Master Thesis

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Contents

Abstract

We consider $\mathbb{R}$ as well as $\dim_{\mathcal{S}}, \dim_{\mathcal{H}} \subseteq \dim_{\mathcal{W}}$.

Chapter 1

An Introduction to the Einstein Relation

In this introductory chapter, we wish to briefly expose the ingredients of the ER - the Hausdorff dimension $\dim_{\mathcal{H}}$, the spectral dimension $\dim_{\mathcal{S}}$, and the walk dimension $\dim_{\mathcal{W}}$ - and state some of their properties.

1.1 Hausdorff measure and Hausdorff dimension

Although the concepts of Hausdorff measure and dimension are well-known, we give the definitions in the interest of completeness. In what follows, let (E, d) be a metric space.

Definition 1.1 (Hausdorff outer measure)**.** For fixed $s \geq 0$, any subset $S \subseteq E$ and any $\delta > 0$, let

$$\mathcal{H}^s_\delta(S) := \inf \left\{ \sum_{i \in I} (\operatorname{diam} U_i)^s : |I| \leq \aleph_0, S \subseteq \bigcup_{i \in I} U_i \subseteq E, \operatorname{diam} U_i \leq \delta \right\},$$

i.e. the infimum is taken over all countable coverings of S with diameter at most δ. The s-dimensional Hausdorff outer measure of S is now defined to be

$$\mathcal{H}^s(S) := \lim_{\delta \searrow 0} \mathcal{H}^s_\delta(S). \tag{1.1}$$

Observe that the limit in (1.1) exists or equals ∞, since $\mathcal{H}^s_\delta(S)$ is monotonically nonincreasing in δ, yet bounded from below by 0. Furthermore, it can be shown that $\mathcal{H}^s$ defines a metric outer measure on E, thus restricting to a measure on a σ-algebra containing the Borel σ-algera $\mathscr{B}(E)$ (cf. [Mat99, p.54ff]). By definition, the obtained measure then is the s-dimensional Hausdorff measure which we will denote by $\mathcal{H}^s$ as well.

In the special case of (E, d) being an Euclidean space, Hausdorff measures interpolate between the usual Lebesgue measures λ^n: For $s = 0$, we have simply $\mathcal{H}^0(S) = \#S$, whereas for any integer $n > 0$, it can be shown that there exists a constant $c_n > 0$ depending only on n such that $\mathcal{H}^n = c_n \lambda^n$, where the constant evaluates to the volume of the unit ball.

Since exponential functions are monotonically increasing, the Hausdorff measures' dependence on s exhibits the same behaviour for a fixed set S. At the same time, simple estimates yield that if $\mathcal{H}^s(S)$ is finite for some s, it vanishes for all $s' < s$, and conversely, if $\mathcal{H}^s(S) > 0$, then $\mathcal{H}^{s'}(S) = \infty$ for all $s' > s$. Therefore, there exists precisely one real number s where $\mathcal{H}^{\cdot}(S)$ jumps from 0 to ∞ (by possibly attaining any value of $[0, \infty]$ there). This motivates the following definition of Hausdorff dimension:

Definition 1.2. The Hausdorff dimension $\dim_{\mathcal{H}}(S)$ of $S \subseteq E$ is defined as

$$\dim_{\mathcal{H}}(S) := \inf\{s \geq 0 : \mathcal{H}^s(S) > 0\}.$$

Due to the above discussion, we have the following equalities:

$$\begin{aligned}\dim_{\mathcal{H}}(S) &= \inf\{s \geq 0 : \mathcal{H}^s(S) > 0\} = \inf\{s \geq 0 : \mathcal{H}^s(S) = \infty\} \\ &= \sup\{s \geq 0 : \mathcal{H}^s(S) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(S) < \infty\},\end{aligned}$$

providing some alternative characterisations of the Hausdorff dimension.

