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95/100 Very Good!

EE 520 Midterm Exam, 18 October 2022

Do all parts of all problems. Write your answers on the exam. Note that the point value of each problem is marked. This exam is open book/open notes. You may refer to any source that you like, but you **cannot consult with other people or collaborate on this exam**. If you use scratch paper, you may hand it in with your exam for consideration of partial credit, but it is not required. Your completed exam must be uploaded as a single PDF file through the Blackboard site before **1 pm on Wednesday 19 October 2022**. You can take as much time as you want on this exam, so long as it is handed in on time, but it should not require more than about two hours to complete.

• Red box is my answer

Part I

CIRCLE the *best* answer for each of the following questions. There is only one best answer for each question. No partial credit. (3 points each.)

1. Which of the following expressions is true for general operators A and B?

   (a) $e^{A\otimes B} = e^A \otimes e^B$.

   (b) $e^{(A+B)} = e^A e^B$. Not true if A & B dont commute

   (c) $\boxed{e^{(A\otimes I + I\otimes B)} = e^A \otimes e^B.}$

   (d) All of the above.

   (e) Both (b) and (c).

2. Which of the following formulas represents the same Boolean function as $(x\&\neg y) \vee (x\&y) \vee (y\&\neg y)$?

   (a) $x \vee y$ (b) $x\&y$ (c) $x\&\neg y$ $\boxed{\text{(d) } x}$

3. How many *invertible* functions are there that map a string of n input bits to a string of n output bits?

   -3

   (a) 2^n (b) $(2^n)!$ (c) 2^{2^n} $\boxed{\text{(d) } 2^{n2^n}}$

4. Which of the following represents a widely-believed relationship among computational complexity classes?

   (a) PSPACE $\subset$ P $\subset$ NP.

   (b) P $\subset$ NP-complete $\subset$ BPP $\subset$ PSPACE.

   (c) $\boxed{\text{P} \subset \text{BPP} \subset \text{NP} \subset \text{PSPACE.}}$

   (d) P $\subset$ NP $\subset$ NP-complete $\subset$ PSPACE.

5. Let A and B be two operators, and suppose A has eigenvalues α_1 and α_2, and B has eigenvalues β_1 and β_2. Which of the following must be an eigenvalue of $A \otimes B$?

   $\boxed{\text{(a) } \alpha_1\beta_2}$ (b) α_1 (c) $\alpha_1 + \beta_1$ (d) Not enough information to answer.

$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\\vdots\\0\\0\\-1\end{bmatrix}$

6. The GHZ state is an entangled state of three qubits: $|\Psi_{\text{GHZ}}\rangle = \sqrt{1/2}(|000\rangle - |111\rangle)$. For which of the following operators is $|\Psi_{\text{GHZ}}\rangle$ an eigenstate?

   (a) $X \otimes X \otimes X$

   (b) $Z \otimes Z \otimes Z$

   (c) $Y \otimes Y \otimes X$

   (d) All of the above.

   (e) Both (a) and (c).

7. Consider the operator $H = i\hbar\omega(|1\rangle\langle 0| - |0\rangle\langle 1|)$, where $\hbar$ and ω are real. Which of the following statements are true?

   $\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$ $\quad\begin{bmatrix} & \end{bmatrix}$

   (a) H is Hermitian.

   (b) H has eigenvalues $\pm\hbar\omega$.

   (c) $H = \hbar\omega Y$.

   $\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$

   (d) All of the above are true.

   (e) Both (a) and (b) are true.

8. What is the unit vector $\vec{n} = (n_x, n_y, n_z)$ on the Bloch sphere corresponding to the qubit state $|\psi\rangle = (1/\sqrt{2})|0\rangle + ((1+i)/2)|1\rangle$?

