The Einstein Relation on Metric Measure Spaces

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Introduction

When regarding an open bounded domain U in $\mathbb{R}^n$, the Einstein relation is an equation expressing that the geometric behavior – expressed in the asymptotic scaling of mass for small balls – is nicely compatible with the analytic structure given by the Dirichlet-Laplace operator Δ on U – expressed in the asymptotic behaviour of its eigenvalue counting function – and with the asymptotic velocity of the stochastic process induced by Δ, namely Brownian motion. With the development of analytic and stochastic theory on (mainly self-similar) fractals, it was also discoverd that the same relation holds on some fractals, most prominently on the Sierpinski gasket SG.

The main goal of this thesis is to provide a general framework for the Einstein relation. To achieve this, we consider metric measure spaces (X, d_X, μ_X), where (X, d_X) is a complete, separable, locally compact, and path-connected metric measure space (not consisting of only a single point) with an everywhere supported Radon measure μ_X on it.

The purpose of the first chapter is to formally introduce the Hausdorff dimension $\dim_{\mathcal{H}}$, the spectral dimension $\dim_{\mathcal{S}}$ and the walk dimension $\dim_{\mathcal{W}}$. For $\dim_{\mathcal{H}}$, we give a short sketch of its definition and some of its properties, including Hutchinson's theorem on the Hausdorff dimension of self-similar sets in $\mathbb{R}^n$. The walk dimension is first motivated by Weyl's result on the asymptotic growth of the Dirichlet-Laplacian's eigenvalue counting function and then defined for operators A on $L^2(X, \mu_X)$ that satisfy certain conditions. These conditions also ensure that there exists an essentially unique Hunt process having A as its inifinitesimal generator. The outline of this theory relating processes, operator semigroups, generators and Dirichlet forms is then given in section 1.2 and 1.3 before the (local) walk dimension and then the Einstein relation are defined.

In the second chapter, we begin by examining the above mentioned classical case of a domain in Euclidean space with Dirichlet-Laplace operator, and continue by presenting the constructions of the standard Laplace operator on the Sierpinski gasket as well as the construction of the Brownian motion on SG. Both constructions rely heavily on the self-similarity and on the fact that SG can be approximated by a sequence of graphs. This also requires a different approach to the walk dimension than the local one from section 1.3, as the vertex set of a graph is always discrete. To see why this graph-theoretic walk-dimension can not be directly adapted to metric measure spaces, we present a counterexample in section 2.3.

The third chapter begins by defining two different types of morphisms between mm-spaces, namely contractions and Lipschitz-maps, both of which give rise to a notion of isomorphy, mm-isomorphism and Lipschitz-isomorphisms, respectively. We then continue to investigate how we can transport the structure needed for the Einstein relation alongside maps $\varphi : (X, d_X, \mu_X) \to$

(Y, d_Y, μ_Y) and prove that the Einstein relation is invariant under Lipschitz-isomorphisms. We proceed by looking at Hölder continuous transformations and manage to proof upper bounds for the walk dimension and apply this to Brownian motions running on the graphs of independent fractional Brownian motion, which generates a family of examples where the Einstein relation might hold with a constant factor different from 1.

The concluding discussion contains several open questions that aim to further a general theory of the Einstein relation as an invariant of metric measure spaces.

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Chapter 1

Fractal Dimensions and the Einstein Relation

In this introductory chapter, we wish to briefly expose the ingredients of the Einstein relation – the Hausdorff dimension $\dim_{\mathcal{H}}$, the spectral dimension $\dim_{\mathcal{S}}$, and the walk dimension $\dim_{\mathcal{W}}$ – and state some of their properties.

1.1 Hausdorff measure and Hausdorff dimension

Although the concepts of Hausdorff measure and dimension are well-known, we give the definitions in the interest of completeness. In what follows, let (X, d) be a metric space.

Definition 1.1 (Hausdorff outer measure)**.** For fixed $s \geq 0$, any subset $S \subseteq X$ and any $\delta > 0$, let

$$\mathcal{H}^s_\delta(S) := \inf \left\{ \sum_{i \in I} (\operatorname{diam} U_i)^s : |I| \leq \aleph_0, S \subseteq \bigcup_{i \in I} U_i \subseteq X, \operatorname{diam} U_i \leq \delta \right\},$$

i.e. the infimum is taken over all countable coverings of S with diameter at most δ. The s-dimensional Hausdorff outer measure of S is now defined to be

$$\mathcal{H}^s(S) := \lim_{\delta \searrow 0} \mathcal{H}^s_\delta(S). \tag{1.1}$$

Observe that the limit in (1.1) exists or equals ∞, since $\mathcal{H}^s_\delta(S)$ is monotonically nonincreasing in δ and bounded from below by 0. Furthermore, it can be shown that $\mathcal{H}^s$ defines a metric outer measure on X, and thus restricts to a measure on a σ-algebra containing the Borel σ-algera $\mathscr{B}(X)$ (cf. [Mat99, p.54ff]). By definition, the obtained measure then is the s-dimensional Hausdorff measure which we will denote by $\mathcal{H}^s$ as well. Note that for $\mathcal{H}^s$ to be a Radon measure, i.e. locally finite and inner regular, $\mathcal{H}^s(X) < \infty$ is sufficient.

In the special case of (X, d) being an Euclidean space, Hausdorff measures interpolate between the usual Lebesgue measures λ^n: For $s = 0$, we have simply $\mathcal{H}^0(S) = \#S$, whereas for any integer $n > 0$, it can be shown that there exists a constant $c_n > 0$ depending only on n such that $\mathcal{H}^n = c_n \lambda^n$, where the constant is the volume of the n-dimensional unit ball.

It can be seen by simple estimates that the map $s \mapsto \mathcal{H}^s(S)$ for fixed $S \subseteq X$ is monotonically nonincreasing. More specifically, if $\mathcal{H}^s(S)$ is finite for some s then it vanishes for all $s' > s$, and conversely, if $\mathcal{H}^s(S) < \infty$ then $\mathcal{H}^{s'}(S) = \infty$ for all $s' < s$. Therefore, there exists precisely one real number s where $\mathcal{H}^{\cdot}(S)$ jumps from ∞ to 0 (by possibly attaining any value of $[0, \infty]$ there). This motivates the following definition of Hausdorff dimension:

Definition 1.2. The Hausdorff dimension $\dim_{\mathcal{H}}(S)$ of $S \subseteq X$ is defined as

$$\dim_{\mathcal{H}}(S) := \inf\{s \geq 0 : \mathcal{H}^s(S) < \infty\}.$$

Due to the above discussion, we have the following equalities:

$$\dim_{\mathcal{H}}(S) = \inf\{s \geq 0 : \mathcal{H}^s(S) < \infty\} = \inf\{s \geq 0 : \mathcal{H}^s(S) = 0\}$$
$$= \sup\{s \geq 0 : \mathcal{H}^s(S) = \infty\} = \sup\{s \geq 0 : \mathcal{H}^s(S) > 0\},$$

providing some alternative characterisations of the Hausdorff dimension.

We further collect some important facts. To this end, let S, S' and $S_1, S_2, ...$ be subsets of X as before. Then, the following properties hold (cf. [Fal07, p.32f] for a discussion in the Euclidean setting; however all arguments adapt to our more general situation without complication):

Monotonicity. If $S \subseteq S'$ then $\dim_{\mathcal{H}}(S) \leq \dim_{\mathcal{H}}(S')$.

Countable Stability. For a sequence $(S_n)_{n \geq 1}$, we have the equality

$$\dim_{\mathcal{H}} \left(\bigcup_{n \geq 1} S_n \right) = \sup_{n \geq 1} \dim_{\mathcal{H}}(S_n).$$

Countable Sets. If $|S| \leq \aleph_0$ then $\dim_{\mathcal{H}}(S) = 0$.

Hölder continuous maps. If (X', d') is another metric space and $f : X \to X'$ is α-Hölder continuous for some $\alpha \in (0, 1]$ then $\dim_{\mathcal{H}}(f(S)) \leq \alpha^{-1} \dim_{\mathcal{H}}(S)$. In particular, the Hausdorff dimension is invariant under a bi-Lipschitz transformation (i.e. an invertible map f with Hölder exponent $\alpha = 1$ for both f and f^{-1}).

Euclidean Case. If (X, d) happens to be an Euclidean space (or more generally a continuously differentiable manifold) of dimension n and S is an open subset then $\dim_{\mathcal{H}}(S) = n$.

We conclude this section by discussing Hutchinson's theorem about the Hausdorff dimension of self-similar sets. For this, we recall that a map $F : X \to X$ on a metric space (X, d) is a strict contraction if its Lipschitz constant satisfies

$$\mathrm{Lip}_F := \sup_{\substack{x,y \in X \\ x \neq y}} \frac{d(F(x), F(y))}{d(x, y)} < 1. \tag{1.2}$$

If the stronger condition $d(F(x), F(y)) = \mathrm{Lip}_F \, d(x, y)$ holds for all $x, y \in X$, we call F a similitude with contraction factor Lip_F.

Theorem 1.3 (Hutchinson, [Hut81])**.** *Let $\mathscr{S} = \{S_1, ..., S_N\}$ be a finite set of strict contractions on the Euclidean space $\mathbb{R}^n$. Then there exists a unique nonempty compact set denoted*

by $|\mathscr{S}|$ which is invariant under $\mathscr{S}$, i.e.

$$|\mathscr{S}| = \bigcup_{i=1}^{N} S_i(|\mathscr{S}|).$$

Furthermore, assume that $|\mathscr{S}|$ satisfies the open set condition (OSC) meaning that there exists a nonempty open set $O \subseteq X$ with the properties $S_i(O) \subseteq O$ and $S_i(O) \cap S_j(O) = \emptyset$ for all $i, j = 1, ..., N$ with $i \neq j$. Also assume that the maps S_i are similitudes with contraction factor $r_i \in (0, 1)$. Then, $s = \dim_{\mathcal{H}}(|\mathscr{S}|)$ is the unique solution to the equation

$$\sum_{i=1}^{N} r_i^s = 1$$

and we have $0 < \mathcal{H}^s(|\mathscr{S}|) < \infty$.

The proof is based on two different ideas: Existence and uniqueness can be shown in any complete metric space (X, d) by envoking the Banach fixed point theorem for the map $\bigcup_{i=1}^{N} S_i$ acting on the space $(\mathcal{F}(X), d_H)$ of all closed subsets of X equipped with Hausdorff distance:

$$d_H(A, A') := \inf \left\{ \varepsilon > 0 : A' \subseteq B(A, \varepsilon) \text{ and } A \subseteq B(A', \varepsilon) \right\} \quad \text{for} \quad A, A' \in \mathcal{F}(X).$$

Indeed, d_H is a complete metric if d is. The statement about the Hausdorff dimension and measure relies on a rather easy upper estimate on $\dim_{\mathcal{H}}$ and an application of the mass distribution principle to get $\mathcal{H}^s(|\mathscr{S}|) > 0$:

Lemma 1.4 (Mass distribution principle, Frostmann, [Mat99, Thm 8.8]). *For a Borel set $X \subseteq \mathbb{R}^n$, we have $\mathcal{H}^s(X) > 0$ if and only if there exists a Borel measure μ on X such that $\mu(B(x, r)) \leq r^s$ for $x \in \mathbb{R}^n$ and $r > 0$.*

While uniqueness and existence of $|\mathscr{S}|$ in theorem 1.3 are still ensured for maps on a complete metric space, the open set condition is not sufficient for statements about the Hausdorff dimension, see [Sch96] for further discussion.

1.2 Weyl asymptotics and spectral dimension

The idea of introducing spectral dimension is inspired by Weyl's law for the eigenvalues of the Dirichlet-Laplace operator which we will discuss here shortly before defining a larger class of operators that have similar spectral properties and are infinitesimal generators of Markov processes.

