Ingeniería Eléctrica
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
UNIVERSIDAD DE CHILE

FI 2002 ELECTROMAGNETISMO
Clase 14
Magnetostática I

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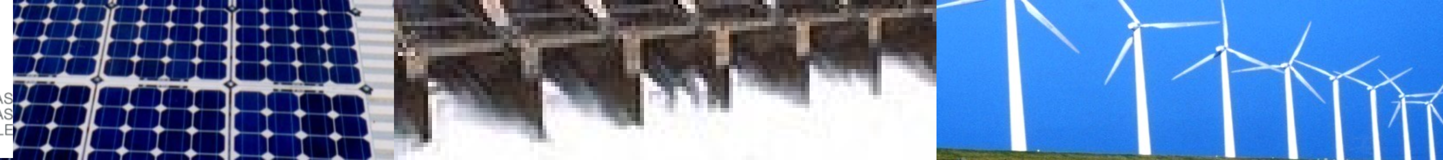

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Wassily Kandinsky, “Amarillo, rojo y azul”, 1925

Introducción

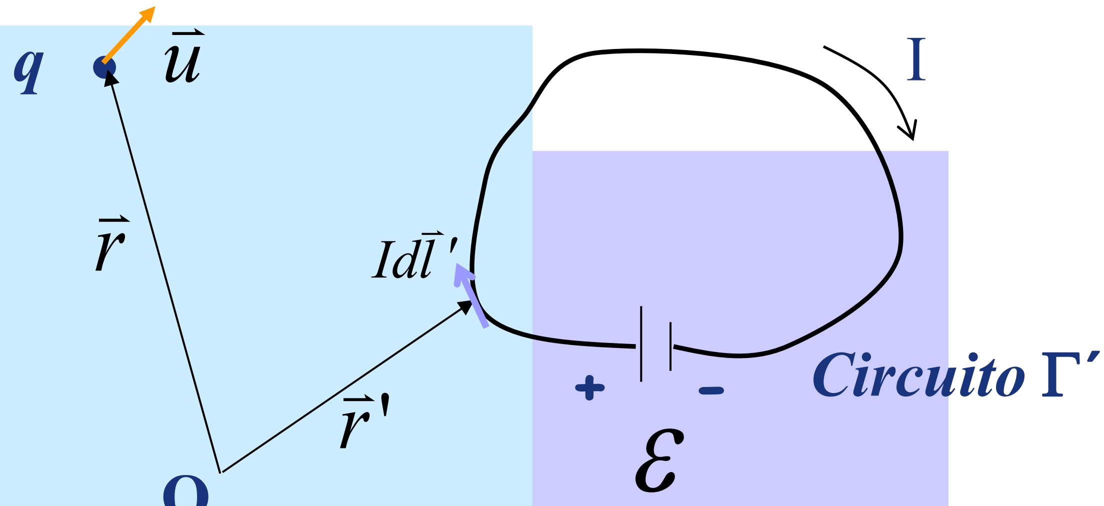
$$\frac{\partial \rho}{\partial t} = 0 \Rightarrow \nabla \cdot \vec{J} = 0$$

Corrientes constantes

En magnetostática estudiaremos régimen estacionario, i.e., corrientes constantes

Fuerza sobre una carga

Fuerza sobre q que se mueve con velocidad $\vec{u}$



Se encuentra experimentalmente que

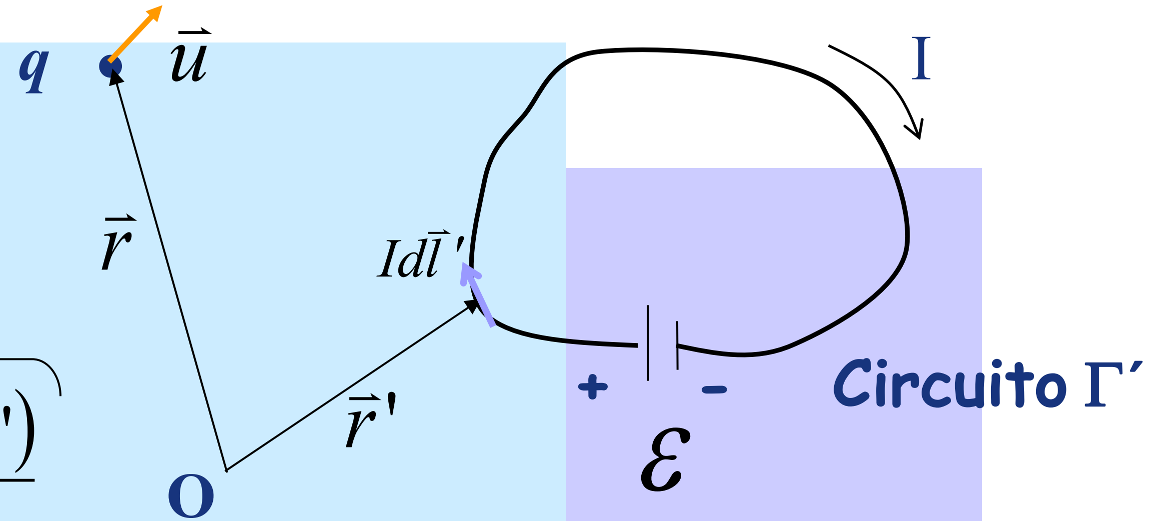
$$\vec{F} = q\vec{u} \times \oint_{\Gamma'} \frac{\mu_0 I d\vec{l}\,' \times (\vec{r} - \vec{r}\,')}{4\pi \|\vec{r} - \vec{r}\,'\|^3}$$

$\mu_0 = 4\pi \times 10^{-7} [H/m]$ permeabilidad del aire (constante)

Campo magnético

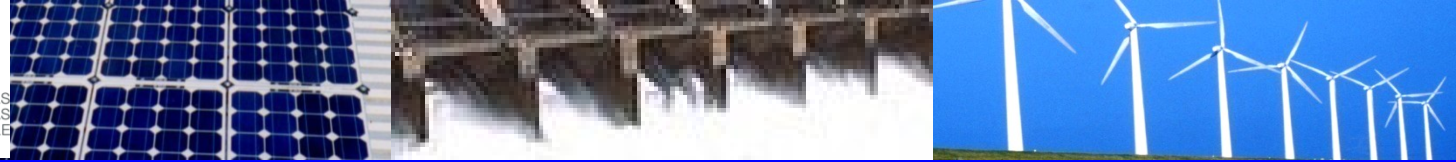
Sólo depende del circuito Γ′

$$\vec{F} = q\vec{u} \times \underbrace{\oint_{\Gamma'} \frac{\mu_0 I d\vec{l}\,' \times (\vec{r} - \vec{r}\,')}{4\pi \left\| \vec{r} - \vec{r}\,' \right\|^3}}$$



Se define el campo magnético producido por circuito Γ′

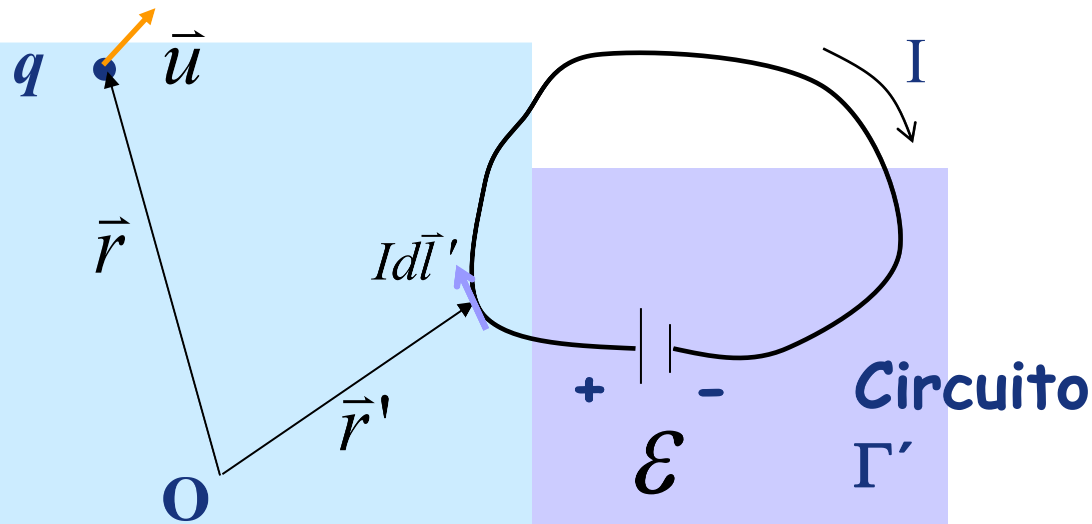
$$\boxed{\vec{B} = \oint_{\Gamma'} \frac{\mu_0 I d\vec{l}\,' \times (\vec{r} - \vec{r}\,')}{4\pi \left\| \vec{r} - \vec{r}\,' \right\|^3}}$$



Fuerza de Lorentz

Fuerza de Lorentz

$$\vec{F} = q\vec{u} \times \vec{B}$$



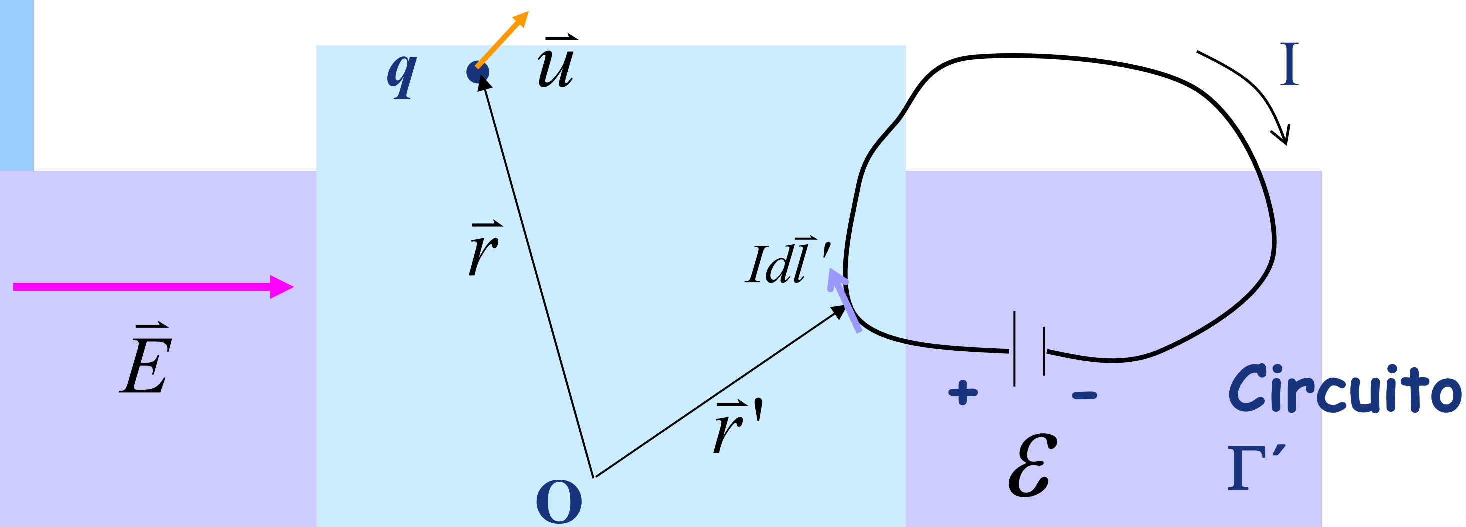
Unidades del campo

$$[F] = [q][V][B] \Rightarrow [B] = \frac{[F]}{[q][V]} = \frac{[N]}{[C][m/seg]}$$

$$1\ Tesla = [T] = \left[\frac{N}{C \times m/seg}\right]$$

$$1[T]Tesla = 10^4[G]Gauss$$

Fuerza de Lorentz



Cuando además hay un campo eléctrico la Fuerza de Lorentz es

$$\vec{F} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

Ejemplo: Fuerza sobre una carga

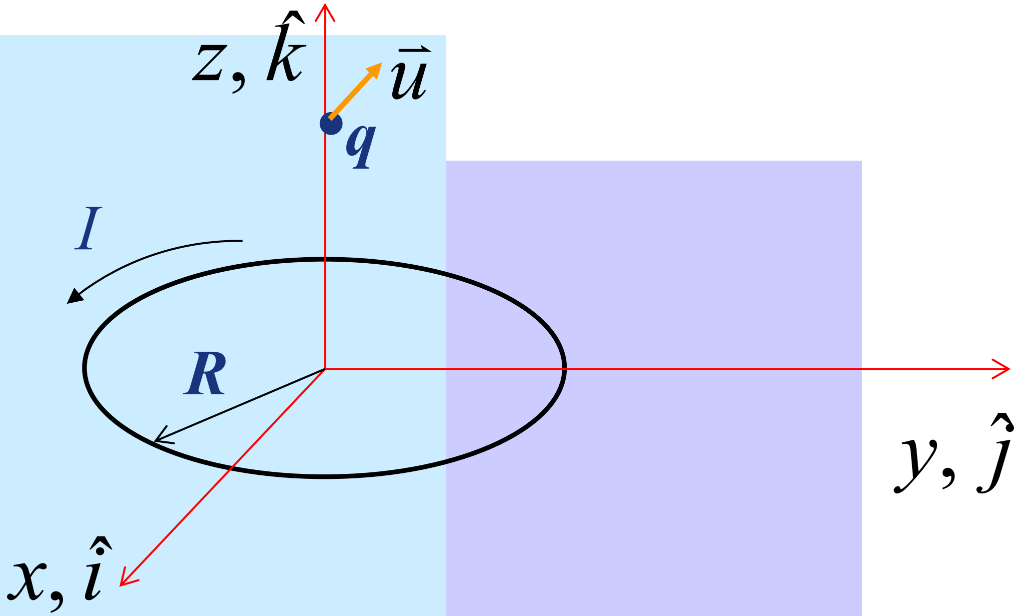
Calcular la fuerza sobre la carga q en los casos:

$$\vec{u} = 0$$

$$\vec{u} = v_o \hat{k}$$

$$\vec{u} = v_o \hat{j}$$

$$\vec{u} = v_o \hat{i}$$



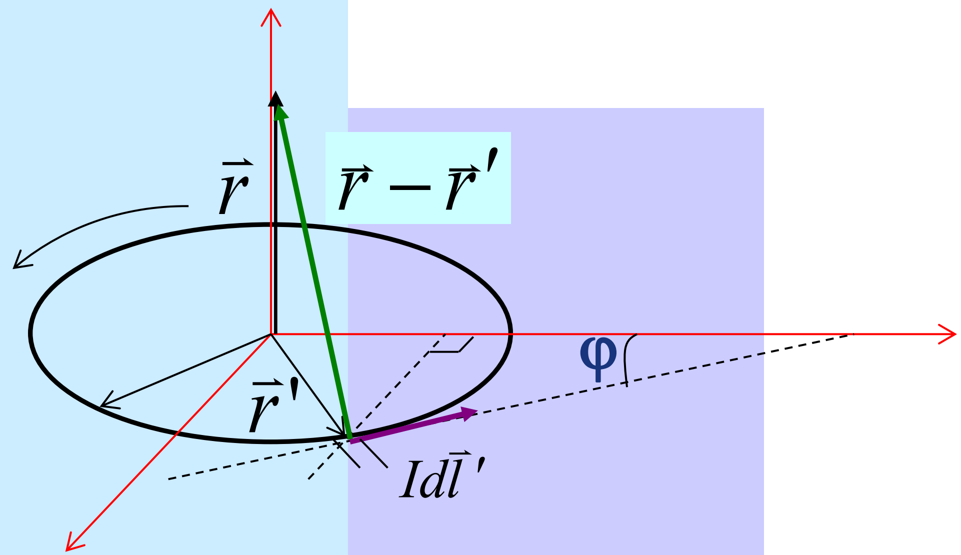
$$\vec{F} = q\vec{u} \times \vec{B}$$

$$\vec{B} = \oint_{\Gamma'} \frac{\mu_0 I d\vec{l}' \times (\vec{r} - \vec{r}')}{4\pi \|\vec{r} - \vec{r}'\|^3}$$

