Date: Name: عبدالقدیر فسرتاش Subject: ID: 99243100

a) $T(n) = 2T(\lfloor n/3 \rfloor) - T(n/9) + n * \log n$ (جواب 1

$T(n) = 2T(n/3) - T(n/9) + n * \log n$

$n = 3^m \rightarrow T(3^m) = 2T(3^{m-1}) - T(3^{m-2}) + 3^m * m$

$T(3^m) = F(m) \Rightarrow F(m) = 2F(m-1) - F(m-2) + 3^m * m$

$\Rightarrow F(m) - 2F(m-1) + F(m-2) = 3^m m \rightarrow$ ناهمگن

$\Rightarrow P(x) = (x^2 - 2x + 1)(x-3)^2$

$x \quad x \quad -1 \quad -1$

$\Rightarrow (x-1)(x-1)(x-3)(x-3) = 0 \rightarrow r_1 = 1, \; r_2 = 1, \; r_3 = 3, \; r_4 = 3$

$F(m) = C_1 \cdot 1^m + m \cdot C_2 1^m + C_3 \cdot 3^m + m \cdot C_4 \cdot 3^m$

به حالت اول برمی گردانیم

$\hookrightarrow T(3^m) = C_1 1^m + m C_2 1^m + C_3 \cdot 3^m + m \cdot C_4 \cdot 3^m$

$\hookrightarrow 3^m = n$

$\hookrightarrow T(n) = C_1 \cdot 1^{\log_3 n} + \log_3 n \cdot C_2 \cdot 1^{\log_3 n} + C_3 n + \log_3 n \cdot n \cdot C_4$

$\Rightarrow T(n) = C_1 \cdot n^0 + C_2 \cdot \log_3 n \cdot n^0 + C_3 n + \log_3 n \cdot n \cdot C_4$

جواب خام $\Rightarrow T(n) = C_1 + C_2 \cdot \log_3 n + C_3 n + \log_3 n \cdot n \cdot C_4$

برای یافتن ci ها قیمت دلخواه زیر را درنظر گیریم. $\rightarrow T(1) = 0, \quad T(3) = 1, \quad T(9) = 2, \quad T(27) = 3$

یافتن ci ها $\rightarrow 0 = C_1 + C_3 \Rightarrow C_3 = -C_1$ ✓ ... I

$1 = \cancel{-C_3} + C_2 + 3C_3 + 3C_4$

$1 = C_2 + 2C_3 + 3C_4$ ✓ ... II

$2 = 2C_2 + 8C_3 + 18C_4$

$\cancel{2} = \cancel{2}(C_2 + 4C_3 + 9C_4)$

$1 = C_2 + 4C_3 + 9C_4$ ✓ ... III

$\cancel{6}C_4 = -\cancel{2}C_3$

$3C_4 = -C_3$ ... IV ✓

$\cancel{C_2} + 4C_3 + 9C_4 = 1$

III − II = $\cancel{C_2} + 2C_3 + 3C_4 = 1$

$2C_3 + 6C_4 = 0$

$$T(27) = 3 = 3C_2 + 26C_3 + \overset{3C_4}{27C_3} = 3$$

$$\Rightarrow -3\cancel{C_2} + C_3 = -3 \qquad \Rightarrow C_3 = 0 \rightarrow 3C_4 = -C_3$$

$$3\cancel{C_2} + 9C_3 = 3 \qquad C_4 = 0/3 = 0$$

$$0 + 12C_3 = 0 \qquad C_3 = -C_1 = 0$$

$$\Rightarrow C_2 = 1$$

$$\begin{cases} C_1 = 0 \\ C_2 = 1 \\ C_3 = 0 \\ C_4 = 0 \end{cases}$$

مقدار C_i ها براساس مقدار دلخواه که قبل انتخاب شده بود بدست آمده

$$\Rightarrow T(n) = \log_3 n$$

کلی $T(n) = C_1 + C_2 \cdot \log_3 n + C_3 \cdot n + C_4 \cdot n \cdot \log n$ Condition$(n = 3^m)$

