Water Heater Agent

R. Pratt / Ver 1.1 / 2-5-2019

General Assumptions

1) All water heaters are assumed to be electric resistance (numbers of heat pumps or thermal solar systems are negligible with respect to the DSO+T study).
2) Water heaters are assumed to have upper and lower heating elements of equal power rating (4.5 kW), located at 2/3 of the way toward the top and very near the bottom of the tank, with a lockout control allowing only one element to operate at a given time with the upper element having priority (this is standard design for today's water heaters).
3) There is a presumption that a policy decision has been made whereby, via code or market transformation programs, each water heater in the service territory is equipped (at correspondingly modest cost to the consumer) with
   a) a passive, thermoelectric mixing valve with a constant, non-electronic setpoint set manually at installation time to the same temperature as the setpoint for the upper heating element of the WH
   b) a communicating controller with sensors for the water temperature in the tank at the height of the upper and lower elements with are installed on all new and replacement water heaters (these can be instead of or in addition to the normal thermoelectric switches)
   c) a hot water flow meter is not available (deemed cost prohibitive).
4) Ideally, the transactive agent design should apply whether there is a mixing valve or not, since near-term demonstrations will likely not be able to afford retrofits involving plumbing.
5) Control will be based on
   a) Setpoint for upper element thermostat (T_{set2}) will not be changed from the normal user setting for the WH, to ensure at least a minimal supply of fully-heated water is almost always available.
   b) When a mixing valve is present, and the WH is not in a demand response mode, the WH will be pre-heated to higher than normal temperature (T_{max}, i.e. Tset2 = Tset1 = by the day-ahead algorithm (Tset2, and sufficient fraction of cold water will be mixed (at T_{cold}) to deliver hot water for end use (at T_{hot}) at the regular setpoint for the upper element thermostat (, i.e., $T_{hot} = T_{set2}$).
   c) Demand response in the form of deferred consumption of electricity is achieved by setting the lower element thermostat (T_{set1}) to a temperature below its normal setting.
   d) , equal to a fixed minimum temperature (T_{min}, i.e., for any time t, $T_{set}1(t) = T_{min}$ or T_{set_upper}).
   e) Demand response is terminated be returning $T_{set\_lower(t)}$ to T_{set_upper}.

Water Schedule

[This section documents the water draw schedule used in the Excel spreadsheet prototyping the agent. Note that the water draw schedule definition used by GridLAB-D may differ.]

Assume there is at most one water draw n defined that occurs within any given simulation interval i.

Assume the water draw schedule is represented by a sequence of water draws (in gallons). Further, assume each water draw (n) is defined as having a start time (Start(n)), and end time (End(n), and a water draw volume (V_{draw}(n)). The water draw flow rate for interval n can then be computed as:

$\dot{V}(n) = V(n) / (End(n) – Start(n)$

Assume there is a simulation model of the water heater (see next section) that operates with a regular time interval (Δt) at each time step (i). Designate the starting and ending times of the interval as IntStart(i) and IntEnd(i), respectively.

Then, the simulation interval i includes water draw from schedule period n when any of three conditions (A, B, or C) exist:

A: schedule period n straddles the interval start time i, i.e., if:
IntStart(i) >= Start(n) <AND> IntStart(i) < End(n)
<OR>
B: schedule period n straddles interval end time i, i.e., if:
IntEnd(i) > Start(n) <AND> IntEnd(i) <= End(n)
<OR>
C: schedule period is entirely within interval i, i.e., if:
IntStart(i) < Start(n) <AND> IntEnd(i) > End(n)

The start of a water draw n that is <u>within interval i</u> (DrawStart(i)) is the maximum of the start time of the interval i and the start time of the water draw:

DrawStart(i) = MAX{ IntStart(i), Start(n) }

and the end of a water n that is <u>within interval i</u> (DrawEnd(i)) is the maximum of the start time of the interval i and the start time of the water draw:

DrawEnd(i) = MIN{ IntEnd(i), End(n) }

Under any condition, the water draw that occurs <u>within interval i</u> associated with water draw (n) is the product of the water draw flow rate and the duration of the draw that lies within interval i:

$V(i) = \dot{V}(n)$ (DrawEnd(i) - DrawStart(i))

Physical Model of Water Heater – 2-Layer Moving Thermocline

Assumptions. The day-ahead and real-time transactive agent(s) will assume the WH can be reasonably represented by modeling the position of a thermocline between two fully mixed layers of water in the tank, with no mixing between layers. Each layer is assumed to be fully mixed, and the temperature of water above the thermocline at the beginning of any interval i is T_{upper}(i) and below the thermocline is T_{lower}(i).

Further assume there is a single heating element at the bottom of the tank with a setpoint that may vary with time (T_{set}(i)).

[Some material for a double thermocline model here.] There are upper and lower heating elements located at know locations in the tank at x_{el_top} and x_{el_bottom}, respectively, with the latter equal to zero. Each has a setpoint than is allowed to vary over time ($T_{set\_top}[i]$ and $T_{set\_bottom}(i)$, respectively). In good practice the upper setpoint is always greater than or equal to the lower setpoint, but in practice this cannot always be assumed to be true, due to human or calibration errors.

The position of the thermocline at the beginning of any interval i (x(i)) is defined as the volume of the lower layer as a fraction of the total water volume of the water tank (i.e., for uniformly tubular or prismatic tanks, this is equal to the height of the top of the lower layer from the bottom of the tank).

The effect of heat loss from the upper layer of the tank is assumed to reduce the temperature of the layer.

The effect of heat loss from the lower layer of the tank is assumed to increase the position of the thermocline, since the basic assumption of the model is that the water in the tank is represented by two and only two fully-mixed layers.

Control of the Tank Heating Element. The power to the heating element is on at the <u>beginning</u> of an interval i ($P_{element}(i)$) if any one of four conditions exist:

IFF

1. the thermocline position is greater than zero, i.e.,
   $x(i) > 0$
2. a water draw begins simultaneously with beginning of the interval, i.e.,
   DrawStart(i) = IntStart(i)
3. the temperature of the upper layer is less that the setpoint minus the deadband, i.e.
   $T_{upper}(i) < T_{set}(i) – \Delta T_{band}$
4. the power to the heating element was on at the <u>end</u> of the previous time period, i.e., there was electrical energy consumed by the water heater (Q_{hw}) in the previous interval <u>and</u> the temperature of the upper layer at the end of the interval is less that the setpoint, i.e.,
   $Q_{hw}(i-1) > 0$ and $T_{upper}(i) < T_{set}(i)$

THEN

$P_{element}(i) = P_{hw}$

ELSE

$P_{element}(i) = 0$

Tank Heat Losses. The portion of the tank heat loss coefficient allocated to each layer of the tank to the ambient is assumed to be proportional to their share of each layer's volume to the total tank volume. Over the simulation interval beginning at i and ending at i+1, the heat loss from each layer is the product of the heat loss coefficient of the tank (UA_{tank}), the fraction of the tank represented by the thermocline at x(i), the temperature difference between the layer and the ambient air, and the duration of the interval. These relationships are expressed as:

$$Q_{loss\_upper}(i) = \dot{Q}_{loss\_upper}(i)\,\Delta t = [1 – x(i)]\,UA_{tank}\,[T_{upper}(i) – T_{amb}(i)]\,\Delta t$$

and

$$Q_{loss\_lower}(i) = \dot{Q}_{loss\_lower}(i)\, \Delta t = x(i)\, UA_{tank}\, [T_{lower}(i) - T_{amb}(i)]\, \Delta t$$

Note the above equations imply that, if the thermocline is at zero ($x(i) = 0$), there is zero heat loss from the lower layer, and under this condition the heat loss from the upper layer is equal to the entire heat loss from the tank (i.e., the entire volume of the tank is in the upper layer). Similarly, if the thermocline is at one ($x(i)=1$), then there is zero heat loss from the upper tank.

Operation of a Mixing Valve When Present. A thermal mixing valve automatically mixes cold water with the hot water supplied from the tank (V_{hw}) to maintain the temperature of the hot water delivered to the user (V) at a temperature equal to (or less, if the hot water tank is depleted) below the mixing valve's setpoint temperature (T_{hot}).

The operation of a mixing valve is portrayed in Figure 1, where the temperatures and flow rates of the mixed water delivered to the end use, the hot water supplied from the tank, and the cold water are $\dot{V}$, $\dot{V}_{hw}$, and $\dot{V}_{cold}$, and T_{mix}, T_{hw}, and T_{cold}, respectively.

Conservation of mass for a control volume around the mixing valve requires:

$$\dot{V} = \dot{V}_{hw} + \dot{V}_{cold}$$

The mixing fraction (f) is defined as:

$$f = \dot{V}_{hw} / \dot{V}$$

Note that, given conservation of mass, the mixing fraction is constrained to have values with the range of 0 and 1.

Conservation of energy in the form of sensible heat for a control volume around the mixing valve requires:

$$\rho\, c_p\, \dot{V}\, T_{mix} = \rho\, c_p\, \dot{V}\, T_{hw}\, \dot{V}_{hw} + \rho\, c_p\, \dot{V}_{cold}\, T_{cold}$$

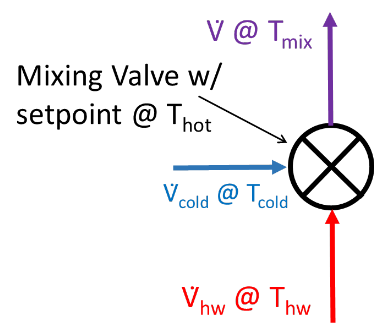

Figure 1. Mixing Valve Operation

Using the conservation of mass to substitute for the volumetric flow rate of the cold water ($\dot{V}_{cold}$), the definition of the mixing fraction (f) can be expressed as:

$$f = (T_{mix} - T_{cold}] \;/\; (T_{hw} - T_{cold})$$

If the mixing valve setpoint (T_{hot}) is greater than the temperature of the water being supplied from the tank, the valve will open completely and the mixed water supplied to the end use will be at the temperature of the water being supplied from the tank:

$$f = (T_{hot} - T_{cold}) / (T_{hw} - T_{cold}) \qquad \text{IFF } T_{hot} \le T_{hw}$$

$$f = 1 \qquad \text{IFF } T_{hot} > T_{hw}$$

Over an interval i, the volume of water draw (V(i)) <u>may</u> exceed the volume of the upper layer of the tank, given the ability of the heating element to reheat the water, including heat losses from the lower layer of the tank.