   (a) $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ (b) $(1/2, 1/2, 1/\sqrt{2})$ (c) $(1/\sqrt{2}, 1/2, 1/2)$ (d) $(1/\sqrt{2}, 1/\sqrt{2}, 0)$

9. Which of the following states is entangled?

   (a) $(1/\sqrt{2})(|01\rangle + |11\rangle)$. —— $\propto$

   (b) $(1/2)(|00\rangle + |01\rangle - |10\rangle - |11\rangle$. —— $-\frac{1}{4}, -\frac{1}{4}$ $\propto$

   (c) $0.36|00\rangle + 0.48|01\rangle - 0.48|10\rangle + 0.64|11\rangle$. —— $0.2304, -0.2304$

   (d) $(1/2)(|00\rangle - |01\rangle - |10\rangle + |11\rangle$. —— $\frac{1}{4}, \frac{1}{4}$ $\propto$

   (e) None of the above are entangled.

   (f) Both (b) and (d) are entangled.

10. Which of the following operators is NOT a valid gate?

   (a) $(X + Z)/\sqrt{2}$.

   (b) $(X + iY)/\sqrt{2}$.

   (c) $(I + iZ)/\sqrt{2}$.

   (d) X.

   (e) These are all valid gates.

Part II

11. (5 points.) Consider two 2×2 matrices A and B:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Show that $\mathrm{Tr}\{A \otimes B\} = \mathrm{Tr}\{A\}\mathrm{Tr}\{B\}$.

$Tr(A) = a+d$

$Tr(B) = e+h$

$$A \otimes B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{bmatrix}$$

$Tr(A)\,Tr(B) = (a+d)(e+h)$

$= ae + ah + de + dh$ — (1)

$\therefore Tr(A \otimes B) = ae + ah + de + dh$

$= Tr(A)\,Tr(B)$

$\boxed{\text{Hence proved.}}$

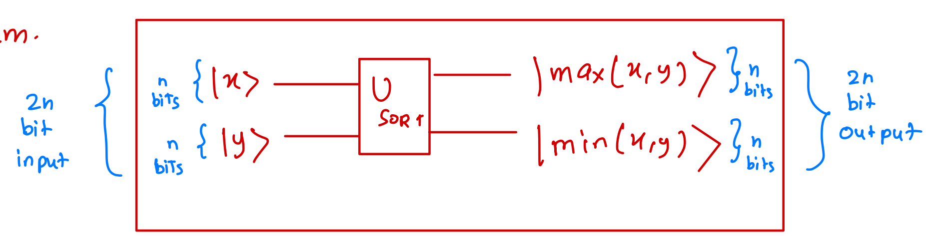
12. (5 points.) Let x and y be two n-bit integers, $x, y \in [0, 2^n - 1]$. We can store them as binary strings in two n-qubit quantum registers, $|x\rangle \otimes |y\rangle$. Is there a unitary operator U_{SORT} that sorts x and y into numerical order:

$$U_{\mathrm{SORT}}(|x\rangle \otimes |y\rangle) = |\max(x, y)\rangle \otimes |\min(x, y)\rangle?$$

If yes, write U_{SORT} as a sum over outer products; if no, explain why not.

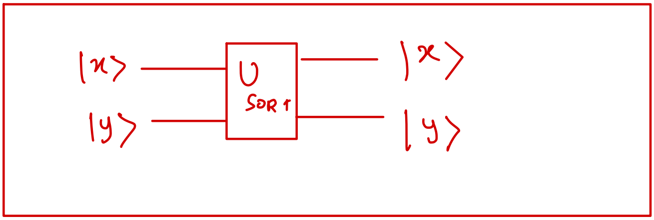
- No, Unitary operator $U_{SORT}(|x\rangle \otimes |y\rangle) = |\max(x,y)\rangle \otimes |\min(x,y)\rangle$ is NOT POSSIBLE.

- Assume a black box for U_{SORT} which gives the output as described in the problem.

