⑨ $$\begin{cases} -U''(x) = f(x) & 0<x<1 \\ U'(0) = 1 \\ U'(1) = 1 \end{cases}$$

Veamos la formulación débil:

$$-U''(x) = f(x) \implies -U''(x)V(x) = f(x)V(x)$$

$$\implies \int_0^1 -U''(x)V(x) = \int_0^1 f(x)V(x)$$

Notar que: $$-\int_0^1 U''(x)V(x)\,dx = -U'(x)V(x)\Big|_0^1 + \int_0^1 U'(x)V'(x)\,dx$$

$$= -(U'(1)V(1) - U'(0)V(0)) + \int_0^1 U'(x)V'(x)\,dx$$

$$= V(0) - V(1) + \int_0^1 U'(x)V'(x)\,dx$$

$$\implies V(0) - V(1) + \int_0^1 U'(x)V'(x)\,dx = \int_0^1 f(x)V(x)\,dx$$

Generando una grilla $x_i = ih$ con $i=0,\dots,n$ y $h=\frac{1}{n}$.

Defino como funciones base:

$$\phi_i(x) = \begin{cases} \dfrac{x-x_{i-1}}{h} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1}-x}{h} & x \in [x_i, x_{i+1}] \\ 0 & \text{c.c.} \end{cases}$$

Luego:

$$\phi_i'(x) = \begin{cases} \frac{1}{h} & x \in [x_{i-1}, x_i] \\ \frac{-1}{h} & x \in [x_i, x_{i+1}] \\ 0 & c.c. \end{cases}$$

Definimos $U_h = \sum_{j=1}^{n-1} c_j \phi_j(x)$. Por lo tanto:

- $U_h' = c_1 \phi_1'(x) + c_2 \phi_2'(x) + \cdots + c_{n-1} \phi_{n-1}'(x)$

- $U_h'(0) = c_1 \underbrace{\phi_1'(0)}_{\frac{1}{h}} + c_2 \underbrace{\phi_2'(0)}_{=0} + \cdots + c_{n-1} \underbrace{\phi_{n-1}'(0)}_{=0}$

$$= c_1 \cdot \frac{1}{h} = 1 \iff \boxed{c_1 = h}$$

- $U_h'(1) = c_1 \underbrace{\phi_1'(1)}_{=0} + c_2 \underbrace{\phi_2'(1)}_{=0} + \cdots + c_{n-1} \underbrace{\phi_{n-1}'(1)}_{=\frac{1}{h}}$

$$= c_{n-1} \frac{1}{h} = 1 \iff \boxed{c_{n-1} = h}$$

Luego faltan determinar $c_2, \cdots, c_{n-2}$.

Tomemos $V(x) \in \text{SPAN}\{\phi_1, \cdots, \phi_n\}$

Dado que $\phi_1(0) = \phi_2(0) = \cdots = \phi_{n-1}(0) = 0 \implies V(0) = 0$

$\phi_1(1) = \phi_2(1) = \cdots = \phi_{n-1}(1) = 0 \implies V(1) = 0$

Por lo tanto tenemos:

$$\underbrace{v(0) - v(1)}_{=0} + \int_0^1 u'(x)\, v'(x)\, dx = \int_0^1 f(x)\, v(x)\, dx$$

$$\Rightarrow \int_0^1 u_h'(x)\, v'(x)\, dx = \int_0^1 f(x)\, v(x)\, dx$$

Luego:

$$\sum_{j=1}^{n-1} c_j \int_0^1 \phi_j' \phi_i' \, dx = \int_0^1 f(x)\, \phi_i(x)\, dx \qquad \forall i = 1, \dots, n-1$$

Definiendo $a(\phi_i, \phi_j) = \int_0^1 \phi_i'(x)\, \phi_j'(x)\, dx$

$\langle f, \phi_i \rangle = \int_0^1 f(x)\, \phi_i(x)\, dx$

Nos queda:

$$\underbrace{\begin{pmatrix} a(\phi_1, \phi_1) & \cdots & a(\phi_1, \phi_{n-1}) \\ \vdots & \ddots & \vdots \\ a(\phi_{n-1}, \phi_{n-1}) & \cdots & a(\phi_{n-1}, \phi_{n-1}) \end{pmatrix}}_{n-1 \times n-1} \underbrace{\begin{pmatrix} h \\ c_2 \\ \vdots \\ c_{n-2} \\ h \end{pmatrix}}_{n-1} = \underbrace{\begin{pmatrix} \langle f, \phi_1 \rangle \\ \\ \vdots \\ \\ \langle f, \phi_{n-1} \rangle \end{pmatrix}}_{n-1}$$

Despejando los valores determinados C_1, C_{n-1}:

$$\underbrace{\begin{pmatrix} a(\phi_1,\phi_2) & \cdots & a(\phi_1,\phi_{n-2}) \\ \vdots & & \\ a(\phi_{n-1},\phi_2) & & a(\phi_{n-1},\phi_{n-2}) \end{pmatrix}}_{A \;\to\; n-1 \times n-3} \underbrace{\begin{pmatrix} c_2 \\ \vdots \\ c_{n-2} \end{pmatrix}}_{x \;\to\; n-3} = \underbrace{\begin{pmatrix} \langle f,\phi_1\rangle - h\left(a(\phi_1,\phi_1) + a(\phi_1,\phi_{n-1})\right) \\ \vdots \\ \langle f,\phi_{n-1}\rangle - h\left(a(\phi_{n-1},\phi_1) + a(\phi_{n-1},\phi_{n-1})\right) \end{pmatrix}}_{b \;\to\; n-1}$$

En donde:

$$A = \frac{1}{h}\begin{pmatrix} -1 & 0 & 0 & & & 0 & 0 \\ 2 & -1 & 0 & & & \vdots & \vdots \\ -1 & 2 & -1 & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & & & \\ & & & & \ddots & & \\ 0 & 0 & \cdots & & -1 & 2 & -1 \end{pmatrix} \in \mathbb{R}^{n-1 \times n-3}$$