EE 810 : Homework #B

Name : Onkar Vivek Apte

UBO ID : [redacted]

2.1 (8)

8. Which of the following descriptions are correct? The solutions x of

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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form

(a) a plane.
(b) a line.
(c) a point.
(d) a subspace.
(e) the nullspace of A.
(f) the column space of A.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x_1 + x_2 + x_3 = 0 \quad$ —— ①

$\therefore \quad x_1 + 2x_3 = 0 \quad$ —— ②

$\therefore \quad x_1 = -2x_3$

Substituting this in eq ①

$-2x_3 + x_2 + x_3 = 0.$

$x_2 - x_3 = 0 \longrightarrow$ This is a equa^n of a line in R^3

Also, by definition,

$L \{ x \mid Ax = 0 \}$ is a nullspace, which is also a spacespace in R^3.

$\therefore$

(A) Plane ✗
(B) Line ✓
(C) Point ✗
(d) Subspace ✓
(e) Nullspace ✓
(f) coloumnspace ✗

2.1 (15)

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15. Let **P** be the plane in $\mathbf{R}^3$ with equation $x+y-2z=4$. The origin $(0,0,0)$ is not in **P**! Find two vectors in **P** and check that their sum is not in **P**.

cosider vector $A = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \longrightarrow$

$x + y - 2z = 4$

$\therefore\ 2 + 2 - 2(0) = 4$

$\therefore\ 4 = 4 \rightarrow$ True

$\therefore\ A \in P.$

cosider vector $B = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} \longrightarrow$

$x + y - 2z = 4$

$\therefore\ 0 + 4 - 2(0) = 4$

$4 = 4 \longrightarrow$ True.

$\therefore\ B \in P.$

$\therefore$ Now consider vector $(A+B) = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} \rightarrow$

$x + y - 2z = 4$

$\therefore\ 2 + 6 - 2(0) = 4$

$\therefore\ 8 = 4 \rightarrow$ false

$\therefore\ (A+B) \notin P.$

$\therefore$ vector $A = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ & vector $B = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$ are two vectors, which lie in the given plane P & their addition, vector $A+B = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}$ does not lie in the plane P.

2.1 (22)

22. For which right-hand sides (find a condition on b_1, b_2, b_3) are these systems solvable?

(a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

$A \quad x = B \qquad\qquad A \quad x = B$

(a) consider: $[A \mid B]$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 0 & 0 & b_3+b_1 \end{array}\right]$$

$\therefore$ This system is solvable only if $b_2 - 2b_1 = 0$ & $b_3 + b_1 = 0$

(b) consider: $[A \mid B]$

$$\left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{array}\right]$$

$\therefore$ $R_2 \to R_2 - 2R_1$

& $R_3 \to R_3 + R_1$

$$\left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & b_2-2b_1 \\ 0 & 0 & b_3+b_1 \end{array}\right]$$

$\therefore$ This system is solvable, only if $b_3 + b_1 = 0$

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2. 1(27)

27. If A is any 8 by 8 invertible matrix, then its column space is ____. Why?

If 8×8 matrix is invertible, means it's determenant is zero.

& ∵ the determinant is zero, there are NO linearly dependant vectors.

∴ $\boxed{\text{Coloumn space will be } R^8}$

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$\boxed{2.2(10)}$

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10. Find a 2 by 3 system $Ax = b$ whose complete solution is

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

Find a 3 by 3 system with these solutions exactly when $b_1 + b_2 = b_3$.

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longrightarrow \therefore Ax = b$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1+w \\ 2+3w \\ w \end{bmatrix} \longrightarrow \begin{array}{l} \therefore x_1 = 1+w \rightarrow x_1 - w = 1 \\ x_2 = 2+3w \rightarrow x_2 - 3w = 2 \end{array} \Bigg\} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$x_3 = w$: x_3 is a free variable.

$\therefore \quad A \quad x \quad = \quad b$

$$\boxed{\therefore \begin{array}{c} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \end{bmatrix} \\ \begin{array}{ccc} x_1 & x_2 & x_3/w \end{array} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}} \longrightarrow \text{This is the required } 2 \times 3 \text{ system.}$$

$\therefore$ 3x3 matrix would be,

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$\therefore$ now for case $b_1 + b_2 = b_3$,

Perform $R_3 \to R_3 + R_2 + R_1$

$$\therefore \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$b_1 + b_2 = b_3$
$1 + 2 = 3$

$\therefore$ This is the required 3×3 system when $b_1 + b_2 = b_3$

2.2(45)

45. Write all known relations between r and m and n if $Ax = b$ has

(a) no solution for some b.