Assuming a two-layer/thermocline model of the tank, and the general case of a water draw that starts <u>after</u> the start of an interval i, there are as many as two distinct periods in the interval with different, mixing fractions: that which can be supplied by the upper layer, and the remainder that must be supplied by the lower layer.

When the thermocline position is less than or equal to one, the temperature of the hot water supplied from the tank is T_{upper} and mixing fraction is

$$f_{upper}(i) = (T_{hot} - T_{cold}] / (T_{upper} - T_{cold}) \qquad \text{IFF } T_{hot} <= T_{upper}(i)$$

$$f_{upper}(i) = 1 \qquad \text{IFF } T_{hot} > T_{upper}(i)$$

and when the thermocline position reaches one, the temperature of the hot water supplied from the tank is T_{lower} and mixing fraction is

$$f_{lower}(i) = (T_{hot} - T_{cold}] / (T_{lower} - T_{cold}) \qquad \text{IFF } T_{hot} <= T_{lower}(i)$$

$$f_{lower}(i) = 1 \qquad \text{IFF } T_{hot} > T_{lower}(i)$$

The volume supplied by the water heater over the time interval i is dependent on the relative proportions of the draw supplied at mixing fractions $f_{upper}(i)$ and $f_{lower}(i)$.

Rate of Change of Thermocline Position During a Water Draw. At any time during a water draw, the rate of change of the thermocline is the sum of the rates of change due to a) the water draw (dx/dt_{draw}), b) losses from the lower layer of the tank (dx/dt_{loss_lower}), and c) power input from the heating element ($dx/dt_{element}$):

$$dx/dt = dx/dt_{draw} + dx/dt_{loss\_lower} + dx/dt_{element}$$

The rate of change of the thermocline position due to the water supplied from the tank is:

$$dx/dt_{draw} = \Delta x/\Delta t = (\dot{V}_{hw}\, dt / V_{tank}) / dt = \dot{V}_{hw} / V_{tank} = f\, \dot{V} / V_{tank}$$

The rate of change of the thermocline position due to losses from the lower layer of the tank is determined from the expression

$$\dot{Q}_{loss\_lower} = \rho\, c_p\, dx/dt_{loss\_lower}\; V_{tank}\, (T_{upper} - T_{lower})$$

So the rate of change of the thermocline position due to the losses from the lower layer of the tank is:

$$dx/dt_{loss\_lower} = \dot{Q}_{loss\_lower} / [\rho\, c_p\, V_{tank}\, (T_{upper} - T_{lower})]$$

The rate of change of the thermocline position due power input from the heating element is determined from the expression

$$P_{element} = -\rho\, c_p\, dx/dt_{element}\, V_{tank}\, (T_{upper} - T_{lower})$$

So the rate of change of the thermocline position due to the operation of the heating element is:

$$dx/dt_{element} = -P_{element} / [\rho\, c_p\, V_{tank}\, (T_{upper} - T_{lower})]$$

The total rate of change of the thermocline due to all three effects is:

$$dx/dt = f\, \dot{V} / V_{tank} + (\dot{Q}_{loss\_lower} - P_{element}) / [\rho\, c_p\, V_{tank}\, (T_{upper} - T_{lower})]$$

Adjustment of Water Draw When a Mixing Valve is Present. At the time the water draw in interval i starts (DrawStart(i)), the thermocline position is

$$x_{start}(i) = x(i) + dx/dt\ (DrawStart(i) – IntStart(i))$$

or

$$x_{start}(i) = x(i) + [\dot{Q}_{loss\_lower}(i) – P_{element}(i)]\ [DrawStart(i) – IntStart(i)] / \{\rho\ c_p\ V_{tank}\ [T_{upper}(i) – T_{lower}(i)]\}$$

The maximum elapsed time from the start of the water draw over which the draw can be supplied from the upper layer ($\Delta t_{upper}(i)$) is equal to the elapsed time for the thermocline to reach one after the water draw starts. This is expressed in the relationship

$$1 = x_{start}(i) + dx/dt|_{upper}(i)\ \Delta t_{upper}(i)$$

The power to the heating element turns on at the same time the draw starts. So, the rate of change of the thermocline during the part of the draw supplied from the upper layer ($dx/dt|_{upper}(i)$) is:

$$dx/dt|_{upper}(i) = f_{upper}(i)\ \dot{V}(i) / V_{tank} + [\dot{Q}_{loss\_lower}(i) – P_{hw}(i)] / \{\rho\ C_p\ V_{tank}\ [T_{upper}(i) – T_{lower}(i)]\}$$

So, the elapsed time from the start of the water draw over which the draw is supplied from the upper layer ($\Delta t_{upper}(i)$) during the interval is the minimum of the duration of the water draw and the elapsed time from the start of the water draw to the time when the thermocline position reaches one:

$$\Delta t_{upper}(i) = MIN\{ DrawEnd(i) – DrawStart(i),\ [1 – x_{start}(i)] / dx/dt|_{upper}(i) \} \qquad \text{IFF } dx/dt|_{upper}(i) > 0$$

$$\Delta t_{upper}(i) = DrawEnd(i) – DrawStart(i) \qquad \text{IFF } dx/dt|_{upper}(i) <= 0$$

and the elapsed time from the start of the water draw over which the draw is supplied from the lower layer ($\Delta t_{lower}(i)$) during the interval is:

$$\Delta t_{lower}(i) = DrawEnd(i) – DrawStart(i) – \Delta t_{upper}(i)$$

The volume of the hot water supplied in interval i ($V_{draw}(i)$) is the sum of that supplied from the upper and lower layers of the tank:

$$V_{draw}(i) = f_{upper}(i)\ \dot{V}(i)\ \Delta t_{upper}(i) + f_{lower}(i)\ \dot{V}(i)\ \Delta t_{lower}(i)$$

and the change in the thermocline position due to the water draw (without losses or heat from the heating element) is the ratio of the draw from the upper layer during the interval ($V_{draw}(i)$) to the volume of the tank:

$$\Delta x_{draw}(i) = f_{upper}(i)\ V_{draw}(i) / V_{tank}$$

The energy of the hot water draw in interval i ($Q_{draw}(i)$) is the sum of that supplied from the upper and lower layers of the tank:

$$Q_{draw}(i) = \rho\ c_p\ f_{upper}(i)\ \dot{V}(i)\ [T_{upper}(i) – T_{cold}]\ \Delta t_{upper}(i) + \rho\ c_p\ f_{lower}(i)\ \dot{V}(i)\ [T_{lower}(i) – T_{cold}]\ \Delta t_{lower}(i)$$

(Q_{draw} is not used in the water heater model other than as a check on the overall heat balance.)

Energy Required to Reheat the Tank to the Setpoint. The heating element operating at full power (P_{hw}) for the duration of the entire interval Δt may not be able to supply the energy needed to reheat both layers by the end of interval i. Assume that the addition of energy from the heating element first lowers the position of the thermocline to zero, i.e., heats the lower layer until its temperature (T_{lower}) reaches the temperature of the upper layer (T_{upper}), then heats the water in the tank from T_{upper} to the setpoint (T_{set}).

The energy required to reheat the lower tank to the temperature of the upper layer ($T_{upper}(i)$), i.e., such that the thermocline position equals zero, including losses from the lower layer and any water draw during the interval i, is:

$$Q_{reheat\_lower}(i) = \rho\, c_p\, [x(i) + \Delta x_{draw}(i)]\, V_{tank}\, [T_{upper}(i) - T_{lower}(i)] + Q_{loss\_lower}(i)$$

The energy required to reheat the <u>entire</u> tank ($Q_{reheat\_upper}(i)$) from $T_{upper}(i)$ to the setpoint ($T_{set}(i)$, including losses from the upper layer during interval i, is:

$$Q_{reheat\_upper}(i) = \rho\, c_p\, V_{tank}\, [T_{set}(i) - T_{upper}(i)] + Q_{loss\_upper}(i)$$

(Note the losses from the lower layer were included in Q_{reheat_lower}.)

Maximum Potential Duration of Heating Element Operation in an Interval. The heat energy from the heating element is assumed to be delivered at constant rated power (P_{hw}) over the time power to it is on. If power to the heating element is on at the beginning of interval i (i.e., $P_{element}(i) > 0$), the power could potentially stay on for the entire period if needed. If the power is off at the beginning of the period, it is assumed will not turn on within the period unless a water draw occurs, in which case the longest the power could stay on is the difference between the times at end of the interval and the start of the water draw.

