EE 810: Homework #6

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5.1 (1)

1. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Verify that the trace equals the sum of the eigenvalues, and the determinant equals their product.

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$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

$\therefore$ Calculating $|A - I\lambda| = 0$

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda) - (-1 \times 2) = 0$$
$$4 - 4\lambda - \lambda + \lambda^2 + 2 = 0$$
$$\lambda^2 - 5\lambda + 6 = 0$$
$$\lambda^2 - 3\lambda - 2\lambda + 6 = 0$$
$$\lambda(\lambda-3) - 2(\lambda-3) = 0$$
$$(\lambda-3)(\lambda-2) = 0 \Rightarrow \lambda_1 = 3, \quad \lambda_2 = 2$$

$\therefore$ Eigenvalues of given matrix A are 3 & 2

Now, $\text{trace}(A) = 1 + 4 = \underline{\underline{5}}$ & Sum of eigenvalues $= \lambda_1 + \lambda_2 = 3 + 2 = \underline{\underline{5}}$

$\therefore$ Hence shown that trace equals sum of eigenvalues.

Now, $|A| = \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} = 4 - (-2) = \underline{\underline{6}}$ & Product of eigenvalues $= \lambda_1 \lambda_2 = 3 \times 2 = \underline{\underline{6}}$

$\therefore$ Hence shown that determinant equals product of eigenvalues.

Now, to find eigenvectors, Consider $\underline{(A-\lambda I)X = 0}$

For $\lambda = 3$, $A - I\lambda = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow R_2 \to R_2 + R_1$$

$$\begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore -2x_1 - x_2 = 0$$

$$2x_1 = -x_2$$

$$x_1 = \frac{-1}{2} x_2$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{-1}{2} \\ 1 \end{bmatrix}$$

For $\lambda = 2$, $A - I\lambda = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$

$$\therefore \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow R_2 \to R_2 + 2R_1$$

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore -x_1 - x_2 = 0$$

$$\therefore x_1 = -x_2$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\therefore$ eigenvectors for the given matrix are

$$\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}_{\lambda=3} \quad \& \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{\lambda=2}$$

$\boxed{5.1(5)}$

5. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Check that $\lambda_1 + \lambda_2 + \lambda_3$ equals the trace and $\lambda_1 \lambda_2 \lambda_3$ equals the determinant.

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$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Consider $\quad |A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 3-\lambda & 4 & 2 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$\therefore (3-\lambda) \begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$\therefore (3-\lambda)(1-\lambda)(-\lambda) = 0.$$

$$\therefore (3-\lambda)(\lambda-1)(\lambda) = 0. \quad \Rightarrow \quad \boxed{\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 1 \\ \lambda_3 &= 3 \end{aligned}}$$

Now, $\quad \text{trace}(A) = 3+1+0 = \underline{\underline{4}} \quad$ & $\quad \lambda_1 + \lambda_2 + \lambda_3 = 0+1+3 = \underline{\underline{4}}$

& $\quad \begin{vmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 3 \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = \underline{\underline{0}} \quad$ & $\quad \lambda_1 \lambda_2 \lambda_3 = (0)(1)(3) = \underline{\underline{0}}$

$\therefore$ $\boxed{\text{Hence, shown that trace equals } \lambda_1 + \lambda_2 + \lambda_3 \text{ \& determinant equals } \lambda_1 \lambda_2 \lambda_3.}$

Now, to find eigenvectors, consider $(A - I\lambda)x = 0$

for $\lambda = 0$:

$$\left[\begin{array}{ccc|c} 3 & 4 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore 3x_1 + 4x_2 + 2x_3 = 0$

$x_2 + 2x_3 = 0$

$\therefore x_2 = -2x_3$

$x_3 = x_3$

& $3x_1 - 8x_3 + 2x_3 = 0.$

$\therefore 3x_1 - 6x_3 = 0$

$\therefore x_1 = 2x_3$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

for $\lambda = 1$,

$$\left[\begin{array}{ccc|c} 2 & 4 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + 0.5R_2} \left[\begin{array}{ccc|c} 2 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$\therefore 2x_1 + 4x_2 + 2x_3 = 0$

