EE 810: Homework #9

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7.2(3)

10/10

3. Explain why $\|ABx\| \le \|A\|\|B\|\|x\|$, and deduce from equation (5) that $\|AB\| \le \|A\|\|B\|$. Show that this also implies $c(AB) \le c(A)c(B)$.

$\because$ Norm of A is the number: $\|A\|^2 = \max\limits_{x \neq 0} \dfrac{\|Ax\|^2}{\|x\|^2}$,

$\therefore \|A\|^2 \geqslant \dfrac{\|Ax\|^2}{\|x\|^2}$ — $(x \neq 0)$

$\therefore$ Norm when non-negative quantity, $\|A\| \geqslant \dfrac{\|Ax\|}{\|x\|}$ — $(x \neq 0)$

$\therefore \|Ax\| \leqslant \|A\| \cdot \|x\|$

$\therefore \|ABx\| = \|A(Bx)\| \leq \|A\| \cdot \|Bx\|$

$\leq \underline{\|A\| \cdot \|B\| \cdot \|x\|}$

$\because \|ABx\| \leq \|A\| \cdot \|B\| \cdot \|x\| \longrightarrow \dfrac{\|ABx\|}{\|x\|} \leq \|A\| \cdot \|B\|$

$\therefore \max\limits_{x=0} \dfrac{\|(AB)x\|}{\|x\|} \leqslant \|A\| \cdot \|B\|$

$\therefore \underline{\|AB\| \leqslant \|A\| \cdot \|B\|}$

Now $c(AB) = \|AB\| \cdot \|(AB)^{-1}\|$

$= \|AB\| \cdot \|B^{-1}A^{-1}\|$

$\left(\because \|B^{-1}A^{-1}\| \leqslant \|B^{-1}\| \cdot \|A^{-1}\|\right)$ $\|AB\| \cdot \|B^{-1}A^{-1}\| \leq \|A\| \cdot \|B\| \cdot \|B^{-1}\| \cdot \|A^{-1}\|$

$= \|A\| \cdot \|A^{-1}\| \cdot \|B\| \cdot \|B^{-1}\|$

$\left(\& \because \begin{array}{l} c(A) = \|A\| \cdot \|A^{-1}\| \\ c(B) = \|B\| \cdot \|B^{-1}\|, \end{array}\right)$

$= \underline{c(A) \cdot c(B)}$

Hence showed: $\|ABx\| \leq \|A\| \cdot \|B\| \cdot \|x\|$
$\|AB\| \leq \|A\| \cdot \|B\|$
$c(AB) \leq c(A)\, c(B)$

7.2(4)

4. For the positive definite $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, compute $\|A^{-1}\| = 1/\lambda_1$, $\|A\| = \lambda_2$, and $c(A) = \lambda_2/\lambda_1$. Find a right-hand side b and a perturbation δb so that the error is the worst possible, $\|\delta x\|/\|x\| = c\|\delta b\|/\|b\|$.

$$|A - I\lambda| = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$\therefore \; 4 - 4\lambda + \lambda^2 - 1 = 0$

$\therefore \; \lambda^2 - 4\lambda + 3 = 0 \;\nearrow\; \lambda_1 = 3, \; \lambda_2 = 1$

$\therefore \lambda_{min} = 1$, $\lambda_{max} = 3$

$\therefore \; \|A^{-1}\| = \dfrac{1}{\lambda_{min}} = \underline{1}$

$\therefore \; \|A\| = \lambda_{max} = \underline{3}$

$\therefore \; c(A) = \|A\| \cdot \|A^{-1}\|$
$= \underline{3}$

Now, for $\lambda = 1$, $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\therefore \; x_1 - x_2 = 0$

$\vec{x_1} : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longleftarrow \; \therefore \; x_1 = x_2$

Now, for $\lambda = 3$, $\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\therefore \; -x_1 - x_2 = 0$

$\vec{x_2} : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \longleftarrow \; \therefore \; x_1 = -x_2$