1.2.1 The classical case

Given a bounded open domain $U \subseteq \mathbb{R}^n$, consider the Laplace operator Δ on X acting on functions satisfying the Dirichlet boundary condition $u \equiv 0$ on ∂U. Then, the spectrum of $-\Delta$ consists of non-negative eigenvalues with a single accumulation point at ∞. Hence we can order them in a

non-increasing way, counting the geometric multiplicities, as

$$0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ... \quad \text{with} \quad \lambda_n \nearrow \infty. \tag{1.3}$$

In this setting, it makes sense to define the eigenvalue counting function via

$$N_{-\Delta}(x) := \max\{n \in \mathbb{N} : \lambda_n \leq x\}, \ x \in \mathbb{R}_{\geq 0}. \tag{1.4}$$

Weyl's law now states that there is the asymptotic equivalence[1]

$$N_{-\Delta}(x) \sim C_n \mathcal{H}^n(U) x^{n/2}, \quad x \nearrow \infty, \tag{1.5}$$

where the constant C_n is independent of the domain U (see [Wey11] and [Wey12] for the original publications). Motivated by (1.5), we define the spectral dimension of $-\Delta$ on U by

$$\dim_{\mathcal{S}}(U, -\Delta) := \lim_{x \to \infty} \frac{\log N_{-\Delta}(x)}{\log x} \tag{1.6}$$

which yields $n/2$ in the situation examined by Weyl's law. Note that the usual definition of $\dim_{\mathcal{S}}$ differs by a factor of 2 (cf. [KL93],[HKK02]) so that $\dim_{\mathcal{S}}(U, \Delta)$ normally coincides with $\dim_{\mathcal{H}}(U) = n$. However, this comes at the cost of an additional factor in the Einstein relation. Moreover, it can be argued that the spectral dimension is rather a property of the operator $-\Delta$ than of the underlying space U. Therefore, we take the liberty to deviate from the established convention in this minor aspect.

1.2.2 The general case

How can we generalise the concepts just introduced to sets which are not bounded open subsets of $\mathbb{R}^n$? For this, suppose we are given a metric measure space (X, d, μ), where (X, d) is a locally compact separable metric space and μ is a Radon measure on X.

Of course, the notion of an eigenvalue counting function as outlined above works for any operator A whose set of eigenvalues possesses only one limit point at $+\infty$. However, as we will explain in the next section, we also wish to associate a reasonably well-behaved Markov process with state space X to A. This requires several additional assumptions that will be motivated in this and the next section. More precisely, we choose to impose the following conditions on A:

Assumptions 1.5. For an operator $A : L^2(X, d) \supseteq \mathscr{D}(A) \to L^2(X, d)$, we assume the following holds:

Self-adjointness. A is a densely defined, self-adjoint (and therefore closed) operator on the Hilbert space $L^2(X, \mu)$.

Eigenvalues. The spectrum is contained in $\mathbb{R}_{\geq 0}$ and the set of eigenvalues can be enumerated as in (1.3).

Regularity (of the corresponding Dirichlet form). The set[2] $\mathscr{D}(\sqrt{A}) \cap C_c(X)$ is dense in $C_c(X)$ with respect to the sup-norm and is dense in the domain $\mathscr{D}(\sqrt{A})$ endowed

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[1] We adopt the notation $f \sim g$ for the equivalence relation given by $\lim \frac{f}{g} = 1$.

[2] We denote by $C_c(X)$ the space of all compactly supported continuous functions on X

with the graph norm

$$\|f\|_{\sqrt{A}} := \|f\|_{L^2} + \|\sqrt{A}f\|_{L^2}, \quad f \in \mathscr{D}(\sqrt{A})$$

Dissipativeness. $-A$ is dissipative. In other words, for all $f \in \mathscr{D}(A)$ and all $\lambda > 0$, we have $\|(\lambda + A)f\| \geq \lambda\|f\|$.

Here, $\sqrt{A}$ denotes an operator satisfying $\sqrt{A} \circ \sqrt{A} = A$ that can be defined via a standard spectral-theoretic construction. We will not go into greater detail here and refer to [QUXLLX!] instead.

The first of these assumptions guarantees that A is a closed operator, whereas the second ensures that $\lambda + A$ is surjective for at least one $\lambda > 0$. Thus, the Hille-Yosida theorem states that there is a strongly continuous semigroup of contractive linear operators T_t on H such that $-A$ is its infinitesimal generator. That is to say:

Definition 1.6. A strongly continuous semigroup $(T_t)_{t \geq 0}$ on a Hilbert space H is a monoid homomorphism $t \mapsto T_t$ from $(\mathbb{R}_{\geq 0}, +)$ to the space of bounded linear operators $(\mathbb{B}(H), \cdot)$ on H (equipped with composition) satisfying for all $f \in H$ the additional property

$$\lim_{t \searrow 0} \|T_t f - f\| = 0.$$

The infinitesimal generator $(-A, \mathscr{D}(A))$ of $(T_t)_{t \geq 0}$ is defined via

$$(-A)f = \lim_{t \searrow 0} \frac{1}{t}(T_t f - f), \ f \in \mathscr{D}(A),$$

where $\mathscr{D}(A)$ is the set of elements in H for which this limit exists.

Theorem 1.7 (Hille–Yosida,[MR12])**.** *An operator $(-A, \mathscr{D}(A))$ is the generator of a strongly continuous semigroup $(T_t)_{t \geq 0}$ with $\|T_t\| \leq 1$ for all $t \geq 0$ if and only if $-A$ is a densely defined, closed, dissipative operator such that for some $\lambda > 0$, the map $\lambda + A$ is surjective.*

It can be shown that there is a one-to-one correspondence between contractive semigroups and operators that satisfy the Hille-Yosida theorem, that is, the semigroup in the above theorem is uniquely determined by A.

Remark 1.8. In the above list of assumptions, dissipativeness is redundant as we regard operators on Hilbert spaces. In this setting, $-A$ is dissipative if A is a positive operator since

$$\|(\lambda + A)f\|^2 - \lambda^2\|f\|^2 = \|Af\|^2 + 2\lambda \langle f, Af \rangle \geq 0$$

Having discussed the motivation for the assumptions 1.5, we now proceed to adapt the definitions made in (1.4) and (1.6) in a rather straightforward way:

Definition 1.9. Given an operator $(A, \mathscr{D}(A))$ on $L^2(X, \mu)$ satisfying the assumptions 1.5, its eigenvalue counting function is defined by

$$N_A(x) := \max\{n \in \mathbb{N} : \lambda_n \leq x\}, \ x \in \mathbb{R}_{\geq 0}, \tag{1.7}$$

and the spectral dimension of A by

$$\dim_{\mathcal{S}}(X, A) := \lim_{x \to \infty} \frac{\log N_A(x)}{\log x} \tag{1.8}$$

if the limit exists.

1.3 Markov processes and walk dimension

1.3.1 From Dirichlet forms to Markov processes

The theory presented here is mostly taken from [FOT11] and [MR12, ch. 4]. Set $H = L^2(X, \mu)$ where μ is a σ-finite Borel-measure on X.

Definition 1.10. A map $\mathcal{E} : \mathscr{D}(\mathcal{E}) \times \mathscr{D}(\mathcal{E}) \to \mathbb{R}$ is a Dirichlet form if it satisfies the following conditions:

  i. The domain $\mathscr{D}(\mathcal{E}) \subseteq H$ of $\mathcal{E}$ is a dense linear subspace.
  ii. $\mathcal{E}$ is a symmetric, non-negative definite bilinear form.
  iii. This form is closed, that is, the inner product space $(\mathscr{D}(\mathcal{E}), \mathcal{E}_\alpha)$ equipped with the scalar product

$$\mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha \langle u, v \rangle \quad \text{for} \quad u, v \in \mathscr{D}(\mathcal{E}_\alpha) = \mathscr{D}(\mathcal{E}), \ \alpha > 0,$$

  is complete (and thus itself a Hilbert space).
  iv. $\mathcal{E}$ is a Markovian form, i.e. for all $u \in \mathscr{D}(\mathcal{E})$, $v := (0 \vee u) \wedge 1 \in \mathscr{D}(\mathcal{E})$ and we have $\mathcal{E}[v] \leq \mathcal{E}[u]$ for the quadratic form of $\mathcal{E}$.

We remark that the choice of $\alpha > 0$ is irrelevant for the completeness of $(\mathscr{D}(\mathcal{E}), \mathcal{E}_\alpha)$ since all induced norms are equivalent to each other.

Definition 1.11. A Dirichlet form $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ on $L^2(X, \mu)$ is said to be

  i. regular if it possesses a core, that is, the space $\mathscr{D}(\mathcal{E}) \cap C_c(X)$ is simultaneously dense in $\mathscr{D}(\mathcal{E})$ with respect to the $\mathcal{E}_1$-norm and in $C_c(X)$ with respect to the uniform norm.
  ii. local if $\mathcal{E}(u, v) = 0$ whenever $u, v \in \mathscr{D}(\mathcal{E})$ have disjoint compact support.
  iii. strongly local if $\mathcal{E}(u, v) = 0$ whenever $u, v \in \mathscr{D}(\mathcal{E})$ have compact support and v is constant on a neighbourhood of $\text{supp}(u)$.

If additionally $\mu(X) < \infty$, we say that $\mathcal{E}$ is

  iv. conservative if $1 \in \mathscr{D}(\mathcal{E})$ and $\mathcal{E}[1] = 0$.
  v. irreducible if it is conservative and $\mathcal{E}[f]$ implies that f is constant.

We can uniquely attach a positively semidefinite operator A to a Dirchlet form (and vice versa) via the relation

$$\mathcal{E}(u, v) = \langle Au, v \rangle, \ u \in \mathscr{D}(A), v \in \mathscr{D}(\mathcal{E}). \tag{1.9}$$

In particular, if A meets the requirements of 1.5, we not only have precisely one strongly continuous contraction semigroup on H as explained by theorem 1.7, but also a unique Dirichlet form thanks to (1.9). In similar style, we would also like to attach a unique Markov process to A - or, equivalently, to the semigroup or the Dirichlet form.

To define a suitable stochastic process with values in X, we first adjoin a cemetary state $*$ in such a way that if X is non-compact, $X_* := X \sqcup \{*\}$ is the one-point compactification of X, whereas $*$ is supposed to be an isolated point if X is compact. Let $X = (\Omega, \mathscr{A}, (X_t)_{t \geq 0}, (\mathbf{P}_x)_{x \in X_*})$ be a stochastic process on a measurable space $(\Omega, \mathscr{A})$ with values in X_*, where we adapt the notation that $\mathbf{P}_x[X_0 = x] = 1$ for all $x \in X_*$ and $\mathbf{P}_*[X_t = *] = 1$ for all $t \geq 0$. Note that X induces a filtration $\mathscr{F} = (\mathscr{F}_t)_{t \geq 0}$ on $\mathscr{A}$ by

$$\mathscr{F}_t = \bigcap_{\mathbf{P} \in \mathcal{M}_1^+(\Omega, \mathscr{A})} \left(\sigma\{X_s : 0 \leq s \leq t\} \right)^{\mathbf{P}}.$$

Here, $\mathcal{M}_1^+$ denotes the set of all probability measures on $(\Omega, \mathscr{A})$, $\sigma\{\cdot\}$ denotes the the σ-algebra generated by $\{\cdot\}$ and $\mathscr{B}^{\mathbf{P}}$ denotes the completion of a σ-algebra $\mathscr{B}$ with respect to the measure $\mathbf{P}$. Henceforth, we will only consider stochastic processes X that satisfy the strong Markov property with respect to $\mathscr{F}$ and are time-homogenous. Such X is called Hunt process if it additionally has right-continuous trajectories with left-limits and is quasi-left-continuous, i.e. any sequence $\tau_n \nearrow \tau$ of $\mathscr{F}$-stopping times satisfies

$$\mathbf{P}_\alpha \left[\lim_{n \to \infty} X_{\tau_n} = X_\tau, \tau < \infty \right] = \mathbf{P}_\alpha[\tau < \infty]$$

for any initial distribution α. We can now translate Markov processes to contractive semigroups by setting

$$(T_t f)(x) := \mathbf{E}_x \left[f(X_t) \right], \ t \geq 0. \tag{1.10}$$

The other direction is more involved, and the process attached to a Dirichlet form is generally non-unique. We have, however, (cf. [FOT11, theorems 7.2.1 and 7.2.2])

Theorem 1.12. *Let $\mathcal{E}$ be a regular Dirichlet form on $L^2(X, \mu)$. Then, there exists a Hunt process X on (X, d) such that the operators $T_t, t \geq 0$, from (1.10) are symmetric and $\mathcal{E}$ is the Dirichlet form belonging to this semigroup.*

Moreover, if $\mathcal{E}$ is local, X is a diffusion process.

