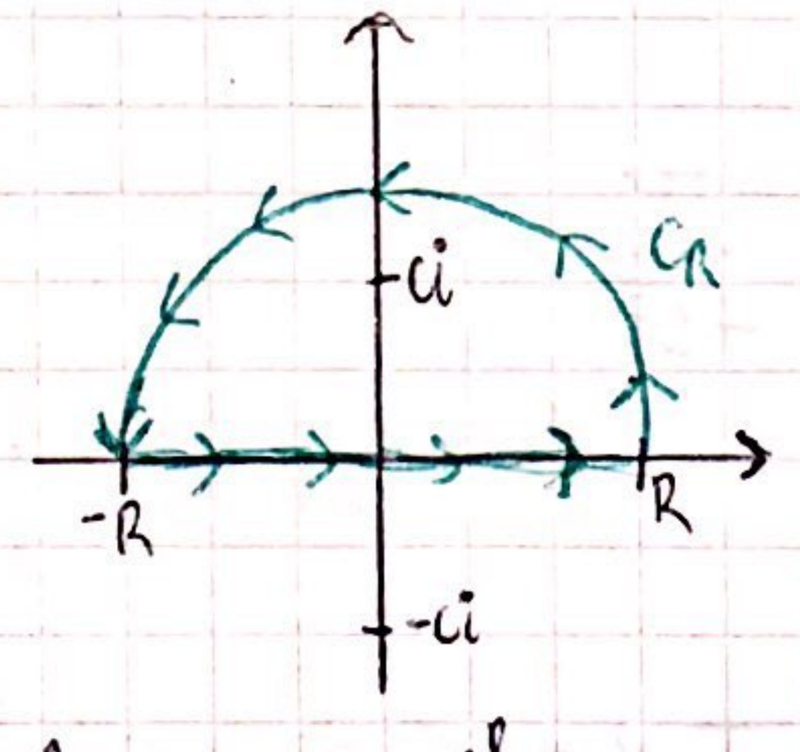
$$\Longrightarrow\int_{0}^{\infty}\frac{1}{(x^2+1)^2}=\frac{\pi}{4}$$

11

3

Sea $f(z)=\frac{1}{(z^2+c^2)^2}$. f tiene singularidades en $z=\pm ci$

Sea $R>c$ y definimos $C=[-R,R]+C_R$



Y sabemos que $\int_C f(z)dz=\int_{-R}^{R}f(x)dx+\int_{C_R}f(z)dz=2\pi i.\operatorname{Res}_{z=ci}f(z)$

Dado que $f(z)=\frac{\frac{1}{(z+ci)^2}}{(z-ci)^2}$, tomando $\varphi(z)=\frac{1}{(z+ci)^2}$, φ es analítica y no nula en $z=ci$

$\Longrightarrow z=ci$ es un polo de orden 2

$$\varphi'(z) = (-2)(z+ci)^{-3}$$

$$\Longrightarrow \operatorname*{Res}_{z=ci} f(z) = \varphi'(ci) = \frac{-2}{(ci+ci)^3} = \frac{-2}{-8c^3 i} = -\frac{i}{4c^3}$$

Por otro lado sea $z \in C_R$:

$$|z^2+c^2|^2 \geq \left| |z|^2 - |c^2| \right|^2 = (R^2-c^2)^2$$

$$\therefore \; |f(z)| \leq \frac{1}{(R^2-c^2)^2} \quad \forall z \in C_R$$

$$\Longrightarrow \left| \int_{C_R} f(z)\,dz \right| \leq \frac{\pi R}{(R^2-c^2)^2}$$

$$\Longrightarrow \lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$$

Luego:

$$\lim_{R\to\infty} \left(\int_{-R}^{R} \frac{1}{(x^2+c^2)^2}\,dx + \int_{C_R} f(z)\,dz \right) = 2\pi i\left(-\frac{i}{4c^3}\right) = \frac{\pi}{2c^3}$$

