Harmonic Maps and Conformal Maps

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Harmonic Maps

Harmonic Map

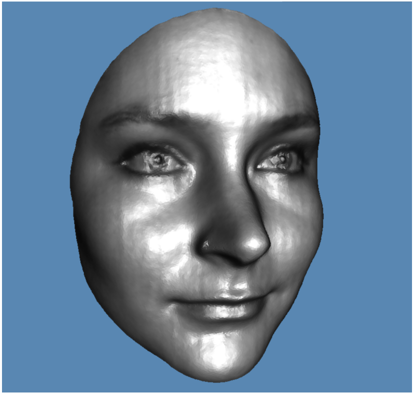


Figure: Harmonic map between topological disks.

Harmonic Map

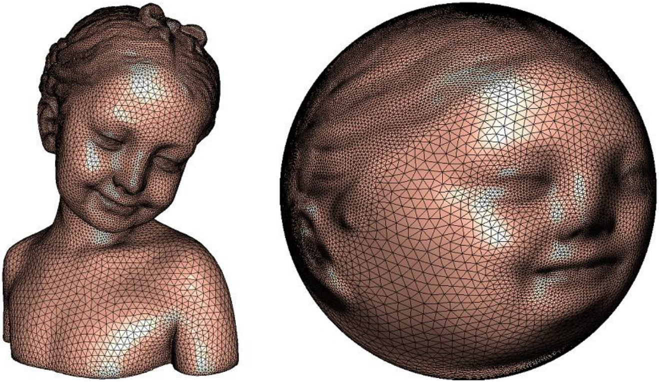


Figure: Harmonic map between topological spheres.

Harmonic Map

Figure: Harmonic map induced foliations.

Harmonic Function

Given a planar domain $\Omega \subset \mathbb{R}^2$, consider the electric potential $u : \Omega \to \mathbb{R}$. The gradient of the potential induces electric currents, and produces heat. The heat power is represented as *harmonic energy*

$$E(u) := \int_\Omega \langle \nabla u, \nabla u \rangle dxdy.$$

In nature, the distribution of u minimizes the heat power, and is called a harmonic function. Assume $h \in C_0^\infty(\Omega)$, then $E(u + \varepsilon h) \geq E(u)$,

$$\frac{d}{d\varepsilon} \int_\Omega \langle \nabla u + \varepsilon \nabla h, \nabla u + \varepsilon \nabla h \rangle dxdy \Big|_{\varepsilon=0} = 2 \int_\Omega \langle \nabla u, \nabla h \rangle dxdy = 0.$$

Harmonic Function

By relation

$$\nabla \cdot (h\nabla u) = \langle \nabla h, \nabla u\rangle + h\nabla \cdot \nabla u,$$

we obtain

$$\int_\Omega \nabla u, \nabla h\rangle = \int_\Omega h\Delta u dxdy - \int_\Omega \nabla \cdot (h\nabla u) dxdy = \int_\omega h\Delta u dxdy,$$

We obtain Laplace equation

$$\boxed{\begin{cases} \Delta u & \equiv & 0 \\ u|_{\partial\Omega} & = & g \end{cases}}$$

Steady temperature field, static electric field, elastic deformation, diffusion field, all are governed by the Laplace equation.

Harmonic Function

Theorem (Mean Value)

Assume $\Omega \subset \mathbb{R}^2$ *is a planar open set,* $u : \Omega \to \mathbb{R}$ *is a harmonic function, then for any* $p \in \Omega$

$$u(p) = \frac{1}{2\pi\varepsilon} \oint_{\gamma} u(q) ds, \tag{1}$$

where γ *is a circle centered at* p*, with radius* ε*.*

Proof.

u is harmonic, du is a harmonic 1-form, its Hodge star ${}^*du$ is also harmonic. Define the conjugate function v, $dv = {}^*du$, then $\varphi(z) := u + \sqrt{-1}v$ is holomorphic. By Cauchy integration formula,

$$\varphi(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{\varphi(\zeta)}{\zeta - z} dz \tag{2}$$

Hence, we obtain the mean value property of harmonic function. □

Harmonic Function

Corollary (Maximal value principle)

Assume $\Omega \subset \mathbb{R}^2$ is a planar domain, and $u : \overline{\Omega} \to \mathbb{R}$ is a non-constant harmonic function, then u can't reach extremal values in the interior of Ω.