$$T(n) = \log_3 n$$

چون $T(n)$ همواره غیر کاهشی است

پس برای همه جواب ها صحت دارد

b) $\sqrt{T(n)} = \sqrt{T(n-1)} + 2\sqrt{T(n-2)}$, $T(0) = T(1) = 1$

تغییر متغیر $\rightarrow \sqrt{T(n)} = F(n) \rightarrow F(n) = F(n-1) + 2F(n-2) \rightarrow$ همگن

$$F(n) - F(n-1) - 2F(n-2) = 0$$

$$p(x) = x^2 - 1x - 2 = 0$$

$$x \quad x \qquad +1 \quad -2$$

$$(x+1)(x-2) \rightarrow r_1 = -1$$

$$\rightarrow r_2 = 2$$

$$\Rightarrow F(n) = C_1 (-1)^n + C_2 (2)^n$$

$$F(0) \Rightarrow 1 = \cancel{C_1} + C_2 \qquad C_2 = 2/3$$

$$F(1) \Rightarrow 1 = -\cancel{C_1} + 2C_2 \qquad C_1 = 1/3 \qquad \Rightarrow F(n) = 1/3 (-1)^n + \frac{2}{3}(2)^n$$

$$\frac{2}{3} = \cancel{3}C_2/\cancel{3}$$

Kian

$$\sqrt{T(n)} = \frac{1}{3}(-1)^n + \frac{2}{3}(2)^n$$

$$T(n) = \sqrt{1/3 (-1)^n + 2/3 (2)^n}$$

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c) $T(n) = 3T(n-1) + 10T(n-2) + 5^n$ ناهمگن

$T(0)=0$

$T(1)=11$

$T(2)=20$

$T(n) - 3T(n-1) + 10T(n-2) = 5^n$

$p(x) = (x^2 - 3x + 10)(x-5)$

$+2 \; -5$

$p(x) = (x+2)(x-5)(x-5) \rightarrow$

$r_1 = -2$, $r_2 = 5$, $r_3 = 5$

$T(n) = C_1 (-2)^n + C_2 (5)^n + C_3 \cdot n (5)^n \Rightarrow$ معادله کلی

$T(0) \rightarrow 0 = C_1 + C_2 + 0 \rightarrow C_1 = -C_2$ ... I

$T(1) \rightarrow 11 = -2C_1 + 5C_2 + 5C_3$

$-C_2$

$\Rightarrow 11 = 7C_2 + 5C_3$ ... II

$-C_2$

$20 = 4C_1 + 25C_2 + 2 \cdot 25 C_3$

$20 = 21C_2 + 50C_3$ ... III $\quad$ (III − 3II) $\Rightarrow$

$20 = \cancel{21C_2} + 50C_3$

$3(11) = 3(7C_2 + 15C_3)$

$33 = \cancel{21C_2} + 15C_3$

$\frac{13}{-35} = \frac{-35 C_3}{-35}$

$C_3 = -13/35$ ✓

$\Rightarrow 11 = 7C_2 + 5(-13/35)$

$\Rightarrow 7C_2 = 11 + 65/35$

$\Rightarrow C_2 = \frac{90}{49}$ ✓

$\Rightarrow C_1 = -C_2 = -90/49$ ✓

$\Rightarrow T(n) = -90/49 (-2)^n + 90/49 (5)^n + (-13/35) \cdot n \cdot (5)^n$

test: $T(0)=0 \rightarrow 0 = -90/49 (-2)^0 + 90/49 (5)^0 + (-13/35)(0) \cdot (5)^0 = 0$ ✓

$T(1) = 11 \rightarrow 11 = -90/49 (-2) + 90/49 (5) + (-13/35)(1)(5)$

$+180/49 + \frac{450}{49} - \frac{13}{7} = \frac{180 + 450 - 91}{49} = \frac{539}{49} = 11$ ✓

Kian $T(2) = 20 \rightarrow 11 = \frac{-360}{49} + \frac{2250}{49} + \frac{-130}{7} = \frac{-360 + 2250 + 910}{49} = \frac{180}{49}$

20 ✓ ✗

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d) $T(n) = \frac{2}{\sqrt{n}} \cdot T(\sqrt{n}) + \log^{n}\sqrt{n} \Rightarrow T(n) = \frac{2}{\sqrt{n}} \cdot T(\sqrt{n}) + \frac{1}{n} \cdot \log(n)$

