The Einstein Relation on Fractals and Other Singular Sets

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Preface: Beware of construction works!

This is an early draft of my Master thesis on the so-called Einstein relation. "Early" because I started working on the topic after the German winter term finished (which was mid-February at my university) and since spent most of my time reading literature to understand the existing theory. Accordingly, a lot of the ideas I currently have are not yet worked out completely, a lot of the text is still unrefined, pictures for the examples in chapter 2 are non-existent, there are several gaps in the material that still have to be filled out - and even the title might be subject to future changes. However, I added short remarks denoted by "TO DO" to signify my overall plans, and I do hope that this current draft - despite admittedly resembling a desolate construction site more than a newly finished building - fulfills the needs as part of this application.

Fabian Burghart, 22nd April 2018.

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Chapter 1

Fractal Dimensions and the Einstein Relation

In this introductory chapter, we wish to briefly expose the ingredients of the Einstein relation - the Hausdorff dimension $\dim_{\mathcal{H}}$, the spectral dimension $\dim_{\mathcal{S}}$, and the walk dimension $\dim_{\mathcal{W}}$ - and state some of their properties.

1.1 Hausdorff measure and Hausdorff dimension

Although the concepts of Hausdorff measure and dimension are well-known, we give the definitions in the interest of completeness. In what follows, let (X, d) be a metric space.

Definition 1.1 (Hausdorff outer measure)**.** For fixed $s \geq 0$, any subset $S \subseteq X$ and any $\delta > 0$, let

$$\mathcal{H}^s_\delta(S) := \inf \left\{ \sum_{i \in I} (\operatorname{diam} U_i)^s : |I| \leq \aleph_0, S \subseteq \bigcup_{i \in I} U_i \subseteq E, \operatorname{diam} U_i \leq \delta \right\},$$

i.e. the infimum is taken over all countable coverings of S with diameter at most δ. The s-dimensional Hausdorff outer measure of S is now defined to be

$$\mathcal{H}^s(S) := \lim_{\delta \searrow 0} \mathcal{H}^s_\delta(S). \tag{1.1}$$

Observe that the limit in (1.1) exists or equals ∞, since $\mathcal{H}^s_\delta(S)$ is monotonically nonincreasing in δ and bounded from below by 0. Furthermore, it can be shown that $\mathcal{H}^s$ defines a metric outer measure on X, and thus restricts to a measure on a σ-algebra containing the Borel σ-algera $\mathscr{B}(X)$ (cf. [Mat99, p.54ff]). By definition, the obtained measure then is the s-dimensional Hausdorff measure which we will denote by $\mathcal{H}^s$ as well. Note that for $\mathcal{H}^s$ to be a Radon measure, i.e. locally finite and inner regular, $\mathcal{H}^s(X) < \infty$ is sufficient.

In the special case of (X, d) being an Euclidean space, Hausdorff measures interpolate between the usual Lebesgue measures λ^n: For $s = 0$, we have simply $\mathcal{H}^0(S) = \#S$, whereas for any integer $n > 0$, it can be shown that there exists a constant $c_n > 0$ depending only on n such that $\mathcal{H}^n = c_n \lambda^n$, where the constant is the volume of the n-dimensional unit ball.

It can be seen by simple estimates that the map $s \mapsto \mathcal{H}^s(S)$ for fixed $S \subseteq X$ is monotonically nonincreasing. More specifically, if $\mathcal{H}^s(S)$ is finite for some s then it vanishes for all $s' > s$, and conversely, if $\mathcal{H}^s(S) < \infty$ then $\mathcal{H}^{s'}(S) = \infty$ for all $s' < s$. Therefore, there exists precisely one real number s where $\mathcal{H}^{\cdot}(S)$ jumps from ∞ to 0 (by possibly attaining any value of $[0, \infty]$ there). This motivates the following definition of Hausdorff dimension:

Definition 1.2. The Hausdorff dimension $\dim_{\mathcal{H}}(S)$ of $S \subseteq X$ is defined as

$$\dim_{\mathcal{H}}(S) := \inf\{s \geq 0 : \mathcal{H}^s(S) < \infty\}.$$

Due to the above discussion, we have the following equalities:

$$\begin{aligned} \dim_{\mathcal{H}}(S) &= \inf\{s \geq 0 : \mathcal{H}^s(S) < \infty\} = \inf\{s \geq 0 : \mathcal{H}^s(S) = 0\} \\ &= \sup\{s \geq 0 : \mathcal{H}^s(S) = \infty\} = \sup\{s \geq 0 : \mathcal{H}^s(S) > 0\}, \end{aligned}$$

providing some alternative characterisations of the Hausdorff dimension.

We further collect some important facts. To this end, let S, S' and $S_1, S_2, ...$ be subsets of E as before. Then, the following properties hold (cf. [Fal07, p.32f] for a discussion in the Euclidean setting; however all arguments adapt to our more general situation without complication):

Monotonicity. If $S \subseteq S'$ then $\dim_{\mathcal{H}}(S) \leq \dim_{\mathcal{H}}(S')$.

Countable Stability. For a sequence $(S_n)_{n\geq 1}$, we have the equality

$$\dim_{\mathcal{H}}\left(\bigcup_{n\geq 1} S_n\right) = \sup_{n\geq 1} \dim_{\mathcal{H}}(S_n).$$

Countable Sets. If $|S| \leq \aleph_0$ then $\dim_{\mathcal{H}}(S) = 0$.

Hölder continuous maps. If (X', d') is another metric space and $f : X \to X'$ is α-Hölder continuous for some $\alpha \in (0, 1]$ then $\dim_{\mathcal{H}}(f(S)) \leq \alpha^{-1} \dim_{\mathcal{H}}(S)$. In particular, the Hausdorff dimension is invariant under a bi-Lipschitz transformation (i.e. an invertible map f with Hölder exponent $\alpha = 1$ for both f and f^{-1}).

Euclidean Case. If (X, d) happens to be an Euclidean space (or more generally a continuously differentiable manifold) of dimension n and S is an open subset then $\dim_{\mathcal{H}}(S) = n$.

We conclude this section by citing Hutchinson's theorem about the Hausdorff dimension of self-similar sets which will provide us with a plethora of interesting examples. For this, we recall that a map $F : X \to X$ on a metric space (X, d) is a strict contraction if its Lipschitz constant satisfies

$$\mathrm{Lip}_F := \sup_{\substack{x,y\in X \\ x\neq y}} \frac{d(F(x), F(y))}{d(x, y)} < 1. \tag{1.2}$$

If the stronger condition $d(F(x), F(y)) = \mathrm{Lip}_F\, d(x, y)$ holds for all $x, y \in X$, we call F a similitude with contraction factor Lip_F.