$\therefore$ worst case error : $(1,1)$ $\therefore$ value of $\delta b = \varepsilon(1,-1)$	$\therefore \|A^{-1}\| = 1$ $\therefore \|A\| = 3$	& $c(A) = 3$

$\boxed{7.2(9)}$

9. Show that $\max|\lambda|$ is not a true norm, by finding 2 by 2 counterexamples to $\lambda_{\max}(A+B) \leq \lambda_{\max}(A)+\lambda_{\max}(B)$ and $\lambda_{\max}(AB) \leq \lambda_{\max}(A)\lambda_{\max}(B)$.

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\therefore$ $\lambda_{max}\, A = 0$

$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\therefore$ $\lambda_{max}\, B = 0$

$A+B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\therefore$ $|A+B - I\lambda| = 0$

$\therefore$ $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$

$\therefore$ $\lambda^2 - 1 = 0$

$\therefore$ $\lambda^2 = 1$

$\therefore$ $\lambda = \pm 1$

$\therefore$ $\lambda_{max}\,(A+B) = +1$

$\therefore$ $\lambda_{max}\,(A+B) > \lambda_{max}\, A + \lambda_{max}\, B$

$(1 > 0 + 0)$

Now, $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\therefore$ $\lambda_{max}\, AB = 1$

$\therefore$ $\lambda_{max}\, AB > \lambda_{max}\, A + \lambda_{max}\, B$

$(1 > 0 + 0)$

$\therefore$ Counter examples for given statements are shown.

7.2 (13)

13. Find the norms and condition numbers from the square roots of $\lambda_{\max}(A^TA)$ and $\lambda_{\min}(A^TA)$:

$$① \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \quad ② \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad ③ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

① $A^TA = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ $\quad \therefore \lambda_{max} = 4$, $\lambda_{min} = 4$

$\therefore \|A\| = \sqrt{\lambda_{max}(A^TA)} = \sqrt{4} = \underline{2}$ $\quad$ & $\quad c(A) = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}} = \sqrt{\frac{4}{4}} = \underline{1}$

② $A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad \therefore \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$

$\therefore (1-\lambda)^2 - 1 = 0$

$\therefore 1 - 2\lambda + \lambda^2 - 1 = 0$

$\therefore \lambda(\lambda - 2) = 0$

$\therefore \lambda_{max} = 2$

$\therefore \lambda_{min} = 0$

$\therefore \|A\| = \sqrt{\lambda_{max}(A^TA)} = \underline{\sqrt{2}}$ $\quad$ & $\quad c(A) = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}} = \sqrt{\frac{2}{0}} = \underline{\infty}$

③ $A^TA = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad \therefore \lambda_{max} = 2$, $\lambda_{min} = 2$

$\therefore \|A\| = \sqrt{\lambda_{max}(A^TA)} = \sqrt{2}$ $\quad$ & $\quad c(A) = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}} = \sqrt{\frac{2}{2}} = \underline{1}$

$\therefore$

Matrix $\boxed{1}$: $\|A\| = 2$, $c(A) = 1$	• Matrix ②: $\|A\| = \sqrt{2}$, $c(A) = \infty$	• Matrix $\boxed{3}$: $\|A\| = \sqrt{2}$, $c(A) = 1$

$\boxed{7.3\ (1)}$ 10/10

1. For the matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$, apply the power method $u_{k+1} = Au_k$ three times to the initial guess $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What is the limiting vector u_∞?

