Geodesics and Harmonic Maps

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Computation under Isothermal Coordinates

Isothermal Coordinates

Lemma (Isothermal Coordinates)

Let $(S, \mathbf{g})$ *be a metric surface, use isothermal coordinates*

$$\mathbf{g} = e^{2u(x,y)}(dx^2 + dy^2).$$

Then we obtain

$$\boxed{\omega_1 = e^u dx \quad \omega_2 = e^u dy}$$

and the orthonormal frame is

$$\boxed{\mathbf{e_1} = e^{-u}\partial_x \quad \mathbf{e_2} = e^{-u}\partial_y}$$

and the connection

$$\boxed{\omega_{12} = -u_y dx + u_x dy}$$

Gaussian Curvature

Proof.

By direct computation, $ds^2 = \omega_1^2 + \omega_2^2$,

$$
\begin{aligned}
d\omega_1 &= de^u \wedge dx \\
&= e^u(u_x dx + u_y dy) \wedge dx \\
&= e^u u_y dy \wedge dx
\end{aligned}
\qquad
\begin{aligned}
d\omega_2 &= de^u \wedge dy \\
&= e^u(u_x dx + u_y dy) \wedge dy \\
&= e^u u_x dx \wedge dy.
\end{aligned}
$$

therefore

$$
\begin{aligned}
\omega_{12} &= \frac{d\omega_1}{\omega_1 \wedge \omega_2}\omega_1 + \frac{d\omega_2}{\omega_1 \wedge \omega_2}\omega_2 \\
&= \frac{e^u u_y dy \wedge dx}{e^{2u} dx \wedge dy} e^u dx + \frac{e^u u_x dx \wedge dy}{e^{2u} dx \wedge dy} e^u dy \\
\omega_{12} &= -u_y dx + u_x dy.
\end{aligned}
$$

□

Lemma (Gaussian curvature)

Under the isothermal coordinates, the Gaussian curvautre is given by

$$\boxed{K = -\frac{1}{e^{2u}}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u.}$$

Proof.

From

$$\omega_{12} = -u_y dx + u_x dy$$

we get

$$K = -\frac{d\omega_{12}}{\omega_1\wedge\omega_2} = -\frac{(u_{xx}+u_{yy})dx\wedge dy}{e^{2u}dx\wedge dy} = -\frac{1}{e^{2u}}\Delta u.$$

□

Gaussian Curvature

Example

The unit disk $|z| < 1$ equipped with the following metric

$$ds^2 = \frac{4dzd\bar{z}}{(1 - z\bar{z})^2},$$

the Gaussian curvature is -1 everywhere.

Proof.

$e^{2u} = \frac{4}{1-x^2-y^2}$, then $u = \log 2 - \log(1 - x^2 - y^2)$.

$$u_x = -\frac{-2x}{1 - x^2 - y^2} = \frac{2x}{1 - x^2 - y^2}.$$

$\square$

Gaussian Curvature

Proof.

then

$$u_{xx} = \frac{2(1-x^2-y^2) - 2x(-2x)}{(1-x^2-y^2)^2} = \frac{2+2x^2-2y^2}{(1-x^2-y^2)^2}$$

similarly

$$u_{yy} = \frac{2+2y^2-2x^2}{(1-x^2-y^2)^2}$$

so

$$u_{xx} + u_{yy} = \frac{4}{(1-x^2-y^2)} = e^{2u}, K = -\frac{1}{e^{2u}}(u_{xx}+u_{yy}) = -1.$$

□

Yamabe Equation

Lemma (Yamabe Equation)

Conformal metric deformation $\mathbf{g} \to e^{2\lambda}\mathbf{g} = \tilde{\mathbf{g}}$, *then*

$$\boxed{\tilde{K} = \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda)}$$

Proof.

Use isothermal parameters, $\mathbf{g} = e^{2u}(dx^2 + dy^2)$, $K = -e^{2u}\Delta u$, similarly $\tilde{\mathbf{g}} = e^{2\tilde{u}}(dx^2 + dy^2)$, $\tilde{K} = -e^{2\tilde{u}}\Delta\tilde{u}$, $\tilde{u} = u + \lambda$,

$$\begin{aligned}
\tilde{K} &= -\frac{1}{e^{2(u+\lambda)}}\Delta(u+\lambda) \\
&= \frac{1}{e^{2\lambda}}\left(-\frac{1}{e^{2u}}\Delta u - \frac{1}{e^{2u}}\Delta\lambda\right) \\
&= \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda).
\end{aligned}$$

Geodesics

Geodesic Equation

Lemma (Geodesic Equation on a Riemann Surface)

Suppose S is a Riemann surface with a metric, $\rho(z)dzd\bar{z} = e^{2u(z)}dzd\bar{z}$, then a geodesic γ with local representation $z(t)$ satisfies the equation:

$$\boxed{\ddot{\gamma} + \frac{2\rho_\gamma}{\rho}\dot{\gamma}^2 \equiv 0.}$$

equivalently,

$$\ddot{\gamma} + 4u_\gamma\dot{\gamma}^2 \equiv 0.$$

Geodesic Equation

Proof.

Assume the velocity vector is $\dot{\gamma} = f_1\mathbf{e_1} + f_2\mathbf{e_2}$, which is parallel along γ, by parallel transport ODE,

$$\left\{\begin{array}{rcl} \frac{df_1}{dt} - f_2\frac{\omega_{12}}{dt} & = & 0 \\ \frac{df_2}{dt} + f_1\frac{\omega_{12}}{dt} & = & 0 \end{array}\right.$$

Suppose the geodesic has local representation $\gamma(t) = (x(t), y(t))$, then $d\gamma = \dot{x}\partial_x + \dot{y}\partial_y = e^u\dot{x}\mathbf{e_1} + e^u\dot{y}\mathbf{e_2}$, $\omega_{12}/dt = -u_y\dot{x} + u_x\dot{y}$, $\rho = e^u$,

$$\frac{d}{dt}(\rho\dot{x}) - (\rho\dot{y})(-u_y\dot{x} + u_x\dot{y}) = 0$$
$$\frac{d}{dt}(\rho\dot{y}) + (\rho\dot{x})(-u_y\dot{x} + u_x\dot{y}) = 0$$

□

Geodesic Equation

continued

in turn,

$$\rho\ddot{x} + \dot{\rho}\dot{x} - \dot{y}(-\rho_y\dot{x} + \rho_x\dot{y}) = \rho\ddot{x} + (\rho_x\dot{x} + \rho_y\dot{y})\dot{x} - \dot{y}(-\rho_y\dot{x} + \rho_x\dot{y}) = 0$$
$$\rho\ddot{y} + \dot{\rho}\dot{y} + \dot{x}(-\rho_y\dot{x} + \rho_x\dot{y}) = \rho\ddot{y} + (\rho_x\dot{x} + \rho_y\dot{y})\dot{y} + \dot{x}(-\rho_y\dot{x} + \rho_x\dot{y}) = 0$$

namely

$$\rho\ddot{x} + \rho_x(\dot{x}^2 - \dot{y}^2) + 2\rho_y\dot{x}\dot{y} = 0$$
$$\rho\ddot{y} - \rho_y(\dot{x}^2 - \dot{y}^2) + 2\rho_x\dot{x}\dot{y} = 0$$

The first row plus $\sqrt{-1}$ times the second row,

$$\rho(\ddot{x} + \sqrt{-1}\ddot{y}) + (\rho_x - \sqrt{-1}\rho_y)(\dot{x} + \sqrt{-1}\dot{y})^2 = 0.$$

Geodesic Equation

continued.

Represent $\gamma(t) = z(t)$, where $z = x + \sqrt{-1}y$, $\rho_z = \frac{1}{2}(\rho_x - \sqrt{-1}\rho_y)$, we obtain the equation for geodesic on complex domain,

$$\ddot{\gamma} + \frac{2\rho_\gamma}{\rho}\dot{\gamma}^2 \equiv 0.$$

$\square$

Geodesic Curvature

Lemma

Given a curve γ on a surface $(S, \mathbf{g})$, with isothermal coordinates (x, y), the angle between ∂_x and $\dot{\gamma}$ is θ, then

$$\boxed{k_g(s) = \frac{d\theta}{ds} + \frac{\omega_{12}}{ds}}$$

Proof.

