EC 880 Homework 6

30/30

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$\boxed{10.2}$

Exercise 10.2: The action of the bit flip channel can be described by the quantum operation $\mathcal{E}(\rho) = (1-p)\rho + pX\rho X$. Show that this may be given an alternate operator-sum representation, as $\mathcal{E}(\rho) = (1-2p)\rho + 2pP_+\rho P_+ + 2pP_-\rho P_-$ where P_+ and P_- are projectors onto the $+1$ and -1 eigenstates of X, $(|0\rangle + |1\rangle)/\sqrt{2}$ and $(|0\rangle - |1\rangle)/\sqrt{2}$, respectively. This latter representation can be understood as a model in which the qubit is left alone with probability $1-2p$, and is 'measured' by the environment in the $|+\rangle, |-\rangle$ basis with probability $2p$.

we know $\quad P_+ = |+\rangle\langle+|$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)$$

$$= \frac{1}{2}\Big(\underbrace{|0\rangle\langle 0| + |1\rangle\langle 1|}_{I} + \underbrace{|0\rangle\langle 1| + |1\rangle\langle 0|}_{X}\Big)$$

$$\therefore \quad P_+ = \frac{1}{2}(I + X) \quad \text{———— (1)}$$

similarly; $\quad P_- = |-\rangle\langle -|$

$$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)$$

$$= \frac{1}{2}\Big(\underbrace{|0\rangle\langle 0|}_{+I} \underbrace{- |0\rangle\langle 1| - |1\rangle\langle 0|}_{-X} + \underbrace{|1\rangle\langle 1|}\Big)$$

$$= \frac{1}{2}(I - X) \quad \text{———— (2)}$$

$$\therefore \quad \mathcal{E}(\rho) = (1-2p)\rho + 2p\,P_+\rho P_+ + 2p\,P_-\rho P_- \quad \text{———— (Given)} \quad (3)$$

substituting eq (1) & (2) gives us

$$= (1-2p)\rho + 2p\left(\frac{1}{2}(I+X)\right)\rho\left(\frac{1}{2}(I+X)\right) + 2p\left(\frac{1}{2}(I-X)\right)\rho\left(\frac{1}{2}(I-X)\right)$$

$$= (1-2p)\rho + \frac{p}{2}(I+X)(\rho + \rho X) + \frac{p}{2}(I-X)(\rho - \rho X)$$

$$= (1-2p)\rho + \left[\frac{p}{2}(\rho + \rho X + \rho X + X\rho X) + \frac{p}{2}(\rho - \rho X - X\rho + X\rho X)\right]$$

$$= \rho - 2p\rho + \frac{p}{2}\left(\rho + \rho + \cancel{\rho X} - \cancel{\rho X} + \cancel{X\rho} - \cancel{X\rho} + X\rho X + X\rho X\right)$$

$$= \rho - 2p\rho + \frac{p}{2}\left[2\rho + 2X\rho X\right]$$

$$= \rho - 2p\rho + p\rho + pX\rho X$$
$$= \rho - p\rho + pX\rho X \quad \text{——— (4)}$$

However, it is also given that:

$$\varepsilon(\rho) = (1-p)\rho + pX\rho X$$
$$= \rho - p\rho + pX\rho X \quad \text{——— (5)}$$

$\therefore$ from (4) & (5), we can conclude that the given both representations are the same.

10.3

Exercise 10.3: Show by explicit calculation that measuring Z_1Z_2 followed by Z_2Z_3 is equivalent, up to labeling of the measurement outcomes, to measuring the four projectors defined by (10.5)–(10.8), in the sense that both procedures result in the same measurement statistics and post-measurement states.

$$P_0 \equiv |000\rangle\langle 000| + |111\rangle\langle 111| \text{ no error} \tag{10.5}$$
$$P_1 \equiv |100\rangle\langle 100| + |011\rangle\langle 011| \text{ bit flip on qubit one} \tag{10.6}$$
$$P_2 \equiv |010\rangle\langle 010| + |101\rangle\langle 101| \text{ bit flip on qubit two} \tag{10.7}$$
$$P_3 \equiv |001\rangle\langle 001| + |110\rangle\langle 110| \text{ bit flip on qubit three.} \tag{10.8}$$

$\because$ $(Z_1Z_2)_+ = (|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes I$

