EE 810: Homework #8

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1.4 (42)

(a) If A^2 is defined then A is necessarily square.

(b) If AB and BA are defined then A and B are square.

(c) If AB and BA are defined then AB and BA are square.

(d) If $AB = B$ then $A = I$.

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(A) TRUE

Assume A with dimention $m \times n$.

$\therefore \quad A \times A = A^2$

$(m \times n) \times (m \times n)$

for this multiplication to be valid, m must be equal to n. — $m = n$

$\therefore \quad A \times A = A^2$

$(m \times m) \times (m \times n) = (m \times m)$

Square Matrix

(B) FALSE

Assume $A \rightarrow$ dim $m \times n$

$B \rightarrow$ dim $n \times q$

Must be same for AB to Exist

$\therefore \quad A \cdot B = AB$

$(m \times n) \quad (n \times q) \quad\quad (m \times q)$

However, if BA also exist, then

$B \cdot A = BA$

$(n \times q) \cdot (m \times n) \quad\quad (n \times n)$

For BA to be valid, $q = m$.

$\therefore$ Possible dimension of A & B are $A_{m \times n}$ & $B_{n \times m}$ for both AB & BA to exist, However m & n can be any thing.

(C) $\boxed{\text{TRUE}}$ As proven in part (B) of this question, if AB & BA exist, then A : dim $m \times n$
B : dim $n \times m$.

$$\therefore \underset{m \times n}{A} \cdot \underset{n \times m}{B} = \underset{m \times m}{AB} \longrightarrow \text{square matrix}$$

$$\therefore \underset{n \times m}{B} \cdot \underset{m \times n}{A} = \underset{n \times n}{BA} \longrightarrow \text{square matrix}$$

(D) $\boxed{\text{False}}$ If $AB = B$, does not have to I in all the cases.

If B is a matrix with all zeros, then A can be anything & AB will still equal B; i.e. matrix with zeros.

$\boxed{1.4\ (49)}$

$\frac{16}{16}$

49. *Elimination for a 2 by 2 block matrix*: When $A^{-1}A = I$, multiply the first block row by CA^{-1} and subtract from the second row, to find the "*Schur complement*" S:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix}.$$

given $A^{-1}A = I$

$\therefore \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix}$

$\therefore \begin{bmatrix} A & B \\ -CA^{-1}A + C & -CA^{-1}B + D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix}$

$\therefore \begin{bmatrix} A & B \\ -C + C & -CA^{-1}B + D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix}$

$\therefore \begin{bmatrix} A & B \\ 0 & -CA^{-1}B + D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix}$

$\therefore \boxed{S = -CA^{-1}B + D}$

1.5 (1)

1. When is an upper triangular matrix nonsingular (a full set of pivots)?

Consider upper triangular matrix $A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$

If A is singular, $\det(A) = 0$

$$\begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix} = 0$$

$$a\begin{vmatrix} d & e \\ 0 & f \end{vmatrix} - b\begin{vmatrix} 0 & e \\ 0 & f \end{vmatrix} + c\begin{vmatrix} 0 & e \\ 0 & f \end{vmatrix} = 0$$

$$a(bc) - b(0) + c(0) = 0$$

$$abc = 0$$

Means atleast one of the element a, b or c is zero.

∴ Simillarly determinant of any upper triangular $n \times n$ matrix is zero, if at least one of the diagonal element of the matrix is zero.

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∴ $\boxed{\text{upper triangular matrix is nonsingular, i.e. it has a full set of pivot, if \& only if all the diagonal elements are non-zero}}$

1.5(11)

11. Solve as two triangular systems, without multiplying LU to find A:

$$LUx = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

$L \quad U \quad x \quad = \quad b$ —— (given)

- Say, $Lc = b$ where $c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$\therefore c_1 = 2$
$\therefore c_1 + c_2 = 0$
$\therefore c_3 + c_3 = 6$

$\longrightarrow$

$\therefore c_1 = 2$
$\therefore c_2 = -2$
$\therefore c_3 = 0$

$\rightarrow \therefore c = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$

- Now, $\because Ux = c$,

$$\begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

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$\therefore 2u + 4v + 4w = 2$
$\therefore v + 2w = -2$
$\therefore w = 0$

$\longrightarrow$

$\therefore u = 5$
$\therefore v = -2$
$\therefore w = 0$

$\longrightarrow$

Hence,

$$x = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$$

1.5 (13) 10/10

13. Solve by elimination, exchanging rows when necessary:

$$① \quad \begin{aligned} u + 4v + 2w &= -2 \\ -2u - 8v + 3w &= 32 \\ v + w &= 1 \end{aligned} \quad \text{and} \quad ② \quad \begin{aligned} v + w &= 0 \\ u + v &= 0 \\ u + v + w &= 1. \end{aligned}$$

① Representing the given three equations in a matrix form.