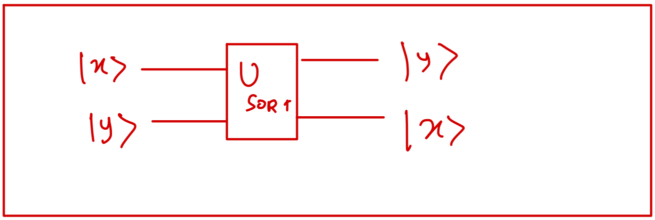


There can be 2 cases for this:

- case ① $x > y$, then U_{SORT} will give :

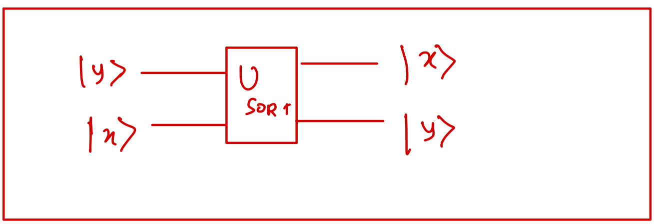


- case ② $y > x$, then U_{SORT} will give :

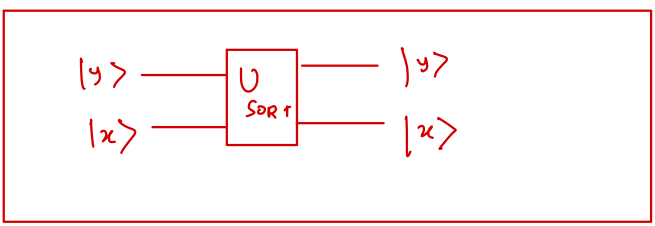


- However, in case ①, even if we switched the input, our block box will give same answer.

$x > y$



- Similarly if $y > x$, → this is also true :



- which means, for two different inputs, our blackbox U_{SORT} gives the same output.

$\therefore$ U_{SORT} cannot be reversible.

$\therefore$ U_{SORT} is not a unitary operator.

$\therefore$ $\boxed{\text{NO such unitary operator } U_{SORT} \text{ is possible.}}$

13. (10 points.) Let us define two normalized state vectors for a qubit:

$$|\psi_1\rangle = \cos(\theta_1)|0\rangle + \sin(\theta_1)|1\rangle,$$
$$|\psi_2\rangle = \cos(\theta_2)|0\rangle + \sin(\theta_2)|1\rangle.$$

Find a condition on θ_1 and θ_2 such that $|\psi_1\rangle + |\psi_2\rangle$ is *also* a normalized state vector.

$$|\psi_1\rangle + |\psi_2\rangle = \cos(\theta_1)|0\rangle + \sin(\theta_1)|1\rangle + \cos(\theta_2)|0\rangle + \sin(\theta_2)|1\rangle$$
$$= \left[\cos(\theta_1) + \cos(\theta_2)\right]|0\rangle + \left[\sin(\theta_1) + \sin(\theta_2)\right]|1\rangle$$

$\therefore$ Normaling $|\psi_1\rangle + |\psi_2\rangle \rightarrow \| \, |\psi_1\rangle + |\psi_2\rangle \, \| = 1$

$$\therefore \sqrt{(\cos(\theta_1) + \cos(\theta_2))^2 + (\sin(\theta_1) + \sin(\theta_2))^2} = 1$$

$$\therefore \sqrt{\cos^2(\theta_1) + \underline{2\cos(\theta_1)\cos(\theta_2)} + \cos^2(\theta_2) + \sin^2(\theta_1) + \underline{2\sin(\theta_1)\sin(\theta_2)} + \sin^2(\theta_2)} = 1$$

$\because \cos^2(\theta_1) + \sin^2(\theta_1) = 1$
& $\cos^2(\theta_2) + \sin^2(\theta_2) = 1$

$$\therefore \sqrt{2 + \underline{\cancel{\cos(\theta_1+\theta_2)} + \cos(\theta_1-\theta_2)} + \underline{\cos(\theta_1-\theta_2) - \cancel{\cos(\theta_1+\theta_2)}}} = 1$$

