EC 880 Homework 8

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2.63

Exercise 2.63: Suppose a measurement is described by measurement operators M_m. Show that there exist unitary operators U_m such that $M_m = U_m\sqrt{E_m}$, where E_m is the POVM associated to the measurement.

$\because$ $M_m = U_m\sqrt{E_m}$, consider, $M_m^\dagger M_m$

$M^\dagger_m = \sqrt{E_m}\, U_m^\dagger$

$$M_m^\dagger M_m = \sqrt{E_m}\,\underline{U_m^\dagger U_m}\sqrt{E_m}$$
$$= \sqrt{E_m}\, I \sqrt{E_m}$$
$$= E_m$$

$\because$ $E_m \longrightarrow$ POVM associated to measurement,

for unitary U, $M_m^\dagger M_m$ is also POVM.

2.64

Exercise 2.64: Suppose Bob is given a quantum state chosen from a set $|\psi_1\rangle, \ldots, |\psi_m\rangle$ of linearly independent states. Construct a POVM $\{E_1, E_2, \ldots, E_{m+1}\}$ such that if outcome E_i occurs, $1 \le i \le m$, then Bob knows with certainty that he was given the state $|\psi_i\rangle$. (The POVM must be such that $\langle\psi_i|E_i|\psi_i\rangle > 0$ for each i.)

- For every $|\psi_i\rangle \longrightarrow$ consider $|\psi_i'\rangle$

  where $\rightarrow$ $\langle \psi_j | \psi_i' \rangle = 0$ – (for $j \ne i$)

- Now, Let $E_i = \frac{1}{m} |\psi_i'\rangle\langle\psi_i'|$ —— $(i = 1 \to m)$

  & $E_{m+1} = I - \sum_{i=1}^{m} E_i$ —— 1

  $\therefore$ If E_i is the outcome, then the probability of before measurement, state being $|\psi_j\rangle$ for some $j \ne i$ is 0.

$\therefore$ $E_1, E_2, \ldots E_{m+1}$ are POVM elements if

(i) Every E_i is positive.

& (ii) $E_1 + E_2 + \ldots + E_{m+1} = I$

- From equation 1, we can see, codition (ii) is true.
- For condition (i) :-

  - $E_1, \ldots E_m$ are positive because E_i is a outer product of $|\psi_i\rangle$.
  - For $E_{m+1} \rightarrow$ consider: vector $|\phi\rangle$

$$\langle \phi | E_{m+1} | \phi \rangle = \langle \phi | I | \phi \rangle - \sum_{i=1}^{m} \langle \phi | E_i | \phi \rangle$$

substituting equation 1

$$= |\phi|^2 - \underbrace{\sum_{i=1}^{m} \frac{1}{m} \langle \phi | \psi_i' \rangle \langle \psi_i' | \phi \rangle}_{\text{always position}}$$

— (for any $|\phi\rangle$ & $|\psi\rangle$, $\langle \phi | \psi_i' \rangle \langle \psi_i' | \phi \rangle \leq |\phi|^2$)

Hence we have satisfied both (i) & (ii).

$\therefore$ required POVM $\{E_1, E_2, \dots E_{m+1}\}$ is

1 to m   m+1

$\boxed{E_i = \frac{1}{m} |\psi_i'\rangle \langle \psi_i'|}$, $\boxed{E_{m+1} = I - \sum E_i}$

4.2

Exercise 4.2: Let x be a real number and A a matrix such that $A^2 = I$. Show that

$$\exp(iAx) = \cos(x)I + i\sin(x)A. \tag{4.7}$$

Expanding exponential function,

$$e^{ixA} = \sum_{n=0}^{\infty} \frac{(ixA)^n}{n!}$$

$$= \frac{(ixA)^0}{0!} + \frac{(ixA)^1}{1!} + \frac{(ixA)^2}{2!} + \frac{(ixA)^3}{3!} + \frac{(ixA)^4}{4!} + \frac{(ixA)^5}{5!} \cdots$$

$$= \left(\frac{(ixA)^0}{0!} + \frac{(ixA)^2}{2!} + \frac{(ixA)^4}{4!} \cdots\right) + \left(\frac{(ixA)^1}{1!} + \frac{(ixA)^3}{3!} + \frac{(ixA)^5}{5!} \cdots\right)$$

$$= \sum_{n=0}^{\infty} \frac{(ixA)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(ixA)^{2n+1}}{(2n+1)!}$$

