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# Flicker noise in accretion discs

Yu. E. Lyubarskii

[1]*Institute of Radio Astronomy, 4 Krasncznamyonnaya, 310002 Kharkov, Ukraine*
[2]*Goddard Space Flight Center, Greenbelt, MD 20771, USA*

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## ABSTRACT

An accretion disc with an $\alpha$ parameter fluctuating at different radii is considered. The relation between the $\alpha$ fluctuations and the resulting fluctuations of the accretion rate near the accreting body is obtained. Fluctuations of the accretion rate and, consequently, of the luminosity proved to be of the flicker-noise type provided that the characteristic time-scale of the $\alpha$ fluctuations is of the order of the local viscous time-scale, and that the fluctuations at different radial scales are uncorrelated.

**Key words:** accretion, accretion discs – X-rays: general.

## 1 INTRODUCTION

It is well-known that both Galactic and extragalactic accretion-powered X-ray sources exhibit random fluctuations in their flux (see review by van der Klis 1989). Fourier power spectra of such fluctuations are generally represented by a power law with power indices of 1–2 (flicker noise). Flicker noise is characterized by correlations extending over a wide range of time-scales and is a common phenomenon for systems in a critical state. A simple cellular-automation model of such critical behaviour was proposed by Bak, Tang & Wiesenfeld (1988). An attempt to apply this model to the accretion discs around black holes was made by Mineshige, Ouchi & Nishimori (1994). Takeuchi, Mineshige & Negoro (1995) extended this discrete model to include a gradual mass diffusion. Hoshino & Takeshima (1993) attributed flicker noise in X-ray pulsars to the density fluctuation spectra in magnetohydrodynamic turbulence. Nowak & Wagoner (1995) assumed that the low-frequency fluctuations could be caused by weak variations of the disc flux on local viscous time-scales, because variations at different radii occur at different time-scales. However, radiation from different radii gets into different spectral bands, therefore this model failed to reproduce the observed flicker noise when a bandpass was included.

Now it should be taken into account that any variation of disc parameters at some radius with a local viscous time-scale implies a variation in the accretion rate at smaller radii and, therefore, near the inner radius of the disc where most of the energy is released. Since the radiation observed in the X-ray band is generated in this zone, fluctuations in the observed flux are proportional to the fluctuations in the accretion rate in this zone. So variations of the disc parameters at different radii result in variations of the accretion rate in the energy release zone and, consequently, of the observed flux at different time-scales. An accretion disc with fluctuating parameters may be considered in the scope of the standard $\alpha$ theory (Shakura 1972; Shakura & Sunyaev 1973) if one assumes that $\alpha$ is fluctuating. The nature of these fluctuations is not considered here. One can assume, for example, that the fluctuating part is associated with the magnetic stresses which may be variable as a result of dynamo processes. Although the characteristic dynamo time-scale is short, in the saturated regime magnetic fields may vary at much larger time-scales (the Sun is an example). $\alpha$ fluctuations may be also caused by some disc instabilities. Thermal instability is not a good candidate because it develops only in the internal radiation-dominated zone of the disc (Shakura & Sunyaev 1976). More promising is the viscous instability with respect to the elongation of the orbits of particles (Lyubarskii, Postnov & Prokharov 1994). Local fluctuations of the eccentricities of particle orbits may result in necessary fluctuations of the effective viscosity.

The aim of this research is to show that the flicker noise in the accretion rate near the inner radius of the disc and, consequently, in the luminosity of the source, may be caused by the viscosity fluctuating independently at different radii of the disc on local viscous time-scales. The important point is that all the consideration is carried out in the scope of viscous hydrodynamics without any ad hoc discretization. Although the energy release near the inner radius of the disc possibly does occur in the form of separate flares (due to magnetic reconnection), the characteristic time-scale of the flicker noise exceeds the accretion time-scale in the energy-release zone of the disc. Therefore the flicker noise should be attributed to the processes at radii that are large compared to the inner radius of the disc. Any such processes may affect the luminosity of the disc only via variations of the accretion rate and should be considered as variations of the continuous flow.

## 2 PRELIMINARY CONSIDERATIONS

The accretion disc will be considered in the scope of the standard $\alpha$ prescription for viscosity (Shakura 1972; Shakura & Sunyaev 1973) assuming that $\alpha$ has a small fluctuating part

$$\alpha=\alpha_0[1+\beta(t, r)], \tag{1}$$

where $\beta \ll 1$. It is assumed that these fluctuations are independent at different radial scales, $\langle\beta(t, r_1)\beta(t, r_2)\rangle\to 0$ when $r_1/r_2\to 0$ or $r_1/r_2\to\infty$ and that the characteristic time-scale of these fluctuations is of the order of accretion time, $\tau_a(r)$, at the corresponding radius, $\langle\beta(t_1, r)\beta(t_2, r)\rangle\to 0$ at $|t_1-t_2| \gg \tau_a$. The accretion time may be estimated as (Shakura & Sunyaev 1973)

$$\tau_a=\left[\alpha\left(\frac{H}{r}\right)^2\Omega_K\right]^{-1}, \tag{2}$$

where $H$ is the disc width and $\Omega_K$ is the Keplerian angular velocity.