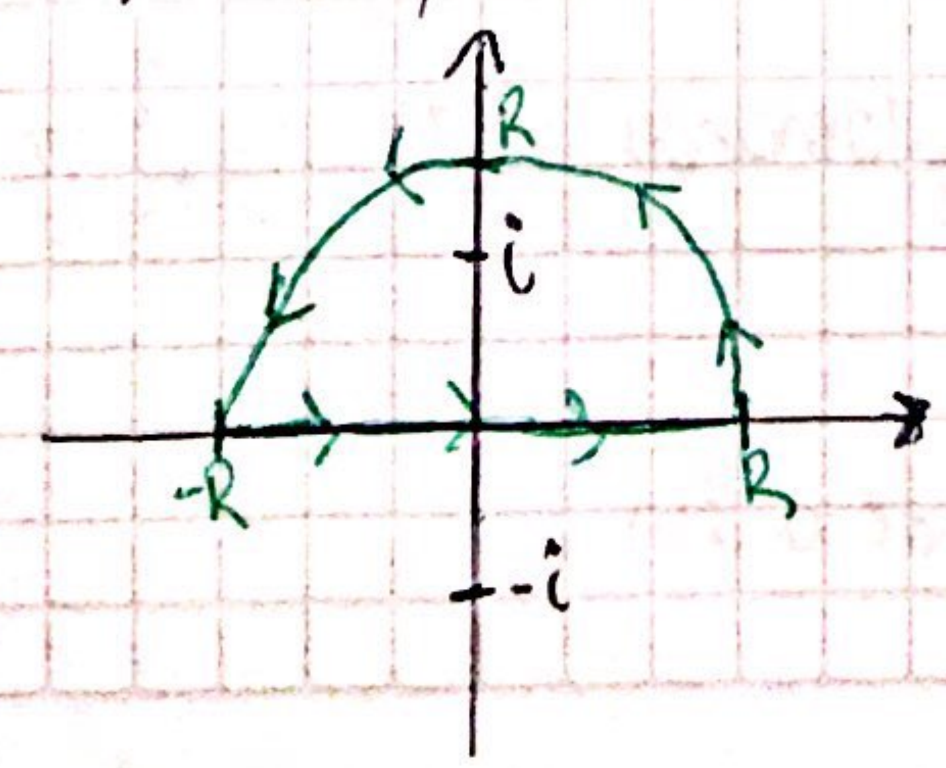
$$\Longrightarrow \int_{-\infty}^{\infty} \frac{1}{(x^2+c^2)^2}\,dx = \frac{\pi}{2c^3}$$

$$\Longrightarrow \int_{0}^{\infty} \frac{1}{(x^2+c^2)^2}\,dx = \frac{\pi}{4c^3}$$

6

Sea $f(z) = \frac{1}{z^2+1}$, f tiene singularidades en $z = \pm i$

Tomamos $R>1$ y definimos $C = [-R,R] + C_R$



Y Sabemos que $\int_{-R}^{R}\frac{e^{ix}}{x^2+1}dx + \int_{C_R}\frac{e^{iz}}{z^2+1}dz = 2\pi i \operatorname{Res}_{z=i}(f(z)e^{iz})$

Dado que $f(z)e^{iz} = \frac{\frac{e^{iz}}{z+i}}{z-i}$, tomando $\varphi(z) = \frac{e^{iz}}{z+i}$, φ es analítica y no nula en $z=\pm i$.

$\Longrightarrow z=i$ es un polo simple

$\Longrightarrow \operatorname{Res}_{z=i} f(z)e^{iz} = \varphi(i) = \frac{e^{-1}}{2i} = -\frac{i}{2e}$

Por lo tanto

$$\underbrace{\int_{-R}^{R}\frac{e^{ix}}{x^2+1}dx + \int_{C_R}\frac{e^{iz}}{z^2+1}dz}_{=\rho} = 2\pi i\left(-\frac{i}{2e}\right) = \frac{\pi}{e}$$

$$\Longrightarrow \int_{-R}^{R}\frac{\cos(x)}{x^2+1}dx + \operatorname{Re}\left(\int_{C_R}\frac{e^{iz}}{z^2+1}dz\right) = \frac{\pi}{e}$$

Tomo parte real de ρ

Notar que:

$|e^{iz}| = e^{-y} \leq 1 \quad \forall y \geq 0$

Por lo tanto si $z \in C_R$:

$$|z^2+1| \geq \big||z|^2 - |1|\big| = R^2-1$$

$$\Longrightarrow \left|\frac{e^{iz}}{z^2+1}\right| \leq \frac{1}{R^2-1}$$

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$$\Longrightarrow \left|\operatorname{Re}\left(\int_{C_R}\frac{e^{iz}}{z^2+1}\,dz\right)\right| \leq \left|\int_{C_R}\frac{e^{iz}}{z^2+1}\right| \leq \frac{\overbrace{\pi . R}^{\text{long}(C)}}{R^2-1}$$

$$\Longrightarrow \operatorname{Re}\left(\int_{C_R}\frac{e^{iz}}{z^2+1}\,dz\right) \xrightarrow[R\to\infty]{} 0$$

Por lo tanto

$$\lim_{R\to\infty}\left(\int_{-R}^{R}\frac{\cos(x)}{x^2+1}\,dx + \operatorname{Re}\left(\int_{C_R}\frac{e^{iz}}{z^2+1}\,dz\right)\right) = \frac{\pi}{e}$$

$$\Longrightarrow \int_{-\infty}^{\infty}\frac{\cos(x)}{x^2+1}\,dx = \frac{\pi}{e}$$

$$\xRightarrow[\text{es par}]{} \int_{0}^{\infty}\frac{\cos(x)}{x^2+1}\,dx = \frac{\pi}{2e}$$