Master Thesis

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Contents

Abstract

We consider $\mathbb{R}$ as well as $\dim_{\mathcal{S}}, \dim_{\mathcal{H}} \subseteq \dim_{\mathcal{W}}$.

Chapter 1

Fractal Dimensions and the Einstein Relation

In this introductory chapter, we wish to briefly expose the ingredients of the Einstein relation - the Hausdorff dimension $\dim_{\mathcal{H}}$, the spectral dimension $\dim_{\mathcal{S}}$, and the walk dimension $\dim_{\mathcal{W}}$ - and state some of their properties.

1.1 Hausdorff measure and Hausdorff dimension

Although the concepts of Hausdorff measure and dimension are well-known, we give the definitions in the interest of completeness. In what follows, let (E, d) be a metric space.

Definition 1.1 (Hausdorff outer measure)**.** For fixed $s \geq 0$, any subset $S \subseteq E$ and any $\delta > 0$, let

$$\mathcal{H}^s_\delta(S) := \inf \left\{ \sum_{i \in I} (\operatorname{diam} U_i)^s : |I| \leq \aleph_0, S \subseteq \bigcup_{i \in I} U_i \subseteq E, \operatorname{diam} U_i \leq \delta \right\},$$

i.e. the infimum is taken over all countable coverings of S with diameter at most δ. The s-dimensional Hausdorff outer measure of S is now defined to be

$$\mathcal{H}^s(S) := \lim_{\delta \searrow 0} \mathcal{H}^s_\delta(S). \tag{1.1}$$

Observe that the limit in (1.1) exists or equals ∞, since $\mathcal{H}^s_\delta(S)$ is monotonically nonincreasing in δ, yet bounded from below by 0. Furthermore, it can be shown that $\mathcal{H}^s$ defines a metric outer measure on E, and thus restricts to a measure on a σ-algebra containing the Borel σ-algera $\mathscr{B}(E)$ (cf. [Mat99, p.54ff]). By definition, the obtained measure then is the s-dimensional Hausdorff measure which we will denote by $\mathcal{H}^s$ as well. Note that for $\mathcal{H}^s$ to be a Radon measure, i.e. locally finite and inner regular, $\mathcal{H}^s(E) < \infty$ is sufficient.

In the special case of (E, d) being an Euclidean space, Hausdorff measures interpolate between the usual Lebesgue measures λ^n: For $s = 0$, we have simply $\mathcal{H}^0(S) = \#S$, whereas for any integer $n > 0$, it can be shown that there exists a constant $c_n > 0$ depending only on n such that

$\mathcal{H}^n = c_n\lambda^n$, where the constant evaluates to the volume of the unit ball.

Since exponential functions are monotonically increasing, the Hausdorff measures' dependence on s exhibits the same behaviour for a fixed set S. At the same time, simple estimates yield that if $\mathcal{H}^s(S)$ is finite for some s, it vanishes for all $s' < s$, and conversely, if $\mathcal{H}^s(S) > 0$, then $\mathcal{H}^{s'}(S) = \infty$ for all $s' > s$. Therefore, there exists precisely one real number s where $\mathcal{H}^{\cdot}(S)$ jumps from 0 to ∞ (by possibly attaining any value of $[0, \infty]$ there). This motivates the following definition of Hausdorff dimension:

Definition 1.2. The Hausdorff dimension $\dim_{\mathcal{H}}(S)$ of $S \subseteq E$ is defined as

$$\dim_{\mathcal{H}}(S) := \inf\{s \geq 0 : \mathcal{H}^s(S) > 0\}.$$

Due to the above discussion, we have the following equalities:

$$\begin{aligned}\dim_{\mathcal{H}}(S) &= \inf\{s \geq 0 : \mathcal{H}^s(S) > 0\} = \inf\{s \geq 0 : \mathcal{H}^s(S) = \infty\} \\ &= \sup\{s \geq 0 : \mathcal{H}^s(S) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(S) < \infty\},\end{aligned}$$

providing some alternative characterisations of the Hausdorff dimension.

We further collect some important facts. To this end, let S, S' and $S_1, S_2, ...$ be subsets of E as before. Then, the following properties hold (cf. [Fal07, p.32f] for a discussion in the Euclidean setting; however all arguments adapt to our more general situation without complication):

Monotonicity. If $S \subseteq S'$ then $\dim_{\mathcal{H}}(S) \leq \dim_{\mathcal{H}}(S')$.