Small variations in $\alpha$ result in small variations in accretion rate,

$$\dot{M}=\dot{M}_0[1+\dot{m}(r, t)], \tag{3}$$

where $\dot{m} \ll 1$. In the non-steady case, the accretion rate depends in general on radius and time; however, if some variations occur at a radius $r_1$ at the time-scale $\tau_a(r_1)$, the disc is adjusted at radii $r \ll r_1$ and the accretion rate here proves to be dependent not on $r$ but only on time. The reason is that the characteristic accretion time-scale $\tau_a$ decreases with decreasing radius and any variations at large radii result, at small radii, in a quasi-stationary disc with accretion rate slowly varying with time. This was demonstrated by Lyubarskii & Shakura (1987) for the general non-linear evolution of the accretion disc; for the small fluctuations around a stationary state this effect will be demonstrated in the next section.

Let us consider first the accretion disc in which $\alpha$ varies near the points $r_1 \gg r_2 \gg \dots r_i \gg \dots$ independently with characteristic times $\tau_a(r_i)$. If $\alpha$ increases in some strip near $r_i$, the redistribution of the angular momentum increases in this strip. The local accretion rate then increases in the inner part of the strip, because larger angular momentum is transferred from the gas in the inner part to the gas in the outer part of the strip. The region of the increased accretion rate propagates toward the centre of the disc with increasing velocity because the smaller the radius, the larger the rate of the angular momentum redistribution. Therefore if $\alpha$ varies near $r_i$ with the time-scale $\tau_a(r_i)$, the disc at $r \ll r_i$ has a steady-state structure with the accretion rate slowly varying together with variations of $\alpha$ near $r_i$. Variations of $\alpha$ at $r_j \ll r_i$ produce variations in accretion rate relative to the new level established by variations at $r_i$. Hence variations of the accretion rate near the central body, where most energy is released, may be represented as a sum of independent variations produced at different radii,

$$\dot{m}=\sum_i \dot{m}_i.$$

Since different $\dot{m}_i$ vary at different time-scales, the luminosity of the source, which is proportional to the accretion rate near the central body, varies at a large range of time-scales.

The accretion rate may be roughly estimated as

$$\dot{M}\sim Ur^2/\tau_a,$$

where $U$ is the surface density in the disc. Variations in $\alpha$ at the time-scale of $\tau_a$ result in variation in the accretion rate,

$$\dot{m}\sim\beta.$$

So if amplitudes of $\alpha$ variations are the same at different radii then amplitudes of accretion rate variations are the same at different time-scales. This means that the power spectrum of the luminosity fluctuations has a power-law index of unity. Different indexes may be obtained if the amplitudes of $\alpha$ fluctuations change with radius. Now let us consider the problem in a more rigorous manner.

## 3 ACCRETION WITH A SLIGHTLY VARYING VISCOSITY PARAMETER $\alpha$

The disc accretion may be treated as a slow diffusion of angular momentum caused by shear stresses in differentially rotating gas (Lightman 1974; Lynden-Bell & Pringle 1974; Bath & Pringle 1981; Filipov 1984; Lyubarskii & Shakura 1987). The angular momentum balance can be written in the form

$$\dot{M}\frac{d\Omega_K r^2}{dr}=\frac{\partial}{\partial r}(2\pi W_{r\varphi}r^2), \tag{4}$$

where $W_{r\varphi}$ is the shear stress integrated over the disc width. Note that in the general case $\dot{M}$ is a function of time and radius and has the meaning of the total flux of material passing through radius $r$ towards the centre.

Making use of the continuity equation integrated over the disc width,

$$\frac{\partial U}{\partial t}=\frac{1}{2\pi r}\frac{\partial\dot{M}}{\partial r}, \tag{5}$$

one can immediately obtain a diffusion-type equation which describes the time evolution of the disc. In variables

$$h=\Omega_K r^2=\sqrt{GMr},$$

where $M$ is the mass of the central body, and

$$F=W_{r\varphi}r^2,$$

it takes a form

$$\frac{\partial U}{\partial t}=\frac{(GM)^2}{2h^3}\frac{\partial^2 F}{\partial h^2}. \tag{6}$$

The relationship between $U$ and $F$ may be found by making use of the $\alpha$-prescription for the viscous stresses and the local heat balance in the disc. In quite a general form it can be written as (Filipov 1984; Lyubarskii & Shakura 1987)

$$U=\frac{(GM)^2F^{1-m}}{2(1-m)Dh^{3-n}}, \tag{7}$$

where exponents $m$ and $n$ are determined by the properties of the gas in the disc. For a disc with Thomson scattering playing the predominant role in the opacity $m=2/5$, $n=6/5$; but if the opacity is determined by free–free and free–bound transitions $m=3/10$, $n=4/5$. The factor $D$ contains the mass of the central body, the universal constants and $\alpha$, namely $D\propto\alpha^{4/5}$.

