EE511 Project:6 ZHANG Fan USC ID 1417-68-5115

Question 1:

Generate 1000 samples of the random variable $A = X + Y$ where $X \sim N(1,4)$ and $Y \sim N(2,9)$. Use the Box-Muller method to generate independent random samples from the component normal distributions. Use every sample generated by the Box Muller method (do not throw one of the pair away). Estimate the covariance between X and Y in your 1000 samples. Generate a histogram for A. Overlay the theoretical p.d.f. the histogram. Calculate the sample mean and sample variance for your samples and compare you estimates with the theoretical values.

The Polar Marsaglia method is an alternate method to generate samples from normal random variables. The method works by choosing a random point in the square $-1 < x < 1$ and $-1 < y < 1$ until $s = x^2 + y^2 < 1$ and then returning $R_1 = x\sqrt{\frac{-2\ln s}{s}}$ and $R_2 = y\sqrt{\frac{-2\ln s}{s}}$. Simulate 1,000,000 pairs of independent samples from a standard normal random variable using the Polar Marsaglia method. Compute the sample mean, sample variance, and covariance between the paired random samples. Repeat the experiment many times and compare the computational time required to generate 1,000,000 pairs of independent samples using the Polar Marsaglia method and the Box-Muller method.

Analysis:

- ### Box-muller method
  - it takes u1, u2, two independent uniformly distributed random variables on (0, 1)
  - defines X = sqrt(−2 log(u1) cos(2π*u2)), Y= sqrt(−2 log(u1) sin(2π*u2))It can be proved that X and Y are N(0, 1) random variables, and independent.
  - Scale them to a particular mean and variance: if Z ∼ N(0, 1) then X := μ + σZ ∼ N(μ, σ^2)
  - Calculate covariance, variance, and mean of x and y.
  - Generate histogram for A. Pay attention to scale the width and normalization.
  - Generate theoretical pdf. The theoretical pdf **A~N(M1+M2,var(V1+V2)=Var(V1)+Var(V2)+Cov(V1,V2))**
  - Compare with theoretical mean, variance, and overlay the figure.
- Then for the Marsaglia’s polar method,
  - Take u1, u2 from uniform distribution on (− 1, 1)
  - Accept if s = u1^2 + u2^2 < 1, otherwise get new u1, u2
  - Let X = sqrt(−2 log(s)/s)*u1, Y = sqrt(−2 log(s)/s)*u2
- Compare the time elapse.

Code for Box-muller method

```
tic;% Start the timer
clear all; close all; clc;

M1 = 1; % Mean of X
M2 = 2; % Mean of Y
V1 = 4; % Variance of X
V2 = 9; % Variance of Y

u1 = rand(1000000,1);
u2 = rand(1000000,1);

% Geberate X and Y that are N(0,1) random variables and independent
X = sqrt( - 2*log(u1)).*cos(2*pi*u2 );
```

```
Y = sqrt( - 2*log(u1)).*sin(2*pi*u2 );

% Scale them to a particular mean and variance
x = sqrt(V1)*X + M1; % x~ N(M1,V1)
y = sqrt(V2)*Y + M2; % y~ N(M2,V2)
toc; % Read elapsed time from stopwatch(Q1_6 compare the computational time)

A=x+y;  h=histogram(A,'Normalization','probability','BinWidth',1);hold on; %Q1_2, generate
histogram for A
NormX=-5:0.1:15;%Q1_3 overlay theoretical norm distribution plot
plot(NormX,normpdf(NormX,M1+M2,sqrt(V1+V2)));
Cov_XY=cov(x,y)%Q1_1: calculate the covariance of X and Y
Meanx=mean(x)%Q1_4 sample mean and var and compare
Meany=mean(y)
```

Code for the Marsaglia's polar method

```
tic;% Start the timer
clear all; close all; clc;

M1 = 1; % Mean of X
M2 = 2; % Mean of Y
V1 = 4; % Variance of X
V2 = 9; % Variance of Y
i = 0; % the random number generated by the algorithm

% Geberate X and Y that are N(0,1) random variables and independent
while(i<=999999)
    u1 = 2*rand()-1;
    u2 = 2*rand()-1;
    s = u1^2 + u2^2;
    if(s < 1)
        i = i + 1;
        X(i) = sqrt(-2*log(s)/s)*u1;
        Y(i) = sqrt(-2*log(s)/s)*u2;
    end
end
% Scale them to a particular mean and variance
x = sqrt(V1)*X + M1; % x~ N(M1,V1)
y = sqrt(V2)*Y + M2; % y~ N(M2,V2)
toc; % Read elapsed time from stopwatch,(Q1_6 compare the computational time)
cov(x,y)%Q_5 mean, variance, cov of x and y
meanx=mean(x)
meany=mean(y)
```

