Discrete Mathematics

Lecture 7

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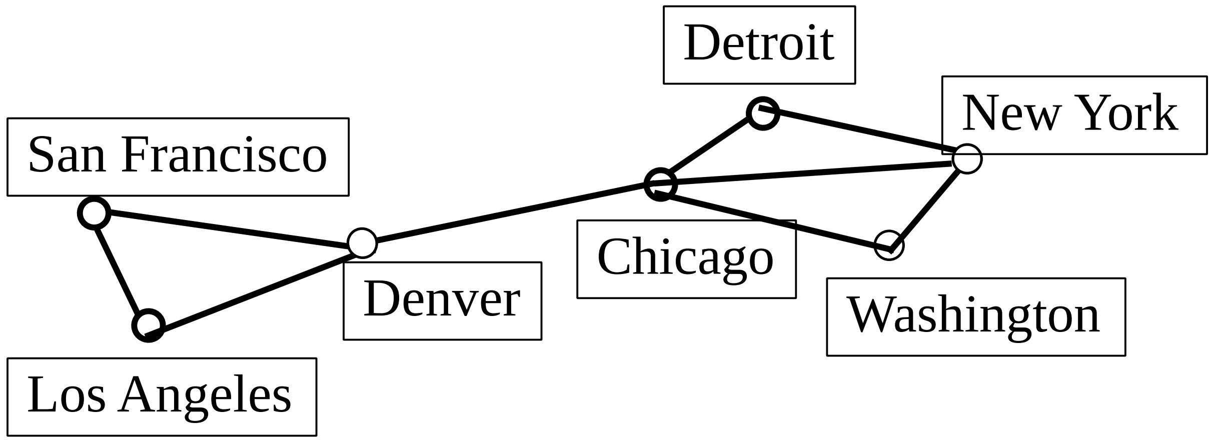
Graphs

Chapter 10

Simple Graph

- A *simple graph* consists of
  - a nonempty set of *vertices* called V
  - a set of edges (unordered pairs of distinct elements of V) called E
- Notation: $G = (V,E)$

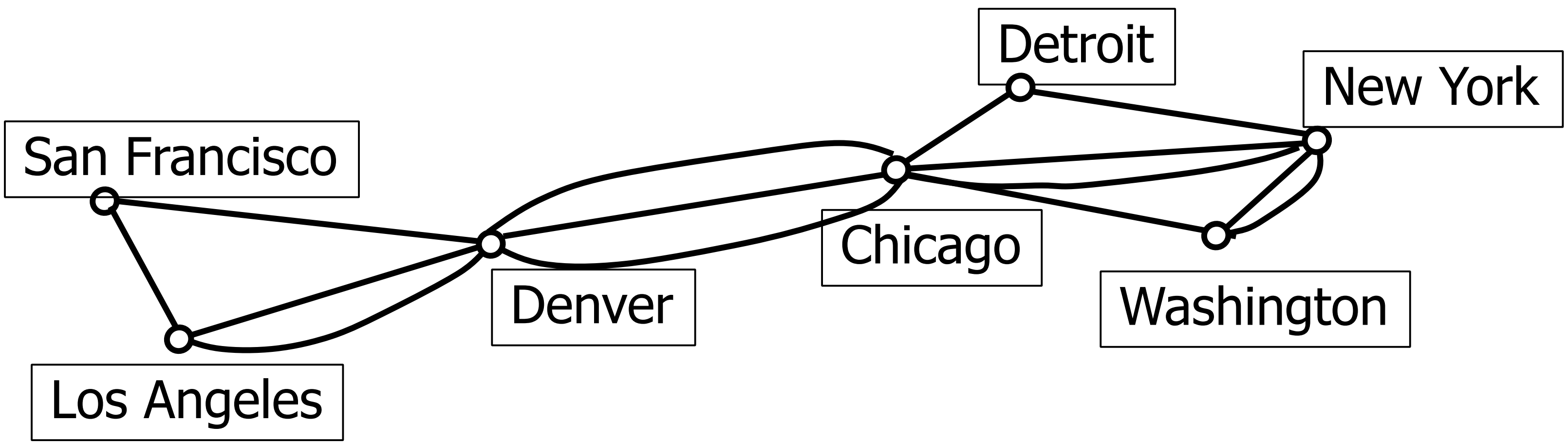
Simple Graph Example



- This simple graph represents a network.
- The network is made up of computers and telephone links between computers

Multigraph

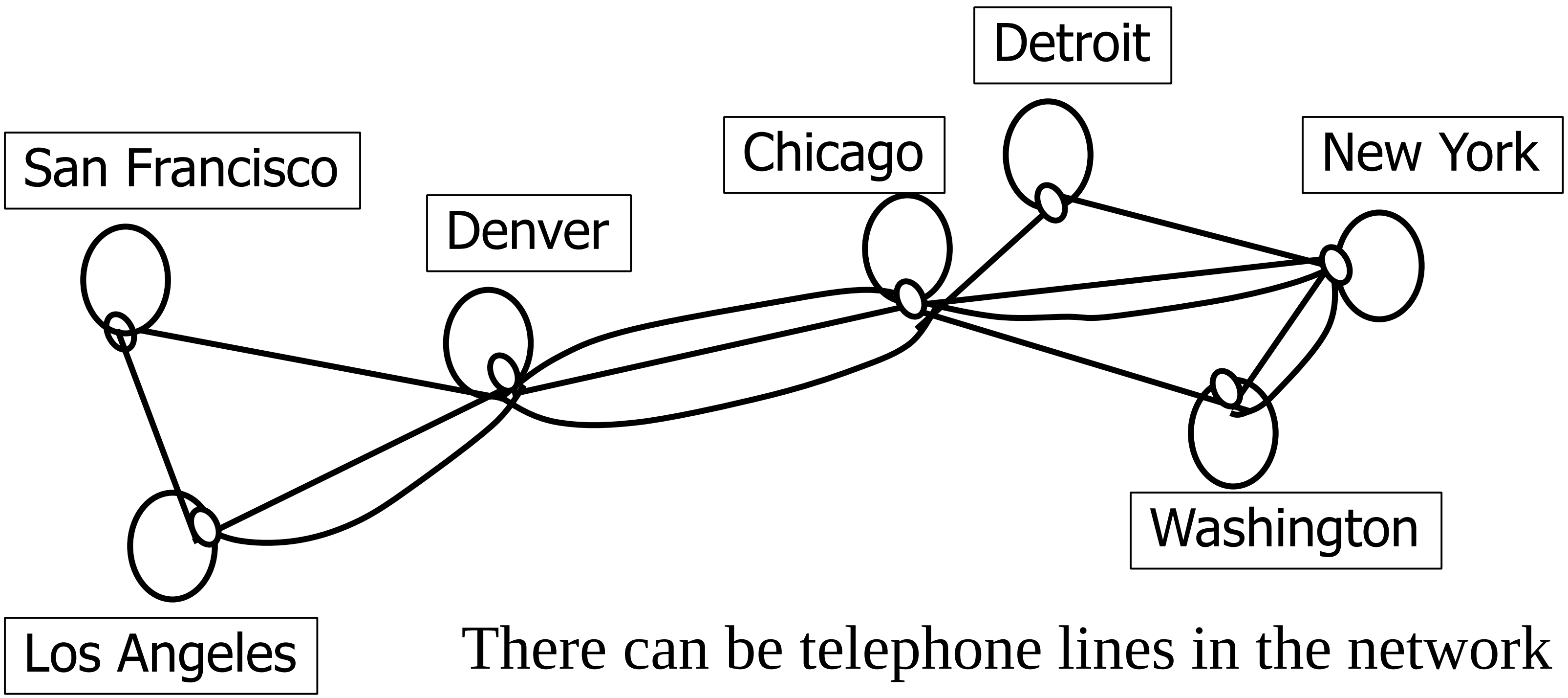
- A *multigraph* can have *multiple edges* (two or more edges connecting the same pair of vertices).