$\frac{T(n)}{\sqrt{n}} = \frac{2}{\sqrt{n}\cdot\sqrt{n}} T(\sqrt{n}) + \frac{1}{\sqrt{n}\cdot n} \cdot \log(n) \to \frac{T(n)}{\sqrt{n}} = 2\frac{T(\sqrt{n})}{n} + \frac{1}{\sqrt{n}\cdot n} \cdot \log(n)$

$(\cancel{\sqrt{n}}\cdot n) \cdot \frac{T(n)}{\cancel{\sqrt{n}}} = 2(\sqrt{n}\cdot\cancel{n}) \cdot \frac{T(\sqrt{n})}{\cancel{n}} + \frac{(\sqrt{n}\cdot n)}{(\sqrt{n}\cdot n)} \cdot \log(n)$

$n \cdot T(n) = 2\sqrt{n} \cdot T(\sqrt{n}) + \log n \Rightarrow F(n) = n \cdot T(n) \Rightarrow$ تغییر متغیر

$F(n) = 2F(\sqrt{n}) + \log(n) \Rightarrow n = 2^{2^m}$

$F(2^{2^m}) = 2F(2^{2^{m-1}}) + \log(2^{2^m}) \Rightarrow F(2^{2^m}) = F(m)$

$F(m) = 2F(m-1) + 2^m \to$ ناهمگن

$p(x) = (x-2)(x-2) = 0$ ✓ $T(n) = \frac{1}{n}\log(\log n)\cdot\log n$

$r_1 = 2 \Rightarrow F(m) = C_1 \cdot (2)^m + C_2 \cdot m \cdot 2^m$

$r_2 = 2$ $F(2^{2^m}) = C_1 \cdot (2)^m + C_2 m \cdot 2^m$

$F(n) = C_1 \cdot 2^{(\log(\log n))} + C_2 \cdot \log(\log n) \cdot 2^{\log(\log n)}$

شرایط اولیه $F(n) = C_1 \cdot \log(n) + C_2 \cdot \log(\log(n)) \cdot \log(n)$

$F(2) = 0$ | $F(2) = 0 = C_1 + 0 \Rightarrow C_1 = 0$ ✓ $F(n) = \log(\log n) \cdot \log n$

$F(4) = 2$ | $F(4) = 2 \to 0 \cdot (2) + C_2 2 = 2 \to C_2 = 1$ ✓ $\cancel{n} \cdot T(n) = \frac{1}{n}\log(\log n) \cdot \log n$

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e) $T(n) = T(\sqrt{n}) - T(\sqrt[4]{n}) + T(\sqrt[8]{n}) + \log(n)$

$T(n) = T(n^{1/2}) - T(n^{1/4}) + T(n^{1/8}) + \log(n)$

$T(n) = T(n^{(1/2)^1}) - T(n^{(1/2)^2}) + T(n^{(1/2)^3}) + \log(n)$

$n = 2^{2^m}$ ← تغییر متغیر $\quad T(2^{2^m}) = T(2^{2^m \cdot (\frac{1}{2})^1}) - T(2^{2^m \cdot (\frac{1}{2})^2}) + T(2^{2^m \cdot (\frac{1}{2})^3}) + 2^m$

$T(2^{2^m}) = T(2^{2^m \cdot 2^{-1}}) - T(2^{2^m \cdot 2^{-2}}) + T(2^{2^m \cdot 2^{-3}}) + 2^m$

$T(2^{2^m}) = F(m) \Rightarrow F(m) = F(m-1) - F(m-2) + F(m-3) + 2^m \rightarrow$ ناهمگن

$F(m) - F(m-1) + F(m-2) - F(m-3) = 2^m$

$p(x) = (x^3 - x^2 + x - 1)(x-2) = 0$

$(x-1)(x^2+1)(x-2) = 0$

$\Rightarrow r_1 = 1$
$r_2 = 2$
$r_3 = -\sqrt{-1} = -i$
$r_4 = \sqrt{-1} = +i$

$F(m) = C_1 \cdot 1^m + C_2 \cdot 2^m + C_3(-i)^m + C_4 \cdot (i)^m$

$i = \sqrt{-1} < -1 < 0$

$\Rightarrow i \in O(0) \Rightarrow F(m) = C_1 \cdot 1^m + C_2 \cdot 2^m$

چون order ، i ها کوچک‌تر از صفر است پس این‌ها صفر در نظر می‌گیریم

$F(m) = C_1 \cdot 1^m + C_2 \cdot 2^m$

$T(2^{2^m}) = C_1 \cdot 1^m + C_2 \cdot 2^m$

$n = 2^{2^m} \rightarrow T(n) = C_1 + C_2 \cdot 2^{\log(\log n)}$

$\Rightarrow T(2) = 0$ | دو مقدار دلخواه برای یافتن C_i $\Rightarrow 0 = \cancel{C_1} \pm C_2 \rightarrow C_1 = -C_2$

