EE 810: Homework #5

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4.2 (1)

1. If a 4 by 4 matrix has $\det A = \frac{1}{2}$, find $\det(2A)$, $\det(-A)$, $\det(A^2)$, and $\det(A^{-1})$.

4×4, $n = 4$

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- $\det(2A) = 2^4 \cdot \det A$
  $= 16 \cdot (\frac{1}{2})$
  $= 8$

- $\det(-A) = (-1)^4 \det(A)$
  $= \frac{1}{2}$

- $\det(A^2) = \det(A) \cdot \det(A)$
  $= \frac{1}{2} \cdot \frac{1}{2}$
  $= \frac{1}{4}$

- $\det(A^{-1}) = \frac{1}{\det(A)}$
  $= \frac{1}{(1/2)}$
  $= 2$

$\therefore$
$$\boxed{\begin{aligned} \det(2A) &= 8 \\ \det(-A) &= 1/2 \\ \det(A^2) &= 1/4 \\ \det(A^{-1}) &= 2 \end{aligned}}$$

$\boxed{4.2\,(5)}$

5. Count row exchanges to find these determinants:

① $\det\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}=\pm 1$ and ② $\det\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}=-1.$

①

$$\begin{vmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{vmatrix}\xrightarrow{R_4\leftrightarrow R_1\ ^{1}} = (-1)\begin{vmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{vmatrix}$$

$$\downarrow R_2\leftrightarrow R_3\ ^{2}$$

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$$=(-1)(-1)\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}$$

$$=(+1)\,(1)(1)(1)(1)$$

$$=\underline{\underline{+1}}$$

②

$$\begin{vmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{vmatrix}\xrightarrow{R_1\leftrightarrow R_4\ ^{1}}(-1)\begin{vmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{vmatrix}\xrightarrow{R_4\leftrightarrow R_2\ ^{2}}(-1)^2\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{vmatrix}$$

$$R_3\leftrightarrow R_4\ ^{3}\ \downarrow$$

$$=(-1)^3\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}$$

$$=(-1)(1)(1)(1)(1)$$

$$=\underline{\underline{-1}}$$

$\therefore$ No. of row exchanges for Q① = 2
No. of row exchanges for Q② = 3

4.2 (10)

10. If Q is an orthogonal matrix, so that $Q^{\mathrm{T}}Q = I$, prove that $\det Q$ equals $+1$ or -1. What kind of box is formed from the rows (or columns) of Q?

Given : $Q^T Q = I$

$\therefore \det(Q^T Q) = \det(I)$

$\therefore \det(Q^T)\det(Q) = \det(I)$

$\therefore \det(Q)\det(Q) = 1$

$\therefore [\det(Q)]^2 = 1$

$\therefore \boxed{\det(Q) = \pm 1}$

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$\because$ det(Q) = +1 or −1, the box formed by the rows or coloumns of the Q as vector will have volume = one.

Volume of box = 1

4.2 (17)

17. Find the determinants of

① $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$, ② $A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$, ③ $A - \lambda I = \begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}$.

For which values of λ is $A - \lambda I$ a singular matrix?

① $\det(A) = \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = (4\times 3) - (2\times 1) = 12 - 2 = \underline{10}$

② $\det(A^{-1}) = \left(\frac{1}{10}\right)^2 \begin{vmatrix} 3 & -2 \\ -1 & 4 \end{vmatrix} = \frac{1}{100}(12-2) = \frac{1}{100}(10) = \underline{\frac{1}{10}}$

③ $\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - (2)(1)$

$= (12 - 4\lambda - 3\lambda + \lambda^2) - 2$

$= \lambda^2 - 7\lambda + 10$

$= \lambda^2 - 5\lambda - 2\lambda + 10$

$= \lambda(\lambda - 5) - 2(\lambda - 5)$

$= (\lambda - 5)(\lambda - 2)$

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$\therefore$ for $\underline{\lambda = 5 \text{ or } \lambda = 2}$,
matrix $A - \lambda I$ is singular.