~ squaring both sides,

$$\therefore 2 + 2\cos(\theta_1 - \theta_2) = 1$$

$$\therefore 2\cos(\theta_1 - \theta_2) = -1$$

$$\therefore \cos(\theta_1 - \theta_2) = \frac{-1}{2}$$

$$\therefore \theta_1 - \theta_2 = \cos^{-1}\left(\frac{-1}{2}\right)$$

$$\therefore \underline{\underline{\theta_1 - \theta_2 = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}}}$$

$\therefore$ Normalizing condition for $|\psi_1\rangle + |\psi_2\rangle$ is

$$\boxed{\theta_1 - \theta_2 = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}}$$

14. (15 points total.) The SWAP operation on two qubits can be treated as a gate, with the matrix

$$U_{\mathrm{SWAP}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

(a) (5 points.) Verify that

$$U_{\mathbf{SWAP}} (|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle,$$

for any arbitrary pair of states $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\phi\rangle = \gamma|0\rangle + \delta|1\rangle$. So the SWAP gate cannot create entanglement.

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad |\phi\rangle = \begin{bmatrix} \gamma \\ \delta \end{bmatrix}, \quad \therefore |\psi\rangle \otimes |\phi\rangle = |\psi\phi\rangle = \begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{bmatrix}$$

$\underline{LHS} = U_{SWAP}(|\psi\rangle \otimes |\phi\rangle)$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{bmatrix} = \begin{bmatrix} \alpha\gamma \\ \beta\gamma \\ \alpha\delta \\ \beta\delta \end{bmatrix} = \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \otimes \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$= |\phi\rangle \otimes |\psi\rangle$$

$$= \underline{\underline{RHS}}$$

$$\boxed{\text{Hence, verified that } U_{SWAP}(|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle}$$

(b) (5 points.) Now find an operator U_{SQRT} such that $U_{\text{SQRT}}^2 = U_{\text{SWAP}}$. This is the "square root of SWAP" gate.

$$U_{SQRT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1+i}{2} & \frac{1-i}{2} & 0 \\ 0 & \frac{1-i}{2} & \frac{1+i}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) (5 points.) Find a pair of qubit states $|\psi\rangle$ and $|\phi\rangle$ such that $U_{\text{SQRT}}(|\psi\rangle \otimes |\phi\rangle)$ is an entangled state. This shows that the square root of SWAP can create entanglement even though SWAP cannot.

$\because$ from U_{SQRT} matrix, we can see that

- input $|00\rangle$ gives $\rightarrow |00\rangle$
- input $|11\rangle$ gives $\rightarrow |11\rangle$
- However $|01\rangle$ gives $\rightarrow \begin{bmatrix} 0 \\ \frac{1+i}{2} \\ \frac{1-i}{2} \\ 0 \end{bmatrix}$

  $|01\rangle \downarrow \boxed{|0\rangle \otimes |1\rangle}$

- & $|10\rangle$ gives $\rightarrow \begin{bmatrix} 0 \\ \frac{1-i}{2} \\ \frac{1+i}{2} \\ \end{bmatrix}$

  $|10\rangle \downarrow \boxed{|1\rangle \otimes |0\rangle}$

which are entangled states!

15. (20 points total.) The "qubit trine states" are three states of a single qubit that are symmetrically spread out in the x-z plane of the Bloch sphere:

$$
\begin{aligned}
|\psi_0\rangle &= |0\rangle\,, \\
|\psi_1\rangle &= -\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\,, \quad = \frac{-1}{2}\begin{bmatrix}1\\ \sqrt{3}\end{bmatrix} \\
|\psi_2\rangle &= -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\,. \quad = \frac{1}{2}\begin{bmatrix}-1\\ \sqrt{3}\end{bmatrix}
\end{aligned}
$$

We will define some generalized measurements based on these states.