So, the maximum potential time of heating element operation during interval i ($\Delta t_{max}(i)$) is:

```
IF P_element(i) > 0
        Δt_max(i) = Δt
ELSE
        IF V(i) > 0
                Δt_max(i) = IntEnd(i) – DrawStart(i)
        ELSE
                Δt_max(i) = 0
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State of Water Heater Tank at the End of an Interval. The energy delivered to the lower layer of the tank during the interval i ($Q_{lower}(i)$) is:

$$Q_{lower}(i) = \mathrm{MIN}\{ P_{hw}\, \Delta t_{max},\ Q_{reheat\_lower}(i) \}$$

The position of the thermocline at the end of the interval i (x(i+1)) is then expressed by:

$$Q_{lower}(i) - Q_{loss\_lower}(i) = \rho\, c_p\, [x(i) + \Delta x_{draw}(i) - x(i+1)]\, V_{tank}\, [T_{upper}(i) - T_{lower}(i)]$$

Or, since the thermocline position is limited to 1, even when the water supplied by the tank is greater than the volume of the upper layer:

$$x(i+1) = \mathrm{MIN}\{\ 1,\ x(i)] + \Delta x_{draw}(i) - [Q_{lower}(i) - Q_{loss\_lower}(i)] / \{\rho\, c_p\, V_{tank}\, [T_{upper}(i) - T_{lower}(i)]\}\ \}$$

The energy delivered to the upper layer ($Q_{upper}(i)$) is expressed as:

$$Q_{lower}(i) + Q_{upper}(i) = \mathrm{MIN}\{ P_{hw}\, \Delta t,\ Q_{reheat\_lower}(i) + Q_{reheat\_upper}(i) \}$$

or

$$Q_{upper}(i) = \mathrm{MIN}\{ P_{hw}\, \Delta t_{max},\ Q_{reheat\_lower}(i) + Q_{reheat\_upper}(i) \} - Q_{lower}(i)$$

The resulting temperature of the upper layer at the end of the interval i ($T_{upper}(i+1)$) is related to the energy delivered to the upper layer over the interval i by the relationship:

$$Q_{upper}(i) - Q_{loss\_upper}(i) = \rho\, c_p\, [1 - x(i+1)]\, V_{tank}\, [T_{upper}(i+1) - T_{upper}(i)]$$

or, since whenever in the sequence of operations $Q_{upper}(i)$ is greater than zero the thermocline position has been reduced to zero first, x(i+1) = 0 and:

$$T_{upper}(i+1) = T_{upper}(i) + [Q_{upper}(i) - Q_{loss\_upper}(i)] / \{ \rho\, c_p\, [1 - x(i+1)]\, V_{tank} \}$$

Energy Consumption During an Interval. The electrical energy consumed during the end of interval i ($Q_{hw}(i)$) is the lesser of the energy of the water draw plus the energy to reheat the tank and the maximum deliverable energy:

$$Q_{hw}(i)) = Q_{lower}(i) + Q_{upper}(i)$$

Physical Model of Water Heater – 2-Node Water Heater Model

Assumptions. Assume a water heater can be reasonably represented by two such nodes, one for the upper portion of the tank from the upper heating element to the top, and one for the lower portion of the tank from the bottom to just below the upper heating element. These will be denoted by subscripts 1 (lower) and 2 (upper), as necessary. Assume each node is always fully mixed, and therefore has a uniform temperature (T_n). Assume each heating element has a setpoint that may vary with time ($T_{set\_n}(i)$)

The water heater tank is assumed to be uniformly tubular or prismatic, and the vertical position of the upper heating element (x_2) is defined in terms of the fraction of its height to the total height of the tank. It's position also defines the boundary between the two nodes.

Thus, the volume of the lower node of the tank (V_1) is:

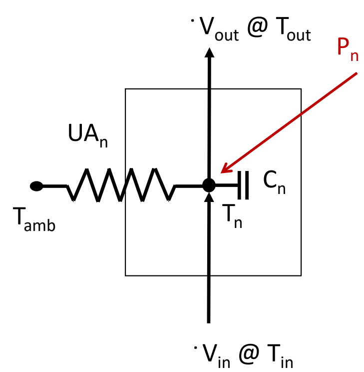
$$V_1 = x_1 V_{tank}$$

And the volume of the upper node of the tank (V_2) is:

$$V_2 = (1 - x_2) V_{tank}$$

For a generic, stationary, fixed-volume water heater node (See Figure 1), conservation of mass (for an incompressible liquid) requires the volumetric flow rates all be equal:

$$\dot{V}_{hw} = \dot{V}_{in} = \dot{V}_{out}$$



Defining the thermal mass of the node (C_n) as the product of the density of water, its specific heat, and the volume of the node (V_n):

$$C_n = \rho\, c_p\, V_n$$

and the (sensible) energy balance on the node is:

$$P_n + \rho\, c_p \dot{V}_{hw} T_{in} = \rho\, c_p \dot{V}_{hw} T_n + \rho\, c_p V_n\, dT_n/dt + UA_n (T_n - T_{amb})$$

where P_n is the power to the node from a heating element.

Differential Equation Solution or Node Temperature. The heat balance can be rearranged as a 1st-order ordinary differential equation describing the node temperature into the form:

$$0 = dT_n/dt + [(\dot{V}_{hw}/V_n + UA_n / (\rho\, c_p V_n)] T_n - [UA_n T_{amb} / (\rho\, c_p V_n) + P_n / (\rho\, c_p V_n) + T_{in} \dot{V}_{hw} / V_n]$$

or:

$$0 \;=\; a_n\, dT_n/dt \;+\; b_n\, T_n \;+\; c_n$$

where: $a_n = 1$

$b_n = \dot{V}_{hw} / V_n \;+\; UA_n / (\rho\, c_p\, V_n) \;=\; \rho\, c_p\, \dot{V}_{hw} \;+\; UA_n$

$c_n \;=\; [-UA_n\, T_{amb} \;-\; P_n] / (\rho\, c_p\, V_n) \;-\; T_{in}\, \dot{V}_{hw} / V_n \;=\; -(UA_n\, T_{amb} \;+\; P_n + \rho\, c_p\, \dot{V}_{hw}\, T_{in})$

Over an interval of time over which P_n, T_{amb}, T_{in} and $\dot{V}_{hw}$ are known and constant, the solution has the form:

$$T_n \;=\; A_n\, e^{-bnt} \;+\; K_n$$

So:

$$dT_n/dt \;=\; -A_n\, b_n\, e^{-bnt}$$

Substituting both into the original differential equation:

$$0 \;=\; -A_n\, n_n\, e^{-bnt} \;+\; b_n\,(A_n\, e^{-bnt} \;+\; K_n) \;+\; c_n \;=\; b_n\, K_n \;+\; c_n$$

So:

$$K_n \;=\; -c_n / b_n$$

At time t=0 in the interval the initial node temperature is $T_1(0)$ is a known boundary condition so:

$$T_n(0) \;=\; A_n \;+\; K_n \;=\; A_n \;-\; c_n / b_n$$

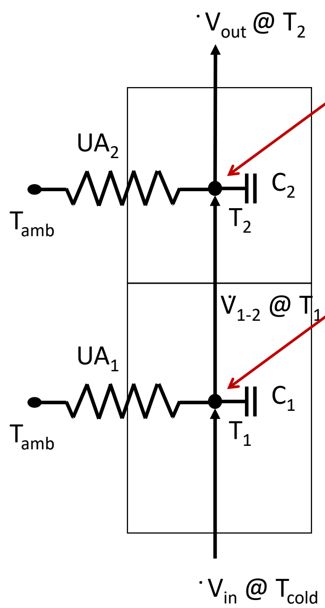
and:

$$A_n \;=\; T_n(0) \;+\; c_n / b_n$$

and the solution for T_n, over a time interval where the boundary conditions are known and constant, is:

$$T_n \;=\; A_n\, e^{-bnt} \;+\; K_n \;=\; [T_n(0) \;+\; c_n / b_n]\, e^{-bnt} \;-\; c_n / b_n$$

Solution for a Two-Node Water Tank. Assume a water heater can be modeled reasonably accurately by two nodes, as in Figure 2.



Conservation of mass (for an incompressible liquid) requires that:

$$\dot{V}_{hw} = \dot{V}_{1\text{-}2} = \dot{V}_{in} = \dot{V}_{out}$$

The thermal mass of each of the nodes is

$$C_1 = \rho\, c_p\, V_1$$

$$C_2 = \rho\, c_p\, V_2$$

Noting that T_{in} for the first (lower) node is T_{cold}, and for the second node is T_1, the (sensible) energy balance for each of the nodes is:

$$P_1 + \rho\, c_p\, \dot{V}_{hw}\, T_{cold} = \rho\, c_p\, \dot{V}_{hw}\, T_1 + \rho\, c_p\, V_1\, dT_1/dt + UA_1\, (T_1 - T_{amb})$$

$$P_2 + \rho\, c_p\, \dot{V}_{hw}\, T_1 = \rho\, c_p\, \dot{V}_{hw}\, T_2 + \rho\, c_p\, V_2\, dT_2/dt + UA_2\, (T_2 - T_{amb})$$

Over a period of time when P_1, P_2, T_{amb}, T_{cold} and $\dot{V}_{hw}$ are known and constant, the heat balance equation for the first (lower) node can be solved as in the previous section:

$$T_1 = [T_1(0) + c_1 / b_1]\, e^{-b1t} - c_1 / b_1$$

The temperature of the lower node is the temperature for water coming in to the upper node, so input into the heat balance equation for the upper node:

$$P_2 + \rho\, c_p\, \dot{V}_{hw}\, T_1 = \rho\, c_p\, \dot{V}_{hw}\, T_2 + \rho\, c_p\, V_2\, dT_2/dt + UA_2\, (T_2 - T_{amb})$$

or, rearranging:

$$0 = dT_2/dt + \dot{V}_{hw}/V_2\, T_2 + UA_2\, (T_2 - T_{amb}) / (\rho\, c_p\, V_2) - P_2 / (\rho\, c_p\, V_2) - \dot{V}_{hw}/V_2\, T_1$$

$$0 = dT_2/dt + [\dot{V}_{hw}/V_2 + UA_2 / (\rho\, c_p\, V_2)]\, T_2 - UA_2\, T_{amb} / (\rho\, c_p\, V_2) - P_2 / (\rho\, c_p\, V_2) - \dot{V}_{hw}/V_2\, T_1$$

which, as before, has the solution:

$$T_2 = A_2 e^{-b2t} + K_2 = [T_2(i) + c_2 / b_2] e^{-b2t} - c_2 / b_2$$

where: $b_2 = \rho c_p \dot{V}_{hw} + UA_2$

$c_2 = -UA_2 T_{amb} - P_2 - \rho c_p \dot{V}_{hw} [(T_1(i) + c_1 / b_1) e^{-b1t} - c_1 / b_1)]$

Linearized Two-Node Solution with Mixing Valve. With a mixing valve and its temperature setpoint (T_{hot}), then the flow rate of the water draw ($\dot{V}$) is a function of the instantaneous temperature of the upper node (T_2). This in turn affects the solution for the temperature in the lower tank (T_1), and so on. So, the differential equation becomes highly non-linear.