Applying power method 3 times $\rightarrow$

$$\therefore\ u_1 = Au_0 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\therefore\ u_2 = Au_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

$$\therefore\ u_3 = Au_3 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 14 \\ -13 \end{bmatrix}$$

& it is given : $\lambda_2 = 3$ i.e. $> \lambda_1 = 1$

$\therefore$ limiting vector u_∞ is a multiple of eigenvector corresponding to $\lambda = 3$.

$$\therefore\ \left[\begin{array}{cc|c} 2-3 & -1 & 0 \\ -1 & 2-3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore\ -x_1 - x_2 = 0$

$\therefore\ x_1 = -x_2$ $\qquad \therefore\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\boxed{\therefore\ \text{limiting vector is a multiple of } \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

$\boxed{7.4\ (1)}$

1. This matrix has eigenvalues $2-\sqrt{2}$, 2, and $2+\sqrt{2}$:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Find the Jacobi matrix $D^{-1}(-L-U)$ and the Gauss-Seidel matrix $(D+L)^{-1}(-U)$ and their eigenvalues, and the numbers ω_{opt} and λ_{max} for SOR.

$\because \quad A = L + D + U$

$$\therefore \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{L} + \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{U}$$

$$\therefore D^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad \& \quad -L-U = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\therefore D^{-1}(-L-U) = \frac{1}{2} I \cdot (-L-U)$$

$$= \frac{1}{2}(-L-U) = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

Clearly,

$$\begin{cases} \lambda_1 = \frac{1}{\sqrt{2}} \\ \lambda_2 = \frac{-1}{\sqrt{2}} \\ \lambda_3 = 0 \end{cases}$$

$$\therefore \mu_{max} = \frac{1}{\sqrt{2}}$$

Now, $$D + L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\therefore \quad (D+L)^{-1} = \frac{1}{8}\begin{bmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\therefore \quad (D+L)^{-1}U = \frac{1}{8}\begin{bmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 0 & -4 & 0 \\ 0 & -2 & -4 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\therefore \quad \lambda_1 = \frac{-1}{2}, \quad \lambda_2 = 0, \quad \lambda_3 = 0$$

$\therefore$ Now,

$$\omega_{opt} = \frac{2\left(1-\sqrt{1-\mu_{max}^2}\right)}{\mu_{max}^2} = \frac{2\left(1-\sqrt{1-\frac{1}{2}}\right)}{\frac{1}{2}}$$

$$= 4\left(1-\sqrt{\tfrac{1}{2}}\right)$$

$$= 4\left(1-\frac{1}{\sqrt{2}}\right)$$

$$= \frac{4}{\sqrt{2}}\left(\sqrt{2}-1\right)$$

$$= \frac{4\left(2-\sqrt{2}\right)}{2}$$

$$= 4-2\sqrt{2}$$

$$\therefore \quad \lambda_{max} = \omega_{opt} - 1$$

$$= 3-2\sqrt{2}$$

$$\therefore \quad \boxed{\begin{aligned} \omega_{opt} &= 4-2\sqrt{2} \\ \lambda_{max} &= 3-2\sqrt{2} \end{aligned}}$$

7.4 (4)

4. The matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 4 & 1 \\ 2 & 2 & 5 \end{bmatrix}$$

is called *diagonally dominant* because every $|a_{ii}| > r_i$. Show that zero cannot lie in any of the circles, and conclude that A is nonsingular.

$\therefore$ eigenvalues:

Input

eigenvalues $\begin{pmatrix} 3 & 1 & 1 \\ 0 & 4 & 1 \\ 2 & 2 & 5 \end{pmatrix}$

Results

$\lambda_1 \approx 6.56155$

$\lambda_2 = 3$

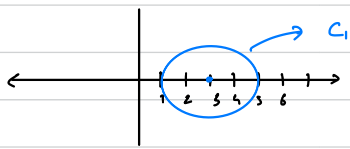
$\lambda_3 \approx 2.43845$

The circles that bound eigenvalues are C_1, C_2, C_3.