Results and comment:

- Box-muller method

Mean, covariance and variance of sample X and Y:

```
Cov_XY =

    4.1940    0.0288
    0.0288    9.2934
```

```
Meanx =

    0.9791

Meany =

    2.0736
```

Mean of sample x: 0.9791; theoretical mean of x: 1;

Mean of sample y: 2.0736; theoretical mean of y: 2;

Variance of sample x: 4.1940; theoretical variance of x: 4;

Variance of sample y: 9.2934; theoretical variance of y: 9;

Covariance of x and y: 0.0288≈0. Therefore, x and y is unrelated.

And since x and y is independent, so the theoretical covariance is 0. So the result is correct.

The histogram of A is as the following figure shows, as well as the theoretical pdf of A:

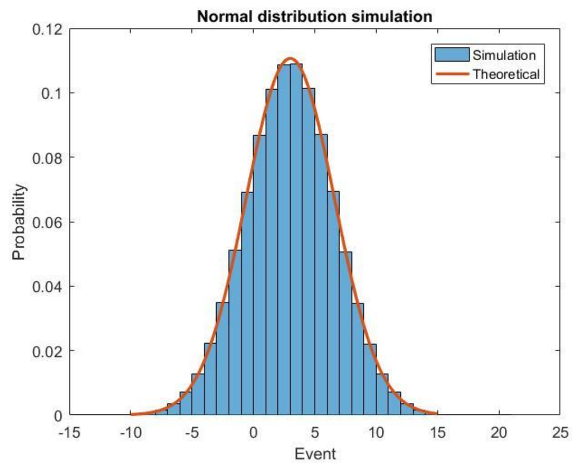

Figure 1: overlay of theoretical and simulation pdf of A

From the figure above shows we can see that the theoretical one and the histogram of sample A is very close.

- Marsaglia's polar method

```
ans =

    4.0060    0.0061
    0.0061    9.0066


meanx =

    0.9989


meany =

    1.9988
```

From the figure left we can see that:

Mean of sample x: 0.9989; compared to the theoretical mean of x: 1;

Mean of sample y: 1.9988; compared to the theoretical mean of y: 2;

Variance of sample x: 4.0060; compared to the theoretical variance of x: 4;

Variance of sample y: 9.0066; compared to the theoretical variance of y: 9;

Covariance of x and y: 0.0061≈0. Therefore, x and y is unrelated.

And since x and y is independent, so the theoretical covariance is 0. So the result is correct.

Plus, this results obviously are more accurate than those in Box-muller method, which are because the sample number is 1000000 compared to 1000 in Box-muller method.

- Compare the time elapse.

Box-muller method:

```
Elapsed time is 0.150685 seconds.
```

Marsaglia's polar method:

```
Elapsed time is 0.547500 seconds.
```

Therefore, the Box-muller method is much efficient than the Marsaglia's polar method.

Question 2:

Consider sampling from a $Gamma(\theta, 1)$ random variable. If θ is an integer then you can perform the sampling by summing θ different $Exp(1)$ random variables. But if θ is not an integer then its more difficult. Generate 1000 samples from $Gamma(5.5,1)$ using an accept-reject method (i.e. do not use built-in functions to generate from the gamma distribution). Generate a histogram and overlay the theoretical p.d.f. the histogram. Comment on the acceptance rate and your overall fit.

Analysis and algorithm:

The algorithm works as follows for $\mathbf{X} \sim \mathrm{Gamma}(\alpha, 1)$ for $\alpha \geq 1$:

1. Set $d = \alpha - 1/3$ and $c = 1/\sqrt{9d}$.
2. Generate $Z \sim \mathrm{N}(0,1)$ and $U \sim \mathrm{U}(0,1)$ independently.
3. If $Z > -1/c$ and $\log U < \frac{1}{2}Z^2 + d - dV + d \times lnV$, where $V = (1 + cZ)^3$, return $\mathbf{X} = d \times V$; otherwise, go back to Step 2.

