Solution

EE 520 Final Exam, 8 December 2020

Do all parts of all problems. Write your answers on the exam. Note that the point value of each problem is marked. This exam is open book/open notes. You may refer to any source that you like, but you **cannot consult with other people or collaborate on this exam**. If you use scratch paper, you may hand it in with your exam for consideration of partial credit, but it is not required. Your completed exam must be uploaded as a single PDF file through the Blackboard site before **9 am on Wednesday 9 December 2020**. You can take as much time as you want on this exam, so long as it is handed in on time, but it should not require more than about two hours to complete.

Part I

CIRCLE the *best* answer for each of the following questions. There is only one best answer for each question. No partial credit. (3 points each.)

1. Which of the following sets of computational problems is believed to be the most *difficult*?

   (a) PSPACE (b) P (c) NP-complete (d) NP (e) BQP

2. Consider the two-qubit pure state $|\psi\rangle = (|00\rangle - |01\rangle - |10\rangle + |11\rangle)/2$. Which of the following is the reduced density matrix $\rho_A = \mathrm{Tr}_B\{|\psi\rangle\langle\psi|\}$?

   (a) $\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ (c) $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (d) $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

3. Let $\hat{O} = (-\hat{X} - \hat{Y} + \hat{Z})/\sqrt{3}$ be an observable. Its eigenvalues are ± 1.What is the probability that measuring $\hat{O}$ will give the result $+1$ if the initial state is $|0\rangle$?

   (a) $2/3$ (b) $(1 + 1/\sqrt{3})/2$ (c) 0 (d) $(1 - 1/\sqrt{3})/2$

4. Suppose we have a Hamiltonian $\hbar\omega\hat{X} \otimes \hat{X}$ on two quantum bits, which we turn on for a time $\tau = \pi/(2\omega)$. What unitary operator is produced by this?

   (a) $(\hat{I} + i\hat{X} \otimes \hat{X})/\sqrt{2}$
   (b) $-\hat{I}$
   (c) $-i\hat{X} \otimes \hat{X}$
   (d) $\hat{R}_X(\pi/2) \otimes \hat{R}_X(\pi/2)$

5. Which of the following states has been written in a Schmidt decomposition?

   (a) $\sqrt{1/6}\,|0\rangle \otimes (|0\rangle + |1\rangle) + \sqrt{1/3}(|0\rangle + |1\rangle) \otimes |1\rangle$
   (b) $\sqrt{1/2}\,|0\rangle \otimes (|0\rangle + |1\rangle)/\sqrt{2} + \sqrt{1/2}\,|0\rangle \otimes (|0\rangle - |1\rangle)/\sqrt{2}$
   (c) $\sqrt{1/3}\,|0\rangle \otimes (|0\rangle + |1\rangle)/\sqrt{2} + \sqrt{2/3}\,|1\rangle \otimes (|0\rangle - |1\rangle)/\sqrt{2}$
   (d) $\sqrt{1/2}\,|0\rangle \otimes (|0\rangle + |1\rangle)/\sqrt{2} + \sqrt{1/2}\,|1\rangle \otimes (|0\rangle + |1\rangle)/\sqrt{2}$

6. Which of the following is NOT a completely positive, trace-preserving (CPTP) map?

   (a) $\rho \to (1-p)\rho + p\hat{X}\rho\hat{X}$.

   (b) $\rho \to \hat{U}\rho\hat{U}^\dagger, \quad \hat{U}^\dagger\hat{U} = \hat{I}$.

   (c) $\rho \to (1-p)\rho + (p/3)(\hat{X}\rho\hat{X} + \hat{Y}\rho\hat{Y} + \hat{Z}\rho\hat{Z})$.

   (d) $\rho \to |0\rangle\langle 0|\,\rho\,|0\rangle\langle 0| + |+\rangle\langle +|\,\rho\,|+\rangle\langle +|$.

7. In computational complexity theory, if we say that problem A *reduces* to problem B, what does that mean?

   (a) The ability to solve A implies the ability to solve B.

   (b) A and B can both be solved efficiently.

