CSE 2202

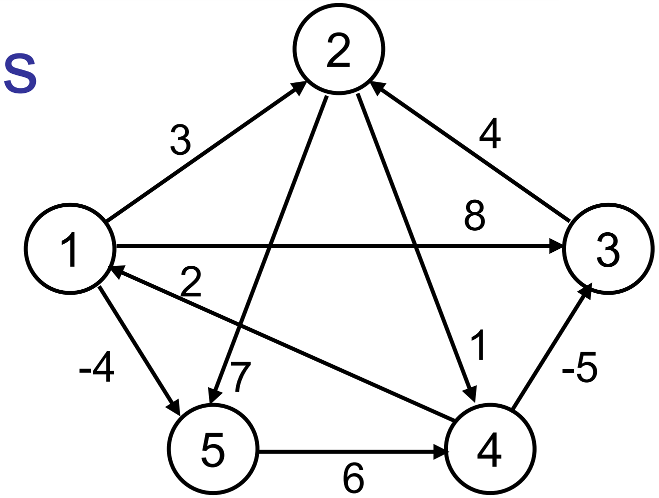
Design and Analysis of Algorithms – I

All Pair Shortest Path

All-Pairs Shortest Paths - Solutions

- Run **BELLMAN-FORD** once from each vertex:
  - $O(V^2E)$, which is $O(V^4)$ if the graph is dense ($E = \Theta(V^2)$)
- If no negative-weight edges, could run **Dijkstra's** algorithm once from each vertex:
  - $O(VE\lg V)$ with binary heap, $O(V^3\lg V)$ if the graph is dense
- We can solve the problem in $O(V^3)$, with no elaborate data structures

All-Pairs Shortest Paths



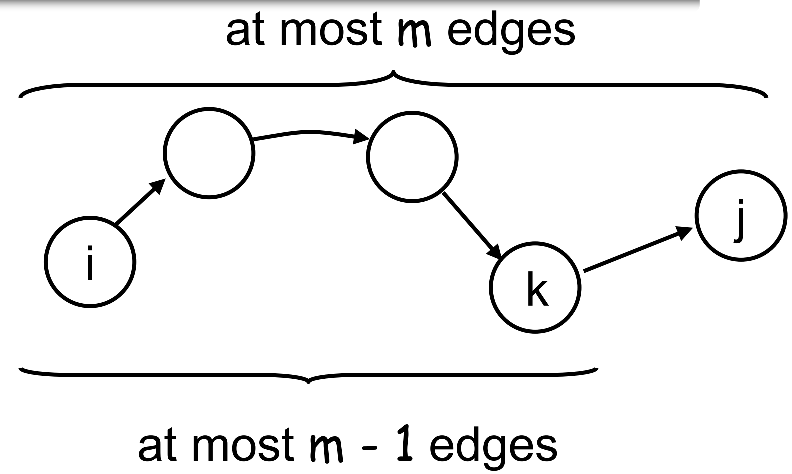
- Assume the graph (G) is given as adjacency matrix of weights
  - $W = (w_{ij})$, $n \times n$ matrix, $|V| = n$
  - Vertices numbered 1 to n

$$w_{ij} = \begin{cases} 0 & \text{if } i = j \\ \text{weight of } (i, j) & \text{if } i \neq j,\ (i, j) \in E \\ \infty & \text{if } i \neq j,\ (i, j) \notin E \end{cases}$$

- Output the result in an $n \times n$ matrix
  $D = (d_{ij})$, where $d_{ij} = \delta(i, j)$
- Solve the problem using dynamic programming

Optimal Substructure of a Shortest Path

- All subpaths of a shortest path are shortest paths
- Let p: a shortest path p from vertex i to j that contains at most m edges
- If i = j
  - w(p) = 0 and p has no edges

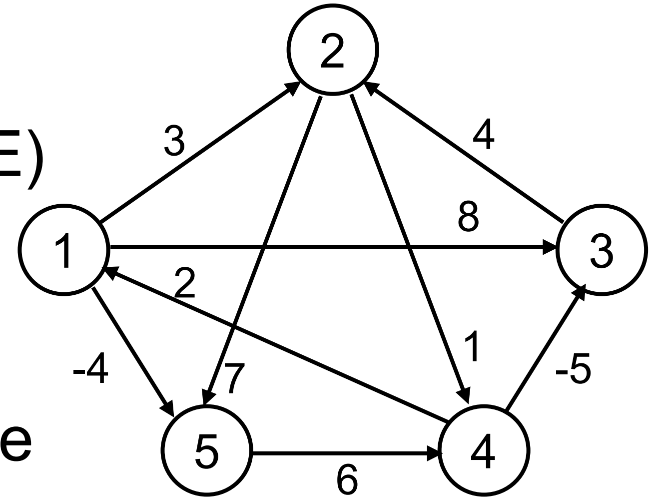


- If $i \neq j$: $p = i \overset{p'}{\rightsquigarrow} k \rightarrow j$
  - p′ has at most m-1 edges
  - p′ is a shortest path

