5 Eigenvalues and Eigenvectors

5.1

EIGENVECTORS AND EIGENVALUES

□ Could be replaced by {R}

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EIGENVECTORS AND EIGENVALUES

- **Definition:** An **eigenvector** of an $n \times n$ matrix A is a nonzero vector **x** such that $A\mathrm{x} = \lambda\mathrm{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution **x** of $A\mathrm{x} = \lambda\mathrm{x}$; such an **x** is called an *eigenvector corresponding to λ*.
- λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)\mathrm{x} = 0 \quad \text{----(1)}$$

has a nontrivial solution.
- The set of *all* solutions of (1) is just the null space of the matrix $A - \lambda I$.

EIGENVECTORS AND EIGENVALUES

- So this set is a *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to λ.

- The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.

- **Example 1:** Show that 7 is an eigenvalue of matrix

$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and find the corresponding eigenvectors.

EIGENVECTORS AND EIGENVALUES

- **Solution:** The scalar 7 is an eigenvalue of A if and only if the equation

$$A\mathrm{x} = 7\mathrm{x} \qquad \text{----(2)}$$

has a nontrivial solution.

- But (2) is equivalent to $A\mathrm{x} - 7\mathrm{x} = 0$, or

$$(A - 7I)\mathrm{x} = 0 \qquad \text{----(3)}$$

- To solve this homogeneous equation, form the matrix

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

EIGENVECTORS AND EIGENVALUES

- The columns of $A-7I$ are obviously linearly dependent, so (3) has nontrivial solutions.
- To find the corresponding eigenvectors, use row operations:

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- The general solution has the form $x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- Each vector of this form with $x_2 \neq 0$ is an eigenvector corresponding to $\lambda = 7$.

EIGENVECTORS AND EIGENVALUES

- **Example 2:** Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.
- **Solution:** Form

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

and row reduce the augmented matrix for $(A - 2I)\mathrm{x} = 0$.

EIGENVECTORS AND EIGENVALUES

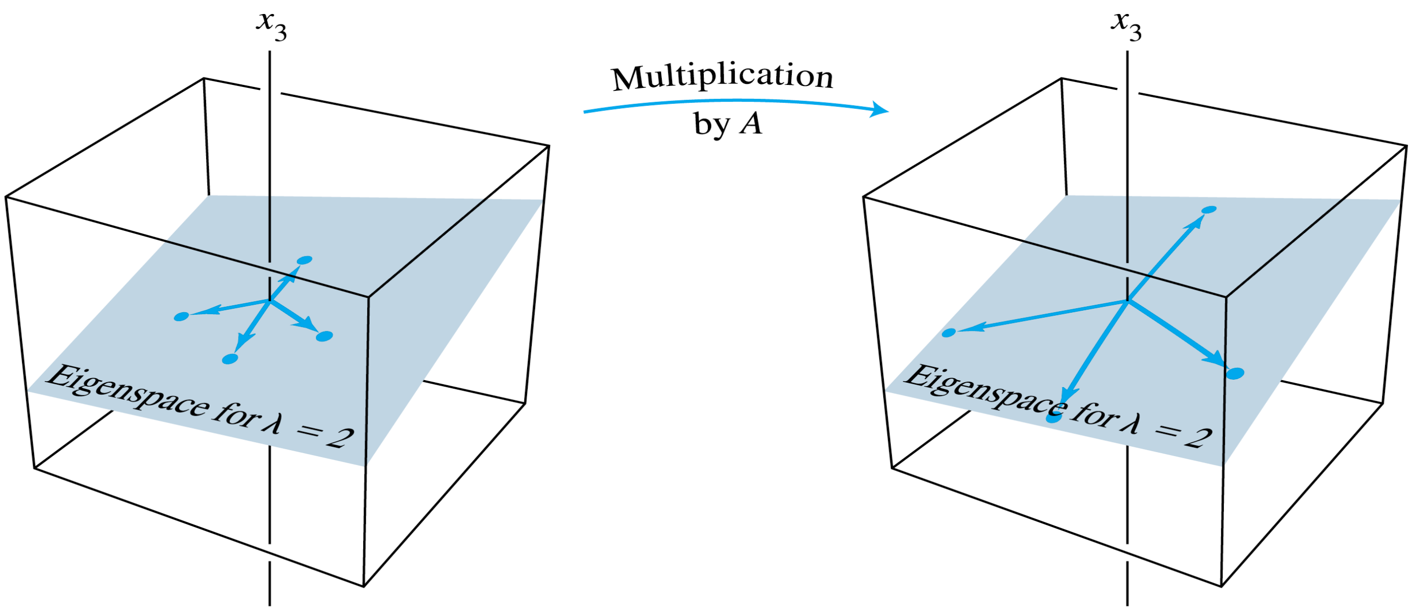
$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- At this point, it is clear that 2 is indeed an eigenvalue of A because the equation $(A-2I)\mathrm{x}=0$ has free variables.
- The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \ x_2 \text{ and } x_3 \text{ free.}$$

EIGENVECTORS AND EIGENVALUES

- The eigenspace, shown in the following figure, is a two-dimensional subspace of $\mathbb{R}^3$.



