EE 810: Homework #$\triangle$

Name: Onkar Vivek Apte

ODU ID: [redacted]

3.1(6)

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6. Find all vectors in $\mathbf{R}^3$ that are orthogonal to $(1,1,1)$ and $(1,-1,0)$. Produce an orthonormal basis from these vectors (mutually orthogonal unit vectors).

$\therefore$ Let $a = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $b = \begin{bmatrix}1\\-1\\0\end{bmatrix}$, $c = \begin{bmatrix}a\\b\\c\end{bmatrix}$

If a, b & c are orthogonal, $a \cdot c = 0$
& $b \cdot c = 0$

$\therefore a + b + c = 0$
& $a - b = 0. \longrightarrow \underline{a = b}.$
$\therefore a + a + c = 0,$
$\therefore \underline{-2a = c}$ $\therefore c = \begin{bmatrix}a\\a\\-2a\end{bmatrix} = a\begin{bmatrix}1\\1\\-2\end{bmatrix}$

$\therefore a, b$ & c are orthogonal.

To make them unit, $|a| = \sqrt{1+1+1} = \sqrt{3}$
$|b| = \sqrt{1+1} = \sqrt{2}$
$|c| = \sqrt{a^2+a^2+4a^2} = a\sqrt{6}$

$\therefore$ Let $x = \frac{a}{|a|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$

$\therefore$ Let $y = \frac{b}{|b|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$

$\therefore$ Let $z = \frac{c}{|c|} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$

$\therefore$ Vectors orthogonal to $\begin{bmatrix}1\\1\\1\end{bmatrix}$ & $\begin{bmatrix}1\\-1\\0\end{bmatrix}$ are $a\begin{bmatrix}1\\1\\-2\end{bmatrix}$, $a \in \mathbb{R}$

Orthonormal basis required is $\{x, y, z\} = \left\{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}\right\}$

3.1 (7)

7. Find a vector x orthogonal to the row space of A, and a vector y orthogonal to the column space, and a vector z orthogonal to the nullspace:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}.$$

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$$A \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to -R_3 + R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Let $x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be orthogonal to row space of A.

$$\therefore \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $a + 2b + c = 0$

$c = 0$

$a + 2b = 0. \longrightarrow a = -2b$

$$\therefore x = \begin{bmatrix} -2b \\ b \\ 0 \end{bmatrix} = b \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ Vectors orthogonal to Row space of (A) are given by $x = b \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ where $b \in \mathbb{R}$

Now, $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 4 \end{bmatrix}$

$$\Big\downarrow \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Let $Y = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be orthogonal to coloumn space.

$$\therefore \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad a + 2b + 3c = 0.$

$b + c = 0. \longrightarrow \quad b = -c$

$\therefore \quad a + 2b - 3c = 0. \rightsquigarrow \quad a = b.$

$$\therefore \quad Y = \begin{bmatrix} b \\ b \\ -b \end{bmatrix} = b \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$\therefore$ vectors orthogonal to coloumn space of A are given by $y = b \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ where $b \in R$

$\therefore$ for null space vector,

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $Z = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$ be orthogonal to the nullspace.

to find nullspace vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad a + 2b + 3c = 0$

$c = 0.$

$\therefore \quad a + 2b = 0.$

$\therefore \quad a = -2b$

$\therefore$ Nullspace vector $= \begin{bmatrix} -2b \\ b \\ 0 \end{bmatrix}$

$= b \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$\because$ Z is orthogonal to Nullspace vector,

$-2a + b + 0(c) = 0. \longrightarrow \therefore \; Z = \begin{bmatrix} k \\ 2k \\ 1 \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\therefore$ $\boxed{\text{vector orthogonal to nullspace is } Z = k \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ where } k \in R}$

3.1 (15)

15. Find a matrix whose row space contains $(1,2,1)$ and whose nullspace contains $(1,-2,1)$, or prove that there is no such matrix.

- vectors in nullspace are orthogonal to vectors in row space.

$$\therefore \quad \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 0$$

$\therefore \quad (1)(1) + (2)(-2) + (1)(1) = 0.$

$\therefore \quad 1 - 4 + 1 = 0.$

$\underline{-2 = 0}$ $\longrightarrow$ false.

which means $\rightarrow$ there exist no such matrix whose row space contains $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ & null space contains $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

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$\boxed{3.2(5)}$

5. In n dimensions, what angle does the vector $(1, 1, \ldots, 1)$ make with the coordinate axes? What is the projection matrix P onto that vector?