There can be multiple telephone lines between two computers in the network.

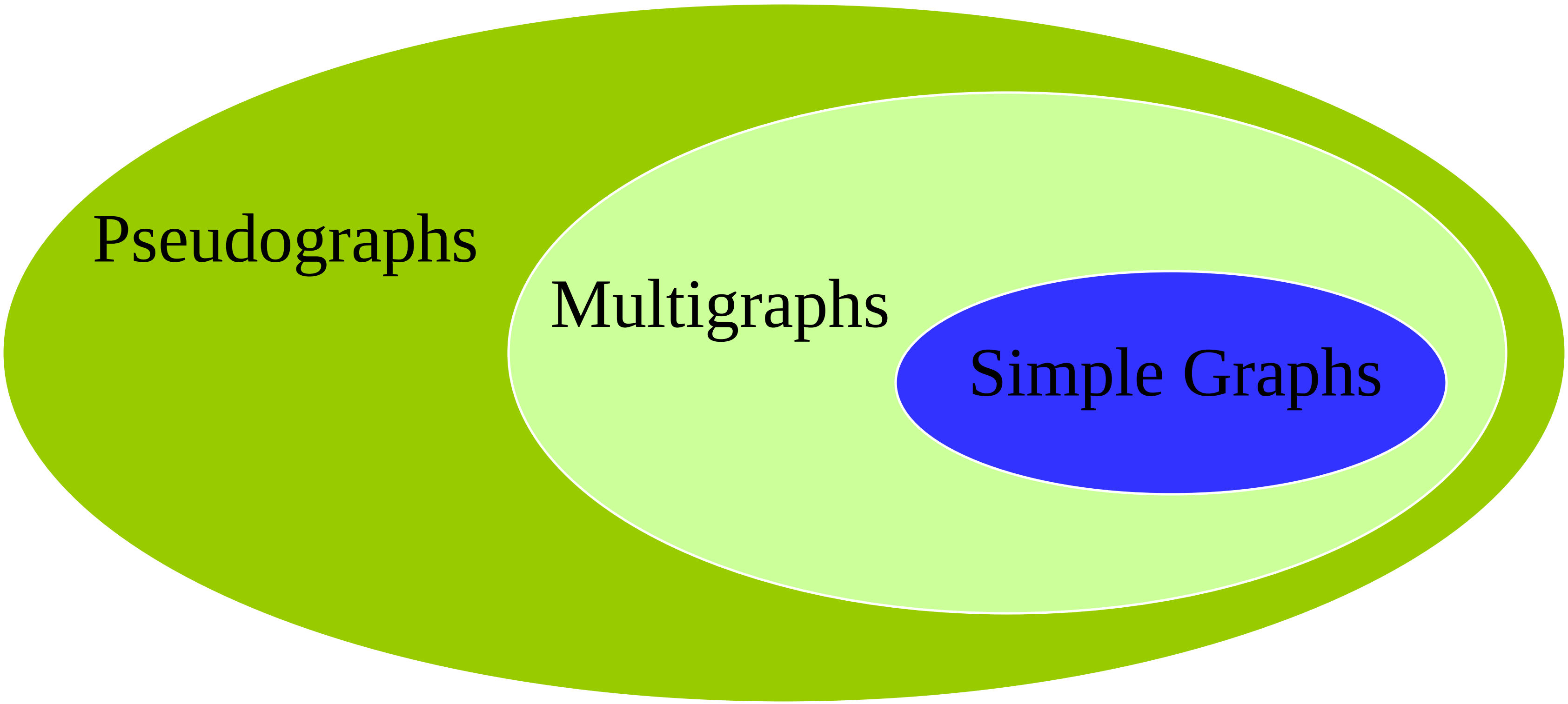
Pseudograph

- A *Pseudograph* can have multiple edges and *loops* (an edge connecting a vertex to itself).



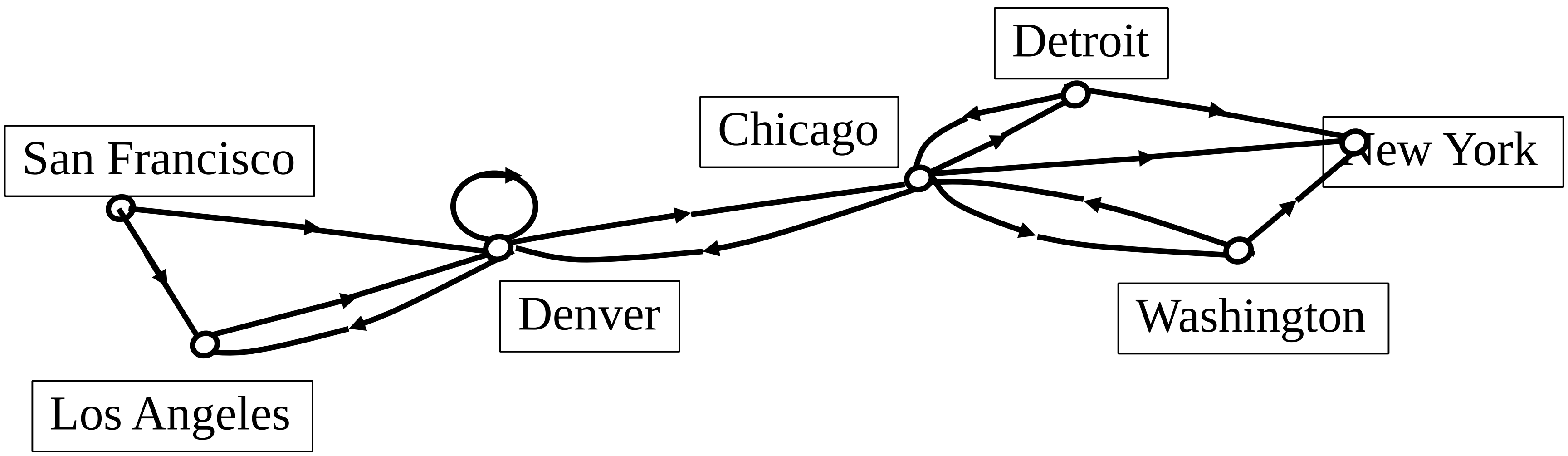
There can be telephone lines in the network from a computer to itself.