$(Z_1Z_2)_- = (|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I$

$(Z_2Z_3)_+ = I \otimes (|00\rangle\langle 00| + |11\rangle\langle 11|)$

$(Z_2Z_3)_- = I \otimes (|01\rangle\langle 01| + |10\rangle\langle 10|)$

$\because$ measurement operator $= (Z_2Z_3)(Z_1Z_2)$,

Possible outcomes: — ① $(Z_2Z_3)^+(Z_1Z_2)^+$
② $(Z_2Z_3)^+(Z_1Z_2)^-$
③ $(Z_2Z_3)^-(Z_1Z_2)^+$
④ $(Z_2Z_3)^-(Z_1Z_2)^-$

Now, $(Z_2Z_3)_+(Z_1Z_2)_+ = [I \otimes (|00\rangle\langle 00| + |11\rangle\langle 11|)][(|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes I]$

$= |000\rangle\langle 000| + |111\rangle\langle 111|$

$= P_0$ ———— ①

Now, $(Z_2Z_3)_+(Z_1Z_2)_- = [I \otimes (|00\rangle\langle 00| + |11\rangle\langle 11|)][(|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I]$

$= |100\rangle\langle 100| + |011\rangle\langle 011|$

$= P_1$ ———— ②

Now, $(Z_2 Z_3)_- (Z_1 Z_2)_+ = \left[I \otimes \left(|01\rangle\langle 01| + |10\rangle\langle 10| \right) \right] \left[\left(|00\rangle\langle 00| + |11\rangle\langle 11| \right) \otimes I \right]$

$$= |001\rangle\langle 001| + |110\rangle\langle 110|$$

$$= P_3 \qquad \text{———} \quad (3)$$

Now, $(Z_2 Z_3)_- (Z_1 Z_2)_- = \left[I \otimes \left(|01\rangle\langle 01| + |10\rangle\langle 10| \right) \right] \left[\left(|01\rangle\langle 01| + |10\rangle\langle 10| \right) \otimes I \right]$

$$= |010\rangle\langle 010| + |101\rangle\langle 101|$$

$$= P_2 \qquad \text{———} \quad (4)$$

$\therefore$ from (1), (2), (3) & (4), we showed that,

measuring $Z_1 Z_2$ followed by $Z_2 Z_3$ is same as measuring projectors defined in eq (10.5)–(10.8)

$\boxed{10.5}$

Exercise 10.5: Show that the syndrome measurement for detecting phase flip errors in the Shor code corresponds to measuring the observables $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$.

codeword's for Shor's code are :

$$|0_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)$$

$$|1_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)$$

Given: $M_1 = X_1 X_2 X_3 X_4 X_5 X_6$

$M_2 = X_4 X_5 X_6 X_7 X_8 X_9$

E_1 = phase - flip error on 1^{st} block.

Say E_1 occurs. then :

$$|0_L'\rangle = \frac{1}{2\sqrt{2}}(|000\rangle-|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)$$

$$|1_L'\rangle = \frac{1}{2\sqrt{2}}(|000\rangle+|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)$$

$$\therefore M_1|0_L'\rangle = X_1X_2X_3X_4X_5X_6|0_L'\rangle$$

$$= (|111\rangle-|000\rangle)(|111\rangle+|000\rangle)(|000\rangle+|111\rangle)\cdot\frac{1}{2\sqrt{2}}$$

$$= \frac{-1}{2\sqrt{2}}(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)$$

$$= -|0_L\rangle$$

Now, $M_2 |0_L'\rangle = X_4 X_5 X_6 X_7 X_8 X_9 |0_L'\rangle$

$$= (|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle\ |111\rangle) / 2\sqrt{2}$$

$$= |0_L\rangle$$

Similarly, $M_1 |1_L'\rangle = \frac{1}{\sqrt{8}} (|111\rangle + |000\rangle)(|111\rangle - |000\rangle)(|000\rangle - |111\rangle)$

$$= -|1_L\rangle$$

& also $M_2 |1_L'\rangle = \frac{1}{\sqrt{8}} (|000\rangle + |111\rangle)(|111\rangle - |000\rangle)(|111\rangle - |000\rangle)$

$$= |1_L\rangle$$

$$\therefore \boxed{\text{for } E_1, \quad \begin{array}{l} M_1 = -1 \\ M_2 = +1 \end{array}}$$

E_2 = phase-flip error on 2^{nd} block.
Say E_2 occurs. then:

$$|0_L'\rangle = \frac{1}{2\sqrt{2}} (|000\rangle + |111\rangle)(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)$$

$$|1_L'\rangle = \frac{1}{2\sqrt{2}} (|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)$$