Code:

```
clear
theta=5.5
d=theta-1/3;
c=1/sqrt(9*d);
i=0;
while(i<1000)
Z=normrnd(0,1);
U=rand();
V=(1+c*Z)^3;
    if(Z>-1/c && log10(U)<1/2*Z^2+d-d*V+d*log(V))
        X(i+1)=d*V;
        i=i+1;
    end
end
%theoretic gamma
histogram(X,'Normalization','probability','BinWidth',1)
hold on;
x=-5:0.1:15;
plot(x,gampdf(x,5.5,1),'LineWidth',2)
%set attributes of the figure
title('Gamma distribution simulation and theoretical curve')
xlabel('Event');
ylabel('Probability')
legend('Simulation','Theoretical')
```

Results:

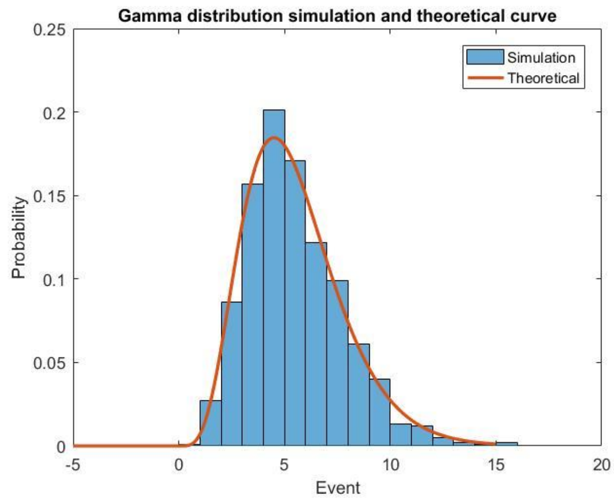

Figure 2: overlay of theoretical and simulation pdf of Gamma distribution

Comment

My acceptance rate is set according to relationship of the $\mathrm{N}{\sim}(0,1)$ and $\mathrm{Gamma}{\sim}(5.5,1)$, since normal distribution is a kind of special Gamma distribution. Therefore, we can calculate a value to multiply U~(0,1), and compare it with the original normal random variable. If N is larger than U, then accept it; or, reject it; at last, we return N.

The overall fit is close to the theoretical distribution but a kind of inaccurate when the event is near the Θ.

Question 3

The Chambers-Mallows-Stuck method describes how to generate samples from an arbitrary alpha stable distribution [1]. Use the Proposition 2.1 and Theorem 3.1 from the formulation by Weron [2] to sample from symmetric ($\beta = 0$) alpha stable pdfs using for four different values of α ($\alpha = 0.5$, 1, 1.8, 2.0).

For each value of α produce a histogram and a time series plot. Comment on the sample magnitude as a function of α. Use the Matlab function stblpdf.m (https://github.com/markveillette/stbl) to overlay the corresponding theoretical alpha-stable pdf on your histogram. Repeat the above procedure assuming a right-skewed alpha stable distribution with $\beta = 0.75$.

Introduction

The variable Z is usually called **a standard alpha-stable random variable** (but keep in mind the word "standard" depends on the choice of parameterization!). The alpha-stable distribution is a four-parameter family of distributions and is (usually) denoted by $S(\alpha, \beta, \gamma, \delta)$.

- The first parameter $\alpha \in (0, 2]$ is called the characteristic exponent, and describes the tail of the distribution.

- The second, $\beta \in [-1, 1]$ is the skewness, and as the name implies, specifies if the distribution is right- ($\beta > 0$) or left- ($\beta < 0$) skewed.
- The last two parameters are the scale, $\gamma > 0$, and the location $\delta \in R$. One can think of these two as being similar to the variance and mean in the normal distribution in the following sense - if $Z \sim S(\alpha, \beta, 1, 0)$, then if $\alpha = 1$, $\gamma Z + \delta \sim S(\alpha, \beta, \gamma, \delta)$.

  The family of alpha-stable distributions is a rich class, nd includes the following distributions as subclasses:

- For α = 2 the distribution reduces to a Gaussian distribution with variance σ2 = 2c2 and mean μ; the skewness parameter β has no effect.
- For α = 1 and β = 0 the distribution reduces to a Cauchy distribution with scale parameter c and shift parameter μ.
- For α = 1/2 and β = 1 the distribution reduces to a Lévy distribution with scale parameter c and shift parameter μ.