Construct an orthonormal frame $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2\}$ by rotating $\{\mathbf{e}_1, \mathbf{e}_2\}$ by angle θ, hence $\bar{\mathbf{e}}_1$ is the tangent vector of γ.

$$\left\{\begin{array}{rcr} \bar{\mathbf{e}}_1 &=& \cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2 \\ \bar{\mathbf{e}}_2 &=& -\sin\theta \mathbf{e}_1 + \cos\theta \mathbf{e}_2 \end{array}\right.$$

$$\begin{aligned} d\bar{\mathbf{e}}_1 &= -\sin\theta d\theta \mathbf{e}_1 + \cos\theta d\mathbf{e}_1 + \cos\theta d\theta \mathbf{e}_2 + \sin\theta d\mathbf{e}_2 \\ &= (-\sin\theta d\theta - \sin\theta\omega_{12})\mathbf{e}_1 + (\cos\theta\omega_{12} + \cos\theta d\theta)\mathbf{e}_2 \\ &+ (\cos\theta\omega_{13} + \sin\theta\omega_{23})\mathbf{e}_3 \end{aligned}$$

Geodesic Curvature

continued

$$\begin{aligned}\bar{\omega}_{12} &= \langle d\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2 \rangle \\ &= (-sin\theta d\theta - \sin\theta\omega_{12})(-\sin\theta) + (\cos\theta\omega_{12} + \cos\theta d\theta)\cos\theta \\ &= d\theta + \omega_{12}.\end{aligned}$$

Therefore

$$k_g = \frac{\bar{\omega}_{12}}{ds} = \frac{d\theta}{ds} + \frac{\omega_{12}}{ds}$$

□

Geodesic Curvature

Lemma (Geodesic Curvature)

Under the isothermal coordinates, the geodesic curvature is given by

$$\boxed{k_g = e^{-u}(k - \partial_{\mathbf{n}} u)}$$

where k is the curvature on the parameter plane, $\mathbf{n}$ *is the exterior normal to the cure on the parameter plane.*

Proof.

We have $\omega_{12} = -u_y dx + u_x dy$. On the parameter plane, the arc length is dt, then $ds = e^u dt$. The parameterization preserves angle, therefore

$$\begin{aligned} k_g &= \frac{d\theta}{ds} + \frac{-u_y dx + u_x dy}{ds} = \frac{dt}{ds}\left(\frac{d\theta}{dt} + \frac{-u_y dx + u_x dy}{dt}\right) \\ &= e^{-u}(k - \langle \nabla u, n \rangle) \\ &= e^{-u}(k - \partial_{\mathbf{n}} u) \end{aligned}$$

Geodesic Curvature

Lemma

Given a metric surface $(S, \mathbf{g})$*, under conformal deformation,* $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$*, the geodesic curvature satisfies*

$$\boxed{k_{\bar{\mathbf{g}}} = e^{-\lambda}(k_{\mathbf{g}} - \partial_{\mathbf{n},\mathbf{g}}\lambda)}$$

Proof.

$$\begin{aligned} k_{\mathbf{g}} &= e^{-(u+\lambda)}(k - \partial_{\mathbf{n}}(u+\lambda)) \\ &= e^{-\lambda}(e^{-u}(k - \partial_{\mathbf{n}}u) - e^{-u}\partial_{\mathbf{n}}\lambda) \\ &= e^{-\lambda}(k_{\mathbf{g}} - \partial_{\mathbf{n},\mathbf{g}}\lambda) \end{aligned}$$

□

Definition (geodesic)

Given a metric surface $(S, \mathbf{g})$, a curve $\gamma : [0,1] \to S$ is a geodesic if $k_{\mathbf{g}}$ is zero everywhere.

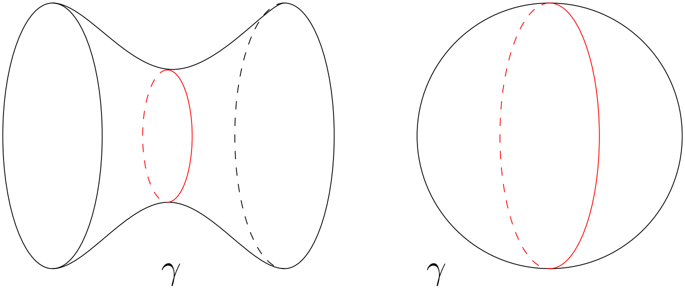


Figure: Stable and unstable geodesics.

Lemma (geodesic)

If γ is the shortest curve connecting p and q, then γ is a geodesic.

Proof.

Consider a family of curves, $\Gamma : (-\varepsilon, \varepsilon) \to S$, such that $\Gamma(0,t) = \gamma(t)$, and

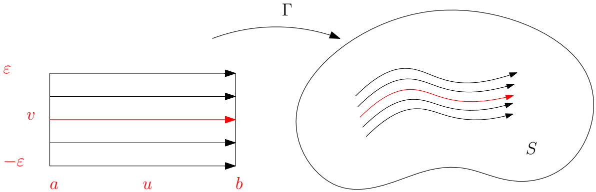
$$\Gamma(s,0) = p, \Gamma(s,1) = q, \frac{\partial \Gamma(s,t)}{\partial s} = \varphi(t)\mathbf{e_2}(t),$$

where $\varphi : [0,1] \to \mathbb{R}$, $\varphi(0) = \varphi(1) = 0$. Fix parameter s, curve $\gamma_s := \Gamma(s,\cdot)$, $\{\gamma_s\}$ for a variation. Define an energy,

$$L(s) = \int_0^1 \left| \frac{d\gamma_s(t)}{dt} \right| dt, \quad \frac{\partial L(s)}{\partial s} = -\int_0^1 \varphi k_{\mathbf{g}}(\tau) d\tau.$$

□

First Variation of arc length



Let $\gamma_v : [a,b] \to M$, where $v \in (-\varepsilon, \varepsilon) \in \mathbb{R}$ be a 1-parameter family of paths. We define the map $\Gamma : [a,b] \times [0,1] \to M$ by

$$\Gamma(u,v) := \gamma_v(u).$$

Define the vector fields $\mathbf{u}$ and $\mathbf{v}$ along γ_v by

$$\mathbf{u} := \frac{\partial \Gamma}{\partial u} = \Gamma_*(\partial_u), \quad \text{and} \quad \mathbf{v} := \frac{\partial \Gamma}{\partial v} = \Gamma_*(\partial_v),$$

We call $\mathbf{u}$ the *tangent vector field* and $\mathbf{v}$ the *variation vector field*.

First Variation of arc length

Lemma (First variation of arc length)

If The length of γ_v is given by

$$L(\gamma_v) := \int_a^b |\mathbf{u}(\gamma_v(u))| du.$$

γ_0 *is parameterized by arc length, that is,* $|\mathbf{u}(\gamma_0(u))| \equiv 1$, *then*

$$\frac{d}{dv}\Big|_{v=0} L(\gamma_v) = -\int_a^b \langle D_{\mathbf{u}}\mathbf{u}, \mathbf{v}\rangle du + \langle \mathbf{u}, \mathbf{v}\rangle\Big|_a^b.$$

If we choose $\mathbf{u} = \mathbf{e_1}$, the tangent vector of γ, $\mathbf{v} = \mathbf{e_2}$ orthogonal to $\mathbf{e_1}$, and fix the starting and ending points of paths, then

$$\frac{d}{dv} L(\gamma_v) = -\int_a^b k_g ds.$$

First variation of arc length

Proof.