Regla de la mano derecha

$$\vec{B} = \oint_{\Gamma'} \frac{\mu_0 I d\vec{l}\,' \times (\vec{r} - \vec{r}\,')}{4\pi \left\| \vec{r} - \vec{r}\,' \right\|^3}$$

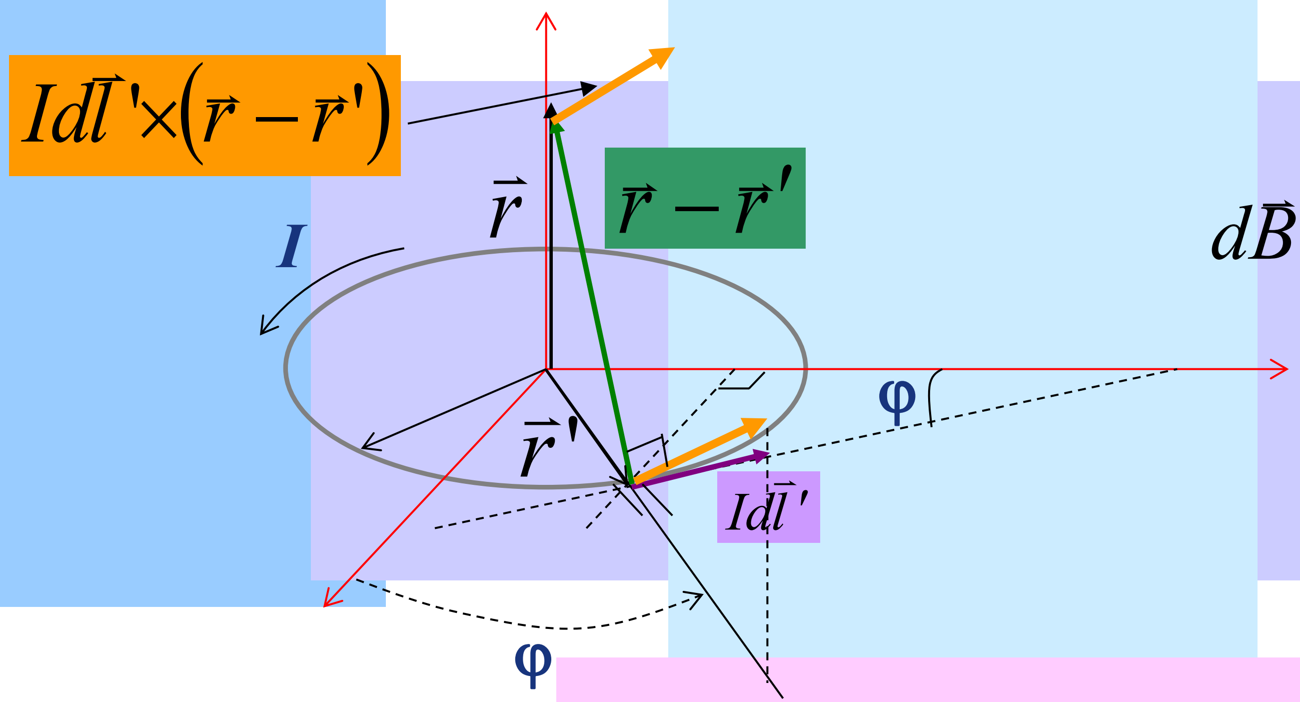
$$d\vec{B} = \frac{\mu_0 I d\vec{l}\,' \times (\vec{r} - \vec{r}\,')}{4\pi \left\| \vec{r} - \vec{r}\,' \right\|^3}$$



Dirección de campo está dado por el producto

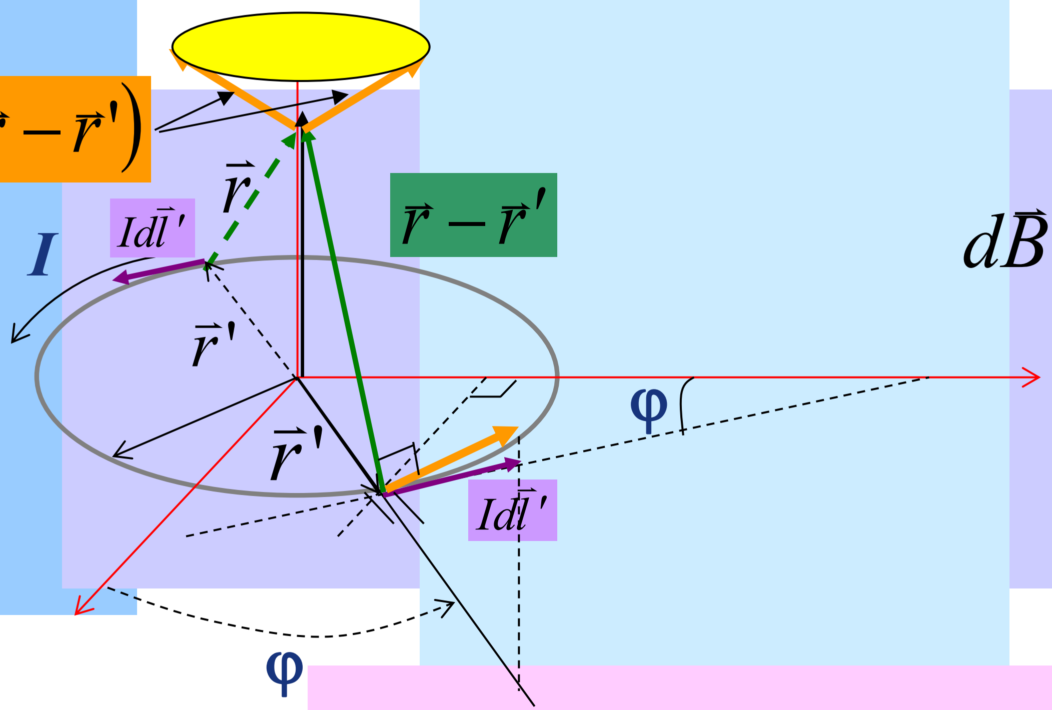
$$I d\vec{l}\,' \times (\vec{r} - \vec{r}\,')$$

Regla de la mano derecha



$$d\vec{B}=\frac{\mu_0 Id\vec{l}\,'\times(\vec{r}-\vec{r}\,')}{4\pi\left\|\vec{r}-\vec{r}\,'\right\|^3}$$

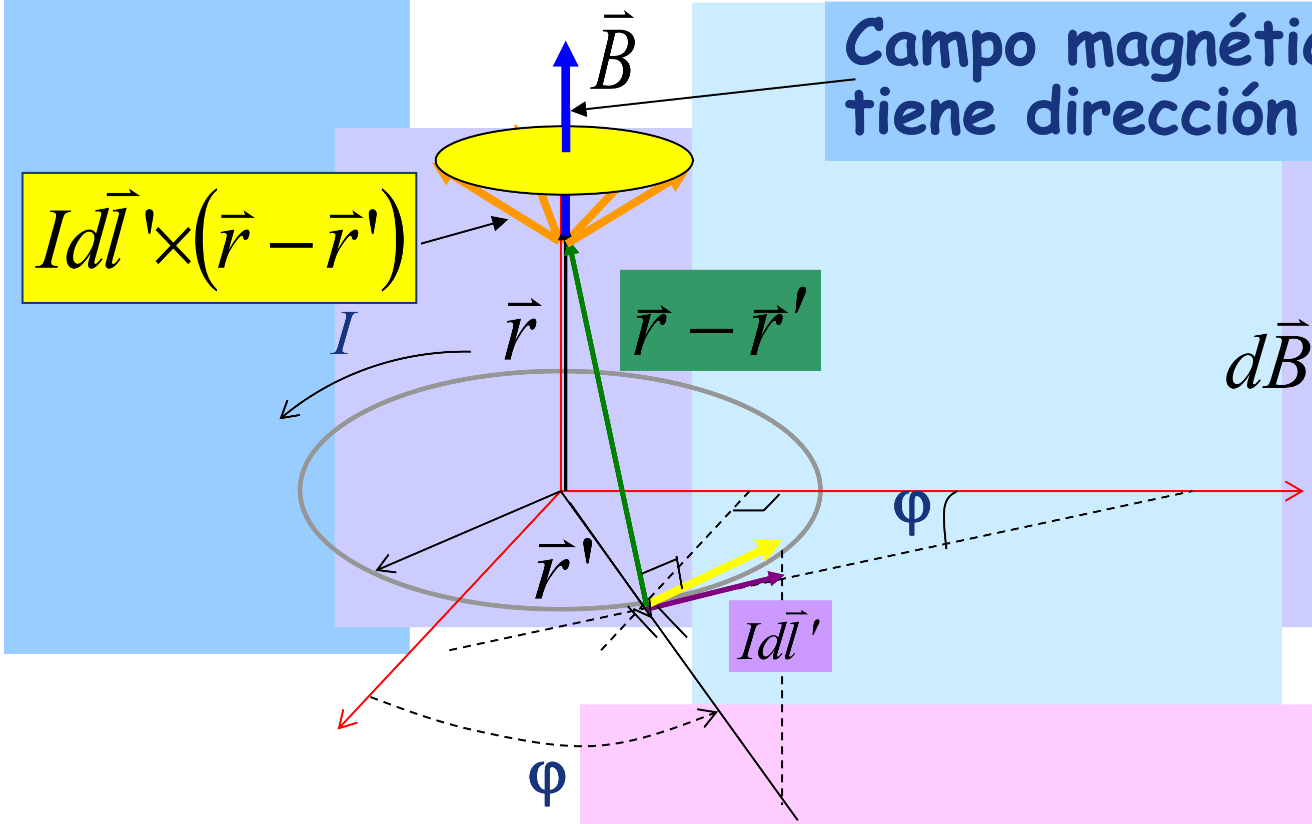
Regla de la mano derecha



$$d\vec{B}=\frac{\mu_0 I d\vec{l}'\times(\vec{r}-\vec{r}')}{4\pi\|\vec{r}-\vec{r}'\|^3}$$

Regla de la mano derecha

Campo magnético resultante sólo tiene dirección según eje z.

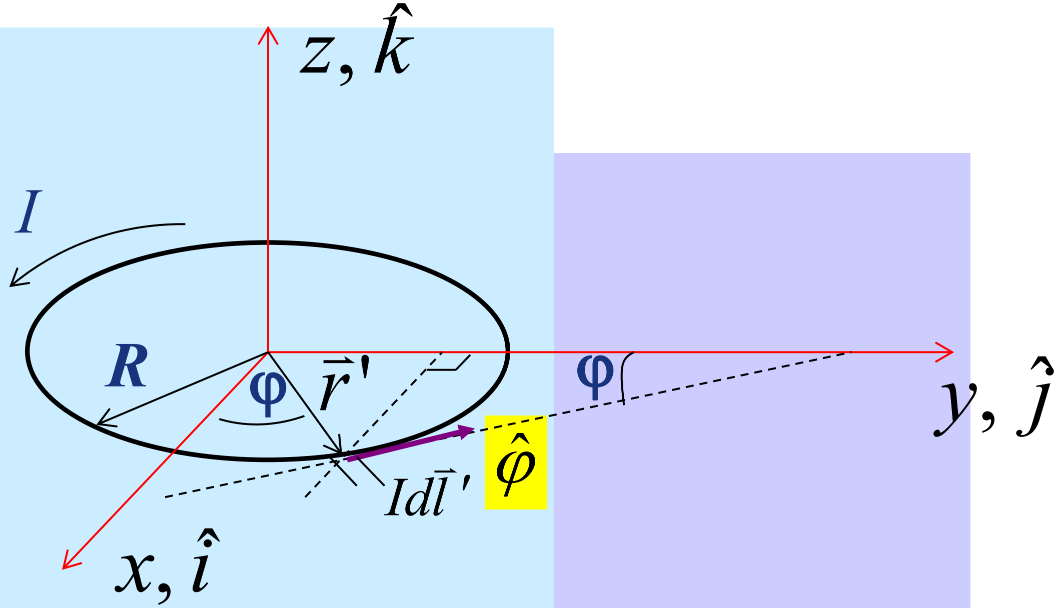


$$d\vec{B} = \frac{\mu_0 I d\vec{l}\,'\times(\vec{r}-\vec{r}\,')}{4\pi\left\|\vec{r}-\vec{r}\,'\right\|^3}$$

Ejemplo

Calculemos $\vec{B}$:

$$\vec{B} = \oint_{\Gamma'} \frac{\mu_0 I d\vec{l}\,' \times (\vec{r} - \vec{r}\,')}{4\pi \|\vec{r} - \vec{r}\,'\|^3}$$



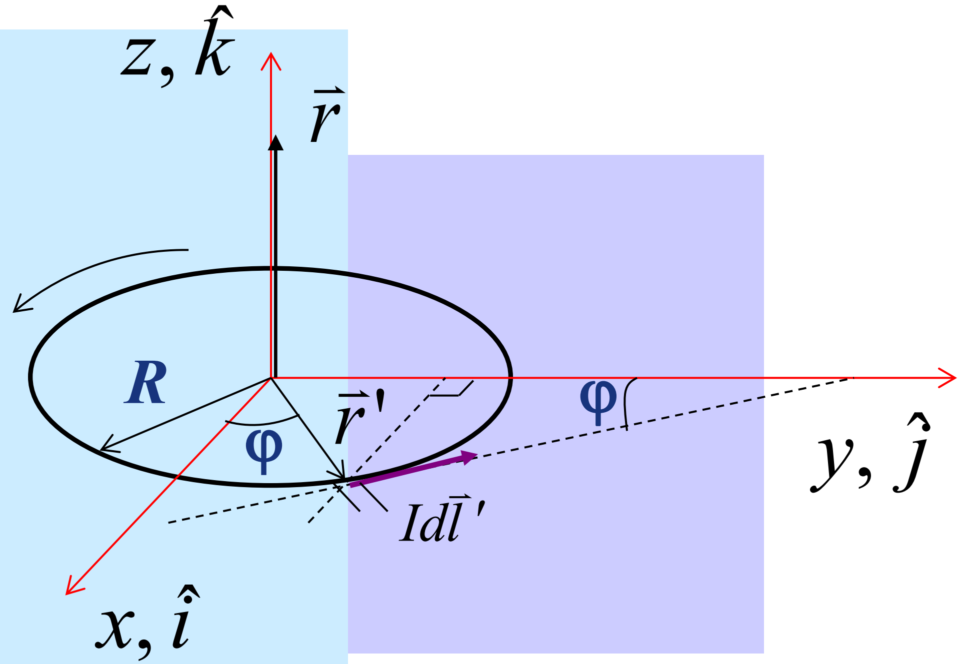
$$I d\vec{l}\,' = I d l' \hat{\varphi} = I d l' \left(-\sin\varphi\, \hat{i} + \cos\varphi\, \hat{j} \right)$$

$$I d\vec{l}\,' = I R d\varphi \left(-\sin\varphi\, \hat{i} + \cos\varphi\, \hat{j} \right)$$

Ejemplo

Calculemos $\vec{B}$:

$$\vec{B} = \oint_{\Gamma'} \frac{\mu_0 I d\vec{l}\,' \times (\vec{r} - \vec{r}\,')}{4\pi \left\| \vec{r} - \vec{r}\,' \right\|^3}$$



