Combinatorics

Sums

$\sum_{k=0}^{n} k = n(n+1)/2$ $\quad$ $\sum_{k=a}^{b} k = (a+b)(b-a+1)/2$
$\sum_{k=0}^{n} k^2 = n(n+1)(2n+1)/6$ $\quad$ $\sum_{k=0}^{n} k^3 = n^2(n+1)^2/4$
$\sum_{k=0}^{n} k^4 = (6n^5+15n^4+10n^3-n)/30$ $\quad$ $\sum_{k=0}^{n} k^5 = (2n^6+6n^5+5n^4-n^2)/12$
$\sum_{k=0}^{n} x^k = (x^{n+1}-1)/(x-1)$ $\quad$ $\sum_{k=0}^{n} kx^k = (x-(n+1)x^{n+1}+nx^{n+2})/(x-1)^2$
$1+x+x^2+\cdots = 1/(1-x)$

Binomial coefficients

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1	1											
2	1	2	1										
3	1	3	3	1									
4	1	4	6	4	1								
5	1	5	10	10	5	1							
6	1	6	15	20	15	6	1						
7	1	7	21	35	35	21	7	1					
8	1	8	28	56	70	56	28	8	1				
9	1	9	36	84	126	126	84	36	9	1			
10	1	10	45	120	210	252	210	120	45	10	1		
11	1	11	55	165	330	462	462	330	165	55	11	1	
12	1	12	66	220	495	792	924	792	495	220	66	12	1
	0	1	2	3	4	5	6	7	8	9	10	11	12

$\binom{n}{k} = \frac{n!}{(n-k)!k!}$

$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$\binom{n+1}{k} = \frac{n+1}{n-k+1}\binom{n}{k}$

$\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}$

$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$

$\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$

$12! \approx 2^{28.8}$
$20! \approx 2^{61.1}$

Number of ways to pick a multiset of size k from n elements: $\binom{n+k-1}{k}$
Number of n-tuples of non-negative integers with sum s: $\binom{s+n-1}{n-1}$, at most s: $\binom{s+n}{n}$
Number of n-tuples of positive integers with sum s: $\binom{s-1}{n-1}$
Number of lattice paths from $(0,0)$ to (a,b), restricted to east and north steps: $\binom{a+b}{a}$

Multinomial theorem. $(a_1+\cdots+a_k)^n = \sum \binom{n}{n_1,\ldots,n_k} a_1^{n_1}\ldots a_k^{n_k}$, where $n_i \geqslant 0$ and $\sum n_i = n$.
$\binom{n}{n_1,\ldots,n_k} = M(n_1,\ldots,n_k) = \frac{n!}{n_1!\ldots n_k!}$. $M(a,\ldots,b,c,\ldots) = M(a+\cdots+b,c,\ldots)M(a,\ldots,b)$

Catalan numbers. $C_n = \frac{1}{n+1}\binom{2n}{n}$. $\quad C_0 = 1$, $C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$. $\quad C_{n+1} = C_n\frac{4n+2}{n+2}$.
$C_0, C_1, \ldots = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, \ldots$
C_n is the number of: properly nested sequences of n pairs of parentheses; rooted ordered binary trees with $n+1$ leaves; triangulations of a convex $(n+2)$-gon.

Derangements. Number of permutations of $n = 0, 1, 2, \ldots$ elements without fixed points is $1, 0, 1, 2, 9, 44, 265, 1854, 14833, \ldots$ Recurrence: $D_n = (n-1)(D_{n-1}+D_{n-2}) = nD_{n-1} + (-1)^n$. Corollary: number of permutations with exactly k fixed points is $\binom{n}{k}D_{n-k}$.

Stirling numbers of 1^{st} kind. $s_{n,k}$ is $(-1)^{n-k}$ times the number of permutations of n elements with exactly k permutation cycles. $|s_{n,k}| = |s_{n-1,k-1}| + (n-1)|s_{n-1,k}|$. $\quad \sum_{k=0}^{n} s_{n,k}\, x^k = x^{\underline{n}}$

Stirling numbers of 2^{nd} kind. $S_{n,k}$ is the number of ways to partition a set of n elements into exactly k non-empty subsets. $S_{n,k} = S_{n-1,k-1} + kS_{n-1,k}$. $S_{n,1} = S_{n,n} = 1$. $x^n = \sum_{k=0}^{n} S_{n,k}\, x^{\underline{k}}$

Bell numbers. B_n is the number of partitions of n elements. $B_0, \ldots = 1, 1, 2, 5, 15, 52, 203, \ldots$
$B_{n+1} = \sum_{k=0}^{n}\binom{n}{k}B_k = \sum_{k=1}^{n} S_{n,k}$. Bell triangle: $B_r = a_{r,1} = a_{r-1,r-1}$, $a_{r,c} = a_{r-1,c-1} + a_{r,c-1}$.