Fixing $u \in [a, b]$, we may consider **u** and **v** as vector fields along the path $v \mapsto \gamma_v(u)$. Then

$$
\begin{aligned}
\frac{\partial}{\partial v}|\mathbf{u}(\gamma_v(u))| &= \frac{\partial}{\partial v}\sqrt{|\mathbf{u}(\gamma_v(u))|^2} \\
&= \frac{1}{2|\mathbf{u}(\gamma_v(u))|}\frac{\partial}{\partial v}|\mathbf{u}(\gamma_v(u))|^2 \\
&= \frac{1}{2|\mathbf{u}|}\mathbf{v}|\mathbf{u}|^2 = |\mathbf{u}|^{-1}\langle D_{\mathbf{v}}\mathbf{u}, \mathbf{u}\rangle_{\mathbf{g}} = \langle D_{\mathbf{v}}\mathbf{u}, \mathbf{u}\rangle_{\mathbf{g}}
\end{aligned}
$$

$\square$

First variation of arc length

Proof.

$$\frac{d}{dv}L(\gamma_v) = \int_a^b \frac{\partial}{\partial v}|\mathbf{u}(\gamma_v(u))|du = \int_a^b \langle D_{\mathbf{v}}\mathbf{u}, \mathbf{u}\rangle_{\mathbf{g}} du$$

Since $D_{\mathbf{v}}\mathbf{u} - D_{\mathbf{u}}\mathbf{v} = [\mathbf{v}, \mathbf{u}]$, and $[\mathbf{v}, \mathbf{u}] = \Gamma_*([\partial_v, \partial_u]) = 0$,

$$\begin{aligned}
\frac{d}{dv}L(\gamma_v) &= \int_a^b \langle D_{\mathbf{u}}\mathbf{v}, \mathbf{u}\rangle_{\mathbf{g}} du \\
&= \int_a^b \left(\frac{d}{du}\langle \mathbf{u}, \mathbf{v}\rangle_{\mathbf{g}} - \langle \mathbf{v}, D_{\mathbf{u}}\mathbf{u}\rangle_{\mathbf{g}} \right) du \\
&= \langle \mathbf{u}, \mathbf{v}\rangle_{\mathbf{g}}\Big|_a^b - \int_a^b \langle \mathbf{v}, D_{\mathbf{u}}\mathbf{u}\rangle_{\mathbf{g}} du.
\end{aligned}$$

$\square$

The second derivative of the length variation $L(s)$ depends on the Gaussian curvature of the underlying surface. If $K < 0$, then the second derivative is positive, the geodesic is stable; if $K > 0$, then the secondary derivative is negative, the geodesic is unstable.

Geodesics

Lemma (Uniqueness of geodesics)

Suppose $(S, \mathbf{g})$ *is a closed oriented metric surface,* $\mathbf{g}$ *induces negative Gaussian curvature everywhere, then each homotopy class has a unique geodesic.*

Proof.

The existence can be obtained by variational method. The uniqueness is by Gauss-Bonnet theorem. Assume two geodesics $\gamma_1 \sim \gamma_2$, then they bound a topological annulus Σ, by Gauss-Bonnet,

$$\int_{\Sigma} K dA + \int_{\partial \Sigma} k_g ds = \chi(\Sigma),$$

The first term is negative, the second is along the geodesics, hence 0, $\chi(\Sigma) = 0$. Contradiction. $\square$

Algorithm: Homotopy Detection

Input: A high genus closed mesh M, two loops γ_1 and γ_2;
Output: Whether $\gamma_1 \sim \gamma_2$;

1. Compute a hyperbolic metric of M, using Ricci flow;
2. Homotopically deform γ_k to geodesics, $k = 1, 2$;
3. if two geodesics coincide, return true; otherwise, return false;

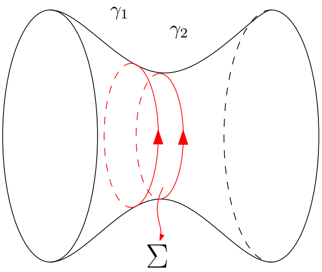


Figure: Geodesics uniqueness.

Algorithm: Shortest Word

Input: A high genus closed mesh M, one loop γ

1. Compute a hyperbolic metric of M, using Ricci flow;
2. Homotopically deform γ to a geodesic;
3. Compute a set of canonical fundamental group basis;
4. Embed a finite portion of the universal covering space onto the Poincaré disk;
5. Lift γ to the universal covering space $\tilde{\gamma}$. If $\tilde{\gamma}$ crosses $b_i^{\pm}$, append $a_i^{\pm}$; crosses $a_i^{\pm}$, append $b_i^{\mp}$.

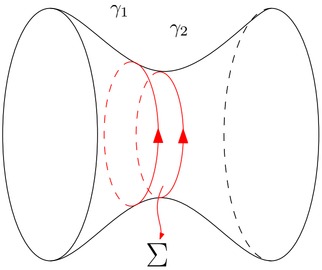


Figure: Geodesics uniqueness

Hyperbolic Geodesics

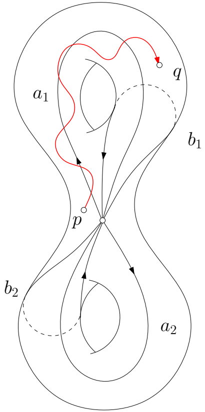
Lemma

Let Σ be a compact hyperbolic Riemann surface, $K \equiv -1$, $p, q \in \Sigma$, then there exists a unique geodesic in each homotopy class, the geodesic depends on p and q continuously.

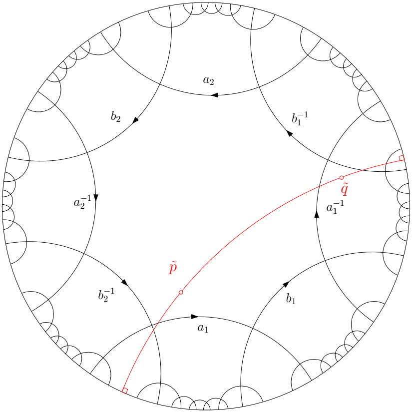
Proof.

Given a path $\gamma : [0,1] \to \Sigma$ connecting p and q. Let $\pi : \mathbb{H}^2 \to \Sigma$ be the universal covering space of Σ. Fix one point $\tilde{p} \in \pi^{-1}(p)$, then there exists a unique lifting of γ, $\tilde{\gamma} : [0,1] \to \mathbb{H}^2$, $\tilde{\gamma}(0) = \tilde{p}$ and $\tilde{\gamma}(1) = \tilde{q}$. On the hyperbolic plane, the geodesic between $\tilde{p}$ and $\tilde{q}$ exists and is unique, $\tilde{\gamma}$ depends on $\tilde{p}$ and $\tilde{q}$ continuously. $\square$

Hyperbolic Geodesic



geodesic on surface



Poincaré's disk model

Harmonic Maps

Harmonic Map

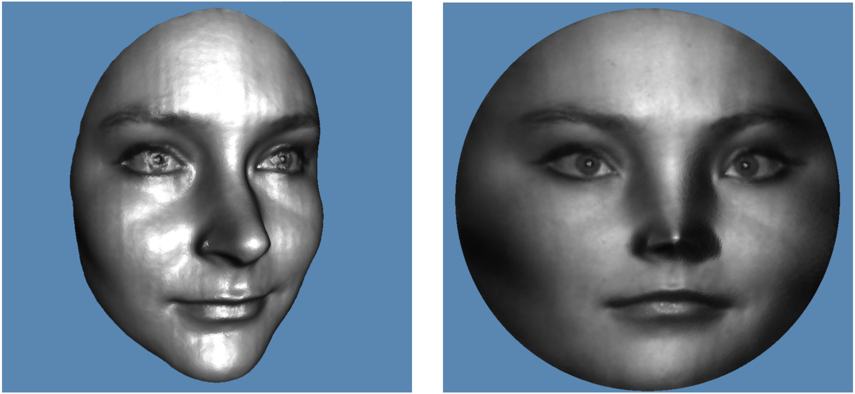


Figure: Harmonic map between topological disks.

Harmonic Map

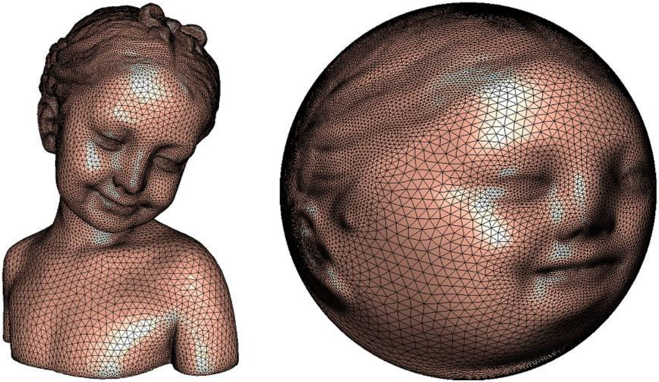


Figure: Harmonic map between topological spheres.