Proof.

Assume p is an interior point in Ω, p is a maximal point of u, $u(p) = C$. By mean value property, we obtain for any point q on the circle $B(p, \varepsilon)$, $u(q) = C$, where ε is arbitrary, therefore u is constant in a neighborhood of p. Therefore $u^{-1}(C)$ is open. On the other hand, u is continuous, $u^{-1}(C)$ is closed, hence $u^{-1}(C) = \Omega$. Contradiction. $\square$

Uniqueness of Harmonic Functions

Corollary

Suppose $\Omega \subset \mathbb{R}^2$ is a planar domain, $u_1, u_2 : \Omega \to \mathbb{R}$ are harmonic functions with the same boundary value, $u_1|_{\partial\Omega} = u_2|_{\partial\Omega}$, then $u_1 = u_2$ on Ω.

Proof.

$u_1 - u_2$ is also harmonic, with 0 boundary value, therefore the maximal and minimal values of $u_1 - u_2$ must be on the boundary, namely they are 0, hence u_1, u_2 are equal in Ω. $\square$

Disk Harmonic Maps

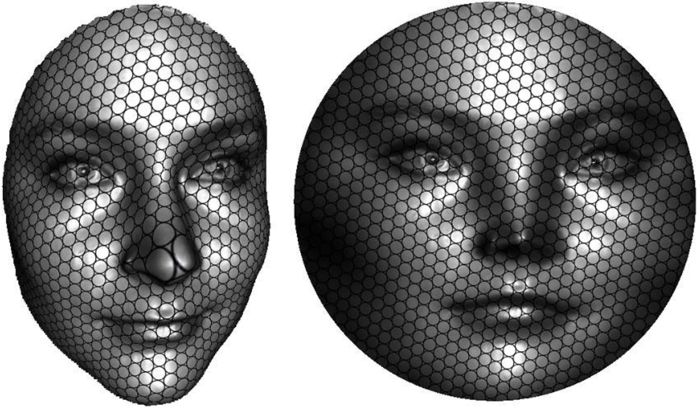


Figure: Harmonic map between topological disks.

Diffeomorphic Property of Disk Harmonic Maps

Theorem (Rado)

Suppose a harmonic map $\varphi : (S, \mathbf{g}) \to (\Omega, dx^2 + dy^2)$ *satisfies:*

1. *planar domain* Ω *is convex*
2. *the restriction of* $\varphi : \partial S \to \partial \Omega$ *on the boundary is homoemorphic,*

then u *is diffeomorphic in the interior of* S.

Proof.

By regularity theory of harmonic maps, we get the smoothness of the harmonic map. Assume $\varphi : (x, y) \to (u, v)$ is not homeomorphic, then there is an interior point $p \in \Omega$, the Jacobian matrix of φ is degenerated as p, there are two constants $a, b \in \mathbb{R}$, not being zeros simultaneously, such that

$$a\nabla u(p) + b\nabla v(p) = 0.$$

By $\Delta u = 0, \Delta v = 0$, the auxiliary function $f(q) = au(q) + bv(q)$ is also harmonic. □

Diffeomorphic Property of Disk Harmonic Maps

continued

By $\nabla f(p) = 0$, p is an saddle point of f. Consider the level set of f near p

$$\Gamma = \{q \in \Omega | f(q) = f(p) - \varepsilon\}$$

Γ has two connected components, intersecting ∂S at 4 points.
But Ω is a planar convex domain, $\partial\Omega$ and the line $au + bv = const$ have two intersection points. By assumption, the mapping φ restricted on the boundary $\varphi : \partial S \to \partial\Omega$ is homeomorphic. Contradiction.

