EE 810 : Homework #8

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6.1 (5)(a)(c)(d)

5. (a) For which numbers b is the matrix $A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix}$ positive definite?

(c) Find the minimum value of $\frac{1}{2}(x^2 + 2bxy + 9y^2) - y$ for b in this range.

(d) What is the minimum if $b = 3$?

10/10

(a) $A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix}$. For being positive definite, $(1)(9) - (b)(b) > 0$

$9 - b^2 > 0$

$(b-3)(b+3) > 0$

if $b^2 - 9 < 0$,

$(b+3)(b-3) < 0$

$\therefore$ range : $-3 < b < 3$

$$\boxed{\text{For positive definitness of } A\text{, value of } b \text{ should obey the constraint} : -3 < b < 3}$$

(c) $F_x = \frac{1}{2}(2x + 2by)$

$= x + by$

$F_{xx} = 1 > 0$

$a + c > 2b$

$ac < b^2$

$F_y = \frac{1}{2}(2bx + 18y) - 1$

$F_{yy} = 9$

$F_{xy} = b$

Now, $F_{xx} \cdot F_{yy} = 9$

$(F_{xy})^2 = b^2$

$\because \quad (F_{xx})(F_{yy}) - (F_{xy})^2 > 0,$

$$9 > b^2$$

$$\therefore b^2 - 9 < 0$$

$$\therefore -3 < b < 3$$

$$\therefore F_{xx} > 0$$

$$\& \quad F_{xx} \cdot F_{yy} - (F_{xy})^2 = 0$$

Now, to Find minimum value of F,

$F_x = 0$ ———— $x + by = 0$

$F_y = 0$ ———— $bx + 9y - 1 = 0.$

$$\therefore x = -by$$

$$\& \quad bx + 9y - 1 = 0$$

$$\therefore b(-by) + 9y - 1 = 0$$

$$\therefore (9 - b^2) y = 1$$

$$\therefore y = \frac{1}{9 - b^2}$$

$\therefore$ Minimum value of F will be :

$$= \frac{1}{2}\left(\frac{b^2}{(9-b^2)^2} + 2b\left(\frac{-b}{9-b^2}\right)\left(\frac{1}{9-b^2}\right) + 9\left(\frac{1}{9-b^2}\right)^2\right) - \frac{1}{9-b^2}$$

$$= \frac{1}{2}\left(\frac{b^2+9}{(9-b^2)^2}\right) - \frac{1}{9-b^2}$$

$$= \frac{-1}{2(9-b^2)}$$

$\therefore$ Minimum value of the function is $\dfrac{-1}{2(9-b^2)}$ for b in its range.

$\because$ minimum value of F will be at $\left(\dfrac{-b}{9-b^2}, \dfrac{1}{9-b^2}\right)$

min of F = $\dfrac{-1}{2(9-b^2)}$

For given $b = 3$, min value $= \dfrac{-1}{0} = -\infty$

$$x + by = 0$$
$$x + 3y = 0$$
$$x = -3y$$

if $y \to \infty$, & $x = -3y$, $x - y \to -\infty$

$\therefore$ No minimum will be present.

minimum value of the function found to be $-\infty$, then no minimum exists when $y \to \infty$, $x = -3y$, so $x - y \to -\infty$.

6.1(8)

8. If $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive definite, test $A^{-1} = \begin{bmatrix} p & q \\ q & r \end{bmatrix}$ for positive definiteness.

$\because \quad A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad A^{-1} = \dfrac{1}{ac-b^2}\begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$

$\therefore$ compairing w/ $A^{-1} = \begin{bmatrix} p & q \\ q & r \end{bmatrix}$, $\quad p = \dfrac{c}{ac-b^2}$

$$q = \frac{-b}{ac-b^2}$$

$$r = \frac{a}{ac-b^2}$$

Now, for positive definiteness:

① $p > 0$

$\therefore \dfrac{c}{ac-b^2} > 0$

$\therefore \underline{c > 0}$

② $pr > q^2$

$\therefore \dfrac{ca}{(ac-b^2)^2} > \dfrac{b^2}{(ac-b^2)^2}$

$\therefore \dfrac{\cancel{ac-b^2}}{(ac-b^2)^{\cancel{2}}} > 0$

$\therefore \underline{ac > b^2}$

$\therefore$ $\boxed{\text{if } a > 0,\ c > 0,\ ac > b^2 \longrightarrow \text{then } A^{-1} \text{ is positive definite.}}$

