Name : Fadi Alahmad Alomar

ID : 1200180049

Assignment 2: Recurrences

Question 1

- $\because$ We are looking for $T(n,n)$ the recurrence equation becomes
  $T(n,n) = T(\frac{n}{2}, \frac{n}{2}) + \Theta(n)$ and $T(1,1) = \Theta(1)$
- Which can be further simplifed into
  $T'(n) = T'(\frac{n}{2}) + \Theta(n)$
- Then using the master theorem we get
  $b = 2,\ a = 1,\ f(n) = n$
  $c_{crit} = \log_b a = \log_2 1 = 0$
  $f(n) = n = \Omega(n^c)$ where $c = 1$
  $\because c > c_{crit}$
  $\therefore T'(n) = \Theta(n)$

Question 2

(a) $T(n) = 3T(\frac{n}{2}) + n^2$

- using the master theorem we get
  $b = 2,\ a = 3,\ f(n) = n^2$
  $c_{crit} = \log_b a = \log_2 3 = 1.585$
  $f(n) = n^2 = \Omega(n^c)$ where $c = 2$
  $\because c > c_{crit}$
  $\therefore T(n) = \Theta(n^2)$

(b) $T(n) = 4T(\frac{n}{2}) + n^2$

- using the master theorem we get
  $b = 2,\ a = 4,\ f(n) = n^2$
  $c_{crit} = \log_b a = \log_2 4 = 2$

$\because f(n) = n^2 = \Theta(n^{c_{crit}} \log^k n)$ where $k = 0$
$\therefore T(n) = \Theta(n^{c_{crit}} \log^{k+1} n) = \Theta(n^2 \log n)$

(c) $T(n) = T(\frac{n}{2}) + 2^n$

- using the master theorem we get
  $b = 2,\ a = 1,\ f(n) = 2^n$
  $c_{crit} = \log_b a = \log_2 1 = 0$
  $f(n) = 2^n = \Omega(n^c)$ where $c \geq 1$
  $\because c > c_{crit}$
  $\therefore T(n) = \Theta(f(n)) = \Theta(2^n)$

(d) $T(n) = 2^n T(\frac{n}{2}) + n^n$

- the master theorem can not be applied as a is not a constant

(e) $T(n) = 16T(\frac{n}{4}) + n$

- using the master theorem we get
  $b = 4,\ a = 16,\ f(n) = n$
  $c_{crit} = \log_b a = \log_4 16 = 2$
  $f(n) = n = O(n^c)$ where $c = 1$
  $\because c_{crit} > c$
  $\therefore T(n) = \Theta(n^{c_{crit}}) = \Theta(n^2)$

(f) $T(n) = 2T(\frac{n}{2}) + \frac{n}{\log n}$

- at first the master theorem seems to work and it is in the secound case as $c = c_{crit} = 1$ but $f(n) = \Theta(n^{c_{crit}} \log^k n)$ where $k = -1$ and k has the range $k \geq 0$ thus it is out of range.
  and $\frac{f(n)}{n^{\log_b a}} = \frac{n \log^{-1} n}{n} = \frac{1}{\log n}$ which is smaller than n^ϵ for any ϵ

(g) $T(n) = 2T(\frac{n}{4}) + n^{0.51}$

- using the master theorem we get
  $b = 4,\ a = 2,\ f(n) = n^{0.51}$
  $c_{crit} = \log_b a = \log_4 2 = 0.5$
  $f(n) = n^{0.51} = O(n^c)$ where $c = 0.51$
  $\because c > c_{crit}$
  $\therefore T(n) = \Theta(f(n)) = \Theta(n^{0.51})$

(h) $T(n) = 0.5T(\frac{n}{2}) + n^{-1}$

- master theorem does not apply as $a < 1$

(i) $T(n) = 16T(\frac{n}{4}) + n!$

- using the master theorem we get
  $b = 4,\ a = 16,\ f(n) = n!$
  $c_{crit} = \log_b a = \log_4 16 = 2$
  $f(n) = n! = \Omega(n^c)$ where $c \geq 1$
  $\because c > c_{crit}$
  $\therefore T(n) = \Theta(f(n)) = \Theta(n!)$