Computational Algorithm for Disk Harmonic Maps

Input:A topological disk M;
Output:A harmonic map $\varphi : M \to \mathbb{D}^2$

1. Construct boundary map to the unit circle, $g : \partial M \to \mathbb{S}^1$, g should be a homeomorphism;
2. Compute the cotangent edge weight;
3. for each interior vertex $v_i \in M$, compute Laplacian
$$\Delta\varphi(v_i) = \sum_{v_j \sim v_i} w_{ij}(\varphi(v_i) - \varphi(v_j)) = 0;$$
4. Solve the linear system, to obtain φ.

General Harmonic Map

Definition (Harmonic Energy)

Let (Σ_1, z) and (Σ_2, u) be two Riemann surfaces, with Riemannian metrics $\sigma(z)dzd\bar{z}$ and $\rho(u)dud\bar{u}$. Given a C^1 map $u : \Sigma_1 \to \Sigma_2$, then the harmonic energy of u is defined as

$$E(z, \rho, u) := \int_{\Sigma_1} \rho^2(u)(u_z \bar{u}_{\bar{z}} + \bar{u}_z u_{\bar{z}}) \frac{i}{2} dz d\bar{z}$$

where $u_z := \frac{1}{2}(u_x - iu_y)$, $u_{\bar{z}} := \frac{1}{2}(u_x + iu_y)$ and $dz \wedge d\bar{z} = -2idx \wedge dy$.

Definition (Harmonic Map)

If the C^1 map $u : \Sigma_1 \to \Sigma_2$ minimizes the harmonic energy, then u is called a harmonic map.

General Harmonic Map

Theorem (Euerl-Larange Equation for Harmonic Maps)

Suppose $u:\Sigma_1\to\Sigma_2$ is a C^2 harmonic map between Riemannian surfaces, then

$$\boxed{u_{z\bar{z}}+\frac{2\rho_u}{\rho}u_z u_{\bar{z}}=0}$$

Geodesics are special harmonic maps, harmonic maps are generalized geodesics:

$$\boxed{\ddot{\gamma}+\frac{2\rho_\gamma}{\rho}\dot{\gamma}^2\equiv 0 \quad u_{z\bar{z}}+\frac{2\rho_u}{\rho}u_z u_{\bar{z}}\equiv 0}$$

Proof.

Suppose u is harmonic, u_t is a variation in a local coordinates system,

$$u + t\varphi, \quad \varphi \in C^0 \cap W_0^{1,2}(\Sigma_1, \Sigma_2)$$

we obtain

$$\frac{d}{dt} E(u + t\varphi)\Big|_{t=0} = 0,$$

□

continued

$$
\begin{aligned}
0 =& \frac{d}{dt}\left\{ \int \rho^2(u+t\varphi)((u+t\varphi)_z(\bar{u}+t\bar{\varphi})_{\bar{z}} \right.\\
&\left.\left. + (\bar{u}+t\bar{\varphi})_z(u+t\varphi)_{\bar{z}}) idzd\bar{z} \right\}\right|_{t=0}\\
=& \int \left\{ \rho^2(u)(u_z\bar{\varphi}_{\bar{z}} + \bar{u}_{\bar{z}}\varphi_z + \bar{u}_z\varphi_{\bar{z}} + u_{\bar{z}}\bar{\varphi}_z) \right.\\
&\left. + 2\rho(\rho_u\varphi + \rho_{\bar{u}}\bar{\varphi})(u_z\bar{u}_{\bar{z}} + \bar{u}_z u_{\bar{z}}) \right\} idzd\bar{z}.
\end{aligned}
$$

continued

We set $\varphi = \frac{\psi}{\rho^2(u)}$,

$$\rho^2\varphi_z = \psi_z - \frac{2\psi}{\rho}(\rho_u u_z + \rho_{\bar{u}}\bar{u}_z)$$

$$\rho^2\varphi_{\bar{z}} = \psi_{\bar{z}} - \frac{2\psi}{\rho}(\rho_u u_{\bar{z}} + \rho_{\bar{u}}\bar{u}_{\bar{z}})$$

$$\rho^2\bar{\varphi}_z = \bar{\psi}_z - \frac{2\bar{\psi}}{\rho}(\rho_u u_z + \rho_{\bar{u}}\bar{u}_z)$$

$$\rho^2\bar{\varphi}_{\bar{z}} = \bar{\psi}_{\bar{z}} - \frac{2\bar{\psi}}{\rho}(\rho_u u_{\bar{z}} + \rho_{\bar{u}}\bar{u}_{\bar{z}})$$