$\because A^{2n} = I$
$\because A^{2n+1} = I$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} I}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} A}{(2n+1)!}$$

$$= \cos(x) I + i \sin(x) A$$

4.3

Exercise 4.3: Show that, up to a global phase, the $\pi/8$ gate satisfies $T = R_z(\pi/4)$.

we know $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$

Now, $R_z\left(\frac{\pi}{4}\right) = e^{\frac{-i\frac{\pi}{4}Z}{2}}$

$$= \cos\left(\frac{\pi}{8}\right) I - i \sin\left(\frac{\pi}{8}\right) Z$$

$$= \begin{bmatrix} \cos\left(\frac{\pi}{8}\right) - i\sin\left(\frac{\pi}{8}\right) & 0 \\ 0 & \cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right) \end{bmatrix}$$

$$= \begin{bmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{bmatrix}$$

multiplying by a global phase $\frac{\pi}{8}$,

$$= \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$$

this is nothing but T gate.

$\therefore$ Hence proved, that upto a global phase, the $\frac{\pi}{8}$ gate satisfy $T = R_z(\pi/4)$

4.5

If $\hat{n} = (n_x, n_y, n_z)$ is a real unit vector in three dimensions then we generalize the previous definitions by defining a rotation by θ about the $\hat{n}$ axis by the equation

$$R_{\hat{n}}(\theta) \equiv \exp(-i\theta\, \hat{n} \cdot \vec{\sigma}/2) = \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right) (n_x X + n_y Y + n_z Z), \quad (4.8)$$

where $\vec{\sigma}$ denotes the three component vector (X, Y, Z) of Pauli matrices.

Exercise 4.5: Prove that $(\hat{n} \cdot \vec{\sigma})^2 = I$, and use this to verify Equation (4.8).

Let $Q = \vec{n} \cdot \hat{\vec{q}}$ $\longrightarrow$ $\because$ it has eigenvalues ± 1,

using spectral decomposition,

$$Q = |q_1\rangle\langle q_1| + |q_2\rangle\langle q_2| \quad \rightarrow \quad \text{where:}$$

$\langle q_1|q_1\rangle = 1$

$\langle q_2|q_2\rangle = 1$

$\langle q_1|q_2\rangle = 0$

$$\therefore\ Q = \Big(|q_1\rangle\langle q_1| - |q_2\rangle\langle q_2|\Big)\Big(|q_1\rangle\langle q_1| - |q_2\rangle\langle q_2|\Big)$$

$$= |q_1\rangle \underbrace{\langle q_1|q_1\rangle}_{1} \langle q_1| - \cancel{|q_2\rangle \underbrace{\langle q_2|q_1\rangle}_{0} \langle q_1|} - \cancel{|q_1\rangle \overbrace{\langle q_1|q_2\rangle}^{0} \langle q_2|} + |q_2\rangle \underbrace{\langle q_2|q_2\rangle}_{1} \langle q_2|$$

$$= |q_1\rangle\langle q_2| + |q_2\rangle\langle q_2| \quad \text{— which is identity.}$$

$$= I.$$

$\frown$ Hence proved, $(\vec{n} \cdot \hat{\vec{\sigma}})^2 = I$

4.8

Exercise 4.8: An arbitrary single qubit unitary operator can be written in the form

$$U = \exp(i\alpha)R_{\hat{n}}(\theta) \tag{4.9}$$

for some real numbers α and θ, and a real three-dimensional unit vector $\hat{n}$.