$T(4) = 1$ | $1 = \cancel{C_1} + 2C_2 \rightarrow C_1 = -1$

$C_2 = 1$

$\begin{cases} C_1 = -1 \\ C_2 = 1 \end{cases}$

چون 2^{2^m} غیرکاهشی و هموار است پس برای هر عدد دیگر جواب صحت دارد.

$\Rightarrow T(n) = (2)^{\log(\log n)} - 1$ ✓

$T(n) = (\log n)^{\log 2} \Rightarrow T(n) = \log(n) - 1$ ✓

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جواب 2)

a) $T(n) = 2/n \; (T(0) + T(1) + \dots + T(n-1)) + n$

$\sum_{K=0}^{n-1} T(K) = T(0) + T(1) + \dots T(n-1) \Rightarrow T(n) = 2/n \; (\sum_{K=0}^{n-1} T(K)) + n$

$\Rightarrow n \cdot T(n) = \sum_{K=0}^{n-1} T(K) + n \dots I \rightarrow n \rightarrow n+1 \Rightarrow (n+1)\, T(n+1) = \sum_{K=0}^{n} T(K) + n + 1 \dots II$

$\Rightarrow II - I \Rightarrow (n+1)\, T(n+1) - n\, T(n) = 2T(n) + 2n + 1$

$\Rightarrow (n+1)\, T(n+1) = (n+2)\, T(n) + 2n + 1$

$\Rightarrow \frac{T(n+1)}{(n+2)} = (\frac{1}{n+1})\, T(n) + \frac{2n+1}{(n+1)(n+2)} \rightarrow F(n) = \frac{T(n)}{n+1}$ *

$\Rightarrow F(n+1) = F(n) + \frac{2n+1}{(n+1)(n+2)}$

اگر روابط بازگشتی را باز کنیم : $g(n+1) = (\frac{2n+1}{(n+1)(n+2)}) + \frac{2n-1}{n(n+1)} + \dots \frac{3}{2(3)} + g(1)$

$\Rightarrow 2(\frac{1}{n+1} + \frac{1}{n} + \dots + \frac{1}{3}) + \frac{3}{2} - \frac{2n+1}{n+1} \Rightarrow g(n+1) \in \Theta(\log(n)) \Rightarrow T(n) \in \Theta(n \cdot \log n)$ ✓

$H_n = \frac{1}{n+1} + \frac{1}{n} + \dots + \frac{1}{3} \in \Theta(\log(n))$

b) $F(n) = 4T(n/2) + n^2 \cdot \sqrt{n}$

با استفاده قضیه اصلی (Mester) حل می‌کنیم :

$a = 4$

$b = 2$

$F(n) = n^2 \sqrt{n}$

$\Rightarrow n^{\log_b^a} \; \boxed{?} \; F(n) \rightarrow n^{\log_2^4} = n^2$

$\Rightarrow n^2 < n^2 \cdot \sqrt{n}$

$\Rightarrow T(n) \in \Theta(n^2 \cdot \sqrt{n})$ ✓

c) $T(n) = 4T(n/3) + n \log(n)$

$a = 4$

$b = 3$

$F(n) = n \cdot \log(n)$

$n^{\log_3^4 - \varepsilon} < n \cdot \log(n)$

$\Rightarrow T(n) \in \Theta(n \cdot \log(n))$

d) $T(n) = T(n/2 + \sqrt{n}) + n$

$\sqrt{n} \in O(n) \rightarrow T(n) = T(n/2) + n$

$a = 1$

$b = 2$

$K = 1$

$1 < 2^1 \Rightarrow$

$T(n) \in \Theta(n)$ ✓

Kian

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جواب 3)

$T(n) = \sqrt{n \times 99 \times T(\sqrt{n}) + 100 \cdot n}$

اثبات کنید از مرتبه $O(n^{2/3})$ است.