A simplifying assumption is to linearize the solution. Solving the heat balance equation for dT_n/dt:

$$dT_n/dt = [P_n - UA_n(T_n - T_{amb})] / (\rho c_p V_n) - \dot{V}_{hw}/V_n (T_n - T_{in})$$

Assuming the solution interval is short enough such that the node temperature does not change appreciably with respect to the ambient and inlet temperatures (T_{amb}, and T_{in}), then

$$T_n(t) \approx T_n(t=0) = T_n(0)$$

Substituting this into the equation for rate of change of the node temperature:

$$dT_n/dt \approx [P_n - UA_n(T_n(0) - T_{amb})] / (\rho c_p V_n) - \dot{V}_{hw}/V_n (T_n(0) - T_{in})$$

The node temperature at the end of the interval Δt_{sub} is then:

$$T_n(\Delta t_{sub}) = dT_n/dt \, \Delta t_{sub} + T_n(0)$$

Physical Model of Water Heater – Moving Layers Model

Assumptions. Assume a water heater is modeled as a set of (moving) layers each at a uniform temperature (e.g., fully mixed).

As far as the GridLAB-D (GLD) simulation is concerned, the state of the water heater object changes only when a heating element turns on or off.

Note that GLD is inherently a variable time step simulation, allowing any simulated object to limit the overall GLD simulation time step. Hence, each GLD object must predict the time of its next state change in advance of being asked by the GLD to execute a time step.

The GLD's water draw schedule is a separate object from the associated water heater object, so any time the water draw changes is a state change with respect to GLD, and it will restrict the simulation time step to the time of the water draw state change.

Hence, during any simulation time step, the water draw is known and at a constant flow rate.

A moving layers model is in essence a hybrid of a layered model (an example is the 2-layer model described above) and a nodal model (an example is the 2-node model described above).

As will be developed below, this implies there are three flow regimes and associated modes of simulation that must be used, as follows:

1. *Static*, when no water draw is occurring, mixing within a layer is dominated by natural convection phenomena (heated water from the element rising, cold near the tank wall falling), and a nodal method applies.
2. *Slug flow*, when the flow rate of water through the tank is high enough that forced convection phenomena dominate the heat transfer from the heating elements to the water (i.e., is above the rate at which a heating element can heat it to the associated setpoint), and a layered method applies
3. *Circulation flow*, when the flow rate of water through the tank is slow enough that heat is transferred from the heating element, gain dominated by natural convection phenomena, and a nodal method applies but must account for the fact that the layers are moving through the tank and past the heating elements.

It should be noted that the *circulation flow regime* occurs in a relative narrow band of low flow rate water draws, as shown below, that also tend to be small in total volume and hence small in terms of the total energy withdrawn from the tank over the course of time. Hence, more liberal approximations to the solution in this regime (to linearize it, for example) may be acceptable.

When the flow regime changes the model and the equations that represent it change. Thus, the water heater simulation must halt at a *regime change*, change its solution method, and then continue. As far as GLD is concerned, this is not a state change, although there may be one associated with a regime change when the water draw changes.

So, during any GLD simulation time step, the power to the elements and the water draw are known and constant as long as Layer N is in the slug flow regime, i.e., does not extend to or below the upper heating element.

A water heater model comprised of a set of moving layers must account for the effect that layers:

1. are created (and subsequently expand) at the bottom of the tank when a cold water flow occurs
2. vanish at the top of the tank when they are depleted
3. merge, when heat is added to the lower of a pair of layers, and the lower layer reaches the same temperature as the upper layer of the pair
4. merge, when more rapid heat loss from the upper layer of a pair results in its temperature eventually reaching the same temperature as the lower layer of the pair
5. the upper element also creates a new layer when its power is on.

Thus there are an arbitrary, and constantly changing, number of layers in the tank that must be tracked by the model.

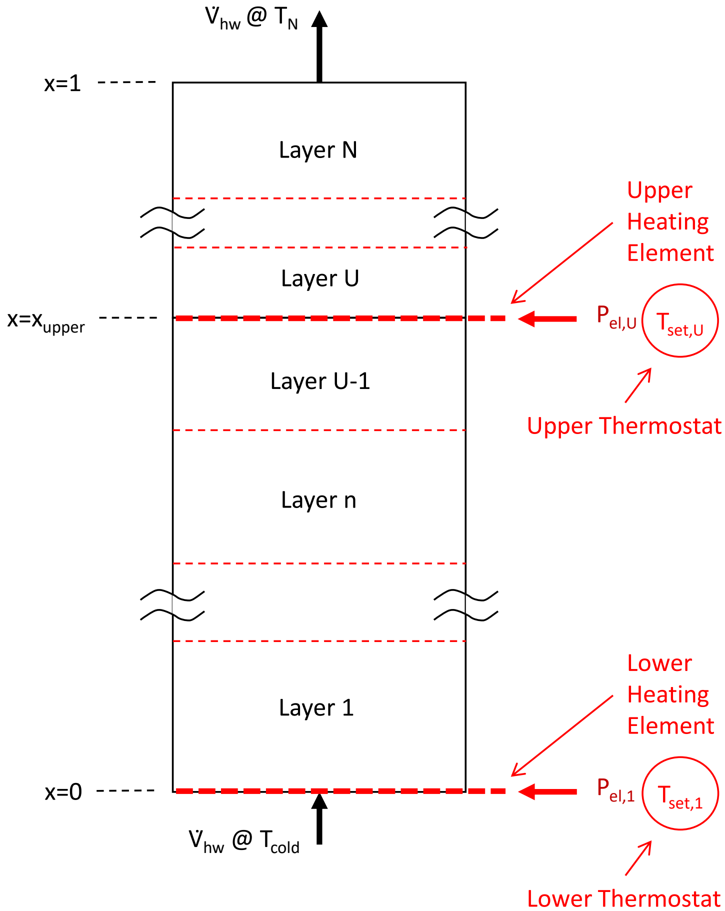
Each of the above transitions in the layer structure is a change in the model *topology*. Like a regime change, a *topology change* is not a state change as far as GLD is concerned (although one may share a root cause it), but the water heater model must halt, change its topology, and then continue its solution.

Types and Movement of Layers. The rate of movement of a layer of fixed volume (dx/dt) is common throughout a cylindrical or prismatic water tank. As before, the rate of movement upward of all layers in the tank is a function of the instantaneous flow rate of water supplied by the tank ($\dot{V}_{hw}$), which in turn is related to the instantaneous mixing fraction (f) and the volumetric flow rate of the water draw ($\dot{V}$):

$$dx_{draw}/dt = \dot{V}_{hw} / V_{tank} = f\,\dot{V} / V_{tank}$$

A water heater tank with N layers is illustrated in the figure below, with Layer 1 at the bottom and Layer N at the top. Note there are potentially five types of layers in the tank:

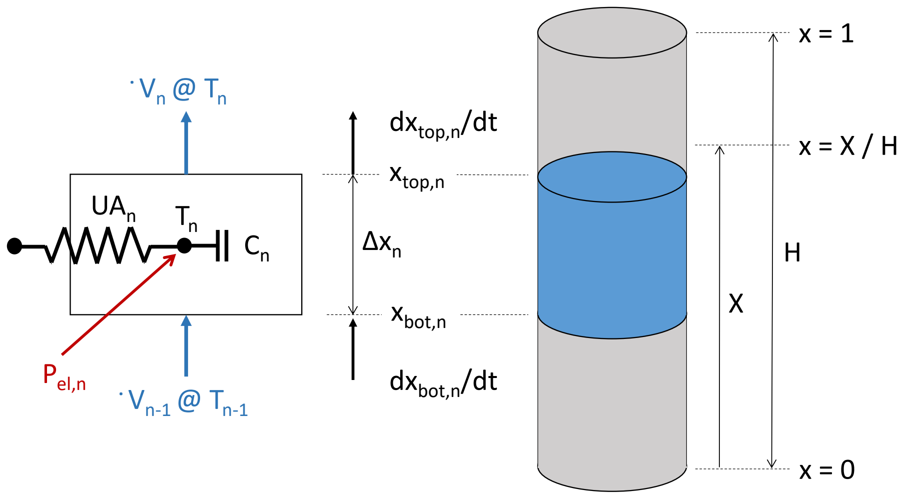
1. Layer 1 – the lower-most layer in the tank (the bottom of Layer 1 is always at x=0, while the position of its top increases by dx/dt)
2. Layer N – upper-most layer in the tank (the top of Layer N is always at x=1, while the position of its bottom increases by dx/dt)
3. Layer U – the layer whose top is above the upper element (when the upper heating element is on, the bottom of layer U is always at $x=x_{upper}$, while the position of its top increases by dx/dt)
4. Layer U-1 – the layer whose bottom is below the upper element (when the upper heating element is on, the top of layer U-1 is always at $x=x_{upper}$, while the position of its bottom increases by dx/dt)
5. Layer n – a generic layer that is not Layer 1, N, U, or U-1.



Note that Layers U and U-1 exist only in relation to an upper heating element that is on during a water flow. Otherwise they simply move upward in the tank as fixed height layers of type Layer n.

Note that not all five types of layers exist in the tank at any given time, depending on conditions. For example, there can be only a single layer in a tank whose upper and lower element setpoints are identical and the tank has not undergone a water draw for a long period of time. The rules for the movement of the layers with regards to their creation, dissolution, and merging are describe in a subsequent section.

Conservation of Mass for a Water Layer. Conservation of mass and energy for a generic water layer (n) of height Δx_n (where x is dimensionless; normalized by the tank height, H_{tank}) whose top is at $x_{top,n}$ and whose bottom is at height $x_{bot,n}$, in a tank whose volume is V_{tank}, is illustrated in the figure below.



Accounting for the fact that the layer's height (Δx_n) will be changing if the rate of change of the position of the top of the layer ($dx_{top,n}/dt$) is not equal to the rate of change of the position of the top of the layer ($dx_{bot,n}/dt$), conservation of mass for the layer n with cross sectional area A_{tank} requires:

$$\rho \dot{V}_n - \rho \dot{V}_{n-1} = \rho\, dV_n/dt = \rho A_n H_{tank} (dx_{top,n} - dx_{bot,n})/dt = \rho V_{tank} (dx_{top,n} - dx_{bot,n})/dt$$