(b) infinitely many solutions for every b.

(c) exactly one solution for some b, no solution for other b.

(d) exactly one solution for every b.

(a) $Ax = b$ has no solutions for some b.

· In case of no solutions for any b,
rank must be less than rows.

but no solution for some b means there is a $b \in R^m$
for which $x \notin R^n$.

$\boxed{r < m}$

(b) $Ax = b$ have infinetly many solutions.

$\therefore$ rank = rows, & rows < coloums.

$\therefore$ $m = r$

& $m < n$

(c) Exactly one solution for some b & No sol^n for others.

$\because$ No solution for b is when rank is greater than rows,

$\because$ for some b we get a valid unique solution,

rank must be equal to colomns

$\therefore$ $r = n$

& $m > n$

(d) Exactly one solution for every b

∴ rank is same as rows & Matrix must be square.

∴ rows & colouns are equal, which is also rank.

∴ $\boxed{r = m = n}$

2.3 (8)

8. If w_1, w_2, w_3 are independent vectors, show that the sums $v_1 = w_2 + w_3$, $v_2 = w_1 + w_3$, and $v_3 = w_1 + w_2$ are *independent*. (Write $c_1v_1 + c_2v_2 + c_3v_3 = 0$ in terms of the w's. Find and solve equations for the c's.)

$\because$ w_1, w_2, w_3 are linearly independant and,

$v_1 = w_2 + w_3$

$v_2 = w_1 + w_3$

$v_3 = w_1 + w_2$

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Let $c_1v_1 + c_2v_2 + c_3v_3 = 0.$

$\therefore c_1(w_2+w_3) + c_2(w_1+w_3) + c_3(w_1+w_2) = 0.$

$\therefore w_1(c_2+c_3) + w_2(c_1+c_3) + w_3(c_1+c_2) = 0$

$\because$ w_1, w_2 & w_3 are independant,

$c_2 + c_3 = 0$ — (1)

$c_1 + c_3 = 0$ — (2)

$c_1 + c_2 = 0$ — (3)

$\downarrow$

$c_1 = -2c_2.$ $\rightarrow$ substitute in (3)

$-2c_2 + c_3 = 0.$

$c_3 = 2c_2$ $\rightarrow$ substitute in (1).

$c_2 + 2c_2 = 0$

$\therefore 3c_2 = 0.$

$c_2 = 0$

$\therefore c_1 = 0$

$\therefore c_3 = 0.$

$\therefore$ since $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, v_1, v_2 & v_3 are also linearly independant

2.3 (19)

19. If $v_1, \ldots, v_n$ are linearly independent, the space they span has dimension ____. These vectors are a ____ for that space. If the vectors are the columns of an m by n matrix, then m is ____ than n.

$\because$ $V_1, V_2 \ldots\ldots V_n$ are linearly independant, they can be represented by a matrix with each of V_s as a coloum.

$\therefore$ This matrix will have n coloums

$\therefore$ V_s span the space R^n. $\rightarrow$ space w/ dim n.

$\because$ They are linearly independant, they can be represented as basis for R^n.

$\because$ They span R^n, the matrix formed by V_s cannot have coloums more than than rows.

$\therefore$ $m \geq n$.

$\therefore$ If $V_1, \ldots, V_n$ are linearly independant, the space they span has dimension $\boxed{n}$. These vectors are a $\boxed{\text{basis}}$ for that space. If the vectors are the columns of an m by n matrix, then m is $\boxed{\text{greater than or equal}}$ to n.

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2.3 (22)

22. Suppose $v_1, v_2, \ldots, v_6$ are six vectors in $\mathbf{R}^4$.

(a) Those vectors (do)(do not)(might not) span $\mathbf{R}^4$.
(b) Those vectors (are)(are not)(might be) linearly independent.
(c) Any four of those vectors (are)(are not)(might be) a basis for $\mathbf{R}^4$.
(d) If those vectors are the columns of A, then $Ax = b$ (has) (does not have) (might not have) a solution.

(a) It is a possibility that all Vs lie in a single line.

$\therefore$ Those vectors $\boxed{\text{might not}}$ span R^4

(b) $\because$ They are in R^4, there can only be possible combination of 4 linearly independent vector.

$\therefore$ best case scnorio, 4 of the given vectors can be linearly independant but the remaining 2 must be linearly dependant.

