$$\sum_i \sum_j \pi_i \pi_j \, d^2(m_i, m_j) = \sum_i \sum_j \sum_n \pi_i \pi_j \underbrace{\pi_n \langle\!\langle x_n - \mu, x_i - x_j \rangle\!\rangle^2}$$

Start by looking
at this (next page)

with (come back to $\sum_j \sum_i \pi_i \pi_j$ prob.

$\langle\langle x_n - \mu, x_i - x_j \rangle\rangle \langle\langle x_n - \mu, x_i - x_j \rangle\rangle$

~~$\langle\langle x_n, x_i - x_j \rangle\rangle$~~

~~ed to ... $\langle\langle x_i - \mu, x_i - \mu \rangle\rangle \langle\langle x_i - \mu, x_j - \mu \rangle\rangle$~~

$$\sum_n \pi_n \left(\int_{T_p} (x_n^{(p)}(t_p) - \mu^{(p)}(t_p))(x_i^{(p)}(t_p) - x_j^{(p)}(t_p))\, dt_p \right)$$

$$\left(\int_{T_q} (x_n^{(q)}(s_q) - \mu^{(q)}(s_q))(x_i^{(q)}(s_q) - x_j^{(q)}(s_q))\, ds_q \right)$$

$$= \sum_n \pi_n \int_{T_p} \int_{T_q} (x_n^{(p)}(t_p) - \mu^{(p)}(t_p))(x_i^{(p)}(t_p) - x_j^{(p)}(t_p))$$

$$(x_n^{(q)}(s_q) - \mu^{(q)}(s_q))(x_i^{(q)}(s_q) - x_j^{(q)}(s_q))\, ds_q$$

$$= \int_{T_p} \int_{T_q} (x_i^{(p)}(t_p) - x_j^{(p)}(t_p))(x_i^{(q)}(s_q) - x_j^{(q)}(s_q)) \cancel{\sum_n \pi_n (x_n}$$

$$\sum_n \pi_n (x_n^{(p)}(t_p) - \mu^{(p)}(t))(x_n^{(q)}(s_q) - \mu^{(q)}(s_q))\, ds_q\, dt_q$$

$$= \underbrace{\int_{T_p} \int_{T_q} (x_i^{(p)}(t_p) - x_j^{(p)}(t_p))(x_i^{(q)}(s_q) - \mu^{(q)}(s_q))\, C_{pq}(t_p, s_q)}\, ds_q\, ds_p$$

now combine $\sum_j \sum_i \pi_i \pi_j$

Look at:

$$\sum_i \sum_j \pi_i \pi_j \left(x_i^{(p)}(t_p) - x_j^{(p)}(t_p) \right)\left(x_i^{(q)}(s_q) - x_j^{(q)}(s_q) \right)$$

$$= \sum_i \sum_j \pi_i \pi_j \left(x_i^{(p)}(t_p) x_i^{(q)}(s_q) - x_i^{(p)}(t_p) x_j^{(q)}(s_q) \right.$$
$$\left. x_j^{(p)}(t_p) x_i^{(q)}(s_q) + x_j^{(p)}(t_p) x_j^{(q)}(s_q) \right)$$

$$= \sum_i \pi_i x_i^{(p)}(t_p) x_i^{(q)}(s_q) - 2 \sum_i \sum_j \pi_i \pi_j x_i^{(p)}(t_p) x_j^{(q)}(s_q)$$
$$+ \sum_j \pi_j x_j^{(p)}(t_p) x_j^{(q)}(s_q)$$

$$= 2 \left(\sum_i \pi_i x_i^{(p)}(t_p) x_i^{(q)}(s_q) - \sum_i \sum_j \pi_i \pi_j x_i^{(p)}(t_p) x_j^{(q)}(s_q) \right)$$

$$= 2 \left(\sum_i \pi_i x_i^{(p)}(t_p) x^{(q)}(s_q) - \underbrace{\sum_i \pi_i x_i^{(p)}(t_p)}_{= u^{(p)}(t_p)} \underbrace{\sum_j \pi_j x^{(q)}(s_q)}_{} \right)$$

$$= \boxed{2 C_{p,q}(t_p, t_q)}$$

$\Rightarrow$ sub back in

$$\Rightarrow \sum_i \sum_j \sum_n \pi_i \pi_j \pi_n \langle\langle x_n - u, x_i - x_j \rangle\rangle^2$$

$$= 2 \iint_{T_0 \cup T_a} C_{pq}(s_q, t_p)\, ds_q\, dt_p$$ and is ok!