Noting that

$$V_n = V_{tank} (x_{top,n} - x_{top,n})$$

The conservation of mass is equivalent to:

$$\rho \dot{V}_n - \rho \dot{V}_{n-1} = \rho V_{tank} (dx_{top,n} - dx_{bot,n})/dt$$

or:

$$\dot{V}_n - \dot{V}_{n-1} = V_{tank} (dx_{top,n}/dt - dx_{bot,n}/dt)$$

Heat Balance for a Water Layer. The heat balance for a generic water layer (n) of height Δx_n, whose top is at $x_{top,n}$ moving at a rate $dx_{top,n}/dt$ and whose bottom is at height $x_{bot,n}$ moving at a rate $dx_{bot,n}/dt$, is illustrated in the figure above, where T_n is the temperature of the node, UA_n is the heat loss coefficient for the node, C_n is the thermal mass of the node, and $P_{el,n}$ is the power to the heating element in the node.

The instantaneous heat balance for the layer is:

$$P_{el,n} = d(C_n T_n)/dt + UA_n (T_n - T_{amb})] + \rho c_p \dot{V}_n T_n - \rho c_p \dot{V}_{n-1} T_{n-1}$$

Noting that the thermal mass of the node C_n is:

$$C_n = \rho c_p V_n$$

Further, note that in the general case of a layer where the top and bottom layers are not moving at the same rate, the heat loss coefficient is varying with time. At any instant, the layer's heat loss coefficient is:

$$UA_n = UA_{tank} (x_{top,n} - x_{bot,n})$$

Substituting these relationships into the heat balance equation:

$$P_{el,n} = \rho\, c_p\, d(V_n\, T_n)/dt + UA_{tank}\,(x_{top,n} - x_{bot,n})\,(T_n - T_{amb}) + \rho\, c_p\, \dot{V}_n\, T_n - \rho\, c_p\, \dot{V}_{n-1}\, T_{n-1}$$

Differentiating using the Chain Rule:

$$P_{el,n} = \rho\, c_p\,(V_n\, dT_n/dt + T_n\, dV_n/dt) + UA_{tank}\,(x_{top,n} - x_{bot,n})\,(T_n - T_{amb}) + \rho\, c_p\, \dot{V}_n\, T_n - \rho\, c_p\, \dot{V}_{n-1}\, T_{n-1}$$

From conservation of mass, substituting for V_n and dV_n/dt in term of the volume of the tank (V_{tank}) and the positions and rates of change of the top and bottom of layer n ($dx_{top,n}/dt$ and $dx_{bot,n}/dt$, respectively):

$$\begin{aligned} P_{el,n} = {} & \rho\, c_p\, V_{tank}\,(x_{top,n} - x_{bot,n})\, dT_n/dt + \rho\, c_p\, T_n\, V_{tank}\,(dx_{top,n}/dt - dx_{bot,n}/dt) + \\ & UA_{tank}\,(x_{top,n} - x_{bot,n})\,(T_n - T_{amb}) + \rho\, c_p\, \dot{V}_n\, T_n - \rho\, c_p\, \dot{V}_{n-1}\, T_{n-1} \end{aligned}$$

Defining the heat capacitance of the tank (C_{tank}) as:

$$C_{tank} = \rho\, c_p\, V_{tank}$$

Simplifies the heat balance to:

$$\begin{aligned} P_{el,n} = {} & C_{tank}\,(x_{top,n} - x_{bot,n})\, dT_n/dt + C_{tank}\, T_n\,(dx_{top,n}/dt - dx_{bot,n}/dt) + \\ & UA_{tank}\,(x_{top,n} - x_{bot,n})\,(T_n - T_{amb}) + \rho\, c_p\, \dot{V}_n\, T_n - \rho\, c_p\, \dot{V}_{n-1}\, T_{n-1} \end{aligned}$$

or, since

$$\dot{V}_n = \dot{V}_{n-1} = V_{tank}\, dx_{draw}/dt$$

$$\begin{aligned} 0 = {} & C_{tank}\,(x_{top,n} - x_{bot,n})\, dT_n/dt + C_{tank}\, T_n\,(dx_{top,n}/dt - dx_{bot,n}/dt) + \\ & UA_{tank}\,(x_{top,n} - x_{bot,n})\,(T_n - T_{amb}) + C_{tank}\, dx_{draw}/dt\,(T_n - T_{n-1}) - P_{el,n} \end{aligned}$$

This is a first order ordinary differential equation that can be readily solved for a period of time over which the flow of hot water from the tank ($\dot{V}_{hw}$) is known and constant. Rearranging the heat balance equation into a convenient form for solution:

$$\begin{aligned} 0 = {} & dT_n/dt + \\ & [(dx_{top,n}/dt - dx_{bot,n}/dt + dx_{draw}/dt) / (x_{top,n} - x_{bot,n}) + UA_{tank}/C_{tank}]\, T_n \\ & - UA_{tank}/C_{tank}\, T_{amb} - dx_{draw}/dt / (x_{top,n} - x_{bot,n})\, T_{n-1} - P_{el,n} / C_{tank}\,(x_{top,n} - x_{bot,n}) \end{aligned}$$

or:

$$0 = a_n\, dT_n/dt + b_n\, T_n + c_n$$

where: $a_n = 1$

$b_n = (dx_{top,n}/dt - dx_{bot,n}/dt + dx_{draw}/dt) / () + UA_{tank}/C_{tank}$

$c_n = -UA_{tank}/C_{tank}\, T_{amb} - dx_{draw}/dt / (x_{top,n} - x_{bot,n})\, T_{n-1} - P_{el,n} / C_{tank}\,(x_{top,n} - x_{bot,n})$

Over an interval of time over which $P_{el,n}$, T_{amb}, T_{n-1} and $\dot{V}_{hw}$ are known and constant, and noting that from $\dot{V}_{hw}$ and the spatial properties of layer n the quantities $x_{top,n} - x_{bot,n}$, $dx_{top,n}/dt$, and $dx_{bot,n}/dt$ are all known and constant, the solution has the form:

$$T_n = A_n\, e^{-bnt} + K_n$$

So:

$$dT_n/dt \;=\; -A_n\, b_n\, e^{-bnt}$$

Substituting both into the original differential equation:

$$0 \;=\; -A_n\, n_n\, e^{-bnt} \;+\; b_n\,(A_n\, e^{-bnt} \;+\; K_n) \;+\; c_n \;=\; b_n\, K_n \;+\; c_n$$

So:

$$K_n \;=\; -c_n \,/\, b_n$$

At time t=0 in the interval the initial node temperature is T^{-}_{1} is a known boundary condition so:

$$T^{-}_{n} \;=\; A_n \;+\; K_n \;=\; A_n \;-\; c_n \,/\, b_n$$

and:

$$A_n \;=\; T^{-}_{n} \;+\; c_n \,/\, b_n$$

and the general solution for T_n, over a time interval where the boundary conditions are known and constant, is:

$$T_n \;=\; A_n\, e^{-bnt} \;+\; K_n \;=\; (T^{-}_{n} \;+\; c_n/b_n)\, e^{-bnt} \;-\; c_n/b_n$$

However, if a mixing valve is present and operating at a mixing fraction less than 100%, the volumetric flow rate from the tank ($\dot{V}_{hw}$) is dependent on the instantaneous temperature in the uppermost layer of the tank (T_N). That varies over such a time period for two reasons:

1. Heat from any heating element in Layer N that is on, i.e., when Layer N extend all the way down to an upper heating element whose power is on (and hence is synonymous with Layer U), or extends all the way to the bottom of the tank (and hence is synonymous with Layer 1) and with the power to the lower heating element on.
2. Heat losses from Layer N (which is a relatively minor effect)

Reduced Heat Balance Equation for Various Types of Layers. A simpler heat balance for each of the eight possible types of layers of a water tank, given their various boundary conditions, is developed in this section.

Note the heat balance equation for the various layers involves four types of boundary conditions:

A. Fixed height, no flows in <u>or</u> out of layer
B. Fixed height, flows <u>both</u> in <u>and</u> out of layer
C. Fixed bottom, floating top of layer
D. Fixed top, floating bottom of layer.

<u>If the power to the upper heating element is not on</u>, a new layer is created with temperature T_{cold} at the bottom of the tank, and all layers are moving through the tank at the rate dx_{draw}/dt. Unless the water in the tank is already at T_{cold}, the result is that there will be at least two layers in the tank. With the heating element off, a new layer is <u>not</u> created by the heat addition from the upper element, and the boundary conditions that distinguish Layers U and U-1 from the generic layer n vanish. Under some conditions it is possible for the bottom layer to extend all the way to the top of the tank, so there are four possible

types of layers in a tank without the upper heating element on, as indicated in the table below. Only the positions of the bottom of Layer 1 and the top of Layer N are fixed, and since the upper heating element is off, Layers U and U-1 are not different than the generic layer n, so the following boundary conditions apply:

Table Xa. Boundary Conditions for Various Layers (Upper Heating Element Off)

Layer	Type	x_{top}	x_{bot}	$x_{top}-x_{bot}$	dx_{top}/dt	dx_{bot}/dt	$\dot{V}_{top}$	$\dot{V}_{bot}$	T_{n-1}
1 ≠ N	C	$x_{top,1}$	0	varies	dx_{draw}/dt	0	0	$V_{tank}\ dx_{draw}/dt$	T_{cold}
n ≠ {1,N)	A	$x_{top,n}$	$x_{bot,n}$	Δx_n	dx_{draw}/dt	dx_{draw}/dt	0	0	T_{n-1}
N ≠ 1	D	1	$x_{bot,1}$	varies	0	dx_{draw}/dt	$V_{tank}\ dx_{draw}/dt$	0	T_{N-1}
N = 1	B	1	0	1	0	0	$V_{tank}\ dx_{draw}/dt$	$V_{tank}\ dx_{draw}/dt$	T_{cold}

With the upper heating element off, the heat balance equation can be reduced to much simpler forms for these four layers. In particular, the generic layer that is neither Layer 1 or Layer N where the height of the layer ($x_{top,n} – x_{bot,n}$) is a constant (Δx_n) is quite simple, as shown in the table below.