As hinted above, those processes are not unique: One can modify X to $\tilde{X}$ by killing the process on a polar set and obtain the same semigroup for both. See section 7.2.2. in [FOT11] for further discussion.

1.3.2 Local walk dimension and Einstein relation

The walk dimension is meant to quantify how fast a given Markov process on X moves away from its starting point x. This is best expressed in terms of the stopping time $\tau_{B(x,r)}$, which is supposed to be the first exit time of the Ball $B(x, r) = \{y \in X : d(x, y) < r\}$. Note that this is indeed an $\mathscr{F}$-stopping time by right continuity of the process in question and by [Kal02, Lemma 7.6].

For the next definition to make sense, we need to impose some additional assumption on the metric space (X, d). We choose to demand for now that X is path connected, but will discuss other scenarios in the next chapter.

Definition 1.13. (Cf. [HKK02]) We define the quantity

$$\dim_{\mathcal{W}}(X, X, x) = \lim_{r \searrow 0} \frac{\log \mathbf{E}_x \left[\tau_{B(x,r)} \right]}{\log r}$$

and call it the (local) walk dimension of (X, d) at $x \in X$ with respect to the Markov process $(X_t)_{t \geq 0}$. If $\dim_{\mathcal{W}}(X, X, x)$ is μ–a.e. constant on X, we shorten our notation to $\dim_{\mathcal{W}}(X, X)$.

We are now finally able to state the Einstein relation:

Definition 1.14. Let (X, d, μ) be a locally compact separable metric measure space and let $(A, \mathscr{D}(A))$ be an operator on $L^2(X, \mu)$ satisfying assumptions 1.5. Suppose $X = ((X_t)_{t \geq 0}, (\mathbf{P}_x)_{x \in X_*})$ is a Markov process associated to A via the Dirichlet form $\mathcal{E}(\cdot, \cdot) = \langle A \cdot, \cdot \rangle$. We then say that the Einstein relation with constant c holds on X with respect to A if

$$\dim_{\mathcal{H}}(X) = c \dim_{\mathcal{S}}(X, A) \dim_{\mathcal{W}}(X, X). \tag{1.11}$$

We omit mentioning the constant if $c = 1$.

1.4 Other versions of the Einstein relation

TO DO: A discussion of further literature, such as Telcs, Mandelbrot, etc... is planned to be included here. Also, hints to the physical background.

Chapter 2

Examples and Non-examples

In this chapter, we will discuss the necessity of some of the restrictive assumptions made previously and explore the Einstein relation by examining some examples and – by doing so – will motivate some of the more general results of the next chapter.

2.1 Euclidean Space

We start by examining the classical setting of paragraph 1.2.1 in greater detail: Let once again $X \subseteq \mathbb{R}^n$ be an open, bounded, non-empty domain, equipped with Euclidean metric and n-dimensional Lebesgue-measure λ^n. Trivially, $\dim_{\mathcal{H}}(X) = n$.

For the Dirichlet-Laplace operator as introduced earlier, we obtain $\dim_{\mathcal{S}}(X, -\Delta) = \frac{n}{2}$ due to (1.4)-(1.6). Simultaneously, it is well-known that $\frac{1}{2}\Delta$ is the generator of the n-dimensional Brownian motion B_t, which can easily be seen as follows:

By (1.10), the semigroup $(T_t)_{t \geq 0}$ induced by B_t reads

$$T_t f(x) = \mathbf{E}_x\left[f(B_t)\right] = \frac{1}{(2\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{2t}\right) f(y) \, \mathrm{d}y, \ f \in L^2(\mathbb{R}^n, \lambda^n) \cap C_c(E),$$

Comparing this expression to the well-known convolution formula (see [Eva10, p.47])

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y) \, \mathrm{d}y, \ x \in \mathbb{R}^n, t \geq 0,$$

for the solution of the heat equation $\Delta u = \partial_t u$ on $\mathbb{R}^n$ with initial value $u(x,0) = f(x)$, we can quickly derive that $\Delta T_t f(x) = 2\partial_t T_t f(x)$. Since imposing the Dirichlet boundary conditions on Δ corresponds to killing B_t at the boundary of X, we thus obtain by definition 1.6 the generator

$$Af = \frac{1}{2}\Delta f,$$

extended to its maximal domain $H_0^1(E, \mu)$. We conclude that the Markov process associated with Δ is $2B_t$.

It remains to determine the walk dimension of the n-dimensional Brownian motion. This can

be done in several ways, for example by appealing to Brownian scaling or by invoking Dynkin's formula after a standard truncation argument: By applying [Kal02, Lemma 19.21] to the function $u_x(y) = \frac{1}{2}|y - x|^2$, we get

$$\mathbf{E}_x\left[\tau_{B(x,r)}\right] = \mathbf{E}_x\left[\int_0^{\tau_{B(x,r)}} \Delta u_x(2B_s) \, \mathrm{d}s\right] = \mathbf{E}_x\left[u_x\left(2B_{\tau_{B(x,r)}}\right) - u_x(0)\right] = \mathbf{E}_x\left[\frac{2r^2}{n}\right] = \frac{2r^2}{n}$$

Therefore, by definition 1.13, we obtain $\dim_{\mathcal{W}}(E, 2B_t) = 2$ which implies together with the results obtained previously that the Einstein relation (with constant 1) holds on E.

2.2 The Sierpinski Gasket

The Sierpinski Gasket is a simple example of an iterated function fractal and can be described according to theorem 1.3 as the unique non-empty compact set $\mathrm{SG} \subseteq \mathbb{R}^2$ which is invariant under the three similitudes

$$S_1(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right), \ \ S_2(x, y) = \left(\frac{x+1}{2}, \frac{y}{2}\right), \ \ S_3(x, y) = \left(\frac{2x+1}{4}, \frac{2y + \sqrt{3}}{4}\right),$$

see PICTURE! Since SG satisfies the (OSC), e.g. by taking the open equilateral triangle with corners $(0, 0), (0, 1)$ and $(1/2, \sqrt{3}/2)$, we obtain both

$$s = \dim_{\mathcal{H}}(\mathrm{SG}) = \frac{\ln 3}{\ln 2} \tag{2.1}$$

and $\mathcal{H}^s(\mathrm{SG}) \in (0, \infty)$ by a second appeal to Hutchinson's theorem.

We will use the remainder of this section to establish the validity of the Einstein relation on SG with respect to the standard Laplace operator on SG, which can be obtained in two different ways. In order to describe these constructions, we first need to fix some notation.

Let $\mathscr{S} = \{S_1, S_2, S_3\}$ as in section 1.1, set $\Sigma = \{1, 2, 3\}$ and denote by

$$\Sigma^* := \{\varepsilon\} \cup \bigcup_{n \geq 1} \Sigma^n$$

the free monoid consisting of all finite words over the alphabet Σ, where the monoid operation is given by concatenation and ε is supposed to represent the empty word. Using the monoid isomorphism $\Sigma^* \cong (\mathscr{S}^*, \circ)$ given by extending $\Sigma \ni i \mapsto S_i \in \mathscr{S}$, we can identify a word of length l,

$$w = w_1...w_l \in \Sigma^l,$$

with the composition

$$S_w := S_{w_1} \circ ... \circ S_{w_l}.$$

By abuse of notation, we will therefore write $w(x)$ instead of $S_w(x)$. By an n-cell we understand the set $w(\mathrm{SG}) \subseteq \mathrm{SG}$, where w is a word of length n. Note that two different cells are either disjoint, or intersect in a single point which we will then call conjunction point, or one of them is completely contained in the other one.

It is possible to approximate SG by a sequence of graphs G_n. Those graphs can be thought of as planar graphs with a triangle for each n-cell, where the vertices are the conjunction points between them. More precisely, let G_n be the graph embedded in $\mathbb{R}^2$ with vertex set V_n inductively defined by

$$V_0 := \left\{ (0,0), (0,1), \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right) \right\}$$

$$V_{n+1} := (S_1 \cup S_2 \cup S_3)(V_n), \quad n \geq 0.$$

Additionally, set

$$V* := \bigcup_{n \geq 0} V_n.$$

Note that $V_0 \subseteq V_1 \subseteq ... \subseteq V_n \subseteq ...$ and that

$$V_n = \bigcup_{w \in \Sigma^n} w(V_0).$$

It also follows from the proof of Hutchinson's theorem that $d_H(V_n, \text{SG}) \to 0$ as $n \to \infty$, so indeed, the graphs G_n approximate SG.

In G_n, connect two vertices $x, y \in V_n$ by a straight edge xy iff they belong to the same n-cell, cf PICTURE!!, in which case we will call them neighbours and write $x \sim_n y$. By E_n we mean the set of all edges in G_n.

2.2.1 Approximation by Dirichlet forms[1]

This analytic approach works by establishing so-called energy forms on graphs G_n – these are graph-theoretic discretisations of the Dirichlet form attached to the Laplace operator. For $n \in \mathbb{N}$, define a bilinear form $\tilde{\mathcal{E}}_n$ on $L^2(V_n) \cong \mathbb{R}^{\#V_n}$ by

$$\tilde{\mathcal{E}}_n(f, g) := \sum_{xy \in E_n} (f(x) - f(y))(g(x) - g(y)), \quad f, g \in L^2(V_n).$$

It is easy to check that this defines a local Dirichlet form on $L^2(V_n)$. As we will see, these bilinear forms are compatible with each other after a suitable renormalisation. Starting from $\tilde{\mathcal{E}}_0$, consider a function $u \in \mathbb{R}^3 \cong L^2(V_0)$. Under all possible extensions $\tilde{u}$ to $\mathbb{R}^6 \cong L^2(V_1)$, which function is the harmonic extension of u, i.e. minimises $\tilde{\mathcal{E}}_1[\tilde{u}]$?

To answer this question, we label each vertex in V_0 by its value under u, say a, b, c and each vertex in $V_1 \setminus V_0$ by its value under $\tilde{u}$, say x, y, z as in PICTURE 3!!!. Since we assume u to be a fixed given function, the values a, b, c are fixed, and we obtain

$$\tilde{\mathcal{E}}_1[\tilde{u}] = \Big[(x - b)^2 + (x - c)^2 + (y - a)^2 + (y - c)^2 + (z - a)^2 + (z - b)^2$$
$$+ (x - y)^2 + (y - z)^2 + (z - x)^2 \Big] \quad (2.2)$$

Finding the minimising values for x, y, z now becomes an exercise in multivariable calculus.

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[1]The material in this section is an overview of the construction given in [Str06, chapter I]

Setting the partial derivatives equal to 0 yields the following system of linear equations:

$$8x = 2(b + c + y + z) \qquad 5x = a + 2b + 2c$$
$$8y = 2(a + c + x + z) \iff 5y = 2a + b + 2c$$
$$8z = 2(a + b + x + y) \qquad 5z = 2a + 2b + c$$

Plugging this solution as $\tilde{u}$ in (2.2) to evaluate $\tilde{\mathcal{E}}_1[\tilde{u}]$ gives

$$\tilde{\mathcal{E}}_1[\tilde{u}] = \frac{1}{25}\Big[(a - 3b + 2c)^2 + (a + 2b - 3c)^2 + (-3a + b + 2c)^2 + (2a + b - 3c)^2$$
$$+ (-3a + 2b + c)^2 + (2a - 3b + c)^2 + (b - a)^2 + (c - b)^2 + (a - c)^2\Big]$$
$$= \frac{30}{25}\Big[a^2 + b^2 + c^2 - ab - ac - bc\Big]$$
$$= \frac{3}{5}\tilde{\mathcal{E}}_0[u]$$

Using the fact that the vertices in $V_{n+1} \setminus V_n$ are in G_n only adjacent to vertices in V_n and that $\tilde{\mathcal{E}}_{n+1}$ is local on n-cells it is possible to show by induction that each $u \in L^2(V_n)$ allows for a unique harmonic extension $\tilde{u} \in L^2(V_{n+1})$ and that

$$\tilde{\mathcal{E}}_{n+1}[\tilde{u}] = \frac{3}{5}\tilde{\mathcal{E}}_n[u]$$

for all $n \geq 0$. This allows us to renormalise the bilinear form $\tilde{\mathcal{E}}_n$ via

$$\mathcal{E}_{n+1} := \left(\frac{3}{5}\right)^n \tilde{\mathcal{E}}_n$$

thus ensuring $\mathcal{E}_{n+1}[\tilde{u}] = \mathcal{E}_n[u]$ and therefore $\mathcal{E}_{n+1}[\hat{u}] \geq \mathcal{E}_n[u]$ for any extension $\hat{u}$ of u.