$-x_3 = 0$

$x_2 = x_2$

$\therefore 2x_1 + 4x_2 + 0 = 0$

$x_1 = -2x_2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

for $\lambda = 3$,

$$\left[\begin{array}{ccc|c} 0 & 4 & 2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|c} 0 & 0 & 6 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

$\therefore 6x_3 = 0$

$-2x_2 + 2x_3 = 0$

$-3x_3 = 0.$

$\therefore x_3 = 0$

$x_1 = x_1$

$x_2 = 0$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \boxed{\text{Eigenvectors for given matrix } A \text{ are : } \underset{\lambda=0}{\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}}, \underset{\lambda=1}{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}, \underset{\lambda=3}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}}$$

Now,

$$B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$\therefore$ Consider $\quad |A - \lambda I| = 0$

$$\therefore \begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = 0$$

$\therefore -\lambda(2-\lambda)(-\lambda) + 2(-2)(2-\lambda) = 0$

$\therefore \lambda^2(2-\lambda) - 4(2-\lambda) = 0$

$\therefore (2-\lambda)(\lambda^2 - 4) = 0$

$= (2-\lambda)(\lambda-2)(\lambda+2) = 0 \quad \Rightarrow \quad \boxed{\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 2 \\ \lambda_3 &= -2 \end{aligned}}$

Now, $\quad \text{trace}(A) = 0 + 2 + 0 = \underline{\underline{2}} \quad$ & $\quad \lambda_1 + \lambda_2 + \lambda_3 = 2 + 2 + (-2) = \underline{\underline{2}}$

& $\quad \begin{vmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 2 \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = \underline{\underline{-8}} \quad$ & $\quad \lambda_1 \lambda_2 \lambda_3 = (2)(2)(-2) = \underline{\underline{-8}}$

$\therefore$ $\boxed{\text{Hence, shown that trace equals } \lambda_1 + \lambda_2 + \lambda_3 \text{ \& determinant equals } \lambda_1 \lambda_2 \lambda_3.}$

Now, to find eigenvectors, consider $(A - I\lambda)x = 0$

for $\lambda = 2$:

$$\left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right]$$

$\therefore$ $-2x_1 + 2x_3 = 0$

$2x_1 - 2x_3 = 0.$

$\therefore$ $x_2 = 0$

$x_1 = x_3$

$x_3 = x_3$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \longleftarrow$$

for $\lambda = -2$

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 4 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right]$$

$\therefore$ $2x_1 + 2x_3 = 0$

$4x_2 = 0$

$\therefore$ $2x_1 + 2x_3 = 0$

$\therefore$ $x_1 = x_1$

$x_2 = 0$

$x_3 = -x_1$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Also $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\therefore$ $\boxed{\text{Eigenvectors for given matrix A are : } \underset{\lambda = 2}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \underset{\lambda = -2}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}}$

5.1 (17)

17. Choose the third row of the "companion matrix"

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \cdot & \cdot & \cdot \end{bmatrix}$$

so that its characteristic polynomial $|A - \lambda I|$ is $-\lambda^3 + 4\lambda^2 + 5\lambda + 6$.

Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}$

$\therefore$ $|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ a & b & c-\lambda \end{vmatrix}$

$= -\lambda \begin{vmatrix} -\lambda & 1 \\ b & c-\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ a & c-\lambda \end{vmatrix}$

$= -\lambda(\lambda^2 - c\lambda - b) - 1(a)$

$= -\lambda^3 + c\lambda^2 + b\lambda + a$

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$\therefore$ comparing this with $-\lambda^3 + 4\lambda^2 + 5\lambda + 6$,

we get $\boxed{\begin{array}{c} a = 6 \\ b = 5 \\ c = 4 \end{array}}$

$\therefore$ third row of Matrix A should be $(6, 5, 4)$ such that it satisfies given condition.