$$\delta(i, j) = \delta(i, k) + w_{kj}$$

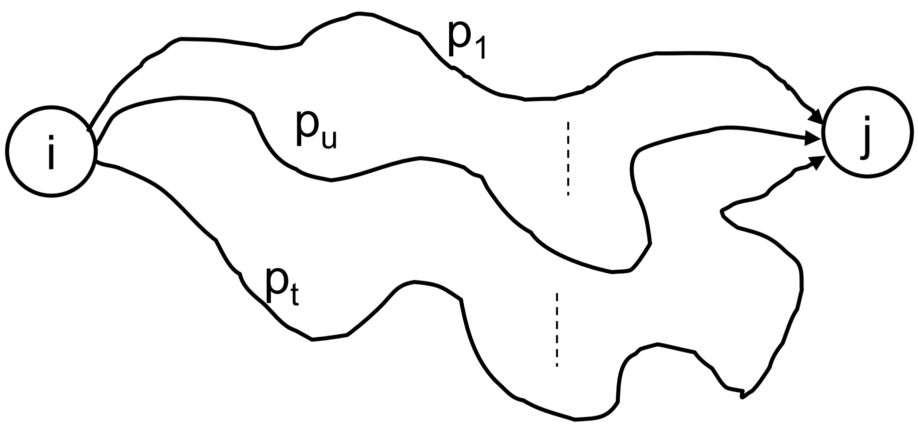
The Floyd-Warshall Algorithm

- **Given:**
  - Directed, weighted graph G = (V, E)
  - Negative-weight edges may be present
  - No negative-weight cycles could be present in the graph
- **Compute:**
  - The shortest paths between all pairs of vertices in a graph



The Structure of a Shortest Path

- For any pair of vertices $i, j \in V$, consider all paths from i to j whose intermediate vertices are all drawn from a subset {1, 2, …, k}
  - Find p, a minimum-weight path from these paths

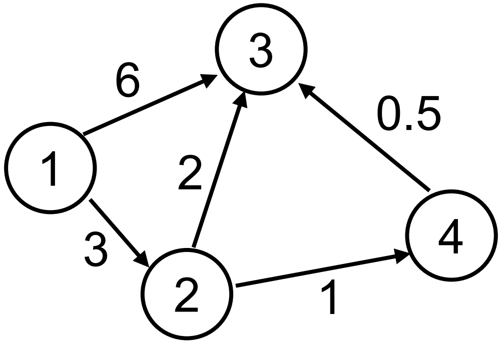


No vertex on these paths has index > k

Example

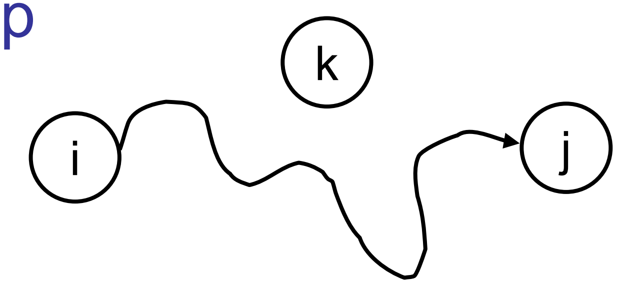
$d_{ij}^{(k)}$ = the weight of a shortest path from vertex i to vertex j with all intermediary vertices drawn from $\{1, 2, \ldots, k\}$

- $d_{13}^{(0)} =$ 6
- $d_{13}^{(1)} =$ 6
- $d_{13}^{(2)} =$ 5
- $d_{13}^{(3)} =$ 5
- $d_{13}^{(4)} =$ 4.5

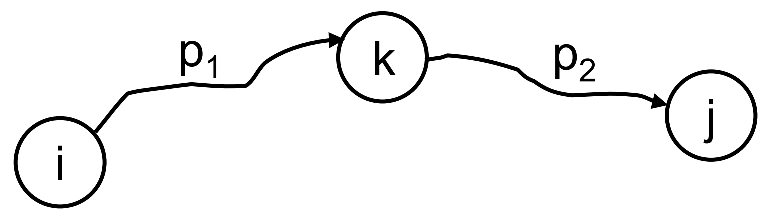


The Structure of a Shortest Path

- k is not an intermediate vertex of path p
  - Shortest path from i to j with intermediate vertices from {1, 2, …, k} is a shortest path from i to j with intermediate vertices from {1, 2, …, k - 1}



- k is an intermediate vertex of path p
  - p_1 is a shortest path from i to k
  - p_2 is a shortest path from k to j
  - k is not intermediary vertex of p_1, p_2
  - p_1 and p_2 are shortest paths from i to k with vertices from {1, 2, …, k - 1}



A Recursive Solution (cont.)

