B Derivation of Eigencomponent.

$$\text{LHS} = \overbrace{\begin{pmatrix} X_1^{(1)} - \mu^{(1)} & \cdots & X_N^{(1)} - \mu^{(1)} \\ \vdots & \ddots & \vdots \\ X_1^{(p)} - \mu^{(p)} & \cdots & X_N^{(p)} - \mu^{(p)} \end{pmatrix}}^{X^T} \overbrace{\begin{pmatrix} M_{11} & \cdots & M_{1N} \\ \vdots & \ddots & \vdots \\ M_{N1} & & M_{NN} \end{pmatrix}}^{M}$$

$$\underbrace{\qquad\qquad}_{P \times N} \qquad \underbrace{\qquad\qquad}_{N \times N}$$

$$= \overbrace{\begin{pmatrix} \sum_{i=1}^{N} (X_i^{(1)} - \mu^{(1)}) M_{i1} & \cdots & \sum_{i=1}^{N} (x_i^{(1)} - \mu^{(1)}(t)) M_{iN} \\ \vdots & & \\ \sum_{i=1}^{N} (X_i^{(p)} - \mu^{(p)}) M_{i1} & \cdots & \sum_{i=1}^{N} (x_i^{(p)} - \mu^{(p)}) M_{iN} \end{pmatrix}}^{X^T M}$$

$$\underbrace{\qquad\qquad}_{P \times N}$$

$$\Rightarrow (X^T M)_{p,j} = \sum_{i=1}^{N} (x_i^{(p)} - \mu^{(p)}) M_{jp}$$

p, j th elmt.

Now

$$X^T M V_k =$$

$$= \underbrace{\begin{pmatrix} \sum_{i=1}^{N} (x_i^{(1)} - \mu_1^{(1)}) M_{i1} & \cdots & \sum_{i=1}^{N} (x_i^{(1)} - \mu^{(1)}) M_{iN} \\ \vdots & & \vdots \\ \sum_{i=1}^{N} (x_i^{(p)} - \mu^{(p)}) M_{i1}^{(p)} & \cdots & \sum_{i=1}^{N} (x_i^{(p)} - \mu^{(p)} M_{iN} \end{pmatrix}}_{P \times N} \underbrace{\begin{pmatrix} [V_k]_1 \\ \vdots \\ [V_k]_N \end{pmatrix}}_{N \times 1}$$

$$P \times 1)$$

$$= \underbrace{\begin{pmatrix} \sum_{j=1}^{N} (x_i^{(1)} - \mu^{(1)}) M_{ij} [V_k]_j \\ \vdots \\ \sum_{j=1}^{N} (x_i^{(p)} - \mu^{(p)}) M_{ij} [V_k]_j \end{pmatrix}}_{P \times 1}$$

$$\Rightarrow [X^T M V_k]_p = \sum_{j=1}^{N} \sum_{i=1}^{N} (x_i^{(p)} - \mu^{(p)}) M_{ij} [V_k]_j$$

Where $M_{ij} = \langle\langle x_i - \mu, x_j - \mu \rangle\rangle$,

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} [V_k]_j (x_i^{(p)} - \mu^{(p)}) \langle\langle x_i - \mu, x_j - \mu \rangle\rangle.$$

<u>RHS</u>

$$X^T V_k = l_k \underbrace{\begin{pmatrix} x_1^{(1)} - \mu^{(1)} & \cdots & x_N^{(1)} - \mu^{(1)} \\ \vdots & & \\ x_1^{(p)} - \mu^{(p)} & \cdots & x_N^{(p)} - \mu^{(p)} \end{pmatrix}}_{p \times N} \underbrace{\begin{pmatrix} [V_k]_1 \\ \vdots \\ [V_k]_N \end{pmatrix}}_{N \times 1}$$

$$= l_k \begin{pmatrix} \sum_{i=1}^{N} (x_i^{(1)} - \mu^{(1)}) [V_k]_i \\ \vdots \\ \sum_{i=1}^{N} (x_i^{(p)} - \mu^{(p)}) [V_k]_i \end{pmatrix}$$

$$\Rightarrow [l_k X^T V_k]_p = l_k \sum_{n=1}^{N} [V_k]_n (x_n^{(p)} - \mu^{(p)})$$

$\Rightarrow$ Eq^{n} 14 is correct.