6.1(20)

20. For $F_1(x,y) = \frac{1}{4}x^4 + x^2y + y^2$ and $F_2(x,y) = x^3 + xy - x$, find the second derivative matrices A_1 and A_2:

$$A = \begin{bmatrix} \partial^2 F/\partial x^2 & \partial^2 F/\partial x \partial y \\ \partial^2 F/\partial y \partial x & \partial^2 F/\partial y^2 \end{bmatrix}.$$

$$F_1(x,y) = \frac{1}{4}x^4 + x^2y + y^2$$

taking partial derivative,

$$\frac{\partial F_1(x,y)}{\partial x} = x^3 + 2xy \qquad \frac{\partial F_1(x,y)}{\partial y} = x^2 + 2y$$

$$\frac{\partial^2 F_1(x,y)}{\partial x^2} = 3x^2 + 2y \qquad \frac{\partial^2 F_1(x,y)}{\partial y^2} = 2$$

$$\frac{\partial^2 F_1(x,y)}{\partial x \, dy} = 2x \qquad \frac{\partial^2 F_1(x,y)}{\partial y \, \partial x} = 2x$$

$\therefore$ Second derivative of A_1 =

$$\begin{bmatrix} \frac{\partial^2 F_1(x,y)}{\partial x^2} & \frac{\partial^2 F_1(x,y)}{\partial x \partial y} \\ \frac{\partial^2 F_1(x,y)}{\partial y \, \partial x} & \frac{\partial^2 F_1(x,y)}{\partial y^2} \end{bmatrix}$$

$$= \begin{bmatrix} 3x^2 + 2y & 2x \\ 2x & 2 \end{bmatrix}$$

Now, for $F_2(x,y) = x^3 + xy - x$

$$\frac{\partial F_2(x,y)}{\partial x} = 3x^2 + y - 1 \qquad \frac{\partial F_2(x,y)}{\partial y} = x$$

$$\frac{\partial^2 F_2(x,y)}{\partial x^2} = 6x \qquad \frac{\partial^2 F_2(x,y)}{\partial y^2} = 0$$

$$\frac{\partial^2 F_2(x,y)}{\partial x \partial y} = 1 \qquad \frac{\partial^2 F_2(x,y)}{\partial y \partial x} = 1$$

$\therefore$ 2nd derivative matrix $A_2 = \begin{bmatrix} \frac{\partial^2 F_2(x,y)}{\partial x^2} & \frac{\partial^2 F_2(x,y)}{\partial x \partial y} \\ \frac{\partial^2 F_2(x,y)}{\partial y \partial x} & \frac{\partial^2 F_2(x,y)}{\partial y^2} \end{bmatrix}$

$$= \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix}$$

A_1 is positive definite, so F_1 is concave up (= convex). Find the minimum point of F_1 and the saddle point of F_2 (look where first derivatives are zero).

For minimum point of F_1, $\frac{\partial F_1(x,y)}{\partial x} = 0$

$\therefore \; x^3 + 2xy = 0$

$$\frac{\partial F_1(x,y)}{\partial y} = 0$$

$$x^2 + 2y = 0$$

$\therefore \; (x,y) \equiv (0,0)$ is minimum point of F_1.

Now, for saddle point,

$$\frac{\partial f}{\partial x} = 0 \quad , \quad \frac{\partial f}{\partial y} = 0 \quad , \quad \left(\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 < 0$$

$$\therefore \frac{\partial F_2(x,y)}{\partial x} = 3x^2 + y - 1 = 0$$

$$\& \frac{\partial F_2(x,y)}{\partial y} = x = 0$$

and $$\left(\frac{\partial^2 F_2(x,y)}{\partial x^2} \cdot \frac{\partial^2 F_2(x,y)}{\partial y^2}\right) - \left(\frac{\partial^2 F_2(x,y)}{\partial x \partial y}\right)^2 < 0$$

Solving this gives us $x = 0$, $y = 1$ —— This is the required saddle point.

$$\therefore \quad A_1 = \begin{bmatrix} 3x^2 + 2y & 2x \\ 2x & 2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix}$$

minimum point for F_1 : $x = 0$, $y = 0$

Saddle pt for F_2 : $x = 0$, $y = 1$

6.2(1)