Consider now any function $u : V^* \to \mathbb{R}$ and denote its restriction to V_n by u_n. Then, $\mathcal{E}_n[u_n]$ is a nondecreasing sequence of nonnegative real numbers and therefore converges against

$$\mathcal{E}[u] := \lim_{n\to\infty} \mathcal{E}_n[u_n] \in [0, \infty].$$

Define $\mathscr{D}(\mathcal{E})$ to be the set of all functions $u : V^* \to \mathbb{R}$ for which this limit is finite. It can be shown that $u \in \mathcal{E}$ implies that u is Hölder continuous and can therefore be uniquely extended to a continuous function on all of SG. By abuse of notation we shall denote this extension by u as well and set $\mathcal{E}[u] := \mathcal{E}[u|_{V^*}]$ whenever the right-hand side is defined. Hence, by polarisation, we obtain a bilinear form $\mathcal{E}(\cdot, \cdot)$ on $\mathscr{D}(\mathcal{E})$.

We further introduce the measure $\mu(\cdot) := \mathcal{H}^s(SG)^{-1}\mathcal{H}^s(\cdot)$ on SG. It is possible to show that the biliner form $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ is a local regular Dirichlet form on $L^2(SG, \mu)$ which in turn is attached to an operator as in equation (1.9). This operator is what is known as standard Laplacian on SG, and its spectral dimension is known to be $\frac{\log 3}{\log 5}$ (see e.g. [Str06, section 3.5] or [KL93]).

2.2.2 Approximation by random walks[2]

It remains to discuss the walk dimension of the diffusion process generated by the standard Laplace operator on SG. Once again, this construction uses the approximating graphs G_n.

For $n \geq 0$, let $Y_k^{(n)}, k \in \mathbb{N}_0$ be a simple random walk on G_n, i.e. given $Y_k^{(n)} = v \in V_n$, the process has equal probability to jump to each of the neighbours of v in V_n.

Take $v \in V_{n-1}$. Then, v is contained in either 1 or 2 $(n-1)$-cells with conjunction points $\{x_1, ..., x_i\} \subseteq V_n \setminus \{v\}$, $i \in \{2, 4\}$ depending on whether or not $v \in V_0$. Assume $Y_0^{(n)} = v$, $n \geq 1$. The only way for $Y^{(n)}$ to leave the $(n-1)$-cells containing v is via the points $\{x_1, ..., x_i\}$. However, due to the symmetry, the probabilities of hitting $x_j, j = 1, ..., i$ first are equal. If we set

$$T_0^{n,m} = \inf \left\{ t \in \mathbb{N}_0 : Y_t^{(n)} \in V_m \right\}$$

$$T_{k+1}^{n,m} = \inf \left\{ t \in \mathbb{N}_0, t > T_k^{n+1,m} : Y_t^{(n)} \in V_m \setminus \left\{ Y_{T_k^{n,m}}^{(n)} \right\} \right\}, \quad k \geq 0,$$

for $n > m$, then $\left(Y_{T_k^{n,m}}^{(n)} \right)_{k \in \mathbb{N}_0}$ is a standard random walk on G_m and therefore equal to $\left(Y_k^{(m)} \right)_{k \in \mathbb{N}_0}$ in distribution. Using self-similarity and standard arguments for finite Markov-chains, it can be shown that $\mathbf{E}\left[T_{k+1}^{n,m} - T_k^{n,m} \right] = 5^{n-m}$ and that $Y^{(n)}$ overcomes a distance (in graph metric) of 2^{n-m}. The renormalised processes $X_k^{(n)} := Y_{5^n k}^{(n)}$, $k \in \mathbb{N}_0$ furthermore converge almost surely and uniformly on compact intervals to a continuous limit process $(X_t)_{t \geq 0}$ with values in SG. Additionally, the infinitesimal generator of X coincides with the SG-Laplacian introduced above. By construction, this process possesses a self-similarity, in the sense that $(X_{5t})_{t \geq 0} = (2X_t)_{t \geq 0}$ in distribution for sufficiently small t, and it is not hard to show that $\dim_{\mathcal{W}}(\text{SG}, X) = \frac{\log 5}{\log 2}$

Putting everything together, we obtain for the Sierpinski gasket:

$$\frac{\log 3}{\log 2} = \dim_{\mathcal{H}}(\text{SG}) = \dim_{\mathcal{S}}(\text{SG}, \Delta) \dim_{\mathcal{W}}(\text{SG}, X) = \frac{\log 3}{\log 5} \frac{\log 5}{\log 2},$$

so indeed, the Einstein relation holds on SG.

2.3 The real line as bounded metric space

$$\lim_{R \nearrow \infty} \frac{\log \mathbf{E}_x \left[\tau_{B(x,R)} \right]}{\log R}. \tag{2.3}$$

Bounded metric spaces form the most important class of spaces for which too naive of an adaption of (2.3) does not yield useful results. Indeed, consider the metric measure space $E = (\mathbb{R}, d_{\arctan}, \lambda^1)$, where the metric is defined as $d_{\arctan}(x, y) = |\arctan(x) - \arctan(y)|$. Since

$$\tan : \left(\left(-\frac{\pi}{2}, \frac{\pi}{2} \right), |\cdot| \right) \to (\mathbb{R}, d_{\arctan})$$

provides an isometry, we have $\dim_{\mathcal{H}}(E) = 1$. On this space, we consider the negative of the usual weak Laplace operator, $-\Delta_{\lambda^1}$, defined by mapping a function $u \in H_0^1(\mathbb{R}, \lambda^1)$ to the unique

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[2]The material in this section is an overview of the construction given in [Bar98, chapter II]

$g \in L^2(\mathbb{R}, \lambda^1)$ such that

$$\int_{\mathbb{R}} g\varphi \, \mathrm{d}\lambda^1 = \int_{\mathbb{R}} \partial_x u \partial_x \varphi \, \mathrm{d}\lambda^1$$

holds for all $\varphi \in H_0^1(\mathbb{R}, \lambda^1)$. Notice how this does not differ from the negative weak Laplace operator on $(\mathbb{R}, |\cdot|, \lambda^1)$ since we did not change the measure and both metrics induce the same topology. Thus, we get from Weyl's classical result $\dim_{\mathcal{S}}(E, -\Delta_{\lambda^1}) = \frac{1}{2}$ and from the arguments developed in section 2.1 that the associated Markov process is $2(B_t)_{t \geq 0}$.

It is now easy to see that (2.3) does not provide a useful notion of a walk dimension: Since $B_{\arctan}(x, R) = \mathbb{R}$ for every radius $R \geq \pi$, the expression diverges to ∞. Even the more careful approach

$$\lim_{R \nearrow \frac{\pi}{2}} \frac{\log \mathbf{E}_0 \left[\tau_{B(x,R)} \right]}{\log R}$$

runs into similar problems: Using the formula $\mathbf{E}\left[\tau_{[a,b]} \right] = -ab$ for the exit time of a standard Brownian motion from the interval $[a, b] \ni 0$, we get

$$\log \mathbf{E}_0 \left[\tau_{B_{\arctan}(0,R)} \right] = 2 \log \tan R = \infty.$$

However, the local walk dimension from definition 1.13 works out quite elgantly: Setting $y = \arctan x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$, we obtain for some $\xi_1 \in (y, y+r), \xi_2 \in (y-r, y)$

$$\frac{\log \mathbf{E}_0 \left[\tau_{B_{\arctan}(0,R)} \right]}{\log r} = \frac{\log \left(\tan(y+r) - \tan y \right)}{\log r} + \frac{\log \left(\tan y - \tan(y-r) \right)}{\log r}$$

$$= \frac{\log \frac{r}{\cos^2 \xi_1}}{\log r} + \frac{\log \frac{r}{\cos^2 \xi_2}}{\log r} = 2 - \frac{\log \cos^2 \xi_1}{\log r} - \frac{\log \cos^2 \xi_2}{\log r}$$

by using the mean value theorem. Taking the limit for $r \searrow 0$ on both sides implies $\xi_1, \xi_2 \to y$ and thus $\dim_{\mathcal{W}}(E, 2B_t, x) = 2$.

Chapter 3

The Einstein Relation on Metric Measure Spaces

This chapter is devoted to the investigation of the Einstein relation in the setting of an abstract mm-space. First, we focus on the behaviour under morphisms between mm-spaces to derive invariance properties of the Einstein relation.

3.1 The Einstein Relation under Lipschitz-isomorphisms

3.1.1 Lipschitz and mm-isomorphisms

We will use this section to introduce two different categories MM_L and $\mathsf{MM}_{\leq 1}$ whose objects are mm-spaces, but with different morphisms:

- In MM_L, the set $\mathsf{MM}_L(X, Y)$ of morphisms from an object $X = (X, d_X, \mu_X)$ to another object $Y = (Y, d_Y, \mu_Y)$ is the set of all Lipschitz-continuous functions

$$\varphi : \operatorname{supp} \mu_X \to \operatorname{supp} \mu_Y$$

  satisfying $\varphi_* \mu_X = \mu_Y$.

- In $\mathsf{MM}_{\leq 1}$, the set $\mathsf{MM}_{\leq 1}(X, Y)$ of morphisms from an object $X = (X, d_X, \mu_X)$ to another object $Y = (Y, d_Y, \mu_Y)$ is the subset of $\mathsf{MM}_L(X, Y)$ consisting of all contraction maps (i.e. Lipschitz-continuous functions f with $\operatorname{Lip}_f \leq 1$, cf. (1.2)).

In both of those categories, composition of morphisms is to be understood as the usual composition of maps. By definition, $\mathsf{MM}_{\leq 1}$ is a subcategory of MM_L. Considering the usual notion of isomorphism, both categories give rise to a meaningful concept of isomorphy for mm-spaces:

Definition 3.1. A Lipschitz-isomorphism between mm-spaces (X, d_X, μ_X) and (Y, d_Y, μ_Y) is a map $\varphi : \operatorname{supp} \mu_X \to \operatorname{supp} \mu_Y$ with $\varphi_* \mu_X = \mu_Y$ satisfying the bi-Lipschitz condition

$$\frac{1}{C} d_X(x, y) \leq d_Y(\varphi(x), \varphi(y)) \leq C d_X(x, y)$$

for all $x, y \in \operatorname{supp} \mu_X$ and a constant $C \in [1, \infty)$ not depending on x, y.

Similarly, an mm-isomorphism is defined to be a Lipschitz-isomorphism with constant $C = 1$. (This coincides with definition 2.8 in [Shi16])

As it turns out, Lipschitz-isomorphisms are precisely the isomorphisms in MM_L, whereas mm-isomorphisms are the ones in $\mathsf{MM}_{\leq 1}$.