(a) (5 points.) Define three measurement operators $\{M_0, M_1, M_2\}$, where

$$M_0 = \sqrt{\frac{2}{3}}|\psi_0\rangle\langle\psi_0|\,, \quad M_1 = \sqrt{\frac{2}{3}}|\psi_1\rangle\langle\psi_1|\,, \quad M_2 = \sqrt{\frac{2}{3}}|\psi_2\rangle\langle\psi_2|\,,$$

Write down the 2×2 matrices for these three operators $\{M_0, M_1, M_2\}$, and show that this is a valid generalized measurement with three possible outcomes (i.e., that $M_0^\dagger M_0 + M_1^\dagger M_1 + M_2^\dagger M_2 = I$).

$$M_0 = \sqrt{\frac{2}{3}} \cdot \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \sqrt{\frac{2}{3}}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \checkmark$$

$$M_1 = \sqrt{\frac{2}{3}} \cdot \frac{-1}{2} \cdot \frac{-1}{2} \begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}\begin{bmatrix}1 & \sqrt{3}\end{bmatrix} = \frac{1}{4}\sqrt{\frac{2}{3}}\begin{bmatrix}1 & \sqrt{3}\\ \sqrt{3} & 3\end{bmatrix} \checkmark$$

$$M_2 = \sqrt{\frac{2}{3}} \cdot \frac{1}{2} \cdot \frac{1}{2} \begin{bmatrix}-1\\ \sqrt{3}\end{bmatrix}\begin{bmatrix}-1 & \sqrt{3}\end{bmatrix} = \frac{1}{4}\sqrt{\frac{2}{3}}\begin{bmatrix}1 & -\sqrt{3}\\ -\sqrt{3} & 3\end{bmatrix} \checkmark$$

$\therefore$ Now, $M_0^\dagger = \sqrt{\frac{2}{3}}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$

$M_1^\dagger = \frac{1}{4}\sqrt{\frac{2}{3}}\begin{bmatrix}1 & \sqrt{3}\\ \sqrt{3} & 3\end{bmatrix}$

$M_2^\dagger = \frac{1}{4}\sqrt{\frac{2}{3}}\begin{bmatrix}1 & -\sqrt{3}\\ -\sqrt{3} & 3\end{bmatrix}$

$\therefore$ Now,

$$M_0^{+} M_0 = \sqrt{\tfrac{2}{3}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \sqrt{\tfrac{2}{3}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M_1^{+} M_1 = \frac{1}{4}\sqrt{\tfrac{2}{3}} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix} \cdot \frac{1}{4}\sqrt{\tfrac{2}{3}} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix}$$

$$= \frac{1}{\cancel{16}_{8}} \cdot \frac{\cancel{2}1}{3} \begin{bmatrix} 4 & \sqrt{3}+3\sqrt{3} \\ \sqrt{3}+3\sqrt{3} & 12 \end{bmatrix}$$

$$= \frac{1}{24} \begin{bmatrix} 4 & 4\sqrt{3} \\ 4\sqrt{3} & 12 \end{bmatrix}$$

$$M_2^{+} M_2 = \frac{1}{4}\sqrt{\tfrac{2}{3}} \begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix} \cdot \frac{1}{4}\sqrt{\tfrac{2}{3}} \begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix}$$

$$= \frac{1}{24} \begin{bmatrix} 4 & -4\sqrt{3} \\ -4\sqrt{3} & 12 \end{bmatrix}$$

$$\therefore M_0^{+} M_0 + M_1^{+} M_1 + M_2^{+} M_2 = \frac{2}{3}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{24}\begin{bmatrix} 4 & 4\sqrt{3} \\ 4\sqrt{3} & 12 \end{bmatrix} + \frac{1}{24}\begin{bmatrix} 4 & -4\sqrt{3} \\ -4\sqrt{3} & 12 \end{bmatrix}$$

$$= \frac{2}{3}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{24}\begin{bmatrix} 8 & 0 \\ 0 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