Theorem 1.3 (Hutchinson, [Hut81])**.** *Let $\mathscr{S} = \{S_1, ..., S_N\}$ be a finite set of strict contractions on the Euclidean space $\mathbb{R}^n$. Then there exists a unique nonempty compact set denoted* *by $|\mathscr{S}|$ which is invariant under $\mathscr{S}$, i.e.*

$$|\mathscr{S}| = \bigcup_{i=1}^{N} S_i(|\mathscr{S}|).$$

Furthermore, assume that $|\mathscr{S}|$ satisfies the open set condition (OSC) meaning that there exists a nonempty open set $O \subseteq E$ with the properties $S_i(O) \subseteq O$ and $S_i(O) \cap S_j(O) = \emptyset$ for all $i, j = 1, ..., N$ with $i \neq j$. Also assume that the maps S_i are similitudes with contraction factor $r_i \in (0, 1)$. Then, $s = \dim_{\mathcal{H}}(|\mathscr{S}|)$ is the unique solution to the equation

$$\sum_{i=1}^{N} r_i^s = 1$$

and we have $0 < \mathcal{H}^s(|\mathscr{S}|) < \infty$.

While uniqueness and existence of $|\mathscr{S}|$ are still ensured for maps on a complete metric space, the open set condition is not sufficient for statements about the Hausdorff dimension, see [Sch96] for further discussion.

1.2 Weyl asymptotics and spectral dimension

The idea of introducing spectral dimension is inspired by Weyl's law for the eigenvalues of the Dirichlet-Laplace operator which we will discuss here shortly.

1.2.1 The classical case

Given a bounded open domain $E \subseteq \mathbb{R}^n$, consider the Laplace operator Δ on E acting on functions satisfying the Dirichlet boundary condition $u \equiv 0$ on ∂E. Then, the spectrum of $-\Delta$ consists of non-negative eigenvalues with a single limit point at ∞. Hence we can order them in a non-increasing way, counting the geometric multiplicities, as

$$0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ... \text{ with } \lambda_n \nearrow \infty. \tag{1.3}$$

In this setting, it makes sense to define the eigenvalue counting function via

$$N_{-\Delta}(x) := \max\{n \in \mathbb{N} : \lambda_n \leq x\}, \; x \in \mathbb{R}_{\geq 0}. \tag{1.4}$$

Weyl's law now states that there is the asymptotic equivalence[1]

$$N_{-\Delta}(x) \sim C_n \mathcal{H}^n(E) x^{n/2}, \quad x \nearrow \infty, \tag{1.5}$$

where the constant C_n is independent of the domain E (see [Wey11] and [Wey12] for the original publications). Motivated by (1.5), we define the spectral dimension of $-\Delta$ on E by

$$\dim_{\mathcal{S}}(E, -\Delta) := \lim_{x \to \infty} \frac{\log N_{-\Delta}(x)}{\log x} \tag{1.6}$$

[1]We adopt the notation $f \sim g$ for the equivalence relation given by $\lim \frac{f}{g} = 1$.

which yields $n/2$ in the situation examined by Weyl's law. Note that the usual definition of $\dim_{\mathcal{S}}$ differs by a factor of 2 (cf. [KL93],[HKK02]) so that $\dim_{\mathcal{S}}(E, \Delta)$ normally coincides with $\dim_{\mathcal{H}}(E) = n$, however coming at the cost of an additional factor in the Einstein relation. Moreover, it can be argued that the spectral dimension is rather a property of the operator $-\Delta$ than of the underlying space E. Therefore, we take the liberty to deviate from the established convention in this minor aspect.

1.2.2 The general case

How can we generalise the concepts just introduced to sets E which are not bounded open subsets of $\mathbb{R}^n$? For this, suppose we are given a metric measure space (E, d, μ), where (E, d) is a locally compact separable metric space and μ is a Radon measure on E.

Of course, the notion of an eigenvalue counting function as outlined above works for any operator A whose set of eigenvalues possesses only one limit point at $+\infty$. However, as we will explain in the next section, we also wish to associate a reasonably well-behaved Markov process with state space E to A. Therefore, we choose to impose the following conditions on A:

Assumptions 1.4. For an operator $A : L^2(E, d) \supseteq \mathcal{D}(A) \to L^2(E, d)$, we assume the following holds:

Self-adjointness. A is a densely defined, self-adjoint operator on the Hilbert space $L^2(E, \mu)$.

Eigenvalues. The spectrum is contained in $\mathbb{R}_{\geq 0}$ and the set of eigenvalues can be enumerated as in (1.3).

Dissipativeness. $-A$ is dissipative. In other words, for all $f \in \mathcal{D}(A)$ and all $\lambda > 0$, we have $\|(\lambda + A)f\| \geq \lambda \|f\|$.

The first of these assumptions guarantees that A is a closed operator, whereas the second ensures that $\lambda + A$ is surjective for at least one $\lambda > 0$. Thus, the Hille-Yosida theorem states that there is a strongly continuous semigroup of contractive linear operators T_t on H such that $-A$ is its infinitesimal generator. That is to say:

Definition 1.5. A strongly continuous semigroup $(T_t)_{t\geq 0}$ on a Hilbert space H is a monoid homomorphism $t \mapsto T_t$ from $(\mathbb{R}_{\geq 0}, +)$ to the space of bounded linear operators $(\mathbb{B}(H), \cdot)$ on H (equipped with composition) satisfying for all $f \in H$ the additional property

$$\lim_{t \searrow 0} \|T_t f - f\| = 0.$$

The infinitesimal generator $(-A, \mathcal{D}(A))$ of $(T_t)_{t\geq 0}$ is defined via

$$(-A)f = \lim_{t \searrow 0} \frac{1}{t}(T_t f - f), \; f \in \mathcal{D}(A),$$

where $\mathcal{D}(A)$ is the set of elements in H for which this limit exists.

Theorem 1.6 (Hille–Yosida)**.** *An operator $(-A, \mathcal{D}(A))$ is the generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ with $\|T_t\| \leq 1$ for all $t \geq 0$ if and only if $-A$ is a densely defined, closed, dissipative operator such that for some $\lambda > 0$, the map $\lambda + A$ is surjective.*

It can be shown that there is a one-to-one correspondence between contractive semigroups and operators that satisfy the Hille-Yosida theorem, that is, the semigroup in the above theorem is uniquely determined by A.