Harmonic Map

Figure: Harmonic map induced foliations.

Harmonic Function

Given a planar domain $\Omega \subset \mathbb{R}^2$, consider the electric potential $u : \Omega \to \mathbb{R}$. The gradient of the potential induces electric currents, and produces heat. The heat power is represented as *harmonic energy*

$$E(u) := \int_\Omega \langle \nabla u, \nabla u \rangle dxdy.$$

In nature, the distribution of u minimizes the heat power, and is called a harmonic function. Assume $h \in C_0^\infty(\Omega)$, then $E(u + \varepsilon h) \geq E(u)$,

$$\frac{d}{d\varepsilon} \int_\Omega \langle \nabla u + \varepsilon \nabla h, \nabla u + \varepsilon \nabla h \rangle dxdy \Big|_{\varepsilon=0} = 2 \int_\Omega \langle \nabla u, \nabla h \rangle dxdy = 0.$$

Harmonic Function

By relation

$$\nabla \cdot (h\nabla u) = \langle \nabla h, \nabla u \rangle + h\nabla \cdot \nabla u,$$

we obtain

$$\int_\Omega \nabla u, \nabla h\rangle = \int_\Omega h\Delta u dxdy - \int_\Omega \nabla \cdot (h\nabla u) dxdy = \int_\omega h\Delta u dxdy,$$

We obtain Laplace equation

$$\boxed{\left\{ \begin{array}{lcl} \Delta u & \equiv & 0 \\ u|_{\partial\Omega} & = & g \end{array} \right.}$$

Steady temperature field, static electric field, elastic deformation, diffusion field, all are governed by the Laplace equation.

Harmonic Function

Theorem (Mean Value)

Assume $\Omega \subset \mathbb{R}^2$ *is a planar open set,* $u : \Omega \to \mathbb{R}$ *is a harmonic function, then for any* $p \in \Omega$

$$u(p) = \frac{1}{2\pi\varepsilon} \oint_{\gamma} u(q) ds, \tag{1}$$

where γ *is a circle centered at* p*, with radius* ε*.*

Proof.

u is harmonic, du is a harmonic 1-form, its Hodge star ${}^*du$ is also harmonic. Define the conjugate function v, $dv = {}^*du$, then $\varphi(z) := u + \sqrt{-1}v$ is holomorphic. By Cauchy integration formula,

$$\varphi(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{\varphi(\zeta)}{\zeta - z} dz \tag{2}$$

Hence, we obtain the mean value property of harmonic function. □

Corollary (Maximal value principle)

Assume $\Omega \subset \mathbb{R}^2$ is a planar domain, and $u : \overline{\Omega} \to \mathbb{R}$ is a non-constant harmonic function, then u can't reach extremal values in the interior of Ω.

Proof.

Assume p is an interior point in Ω, p is a maximal point of u, $u(p) = C$. By mean value property, we obtain for any point q on the circle $B(p, \varepsilon)$, $u(q) = C$, where ε is arbitrary, therefore u is constant in a neighborhood of p. Therefore $u^{-1}(C)$ is open. On the other hand, u is continuous, $u^{-1}(C)$ is closed, hence $u^{-1}(C) = \Omega$. Contradiction. $\square$

Uniqueness of Harmonic Functions

Corollary

Suppose $\Omega \subset \mathbb{R}^2$ is a planar domain, $u_1, u_2 : \Omega \to \mathbb{R}$ are harmonic functions with the same boundary value, $u_1|_{\partial\Omega} = u_2|_{\partial\Omega}$, then $u_1 = u_2$ on Ω.

Proof.

$u_1 - u_2$ is also harmonic, with 0 boundary value, therefore the maximal and minimal values of $u_1 - u_2$ must be on the boundary, namely they are 0, hence u_1, u_2 are equal in Ω. $\square$

Disk Harmonic Maps

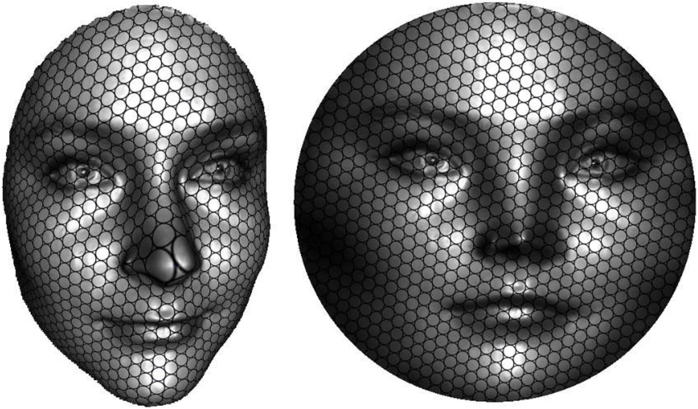


Figure: Harmonic map between topological disks.

Diffeomorphic Property of Disk Harmonic Maps

Theorem (Rado)

Suppose a harmonic map $\varphi : (S, \mathbf{g}) \to (\Omega, dx^2 + dy^2)$ *satisfies:*

1. *planar domain* Ω *is convex*
2. *the restriction of* $\varphi : \partial S \to \partial \Omega$ *on the boundary is homoemorphic,*

then u *is diffeomorphic in the interior of* S*.*

Proof.

By regularity theory of harmonic maps, we get the smoothness of the harmonic map. Assume $\varphi : (x, y) \to (u, v)$ is not homeomorphic, then there is an interior point $p \in \Omega$, the Jacobian matrix of φ is degenerated as p, there are two constants $a, b \in \mathbb{R}$, not being zeros simultaneously, such that

$$a\nabla u(p) + b\nabla v(p) = 0.$$

By $\Delta u = 0, \Delta v = 0$, the auxiliary function $f(q) = au(q) + bv(q)$ is also harmonic. □

Diffeomorphic Property of Disk Harmonic Maps

continued

By $\nabla f(p) = 0$, p is an saddle point of f. Consider the level set of f near p

$$\Gamma = \{q \in \Omega | f(q) = f(p) - \varepsilon\}$$

Γ has two connected components, intersecting ∂S at 4 points.
But Ω is a planar convex domain, $\partial\Omega$ and the line $au + bv = const$ have two intersection points. By assumption, the mapping φ restricted on the the boundary $\varphi : \partial S \to \partial\Omega$ is homeomorphic. Contradiction.

Computational Algorithm for Disk Harmonic Maps

Input:A topological disk M;
Output:A harmonic map $\varphi : M \to \mathbb{D}^2$

1. Construct boundary map to the unit circle, $g : \partial M \to \mathbb{S}^1$, g should be a homeomorphism;
2. Compute the cotangent edge weight;
3. for each interior vertex $v_i \in M$, compute Laplacian

$$\Delta\varphi(v_i) = \sum_{v_j \sim v_i} w_{ij}(\varphi(v_i) - \varphi(v_j)) = 0;$$

4. Solve the linear system, to obtain φ.

General Harmonic Map

Definition (Harmonic Energy)

Let (Σ_1, z) and (Σ_2, u) be two Riemann surfaces, with Riemannian metrics $\sigma(z)dzd\bar{z}$ and $\rho(u)dud\bar{u}$. Given a C^1 map $u : \Sigma_1 \to \Sigma_2$, then the harmonic energy of u is defined as

$$E(z, \rho, u) := \int_{\Sigma_1} \rho^2(u)(u_z \bar{u}_{\bar{z}} + \bar{u}_z u_{\bar{z}}) \frac{i}{2} dzd\bar{z}$$

where $u_z := \frac{1}{2}(u_x - iu_y)$, $u_{\bar{z}} := \frac{1}{2}(u_x + iu_y)$ and $dz \wedge d\bar{z} = -2idx \wedge dy$.

Definition (Harmonic Map)

If the C^1 map $u : \Sigma_1 \to \Sigma_2$ minimizes the harmonic energy, then u is called a harmonic map.

