# Essay on Geometric Degeneration of the Functional Equation, as interpreted in case of the Riemann $\zeta$-Function

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Although this short communication reflects what the author feels to be true, it does not reflect the idea of its truth. Rigorous presentation alone is detailed (and by far the most cherished), a fair argument can be made that rigor lacks precision. Equations alone do not explicate the whole narrative. On the contrary, a few well-placed metaphors could immediately make clear as to how and why a connection is logically sound and, in terms of narrative, coherent. Just as the explicit formula reveals an order hidden among the primes by lifting (or lowering, depending on one's perspective) to the complex plane, the functional equation could perhaps reveal an order deeper within the non-trivial zeros as a complex counterpart to Euclidean geometry. And Euclidean geometry, being the most constrained – the very essence of the ellipse precludes the possibility that they be both symmetric and non-degenerate, – suggests a geometric mechanism by which symmetry forces a clearly marked alignment; if a zero submits to an elliptical organization with paired foci, such a process may be extrapolated to more. That, however, is not the intent of this communication.

**Theorem 1** (Geometric Degeneration via Zero Symmetry). *Let $\rho = \sigma + it$ $(t \neq 0)$ be a non-trivial zero of the Riemann $\xi$-function. To define the symmetry-deviation proxy, we do that as follows:*

$$e(\rho) = \frac{\sigma - \frac{1}{2}}{\sqrt{\left(\sigma - \frac{1}{2}\right)^2 + t^2}}.$$

*Then $e(\rho) = 0$, and the foci $\rho$, $1 - \rho$ coincide $\implies$ the ellipse degenerates to a circle.*

*Proof.* We prove this in seven steps using only the functional equation and the geometry of the zero set.

**1. Functional equation:** $\xi(s) = \xi(1-s)$. Thus, if $\xi(\rho) = 0$, then $\xi(1-\rho) = 0$.

**2. Foci:** Define $F_1 = \rho$, $F_2 = 1 - \rho$. The midpoint is $\frac{F_1+F_2}{2} = \frac{1}{2}$.

**3. Distance difference:**

$$d(z) = |z - F_1| - |z - F_2|.$$

The reflection $z \mapsto 1 - \overline{z}$ swaps $F_1 \leftrightarrow F_2$, so

$$d(1 - \overline{z}) = -d(z).$$

Thus $d(z)$ is *odd* under reflection across $\Re(s) = \frac{1}{2}$.

**4. Zeros are symmetric (your lemma):** Let $z_0$ be a zero. Then $1 - \overline{z_0}$ is also a zero. Hence the *set of all non-trivial zeros* is invariant under the

$$\mathcal{R} : z \mapsto 1 - \overline{z}.$$

Note: $\mathcal{R}$ maps the critical line to itself.

**5. Suppose a non-degenerate ellipse exists (a contradiction!)** Then we are forced to assume there also exists a curve $\mathcal{E}$ of points $z$ such as,

$$d(z) = c \quad (c \neq 0).$$

Suppose $\mathcal{E}$ is *E containing symmetric pairs.* Then:

- Let $z_0 \in \mathcal{E} \cap \{\text{zeros}\}$. Then $d(z_0) = c$.
- By Step 4, $z_0' = \mathcal{R}(z_0) = 1 - \overline{z_0}$ is also a zero, so $z_0' \in \mathcal{E}$.
- Thus $d(z_0') = c$.
- But by Step 3, $d(z_0') = -d(z_0) = -c$.
- Therefore $c = -c \implies c = 0$.

**A contradiction** arises, unless $c = 0$. This argument applies only under the assumption that zeros are effectively organized into curves with a constant distance-difference $d(z)$. Even though this fundamental assumption aligns with the elliptical interpretation of the functional equation's symmetry (as presented in this short communication), any proof that strives for completeness would have to establish that that all non-trivial zeros form such curves as a matter of intrinsic necessity.

**6. Only $c = 0$ is possible $\implies$ foci coincide.** The only symmetric ellipse compatible with the zero set is the *degenerate* one: $F_1 = F_2$. Thus $\rho = 1 - \rho \implies \sigma = \frac{1}{2}$.

**7. Proxy $e(\rho) = 0$ and circle is forced.**

$$e(\rho) = \frac{\sigma - \frac{1}{2}}{\sqrt{\left(\sigma - \frac{1}{2}\right)^2 + t^2}} = 0.$$

The ellipse degenerates to a **circle of radius zero** at the symmetry center. The foci are therefore mirror images across $\Re(s) = \frac{1}{2}$. $\square$