(j) $T(n) = \sqrt{2}T(\frac{n}{2}) + \log n$

- using the master theorem we get
  $b = 2,\ a = \sqrt{2},\ f(n) = \log n$
  $c_{crit} = \log_b a = \log_2 \sqrt{2} = 0.5$
  $f(n) = \log n = O(n^c)$ where $c = 0.4$
  $\because c_{crit} > c$
  $\therefore T(n) = O(n^{c_{crit}}) = O(n^{0.5})$

(k) $T(n) = 6T(\frac{n}{3}) + n^2 \log n$

- using the master theorem we get
  $b = 3,\ a = 6,\ f(n) = n^2 \log n$
  $c_{crit} = \log_b a = \log_3 6 = 1.631$
  $f(n) = n^2 \log n = \Omega(n^c)$ where $c = 2$
  $\because c > c_{crit}$
  $\therefore T(n) = \Theta(f(n)) = \Theta(n^2 \log n)$

(l) $T(n) = 4T(\frac{n}{2}) + \frac{n}{\log n}$

- using the master theorem we get
  $b = 2,\ a = 4,\ f(n) = \frac{n}{\log n}$
  $c_{crit} = \log_b a = \log_2 4 = 2$
  $f(n) = \frac{n}{\log n} = O(n^c)$ where $c = 1$
  $\because c_{crit} > c$
  $\therefore T(n) = O(n^{c_{crit}}) = O(n^2)$

(m) $T(n) = 64T(\frac{n}{8}) - n^2 \log n$

- the master theorem can not be applied as $f(n)$ is a negative value

(n) $T(n) = 7T(\frac{n}{3}) + n^2$

- using the master theorem we get
  $b = 3,\ a = 7,\ f(n) = n^2$
  $c_{crit} = \log_b a = \log_3 7 = 1.771$
  $f(n) = n^2 = \Omega(n^c)$ where $c = 2$
  $\because c > c_{crit}$
  $\therefore T(n) = \Theta(f(n)) = \Theta(n^2)$

(o) $T(n) = 4T(\frac{n}{2}) + \log n$

- using the master theorem we get
  $b = 2,\ a = 4,\ f(n) = \log n$
  $c_{crit} = \log_b a = \log_2 4 = 2$
  $f(n) = \log n = O(n^c)$ where $c = 1$
  $\because c_{crit} > c$
  $\therefore T(n) = O(n^{c_{crit}}) = O(n^2)$

(p) $T(n) = T(\frac{n}{2}) + n(2 - \cos n)$

- the master theorm should apply and use the third case but it does not as $f(n)$ cannot be a complexity function because it breaks the regularity condition where $af(\frac{n}{b}) \leq kf(n)$ where $k < 1$.
  if we take $k = \lim_{x \to 1, x \neq 1} x$ and put $n = 6$ we get $f(n) = 6.239,\ f(\frac{n}{2}) = 8.9699$

Question 3

- $\because$ We are looking for $T(n, n)$ the recurrence equation becomes
  $T(n, n) = S(n, \frac{n}{2}) + \Theta(n) = T(\frac{n}{2}, \frac{n}{2}) + \Theta(n) + \Theta(n)$
- Which can be further simplifed into
  $T'(n) = T'(\frac{n}{2}) + 2\Theta(n) = T'(\frac{n}{2}) + \Theta(n)$
- Then using the master theorem we get
  $b = 2,\ a = 1,\ f(n) = n$
  $c_{crit} = \log_b a = \log_2 1 = 0$
  $f(n) = n = \Omega(n^c)$ where $c = 1$
  $\because c > c_{crit}$
  $\therefore T'(n) = \Theta(n)$