General Harmonic Map

Theorem (Euerl-Larange Equation for Harmonic Maps)

Suppose $u : \Sigma_1 \to \Sigma_2$ is a C^2 harmonic map between Riemannian surfaces, then

$$\boxed{u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_z u_{\bar{z}} = 0}$$

Geodesics are special harmonic maps, harmonic maps are generalized geodesics:

$$\boxed{\ddot{\gamma} + \frac{2\rho_\gamma}{\rho}\dot{\gamma}^2 \equiv 0 \quad u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_z u_{\bar{z}} \equiv 0}$$

Proof.

Suppose u is harmonic, u_t is a variation in a local coordinates system,

$$u + t\varphi, \quad \varphi \in C^0 \cap W_0^{1,2}(\Sigma_1, \Sigma_2)$$

we obtain

$$\frac{d}{dt} E(u + t\varphi)\Big|_{t=0} = 0,$$

□

General Harmonic Map

continued

$$
\begin{aligned}
0 =& \frac{d}{dt}\left\{ \int \rho^2(u+t\varphi)((u+t\varphi)_z(\bar{u}+t\bar{\varphi})_{\bar{z}} \right.\\
& \left.\left. + (\bar{u}+t\bar{\varphi})_z(u+t\varphi)_{\bar{z}}) i dz d\bar{z} \right\}\right|_{t=0}\\
=& \int \left\{ \rho^2(u)(u_z\bar{\varphi}_{\bar{z}} + \bar{u}_{\bar{z}}\varphi_z + \bar{u}_z\varphi_{\bar{z}} + u_{\bar{z}}\bar{\varphi}_z) \right.\\
& \left. + 2\rho(\rho_u\varphi + \rho_{\bar{u}}\bar{\varphi})(u_z\bar{u}_{\bar{z}} + \bar{u}_z u_{\bar{z}}) \right\} i dz d\bar{z}.
\end{aligned}
$$

continued

We set $\varphi = \frac{\psi}{\rho^2(u)}$,

$$\rho^2 \varphi_z = \psi_z - \frac{2\psi}{\rho}(\rho_u u_z + \rho_{\bar{u}} \bar{u}_z)$$

$$\rho^2 \varphi_{\bar{z}} = \psi_{\bar{z}} - \frac{2\psi}{\rho}(\rho_u u_{\bar{z}} + \rho_{\bar{u}} \bar{u}_{\bar{z}})$$

$$\rho^2 \bar{\varphi}_z = \bar{\psi}_z - \frac{2\bar{\psi}}{\rho}(\rho_u u_z + \rho_{\bar{u}} \bar{u}_z)$$

$$\rho^2 \bar{\varphi}_{\bar{z}} = \bar{\psi}_{\bar{z}} - \frac{2\bar{\psi}}{\rho}(\rho_u u_{\bar{z}} + \rho_{\bar{u}} \bar{u}_{\bar{z}})$$

continued

$$\bar{u}_{\bar{z}}\rho^2\varphi_z = \psi_z\bar{u}_{\bar{z}} - \frac{2\psi}{\rho}(\rho_u u_z\bar{u}_{\bar{z}} + \rho_{\bar{u}}\bar{u}_z\bar{u}_{\bar{z}})$$

$$\bar{u}_z\rho^2\varphi_{\bar{z}} = \psi_{\bar{z}}\bar{u}_z - \frac{2\psi}{\rho}(\rho_u u_{\bar{z}}\bar{u}_z + \rho_{\bar{u}}\bar{u}_{\bar{z}}\bar{u}_z)$$

$$u_{\bar{z}}\rho^2\bar{\varphi}_z = \bar{\psi}_z u_{\bar{z}} - \frac{2\bar{\psi}}{\rho}(\rho_{\bar{u}}\bar{u}_z u_{\bar{z}} + \rho_u u_z u_{\bar{z}})$$

$$u_z\rho^2\bar{\varphi}_{\bar{z}} = \bar{\psi}_{\bar{z}} u_z - \frac{2\bar{\psi}}{\rho}(\rho_{\bar{u}}\bar{u}_{\bar{z}} u_z + \rho_u u_{\bar{z}} u_z)$$

General Harmonic Map

continued

$$
\begin{aligned}
&\frac{2}{\rho}(\rho_u \psi + \rho_{\bar{u}} \bar{\psi})(u_z \bar{u}_{\bar{z}} + \bar{u}_z u_{\bar{z}}) \\
&= \frac{2\psi}{\rho}\rho_u(u_z \bar{u}_{\bar{z}} + \bar{u}_z u_{\bar{z}}) + \frac{2\bar{\psi}}{\rho}\rho_{\bar{u}}(\bar{u}_z u_{\bar{z}} + u_z \bar{u}_{\bar{z}})
\end{aligned}
$$

Take summation,

$$
\begin{aligned}
\bar{u}_{\bar{z}}\rho^2\varphi_z + u_z\rho^2\bar{\varphi}_{\bar{z}} &= \left(\psi_z \bar{u}_{\bar{z}} - \frac{2\psi}{\rho}\rho_{\bar{u}}\bar{u}_z\bar{u}_{\bar{z}}\right) + \left(\bar{\psi}_{\bar{z}} u_z - \frac{2\bar{\psi}}{\rho}\rho_u u_{\bar{z}} u_z\right) \\
\bar{u}_z\rho^2\varphi_{\bar{z}} + u_{\bar{z}}\rho^2\bar{\varphi}_z &= \left(\psi_{\bar{z}} \bar{u}_z - \frac{2\psi}{\rho}\rho_{\bar{u}}\bar{u}_{\bar{z}}\bar{u}_z\right) + \left(\bar{\psi}_z u_{\bar{z}} - \frac{2\bar{\psi}}{\rho}\rho_u u_z u_{\bar{z}}\right)
\end{aligned}
$$

continued

The above equation becomes

$$
\begin{aligned}
0 =& 2\Re\int\left(\bar{\psi}_{\bar{z}}u_z - \frac{2\bar{\psi}}{\rho}\rho_u u_{\bar{z}}u_z\right) idzd\bar{z}\\
&+2\Re\int\left(\psi_{\bar{z}}\bar{u}_z - \frac{2\psi}{\rho}\rho_{\bar{u}}\bar{u}_{\bar{z}}\bar{u}_z\right) idzd\bar{z}
\end{aligned}
$$

If $u \in C^2$, we can integrate by parts, $(u_z\bar{\psi})_{\bar{z}} = u_{z\bar{z}}\bar{\psi} + u_z\bar{\psi}_{\bar{z}}$,

$$
\begin{aligned}
0 =& 2\Re\int\left(u_{z\bar{z}} + \frac{2\rho_u}{\rho}u_{\bar{z}}u_z\right)\bar{\psi} idzd\bar{z}\\
&+2\Re\int\left(\bar{u}_{z\bar{z}} + \frac{2\rho_{\bar{u}}}{\rho}\bar{u}_{\bar{z}}\bar{u}_z\right)\psi idzd\bar{z}
\end{aligned}
$$

General Harmonic Map

continued

Therefore

$$0 = 2\Re \int \left(u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_{\bar{z}} u_z \right) \bar{\psi} i dz d\bar{z}$$

□

Hopf Differential of Harmonic Maps

Theorem (Hopf Diffential of Harmonic Maps)

Let $u : (\Sigma_1, \lambda^2(z)dzd\bar{z}) \to (\Sigma_2, \rho^2(u)dud\bar{u})$ *is harmonic, then the Hopf differential of the map*

$$\boxed{\Phi(u) := \rho^2 u_z \bar{u}_z dz^2}$$

is holomorphic quadratic differential on Σ_1. *Furthermore* $\Phi(u) \equiv 0$, *if and only if* u *is holomorphic or anti-holomorprhic.*

Proof.