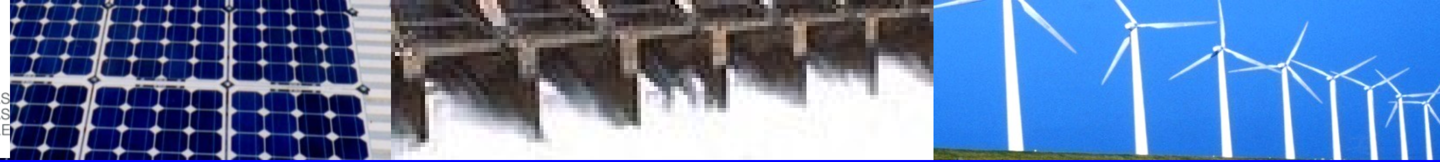
$$\vec{r}\,' = R\hat{\rho} = R(\cos\varphi\,\hat{i} + \sin\varphi\,\hat{j})$$

$$\vec{r} = z\hat{k}$$

$$\vec{r} - \vec{r}\,' = -R\cos\varphi\,\vec{i} - R\sin\varphi\,\hat{j} + z\hat{k}$$

$$\left\| \vec{r} - \vec{r}\,' \right\| = \left[R^2\cos^2\varphi + R^2\sin^2\varphi + z^2 \right]^{1/2}$$

$$\left\| \vec{r} - \vec{r}\,' \right\| = \left[R^2 + z^2 \right]^{1/2}$$

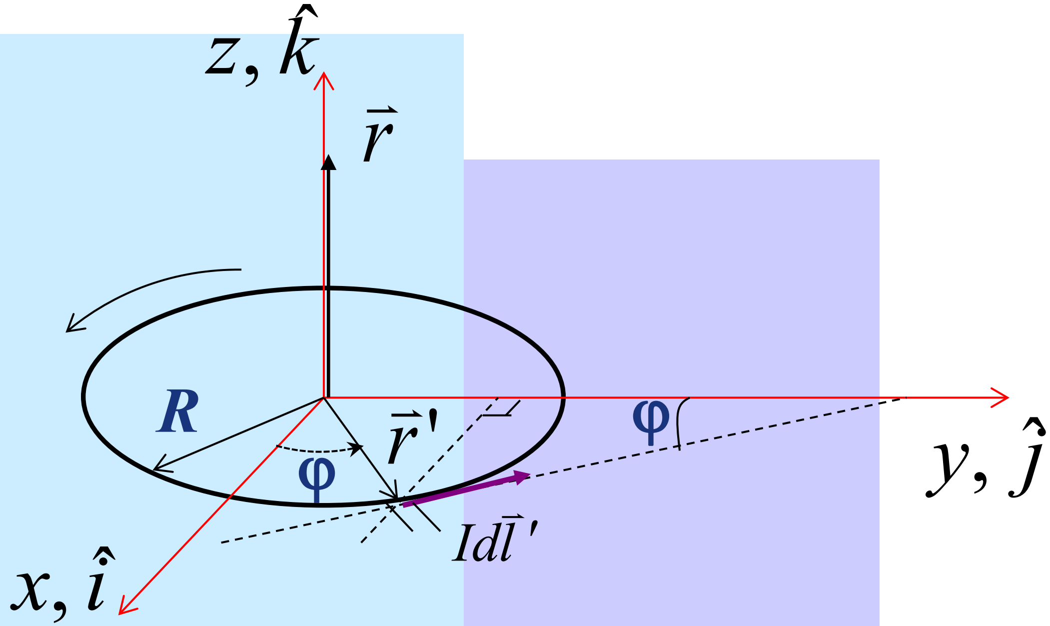


Ejemplo

Calculemos $\vec{B}$

$$\vec{B} = \oint_{\Gamma'} \frac{\mu_0 I d\vec{l}\,' \times (\vec{r} - \vec{r}\,')}{4\pi \left\| \vec{r} - \vec{r}\,' \right\|^3}$$

Circuito Γ': $\varphi = [0, 2\pi]$

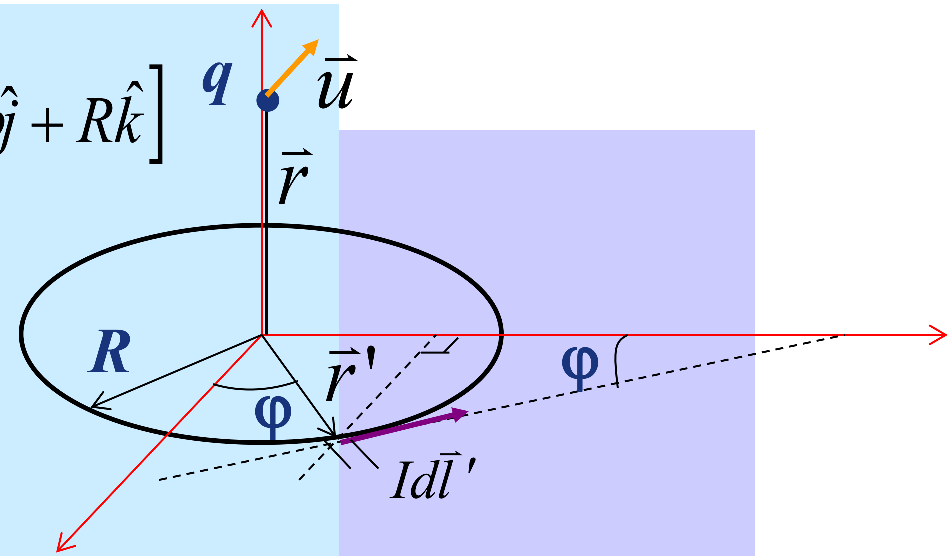


$$\oint_{\Gamma'} = \int_0^{2\pi} \frac{\mu_0}{4\pi} \frac{IRd\varphi}{\left[R^2 + z^2\right]^{3/2}} \left(-\sin\varphi\hat{i} + \cos\varphi\hat{j}\right) \times \left(-R\cos\varphi\hat{i} - R\sin\varphi\hat{j} + z\hat{k}\right)$$

$$\Rightarrow \oint_{\Gamma'} = \int_0^{2\pi} \frac{\mu_0 IRd\varphi}{4\pi\left[R^2 + z^2\right]^{3/2}} \left[R\sin^2\varphi\hat{k} + z\sin\varphi\hat{j} + R\cos^2\varphi\hat{k} + z\cos\varphi\hat{i}\right]$$

Ejemplo

$$\oint_{\Gamma'} = \int_0^{2\pi} \frac{\mu_0 IR d\varphi}{4\pi\left[R^2+z^2\right]^{3/2}}\left[z\cos\varphi\hat{i} + z\sin\varphi\hat{j} + R\hat{k}\right]$$



$$\Rightarrow \vec{B} = \frac{\mu_0 IR^2 2\pi}{4\pi\left[R^2+z^2\right]^{3/2}}\hat{k}$$

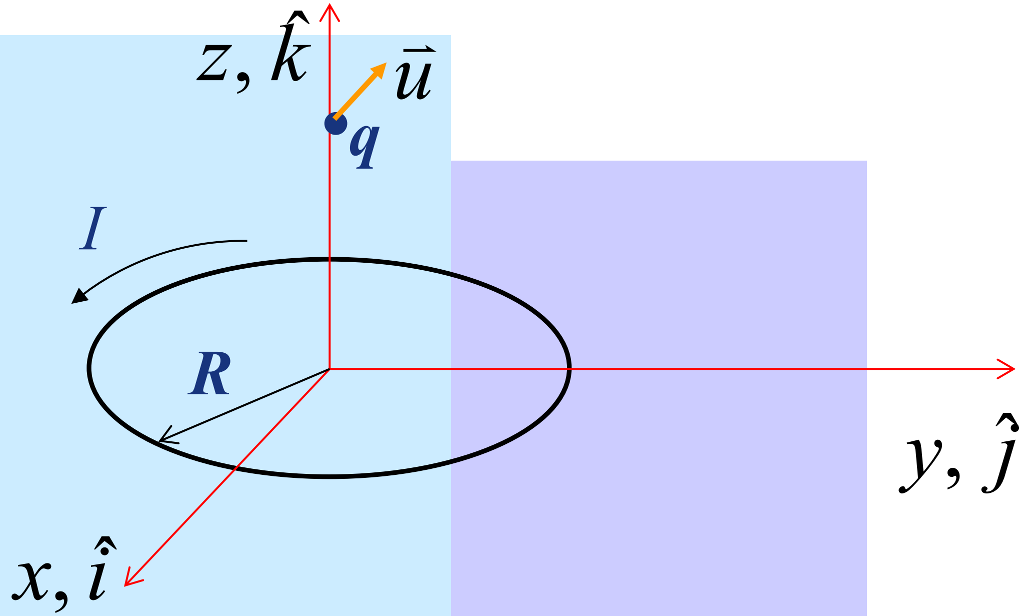
Campo sólo tiene componente según z!

$$\Rightarrow \vec{F} = q\vec{u} \times \frac{\mu_0 IR^2}{2\left[R^2+z^2\right]^{3/2}}\hat{k}$$

$$\therefore \quad \vec{F} = \frac{q\mu_0 IR^2}{2\left[R^2+z^2\right]^{3/2}}\vec{u}\times\hat{k}$$

Ejemplo: Fuerza sobre una carga

$$\vec{F} = \frac{q\mu_0 I R^2}{2\left[R^2 + z^2\right]^{3/2}} \vec{u} \times \hat{k}$$



La fuerza sobre la carga q en cada caso es:

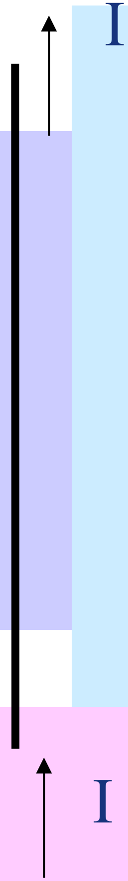
$$\vec{u} = 0 \quad \Rightarrow \vec{F} = 0$$

$$\vec{u} = v_o \hat{k} \quad \Rightarrow \vec{F} = 0$$

$$\vec{u} = v_o \hat{j} \Rightarrow \vec{F} = \frac{q\mu_0 I R^2}{2\left[R^2 + z^2\right]^{3/2}} v_0 \hat{i}$$

$$\vec{u} = v_o \hat{i} \quad \Rightarrow \vec{F} = -\frac{q\mu_0 I R^2}{2\left[R^2 + z^2\right]^{3/2}} v_0 \hat{j}$$

Campo Magnético Linea Infinita

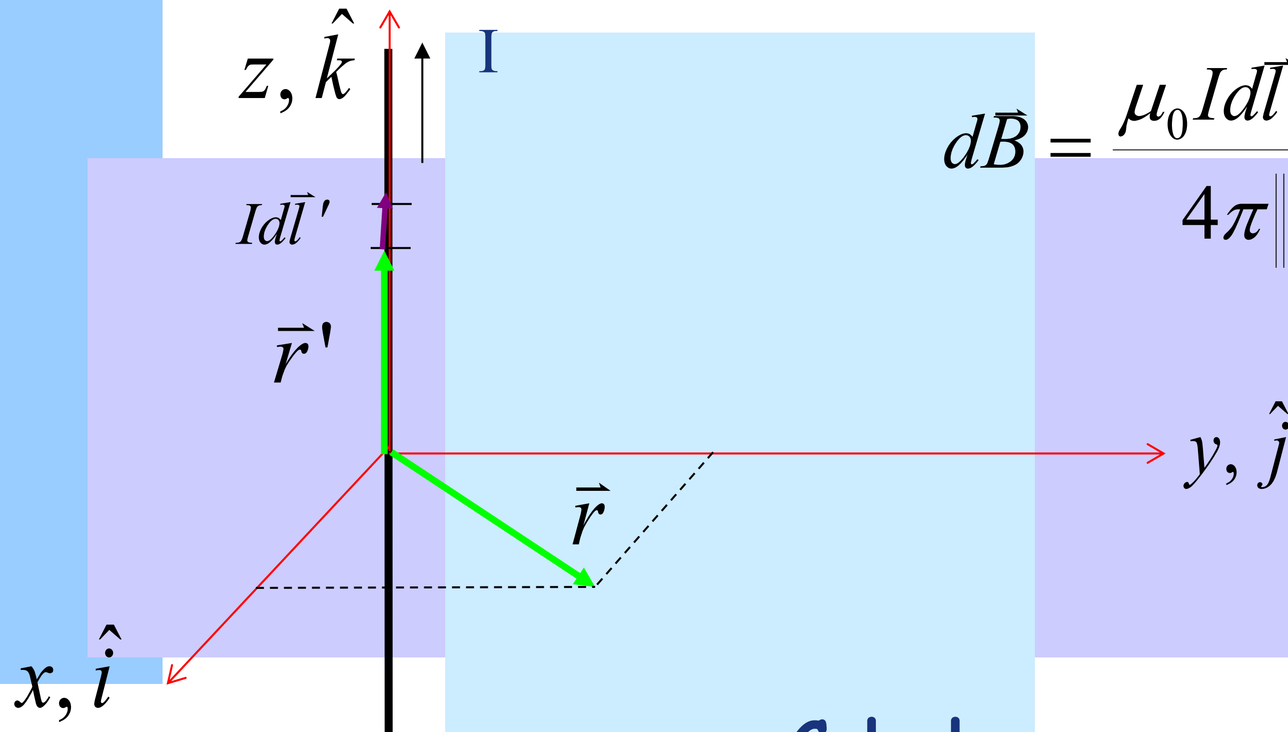


$$\vec{B} = ?$$

Dado que el conductor es infinito, el campo no varía según z.

Campo Magnético Linea Infinita

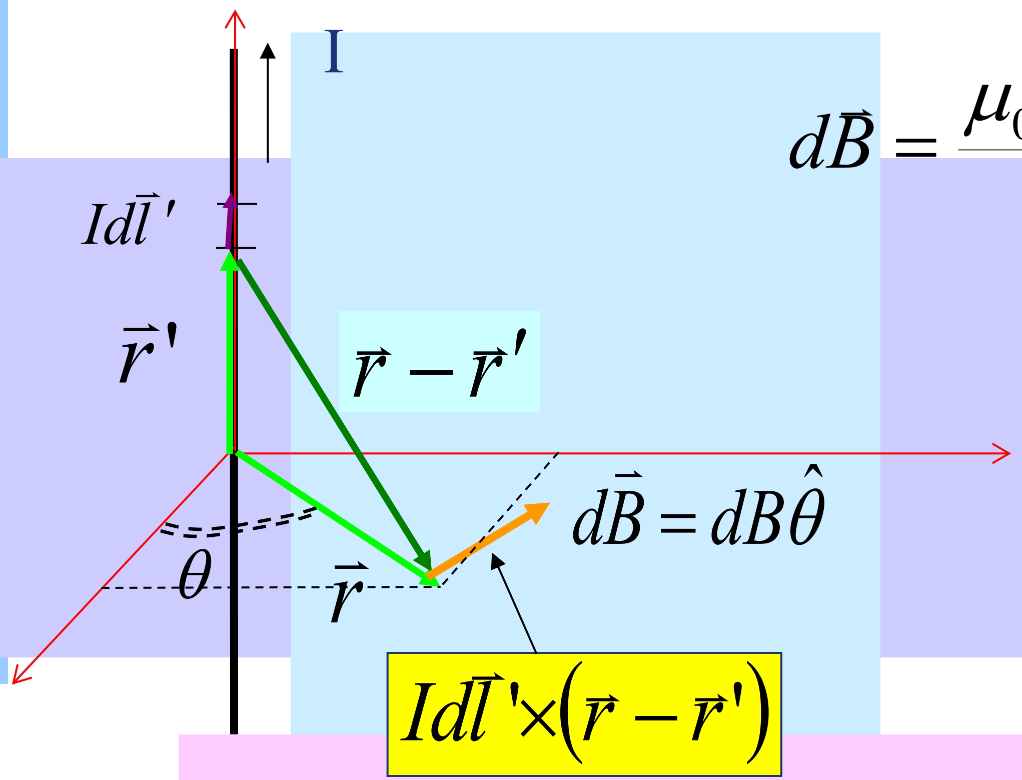
$$d\vec{B} = \frac{\mu_0 I d\vec{l}\,' \times (\vec{r} - \vec{r}\,')}{4\pi \left\| \vec{r} - \vec{r}\,' \right\|^3}$$