A acts as a dilation on the eigenspace.

- A basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

EIGENVECTORS AND EIGENVALUES

- **Theorem 1:** The eigenvalues of a triangular matrix are the entries on its main diagonal.

EIGENVECTORS AND EIGENVALUES

- **Theorem 2:** If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

$\mathbf{v}_{p+1}$

5 Eigenvalues and Eigenvectors

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THE CHARACTERISTIC EQUATION

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DETERMINANATS

- Let A be an $n \times n$ matrix, let U be any echelon form obtained from A by row replacements and row interchanges (without scaling), and let r be the number of such row interchanges.

- Then the determinant of A, written as det A, is $(-1)^r$ times the product of the diagonal entries $u_{11}, \ldots, u_{nn}$ in U.

- If A is invertible, then $u_{11}, \ldots, u_{nn}$ are all pivots (because $A \square I_n$ and the u_{ii} have not been scaled to 1's).

DETERMINANATS

- Otherwise, at least u_{nn} is zero, and the product $u_{11} \ldots u_{nn}$ is zero.

- Thus

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix}, & \text{when A is invertible} \\ 0, & \text{when A is not invertible} \end{cases}$$

DETERMINANATS

- **Example 1:** Compute $\det A$ for $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

- **Solution:** The following row reduction uses one row interchange:

$$A \square \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix} \square \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & -6 & -1 \end{bmatrix} \square \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = U_1$$

DETERMINANATS

- So det A equals $(-1)^1(1)(-2)(-1) = -2$.
- The following alternative row reduction avoids the row interchange and produces a different echelon form.
- The last step adds $-1/3$ times row 2 to row 3:

$$A \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & 0 & 1/3 \end{bmatrix} = U_2$$

- This time det A is $(-1)^0(1)(-6)(1/3) = -2$, the same as before.

THE INVERTIBLE MATRIX THEOREM (CONTINUED)

- **Theorem:** Let A be an $n \times n$ matrix. Then A is invertible if and only if:
  s. The number 0 is *not* an eigenvalue of A.
  t. The determinant of A is *not* zero.

- **Theorem 3: Properties of Determinants**
- Let A and B be $n \times n$ matrices.
  a. A is invertible if and only if $\det A \neq 0$.
  b. $\det AB = (\det A)(\det B)$.
  c. $\det A^T = \det A$.

PROPERTIES OF DETERMINANTS

d. If A is triangular, then det A is the product of the entries on the main diagonal of A.

e. A row replacement operation on A does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

THE CHARACTERISTIC EQUATION

- Theorem 3(a) shows how to determine when a matrix of the form $A-\lambda I$ is *not* invertible.

- The scalar equation $\det(A-\lambda I)=0$ is called the **characteristic equation** of A.

- A scalar λ is an eigenvalue of an $n\times n$ matrix A if and only if λ satisfies the characteristic equation

$$\det(A-\lambda I)=0$$

THE CHARACTERISTIC EQUATION

- **Example 2:** Find the characteristic equation of

$$A=\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Solution:** Form $A-\lambda I$, and use Theorem 3(d):

THE CHARACTERISTIC EQUATION

$$\det(A-\lambda I)=\det\begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

$$=(5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

- The characteristic equation is

$$(5-\lambda)^2(3-\lambda)(1-\lambda)=0$$

or

$$(\lambda-5)^2(\lambda-3)(\lambda-1)=0$$

THE CHARACTERISTIC EQUATION

- Expanding the product, we can also write

$$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$$

- If A is an $n \times n$ matrix, then $\det(A - \lambda I)$ is a polynomial of degree n called the **characteristic polynomial** of A.
- The eigenvalue 5 in Example 2 is said to have *multiplicity* 2 because $(\lambda - 5)$ occurs two times as a factor of the characteristic polynomial.
- In general, the (**algebraic**) **multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic equation.

SIMILARITY

- If A and B are $n \times n$ matrices, then A **is similar to** B if there is an invertible matrix P such that $P^{-1}AP = B$, or, equivalently, $A = PBP^{-1}$.

- Writing Q for P^{-1}, we have $Q^{-1}BQ = A$.

- So B is also similar to A, and we say simply that A and B **are similar**.