If u is harmonic, then

$$\begin{aligned}\frac{\partial}{\partial \bar{z}}(\rho^2 u_z \bar{u}_z) &= \rho^2 u_{z\bar{z}} \bar{u}_z + \rho^2 u_z \bar{u}_{z\bar{z}} + 2\rho\rho_u u_{\bar{z}} u_z \bar{u}_z + 2\rho\rho_{\bar{u}} \bar{u}_{\bar{z}} u_z \bar{u}_z \\ &= (\rho^2 u_{z\bar{z}} + 2\rho\rho_u u_{\bar{z}} u_z)\bar{u}_z + (\rho^2 \bar{u}_{z\bar{z}} + 2\rho\rho_{\bar{u}} \bar{u}_{\bar{z}} \bar{u}_z) u_z = 0.\end{aligned}$$

Therefore $\Phi(u)$ is holomorphic. □

Hopf Differential of Harmonic Maps

Proof.

If $\Phi(u) = \rho^2 u_z \bar{u}_z \equiv 0$, then either $u_z = 0$ or $\bar{u}_z = 0$. Since the Jacobian determinant equals to

$$|u_z|^2 - |u_{\bar{z}}|^2 > 0,$$

therefore $\bar{u}_z = 0$, namely $u_{\bar{z}} = 0$, u is holomorphic or anti-holomorphic. u is holomorphic, equivalent to $L \equiv 0$; u is anti-holomorphic, equivalent to $H \equiv 0$. We know H and L have isolated zeros, unless they are zero everywhere. Hence u is entirely holomorphic or anti-holomorphic. $\square$

Spherical Harmonic Map

Lemma

A holomorphic quadratic differential ω is on the unit sphere, then ω is zero.

Proof.

Choose two charts z and $w = \frac{1}{z}$. Let $\omega = \varphi(z)dz^2$, then

$$\varphi(z)dz^2 = \varphi\left(\frac{1}{w}\right)\left(\frac{dz}{dw}\right)^2 dw^2 = \varphi\left(\frac{1}{w}\right)\frac{1}{w^4}dw^2.$$

since ω is globally holomorphic, when $w \to 0$,

$$\varphi\left(\frac{1}{w}\right)\frac{1}{w^4} < \infty,$$

hence $z \to \infty$, $\varphi(z) \to 0$. By Liouville theorem, $\varphi \equiv 0$. □

Spherical Harmonic Map

Theorem (Spherical Harmonic Maps)

Harmonic maps between genus zero closed metric surfaces must be conformal.

Proof.

Suppose $u : \Sigma_1 \to \Sigma_2$ is a harmonic map, then $\Phi(u)$ must be a holomorphic quadratic differential. Since Σ_1 is of genus zero, therefore $\Phi(u) \equiv 0$. Hence u is holomorphic. $\square$

Spherical Harmonic Map

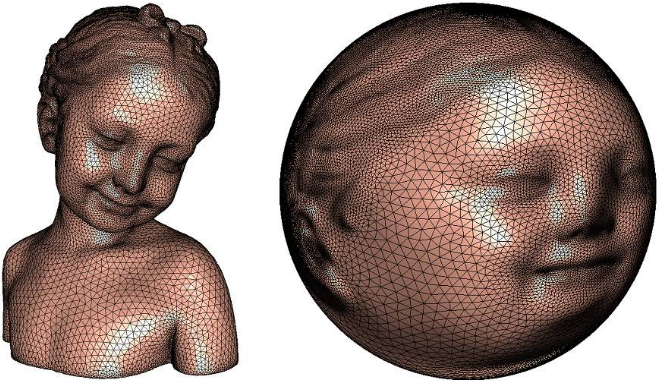


Figure: Spherical Harmonic Map

Uniqueness Spherical Harmonic Map

Definition (Möbius Transformation)

A Möbius transformation $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has the form

$$z \mapsto \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, \quad ad-bc=1.$$

Given $\{z_0, z_1, z_2\}$, there is a unique Möbius transformation, that maps them to $\{0, 1, \infty\}$,

$$z \mapsto \frac{z-z_0}{z-z_2}\frac{z_1-z_2}{z_1-z_0}.$$

Theorem (Uniquess of Spherical Conformal Automorphisms)

Suppose $f : \mathbb{S}^2 \to \mathbb{S}^2$ is a biholomorphic automorphism, then f must be a Möbius transformation.

Uniqueness of Spherical Harmonic Map

Proof.

By stereo-graphic projection, we map the sphere to the extened complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. First, the poles of f must be finite. Suppose there are infinite poles of f, because $\mathbb{S}^2$ is compact, there must be accumulation points, then f must be a constant value function.

Let $z_1, z_2, \dots, z_n$ be the finite poles of f, with degrees $e_1, e_2, \dots, e_n$. Let $g = \Pi_i (z - z_i)^{e_i}$, then fg is a holomorphic function on $\mathbb{C}$, therefore fg is entire, namely, fg is a polynomial. Therefore

$$f = \frac{\sum_{i=1}^{n} a_i z^i}{\sum_j b_j z^j},$$

if $n > 1$ then f has multiple zeros, contradict to the condition that f is an automorphism. Therefore $n = 1$. Similarly $m = 1$. □

Computational Algorithm for Spherical Harmonic Map

Input: A genus zero closed mesh M;
Output: A spherical harmonic map $\varphi : M \to \mathbb{S}^2$;

1. Compute Gauss map $\varphi : M \to \mathbb{S}^2$, $\varphi(v) \leftarrow \mathbf{n}(v)$;
2. Compute the cotangent edge weight, compute Laplacian
$$\Delta\varphi(v_i) = \sum_{v_i \sim v_j} w_{ij}(\varphi(v_j) - \varphi(v_i)),$$
3. project the Laplacian to the tangent plane,
$$D\varphi(v_i) = \Delta\varphi(v_i) - \langle \Delta\varphi(v_i), \varphi(v_i)\rangle \varphi(v_i)$$
4. for each vertex, $\varphi(v_i) \leftarrow \varphi(v_i) - \lambda D\varphi(v_i)$;
5. compute the mass center $c = \sum A_i \varphi(v_i) / \sum_j A_j$; normalize $\varphi(v_i) = \varphi(v_i) - c / |\varphi(v_i) - c|$;
6. Repeat step 2 through 5, until the Laplacian norm is less than ε.

General theory for Surface Harmonic Maps

Existence of Harmonic Map

Theorem (Existence of Harmonic Maps)

Assume Σ is a Riemann surface, $(N, \rho(u)dud\bar{u})$ is a metric surface, then for any smooth mapping $\varphi : \Sigma \to N$, there is a harmonic map $f : \Sigma \to N$ homotopic to φ.

The can be proven using Courant-Leesgue lemma, which controls the geodesic distance between image points by harmonic energy.

Regularity of Harmonic Map

Theorem (Regularity of Harmonic Maps)

Let $u : \Sigma_1 \to \Sigma_2$ be a (weak) harmonic map between Riemann surfaces, Σ_2 is with hyperbolic metric, the harmonic energy of u is finite, then u is a smooth map.

This is based on the regularity theory of ellptic PDEs.

Diffeomorphic Properties of Harmonic Maps

Theorem (Diffeomorphic Properties of Harmonic Maps)

Let Σ_1 and Σ_2 are compact Riemann surfaces with the same genus, $K_2 \leq 0$. If $u : \Sigma_1 \to \Sigma_2$ is a degree one harmonic map, then u is a diffeomorphism.

Uniqueness of Harmonic Map

Theorem (Uniqueness of Harmonic Map)

Suppose Σ_1 and Σ_2 are compact Riemann surface, Σ_2 is with hyperbolic metric. $u_0, u_1 : \Sigma_1 \to \Sigma_2$ are homotopic harmonic maps. If one of the Jacobian matrix is non-degenerated at a point, then $u_0 \equiv u_1$.

Uniqueness of Harmonic Map

Theorem

Suppose Σ_1 and Σ_2 are Riemann surfaces, the Riemannian metric on Σ_2 induces non-positive curvature K. Let $u \in C^2(\Sigma_1, \Sigma_2)$, $\varphi(z,t)$ is the variation of u, $\dot{\varphi} \neq 0$. If u is harmonic, or for any point $z \in \Sigma_1$, $\varphi(z_1, \cdot)$ is geodesic, then

$$\frac{d}{dt^2} E(u + \varphi(t))\Big|_{t=0} \geq 0. \tag{3}$$

If $K < 0$, then either

$$\frac{d}{dt^2} E(u + \varphi(t))\Big|_{t=0} > 0. \tag{4}$$

or

$$u_z \bar{u}_{\bar{z}} - u_{\bar{z}} \bar{u}_z \equiv 0, \tag{5}$$

Namely the rank of u is ≤ 1 everywhere.