$\therefore M_1 |0_L'\rangle = \frac{1}{\sqrt{8}} (|111\rangle + |000\rangle)(|111\rangle - |000\rangle)(|000\rangle + |111\rangle)$

$$= -|0_L\rangle$$

$$\therefore\ M_1 |1_L'\rangle = (|111\rangle - |000\rangle)(|111\rangle + |000\rangle)(|000\rangle - |111\rangle) \cdot \frac{1}{2\sqrt{2}}$$

$$= -|1_L\rangle$$

$$\therefore\ M_2 |0_L'\rangle = (|000\rangle + |111\rangle)(|111\rangle - |000\rangle)(|111\rangle + |000\rangle) \cdot \frac{1}{2\sqrt{2}}$$

$$= -|0_L'\rangle$$

$$\therefore\ M_2 |1_L'\rangle = (|000\rangle - |111\rangle)(|111\rangle + |000\rangle)(|111\rangle - |000\rangle) \cdot \frac{1}{2\sqrt{2}}$$

$$= -|1_L\rangle$$

$$\therefore\ \boxed{\text{for } E_2 : \begin{array}{l} M_1 = -1 \\ M_2 = -1 \end{array}}$$

Now,

E_3 = phase-flip error on 3rd block.

Say E_3 occurs. then:

$$|0_L'\rangle = \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)$$

$$|1_L'\rangle = \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)$$

$$\therefore\ M_1 |0_L'\rangle = (|111\rangle + |000\rangle)(|111\rangle + |000\rangle)(|000\rangle - |111\rangle) \frac{1}{\sqrt{8}}$$

$$= |0_L\rangle$$

$$\therefore\ M_1 |1_L'\rangle = (|111\rangle - |000\rangle)(|111\rangle - |000\rangle)(|111\rangle + |000\rangle) \frac{1}{\sqrt{8}}$$

$$= |1_L\rangle$$

& $M_2|0_L'\rangle = (|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|111\rangle - |000\rangle)\frac{1}{\sqrt{8}}$

$= -|0_L\rangle$

$M_2|1_L'\rangle = (|000\rangle - |111\rangle)(|111\rangle - |000\rangle)(|111\rangle + |000\rangle)\frac{1}{\sqrt{8}}$

$= -|1_L\rangle$

$\therefore$ for E_3, $M_1 = +1$
$M_2 = -1$

→ Hence we have showed measurement for detecting bit flip, be 1^{st}, 2^{nd} or 3^{rd} corrosponds to measuring the observables, $X_1X_2X_3X_4X_5X_6$ & $X_4X_5X_6X_7X_8X_9$

10.6

Exercise 10.6: Show that recovery from a phase flip on any of the first three qubits may be accomplished by applying the operator $Z_1Z_2Z_3$.

Phase flip on any of the first three qubits is given by — say E_1 (qes 10.5)

for E_1, $|0_L'\rangle = \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$

$|1_L'\rangle = \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$

∴ Now, say $\mathcal{Z} = Z_1Z_2Z_3$.

$\mathcal{Z}|0_L'\rangle = (|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)/2\sqrt{2}$
$= |0_L\rangle$

& $\mathcal{Z}|1_L'\rangle = (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$
$= |1_L\rangle$

Hence, we showed that by applying $\mathcal{Z}$, i.e. $Z_1Z_2Z_3$, we can do recovery on phase flip on any of the first three qubits.

10.8

Exercise 10.8: Verify that the three qubit phase flip code $|0_L\rangle = |+++\rangle, |1_L\rangle = |---\rangle$ satisfies the quantum error-correction conditions for the set of error operators $\{I, Z_1, Z_2, Z_3\}$.

Given code : $P = |+++\rangle\langle+++| + |---\rangle\langle---|$

$E = \{I, Z_1, Z_2, Z_3\}$

$PIZ_1P = PZ_1P = PZ_1(|+++\rangle\langle+++| + |---\rangle\langle---|)$

$= P(|-++\rangle\langle-++| + |+--\rangle\langle+--|)$

$= (|+++\rangle\langle+++| + |---\rangle\langle---|)(|-++\rangle\langle-++| + |+--\rangle\langle+--|)$

$= 0$

Simillarly,

$PIZ_2P = PZ_2P = PZ_2(|+++\rangle\langle+++| + |---\rangle\langle---|)$

$= (|+++\rangle\langle+++| + |---\rangle\langle---|)(|+-+\rangle\langle+-+| + |-+-\rangle\langle-+-|)$

$= 0$

Simillarly,

$PIZ_3P = PZ_3P = PZ_3(|+++\rangle\langle+++| + |---\rangle\langle---|)$

$= (|+++\rangle\langle+++| + |---\rangle\langle---|)(|++-\rangle\langle++-| + |--+\rangle\langle--+|)$

$= 0$

Also, $PIIP = PI^2P = PP = P^2 = P$

& also
$$\left.\begin{aligned} PZ_1IP &= PZ_1P = 0 \\ PZ_2IP &= PZ_2P = 0 \\ PZ_3IP &= PZ_2P = 0 \end{aligned}\right\}$$ shown above.