Having discussed the motivation for the assumptions 1.4, we now proceed to adapt the definitions made in (1.4) and (1.6) in a rather straightforward way:

Definition 1.7. Given an operator $(A, \mathcal{D}(A))$ on $L^2(E, \mu)$ satisfying the assumptions 1.4, its eigenvalue counting function is defined by

$$N_A(x) := \max\{n \in \mathbb{N} : \lambda_n \leq x\}, \ x \in \mathbb{R}_{\geq 0}, \tag{1.7}$$

and the spectral dimension of A by

$$\dim_{\mathcal{S}}(E, A) := \lim_{x \to \infty} \frac{\log N_A(x)}{\log x}. \tag{1.8}$$

TO DO: Provide examples for operators satisfying assumptions 1.4 - e.g. one-sided inverses of injective bounded compact operators. Also, streamline those assumptions, as of now, they are somewhat redundant. Stronger emphasis on Feller semigroups might also be needed, since Dynkin's probabilistic characterisation of the generator [Kal02, Theorem 19.23] might help in providing some general results concerning the validity of the Einstein relation.

1.3 Markov processes and walk dimension

1.3.1 From Dirichlet forms to Markov processes

The theory presented here is mostly taken from [FOT11] and [MR12, ch. 4]. Set $H = L^2(E, \mu)$ where μ is a σ-finite Borel-measure on E.

Definition 1.8. A map $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$ is a Dirichlet form if it satisfies the following conditions:

i. The domain $\mathcal{D}(\mathcal{E}) \subseteq H$ of $\mathcal{E}$ is a dense linear subspace.
ii. $\mathcal{E}$ is a symmetric, non-negative definite bilinear form.
iii. This form is closed, that is, the inner product space $(\mathcal{D}(\mathcal{E}), \mathcal{E}_\alpha)$ equipped with the scalar product

$$\mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha \langle u, v \rangle \quad \text{for} \quad u, v \in \mathcal{D}(\mathcal{E}_\alpha) = \mathcal{D}(\mathcal{E}), \ \alpha > 0,$$

is complete (and thus itself a Hilbert space).
iv. $\mathcal{E}$ is a Markovian form, i.e. for all $u \in \mathcal{D}(\mathcal{E})$, $v := (0 \vee u) \wedge 1 \in \mathcal{D}(\mathcal{E})$ and we have $\mathcal{E}[v] \leq \mathcal{E}[u]$ for the quadratic form of $\mathcal{E}$.

We remark that the choice of $\alpha > 0$ is irrelevant for the completeness of $(\mathcal{D}(\mathcal{E}), \mathcal{E}_\alpha)$ since all induced norms are equivalent to each other.

Definition 1.9. A Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E, \mu)$ is said to be

i. regular if it possesses a core, that is, the space $\mathcal{D}(\mathcal{E}) \cap C_c(E)$ is simultaneously dense in $\mathcal{D}(\mathcal{E})$ with respect to the $\mathcal{E}_1$-norm and in $C_c(E)$ with respect to the uniform norm.
ii. local if $\mathcal{E}(u, v) = 0$ whenever $u, v \in \mathcal{D}(\mathcal{E})$ have disjoint compact support.
iii. strongly local if $\mathcal{E}(u, v) = 0$ whenever $u, v \in \mathcal{D}(\mathcal{E})$ have compact support and v is

constant on a neighbourhood of $\mathrm{supp}(u)$.

If additionally $\mu(E) < \infty$, we say that $\mathcal{E}$ is

iv. conservative if $1 \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}[1] = 0$.

v. irreducible if it is conservative and $\mathcal{E}[f]$ implies that f is constant.

We can uniquely attach a positively semidefinite operator A to a Dirchlet form (and vice versa) via the relation

$$\mathcal{E}(u, v) = \langle Au, v\rangle\,,\; u \in \mathcal{D}(A), v \in \mathcal{D}(\mathcal{E}). \tag{1.9}$$

In particular, if A meets the requirements of 1.4, we not only have precisely one strongly continuous contraction semigroup on H as explained by theorem 1.6, but also a unique Dirichlet form thanks to (1.9). In similar style, we would also like to attach a unique Markov process to A - or, equivalently, to the semigroup or the Dirichlet form.

To define a suitable stochastic process with values in E, we first adjoin a cemetary state $*$ in such a way that if E is non-compact, $E_* := E \sqcup \{*\}$ is the one-point compactification of E, whereas $*$ is supposed to be an isolated point if E is compact. Let $X = (\Omega, \mathscr{A}, (X_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in E_*})$ be a stochastic process on a measurable space $(\Omega, \mathscr{A})$ with values in E_*, where we adapt the notation that $\mathbf{P}_x[X_0 = x] = 1$ for all $x \in E_*$ and $\mathbf{P}_*[X_t = *] = 1$ for all $t \geq 0$. Note that X induces a filtration $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$ on $\mathscr{A}$ by

$$\mathscr{F}_t = \bigcap_{\mathbf{P}\in\mathcal{M}_1^+(\Omega,\mathscr{A})} \left(\sigma\{X_s : 0 \leq s \leq t\}\right)^{\mathbf{P}}.$$

Here, $\mathcal{M}_1^+$ denotes the set of all probability measures on $(\Omega, \mathscr{A})$, $\sigma\{\cdot\}$ denotes the the σ-algebra generated by $\{\cdot\}$ and $\mathscr{B}^{\mathbf{P}}$ denotes the completion of a σ-algebra $\mathscr{B}$ with respect to the measure $\mathbf{P}$. Henceforth, we will only consider stochastic processes X that satisfy the strong Markov property with respect to $\mathscr{F}$ and are time-homogenous. Such X is called Hunt process if it additionally has right-continuous trojectories and is quasi-left-continuous, i.e. any sequence $\tau_n \nearrow \tau$ of $\mathscr{F}$-stopping times satisfies

$$\mathbf{P}_\alpha\left[\lim_{n\to\infty} X_{\tau_n} = X_\tau, \tau < \infty\right] = \mathbf{P}_\alpha[\tau < \infty]$$

for any initial distribution α. We can now easily translate Markov processes to contractive semigroups by setting

$$(T_t f)(x) := \mathbf{E}_x\left[f(X_t)\right],\; t \geq 0. \tag{1.10}$$

The other direction is more involved, and the process attached to a Dirichlet form is generally non-unique. We have, however, (cf. [FOT11, theorems 7.2.1 and 7.2.2])

Theorem 1.10. *Let $\mathcal{E}$ be a regular Dirichlet form on $L^2(X, \mu)$. Then, there exists a Hunt process X on (E, d) such that the operators $T_t, t \geq 0$, from* (1.10) *are symmetric and $\mathcal{E}$ is the Dirichlet form belonging to this semigroup.*

Moreover, if $\mathcal{E}$ is local, X is a diffusion process.