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & -2 \\ -2 & -8 & 3 & 32 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$\downarrow R_2 \to R_2 + 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & -2 \\ 0 & 0 & 7 & 28 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$\downarrow R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 7 & 28 \end{array}\right]$$

$\downarrow R_3 \to R_3/7$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right] \longrightarrow \begin{array}{l} \therefore w = 4 \\ \therefore v + w = 1 \\ \therefore u + 4v + 2w = -2 \end{array} \longrightarrow \begin{array}{l} \therefore u = 2 \\ \therefore v = -3 \\ \therefore w = 4 \end{array}$$

$$\boxed{\therefore \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \quad \& \because \text{we did } R_2 \leftrightarrow R_3, \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$$

(2) Representing the given three equations in a matrix form.

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

$$\Big\downarrow R_3 \leftrightarrow R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\Big\downarrow R_2 \to R_2 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\Big\downarrow R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{array}\right] \longrightarrow \begin{array}{c} -w = -1 \\ v + w = 0 \\ u + v + w = 1 \end{array} \longrightarrow \begin{array}{c} w = 1 \\ v = -1 \\ u = 1 \end{array}$$

$$\therefore \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

& $\because$ we did $R_1 \leftrightarrow R_3$

& then $R_2 \leftrightarrow R_3$

we did $\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \longrightarrow \begin{bmatrix} R_3 \\ R_2 \\ R_1 \end{bmatrix} \rightarrow \begin{bmatrix} R_3 \\ R_1 \\ R_2 \end{bmatrix}$

1.6 (2)

(a) Find the inverses of the permutation matrices

$$\boxed{1} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boxed{2} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

$\boxed{1}$

$$\det(P) = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$= -1$ —— $\therefore P$ is non-singular

$\therefore P^{-1}$ exists.

$\therefore$ by gauss-jordan method,

$$[P \mid I] \rightarrow \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow R_3 \leftrightarrow R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

P^{-1}

$$\therefore \boxed{P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$$

$\boxed{2}$

$$\det(P) = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$= 1$ ——— $\therefore$ P is non-singular

$\therefore P^{-1}$ exists.

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$\therefore$ by gauss-jordan method,

$$[P \mid I] \rightarrow \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow R_2 \leftrightarrow R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix}}_{P^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

$$\therefore \boxed{P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}$$

(b) Explain for permutations why P^{-1} *is always the same as* P^{T}. Show that the 1s are in the right places to give $PP^{T} = I$.

- What permutation do is to exchanges rows of a matrix.

- Inverse of a matrix always undo the transformation. ∴ for permutations, P^{-1} should un-exchange the rows exchanged by P.

- consider :

$$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

∵ we have calculated in 1.6 (2) (a) $\boxed{1}$ & $\boxed{2}$,

$$P_1^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad P_2^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

but $$P_1^{T} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad P_2^{T} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

∴ $\underline{\underline{P^{-1} = P^{T}}}$

consider permutation matrix $P = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$\therefore PP^T = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$= \begin{bmatrix} a^2+b^2+c^2 & ad+be+cf & ag+bh+ci \\ ad+be+cf & d^2+e^2+f^2 & dg+eh+fi \\ ag+bh+ci & dg+eh+fi & g^2+h^2+i^2 \end{bmatrix}$$

$\because$ P is a permutation matrix, one element in each row is 1.

$\therefore$ all three a, b & c cannot be 0s at same time.
$\therefore$ all three e, f & g cannot be 0s at same time.
$\therefore$ all three h, i & j cannot be 0s at same time.