Eq 15

$$[X^T M V_k]_p = \sum_{i=1}^{N}\sum_{j=1}^{N} [V_k]_j \left(X_i^{(p)}(\cdot) - \mu^{(p)}(\cdot)\right) \langle\langle X_i - \mu, X_j - \mu \rangle$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} [V_k]_j \left(X_i^{(p)}(\cdot) - \mu^{(p)}(\cdot)\right) \sum_{q=1}^{P} \int_{T_p} \left(X_i^{(q)}(s_q) - \mu^{(q)}(s_q)\right)\left(X_i^{(q)}(s_q) - \mu^{(q)}(s_q)\right) ds_q$$

$$\cancel{= \sum_{i=1}^{N}\sum_{j=1}^{N} [V_k]_j}$$

$$\cancel{\sum_{i=1}^{N}\sum_{j=1}^{N} [V_k]_j}$$

$$= \sum_{q=1}^{P} \int_{T_q} \overbrace{\sum_{i=1}^{N} \left(X_i^{(p)}(\cdot) - \mu^{(p)}(\cdot)\right)\left(X_i^{(q)}(s_q) - \mu^{(q)}(s_q)\right)}^{\text{if } \pi_i = 1/N \ \forall i \text{ then } = C_{pq}(\cdot, s_q)}$$

$$\sum_{j=1}^{N} [V_k]_j \left(X_j^{(q)}(s_q) - \mu^{(q)}(s_q)\right) ds_q$$

(if $\pi_i = 1/N \ \forall i$)

$$= \sum_{q=1}^{P} \int N\, C_{pq}(\cdot, s_q) \sum_{j=1}^{N} [V_k]_j \left(X_j^{(q)}(s_q) - \mu^{(q)}(s_q)\right) ds_q$$

$$= N \sum_{j=1}^{N} \langle\langle C_{p\cdot}(\cdot,\cdot), [V_k]_j (X_j - \mu) \rangle\rangle$$

$\Rightarrow$ Eq^{n} 15 seems ok only if $\pi_i = \frac{1}{N} \ \forall n$

If $M_{ij} = \pi_i \pi_j \langle\langle X_i - \mu, X_j - \mu \rangle\rangle$:

$$[X^T M V_k]_p = \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_i \pi_j [V_k]_j \left(X_i^{(p)}(\cdot) - \mu^{(p)}(\cdot) \right) \langle\langle X_i - \mu, X_j - \mu \rangle\rangle$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_i \pi_j [V_k]_j \left(X_i^{(p)}(\cdot) - \mu^{(p)}(\cdot) \right) \sum_{q=1}^{p} \int_{T_q} \left(X_i^{(q)}(s_q) - \mu^{(q)}(s_q) \right) \left(X_j^{(q)}(s_q) - \mu^{(q)}(s_q) \right) ds_q$$

$$= \sum_{q=1}^{p} \int_{T_q} \underbrace{\sum_{i=1}^{N} \pi_i \left(X_i^{(p)}(\cdot) - \mu^{(p)}(\cdot) \right) \left(X_i^{(q)}(s_q) - \mu^{(q)}(s_q) \right)}_{= C_{pq}(\cdot, s_q)} \sum_{j=1}^{N} [V_k]_j \pi_j \left(X_j^{(q)}(s_q) - \mu^{(q)}(s_q) \right) ds_q$$

$$= \sum_{q=1}^{p} \int_{T_q} C_{pq}(\cdot, s_q) \sum_{j=1}^{N} \pi_j [V_k]_j \left(X_j^{(q)}(s_q) - \mu^{(q)}(s_q) \right) ds_q$$

$$= \sum_{j=1}^{N} \pi_j \left(\sum_{q=1}^{p} \int C_{pq}(\cdot, s_q) \left(X_j^{(q)}(s_q) - \mu^{(q)}(s_q) \right) ds_q \right)$$

$$= \sum_{j=1}^{N} \langle\langle C_{p\cdot}(\cdot, \cdot), \pi_j [V_k]_j X_j - \mu \rangle\rangle$$