As hinted above, those processes are not unique: One can modify X to $\tilde{X}$ by killing the process on a polar set and obtain the same semigroup for both. See section 7.2.2. in [FOT11] for further discussion.

1.3.2 Local walk dimension and Einstein relation

The walk dimension is meant to quantify how fast a given Markov process on E moves away from its starting point x. This is best expressed in terms of the stopping time $\tau_{B(x,r)}$, which is supposed to be the first exit time of the Ball $B(x,r) = \{y \in E : d(x,y) < r\}$. Note that this is indeed an $\mathscr{F}$-stopping time by right continuity of the process in question and by [Kal02, Lemma 7.6].

For the next definition to make sense, we need to impose some additional assumption on the metric space (E,d). We choose to demand for now that E is path connected, but will discuss other scenarios in the next chapter.

Definition 1.11. (Cf. [HKK02]) We define the quantity

$$\dim_{\mathcal{W}}(E,X,x) = \lim_{r \searrow 0} \frac{\log \mathbf{E}_x \left[\tau_{B(x,r)}\right]}{\log r}$$

and call it the (local) walk dimension of (E,d) at $x \in E$ with respect to the Markov process $(X_t)_{t\geq 0}$. If $\dim_{\mathcal{W}}(E,X,x)$ is μ–a.e. constant on E, we shorten our notation to $\dim_{\mathcal{W}}(E,X)$.

We are now finally able to state the Einstein relation:

Definition 1.12. Let (E,d,μ) be a locally compact separable metric measure space and let $(A,\mathcal{D}(A))$ be an operator on $L^2(E,\mu)$ satisfying assumptions 1.4. Suppose $X = \big((X_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in E_*}\big)$ is a Markov process associated to A via the Dirichlet form $\mathcal{E}(\cdot,\cdot) = \langle A\,\cdot,\cdot\rangle$. We then say that the Einstein relation with constant c holds on E with respect to A if

$$\dim_{\mathcal{H}}(E) = c \dim_{\mathcal{S}}(E,A) \dim_{\mathcal{W}}(E,X). \tag{1.11}$$

We omit mentioning the constant if $c = 1$.

1.4 Other versions of the Einstein relation

TO DO: A discussion of further literature, such as Telcs, Mandelbrot, etc... is planned to be included here. Also, hints to the physical background.

Chapter 2

Examples and Non-examples

In this chapter, we will discuss the necessity of some of the restrictive assumptions made previously and explore the Einstein relation by examining some examples and will motivate some of the more general results of the next chapter.

2.1 Euclidean Space

We start by examining the classical setting of paragraph 1.2.1 in greater detail: Let once again $E \subseteq \mathbb{R}^n$ be an open, bounded, non-empty domain, equipped with Euclidean metric and n-dimensional Lebesgue-measure λ^n. Trivially, $\dim_{\mathcal{H}}(E) = n$.

2.1.1 The Dirichlet–Laplace operator

TO DO: The Dirichlet boundary conditions translate to a BM killed on the boundary, I should pay some more attention to that (results hold regardless).

For the Dirichlet–Laplace operator as introduced earlier, we obtain $\dim_{\mathcal{S}}(E, -\Delta) = \frac{n}{2}$ due to (1.4)-(1.6). Simultaneously, it is well-known that $\frac{1}{2}\Delta$ is the generator of the n-dimensional Brownian motion B_t, which can easily be seen as follows:

By (1.10), the semigroup $(T_t)_{t\geq 0}$ induced by B_t reads

$$T_t f(x) = \mathbf{E}_x\left[f(X_t)\right] = \frac{1}{(2\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{2t}\right) f(y)\, \mathrm{d}y,\ f \in L^2(\mathbb{R}^n, \lambda^n) \cap C_c(E),$$

Comparing this expression to the well-known convolution formula

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y)\, \mathrm{d}y,\ x \in \mathbb{R}^n, t \geq 0,$$

for the solution of the heat equation $\Delta u = \partial_t u$ with initial value $u(x,0) = f(x)$, we can quickly derive that $\Delta T_t f(x) = 2\partial_t T_t f(x)$ and thus obtain by definition 1.5 the generator

$$Af = \frac{1}{2}\Delta f,$$

extended to its maximal domain $H_0^1(E,\mu)$. We conclude that the Markov process associated with Δ is $2B_t$.

It remains to determine the walk dimension of the n-dimensional Brownian motion. This can be done in several ways, for example by appealing to Brownian scaling or by invoking Dynkin's formula after a standard truncation argument: By applying [Kal02, Lemma 19.21] to the function $u_x(y) = \frac{1}{2}|y|^2 - x$, we get

$$\mathbf{E}_x\left[\tau_{B(x,r)}\right] = \mathbf{E}_x\left[\int_0^{\tau_{B(x,r)}} \Delta u_x(2B_s)\ \mathrm{d}s\right] = \mathbf{E}_x\left[u_x\left(2B_{\tau_{B(x,r)}}\right) - u_x(0)\right] = \mathbf{E}_x\left[\frac{2r^2}{n}\right] = \frac{2r^2}{n}$$

Therefore, by definition 1.11, we obtain $\dim_{\mathcal{W}}(E, 2B_t) = 2$ which implies together with the results obtained previously that the Einstein relation (with constant 1) holds on E.

2.1.2 Other partial differential operators

TO DO: Get spectral estimates for more general second-order partial differential operators; afterwards, the walk dimension should follow again from Dynkin's formula.