Now, $PZ_1Z_1P = PZ_1^2P = PIP = P^2 = P$

Similarly $PZ_2Z_2P = PZ_3Z_3P = P$

Now, $P Z_1 Z_2 P = P Z_1 Z_2 \left(|+++\rangle\langle+++| + |---\rangle\langle---| \right)$

$= P\left(|--+\rangle\langle--+| + |+--\rangle\langle++-| \right) \rightarrow$ (Same as $P Z_3 P$)

$= 0$

$\therefore \quad P Z_i Z_j P = 0$ —— given $i \neq j$.

if $i = j$, $P Z_i Z_i P = P$

The quantum EC condition is $P E_i^{\dagger} E_j P = \alpha_{ij} P \quad \forall\, i,j$

↳ Hermitian

Here: $\alpha = \begin{array}{c|c} & \begin{array}{cccc} I & Z_1 & Z_2 & Z_3 \end{array} \\ \begin{array}{c} I \\ Z_1 \\ Z_2 \\ Z_3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \longrightarrow = I.$

$\because I$ is hermitian

$\boxed{\text{It has been verified that given code satisfies the quantum EC conditions for set of errors : } \{I, Z_1, Z_2, Z_3\}}$

10.9

Exercise 10.9: Again, consider the three qubit phase flip code. Let P_i and Q_i be the projectors onto the $|0\rangle$ and $|1\rangle$ states, respectively, of the ith qubit. Prove that the three qubit phase flip code protects against the error set $\{I, P_1, Q_1, P_2, Q_2, P_3, Q_3\}$.

$P_i = |0\rangle\langle 0|_i$, $E = \{I, P_1, Q_1, P_2, Q_2, P_3, Q_3\}$

$Q_i = |1\rangle\langle 1|_i$

Given code : $P = |{+}{+}{+}\rangle\langle{+}{+}{+}| + |{-}{-}{-}\rangle\langle{-}{-}{-}|$

Now,

$$PIIP = PI^2P = P$$

$$PP_1P_1P = PP_1^2P = PP_1P$$

$$= P\left(|0\rangle\langle 0|_1\right)\left(|{+}{+}{+}\rangle\langle{+}{+}{+}| + |{-}{-}{-}\rangle\langle{-}{-}{-}|\right)$$

$$= \left(|{+}{+}{+}\rangle\langle{+}{+}{+}| + |{-}{-}{-}\rangle\langle{-}{-}{-}|\right)\cdot\frac{1}{\sqrt{2}}\left(|0{+}{+}\rangle\langle 0{+}{+}| + |0{-}{-}\rangle\langle 0{-}{-}|\right)$$

$$= \frac{1}{2}\left(|{+}{+}{+}\rangle\langle{+}{+}{+}| + |{-}{-}{-}\rangle\langle{-}{-}{-}|\right) = \frac{1}{2}P$$

Now,

$$PP_2P_2P = PP_2^2P = PP_2P$$

$$= P\left(|0\rangle\langle 0|_2\right)\left(|{+}{+}{+}\rangle\langle{+}{+}{+}| + |{-}{-}{-}\rangle\langle{-}{-}{-}|\right)$$

$$= \left(|{+}{+}{+}\rangle\langle{+}{+}{+}| + |{-}{-}{-}\rangle\langle{-}{-}{-}|\right)\cdot\frac{1}{\sqrt{2}}\left(|{+}0{+}\rangle\langle{+}0{+}| + |{-}0{-}\rangle\langle{-}0{-}|\right)$$

$$= \frac{1}{2}\left(|{+}{+}{+}\rangle\langle{+}{+}{+}| + |{-}{-}{-}\rangle\langle{-}{-}{-}|\right) = \frac{1}{2}P$$