1. For what range of numbers a and b are the matrices A and B positive definite?

$$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{bmatrix}.$$

For A, for positive definiteness,

$$|a| > 0$$

$$\begin{vmatrix} a & 2 \\ 2 & a \end{vmatrix} > 0 \longrightarrow \begin{matrix} a^2 - 4 > 0 \\ (a+2)(a-2) > 0 \end{matrix} \longrightarrow \begin{matrix} \therefore a > 2 \\ \& \ a < -2 \end{matrix}$$

$$\begin{vmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{vmatrix} > 0$$

So when $\begin{vmatrix} a & 2 \\ 2 & a \end{vmatrix} > 0$,

$a > 2$ & $a < -2$

but here take the range $a > 2$

$\because |a| > 0$ for matrix to be positive definite.

$$a(a^2-4) - 2(2a-4) + 2(4-2a) > 0$$

$$a^3 - 12a + 16 > 0$$

$$(a-2)(a+4) > 0 \longrightarrow a > 2 \text{ or } a < -4$$

But here take the range : $a > 2$

$\because |a| > 0$ for the matrix to be positive definiteness

For matrix B,

$|1| > 0$ — true already

$\begin{vmatrix} 1 & 2 \\ 2 & b \end{vmatrix} > 0 \longrightarrow 4 - b > 0$

$\therefore b > 0$

$\begin{vmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{vmatrix} > 0 \longrightarrow 1(7b - 64) - 2(14 - 32) + 4(16 - 4b) > 0$

$-9b + 36 > 0$

$9b < 36$

$b < 4$

here $b > 4$ & $b < 4$ are conditions.

$\therefore$ No common range.

$\therefore$ B cannot be positive definite.

a>2 as you showed above

- For $a > 0$, matrix A can be positive definite
- Matrix B can never be positive definite.

6.2 (5)

5/5

5. *If A and B are positive definite, then A + B is positive definite*. Pivots and eigenvalues are not convenient for $A+B$. Much better to prove $x^T(A+B)x > 0$.

Given : if A, B : Positive definite $\longrightarrow$ $A+B$: Positive definite.

we have to show, $x^T(A+B)x > 0$

$\because$ A : Positive definite, $x^T A x > 0$ $(x \neq 0)$
$\because$ B : Positive definite, $x^T B x > 0$ $(x \neq 0)$

$\therefore$ for any $x \neq 0$, $x^T(A+B)x = x^T A x + x^T B x$
> 0

$\therefore$ $\boxed{\text{Hence proved that } x^T(A+B)x > 0 \text{ — (for any } x \neq 0)}$

6.2(7)

7. If $A = Q\Lambda Q^{\mathrm{T}}$ is symmetric positive definite, then $R = Q\sqrt{\Lambda}Q^{\mathrm{T}}$ is its *symmetric positive definite square root*. Why does R have positive eigenvalues? Compute R and verify $R^2 = A$ for

$$A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix}.$$

For $A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$, $R = Q\sqrt{\Lambda}\,Q^T$ is positive definite,
then for all $x \neq 0$,
$x^T R x > 0$

Now, consider: $Rx = \lambda x$
$$x^T R x = x^T \lambda x$$
$$x^T R x = \lambda |x|^2$$

$(\because x^T R x > 0,)$
$\lambda > 0.$

$$\boxed{\therefore R \text{ have positive eigenvalues } \lambda > 0.}$$

Now, to Find R,

Computing Q, $A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} \xrightarrow{R_2 - \frac{6}{10}R_1} \begin{bmatrix} 10 & 6 \\ 0 & \frac{32}{5} \end{bmatrix} = U$

similarly, $L = \begin{bmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{bmatrix}$

$\therefore \quad LDU = \begin{bmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{bmatrix} \begin{bmatrix} 10 & 6 \\ 0 & \frac{32}{5} \end{bmatrix}$

$R_1 \to R_1/10$
$R_2 \to R_2/(32/5)$

$= \begin{bmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{bmatrix} \begin{bmatrix} 1 & \frac{6}{10} \\ & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{bmatrix}^T$

comparing this LDU w/ $Q\sqrt{\Lambda}\,Q^T$,

$Q = \begin{bmatrix} 1 & 0 \\ 6/10 & 1 \end{bmatrix} = L$

$\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & 32/5 \end{bmatrix} = D$

$Q^T = \begin{bmatrix} 1 & \frac{6}{10} \\ 0 & 1 \end{bmatrix} = L^T$

here, $A = Q \Lambda Q^T$ is symmetric & positive definite

& $R = Q\sqrt{\Lambda}\,Q^T$ is symmetric positive definite sq. root.