Types of Undirected Graphs



Directed Graph

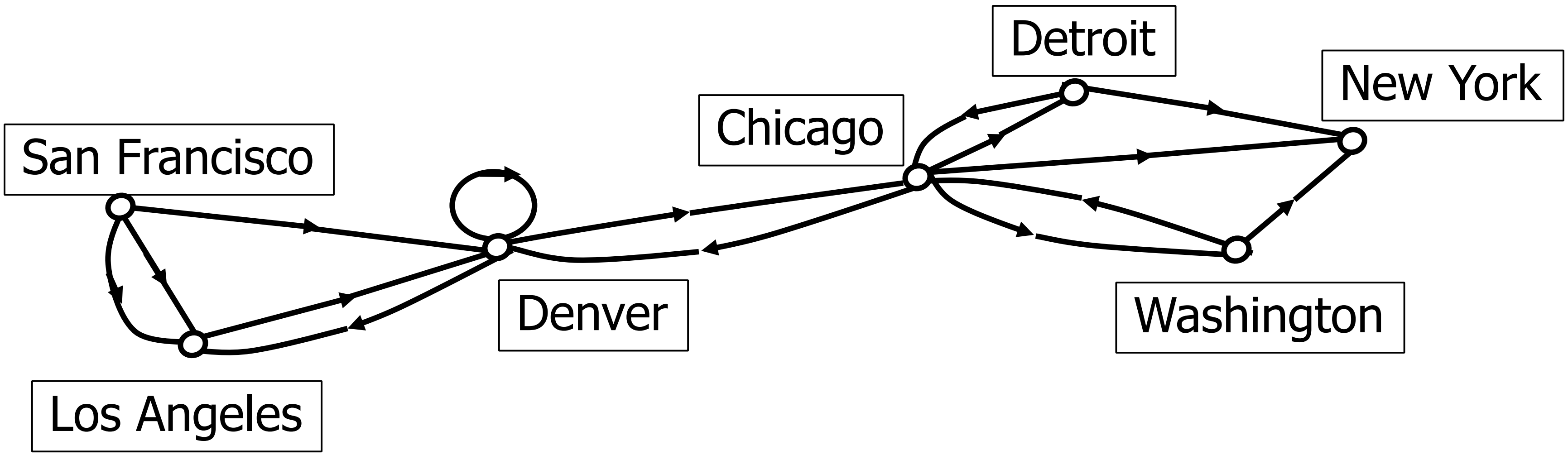
The edges are ordered pairs of (not necessarily distinct) vertices.



Some telephone lines in the network may operate in only one direction. Those that operate in two directions are represented by pairs of edges in opposite directions.

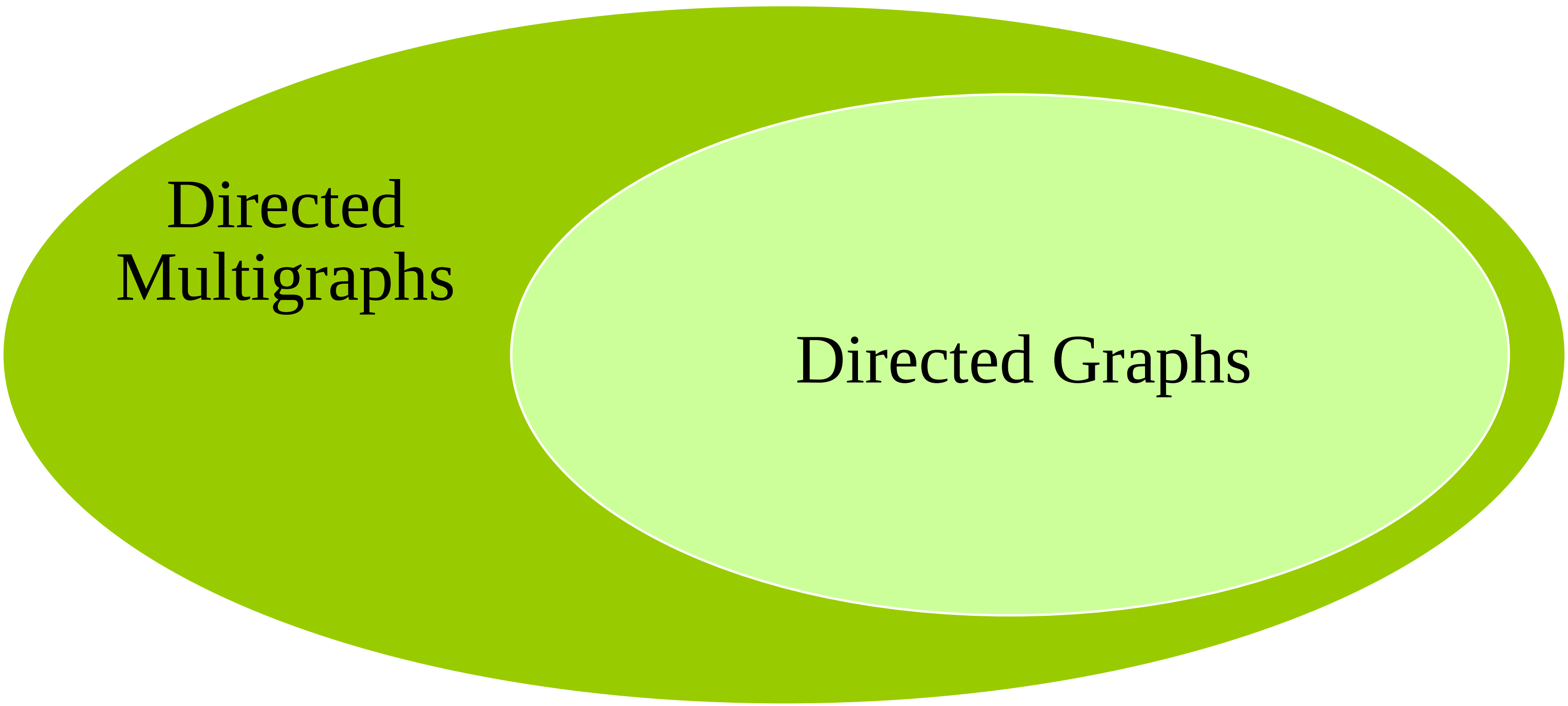
Directed Multigraph

A directed multigraph is a directed graph with multiple edges between the same two distinct vertices.



There may be several one-way lines in the same direction from one computer to another in the network.

Types of Directed Graphs



Summary

Type	Edges	Loops	Multiple Edges
Simple Graph	Undirected	NO	NO
Multigraph	Undirected	NO	YES
Pseudograph	Undirected	YES	YES
Directed Graph	Directed	YES	NO

Chapter 10.2

Graph Terminology

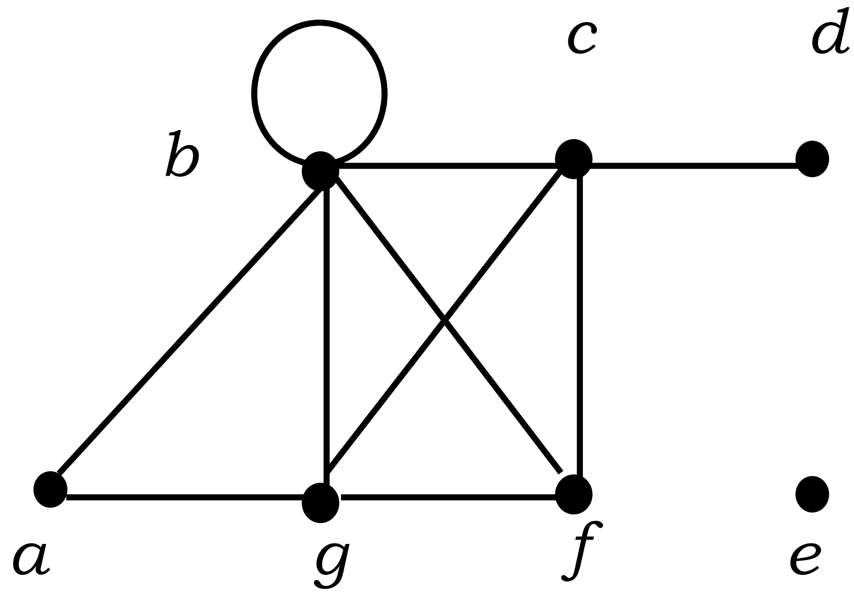
Adjacent Vertices in Undirected Graphs

- Two vertices, u and v in an undirected graph G are called *adjacent* (or neighbors) in G, if $\{u,v\}$ is an edge of G.
- An edge e connecting u and v is called *incident* with vertices u and v, or is said to connect u and v.
- The vertices u and v are called *endpoints* of edge $\{u,v\}$.

