Ejercicio 8:

a) $-((1+x)\,u'(x))' = 0$

$-((1+x)'\,u'(x) + (1+x)\,u''(x)) = 0$

$-u'(x) + (1+x)\,u''(x) = 0 \longrightarrow u(x) = c_1 \ln(1+x) + c_2$

Cómo $u(x)$ debe satisfacer $u(0)=0$, $u'(1)=1$, definimos c_1 y c_2

$u(0) = c_2 = 0$, $\therefore$ $u(x) = c_1 \ln(1+x)$

$u'(x) = \dfrac{c_1}{1+x}$, $u'(1) = \dfrac{c_1}{1+1} = \dfrac{c_1}{2} = 1 \Rightarrow c_2 = 2$

$\therefore$ $u(x) = 2\ln(1+x)$

b) Tenemos que resolver $\displaystyle\int_0^1 -((1+x)\,u'(x))'\, v \, dx = \int_0^1 f v \, dx$

Aplicando integración por partes:

$$-(1+x)\,u'(x)\,v(x)\Big|_0^1 + \int_0^1 (1+x)\,u'(x)\,v'(x)\,dx = \int_0^1 f\,v\,dx$$

$$-2\,u'(1)\,v(1) + 1 \cdot u'(0)\underbrace{v(0)}_{=0} + \int_0^1 (1+x)\,u'(x)\,v'(x)\,dx = \int_0^1 f\,v\,dx$$

c) Cómo dividimos en tres subintervalos, el vector de carga está dado por

$$b = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Veamos la matriz de rigidez

$$\int_0^1 (1+x)\, u'(x)\, v'(x)\, dx = \int_0^1 (1+x)\, \phi_i'(x)\, \phi_j'(x)\, dx$$

Supongamos $i=j \Rightarrow \int_0^1 (1+x)\left(\phi_j'(x)\right)^2 dx = \int_{x_{i-1}}^{x_i} (1+x)\frac{1}{h^2}\,dx + \int_{x_i}^{x_{i+1}} (1+x)\left(-\frac{1}{h}\right)^2 dx$

$i = 1/3 = j$

$h = 1/3$

$$\int_0^{1/3} \frac{(1+x)}{h^2}\,dx + \int_{1/3}^{2/3} \frac{(1+x)}{h^2}\,dx = \frac{1}{h^2}\left(x + \frac{x^2}{2}\right)\Bigg|_0^{2/3} = \frac{1}{h^2}\left(\frac{2}{3} + \frac{4}{9\cdot 2}\right) = 8$$

$i = 2/3 = j$

$$\int_{1/3}^{2/3} \frac{(1+x)}{h^2}\,dx + \int_{2/3}^{1} \frac{(1+x)}{h^2}\,dx = \frac{1}{h^2}\left(x + \frac{x^2}{2}\right)\Bigg|_{1/3}^{1} = \frac{1}{h^2}\left(1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{9\cdot 2}\right) = 10$$

$i = 1 = j$

$$\int_{2/3}^{1} \frac{(1+x)}{h^2}\,dx = \frac{1}{h^2}\left(x + \frac{x^2}{2}\right)\Bigg|_{2/3}^{1} = \frac{1}{h^2}\left(1 + \frac{1}{2} - \frac{2}{3} - \frac{4}{9\cdot 2}\right) = \frac{11}{2}$$

Supongamos ahora $i \neq j \Rightarrow \int_0^1 (1+x)\, \phi_j'(x)\, \phi_i'(x)\, dx$

$i=0,\ j=1/3$ ó $i=1/3,\ j=0$

$$\int_{1/3}^{2/3} (1+x)\left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) dx = -\frac{1}{h^2}\left(x + \frac{x^2}{2}\right)\Bigg|_{1/3}^{2/3} = -\frac{1}{h^2}\left(\frac{2}{3} + \frac{4}{9\cdot 2} - \frac{1}{3} - \frac{1}{9\cdot 2}\right) = -\frac{9}{2}$$

$i = 2/3,\ j = 1 \quad o \quad i = 1,\ j = 2/3$

$$\int_{2/3}^{1} (1+x)\left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) dx = -\frac{1}{h^2}\left(x + \frac{x^2}{2}\right)\Bigg|_{2/3}^{1} = -\frac{1}{h^2}\left(1 + \frac{1}{2} - \frac{2}{3} - \frac{4}{9\;2}\right) = -\frac{11}{2}$$

Otro caso, vale 0

$\therefore$ La matriz de rigidez es:

$$A = \frac{1}{2}\begin{bmatrix} 16 & -9 & 0 \\ -9 & 20 & -11 \\ 0 & -11 & 11 \end{bmatrix}$$