EE 810: Homework # 7

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5.6(1)

1. If B is similar to A and C is similar to B, show that C is similar to A. (Let $B = M^{-1}AM$ and $C = N^{-1}BN$.) Which matrices are similar to I?

$\because$ B is similar to A : $B = M^{-1}AM$
$\because$ C is similar to B : $C = N^{-1}BN$

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$$\therefore C = N^{-1}(M^{-1}AM)N$$
$$= N^{-1}M^{-1}A\,MN$$
$$= (NM)^{-1}A\,MN$$

say $X = MN$

$$\therefore \underline{C = X^{-1}AX}$$

$\therefore$ $\boxed{\text{C is similar to A.}}$

Let Y be any non singular matrix.

$$\underline{I = Y^{-1}IY}$$

$\therefore$ $\boxed{\text{I is only similar to I.}}$

5.6(3)

3. Explain why A is never similar to $A+I$.

Consider B is similar to A.

$\therefore$ $B = M^{-1}AM$ such that A & B have same λs.

$\therefore$ λs of A will be given by $|A - \lambda I| = 0$

$\underline{\lambda = \lambda_1, \lambda_2 \dots \lambda_n.}$ —— ①

Now consider $A+I$.

$\therefore$ $|(A+I) - \lambda I| = 0$

$\therefore$ $|A - (\lambda - 1) I| = 0.$ $\therefore$ It's roots will be $\underline{\lambda_1 - 1, \lambda_2 - 1, \dots \lambda_3 - 1}$ —— ②

From ① & ②, we see that eigenvalues of A & $A+I$ can never be the same.

$\therefore$ $\boxed{A \text{ \& } A+I \text{ are never similar.}}$

5.6(8)

8. What matrix M changes the basis $V_1 = (1,1)$, $V_2 = (1,4)$ to the basis $v_1 = (2,5)$, $v_2 = (1,4)$? The columns of M come from expressing V_1 and V_2 as combinations $\sum m_{ij} v_i$ of the v's.

$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad V_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $\quad v_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\because$ here, $V_2 = v_2$

& $v_1 - v_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = V_1$

$\therefore$ $V_1 = (1)\, v_1 + (-1)\, v_2$

$V_2 = (0)\, v_1 + (1)\, v_2$

$\therefore$ required matrix for basis change : $M = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

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$\boxed{5.6\,(13)}$

13. The derivative of $a + bx + cx^2$ is $b + 2cx + 0x^2$.

(a) Write the 3 by 3 matrix D such that

$$D \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix}.$$

(b) Compute D^3 and interpret the results in terms of derivatives.

(c) What are the eigenvalues and eigenvectors of D?

writing $b, 2c$ & 0 as linear combinations of a, b, c will give us matrix D.

$\therefore\ b = 0a + 1b + 0c$

$\therefore\ 2c = 0a + 0b + 2c$

$\therefore\ 0 = 0a + 0b + 0c$

$$\therefore\ \boxed{D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$$

Now, $D^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$$\boxed{D^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

- D is a differentiation matrix.
- Multiplying D^3 by a 3×1 vector will give zero vector since it is equivalent to differentiating a polynomial of degree 2 three times.

Now,

$$|D - I\lambda| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix}$$

$$= (-\lambda)^3$$

equating this with 0 gives $\boxed{\begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{matrix}}$

$\therefore$ for $\lambda = 0$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\therefore \begin{bmatrix} x_2 \\ 2x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $x_2 = 0$

$2x_3 = 0$

$0 = 0$

$\therefore$ $\boxed{\text{eigen vector of } D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$

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$\boxed{5\text{-}6(15)}$

15. On the space of 2 by 2 matrices, let T be the transformation that *transposes every matrix*. Find the eigenvalues and "eigenmatrices" for $A^T = \lambda A$.

$\because$ T is the transformation matrix.