5.1 (20)

20. Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$ has the ____ eigenvectors as A. Its eigenvalues are ____ by 1.

$$|A - I\lambda| = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8$$

$$= 3 - 3\lambda - \lambda + \lambda^2 - 8$$

$$= \lambda^2 - 4\lambda - 5 \longrightarrow \quad \therefore \lambda_1 = -1, \quad \lambda_2 = -5$$

$$= \lambda^2 - 5\lambda + \lambda - 5$$

$$= \lambda(\lambda - 5) + 1(\lambda - 5)$$

$$= (\lambda + 1)(\lambda - 5)$$

for $\lambda = -1$,

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 2 & 4 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore 2x_1 + 4x_2 = 0$

$x_2 = x_2$

$\therefore x_1 = -2x_2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

for $\lambda = 5$,

$$\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 2 & -2 & 0 \end{array}\right] \xrightarrow{R_2 \to 2R_2 + R_1} \left[\begin{array}{cc|c} -4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore -4x_1 + 4x_2 = 0$

$\therefore x_1 = x_2$

$x_2 = x_2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ Eigenvectors for eigenvalues -1 & 5 are $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

Let $A+I$ be B.

$$|B - I\lambda| = \begin{vmatrix} 2-\lambda & 4 \\ 2 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 8$$
$$= 8 - 4\lambda - 2\lambda + \lambda^2 - 8$$
$$= \lambda^2 - 6\lambda$$
$$= \lambda(\lambda - 6) \longrightarrow \therefore \lambda_1 = 0, \quad \lambda_2 = 6$$

for $\lambda = 0$,
$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 2 & 4 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore 2x_1 + 4x_2 = 0$

$x_2 = x_2$

$\therefore x_1 = -2x_2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

for $\lambda = 6$,
$$\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 2 & -2 & 0 \end{array}\right] \xrightarrow{R_2 \to 2R_2 + R_1} \left[\begin{array}{cc|c} -4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore -4x_1 + 4x_2 = 0$

$\therefore x_1 = x_2$

$x_2 = x_2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ Eigenvectors for eigenvalues 0 & 6 are $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

$\therefore$ $A+I$ has the same eigenvectors as A.
Its eigenvalues are increased by 1.

5/5

5.1(29)

4/5

29. A 3 by 3 matrix B is known to have eigenvalues 0, 1, 2, This information is enough to find three of these:

(a) the rank of B,

(b) the determinant of B^TB,

(c) the eigenvalues of B^TB, and

(d) the eigenvalues of $(B+I)^{-1}$.

from the given information, we can safely assume

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) we can clearly see, 1^{st} coloumn is dependent.

$\therefore$ $\boxed{\text{Rank of } B \text{ is } 2}$

(b) $\because$ $|b| = 0$, $\boxed{|b^Tb| = 0}$

(c) $b^Tb = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$\therefore$ $\boxed{\text{Eigenvalues of } b^Tb \text{ are } 0, 1, 4}$ X

(d) Now, $B + I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\therefore$ to find $(B+I)^{-1}$,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to \left(\frac{R_3}{3}\right)]{R_2 \to \left(\frac{R_2}{2}\right)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$$

$(B+I)^{-1}$

$\therefore$ $\boxed{\text{Eigenvalues of } (B+I)^{-1} \text{ are } 1, \tfrac{1}{2}, \tfrac{1}{3}}$

5.2(7)

7. If $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$, find A^{100} by diagonalizing A.

$$\therefore \quad |A - I\lambda| = \begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = (4-\lambda)(2-\lambda) - 3$$

$$= 8 - 2\lambda - 4\lambda + \lambda^2 - 3$$

$$= \lambda^2 - 6\lambda + 5$$

$$= \lambda^2 - 5\lambda - \lambda + 5 \quad \longrightarrow \therefore \lambda_1 = 1$$

$$= \lambda(\lambda - 5) - 1(\lambda - 5) \qquad \lambda_2 = 5$$

$$= (\lambda - 1)(\lambda - 5)$$

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for $\lambda = 1$:

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to 3R_2 - R_1} \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\therefore \quad 3x_1 + 3x_2 = 0$$

$$x_2 = x_2 \qquad \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$x_1 = -x_2$$

for $\lambda = 5$:

$$\left[\begin{array}{cc|c} -1 & 3 & 0 \\ 1 & -3 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} -1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\therefore \quad -x_1 + 3x_2 = 0$$

$$x_2 = x_2 \qquad \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$x_1 = 3x_2$$

$$\therefore \quad S = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \qquad \therefore \quad S^{-1} = \frac{1}{4} \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$\therefore \quad A = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1/4 & 3/4 \\ 1/4 & 1/4 \end{bmatrix}$$

$$\therefore \quad \boxed{A^{100} = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5^{100} \end{bmatrix} \begin{bmatrix} -1/4 & 3/4 \\ 1/4 & 1/4 \end{bmatrix}}$$