$\therefore$ Those vectors $\boxed{\text{are not}}$ linearly independant.

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(c) It is possible that any 4 out of 6 given vectors are linearly independant & hence can form basis in R^4.

$\therefore$ Any four of those vectors $\boxed{\text{might be}}$ a basis for R^4

(d) It is possible that in some case, $Ax = b$ might have a solution & in some cases it might not

$\therefore$ $Ax = b$ $\boxed{\text{might not have}}$ a solution.

2.4 (2)

2. Find the dimension and construct a basis for the four subspaces associated with each of the matrices

$$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Row space of A : basis = non zero rows of U

$$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix} \xrightarrow[U]{} \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore An = 0$ ↓ then $Un = 0$

Now,

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow x_2 + 4x_3 = 0$$

$\therefore x_2 = -4x_3$

$A \quad n \quad = \quad 0$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ -4x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore N(A) = N(U) = \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; \begin{matrix} x_1 \in R \\ x_3 \in R \\ x_4 \in R \end{matrix} \right\}$$

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basis : $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$; Dimension of N(A), N(U) = 3

$\because$ U is row echelon of A,

$$C(A) = \left\{ x \begin{bmatrix} 1 \\ 2 \end{bmatrix} \;\middle|\; x \in R \right\}$$

$$C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$ (pivot coloumn of A)

Dim = 1

Similarly of U, $U(A) = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ Dim = 1

Now, cosider $A^T = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 4 & 8 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore A^T x = 0.$

$$\therefore \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 + 2x_2 = 0.$

$\therefore x_1 = -2x_2$

$\therefore$ vector $x = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix}$ for $A^T x = 0.$

$$\therefore N(A^T) = \left\{ x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \;\middle|\; x_2 \in R \right\}$$ Dim = 1

& $C(A^T) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}$ coloumn of pivot Dim = 1 similarly, $C(U^T) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}$ dim = 1

Now, $U^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 4 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_3 - 4R_1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$U^T x = 0$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x_1 = 0.$

$x_2 \longrightarrow$ free variable.

$\therefore x = \begin{bmatrix} 0 \\ t \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore N(U^T) = \left\{ x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \;\middle|\; x_2 \in R \right\}$ dim = 1

2.4(4)

4. Describe the four subspaces in three-dimensional space associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$\because$ $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, The basis for $C(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\therefore$ The coloumn space of A is defined as : $C(A) = \left\{ x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$

Now, for Null space, $Ax = 0$

$$\therefore \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$\therefore$ $x_2 = 0$

$\therefore$ $x_3 = 0$

& x_1 is a free variable, say $x_1 = t$

$\therefore$ vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ basis of null space of A is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

& $N(A) = \left\{ x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \;\middle|\; x \in \mathbb{R} \right\}$

Now, $A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$\therefore$ basis for the coloumn space is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

and $\boxed{C(A^T) = \left\{ x \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}}$

$\therefore$ for null space, $A^T x = 0.$

$\therefore \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ $x_1 = 0$

$x_2 = 0$

& x_3 is a free varlable, say $x_3 = t$.

$\therefore$ Vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ basis for $N(A^T)$ are $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

& $\boxed{N(A^T) = \left\{ x \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; x \in \mathbb{R} \right\}}$

2.4 (14)

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14. Find a left-inverse and/or a right-inverse (when they exist) for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}.$$

① $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$\therefore$ Rank $= 2$

$\therefore$ Matrix A is full row rank

$\therefore$ A has right inverse C.

$$\therefore \quad C = A^T (A A^T)^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\boxed{C = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}}$$

② $m = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$

∴ Rank = 2

∴ Matrix m is full coloumn rank.

∴ Left inverse of m exists.

$$\therefore \quad B = (M^T M)^{-1} M$$

$$= \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\boxed{B = \frac{1}{3} \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix}}$$

(c) $T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ — if $a = 0$,

$\boxed{\text{T does not have inverse.}}$

— if $a \neq 0$,

rank = 2

$\therefore$ T is full row rank as well as full coloumn rank.

$$\therefore \boxed{\text{Left inverse} = \text{Right inverse} = \frac{1}{a^2}\begin{bmatrix} a & -b \\ 0 & a \end{bmatrix}}$$

2.6(4)

4. Every straight line remains straight after a linear transformation. If z is halfway between x and y, show that Az is halfway between Ax and Ay.

