Merge Sort

- Merge sort is a **divide-and-conquer** algorithm that sorts an array by **repeatedly dividing it in half** and **merging the sorted sub-arrays**.

- The time complexity of merge sort is $O(n\log n)$, where n is the number of elements in the array.
  - Making it more efficient than other sorting algorithms with a **quadratic time** complexity (i.e., bubble sort and insertion sort.)
  - The polynomial equation whose highest degree is two is called a quadratic equation.
  - Form of Quadratic equation
    - $ax^2 + bx + c = 0$; where a, b, c are real numbers and $a \neq 0$.

Merge Sort is a stable sort?

Merge sort is a **stable sort**, meaning that it preserves the relative order of elements with equal keys.

- if two elements have the same key value, their relative order in the sorted array will be the same as their relative order in the original unsorted array

Example

- Sort an array of people with their **ages as the key** (i.e., by age)
  - a stable sort will keep people of the same age in the same order as they were in the original array.
  - In contrast, an unstable sort may change the relative order of elements with the same key value.

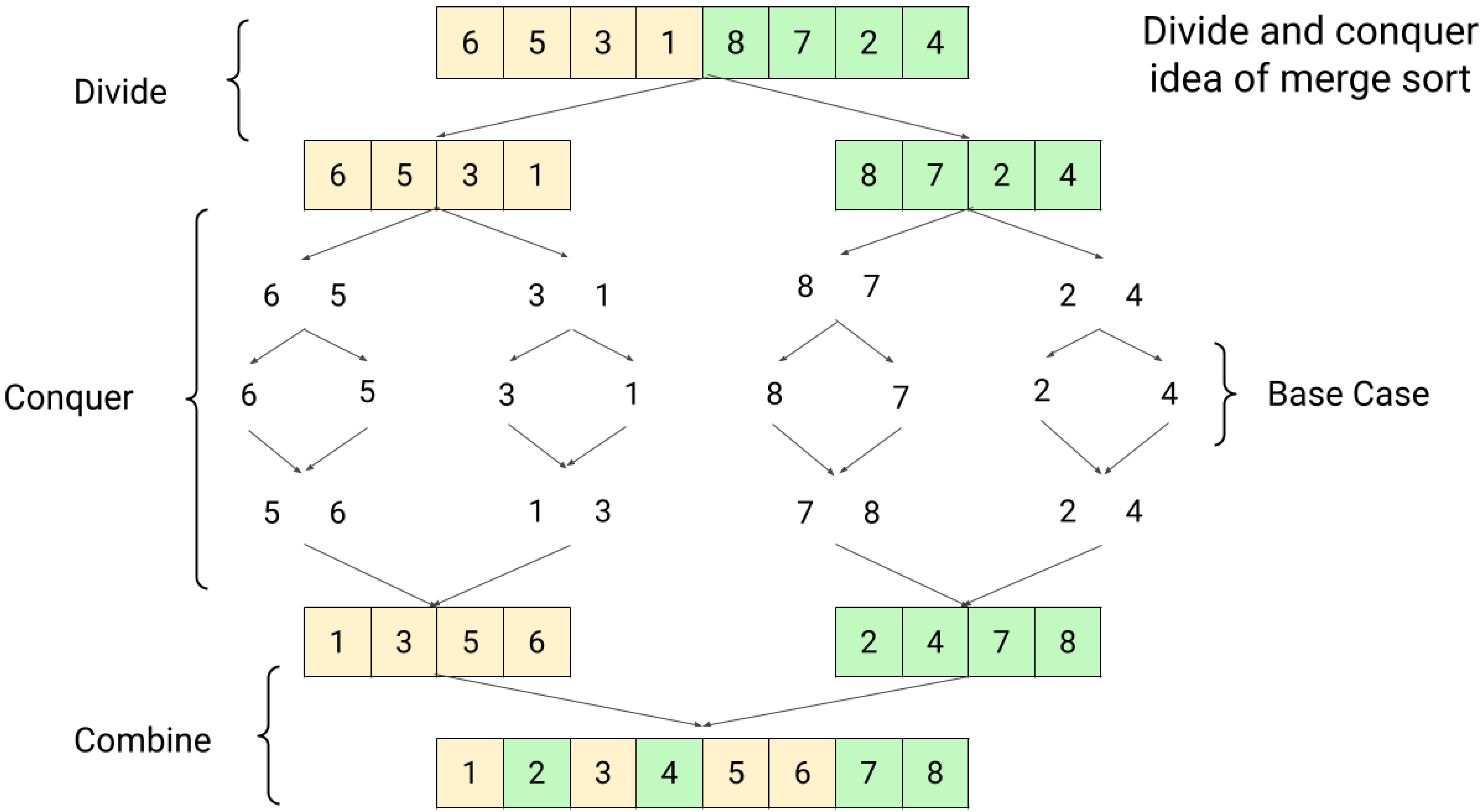
Merge Sort is a divide-and-conquer algorithm?

Merge sort is a sorting algorithm that works by

- dividing an array into smaller subarrays,
- sorting each subarray,
- merging the sorted subarrays back together to form the final sorted array.

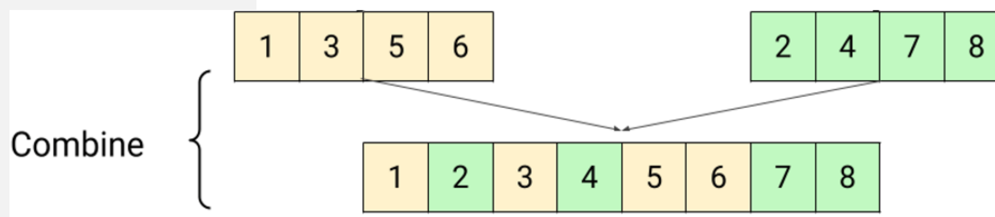
1. ***Divide:*** If S has zero or one element, return S immediately; it is already sorted. Otherwise (S has at least two elements), remove all the elements from S and put them into two sequences, S_1 and S_2, each containing about half of the elements of S; that is, S_1 contains the first $\lfloor n/2 \rfloor$ elements of S, and S_2 contains the remaining $\lceil n/2 \rceil$ elements.
2. ***Conquer:*** Recursively sort sequences S_1 and S_2.
3. ***Combine:*** Put back the elements into S by merging the sorted sequences S_1 and S_2 into a sorted sequence.