Indeed, consider a Lipschitz-isomorphism $\varphi : X \supseteq \operatorname{supp} \mu_X \to \operatorname{supp} \mu_Y \subseteq Y$. By definition, this is an injective morphism from $\mathsf{MM}_L(X, Y)$. We need to show that φ is surjective to ensure the existence of a two-sided inverse in $\mathsf{MM}_L(Y, X)$. To this end, suppose there exists $y \in \operatorname{supp} \mu_Y \setminus \varphi(\operatorname{supp} \mu_X) =: Z$. Since $\operatorname{supp} \mu_X$ is closed, so is its image under the homeomorphism φ, and hence $Z \subseteq \operatorname{supp} \mu_Y$ is open. As every open subset of $\operatorname{supp} \mu_Y$ is required to have positive measure, we obtain the contradiction

$$0 < \mu_Y(Z) = \varphi_* \mu_X(Z) = \mu_X \left(\varphi^{-1} \left(\operatorname{supp} \mu_Y \setminus \varphi(\operatorname{supp} \mu_X) \right) \right) = 0.$$

Hence, φ is indeed a bijection. Conversely, if φ is an isomorphism from $\mathsf{MM}_L(X, Y)$ then we get the lower Lipschitz-bound from the Lipschitz-continuity of $\varphi^{-1} \in \mathsf{MM}_L(Y, X)$, thus showing that φ is also a Lipschitz-isomorphism. Analogously, the corresponding statement for mm-isomorphisms can be derived.

We will write $(X, d_X, \mu_X) \simeq (Y, d_Y, \mu_Y)$ if X and Y are Lipschitz-isomorphic, and $(X, d_X, \mu_X) \cong (Y, d_Y, \mu_Y)$ if they are mm-isomorphic. Trivially, $X \cong Y$ implies $X \simeq Y$.

In what follows, we will always assume $\operatorname{supp} \mu_X = X$.

Remark 3.2. Of course, we always have $(X, d_X, \mu_X) \cong (\operatorname{supp} \mu_X, d_X, \mu_X)$ by virtue of $\operatorname{id} : X \supseteq \operatorname{supp} \mu_X \to \operatorname{supp} \mu_X$. The restriction $\operatorname{supp} \mu_X = X$ becomes necessary for the Einstein relation since $\dim_{\mathcal{H}}(\operatorname{supp} \mu_X)$ might be strictly smaller than $\dim_{\mathcal{H}}(X)$, the term appearing in the Einstein relation (1.11). We will later see (Proposition 3.4) that the Einstein relation is invariant under Lipschitz-isomorphisms which provides some motivation to circumvent this restriction by considering the relation

$$\dim_{\mathcal{H}}(\operatorname{supp} \mu_X) = c \dim_{\mathcal{S}}(\operatorname{supp} \mu_X, A) \dim_{\mathcal{W}}(\operatorname{supp} \mu_X, M)$$

instead of (1.11).

3.1.2 Transport of structure

Given two mm-spaces (X, d_X, μ_X) and (Y, d_Y, μ_Y) with a map $\varphi : X \to Y$, where a suitable operator $A : L^2(X, \mu_X) \supseteq \mathscr{D}(A) \to L^2(X, \mu_X)$ satisfies the Einstein relation with constant c on X. How can we transport A alongside φ to become an operator on $L^2(Y, \mu_Y)$, and which restrictions do we need to impose on φ to ensure that this transport of structure is compatible with the theory from chapter 1?

Note first that any bimeasurable bijection $\varphi : (X, d_X, \mu_X) \to (Y, d_Y, \mu_Y)$ induces by precomposition an operator

$$\varphi_* : L^2(Y, \nu) \to L^2(X, \mu)$$
$$f(y) \mapsto (f \circ \varphi)(x)$$

which is an isometric isomorphism because $\varphi^{-1} : Y \to X$ induces its inverse and because of

$$\|\varphi_* f\|_{L^2(X,\mu_X)}^2 = \int_X |f(\varphi(x))|^2 \, \mathrm{d}\mu_X = \int_Y |f|^2 \, \mathrm{d}\varphi_* \mu_X = \int_Y |f|^2 \, \mathrm{d}\mu_Y = \|f\|_{L^2(Y,\mu_Y)}^2, \quad (3.1)$$

by the change of variables formula for Lebesgue integrals.

Denote by $\mathbb{L}(H)$ the set of all partially defined linear maps (not necessarily bounded) on a Hilbert space H. Given an operator $A \in \mathbb{L}(L^2(X,\mu_X))$, we can now contruct an operator $\varphi_{\mathbb{L}} A \in \mathbb{L}(L^2(Y,\mu_Y))$ by conjugating with φ_*. More explicitly, we define the map

$$\varphi_{\mathbb{L}} : \mathbb{L}(L^2(X,\mu)) \to \mathbb{L}(L^2(Y,\mu))$$

where $(\varphi_{\mathbb{L}} A)f := (\varphi_*^{-1} \circ A \circ \varphi_*)f$ and $\mathscr{D}(\varphi_{\mathbb{L}} A) = \varphi_*^{-1}(\mathscr{D}(A))$. Note that φ_L is again a bijection with inverse given by $\varphi_L^{-1} = (\varphi^{-1})_L$ and that this bijection restricts to the spaces of bounded linear operators.

It follows immediately that $\mathscr{D}(\varphi_{\mathbb{L}} A)$ is dense iff $\mathscr{D}(A)$ is, and $\varphi_{\mathbb{L}} A$ is self-adjoint iff A is. Indeed, consider arbitrary $f, g \in \mathscr{D}(\varphi_{\mathbb{L}} A)$ with $f = \varphi_*^{-1}(\bar{f})$ and $g = \varphi_*^{-1}(\bar{g})$, where $\bar{f}, \bar{g} \in \mathscr{D}(A)$. Then, applying (3.1), we have

$$\langle (\varphi_{\mathbb{L}} A)f, g \rangle_{L^2(Y,\mu_Y)} = \left\langle \varphi_*^{-1} A \varphi_* \varphi_*^{-1} \bar{f}, \varphi_*^{-1} \bar{g} \right\rangle_{L^2(Y,\mu_Y)} = \left\langle A\bar{f}, \bar{g} \right\rangle_{L^2(X,\mu_X)}$$

and we can perform the same calculations for $\langle f, (\varphi_{\mathbb{L}} A)g \rangle_{L^2(Y,\mu_Y)}$, thus establishing the claimed equivalence. It is equally straightforward to check that the resolvent sets and the eigenvalues of A and $\varphi_{\mathbb{L}} A$ coincide: Consider $\lambda \in \rho(A)$, that is, $(\lambda - A)^{-1}$ is a bounded linear operator on $L^2(X,\mu_X)$. To show $\lambda \in \rho(\varphi_L A)$, we consider

$$(\lambda - \varphi_L A)^{-1} = \left(\lambda - \varphi_*^{-1} A \varphi_* \right)^{-1} = \left(\varphi_*^{-1} (\lambda - A) \varphi_* \right)^{-1} = \varphi_*^{-1} (\lambda - A)^{-1} \varphi_* = \varphi_L (\lambda - A)^{-1}$$

which is a bounded linear operator on $L^2(Y,\mu_Y)$. If $\lambda \in \mathbb{C}$ happens to be an eigenvalue of A with eigenfunction $f \in L^2(X,\mu_X)$, then – as might have been expected – $\varphi_*^{-1} f$ is an eigenfunction of $\varphi_L A$ to the eigenvalue λ as well. This can easily be checked by calculating

$$(\varphi_L A)(\varphi_*^{-1} f) = \varphi_*^{-1} A f = \lambda(\varphi_*^{-1} f).$$

Moreover, $\varphi_{\mathbb{L}}$ respects operator semigroups: If $(T_t)_{t \geq 0}$ is a strongly continuous contraction semigroup on $L^2(X,\mu)$ with generator $(-A, \mathscr{D}(A))$ then $(\varphi_{\mathbb{L}} T_t)_{t \geq 0}$ is a semigroup with the same properties on $L^2(Y,\mu_Y)$ and with generator $(-\varphi_{\mathbb{L}} A, \mathscr{D}(\varphi_{\mathbb{L}} A))$. Indeed, the semigroup property is trivial to check. For contractiveness, note that for $L^2(Y,\mu_Y) \ni f = \varphi_*^{-1} \bar{f}$ with $\bar{f} \in L^2(X,\mu_X)$

$$\|(\varphi_{\mathbb{L}} T_t)f\|_{L^2(Y,\mu_Y)} = \left\| \varphi_*^{-1} T_t \varphi_* \varphi_*^{-1} \bar{f} \right\|_{L^2(Y,\mu_Y)} = \left\| T_t \bar{f} \right\|_{L^2(X,\mu_X)} \leq \|\bar{f}\|_{L^2(X,\mu_X)} = \|f\|_{L^2(Y,\mu_Y)}.$$

For strong continuity, we calculate

$$\|\varphi_{\mathbb{L}} T_t f - \varphi_{\mathbb{L}} T_0 f\| = \left\| \varphi_*^{-1} (T_t \varphi_* f - \varphi_* f) \right\| = \|T_t(\varphi_* f) - (\varphi_* f)\| \to 0$$

for $t \searrow 0$ and arbitrary $f \in L^2(X,\mu)$, and verifying the generator works analogously.

Note however that a bi-measurable bijection φ does not respect enough structure to ensure that $\varphi_{\mathbb{L}}\sqrt{A}$ generates a regular Dirichlet form if and only if $\sqrt{A}$ does – recall that this means the density of $\mathscr{D}(\sqrt{A}) \cap C_c(Y)$ in both $\mathscr{D}(\sqrt{A})$ and $C_c(Y)$. To this end, suppose now that $\varphi : X \to Y$ is a homeomorphism between X and Y (since both spaces are equipped with their Borel σ-algebras, such φ is automatically bi-measurable and bijective). Similar to the case of L^2-spaces, this induces an isometric isomorphism $\varphi_* : C_0(Y) \to C_0(X), \varphi_*(f) = f \circ \varphi$ between algebras of continuous functions vanishing at infinity, equipped with sup-norm $\|\cdot\|_{C_0}$. This isomorphism restricts to the subalgebras of compactly supported continuous functions $C_c(X)$ resp. $C_c(Y)$.

Lemma 3.3. *With the notation just introduced, if the Dirichlet form on $L^2(X, \mu_X)$ defined by*

$$\mathcal{E}(f,g) := \left\langle \sqrt{A}f, \sqrt{A}g \right\rangle_{L^2(X,\mu_X)} \quad \text{for } f, g \in \mathscr{D}(\sqrt{A})$$

is regular then so is the Dirichlet form on $L^2(Y, \mu_Y)$ defined by

$$(\varphi_{\mathbb{L}})^* \mathcal{E}(\bar{f}, \bar{g}) := \left\langle (\varphi_{\mathbb{L}}\sqrt{A})\bar{f}, (\varphi_{\mathbb{L}}\sqrt{A})\bar{g} \right\rangle_{L^2(Y,\mu_Y)} \quad \text{for } \bar{f}, \bar{g} \in \varphi_*^{-1}(\mathscr{D}(\sqrt{A})).$$

Proof. We need to show that the intersection of $\mathscr{D}(\varphi_{\mathbb{L}}\sqrt{A}) = \varphi_*^{-1}(\mathscr{D}(\sqrt{A}))$ and $C_c(Y)$ is dense both in $C_c(Y)$ w.r.t $\|\cdot\|_{C_0(Y)}$ and in $\mathscr{D}(\varphi_{\mathbb{L}}\sqrt{A})$ w.r.t. $((\varphi_{\mathbb{L}})^*\mathcal{E})_1$ as introduced in definition 1.10. For the first part, take $C_c(Y) \ni f = \varphi_*^{-1}g$ for $g \in C_c(X)$. Then, there exists a sequence $(g_n)_{n\in\mathbb{N}} \subseteq \mathscr{D}(\sqrt{A}) \cap C_c(X)$ with $\|g_n - g\|_{C_0(X)} \to 0$ as $n \to \infty$. Since φ_*^{-1} is isometric, we conlude that $f_n := \varphi_*^{-1}g_n \in \mathscr{D}(\varphi_{\mathbb{L}}\sqrt{A}) \cap C_c(Y)$ converges to f in $\|\cdot\|_{C_0(Y)}$.