Substitution of the expression (7) into equation (6) yields

$$\frac{\partial F}{\partial t}=\frac{DF^m}{h^n}\frac{\partial^2F}{\partial h^2}+\frac{F}{(1-m)D}\frac{\partial D}{\partial t}. \tag{8}$$

Since the $\alpha$ variations are small, one can consider small variations around a steady state,

$$D=D_0\left(1+\frac{4}{5}\beta\right),\qquad F=\frac{\dot{M}_0h}{2\pi}+\psi.$$

Linearizing equation (8) one can get

$$\frac{\partial\psi}{\partial t}=\frac{D_0}{h^{n-m}}\left(\frac{\dot{M}_0}{2\pi}\right)^m\frac{\partial^2\psi}{\partial h^2}+\frac{2\dot{M}_0h}{5\pi(1-m)}\frac{\partial\beta}{\partial t}. \tag{9}$$

This is a linear diffusion equation with a source term. It can be resolved immediately making use of the Green function (Lynden-Bell & Pringle 1974)

$$\psi(t,x)-\psi(0,x)=\frac{\dot{M}_0\kappa^2x^l}{5\pi(1-m)}\int_0^t \mathrm{d}s\int_0^\infty \mathrm{d}x_1$$

$$\times\frac{x_1^{l+1}}{t-s}\exp\left[-\frac{(x^2+x_1^2)\kappa^2}{4(t-s)}\right]I_l\left[\frac{\kappa^2xx_1}{2(t-s)}\right]\frac{\partial\beta(s,x_1)}{\partial s}.$$

Here

$$l=\frac{1}{2+n-m},\qquad x=h^{1/2l},$$

$$\frac{4}{(2+n-m)\kappa^2}=D_0\left(\frac{\dot{M}_0}{2\pi}\right)^m,$$

and $I_\nu(z)$ is the Bessel function of the imaginary argument. In the new variables, characteristic accretion time may be estimated as

$$\tau_a\sim(\kappa x)^2.$$

We are interested in the accretion rate; according to equation (4) it is proportional to the radial derivative of $F$, so one can find

$$\dot{m}(t,x)-\dot{m}(0,x)=\int_0^t\int_0^\infty G(t,x;s,x_1)\frac{\partial\beta(s,x_1)}{\partial s}\,\mathrm{d}s\,\mathrm{d}x_1; \tag{10}$$

$$G(t,x;s,x_1)=\frac{\kappa^4x^{(1-l)}x_1^{l+1}}{10l(1-m)(t-s)^2}\exp\left\{-\frac{(x^2+x_1^2)\kappa^2}{4(t-s)}\right\}$$

$$\times\left\{x_1I_{l-1}\left[\frac{\kappa^2xx_1}{2(t-s)}\right]-xI_l\left[\frac{\kappa^2xx_1}{2(t-s)}\right]\right\}.$$

Let, for example, $\beta$ be non-zero at $x_1\gg x$. The exponent in $G(t,x;s,x_1)$ suppresses the contribution of the region $t-s\ll(\kappa x_1)^2$ in the integral. Therefore the argument of the Bessel function is small and the expansion

$$I_\nu(z)=(z/2)^\nu \tag{11}$$

may be substituted into the integral. Then the $x$ dependence of $\dot{m}$ is cancelled to the first order in $x/x_1$. So we confirm the qualitative statement of the previous section about the independence of the accretion rate on radius in case disc parameters vary at large radii. However, now one can find the power spectrum of $\dot{M}$ fluctuations in quite a formal way.