5.2(10)

10. Suppose A has eigenvalues 1, 2, 4. What is the trace of A^2? What is the determinant of $(A^{-1})^T$?

If A has eigenvalues 1, 2, 4,
A^2 will have eigenvalues 1, 4, 16

$\therefore$ $\boxed{\text{trace}(A^2) = 1+4+16 = 21}$

Now, $|A| = (1)(2)(4)$
$= 8$

$\therefore$ $|A^{-1}| = \frac{1}{8}$ $\longrightarrow$ $\boxed{|(A^{-1})^T| = |A^{-1}| = \frac{1}{8}}$

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5.2 (18)

18. Suppose $A = S\Lambda S^{-1}$. What is the eigenvalue matrix for $A+2I$? What is the eigenvector matrix? Check that $A+2I = (\quad)(\quad)(\quad)^{-1}$.

$$A = S\Lambda S^{-1}$$

$$\therefore \quad \Lambda = S^{-1}AS$$

for $A+2I$, eigenvalue matrix — say Λ',

$$\begin{aligned}\Lambda' &= S^{-1}(A+2I)S \\ &= (S^{-1}A + 2S^{-1}I)S \\ &= S^{-1}AS + 2S^{-1}IS \\ &= \Lambda + 2I\end{aligned}$$

Hence, eigenvalue matrix for $A+2I$ is $\Lambda + 2I$.

$$\therefore \quad \boxed{A + 2I = S(\Lambda + 2I)S^{-1}}$$

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5.1 (13)

5|5

13. If B has eigenvalues 1, 2, 3, C has eigenvalues 4, 5, 6, and D has eigenvalues 7, 8, 9, what are the eigenvalues of the 6 by 6 matrix $A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$?

∴ from the given information, we can assume,

$$A = \begin{bmatrix} 1 & 0 & 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 & 5 & 0 \\ 0 & 0 & 3 & 0 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

∴ we can clearly see, $\boxed{\text{eigenvalues of } A \text{ are } 1, 2, 3, 7, 8, 9}$

5.1(22)

22. Compute the eigenvalues and eigenvectors of A and A^2:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

A^2 has the same _____ as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues _____.

5/5

$$|A - I\lambda| = \begin{vmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{vmatrix} = (-1-\lambda)(-\lambda) - 6$$

$$= \lambda + \lambda^2 - 6$$

$$= \lambda^2 + \lambda - 6 \longrightarrow \therefore \boxed{\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= -3 \end{aligned}}$$

$$= \lambda^2 + 3\lambda - 2\lambda - 6$$

$$= \lambda(\lambda+3) - 2(\lambda+3)$$

$$= (\lambda-2)(\lambda+3)$$

for $\lambda = -3$,

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 2 & 3 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore 2x_1 + 3x_2 = 0$

$x_2 = x_2$

$\therefore \; x_1 = \left(\frac{-3}{2}\right)x_2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-3}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{-3}{2} \\ 1 \end{bmatrix} \longleftarrow$$

for $\lambda = 2$,

$$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ 2 & -2 & 0 \end{array}\right] \xrightarrow{R_2 \to 2R_2 + R_1} \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore -3x_1 + 3x_2 = 0$

$\therefore \; x_1 = x_2$

$x_2 = x_2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longleftarrow$$

$\therefore$ Eigenvectors for eigenvalues -3 & 2 are $\begin{bmatrix} \frac{-3}{2} \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

Now, for A^2,

$$|A^2 - I\lambda| = \begin{vmatrix} 7-\lambda & -3 \\ -2 & 6-\lambda \end{vmatrix} = (7-\lambda)(6-\lambda) - 6$$

$$= 42 - 6\lambda - 7\lambda + \lambda^2 - 6$$

$$= \lambda^2 - 13\lambda + 36$$

$$= (\lambda - 4)(\lambda - 9) \rightarrow \therefore \boxed{\begin{matrix} \lambda_1 = 4 \\ \lambda_2 = 9 \end{matrix}}$$

for $\lambda = 9$,

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 2 & 3 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore 2x_1 + 3x_2 = 0$