Calculemos campo en plano x-y para z=0

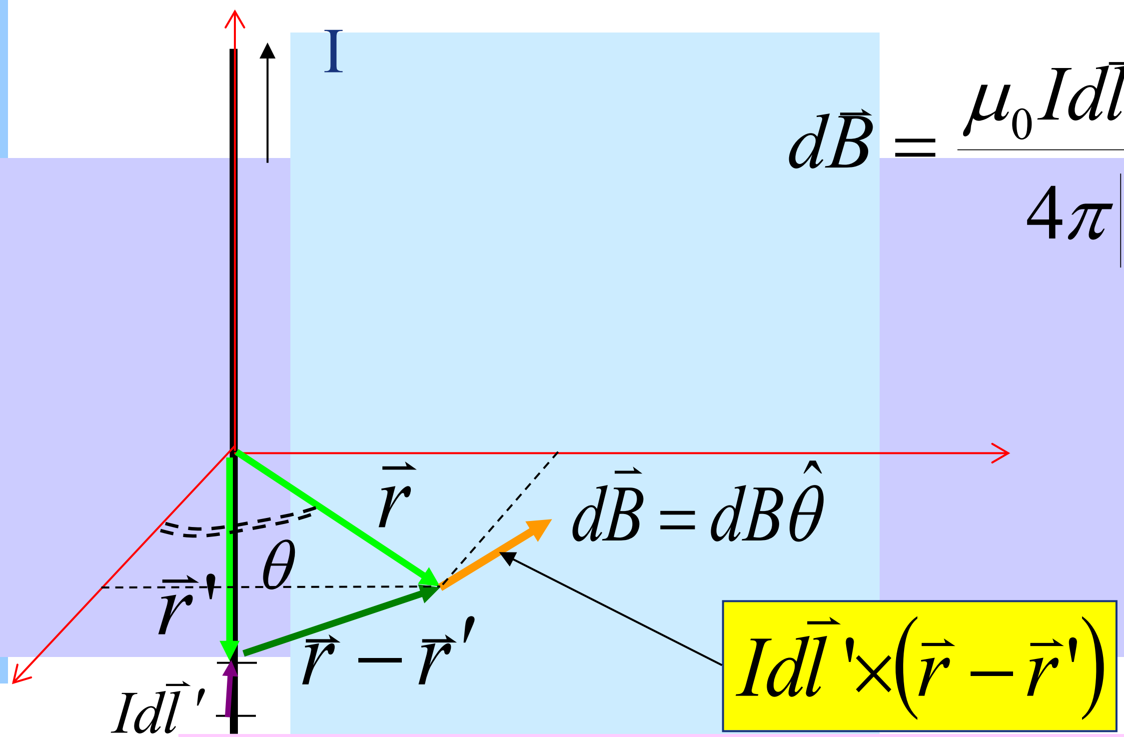
Campo Magnético Linea Infinita

$$d\vec{B} = \frac{\mu_0 I d\vec{l}\,' \times (\vec{r} - \vec{r}\,')}{4\pi \|\vec{r} - \vec{r}\,'\|^3}$$

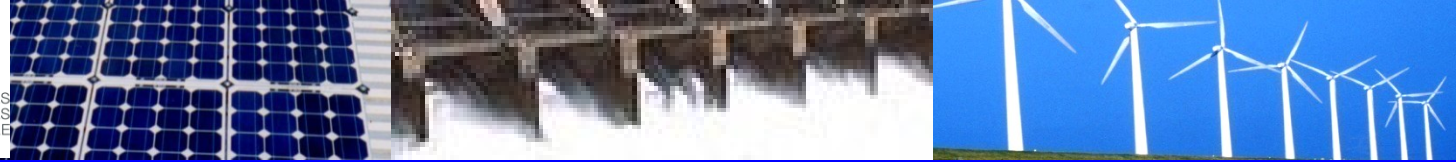


Campo Magnético Linea Infinita

$$d\vec{B} = \frac{\mu_0 I d\vec{l}\,' \times (\vec{r} - \vec{r}\,')}{4\pi \|\vec{r} - \vec{r}\,'\|^3}$$

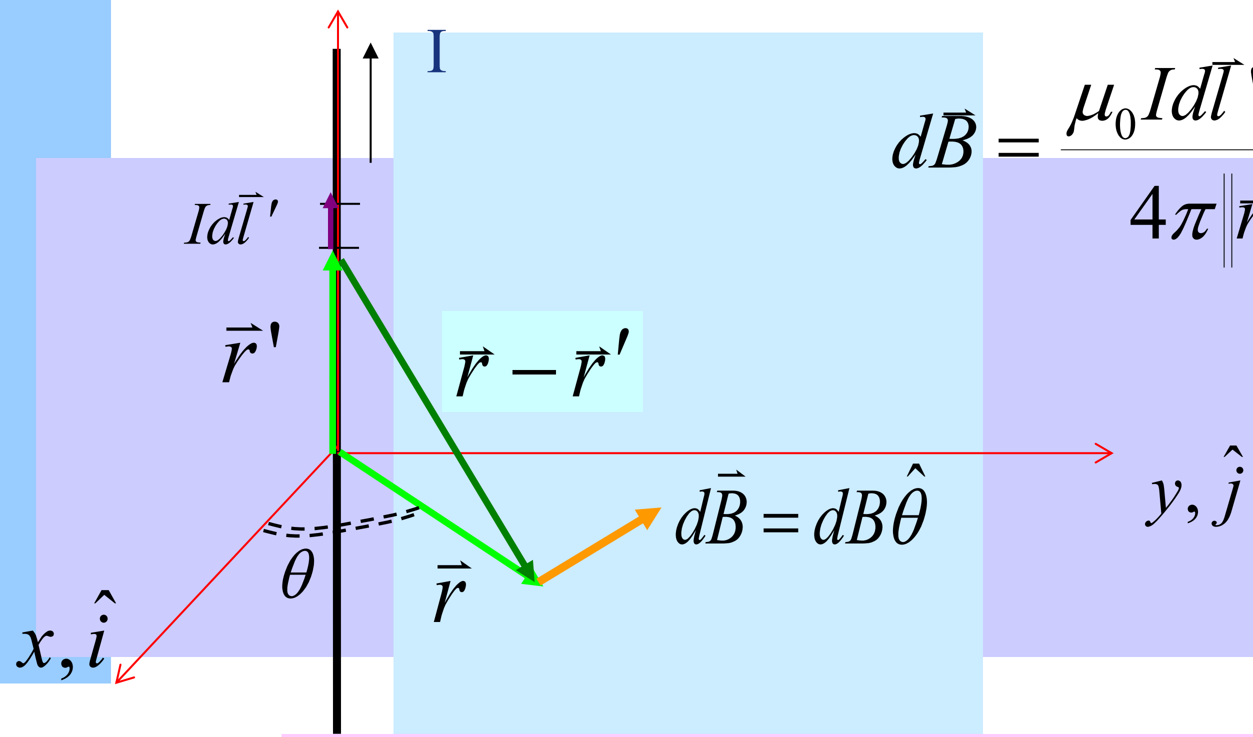


Notar que la contribución de todos los elementos de corriente tiene la misma dirección según $\hat{\theta}$

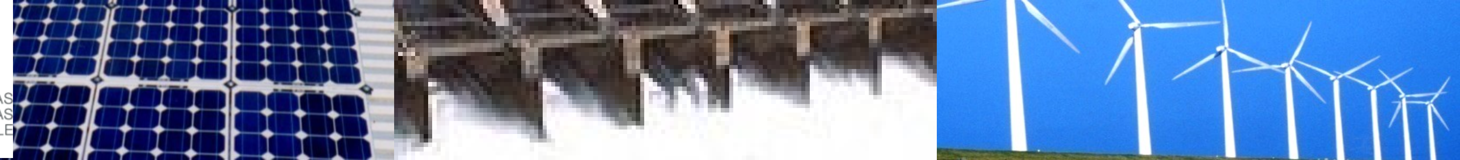


Campo Magnético Linea Infinita

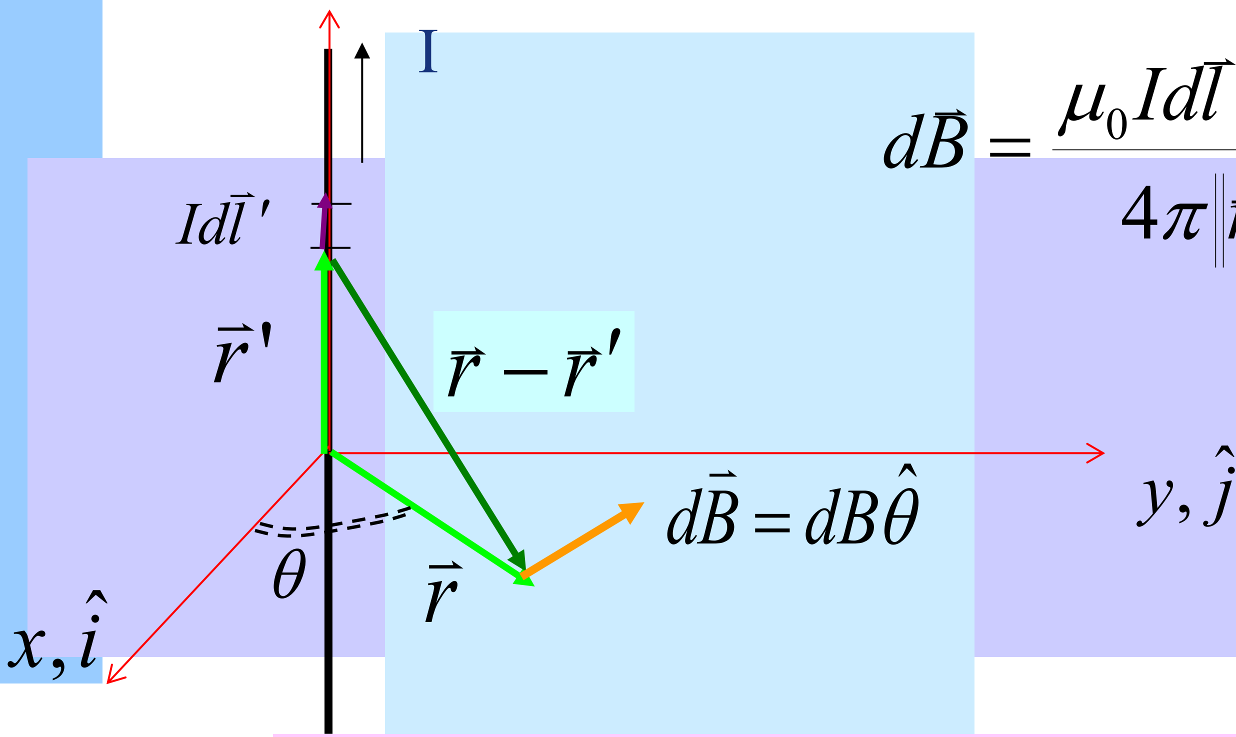
$$d\vec{B} = \frac{\mu_0 I d\vec{l}\,' \times (\vec{r} - \vec{r}\,')}{4\pi \left\| \vec{r} - \vec{r}\,' \right\|^3}$$



$$\left.\begin{aligned} \vec{r}\,' &= z'\hat{k} \\ \vec{r} &= r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} \\ Id\vec{l}\,' &= Idz'\hat{k} \end{aligned}\right\} \quad d\vec{B} = \frac{\mu_0 I dz'\hat{k} \times \left(r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} - z'\hat{k}\right)}{4\pi\left[r^2 + z'^2\right]^{3/2}}$$

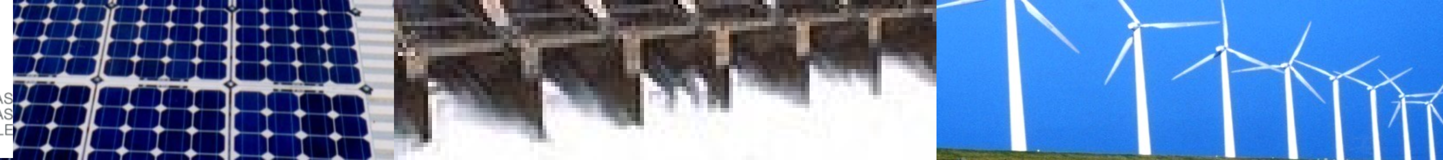


Campo Magnético Linea Infinita



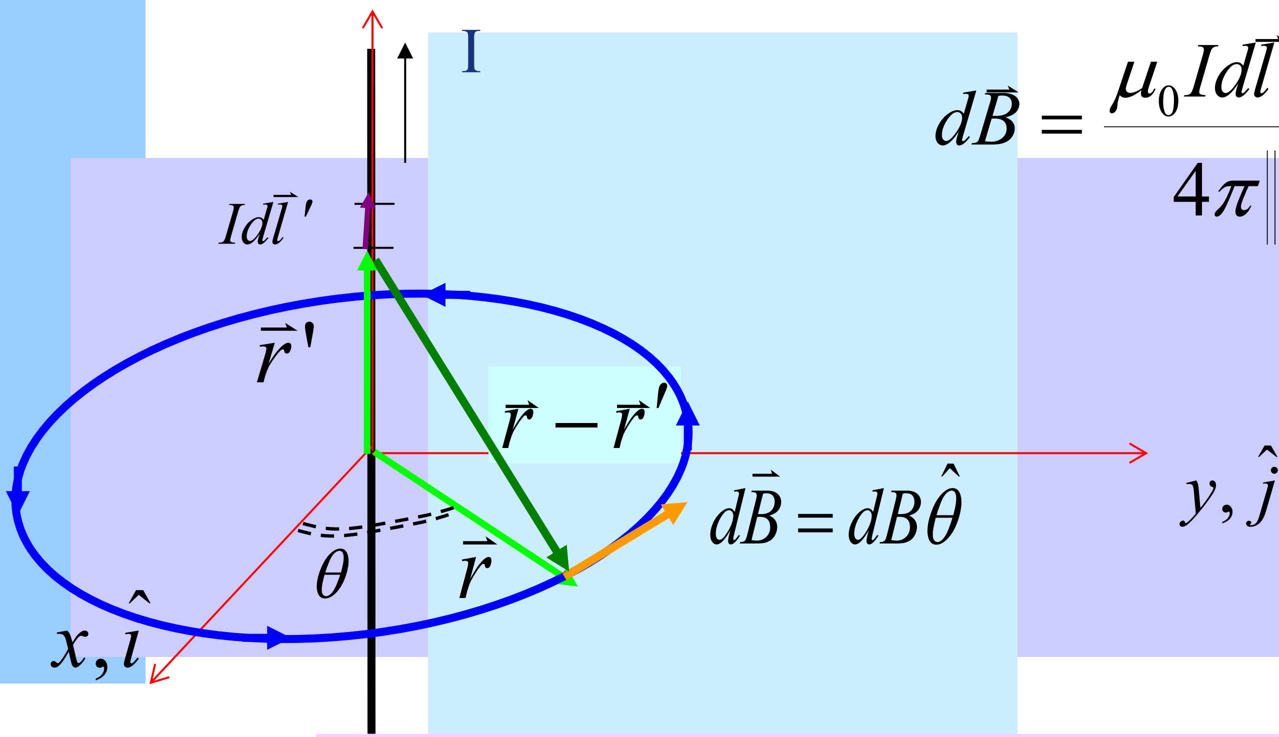
$$d\vec{B}=\frac{\mu_0 I d\vec{l}\,'\times(\vec{r}-\vec{r}\,')}{4\pi\left\|\vec{r}-\vec{r}\,'\right\|^3}$$

$$\vec{B}=\int_{z'=-\infty}^{z'=\infty}\frac{\mu_0 I\, r\left(\cos\theta\,\hat{j}-\sin\theta\,\hat{i}\right)dz'}{4\pi\left[r^2+z'^2\right]^{3/2}}$$