* تغییر متغیر $\Rightarrow T(n) = n^K$ *

$T(n) = \sqrt{n} \cdot \sqrt{99\,T(\sqrt{n}) + 100}$

$n^K = \sqrt{n} \cdot \sqrt{99 \cdot n^{K/2} + 100} \rightarrow$

عدد ثابت در پیچیدگی اهمیت ندارد

$\Rightarrow n^K = n^{1/2} \cdot \sqrt{n^{K/2}}$

$\Rightarrow n^K = n^{1/2} \cdot n^{\frac{K}{4}} \rightarrow n^K = n^{\frac{K}{4} + \frac{1}{2}}$

$\Rightarrow n = n \Rightarrow K = \frac{K}{4} + \frac{1}{2}$

$\frac{K}{1} - \frac{K}{4} = \frac{1}{2} \rightarrow \frac{4K - K}{4} = \frac{1}{2}$

$\rightarrow 4\,\frac{3K}{4} = \frac{1}{2} \cdot 4 \Rightarrow K = 2/3$ ✓

$T(n) = n^K \rightarrow T(n) = n^{2/3} \rightarrow$ اثبات شد

$T(n) \in O(n^{2/3})$ ✓

جواب 4)

Merge sort حالت اصلی $\rightarrow T(n) = 2T(n/2) + O(n)$

حالت خواسته شده $\rightarrow T(n) = 3T(n/3) + O(n) \sim n$ ✓

$\Rightarrow T(n) = 3T(n/3) + n$ با استفاده از قضیه مستر:

$$\begin{cases} a = 3 \\ b = 3 \\ K = 1 \end{cases}$$

پس با هم مساوی داریم $3 = 3^1$

Kian $T(n) = n \cdot \log(n)$ ✓

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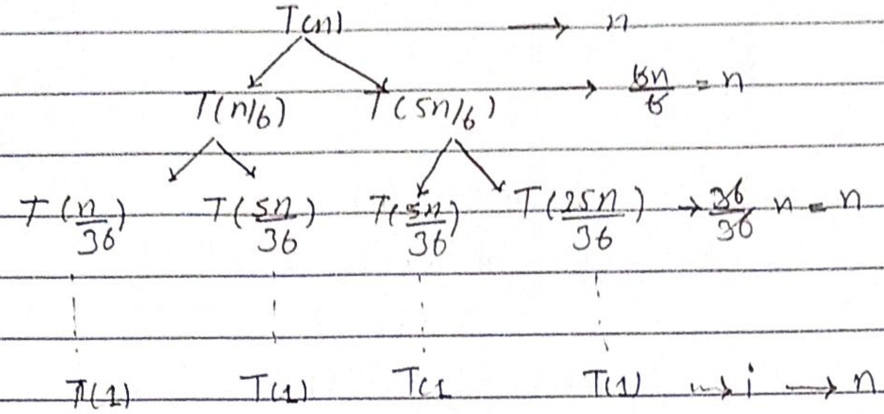
a) $T(n) = T(n/6) + T(5n/6) + n$ (جواب 5