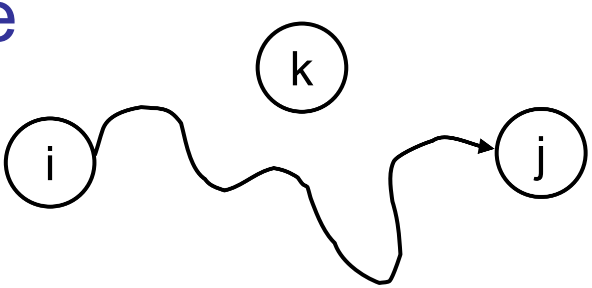
$d_{ij}^{(k)}$ = the weight of a shortest path from vertex i to vertex j with all intermediary vertices drawn from $\{1, 2, \ldots, k\}$

- $k = 0$
- $d_{ij}^{(k)} = w_{ij}$

A Recursive Solution (cont.)

$d_{ij}^{(k)}$ = the weight of a shortest path from vertex i to vertex j with all intermediary vertices drawn from $\{1, 2, \ldots, k\}$

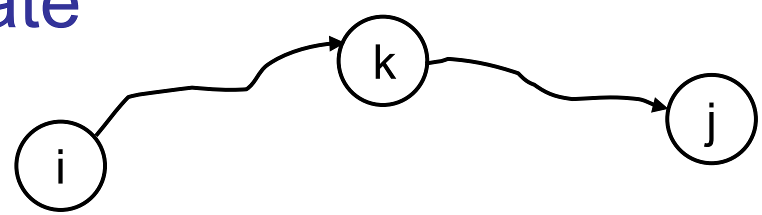
- $k \geq 1$
- **Case 1:** k is not an intermediate vertex of path p
- $d_{ij}^{(k)} = d_{ij}^{(k-1)}$



A Recursive Solution (cont.)

$d_{ij}^{(k)}$ = the weight of a shortest path from vertex i to vertex j with all intermediary vertices drawn from $\{1, 2, \ldots, k\}$

- $k \geq 1$
- **Case 2:** k is an intermediate vertex of path p

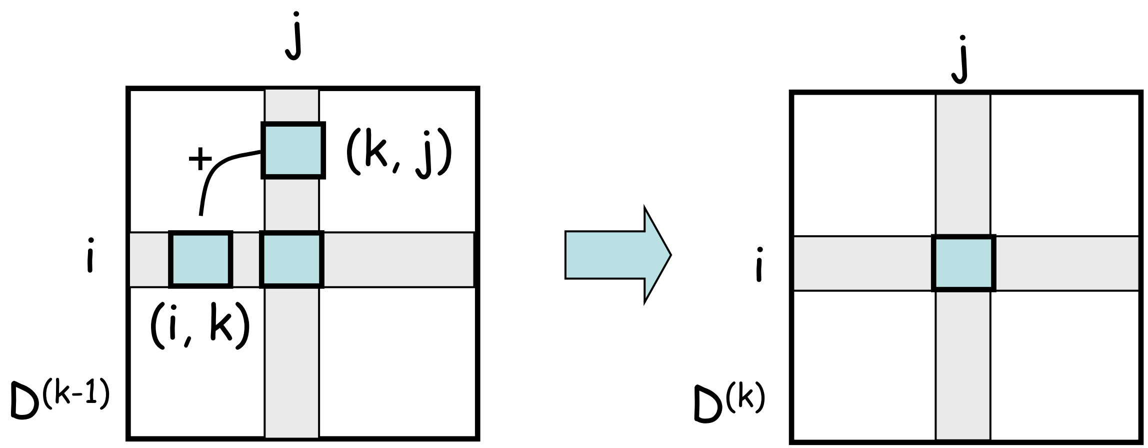


- $d_{ij}^{(k)} = \quad d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

Computing the Shortest Path Weights

- $d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 \\ \min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\} & \text{if } k \geq 1 \end{cases}$
- The final solution: $D^{(n)} = (d_{ij}^{(n)})$:

$$d_{ij}^{(n)} = \delta(i, j) \ \forall \ i, j \in V$$



The Floyd-Warshall algorithm

```
Floyd-Warshall(W[1..n][1..n])
D ← W      // D^(0)
for k ← 1 to n do // compute D^(k)
   for i ←1 to n do
      for j ←1 to n do
         if D[i][k] + D[k][j] < D[i][j] then
            D[i][j] ←D[i][k] + D[k][j]
return D
```

Running Time: O(n^3)

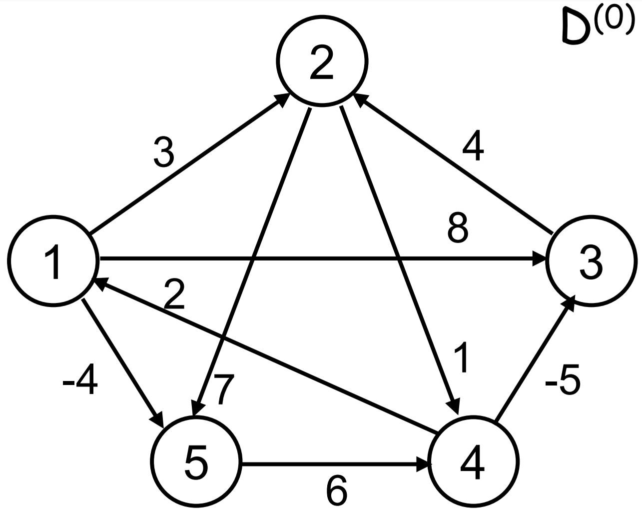
Computing predecessor matrix

- *How do we compute the predecessor matrix?*
  - Initialization: $p^{(0)}(i,j) = \begin{cases} nil & \text{if } i = j \text{ or } w_{ij} = \infty \\ i & \text{if } i \neq j \text{ and } w_{ij} < \infty \end{cases}$

```
Floyd-Warshall(W[1..n][1..n])
…
for k ←1 to n do // compute D(k)
    for i ←1 to n do
        for j  ←1 to n do
            if D[i][k] + D[k][j] < D[i][j] then
                D[i][j] ←D[i][k] + D[k][j]
                P[i][j] ← k
return D
```