For the second part, we analogously take $\mathscr{D}(\varphi_{\mathbb{L}}\sqrt{A}) \ni f = \varphi_*^{-1}g$ for $g \in \mathscr{D}(\sqrt{A})$. By regularity of $\mathcal{E}$, there exists again a sequence $(g_n)_{n\in\mathbb{N}} \subseteq \mathscr{D}(\sqrt{A}) \cap C_c(X)$ with $\mathcal{E}_1[g_n - g] \to 0$ as $n \to \infty$. Setting $f_n := \varphi_*^{-1}g_n \in \mathscr{D}(\varphi_{\mathbb{L}}\sqrt{A}) \cap C_c(Y)$, we obtain

$$\begin{aligned}
((\varphi_{\mathbb{L}})^*\mathcal{E})_1[f_n - f] &= \left\| \left(\varphi_{\mathbb{L}}\sqrt{A}\right)(f_n - f) \right\|_{L^2(Y,\mu_Y)}^2 + \|f_n - f\|_{L^2(Y,\mu_Y)}^2 \\
&= \left\| \varphi_*^{-1}\sqrt{A}\varphi_*\varphi_*^{-1}(g_n - g) \right\|_{L^2(Y,\mu_Y)}^2 + \|\varphi_*^{-1}(g_n - g)\|_{L^2(Y,\mu_Y)}^2 \\
&= \left\| \sqrt{A}(g_n - g) \right\|_{L^2(X,\mu_X)}^2 + \|g_n - g\|_{L^2(X,\mu_X)}^2 \\
&= \mathcal{E}_1[g_n - g] \to 0
\end{aligned}$$

which concludes the proof. $\qquad\square$

Putting everything together, we observe that A satisfies the assumptions in 1.5 iff $\varphi_{\mathbb{L}}A$ does whenever $\varphi : X \to Y$ is a homeomorphism, and then $\dim_{\mathcal{S}}(X, A) = \dim_{\mathcal{S}}(Y, \varphi_{\mathbb{L}}A)$. The spectral dimension is therefore stable under a very large class of transformations. As it turns out, this will not be the case for Hausdorff and walk dimension.

Proposition 3.4. *Let (X, d_X, μ) and (Y, d_Y, ν) be complete separable locally compact path-connected metric measure spaces with $\operatorname{supp}\mu = X$ and $\operatorname{supp}\nu = Y$ that are Lipschitz-isomorphic by virtue of the map $\varphi : X \to Y$. Suppose the Einstein relation with constant c holds on X with respect to an operator $(A, \mathscr{D}(A))$ satisfying assumptions 1.5. Then, the Einstein relation also holds on Y with the same constant c and with respect to $\varphi_{\mathbb{L}}A$.*

Proof. As the Hausdorff dimension is invariant under bi-Lipschitz maps we obtain $\dim_{\mathcal{H}}(X) = \dim_{\mathcal{H}}(Y)$, and as observed above, $\dim_{\mathcal{S}}(X, A) = \dim_{\mathcal{S}}(Y, \varphi_{\mathbb{L}} A)$. So, it remains to show $\dim_{\mathcal{W}}(X, M) = \dim_{\mathcal{W}}(X, M^{(\varphi)})$ where M is a Hunt process associated to A and $M^{(\varphi)}$ is one associated to $\varphi_{\mathbb{L}} A$.

We consider the process $N_t := \varphi(M_t)$. This process is a Hunt process with values in Y, and possesses the semigroup

$$T_t^{(N)} f = \mathbf{E}. \left[f(N_t) \right] = \mathbf{E}_{\varphi^{-1}(\cdot)} \left[(f \circ \varphi)(M_t) \right] = T_t[\varphi_* f](\varphi^{-1}(\cdot)) = \varphi_*^{-1} T_t \varphi_* f = (\varphi_{\mathbb{L}} T_t) f$$

where we used the notation from the discussion above.

Thus, due to theorem 1.12, the processes N and $M^{(\varphi)}$ coincide up to their behaviour on a polar set. It is therefore enough to determine the walk dimension for N_t. By the bi-Lipschitz continuity of φ, we obtain

$$\varphi\left(B_X \left(x, C^{-1} r \right) \right) \subseteq B_Y \left(\varphi(x), r \right) \subseteq \varphi(B_X(x, Cr))$$

where $C > 0$ is the two-sided Lipschitz constant of φ. Hence, $\tau_{C^{-1}r}^{(M)} \leq \tau_r^{(N)} \leq \tau_{Cr}^{(M)}$. From this, we get for all sufficiently small $r > 0$

$$\frac{\log Cr}{\log r} \cdot \frac{\log \mathbf{E}_x \left[\tau_{Cr}^{(M)} \right]}{\log Cr} \leq \frac{\log \mathbf{E}_{\varphi(x)} \left[\tau_r^{(N)} \right]}{\log r} \leq \frac{\log C^{-1}r}{\log r} \cdot \frac{\log \mathbf{E}_x \left[\tau_{C^{-1}r}^{(M)} \right]}{\log C^{-1}r}.$$

Taking the limit for $r \searrow 0$ and applying a standard squeezing argument, we obtain $\dim_{\mathcal{W}}(X, M) = \dim_{\mathcal{W}}(Y, N)$. $\qquad\square$

Remark 3.5. Note that we required the bi-Lipschitz property for determining $\dim_{\mathcal{H}}$ and $\dim_{\mathcal{W}}$, whereas we only needed φ to be a homeomorphism in order to show that $M^{(\varphi)}$ and N share the same semigroup. This allows us in the following sections – given a homeomorphism $\varphi : (X, d_X, \mu_X) \to (Y, d_Y)$ – to transport the complete structure needed for the Einstein relation by

- Endowing (Y, μ_Y) with the push-forward measure $\varphi_* \mu_X$.

- Mapping the generator $(A, \mathscr{D}(A))$ to $(\varphi_L A, \varphi_*^{-1} \mathscr{D}(A))$, thus also mapping the generated semigroup T_t to $\varphi_L T_t$.

- Sending the Hunt process M to $\varphi(M)$.

What we did so far ensures that all these constructions are compatible with each other.

From proposition 3.4 we immediately obtain the following two corollaries:

Corollary 3.6. *If* $(X, d_X, \mu_X) \cong (Y, d_Y, \mu_Y)$ *and the Einstein relation with constant c holds on X w.r.t.* $(A, \mathscr{D}(A))$ *is an operator on* $L^2(X, \mu)$, *then it also holds on Y with the same constant w.r.t.* $\varphi_{\mathbb{L}} A$.

Corollary 3.7. *If* $X \subseteq \mathbb{R}^n$ *and* d_1 *and* d_2 *are metrics which are induced by norms, then* $\mathrm{id}_X : (X, d_1, \mu) \to (X, d_2, \mu)$ *will preserve the constant in the Einstein relation.*

The second corollary follows from the well-known fact that all norms on a finite-dimensional Banach space are equivalent.

3.2 Hölder regularity and graphs of functions

A natural question arising at this point is whether the invariance of the Einstein relation of propsition 3.4 can be extended to a larger class of transformations. In particular, what happens if φ is only a Hölder continuous map instead of a bi-Lipschitz one?

As we saw in the previous section, such φ does not impede the spectral dimension, but it is well-known that α-Hölder continuous transformations are not compatible with the Hausdorff dimension, besides the general estimate $\dim_{\mathcal{H}}(\varphi(X)) \leq \alpha^{-1} \dim_{\mathcal{H}}(X)$ mentioned in chapter 1. We will see that a similar picture occurs for the walk dimension.

Definition 3.8. Let $\alpha \in (0,1]$. We say that a map $\varphi : (X, d_X) \to (Y, d_Y)$ between two metric spaces is locally α-Hölder continuous at $x \in X$ if there exists an open neighbourhood $U \subseteq X$ of x and a constant $C > 0$ such that

$$d_Y(\varphi(x), \varphi(y)) \leq C d_X(x, y)^\alpha$$

for all $y \in U$. If this holds for all $x \in X$ we call φ locally α-Hölder continuous on X.

Note that if φ is α-Hölder continuity then it is also β-Hölder continuous for any $\beta < \alpha$ and that for $\alpha = 1$, we get back the definition of Lipschitz continuity. This allows us to define Hölder regularity as precisely the parameter α at which the phase transition betweeen being Hölder continuous and not being Hölder continuous occurs.

Definition 3.9. In extension of the previous definition, we say that φ is locally α-Hölder regular at $x \in X$ if α is the supremum of all $\beta > 0$ for which φ is locally β-Hölder continuous at x. Equivalently, such α is the infimum of 1 and all $\gamma \leq 1$ for which φ is not locally γ-Hölder continuous at x.

Being prepared with these definitions, we can now take...

3.2.1 A closer look at the walk dimension

Lemma 3.10. *Let $(M_t^x)_{t \geq 0}$ be a right-continuous stochastic process on (X, d_X) starting in $x \in X$ and let $\varphi : (X, d_X) \to (Y, d_Y)$ be a map which is locally α-Hölder regular at x for some $\alpha \in (0, 1)$. Suppose further that the local walk dimension of M at x exists. Then the upper local walk dimension, defined by*

$$\overline{\dim_{\mathcal{W}}}(X, M; x) := \limsup_{r \searrow 0} \frac{\log \mathbf{E}_x\left[\tau_r^{(M)}\right]}{\log r},$$

satisfies

$$\overline{\dim_{\mathcal{W}}}(Y, \varphi(M); \varphi(x)) \leq \frac{1}{\alpha} \dim_{\mathcal{W}}(X, M; x). \tag{3.2}$$

Proof. Let $0 < \beta < \alpha$. Then, φ is locally β-Hölder continuous at x and therefore, there exists a constant $C > 0$ such that

$$\varphi\left(B_X(x, r)\right) \subseteq B_Y\left(\varphi(x), C r^\beta\right)$$

for all sufficiently small $r > 0$. Thus, if $\varphi(M)$ exits $B_Y\left(\varphi(x), C r^\beta\right)$, it already left $\varphi(B_X(x, r))$.

Since $\varphi(M_t) \notin \varphi(B_X(x,r))$ implies $M_t \notin B_X(x,r)$ by comparing the preimages, we obtain the inequality

$$\tau_r^{(M)} \leq \tau_{\varphi(B_X(x,r))}^{(\varphi(M))} \leq \tau_{Cr^\beta}^{(\varphi(M))}.$$

This implies for all r small enough

$$\frac{\log \mathbf{E}_{\varphi(x)}\left[\tau_{Cr^\beta}^{(\varphi(M))}\right]}{\log Cr^\beta} \leq \frac{\log \mathbf{E}_x\left[\tau_r^{(M)}\right]}{\log r} \cdot \frac{\log r}{\log Cr^\beta},$$

where the right-hand side converges to $\beta^{-1} \dim_{\mathcal{W}}(X,M;x)$ as $r \searrow 0$, thus showing

$$\overline{\dim_{\mathcal{W}}}(Y,\varphi(M);\varphi(x)) \leq \frac{1}{\beta} \dim_{\mathcal{W}}(X,M;x).$$

As all estimates are valid for every $\beta < \alpha$ and since $\overline{\dim_{\mathcal{W}}}(Y,\varphi(M);\varphi(x)$ does not depend on β, we can take the supremum over all $\beta < \alpha$ to obtain

$$\overline{\dim_{\mathcal{W}}}(Y,\varphi(M);\varphi(x)) \leq \frac{1}{\alpha} \dim_{\mathcal{W}}(X,M;x)$$

which concludes the proof. $\qquad\square$

In general, equality in (3.2) does not hold. Fix $0 < \alpha < 1$. We consider the measure space $(\mathbb{R}^2, \lambda^2)$ endowed with two different metrics – first with the metric

$$d_1^{(\alpha)}(x,y) = |x_1 - x_2|^\alpha + |x_2 - y_2| \quad \text{for} \quad x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2$$

and second with the metric d_1 induced by the 1-norm. That is, we set $X = \left(\mathbb{R}^2, d_\infty^{(\alpha)}, \lambda^2\right)$ and $Y = (\mathbb{R}^2, d_1, \lambda^2)$. By definition, $\mathrm{id}: X \to Y$ provides a homeomorphism that is everywhere locally α-Lipschitz continuous. Let $(W_t)_{t \geq 0}$ be a 1-dimensional standard Wiener process, and regard $(0, W_t)$ as a process in X which has $\dim_{\mathcal{W}}(X,(0,W_t),0) = 2$. Therefore,

$$\dim_{\mathcal{W}}(Y,(0,W_t),0) = 2 < \frac{2}{\alpha}.$$

Despite this counterexample, we get equality in (3.2) in the following setting:

Lemma 3.11. *Let (X,d_X) be a path-connected metric space not consisting of more than one single point and let $M = (M_t^x)_{t \geq 0}$ be an X-valued continuous stochastic process starting in $x \in X$. Let $\varphi: (X,d_X) \to (Y,d_Y)$ be locally α-Hölder regular at x. Suppose further that $\dim_{\mathcal{W}}(X,M;x)$ exists. Then*

$$\overline{\dim_{\mathcal{W}}}(Y,\varphi(M);\varphi(x)) = \frac{1}{\alpha} \dim_{\mathcal{W}}(X,M;x)$$

holds, provided there exists a constant $C > 1$ and a sequence $(r_n)_{n \in \mathbb{N}}$ with $r_n \searrow 0$ such that for all $n \in \mathbb{N}$, there exists a set $\Gamma_n \subseteq B_X(x, Cr_n) \setminus B_X(x, r_n)$ subject to the following two conditions:

i. For all $y \in \Gamma_n$, φ violates an α-Hölder estimate: $d_Y(\varphi(x),\varphi(y)) > d_X(x,y)^{\alpha+\varepsilon}$.
ii. The complement of Γ_n, $X \setminus \Gamma_n$, splits into at least two non-empty path-connected components.