Let $a = \underset{1\times n}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}$. Consider coordinate axis $x = \underset{1\times n}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$

- Finding $|a| = \sqrt{\underbrace{1^2 + 1^2 \cdots \cdots 1^2}_{n \text{ times}}}$

$= \sqrt{\underbrace{1 + 1 + 1 \cdots + 1}_{n \text{ times}}}$

$= \sqrt{n}$

- Finding $|x| = \sqrt{1^2 + 0^2 + 0^2 \cdots + 0^2}$

$= \sqrt{1}$

$= 1$

- Finding $a \cdot b = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = (1)(1) + (1)(0) + \cdots (1)(0)$

$= 1 + 0 + 0 \cdots \cdots 0$

$= 1$

$\therefore$ to find angle between a & x, $\cos\theta = \dfrac{a \cdot b}{|a|\,|b|}$

$= \dfrac{1}{\sqrt{n} \cdot 1} = \dfrac{1}{\sqrt{n}} = \underline{\underline{n^{-\frac{1}{2}}}}$

$\boxed{\therefore \text{ In } n \text{ dimensions, angle between } \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \text{ \& a coordinate axis } = \cos^{-1}(n^{-\frac{1}{2}})}$

• The projection matrix will be given as:

$$P = \frac{a a^T}{|a|^2}$$

$$= \frac{\underset{1\times n}{\begin{bmatrix} 1 \\ 1 \\ \vdots \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}}}{|a|^2}$$

$$= \frac{\underset{n\times n}{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \end{bmatrix}}}{\sqrt{\underbrace{1^2 + 1^2 \cdots\cdots 1^2}_{n \text{ times}}}}$$

$$= \frac{1}{\sqrt{n}} \underset{n\times n}{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \end{bmatrix}}$$

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$$\boxed{\therefore \text{ Projection matrix onto } \underset{1\times n}{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}} \text{ is given by } P = \frac{1}{\sqrt{n}} \underset{n\times n}{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \end{bmatrix}}}$$

3.2 (7)

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7. By choosing the correct vector b in the Schwarz inequality, prove that

$$(a_1+\cdots+a_n)^2 \le n(a_1^2+\cdots+a_n^2).$$

When does equality hold?

Let $a = \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{1\times n}$ & $b = \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{1\times n}$

$\therefore$ using schwars inequality, we get:

$$a^T b \leqslant |a|\,|b|$$

$$\therefore \underbrace{\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}}_{n\times 1} \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{1\times n} \leqslant \sqrt{a_1^2+a_2^2 \cdots \; a_n^2} \cdot \sqrt{\underbrace{1^2+1^2 \cdots + 1^2}_{n \text{ times}}}$$

$$\therefore a_1 + a_2 + \cdots \; a_n \leqslant \sqrt{a_1^2+a_2^2+\cdots+a_n^2} \cdot \sqrt{n}$$

— (squaring)

$$\therefore (a_1+a_2+\cdots \; a_n)^2 \leqslant n(a_1^2+a_2^2+\cdots+a_n^2)$$

$\because$ schwarz inequality becomes equality when a & b lie on the same line, for this case, it is only possible if $a_1 = a_2 = a_3 = \cdots = a_n$.

$\therefore$ $\boxed{(a_1+\cdots+a_n)^2 \leqslant n\,(a_1^2+\cdots+a_n^2) \text{ is true if } a_1 = a_2 = \cdots = a_n}$

3.2 (11)

11. (a) Find the projection matrix P_1 onto the line through $a = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and also the matrix P_2 that projects onto the line perpendicular to a.

(b) Compute $P_1 + P_2$ and $P_1 P_2$ and explain.