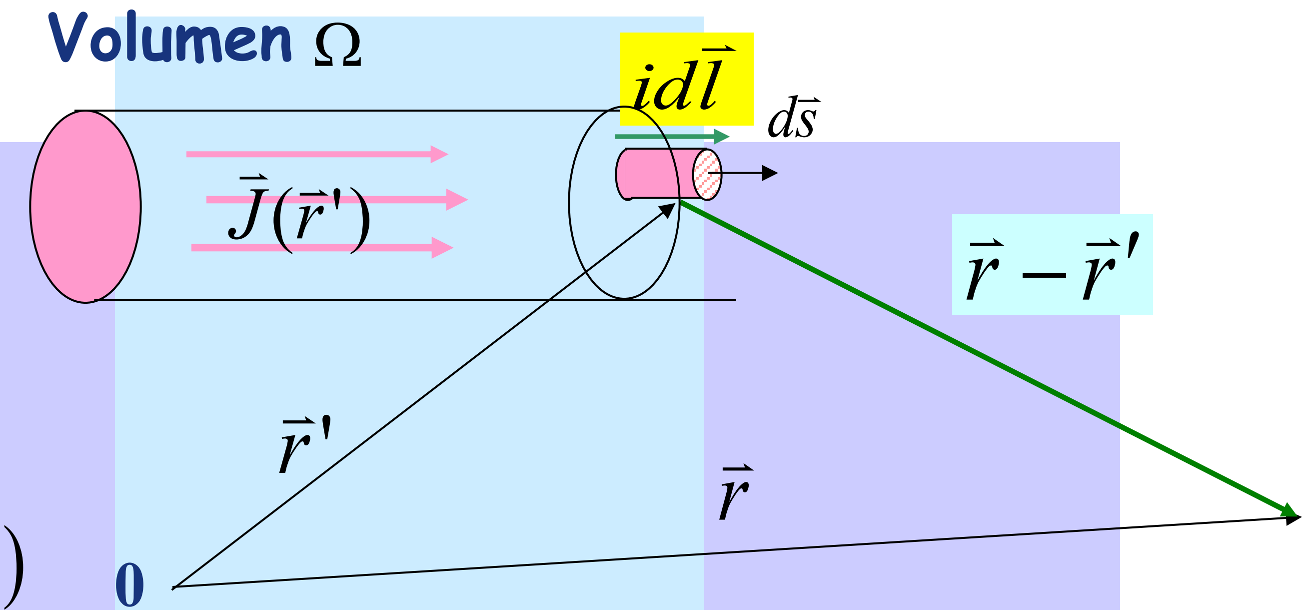
Campo Magnético Linea Infinita

$$d\vec{B} = \frac{\mu_0 I d\vec{l}\,' \times (\vec{r} - \vec{r}\,')}{4\pi \left\| \vec{r} - \vec{r}\,' \right\|^3}$$



Se obtiene finalmente $\therefore \quad \vec{B} = \dfrac{\mu_0 I}{2\pi r}\hat{\theta}$

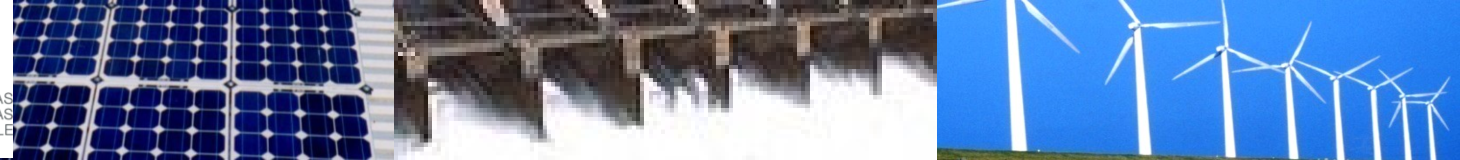
Campo magnético de distribuciones de corriente



$$d\vec{B} = \frac{\mu_0 I d\vec{l}\,' \times (\vec{r}-\vec{r}\,')}{4\pi \|\vec{r}-\vec{r}\,'\|^3}$$

$$id\vec{l} = \vec{J}\cdot d\vec{s}\cdot d\vec{l} = \vec{J}dv'$$

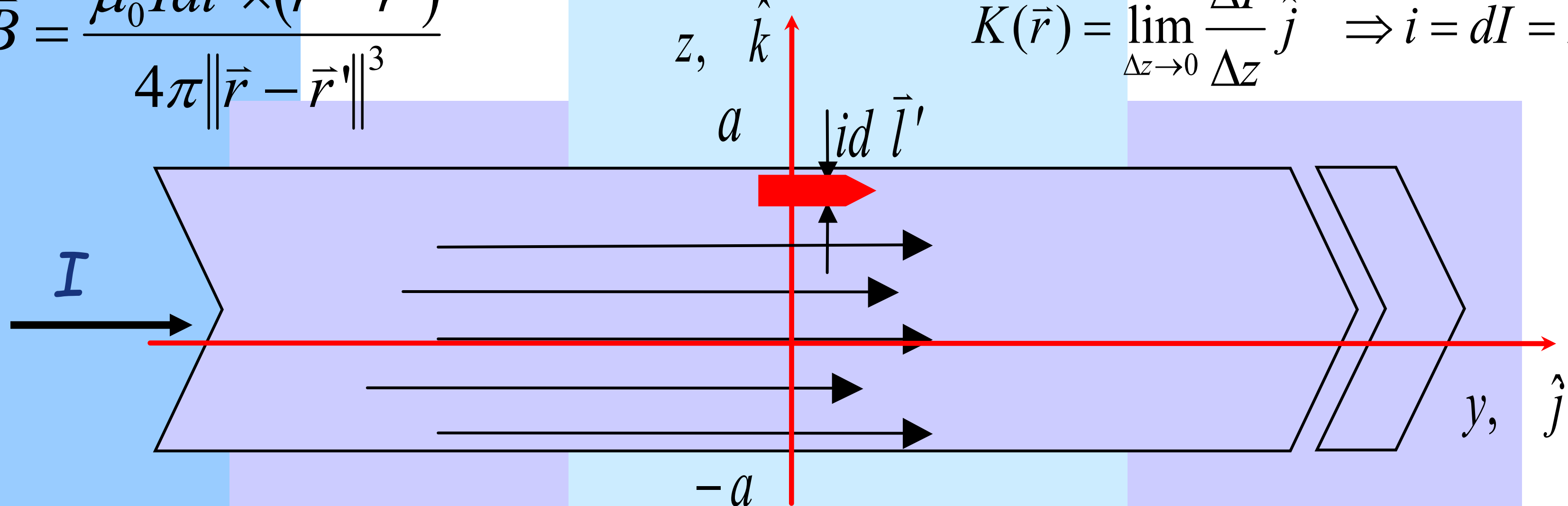
$$\vec{B} = \frac{\mu_0}{4\pi}\iiint_{\Omega} \frac{\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|^3}dv'$$



Densidad de Corriente Superficial

$$d\vec{B} = \frac{\mu_0 I d\vec{l}\,' \times (\vec{r} - \vec{r}\,')}{4\pi \|\vec{r} - \vec{r}\,'\|^3}$$

$$\vec{K}(\vec{r}) = \lim_{\Delta z \to 0} \frac{\Delta I}{\Delta z}\hat{j} \quad \Rightarrow i = dI = \vec{K} \bullet dz\hat{j}$$



Sólo hay corriente en la superficie S del plano *y-z*

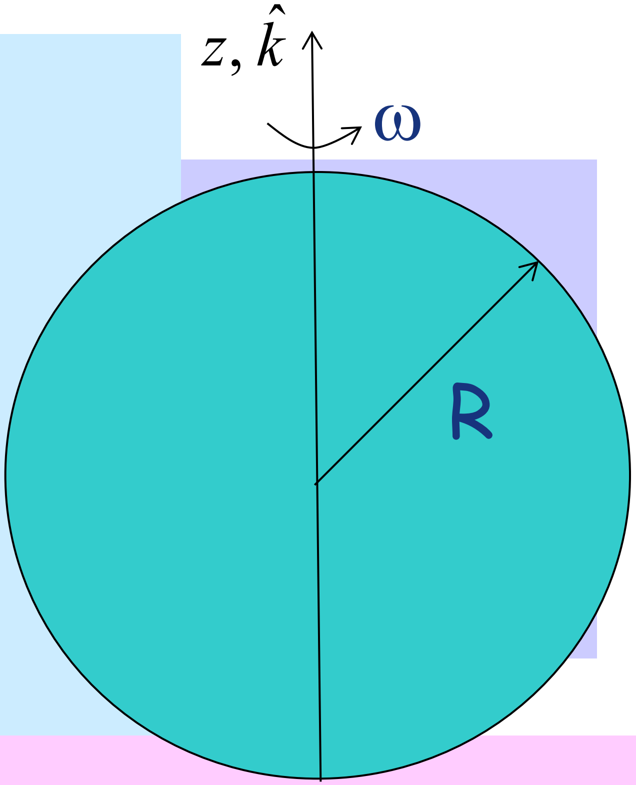
$$i d\vec{l}\,' = (\vec{K} \cdot dz\hat{j})dy\hat{j} = (K\hat{j} \cdot \hat{j})dzdyj = Kdzdy\hat{j} = \vec{K}ds$$

$$\vec{B} = \frac{\mu_0}{4\pi} \iint_S \frac{\vec{K}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3} ds'$$

Campo de Densidad de Corriente Superficial

Casquete esférico cargado con carga superficial σ y girando a velocidad angular ω

- Calcular densidad de corriente superficial
- Calcular campo magnético en el eje de rotación para $z >> R$



Densidad de Corriente Superficial

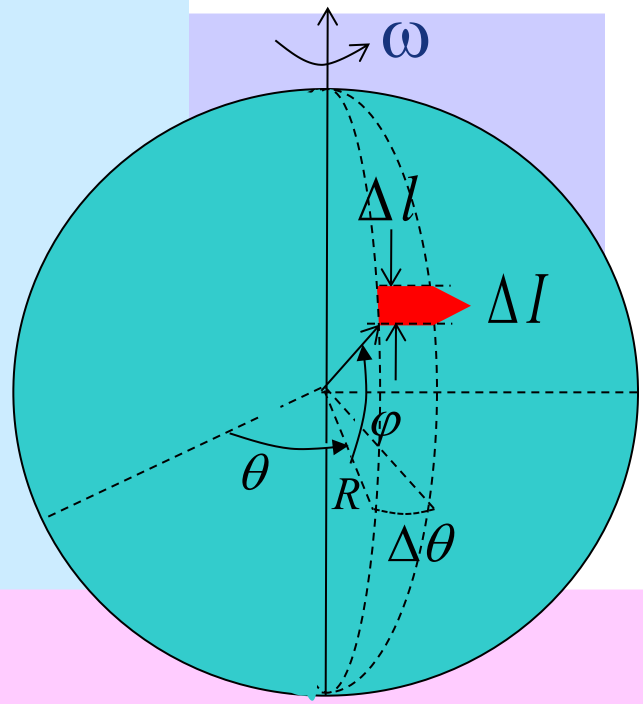
- Densidad de corriente superficial

$$\vec{K}(\vec{r}) = \lim_{\Delta l \to 0} \frac{\Delta I}{\Delta l} \hat{\theta} \quad \left[\frac{A}{m}\right]$$

$$\Delta q = \sigma \Delta s \qquad \Delta s = R \sin\varphi \Delta\theta \Delta l$$

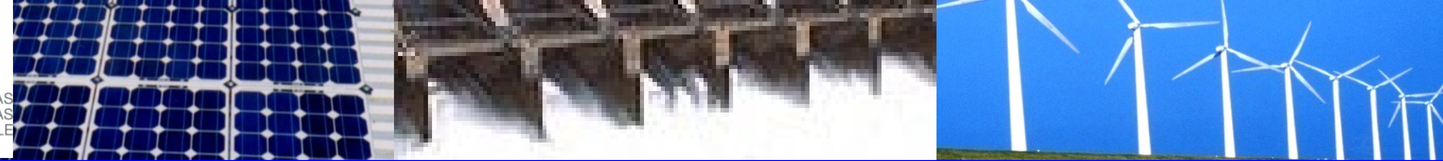
$$\Delta l = R\Delta\varphi \qquad \Delta s = R \sin\varphi \Delta\theta R \Delta\varphi$$

$$\Delta q = \sigma R^2 \sin\varphi \Delta\theta \Delta\varphi$$



$$\Delta I = \frac{\Delta q}{\Delta t} = \frac{\sigma R^2 \sin\varphi \Delta\theta \Delta\varphi}{\Delta t}$$

$$\Delta I = \sigma R^2 \sin\varphi \Delta\varphi \frac{\Delta\theta}{\Delta t} = \sigma R^2 \omega \sin\varphi \Delta\varphi \qquad \Rightarrow \vec{K}(\vec{r}) = \sigma R \omega \sin\varphi \hat{\theta} \left[\frac{A}{m}\right]$$



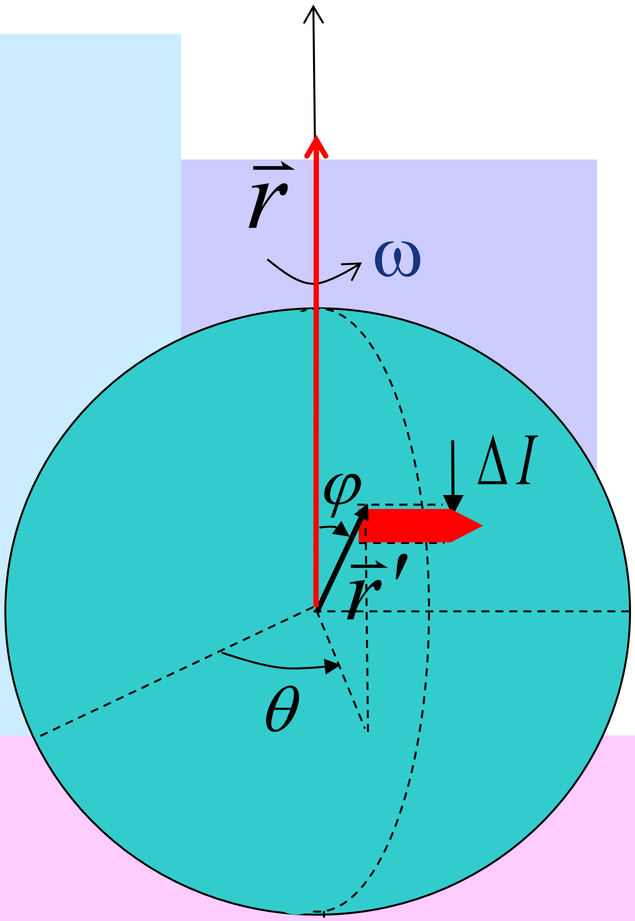
Densidad de Corriente Superficial

- **Campo magnético en el eje**

$$\vec{B} = \frac{\mu_0}{4\pi}\iint_S \frac{\vec{K}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|^3}\,ds'$$

$$\vec{K}(\vec{r}) = \sigma R\omega \sin\varphi\,\hat{\theta}$$

$$\vec{K}(\vec{r}) = \sigma R\omega \sin\varphi(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

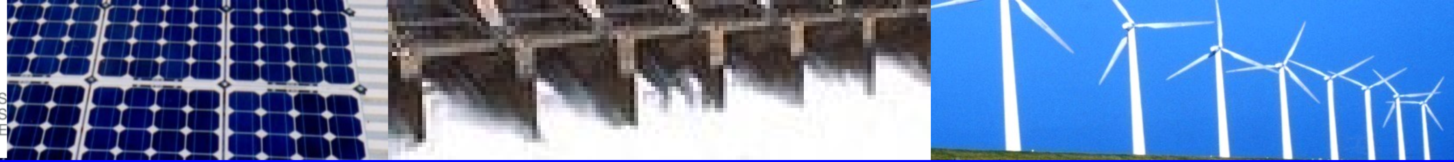


$$\vec{r}\,' = R\hat{\rho}$$

$$\vec{r}\,' = R\sin\varphi(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) + R\cos\varphi\,\hat{k}$$

$$\vec{r} = z\hat{k}$$

$$\vec{r}-\vec{r}\,' = -R\sin\varphi\cos\theta\,\hat{i} - R\sin\varphi\sin\theta\,\hat{j} + (z - R\cos\varphi)\hat{k}$$