2.2 Sierpinski Gasket

The Sierpinski Gasket is a simple example of an iterated function fractal and can be described according to theorem 1.3 as the unique non-empty compact set $\mathrm{SG} \subseteq \mathbb{R}^2$ which is invariant under the three similitudes

$$S_1(x,y) = \left(\frac{x}{2}, \frac{y}{2}\right),\ S_2(x,y) = \left(\frac{x+1}{2}, \frac{y}{2}\right),\ S_3(x,y) = \left(\frac{2x+1}{4}, \frac{2y+\sqrt{3}}{4}\right),$$

see PICTURE! Since SG satisfies the (OSC), e.g. by taking the open equilateral triangle with corners $(0,0), (0,1)$ and $(1/2, \sqrt{3}/2)$, we obtain both

$$s = \dim_{\mathcal{H}}(\mathrm{SG}) = \frac{\ln 3}{\ln 2} \tag{2.1}$$

and $\mathcal{H}^s(\mathrm{SG}) \in (0,\infty)$ by a second appeal to Hutchinson's theorem.

We will use the remainder of this section to establish the validity of the Einstein relation on SG with respect to the standard Laplace operator on SG, which can be obtained in two different ways:

TO DO: Give an overview of these constructions, [Str06] for the approach via graph approximation and [Bar98] for the probabilistic approach.

2.3 Combs and inhomogenous graphs

In this section, we start by considering the graph $\mathbf{C}_2$, called the two-dimensional integer comb, with vertex set $\mathbb{Z}^2$ and edge set given by

$$\left\{\{(n_1,n_2),(m_1,m_2)\} \in \mathbb{Z}^2 \times \mathbb{Z}^2 : |n_1 - m_1| = 1, n_2 = m_2 = 0 \ \text{ or } \ |n_2 - m_2| = 1, n_1 = m_1\right\}$$

TO DO: This example satisfies the Einstein relation with a constant $c \neq 1$ [Ber06], but it also requires an adapted definition of wak dimension, since the vertex set is not path connected. The usual way to solve this problem is by considering the quantity (cf. [Fre12])

$$\lim_{R \nearrow \infty} \frac{\log \mathbf{E}_x \left[\tau_{B(x,R)}\right]}{\log R}. \tag{2.2}$$

On a countable connected graph, this limit will be independent of x.

2.4 Bounded metric spaces

Bounded metric spaces form the most important class of spaces for which too naive of an adaption of (2.2) does not yield useful results. Indeed, consider the metric measure space $E = (\mathbb{R}, d_{\mathrm{arctan}}, \lambda^1)$, where the metric is defined as $d_{\mathrm{arctan}}(x,y) = |\arctan(x) - \arctan(y)|$. Since

$$\tan : \left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), |\cdot|\right) \to (\mathbb{R}, d_{\mathrm{arctan}})$$

provides an isometry, we have $\dim_{\mathcal{H}}(E) = 1$. On this space, we consider the negative of the usual weak Laplace operator, $-\Delta_{\lambda^1}$, defined by mapping a function $u \in H_0^1(\mathbb{R}, \lambda^1)$ to the unique $g \in L^2(\mathbb{R}, \lambda^1)$ such that

$$\int_{\mathbb{R}} g\varphi \ \mathrm{d}\lambda^1 = \int_{\mathbb{R}} \partial_x u \partial_x \varphi \ \mathrm{d}\lambda^1$$

holds for all $\varphi \in H_0^1(\mathbb{R}, \lambda^1)$. Notice how this does not differ from the negative weak Laplace operator on $(\mathbb{R}, |\cdot|, \lambda^1)$ since we did not change the measure and both metrics induce the same topology. Thus, we get from Weyl's classical result $\dim_{\mathcal{S}}(E, -\Delta_{\lambda^1}) = \frac{1}{2}$ and from the arguments developed in section 2.1 that the associated Markov process is $2(B_t)_{t\geq 0}$.

It is now easy to see that (2.2) does not provide a useful notion of a walk dimension: Since $B_{\mathrm{arctan}}(x,R) = \mathbb{R}$ for every radius $R \geq \pi$, the expression diverges to ∞. Even the more careful approach

$$\lim_{R \nearrow \frac{\pi}{2}} \frac{\log \mathbf{E}_0 \left[\tau_{B(x,R)}\right]}{\log R}$$

runs into similar problems: Using the formula $\mathbf{E}\left[\tau_{[a,b]}\right] = -ab$ for the exit time of a standard Brownian motion from the intervall $[a,b] \ni 0$, we get

$$\log \mathbf{E}_0 \left[\tau_{B_{\mathrm{arctan}}(0,R)}\right] = 2 \log \tan R = \infty.$$

However, the local walk dimension from definition 1.11 works out quite elgantly: Setting

$y = \arctan x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we obtain for some $\xi_1 \in (y, y+r), \xi_2 \in (y-r, y)$

$$\begin{aligned}
\frac{\log \mathbf{E}_0 \left[\tau_{B_{\arctan}(0,R)}\right]}{\log r} &= \frac{\log\left(\tan(y+r) - \tan y\right)}{\log r} + \frac{\log\left(\tan y - \tan(y-r)\right)}{\log r} \\
&= \frac{\log \frac{r}{\cos^2 \xi_1}}{\log r} + \frac{\log \frac{r}{\cos^2 \xi_2}}{\log r} = 2 - \frac{\log \cos^2 \xi_1}{\log r} - \frac{\log \cos^2 \xi_2}{\log r}
\end{aligned}$$

by using the mean value theorem. Taking the limit for $r \searrow 0$ on both sides implies $\xi_1, \xi_2 \to y$ and thus $\dim_{\mathcal{W}}(E, 2B_t, x) = 2$.

TO DO: At present, this is still a bit surprising to me, because it suggests that the ER is more stable under a deformation of the underlying space than I expected it. I suspect that the reason for this behavior is that tan is locally Lipschitz-continuous.

Also TO DO: Prof. Freiberg suggested to look at a metric variant of the Laplace operator on $(\mathbb{R}, d_{\arctan}, \lambda^1)$ instead, defined as in [Shi16, def. 7.43]. Moreover, this example is open to generalisations to other weak differential operators, much in the spirit of my plans for 2.1.2

Chapter 3

The Einstein Relation on Metric Measure Spaces

This chapter is devoted to the investigation of the Einstein relation in the setting of an abstract mm-space. First, we focus on the behaviour under morphisms between mm-spaces to derive invariance properties of the Einstein relation. We will then use some of these properties to construct a sequence of mm-spaces each satisfying the Einstein relation with constants that diverge, thus showing that these constants can become arbitrarily large.