$x_2 = x_2$

$\therefore x_1 = \left(\frac{-3}{2}\right)x_2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-3}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{-3}{2} \\ 1 \end{bmatrix}$$

for $\lambda = 4$,

$$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ 2 & -2 & 0 \end{array}\right] \xrightarrow{R_2 \to 2R_2 + R_1} \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore -3x_1 + 3x_2 = 0$

$\therefore x_1 = x_2$

$x_2 = x_2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ Eigenvectors for eigenvalues 9 & 4 are $\begin{bmatrix} \frac{-3}{2} \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

$\therefore$ A^2 has same $\boxed{\text{eigenvectors}}$ as A.

when A has eigenvalues λ_1 & λ_2,

& A^2 has eigenvalues $\boxed{\lambda_1^2 \text{ \& } \lambda_2^2}$.

5.2 (11)

11. If the eigenvalues of A are 1, 1, 2, which of the following are certain to be true? Give a reason if true or a counterexample if false:

(a) A is invertible.

(b) A is diagonalizable.

(c) A is not diagonalizable.

• for A, $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 2$ $\longrightarrow$ $\therefore |A| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = (1)(1)(2) = 2 \neq 0$

$\therefore$ $\boxed{A \text{ is invertible} = \text{True}}$

$\because$ A has repeated eigenvalue, for 3×3 matrix, it will yield only two independant eigenvetors.

$\therefore$ A is diagonalizable = False

A is not diagonalizable = True

Ex. : $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

has eigenvalues 1, 1, 2.

→ but it is NOT diagonalizable.

Matrix $n \times n$ is diagonalizable iff & only if it has n Linearly idependant eigenvectors.

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5.2 (29)

29. $A^k = S\Lambda^k S^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than ____. Does $A^k \to 0$ or $B^k \to 0$?

$$A = \begin{bmatrix} .6 & .4 \\ .4 & .6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

Consider $|A - I\lambda| = \begin{vmatrix} 0.6-\lambda & 0.4 \\ 0.4 & 0.6-\lambda \end{vmatrix} = \lambda^2 - 1.2\lambda + 0.36 - 0.16$

$= \lambda^2 - \lambda - 0.12\lambda + 0.2$

$= (\lambda - 1)(\lambda - 0.2)$

$\therefore \lambda_1 = 1$

$\lambda_2 = 0.2$

$\therefore$ After diagonalization,

$\lambda_2 \to 0$ for A^k, $k \to \infty$

but $\lambda_1 \to 1$ for $k \to \infty$.

$\therefore$ $\boxed{A^k \text{ does not approach } 0 \text{ as } k \to \infty}$

Consider, $|B - \lambda I| = \begin{vmatrix} 0.6-\lambda & 0.9 \\ 0.1 & 0.6-\lambda \end{vmatrix} = \lambda^2 - 1.2\lambda + 0.36 - 0.09$

$= \lambda^2 - 0.9\lambda - 0.3\lambda + 0.27$

$= (\lambda - 0.9)(\lambda - 0.3)$

$\therefore \lambda_1 = 0.9$

$\lambda_2 = 0.3$

$\therefore$ for B^k where $k \to \infty$,

$\lambda_1 \to 0$

$\lambda_2 \to 0$. $\therefore B^k \to 0$.

$\therefore$ $\boxed{\text{As } k \to \infty,\ B^k \to 0}$

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$A^k = S\Lambda^k S^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than $\boxed{1}$.

5-2(33)

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33. Diagonalize B and compute $S\Lambda^k S^{-1}$ to prove this formula for B^k:

$$B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{has} \quad B^k = \begin{bmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{bmatrix}.$$

$$|B - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 0 = (3-\lambda)(2-\lambda)$$

$\therefore \lambda_1 = 3$, $\lambda_2 = 2$

for $\lambda = 2$: $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\therefore x_1 + x_2 = 0$, $x_2 = 0$ $\therefore x_1 = -x_2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

for $\lambda = 3$: $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right]$ $\therefore x_1 = 0$, $-x_2 = 0$

$\therefore$ for $x_1 = 1$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\therefore S = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \therefore S^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\therefore B^k = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2^k & 3^k \\ 2^k & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^k & -2^k + 3^k \\ 0 & 2^k \end{bmatrix} = \begin{bmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{bmatrix}$$

- Hence, proved the value of B^k

$\boxed{5.3(4)}$

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4. Suppose each "Gibonacci" number G_{k+2} is the *average* of the two previous numbers G_{k+1} and G_k. Then $G_{k+2} = \frac{1}{2}(G_{k+1} + G_k)$:

$$\begin{aligned} G_{k+2} &= \tfrac{1}{2}G_{k+1} + \tfrac{1}{2}G_k \\ G_{k+1} &= G_{k+1} \end{aligned} \quad \text{is} \quad \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Find the limit as $n \to \infty$ of the matrices $A^n = S\Lambda^n S^{-1}$.