1. Prove this fact.
2. Find values for α, θ, and $\hat{n}$ giving the Hadamard gate H.
3. Find values for α, θ, and $\hat{n}$ giving the phase gate

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}. \tag{4.10}$$

① $U = e^{i\alpha} R_n(\theta) = e^{i\alpha} e^{-i\left(\frac{\theta}{2}\right)(n.\sigma)}$

Now, taking adjoint,

$$U^\dagger = \left(e^{i\alpha}\right)^\dagger \left(e^{-i\left(\frac{\theta}{2}\right)(n.\sigma)}\right)^\dagger$$

$$= e^{-i\alpha} e^{i\frac{\theta}{2}(n.\sigma)}$$

$\therefore$ $UU^\dagger = I$. — & $R_{\hat{n}}(\theta)$ gives rotation around every possible axis $\vec{n}$ & $e^{i\alpha}$ can control the global phase, U is unitary which can represent any rotation on bloch sphere.

② $\because$ $H = \dfrac{X + Z}{\sqrt{2}}$, choosing

$$\boxed{\begin{aligned} \alpha &= -\pi/2 \\ \theta &= \pi/2 \\ \vec{n} &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}/\sqrt{2} \end{aligned}}$$

$\pi/2$

π

gives:

$$U = -i\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\left(\frac{X+Z}{\sqrt{2}}\right)\right)$$

$$= \frac{(X+Z)}{\sqrt{2}} \qquad = H.$$

3 choosing $\boxed{\begin{array}{l} \alpha = \pi/4 \\ \theta = -\pi/4 \\ \vec{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{array}}$ $\pi/2$ gives us :

-3

$$U = e^{i\frac{\pi}{4}} e^{-i\frac{\pi}{4} Z}$$

$$= S$$

4.17

Exercise 4.17: (Building CNOT from controlled-Z gates) Construct a CNOT gate from one controlled-Z gate, that is, the gate whose action in the computational basis is specified by the unitary matrix

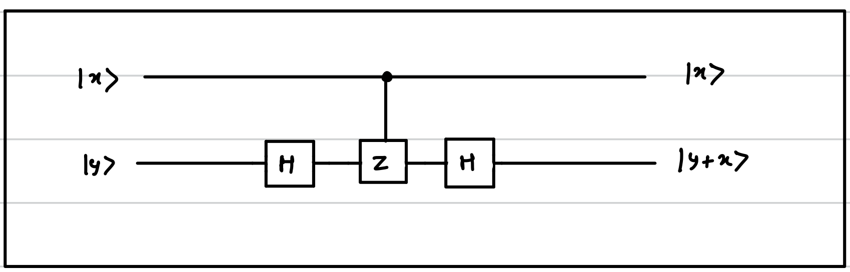
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

and two Hadamard gates, specifying the control and target qubits.

$\because \quad X = HZH,$

$$\begin{aligned} \text{CNOT}\,|x,y\rangle &= |x\rangle + X^{x}|y\rangle \\ &= |x\rangle + (HZH)^{x}|y\rangle \\ &= |x\rangle + HZ^{x}H\,|y\rangle \\ &= \underbrace{I\otimes H \cdot C^{1}(Z) \cdot I \otimes H \;\; |x,y\rangle} \end{aligned}$$

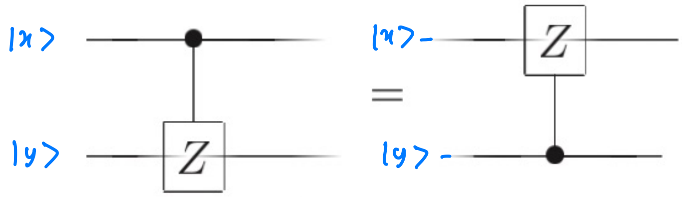
Reprenting this in circuit diagram :



4.18

Exercise 4.18: Show that

Assume:



This circuit is nothing but tensor product of $|x\rangle \otimes Z^{x}|y\rangle$

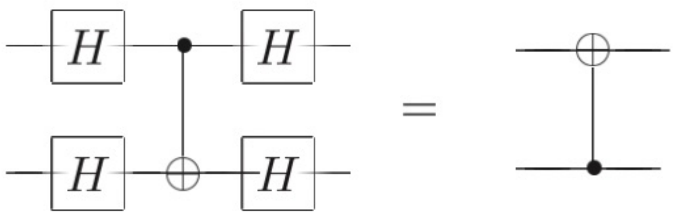
$= |x\rangle \otimes (-1)^{xy}|y\rangle$

$= (-1)^{xy}|x,y\rangle$

$= Z^{y}|x\rangle \otimes |y\rangle$ —— which is what second circuit is representing.