General Harmonic Map

continued

$$
\begin{aligned}
\bar{u}_{\bar{z}}\rho^2\varphi_z &= \psi_z\bar{u}_{\bar{z}} - \frac{2\psi}{\rho}(\rho_u u_z\bar{u}_{\bar{z}} + \rho_{\bar{u}}\bar{u}_z\bar{u}_{\bar{z}})\\
\bar{u}_z\rho^2\varphi_{\bar{z}} &= \psi_{\bar{z}}\bar{u}_z - \frac{2\psi}{\rho}(\rho_u u_{\bar{z}}\bar{u}_z + \rho_{\bar{u}}\bar{u}_{\bar{z}}\bar{u}_z)\\
u_{\bar{z}}\rho^2\bar{\varphi}_z &= \bar{\psi}_z u_{\bar{z}} - \frac{2\bar{\psi}}{\rho}(\rho_{\bar{u}}\bar{u}_z u_{\bar{z}} + \rho_u u_z u_{\bar{z}})\\
u_z\rho^2\bar{\varphi}_{\bar{z}} &= \bar{\psi}_{\bar{z}} u_z - \frac{2\bar{\psi}}{\rho}(\rho_{\bar{u}}\bar{u}_{\bar{z}} u_z + \rho_u u_{\bar{z}} u_z)
\end{aligned}
$$

General Harmonic Map

continued

$$\frac{2}{\rho}(\rho_u\psi + \rho_{\bar{u}}\bar{\psi})(u_z\bar{u}_{\bar{z}} + \bar{u}_z u_{\bar{z}})$$
$$= \frac{2\psi}{\rho}\rho_u(u_z\bar{u}_{\bar{z}} + \bar{u}_z u_{\bar{z}}) + \frac{2\bar{\psi}}{\rho}\rho_{\bar{u}}(\bar{u}_z u_{\bar{z}} + u_z\bar{u}_{\bar{z}})$$

Take summation,

$$\bar{u}_{\bar{z}}\rho^2\varphi_z + u_z\rho^2\bar{\varphi}_{\bar{z}} = \left(\psi_z\bar{u}_{\bar{z}} - \frac{2\psi}{\rho}\rho_{\bar{u}}\bar{u}_z\bar{u}_{\bar{z}}\right) + \left(\bar{\psi}_{\bar{z}}u_z - \frac{2\bar{\psi}}{\rho}\rho_u u_{\bar{z}}u_z\right)$$
$$\bar{u}_z\rho^2\varphi_{\bar{z}} + u_{\bar{z}}\rho^2\bar{\varphi}_z = \left(\psi_{\bar{z}}\bar{u}_z - \frac{2\psi}{\rho}\rho_{\bar{u}}\bar{u}_{\bar{z}}\bar{u}_z\right) + \left(\bar{\psi}_z u_{\bar{z}} - \frac{2\bar{\psi}}{\rho}\rho_u u_z u_{\bar{z}}\right)$$

continued

The above equation becomes

$$
\begin{aligned}
0 =& 2\Re \int \left(\bar{\psi}_{\bar{z}} u_z - \frac{2\bar{\psi}}{\rho} \rho_u u_{\bar{z}} u_z \right) idzd\bar{z} \\
&+ 2\Re \int \left(\psi_{\bar{z}} \bar{u}_z - \frac{2\psi}{\rho} \rho_{\bar{u}} \bar{u}_{\bar{z}} \bar{u}_z \right) idzd\bar{z}
\end{aligned}
$$

If $u \in C^2$, we can integrate by parts, $(u_z\bar{\psi})_{\bar{z}} = u_{z\bar{z}}\bar{\psi} + u_z\bar{\psi}_{\bar{z}}$,

$$
\begin{aligned}
0 =& 2\Re \int \left(u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_{\bar{z}} u_z \right) \bar{\psi} idzd\bar{z} \\
&+ 2\Re \int \left(\bar{u}_{z\bar{z}} + \frac{2\rho_{\bar{u}}}{\rho} \bar{u}_{\bar{z}} \bar{u}_z \right) \psi idzd\bar{z}
\end{aligned}
$$