Below is the general heat balance equation in a format useful for developing entries in the tables in the next section:

$P_{el,n} = C_{tank}\ (x_{top,n} – x_{bot,n})\ dT_n/dt\ +$
$C_{tank}\ T_n\ (dx_{top,n}/dt - dx_{bot,n}/dt)\ +$
$UA_{tank}\ (x_{top,n} – x_{bot,n})\ (T_n – T_{amb})\ +$
$\rho\ c_p\ \dot{V}_n\ T_n\ -\ \rho\ c_p\ \dot{V}_{n-1}\ T_{n-1}$

Table Ya. Reduced Heat Balance Equations for Various Layers (Upper Heating Element Off)

Layer	Bo.Co. Type	Heat Balance Equation
1 ≠ N	C	$P_{el,1} = C_{tank}\ x_{top,1}\ dT_1/dt + C_{tank}\ dx_{draw}/dt\ T_1 + UA_{tank}\ x_{top,1}\ (T_1 – T_{amb}) - C_{tank}\ dx_{draw}/dt\ T_{cold}$
n ≠ {1,N)	A	$0 = C_{tank}\ \Delta x_n\ dT_n/dt + UA_{tank}\ \Delta x_n\ (T_n – T_{amb})$
N ≠ 1	D	$0 = C_{tank}\ (1 – x_{bot,N})\ dT_N/dt + UA_{tank}\ (1 – x_{bot,N})\ (T_N – T_{amb})$
N = 1	B	$P_{el,U} + P_{el,1} = C_{tank}\ dT_N/dt + UA_{tank}\ (T_N – T_{amb}) + C_{tank}\ dx_{draw}/dt\ T_N - C_{tank}\ dx_{draw}/dt\ T_{cold}$

In the most general case, the power to the upper heating element may be on. If so, a new layer (U) is also created at the position of the upper heating element during a water draw. Note that under some conditions Layer U can extend all the way to the top of the tank, i.e., Layers U and N are synonymous. Similarly, under some conditions Layer 1 can extend all the way to the upper heating element (at x_{upper}), i.e., Layers 1 and U-1 are synonymous. So, in the general case, including when power to the upper heating element is on, the following boundary conditions apply:

Table Xb. Boundary Conditions for Various Layers (General Case)

Layer	Type	x_{top}	x_{bot}	$x_{top} - x_{bot}$	dx_{top}/dt	dx_{bot}/dt	$\dot{V}_{top}$	$\dot{V}_{bot}$	T_{n-1}
1 ≠ {U-1,N}	C	$x_{top,1}$	0	varies	dx_{draw}/dt	0	0	$V_{tank}\ dx_{draw}/dt$	T_{cold}
n ≠ {1,U-1,U,N}	A	$x_{top,n}$	$x_{bot,n}$	Δx_n	dx_{draw}/dt	dx_{draw}/dt	0	0	T_{n-1}
U-1 ≠ 1	D	x_{upper}	$x_{bot,n}$	varies	0	dx_{draw}/dt	$V_{tank}\ dx_{draw}/dt$	0	T_{cold}
U-1 = 1	B	x_{upper}	0	x_{upper}	0	0	$V_{tank}\ dx_{draw}/dt$	$V_{tank}\ dx_{draw}/dt$	T_{cold}
U ≠ N	C	$x_{top,U}$	x_{upper}	varies	dx_{draw}/dt	0	0	$V_{tank}\ dx_{draw}/dt$	T_{U-1}
N ≠ {1,U}	D	1	$x_{bot,N}$	varies	0	dx_{draw}/dt	$V_{tank}\ dx_{draw}/dt$	0	T_{N-1}
N = U	B	1	x_{upper}	1 - x_{upper}	dx_{draw}/dt	0	0	$V_{tank}\ dx_{draw}/dt$	T_{U-1}
N = 1	B	1	0	1	0	0	$V_{tank}\ dx_{draw}/dt$	$V_{tank}\ dx_{draw}/dt$	T_{cold}

and, the corresponding reduced heat balance equations are:

Table Yb. Reduced Heat Balance Equations for Various Layers (General Case)

Layer	Bo.Co. Type	Heat Balance Equation
1 ≠ {U-1,N}	C	$P_{el,1} = C_{tank}\, x_{top,1}\, dT_1/dt + C_{tank}\, dx_{draw}/dt\, T_1 + UA_{tank}\, x_{top,1}\,(T_1 - T_{amb}) - C_{tank}\, dx_{draw}/dt\, T_{cold}$
n ≠ {1,U-1,U,N}	A	$0 = C_{tank}\,\Delta x_n\, dT_n/dt + UA_{tank}\,\Delta x_n\,(T_n - T_{amb})$
U-1 ≠ 1	D	$0 = C_{tank}\,(x_{upper} - x_{bot,U-1})\, dT_{U-1}/dt + UA_{tank}\,(x_{upper} - x_{bot,U-1})\,(T_{U-1} - T_{amb})$
U-1 = 1	B	$P_{el,1} = C_{tank}\, x_{upper}\, dT_1/dt + UA_{tank}\, x_{upper}\,(T_1 - T_{amb}) + C_{tank}\, dx_{draw}/dt\, T_1 - C_{tank}\, dx_{draw}/dt\, T_{cold}$
U ≠ N	C	$P_{el,U} = C_{tank}\,(x_{top,U} - x_{upper})\, dT_U/dt + C_{tank}\, dx_{draw}/dt\, T_U + UA_{tank}\,(x_{top,U} - x_{upper})\,(T_U - T_{amb}) - C_{tank}\, dx_{draw}/dt\, T_{U-1}$
N ≠ {1,U}	D	$0 = C_{tank}\,(1 - x_{bot,N})\, dT_N/dt + UA_{tank}\,(1 - x_{bot,N})\,(T_N - T_{amb})$
N = U	B	$P_{el,U} = C_{tank}\,(1 - x_{upper})\, dT_N/dt + UA_{tank}\,(1 - x_{upper})\,(T_N - T_{amb}) + C_{tank}\, dx_{draw}/dt\, T_N - C_{tank}\, dx_{draw}/dt\, T_{N-1}$
N = 1	B	$P_{el,U} + P_{el,1} = C_{tank}\, dT_N/dt + UA_{tank}\,(T_N - T_{amb}) + C_{tank}\, dx_{draw}/dt\, T_N - C_{tank}\, dx_{draw}/dt\, T_{cold}$

Examination of the table above indicate there are two reduced forms for the solution:

- Form 1 reduces to two terms for the rate of change of the layer temperature and the heat loss, and applies to Boundary Condition Type A (fixed layer height) and Type D (fixed top position).
- Form 2 reduces to four terms, with the additional two relating to the change in energy of the layer due to movement of the lower boundary and the change in the thermal mass in the layer, and applies to Boundary Condition Type B (layer position) and Type C (fixed bottom position).

This Table is the Step-by-Step Derivation of Table Yb (Above) for QA Purposes

Layer	Bo.Co. Type	Heat Balance Equation (Substitute Bo.Cos.)	Heat Balance Equation
1 ≠ {U-1, N}	C	$P_{el,1} = C_{tank} (x_{top,1} - 0)\, dT_1/dt\ +$ $C_{tank}\, T_1 (dx_{draw}/dt - 0)\ +$ $UA_{tank} (x_{top,n} - 0) (T_1 - T_{amb})\ +$ $\rho\, c_p\, 0\, T_1\ -\ \rho\, c_p\, V_{tank}\, dx_{draw}/dt\, T_{cold}$	$P_{el,1} = C_{tank}\, x_{top,1}\, dT_1/dt\ +$ $C_{tank}\, dx_{draw}/dt\, T_1\ +$ $UA_{tank}\, x_{top,n} (T_1 - T_{amb})\ -$ $C_{tank}\, dx_{draw}/dt\, T_{cold}$
n ≠ {1,U-1, U,N}	A	$0 = C_{tank} (x_{top,n} - x_{bot,n})\, dT_n/dt\ +$ $C_{tank}\, T_n (0)\ +$ $UA_{tank} (x_{top,n} - x_{bot,n}) (T_n - T_{amb})\ +$ $\rho\, c_p\, 0\, T_n\ -\ \rho\, c_p\, 0\, T_{n-1}$	$0 = C_{tank} (x_{top,n} - x_{bot,n})\, dT_n/dt\ +$ $UA_{tank} (x_{top,n} - x_{bot,n}) (T_n - T_{amb})$
U-1 ≠ 1	D	$0 = C_{tank} (x_{upper} - x_{bot,U-1})\, dT_n/dt\ +$ $C_{tank}\, T_{U-1} (0 - dx_{bot,U-1}/dt)\ +$ $UA_{tank} (x_{upper} - x_{bot,U-1}) (T_{U-1} - T_{amb})\ +$ $\rho\, c_p\, V_{tank}\, dx_{draw}/dt\, T_{U-1}\ -\ \rho\, c_p\, 0\, T_{U-2}$	$0 = C_{tank} (x_{upper} - x_{bot,U-1})\, dT_{U-1}/dt\ \cancel{-}$ $\cancel{C_{tank}\, T_{U-1}\, dx_{bot,U-1}/dt}\ +$ $UA_{tank} (x_{upper} - x_{bot,U-1}) (T_{U-1} - T_{amb})\ +$ $\cancel{\rho\, c_p\, V_{tank}\, dx_{draw}/dt\, T_{U-1}}$
U-1 = 1	B	$P_{el,1} = C_{tank} (x_{upper} - 0)\, dT_1/dt\ +$ $C_{tank}\, T_1 (0 - 0)\ +$ $UA_{tank} (x_{upper} - 0) (T_1 - T_{amb})\ +$ $\rho\, c_p\, V_{tank}\, dx_{draw}/dt\, T_1\ -$ $\rho\, c_p\, V_{tank}\, dx_{draw}/dt\, T_{cold}$	$P_{el,1} = C_{tank}\, x_{upper}\, dT_1/dt\ +$ $UA_{tank}\, x_{upper} (T_1 - T_{amb})\ +$ $\rho\, c_p\, V_{tank}\, dx_{draw}/dt\, T_1\ -$ $\rho\, c_p\, V_{tank}\, dx_{draw}/dt\, T_{cold}$
U ≠ N	C	$P_{el,U} = C_{tank} (x_{top,U} - x_{upper})\, dT_U/dt\ +$ $C_{tank}\, T_U (dx_{draw}/dt - 0)\ +$ $UA_{tank} (x_{top,U} - x_{upper}) (T_U - T_{amb})\ +$ $\rho\, c_p\, 0\, dx_{draw}/dt\, T_U\ -$ $\rho\, c_p\, V_{tank}\, dx_{draw}/dt\, T_{U-1}$	$P_{el,U} = C_{tank} (x_{top,U} - x_{upper})\, dT_U/dt\ +$ $C_{tank}\, T_U\, dx_{draw}/dt\ +$ $UA_{tank} (x_{top,U} - x_{upper}) (T_U - T_{amb})\ -$ $\rho\, c_p\, V_{tank}\, dx_{draw}/dt\, T_{U-1}$
N ≠ {1,U}	D	$0 = C_{tank} (1 - x_{bot,N})\, dT_N/dt\ +$ $C_{tank}\, T_N (0 - dx_{draw}/dt)\ +$ $UA_{tank} (1 - x_{bot,N}) (T_N - T_{amb})\ +$ $\rho\, c_p\, V_{tank}\, dx_{draw}/dt\, T_N\ -$ $\rho\, c_p\, 0\, T_{N-1}$	$0 = C_{tank} (1 - x_{bot,N})\, dT_N/dt\ \cancel{-}$ $\cancel{C_{tank}\, T_N\, dx_{draw}}\ +$ $UA_{tank} (1 - x_{bot,N}) (T_N - T_{amb})\ \cancel{+}$ $\cancel{\rho\, c_p\, V_{tank}\, dx_{draw}/dt\, T_N}$
N = U	B	$P_{el,U} = C_{tank} (1 - x_{upper})\, dT_N/dt\ +$ $C_{tank}\, T_N (0 - 0)\ +$ $UA_{tank} (1 - x_{upper}) (T_N - T_{amb})\ +$ $\rho\, c_p\, V_{tank}\, dx_{draw}/dt\, T_N\ -$ $\rho\, c_p\, V_{tank}\, dx_{draw}/dt\, T_{N-1}$	$P_{el,U} = C_{tank} (1 - x_{upper})\, dT_N/dt\ +$ $UA_{tank} (1 - x_{upper}) (T_N - T_{amb})\ +$ $\rho\, c_p\, V_{tank}\, dx_{draw}/dt\, T_N\ -$ $\rho\, c_p\, V_{tank}\, dx_{draw}/dt\, T_{N-1}$
N = 1	B	$P_{el,1}\ +\ P_{el,U} = C_{tank} (1 - 0)\, dT_n/dt\ +$ $C_{tank}\, T_n (0 - 0)\ +$ $UA_{tank} (1 - 0) (T_N - T_{amb})\ +$ $\rho\, c_p\, V_{tank}\, V_{tank}\, dx_{draw}/dt\, T_N\ -$ $\rho\, c_p\, V_{tank}\, dx_{draw}/dt\, T_{cold}$	$P_{el,1}\ +\ P_{el,U} = C_{tank}\, dT_n/dt\ +$ $UA_{tank} (T_N - T_{amb})\ +$ $\rho\, c_p\, V_{tank}\, V_{tank}\, dx_{draw}/dt\, T_N\ -$ $\rho\, c_p\, V_{tank}\, dx_{draw}/dt\, T_{cold}$