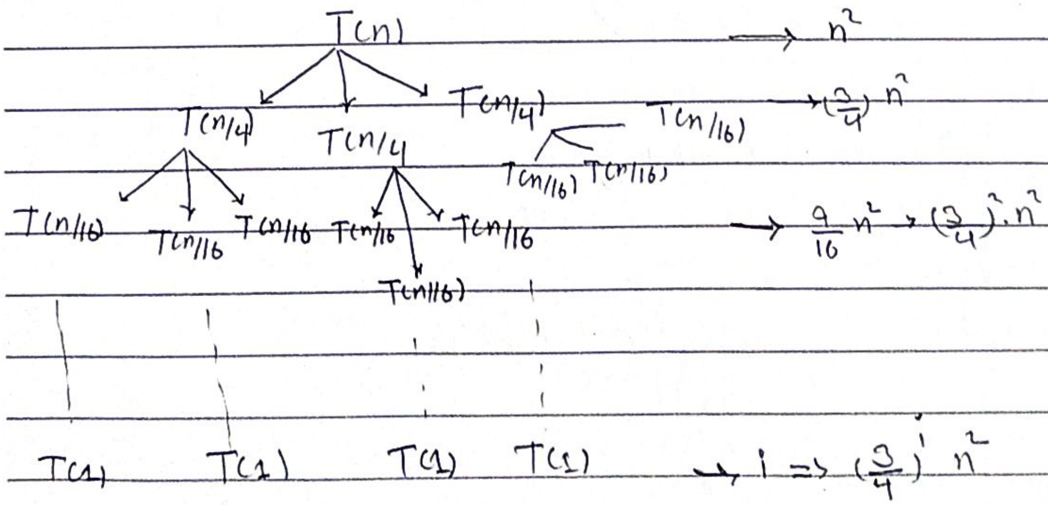


$K = \log_{6/5} n$ ارتفاع درخت

$\Rightarrow T(n) = \sum_{i=0}^{\log_{6/5} n} n \Rightarrow T(n) \in \Theta(n \cdot \log n)$ ✓

b) $T(n) = 3T(n/4) + \Theta(n^2)$

$T(n) = 3T(n/4) + n^2$



ارتفاع درخت $K = \log_4 n \Rightarrow T(n) = \sum_{i=0}^{\log n} (\frac{3}{4})^i n^2 \Rightarrow n^2 \cdot \sum_{i=0}^{\infty} (3/4)^i = n^2 (\frac{1}{1-3/4})$

$T(n) = n^2 (\frac{1}{1/4}) \Rightarrow T(n) \in \Theta(n^2)$

Kian $T(n) \in \Theta(n^2)$ ✓

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(جواب 6

```
void Function (int arr[], int len){
    int temp;
    if (len==1){
        return;
    }
    For (int i=0; i< len-1; i++){
        if (arr[i] > arr[i+1]){
            temp = arr[i];
            arr[i] = arr[i+1];
            arr[i+1] = temp;
        }
    }
    Function (arr, len-1);
```

$\theta(c)$ — int temp; if (len==1){ return; }

$O(len)$ — For loop

رابطه بازگشتی را می‌نویسیم:

$$Function(len) = \begin{cases} len = 1 & \emptyset \\ len > 1 & : Function(arr, len-1) \end{cases}$$

تغییر متغیر $\rightarrow len = n$, $Function = T$ ✓

$$\Rightarrow T(n) = T(n-1) + \underbrace{O(n) + \theta(c)}_{O(n)}$$

$$\Rightarrow T(n) = T(n-1) + O(n)$$

$$\Rightarrow T(n) = T(n-1) + n \rightarrow$$ حل رابطه بازگشتی

$T(n) \longrightarrow n$

$T(n-1) \longrightarrow n$

$T(n-2) \longrightarrow n$

$T(n-n) = T(0) \longrightarrow n$

$K = n$

$$T(n) \Rightarrow \sum_{i=0}^{n}(n) \Rightarrow T(n) = n \cdot \sum_{i=0}^{n}(1)$$

$$T(n) = n \cdot n = n^2$$

$$\Rightarrow T(n) \in \Theta(n^2)$$ ✓

Kian

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def Function (var, idx):

$\quad$ if idx == len(var) - 1

$\theta(c)$ { print(var)
return

For i in range (idx, len(var)):

n { $\theta(c) \leftarrow$ var = swap (var, idx, i)

Function (var, idx+1)

$\theta(c) \leftarrow$ var = swap (var, idx, i)

$\theta(c) + \theta(c) = \theta(k)$

$\Rightarrow n\,(\text{Function}(n-1) + k)$

def ~~Function~~ swap (var, i, j):

$\theta(c)$ { var = list(var)

var[i], var[j] = var[j], var[i];

return ''.join(var)

رابطه بازگشتی $\Rightarrow F(n)$ $\begin{cases} n \in \theta(1) & \text{if } n == 0 \\ \cancel{F(n-1)}+k & \\ n \cdot F(n-1) + k \cdot n & \text{if } n > 0 \end{cases}$

$\Rightarrow F(n) = n \cdot F(n-1) + kn$

$\Rightarrow F(n) = n! \;\; \rightarrow F(n) \in \Theta(n!)$ ✓

Kian