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$$P_1 = \frac{aa^T}{a^Ta} = \frac{\begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \end{bmatrix}} = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

Now, to find P_2 which projects onto a line $\perp$ to $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$,

$$P_2 = I - P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix}$$

$$= \begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$$

$\therefore$ Required matrices are :

$$\boxed{P_1 = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \qquad P_2 = \frac{1}{10}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}}$$

Now,

$$P_1 + P_2 = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} + \frac{1}{10}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{P_1 + P_2 = I \quad \text{—— Sum of two orthogonal projections is identity.}}$$

$$P_1 P_2 = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \cdot \frac{1}{10}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{3}{10}\begin{bmatrix} 9-9 & -3+3 \\ 27-27 & -9+9 \end{bmatrix}$$

$$= \frac{3}{10}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\boxed{P_1 \cdot P_2 = \text{Null Matrix} \quad \text{—— Product of perpendular projectors is Null matrix}}$$

3.2 (12)

12. Find the matrix that projects every point in the plane onto the line $x+2y=0$.

$x + 2y = 0$

$\therefore \quad x = -2y$

$\therefore$ vector associated with this line $= \begin{bmatrix} -2y \\ y \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

for $y = 1$, vector $= \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

· Let the vector on $(x + 2y = 0)$ be $z = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\therefore$ Projector on z will be given by:

$$P = \frac{z z^T}{z^T z} = \frac{\begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \end{bmatrix}}{\begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

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$$= \frac{1}{5} \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

$\therefore$ $\boxed{\text{Matrix that projects every point onto } x+2y=0 \text{ is } \frac{1}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}}$

3.3 (1)

1. Find the best least-squares solution $\hat{x}$ to $3x = 10$, $4x = 5$. What error E^2 is minimized? Check that the error vector $(10 - 3\hat{x}, 5 - 4\hat{x})$ is perpendicular to the column $(3, 4)$.

Let $a = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $b = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

$\therefore$ Least-square solution $\rightarrow$ $\hat{x} = \dfrac{a^T b}{a^T a}$

$$= \frac{\begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix}}{\begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}} = \frac{30 + 20}{9 + 16} = \frac{50}{25}$$

$$= \underline{\underline{2}}$$

Now, to minimize error,

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$$E^2 = |a\hat{x} - b|^2$$
$$= (a_1\hat{x} - b_1)^2 + (a_2\hat{x} - b_2)^2$$
$$= (3 \cdot 2 - 10)^2 + (4 \cdot 2 - 5)^2$$
$$= (-4)^2 + (3)^2$$
$$= \underline{\underline{25}}$$

Now, cheking the perpendicularity of $e = \begin{bmatrix} 10 - 3\hat{x} \\ 5 - 4\hat{x} \end{bmatrix}$,

$(e \cdot a)$
dot product

$$= \begin{bmatrix} 10 - 3\hat{x} & 5 - 4\hat{x} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 - (3 \times 2) & 5 - (4 \times 2) \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \underline{\underline{0}}$$

$\therefore$ $e \perp a$.

$\therefore$ $\boxed{\begin{array}{l} \text{Best least square solution} = 2 \\ \text{Value of minimized error} = 25 \\ \begin{bmatrix} 10 - 3\hat{x} \\ 5 - 4\hat{x} \end{bmatrix} \text{ is perpendicular to } \begin{bmatrix} 3 \\ 4 \end{bmatrix} \end{array}}$

$\boxed{3.3\ (6)}$

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6. Find the projection of b onto the column space of A:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}.$$

Finding $A^T A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix}$

$$\therefore (A^T A)^{-1} = \frac{1}{44} \begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix}$$

Now, calculating $A^T b = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} -11 \\ 27 \end{bmatrix}$

$\therefore$ Now, Projection of b onto coloumn space of A will be given by :

$$P = A \cdot (A^T A)^{-1} \cdot A^T b$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \cdot \frac{1}{44} \begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix} \cdot \begin{bmatrix} -11 \\ 27 \end{bmatrix}$$

$$= \frac{1}{44} \begin{bmatrix} 92 \\ -56 \\ 260 \end{bmatrix}$$

$\therefore$ Projection of b onto coloumn space of A is $\frac{1}{44} \begin{bmatrix} 92 \\ -56 \\ 260 \end{bmatrix}$

Split b into $p+q$, with p in the column space and q perpendicular to that space. Which of the four subspaces contains q?