## 4 POWER SPECTRUM OF THE FLUCTUATIONS OF THE ACCRETION RATE

To find the power spectrum of the fluctuations we need an appropriate squared average function of the fluctuating value. The most important such function is the covariance, which is directly linked to the power spectrum via the Fourier transformation. However, the covariance does not exist for fluctuations with power-law indexes larger than or equal to unity becasue low-frequency fluctuations contribute too much and the Fourier integral diverges. The most convenient function is the structure function (see, e.g., Rytov, Kravtsov & Tatarskii 1988) which is determined as

$$D(\tau,x)=\langle[\dot{m}(t+\tau,x)-\dot{m}(t,x)]^2\rangle. \tag{12}$$

For the stationary process one can evidently put $t=0$ in this definition. This function is linked to the power spectrum $p(f)$ by the transformations

$$D(\tau)=2\int_{-\infty}^{\infty}(1-\cos 2\pi f\tau)p(f)\,\mathrm{d}f; \tag{13}$$

$$p(f)=\frac{1}{(2\pi)^2f}\int_0^\infty \sin 2\pi\tau\frac{\mathrm{d}D}{\mathrm{d}\tau}\,\mathrm{d}\tau. \tag{14}$$

Now one can directly calculate the power spectrum of the accretion-rate fluctuations.

Squaring and averaging equation (10) yields

$$D(t,x)=\int_0^t\mathrm{d}s_1\int_0^t\mathrm{d}s_2\int_0^\infty\mathrm{d}x_1\int_0^\infty\mathrm{d}x_2$$

$$\times G(t,x;s_1,x_1)G(t,x;s_2,x_2)\frac{\partial^2K(s_1,s_2,x_1,x_2)}{\partial s_1\,\partial s_2}, \tag{15}$$

where

$$K(s_1,s_2,x_1,x_2)=\langle\beta(s_1,x_1)\beta(s_2,x_2)\rangle \tag{16}$$

is the covariance of $\alpha$ fluctuations. Now one can find fluctuations of the accretion rate and, consequently, of the luminosity for any fluctuations of the $\alpha$ parameter.

Let us assume that $\alpha$ fluctuations are de-correlated at different radial scales and the correlation time-scale at some radius is of the order of the viscous time-scale $\tau_a$ at this radius. Then one can represent $K$ in the form

$$K(s_1, s_2, x_1, x_2) = U\left(\frac{x_1}{x_2}\right) W\left[\frac{s_1 - s_2}{\kappa^2(x_1^2 + x_2^2)}\right], \tag{17}$$

where $U(z)\to 0$ at $z\to 0$ or $z\to\infty$ and $W(z)\to 0$ at $z\to\pm\infty$.

The power spectrum is expressed through the time derivative of $D$. Introducing new integration variables

$$z_1 = \frac{x_1}{x}; \qquad z_2 = \frac{x_2}{x}; \qquad y_1 = \frac{t - s_1}{(\kappa x)^2}; \qquad y_2 = \frac{t - s_2}{(\kappa x)^2}$$

and taking the derivative, one can obtain

$$\frac{\partial D}{\partial t} = -\left[\frac{\kappa x}{10 l t(1-m)}\right]^2 \int_0^{t/\kappa x^2} \frac{\mathrm{d}y_1}{y_1^2} \int_0^\infty \mathrm{d}z_1 \int_0^\infty \mathrm{d}z_2$$
$$\times \frac{(z_1 z_2)^{(1+l)}}{(z_1^2 + z_2^2)^2}\left[z_2 I_{l-1}\left(\frac{z_2 \kappa^2 x^2}{2t}\right) - I_l\left(\frac{z_2 \kappa^2 x^2}{2t}\right)\right]$$
$$\times \left[z_1 I_{l-1}\left(\frac{z_1}{2y_1}\right) - I_l\left(\frac{z_1}{2y_1}\right)\right] W''\left[\frac{t/(\kappa x)^2 - y_1}{z_1^2 + z_2^2}\right]$$
$$\times \exp\left[-\frac{z_1^2+1}{4y_1} - \frac{(z_2^2+1)\kappa^2 x^2}{4t}\right]\left[U\left(\frac{z_1}{z_2}\right) + U\left(\frac{z_2}{z_1}\right)\right]. \tag{18}$$

Note that $W''$ is negative so $(\partial D)/(\partial t)$ is positive.

Now let us consider the asymptotic form of the integral at times that are large compared to the viscous time-scale near the inner boundary of the disc, $t \gg (\kappa x)^2$. In this case the region $z_2 \sim \sqrt{t}/\kappa x \gg 1$ contributes predominantly to the integral. Function $U$ leaves only region $z_1 \sim z_2$ to contribute to the integral, therefore one can neglect unity as compared to both $z_1$ and $z_2$ in the exponent. The exponent also suppresses contribution from the region $y_1 \ll z_1^2$ to the integral, therefore arguments of the Bessel functions may be considered as small and expression (11) may be substituted. Then introducing new integration variables

$$p_1^2 = \frac{z_1^2}{y_1}; \qquad p_2^2 = \frac{(\kappa x z_2)^2}{t}; \qquad v = \frac{y_1(\kappa x)^2}{t},$$

one can get finally

$$\frac{\partial D}{\partial t} = \frac{A}{t}; \tag{19}$$

$$A = -\frac{2^{1-4l}}{5l^2(1-m)^2}\int_0^1 \mathrm{d}v \int_0^\infty \mathrm{d}p_1 \int_0^\infty \mathrm{d}p_2$$
$$\times \frac{(p_1 p_2)^{1+2l}}{(p_1^2 v + p_2^2)^2} W''\left(\frac{1-v}{vp_2^2 + p_1^2}\right)$$
$$\times \exp\left(-\frac{p_1^2+p_2^2}{4}\right)\left[U\left(\frac{p_1\sqrt{v}}{p_2}\right) + U\left(\frac{p_2}{p_1\sqrt{v}}\right)\right].$$