4.20

Exercise 4.20: (CNOT **basis transformations**) Unlike ideal classical gates, ideal quantum gates do not have (as electrical engineers say) 'high-impedance' inputs. In fact, the role of 'control' and 'target' are arbitrary – they depend on what basis you think of a device as operating in. We have described how the CNOT behaves with respect to the computational basis, and in this description the state of the control qubit is not changed. However, if we work in a different basis then the control qubit *does* change: we will show that its phase is flipped depending on the state of the 'target' qubit! Show that



· both hadamard gates acting here will be :

$$H \otimes H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

CNOT will be : $\quad \text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$\therefore$ total circuit $= (H \otimes H)\ \text{CNOT}\ (H \otimes H)$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

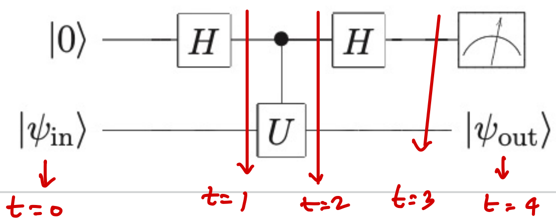
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\longrightarrow$ This is nothing but a CNOT where control & target bit's are assigned different.

→ Hence it is shown that given circuits are equivalent.

4.34

Exercise 4.34: (Measuring an operator) Suppose we have a single qubit operator U with eigenvalues ± 1, so that U is both Hermitian and unitary, so it can be regarded both as an observable and a quantum gate. Suppose we wish to measure the observable U. That is, we desire to obtain a measurement result indicating one of the two eigenvalues, and leaving a post-measurement state which is the corresponding eigenvector. How can this be implemented by a quantum circuit? Show that the following circuit implements a measurement of U:



Let states of the system at these times be:

$t=0 \longrightarrow |\psi_0\rangle$

$t=1 \longrightarrow |\psi_1\rangle$

$t=2 \rightarrow |\psi_2\rangle$

$t=3 \rightarrow |\psi_3\rangle$

$$\therefore \; |\psi_0\rangle = |0\rangle \otimes |\psi_{in}\rangle$$

$$\therefore \; |\psi_1\rangle = (H \otimes I)(|0\rangle \otimes |\psi_{in}\rangle)$$

$$= H|0\rangle \otimes |\psi_{in}\rangle$$

$$= \frac{1}{\sqrt{2}} |0\rangle \otimes |\psi_{in}\rangle + \frac{1}{\sqrt{2}} |1\rangle \otimes |\psi_{in}\rangle$$

$$= \frac{1}{\sqrt{2}} \left(|0\rangle|\psi_{in}\rangle + |1\rangle|\psi_{in}\rangle \right)$$

$$\therefore \; |\psi_2\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle|\psi_{in}\rangle + |1\rangle U|\psi_{in}\rangle \right)$$

$$\therefore \; |\psi_3\rangle = \frac{1}{\sqrt{2}} \left[\frac{1}{\sqrt{2}} \left((|0\rangle + |1\rangle)|\psi_{in}\rangle \right) + \frac{1}{\sqrt{2}} \left((|0\rangle - |1\rangle) U|\psi_{in}\rangle \right) \right]$$

$$\therefore \quad |\psi_3\rangle = \frac{1}{2}\Big[\,|0\rangle (I+U)|\psi_{in}\rangle + |1\rangle (I-U)|\psi_{in}\rangle\,\Big]$$

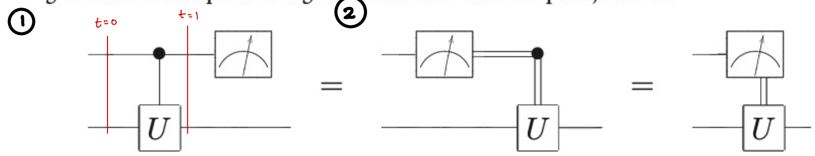
- If the measurement gives us 0 $(\lambda_1 = +1)$, we have corresponding eigenvector $(I+U)|\psi_{in}\rangle$.