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$\because$ z is halfway between x & y,

$$z = \frac{x+y}{2} = \frac{1}{2}(x+y)$$

$\because$ Every line remains a line after linear transformation, apply A on both sides,

$$Az = A \cdot \left(\frac{1}{2}(x+y)\right)$$

$$Az = \frac{1}{2}(Ax + Ay)$$ —— (Possible since A is linear transforma^n)

$\therefore$ $\boxed{Az \text{ is also halfway between } Ax \text{ \& } Ay}$

2.6 (14)

14. Prove that T^2 is a linear transformation if T is linear (from $\mathbf{R}^3$ to $\mathbf{R}^3$).

assume a general vector in R_3,

$(a_1x_1 + a_2x_2 + a_3x_3)$, where, a_1, a_2, a_3 are scalars.
$a_1, a_2, a_3 \in R$
& x_1, x_2, x_3 are a set of basis, $x_1, x_2, x_3 \in R^3$

Applying Linear transformation T,

$$T(a_1x_1 + a_2x_2 + a_3x_3)$$
$$= T(a_1x_1) + T(a_2x_2) + T(a_3x_3)$$
$$= a_1T(x_1) + a_2T(x_2) + a_3T(x_3)$$

Now, Apply a transformation T^2,

$$T^2(a_1x_1 + a_2x_2 + a_3x_3)$$
$$= T(T(a_1x_1 + a_2x_2 + a_3x_3))$$
$$= T(a_1T(x_1) + a_2T(x_2) + a_3T(x_3))$$
$$= a_1T^2(x_1) + a_2T^2(x_2) + a_3T^2(x_3)$$

where, $a_1, a_2, a_3 \in R$ & $x_1, x_2, x_3 \in \mathbb{R}^3$

T^2 also preserves addition & scalar multiplication

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$\therefore$ $\boxed{T^2 \text{ also gives a linear transformation.}}$

$\boxed{2.6\ (22)}$

22. Which of these transformations is not linear? The input is $v = (v_1, v_2)$.

(a) $T(v) = (v_2, v_1)$. (b) $T(v) = (v_1, v_1)$.
(c) $T(v) = (0, v_1)$. (d) $T(v) = (0, 1)$.

consider

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}, \quad cx = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$$

(a) $$T(x+y) = \begin{bmatrix} x_2 + y_2 \\ x_1 + y_1 \end{bmatrix}$$

$$T(x) = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}, \quad T(y) = \begin{bmatrix} y_2 \\ y_1 \end{bmatrix}, \quad T(x) + T(y) = \begin{bmatrix} x_2 + y_2 \\ x_1 + y_1 \end{bmatrix}$$

$\because T(x+y) = T(x) + T(y)$, given transformation preserves vector addition.

$$T(cx) = \begin{bmatrix} cx_2 \\ cx_1 \end{bmatrix}$$

$$c \cdot T(x) = c \cdot \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} cx_2 \\ cx_1 \end{bmatrix}$$

$\because T(cx) = cT(x)$, given transformation preserves scalar multiplication.

$\therefore$ $\boxed{T(v) = (v_2, v_1) \text{ is a linear transformation}}$

(b) $$T(x) = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}, \quad T(y) = \begin{bmatrix} y_1 \\ y_1 \end{bmatrix}, \quad T(x) + T(y) = \begin{bmatrix} x_1 + y_1 \\ x_1 + y_1 \end{bmatrix}$$

$$T(x+y) = \begin{bmatrix} x_1 + y_1 \\ x_1 + y_1 \end{bmatrix}$$

Now, $$T(x) = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}, \quad cT(x) = c\begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_1 \end{bmatrix}$$

$$T(cx) = \begin{bmatrix} cx_1 \\ cx_1 \end{bmatrix}$$

$\therefore$ $\boxed{T(v) = (v_1, v_1) \text{ is a linear transforman}}$

(c) $T(x) = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$, $T(y) = \begin{bmatrix} 0 \\ y_1 \end{bmatrix}$, $T(x) + T(y) = \begin{bmatrix} 0 \\ x_1 + y_1 \end{bmatrix}$

$$T(x+y) = \begin{bmatrix} 0 \\ x_1 + y_1 \end{bmatrix}$$

$$c\,T(x) = c \cdot \begin{bmatrix} 0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ cx_1 \end{bmatrix}$$

$$T(cx) = \begin{bmatrix} 0 \\ cx_1 \end{bmatrix}$$

$\therefore$ $\boxed{T(v) = (0, v_1) \text{ is a linear transformation}}$

(d) $T(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $T(y) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $T(x) + T(y) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$T(x+y) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \therefore \underline{T(x) + T(y) \neq T(x+y)}$

$$c\,T(x) = c \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

$T(cx) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \therefore \underline{c \cdot T(x) \neq T(cx)}$