$\therefore \quad R = \sqrt{\Lambda}\,Q^T$ & $\sqrt{\Lambda} = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{32/5} \end{bmatrix}$

$\therefore \quad R = \begin{bmatrix} \sqrt{10} & 3/5\,\sqrt{10} \\ 0 & 4/5\,\sqrt{10} \end{bmatrix}$

$$\therefore \boxed{\begin{matrix} R^2 = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} = A. \\ \\ \therefore \quad R^2 = A \end{matrix}}$$

Now, for $A = \begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix}$

$$\downarrow R_2 \to R_2 - \frac{6}{10} R_1$$

$$= \begin{bmatrix} 10 & -6 \\ 0 & 32/5 \end{bmatrix} \xrightarrow[R_2/(32/5)]{R_1/10} \begin{bmatrix} 1 & \frac{-6}{10} \\ 0 & 1 \end{bmatrix} = U$$

Similarly, $L = \begin{bmatrix} 1 & 0 \\ \frac{-6}{10} & 1 \end{bmatrix}$

$$\therefore \quad LDU = \begin{bmatrix} 1 & 0 \\ \frac{-6}{10} & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 32/5 \end{bmatrix} \begin{bmatrix} 1 & \frac{-6}{10} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \frac{-6}{10} & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 32/5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{-6}{10} & 1 \end{bmatrix}^T$$

Comparing w/ $Q \sqrt{\Lambda} Q^T$,

$$Q = \begin{bmatrix} 1 & 0 \\ -6/10 & 1 \end{bmatrix} = L$$

$$\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & 32/5 \end{bmatrix} = D$$

$$Q^T = \begin{bmatrix} 1 & \frac{-6}{10} \\ 0 & 1 \end{bmatrix} = L^T$$

Now, $\because \sqrt{\Lambda} = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{32/5} \end{bmatrix}$

$$R = \sqrt{\Lambda}\, Q^T$$

$$= \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{32/5} \end{bmatrix} \begin{bmatrix} 1 & -6/10 \\ 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{10} & \frac{-3}{5}\sqrt{10} \\ 0 & \frac{4}{5}\sqrt{10} \end{bmatrix}$$

$$\therefore \boxed{R^2 = \begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix} = A \qquad \therefore R^2 = A}$$

6.2(15)

15. Suppose A is symmetric positive definite and Q is an orthogonal matrix. True or false:

(a) Q^TAQ is a diagonal matrix.
(b) Q^TAQ is symmetric positive definite.
(c) Q^TAQ has the same eigenvalues as A.
(d) e^{-A} is symmetric positive definite.

Give A : symmetric positive definite
Q : orthogonal Mx.

(a) Q^TAQ : $\because$ Q is orthogonal matrix, Q must contain eigenvectors of A.
$\therefore$ Q^TAQ cannot be a diagonal matrix.
$\therefore$ $\boxed{(a) = \text{FALSE}}$

(b) $\because$ Q is orthogonal, $Q^T = Q^{-1}$.
$\because$ $Q^{-1}AQ$ is similar matrix to A, they have same eigen values.
$\therefore$ Q^TAQ & A have same eigenvalues.
$\therefore$ A: symmatrix positive definite implies $\longrightarrow$ Q^TAQ is also symmatrix positive definite. $\boxed{(b) = \text{TRUE}}$

(c) as argued in (b), $\boxed{(c) = \text{TRUE}}$

(d) $\because$ A is symmetric, $-A$ is also symmetric, e^{-A} is also symmetric.
eigenvalues of e^{-A} are in form $e^{-\lambda}$, i.e a positive number.
$\therefore$ $\boxed{(d) = \text{TRUE}}$

6.2 (22)

22. A diagonal entry a_{jj} of a symmetric matrix cannot be smaller than all λ's. If it were, then $A - a_{jj}I$ would have ____ eigenvalues and would be positive definite. But $A - a_{jj}I$ has a ____ on the main diagonal.

- A diagonal entry a_{jj} of symmetric mx cannot be smaller than all eigenvalues.
- If it were, then $A - a_{jj}I$ would have positive eigenvalues value, thus being a positive definite.