   (c) Given an oracle for B, we can solve A in polynomial time.

   (d) B is NP-complete.

8. Suppose that two physically adjacent qubits interact with each other, so that their energy levels are $E_{00} = 0 < E_{01} < E_{10} < E_{11}$, and $E_{11} \neq E_{01} + E_{10}$. If you wish to produce a CNOT gate by *resonant driving*, what driving frequency should you use?

   (a) $(E_{11} - E_{10})/\hbar$, (b) $(E_{10} - E_{01})/\hbar$, (c) $(E_{10} - E_{00})/\hbar$, (d) $E_{11}/\hbar$.

9. For what kinds of states does the von Neumann entropy $S(\rho) = -\mathrm{Tr}\{\rho \log \rho\}$ take its *minimum* value?

   (a) Entangled states.

   (b) Pure states.

   (c) Maximally mixed states.

   (d) None of the above.

   (e) Both (a) and (c).

10. Suppose that we have the ability to carry out the following quantum gates: CNOT, H (Hadamard gate) and S (phase gate). Which of the following additional gates would make this set *universal*?

    (a) The T ($\pi/8$) gate.

    (b) The X gate.

    (c) The Toffoli gate.

    (d) Either (a) or (c).

    (e) This gate set is already universal.

Part II

11. (5 points.) Write down the single-bit unitary operator corresponding to a Bloch sphere rotation by π about the axis $\vec{n} = (n_x, n_y, n_z)$, with $n_x = n_z = 1/\sqrt{2}$ and $n_y = 0$. Do you recognize this operator?

$$U = \cos(\pi/2) I - i\sin(\pi/2)(\vec{n}\cdot\vec{\sigma})$$

$$= -i\left(\frac{1}{\sqrt{2}}X + \frac{1}{\sqrt{2}}Z\right) = -i\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

This is $-i \times$ Hadamard, where $-i$ is a global phase.

12. (5 points.) Let $\hat{H}_A$ and $\hat{H}_B$ be two Hamiltonians, and let f be a real number with $0 < f < 1$. Show that

$$\exp\left[-\frac{i}{\hbar}\hat{H}_A(f\Delta t)\right] \times \exp\left[-\frac{i}{\hbar}\hat{H}_B((1-f)\Delta t)\right] = \exp\left[-\frac{i}{\hbar}(f\hat{H}_A + (1-f)\hat{H}_B)\Delta t\right] + O(\Delta t^2).$$

$$= \left(I - \frac{i}{\hbar}H_A(f\Delta t) + O(\Delta t^2)\right)\left(I - \frac{i}{\hbar}H_B((1-f)\Delta t) + O(\Delta t^2)\right)$$

$$= I - \frac{i}{\hbar}(fH_A + (1-f)H_B)\Delta t + O(\Delta t^2)$$

$$= \exp\left[-\frac{i}{\hbar}(fH_A + (1-f)H_B)\Delta t\right] + O(\Delta t^2)$$

13. (10 points.) The Bell states, being maximally entangled, have some peculiar properties. Here's a particular example. Consider the Bell state

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right).$$

Show that for any operator $\hat{O}$ that acts on a single qubit,

$$\left(\hat{O}\otimes\hat{I}\right)|\Phi_+\rangle = \left(\hat{I}\otimes\hat{O}^T\right)|\Phi_+\rangle,$$

where $\hat{O}^T$ is the *transpose* of $\hat{O}$.

$$O = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$O^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

This is easy to show in matrix form:

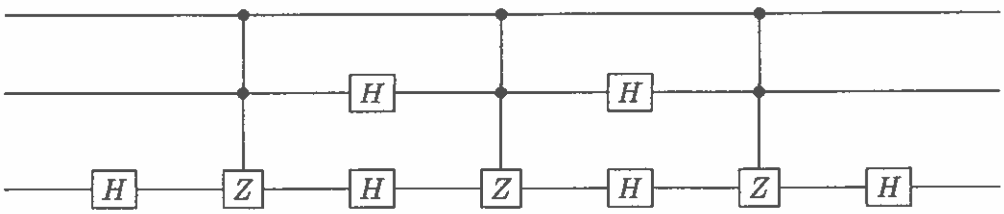
$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \quad O\otimes I = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad I\otimes O^T = \begin{pmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{pmatrix}$$