$\therefore$

- $\det(A) = 10$
- $\det(A^{-1}) = 1/10$
- $\det(A - \lambda I) = (\lambda - 5)(\lambda - 2)$
- $A - \lambda I$ is singular for $\lambda = 5$ or $\lambda = 2$

$\boxed{4.2(20)}$

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20. Do these matrices have determinant 0, 1, 2, or 3?

① $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ② $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ ③ $C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

① $\det(A) = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

$= 0 - 0 + 1\left[(1)(1) - (0)(0)\right]$

$= +1$

② $\det(B) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$

$= 0 - 1\left[(1)(0) - (1)(1)\right] + 1\left[(1)(1) - (1)(0)\right]$

$= -1[-1] + 1[1]$

$= +1 + 1$

$= 2$

③ $\det(C) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\cancel{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}} + 1\cancel{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}$

$= 1\left[(1)(1) - (1)(1)\right]$

$= 1[0]$

$= 0$

$\therefore \boxed{\begin{aligned} \det(A) &= 1 \\ \det(B) &= 2 \\ \det(C) &= 0 \end{aligned}}$

4.2(25)

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25. Elimination reduces A to U. Then $A = LU$:

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = LU.$$

Find the determinants of L, U, A, $U^{-1}L^{-1}$, and $U^{-1}L^{-1}A$.

① L is triangular matrix. $\det(L) = (1)(1)(1) = 1$

② U is triangular matrix. $\det(U) = (3)(2)(-1) = -6$

③ $\det(A) = \det(L) \cdot \det(U) = 1 \cdot (-6) = -6$

③ $\det(U^{-1}L^{-1}) = \det(U^{-1}) \cdot \det(L^{-1}) = \left(\frac{1}{\det(U)}\right) \cdot \left(\frac{1}{\det(L)}\right) = \frac{1}{-6} \cdot \frac{1}{1} = \frac{1}{-6}.$

④ $\det(U^{-1}L^{-1}A) = \det(U^{-1}L^{-1}) \cdot \det(A) = \frac{1}{-6} \cdot (-6) = 1$

$\therefore$ $\boxed{\begin{aligned} \det(L) &= 1 \\ \det(U) &= -6 \\ \det(A) &= -6 \\ \det(U^{-1}L^{-1}) &= -1/6 \\ \det(U^{-1}L^{-1}A) &= 1 \end{aligned}}$

$\boxed{4.3(8)}$

8. Compute the determinants of A_2, A_3, A_4. Can you predict A_n?

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Use row operations to produce zeros, or use cofactors of row 1.

- $\det(A_2) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = \underline{\underline{-1}}$

- $\det(A_3) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$

$= -1[0-1] + 1[1-0]$

$= 1 + 1$

$= \underline{\underline{2}}$

- $\det(A_4) = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$

1 row exchange

2 row exchange

$\underbrace{= 0 - (1)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} + 1(-1)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} - 1(-1)^2\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}}_{\text{using row change \& multiply by } (-1).}$

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$

$= 1(-1) - 1(-1) + 1(1) = 1$

substitute back

$\det(A_4) = 0 - 1(1) + 1(-1)(1) - 1(1)(1)$

$= \underline{\underline{-3}}$

$\therefore$ we can see, in general: $\boxed{\det(A_n) = (-1)^{n-1}(n-1)}$

4.3 (34)

34. With 2 by 2 blocks, you cannot always use block determinants!

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D| \quad \text{but} \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|.$$

(a) Why is the first statement true? Somehow B doesn't enter.

(b) Show by example that equality fails (as shown) when C enters.

(c) Show by example that the answer $\det(AD - CB)$ is also wrong.

