EE 810. Home work #4

Name. Onkar Vively Apts,

D80 ID .

6. Find all vectors in
$$\mathbb{R}^3$$
 that are orthogonal to $(1,1,1)$ and $(1,-1,0)$. Produce an orthonormal basis from these vectors (mutually orthogonal unit vectors).

0+b+c=0

To make them unit,
$$|a| = \sqrt{1+1+1} = \sqrt{3}$$

 $|b| = \sqrt{1+1} = \sqrt{2}$
 $|c| = \sqrt{4^2+4^2+4a^2} = a\sqrt{6}$

7. Find a vector x orthogonal to the row space of A, and a vector y orthogonal to the column space, and a vector z orthogonal to the nullspace:

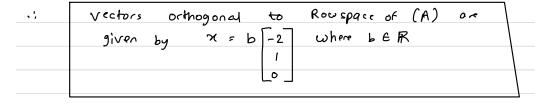
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}.$$

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	2	4	3	
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$$\therefore \qquad Q + 2b + C = 0$$

a - 2 b



Now,
$$A^{T} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 4 \end{bmatrix}$$

$$R_{2} \rightarrow R_{2} - 2R_{1}$$

$$R_{3} \rightarrow R_{3} - R_{1}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Let $Y = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be orthogonal to coloumn space.

$$\therefore \begin{bmatrix} 1 & 2 & 3 \\ b & c \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore A + 2b + 3c = 0.$$

$$b + c = 0. \rightarrow b = -c$$

$$\therefore A + 2b - 3c = 0. \rightarrow a = b.$$

$$\therefore Y = \begin{bmatrix} b \\ b \\ -b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\therefore Vectors orthogonal to coloumn space of A are given by $Y = b \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ where $b \in R$$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_5 \to R_5 - R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & c \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-2a + b + o(c) = o. \longrightarrow : Z = \begin{bmatrix} K \\ 2K \end{bmatrix} = K \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

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_ 15. Find a matrix whose row space contains (1,2,1) and whose nullspace contains (1,-2,1), or prove that there is no such matrix.

· Vectors in nulspace are orthogonal to vectors in rowspace.

(1)(1) + (2)(-2) + (1)(1) = 0

1-4+1 = 0.

 $-2 = 0 \longrightarrow false.$

which means -> there exist no such matrix

whose row space contains [1] & null space contains [1]

2
1



5. In *n* dimensions, what angle does the vector $(1,1,\ldots,1)$ make with the coordinate axes? What is the projection matrix *P* onto that vector?

Let
$$a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
. Consider coordinate axis $\mathcal{X} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

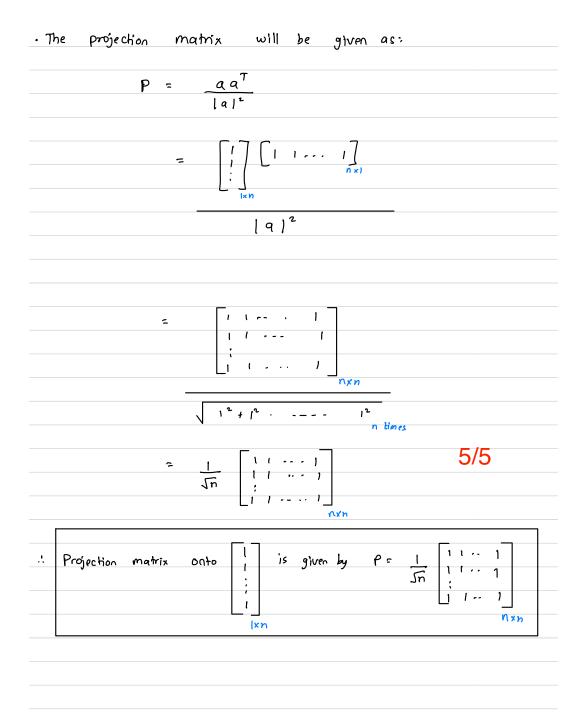
• Finding
$$|a| = |a|^2 + |a|^2 - |a|^2 + |a|^$$

= \n

$$|\mathcal{M}| = \sqrt{|^2 + 0^2 + 0^2 - - + 0^2}$$

Finding
$$[M] = \sqrt{1^2 + 0^2 + 0^2} - + 0^2$$
= $\sqrt{1}$

Finding
$$a.b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1)(1) + (1)(0) +(1)(0)$$



7. By choosing the correct vector b in the Schwarz inequality, prove that

$$(a_1 + \cdots + a_n)^2 \le n(a_1^2 + \cdots + a_n^2).$$

When does equality hold?

Let
$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$
 $b = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

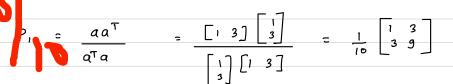
: using schwars inequality, we get:

is schools inequality becomes equality when a k b

lie on the same line, for this case, it is only

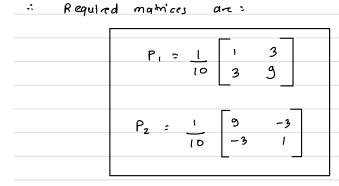
possible if
$$a_1 = a_2 = a_3 = \dots = a_n$$
.