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$\therefore$ $\boxed{T(v) = (0, 1) \text{ is NOT a linear Transformation}}$

$\boxed{T(v) = (0, 1) \text{ is a Non linear transformation}}$

2.6 (28)

28. Find the *range* and *kernel* (those are new words for the column space and nullspace) of T.

(a) $T(v_1, v_2) = (v_2, v_1)$.
(b) $T(v_1, v_2, v_3) = (v_1, v_2)$.
(c) $T(v_1, v_2) = (0, 0)$.
(d) $T(v_1, v_2) = (v_1, v_1)$.

(a) $T(v_1, v_2) = (v_2, v_1)$

$\therefore \quad C(T) = \text{range}(T) = \{ T(v_1, v_2) \mid (v_1, v_2) \in R^2 \}$

$= \{ (v_2, v_1) \mid v_1, v_2 \in R \}$

$\boxed{\text{range}(T) = R^2}$

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$\therefore \quad N(T) = \text{kernal}(T) = \{ (v_1, v_2) \in R^2 \mid T(v_1, v_2) = 0 \}$

$= \{ (v_1, v_2) \in R^2 \mid (v_2, v_1) = 0 \}$

$= \{ (v_1, v_2) \in R^2 \mid v_1 = 0,\ v_2 = 0 \}$

$= \{ (v_1, v_2) \in R^2 \mid (v_1, v_2) = 0 \}$

$\boxed{\text{kernal}(T) = \{0, 0\}}$

(b) $T(v_1, v_2, v_3) = (v_1, v_2)$

$\therefore C(T) = \text{range}(T) = \{T(v_1, v_2, v_3) \mid (v_1, v_2, v_3) \in R^3\}$

$= \{T(v_1, v_2, v_3) \mid v_1, v_2 \in R\}$

$$\boxed{\text{range}(T) = R^2}$$

$\therefore N(T) = \text{kernal}(T) = \{(v_1, v_2, v_3) \in R^3 \mid T(v_1, v_2, v_3) = 0\}$

$= \{(v_1, v_2, v_3) \in R^3 \mid (v_1, v_2) = 0\}$

$= \{(v_1, v_2, v_3) \in R^3 \mid v_1 = 0, v_2 = 0\}$

$= \{(v_1, v_2, v_3) \in R^3 \mid (v_1, v_2, v_3) = (0, 0, v_3)\}$

$$\boxed{\text{kernal}(T) = \{(0, 0, v_3) \mid v_3 \in R\}}$$

(c) $T(v_1, v_2) = (0, 0)$

$\therefore\ C(T) = \text{range}(T) = \{T(v_1, v_2) \mid (v_1, v_2) \in R^2\}$

$= \{(0, 0) \mid v_1, v_2 \in R\}$

$$\boxed{\text{range}(T) = \{(0, 0)\}}$$

$\therefore\ N(T) = \text{kernal}(T) = \{(v_1, v_2) \in R^2 \mid T(v_1, v_2) = 0\}$

$= \{(v_1, v_2) \in R^2 \mid (0, 0) = 0\}$

$= \{(v_1, v_2) \in R^2 \mid (v_1, v_2) \in R\}$

$$\boxed{\text{kernal}(T) = R^2}$$

(D) $T(v_1, v_2) = (v_1, v_1)$

$\therefore\ C(T) = \text{range}(T) = \{ T(v_1, v_2) \mid (v_1, v_2) \in R^2 \}$

$= \{ (v_1, v_1) \mid v_1, v_2 \in R \}$

$$\boxed{\text{range}(T) = \{ (v_1, v_2) \mid v_1 = v_2,\ v_1, v_2 \in R \}}$$

$\therefore\ N(T) = \text{Kernal}(T) = \{ (v_1, v_2) \in R^2 \mid T(v_1, v_2) = 0 \}$

$= \{ (v_1, v_2) \in R^2 \mid (v_1, v_1) = 0 \}$

$= \{ (v_1, v_2) \in R^2 \mid v_1 = 0 \}$

$$\boxed{\text{Kernal}(T) = \{ (0, v_2) \mid v^2 \in R \}}$$