Merge Sort is a divide-and-conquer algorithm?



Merge or Combine Operation

```python
def merge(left, right):
    i = j = 0
    merged = []
    while i < len(left) and j < len(right):
        if left[i] <= right[j]:
            merged.append(left[i])
            i += 1
        else:
            merged.append(right[j])
            j += 1
    while i < len(left):
        merged.append(left[i])
        i += 1
    while j < len(right):
        merged.append(right[j])
        j += 1
    return merged
arr1 = [1, 3, 5, 6]
arr2 = [2, 4, 7, 8]
sorted_arr = merge(arr1, arr2)
print(sorted_arr)
```




```
[1, 2, 3, 4, 5, 6, 7, 8]
```

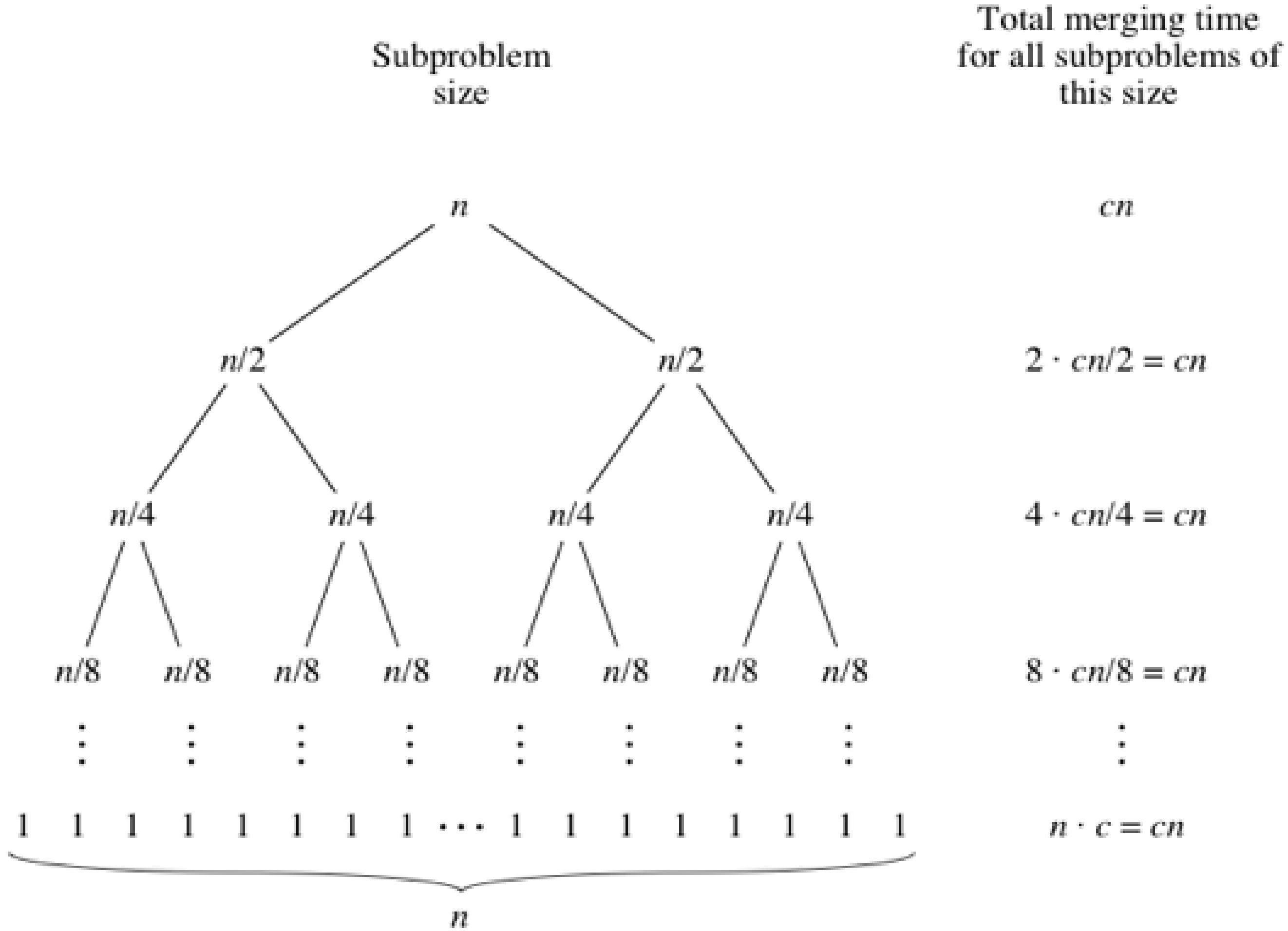
```python
def merge_sort(arr):
    if len(arr) <= 1:
        return arr
    else:
        middle = len(arr) // 2
        left_half = merge_sort(arr[:middle])
        right_half = merge_sort(arr[middle:])
        return merge(left_half, right_half)

def merge(left, right):
    i = j = 0
    merged = []
    while i < len(left) and j < len(right):
        if left[i] <= right[j]:
            merged.append(left[i])
            i += 1
        else:
            merged.append(right[j])
            j += 1
    while i < len(left):
        merged.append(left[i])
        i += 1
    while j < len(right):
        merged.append(right[j])
        j += 1
    return merged
arr = [5, 4, 3, 2, 1]
sorted_arr = merge_sort(arr)
print(sorted_arr)  # [1, 2, 3, 4, 5]
```

Complexity of Merge Sort

We assume that we're sorting a total of *n* elements in the entire array.

1. The divide step takes constant time, regardless of the subarray size. After all, the divide step just computes the midpoint q of the indices p and r. Recall that in big-Θ notation, we indicate constant time by $\Theta(1)$.
2. The conquer step, where we recursively sort two subarrays of approximately $n/2$ elements each, takes some amount of time, but we'll account for that time when we consider the subproblems.
3. The combine step merges a total of n elements, taking $\Theta(n)$ time.