Proof. Due to the previous lemma, it only remains to show "$\geq$". By X_n, we denote the connected component of $X \setminus \Gamma_n$ which contains x. Since $B_X(x, r_n) \subseteq X_n \subseteq B_X(x, Cr_n)$ by definition of Γ_n, assumption *i.* implies that

$$B_Y\left(\varphi(x), r_n^\alpha\right) \subseteq Y_n := \varphi(X_n) \subseteq \varphi\left(B_X(x, Cr_n)\right).$$

Thus,

$$\tau_{\varphi(M)}\left(r_n^\alpha\right) \leq \tau_{\varphi(M)}(Y_n) \leq \tau_{\varphi(M)}\left(\varphi\left(B_X(x, Cr_n)\right)\right) = \tau_M(Cr_n),$$

which in turn yields to

$$\frac{\log \mathbf{E}_x\left[\tau_M(Cr_n)\right]}{\log Cr_n} \cdot \frac{\log Cr_n}{\log r_n^\alpha} \leq \frac{\log \mathbf{E}_{\varphi(x)}\left[\tau_{\varphi(M)}\left(r_n^\alpha\right)\right]}{\log r_n^\alpha}$$

and consequentially for $n \to \infty$ to

$$\frac{1}{\alpha}\dim_{\mathcal{W}}(X, M; x) \leq \liminf_{n \to \infty} \frac{\log \mathbf{E}_{\varphi(x)}\left[\tau_{\varphi(M^x)}\left(r_n^\alpha\right)\right]}{\log r_n^\alpha} =: w. \tag{3.3}$$

Thanks to lemma 3.10 we also have

$$w \leq \overline{\dim_{\mathcal{W}}}(Y, \varphi(M); \varphi(x)) \leq \frac{1}{\alpha}\dim_{\mathcal{W}}(X, M; x)$$

which shows the assertion when combined with (3.3). $\qquad\square$

Remark 3.12. Suppose $N = (N_t^y)_{t \geq 0}$ is a stochastic process with values in (Y, d_Y) that is almost surely (locally) α-Hölder regular and satisfies the assumptions of lemma 3.11 with probability 1. Then we can always use lemma 3.11 to obtain $\overline{\dim_{\mathcal{W}}}(Y, N; \cdot) = \alpha^{-1}$ by setting $(X, d_X) = (\mathbb{R}_{\geq 0}, |\cdot|)$, chosing M deterministically as $M_t = t$ and regard N as the (random) map $\varphi = N : X \to Y$.

In the special case where X is an open domain in the 1-dimensional euclidean space and M is a Brownian motion in X we can disregard condition *ii.* in lemma 3.11 since the exit time for the Brownian motion does only depend on the distance from the starting point. We will not go into greater detail here, but will expand on this idea in the section below.

3.2.2 Graphs of continuous functions

Given a continuous map $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces, its graph

$$\mathrm{gr}(f) := \{(x, f(x)) \in X \times Y : x \in X\}$$

can be equipped with the restriction of the maximum metric on $X \times Y$,

$$d_\infty((x, y), (x', y')) := d_X(x, x') \vee d_Y(y, y'), \quad x, x' \in X, y, y' \in Y,$$

to $\mathrm{gr}(f) \subseteq X \times Y$. This makes $(\mathrm{gr}(f), d_\infty)$ a metric space that comes with a natural map $\varphi : X \to \mathrm{gr}(f)$ sending $x \in X$ to $\varphi(x) := (x, f(x))$. Since f is continuous, it is easy to check that φ provides a homeomorphism between X and $\mathrm{gr}(f)$ with the inverse given by the projection onto

the first coordinate, π. We point out that while π is always Lipschitz-continuous, φ is (locally) α-Hölder continuous if and only if f is. Indeed, we have

$$d_Y(f(x), f(x')) < C d_X(x, x')^\alpha \iff d_X(x, x') \vee d_Y(f(x), f(x')) < (1 \vee C) d_X(x, x')^\alpha$$

whenever $d_X(x, x') < 1$.

This setting is therefore a natural application to the arguments of the previous section. Unfortunately, not much is known about the Hausdorff dimension of these objects,

As deterministic α-Hölder regular functions are rather complicated objects to deal with, we will instead consider random functions, where we have the regularity result of the Kolmogorov-Chentsov theorem.

Consider a 1-dimensional continuous α-self-similar process $(X_t)_{t \in \mathbb{R}}$ with stationary increments over a suitable probability space $(\Omega, \mathscr{A}, \mathbf{P})$. Here, α-self-similar for $0 < \alpha \leq 1$ means that the processes $(X_t)_{t \in \mathbb{R}}$ and $(\xi^{-\alpha} X_{\xi t})_{t \in \mathbb{R}}$ have the same distribution. This condition implies $X_0 = 0$ $\mathbf{P}$-almost surely. Denote by

$$\mathrm{gr}(M) = \mathrm{gr}(X.(\omega)) := \{(t, X_t(\omega)) : t \in \mathbb{R}\} \subseteq \mathbb{R}^2$$

the graph of M for fixed $\omega \in \Omega$.

We endow $\mathrm{gr}(M)$ with the metric d_∞ obtained by restricting the maximum norm $|(x, y)|_\infty = \max\{|x|, |y|\}$ of the surrounding space $\mathbb{R}^2$ to $\mathrm{gr}(M)$. By construction, $\mathrm{gr}(M)$ comes with the natural map $\varphi = \varphi_\omega : \mathbb{R} \to \mathrm{gr}(M)$ sending t to (t, X_t), providing a homeomorphism between $\mathbb{R}$ and $\mathrm{gr}(M)$. While φ^{-1} is always Lipschitz-continuous since it is the projection onto the first coordinate, φ is in general only locally Hölder continuous with an exponent strictly smaller than 1. In fact, φ_ω is γ-Hölder continuous if and only if the trajectory $X.(\omega)$ is.

As pointed out in remark 3.5, we can now transfer the analytic structure of section 2.1 on the real line $\mathbb{R}$ via φ to $\mathrm{gr}(M)$. More explicitly, we have the measure $\varphi_* \lambda^1$ on $\mathrm{gr}(M)$ and an operator $\varphi_L \Delta_{\lambda^1}$ acting on $L^2(\mathrm{gr}(M), \varphi_* \lambda^1)$ that generates the Hunt process $(\varphi(W_s^t))_{s \geq 0}$, where $(W_s^t)_{s \geq 0}$ is a Wiener process independent from M with start in $t \in \mathbb{R}$.

We note the following:

- From [LX98], we get $\dim_{\mathcal{H}}(\mathrm{gr}(M)) = 2 - \alpha$ provided M satisfies the technical conditions that there exist positive constants K, K' and $0 < r_0 < 1/3$ such that

$$p(1, x, B(x, r_0)) \geq K \quad \text{and} \quad p(1, x, B(x, r)) \leq K'(1 \wedge r) \tag{3.4}$$

  holds for all $x \in \mathbb{R}$ and every sufficiently small $r > 0$. Here, $p(t, x, A)$ denotes the transition function for M.

- As discussed in the last section we have

$$\frac{1}{2} = \dim_{\mathcal{S}}\left(\mathbb{R}, -\frac{1}{2}\Delta_{\lambda^1}\right) = \dim_{\mathcal{S}}\left(\mathrm{gr}(B), \Phi\left(-\frac{1}{2}\Delta_{\lambda^1}\right)\right),$$

  where we once again appealed to Weyl's classical results.

- It remains to evaluate the walk dimension for $\varphi(W_s)$ on $\mathrm{gr}(M)$. As will be seen in the

subsequent lemmata, we have $\dim_{\mathcal{W}}(\mathrm{gr}(X.(\omega)), \varphi(W)) = \frac{2}{\alpha}$ **P**-almost surely.

The Einstein relation – despite holding with constant 1 on $\left([0,1], -\frac{1}{2}\Delta_{\lambda^1}, W\right)$ – therefore changes its constant under application of φ to

$$c(\alpha) = \alpha(2-\alpha), \quad \alpha \in (0,1].$$

Lemma 3.13. *With the notation just introduced and under assumption of* (3.4), *we have*

$$\dim_{\mathcal{W}}\left(\mathrm{gr}(X.(\omega)), \varphi_\omega(W), (T, X_T(\omega))\right) = \frac{2}{\alpha} \tag{3.5}$$

P-*almost surely for each* $T > 0$.

Proof. For brevity, set $x = x(\omega) = (T, X_T(\omega)) \in \mathrm{gr}(X.(\omega))$ and chose $r > 0$. Let $B_\infty(x,r)$ denote the open ball of radius r around x with respect to d_∞.

We begin by introducing the random times $\Theta_r^+(X.(\omega), T), \Theta_r^-(X.(\omega), T)$ to denote the time where the process M first exits resp. last enters $B_\infty(x,r)$ – in other words,

$$\Theta_r^+(X.(\omega), T) := \inf\{1 \geq t > T : (t, X_t) \notin B_\infty(x,r)\}$$
$$\Theta_r^-(X.(\omega), T) := \sup\{0 \leq t < T : (t, X_t) \notin B_\infty(x,r)\}. \tag{3.6}$$

By the standard result for the expectation of two-sided exit times for Brownian motion, we now obtain

$$\mathbf{E}_x\left[\tau_{B_\infty(x,r)}^{(\varphi(W))}\right] = -\left(\Theta_r^+(X.(\omega), T) - T\right)\left(\Theta_r^-(X.(\omega), T) - T\right)$$

and consequentially

$$\frac{\log \mathbf{E}_x\left[\tau_{B_\infty(x,r)}^{(\varphi(W))}\right]}{\log r} = \frac{\log\left(\Theta_r^+(X.(\omega), T) - T\right)}{\log r} + \frac{\log\left(T - \Theta_r^-(X.(\omega), T)\right)}{\log r}. \tag{3.7}$$

We will show that both summands on the right-hand side converge to $1/\alpha$ as $r \searrow 0$. To this end, we further introduce the random variables ϑ_r^+ and ϑ_r^- by

$$\vartheta_r^\pm(T) := \inf\{s \geq 0 : X_{T\pm s} \notin (X_T - r, X_T + r)\} = \inf\{s \geq 0 : X_{T\pm s} - X_T \notin (-r, r)\}.$$

It follows from the definitions that $\vartheta_r^\pm \wedge r = |\Theta_r^\pm(X, T) - T|$ which allows for the estimate

$$\vartheta_r^\pm \wedge r^{1/\alpha} \leq |\Theta_r^\pm(X, T) - T| \leq \vartheta_r^\pm. \tag{3.8}$$