Example

$$d_{ij}^{(k)} = \min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}$$



$D^{(0)} = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

$D^{(1)}$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$D^{(2)}$

	1	2	3	4	5
1	0	3	8	4	-4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

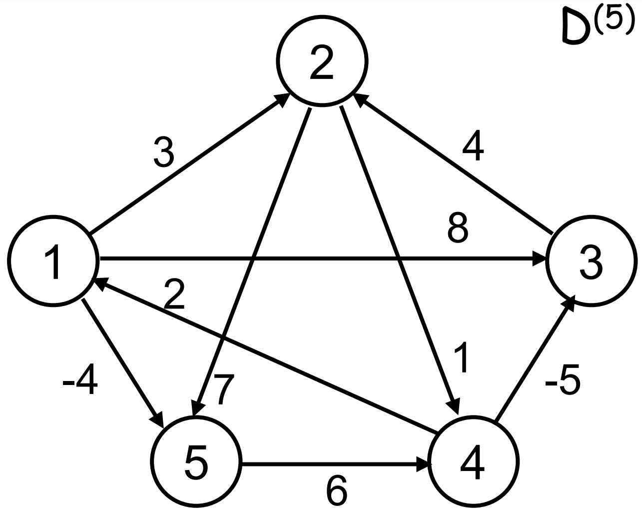
$D^{(3)}$

0	3	8	4	-4
∞	0	∞	1	7
∞	4	0	5	11
2	-1	-5	0	-2
∞	∞	∞	6	0

$D^{(4)}$

0	3	-1	4	-4
3	0	-4	1	-1
7	4	0	5	3
2	-1	-5	0	-2
8	5	1	6	0

Example $d_{ij}^{(k)} = \min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}$



$D^{(5)}$

	1	2	3	4	5
1	0	1	-3	2	-4
2	3	0	-4	1	-1
3	7	4	0	5	3
4	2	-1	-5	0	-2
5	8	5	1	6	0

$P^{(5)}$

	1	2	3	4	5
1	-	3	4	5	1
2	4	-	4	2	1
3	4	3	-	2	1
4	4	3	4	-	1
5	4	3	4	5	-

Source: 5, Destination: 1

Shortest path: 8

Path: 5 …1 : 5…4…1: 5->4…1: 5->4->1

Source: 1, Destination: 3

Shortest path: -3

Path: 1 …3 : 1…4…3: 1…5…4…3: 1->5->4->3

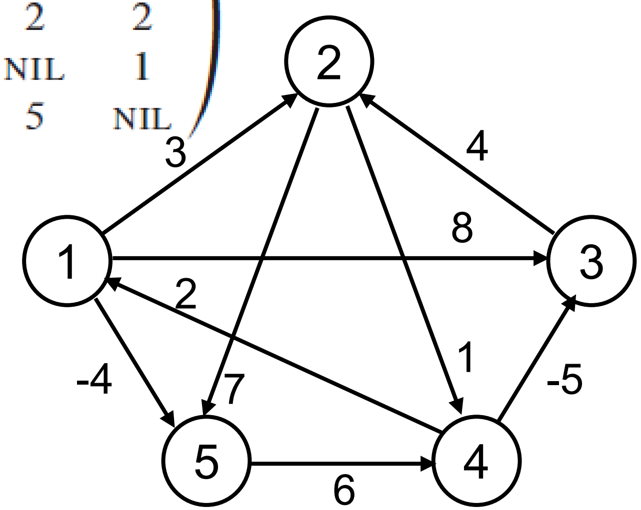
Example

$$d_{ij}^{(k)} = \min \{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}$$

$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(0)} = \begin{pmatrix} \mathrm{NIL} & 1 & 1 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 & 2 \\ \mathrm{NIL} & 3 & \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} \\ 4 & \mathrm{NIL} & 4 & \mathrm{NIL} & \mathrm{NIL} \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 5 & \mathrm{NIL} \end{pmatrix}$$

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(1)} = \begin{pmatrix} \mathrm{NIL} & 1 & 1 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 & 2 \\ \mathrm{NIL} & 3 & \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} \\ 4 & 1 & 4 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 5 & \mathrm{NIL} \end{pmatrix}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(2)} = \begin{pmatrix} \mathrm{NIL} & 1 & 1 & 2 & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 & 2 \\ \mathrm{NIL} & 3 & \mathrm{NIL} & 2 & 2 \\ 4 & 1 & 4 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 5 & \mathrm{NIL} \end{pmatrix}$$