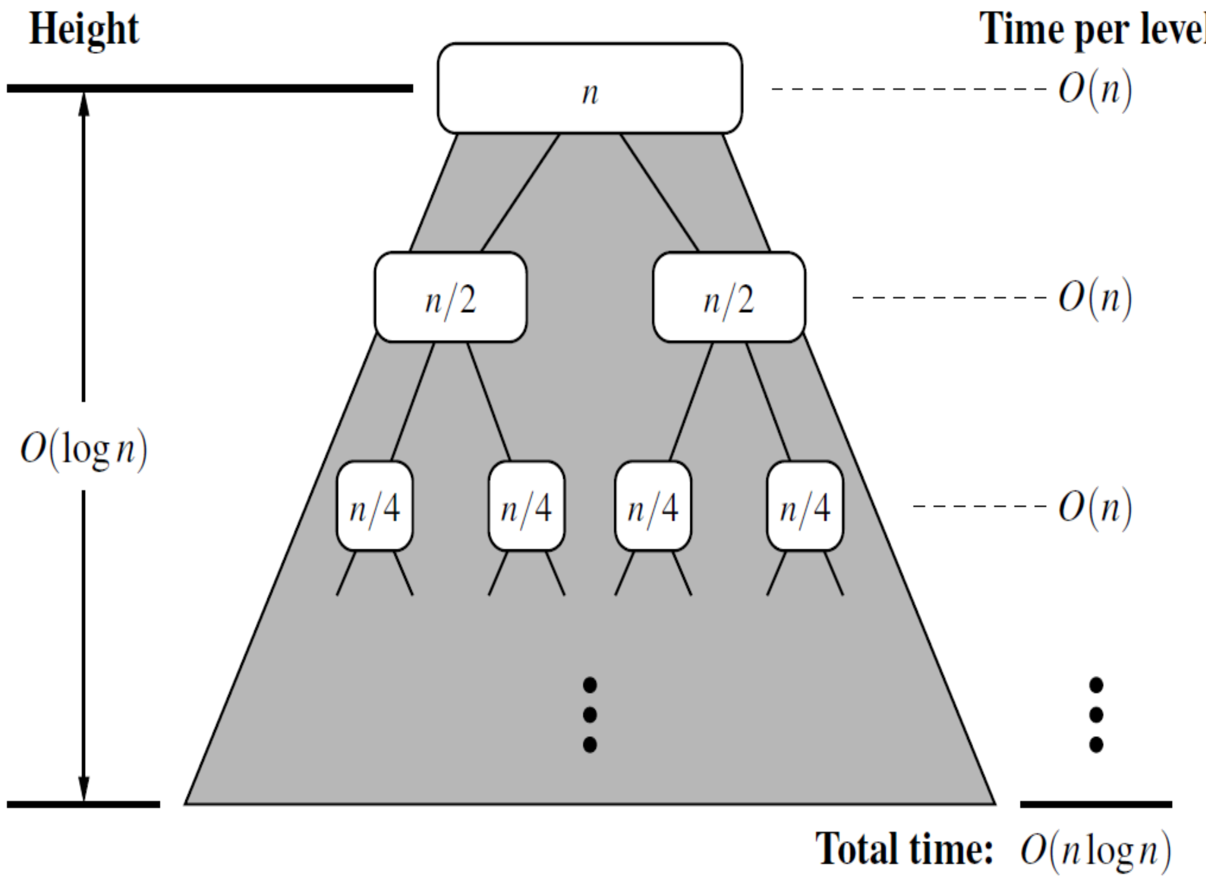
Complexity of Merge Sort



- As the subproblems get smaller, the number of subproblems doubles at each "level" of the recursion, but the merging time halves.
- The doubling and halving cancel each other out, and so the total merging time is *cn* at each level of recursion.

So time complexity of merge sort = Cost sum at each level = O(logn)*O(n) = O(nlogn)

Complexity of Merge Sort



So time complexity of merge sort = Cost sum at each level = O(logn)*O(n) = O(nlogn)

Complexity of Merge Sort

Merge sort space complexity depends on the extra space used by the merging process and the size of recursion call stack used by the recursion.

- Space complexity of merging process = $O(n)$
- Space complexity for recursion call stack = Height of merge sort recursion tree = $O(\log n)$.
- Space complexity of merge sort algorithm = $O(n) + O(\log n) = O(n)$

As we have seen here, space complexity is dominated by the extra space used by the merging process.

Application of Merge Sort

- To sort linked lists in O(nlogn) time:
- Inversion count problem
  - the inversion count problem tells how many pairs need to be swapped in order to get a sorted array. Merge sort works best for solving this problem.

- Merge sort is an external sorting technique.
  - if we have data of 1GB but the available RAM size is 500MB, then we will use merge sort.
  - When the data being sorted is too large to fit in the primary memory (often RAM) of a computing device and must instead reside in the slower external memory (usually a hard drive).
  - Whenever we have an input size larger than the RAM size, we use merge sort. Thus, merge sort is very well suited for larger datasets.

Drawbacks of Merge

- Merge sort is not a space-efficient algorithm. It makes use of an extra O(n) space.
- In the case of smaller input size, merge sort works slower in comparison to other sorting techniques.
- If the data is already sorted, merge sort will be a very expensive algorithm in terms of time and space.
  - This is because it will still traverse the whole array and perform all the operations.

Quick sort Algorithm - Tony Hoare in 1960

- Tony Hoare developed the quicksort algorithm in the early 1960s while working at the National Physical Laboratory in the United Kingdom.
  - Merge sort was one of the sorting algorithms that existed at the time
- In the case of quicksort, the algorithm uses an in-place partitioning technique,
  - which means it doesn't require any additional memory allocation during the sorting process.
- The space complexity of both merge sort and quicksort is O(n) and the stack space used during the recursion is O(log n) - which is considered as constant space.
- Quick sort is not a stable algorithm

Quick sort Algorithm - Tony Hoare in 1960

- Quick Sort follows a recursive algorithm.
- It uses the idea of divide and conquer approach.
- It divides the given array into two sections (i.e. two subarray) using a partitioning element called as pivot.