$\because$ λs of $A - a_{jj}I$ = $\lambda - a_{jj} > 0$ — where λ is some eigenvalue of A.

But $A - a_{jj}I$ has a zero on the main diagonal.

A diagonal entry a_{jj} of a symmetric matrix cannot be smaller than all λ's. If it were, then $A - a_{jj}I$ would have $\boxed{\text{Positive}}$ eigenvalues and would be positive definite. But $A - a_{jj}I$ has a $\boxed{\text{zero}}$ on the main diagonal.

6.2 (23)

5/5

23. Give a quick reason why each of these statements is true:

(a) Every positive definite matrix is invertible.

(b) The only positive definite projection matrix is $P = I$.

(c) A diagonal matrix with positive diagonal entries is positive definite.

(d) A symmetric matrix with a positive determinant might not be positive definite!

(a) Every positive matrix has determinant greater that 0.
∴ Every such matrix is invertible.

(b) Every projection matrix except I have determinant zero.

(c) Diagonal entries of a diagonal matrix are it's eigenvalues.
∴ the given matrix will have positive eigenvalues.

(d) Because, there could be symmetrix matrix wit det > 0
who has upper left sub matrix deteminants which are
not positive.

6.2(32)

32. Apply any three tests to each of the matrices

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix},$$

to decide whether they are positive definite, positive semidefinite, or indefinite.

For A, Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\therefore \quad x^T A x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= (x_1 + x_2 + x_3) x_1 + (x_1 + x_2 + x_3) x_2 + (x_1 + x_2) x_3$$

$$= x_1^2 + 2x_1x_2 + 2x_1x_3 + x_2^2 + 2x_2x_3 + x_1x_3$$

$\therefore$ if $x_1, x_2, x_3 < 0$, $x^T A x < 0$.

Now let $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - 2\lambda^2 - 2\lambda = 0$$

$$\therefore \lambda(\lambda^2 - 2\lambda - 2) = 0$$

$$\therefore \lambda_1 = 0$$

$$\lambda_2 = 1 + \sqrt{3}$$

$$\lambda_3 = 1 - \sqrt{3}$$

Now, upper submatrix of A, $A_1 = [1]$

$\therefore |A_1| = 1$

Simillarly, $A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $|A_2| = 0$

Now, consider matrix B.

$$x^T B x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= (2x_1 + x_2 + 2x_3)x_1 + (x_1 + x_2 + x_3)x_2 + (2x_1 + x_2 + 2x_3)x_3$$

$$= 2x_1^2 + 2x_1x_2 + 4x_1x_3 + x_2^2 + 2x_2x_3 + 2x_3^2$$

Now, for B, $|B - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 2 \\ 1 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore (2-\lambda)\left[(1-\lambda)(2-\lambda)-1\right] - 1\left[(2-\lambda)(1-\lambda)-1\right] + 2\left[1 - 2(1-\lambda)\right] = 0$$

$$\therefore \lambda^3 - 5\lambda^2 + 2\lambda = 0$$

$$\therefore \lambda(\lambda - 5\lambda + 2) = 0$$

$$\therefore \lambda_1 = 0$$

$$\lambda_2 = \frac{5+\sqrt{7}}{2}$$

$$\lambda_3 = \frac{5-\sqrt{7}}{2}$$

Now, upper submatrix of B, $B_1 = [2]$

$\therefore |B_1| = 2$

Simillarly, $B_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $|B_2| = 1$

simillarly, $B_3 = B$, $|B_3| = 2(2-1) - 1(2-2) + 2(1-2)$

$= 0$

$\boxed{\begin{array}{l} \therefore \text{ Matrix } A \text{ is — indefinite.} \\ \& \text{ Matrix } B \text{ is — Positive semidefinite} \end{array}}$

6-3(5)

5. Compute A^TA and AA^T, and their eigenvalues and unit eigenvectors, for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Multiply the three matrices $U\Sigma V^T$ to recover A.

$$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Now, $|AA^T - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$\therefore \ \lambda^2 - 4\lambda + 3 = 0$

$\therefore \ (\lambda-1)(\lambda-3) = 0$ —— $\lambda_1 = 1$, $\lambda_2 = 3$

for $\lambda = 1$, $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$ $R_2 \to R_1 - R_2$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\therefore \ x_1 + x_2 = 0$