- Changing A into $P^{-1}AP$ is called a **similarity transformation**.

SIMILARITY

- **Theorem 4:** If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).
- **Warnings:**
  1. The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar even though they have the same eigenvalues.

SIMILARITY

2. Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E). Row operations on a matrix usually change its eigenvalues.

5 Eigenvalues and Eigenvectors

5.3

DIAGONALIZATION

□ Could be replaced by {R}

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DIAGONALIZATION

- **Example 1:** Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^k, given that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

- **Solution:** The standard formula for the inverse of a 2×2 matrix yields

$$P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

DIAGONALIZATION

- Then, by associativity of matrix multiplication,

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD\underbrace{(P^{-1}P)}_{I}DP^{-1} = PDDP^{-1}$$

$$= PD^2P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

- Again,

$$A^3 = (PDP^{-1})A^2 = (PD\underbrace{P^{-1})P}_{I}D^2P^{-1} = PDD^2P^{-1} = PD^3P^{-1}$$

DIAGONALIZATION

- In general, for $k \geq 1$,

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot 5^k - 3^k & 5^k - 3^k \\ 2\cdot 3^k - 2\cdot 5^k & 2\cdot 3^k - 5^k \end{bmatrix}$$

- A square matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal, matrix D.

THE DIAGONALIZATION THEOREM

- **Theorem 5:** An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

  In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P and n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

  In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$. We call such a basis an **eigenvector basis** of $\mathbb{R}^n$.

THE DIAGONALIZATION THEOREM

- If, in fact, the eigenvectors are linearly independent, then P is invertible (by the Invertible Matrix Theorem), and $AP = PD$ implies that $A = PDP^{-1}$.

DIAGONALIZING MATRICES

- **Example 2:** Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

  That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

- **Solution:** There are four steps to implement the description in Theorem 5.
- *Step 1.* ***Find the eigenvalues of A.***
- Here, the characteristic equation turns out to involve a cubic polynomial that can be factored:

DIAGONALIZING MATRICES

$$0 = \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4$$
$$= -(\lambda - 1)(\lambda + 2)^2$$

- The eigenvalues are $\lambda = 1$ and $\lambda = -2$.
- *Step 2.* ***Find three linearly independent eigenvectors of A.***
- *Three* vectors are needed because A is a 3×3 matrix.
- This is a critical step.
- If it fails, then Theorem 5 says that A cannot be diagonalized.

DIAGONALIZING MATRICES

- Basis for $\lambda = 1$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

- Basis for $\lambda = -2$: $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

- You can check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set.

DIAGONALIZING MATRICES

- *Step 3.* ***Construct P from the vectors in step 2.***
- The order of the vectors is unimportant.
- Using the order chosen in step 2, form

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

- *Step 4.* ***Construct D from the corresponding eigenvalues.***
- In this step, it is essential that the order of the eigenvalues matches the order chosen for the columns of P.

DIAGONALIZING MATRICES

- Use the eigenvalue $\lambda = -2$ twice, once for each of the eigenvectors corresponding to $\lambda = -2$:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- To avoid computing P^{-1}, simply verify that $AD = PD$.
- Compute

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

DIAGONALIZING MATRICES

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

- **Theorem 6:** An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.
- **Proof:** Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be eigenvectors corresponding to the n distinct eigenvalues of a matrix A.
- Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent, by Theorem 2 in Section 5.1.
- Hence A is diagonalizable, by Theorem 5.

MATRICES WHOSE EIGENVALUES ARE NOT DISTINCT

- It is not *necessary* for an $n \times n$ matrix to have n distinct eigenvalues in order to be diagonalizable.
- Theorem 6 provides a *sufficient* condition for a matrix to be diagonalizable.

- If an $n \times n$ matrix A has n distinct eigenvalues, with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and if $P = \begin{bmatrix} \mathrm{v}_1 & \cdots & \mathrm{v}_2 \end{bmatrix}$, then P is automatically invertible because its columns are linearly independent, by Theorem 2.

MATRICES WHOSE EIGENVALUES ARE NOT DISTINCT

- When A is diagonalizable but has fewer than n distinct eigenvalues, it is still possible to build P in a way that makes P automatically invertible, as the next theorem shows.

- **Theorem 7:** Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.
  a. For $1 \le k \le p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

MATRICES WHOSE EIGENVALUES ARE NOT DISTINCT

b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (i) the characteristic polynomial factors completely into linear factors and (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

c. If A is diagonalizable and B_k is a basis for the eigenspace corresponding to B_k for each k, then the total collection of vectors in the sets $B_1, \ldots, B_p$ forms an eigenvector basis for $\mathbb{R}^n$.