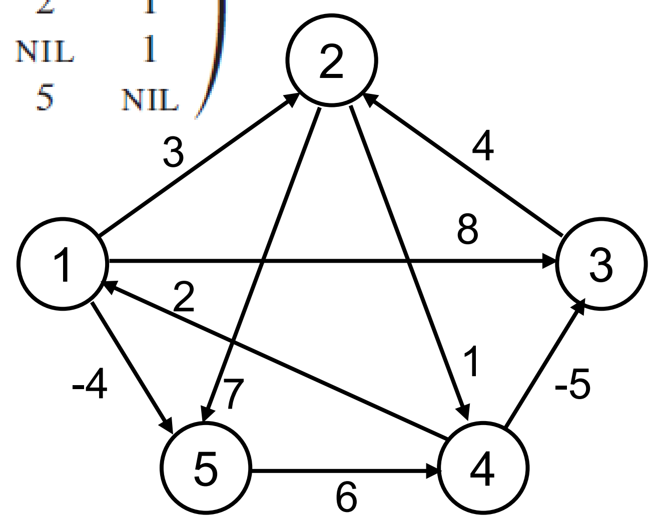
Example

$$d_{ij}^{(k)} = \min \{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(2)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(3)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad \Pi^{(4)} = \begin{pmatrix} \text{NIL} & 1 & 4 & 2 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$

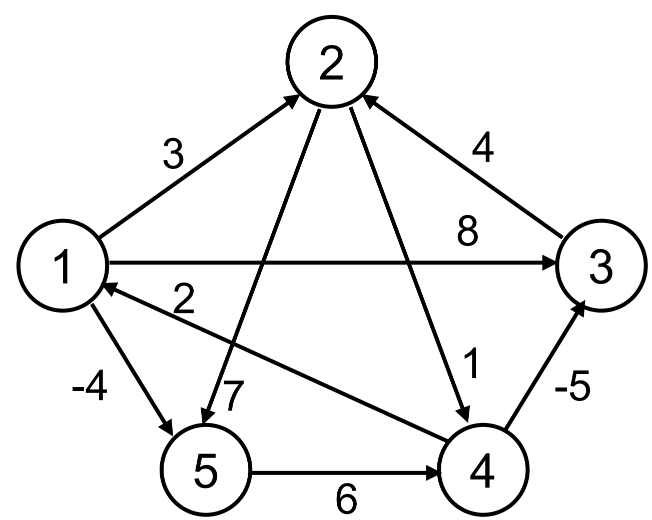


Example

$$d_{ij}^{(k)} = \min \{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}$$

$$D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad \Pi^{(4)} = \begin{pmatrix} \text{NIL} & 1 & 4 & 2 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad \Pi^{(5)} = \begin{pmatrix} \text{NIL} & 3 & 4 & 5 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$



PrintPath for Warshall’s Algorithm

```
PrintPath(s, t)
{
   if(P[s][t]==nil) {print(“No path”);return;}
   else if (P[s][t]==s){
        print(s);
   }
   else{
        print_path(s,P[s][t]);
        print_path(P[s][t], t);
   }
}
Print (t) at the end of the PrintPath(s,t)
```

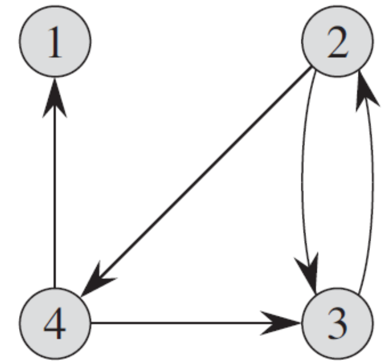
Question

- Why should we use D[i, j] instead of $D^{(k)}[i, j]$?
- Exercise:
  - 25.2-4: Memory $O(n^2)$
  - 25.2-6: Negative weight cycle
  - Find the shortest positive cycle

Transitive closure of the graph

- Input:
  - Un-weighted graph G: $W[i][j] = 1$, if $(i,j) \in E$, $W[i][j] = 0$ otherwise.
- Output:
  - $T[i][j] = 1$, if there is a path from i to j in G, $T[i][j] = 0$ otherwise.
- Algorithm:
  - Just run Floyd-Warshall with weights 1, and make $T[i][j] = 1$, whenever $D[i][j] < \infty$.
  - More efficient: use only Boolean operators

Transitive closure algorithm




```
Transitive-Closure(W[1..n][1..n])
T ← W    // T^(0)
for k ←1 to n do // compute T^(k)
   for i ←1 to n do
      for i ←1 to n do
         T[i][j] ← T[i][j] ∨ (T[i][k] ∧ T[k][j])
return T
```

$$T^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad T^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$$T^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad T^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Complexity

- Bellman-Ford algorithm:

  Running time: O(VE)