3.1 The Einstein Relation under Lipschitz-isomorphisms

3.1.1 Lipschitz and mm-isomorphisms

We will use this section to introduce two different categories MM_L and $\mathsf{MM}_{\leq 1}$ whose objects are mm-spaces, but with different morphisms:

- In MM_L, the set $\mathsf{MM}_L(X,Y)$ of morphisms from an object $X = (X, d_X, \mu_X)$ to another object $Y = (Y, d_Y, \mu_Y)$ is the set of all Lipschitz-continuous functions

$$\varphi : \operatorname{supp} \mu_X \to \operatorname{supp} \mu_Y$$

  satisfying $\varphi_* \mu_X = \mu_Y$.
- In $\mathsf{MM}_{\leq 1}$, the set $\mathsf{MM}_{\leq 1}(X,Y)$ of morphisms from an object $X = (X, d_X, \mu_X)$ to another object $Y = (Y, d_Y, \mu_Y)$ is the subset of $\mathsf{MM}_L(X,Y)$ consisting of all contraction maps (i.e. Lipschitz-continuous functions f with $\mathrm{Lip}_f \leq 1$, cf. (1.2)).

In both of those categories, composition of morphisms is to be understood as the usual composition of maps. By definition, $\mathsf{MM}_{\leq 1}$ is a subcategory of MM_L. Considering the usual notion of isomorphism, both categories give rise to a meaningful concept of isomorphy for mm-spaces:

Definition 3.1. A Lipschitz-isomorphism between mm-spaces (X, d_X, μ_X) and (Y, d_Y, μ_Y) is a map $\varphi : \operatorname{supp} \mu_X \to \operatorname{supp} \mu_Y$ with $\varphi_* \mu_X = \mu_Y$ satisfying the bi-Lipschitz condition

$$\frac{1}{C} d_X(x,y) \leq d_Y(\varphi(x), \varphi(y)) \leq C d_X(x,y)$$

for all $x, y \in \operatorname{supp} \mu_X$ and a constant $C \in [1, \infty)$ not depending on x, y.

Similarly, an mm-isomorphism is defined to be a Lipschitz-isomorphism with constant $C = 1$. (This coincides with definition 2.8 in [Shi16])

As it turns out, Lipschitz-isomorphisms are precisely the isomorphisms in MM_L, whereas mm-isomorphisms are the ones in $\mathsf{MM}_{\leq 1}$.

Indeed, consider a Lipschitz-isomorphism $\varphi : X \supseteq \operatorname{supp} \mu_X \to \operatorname{supp} \mu_Y \subseteq Y$. By definition, this is an injective morphism from $\mathsf{MM}_L(X, Y)$. We need to show that φ is surjective to ensure the existence of a two-sided inverse in $\mathsf{MM}_L(Y, X)$. To this end, suppose there exists $y \in \operatorname{supp} \mu_Y \setminus \varphi(\operatorname{supp} \mu_X) =: Z$. Since $\operatorname{supp} \mu_X$ is closed, so is its image under the homeomorphism φ, and hence $Z \subseteq \operatorname{supp} \mu_Y$ is open. As every open subset of $\operatorname{supp} \mu_Y$ is required to have positive measure, we obtain the contradiction

$$0 < \mu_Y(Z) = \varphi_* \mu_X(Z) = \mu_X \left(\varphi^{-1} \left(\operatorname{supp} \mu_Y \setminus \varphi(\operatorname{supp} \mu_X) \right) \right) = 0.$$

Hence, φ is indeed a bijection. Conversely, if φ is an isomorphism from $\mathsf{MM}_L(X, Y)$ then we get the lower Lipschitz-bound from the Lipschitz-continuity of $\varphi^{-1} \in \mathsf{MM}_L(Y, X)$, thus showing that φ is also a Lipschitz-isomorphism. Analogously, the corresponding statement for mm-isomorphisms can be derived.

We will write $(X, d_X, \mu_X) \simeq (Y, d_Y, \mu_Y)$ if X and Y are Lipschitz-isomorphic, and $(X, d_X, \mu_X) \cong (Y, d_Y, \mu_Y)$ if they are mm-isomorphic. Trivially, $X \cong Y$ implies $X \simeq Y$.

In what follows, we will always assume $\operatorname{supp} \mu_X = X$.

Remark 3.2. Of course, we always have $(X, d_X, \mu_X) \cong (\operatorname{supp} \mu_X, d_X, \mu_X)$ by virtue of $\mathrm{id} : X \supseteq \operatorname{supp} \mu_X \to \operatorname{supp} \mu_X$. The restriction $\operatorname{supp} \mu_X = X$ becomes necessary for the Einstein relation since $\dim_{\mathcal{H}}(\operatorname{supp} \mu_X)$ might be strictly smaller than $\dim_{\mathcal{H}}(X)$, the term appearing in the Einstein relation (1.11). We will later see (Proposition 3.3) that the Einstein relation is invariant under Lipschitz-isomorphisms which provides some motivation to circumvent this restriction by considering the relation

$$\dim_{\mathcal{H}}(\operatorname{supp} \mu_X) = c \dim_{\mathcal{S}}(\operatorname{supp} \mu_X, A) \dim_{\mathcal{W}}(\operatorname{supp} \mu_X, M)$$

instead of (1.11).

3.1.2 Transport of structure

Given two mm-spaces (X, d_X, μ_X) and (Y, d_Y, μ_Y) with a map $\varphi : X \to Y$, where a suitable operator $A : L^2(X, \mu_X) \supseteq \mathcal{D}(A) \to L^2(X, \mu_X)$ satisfies the Einstein relation with constant c on X. How can we transport A alongside φ to become an operator on $L^2(Y, \mu_Y)$, and which restrictions do we need to impose no φ to ensure that this transport of structure is compatible with the theory from chapter 1?

Note first that any bimeasurable bijection $\varphi : (X, d_X, \mu_X) \to (Y, d_Y, \mu_Y)$ induces by precomposition an operator

$$\begin{aligned}\varphi_* : L^2(Y,\nu) &\to L^2(X,\mu)\\ f(y) &\mapsto (f\circ\varphi)(x)\end{aligned}$$

which is an isometric isomorphism because $\varphi^{-1} : Y \to X$ induces its inverse and because of

$$\|\varphi_* f\|^2_{L^2(X,\mu_X)} = \int_X |f(\varphi(x))|^2 \,\mathrm{d}\mu_X = \int_Y |f|^2 \,\mathrm{d}\varphi_*\mu_X = \int_Y |f|^2 \,\mathrm{d}\mu_Y = \|f\|^2_{L^2(Y,\mu_Y)}, \tag{3.1}$$

by the change of variables formula.