$\therefore a^2+b^2+c^2 = 1$
$\therefore e^2+f^2+g^2 = 1$ $\longrightarrow$ but these are entries along the diagonal of PP^T
$\therefore h^2+i^2+j^2 = 1$

Also, all the other entries except diagonal must be 0.

$$\therefore PP^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{I}$$

$\boxed{\text{Hence, the 1s are in right place to give } PP^T = \underline{I}}$

1.6 (10)

10. Find the inverses (in any legal way) of

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}.$$

(1) using gauss - Jordan method,

$$[\, A_1 \mid I \,] = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow \; \begin{aligned} R_2 &= R_2/2 \\ R_3 &= R_3/3 \\ R_4 &= R_4/4 \end{aligned}$$

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 \end{array}\right]$$

$$\downarrow \; \begin{aligned} R_1 &\leftrightarrow R_4 \\ R_2 &\leftrightarrow R_3 \end{aligned}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

A_1^{-1}

$$\therefore \boxed{A_1^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}$$

(2) Using gauss - Jordan method,

$$[A_2 \mid I] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{-2}{3} & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 + 1/2 R_1$

$R_3 \to R_3 + 2/3 R_2$

$R_4 \to R_4 + 3/4 R_3$

only for this specific question, we can perform the operations one after other without computing full matrix after every operation — $\because$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/4 & 2/4 & 3/4 & 0 \end{array}\right]$$

A_2^{-1}

$$\therefore \boxed{A_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 \\ 1/4 & 2/4 & 3/4 & 0 \end{bmatrix}}$$

③ using gauss-jordan method,

$$[A_3 \mid I] = \left[\begin{array}{cccc|cccc} a & b & 0 & 0 & 1 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & 1 & 0 \\ 0 & 0 & c & d & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow R_1 \to R_1/a$$

$$\left[\begin{array}{cccc|cccc} 1 & b/a & 0 & 0 & 1/a & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & 1 & 0 \\ 0 & 0 & c & d & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow R_2 \to R_2 - cR_1$$

$$\left[\begin{array}{cccc|cccc} 1 & b/a & 0 & 0 & 1/a & 0 & 0 & 0 \\ 0 & \frac{ad-cb}{a} & 0 & 0 & -c/a & 1 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & 1 & 0 \\ 0 & 0 & c & d & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow R_2 \to R_2 \times \frac{a}{ad-cb}$$

$$\left[\begin{array}{cccc|cccc} 1 & b/a & 0 & 0 & 1/a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & 1 & 0 \\ 0 & 0 & c & d & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow R_1 \to R_1 - \frac{1}{ab} R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{d}{ad-bc} & \frac{-b}{aa-bc} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & 1 & 0 \\ 0 & 0 & c & d & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow R_3 = \frac{R_3}{a}$$

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$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 \\ 0 & 0 & 1 & b/a & 0 & 0 & 1/a & 0 \\ 0 & 0 & c & d & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \; R_4 = R_4 - cR_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 \\ 0 & 0 & 1 & b/a & 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & \frac{ad-bc}{a} & 0 & 0 & \frac{-c}{a} & 1 \end{array}\right]$$

$$\Big\downarrow \; R_4 = \frac{a}{ad-bc}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 \\ 0 & 0 & 1 & b/a & 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\Big\downarrow \; R_3 \to R_3 - \frac{1}{ab} R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

A_3^{-1}

$$\boxed{\therefore A_3^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b & 0 & 0 \\ -c & a & 0 & 0 \\ 0 & 0 & d & -b \\ 0 & 0 & -c & a \end{bmatrix}}$$

$\boxed{1.6\ (20)}$

20. Find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}.$$

using gauss - jordan method,

$$[A \mid I] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1/3 & 1/3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} R_2 \to R_2 - R_1/4 \\ R_3 \to R_3 - R_1/3 \\ R_4 \to R_4 - R_1/2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/4 & 1 & 0 & 0 \\ 0 & 1/3 & 1 & 0 & -1/3 & 0 & 1 & 0 \\ 0 & 1/2 & 1/2 & 1 & -1/2 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} R_4 \to R_4 - R_2/2 \\ R_3 \to R_3 - R_2/3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/4 & -1/3 & 1 & 0 \\ 0 & 0 & 1/2 & 1 & -3/8 & -1/2 & 0 & 1 \end{array}\right]$$

$$\swarrow \quad R_4 \to R_4 - R_3/2$$

$$
\left[\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1/4 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1/4 & -1/3 & 1 & 0 \\
0 & 0 & 0 & 1 & -1/4 & -1/3 & -1/2 & 1
\end{array}\right]
$$