Degree of a Vertex

- The *degree of a vertex* in an undirected graph is the number of edges incident with it
  - except that a loop at a vertex contributes twice to the degree of that vertex
- The degree of a vertex v is denoted by $\deg(v)$.

Example



- Find the degrees of all the vertices:

deg(a) = 2, deg(b) = 5, deg(c) = 4, deg(d) = 1,

deg(e) = 0, deg(f) = 3, deg(g) = 4

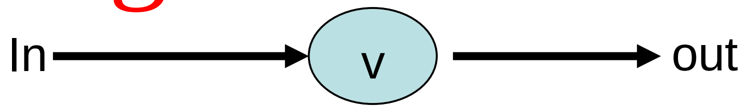
Adjacent Vertices in Directed Graphs

When (u,v) is an edge of a directed graph G,
u is said to be *adjacent to v* and
v is said to be *adjacent from u.*

(u, v)

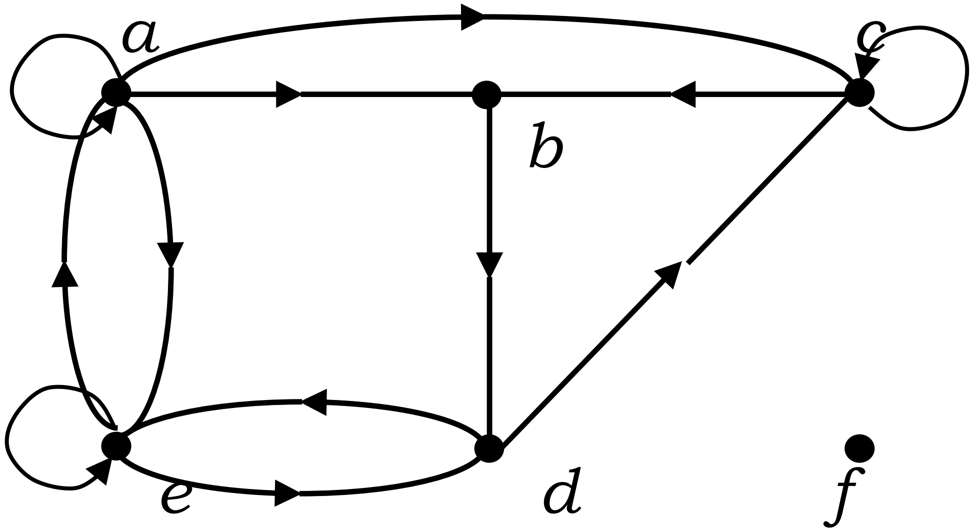
initial vertex   terminal vertex

Degree of a Vertex



- *In-degree* of a vertex v
  - The number of vertices *adjacent to* v (the number of edges with v as their terminal vertex
  - Denoted by $\deg^{-}(v)$
- *Out-degree* of a vertex v
  - The number of vertices *adjacent from* v (the number of edges with v as their initial vertex)
  - Denoted by $\deg^{+}(v)$
- A loop at a vertex contributes 1 to both the in-degree and out-degree.

Example



Find the in-degrees and out-degrees of this digraph.

In-degrees: $deg^-(a) = 2$, $deg^-(b) = 2$, $deg^-(c) = 3$, $deg^-(d) = 2$, $deg^-(e) = 3$, $deg^-(f) = 0$

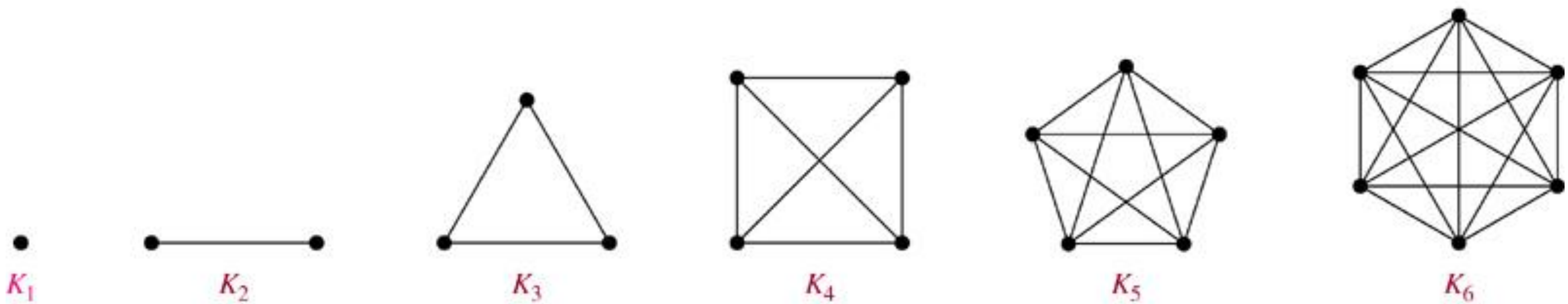
Out-degrees: $deg^+(a) = 4$, $deg^+(b) = 1$, $deg^+(c) = 2$, $deg^+(d) = 2$, $deg^+(e) = 3$, $deg^+(f) = 0$

Theorem 3

- The sum of the in-degrees of all vertices in a digraph = the sum of the out-degrees = the number of edges.
- Let $G = (V, E)$ be a graph with directed edges. Then:

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = |E|$$

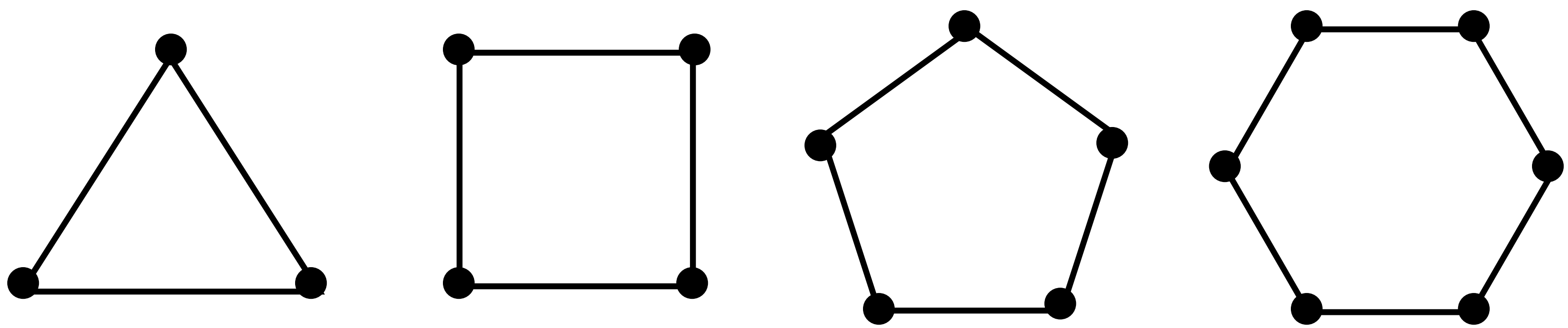
Complete Graph



- The *complete graph* on n vertices (K_n) is the simple graph that contains exactly one edge between each pair of distinct vertices.
- The figures above represent the complete graphs, K_n, for n = 1, 2, 3, 4, 5, and 6.