Given matrix : $\begin{bmatrix} A_{2\times2} & B_{2\times2} \\ O_{2\times2} & D_{2\times2} \end{bmatrix}_{4\times4} = \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ 0 & 0 & d_{11} & d_{12} \\ 0 & 0 & d_{21} & d_{22} \end{bmatrix}$

cofactring along coloumn1

$$\therefore \begin{vmatrix} A & B \\ O & D \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & b_{21} & b_{22} \\ 0 & d_{11} & d_{12} \\ 0 & d_{21} & d_{22} \end{vmatrix} - a_{22} \begin{vmatrix} a_{12} & b_{11} & b_{12} \\ 0 & d_{11} & d_{12} \\ 0 & d_{21} & d_{22} \end{vmatrix}$$

$$= a_{11} \cdot a_{22} \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} - a_{22} \cdot a_{12} \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}$$

$$= (a_{11} \cdot a_{22} - a_{22} \cdot a_{12}) \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}$$

$$= |A||D| \longrightarrow$$

(a)

- Hence, first statement has been proved as True.
- B doesnt enter.

- Now, let

$A = O \longrightarrow |A| = 0$

$B = I \longrightarrow |B| = 1$

$C = I \longrightarrow |C| = 1$

$D = O \longrightarrow |D| = 1$

$|A||D| - |B||C|$

$= 0 - 1$

$= -1$ ——— eq (1)

$$\therefore \begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = 1 \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 1 \cdot 1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= (1)(1)(1-0)$$

$$= 1 \quad \text{—— eq } (2)$$

$\therefore$ from eq (i) & eq (2),

we can see,

(b)

$$\boxed{\begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |B||C|}$$

Hence proved, with example.

Now, Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}$

$$AD - BC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \det(AD - BC) = \begin{vmatrix} 4 & 0 \\ 0 & 0 \end{vmatrix} = 4 \quad \text{—— eq } (3)$$

Now,

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{vmatrix}$$

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$$= 0 \quad \text{———} \quad \text{eq } (4)$$

$\therefore$ from eq (3) & (4), we have proved,

(c)

$$\boxed{\begin{array}{l} \det(AD - BC) \text{ is wrong answer} \\ \text{for } \begin{vmatrix} \underset{2\times2}{A} & \underset{2\times2}{B} \\ \underset{2\times2}{C} & \underset{2\times2}{D} \end{vmatrix}_{4\times4} \end{array}}$$

4.4 (28)

28. A box has edges from $(0,0,0)$ to $(3,1,1)$, $(1,3,1)$, and $(1,1,3)$. Find its volume and also find the area of each parallelogram face.

consider 3 vectors: $a = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

$\therefore$ consider matrix $V = [\, a \,\vdots\, b \,\vdots\, c \,]$

$$= \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

$\therefore$ Volume $= \det(V) = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix}$

$$= 3 \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}$$

$$= 3(9-1) - 1(3-1) + 1(1-3)$$

$$= 3(8) - (2) + 1(-2)$$

$$= 24 - 2 - 2$$

$$= \underline{\underline{20}}$$

- from vectors, we can clearly see, a, b, c have same length, i.e. :

$$\text{length of one side} = \sqrt{1^2 + 1^2 + 3^2}$$
$$= \sqrt{1 + 1 + 9}$$
$$= \sqrt{11}$$

$$\therefore \text{Area of parallelogram} = \begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix}$$

$$= i \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - j \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + k \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$
$$= i(1-3) - j(3-1) + k(9-1)$$
$$= -2i - 2j + 8k$$
$$= \sqrt{(2^2 + 2^2 + 8^2)} = \sqrt{4 + 4 + 64}$$
$$= \sqrt{72}$$
$$= \sqrt{2 \times 36}$$
$$= \underline{\underline{6\sqrt{2}}}$$

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$$\therefore \boxed{\begin{array}{l} \text{Volume of box} = 20 \text{ unit}^3 \\ \text{Area of each face} = 6\sqrt{2} \text{ unit}^2 \end{array}}$$