How is the division performed?

- All the elements to the left side of pivot are smaller than pivot.
- All the elements to the right side of pivot are larger than pivot.

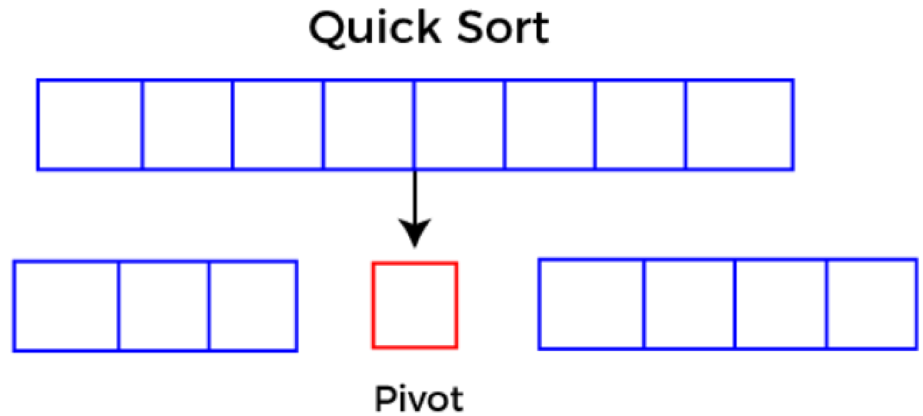
- Then, sub arrays are sorted separately by applying quick sort algorithm **recursively**.

```python
def quicksort(A, low, high):
    # Base case: if the subarray has 0 or 1 element, it is already sorted
    if low < high:
        # Partition the subarray and get the partition point
        p = hoare_partition(A, low, high)
        # Recursively sort the left subarray
        quicksort(A, low, p)
        # Recursively sort the right subarray
        quicksort(A, p + 1, high)
```

```python
# Test
A = [3, 8, 2, 5, 1, 4, 7, 6]
quicksort(A, 0, len(A) - 1)
print(A) # [1, 2, 3, 4, 5, 6, 7, 8]
```

```python
def hoare_partition(A, low, high):
    # Choose the first element as the pivot
    pivot = A[low]
    # Initialize the left pointer just before the first element
    i = low - 1
    # Initialize the right pointer just after the last element
    j = high + 1
    # Continue until the pointers cross
    while True:
        # Increment the left pointer until an element greater than the pivot is found
        i += 1
        while A[i] < pivot:
            i += 1
        # Decrement the right pointer until an element less than the pivot is found
        j -= 1
        while A[j] > pivot:
            j -= 1
        # If the pointers have crossed, the partition is complete
        if i >= j:
            return j
        # Swap the elements at the left and right pointers
        A[i], A[j] = A[j], A[i]
```