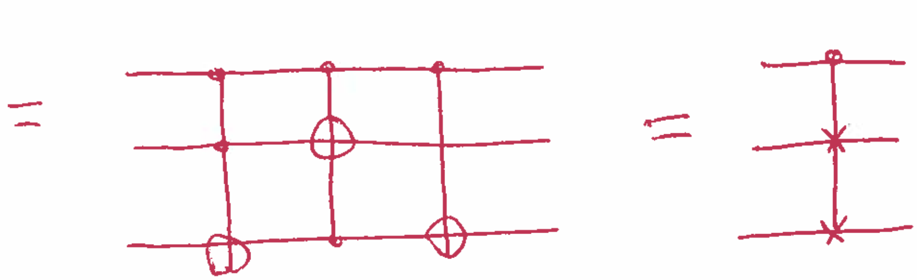
$$(O\otimes I)|\Phi_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$$

$$(I\otimes O^T)|\Phi_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$$

Equal

14. (15 points.) Find the simplest quantum circuit you can that is equivalent to this circuit:





controlled-SWAP or Fredkin gate

Part III

15. (20 points total.) Sometimes quantum noise processes may have special symmetries such that with a properly-chosen quantum error-correcting code the encoded state is actually *immune* to noise. For these special cases, it is only necessary to encode the state, but not to detect or correct errors. Such a special code is called a *decoherence-free subspace.*

Suppose that we encode one *logical* qubit into two *physical* qubits as follows:

$$|0\rangle \rightarrow |0_L\rangle = |01\rangle\,, \quad |1\rangle \rightarrow |1_L\rangle = |10\rangle\,.$$

Our error set is $\mathcal{E} = \{\hat{E}_0 = \hat{I}, \hat{E}_1 = \hat{Z}_1 + \hat{Z}_2, \hat{E}_2 = \hat{Z}_1\hat{Z}_2\}$. That is, $\hat{Z}$ errors always occur *symmetrically* on the two qubits..

(a) (5 points.) Show that any state $\alpha\,|0_L\rangle + \beta\,|1_L\rangle$ in the codespace is an eigenvector of $\hat{E}_0$, $\hat{E}_1$, and $\hat{E}_2$. What are the eigenvalues?

$E_0(\alpha|0_L\rangle + \beta|1_L\rangle) = \alpha|0_L\rangle + \beta|1_L\rangle \quad \lambda_0 = 1$

$E_1(\alpha|0_L\rangle + \beta|1_L\rangle) = (Z_1 + Z_2)(\alpha|01\rangle + \beta|10\rangle)$

$= (\alpha|01\rangle - \alpha|01\rangle - \beta|10\rangle + \beta|10\rangle) = 0 \quad \lambda_1 = 0$

$E_2(\alpha|0_L\rangle + \beta|1_L\rangle) = Z_1Z_2(\alpha|01\rangle + \beta|10\rangle)$

$= -\alpha|01\rangle - \beta|10\rangle = -(\alpha|0_L\rangle + \beta|1_L\rangle) \quad \lambda_2 = -1$

(b) (5 points.) Show that any state $\alpha\,|0_L\rangle + \beta\,|1_L\rangle$ in the codespace is an eigenvector of the operator

$$\exp\left(-i\theta\hat{Z}_1\hat{Z}_2\right),$$

and find the eigenvalue as a function of θ.

Since $\overset{=E_2}{Z_1Z_2}$ has eigenvalue $\lambda_2 = -1$ on any codeword,

$\exp(-i\theta E_2)(\alpha|0_L\rangle + \beta|1_L\rangle) = e^{-i\lambda_2\theta}(\qquad)$

$= e^{i\theta}(\alpha|0_L\rangle + \beta|1_L\rangle)$

So it's an eigenvector with eigenvalue $e^{i\theta}$.

