CPTP Maps & Quantum Correlations without Entanglement - An Overview of Quantum Discord

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1 Quantum Channels & Assumptions

How do we represent the action of a channel, i.e. changes to a system due to noise/environment/bath, mathematically? We do that using a completely-positive trace-preserving (CPTP) map. In reality, the evolution is always unitary, but that unitary operator is on our system combined with the bath, and we use Kraus operator sum representation (OSR) to denote what happens to only our system, ignoring the environment.

Consider a system A and environment E. Let the initial joint state be $\rho_{AE}(0)$. Then, the system will evolve as:

$$\rho_{AE}(t) = U(t)\rho_{AE}(0)U^{\dagger}(t),$$

and

$$\rho_A(t) = \mathrm{Tr}_B[\rho_{AE}(t)].$$

Here, we can mathematically show that our system evolves as:

$$\rho_A(0) \xrightarrow{\Phi} \rho_A(t),$$
$$\rho_A(t) = \Phi[\rho_A(0)] = \sum_{\alpha} K_{\alpha}(t)\rho_A(0)K_{\alpha}^{\dagger}(t),$$

where Φ is a CPTP map. We usually think of the above expression from open quantum systems in quantum information theory as:

$$\underbrace{\rho_A}_{1}\underbrace{\xrightarrow{\Phi}}_{2}\underbrace{\rho_B}_{3}.$$

1 $\mapsto$ System on the Alice's side,
2 $\mapsto$ System passing through a quantum channel,
3 $\mapsto$ System received on the Bob's side, which is now changed compared to what Alice sent him due to the effect of channel (Noise/Environment).

However, to represent a quantum channel as a CPTP map, in the naive derivation for operator sum representation[1], we make a non-trivial assumption that the initial state is completely factorized. That is,

$$\rho_{AE}(0) = \rho_A(0) \otimes \rho_E(0).$$

It means that we always assume, before starting a communication protocol, Alice's system is completely uncorrelated with the environment. Can we relax this assumption and still be able to describe the effect of our system through a quantum channel as a CPTP map?

If we start with a completely general state, which can have any quantum and classical correlations, that is,

$$\rho_{AE}(0) = \sum_{i\mu} \lambda_{i\mu}|i\rangle\langle i|_A \otimes |\mu\rangle\langle\mu|_E,$$

where $\{|i\rangle_A\}$ is a basis for system A and $\{|\mu\rangle_E\}$ is a basis for the environment, it turns out that you cannot describe the evolution of ρ_A as a CPTP map. Maybe it was too ambitious. What if we started with a separable initial state?

$$\rho_{AE}(0) = \sum_i p_i \rho_A^i \otimes \rho_E^i.$$

Now, it may seem that states of this type only have classical correlations and no quantum correlations/entanglement. However, even in this case, we fail to obtain a CP evolution for our system A.
It turns out that there is still some quantumness in the correlations of this state.

2 What is Quantum Discord?

In classical information theory, we define mutual information between two random variables X and Y as:

$$I(Y:X) = H(Y) + H(X) - H(X,Y).$$

According to Bayes' Theorem,

$$\begin{aligned} p(Y,X) &= p(Y|X)p(x), \\ &= p(X|Y)p(y). \end{aligned}$$

Therefore,

$$J(Y:X) = H(Y) - H(Y|X),$$

where $I(Y:X)$ and $J(Y:X)$ denote the same quantity, classical mutual information. However, in the quantum case, measuring system X generally affects system Y if the joint state ρ_{XY} is correlated. So, the asymmetry in the expression of $J(Y:X)$ means that it can be different than $I(Y:X)$.

The first quantum mutual information is:

$$I_Q\left(\rho_{Y:X}\right) = H\left(\rho_Y\right) + H\left(\rho_X\right) - H\left(\rho_{XY}\right).$$

where,

$$\begin{aligned} \rho_{XY} &= \text{total state of system } X \text{ and } Y, \\ \rho_X &= \mathrm{Tr}_Y[\rho_{XY}], \\ \rho_Y &= \mathrm{Tr}_X[\rho_{XY}]. \end{aligned}$$

The second mutual information quantity, J_Q will arise from measuring X. Assume we perform a projective measurement with a set of complete orthonormal projectors $\{\Pi_i\}$ on system X. Therefore, the state will become,

$$\rho_{XY} \xrightarrow{\{\Pi_i\}} \frac{\Pi_i \rho_{XY} \Pi_i}{p_i} = \underbrace{\rho_{Y|\Pi_i}}_{\text{w/ prob } p_i}.$$

Hence, the Von-Neumann entropy will be,

$$\begin{aligned} H(Y|\{\Pi_i\}) &= \sum_i p_i H(\rho_{Y|\Pi_i}), \\ \therefore H(\rho_{Y|X}) &= \min_{\{\Pi_i\}} H(Y|\{\Pi_i\}). \end{aligned}$$

Hence, the second quantity for quantum mutual information is:

$$J_Q\left(\rho_{Y:X}\right) = H\left(\rho_Y\right) - H\left(\rho_Y \mid \rho_X\right),$$

And generally, these two quantities are different, i.e., $I_Q(\rho_{Y:X}) \neq J_Q(\rho_{Y:X})$. Therefore, quantum discord is defined as:

$$\boxed{D_Q(\rho_{XY}) = I_Q(\rho_{Y:X}) - J_Q(\rho_{Y:X}).}$$

It has been shown that in general, $D_Q(\rho_{XY}) \geq 0$, and states for which $D_Q(\rho_{XY}) = 0$, we call them zero-discord states.