Countable Stability. For a sequence $(S_n)_{n\geq 1}$, we have the equality

$$\dim_{\mathcal{H}}\left(\bigcup_{n\geq 1} S_n\right) = \sup_{n\geq 1} \dim_{\mathcal{H}}(S_n).$$

Countable Sets. If $|S| \leq \aleph_0$ then $\dim_{\mathcal{H}}(S) = 0$.

Hölder continuous maps. If (E', d') is another metric space and $f : E \to E'$ is α-Hölder continuous for some $\alpha \in (0, 1]$ then $\dim_{\mathcal{H}}(f(S)) \leq \alpha^{-1} \dim_{\mathcal{H}}(S)$. In particular, the Hausdorff dimension is invariant under a bi-Lipschitz transformation (i.e. an invertible map f with Hölder exponent $\alpha = 1$ for both f and f^{-1}).

Euclidean Case. If (E, d) happens to be an Euclidean space (or more generally a continuously differentiable manifold) of dimension n and S is an open subset then $\dim_{\mathcal{H}}(S) = n$.

We conclude this section by citing Hutchinson's theorem about the Hausdorff dimension of self-similar sets which will provide us with a plethora of interesting examples. For this, we recall that a map $F : E \to E$ on a metric space (E, d) is a strict contraction if its Lipschitz constant satisfies

$$\mathrm{Lip}_F := \sup_{\substack{x,y\in E \\ x\neq y}} \frac{d(F(x), F(y))}{d(x, y)} < 1.$$

If the stronger condition $d(F(x), F(y)) = \mathrm{Lip}_F\, d(x, y)$ holds for all $x, y \in E$, we call F a similitude with contraction factor Lip_F.

Theorem 1.3 (Hutchinson, [Hut81]). *Let $\mathcal{S} = \{S_1, ..., S_N\}$ be a finite set of strict contractions on the Euclidean space $\mathbb{R}^n$. Then there exists a unique nonempty compact set denoted by $|\mathcal{S}|$ invariant under $\mathcal{S}$, i.e.*

$$|\mathcal{S}| = \bigcup_{i=1}^{N} S_i(|\mathcal{S}|).$$

Furthermore, assume that $|\mathcal{S}|$ satisfies the open set condition (OSC) which means that there exists a nonempty open set $O \subseteq E$ with the properties $S_i(O) \subseteq O$ and $S_i(O) \cap S_j(O) = \emptyset$ for all $i, j = 1, ..., N$ with $i \neq j$. Also assume that the maps S_i are similitudes with contraction factor $r_i \in (0, 1)$. Then, $s = \dim_{\mathcal{H}}(|\mathcal{S}|)$ is the unique solution to the equation

$$\sum_{i=1}^{N} r_i^s = 1$$

and we have $0 < \mathcal{H}^s(|\mathcal{S}|) < \infty$.

While uniqueness and existence of $|\mathcal{S}|$ are still ensured for maps on a complete metric space, the open set condition is not sufficient for statements about the Hausdorff dimension, see [Sch96] for further discussion.

1.2 Weyl asymptotics and spectral dimension

The idea of introducing spectral dimension is inspired by Weyl's law for the eigenvalues of the Dirichlet-Laplace operator which we will discuss here shortly:

1.2.1 The classical case

Given a bounded open domain $E \subseteq \mathbb{R}^n$, consider the Laplace operator Δ on E acting on functions satisfying the Dirichlet boundary condition $u \equiv 0$ on ∂E. Then, the spectrum of $-\Delta$ consists of non-negative eigenvalues with a single limit point at ∞. Hence we can order them in a non-increasing way, counting the geometric multiplicities, as

$$0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ... \text{ with } \lambda_n \nearrow \infty.$$

In this setting, it makes sense to define the eigenvalue counting function via

$$N_{-\Delta}(x) := \max\{n \in \mathbb{N} : \lambda_n \leq x\}. \tag{1.2}$$

Weyl's law now states that there is the asymptotic equivalence[1]

$$N_{-\Delta}(x) \sim C_n \mathcal{H}^n(E) x^{n/2}, \quad x \nearrow \infty, \tag{1.3}$$

[1]We adopt the notation $f \sim g$ for the equivalence relation given by $\lim \frac{f}{g} = 1$.