General Harmonic Map

continued

Therefore

$$0 = 2\Re \int \left(u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_{\bar{z}} u_z \right) \bar{\psi} i dz d\bar{z}$$

$\square$

Hopf Differential of Harmonic Maps

Theorem (Hopf Diffential of Harmonic Maps)

Let $u : (\Sigma_1, \lambda^2(z)dzd\bar{z}) \to (\Sigma_2, \rho^2(u)dud\bar{u})$ *is harmonic, then the Hopf differential of the map*

$$\boxed{\Phi(u) := \rho^2 u_z \bar{u}_z dz^2}$$

is holomorphic quadratic differential on Σ_1*. Furthermore* $\Phi(u) \equiv 0$*, if and only if* u *is holomorphic or anti-holomorprhic.*

Proof.

If u is harmonic, then

$$\begin{aligned}\frac{\partial}{\partial \bar{z}}(\rho^2 u_z \bar{u}_z) &= \rho^2 u_{z\bar{z}} \bar{u}_z + \rho^2 u_z \bar{u}_{z\bar{z}} + 2\rho\rho_u u_{\bar{z}} u_z \bar{u}_z + 2\rho\rho_{\bar{u}} \bar{u}_{\bar{z}} u_z \bar{u}_z \\ &= (\rho^2 u_{z\bar{z}} + 2\rho\rho_u u_{\bar{z}} u_z)\bar{u}_z + (\rho^2 \bar{u}_{z\bar{z}} + 2\rho\rho_{\bar{u}} \bar{u}_{\bar{z}} \bar{u}_z) u_z = 0.\end{aligned}$$

Therefore $\Phi(u)$ is holomorphic. □

Hopf Differential of Harmonic Maps

Proof.

If $\Phi(u) = \rho^2 u_z \bar{u}_z \equiv 0$, then either $u_z = 0$ or $\bar{u}_z = 0$. Since the Jacobian determinant equals to

$$|u_z|^2 - |u_{\bar{z}}|^2 > 0,$$

therefore $\bar{u}_z = 0$, namely $u_{\bar{z}} = 0$, u is holomorphic or anti-holomorphic. u is holomorphic, equivalent to $L \equiv 0$; u is anti-holomorphic, equivalent to $H \equiv 0$. We know H and L have isolated zeros, unless they are zero everywhere. Hence u is entirely holomorphic or anti-holomorphic. □

Spherical Harmonic Map

Lemma

A holomorphic quadratic differential ω is on the unit sphere, then ω is zero.

Proof.

Choose two charts z and $w = \frac{1}{z}$. Let $\omega = \varphi(z)dz^2$, then

$$\varphi(z)dz^2 = \varphi\left(\frac{1}{w}\right)\left(\frac{dz}{dw}\right)^2 dw^2 = \varphi\left(\frac{1}{w}\right)\frac{1}{w^4}dw^2.$$

since ω is globally holomorphic, when $w \to 0$,

$$\varphi\left(\frac{1}{w}\right)\frac{1}{w^4} < \infty,$$

hence $z \to \infty$, $\varphi(z) \to 0$. By Liouville theorem, $\varphi \equiv 0$. $\square$

Spherical Harmonic Map

Theorem (Spherical Harmonic Maps)

Harmonic maps between genus zero closed metric surfaces must be conformal.

Proof.