Consider the variation of the mapping u, $u(z)+\varphi(z,t)$, where $\varphi(z,0)\equiv 0$. Let $\dot{\varphi}=\frac{\partial}{\partial t}\varphi$, $\ddot{\varphi}:=\frac{\partial^2}{\partial t^2}\varphi$. $K=-\Delta\log\rho=-\frac{4}{\rho^4}(\rho\rho_{u\bar{u}}-\rho_u\rho_{\bar{u}})$

$$
\begin{aligned}
\frac{d^2}{dt^2}E(u+\varphi(t))\Big|_{t=0} = 2\int \Bigg\{ & \rho^2\left(\dot{\varphi}_z+2\frac{\rho_u}{\rho}u_z\dot{\varphi}\right)\left(\dot{\bar{\varphi}}_{\bar{z}}+2\frac{\rho_{\bar{u}}}{\rho}\bar{u}_{\bar{z}}\dot{\bar{\varphi}}\right)\\
& +\rho^2\left(\dot{\bar{\varphi}}_z+2\frac{\rho_{\bar{u}}}{\rho}\bar{u}_z\dot{\bar{\varphi}}\right)\left(\dot{\varphi}_{\bar{z}}+2\frac{\rho_u}{\rho}u_{\bar{z}}\dot{\varphi}\right)\\
& -\rho^4\frac{K}{2}(u_z\dot{\bar{\varphi}}-\bar{u}_z\dot{\varphi})(\bar{u}_{\bar{z}}\dot{\varphi}-u_{\bar{z}}\dot{\bar{\varphi}})\\
& -(\rho^2\ddot{\varphi}+2\rho\rho_u\dot{\varphi}^2)\left(\bar{u}_{z\bar{z}}+\frac{2\rho_{\bar{u}}}{\rho}\bar{u}_z\bar{u}_{\bar{z}}\right)\\
& -(\rho^2\ddot{\bar{\varphi}}+2\rho\rho_{\bar{u}}\dot{\bar{\varphi}}^2)\left(u_{z\bar{z}}+\frac{2\rho_u}{\rho}u_zu_{\bar{z}}\right)\Bigg\}\,idzd\bar{z}
\end{aligned}
\tag{6}
$$

Uniqueness of Harmonic Map

If u is harmonnic, then

$$u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_z u_{\bar{z}} = 0,$$

or if $\varphi(z, \cdot)$ is geodesic, then

$$\rho^2 \ddot{\varphi} + 2\rho\rho_u \dot{\varphi}^2 = 0.$$

Then, the last two items vanish. Since $K \leq 0$, there first three items are non-negative.
If $K < 0$, then $\frac{d^2}{dt^2} E(u + \varphi(t))|_{t=0}$ is either positive or zero. If it is 0, then the integrands must be 0 everywhere, therefore

$$u_z \dot{\bar{\varphi}} - \bar{u}_z \dot{\varphi} \equiv \bar{u}_{\bar{z}} \dot{\varphi} - u_{\bar{z}} \dot{\bar{\varphi}} \equiv 0. \tag{7}$$

Uniqueness of Harmonic Map

Furthermore

$$\frac{\partial}{\partial z}(\rho^2\dot{\varphi}\dot{\bar{\varphi}}) = (\rho^2\dot{\varphi}_z + 2\rho\rho_u u_z\dot{\varphi})\dot{\bar{\varphi}} + (\rho^2\dot{\bar{\varphi}} + 2\rho\rho_u \bar{u}_z\dot{\bar{\varphi}})\dot{\varphi} = 0. \tag{8}$$

Similarly

$$\frac{\partial}{\partial \bar{z}}(\rho^2\dot{\varphi}\dot{\bar{\varphi}}) = 0. \tag{9}$$

We obtain

$$\rho^2\dot{\varphi}\dot{\bar{\varphi}} \equiv \text{const}. \tag{10}$$

By assumption $\dot{\varphi} \not\equiv 0$, the constant is non-zero, hence $\dot{\varphi}$ and $\dot{\bar{\varphi}}$ are non-zero everywhere, by (7) we get

$$|u_z||\dot{\varphi}| = |\bar{u}_z||\dot{\bar{\varphi}}|$$

hence

$$|u_z| = |\bar{u}_z| = |u_{\bar{z}}|$$

we get (5). $\square$

Uniqueness of Harmonic Map

Theorem (Uniqueness)

Suppose Σ_1 and Σ_2 are compact Riemann surface, Σ_2 is with hyperbolic metric. $u_0, u_1 : \Sigma_1 \to \Sigma_2$ are homotopic harmonic maps. If one of the Jacobian matrix is non-degenerated at a point, then $u_0 \equiv u_1$.

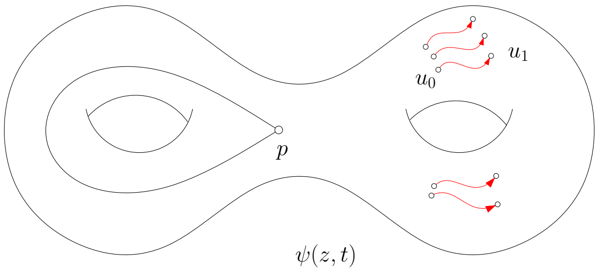
Proof.

Given a homotopy connecting u_0 and u_1, $h(z,t) : \Sigma_1 \times [0,t] \to \Sigma_2$, such that $h(z,0) = u_0(z)$, $h(z,1) = u_1(z)$. Let $\psi(z,t)$ is a geodesic from $u_0(z)$ to $u_1(z)$ and homotopic to $h(z,t)$, with parameter

$$\rho(\psi(z,t))|\dot{\psi}(z,t)| \equiv \text{const}$$

then $u_t(z) := \psi(z,t)$ is also a homotopy connecting u_0 and u_1. □

Uniqueness of Harmonic Map

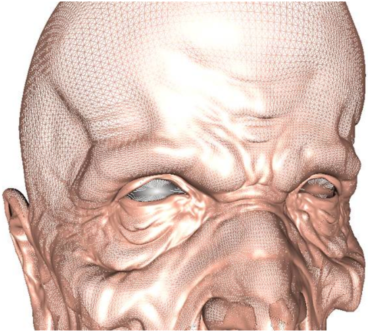


continue

We define function $f(t) := E(u_t)$. By above theorem, $\forall t \in [0,1]$, $\ddot{f}(t) \geq 0$, hence $f(t)$ is convex. Since u_0 and u_1 are harmonic, $\dot{f}(0) = \dot{f}(1) = 0$. By the assumption of the Jacobian matrix, either $\ddot{f}(0) > 0$ or $\ddot{f}(1) > 0$, hence we must have $\dot{\psi}(t) \equiv 0$, namely $u_0 \equiv u_1$. $\square$

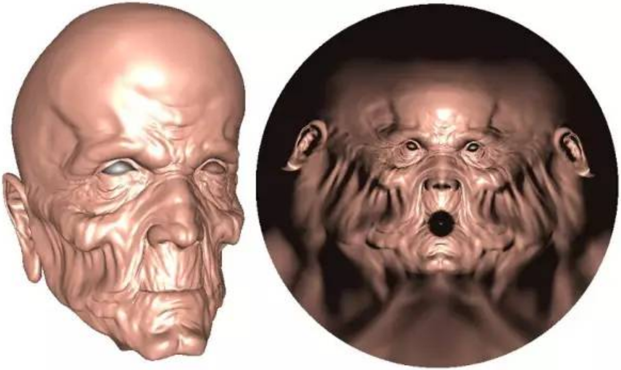
Finite Element Method

Finite Element Method



Given a smooth surface $(S, \mathbf{g})$, we can construct a sequence of triangle meshes $\varphi_n : S \to (M_n, \mathbf{d}_n)$, the pull back metric $\{\varphi_n^* \mathbf{d}_n\}$ converge to $\mathbf{g}$.