Now,

$$P P_3 P_3 P = P P_3^2 P = P P_3 P$$

$$= P \left(|0\rangle\langle 0|_3 \right) \left(|{+}{+}{+}\rangle\langle {+}{+}{+}| + |{-}{-}{-}\rangle\langle {-}{-}{-}| \right)$$

$$= \left(|{+}{+}{+}\rangle\langle {+}{+}{+}| + |{-}{-}{-}\rangle\langle {-}{-}{-}| \right) \cdot \frac{1}{\sqrt{2}} \left(|{+}{+}0\rangle\langle {+}{+}0| + |{-}{-}0\rangle\langle {-}{-}0| \right)$$

$$= \frac{1}{2} \left(|{+}{+}{+}\rangle\langle {+}{+}{+}| + |{-}{-}{-}\rangle\langle {-}{-}{-}| \right) = \frac{1}{2} P$$

Simillary for Q_i, we can say:

$$P Q_1 Q_1 P = P Q_2 Q_2 P = P Q_3 Q_3 P = \frac{1}{2} P$$

Now, $$P P_1 P_2 P = P \left(|0\rangle\langle 0|_1 \right) \left(|0\rangle\langle 0|_2 \right) \left(|{+}{+}{+}\rangle\langle {+}{+}{+}| + |{-}{-}{-}\rangle\langle {-}{-}{-}| \right)$$

$$= \left(|{+}{+}{+}\rangle\langle {+}{+}{+}| + |{-}{-}{-}\rangle\langle {-}{-}{-}| \right) \cdot \frac{1}{2} \cdot \frac{1}{2} \left(|00{+}\rangle\langle 00{+}| + |00{-}\rangle\langle 00{-}| \right)$$

$$= \frac{1}{4} P$$

Now, $$P P_2 P_3 P = P \left(|0\rangle\langle 0|_2 \right) \left(|0\rangle\langle 0|_3 \right) \left(|{+}{+}{+}\rangle\langle {+}{+}{+}| + |{-}{-}{-}\rangle\langle {-}{-}{-}| \right)$$

$$= \left(|{+}{+}{+}\rangle\langle {+}{+}{+}| + |{-}{-}{-}\rangle\langle {-}{-}{-}| \right) \cdot \frac{1}{2} \cdot \frac{1}{2} \left(|{+}00\rangle\langle {+}00| + |{-}00\rangle\langle {-}00| \right)$$

$$= \frac{1}{4} P$$

Now, $$P P_1 P_3 P = P \left(|0\rangle\langle 0|_1 \right) \left(|0\rangle\langle 0|_3 \right) \left(|{+}{+}{+}\rangle\langle {+}{+}{+}| + |{-}{-}{-}\rangle\langle {-}{-}{-}| \right)$$

$$= \left(|{+}{+}{+}\rangle\langle {+}{+}{+}| + |{-}{-}{-}\rangle\langle {-}{-}{-}| \right) \cdot \frac{1}{2} \cdot \frac{1}{2} \left(|0{+}0\rangle\langle 0{+}0| + |0{-}0\rangle\langle 0{-}0| \right)$$

$$= \frac{1}{4} P$$

Similarly, we can show: $P Q_1 Q_2 P = P Q_2 Q_2 P = P Q_1 Q_2 P = \frac{1}{4} P$

Also, $P\, P_i Q_j\, P = P\left(|0\rangle\langle 0|_i \right) \left(|1\rangle\langle 1|_j \right) \left(|{+}{+}{+}\rangle\langle {+}{+}{+}| + |{-}{-}{-}\rangle\langle {-}{-}{-}| \right)$

$= 0$

$\therefore \quad P P_i Q_j P = 0$

$$\therefore \quad \alpha = \begin{array}{c} \\ I \\ P_1 \\ Q_1 \\ P_2 \\ Q_2 \\ P_3 \\ Q_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} I & P_1 & Q_1 & P_2 & Q_2 & P_3 & Q_3 \end{array} \\ \begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 1/4 & 0 & 1/4 & 0 \\ 1/2 & 0 & 1/2 & 0 & 1/4 & 0 & 1/4 \\ 1/2 & 1/4 & 0 & 1/2 & 0 & 1/4 & 0 \\ 1/2 & 0 & 1/4 & 0 & 1/2 & 0 & 1/4 \\ 1/2 & 1/4 & 0 & 1/4 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/4 & 0 & 1/4 & 0 & 1/2 \end{bmatrix} \end{array}$$

we can clearly see that $\alpha = \alpha^{+}$. | $\therefore$ α is hermitian

$\boxed{\text{→ Hence we have proved that 3 qubit phase flip code protect against} - \{I, P_1, Q_1, P_2, Q_2, P_3, Q_3\}}$