$x_1 = -x_2$

$$\therefore \ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = -x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \longleftarrow$$

for $\lambda = 3$, $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$ $R_2 \to -R_1 + R_2$ $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\therefore -x_1 + x_2 = 0$

$\therefore \quad x_1 = x_2$

$\therefore x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now, consider $|A^T A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\therefore \lambda(\lambda - 1)(\lambda - 3) = 0$ $\quad \therefore \lambda_1 = 0$

$\lambda_2 = 1$

$\lambda_3 = 3$

for $\lambda = 0$, $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\therefore x_1 - x_3 = 0 \to x_1 = x_3$

$\therefore x_2 + x_3 = 0 \to x_2 = -x_3$

& $x_3 = x_3$

$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

for $\lambda = 3$,

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore\ x_1 - x_3 = 0$

$x_1 = x_3$

$\therefore\ x_2 - 2x_3 = 0$

$x_2 = 2x_3$

$\therefore\ x_3 = x_3$

$$\therefore\ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore$ Normalizing eigenvectors of AA^T, :

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore\ U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$\therefore$ Normalizing eigenvectors of A^TA :

$$V_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad V_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad V_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore\ V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

Now, eigenvalues of AA^T : $\lambda_1 = 1$

$\lambda_2 = 3$

$$\therefore \quad \sigma_1 = \sqrt{\lambda_1} = 1 \qquad \sigma_2 = \sqrt{\lambda_2} = \sqrt{3}$$

$$\therefore \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$\therefore \quad U\Sigma V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & \sqrt{3}/\sqrt{2} & 0 \\ -1/\sqrt{2} & \sqrt{3}/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= A$$

$$\therefore \quad AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and we verified that $U\Sigma V^T = A$

6.3 (12)

12. (a) If A changes to $4A$, what is the change in the SVD?

(b) What is the SVD for A^T and for A^{-1}?

(a) $\because$ eigenvectors of AA^T are in U & eigenvectors of A^TA are in V,

The r singlur-values on the diagonal of Σ are square roots of nonzero eigenvalues of both A^TA & AA^T.

Now, eigenvalue eq^n : $AA^Tx = \lambda x$

Now, multiplying with constant c : $cAA^Tx = c\lambda x$

$\therefore$ multiplying with c, eigenvectors : remains same.

eigenvalues : $\lambda \longrightarrow c\lambda$

$\therefore$ If we change $A \to 4A$, U & V will stay same.

$\therefore$ Now, for $4A$: $(4A)(4A)^T = 16\,AA^T$

$\therefore$ Eigenvalue eq^n : $16\,AA^Tx = \underset{\sim}{16}\lambda x$

$\because$ $\sigma = \sqrt{\lambda}$, $\sigma = \sqrt{16\lambda}$ $\longrightarrow$ $\big((c\lambda)$ for $4A\big)$

$= 4\sqrt{\lambda}$

$\therefore$ diagonal matrix $\Sigma \longrightarrow 4\Sigma$ for $A \to 4A$.

and $U \to U$ & $V \to V$.

$\therefore$ SVD for $4A = U(4\Sigma)V^T = 4\,(U\Sigma V^T)$

$\therefore$ $\boxed{\begin{array}{l}\text{change in the SVD after } A \to 4A \text{ is} \\ U \Sigma V^T \to 4\, U \Sigma V^T\end{array}}$

(b) $\because$ SVD of $A = U \Sigma V^T$,

$$A^T = (U \Sigma V^T)^T$$

$$A^T = V \Sigma^T U^T$$

also, if A : non-singular,

$$\because A = U \Sigma V^T,$$

$$A^{-1} = (U \Sigma V^T)^{-1}$$

$$= V \Sigma^{-1} U^T$$

($\because$ U, V : orthogonal)

$\therefore$ $\boxed{\begin{array}{l}\text{SVD for } A^T = V \Sigma^T U^T \\ \qquad\quad\; A^{-1} = V \Sigma^{-1} U^T\end{array}}$

6.3 (B)

13. Why doesn't the SVD for $A+I$ just use $\Sigma+I$?

Let SVD for $A+I$ use $\Sigma+I$.

$$\begin{aligned} \therefore \quad A+I &= U(\Sigma+I)V^T \\ &= U\Sigma V^T + UIV^T \\ &= U\Sigma V^T + UV^T \\ &\neq A+I \quad \text{——} \quad \text{contradiction} \end{aligned}$$

The r singular values on diagonal of Σ are sq. root of non-zero λs of A^TA.