Cycle

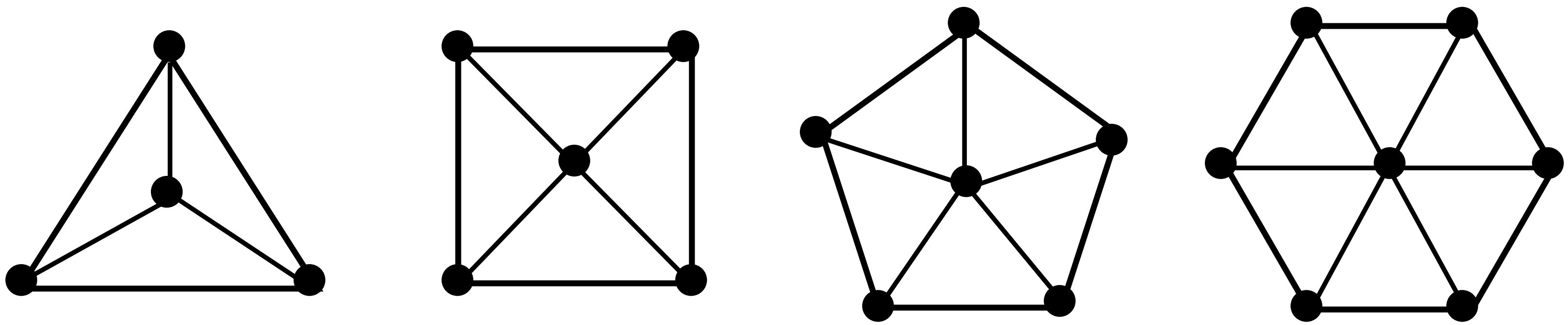
- The *cycle* $\boldsymbol{C_n}$ ($n \geq 3$), consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1,v_2\}, \{v_2,v_3\}, \ldots, \{v_{n-1},v_n\}$, and $\{v_n,v_1\}$.



Cycles: C_3 C_4 C_5 C_6

Wheel

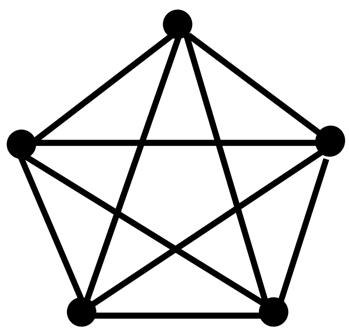
When a new vertex is added to a cycle C_n and this new vertex is connected to each of the n vertices in C_n, we obtain a *wheel* W_n.



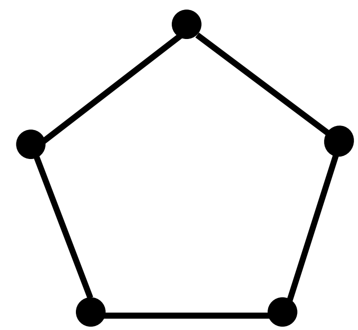
Wheels: W_3 W_4 W_5 W_6

Subgraph

- A *subgraph* of a graph $G = (V,E)$ is a graph $H = (W,F)$ where $W \subseteq V$ and $F \subseteq E$.



K_5

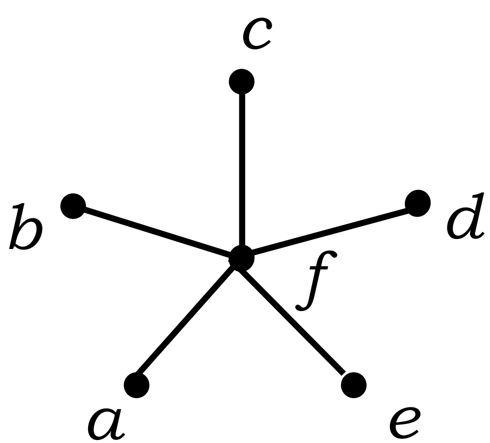


C_5

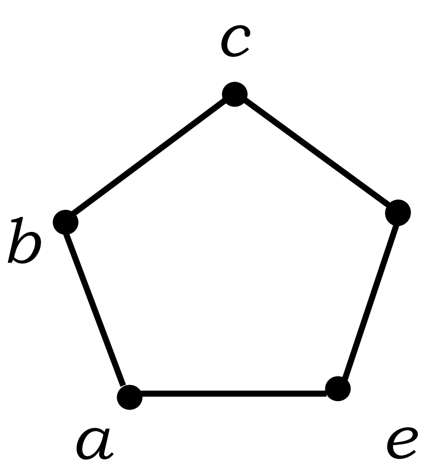
Is C_5 a subgraph of K_5?

Union

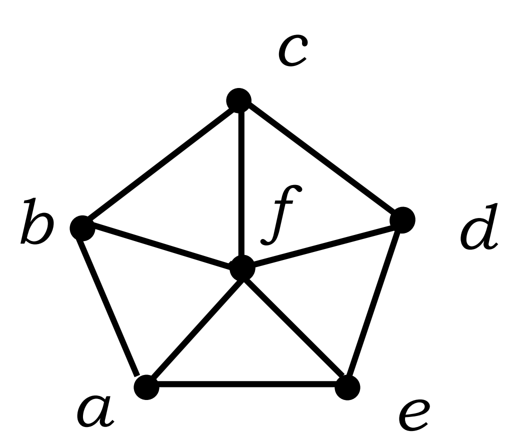
- The *union* of 2 simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. The union is denoted by $G_1 \cup G_2$.



S_5



C_5



W_5

$$S_5 \cup C_5 = W_5$$

Discrete Structures

Chapter 10.3

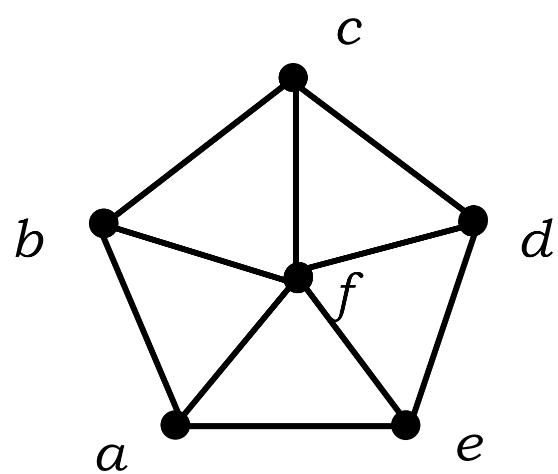
Representing Graphs and Graph Isomorphism

Adjacency Matrix

- A simple graph $G = (V,E)$ with n vertices can be represented by its *adjacency matrix*, A, where the entry a_{ij} in row i and column j is:

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge in } G \\ 0 & \text{otherwise} \end{cases}$$