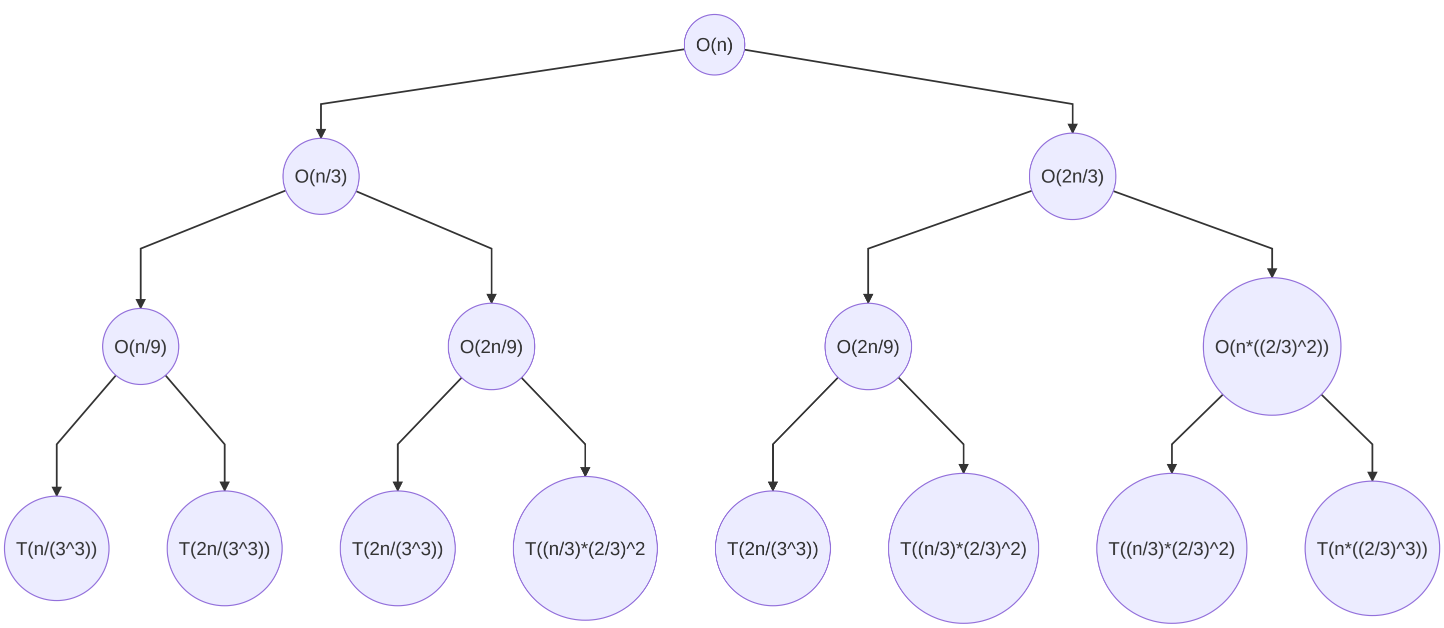
Quesstion 4

(a) $T(n) = 2T(\lfloor\sqrt{n}\rfloor) + \log n$
Let $n = 2^k$
$T(2^k) = 2T(2^{\frac{k}{2}}) + k$
Let $T(2^k) = S(m)$
$\therefore S(m) = 2S(m/2) + m$

- Then using the master theorem we get

$b = 2,\ a = 2,\ f(m) = m$
$c_{crit} = \log_b a = \log_2 2 = 1$
$f(m) = m = \Omega(m^c)$ where $c = 1$
$\because f(m) = m^1 = \Theta(m^{c_{crit}} \log^k m)$ where $k = 0$
$\therefore T(m) = \Theta(m^{c_{crit}} \log^{k+1} m) = \Theta(m \log m)$
substiting m by $\log n$ which is equal to 2^k we get
$T(n) = \Theta(\log n * \log(\log n))$

(b) $T(n) = T(\frac{n}{3}) + T(\frac{2n}{3}) + O(n)$



in every level we do exactly $O(n)$ work.
it can be easily shown from the tree that the longest branch that detemines it is height is the one that is far on the right.
the tree terminates when $T(1)$
$\therefore 1 = n * (\frac{2}{3})^i \rightsquigarrow i = \log_{\frac{3}{2}} n$
$\therefore T(n) = \sum_{i=0}^{\log_{\frac{3}{2}} n} n = n((\log_{\frac{3}{2}} n) + 1) = O(n \log n)$