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$$
\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/4 & 1 & 0 & 0 \\ -1/4 & -1/3 & 1 & 0 \\ -1/4 & -1/3 & -1/2 & 1 \end{bmatrix}}_{A^{-1}}
$$

$$
\therefore \boxed{A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/4 & 1 & 0 & 0 \\ -1/4 & -1/3 & 1 & 0 \\ -1/4 & -1/3 & -1/2 & 1 \end{bmatrix}}
$$

$\boxed{1.6\ (40)}$

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40. True or false (with a counterexample if false and a reason if true):

(a) A 4 by 4 matrix with a row of zeros is not invertible.

(b) A matrix with 1s down the main diagonal is invertible.

(c) If A is invertible then A^{-1} is invertible.

(d) If A^T is invertible then A is invertible.

(A) $\boxed{\text{TRUE}}$

If 4x4 matrix has row of zeros,
∴ its determinant will be zero.
it means it is a singular matrix.
∴ Inverse of singular matrix is not possible.
∴ It is not invertible.

(B) $\boxed{\text{FALSE}}$

Consider a simple matrix

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\det(M) = 0 \longrightarrow \text{singular.}$$

∴ inverse NOT possible

(C) $\boxed{\text{TRUE}}$

∵ A is invertible, A^{-1} exist.

$$\therefore\ A^{-1}A = I$$

but also → $A A^{-1} = I$

This mean inverse of A^{-1} is A.

$$(A^{-1})^{-1} = A.$$

∴ A^{-1} is invertible.

D $\boxed{\text{TRUE}}$

$\because$ A^T is invertible,
it has n pivots. — (A^T: dim $n \times n$)

$\therefore (A^T)^T = A$ also has n pivots
dim $n \times n$

$\because$ A has dim $n \times n$ & has n pivots,
A is also invertible.

Review 1.13

1.13 Solve $Ax = b$ by solving the triangular systems $Lc = b$ and $Ux = c$:

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

What part of A^{-1} have you found, with this particular b?

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$\therefore \quad L\,C = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore \; c_3 = 1$

$c_2 = 0$

$c_1 = 0$

$\therefore \; c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Now,

$$Ux = c$$

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$$\therefore \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore \quad 2x_1 + 2x_2 + 4x_3 = 0$

$\therefore \quad x_2 + 3x_3 = 0$

$\therefore \quad x_3 = 1$

$\longrightarrow$

$\therefore \quad x_3 = 1$

$\therefore \quad x_2 = -3$

$\therefore \quad x_1 = 1$

$\therefore$ solution of given triangular system is $x = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$

$\because$ this $b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, row 1, row 2 empty & row 3 = 1,

we get last coloumn of A^{-1} as x with this b

Review 1.22

1.22 (a) If A is invertible what is the inverse of A^{T}?

(b) If A is also symmetric what is the transpose of A^{-1}?

(c) Illustrate both formulas when $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

(a) consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\therefore \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{— (i)}$$

can be shown easily using matrix identities

Now, $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$$\therefore \quad (A^T)^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \quad \text{— (ii)}$$

∴ from equation (i) & (ii), we can clearly see that inverse of A^T is nothing but transpose of A^{-1}

$$\therefore \boxed{(A^T)^{-1} = (A^{-1})^T} \quad \text{— (given, A is invertible)}$$

proven here

(b) if A is symmetric, ——— $\therefore A^T = A$

$A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$; $A^{-1} = \dfrac{1}{ab - c^2} \begin{bmatrix} b & -c \\ -c & a \end{bmatrix}$ ∵ $(A^T)^{-1} = (A^{-1})^T$

$$\therefore \boxed{(A^{-1})^T = A^{-1}}$$

$A^T = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$; $A^{-1} = \dfrac{1}{ab - c^2} \begin{bmatrix} b & -c \\ -c & a \end{bmatrix}$

(c)

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad A^{-1} = \frac{1}{2-1}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Now, $A^T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad (A^T)^{-1} = \frac{1}{2-1}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\therefore \quad (A^{-1})^T = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = A^{-1}$$

$$\therefore \quad (A^T)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = A^{-1}$$

Hence, both formulas have been illustrated.

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