- Dijkstra's Algorithm

Q	Total
array	$O(V^2)$
binary heap	$O(E \lg V)$
Fibonacci heap	$O(V \lg V + E)$

Complexity

- Run **BELLMAN-FORD** once from each vertex:
  - $O(V^2E)$, which is $O(V^4)$ if the graph is dense ($E = \Theta(V^2)$)
- If no negative-weight edges, could run **Dijkstra's** algorithm once from each vertex:
  - $O(VE\lg V)$ with binary heap, $O(V^3\lg V)$ if the graph is dense
- We can solve the problem in $O(V^3)$, with no elaborate data structures

Johnson's Algorithm

Feature	Johnson's Algorithm	Warshall's (Floyd-Warshall) Algorithm
Time Complexity	$O(VE + V^2 \log V)$	$O(V^3)$
Space Complexity	$O(V + E)$	$O(V^2)$
Negative Weights	Handles negative weights (without negative cycles)	Handles negative weights (without negative cycles)
Optimality	More efficient for sparse graphs	More efficient for dense graphs

Time Complexity: The main steps in the algorithm are Bellman-Ford Algorithm called once and Dijkstra called V times.

Time complexity of Bellman Ford is **O(VE)** and time complexity of Dijkstra is **O(VLogV)**. So overall time complexity is **O(V²log V + VE)**.

Johnson's Algorithm

Johnson's algorithm uses the technique of **reweighting**

- If all edge weights w in a graph G are nonnegative, we can find shortest paths between all pairs of vertices by running Dijkstra's algorithm once from each vertex;
- with the Fibonacci-heap min-priority queue, the running time of this all-pairs algorithm is $O(V^2 \lg V + VE)$.
- If G has negative-weight edges but no negative-weight cycles, we simply compute **a new set of nonnegative edge weights** that allows us to use the same method…

Johnson's Algorithm

The new set of edge weights $\hat{w}$ must satisfy two important properties:

1. For all pairs of vertices $u, v \in V$, a path p is a shortest path from u to v using weight function w if and only if p is also a shortest path from u to v using weight function $\hat{w}$.
2. For all edges (u, v), the new weight $\hat{w}(u, v)$ is nonnegative.

As we shall see in a moment, we can preprocess G to determine the new weight function $\hat{w}$ in $O(VE)$ time.

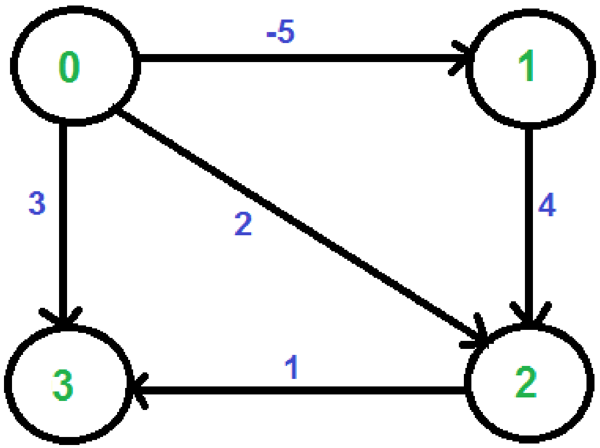
Lemma 25.1 (Reweighting does not change shortest paths)

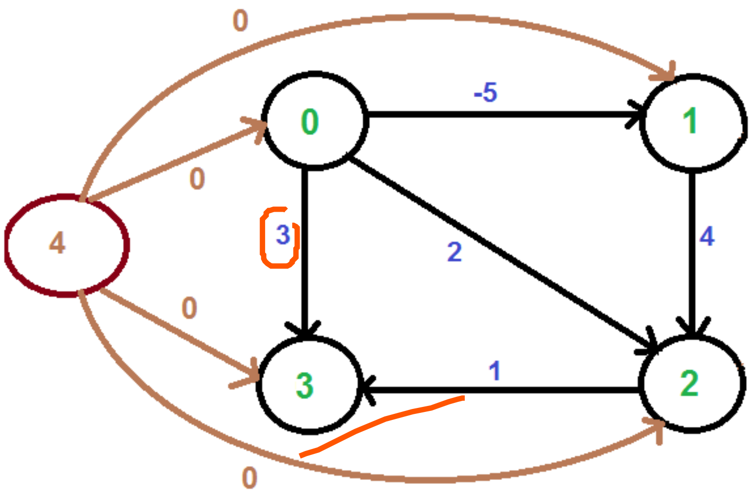
Johnson’s Algorithm

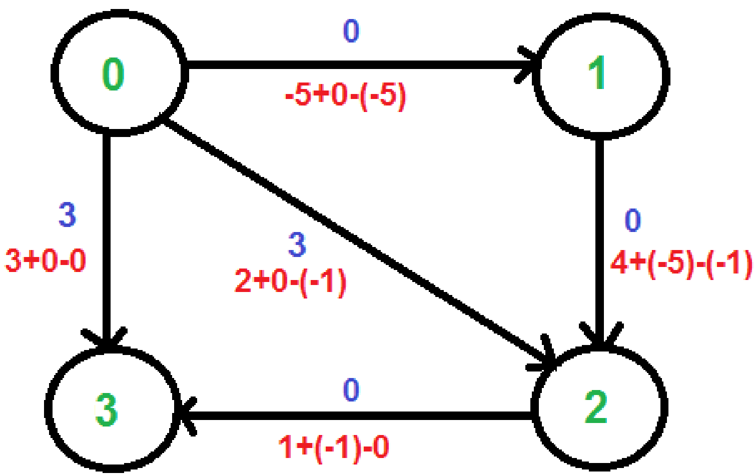
Johnson's algorithm has three main steps.