$\boxed{\rightarrow \text{Hence proved that } M_0^{\dagger}M_0 + M_1^{\dagger}M_1 + M_2^{\dagger}M_2 = I}$

i.e. it is a valid general measurement

<u>where :</u>

$$M_0 = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M_1 = \frac{1}{4}\sqrt{\frac{2}{3}} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix}$$

$$M_2 = \frac{1}{4}\sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix}$$

(b) (5 points.) We will now define a two-outcome generalized measurement with measurement operators $\{A_0, A_1\}$ derived from the previous one:

$$A_0 = M_0, \qquad A_1 = \sqrt{I - M_0^\dagger M_0}.$$

Write down the 2×2 matrices for these two operators $\{A_0, A_1\}$, and show that this is a valid generalized measurement with two possible outcomes.

$$\boxed{A_0 = \sqrt{\frac{2}{3}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}$$

$$\boxed{A_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & 1 \end{bmatrix}}$$

$$A_1 = \sqrt{I - M_0^\dagger M_0} = \sqrt{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 0 \end{bmatrix}}$$

$$= \sqrt{\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & 1 \end{bmatrix}$$

~~(c) (5 points.) Now we define a different two-outcome generalized measurement with measurement operators $\{B_1, B_2\}$:~~

$$\cancel{B_1 = M_1 A_1^{-1}, \qquad B_2 = M_2 A_1^{-1},}$$

~~where A_1^{-1} is the inverse of the operator A_1 from part (b). Write down the 2×2 matrices for these two operators $\{B_1, B_2\}$, and show that this is a valid generalized measurement with two possible outcomes.~~

Answer to 15(b) contineus:

- Now, to show that this is a valid general measurement, we need to prove

$$A_0^\dagger A_0 + A_1^\dagger A_1 = I \longrightarrow A_0^\dagger A_0 = M_0^\dagger M_0$$

we can see clearly from matrix representation of A_1 shown above, that:

$$A_1^\dagger = A_1$$

$$\therefore A_1^\dagger A_1 = A_1^2$$

$$\& \because A_1 = \sqrt{I - M_0^\dagger M_0},$$

$$A_1^2 = I - M_0^\dagger M_0$$

$\therefore$ Our original equation becomes:

$$\text{LHS} = M_0^\dagger M_0 + (I - M_0^\dagger M_0)$$

$$= I = \text{RHS}$$

$\boxed{\text{Hence, we proved that } A_0, A_1 \text{ is a valid general measurement with 2-outcomes}}$

Answer to 15.(c)

(c) (5 points.) Now we define a different two-outcome generalized measurement with measurement operators $\{B_1, B_2\}$:

$$B_1 = M_1 A_1^{-1}, \qquad B_2 = M_2 A_1^{-1},$$

where A_1^{-1} is the inverse of the operator A_1 from part (b). Write down the 2×2 matrices for these two operators $\{B_1, B_2\}$, and show that this is a valid generalized measurement with two possible outcomes.

- First, calculating $A_1^{-1} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore B_1 = \frac{1}{4}\sqrt{\frac{2}{3}} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} = \boxed{\frac{1}{4}\sqrt{\frac{2}{3}} \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 3 & 3 \end{bmatrix} = B_1}$$

$$\therefore B_2 = \frac{1}{4}\sqrt{\frac{2}{3}} \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} = \boxed{\frac{1}{4}\sqrt{\frac{2}{3}} \begin{bmatrix} \sqrt{3} & -\sqrt{3} \\ -3 & 3 \end{bmatrix} = B_2}$$

For this to be valid measurement, $B_1^\dagger B_1 + B_2^\dagger B_2 = I$

$$\therefore B_1^\dagger B_1 = \frac{1}{4}\sqrt{\frac{2}{3}} \begin{bmatrix} \sqrt{3} & 3 \\ \sqrt{3} & 3 \end{bmatrix} \frac{1}{4}\sqrt{\frac{2}{3}} \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 3 & 3 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 12 & 12 \\ 12 & 12 \end{bmatrix}$$

$$\therefore B_2^\dagger B_1 = \frac{1}{4}\sqrt{\frac{2}{3}} \begin{bmatrix} \sqrt{3} & -\sqrt{3} \\ -3 & 3 \end{bmatrix} \frac{1}{4}\sqrt{\frac{2}{3}} \begin{bmatrix} \sqrt{3} & -3 \\ -\sqrt{3} & 3 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 12 & -12 \\ -12 & 12 \end{bmatrix}$$

$$\therefore B_1^\dagger B_1 + B_2^\dagger B_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\boxed{\text{Hence it is proved that it is a valid general measurement.}}$

