Geometric Degeneration of the Functional Equation

For the Riemann ζ-Function

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Theorem 1 (Geometric Degeneration via Zero Symmetry)**.** *Let $\rho = \sigma + it$ ($t \neq 0$) be a non-trivial zero of the Riemann ξ-function. To define the symmetry-deviation proxy, we do that as follows:*

$$e(\rho) = \frac{\sigma - \frac{1}{2}}{\sqrt{\left(\sigma - \frac{1}{2}\right)^2 + t^2}}.$$

Then $e(\rho) = 0$, and the foci ρ, $1 - \rho$ coincide $\implies$ the ellipse degenerates to a circle.

Proof. We prove this in seven steps using only the functional equation and the geometry of the zero set.

1. Functional equation: $\xi(s) = \xi(1 - s)$. Thus, if $\xi(\rho) = 0$, then $\xi(1 - \rho) = 0$.

2. Foci: Define $F_1 = \rho$, $F_2 = 1 - \rho$. The midpoint is $\frac{F_1+F_2}{2} = \frac{1}{2}$.

3. Distance difference:

$$d(z) = |z - F_1| - |z - F_2|.$$

The reflection $z \mapsto 1 - \overline{z}$ swaps $F_1 \leftrightarrow F_2$, so

$$d(1 - \overline{z}) = -d(z).$$

Thus $d(z)$ is *odd* under reflection across $\Re(s) = \frac{1}{2}$.

4. Zeros are symmetric (your lemma): Let z_0 be a zero. Then $1 - \overline{z_0}$ is also a zero. Hence the *set of all non-trivial zeros* is invariant under

$$\mathcal{R} : z \mapsto 1 - \overline{z}.$$

Note: $\mathcal{R}$ maps the critical line to itself.

5. Suppose a non-degenerate ellipse exists (a contradiction!) Then we are forced to assume there also exists a curve $\mathcal{E}$ of points z such as,

$$d(z) = c \quad (c \neq 0).$$

Suppose $\mathcal{E}$ is *E containing symmetric pairs.* Then:

- Let $z_0 \in \mathcal{E} \cap \{\text{zeros}\}$. Then $d(z_0) = c$.
- By Step 4, $z_0' = \mathcal{R}(z_0) = 1 - \overline{z_0}$ is also a zero, so $z_0' \in \mathcal{E}$.
- Thus $d(z_0') = c$.
- But by Step 3, $d(z_0') = -d(z_0) = -c$.

- Therefore $c = -c \implies c = 0$.

Contradiction unless $c = 0$.

6. Only $c = 0$ is possible $\implies$ foci coincide. The only symmetric ellipse compatible with the zero set is the *degenerate* one: $F_1 = F_2$. Thus $\rho = 1 - \rho \implies \sigma = \frac{1}{2}$.

7. Proxy $e(\rho) = 0$ and circle is forced.

$$e(\rho) = \frac{\sigma - \frac{1}{2}}{\sqrt{\left(\sigma - \frac{1}{2}\right)^2 + t^2}} = 0.$$

The ellipse degenerates to a **circle of radius zero** at the symmetry center. The foci are therefore mirror images across $\Re(s) = \frac{1}{2}$. Apart from number-theoretical potential this offers, and of course differential geometry, the truth is mathematics is more often than not a labyrinth of glass and mirrors, fractures, shadows of shadows.

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