1. A new vertex is added to the graph, and it is connected by edges of zero weight to all other vertices in the graph.
2. All edges go through a reweighting process that eliminates negative weight edges.
3. The added vertex from step 1 is removed and Dijkstra's algorithm is run on every node in the graph.

Johnson's Algorithm







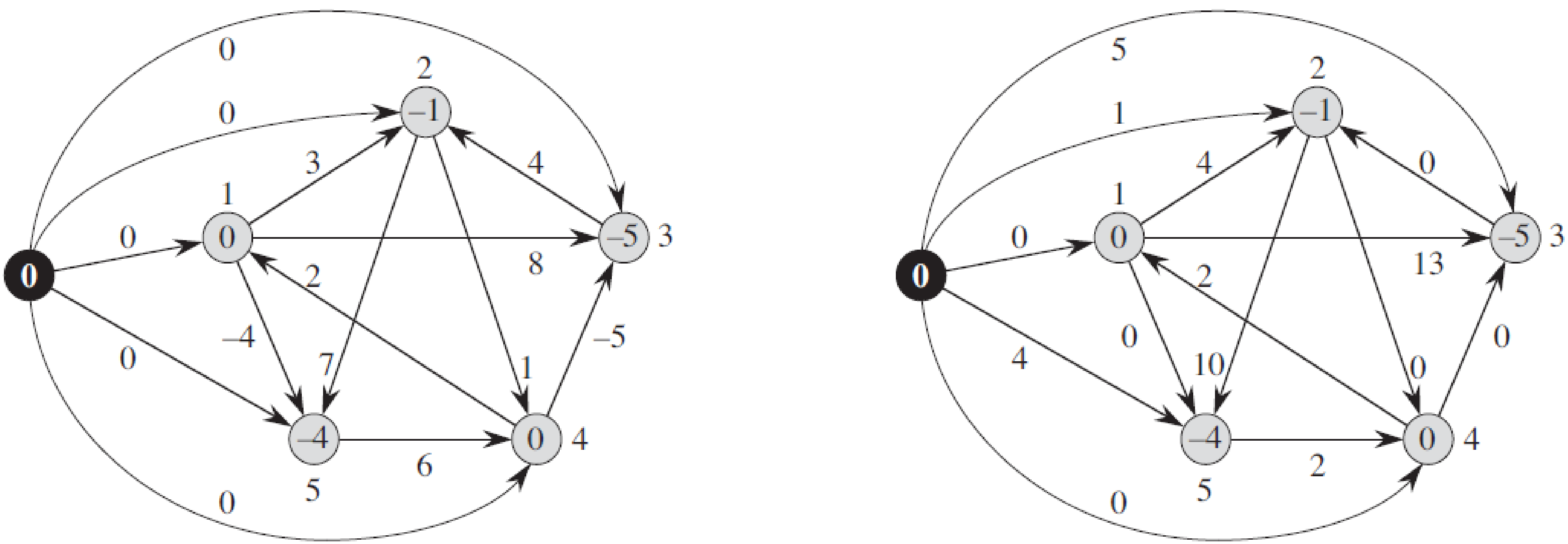
Distances from 4 to 0, 1, 2 and 3 are 0, -5, -1 and 0 respectievely.

- We calculate the shortest distances from 4 to all other vertices using Bellman-Ford algorithm.
- The shortest distances from 4 to 0, 1, 2 and 3 are 0, -5, -1 and 0 respectively,

  i.e., **h[] = {0, -5, -1, 0}**.
- Once we get these distances, we remove the source vertex 4 and reweight the edges using following formula.