Adjacency Matrix Example



W_5

$$\begin{array}{c c} & \text{To} \\ & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \text{From} & \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$\{v_1, v_2\}$

row column

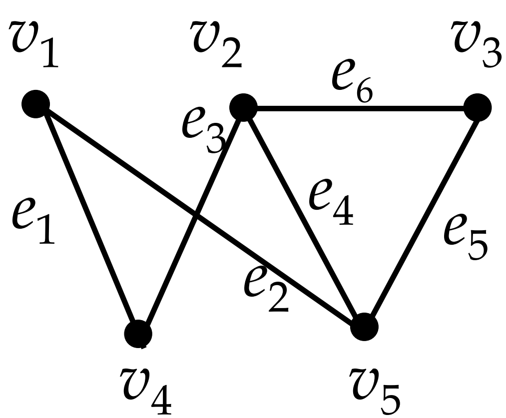
Incidence Matrix

- Let $G = (V,E)$ be an undirected graph. Suppose $v_1,v_2,v_3,\ldots,v_n$ are the vertices and $e_1,e_2,e_3,\ldots,e_m$ are the edges of G. The *incidence matrix* w.r.t. this ordering of V and E is the $n\times m$ matrix $M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$

Incidence Matrix Example

- Represent the graph shown with an incidence matrix.



$$
\begin{array}{c|cccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
\hline
v_1 & 1 & 1 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 0 & 1 & 1 & 0 & 1 \\
v_3 & 0 & 0 & 0 & 0 & 1 & 1 \\
v_4 & 1 & 0 & 1 & 0 & 0 & 0 \\
v_5 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}
$$

edges

vertices

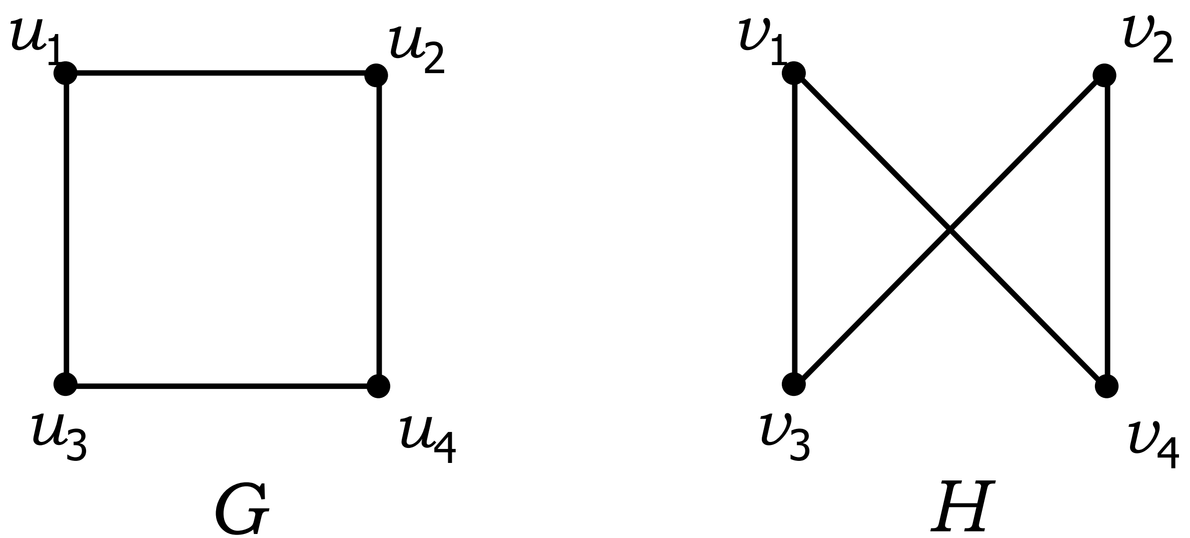
Isomorphism

- Two simple graphs are isomorphic if:
  - there is a one-to one correspondence between the vertices of the two graphs
  - the adjacency relationship is preserved

Isomorphism (Cont.)

- The simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are *isomorphic* if there is a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1.

Example



Are G and H isomorphic?

$$f(u_1) = v_1,\ \ f(u_2) = v_4,\ \ f(u_3) = v_3,\ \ f(u_4) = v_2$$

Discrete Structures

Chapter 10.4
Connectivity

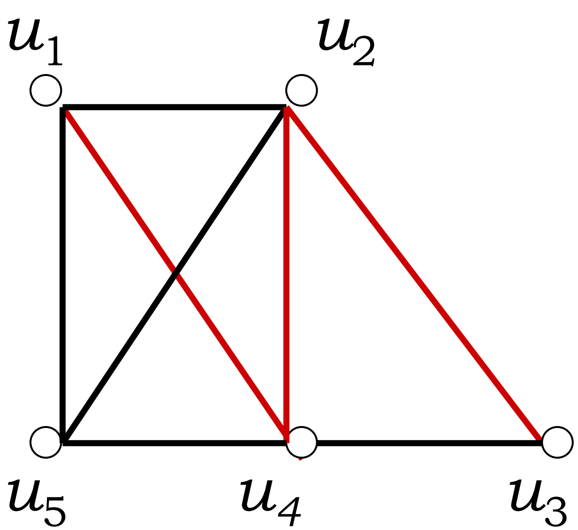
Paths in Undirected Graphs

- There is a *path* from vertex v_0 to vertex v_n if there is a sequence of edges from v_0 to v_n
  - This path is labeled as $v_0, v_1, v_2, \ldots, v_n$ and has a length of n.
- The path is a *circuit* if the path begins and ends with the same vertex.
- A path is *simple* if it does not contain the same edge more than once.

Paths in Undirected Graphs

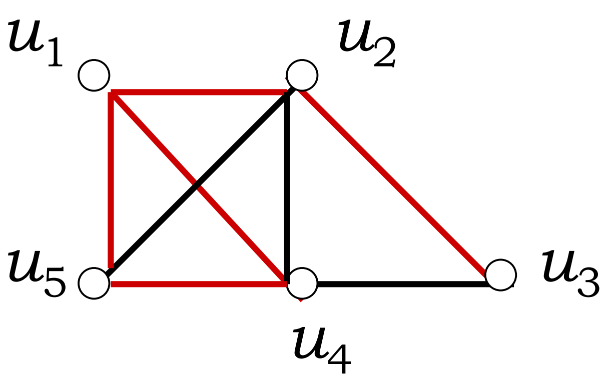
- A path or circuit is said to *pass through* the vertices $v_0, v_1, v_2, \ldots, v_n$ or *traverse* the edges $e_1, e_2, \ldots, e_\mathrm{n}$.