$\therefore$ Centre of $C_1 = (3, 0)$

radius $r_1 = |1| + |1|$

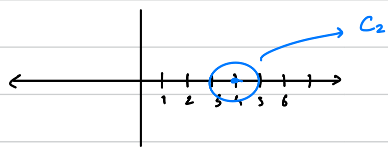
$= 2$



$\therefore$ Centre of $C_2 = (4, 0)$

radius $r_2 = |0| + |1|$

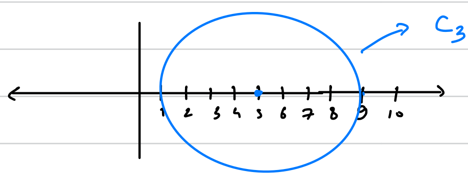
$= 1$



$\therefore$ Centre of $C_3 = (5, 0)$

radius $r_3 = |2| + |2|$

$= 4$



$\boxed{\text{(0,0) does not lie in any of the circles}}$

7.4(5)

5. Write the Jacobi matrix J for the diagonally dominant A of Problem 4, and find the three Gershgorin circles for J. Show that all the radii satisfy $r_i < 1$, and that the Jacobi iteration converges.

$\because \quad A = L + D + U$

$$\therefore \begin{bmatrix} 3 & 1 & 1 \\ 0 & 4 & 1 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad L + U = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix} \quad \& \quad D^{-1} =$$

Input

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}^{-1} \quad \text{(matrix inverse)}$$

Result

$$= \frac{1}{60}\begin{pmatrix} 20 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

Now, $J = D^{-1}(L+U)$

$$= \frac{1}{60}\begin{bmatrix} 20 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 12 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 0 & 0 & 1/4 \\ 2/5 & 2/5 & 0 \end{bmatrix}$$

$\therefore$ Gershgorin points :

$C_1 \doteq (0,0)$, $\quad r_1 = \left|\frac{1}{3}\right| + \left|\frac{1}{3}\right| = \underline{\frac{2}{3}}$

$C_2 \doteq (0,0)$, $\quad r_2 = |0| + \left|\frac{1}{4}\right| = \underline{\frac{1}{4}}$

$C_3 \doteq (0,0)$, $\quad r_3 = \left|\frac{2}{5}\right| + \left|\frac{2}{5}\right| = \underline{\frac{4}{5}}$

Hence, we have shown that $r_1, r_2, r_3 < 0$ & Jacobian iteration converges.

$\boxed{7.4\ (7)}$

12/10

7. For a general 2 by 2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

find the Jacobi iteration matrix $S^{-1}T = -D^{-1}(L+U)$ and its eigenvalues μ_i. Find also the Gauss-Seidel matrix $-(D+L)^{-1}U$ and its eigenvalues λ_i, and decide whether $\lambda_{\max} = \mu_{\max}^2$.

$\because\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ L = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix},\ D = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix},\ U = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$

$\therefore\ D^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{bmatrix}$ & $L+U = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$

$\therefore$ Jacobian Iteration Mx $S^{-1}T = -D^{-1}(L+U) = -\begin{bmatrix} 1/a & 0 \\ 0 & 1/d \end{bmatrix}\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & \frac{-b}{a} \\ \frac{-c}{d} & 0 \end{bmatrix}$$

$\therefore\ \lambda$ of $S^{-1}T \Rightarrow$

Input

eigenvalues $\begin{pmatrix} 0 & -\frac{b}{a} \\ -\frac{c}{d} & 0 \end{pmatrix}$

Results

$\lambda_1 = -\frac{\sqrt{b}\,\sqrt{c}}{\sqrt{a}\,\sqrt{d}}$

$\lambda_2 = \frac{\sqrt{b}\,\sqrt{c}}{\sqrt{a}\,\sqrt{d}}$

$\therefore\ \mu_{max} = \frac{\sqrt{b}\,\sqrt{c}}{\sqrt{a}\,\sqrt{d}}$

Now, $D+L = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$, $\therefore\ (D+L)^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ \frac{-c}{ad} & \frac{1}{d} \end{bmatrix}$

$\therefore\ -(D+L)^{-1}U = \begin{bmatrix} \frac{1}{a} & 0 \\ \frac{-c}{ad} & \frac{1}{d} \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & \frac{-b}{a} \\ 0 & \frac{cb}{ad} \end{bmatrix}$$