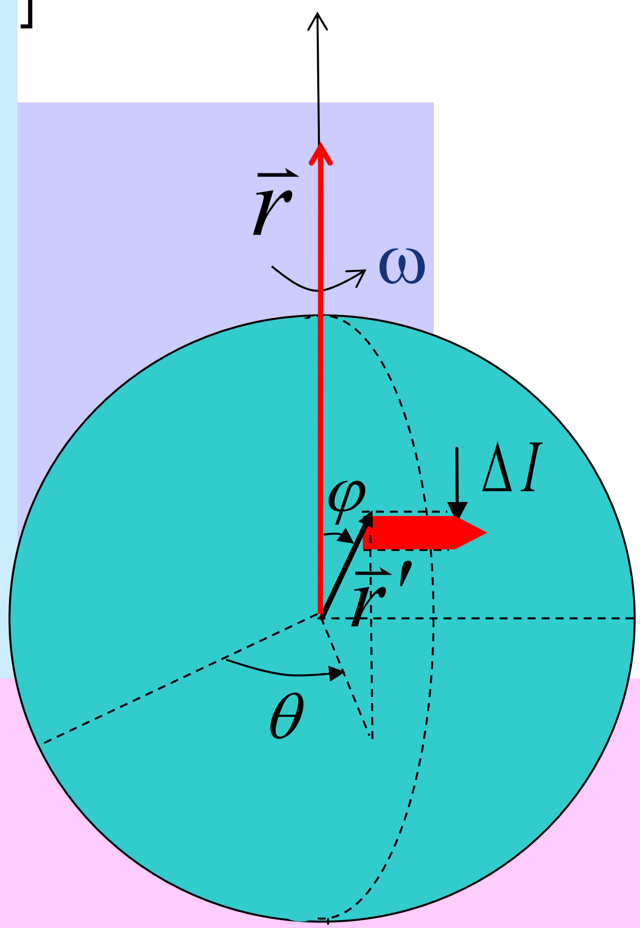


Densidad de Corriente Superficial

$$\|\vec{r}-\vec{r}'\| = \left[R^2\sin^2\varphi\cos^2\theta + R^2\sin^2\varphi\sin^2\theta + (z-R\cos\varphi)^2\right]^{1/2}$$

$$\|\vec{r}-\vec{r}'\| = \left[R^2\sin^2\varphi + z^2 - 2zR\cos\varphi + R^2\cos^2\varphi\right]^{1/2}$$

$$\|\vec{r}-\vec{r}'\| = \left[R^2 + z^2 - 2zR\cos\varphi\right]^{1/2}$$



$$\vec{K}(\vec{r}\,')\times(\vec{r}-\vec{r}\,') = (\sigma R\omega\sin\varphi(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}))\times$$
$$(-R\sin\varphi\cos\theta\,\vec{i} - R\sin\varphi\sin\theta\,\hat{j} + (z-R\cos\varphi))\hat{k}$$

Notar que el campo magnético sólo tiene componente según $\hat{k}$

$$\vec{K}(\vec{r}\,')\times(\vec{r}-\vec{r}\,') = \sigma R\omega\sin\varphi\sin\theta\, R\sin\varphi\sin\theta\,\hat{k} +$$
$$\sigma R\omega\sin\varphi\cos\theta R\sin\varphi\cos\theta\hat{k} \Rightarrow \vec{K}(\vec{r}\,')\times(\vec{r}-\vec{r}\,') = \sigma R^2\omega\sin^2\varphi\,\hat{k}$$

Densidad de Corriente Superficial

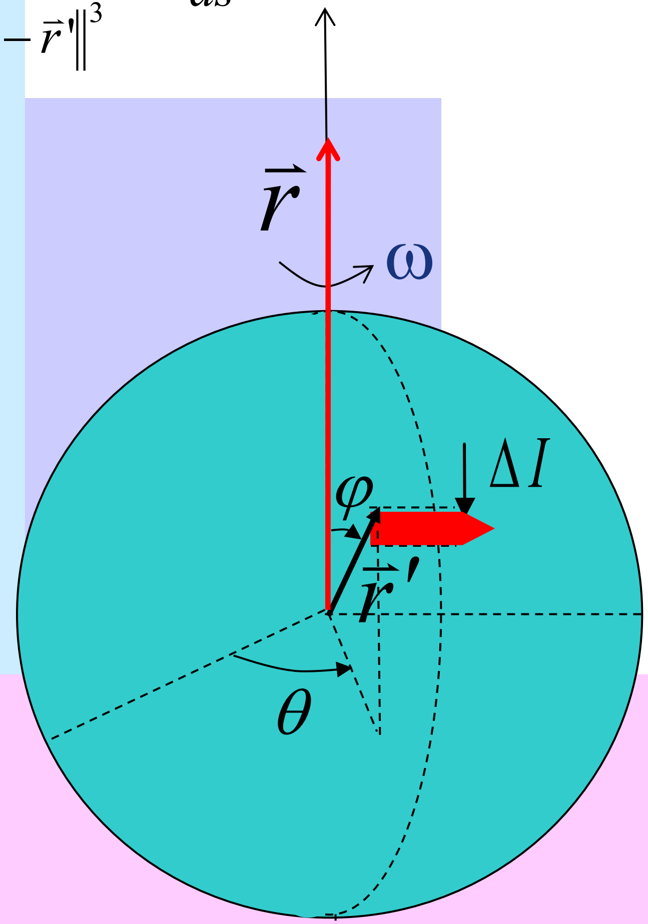
- **Campo magnético en el eje** $\vec{B}=\frac{\mu_0}{4\pi}\iint_S \frac{\vec{K}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|^3}ds'$

$$\vec{B}=\frac{\mu_0}{4\pi}\int_{\varphi=0}^{\pi}\int_{\theta=0}^{2\pi}\frac{\sigma R^2\omega\sin^2\varphi\,\hat{k}}{\left[R^2+z^2-2zR\cos\varphi\right]^{3/2}}R^2\sin\varphi\,d\theta d\varphi$$

$$\vec{B}=\frac{\mu_0}{4\pi}\int_{\varphi=0}^{\pi}2\pi\frac{\sigma R^4\omega\sin^3\varphi\,\hat{k}}{\left[R^2+z^2-2zR\cos\varphi\right]^{3/2}}d\varphi$$

$$\vec{B}=\frac{\mu_0\sigma R^4\omega}{2}\int_{\varphi=0}^{\pi}\frac{\sin^3\varphi}{\left[R^2+z^2-2zR\cos\varphi\right]^{3/2}}d\varphi\,\hat{k}$$

$$\vec{B}=\frac{\mu_0\sigma R^4\omega}{2(R^2+z^2)^{3/2}}\int_{\varphi=0}^{\pi}\frac{\sin^3\varphi}{\left[1-\frac{2zR}{R^2+z^2}\cos\varphi\right]^{3/2}}d\varphi\,\hat{k}$$



Densidad de Corriente Superficial

- Campo magnético en el eje z>>R

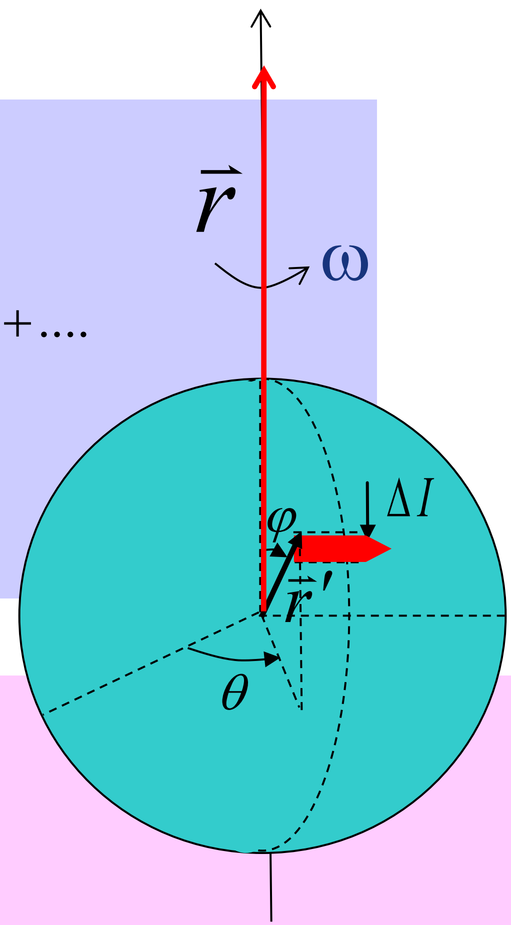
$$\vec{B} = \frac{\mu_0 \sigma R^4 \omega}{2(R^2+z^2)^{3/2}} \int_{\varphi=0}^{\pi} \frac{\sin^3\varphi}{\left[1 - \frac{2zR}{R^2+z^2}\cos\varphi\right]^{3/2}} d\varphi \hat{k}$$

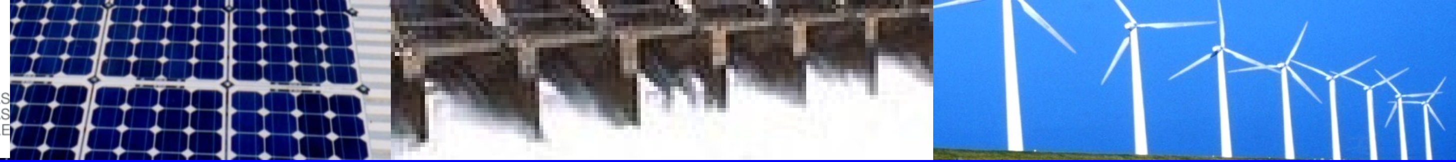
Usando la aproximación $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...$

$$\Rightarrow \left[1 - \frac{2zR}{R^2+z^2}\cos\varphi\right]^{-3/2} = 1 + \frac{3}{2}\frac{2zR}{R^2+z^2}\cos\varphi + ...$$

$$\vec{B} = \frac{\mu_0 \sigma R^4 \omega}{2(R^2+z^2)^{3/2}} \int_{\varphi=0}^{\pi} \sin^3\varphi \left(1 + \frac{3}{2}\frac{2zR}{R^2+z^2}\cos\varphi\right) d\varphi \hat{k}$$

$$\vec{B} = \frac{\mu_0 \sigma R^4 \omega \hat{k}}{2(R^2+z^2)^{3/2}} \left(\int_{\varphi=0}^{\pi} \sin^3\varphi \, d\varphi + \int_{\varphi=0}^{\pi} \frac{3zR}{R^2+z^2}\cos\varphi \sin^3\varphi \, d\varphi \right)$$





Densidad de Corriente Superficial

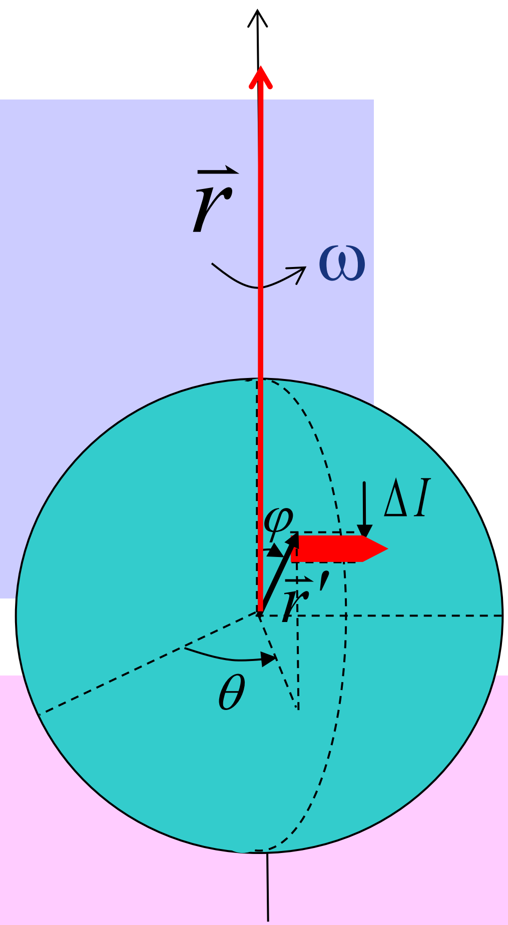
- **Campo magnético en el eje z>>R**

$$\vec{B}=\frac{\mu_0\sigma R^4\omega\hat{k}}{2(R^2+z^2)^{3/2}}\left(\int_{\varphi=0}^{\pi}\sin^3\varphi\,d\varphi+\int_{\varphi=0}^{\pi}\frac{3zR}{R^2+z^2}\cos\varphi\sin^3\varphi\,d\varphi\right)$$

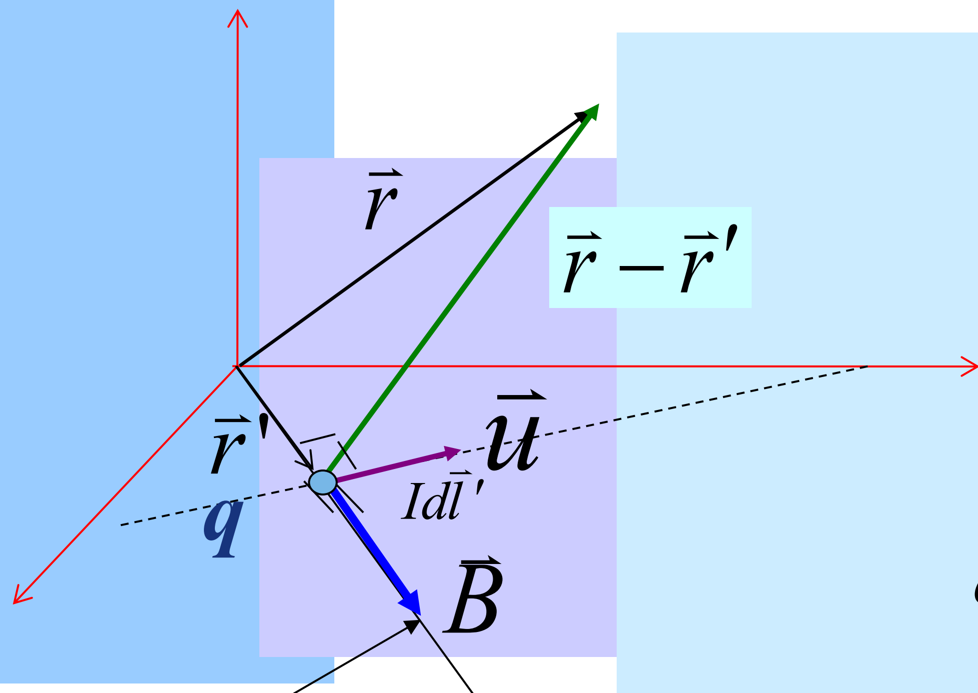
$$\int_{\varphi=0}^{\pi}\sin^3\varphi\,d\varphi=-\frac{\cos\varphi}{3}\left(2+\sin^2\varphi\right)\Big|_0^{\pi}=\frac{2}{3}+\frac{2}{3}=\frac{4}{3}$$

$$\int_{\varphi=0}^{\pi}\cos\varphi\sin^3\varphi\,d\varphi=\frac{\sin^4\varphi}{4}\Big|_0^{\pi}=0$$

$$\therefore\ \vec{B}=\frac{2\mu_0\sigma R^4\omega}{3(R^2+z^2)^{3/2}}\hat{k}$$



Campo Magnético de una Carga Puntual



$$Id\vec{l}\,'\times(\vec{r}-\vec{r}\,')$$

$$d\vec{B}=\frac{\mu_0 Id\vec{l}\,'\times(\vec{r}-\vec{r}\,')}{4\pi\|\vec{r}-\vec{r}\,'\|^3}$$