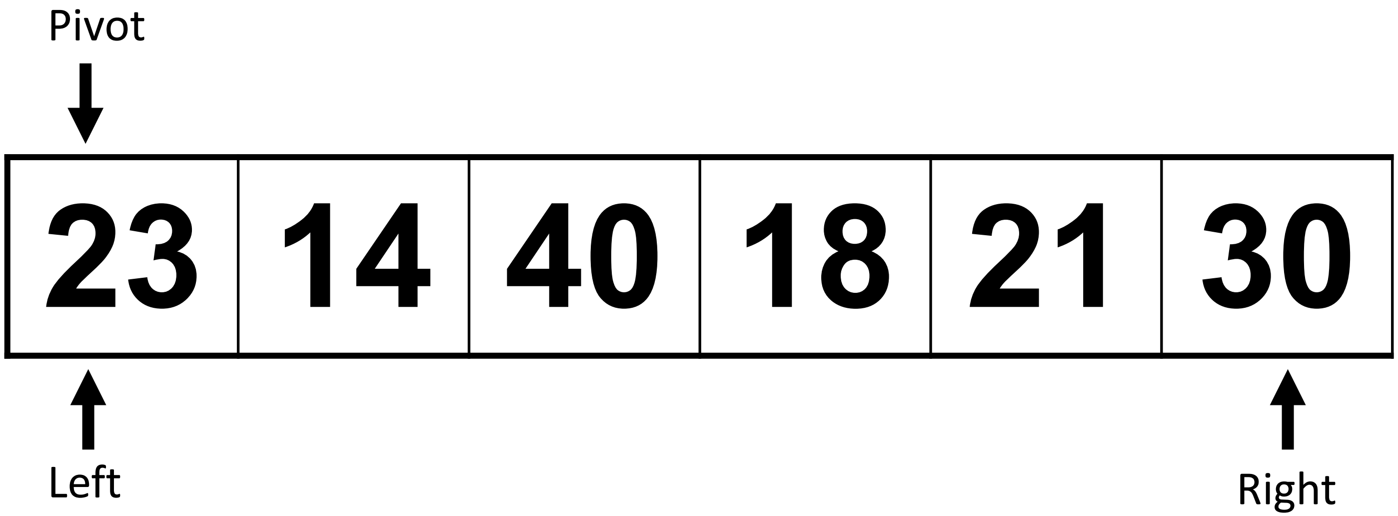
Quick Sort – Worked example



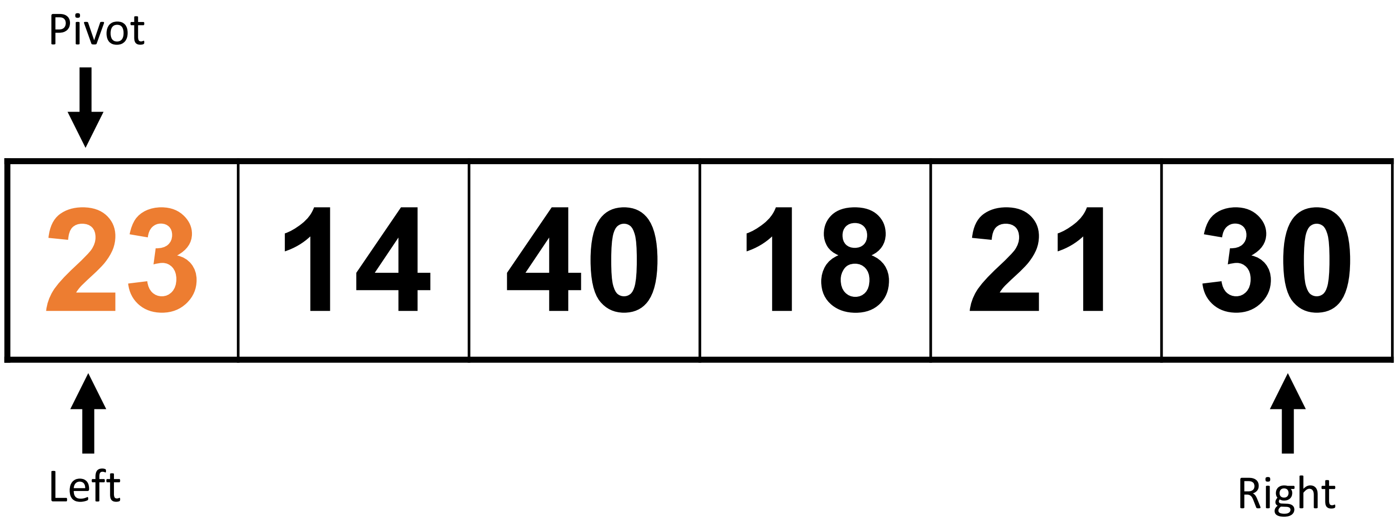
- Pivot can be random, i.e. select the random pivot from the given array.
- Pivot can either be the rightmost element of the leftmost element of the given array.
- Select median as the pivot element.

23	14	40	18	21	30

Quick Sort – Worked example



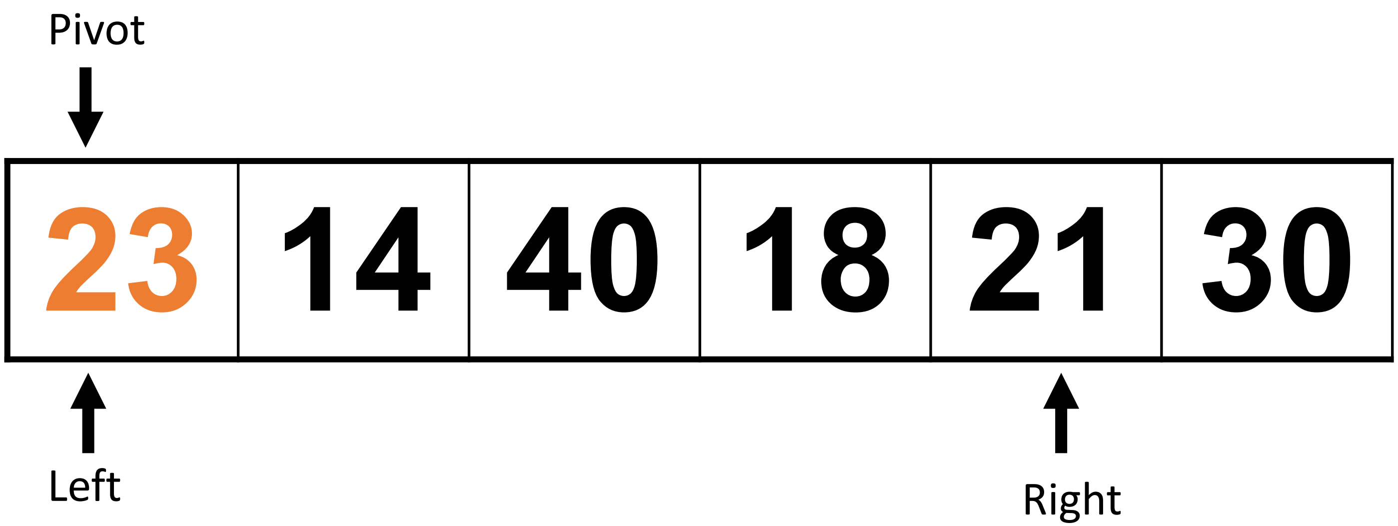
Quick Sort – Worked example



- The pivot is at left, so algorithm starts from right and move towards left.

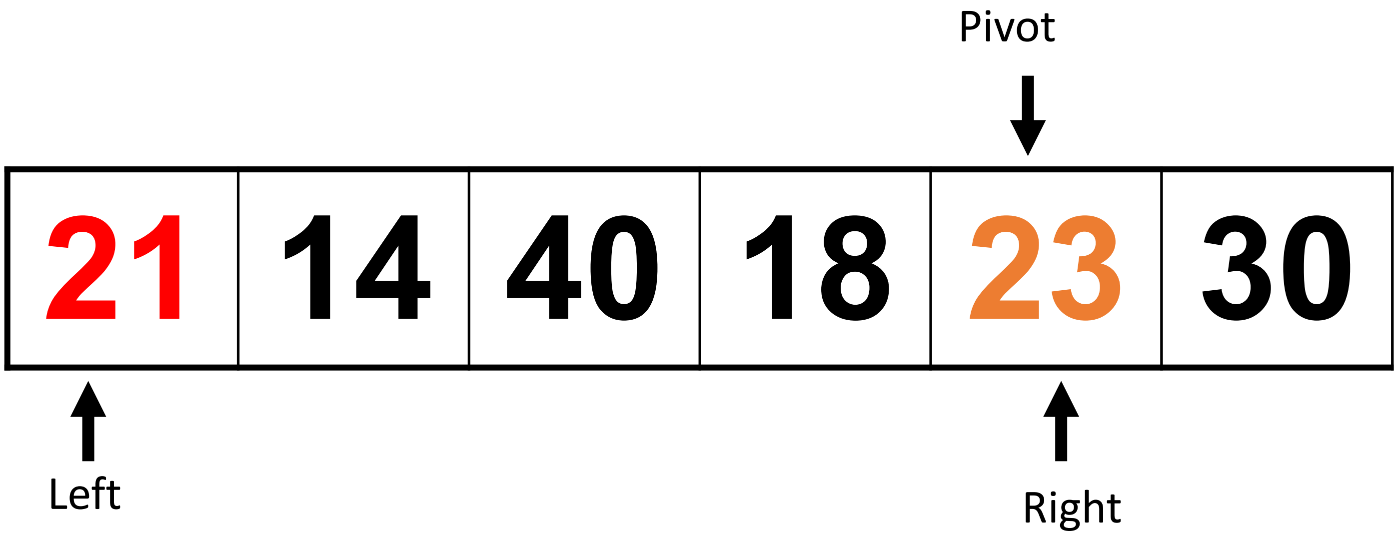
- Now, **A[pivot] < A[right],** so the right pointer moves forward one position towards left