Denote by $\mathbb{L}(H)$ the set of all partially defined linear maps (not necessarily bounded) on a Hilbert space H. Given an operator $A \in \mathbb{L}(L^2(X,\mu_X))$, we can now contruct an operator $\varphi_{\mathbb{L}}A \in \mathbb{L}(L^2(Y,\mu_Y))$ by conjugating with φ_*. More explicitly, we define the map

$$\varphi_{\mathbb{L}} : \mathbb{L}(L^2(X,\mu)) \to \mathbb{L}(L^2(Y,\mu))$$

where $(\varphi_{\mathbb{L}}A)f := (\varphi_*^{-1} \circ A \circ \varphi_*)f$ and $\mathcal{D}(\varphi_{\mathbb{L}}A) = \varphi_*^{-1}(\mathcal{D}(A))$. It follows immediately that $\mathcal{D}(\varphi_{\mathbb{L}}A)$ is dense iff $\mathcal{D}(A)$ is, and $\varphi_{\mathbb{L}}A$ is self-adjoint iff A is. Indeed, consider arbitrary $f, g \in \mathcal{D}(\varphi_{\mathbb{L}}A)$ with $f = \varphi_*^{-1}(\bar{f})$ and $g = \varphi_*^{-1}(\bar{g})$, where $\bar{f}, \bar{g} \in \mathcal{D}(A)$. Then, applying (3.1), we have

$$\langle(\varphi_{\mathbb{L}}A)f, g\rangle_{L^2(Y,\mu_Y)} = \left\langle \varphi_*^{-1}A\varphi_*\varphi_*^{-1}\bar{f}, \varphi_*^{-1}\bar{g}\right\rangle_{L^2(Y,\mu_Y)} = \left\langle A\bar{f}, \bar{g}\right\rangle_{L^2(X,\mu_X)}$$

and we can perform the same calculations for $\langle f, (\varphi_{\mathbb{L}}A)g\rangle_{L^2(Y,\mu_Y)}$, thus establishing the claimed equivalence. It is equally easy to check that the resolvent sets and the eigenvalues of A and $\varphi_{\mathbb{L}}A$ coincide.

Note however that a bi-measurable bijection φ does not respect enough structure to ensure that $\varphi_{\mathbb{L}}\sqrt{A}$ generates a regular Dirichlet form if and only if $\sqrt{A}$ does – recall that this means the density of $\mathcal{D}(\sqrt{A}) \cap C_c(Y)$ in both $\mathcal{D}(\sqrt{A})$ and $C_c(Y)$. To this end, suppose now that $\varphi : X \to Y$ is a homeomorphism between X and Y (since both spaces are equipped with their Borel σ-algebras, such φ is automatically bi-measurable and bijective). Similar to the case of L^2-spaces, this induces an isometric isomorphism $\varphi_* : C_0(Y) \to C_0(X), \varphi_*(f) = f \circ \varphi$ between algebras of continuous functions vanishing at infinity, equipped with sup-norm $\|\cdot\|_{C_0}$. This isomorphism restricts to the subalgebras of compactly supported continuous functions $C_c(X)$, resp. $C_c(Y)$.

so we observe that A satisfies assumptions 1.4 iff $\varphi_{\mathbb{L}}A$ does. This motivates the following proposition:

Proposition 3.3. *Let (X, d_X, μ) and (Y, d_Y, ν) be complete separable metric measure spaces with $\operatorname{supp}\mu = X$ and $\operatorname{supp}\nu = Y$ that are mm–isomorphic by virtue of the map $\phi : X \to Y$. Suppose the Einstein relation with constant c holds on X with respect to an operator $(A, \mathcal{D}(A))$ satisfying assumptions 1.4. Then, the Einstein relation also holds on Y with the same constant c and with respect to $\varphi_{\mathbb{L}}A$.*

Proof. Trivially, $\dim_{\mathcal{H}}(X) = \dim_{\mathcal{H}}(Y)$, and as observed above, $\dim_{\mathcal{S}}(X, A) = \dim_{\mathcal{S}}(Y, \varphi_{\mathbb{L}}A)$. So, it remains to show $\dim_{\mathcal{W}}(X, M) = \dim_{\mathcal{W}}(X, M^{(\varphi)})$ where M is a Hunt process associated to A and $M^{(\varphi)}$ is one associated to $\varphi_{\mathbb{L}}A$.

To this end, we remark first that if $(T_t)_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^2(X,\mu)$ with generator $(A,\mathcal{D}(A))$ then $(\varphi_{\mathbb{L}}T_t)_{t\geq 0}$ is a semigroup with the same properties on $L^2(Y,\nu)$ and with generator $(\varphi_{\mathbb{L}}A,\mathcal{D}(\varphi_{\mathbb{L}}A))$. Indeed, the semigroup property is trivial to check. For strong continuity, we calculate

$$\|\varphi_{\mathbb{L}}T_t f - \varphi_{\mathbb{L}}T_0 f\| = \left\|\varphi_*^{-1}(T_t\varphi_* f - \varphi_* f)\right\| = \|T_t(\varphi_* f) - (\varphi_* f)\| \to 0$$

for $t \searrow 0$ for arbitrary $f \in L^2(X,\mu)$, and verifying the generator works analogously.

In the next step, we consider the process $N_t := \varphi(M_t)$. This process is a Hunt process with values in Y, and possesses the semigroup

$$T_t^{(N)} f(y) = \mathbf{E}_y\left[f(N_t)\right] = \mathbf{E}_{\varphi^{-1}(y)}\left[(f\circ\varphi)(M_t)\right] = T_t[\varphi_* f](\varphi^{-1}(y)) = \varphi_*^{-1}T_t\varphi_* f = \varphi_{\mathbb{L}}T_t.$$

Thus, up to killing on polar sets the processes N_t and $M^{(\varphi)}$ coincide. In particular, we have the equality $\mathbf{E}_x\left[\tau_{B(x,r)}\right] = \mathbf{E}_{\varphi(x)}\left[\tau^{(\varphi)}_{B(\varphi(x),r)}\right]$ for all sufficiently small $r > 0$. □

TO DO: How weak can the assumptions be chosen?