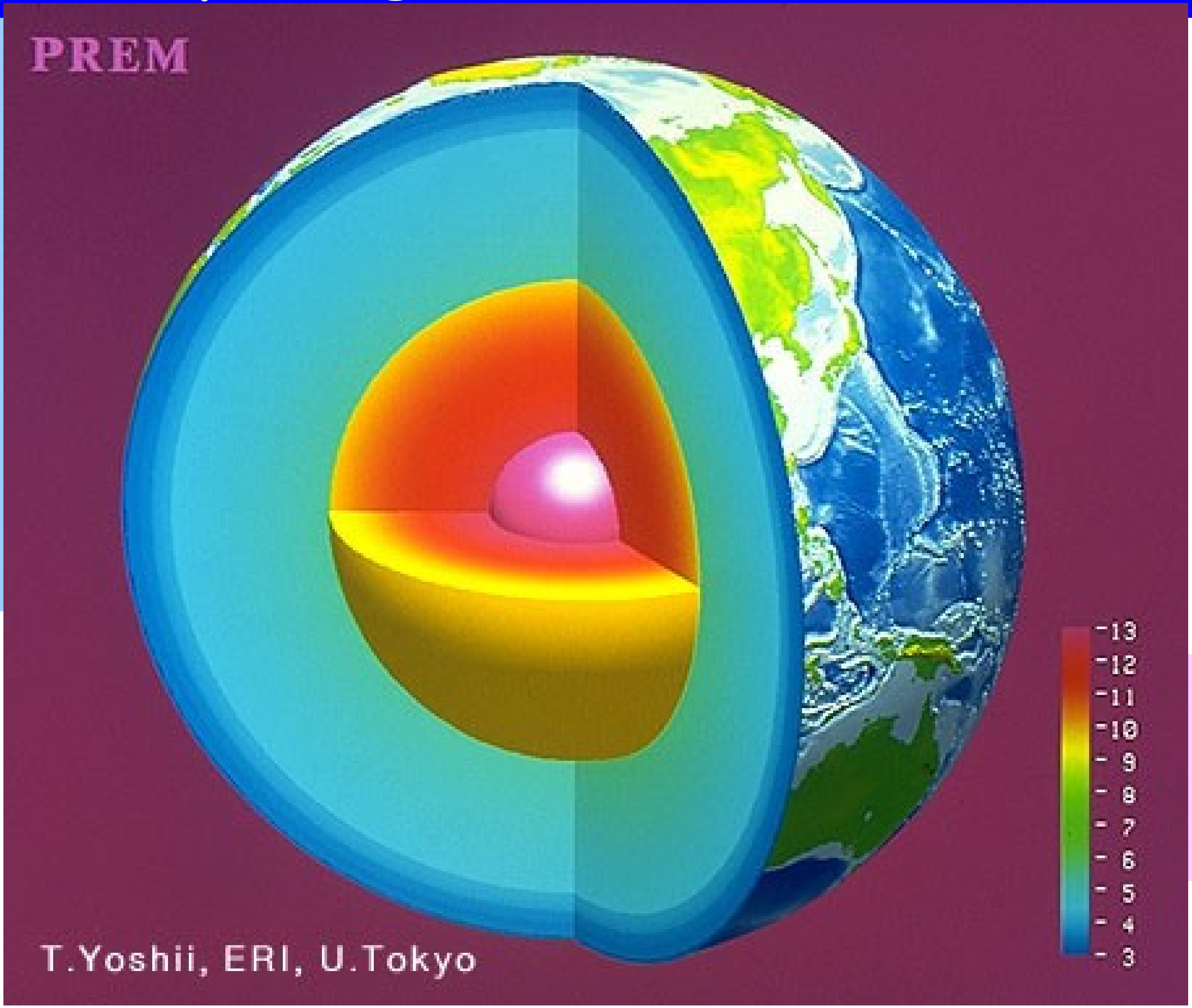
$$id\vec{l}=\frac{dq}{dt}dl\hat{u}=dq\frac{dl}{dt}\hat{u}$$

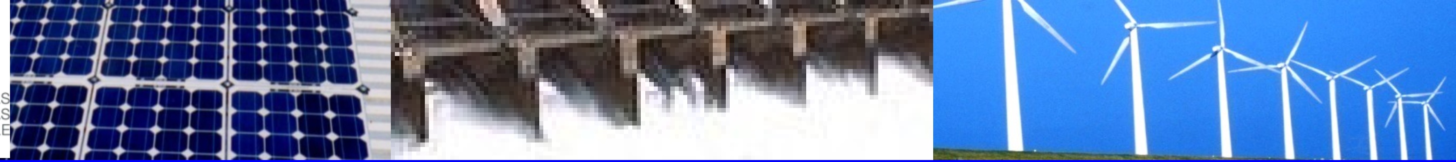
$$dq\rightarrow q,\quad \frac{dl}{dt}=\|\vec{u}\|\quad \text{y}\quad d\vec{B}\rightarrow\vec{B}$$

$$\Rightarrow \vec{B}=\frac{\mu_0}{4\pi}q\vec{u}\times\frac{(\vec{r}-\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|^3}$$

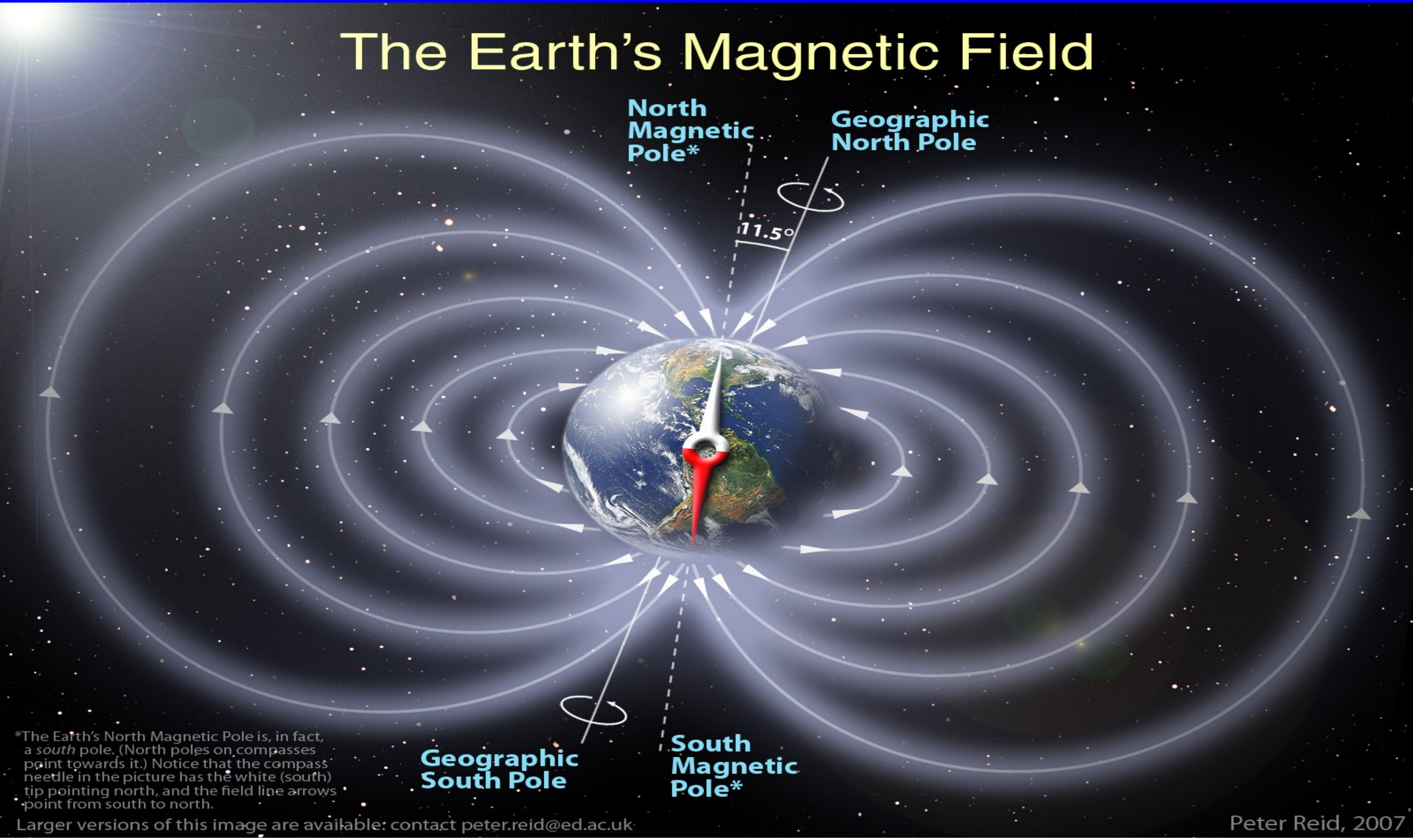
Campo magnético resultante perpendicular a la velocidad

Campo Magnético Terrestre



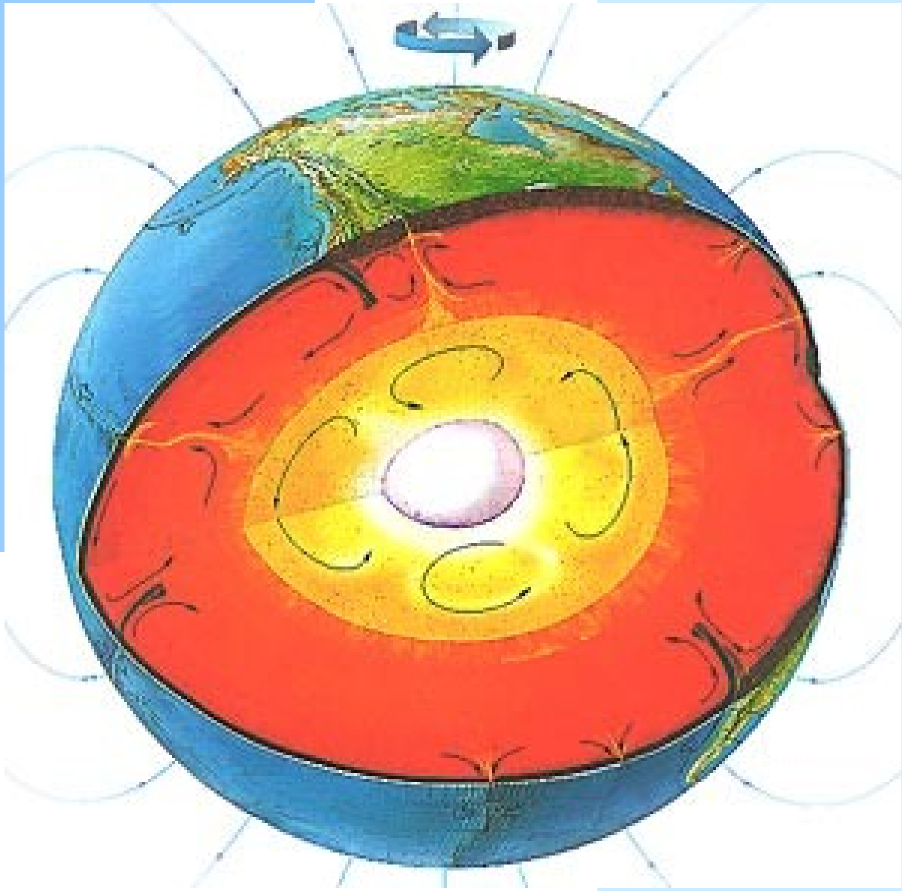


Campo Magnético Terrestre

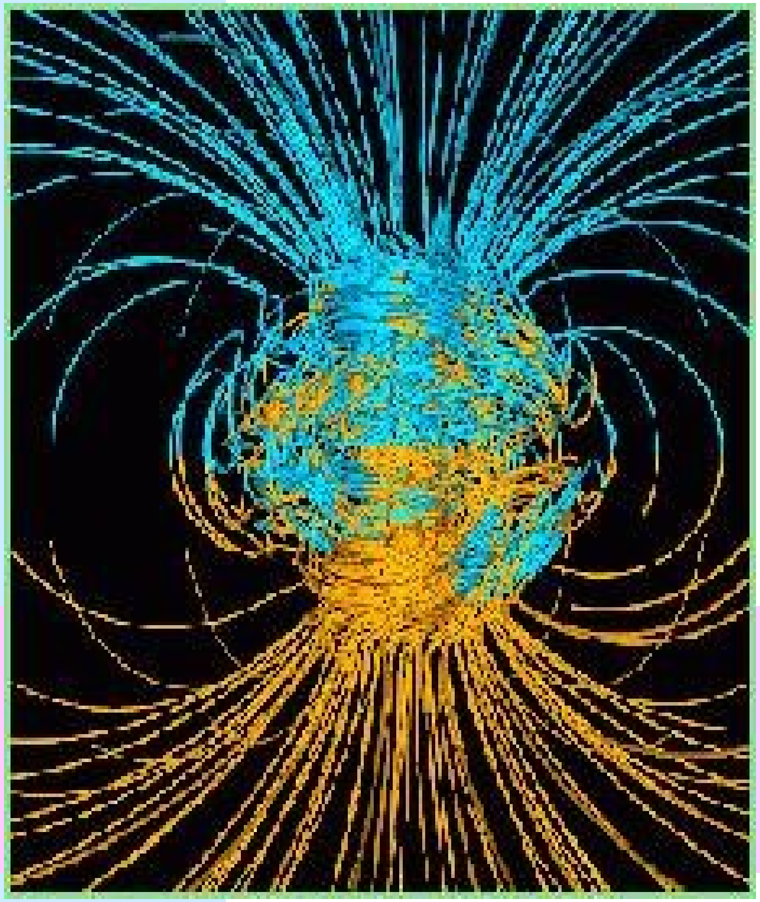


Campo Magnético Terrestre

Convección



Geodínamo

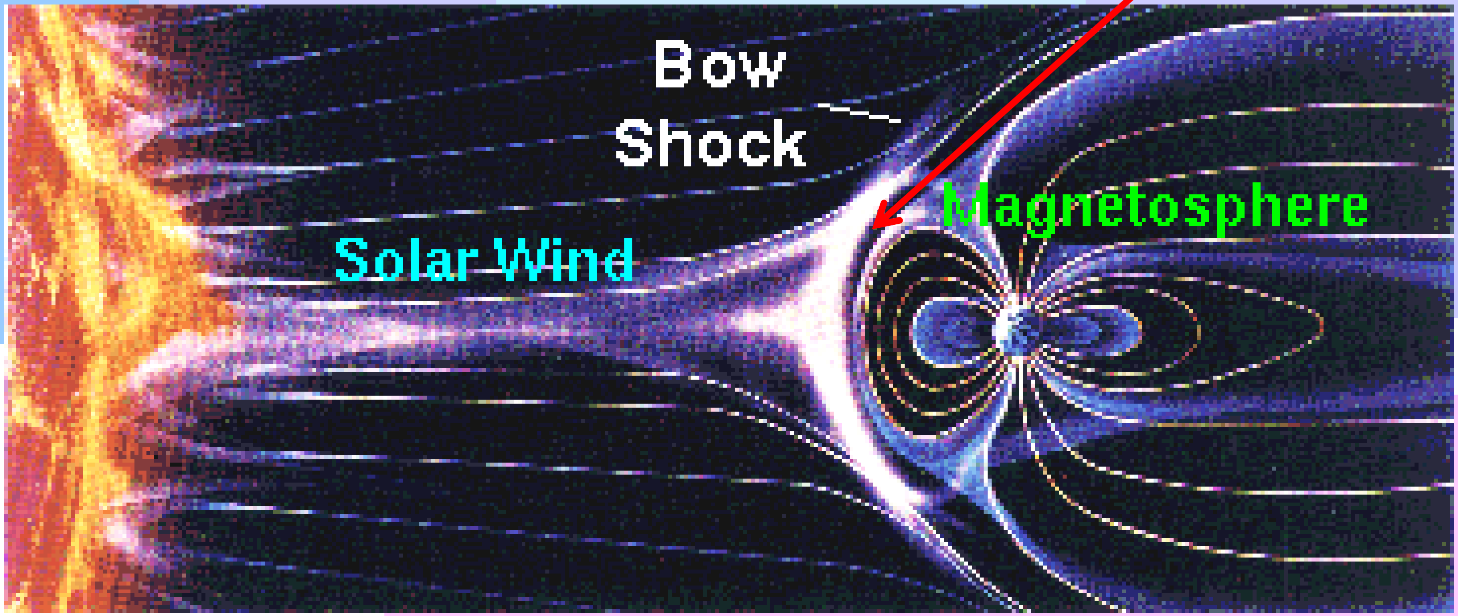


Campo Magnético Terrestre

- Viento solar: Gases ionizados que vienen del sol a 400Km/s
- Varían con la actividad de la superficie solar