(d) (5 points.) Now consider the following measurement procedure:

1. Perform the generalized measurement $\{A_0, A_1\}$.
2. If you get result 0 then STOP. The outcome of the measurement procedure is 0.
3. Otherwise, perform the generalized measurement $\{B_1, B_2\}$.
4. If you get result 1, the outcome of the measurement procedure is 1.
5. If you get result 2, the outcome of the measurement procedure is 2.

Show that carrying out this procedure gives the exact same result as doing the three-outcome generalized measurement from part (a). (That is, the probability of the three outcomes is the same in both cases, and the state after each measurement result is the same in both cases.)

for the generalized measurement $\{M_0, M_1, M_2\}$

Eigenstates of M_0 (unnormalized) : $\sqrt{\frac{2}{3}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Eigenstates of M_1 (unnormalized) : $\frac{1}{4}\sqrt{\frac{2}{3}}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix} \longrightarrow \left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 1 \end{bmatrix}, \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} \right\}$

Eigenstates of M_2 (unnormalized) : $\frac{1}{4}\sqrt{\frac{2}{3}}\begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix} \longrightarrow \left\{ \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ 1 \end{bmatrix}, \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \right\}$

Now, following the procedure given here:

Eigenstates of A_0 (unnormalized) : $\sqrt{\frac{2}{3}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Eigenstates of $B_1 A_1$ (unnormalized) : $\frac{1}{4}\sqrt{\frac{2}{3}}\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{\sqrt{\frac{3}{2}}}{2} \end{bmatrix} \longrightarrow \underline{\left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 1 \end{bmatrix}, \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} \right\}}$$

Eigenstates for B_2A_1 : $\frac{1}{4}\sqrt{\frac{2}{3}} \begin{bmatrix} \sqrt{3} & -\sqrt{3} \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}$

(unnormalized)

Wellllll…this doesn't quite show what you are intending. For generalized measurements, it's not just the eigenvectors that determine the outcome probability or final state.

$$= \begin{bmatrix} \frac{1}{2\sqrt{6}} & \frac{-1}{2\sqrt{2}} \\ \frac{-1}{2\sqrt{2}} & \frac{\sqrt{\frac{3}{2}}}{2} \end{bmatrix} \longrightarrow \underline{\left\{ \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ 1 \end{bmatrix}, \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \right\}}$$

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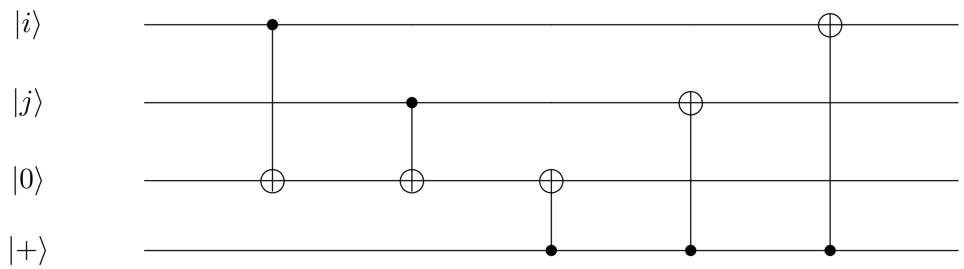
∴ as we can see, in both the cases mentioned in the question, these generalized measurements in both the cases are equivalent.

i.e. ① Probability of outcome measurement is same in both cases.