$\because \quad b = p + q,$

$q = p - b$

$$= \frac{1}{44}\begin{bmatrix} 92 \\ -56 \\ 260 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \frac{1}{44}\begin{bmatrix} 48 \\ -144 \\ -48 \end{bmatrix}$$

$\therefore$ Required splitting $b = p + q$ is :

$$\boxed{\underbrace{\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}}_{b} = \underbrace{\frac{1}{44}\begin{bmatrix} 92 \\ -56 \\ 260 \end{bmatrix}}_{p} + \underbrace{\frac{1}{44}\begin{bmatrix} 48 \\ -144 \\ -48 \end{bmatrix}}_{q}}$$

Now, $\because$ coloumn space and the left null space of A are orthogonal & q is $\perp$ to $C(A)$, we have :

$$q^T A = \frac{1}{44}\begin{bmatrix} 44 & -144 & -48 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{44}\begin{bmatrix} 48 - 144 + 96 \\ 48 + 144 - 192 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ $\boxed{q \text{ is left null space.}}$

3.4 (2)

2. Project $b = (0,3,0)$ onto each of the orthonormal vectors $a_1 = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ and $a_2 = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, and then find its projection p onto the plane of a_1 and a_2.

$$a_1 = \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

$\therefore$ Projecton of b on a_1 $= P_1 = \dfrac{b \cdot a_1}{a_1 \cdot a_1} a_1$

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$$= \frac{(0 \times 2/3) + (3 \times 2/3) + (0 \times -1/3)}{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2} \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

$$= \frac{2}{4/9 + 4/9 + 1/9} \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

$$= \frac{2}{1} \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 4/3 \\ 4/3 \\ -2/3 \end{bmatrix}$$

$\therefore$ $\boxed{\text{Projection of } b \text{ on } a_1 \text{ is } \begin{bmatrix} 4/3 \\ 4/3 \\ -2/3 \end{bmatrix}}$

$\therefore$ Projection of b onto a_2 = $P_2 = \dfrac{b \cdot a_2}{a_2 \cdot a_2} a_2$

$$= \frac{(0) + (3 \times 2/3) + (0)}{\left(\frac{-1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$= 2 \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} -2/3 \\ 4/3 \\ 4/3 \end{bmatrix}$$

$\therefore$ $\boxed{\text{Projection of } b \text{ on } a_2 \text{ is } \begin{bmatrix} -2/3 \\ 4/3 \\ 4/3 \end{bmatrix}}$

Now, Projection of b on plane of a_1 & a_2 = $P = P_1 + P_2$

$$= \begin{bmatrix} 4/3 \\ 4/3 \\ -2/3 \end{bmatrix} + \begin{bmatrix} -2/3 \\ 4/3 \\ 4/3 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 \\ 8/3 \\ 2/3 \end{bmatrix}$$

$\therefore$ $\boxed{\text{Projection of } b \text{ on plane of } a_1 \text{ \& } a_2 \text{ is } \begin{bmatrix} 2/3 \\ 8/3 \\ 2/3 \end{bmatrix}}$

3.4 (5)

5. If u is a unit vector, show that $Q = I - 2uu^T$ is a symmetric orthogonal matrix. (It is a reflection, also known as a Householder transformation.) Compute Q when $u^T = \left[\frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}\ -\frac{1}{2}\right]$.

- It is given that $Q = I - 2uu^T$ where u is unit vector.
- We have to prove Q is symmetric & orthogonal.
- If Q is symmetric, $Q^T = Q$ must be true.
- If Q is orthogonal, $Q^TQ = I$ must be true.

$\therefore$ cosider

$$Q^TQ = (I - 2uu^T)^T(I - 2uu^T)$$
$$= \left[I^T - 2(uu^T)^T\right](I - 2uu^T)$$
$$= \left[I - 2\left((u^T)^T u^T\right)\right](I - 2uu^T)$$
$$= (I - 2uu^T)(I - 2uu^T)$$
$$= I - 2uu^T - 2uu^T + 4u\underline{u^Tu}u^T$$
$$= I - 4uu^T + 4uu^T$$
$$= I$$

— hence proved, $Q^TQ = I$.

$\therefore$ <u>Q is orthogonal.</u> — (1)

$\therefore$ Now, cosider

$$Q^T = (I - 2uu^T)^T = (I^T - 2(uu^T)^T)$$
$$= I^T - 2\left((u^T)^T u^T\right)$$
$$= I - 2uu^T$$
$$= Q.$$

hence proved, $Q^T = Q$.