We further collect some important facts. To this end, let S, S' and $S_1, S_2, \ldots$ be subsets of E as before. Then, the following properties hold (cf. [Fal07, p.32f] for a discussion in the Euclidean setting; however all arguments adapt to our more general situation without complication):

Monotonicity. If $S \subseteq S'$ then $\dim_{\mathcal{H}}(S) \leq \dim_{\mathcal{H}}(S')$.

Countable Stability. For a sequence $(S_n)_{n\geq 1}$, we have the equality

$$\dim_{\mathcal{H}}\left(\bigcup_{n\geq 1} S_n\right) = \sup_{n\geq 1} \dim_{\mathcal{H}}(S_n).$$

Countable Sets. If $|S| \leq \aleph_0$ then $\dim_{\mathcal{H}}(S) = 0$.

Hölder continuous maps. If (E', d') is another metric space and $f : E \to E'$ is α-Hölder continuous for some $\alpha \in (0, 1]$ then $\dim_{\mathcal{H}}(f(S)) \leq \alpha^{-1} \dim_{\mathcal{H}}(S)$. In particular, the Hausdorff dimension is invariant under a bi-Lipschitz transformation (i.e. an invertible map f with Hölder exponent $\alpha = 1$ for both f and f^{-1}).

Euclidean Case. If (E, d) happens to be an Euclidean space (or more generally a continuously differentiable manifold) of dimension n and S is an open subset then $\dim_{\mathcal{H}}(S) = n$.

We conclude this section by citing Hutchinson's theorem about the Hausdorff dimension of self-similar sets which will provide us with a plethora of interesting examples. For this, we recall that a map $F : E \to E$ on a metric space (E, d) is a strict contraction if its Lipschitz constant satisfies

$$\mathrm{Lip}_F := \sup_{\substack{x,y\in E\\ x\neq y}} \frac{d(F(x), F(y))}{d(x, y)} < 1.$$

If the stronger condition $d(F(x), F(y)) = \mathrm{Lip}_F\, d(x, y)$ holds for all $x, y \in E$, we call F a similitude with contraction factor Lip_F.

Theorem 1.3 (Hutchinson, [Hut81]). *Let $\mathcal{S} = \{S_1, ..., S_N\}$ be a finite set of strict contractions on the Euclidean space $\mathbb{R}^n$. Then there exists a unique nonempty compact set denoted by $|\mathcal{S}|$ invariant under $\mathcal{S}$, i.e.*

$$|\mathcal{S}| = \bigcup_{i=1}^{N} S_i(|\mathcal{S}|).$$

Furthermore, assume that $|\mathcal{S}|$ satisfies the open set condition (OSC) which means that there exists a nonempty open set $O \subseteq E$ with the properties $S_i(O) \subseteq O$ and $S_i(O) \cap S_j(O) = \emptyset$ for all $i, j = 1, ..., N$ with $i \neq j$. Also assume that the maps S_i are similitudes with contraction factor $r_i \in (0, 1)$. Then, $s = \dim_{\mathcal{H}}(|\mathcal{S}|)$ is the unique solution to the equation

$$\sum_{i=1}^{N} r_i^s = 1$$

and we have $0 < \mathcal{H}^s(|\mathcal{S}|) < \infty$.

While uniqueness and existence of $|\mathcal{S}|$ are still ensured for maps on a complete metric space, the open set condition is not sufficient for statements about the Hausdorff dimension, see [Sch96] for further discussion.

1.2 Weyl asymptotics and spectral dimension

1.3 Diffusion processes and walk dimension

1.3.1 From Dirichlet forms to Markov processes

We start with the following definition (cf. [MR12, Def. IV.1.13]):

Definition 1.4. Given a filtered probability space $\left(\Omega, \mathscr{A}, \mathscr{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbf{P}\right)$ satisfying the usual conditions, an $\mathscr{F}$-adapted time-homogenous Markov process $X = (X_t)_{t\geq 0}$ with state space E_Δ is called a right process if it satisfies the strong Markov property for all $\mathscr{F}$-stopping times and all its trajectories are right continuous.

1.4 Other versions of the Enistein relation

Telcs, Mandelbrot, etc...

Chapter 2

Examples and Non-examples

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