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$$TA = A^T = \lambda A.$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$$

$\therefore$ $a = \lambda a$ $\quad b = \lambda c$

$d = \lambda d$ $\quad c = \lambda b$

$\therefore$ $\lambda a - a = 0$ $\quad b = \lambda(\lambda b)$ $\longrightarrow$ $a(\lambda - 1) = 0$

$\therefore$ $\lambda d - d = 0$ $\quad b = \lambda^2 b$ $\quad d(\lambda - 1) = 0$

$\lambda^2 b - b = 0$ $\quad b(\lambda^2 - 1) = 0$

$c = \lambda b$

$\therefore$ $a = 0 \rightarrow \lambda = 1$

$b = 0 \longrightarrow \lambda = +1, -1$

$d = 0 \rightarrow \lambda = 1$

$c = \lambda b$

$\therefore$ eigenvalues : $\lambda_1 = 1$

$\lambda_2 = -1$

& $\therefore$ eigen matrices :

for $\lambda = 1$

$$\text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

For $\lambda = -1$

$$\text{span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

5.6 (23)

23. If A has eigenvalues 0, 1, 2, what are the eigenvalues of $A(A-I)(A-2I)$?

Characteristic polynomial for $A(A-I)(A-2I)$:

$$p(\lambda) = \lambda(\lambda-1)(\lambda-2)$$

However,

using cayley - hamilton theorm,

$$p(A) = A(A-I)(A-2I)$$

$$= O \longrightarrow O : \text{zero matrix.}$$

$\therefore$ $\boxed{\text{eigenvalues of } A(A-I)(A-2I) \text{ are } 0, 0, 0}$

5.6 (29)

29. Compute A^{10} and e^A if $A = MJM^{-1}$:

$$A = \begin{bmatrix} 14 & 9 \\ -16 & -10 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}.$$

Using the method of finding power for jordan matrix,

$$A^{10} = (MJM^{-1})^{10}$$

$$= M\, J^{10}\, M^{-1}$$

$$= \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \cdot 2^{10} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$$

$$= 2^{10} \begin{bmatrix} 3 & 13 \\ -4 & -17 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$$

$$= 2^{10} \begin{bmatrix} 61 & 45 \\ -80 & -59 \end{bmatrix}$$

$$J^{10} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}^{10}$$

$$= \begin{bmatrix} 2^{10} & 10 \cdot 2^{10-1} \\ 0 & 2^{10} \end{bmatrix}$$

$$= \begin{bmatrix} 2^{10} & 2 \cdot 5 \cdot 2^{9} \\ 0 & 2^{10} \end{bmatrix}$$

$$= 2^{10} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ Now, $e^A = M e^J M^{-1} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} e^2 & 1 \cdot e^2 \\ 0 & e^2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$

$$= e^2 \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$$

$$= e^2 \begin{bmatrix} 13 & 9 \\ -16 & -11 \end{bmatrix}$$

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$\therefore$ $\boxed{\underline{A^{10}} = 2^{10} \begin{bmatrix} 61 & 45 \\ -80 & -59 \end{bmatrix} \ \& \ \underline{e^A} = e^2 \begin{bmatrix} 13 & 9 \\ -16 & -11 \end{bmatrix}}$

5.6 (38)

38. These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors (find them). But the block sizes don't match and J *is not similar to* K:

$$J = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{and} \quad K = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

For any matrix M, compare JM with MK. If they are equal, show that M is not invertible. Then $M^{-1}JM = K$ is impossible.

consider $M = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$

Now, to compare JM & MK,

$$JM = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now,

$$MK = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a_{12} & a_{13} & 0 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ 0 & a_{42} & a_{43} & 0 \end{bmatrix}$$

Now, assume $JM = MK$. $\therefore$

$a_{21} = a_{24}$

$a_{41} = a_{44}$

$a_{21} = a_{22}$

$a_{41} = a_{42}$

$a_{11} = a_{22}$

$a_{23} = a_{12}$

$a_{31} = a_{42}$

$a_{43} = a_{32}$

$$\therefore M = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & 0 \end{bmatrix}$$

$\because$ 1^{st} column = zero,

$|M| = 0$, matrix is singular.

$\therefore$ $\boxed{M \text{ is not invertible.}}$

$\therefore$ J & K are not similar.

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$\because$ M is not invertible,

$\boxed{MJM^{-1} = K \text{ is not possible.}}$

- <u>Eigenvectors of J are</u> : $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

- <u>Eigenvectors of K are</u> : $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

5.6 (42)

42. Prove that *AB has the same eigenvalues as BA*.

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Let $AB = X$

& $BA = Y$

we can write $X = IX$

$= B^{-1}BX$

$= B^{-1}BAB$

$= B^{-1}(BA)B$

$\underline{X = B^{-1}(Y)B}$ $\longrightarrow$ ∴ X & Y are similar

∴ AB & BA are similar.

$\boxed{\text{∴ } AB \text{ \& } BA \text{ have same eigenvalues.}}$

Review 5.13

5.13 (a) Show that the matrix differential equation $dX/dt = AX + XB$ has the solution $X(t) = e^{At}X(0)e^{Bt}$.