3.2 Hölder regularity and graphs of functions

To see that some assumptions are indeed necessary, we consider the following example: For a suitable probability space $(\Omega, \mathscr{A}, \mathbf{P})$, denote by $\mathscr{G} = \mathscr{G}(\omega) := \{(t, B_t(\omega)) : t \in [0,1]\}$ for fixed $\omega \in \Omega$ the graph of a trajectory of a standard Brownian motion over the intervall $[0,1]$. This comes with the natural map $\varphi = \varphi_\omega : [0,1] \to \mathscr{G}$ sending t to (t, B_t). We equip $\mathscr{G}$ with the metric d_∞ induced by the maximum norm $|(x,y)|_\infty = \max\{|x|,|y|\}$ of the surrounding space $\mathbb{R}^2$ and observe that by this choice, φ is continuous but not Lipschitz-continuous. Finally, $\mathscr{G}$ becomes a metric measure space by endowing it with the push-forward measure $\varphi_*\lambda^1$. We can now observe the following:

- The Hausdorff dimension of $(\mathscr{G}, d_\infty)$ coincides with the Hausdorff dimension with respect to Euclidean metric since all norms on finitely dimensional vector spaces are equivalent. Thus, $\dim_{\mathcal{H}}(\mathscr{G}) = \frac{3}{2}$ $\mathbf{P}$-almost surely, see [Fal07, Theorem 16.7].
- The same arguments as used in the proof of proposition 3.3 yield that
$$\frac{1}{2} = \dim_{\mathcal{S}}\left([0,1], -\frac{1}{2}\Delta_{\lambda^1}\right) = \dim_{\mathcal{S}}\left(\mathscr{G}, \Phi\left(-\frac{1}{2}\Delta_{\lambda^1}\right)\right),$$
where we once again applied Weyl's classical results.
- It remains to evaluate the walk dimension for $\varphi(\bar{B}_t)$ on $\mathscr{G}$, where $\bar{B}$ denotes a Brownian motion independent from B. As will be seen in the subsequent lemma, we have $\dim_{\mathcal{W}}(\mathscr{G}(\omega), \varphi(\bar{B})) = 4$ $\mathbf{P}$-almost surely.

Therefore, the Einstein relation – despite holding with constant 1 on $\left([0,1], -\frac{1}{2}\Delta_{\lambda^1}, 2\bar{B}\right)$ – changes its constant to 4/3 under application of φ.

Lemma 3.4. *With the notation just introduced, we have*
$$\dim_{\mathcal{W}}\left(\mathscr{G}(\omega), \varphi_\omega(\bar{B})\right) = 4 \tag{3.2}$$
$\mathbf{P}$*-almost surely.*

Proof. For brevity, set $x = x(\omega) = (T, B_T(\omega)) \in \mathbb{R}^2$ and chose $r > 0$ small enough for $B_\infty(x,r) \subseteq [0,1] \times \mathbb{R}$, where $B_\infty(x,r)$ stands for the open ball of radius r around x with respect to d_∞. We begin by introducing the random times $\Theta_r^+(B_\cdot(\omega), T), \Theta_r^-(B_\cdot(\omega), T)$ to denote the time where the Brownian motion B first resp. last exits $B_\infty(x,r)$ – in other words,
$$\begin{aligned} \Theta_r^+(B_\cdot(\omega), T) &:= \inf\{1 \geq s > T : B_s \notin B_\infty(x,r)\} \\ \Theta_r^-(B_\cdot(\omega), T) &:= \sup\{0 \leq s < T : B_s \notin B_\infty(x,r)\}. \end{aligned} \tag{3.3}$$

Note first that by the Markov property of Brownian motion as well as by its time symmetry, these random variables are independent and the shifted random variables $\Theta_r^+(B_\cdot(\omega), T) - T, T - \Theta_r^-(B_\cdot(\omega), T)$ share the same distribution.

By the standard result for the expectation of two-sided exit times for Brownian motion, we now obtain
$$\mathbf{E}_x\left[\tau_{B_\infty(x,r)}^{(\varphi(\bar{B}))}\right] = -(T_r^+(\omega) - T)(T_r^-(\omega) - T)$$

and consequentially

$$\frac{\log \mathbf{E}_x\left[\tau_{B_\infty(x,r)}^{(\varphi(\bar{B}))}\right]}{\log r} = \frac{\log\left(\Theta_r^+(B.(\omega),T) - T\right)}{\log r} + \frac{\log\left(T - \Theta_r^-(B.(\omega),T)\right))}{\log r}.$$

Therefore, it will suffice to show that $\mathbf{P}$-almost surely,

$$\lim_{r\searrow 0} \frac{\log\left(\Theta_r^+(B.(\omega),T) - T\right)}{\log r} = \lim_{r\searrow 0} \frac{\log\left(T - \Theta_r^-(B.(\omega),T)\right)}{\log r} = 2, \tag{3.4}$$

where it is enough to prove that one of the limits exist and equals 2. To this end, we consider more generally for a continuous function $f : [0,1] \to \mathbb{R}$ the functionals

$$w : f(T) \mapsto \liminf_{r\searrow 0} \frac{\log\left(\Theta_r^+(f,T) - T\right)}{\log r}$$
$$W : f(T) \mapsto \limsup_{r\searrow 0} \frac{\log\left(\Theta_r^+(f,T) - T\right)}{\log r}$$

where $\Theta_r^\pm(f,T)$ is defined analogously to (3.3). Suppose now that f is α-Hölder continuous ($0 < \alpha \leq 1$) in a given point $T \in [0,1]$, that is, there exists a constant $C > 0$ and a ε-neighbourhood of T such that for all s inside this neighbourhood,

$$|f(T) - f(s)| \leq C|T - s|^\alpha$$

is satisfied. This yields $\Theta_r^+(f,T) \geq (r/C)^{1/\alpha}$ for $r < \varepsilon$ and therefore $(Wf)(T) \leq \frac{1}{\alpha}$. Conversely, suppose that f is not β-Hölder continuous ($\alpha < \beta \leq 1$) in T. In particular, there exists a sequence $s_n = T + r_n \to T$ fulfilling the estimate

$$|f(T) - f(s_n)| > |T - s_n|^\beta = r_n^\beta,$$

from which we deduce $\Theta_{r_n}^+(f,T) \leq r_n^{1/\beta}$ and thus $(wf)(T) \geq \frac{1}{\beta}$. In particular, $(Wf)(T) = (wf)(T) = \frac{1}{c}$ if f is α-Hölder continuous in T for all $\alpha < c$ but not β-Hölder continuous for any $\beta > c$.

Having established this claim, the limit in equation (3.4) is a direct consequence of Paley-Zygmunds regularity theorem for paths of Brownian motion. Moreover, this also shows that the limit does not depend on $T \in (0,1)$. □

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