$\therefore$ Eigenvalues of gauss-sidel matrix, $-(D+L)^{-1}U$

Input

eigenvalues	$\begin{pmatrix} 0 & -\frac{b}{a} \\ 0 & \frac{cb}{ad} \end{pmatrix}$

Results

$\lambda_1 = 0$

$\lambda_2 = \frac{bc}{ad}$

$\therefore \lambda_{max} = \frac{bc}{ad}$

& we calculated $\mu_{max} = \frac{\sqrt{b}\sqrt{c}}{\sqrt{a}\sqrt{d}} = \sqrt{\frac{bc}{ad}}$

$\therefore \lambda_{max} = \mu_{max}^2$

$\therefore$ Jacobian iteration matrix : $S^{-1}T = \begin{bmatrix} 0 & \frac{-b}{a} \\ \frac{-c}{d} & 0 \end{bmatrix}$

& its eigenvalues $\Rightarrow \pm\sqrt{\frac{bc}{ad}}$

$\therefore$ Gauss-sidel matrix $= \begin{bmatrix} 0 & \frac{-b}{a} \\ 0 & \frac{cb}{ad} \end{bmatrix}$

& its eigenvalues : $0, \frac{cb}{ad}$

& $\lambda_{max} = \mu^2_{max}$ —— True

$\boxed{7.4\,(10)}$

10. Show why the iteration $x_{k+1} = (I - A)x_k + b$ does not converge for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

$$I - A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Let $x_k = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Let $b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\therefore\ x_1 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$

$$\therefore\ x_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 + x_1 - x_2 \\ -x + x_2 - x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - 2x_2 \\ -2x_1 + 2x_2 \end{bmatrix}$$

$$\therefore\ x_3 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2x_1 - 2x_2 \\ -2x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} -4x_1 + 4x_2 \\ 4x_1 - 4x_2 \end{bmatrix}$$

$$\therefore\ x_4 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -4x_1 + 4x_2 \\ 4x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 8x_1 - 8x_2 \\ -8x_1 + 8x_2 \end{bmatrix}$$

$$\therefore\ x_k = \begin{bmatrix} (-1)^k \left(2^{k-1}x_1 - 2^{k-1}x_2\right) \\ (-1)^k \left(-2^{k-1}x_1 + 2^{k-1}x_2\right) \end{bmatrix} \longrightarrow$$

Clearly, this does not converge unless $x_1 = x_2$, regardless the value of vector b.

$\therefore$ $x_k = (I - A)x_k + b$ does not converge for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

7.4 (17)

17. What bound on $|\lambda|_{\max}$ does Gershgorin give For these matrices (see Problem 4)? What are the three Gershgorin circles that contain all the eigenvalues?

$$1 \quad A = \begin{bmatrix} .3 & .3 & .2 \\ .3 & .2 & .4 \\ .2 & .4 & .1 \end{bmatrix} \qquad 2 \quad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

The key point for large matrices is that matrix-vector multiplication is much faster than matrix-matrix multiplication. A crucial construction starts with a vector b and computes $Ab, A^2b, \ldots$ (but never A^2!). The first N vectors span the Nth ***Krylov subspace***. They are the columns of the ***Krylov matrix*** K_N:

$$K_N = \begin{bmatrix} b & Ab & A^2b & \cdots & A^{N-1}b \end{bmatrix}.$$

The Arnoldi-Lanczos iteration orthogonalizes the columns of K_N, and the conjugate gradient iteration solves $Ax = b$ when A is symmetric positive definite.