Note that the reduced heat balance equations in the table above all reduce to a common form when there is no water draw because terms containing the rate of change of layers' upper and lower boundary are all zero ($dx_{draw}/dt = 0$).

Effect of Mixing Valve. The operation of a mixing valve has been described previously.

Note that in a moving layer model in the slug flow regime, the hot water supplied from the tank is drawn from the uppermost layer and (except for tank jacket heat losses, which are of secondary importance) is at a constant temperature. Thus, in the presence of a mixing valve, the mixing fraction while water is drawn from a given layer is known and constant. As will be seen, this makes the solution of the moving layer model considerably simpler, except in the *circulation flow regime*.

When in the circulation flow regime, the uppermost layer may have a continuous temperature change during a water draw, and the effect of the mixing valve must be taken into account …

Approximate Solution Based on Linearized Heat Loss (Form 1 – Boundary Condition Types A and D). In order to simplify the heat balance equations even further, and readily predict the time of the next topology or state change, it is useful to linearize the heat balance equation by assuming the heat loss rate is constant over practical solution intervals.

Since the change in the water temperature of a layer due to heat losses over practical solution intervals are deemed small, the rate of heat loss from any layer can reasonably be assumed to be constant over the interval at the initial temperature (T^-_n).

For layers with one fixed edge (either the top or the bottom), its position ($x_{edge,n}$) varies linearly from its initial position ($x_{edge,n}$) over time as a function of ($dx_{edge,n}/dt = dx_{draw}/dt$):

$$x_{edge,n} = x^-_{edge,n} + t\, dx_{draw}/dt$$

Substituting this relationship and the assumption for the layer temperature, simplified heat balance equations of Form 1 (i.e., Boundary Condition Types A and D) are shown in the table below.

Table Za. Linearized Heat Balance Equations for Boundary Condition Types A and D

Layer	Bo. Co. Type	Linearized Heat Balance Equation
$n \neq \{1,U\text{-}1,U,N\}$	A	$0 = C_{tank}\, \Delta x_n\, dT_n/dt + UA_{tank}\, \Delta x_n\, (T^-_n - T_{amb})$
$U\text{-}1 \neq 1$	D	$0 = (x_{upper} - x^-_{bot,n} - t\, dx_{draw}/dt)\, (C_{tank}\, dT_{U-1}/dt + UA_{tank}\, (T^-_{U-1} - T_{amb})]$
$N \neq \{1,U\}$	D	$0 = (1 - x^-_{bot,n} + t\, dx_{draw}/dt)\, [C_{tank}\, dT_N/dt + UA_{tank}\, (T^-_N - T_{amb})]$

These all have solutions of the form:

$$T_n = T^-_n + t\, dT_n/dt$$

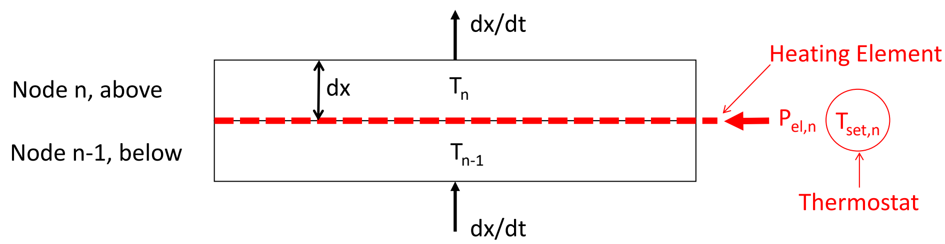
and the solutions for each boundary condition type from solving the linearized heat balance equation are shown in the table below. Note none of these layers has a heating element in it, by definition.

Table Zb. Solutions to the Linearized Heat Balance Equations for Boundary Condition Types A and D

Layer	Type	Rate of Change of Layer Temp.	Solution for Layer Temperature
$n \neq \{1,U\text{-}1,U,N\}$	A	$dT_n/dt = -UA_{tank}/C_{tank}\, (T^-_n - T_{amb})$	$T_n = T^-_n - t\, UA_{tank}/C_{tank}\, (T^-_n - T_{amb})$
$U\text{-}1 \neq 1$	D	$dT_n/dt = -UA_{tank}/C_{tank}\, (T^-_{U-1} - T_{amb})$	$T_n = T^-_n - t\, UA_{tank}/C_{tank}\, (T^-_{U-1} - T_{amb})$
$N \neq \{1,U\}$	D	$dT_n/dt = -UA_{tank}/C_{tank}\, (T^-_N - T_{amb})$	$T_n = T^-_n - t\, UA_{tank}/C_{tank}\, (T^-_N - T_{amb})$

Solution Across a Heating Element for the Slug Flow Regime. Assume the water draw is known and constant. Further assume the mixing fraction for any mixing valve present can be reasonably be assumed constant over a period of time. The latter is equivalent to an assumption that the rate of change of temperature in Layer N is small enough so that its effect on the mixing fraction during the interval is negligible. This is true for nearly all(?) conditions of practical interest in a water heater, over solution intervals of five minutes or less.

Under these conditions a layer of water moving past a heating element whose power is on is illustrated in the figure below for a slug of water (layer n) of height Δx, originating below the element from node n-1 at temperature T_{n-1}, moving at rate dx/dt past a heating element consuming power at rate $P_{el,n}$ controlled by a thermostat set at $T_{set,n}$.



Assuming the flow can reasonably be assumed to at a uniform velocity across the face of the cross-sectional area of the tank (e.g., that the boundary layers at the tank walls can be ignored), and ignoring heat losses for an infinitesimally small dx, the power balance is:

$$P_{el} = \rho \, c_p \, \dot{V}_{hw} \, (T_n - T_{n-1}) = \rho \, c_p \, f \, \dot{V} \, (T_n - T_{n-1})$$ *[Add heat loss rate to Pel?]*

which can be solved for the temperature of the water immediately above the heating element:

$$T_n = T_{n-1} + P_{el} / \rho \, c_p \, f \, \dot{V}$$

The point at which the transition from the slug flow regime to the circulation flow regime occurs in this circumstance is not well understood, with a gradual with a blending of the two types of phenomena over a range of conditions.

Further, note that T_n is physically constrained by the thermostat to be equal to or less than the setpoint (T_{set}), and the heating element would have to cycle on and off rapidly to maintain T_n at T_{set}. So, regardless of the details of the transition between the physical phenomena, the solution regime for a moving layer model must transition from slug flow to circulation flow at a hot water flow rate from the tank ($\dot{V}_{trans}$) defined by:

$$T_n = T_{set} = T_{n-1} + P_{el} / \rho \, c_p \, \dot{V}_{trans}$$

So, the transition flow rate at or above which the slug flow solution applies across the heating element is:

$$\dot{V}_{trans} = P_{el} / [\rho \, c_p \, (T_{set} - T_{n-1})]$$

Note that the transition flow rate is inversely proportional to the temperature difference between the thermostat setpoint and that of the layer immediately below the heating element. Thus, under conditions where the temperature of the layer below the heating element is less than but close to T_{set}

during a water draw, there will be an increasing slug flow transition threshold, and the full solution to the heat balance equation will be[?] required for the layer above the heating element.

Approximate Solution Based on Linearized Heat Loss (Form 2 – Boundary Condition Types B and C). The reduced heat balance equations for Form 2 (i.e., Boundary Condition Types B and C) are, with the linearized heat loss assumption:

Table Zd. Linearized Heat Balance Equations for Boundary Condition Types B and C

Layer	Bo.Co. Type	Heat Balance Equation
1 ≠ {U-1,N}	C	$P_{el,1} = C_{tank}\, x_{top,1}\, dT_1/dt + C_{tank}\, dx_{draw}/dt\, T_1 + UA_{tank}\, x_{top,1}\, (T^-_1 - T_{amb}) - C_{tank}\, dx_{draw}/dt\, T_{cold}$
U-1 = 1	B	$P_{el,1} = C_{tank}\, x_{upper}\, dT_1/dt + UA_{tank}\, x_{upper}\, (T^-_1 - T_{amb}) + C_{tank}\, dx_{draw}/dt\, T_1 - C_{tank}\, dx_{draw}/dt\, T_{cold}$
U ≠ N	C	$P_{el,U} = C_{tank}\, (x_{top,U} - x_{upper})\, dT_U/dt + C_{tank}\, dx_{draw}/dt\, T_U + UA_{tank}\, (x_{top,U} - x_{upper})\, (T^-_U - T_{amb}) - C_{tank}\, dx_{draw}/dt\, T_{U-1}$
N = U	B	$P_{el,U} = C_{tank}\, (1 - x_{upper})\, dT_N/dt + UA_{tank}\, (1 - x_{upper})\, (T^-_N - T_{amb}) + C_{tank}\, dx_{draw}/dt\, T_N - C_{tank}\, dx_{draw}/dt\, T_{N-1}$
N = 1	B	$P_{el,1} = C_{tank}\, dT_N/dt + UA_{tank}\, (T^-_N - T_{amb}) + C_{tank}\, dx_{draw}/dt\, T_N - C_{tank}\, dx_{draw}/dt\, T_{cold}$