Finite Element Method



For each M_n, construct a harmonic map $f_n : M_n \to \mathbb{D}^2$. Then $\{f_n\}$ converge to the smooth harmonic map $f : S \to \mathbb{D}^2$.

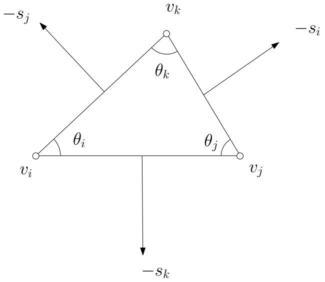
Finite Element Method

Lemma (Discrete Harmonic Energy)

Given a piecewise linear function $f : M \to \mathbb{R}$, then the harmonic energy of f is given by

$$E(f) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij} (f(v_i) - f(v_j))^2.$$

$w_{ij} = \cot \theta_{ij}^k + \cot \theta_{ji}^l.$



Finite Element Method

Definition (Barry-centric Coordinates)

Given a Euclidean triangle with vertices $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k$the bary-centric coordinates of a planar point $\mathbf{p} \in \mathbb{R}^2$ with respect to the triangle are $(\lambda_i, \lambda_j, \lambda_k)$, $\mathbf{p} = \lambda_i \mathbf{v_i} + \lambda_j \mathbf{v_j} + \lambda_k \mathbf{v_k}$, where

$$\lambda_i = \frac{(\mathbf{v}_j - \mathbf{p}) \times (\mathbf{v}_k - \mathbf{p}) \cdot \mathbf{n}}{(\mathbf{v}_j - \mathbf{v}_i) \times (\mathbf{v}_k - \mathbf{v}_i) \cdot \mathbf{n}},$$

the ratio between the area of the triangle $\mathbf{p}, \mathbf{v}_j, \mathbf{v}_k$ and the area of $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k$. λ_j and λ_k are defined similarly.

By direct computation, the sum of the bary-centric coordinates is 1

$$\lambda_i + \lambda_j + \lambda_k = 1.$$

If $\mathbf{p}$ is the interior point of the triangle, then all components of the bary-centric coordinates are positive.

Lemma

Suppose $f : \Delta \to \mathbb{R}$ is a linear function,

$$f(p) = \lambda_i f(v_i) + \lambda_j f(v_j) + \lambda_k f(v_k),$$

then the gradient of the function is

$$\nabla f(p) = \frac{1}{2A}(s_i f(v_i) + s_j f(v_j) + s_k f(v_k)).$$

its harmonic energy is

$$\int_\Delta \langle \nabla f, \nabla f \rangle dA = \frac{\cot\theta_i}{2}(f_j - f_k)^2 + \frac{\cot\theta_j}{2}(f_k - f_i)^2 + \frac{\cot\theta_k}{2}(f_i - f_j)^2. \quad (11)$$

Proof.

We have

$$\mathbf{s}_i + \mathbf{s}_j + \mathbf{s}_k = \mathbf{n} \times \{(\mathbf{v}_k - \mathbf{v}_j) + (\mathbf{v}_i - \mathbf{v}_k) + (\mathbf{v}_j - \mathbf{v}_i)\} = \mathbf{0}$$

therefore

$$\langle \mathbf{s}_i, \mathbf{s}_i \rangle = \langle \mathbf{s}_i, -\mathbf{s}_j - \mathbf{s}_k \rangle = -\langle \mathbf{s}_i, \mathbf{s}_j \rangle - \langle \mathbf{s}_i, \mathbf{s}_k \rangle.$$

pick a point $\mathbf{p}$

$$\mathbf{p} = \lambda_i \mathbf{v}_i + \lambda_j \mathbf{v}_j + \lambda_k \mathbf{v}_k,$$

bary-centric coordinates $|(\mathbf{v}_k - \mathbf{v}_j) \times (\mathbf{p} - \mathbf{v}_j)| = \langle \mathbf{p} - \mathbf{v}_j, \mathbf{s}_i \rangle$

$$\lambda_i = \frac{1}{2A}\langle \mathbf{p} - \mathbf{v}_j, \mathbf{s}_i \rangle, \lambda_j = \frac{1}{2A}\langle \mathbf{p} - \mathbf{v}_k, \mathbf{s}_j \rangle, \lambda_k = \frac{1}{2A}\langle \mathbf{p} - \mathbf{v}_i, \mathbf{s}_k \rangle,$$

where A is the triangle area. $\square$

Finite Element Method

continued

The linear function is

$$
\begin{aligned}
f(\mathbf{p}) &= \lambda_i f_i + \lambda_j f_j + \lambda_k f_k \\
&= \frac{1}{2A}\langle \mathbf{p} - \mathbf{v}_j, f_i \mathbf{s}_i \rangle + \frac{1}{2A}\langle \mathbf{p} - \mathbf{v}_k, f_j \mathbf{s}_j \rangle + \frac{1}{2A}\langle \mathbf{p} - \mathbf{v}_i, f_k \mathbf{s}_k \rangle \\
&= \langle \mathbf{p}, \frac{1}{2A}(f_i \mathbf{s}_i + f_j \mathbf{s}_j + f_k \mathbf{s}_k) \rangle - \frac{1}{2A}(\langle \mathbf{v}_j, f_i \mathbf{s}_i \rangle + \langle \mathbf{v}_k, f_j \mathbf{s}_j \rangle + \langle \mathbf{v}_i, f_k \mathbf{s}_k \rangle).
\end{aligned}
$$

Hence we obtain the gradient

$$
\nabla f = \frac{1}{2A}(f_i \mathbf{s}_i + f_j \mathbf{s}_j + f_k \mathbf{s}_k).
$$

Finite Element Method

continued

we compute the harmonic energy

$$
\begin{aligned}
&\int_{\Delta} \langle \nabla f, \nabla f \rangle dA \\
=&\frac{1}{4A} \langle f_i \mathbf{s}_i + f_j \mathbf{s}_j + f_k \mathbf{s}_k, f_i \mathbf{s}_i + f_j \mathbf{s}_j + f_k \mathbf{s}_k \rangle \\
=&\frac{1}{4A} \left(\sum_i \langle \mathbf{s}_i, \mathbf{s}_i \rangle f_i^2 + 2 \sum_{i<j} \langle \mathbf{s}_i, \mathbf{s}_j \rangle f_i f_j \right) \\
=&\frac{1}{4A} \left(-\sum_i \langle \mathbf{s}_i, \mathbf{s}_j + \mathbf{s}_k \rangle f_i^2 + 2 \sum_{i<j} \langle \mathbf{s}_i, \mathbf{s}_j \rangle f_i f_j \right) \\
=&-\frac{1}{4A} \left(\langle \mathbf{s}_i, \mathbf{s}_j \rangle (f_i - f_j)^2 + \langle \mathbf{s}_j, \mathbf{s}_k \rangle (f_j - f_k)^2 + \langle \mathbf{s}_k, \mathbf{s}_i \rangle (f_k - f_i)^2 \right)
\end{aligned}
$$

Finite Element Method

continued

Since

$$\frac{\langle \mathbf{s}_i, \mathbf{s}_j \rangle}{2A} = -\cot\theta_k, \frac{\langle \mathbf{s}_j, \mathbf{s}_k \rangle}{2A} = -\cot\theta_i, \frac{\langle \mathbf{s}_k, \mathbf{s}_i \rangle}{2A} = -\cot\theta_j.$$

Hence the harmonic energy is

$$\int_\Delta \langle \nabla f, \nabla f \rangle dA = \frac{\cot\theta_i}{2}(f_j - f_k)^2 + \frac{\cot\theta_j}{2}(f_k - f_i)^2 + \frac{\cot\theta_k}{2}(f_i - f_j)^2.$$

$\square$.

Finite Element Method

Lemma (Discrete Harmonic Energy)

Given a piecewise linear function $f : M \to \mathbb{R}$, then the harmonic energy of f is given by

$$E(f) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij}(f(v_i) - f(v_j))^2.$$

$w_{ij} = \cot\theta_{ij}^k + \cot\theta_{ji}^l.$

Proof.

We add the harmonic energies on all faces together, and merge the items associated with the same edge, then each edge contributes $w_{ij}(f_j - f_i)^2$. □