Quick Sort – Worked example



- Now, **A[pivot] > A[right]**, swap A[pivot] with A[right], and then pivot moves to right

Quick Sort – Worked example



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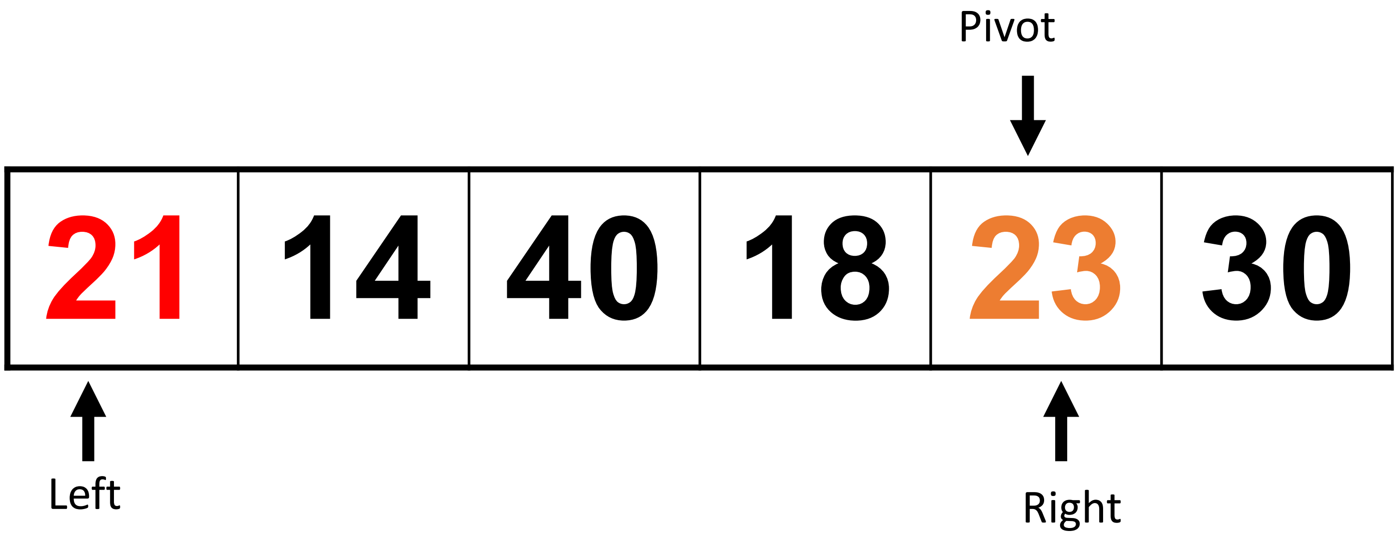

As pivot is at right, so algorithm starts from left and moves to right.

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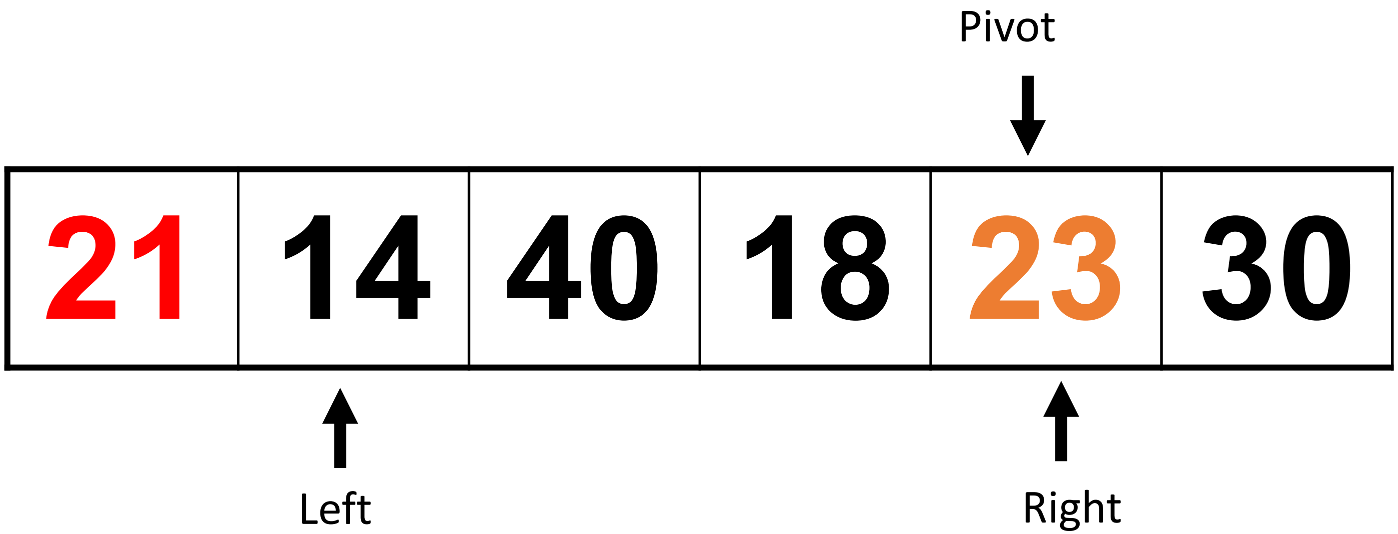