(c) If $G_0 = 0$ and $G_1 = 1$, show that the Gibonacci numbers approach $\frac{2}{3}$.

$\therefore A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$ $\quad \therefore |A - \lambda I| = \begin{vmatrix} 0.5-\lambda & 0.5 \\ 1 & -\lambda \end{vmatrix}$

$= (0.5-\lambda)(-\lambda) - (0.5)$

$= -0.5\lambda + \lambda^2 - 0.5$

$\therefore \lambda_1 = 0.5$ $\quad = \lambda^2 - 0.5\lambda - 0.5$

$\lambda_2 = 1$ $\quad = \lambda^2 + 1\lambda - 0.5\lambda - 0.5$

$= \lambda(\lambda+1) - 0.5(\lambda+1)$

$= (\lambda - 0.5)(\lambda+1)$

for $\lambda = 1$, $\left[\begin{array}{cc|c} -0.5 & 0.5 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{R_1 \to R_2 + 2R_1} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right]$

$\therefore x_1 = x_1$

$x_1 - x_2 = 0$

$x_2 = x_1$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

for $\lambda = -0.5$, $\left[\begin{array}{cc|c} 1 & 0.5 & 0 \\ 1 & 0.5 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & 0.5 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\therefore x_1 + 0.5x_2 = 0$

$x_2 = x_2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \longleftarrow$ $\quad \therefore -x_1 = 0.5x_2$

$\therefore x_1 = x_1$

$x_2 = -2x_1$

$\therefore$ $\boxed{\begin{array}{l}\text{Eigenvalues of } A \text{ are } 1 \;\&\; -0.5. \\ \text{Eigenvectors of } A \text{ are } \begin{bmatrix}1\\1\end{bmatrix} \;\&\; \begin{bmatrix}1\\-2\end{bmatrix}\end{array}}$

Now, $S = \begin{bmatrix}1 & 1\\1 & -2\end{bmatrix}$ $\therefore$ $S^{-1} = \frac{1}{3}\begin{bmatrix}2 & 1\\1 & -1\end{bmatrix}$

$$\therefore A^n = \begin{bmatrix}1 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}1^n & 0\\0 & 0.5^n\end{bmatrix}\begin{bmatrix}2/3 & 1/3\\1/3 & -1/3\end{bmatrix}$$

$\therefore$ as $n \to \infty$, $1^n \to 1$, $0.5^n \to 0$.

$$\therefore A^\infty = \begin{bmatrix}1 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}2/3 & 1/3\\1/3 & -1/3\end{bmatrix} = \begin{bmatrix}2/3 & 1/3\\2/3 & 1/3\end{bmatrix}$$

$$\boxed{\lim_{n\to\infty} A^n = \begin{bmatrix}2/3 & 1/3\\2/3 & 1/3\end{bmatrix}}$$

Now, $G_{k+1} = A^k\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2/3 & 1/3\\2/3 & 1/3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2/3\\1/3\end{bmatrix}$

$\boxed{\text{Hence, it is shown that if } G_0 = 0,\ G_1 = 1, \text{ Gibbonacci number approaches } \left(\tfrac{2}{3}\right)}$

5.3 (17)

17. What values of α produce instability in $v_{n+1} = \alpha(v_n + w_n)$, $w_{n+1} = \alpha(v_n + w_n)$?

$$\therefore \begin{bmatrix} v_{n+1} \\ w_{n+1} \end{bmatrix} = \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix} \begin{bmatrix} v_n \\ w_n \end{bmatrix}$$

$$\therefore |A - \lambda I| = \begin{vmatrix} \alpha-\lambda & \alpha \\ \alpha & \alpha-\lambda \end{vmatrix} = (\alpha-\lambda)^2 - \alpha^2$$

$$= \lambda^2 - 2\alpha\lambda$$

$$= \lambda(\lambda - 2\alpha)$$

$\therefore \lambda_1 = 0 \rightarrow$ No issue.

