EE 810: Homework #11

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16. Use 8H to show that the following equation has no solution, because the alternative holds:

$$\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

8H $Ax = b$ has a solution ***or*** there is a y such that $yA = 0$ and $yb \neq 0$.

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$A = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

→ we have to show $Ax = b$ have no solution,
we can show this using (8H).

→ ley $y = [y_1 \ \ y_2]$

Solving for $[y_1 \ \ y_2] \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} = 0$,

$\therefore [2y_1 + 4y_2 \quad 2y_1 + 4y_2] = 0$

$\therefore y_1 + 2y_2 = 0$.

Solving for $yb \neq 0$, $[y_1 \ \ y_2] \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq 0$

$\therefore y_1 + y_2 \neq 0$

$\because yb \neq 0$, y has many solutions.

$\therefore$ $\boxed{\text{Since there are many } y \text{ such that } yA = 0 \ \& \ yb \neq 0, \text{ it means that } \underline{Ax = b \text{ does not have a solution.}}}$

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17. Use 8I to show that there is no solution $x \geq 0$ (the alternative holds):

$$\begin{bmatrix} 1 & 3 & -5 \\ 1 & -4 & -7 \end{bmatrix} x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

8I $Ax = b$ has a **nonnegative** solution *or* there is a y with $yA \geq 0$ and $yb < 0$.

$Ax = b$, $A = \begin{bmatrix} 1 & 3 & -5 \\ 1 & -4 & -7 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\therefore$ Let $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}$

$\therefore$ solving $yA \geq 0$,

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -5 \\ 1 & -4 & -7 \end{bmatrix} \geq 0$$

$$\therefore \left.\begin{aligned} y_1 + y_2 &\geq 0 \\ 3y_1 - 4y_2 &\geq 0 \\ -5y_1 - 7y_2 &\geq 0 \end{aligned}\right\} \longrightarrow \text{Consider } y = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

this satisfies all 4 equations,

i.e. $yA \geq 0$

& $yb < 0$

Now, solving for $yb < 0$,

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} < 0$$

$$\therefore 2y_1 + 3y_2 < 0$$

$\boxed{\text{Hence, using 8I, we showed that } Ax = b \text{ has no solution.}}$

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18. Show that the alternatives in 8J ($Ax \geq b$, $x \geq 0$, $yA \geq 0$, $yb < 0$, $y \leq 0$) cannot both hold. *Hint*: yAx.

8J $Ax \geq b$ has a solution $x \geq 0$ ***or*** there is a $y \leq 0$ with $yA \geq 0$ and $yb < 0$.

we have given : $Ax \geqslant b$.

$\therefore \quad yAx \leq yb$ ———— [$\because y \leq 0$ — given]

$\because \quad yb < 0$, we can say, $yAx < 0$ ———— (1)

Now, we have also given $yA \geqslant 0$ & $x \geqslant 0$.

$\therefore \quad yAx \geqslant 0$ ———— (2)

$\therefore$ as we can see, eq (1) & eq (2) contradict each other.

$\therefore$ $\boxed{\text{alternatives} \left(\begin{array}{l} Ax \geqslant b, \\ x \geqslant 0, \\ yA \geqslant 0, \\ yb < 0 \\ y \leq 0 \end{array} \right) \text{ both cannot hold true.}}$

Problem 1

Using the Fourier-Motzkin method, check the feasibility of the following system of inequalities:

$x-5y+2z \geq 7$

$x-2y-6z \geq -12$

$-x+6y-4z \geq -10$

$-x+8y-3z \geq -9$

$-10y+z \geq -15$

Now first, lets try to eliminate x.

∴ write all inequalities in terms of x. $\longrightarrow$

$x \geqslant 7 + 5y - 2z$ —— ①

$x \geqslant -12 + 2y + 6z$ —— ②

$x \leqslant 10 + 6y - 4z$ —— ③

$x \leqslant 9 + 8y - 3z$ —— ④

$-10y + z \geqslant -15.$ —— ⑤

∴ from eq ①, ②, ③, ④, we can say:

$7 + 5y - 2z \leqslant 10 + 6y - 4z$

$7 + 5y - 2z \leqslant 9 + 8y - 3z$

$-12 + 2y + 6z \leqslant 10 + 6y - 4z$

$-12 + 2y + 6z \leqslant 9 + 8y - 3z$

$-10y + z \geqslant -15$

$0 \leqslant y - 2z + 3$ —— ⑥

$0 \leqslant 3y - z + 2$ —— ⑦

$0 \leqslant 4y - 10z + 22$ —— ⑧

$0 \leqslant 6y - 9z + 21$ —— ⑨

$0 \leqslant -10y + z + 15$ —— ⑩

writing eq^n (8) to (10) in the form of z gives:

$$z \leq \frac{y+3}{2} \quad \text{———} \quad (11)$$

$$z \leq 3y + 2 \quad \text{———} \quad (12)$$

$$z \leq \frac{4y+22}{10} \quad \text{———} \quad (13)$$

$$z \leq \frac{6y+21}{9} \quad \text{———} \quad (14)$$

$$z \geq -10y - 15 \quad \text{———} \quad (15)$$

from eq^n (11), (12), (13), (14), (15), we can conclude:

$$10y - 15 \leq \frac{y+3}{2} \Rightarrow 20y - 30 \leq y + 3$$

$$10y - 15 \leq 3y + 2 \Rightarrow 7y \leq 17$$

$$10y - 15 \leq \frac{4y+22}{10} \Rightarrow 100y - 150 \leq 4y + 22$$

$$10y - 15 \leq \frac{6y+21}{9} \Rightarrow 90y - 135 \leq 6y + 21$$

$$19y \leq 33$$

$$7y \leq 17$$

$$96y \leq 172$$

$$84y \leq 156$$

$\therefore \quad y \leq \frac{33}{19}$ 1.736

$y \leq \frac{17}{7}$ 2.428

$y \leq \frac{172}{96}$ 1.791

$y \leq \frac{156}{84}$ 1.857

minimum

$$y \leq \frac{33}{19}$$

– Hence, we get a solution for y.