Fuerza de Lorentz

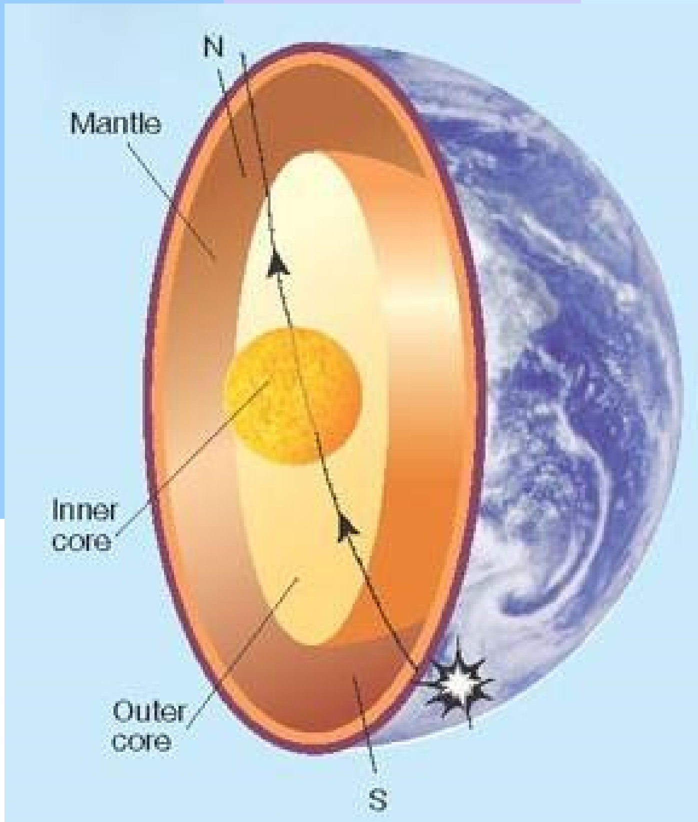
$$\vec{F} = q\vec{u} \times \vec{B}$$



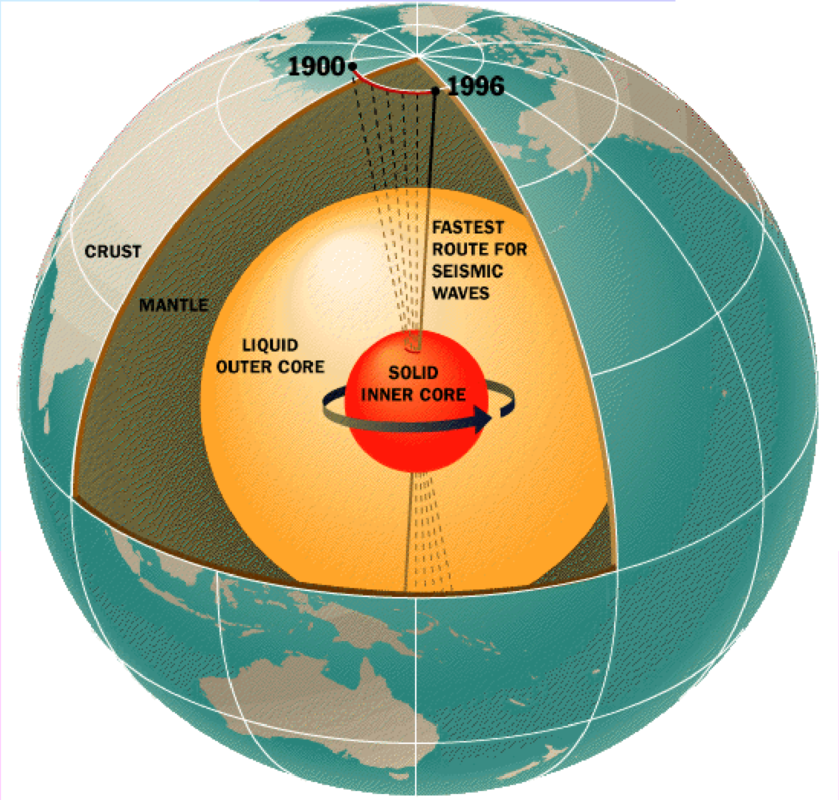
Fuente: National Geophysical Data Center, USA

Campo Magnético Terrestre

Estudio del núcleo



Anisotropía y super-rotación



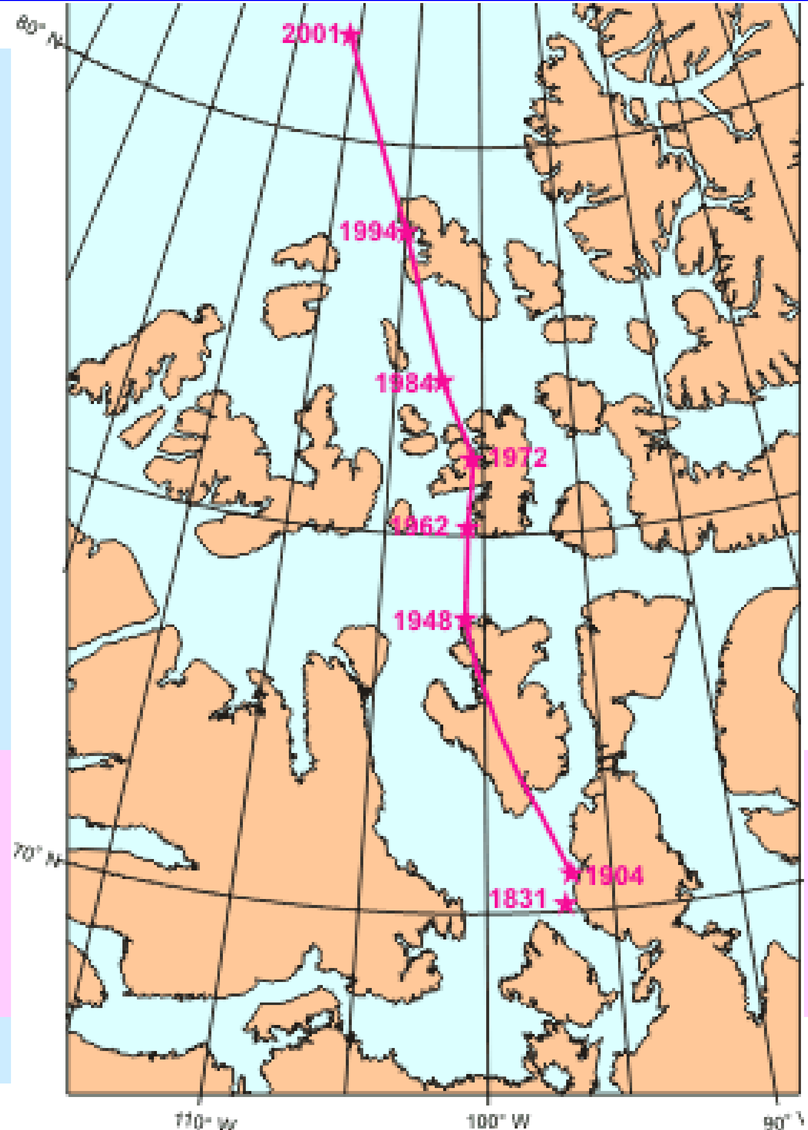
Campo Magnético Terrestre

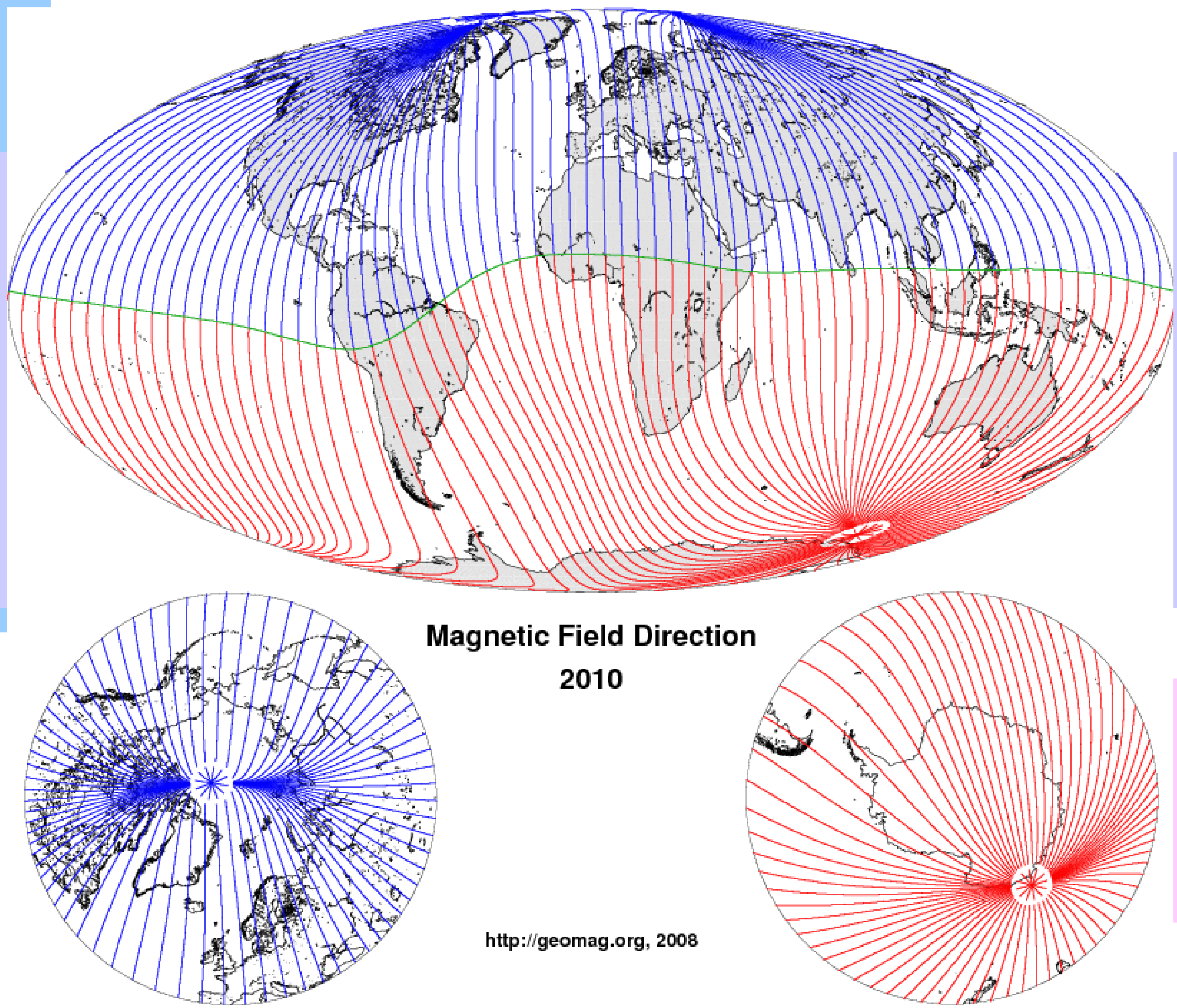
Movimiento del polo norte magnético en el Artico de Canadá en el periodo 1831-2001. Fuente: Geological Survey of Canada.

Se mueve 40 km cada año, approx.

Ultimamente se mueve crecientemente mas rápido.

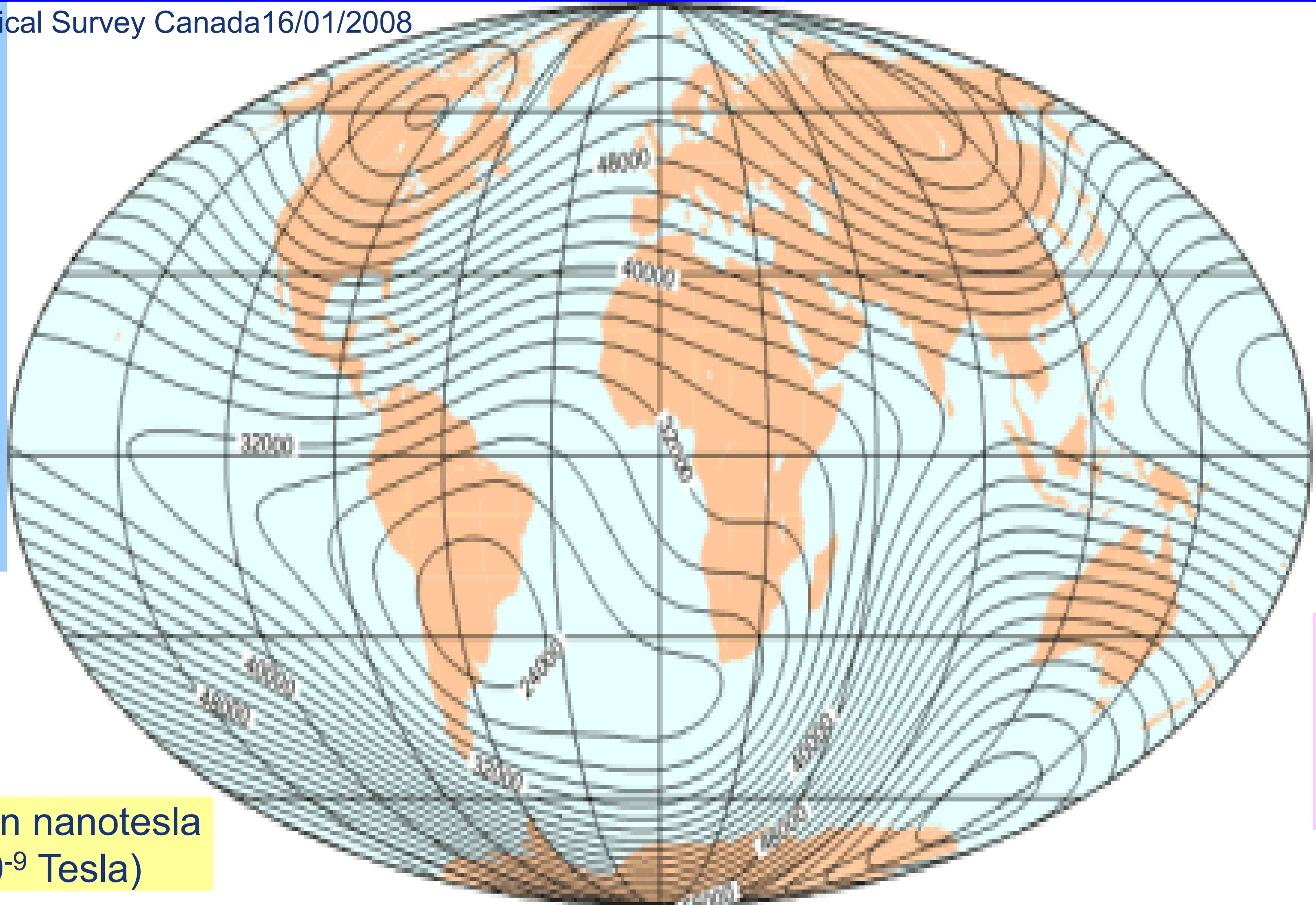
La última vez que el polo magnético se cambio al otro lado de la tierra fue hace 780.000 años




Magnetic Field Direction
2010
http://geomag.org, 2008

Campo Magnético Terrestre

Fuente: Geological Survey Canada16/01/2008



Medido en nanotesla
1 nT = 10^{-9} Tesla)