Simulation algorithm theory:

According to the paper 2,

$$\gamma \sim \mathrm{U}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \gamma_0 = -\frac{\frac{\pi}{2}\beta_2 \mathrm{K}(\alpha)}{\alpha}, \mathrm{W} \sim \exp(1)$$

for $\alpha \neq 1$, $\mathrm{S}_\alpha(1, \beta_2, 0)$:

$$\mathrm{X} = \frac{\sin\alpha(\gamma - \gamma_0)}{(\cos\gamma)^{\frac{1}{\alpha}}}\left(\cos\frac{\gamma - \alpha(\gamma - \gamma_0)}{W}\right)^{\frac{1-\alpha}{\alpha}}$$

for $\alpha = 1$, $\mathrm{S}_1(1, \beta_2, 0)$:

$$\mathrm{X} = \left(\frac{\pi}{2} + \beta_2\gamma\right)\tan\gamma - \beta_2 \log\left(\frac{W\cos\gamma}{\frac{\pi}{2} + \beta_2\gamma}\right)$$

Main code:

```
clear
t=-10:0.1:10;
alpha=[0.5,1,1.8,2.0];%set alpha
beta=0.75;%set beta
gama=1;%why 1?
for z=1:length(alpha)
    for i=1:10000
    X(i)=Sim(alpha(z),beta);
    end
subplot(2,2,z);
histogram(X,'Normalization','probability','Binwidth',1)
hold on;%above code obtain simulation histogram
T=stblpdf(t,alpha(z),beta,gama,0);%obtain theoretical values
plot(t,T,'LineWidth',2)
%set plot attributes
axis([-10 10 0 0.6])
title(['alpha=', num2str(alpha(z)),'    beta=',num2str(beta)])
xlabel('Event');
ylabel('Probability')
legend('Simulation','Theoretical')
```

```
end
```

Function code of simulation theory

```
function X=Sim(alpha,beta2)
    gama=pi*rand()-pi/2;
    W=exprnd(1);
    K=alpha-1+sign(1-alpha);
    gama0=-pi/2*beta2*K/alpha;
if alpha ~=1
    X=sin(alpha*(gama-gama0))/(cos(gama))^(1/alpha)*(cos(gama-alpha*(gama-gama0))/W)^((1-
alpha)/alpha);
    if X>10 X=11;
    elseif X<-10 X=-11;
        %some of too large or too small values are set to a fixed value so that when plot the
figure,
        %they will not influence the other value.
    end
elseif alpha==1
    X=(pi/2+beta2*gama)*tan(gama)-beta2*log10(W*cos(gama)/((pi/2)+beta2*gama));
end
```

Results:

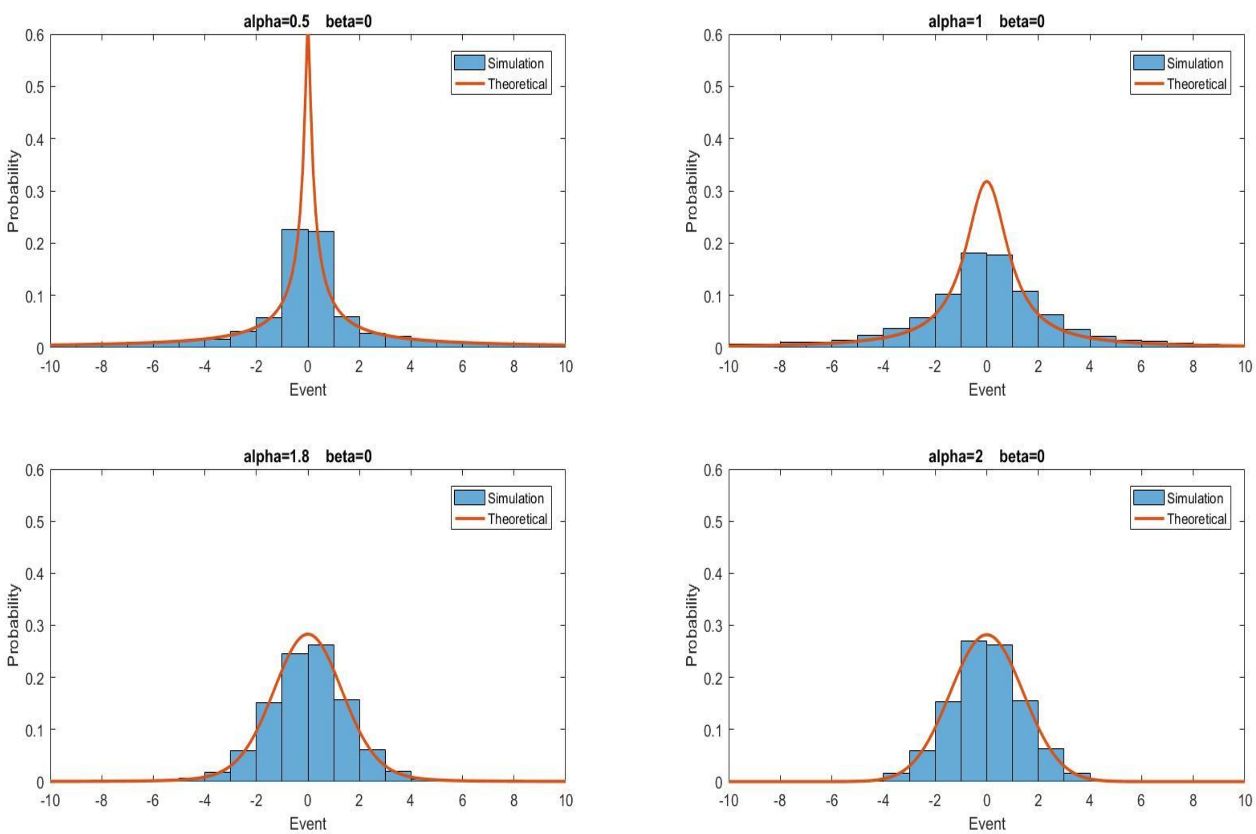

Figure 3: overlay of theoretical and simulation pdf of α-stable distribution ($\beta = 0$)

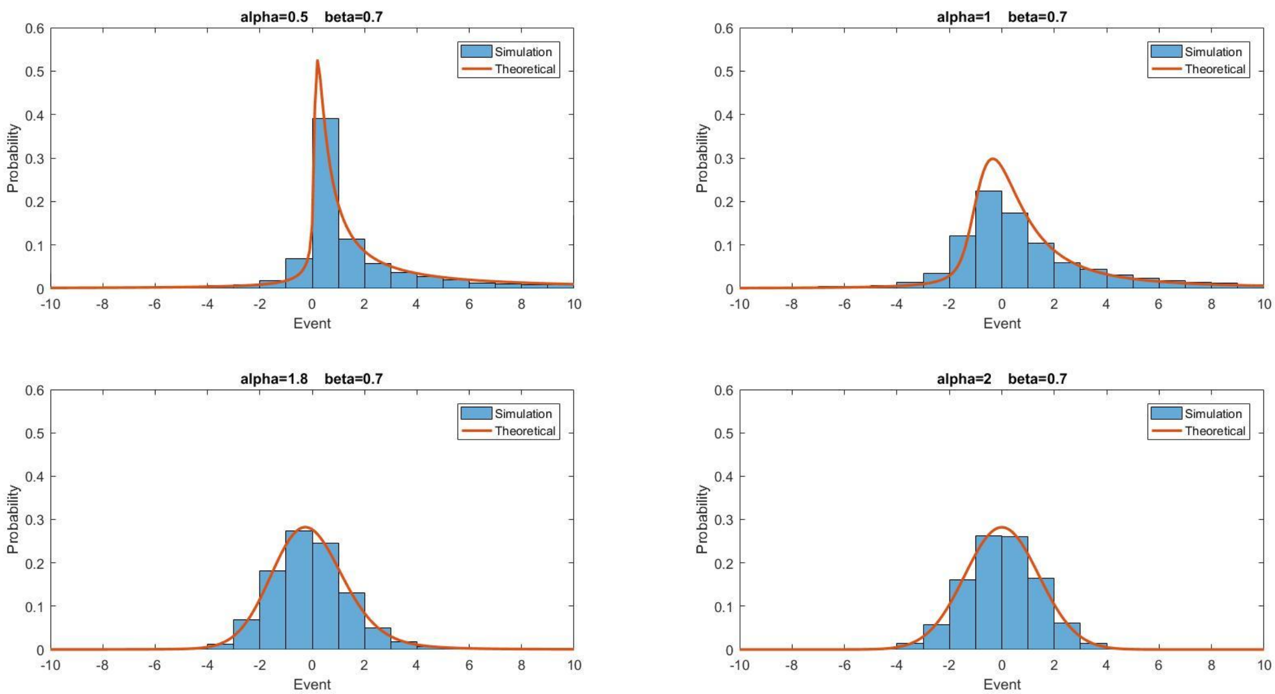

Figure 4: overlay of theoretical and simulation pdf of α-stable distribution ($\beta = 0.7$)

Comments

As we can see from the above two figures, we can draw some conclusions:

- when α<1, the probability of occurrence of too large or too small greatly increases.
- When $\beta = 0$, the figure is symmetric; while $\beta > 0$, the figure skews to right side; and while $\beta < 0$, the figure skews to left side.
- When $\alpha = 1$, the method of the paper will cause a relatively large error while $\alpha \neq 1$,
- the results are more accurate.