$\boxed{\therefore \text{ Given system of inequalities has a feasible solution.}}$

Ex. say $y = 1$ $\quad \therefore$ eqn 11 – 15 become –

$z \leq 2$

$z \leq 5$

$z \leq 26/10$

$z \leq 3$

$z \geq -5$

$\therefore$ say $z = 0$ $\quad \therefore$ eqn 1 – 4 becomes –

$x \geq 12$

$x \geq -10$

$x \leq 16$

$x \leq 17$

$\therefore$ say $x = 14$ ———— $\therefore$ $(14, 1, 0)$ is a solution for the given system of inequalities.

PS B : 1

15/15

1. Find the Jordan forms (in three steps!) of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Finding eigenvalues of A, $|A - \lambda I| = 0$

$\therefore \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$

$\therefore (1-\lambda)^2 - 1 = 0$

$\therefore \lambda^2 - 2\lambda = 0.$

$\therefore \lambda(\lambda - 2) = 0$

$\therefore \lambda = 0, \quad \lambda = 2$ —— $\lambda_1 = 2$, $\lambda_2 = 0$

$\therefore$ Jordon form of A is $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cong \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$\therefore$ finding eigenvalues of X, $|X - I\lambda| = 0.$

$\therefore \begin{vmatrix} 1-\lambda & 2 \\ 0 & 0 \end{vmatrix} = 0.$

$\therefore (1-\lambda)0 + 2(0) = 0.$

$= 0.$ — $0\lambda_1 = 2\lambda_2$

$\therefore$ Jordon form of B is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

PS B : 3

3. For the matrix B in Problem 1, use $Me^{Jt}M^{-1}$ to compute the exponential e^{Bt}, and compare it with the power series $I + Bt + (Bt)^2/2! + \cdots$.

$$B = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The exponential e^{Bt} can be obtained by using :

$$e^{Bt} = M e^{Jt} M^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{0t} & te^{0t} & 0 \\ 0 & e^{0t} & 0 \\ 0 & 0 & e^{0t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{e^{Bt} = \begin{bmatrix} 1 & t & 2t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

We see that $B^2 = 0$

Now, to compare this w/ exponential series,

$\therefore$ all subsequent terms will be 0.

$$\therefore e^{Bt} = I + Bt$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 2t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t & 2t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

same as using $Me^{Jt}M^{-1}$.

PS B : 5

5. Find "by inspection" the Jordan forms of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

by inspection, we can see, A has : $\lambda_1 = 1$

$\lambda_2 = 4$

$\lambda_3 = 6$

$\therefore$ Jordon form of A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

by inspecting, we can see, matrix has rank 1. (row1 = - row2)

$\therefore \lambda_1 = 0$

$\lambda_2 = 0.$

$\therefore$ Jordon form of B is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

PS 8 : 5

6. Find the Jordan form J and the matrix M for A and B (B has eigenvalues $1, 1, 1, -1$). What is the solution to $du/dt = Au$, and what is e^{At}?

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 2 & 0 & 1 \\ -2 & 1 & -1 & 1 \\ 2 & -1 & 2 & 0 \end{bmatrix}.$$

Finding eigenvectors for A,

$Ae_1 = 0e_1$, $Ae_3 = 0e_3 + e_1$, $Ae_3 = 0e_3 + e_3$

$Ae_2 = 0e_2$, $Ae_4 = 0e_4 + e_2$

$\therefore$ row $m = \begin{bmatrix} e_1 & e_3 & e_5 & e_2 & e_4 \end{bmatrix}$

$\therefore$ $J = M^{-1}AM$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\because$ $\dfrac{dy}{dt} = Ay$

$m\dfrac{dy}{dt} = Amy = mJu$

$$\therefore \quad e^{At} = \begin{bmatrix} 1 & t & \frac{t^2}{2} & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad u = M e^{At} M^{-1} (u_0)$$

Now, for B, $\because$ λs are $1, 1, 1, -1$, $J = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

$$\therefore B(x_1, x_2, x_3, x_4) = [x_1 \; x_2 \; x_3 \; x_4] \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

This leads to
$$Bx_1 = x_1$$
$$Bx_2 = x_2 + x_1$$
$$(B-I)\, x_2 = x_1$$
$$Bx_3 = x_3$$
$$Bx_4 = x_4$$

which means that x_1 is eigen vector of B with $\lambda = 1$.

$\therefore$ $x_1 = (1, -1, -1, 1)$

$x_2 = (0, -1/2, 0, -1/2)$

$x_3 = (0, 1, 0, -1)$

$x_4 = (1, 4, 3, -3)$

$$\therefore M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & -\frac{1}{2} & 1 & 1 \\ -1 & 0 & 0 & 3 \\ 1 & -\frac{1}{2} & -1 & 3 \end{bmatrix}$$