1 for row 1, $\lambda_1 = |0.3| + |0.3| + |0.2| = 0.8$

for row 2, $\lambda_2 = |0.3| + |0.2| + |0.4| = 0.9$ $\Big\}$ $\lambda_{max} = 0.9$

for row 3, $\lambda_3 = |0.2| + |0.4| + |0.1| = 0.7$

Eigenvalues of A :

Input

eigenvalues	$\begin{pmatrix} 0.3 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.1 \end{pmatrix}$

Results

$\lambda_1 \approx 0.806318$

$\lambda_2 \approx -0.258992$

$\lambda_3 \approx 0.0526744$

$\therefore C_1 \equiv (0.3, 0)$

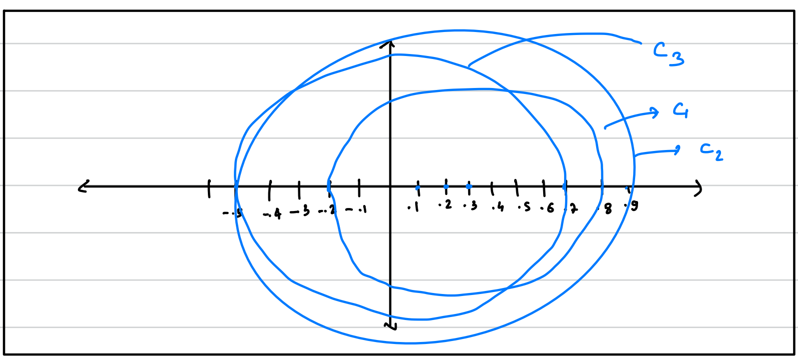
$r_1 = |0.3| + |0.2| = 0.5$

$C_2 \equiv (0.2, 0)$

$r_2 = |0.3| + |0.4| = 0.7$

$C_3 = (0.1, 0)$

$r_3 = |0.2| + |0.4| = 0.6$



• C_3 contains all eigenvalues

• bound on $|\lambda|$

$|-0.5| < |\lambda| < |0.9|$

②

for row 1, $\lambda_1 = |2| + |-1| + |0| = 3$

for row 2, $\lambda_2 = |-1| + |2| + |-1| = 4$

for row 3, $\lambda_3 = |0| + |-1| + |2| = 3$

$\Big\}\ \lambda_{max} = 4$

Eigenvalues of A :

Input

eigenvalues $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

Results

$\lambda_1 = 2+\sqrt{2}$ $\approx$ 3.41

$\lambda_2 = 2$

$\lambda_3 = 2-\sqrt{2}$ $\approx$ 0.5

$\therefore C_1 \equiv (2,0)$

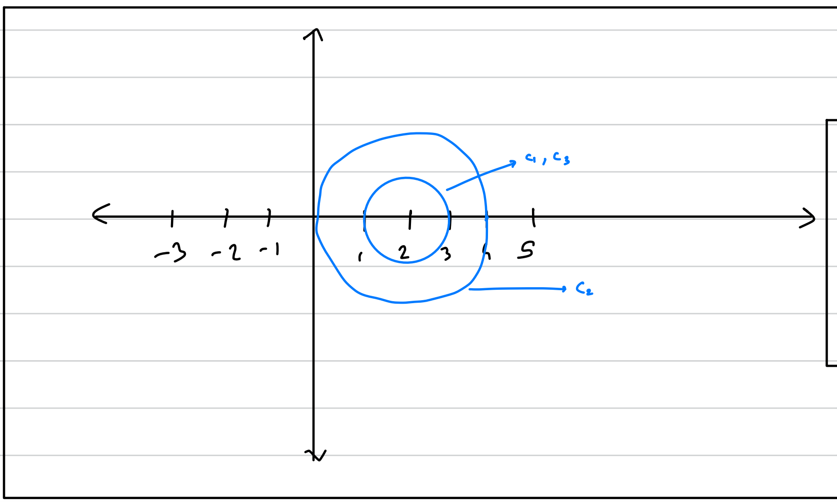
$r_1 = |-1| + |0| = 1$

$C_2 \equiv (2,0)$

$r_2 = |-1| + |-1| = 2$

$C_3 = (2,0)$

$r_3 = |0| + |-1| = 1$



• C_2 contains all eigenvalues

• Bound on $|\lambda|$

$|0| < |\lambda| < |4|$