$\langle\psi|M_0^\dagger M_0|\psi\rangle = \langle\psi|A_0^\dagger A_0|\psi\rangle$
$\langle\psi|M_1^\dagger M_1|\psi\rangle = \langle\psi|(B_1A_1)^\dagger B_1A_1|\psi\rangle$
$\langle\psi|M_2^\dagger M_2|\psi\rangle = \langle\psi|(B_2A_1)^\dagger B_2A_1|\psi\rangle$

② State in which system is left in after the measurement is also same. (one of the eigenstates)

16. (15 points.) Write down the four states produced by the following circuit:



for $ij = 00, 01, 10, 11$, where $|i\rangle$ and $|j\rangle$ are standard Z basis vectors, and

$$|+\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle).$$

$|0\rangle \to |0\rangle \to |0\rangle \to |0\rangle \to |0\rangle$

$|1\rangle \to |1\rangle \to |1\rangle \to |1\rangle$

$|0\rangle \to |0\rangle \to |1\rangle$

$|+\rangle \to |+\rangle \to |+\rangle$

$\frac{|10\rangle + |01\rangle}{\sqrt{2}}$ $\quad \frac{|110\rangle + |001\rangle}{\sqrt{2}}$ $\quad \boxed{\frac{|0110\rangle + |1001\rangle}{\sqrt{2}}}$ ✓

$|1\rangle \to |1\rangle \to |1\rangle \to |1\rangle \to |1\rangle$

$|1\rangle \to |1\rangle \to |1\rangle \to |1\rangle$

$|0\rangle \to |1\rangle \to |0\rangle$

$|+\rangle \to |+\rangle \to |+\rangle$

$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ $\quad \frac{|100\rangle + |011\rangle}{\sqrt{2}}$ $\quad \boxed{\frac{|1100\rangle + |0011\rangle}{\sqrt{2}}}$ ✓

$|0\rangle \to |0\rangle \to |0\rangle \to |0\rangle \to |0\rangle$

$|0\rangle \to |0\rangle \to |0\rangle \to |0\rangle$

$|0\rangle \to |0\rangle \to |0\rangle$

$|+\rangle \to |+\rangle \to |+\rangle$

$\to \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ $\quad \frac{|000\rangle + |111\rangle}{\sqrt{2}}$ $\quad \boxed{\frac{|0000\rangle + |1111\rangle}{\sqrt{2}}}$ ✓

$|1\rangle \to |1\rangle \to |1\rangle \to |1\rangle \to |1\rangle$

$|0\rangle \to |0\rangle \to |0\rangle \to |0\rangle$

$|0\rangle \to |1\rangle \to |1\rangle$

$|+\rangle \to |+\rangle \to |+\rangle$

$\frac{|10\rangle + |01\rangle}{\sqrt{2}}$ $\quad \frac{|010\rangle + |101\rangle}{\sqrt{2}}$ $\quad \boxed{\frac{|1010\rangle + |0101\rangle}{\sqrt{2}}}$ ✓

	Input		Output
1	$\|0\rangle \otimes \|0\rangle \otimes \|0\rangle \otimes \|+\rangle$	$\longrightarrow$	$\frac{\|0000\rangle + \|1111\rangle}{\sqrt{2}}$
2	$\|1\rangle \otimes \|0\rangle \otimes \|0\rangle \otimes \|+\rangle$	$\longrightarrow$	$\frac{\|1010\rangle + \|0101\rangle}{\sqrt{2}}$
3	$\|0\rangle \otimes \|1\rangle \otimes \|0\rangle \otimes \|+\rangle$	$\longrightarrow$	$\frac{\|0110\rangle + \|1001\rangle}{\sqrt{2}}$
4	$\|1\rangle \otimes \|1\rangle \otimes \|0\rangle \otimes \|+\rangle$	$\longrightarrow$	$\frac{\|1100\rangle + \|0011\rangle}{\sqrt{2}}$

$|1\rangle \otimes |1\rangle \otimes |0\rangle \otimes |+\rangle$ ✓