As A[pivot] > A[left], so the left pointer moves one position to right

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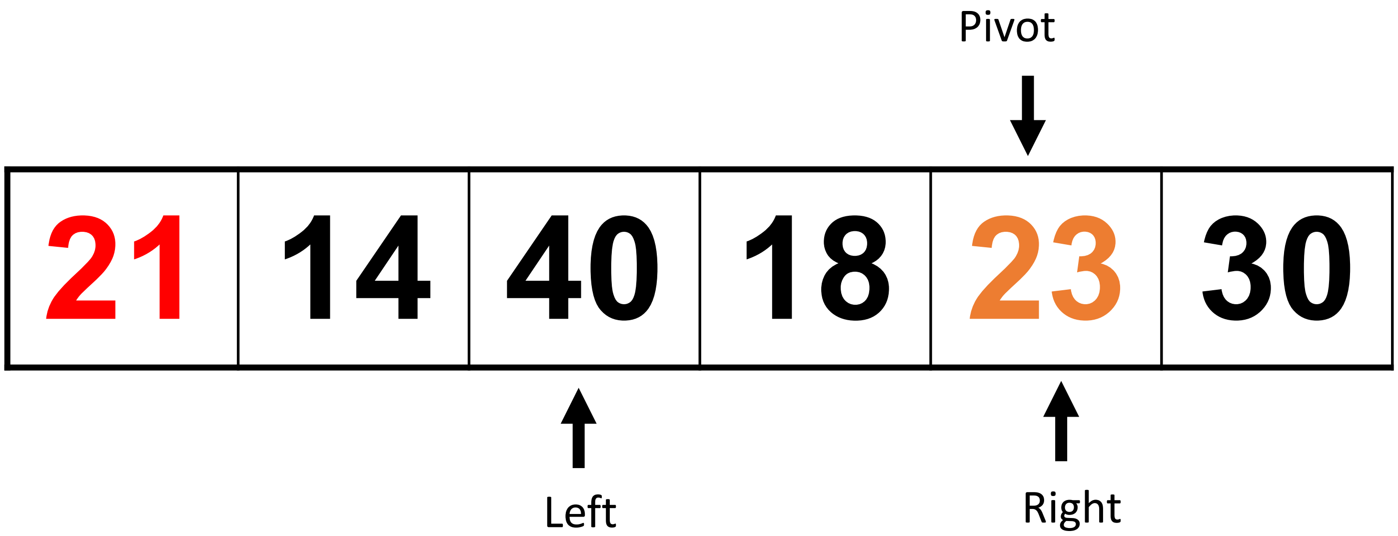

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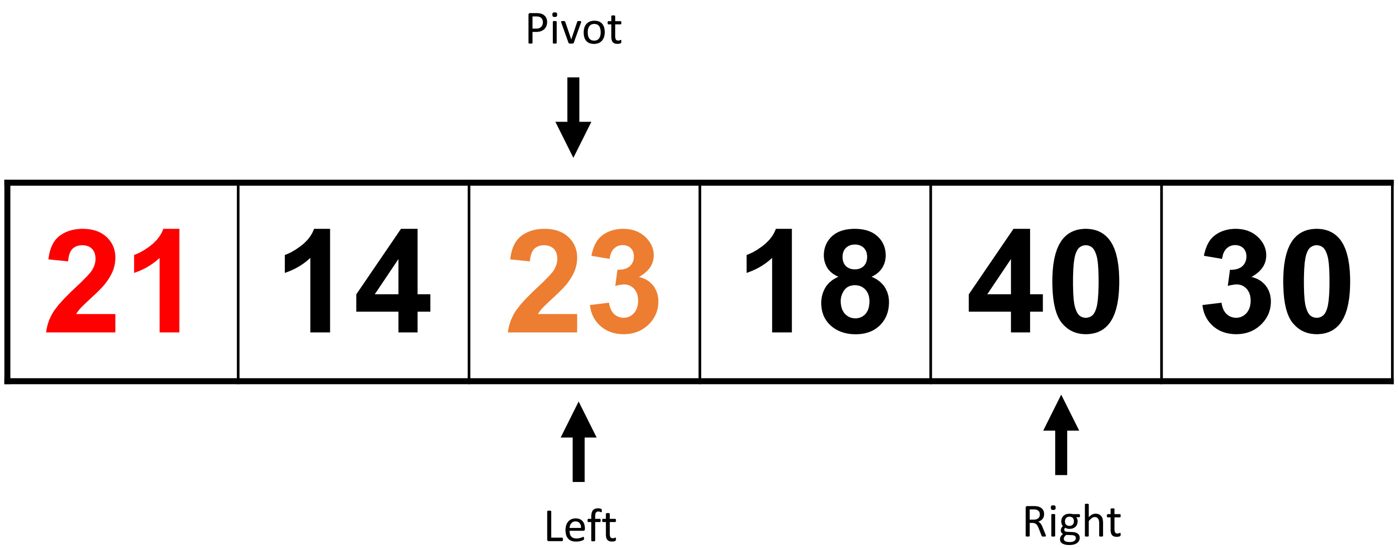
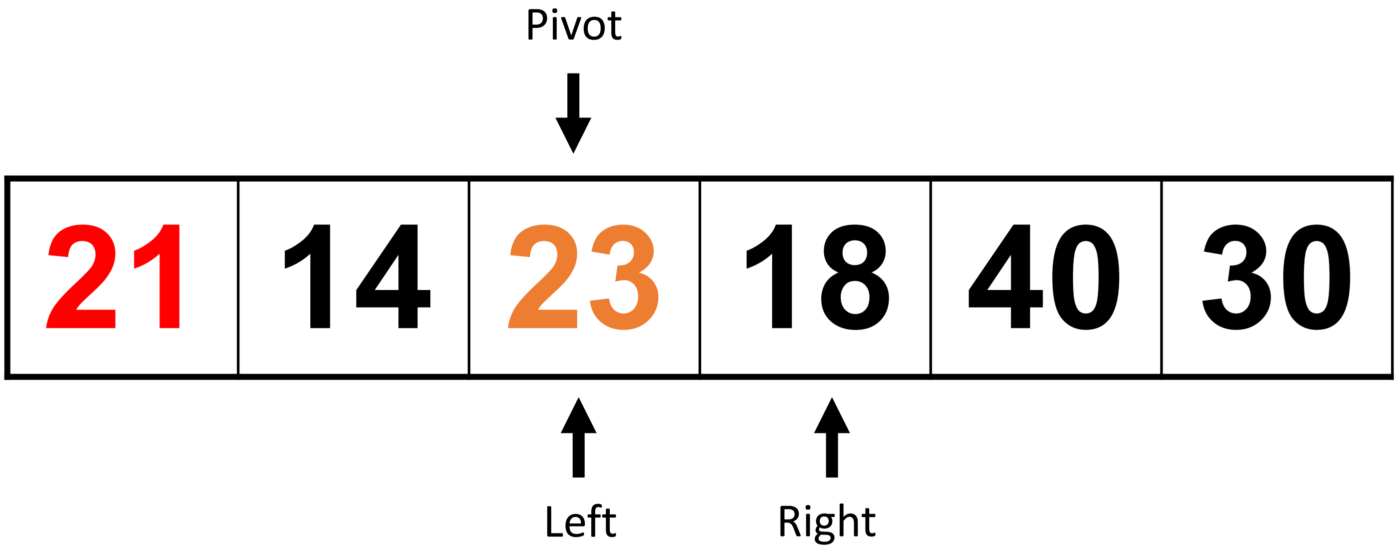