Note that the structure function is independent of the radius. Substitution into equation (14) immediately yields the power spectrum

$$p(f) = \frac{A}{8\pi f}. \tag{20}$$

Thus fluctuations of the $\alpha$ parameter with amplitude independent of the radius result in $1/f$ fluctuations of the luminosity.

More general flicker noise may be obtained if the amplitude of the $\alpha$ fluctuations depends on radius. Let this dependence be a power law

$$\sqrt{\langle\beta^2\rangle} \propto r^b. \tag{21}$$

Then one can take the covariance (16) in the form

$$K(s_1, s_2, x_1, x_2) = \left(\frac{x_1 x_2}{x_0^2}\right)^{4lb} U\left(\frac{x_1}{x_2}\right) W\left[\frac{s_1 - s_2}{\kappa^2(x_1^2 + x_2^2)}\right].$$

Substituting this expression into equation (15) and performing integrations in the same way as before yields

$$p(f) = \frac{A\Gamma(4lb)\sin 2l\pi b}{2\pi(2\pi f)^{1+4lb}},$$

where $\Gamma(x)$ is the Gamma function,

$$A = -\frac{2^{1-4l}}{5l^2(1-m)^2(x_0\kappa)^{8lb}}\int_0^1 \mathrm{d}v \int_0^\infty \mathrm{d}p_1 \int_0^\infty \mathrm{d}p_2$$
$$\times \frac{v^{2lb}(p_1 p_2)^{(1+2l)}}{(p_1^2 v + p_2^2)^2} \exp\left(-\frac{p_1^2+p_2^2}{4}\right)$$
$$\times \left[U\left(\frac{p_1\sqrt{v}}{p_2}\right) + U\left(\frac{p_2}{p_1\sqrt{v}}\right)\right] W''\left(\frac{1-v}{vp_1^2+p_2^2}\right).$$

Hence if the amplitude of the $\alpha$ fluctuations grows with radius, the power spectrum of the luminosity fluctuations has a slope larger than unity.

## 5 ADVECTION-DOMINATED DISCS

Let us now consider geometrically thick, optically thin accretion discs in which energy released as a result of viscosity is transported predominantly by the advection process (Spruit et al. 1987; Narayan & Yi 1994, 1995a,b; Abramowicz et al. 1995; Chen 1995). In such discs, accretion velocity is large and characteristic accretion time is

$$\tau_a = (\alpha\Omega_K)^{-1}, \tag{22}$$

therefore characteristic fluctuation times should be short. The flow in the geometrically thick disc is two-dimensional. Narayan & Yi (1995a) found the stationary self-similar solution of the form

$$V_\varphi = r\Omega_K\Omega(\theta), \qquad V_r = r\Omega_K\mathcal{V}(\theta), \qquad V_\theta = 0, \tag{23}$$

$$\rho = r^{-3/2}\mathcal{R}(\theta), \qquad P = \sqrt{r}\,\Omega_K^2\mathcal{P}(\theta), \tag{24}$$

where $\rho$ is the gas density and $P$ the gas pressure in the disc. Let us look for small fluctuations around this stationary flow.

The two-dimensional non-steady disc accretion is governed by the system of equations which is conveniently written in spherical polar coodinates. The continuity equation gives

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho V_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho V_\theta)=0. \tag{25}$$

The momentum equations are very complicated in general form; for the sake of simplicity let us assume that the viscosity parameter $\alpha$ is small, then the radial velocity $V_r$ is much less than the azimuthal velocity (however, it remains large as compared with the radial velocity in the standard disc). Let us also note that latitudinal velocity $V_\theta$ is zero in the unperturbed flow, therefore it should be small in our case. Then the radial and latitudinal momentum equations include only the balance of the centrifugal, gravitational and pressure forces, namely

$$\rho\frac{V_\varphi^2}{r}-\rho\Omega_K^2 r-\frac{\partial P}{\partial r}=0, \tag{26}$$

$$\cot\theta\rho V_\varphi^2-\frac{\partial P}{\partial\theta}=0. \tag{27}$$

The angular momentum balance equation gives

$$\rho\left(\frac{\partial V_\varphi}{\partial t}+V_r\frac{\partial V_\varphi}{\partial r}+\frac{V_rV_\varphi}{r}+\frac{V_\theta V_\varphi\cot\theta}{r}\right)$$
$$=\frac{\partial}{\partial r}\left[\eta r\frac{\partial}{\partial r}\left(\frac{V_\varphi}{r}\right)\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{\eta\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{V_\varphi}{\sin\theta}\right)\right]$$
$$+\frac{\eta}{r}\left[3r\frac{\partial}{\partial r}\left(\frac{V_\varphi}{r}\right)+\frac{2\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{V_\varphi}{\sin\theta}\right)\right]. \tag{28}$$