$\therefore$ Singular values of $A+I$ are not $\sigma_r + I$, (where σ_r are singular values of A) but they are eigenvalues of $(A+I)^T(A+I)$.

$\therefore$ $\boxed{\text{SVD for } A+I \text{ cannot use } \Sigma+I}$

6.3(15)

15. Find the SVD and the pseudoinverse $V\Sigma^{+}U^{\mathrm{T}}$ of

$$A=\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}, \qquad B=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}, \quad \text{and} \quad C=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}.$$

① $AA^T = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}4\end{bmatrix}$

$\therefore \lambda = 4$

$\therefore$ Corrosponding eigenvector $u_1 = \begin{bmatrix}1\end{bmatrix}$

$A^TA = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix}$

$\therefore \lambda_1 = 4 \qquad \therefore \sigma_1 = \sqrt{4} = 2$

$\lambda_2 = 0$

$\lambda_3 = 0$

$\lambda_4 = 0$

$\therefore$ Corrosponding eigenvectors are : $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\-\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$

normallzing them adds a multiplication factor of $\frac{1}{2}$ to each eigenvector.

$$\therefore \boxed{\text{SVD of } A = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}}$$

Pseudoinverse of A :

$$A^+ = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

$$\boxed{\underbrace{A^+}_{\text{Pseudoinverse of } A} = \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$$

Now, $BB^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore \lambda_1 = 1$ & $u_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\lambda_2 = 1$ $\quad u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now, $B^TB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \therefore \lambda_1 = 1$

$\lambda_2 = 1$

$\lambda_3 = 0$

$\therefore v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

& $\sigma_1 = \sqrt{1} = 1$

$\sigma_2 = \sqrt{1} = 1$

$$\boxed{\therefore \text{ SVD of } B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

Now, Pseudoinverse of B,

$$B^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\boxed{B^+ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \text{Pseudoinverse of } B}$$

Now, $CC^T = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \therefore \lambda_1 = 2$

$\lambda_2 = 0$

$\therefore u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now, $C^T C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad \therefore \lambda_1 = 2$

$\lambda_2 = 0$

$\therefore v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore \sigma_1 = \sqrt{2}$

$\therefore$ $\boxed{\text{SVD of } C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}$

Now, Pseudoinverse of C:

$$C^+ = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{C^+ = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix} = \underline{\text{Pseudoinverse of } C}}$$

6-3(21)

21. Removing zero rows of U leaves $A = \underline{L}\,\underline{U}$, where the r columns or $\underline{L}$ span the column space of A and the r rows of $\underline{U}$ span the row space. Then A^+ has the explicit formula $\underline{U}^T(\underline{U}\,\underline{U}^T)^{-1}(\underline{L}^T\underline{L})^{-1}\underline{L}^T$.

① Why is A^+b in the row space with $\underline{U}^T$ at the front? ② Why does $A^TAA^+b = A^Tb$, so that $x^+ = A^+b$ satisfies the normal equation as it should?

①

$$A^+b = U^T \underbrace{(UU^T)^{-1}(L^TL)^{-1}L^Tb}_{y}$$

$$= U^Ty$$

$$= \begin{bmatrix} u_1 & \cdots & u_r \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_r \end{bmatrix}$$

$$= y_1u_1 + \cdots + y_ru_r$$

$\therefore$ A^+b is a linear combination of the coloumns of U^T which spans the row space of the matrix A.

Thus A^+b is in the rowspace of A.

② Now, using the formula for A, A^+ & associative law, we get:

$$A^TAA^+b = A^T(LU)U^T(UU^T)^{-1}(L^TL)^{-1}L^Tb$$

$$= A^TL\,\underbrace{(UU^T)(UU^T)^{-1}}_{I}(L^TL)^{-1}L^Tb$$

$$= A^TL(L^TL)^{-1}L^Tb$$

$$= A^T\underbrace{(LL^{-1})}_{I}\underbrace{(L^{T^{-1}}L^T)}_{I}b$$

$$= A^T I I b$$
$$= A^T b$$

$\therefore$ $\boxed{\begin{array}{l}\text{$A^+ b$ is a linear combination of the coloumns of} \\ \text{U^T which span the row space of the matrix A.} \\ \\ \text{$A^T A A^+ b = A^T b$ is shown.}\end{array}}$