As A[pivot] < A[left], so swap and pivot is at left

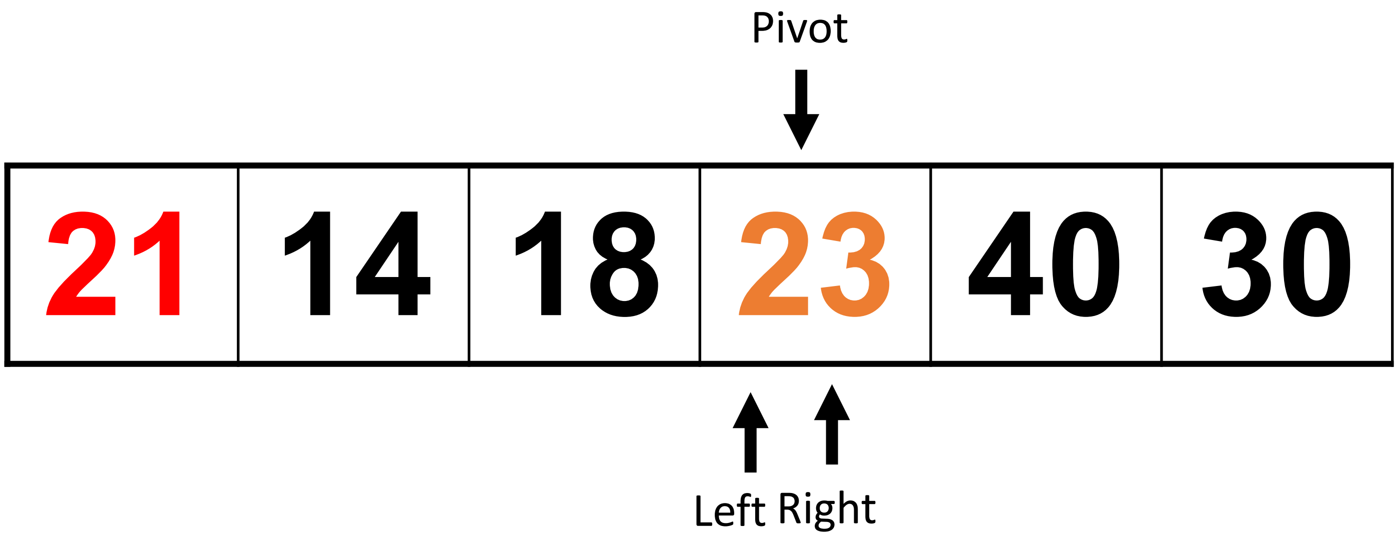
Since, pivot is at left, so algorithm starts from right, and move to left.

Quick Sort – Worked example



As A[pivot] < A[right], right moves one position to left

Quick Sort – Worked example



As A[pivot] > A[left], left moves one position to right

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        j -= 1
        while A[j] > pivot:
            j -= 1
        # If the pointers have crossed, the partition is complete
        if i >= j:
            return j
        # Swap the elements at the left and right pointers
        A[i], A[j] = A[j], A[i]
```

Quick sort – Complexity

- **O(n) operations** are required to find the location of an element that splits the array into two parts, .
  - every element in the array is compared to the partitioning element.
- After the division, each section is examined separately.
- If the array is split approximately in half **(which is not usually!)**, then there will be $\log_2 n$ splits.

- Therefore, total comparisons required are

$$f(n) = n \times \log 2n = O(n\log_2 n).$$

Quick sort – Worst Case Complexity

- Quick Sort is sensitive to the order of input data.
- It gives the worst performance when elements are already in the ascending order.
  - It then divides the array into sections of **1 and (n-1)** elements in each call.
  - Hence, **(n-1)** divisions in all.
- The total comparisons required are

$$f(n) = n \times (n-1) = O(n^2).$$

- Making it worst than merge sort and heap sort in terms of worst-case complexity.
- It is not a **stable sort** i.e., the order of equal elements may not be preserved.

Quick sort – Worst Case Complexity

- Quick sort is called quick because it has a much faster average case performance compared to other sorting algorithms like merge sort.
- While it does have a worst-case time complexity of $O(n^2)$ if the pivot is chosen poorly, in practice the pivot is often chosen randomly which results in a much faster average case performance of $O(n \log n)$.
- Additionally, quick sort has a smaller constant factor than merge sort, which means it can be faster in practice even when the input is already sorted or nearly sorted.