The energy balance equation gives

$$\rho\left(\frac{\partial E}{\partial t}-\frac{P}{\rho^2}\frac{\partial\rho}{\partial t}+V_r\frac{\partial E}{\partial r}-\frac{V_rP}{\rho^2}\frac{\partial\rho}{\partial r}+\frac{V_\theta}{r}\frac{\partial E}{\partial\theta}-\frac{V_\theta P}{r\rho^2}\frac{\partial\rho}{\partial\theta}\right)$$
$$=\eta\left\{\left[r\frac{\partial}{\partial r}\left(\frac{V_\varphi}{r}\right)\right]^2+\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{V_\varphi}{\sin\theta}\right)\right]^2\right\}. \tag{29}$$

The internal energy of the gas may be expressed as

$$E=\frac{1}{\gamma-1}\frac{P}{\rho},$$

where $\gamma$ is the ratio of specific heats. In the geometrically thick accretion disc, the $\alpha$-prescription for the viscosity may be introduced as

$$\eta=\frac{\alpha P}{\Omega_K}. \tag{30}$$

Now let us introduce small deviations of the disc parameters from the stationary parameters

$V_r=\alpha r\Omega_K\mathscr{V}(\theta)(1+v)$,

$V_\varphi=r\Omega_K\Omega(\theta)(1+\omega)$,

$V_\theta=\alpha r\Omega_K u$,

$P=\sqrt{r}\Omega_K^2\mathscr{P}(\theta)(1+p)$,

$\rho=r^{-3/2}\mathscr{R}(\theta)(1+\sigma)$.

Note that the function $\mathscr{V}(\theta)$ is redefined here, as compared to equation (23), to express $\alpha$-dependence explicitly.

Substituting these variables into the system (25–29) together with equation (1), one can get the linear equations. Making use of the Fourier transform,

$$y(r,f)=\int y(r,t)\mathrm{e}^{2\pi f t}\,\mathrm{d}t,$$

where $y$ represents any of the variables $v, w, u, p, b$, one can get

$$\mathscr{V}r\frac{\partial}{\partial r}(\sigma+v)+\frac{\partial u}{\partial\theta}-\frac{2\pi fi}{\alpha_0\Omega_K}\sigma=0, \tag{31}$$

$$\mathscr{P}r\frac{\partial p}{\partial r}-\frac{5}{2}p-2\Omega^2\omega+(1-\Omega^2)\sigma=0, \tag{32}$$

$$\mathscr{P}\frac{\partial p}{\partial\theta}+\frac{\mathrm{d}\mathscr{P}}{\mathrm{d}\theta}p-\mathscr{R}\Omega^2\cot\theta(2\omega+\sigma)=0, \tag{33}$$

$$\mathscr{P}r^2\frac{\partial^2\omega}{\partial r^2}-\frac{3}{2}\mathscr{P}r\frac{\partial p}{\partial r}-\mathscr{R}\mathscr{V}r\frac{\partial\omega}{\partial r}$$
$$+\frac{1}{\Omega}\frac{\partial}{\partial\theta}\left[\mathscr{P}\Omega\frac{\partial\omega}{\partial\theta}+(p+\omega)\mathscr{P}\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\Omega}{\sin\theta}\right)\right]$$
$$+\frac{2\cos\theta\mathscr{P}}{\Omega}\frac{\partial}{\partial\theta}\left(\frac{\Omega\omega}{\sin\theta}\right)+\left(\mathscr{R}\mathscr{V}-\frac{3}{4}\mathscr{P}+\frac{2\pi fi}{\alpha_0\Omega_K}\right)\omega$$
$$-\frac{\mathscr{R}\mathscr{V}}{2}(\sigma+v)-\cot\theta\mathscr{R}\mathscr{P}u\left[\frac{2\cos\theta}{\Omega}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\Omega}{\sin\theta}\right)-\frac{3}{4}\right]p$$
$$=\frac{3}{2}\mathscr{P}r\frac{\partial\beta}{\partial r}-\frac{1}{\Omega}\frac{\partial}{\partial\theta}\mathscr{P}\sin\theta\frac{\partial}{\partial\theta}\left(\frac{\Omega\beta}{\sin\theta}\right)$$
$$-\mathscr{P}\left[2\cos\theta\frac{\partial}{\partial\theta}\left(\frac{\Omega}{\sin\theta}\right)-\frac{9}{2}\Omega\right]\beta, \tag{34}$$