(c) (5 points.) Show that any state $\alpha|0_L\rangle + \beta|1_L\rangle$ in the codespace is an eigenvector of the operator

$$\exp\left(-i\theta\hat{Z}_1\right)\exp\left(-i\theta\hat{Z}_2\right),$$

and find the eigenvalue. (Note that the value of θ is the same in both exponential functions.)

$$\exp(-i\theta Z_1)\exp(-i\theta Z_2) = \exp(-i\theta(Z_1+Z_2))$$
$$= \exp(-i\theta E_1)$$

Since $(\alpha|0_L\rangle+\beta|1_L\rangle)$ is an eigenvector of E_1 w/ eigenvalue of $\exp(-i\theta E_1)$ with eigenvalue

$$e^{-i\theta\lambda_1} = e^{-i\theta\times 0} = 1.$$

(d) (5 points.) Show that this code satisfies the error-correcting condition

$$\hat{\mathcal{P}}\hat{E}_i^\dagger\hat{E}_j\hat{\mathcal{P}} = \alpha_{ij}\hat{\mathcal{P}}$$

for all $\hat{E}_i, \hat{E}_j \in \mathcal{E}$, where the numbers α_{ij} form a Hermitian matrix and the projector $\hat{\mathcal{P}}$ onto the codespace is

$$\hat{\mathcal{P}} = |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|.$$

Since $|0_L\rangle$ and $|1_L\rangle$ are both eigenvectors of E_i with eigenvalues λ_i, then we can see that

$$PE_i^\dagger E_jP = PE_iE_jP = \lambda_j PE_iP = \lambda_i\lambda_j P^2 = \lambda_i\lambda_j P.$$

Here $\alpha_{ij} = \lambda_i\lambda_j$:

$$\alpha = \begin{array}{c} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \\ \begin{array}{ccc} 0 & 1 & 2 \end{array} \end{array}$$

α is Hermitian (as it must be)

16. (15 points total.) Suppose we have a quantum circuit, and the error probability per gate is p. If we use a quantum error-correcting code, the number of gates needed is increased, but any single error in a codeword can be corrected. This makes the probability of an *uncorrectable* error per *logical* gate equal to roughly Cp^2 for some constant C.

    (a) (5 points.) If we concatenate the code k times, what is the uncorrectable error probability per logical gate?

    $$p \to Cp^2 \to C(Cp^2)^2 = C^3p^4$$
    $$\to C(C^3p^4)^2 = C^7p^8, \text{ etc.}$$

    So with k levels of concatenation

    $$p \to (Cp)^{2^k}/C.$$

    (b) (5 points.) For the Steane code, $C \sim 10^4$. How small does p have to be for concatenation to improve the success probability (i.e., what is the *threshold* probability)?

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From the above, to have an improvement we need $Cp<1$, which means $p<1/C$. So we need $p<10^{-4}$ for the Steane code.

(c) (5 points.) Suppose that we wish to carry out a circuit with 10^7 logical gates, and the error probability per physical gate is $p = 10^{-5}$. We will encode all the logical qubits in the concatenated Steane code. Using your answer from part (b) above, how many levels of concatenation are needed for this whole circuit to succeed with probability greater than 99%?

If the number of gates is N and the error rate is p, then the probability of no errors is

$$(1-p)^N.$$

With K levels of concatenation, $p \to (Cp)^{2^K}/C$

So we want

$$(1-(Cp)^{2^K}/C)^N > 0.99$$

Take the natural log of both sides:

$$\ln(1-(Cp)^{2^K}/C)^N = N \ln(1-(Cp)^{2^K}/C)$$

$$\approx -\frac{N}{C}(Cp)^{2^K} > \ln 0.99$$

$$\Rightarrow (Cp)^{2^K} < -\frac{C}{N}\ln 0.99 = \frac{C}{N}\ln\left(\frac{1}{0.99}\right)$$

Take the log of both sides

$$2^K \geq \frac{\ln(C/N) + \ln\ln(1/0.99)}{\ln(Cp)}$$

$C/N = 10^{-3}$, $Cp = 10^{-1}$. Plugging these numbers in,

$$2^K > 5 \Rightarrow \boxed{K \geq 3}$$