3 Why do we care about Quantum Discord?

It has been shown[2] that there exist separable states, such that $D_Q(\rho_{Y:X}) > 0$, which is a shocking result because separable states are by definition non-entangled states. The way you construct them is:

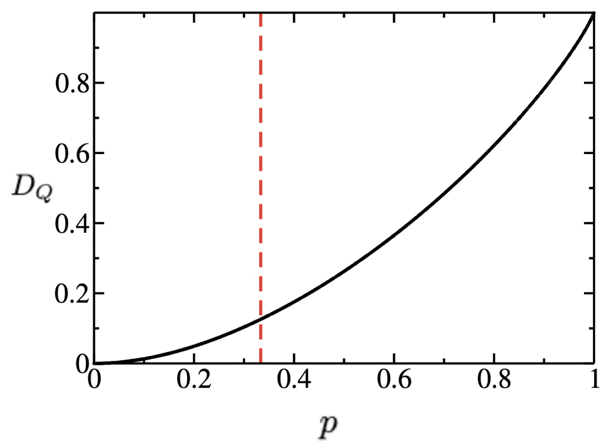
$$\rho_{AE} = \sum_i p_i \rho_A^i \otimes \rho_E^i,$$

You can see that it is a tensor product, so there is no entanglement, and then you just add them up with the weightage of classical probabilities $\{p_i\}$. It seems that this state should only have classical correlations between system A & E and no quantum correlations. However, the result shows that there is something quantum about these kinds of states, and that is captured by non-zero discord. If these states only had classical correlations, then two ways of computing mutual information should've yielded the same result, thus getting zero quantum discord quantity. But there is some difference between the two ways of calculating mutual information, and that difference is due to some additional quantum property that they have.

While quantum discord and entanglement are the same for pure states, it differs for mixed states. Let's see an example of Werner states[2][3], given by:

$$\rho_W = \frac{1-p}{4}\mathbb{I}_2 \otimes \mathbb{I}_2 + p\left|\psi^-\right\rangle\left\langle\psi^-\right|,$$

where $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. The state is separable when $p \leq \frac{1}{3}$. However, $D_Q(\rho_W) > 0$, in the entire range of p except at $p = 0$. Let's see how D_Q varies with p.



The red line at $p = \frac{1}{3}$, shows that the states to its right have entanglements and the states to its left have no entanglement. We can clearly see that Quantum Discord is still non-zero to the left of the red line.

People now try to measure the performance of quantum algorithms in terms of this measure, Quantum Discord, which is very different from entanglement. Entanglement doesn't account for discord because separable states have 0 entanglements. Something beyond entanglement, some residual quantum correlation exists for states like this.

Now, coming back to our attempt in section 1 to relax the assumption of the initial state, it turns out that zero-discord discord states do allow us to generalize the assumption of a factorizable initial state.

Example of zero-discord state: A special class of separable states does have zero discord, and we can write them in the form

$$\rho_{AE} = \sum_i p_i \Pi_A^i \otimes \rho_E^i.$$

So, we know that zero-discord states give rise to CP maps, but there are also discordant states which give rise to CP maps[4]. But not only that, even some entangled states can do so[5]. The main idea is the concept of Information backflow from the environment, which can be quantified using the quantum data processing inequality[6]. A CP map is obtained, even in the presence of initial correlations between the system and the environment, iff such correlations do not allow any backward flow of information from the environment to the system.

4 Properties of Quantum Discord

Now, let's see some properties of Quantum Discord:

1. $D_Q(\rho_{XY}) \geq 0$, since $I_Q(\rho_{Y:X}) \geq J_Q(\rho_{Y:X})$.

2. Quantum discord is not symmetric. Measuring system X and measuring system Y, will generally result in different quantities.

3. Quantum discord is invariant under local unitary transformations.

$$D_Q(\rho_{XY}) = D_Q((U_X \otimes U_Y)\rho_{XY}(U_X \otimes U_Y)^\dagger),$$

   where U_X and U_Y are arbitrary unitaries on subsystems X & Y.

4. Quantum Discord is zero iff there exists a local measurement on X that does not disturb the quantum system.

5. For a bipartite pure state, quantum discord reduces to entanglement, i.e. Von-Neumann entropy of the local density matrix.[6]

References

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