- 11. (a) Find the projection matrix P_1 onto the line through $a = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and also the matrix P_2 that projects onto the line perpendicular to a.
 - (b) Compute $P_1 + P_2$ and P_1P_2 and explain.



Now, to find P2 which projects onto a line 1 to $\begin{bmatrix} 1\\3 \end{bmatrix}$,

P2 = $T - P_1$ = $\begin{bmatrix} 1 & 0\\0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{10} & \frac{3}{10}\\\frac{3}{10} & \frac{9}{10} \end{bmatrix}$ = $\begin{bmatrix} \frac{9}{10} & -\frac{3}{10}\\-\frac{3}{10} & \frac{1}{10} \end{bmatrix}$ = $\frac{1}{10} \begin{bmatrix} 9 & -3\\0 & \frac{3}{10} & \frac{3}{10} \end{bmatrix}$



$$P_1 + P_2 = \frac{1}{10} \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} + \frac{1}{10} \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$$

$$P_1 + P_2 = I$$
 Sum of two orthogonal Projections is identity.

$$\rho, \rho_2 = \frac{1}{10} \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \cdot \frac{1}{10} \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$$

12. Find the matrix that projects every point in the plane onto the line x + 2y = 0.

: x = -2y

: vector associated with this line = \[-2y \] = y \[-2 \]

for y=1, vector = -2

· Let the vector on (n + 2y = 0) be $Z = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

- Projector on Z will be given by:

 $\begin{array}{ccc}
\rho : & 22^{T} \\
\hline
7^{T} & 2
\end{array} = \begin{bmatrix}
-2 \\
1
\end{bmatrix} \begin{bmatrix}
-2 \\
1
\end{bmatrix}$

: Matrix that projects every point onto 2+2y=0is $1 \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$

Nou, to

minimize

E2

1. Find the best least-squares solution \hat{x} to 3x = 10, 4x = 5. What error E^2 is minimized? Check that the error vector $(10 - 3\hat{x}, 5 - 4\hat{x})$ is perpendicular to the column (3, 4).

$$a = \begin{bmatrix} 3 \end{bmatrix}$$
 and $b = \begin{bmatrix} 0 \end{bmatrix}$

$$= |a\hat{n} - b|^2$$

$$= (a_1 \hat{n} - b_1)^2 + (a_2 \hat{n} - b_2)^2$$

$$= (3 \cdot 2 - 10)^2 + (4 \cdot 2 - 5)^2$$

$$= (-4)^2 + (3)^2$$

5/5

(e.a) =
$$[10-3\hat{n}]$$
 5-4 \hat{n}] $[3]$ = $[10-(9x2)]$ 5-(4x2)

$$= \begin{bmatrix} 4 & -3 \end{bmatrix} \begin{bmatrix} \frac{3}{4} \end{bmatrix} = \underbrace{0}$$

3.3 (6)



$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}. \qquad \dots$$

Finding

$$A^{\mathsf{T}} A = \begin{bmatrix} 1 & 1 & -2 \\ & 1 & -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & -8 & 18 \end{bmatrix}$$

$$\therefore \left(A^{T} A\right)^{-1} = \underbrace{1}_{44} \begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix}$$

Now, Calculating
$$A^Tb = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -1 & 4 \end{bmatrix}$$

2. Project b = (0,3,0) onto each of the orthonormal vectors $a_1 = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ and $a_2 = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, and then find its projection p onto the plane of a_1 and a_2 .

$$a_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ -1/3 \end{bmatrix}$$
, $a_2 = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$

.. Projecton of b on $a_1 = P_1 = b \cdot a_1 \quad a_1 \quad a_1 \cdot a_1$



$$= \frac{\left(0 \times \frac{2}{3}\right) + \left(3 \times \frac{2}{3}\right) + \left(0 \times \frac{2}{3}\right)}{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} \begin{bmatrix} 2/2 \\ 2/3 \\ -1/3 \end{bmatrix}$$

$$=\frac{2}{1}\begin{bmatrix} 2/3\\ 2/3\\ -1/3 \end{bmatrix}$$

Projection of b on a, is 4/3
4/3
-2/3

3.4(5)

=. Nou, costder

- 5. If u is a unit vector, show that $Q = I 2uu^{T}$ is a symmetric orthogonal matrix. (It is a reflection, also known as a Householder transformation.) Compute Q when $u^{T} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.
- · It is given that Q = I 2uuT where u is unit vector.
- . We have to prove Q is symmetric k orthogonal.