Suppose $u : \Sigma_1 \to \Sigma_2$ is a harmonic map, then $\Phi(u)$ must be a holomorphic quadratic differential. Since Σ_1 is of genus zero, therefore $\Phi(u) \equiv 0$. Hence u is holomorphic. $\square$

Spherical Harmonic Map

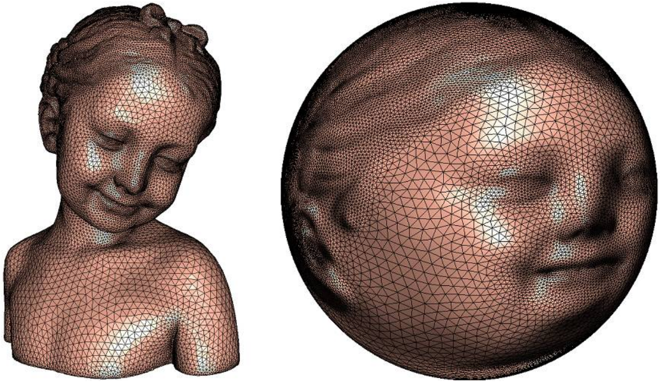


Figure: Spherical Harmonic Map

Uniqueness Spherical Harmonic Map

Definition (Möbius Transformation)

A Möbius transformation $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has the form

$$z \mapsto \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, \quad ad-bc=1.$$

Given $\{z_0, z_1, z_2\}$, there is a unique Möbius transformation, that maps them to $\{0, 1, \infty\}$,

$$z \mapsto \frac{z-z_0}{z-z_2}\frac{z_1-z_2}{z_1-z_0}.$$

Theorem (Uniquess of Spherical Conformal Automorphisms)

Suppose $f : \mathbb{S}^2 \to \mathbb{S}^2$ is a biholomorphic automorphism, then f must be a Möbius transformation.

Uniqueness of Spherical Harmonic Map

Proof.

By stereo-graphic projection, we map the sphere to the extened complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. First, the poles of f must be finite. Suppose there are infinite poles of f, because $\mathbb{S}^2$ is compact, there must be accumulation points, then f must be a constant value function.

Let $z_1, z_2, \dots, z_n$ be the finite poles of f, with degrees $e_1, e_2, \dots, e_n$. Let $g = \Pi_i (z - z_i)^{e_i}$, then fg is a holomorphic function on $\mathbb{C}$, therefore fg is entire, namely, fg is a polynomial. Therefore

$$f = \frac{\sum_{i=1}^{n} a_i z^i}{\sum_j b_j z^j},$$

if $n > 1$ then f has multiple zeros, contradict to the condition that f is an automorphism. Therefore $n = 1$. Similarly $m = 1$. □

Computational Algorithm for Spherical Harmonic Map

Input: A genus zero closed mesh M;
Output: A spherical harmonic map $\varphi : M \to \mathbb{S}^2$;

1. Compute Gauss map $\varphi : M \to \mathbb{S}^2$, $\varphi(v) \leftarrow \mathbf{n}(v)$;
2. Compute the cotangent edge weight, compute Laplacian
$$\Delta\varphi(v_i) = \sum_{v_i \sim v_j} w_{ij}(\varphi(v_j) - \varphi(v_i)),$$
3. project the Laplacian to the tangent plane,
$$D\varphi(v_i) = \Delta\varphi(v_i) - \langle \Delta\varphi(v_i), \varphi(v_i) \rangle \varphi(v_i)$$
4. for each vertex, $\varphi(v_i) \leftarrow \varphi(v_i) - \lambda D\varphi(v_i)$;
5. compute the mass center $c = \sum A_i\varphi(v_i)/\sum_j A_j$; normalize $\varphi(v_i) = \varphi(v_i) - c/|\varphi(v_i) - c|$;
6. Repeat step 2 through 5, until the Laplacian norm is less than ε.

General theory for Surface Harmonic Maps

Existence of Harmonic Map

Theorem (Existence of Harmonic Maps)

Assume Σ is a Riemann surface, $(N, \rho(u)dud\bar{u})$ is a metric surface, then for any smooth mapping $\varphi : \Sigma \to N$, there is a harmonic map $f : \Sigma \to N$ homotopic to φ.

The can be proven using Courant-Leesgue lemma, which controls the geodesic distance between image points by harmonic energy.

Regularity of Harmonic Map

Theorem (Regularity of Harmonic Maps)

Let $u : \Sigma_1 \to \Sigma_2$ be a (weak) harmonic map between Riemann surfaces, Σ_2 is with hyperbolic metric, the harmonic energy of u is finite, then u is a smooth map.