Example



- u_1, u_4, u_2, u_3
  - Is it simple?
  - *yes*
  - What is the length?
  - *3*
  - Does it have any circuits?
  - *no*

Example



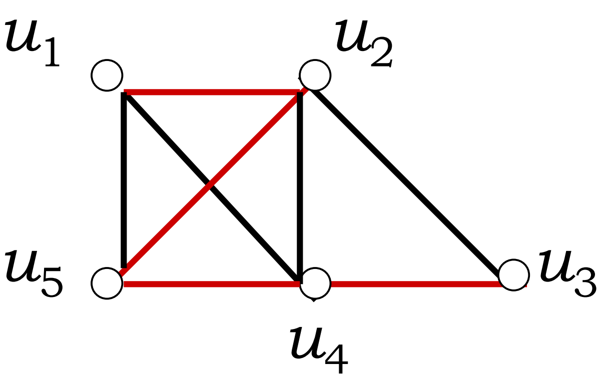
- $u_1, u_5, u_4, u_1, u_2, u_3$

  – Is it simple?
  – *yes*

  – What is the length?
  – *5*

  – Does it have any circuits?
  – Yes; u_1, u_5, u_4, u_1

Example



- u_1, u_2, u_5, u_4, u_3
  - Is it simple?
  - *yes*

  - What is the length?
  - *4*

  - Does it have any circuits?
  - *no*

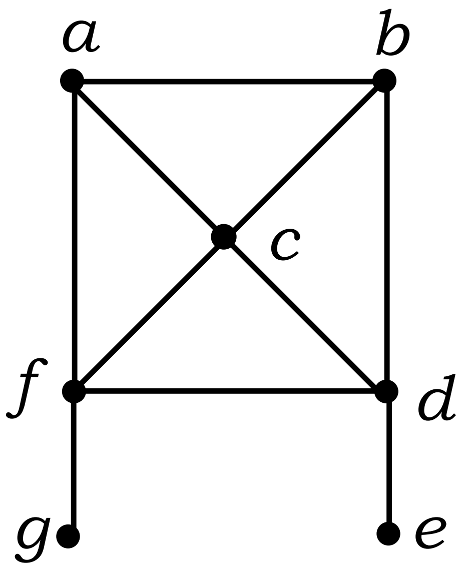
Connectedness

- An undirected graph is called *connected* if there is a path between every pair of distinct vertices of the graph.

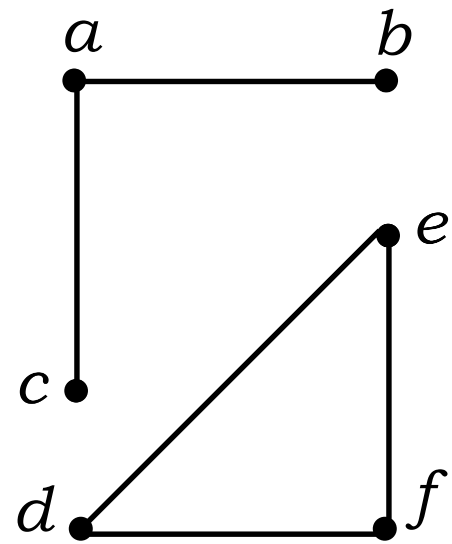
- There is a simple path between every pair of distinct vertices of a connected undirected graph.

Example

Are the following graphs connected?



Yes



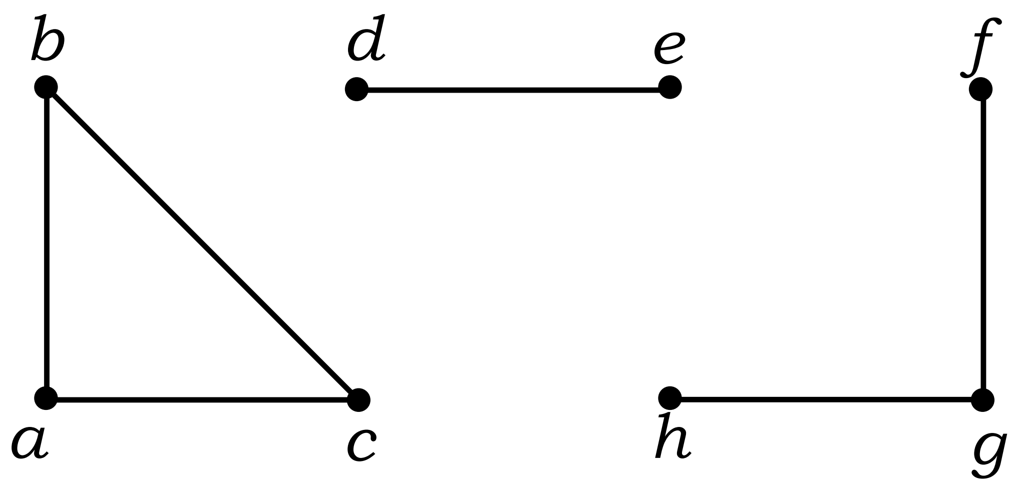
No

Connectedness (Cont.)

- A graph that is not connected is the union of two or more disjoint connected subgraphs (called the *connected components* of the graph).

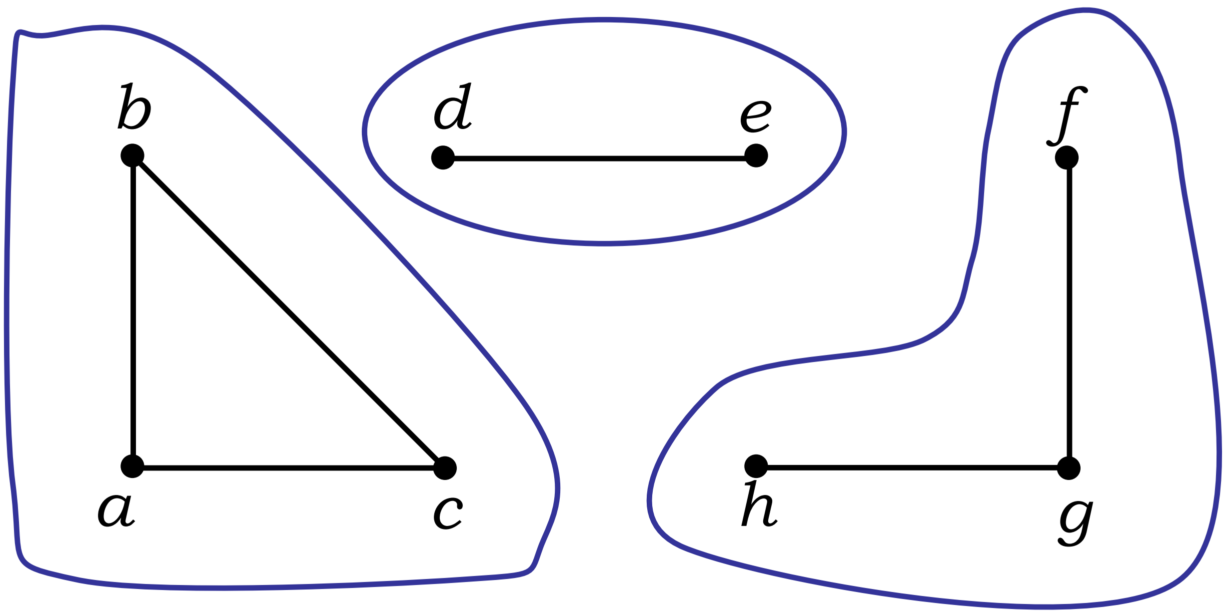
Example

- What are the connected components of the following graph?



Example

- What are the connected components of the following graph?