(b) Prove that the solutions of $dX/dt = AX - XA$ keep the same eigenvalues for all time.

(a) To prove : $\dfrac{dX}{dt} = AX + XB$

$\because \quad X(t) = e^{At} X(0) e^{Bt}$

$\therefore \quad \dfrac{dX}{dt} = A \underbrace{e^{At} X(0) e^{Bt}}_{X} + \underbrace{e^{At} X(0) e^{Bt}}_{X} \cdot B$

$\therefore \quad \boxed{\dfrac{dX}{dt} = AX + XB}$

(b) from the above result, we can conclude,

if $\dfrac{dX}{dt} = AX - XA$,

$X(t) = e^{At} X(0) e^{-At}$

$\therefore \quad X(t) = e^{At} X(0) \left(e^{At}\right)^{-1}$ — (in the form $A = BCB^{-1}$)

$\therefore$ $X(t)$ & $X(0)$ are similar matrix.

$\therefore$ $\boxed{\text{X have same eigen values for all values of t.}}$

Review 5.15

5.15 Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{bmatrix}.$$

What property do you expect for the eigenvectors, and is it true?

Now, $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} -\lambda & -i & 0 \\ i & 1-\lambda & i \\ 0 & -i & -\lambda \end{vmatrix} = 0$$

$$\therefore (-\lambda)\left(-(1-\lambda)\lambda + i^2\right) + i(-i\lambda) = 0$$

$$-\lambda^3 + \lambda^2 + 2\lambda = 0 \quad \Rightarrow \quad \lambda_1 = 0$$

$$\lambda_2 = -1$$

$$\lambda_3 = 2$$

for $\lambda = 0$,

$$\begin{bmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

solving this gives:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \text{say } x_1$$

for $\lambda = -1$:

$$\begin{bmatrix} 1 & -i & 0 \\ i & 2 & i \\ 0 & -i & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this gives :

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix} \longrightarrow \text{say } x_2$$

for $\lambda = 2$: $\begin{bmatrix} -2 & -i & 0 \\ i & -1 & i \\ 0 & -i & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solving this gives : $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2i \\ 1 \end{bmatrix} \longrightarrow$ say x_3

$\because$ $A^{+} = \begin{bmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{bmatrix} = A$, we expect : eigenvectors to be orthogonal to each other.

$\therefore$ $x_1 \cdot x_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix} = (-1)(1) + 0(i) + (1)(1)$

$= 0. \longrightarrow \underline{x_1 \perp x_2}$

$\therefore$ $x_2 \cdot x_3 = \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2i \\ 1 \end{bmatrix} = (1)(1) + (-i)(2i) + (1)(1)$

$= 1 - 2 + 1$

$= 0 \longrightarrow \underline{x_2 \perp x_3}$

$\therefore$ $x_1 \cdot x_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2i \\ 1 \end{bmatrix} = (-1)(1) + (0)(2i) + (1)(1)$

$= 0 \longrightarrow \underline{x_1 \perp x_3}$

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$\therefore$ $\boxed{\text{A has Eigenvalues : } 0, -1, 2}$

$\boxed{\text{Eigenvectors corrosponding to } 0, -1, 2 \text{ are respectively : } \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2i \\ 1 \end{bmatrix}}$

we expect eigenvectors of hermitian matrix to be orthogonal & Yes, as shown above, it is <u>True</u> for the given matrix A.

$x_1 \perp x_2 \perp x_3$

Review 5.30

5.30 What is the limit as $k \to \infty$ (the Markov steady state) of $\begin{bmatrix} .4 & .3 \\ .6 & .7 \end{bmatrix}^k \begin{bmatrix} a \\ b \end{bmatrix}$?

$A = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix} \longrightarrow$

- $|A| = 0.1 \neq 0$
- sum of coloumns = 1

$\therefore$ A is a regular stochastic Mx.

$\therefore$ steady state vector x is unique.

Let x be eigenvector corrosponding to eigenvalue 1.

$$(A - I)x = 0$$

$$\begin{bmatrix} -0.6 & 0.3 \\ 0.6 & -0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x_1 + x_2 = 0$

$x_2 = 2x_1 \quad \Rightarrow \quad x = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

$\therefore$ considering $\begin{bmatrix} a \\ b \end{bmatrix}$ as initial state,

$$\boxed{\lim_{k \to \infty} \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix}^k \begin{bmatrix} a \\ b \end{bmatrix} = (a+b) \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}}$$

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