$$\frac{\mathscr{V}}{\gamma-1}r\frac{\partial}{\partial r}(p-\gamma\sigma)+3\Omega^2r\frac{\partial\omega}{\partial r}-2\sin^2\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\Omega}{\sin\theta}\right)\frac{\partial}{\partial\theta}\left(\frac{\Omega\omega}{\sin\theta}\right)$$
$$-\frac{9}{2}\Omega^2\omega+\frac{\mathscr{R}^2u}{(\gamma-1)\mathscr{P}}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathscr{P}}{\mathscr{R}^2}\right)-\frac{(5-3\gamma)\mathscr{V}}{\gamma-1}v$$
$$-\left[\frac{(5-3\gamma)\mathscr{V}}{\gamma-1}+\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\Omega}{\sin\theta}\right)^2+\frac{9}{4}\Omega^2-\frac{2\pi fi}{(\gamma-1)\alpha_0\Omega_K}\right]p$$
$$+\frac{2\pi f\gamma i}{(\gamma-1)\alpha_0\Omega_K}\sigma=\left\{\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\Omega}{\sin\theta}\right)\right]^2+\frac{9}{4}\Omega^2\right\}\beta. \tag{35}$$

The system of equations is very complicated; however, we do not need the explicit solution. The linearity implies that any linear combination of the variables may be generally represented in the form

$$y(r,\theta,f)=\int G(r,\theta;r_1,\theta_1;f)\beta(r_1,\theta_1,f)\,\mathrm{d}r_1\,\mathrm{d}\theta_1.$$

We are interested in the accretion rate

$$\dot{M}=-\int 2\pi r^2\sin\theta\rho V_{\mathrm{r}}\,\mathrm{d}\theta.$$

The Fourier transform of the small variation of the accretion rate around the steady state may be written as

$$\dot{m}(r,f)=-2\pi\sqrt{GM}\int\rho(\theta)V(\theta)[\sigma(r,\theta,f)+v(r,\theta,f)]\,\mathrm{d}\theta.$$

Note that the radial variable $r$ enters in the right-hand side of the system either in the form $r\,\partial/\partial r$ or in the expression $f/\alpha_0\Omega_{\mathrm{K}}$. Let us introduce the new radial variable

$$x=\left(\frac{f}{\alpha_0\Omega_{\mathrm{K}}}\right)^{2/3}=\left(\frac{f}{\alpha_0\sqrt{GM}}\right)^{2/3}r. \tag{36}$$

Then the left-hand side of the system (31–35) becomes independent of $f$, therefore the Green function of the system is also independent, in the new variable, of $f$. So variations of the accretion rate may be written in the form

$$\dot{m}(r,f)=\int G(x,\theta;x_1,\theta_1)\beta[(\alpha_0\sqrt{GM}f^{-1})^{2/3}x_1,\theta_1,f]\,\mathrm{d}\theta\,\mathrm{d}\theta_1\,\mathrm{d}x_1.$$

The Green function $G$ gives the Fourier transform of the acretion rate in case $\beta\propto\delta(r-r_1)e^{2\pi ift}$. We are interested in the accretion rate in the energy release zone caused by $\alpha$ fluctuations at large radii $r_1\gg r$ with characteristic times

$$f^{-1}\sim\tau_{\mathrm{a}}(r_1)=[\alpha\Omega_{\mathrm{K}}(r_1)]^{-1}\gg[\alpha\Omega_{\mathrm{K}}(r)]^{-1}. \tag{37}$$

Therefore the Green function should be the solution of the system of equations (31–35) with the right-hand side equal to zero; and in the left-hand side, the terms containing $f$ may be neglected since $f$ enters only in the expression $f/\alpha_0\Omega_{\mathrm{K}}(r)$ which is small according to the condition (37). Hence, in this case, the system (31–35) reduces to the stationary system, because terms with $f$ in the Fourier-transformed equations arose from terms with $\partial/\partial t$ in the initial system (25–29). The stationary system has a solution with a constant accretion rate, therefore the disc at small radii is adjusted to the slow fluctuations at large radii in such a way that the accretion rate becomes independent of the radius and may only slowly depend on time. Since the condition (37) corresponds to $x<<1$ this means that there is a finite limit $G(0,\theta;x_1,\theta_1)$, and the accretion rate in the energy-release zone may be presented in the form

$$\dot{m}(f)=\int G(0,\theta;x_1,\theta_1)\beta[(\alpha_0\sqrt{GM}f^{-1})^{2/3}x_1,\theta_1,f]\,\mathrm{d}\theta\,\mathrm{d}\theta_1\,\mathrm{d}x_1. \tag{38}$$