$\lambda_2 = 2\alpha \rightarrow$ for stability, 2α must be less than or equal to 1.

$\therefore \alpha$ must be less than or equal to $\frac{1}{2}$.

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$\therefore$ $\boxed{|\alpha| > \frac{1}{2} \text{ will cause instability.}}$

5.4 (1)

1. Following the first example in this section, find the eigenvalues and eigenvectors, and the exponential e^{At}, for

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 - 1$$

$$= \lambda^2 + 2\lambda$$

$$= \lambda(\lambda+2) \qquad \therefore \lambda_1 = 0$$

$$\lambda_2 = -2$$

For $\lambda = 0$, $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\therefore -x_1 + x_2 = 0$

$x_2 = x_2$

$x_1 = x_2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longleftarrow$$

$\therefore \lambda = -2$, $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\therefore x_1 + x_2 = 0$

$x_2 = x_2$

$x_1 = -x_2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \longleftarrow$$

$$\therefore S = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \& \quad S^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\therefore e^{At} = S e^{\Lambda t} S^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{0t} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -e^{-2t} & 1 \\ e^{-2t} & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\therefore \quad \boxed{e^{At} = \begin{bmatrix} e^{-2t}+1 & -e^{-2t}+1 \\ -e^{-2t}+1 & e^{-2t}+1 \end{bmatrix}}$$

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Eigenvectors of A for eigenvalues 0 & -2 are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ respectively.

5.4 (5) (b)

5. A diagonal matrix like $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ satisfies the usual rule $e^{\Lambda(t+T)} = e^{\Lambda t} e^{\Lambda T}$, because the rule holds for each diagonal entry.

(b) Show that $e^{A+B} = e^A e^B$ is *not true* for matrices, from the example

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \qquad \text{(use series for } e^A \text{ and } e^B\text{)}.$$

$$e^A = \frac{I}{0!} + \frac{A}{1!} + \frac{(A)^2}{2!} + \frac{(A)^3}{3!} \ldots$$

Now, $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ $A^2, A^3, A^4 \ldots$ all equal $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\therefore e^A = I + A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$e^B = I + B + \frac{(B)^2}{2!} + \frac{(B)^3}{3!} \ldots$$

Now, $B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$

$$B^2 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ $B^2, B^3, B^4 \ldots$ all $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\therefore e^B = I + B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\therefore e^A e^B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{—— (1)}$$

Now, $A + B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\therefore |A + B - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

$$\therefore \lambda = \sqrt{1}$$

$$\therefore \lambda = \pm i$$

Now,

$$\because \quad e^{(A+B)t} = \frac{1}{2} e^{it} \begin{bmatrix} 1 \\ -i \end{bmatrix} + \frac{1}{2} e^{-it} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Substituting $\cos(t) + i\sin(t) = e^{it}$

$\cos(t) - i\sin(t) = e^{-it}$

$$\therefore \quad e^{(A+B)t} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$$\therefore \text{ for } t = 1, \quad e^{A+B} = \begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix} \quad \text{—— } ②$$

$\therefore$ from equation ① & ②, we can say:

$$\boxed{e^{A+B} \neq e^A e^B}$$

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5.4 (9) (a, b)

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9. Decide the stability of $u' = Au$ for the following matrices:

(a) $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. (b) $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$.

For A, trace (A) $= \lambda_1 + \lambda_2 = 7$

& $|A| = \lambda_1 \lambda_2 = (10) - (12) = -2$.

$\therefore$ we can conclude, at least one eigenvalue is positive & $\therefore$ other is negative. (No 0 eigenvalues)

$\therefore$ $\boxed{\text{For A, system is unstable.}}$

For B, trace (B) $= \lambda_1 + \lambda_2 = 0$

& $|B| = \lambda_1 \lambda_2 = (-1) - (6) = -7$

$\therefore$ we can conclude, at least one eigenvalue is positive & $\therefore$ other is negative. (no eigenvalue is 0)

$\therefore$ $\boxed{\text{For B, system is unstable.}}$