This is based on the regularity theory of ellptic PDEs.

Diffeomorphic Properties of Harmonic Maps

Theorem (Diffeomorphic Properties of Harmonic Maps)

Let Σ_1 and Σ_2 are compact Riemann surfaces with the same genus, $K_2 \leq 0$. If $u : \Sigma_1 \to \Sigma_2$ is a degree one harmonic map, then u is a diffeomorphism.

Uniqueness of Harmonic Map

Theorem (Uniqueness of Harmonic Map)

Suppose Σ_1 and Σ_2 are compact Riemann surface, Σ_2 is with hyperbolic metric. $u_0, u_1 : \Sigma_1 \to \Sigma_2$ are homotopic harmonic maps. If one of the Jacobian matrix is non-degenerated at a point, then $u_0 \equiv u_1$.

Uniqueness of Harmonic Map

Theorem

Suppose Σ_1 and Σ_2 are Riemann surfaces, the Riemannian metric on Σ_2 induces non-positive curvature K. Let $u \in C^2(\Sigma_1, \Sigma_2)$, $\varphi(z,t)$ is the variation of u, $\dot{\varphi} \neq 0$. If u is harmonic, or for any point $z \in \Sigma_1$, $\varphi(z_1, \cdot)$ is geodesic, then

$$\frac{d}{dt^2} E(u + \varphi(t))\Big|_{t=0} \geq 0. \tag{3}$$

If $K < 0$, then either

$$\frac{d}{dt^2} E(u + \varphi(t))\Big|_{t=0} > 0. \tag{4}$$

or

$$u_z \bar{u}_{\bar{z}} - u_{\bar{z}} \bar{u}_z \equiv 0, \tag{5}$$

Namely the rank of u is ≤ 1 everywhere.

Consider the variation of the mapping u, $u(z)+\varphi(z,t)$, where $\varphi(z,0)\equiv 0$. Let $\dot{\varphi}=\frac{\partial}{\partial t}\varphi$, $\ddot{\varphi}:=\frac{\partial^2}{\partial t^2}\varphi$. $K=-\Delta\log\rho=-\frac{4}{\rho^4}(\rho\rho_{u\bar{u}}-\rho_u\rho_{\bar{u}})$

$$
\begin{aligned}
\frac{d^2}{dt^2}E(u+\varphi(t))\Big|_{t=0} = 2\int \Bigg\{ & \rho^2\left(\dot{\varphi}_z+2\frac{\rho_u}{\rho}u_z\dot{\varphi}\right)\left(\dot{\bar{\varphi}}_{\bar{z}}+2\frac{\rho_{\bar{u}}}{\rho}\bar{u}_{\bar{z}}\dot{\bar{\varphi}}\right)\\
&+\rho^2\left(\dot{\bar{\varphi}}_z+2\frac{\rho_{\bar{u}}}{\rho}\bar{u}_z\dot{\bar{\varphi}}\right)\left(\dot{\varphi}_{\bar{z}}+2\frac{\rho_u}{\rho}u_{\bar{z}}\dot{\varphi}\right)\\
&-\rho^4\frac{K}{2}(u_z\dot{\bar{\varphi}}-\bar{u}_z\dot{\varphi})(\bar{u}_{\bar{z}}\dot{\varphi}-u_{\bar{z}}\dot{\bar{\varphi}})\\
&-(\rho^2\ddot{\varphi}+2\rho\rho_u\dot{\varphi}^2)\left(\bar{u}_{z\bar{z}}+\frac{2\rho_{\bar{u}}}{\rho}\bar{u}_z\bar{u}_{\bar{z}}\right)\\
&-(\rho^2\ddot{\bar{\varphi}}+2\rho\rho_{\bar{u}}\dot{\bar{\varphi}}^2)\left(u_{z\bar{z}}+\frac{2\rho_u}{\rho}u_zu_{\bar{z}}\right)\Bigg\}\, idzd\bar{z}
\end{aligned}
\tag{6}
$$