We also observe that, due to the self-similarity and the stationary increments of X,

$$\begin{aligned}
\vartheta_{\xi r}^{\pm} &= \inf\left\{s \geq 0 : X_{T \pm s} - X_T \notin (-\xi r, \xi r)\right\} \\
&\stackrel{\mathcal{D}}{=} \inf\left\{s \geq 0 : X_{\pm s} \notin (-\xi r, \xi r)\right\} \\
&= \inf\left\{s \geq 0 : \xi^{-1} M_{\pm s} \notin (-r, r)\right\} \\
&\stackrel{\mathcal{D}}{=} \inf\left\{s \geq 0 : X_{\pm \xi^{-1/\alpha} s} \notin (X_T - r, X_T + r)\right\} \\
&= \xi^{1/\alpha} \inf\left\{s \geq 0 : X_{\pm s} \notin (-r, r)\right\} \\
&\stackrel{\mathcal{D}}{=} \xi^{1/\alpha} \vartheta_r^{\pm}
\end{aligned}$$

for all $\xi > 0$. Hence, we can rewrite (3.8) as

$$r^{1/\alpha}\left(\vartheta_1^{\pm} \wedge 1\right) = \vartheta_r^{\pm} \wedge r^{1/\alpha} \leq \left|\Theta_r^{\pm}(M, T) - T\right| \leq \vartheta_r^{\pm} = r^{1/\alpha}\vartheta_1^{\pm} \tag{3.9}$$

After taking logarithms on both sides and dividing by $\log r$, we obtain

$$\frac{1}{\alpha} + \frac{\log \vartheta_1^{\pm}}{\log r} \leq \frac{\log\left|\Theta_r^{\pm}(M, T) - T\right|}{\log r} \leq \frac{1}{\alpha} + \frac{\log\left(\vartheta_1^{\pm} \wedge 1\right)}{\log r}.$$

When taking the limit for $r \searrow 0$, the fractions involving $\vartheta_1^{\pm}$ vanish $\mathbf{P}$-almost surely since, by continuity of X, $\vartheta_1^{\pm} > 0$ with probability 1. This, combined with (3.7), concludes the proof. $\square$

Lemma 3.14. *In the setting of the previous lemma, $\dim_{\mathcal{W}}(\mathrm{gr}(X), \varphi(W)) = \frac{2}{\alpha}$ holds with probability 1.*

Proof. By (3.5) and the σ-additivity of measures, we can asssume

$$\mathbf{P}\left[\dim_{\mathcal{W}}\left(\mathrm{gr}(X), \varphi(W), (T, X_T)\right) = \frac{2}{\alpha} \quad \forall T \in \mathbb{Q}\right] = 1.$$

Suppose $T \in \mathbb{R} \setminus \mathbb{Q}$ and let $\left(W_s^T\right)_{s \geq 0}$ be a Wiener process with start in T. Define $\tau_{\mathbb{Q}}$ to be the infimum of times $s > 0$ with $W_s^T \in \mathbb{Q}$. Due to Blumenthal's 0-1-law, we know that $\inf\left\{s > 0 : W_s^T \neq T\right\} = 0$ with probability 1. At the same time, since the rationals are dense in the reals and the paths of W^T are continuous,

$$0 \leq \tau_{\mathbb{Q}} \leq \inf\left\{s > 0 : W_s^T \neq T\right\} = 0$$

holds almost surely as well. $\square$

Therefore, it will suffice to show that $\mathbf{P}$-almost surely,

$$\lim_{r \searrow 0} \frac{\log\left(\Theta_r^+(B_\cdot(\omega), T) - T\right)}{\log r} = \lim_{r \searrow 0} \frac{\log\left(T - \Theta_r^-(B_\cdot(\omega), T)\right)}{\log r} = 2, \tag{3.10}$$

where it is enough to prove that one of the limits exist and equals 2. To this end, we consider

more generally for a continuous function $f : [0,1] \to \mathbb{R}$ the functionals

$$w : f(T) \mapsto \liminf_{r \searrow 0} \frac{\log\left(\Theta_r^+(f,T) - T\right)}{\log r}$$

$$W : f(T) \mapsto \limsup_{r \searrow 0} \frac{\log\left(\Theta_r^+(f,T) - T\right)}{\log r}$$

where $\Theta_r^{\pm}(f,T)$ is defined analogously to (3.6). Suppose now that f is α-Hölder continuous $(0 < \alpha \leq 1)$ in a given point $T \in [0,1]$, that is, there exists a constant $C > 0$ and a ε-neighbourhood of T such that for all s inside this neighbourhood,

$$|f(T) - f(s)| \leq C|T - s|^{\alpha}$$

is satisfied. This yields $\Theta_r^+(f,T) \geq (r/C)^{1/\alpha}$ for $r < \varepsilon$ and therefore $(Wf)(T) \leq \frac{1}{\alpha}$. Conversely, suppose that f is not β-Hölder continuous $(\alpha < \beta \leq 1)$ in T. In particular, there exists a sequence $s_n = T + r_n \to T$ fulfilling the estimate

$$|f(T) - f(s_n)| > |T - s_n|^{\beta} = r_n^{\beta},$$

from which we deduce $\Theta_{r_n}^+(f,T) \leq r_n^{1/\beta}$ and thus $(wf)(T) \geq \frac{1}{\beta}$. In particular, $(Wf)(T) = (wf)(T) = \frac{1}{c}$ if f is α-Hölder continuous in T for all $\alpha < c$ but not β-Hölder continuous for any $\beta > c$.

Having established this claim, the limit in equation (3.10) is a direct consequence of Paley-Zygmunds regularity theorem for paths of Brownian motion. Moreover, this also shows that the limit does not depend on $T \in (0,1)$.

Discussion

As we saw in the preceeding chapter, the Einstein relation is an invariant of metric measure spaces. Under Hölder continuous tranformations, its behaviour depends on the Hausdorff and walk dimensions, for both of which we have the same upper bound, but in general different behaviour. Besides the open conjecture about Brownian motion on the graph of a fractional Brownian motion, there remains a plethora of further questions to discuss such as (arranged in order of increasing speculativeness):

Is there a general lemma providing lower estimates for the walk dimension? This question is almost self-explanatory, and it aims at a statement that plays a similar role for the walk dimension as the mass-distribution principle does for the Hausdorff dimension.

Is the Hausdorff dimension the "right" fractal dimension for the Einstein relation? There are several alternative ways to define geometric dimensions for fractals, such as the packing dimension or the box-counting dimension. Of course, any reasonably well-behaved notion of dimension should be definable on a large class of metric spaces and should be invariant under isometries. Naturally, then the arguments used in the proof of proposition 3.4 in the stricter setting of mm-isomorphisms show that the Einstein relation will still be an invariant of mm-spaces.

When evaluating how "good" a fractal dimension for this purpose is, two questions should be asked:

1. *Can this variant of the Einstein relation distinguish between nonisomorphic mm-spaces and if so, does it better than other variants?*
2. *Are there general theorems for this variant that give explanations on why the Einstein relation should hold with constant c on interesting classes of spaces?*

Of course, the latter questions are difficult to answer and not much is understood yet even for $\dim_{\mathcal{H}}$.

A similar question is whether there exists a variation to the Einstein relation that can tell apart spaces that are Lipschitz- but not mm-isomorphic.

Is it possible to extend the Einstein relation (1.11) **to graphs in such a way that it is compatible with the discrete version from section 2.2?** We saw in section 2.3 that the local walk dimension is better suited for bounded metric spaces. On the other hand, approximating spaces by a sequence of finite graphs as in the case of the Sierpinski gasket is a useful tool to have. However, for graphs the limit $r \searrow 0$ in the definition of the walk dimension does not make sense.

One way to circumvent these problems with a unified approach might be to consider metric graphs $\mathscr{G}$. Here, a metric graph is a disjoint collection of closed intervals I_i, where either $I_i = [a_i, b_i]$ or

$I_i = [a_i, \infty)$ for $a_i, b_i \in \mathbb{R}$, $i \in \mathscr{I}$ an index set, together with an equivalence relation $\sim$ on the set of boundary points $\{a_i, b_i : i \in \mathscr{I}\}$, where the boundary points are identified according to $\sim$. In other words, $\mathscr{G}$ is the quotient space

$$\Big(\bigsqcup_{i \in \mathscr{I}} I_i \Big) \big/ \sim .$$

As stochastic processes on metric graphs have been investigated in recent years (cf. [Wer16]), it is a natural question to ask whether one can replace the approximation of the Brownian motion on SG by random walks on G_n with an approximation by Brownian motions on $\mathscr{G}_n$, where $\mathscr{G}_n$ are the metric graphs with the metric structure coming from the embedding of G_n in $\mathbb{R}^2$. If this happens to be the case, one can furthermore ask if definition 1.13, applied to the approximating processes on $\mathscr{G}_n$, yields an approximation of the walk dimension on SG.

What are the topological properties of the Einstein relation? This question aims at finding a general setting in which the Einstein relation on a given space can be approximated by Einstein relations on other spaces.

The class of isomorphism classes of (compact!) mm-spaces, $\mathsf{cMM}_{\leq 1}/\cong$, can be endowed with different topologies, perhaps the most well-known way of doing this is via the Gromov-Hausdorff-Prohorov metric, defined in the following way:

Let (X, d_X, μ_X) and (Y, d_Y, μ_Y) be compact mm-spaces. Denote by d_H the usual Hausdorff distance between closed sets in a metric space (cf. section 1.1) and by d_P the Prohorov distance between probability measures ν, ν' on Z,

$$d_P(\nu, \nu') := \inf \{\varepsilon > 0 : \nu(B(A, \varepsilon)) > \nu'(A) - \varepsilon \text{ for any } A \in \mathscr{B}(Z)\}.$$

Then,
$$d_{GHP}(Z; \iota_X, \iota_Y) := \inf_{\iota_X, \iota_Y} \Big(d_H\big(\iota_X(X), \iota_Y(Y)\big) + d_P\big((\iota_X)_* \mu_X, (\iota_Y)_* \mu_Y\big) \Big),$$

where the infimum is taken over all metric spaces (Z, d_Z) and all isometric embeddings $\iota_X : X \hookrightarrow Z$, $\iota_Y : Y \hookrightarrow Z$ of X, Y into Z. This defines a pseudo-metric on $\mathsf{cMM}_{\leq 1}/\cong$.

Now take a subset $\mathsf{X} \subseteq \mathsf{cMM}_{\leq 1}$ and a mapping

$$\mathsf{X} \ni (X, d_X, \mu_X) \mapsto (A_X, \mathscr{D}(A_X)) \in \mathbb{L}(L^2(X, \mu_X))$$

that assigns to each mm-space an operator that satisfies the conditions 1.5. After taking the quotient, we are left with a map that sends each mm-isomorphism class to a linear operators on a representative of this class,

$$(\mathsf{X}/\cong) \ni [(X, d_X, \mu_X)]_\cong \mapsto (A_X, \mathscr{D}(A_X)) \in \mathbb{L}(L^2(X, \mu_X)),$$

where the right-hand side is unique up to the transport of structure induced by mm-isomorphisms as discussed in section 3.1. In particular, we can now regard the constant in the Einstein relation

as a function

$$\mathscr{E} : \mathsf{X}/\!\cong \ \to \mathbb{R}_{\geq 0}$$

$$[(X, d_X, \mu_X)]_{\cong} \mapsto c = \frac{\dim_{\mathcal{H}}(X)}{\dim_{\mathcal{S}}(X, A_X)\dim_{\mathcal{W}}(X, M^{A_X})} \tag{3.11}$$

In general, fixing a topology $\mathscr{T}$ on $\mathsf{cMM}_{\leq 1}/\!\cong$, a set X as above and an assignment A_X (which should, in some sense, depend continuously on $[X]$), what can be said about the topological properties of $\mathscr{E}$? Is it continuous w.r.t $\mathscr{T}$? Are at least the preimages of single points closed?

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Declaration of authorship

I hereby declare that the submitted thesis is the result of my own work written only with the assistance of the indicated sources. I confirm that all direct and indirect quotations of other authors' statements are marked and cited accordingly.

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