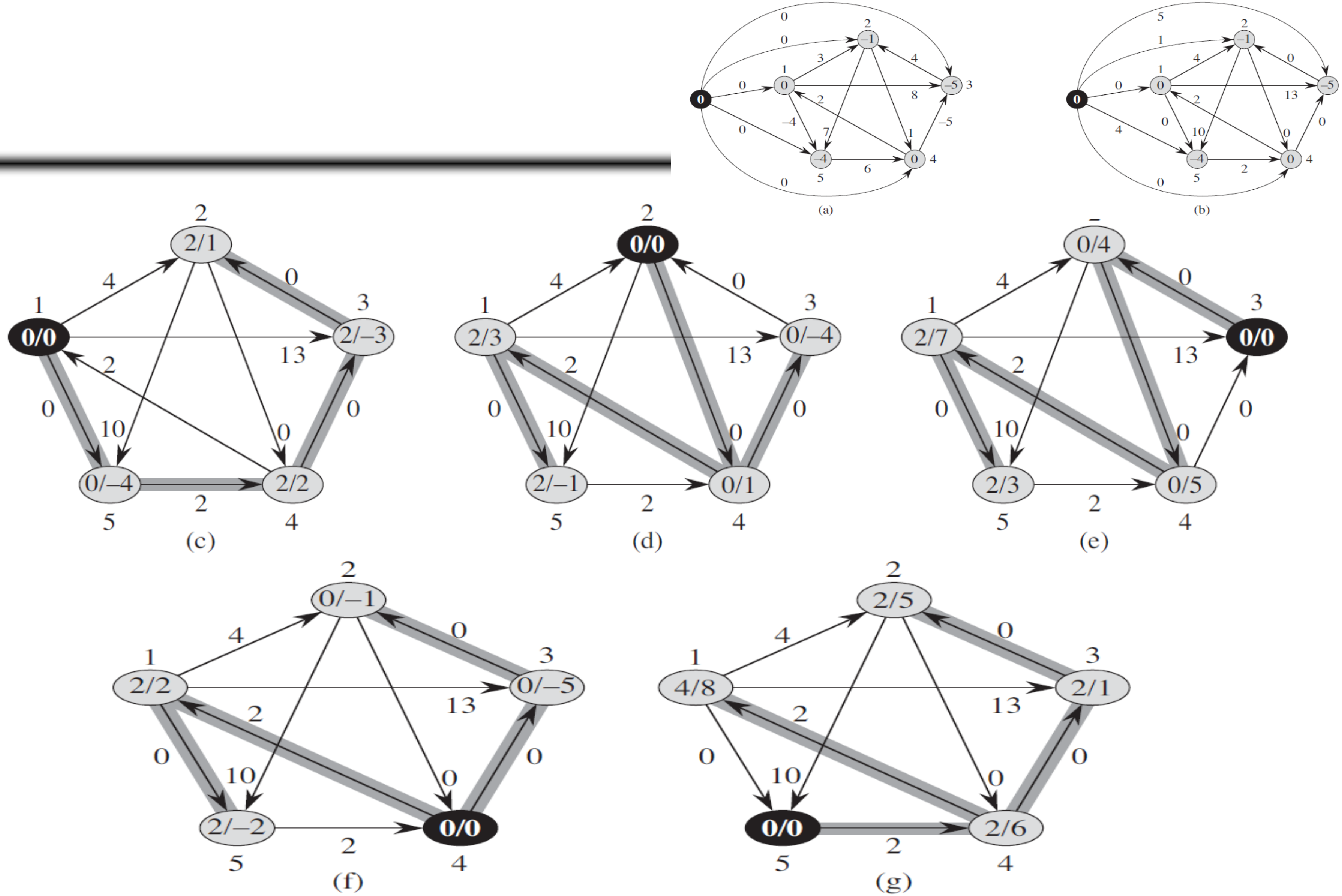
$$\hat{w}(u, v) = w(u, v) + h(u) - h(v)$$

Johnson's Algorithm



(a) The graph G' with the original weight function w. The new vertex s is black. Within each vertex v is $h(v) = \delta(s, v)$.

(b) After reweighting each edge (u, v) with weight function $\hat{w}(u, v) = w(u, v) + h(u) - h(v)$.



Dijkstra's algorithm on each vertex of G using weight function $\hat{w}$. In each part, the source vertex u is black, and shaded edges are in the shortest-paths tree computed by the algorithm. Within each vertex v are the values $\hat{\delta}(u, v)$ and $\delta(u, v)$, separated by a slash. The value $d_{uv} = \delta(u, v)$ is equal to $\hat{\delta}(u, v) + h(v) - h(u)$.

JOHNSON(G, w)

compute G', where $G'.V = G.V \cup \{s\}$,
  $G'.E = G.E \cup \{(s, v) : v \in G.V\}$, and
  $w(s, v) = 0$ for all $v \in G.V$
if BELLMAN-FORD(G', w, s) == FALSE
  print "the input graph contains a negative-weight cycle"
else for each vertex $v \in G'.V$
    set $h(v)$ to the value of $\delta(s, v)$
      computed by the Bellman-Ford algorithm
  for each edge $(u, v) \in G'.E$
    $\boxed{\widehat{w}(u, v) = w(u, v) + h(u) - h(v)}$
  let $D = (d_{uv})$ be a new $n \times n$ matrix
  for each vertex $u \in G.V$
    run DIJKSTRA$(G, \widehat{w}, u)$ to compute $\widehat{\delta}(u, v)$ for all $v \in G.V$
    for each vertex $v \in G.V$
      $\boxed{d_{uv} = \widehat{\delta}(u, v) + h(v) - h(u)}$
  return D

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    for each vertex $v \in G.V$
      $d_{uv} = \hat{\delta}(u, v) + h(v) - h(u)$
  return D

If we implement the min-priority queue in Dijkstra's algorithm by a Fibonacci heap, Johnson's algorithm runs in $\boxed{O(V^2 \lg V + VE)}$ time. The simpler binary min-heap implementation yields a running time of $\boxed{O(VE \lg V)}$, which is still asymptotically faster than the Floyd-Warshall algorithm if the graph is sparse.