Master Thesis

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Contents

Abstract

We consider $\mathbb{R}$ as well as $\dim_{\mathcal{S}}, \dim_{\mathcal{H}} \subseteq \dim_{\mathcal{W}}$.

Chapter 1

An Introduction to the Einstein Relation

In this introductory chapter, we wish to briefly expose the ingredients of the ER - the Hausdorff dimension $\dim_{\mathcal{H}}$, the spectral dimension $\dim_{\mathcal{S}}$, and the walk dimension $\dim_{\mathcal{W}}$ - and state some of their properties.

1.1 Hausdorff measure and Hausdorff dimension

Although the concepts of Hausdorff measure and dimension are well-known, we give the definitions in the interest of completeness. In what follows, let (X, d) be a metric space.

Definition 1 (Hausdorff outer measure)**.** For fixed $s \geq 0$, any subset $S \subseteq X$ and any $\delta > 0$, let

$$\mathcal{H}^s_\delta(S) := \inf \left\{ \sum_{i \in I} (\operatorname{diam} U_i)^s : |I| \leq \aleph_0, S \subseteq \bigcup_{i \in I} U_i \subseteq X, \operatorname{diam} U_i \leq \delta \right\},$$

i.e. the infimum is taken over all countable coverings of S with diameter at most δ. The s-dimensional Hausdorff outer measure of S is now defined to be

$$\mathcal{H}^s(S) := \lim_{\delta \searrow 0} \mathcal{H}^s_\delta(S). \tag{1.1}$$

Observe that the limit in (1.1) exists or equals ∞, since $\mathcal{H}^s_\delta(S)$ is monotonically nonincreasing in δ, yet bounded from below by 0. Furthermore, it can be shown that $\mathcal{H}^s$ defines a metric outer measure on X, thus restricting to a measure on a σ-algebra containing the Borel σ-algera $\mathcal{B}(X)$ (cf. [Mat99, p.54ff]). By definition, the obtained measure then is the s-dimensional Hausdorff measure which we will denote by $\mathcal{H}^s$ as well.

Since exponential functions are monotonically increasing, the Hausdorff measures' dependence on s exhibits the same behavior. At the same time, simple estimates yield that if $\mathcal{H}^s(S)$ is finite for some s, it vanishes for all $s' < s$, and conversely, if $\mathcal{H}^s(S) > 0$, then

$\mathcal{H}^{s'}(S) = \infty$ for all $s' > s$. Therefore, there exists precisely one real number s where $\mathcal{H}^{\cdot}(S)$ jumps from 0 to ∞ (by possibly attaining any value of $[0, \infty]$ there). This motivates the following definition of Hausdorff dimension:

Definition 2. The Hausdorff dimension $\dim_{\mathcal{H}}(S)$ of $S \subseteq X$ is defined as

$$\dim_{\mathcal{H}}(S) := \inf\{s \geq 0 : \mathcal{H}^s(S) > 0\}.$$

Due to the above discussion, we have the following equalities:

$$\begin{aligned} \dim_{\mathcal{H}}(S) &= \inf\{s \geq 0 : \mathcal{H}^s(S) > 0\} = \inf\{s \geq 0 : \mathcal{H}^s(S) = \infty\} \\ &= \sup\{s \geq 0 : \mathcal{H}^s(S) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(S) < \infty\}, \end{aligned}$$

providing some alternative characterisations of the Hausdorff dimension.

1.2 Weyl asymptotics and spectral dimension

1.3 Diffusion processes and walk dimension

Chapter 2

Examples and Non-examples

Bibliography

[Mat99] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge 1999.