 The Q is symmetric, $Q^T = Q$ must be true.
- . If Q is orthogonal, QTQ = I must be true.
- $Cosider \qquad Q^{T}Q = \left(I 2uu^{T}\right)^{T} \left(I 2uu^{T}\right)$
 - $= \left[\int_{-\infty}^{\infty} -2(u u^{T})^{T} \right] \left(\int_{-\infty}^{\infty} 2u u^{T} \right)$
 - $= \left[I 2 \left(\left(u^{T} \right)^{T} u^{T} \right) \right] \left(I 2 u^{T} \right)$
 - = (I 244T) (I 244T)
 - = I 244 244 + 44 utu ut
 - = $T 4uu^{\dagger} + 4uu^{\dagger}$
 - = I hence proved, QTQ = I.
 - .. Q is orthogonal.
 - T (T . . . T)
 - $Q^{T} = \left(T 2uu^{T} \right)^{T} = \left(T^{T} 2\left(uu^{T} \right)^{T} \right)$ $= T^{T} 2\left(u^{T} \right)^{T} u^{T} \right)$
 - = I 2 44^T
 - = Q.
 - Lhence proved, QT = Q.
 - .. Q is symmetric. ©
 - : from statement () k (),
 - It has been proved that Q is a symmetric orthogonal matrix

6. Find a third column so that the matrix

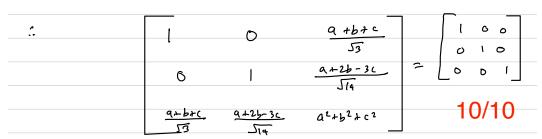
$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} \\ 1/\sqrt{3} & 2/\sqrt{14} \\ 1/\sqrt{3} & -3/\sqrt{14} \end{bmatrix}$$

is orthogonal. It must be a unit vector that is orthogonal to the other columns; how much freedom does this leave? Verify that the rows automatically become orthonormal at the same time.

Let	coloum	3	be	C3 =	[a]	~	<i>.</i> :.	Q =	1/53	1/514	9
					ء ا				1/53	2/514	Ь
									1/53	-3/514	С

~ Q is orthogonal,

1/53	1/53	1/53	1/53	1/514	a		$\lceil \iota \rceil$	۵	6
1/514	2/514	-3/JIA	1/53	2/514	b	(1	٥	1	0
9	Ь	1/53 -3/1/19 C	1/53	-3/514	c _		٥	ο	ι



$$|R_1| = \sqrt{\frac{1}{3} + \frac{1}{14} + \frac{25}{42}} = 1$$

$$|R_2| = \sqrt{\frac{1}{2} + \frac{4}{14} + \frac{16}{42}} = 1$$

$$|R_3| = \sqrt{\frac{1}{3} + \frac{9}{14} + \frac{1}{42}} = 1$$

$$|R_4| = \sqrt{\frac{1}{3} + \frac{9}{14} + \frac{1}{42}} = 1$$

$$R_{1} \cdot R_{3} = \left(\frac{1}{53} \cdot \frac{1}{53}\right) + \left(\frac{1}{514} \cdot \frac{-3}{514}\right) + \left(\frac{-5}{542} \cdot \frac{1}{542}\right)$$

$$= \frac{1}{3} - \frac{3}{4} - \frac{5}{42} = 0$$

$$R_{2}.R_{3} = \left(\frac{1}{53} \cdot \frac{1}{53}\right) + \left(\frac{2}{514} \cdot \frac{-3}{514}\right) + \left(\frac{4}{542} \cdot \frac{1}{542}\right)$$

$$=\frac{1}{3}-\frac{6}{4}+\frac{4}{42}=0$$

$$R_1 \cdot R_2 = \left(\frac{1}{53} \cdot \frac{1}{53}\right) + \left(\frac{1}{515} \cdot \frac{2}{515}\right) + \left(\frac{-5}{542} \cdot \frac{4}{542}\right)$$

we have verified that rows became orthonormal.





$$a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and write the result in the form A = QR.

$$= \frac{q}{(q)} = \frac{1}{\sqrt{\sigma^2 + \sigma^2 + 1^2}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$B = b - (q_1 \cdot b) q_1$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore q_2 : \underline{B} = \frac{1}{\sqrt{o^2 + 1^2 L_0^2}} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore q_3 = \frac{C}{|C|} = \frac{1}{\sqrt{(2+0)^2+6^2}} \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$$

Nov, writing in	A=QR form,
A =	Q R
[a]b;c]=	[q,: q2: q3
	0 9 _{1.6} 9 _{2.} C
	O O 13. C
r. 0 0 1 =	0 0 1 1 1
	010 011
[(1]	(00] 001
	This is the required result
	in A = QR form.

14. From the nonorthogonal a, b, c, find orthonormal vectors q_1, q_2, q_3 :

$$a = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad c = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

$$q_1 = \frac{1}{191} = \frac{1}{\sqrt{1^2 + 1^2 + 6^2}} = \frac{1}{6} = \frac{1}{52} = \frac{1}{6} = \frac{1}{52} = \frac{1}{6}$$

$$\therefore q_{2} = \frac{\beta}{|\beta|} = \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4} + 1}} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{36}} \\ -\frac{1}{\sqrt{36}} \\ \frac{2}{\sqrt{36}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\frac{1}{1} = \frac{1}{|C|} = \frac{1}{\sqrt{|C|^2 + |C|^2 + |C|^2}} = \frac{-1/53}{1/53}$$

: Or thonormal vectors are:
$$(9, 9_2, 9_3) = \{ [1/52], [1/56], [-1/53] \}$$