Three conditions apply to this solution set:

1. The water draw is zero
2. The water flow through the tank is greater than or equal to the slug flow threshold.
3. The water flow through the tank is less than the slug flow threshold.

Each of these conditions are examined here.

Condition 1 (Zero Flow). When there is zero flow, $dx_{draw}/dt=0$ and the solutions for Boundary Condition Types B and C reduce to that of Form 1 for the general layer of fixed height (Layer n ≠ {1,U-1,U,N}).

Condition 2 (Slug Flow). There are two cases here.

Case A. For layers with fixed lower but without fixed upper boundaries in slug flow (i.e., Boundary Condition Type C, i.e., Layer 1, and Layer U if the upper heating element is on).

$$\Sigma_n P_{el,n} - UA_{tank}\, x_{top,n}\, (T^-_n - T_{amb}) = C_{tank}\, dx_{draw}/dt\, T_n - C_{tank}\, dx_{draw}/dt\, T_{n-1}$$

So:

$$T_n = T_{n-1} + [P_{el,n} - UA_{tank}/C_{tank}\, x_{top,n}\, (T^-_n - T_{amb})] / dx_{draw}/dt$$

as long as slug flow conditions are valid, i.e., T_n is less then $T_{set,n}$ if the power to the relevant heating element at the bottom of the layer is on (i.e., $P_{el,n}>0$).

Substituting for the time-dependent position ($x_{top,n}$):

$$T_n = T_{n-1} + [P_{el,n} - UA_{tank}/C_{tank}\, (x^-_{edge,n} + t\, dx_{draw}/dt)\, (T^-_n - T_{amb})] / dx_{draw}/dt$$

Case B. For layers with fixed upper and lower boundaries in slug flow??? In general??? (i.e., Boundary Condition Type B, i.e., Layers, i.e., Layers U=1, N=U, and N=1). Here, we make the argument that these are irrelevant cases …

Note that if the element is on, a new layer U has been forming since it did so. If the upper heating element is off, layer U and U-1 are irrelevant; U becomes synonymous with n or N, and U-1 simply moves up past the element as a slug. Rgp]

If there is flow, the heating element will turn on if the temperature of the layer below it is lower than the setpoint less the deadband and a new layer beneath it is thus formed.

This will not be the case for a Layer U=1 or Layer N=1 unless their temperature is very close to T_{cold}, which can only occur in an almost entirely depleted tank, i.e., after a very high, sustained water draw equal to the size of the tank or more.

This can occur for Layer N=U, since the temperature of Layer U-1 could be above the setpoint (if it has recently been shifted to lower value), or within the deadband of the setpoint in the course of normal operation, in which case the heating element would not turn on and a new layer is not formed. In this case, the Layer N=U becomes Layer N≠U, i.e. there is a topology change, and the Layer that was U=1 retains its integrity and simply moves across the element as a fixed height layer of type n ≠ {1,U-1,U,N}. Thus a Form 1 solution then applies until such time as the heating element turns on.

Otherwise the Form 1 solution then applies, since the layer simply moves upward and no longer *has* a fixed lower boundary. I.e., there has been a topology change, with N=1 and N=U becoming N ≠ {1,U}, or with U=1 becoming a new layer of type U-1 the new with a new Layer 1 rising beneath it.

Or:

$$T_n = T_{n-1} + [\Sigma_n P_{el,n} - UA_{tank}/C_{tank}\, x_{top,n}\, (T^-_n - T_{amb})] / dx_{draw}/dt$$

These all have solutions of the form:

$$T_n = T^-_n + t\, dT_n/dt$$

and the solutions from solving the linearized heat balance equation for each Boundary Condition Type B and C are shown in the table below.

Table Zc. Solutions to the Linearized Heat Balance Equations in Slug Flow for Boundary Condition Types B and C

Layer	Bo. Co. Type	Solution for Layer Temperature (in Slug Flow)
1 ≠ {U-1,N}	C	$T_1 = T_{cold} + [P_{el,1} - UA_{tank}/C_{tank}\, (x^-_{top,n} + t\, dx_{draw}/dt)\, (T^-_n - T_{amb})] / dx_{draw}/dt$
U-1 = 1	B	$T_1 = T_{cold} + [P_{el,1} - UA_{tank}/C_{tank}\, x_{upper}\, (T^-_1 - T_{amb})] / dx_{draw}/dt$
U ≠ N	C	$T_U = T_{U-1} + [P_{el,U} - UA_{tank}/C_{tank}\, (x^-_{top,1} + t\, dx_{draw}/dt)\, (T^-_U - T_{amb})] / dx_{draw}/dt$
N = U	B	$T_N = T_{N-1} + [P_{el,U} - UA_{tank}/C_{tank}\, (T^-_N - T_{amb})] / dx_{draw}/dt$
N = 1	B	$T_n = T_{cold} + [\Sigma_n P_{el,1} - UA_{tank}/C_{tank}\, (T^-_N - T_{amb})] / dx_{draw}/dt$

Solution when flow is less than the slug flow threshold.

When there is non-zero flow below the slug flow threshold with the heating element power on, Layer 1 or a Layer U remain distinct layers with rising temperatures, and the heat balance equation with a linearized heat loss rate is:

$$P_{el,n} - UA_{tank}\,\Delta x_n\,(T^-_n - T_{amb}) + C_{tank}\,dx_{draw}/dt\;T_{n-1} = C_{tank}\,\Delta x_n\,dT_n/dt \;+ C_{tank}\,dx_{draw}/dt\;T_n$$

which remains an ordinary 1st-order differential equation, albeit one that is readily solved.

It has the exact solution (previously developed; see the *Heat Balance for a Water Layer* section, above) of the form:

$$T_n = (T^-_n + c_n/b_n)\,e^{-b_n t} - c_n/b_n$$

Solving for the elapsed time for the layer's temperature (T_n) to reach a heating element setpoint (triggering a topology change, and sometimes also a state change), or reach the temperature of the layer above (triggering a topology change by merging the layers):

$$T_{trigger} + c_n/b_n\,/\,[T^-_n + c_n/b_n] = e^{-b_n t_{trigger}}$$

or

$$t_{trigger} = -\ln(\,T_{trigger} + c_n/b_n\,/\,(T^-_n + c_n/b_n)\,]\,/\,b_n$$

The "problem" with such solutions is that predicting the time it will take for the layer to reach a given temperature requires an iterative procedure. It would be nice to have a reasonably accurate linearized solution for this condition, as well …

Creation, Dissolution, and Merging Of Layers. The following conditions define a topology change for the water heater model, requiring the solution to pause, change topology, and then continue until the next topology change, flow regime change, or state change. Topology changes only occur for layers with a fixed top or bottom position (i.e., Layers 1, U-1, U, and N or combinations thereof.

Topology changes that occur solely due to flow of hot water draw from the tank are:

1. *New Layer 1*. A new Layer 1 is created at the bottom of the tank when there is a water draw and the current Layer 1's temperature is not equal to Tcold. The existing Layer 1 becomes Layer 2.
2. *Layer N Exits*. Layer N vanishes when it fully exits out the top of the tank. The layer beneath it becomes the new Layer N.
3. *Layer U-1 Exits*. Layer U-1 is re-designated as Layer U when it fully passes the upper heating element and when the power to the element is on. The layer beneath it re-designated as Layer U-1.

Topology changes that occur due to mergers with the layer above result from the lower layer of the pair being heated and are:

4. *Layers 1 and 2 Merge*. Layer 1 merges with Layer 2 when heating from the lower heating element raises its temperature to that of Layer 2. All layers above what was Layer 2 potentially are reassigned layer numbers.
5. *Layers U and U+1 Merge.* Layer U merges with Layer U+1 when heating from the upper heating element raises its temperature to that of Layer U+1.

Predicting the Time of the Next State and Topology Change. The *New Layer 1* case only occurs instantly when a water draw begins, i.e., when there has been a state change. No prediction of the time to a topology change is needed.

From the beginning of a time interval over which the flow of hot water from the tank can be assumed to be constant (see discussions above), the elapsed time to the next topology change (Δt^*) is the minimum of:

2. *Layer N Exits*: $\Delta t^* = (1 - x^-_{bot,N}) / dx_{draw}/dt$
3. *Layer U-1 Exits*: $\Delta t^* = (1 - x^-_{bot,U-1}) / dx_{draw}/dt$
4. *Layers 1 and 2 Merge*.
5. *Layers U and U+1 Merge.*