where the constant C_n is independent of the domain E (see [Wey11] and [Wey12] for the original publications). Motivated by (1.3), we define the spectral dimension of Δ on E by

$$\dim_{\mathcal{S}}(E, \Delta) := \lim_{x \to \infty} \frac{\log N_{-\Delta}(x)}{\log x} \tag{1.4}$$

which yields $n/2$ in the situation examined by Weyl's law. Note that the usual definition of $\dim_{\mathcal{S}}$ differs by a factor of 2 (cf. [QUOTE!!]) so that $\dim_{\mathcal{S}}(E, \Delta)$ coincides with $\dim_{\mathcal{H}}(E) = n$, however coming at the cost of an additional factor in the Einstein relation. Therefore, we take the liberty to deviate from the established convention in this minor aspect.

1.3 Diffusion processes and walk dimension

1.3.1 From Dirichlet forms to Markov processes

For the following definition taken from [FOT11, p.3ff], set $H = L^2(E, \mu)$ where μ is a σ-finite Borel-measure on E.

Definition 1.4. A map $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$ is a Dirichlet form if it satisfies the following conditions:

i. The domain $\mathcal{D}(\mathcal{E}) \subseteq H$ of $\mathcal{E}$ is a dense linear subspace.

ii. $\mathcal{E}$ is a symmetric, non-negative definite bilinear form.

iii. This form is closed, that is, the inner product space $(\mathcal{D}(\mathcal{E}), \mathcal{E}_\alpha)$ equipped with the inner product

$$\mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha \langle u, v \rangle \quad \text{for} \quad u, v \in \mathcal{D}(\mathcal{E}_\alpha) = \mathcal{D}(\mathcal{E}),\ \alpha > 0,$$

is complete (and thus itself a Hilbert space).

iv. $\mathcal{E}$ is a Markovian form, i.e. for all $u \in \mathcal{D}(\mathcal{E})$, $v := (0 \vee u) \wedge 1 \in \mathcal{D}(\mathcal{E})$ and we have $\mathcal{E}[v] \leq \mathcal{E}[u]$ for the quadratic form of $\mathcal{E}$.

We remark that the choice of $\alpha > 0$ is irrelevant for the completeness of $(\mathcal{D}(\mathcal{E}), \mathcal{E}_\alpha)$ since all induced norms are equivalent to each other.

We start with the following definition (cf. [MR12, Def. IV.1.13]):

Definition 1.5. Given a filtered probability space $\left(\Omega, \mathscr{A}, \mathscr{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbf{P}\right)$ satisfying the usual conditions, an $\mathscr{F}$-adapted time-homogenous Markov process $X = (X_t)_{t\geq 0}$ with state space E_Δ is called a right process if it satisfies the strong Markov property for all $\mathscr{F}$-stopping times and all its trajectories are right continuous.

1.4 Other versions of the Einstein relation

Telcs, Mandelbrot, etc...

Chapter 2

Examples and Non-examples

In this chapter, we will explore the Einstein relation by examinating some examples and will thusly motivate some of the more general results of the next chapter.

2.1 Euclidean Space

2.2 Sierpinski Gasket

The Sierpinski Gasket is a simple example of an iterated function fractal and can be described according to theorem 1.3 as the unique non-empty compact set $\mathrm{SG} \subseteq \mathbb{R}^2$ which is invariant under the three similitudes

$$S_1(x,y) = \left(\frac{x}{2}, \frac{y}{2}\right),\ S_2(x,y) = \left(\frac{x+1}{2}, \frac{y}{2}\right),\ S_3(x,y) = \left(\frac{2x+1}{4}, \frac{2y+\sqrt{3}}{4}\right),$$

see PICTURE! Since SG satisfies the (OSC), e.g. by taking the open equilateral triangle with corners $(0,0), (0,1)$ and $(1/2, \sqrt{3}/2)$, we obtain both $s = \dim_{\mathcal{H}}(\mathrm{SG}) = \frac{\ln 3}{\ln 2}$ and $\mathcal{H}^s(\mathrm{SG}) \in (0, \infty)$ by a second appeal to Hutchinson's theorem.

2.3 Combs and inhomogenous graphs

2.4 Bounded metric spaces

Chapter 3

The Einstein Relation under Change of Measure and Metric??

Vague ideas only, no plans (yet) :/

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