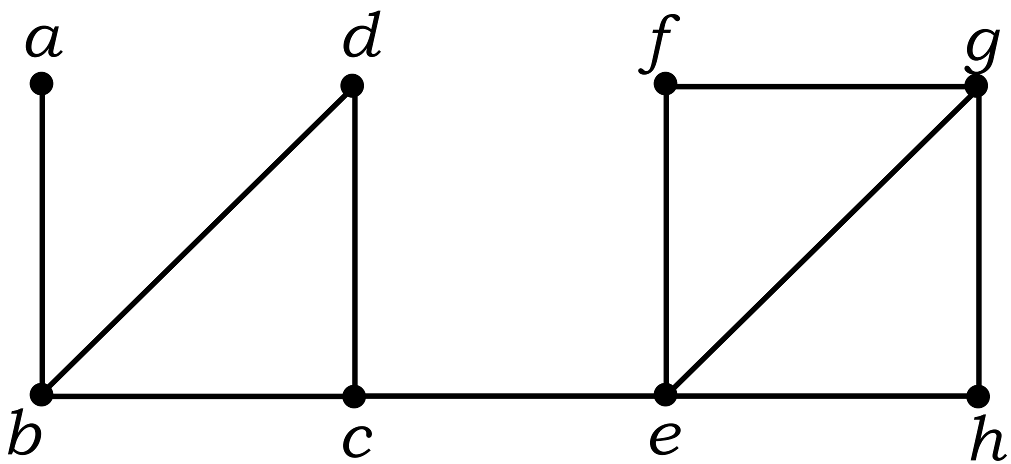
{a, b, c}, {d, e}, {f, g, h}

Cut edges and vertices

- If one can remove a vertex (and all incident edges) and produce a graph with more connected components, the vertex is called a *cut vertex*.
- If removal of an edge creates more connected components the edge is called a *cut edge* or *bridge*.

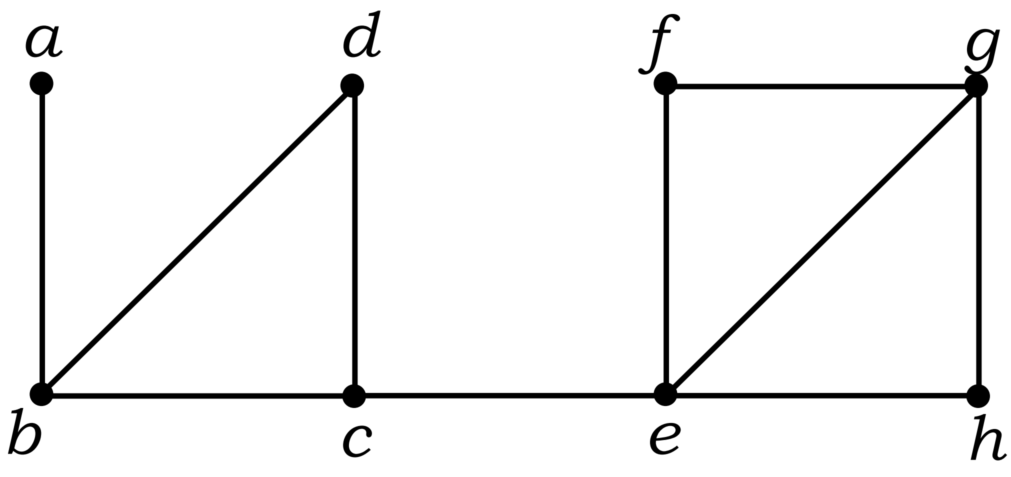
Example

- Find the cut vertices and cut edges in the following graph.



Example

- Find the cut vertices and cut edges in the following graph.



Cut vertices: c and e

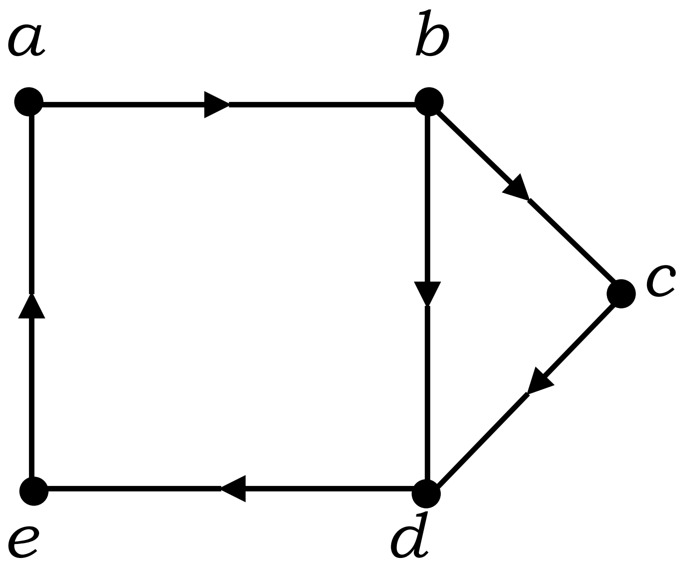
Cut edge: (c, e)

Connectedness in Directed Graphs

- A directed graph is ***strongly connected*** if there is a directed path between every pair of vertices a&b. (from a to b) (from b to a).
- A directed graph is ***weakly connected*** if there is a path between every pair of vertices in the underlying undirected graph, (i.e when the directions are disregarded).

Example

- Is the following graph strongly connected? Is it weakly connected?

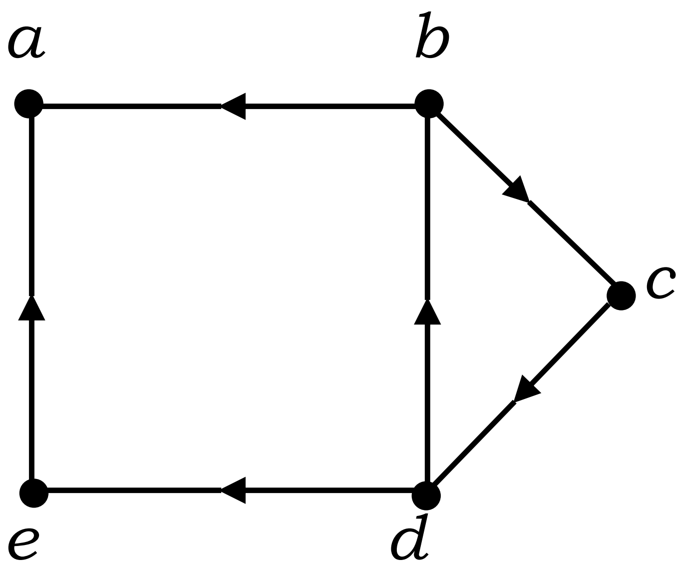


This graph is strongly connected. Why? Because there is a directed path between every pair of vertices.

If a directed graph is strongly connected, then it must also be weakly connected.

Example

- Is the following graph strongly connected? Is it weakly connected?



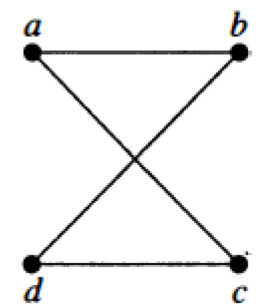
This graph is not strongly connected. Why not? Because there is no directed path between *a* and *b*, *a* and *e*, etc.

However, it *is* weakly connected. (Imagine this graph as an undirected graph.)

Counting Paths Between Vertices

Let G be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering $v_1, v_2, \ldots, v_n$ (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i, j)th entry of $\mathbf{A}^r$.

EXAMPLE



How many paths of length four are there from a to d in the simple graph G

Solution:

the number of paths of length four from a to d is the $(1, 4)$th entry of $\mathbf{A}^4$.

$$\mathbf{A}^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix},$$

there are exactly eight paths of length four from a to d. By inspection of the graph, we see that a, b, a, b, d; a, b, a, c, d; a, b, d, b, d; a, b, d, c, d; a, c, a, b, d; a, c, a, c, d; a, c, d, b, d; and a, c, d, c, d are the eight paths from a to d. ◀