$\therefore$ <u>Q is symmetric.</u> — (2)

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$\therefore$ from statement (1) & (2),

$\boxed{\text{It has been proved that } Q \text{ is a symmetric orthogonal matrix}}$

$\because \quad u^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}$

$\therefore \quad u = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

$\therefore \quad uu^T = \frac{1}{2} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \cdot \frac{1}{2} \cdot \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}$

$= \frac{1}{4} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$

$\therefore \quad 2uu^T = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$

$\therefore \quad I - 2uu^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$

$\therefore \quad \boxed{Q = \frac{1}{2} \begin{bmatrix} +1 & -1 & +1 & +1 \\ -1 & +1 & +1 & +1 \\ +1 & +1 & +1 & -1 \\ +1 & +1 & -1 & +1 \end{bmatrix}}$

3.4 (6)

6. Find a third column so that the matrix

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & \\ 1/\sqrt{3} & 2/\sqrt{14} & \\ 1/\sqrt{3} & -3/\sqrt{14} & \end{bmatrix}$$

is orthogonal. It must be a unit vector that is orthogonal to the other columns; how much freedom does this leave? Verify that the rows automatically become orthonormal at the same time.

Let coloum 3 be $C_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow \therefore Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & a \\ 1/\sqrt{3} & 2/\sqrt{14} & b \\ 1/\sqrt{3} & -3/\sqrt{14} & c \end{bmatrix}$

$\because$ Q is orthogonal,

$$Q^T Q = I$$

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{14} & 2/\sqrt{14} & -3/\sqrt{14} \\ a & b & c \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & a \\ 1/\sqrt{3} & 2/\sqrt{14} & b \\ 1/\sqrt{3} & -3/\sqrt{14} & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 0 & \frac{a+b+c}{\sqrt{3}} \\ 0 & 1 & \frac{a+2b-3c}{\sqrt{14}} \\ \frac{a+b+c}{\sqrt{3}} & \frac{a+2b-3c}{\sqrt{14}} & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$\therefore \left.\begin{aligned} a+b+c &= 0 \\ a+2b-3c &= 0. \\ a^2+b^2+c^2 &= 1 \end{aligned}\right\} \xrightarrow[\text{these}]{\text{Solving}}$$

Input

$\{a+b+c=0,\ a+2b-3c=0,\ a^2+b^2+c^2=1\}$

Alternate forms

$\{a+b+c=0,\ a+2b=3c,\ a^2+b^2+c^2=1\}$

$\{c=-a-b,\ c=\frac{a}{3}+\frac{2b}{3},\ a^2+b^2+c^2=1\}$

Solutions

$a=-\frac{5}{\sqrt{42}},\quad b=2\sqrt{\frac{2}{21}},\quad c=\frac{1}{\sqrt{42}}$

$a=\frac{5}{\sqrt{42}},\quad b=-2\sqrt{\frac{2}{21}},\quad c=-\frac{1}{\sqrt{42}}$

$a = \frac{-5}{\sqrt{42}} \qquad \frac{5}{\sqrt{42}}$

$b = \frac{4}{\sqrt{42}} \quad \text{OR} \quad \frac{-4}{\sqrt{42}}$

$c = \frac{1}{\sqrt{42}} \qquad \frac{-1}{\sqrt{42}}$

$\therefore \quad C_3 = \frac{1}{\sqrt{42}} \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$ or $\frac{-1}{\sqrt{42}} \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$

to make C_3 unit, $|C_3| = \sqrt{\frac{25}{42} + \frac{16}{42} + \frac{1}{42}}$

$= 1$

$\therefore$ C_3 is a unit vector.

Now, to check ortrogonality,

$$C_1 \cdot C_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} -5/\sqrt{42} \\ 4/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix} = 0$$

$$C_2 \cdot C_3 = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ -3/\sqrt{14} \end{bmatrix} \cdot \begin{bmatrix} -5/\sqrt{42} \\ 4/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix} = 0$$

$\therefore$ C_3 is $\perp$ C_2 & C_1

also $-C_3$ is $\perp$ C_2 & C_1.