Uniqueness of Harmonic Map

If u is harmonnic, then

$$u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_z u_{\bar{z}} = 0,$$

or if $\varphi(z,\cdot)$ is geodesic, then

$$\rho^2 \ddot{\varphi} + 2\rho\rho_u \dot{\varphi}^2 = 0.$$

Then, the last two items vanish. Since $K \leq 0$, there first three items are non-negative.
If $K < 0$, then $\frac{d^2}{dt^2} E(u + \varphi(t))|_{t=0}$ is either positive or zero. If it is 0, then the integrands must be 0 everywhere, therefore

$$u_z \dot{\bar{\varphi}} - \bar{u}_z \dot{\varphi} \equiv \bar{u}_{\bar{z}} \dot{\varphi} - u_{\bar{z}} \dot{\bar{\varphi}} \equiv 0. \tag{7}$$

Uniqueness of Harmonic Map

Furthermore

$$\frac{\partial}{\partial z}(\rho^2\dot{\varphi}\dot{\bar{\varphi}}) = (\rho^2\dot{\varphi}_z + 2\rho\rho_u u_z\dot{\varphi})\dot{\bar{\varphi}} + (\rho^2\dot{\bar{\varphi}} + 2\rho\rho_u \bar{u}_z\dot{\bar{\varphi}})\dot{\varphi} = 0. \tag{8}$$

Similarly

$$\frac{\partial}{\partial \bar{z}}(\rho^2\dot{\varphi}\dot{\bar{\varphi}}) = 0. \tag{9}$$

We obtain

$$\rho^2\dot{\varphi}\dot{\bar{\varphi}} \equiv \text{const.} \tag{10}$$

By assumption $\dot{\varphi} \neq 0$, the constant is non-zero, hence $\dot{\varphi}$ and $\dot{\bar{\varphi}}$ are non-zero everywhere, by (7) we get

$$|u_z||\dot{\varphi}| = |\bar{u}_z||\dot{\bar{\varphi}}|$$

hence

$$|u_z| = |\bar{u}_z| = |u_{\bar{z}}|$$

we get (5). □

Uniqueness of Harmonic Map

Theorem (Uniqueness)

Suppose Σ_1 and Σ_2 are compact Riemann surface, Σ_2 is with hyperbolic metric. $u_0, u_1 : \Sigma_1 \to \Sigma_2$ are homotopic harmonic maps. If one of the Jacobian matrix is non-degenerated at a point, then $u_0 \equiv u_1$.

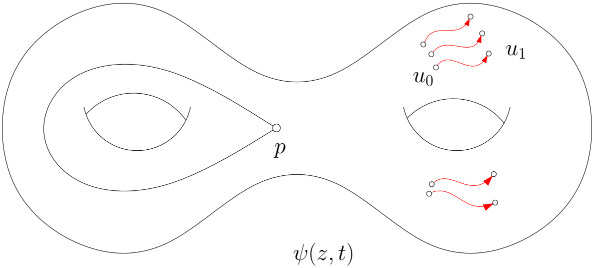
Proof.

Given a homotopy connecting u_0 and u_1, $h(z,t) : \Sigma_1 \times [0,t] \to \Sigma_2$, such that $h(z,0) = u_0(z)$, $h(z,1) = u_1(z)$. Let $\psi(z,t)$ is a geodesic from $u_0(z)$ to $u_1(z)$ and homotopic to $h(z,t)$, with parameter

$$\rho(\psi(z,t))|\dot{\psi}(z,t)| \equiv \text{const}$$

then $u_t(z) := \psi(z,t)$ is also a homotopy connecting u_0 and u_1. □

Uniqueness of Harmonic Map



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We define function $f(t) := E(u_t)$. By above theorem, $\forall t \in [0,1]$, $\ddot{f}(t) \geq 0$, hence $f(t)$ is convex. Since u_0 and u_1 are harmonic, $\dot{f}(0) = \dot{f}(1) = 0$. By the assumption of the Jacobian matrix, either $\ddot{f}(0) > 0$ or $\ddot{f}(1) > 0$, hence we must have $\dot{\psi}(t) \equiv 0$, namely $u_0 \equiv u_1$. $\square$