- If the measurement gives 1 $(\lambda_2 = -1)$, the corresponding eigenvector is $-1\cdot(I-U)|\psi_{in}\rangle$.

- Thus we have showed that the given circuit implements a measurement of U.

4.35

Exercise 4.35: (Measurement commutes with controls) A consequence of the principle of deferred measurement is that measurements commute with quantum gates when the qubit being measured is a control qubit, that is:



(Recall that the double lines represent classical bits in this diagram.) Prove the first equality. The rightmost circuit is simply a convenient notation to depict the use of a measurement result to classically control a quantum gate.

Considering circuit 1 :

at $t=0 \longrightarrow |\psi_0\rangle = (a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle)$
$= (ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle)$

Now at $t=1 \longrightarrow |\psi_1\rangle = (ac|00\rangle + ad|01\rangle + U.bc|10\rangle + U.bd|11\rangle)$

After measurement, qubit #2 $\rightarrow \left[a(c|0\rangle + d|1\rangle) + b(cU|0\rangle + dU|1\rangle)\right]$

• Now, Considering circuit 2 , if qubit #1 $= |0\rangle$,
qubit #2 $= c|0\rangle + d|1\rangle$.

if qubit #1 $= |1\rangle$
qubit #2 $= cU|0\rangle + dU|1\rangle$.

Combining these
↓
we get $a(c|0\rangle + d|1\rangle) + b(cU|0\rangle + dU|1\rangle)$

$\boxed{\text{- Hence all the circuits given are equivalent.}}$

4.36

Exercise 4.36: Construct a quantum circuit to add two two-bit numbers x and y modulo 4. That is, the circuit should perform the transformation $|x, y\rangle \to |x, x + y \bmod 4\rangle$.

$x_0\ x_1\ y_0\ y_1$

$|00,00\rangle \longrightarrow |00,00\rangle$ 0

$|00,01\rangle \longrightarrow |00,01\rangle$ 1

$|00,10\rangle \longrightarrow |00,10\rangle$ 2

$|00,11\rangle \longrightarrow |00,11\rangle$ 3

$|10,00\rangle \longrightarrow |10,10\rangle$ 10

$|10,01\rangle \longrightarrow |10,11\rangle$ 11

$|10,10\rangle \longrightarrow |10,00\rangle$ 8

$|10,11\rangle \longrightarrow |10,01\rangle$ 9

$|01,00\rangle \longrightarrow |01,01\rangle$ 5

$|01,01\rangle \longrightarrow |01,10\rangle$ 6

$|01,10\rangle \longrightarrow |01,11\rangle$ 7

$|01,11\rangle \longrightarrow |01,00\rangle$ 4

$|11,00\rangle \longrightarrow |11,11\rangle$ 15

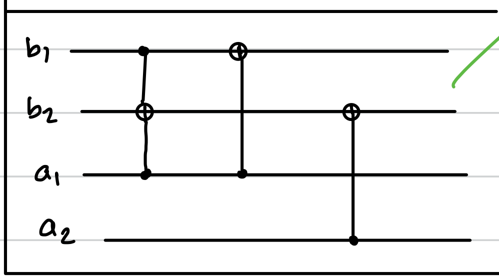
$|11,01\rangle \longrightarrow |11,00\rangle$ 12

$|11,10\rangle \longrightarrow |11,01\rangle$ 13

$|11,11\rangle \longrightarrow |11,10\rangle$ 14

- x_0 & x_1 doesnt change on the output
- y_0 changes if x_0 is

Let 1st Number → $a_1 a_2$
Let 2nd Number → $b_1\ b_2$



This is the required circuit