C_3 is also unit.

$$\boxed{\therefore \text{ required coloumn is } \frac{a}{\sqrt{42}} \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix} \quad \text{where it has freedom of } a = \pm 1}$$

Now, to verify if rows are orthoromal, Let $R_1\ R_2\ R_3$ be rows.

$$\left.\begin{aligned} |R_1| &= \sqrt{\tfrac{1}{3}+\tfrac{1}{14}+\tfrac{25}{42}} = 1 \\ |R_2| &= \sqrt{\tfrac{1}{3}+\tfrac{4}{14}+\tfrac{16}{42}} = 1 \\ |R_3| &= \sqrt{\tfrac{1}{3}+\tfrac{9}{14}+\tfrac{1}{42}} = 1 \end{aligned}\right\} \quad \underline{R_1, R_2, R_3 \text{ are unit}} \quad — \ 1$$

$$R_1 \cdot R_3 = \left(\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{14}} \cdot \frac{-3}{\sqrt{14}}\right) + \left(\frac{-5}{\sqrt{42}} \cdot \frac{1}{\sqrt{42}}\right)$$

$$= \frac{1}{3} - \frac{3}{4} - \frac{5}{42} = 0$$

$$R_2 \cdot R_3 = \left(\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\right) + \left(\frac{2}{\sqrt{14}} \cdot \frac{-3}{\sqrt{14}}\right) + \left(\frac{4}{\sqrt{42}} \cdot \frac{1}{\sqrt{42}}\right)$$

$$= \frac{1}{3} - \frac{6}{14} + \frac{4}{42} = 0$$

$$R_1 \cdot R_2 = \left(\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{14}} \cdot \frac{2}{\sqrt{14}}\right) + \left(\frac{-5}{\sqrt{42}} \cdot \frac{4}{\sqrt{42}}\right)$$

$$= 0$$

$$\therefore \underline{\underline{R_1 \perp R_2 \perp R_3}} \quad — \ 2$$

$\therefore$ from statement 1 4 2,

$$\boxed{\text{we have verified that rows became orthonormal.}}$$

$\boxed{3.4(13)}$

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13. Apply the Gram-Schmidt process to

$$a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and write the result in the form $A = QR$.

Applying gram-schmidt on $A = [\, a \,\vdots\, b \,\vdots\, c \,]$,

$$\therefore q_1 = \frac{a}{|a|} = \frac{1}{\sqrt{0^2+0^2+1^2}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore B = b - (q_1 \cdot b)\, q_1$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore q_2 = \frac{B}{|B|} = \frac{1}{\sqrt{0^2+1^2+0^2}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore C = c - (q_1 \cdot c)\, q_1 - (q_2 \cdot c)\, q_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore q_3 = \frac{C}{|C|} = \frac{1}{\sqrt{1^2+0^2+0^2}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Now, writing in $A = QR$ form,

$$A = QR$$

$$\left[a \,\vdots\, b \,\vdots\, c \right] = \left[q_1 \,\vdots\, q_2 \,\vdots\, q_3 \right] \begin{bmatrix} q_1 \cdot a & q_1 \cdot b & q_1 \cdot c \\ 0 & q_2 \cdot b & q_2 \cdot c \\ 0 & 0 & q_3 \cdot c \end{bmatrix}$$

$$\therefore \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

This is the required result in $A = QR$ form.

$\boxed{3.4(14)}$

14. From the nonorthogonal a, b, c, find orthonormal vectors q_1, q_2, q_3:

$$a = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Applying gram-schmidt on $A = [a \vdots b \vdots c]$,

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$$q_1 = \frac{a}{|a|} = \frac{1}{\sqrt{1^2+1^2+0^2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\therefore B = b - (q_1 \cdot b)\, q_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\therefore q_2 = \frac{B}{|B|} = \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$$\therefore C = c - (q_1 \cdot c)\, q_1 - (q_2 \cdot c)\, q_2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} - \frac{3}{\sqrt{6}}\begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore q_3 = \frac{C}{|C|} = \frac{1}{\sqrt{1^2+1^2+1^2}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\boxed{\therefore \text{Orthonormal vectors are: } \{q_1, q_2, q_3\} \equiv \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}}$$