Now one can find the power spectrum of the fluctuations of the accretion rate via the power spectrum of $\alpha$ fluctuations by squaring equation (38) and making use of the general formula (see e.g. Rytov et al. 1988),

$$\langle y(f)y(f')\rangle=2\pi p_y(f)\delta(f+f'). \tag{39}$$

The power spectrum of $\alpha$ fluctuations may be found as the Fourier transform of the covariance. The latter may be presented in the form (cf. equation 17)

$$K(r_1,r_2,\theta_1,\theta_2,t_1,t_2)=U\left(\frac{r_1}{r_2},\theta_1,\theta_2\right)W\left[\frac{t_1-t_2}{\tau_{\mathrm{a}}(r_1)+\tau_{\mathrm{a}}(r_2)}\right].$$

where $U(z,\theta_1,\theta_2)\to0$ at $z\to0$ or $z\to\infty$ and $W(z)\to0$ at $z\to\pm\infty$ and $\tau_{\mathrm{a}}$ is given by equation (22). Performing a Fourier transform with respect to the time, one can get the power spectrum of $\alpha$ fluctuations in the form

$$p_\alpha(r_1,r_2,\theta_1,\theta_2,f)=U\left(\frac{r_1}{r_2},\theta_1,\theta_2\right)\times\tilde{W}\{[\tau_{\mathrm{a}}(r_1)+\tau_{\mathrm{a}}(r_2)]f\}[\tau_{\mathrm{a}}(r_1)+\tau_{\mathrm{a}}(r_2)], \tag{40}$$

where $\tilde{W}$ is the Fourier transform of the function $W$. Squaring and averaging equation (38), making use of formula (39) and substituting equation (40), in which $r$ should be substituted by $x$ according to equation (36), one can get finally

$$p(f)=\frac{A}{f},$$

$$A=\int G(0,\theta_1;x_1,\theta_3)G(0,\theta_2;x_2,\theta_4)U\left(\frac{x_1}{x_2},\theta_1,\theta_2\right)\times\tilde{W}(x_1^{3/2}+x_2^{3/2})(x_1^{3/2}+x_2^{3/2})\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3\,\mathrm{d}\theta_4\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

In the more general case of $\alpha$ fluctuations depending on radius, according to equation (21) one can easily get the power spectrum of the accretion-rate fluctuations in the form

$$p(f)\propto f^{-1-4/3b}. \tag{41}$$

## 6 DISCUSSION

The luminosity of the accretion disc is determined by the accretion rate near the accreting body. This part follows variations at larger radii if their time-scale is of the order of or larger than the viscous time-scale at these radii. Therefore the energy release zone 'feels' processes which occur at larger radii and the luminosity may vary on different time-scales. In the simplest case considered here, viscosity fluctuates independently at different radii and produces flicker noise fluctuations in the luminosity. The power spectrum of these fluctuations extends from $f_{\mathrm{min}}$ to $f_{\mathrm{max}}$, where $f_{\mathrm{min}}$ is of the order of the inverse accretion time near the outer radius of the disc and $f_{\mathrm{max}}$ is of the order of the inverse accretion time near the inner radius of the disc.

High-frequency flicker noise, with power spectra extended up to hundreds of Hz observed in the low state of black hole objects like Cyg X-1, may be considered as additional evidence that accretion discs in these systems are advection-dominated, because the accretion time-scale in such discs is short. The break in the flicker-noise spectra of these objects is observed near $f_{\min} \sim 0.1$ Hz (Pietsch et al. 1993; Vikhlinin et al. 1994). According to equation (22) the outer radius of the disc should be

$$r_{\rm out} \sim (\alpha\sqrt{GM} f_{\min}^{-1})^{2/3} = 2\times 10^{9}\left(\frac{\alpha}{0.3}\frac{0.1\ \rm Hz}{f_{\min}}\right)^{2/3}\left(\frac{M}{\rm M_{\odot}}\right)^{1/3}\ \rm cm.$$

The accretion time-scale (2) in the standard Shakura–Sunyaev discs is relatively large, therefore these discs should exhibit low-frequency noise such as that observed from low-mass X-ray binaries (see e.g. Hasinger & van der Klis 1989) or black hole objects in the high state (Miyamoto et al. 1993). These sources exhibit flicker-noise power spectra down to 1 mHz without turnover. It would be very interesting to investigate fluctuations at larger time-scales and find the turnover in the power spectrum. This may provide information about the outer parts of the disc.

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