Market Microstructure and Algorithmic Trading

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Lecture notes – HT 2024

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1 Introduction

Recent regulatory changes due to Reg NMS and MiFID along with the automation of algorithmic trading shifted the focus of researchers and practitioners towards liquidity and market microstructure problems. New strands of the mathematical finance literature now focus on new topics such as order execution, market making, systemic risk, and counterparty risk from a microstructure perspective.

Electronic markets are mainly organized around Limit Order Books (LOBs) and Over-The-Counter (OTC) markets to exchange securities between market participants. This course introduces the literature on optimal trading in high frequency markets with a focus on optimal order execution in LOBs and optimal market making in OTC markets. Optimal trading problems share a common structure that will be followed throughout this course. In each section, we (i) identify a decision problem motivated by practical situations faced by market operators, we (ii) propose a parsimonious model which summarises the environment in key variables that must be considered, we (iii) frame the decision problem as an optimisation problem which can be addressed using classical mathematical tools, and finally we (iv) obtain a solution (often in closed-form) which we study through simulations and discussions.

Next, we describe the mechanisms of LOBs and OTC markets. LOBs allow traders to submit different types of orders that indicate the price, the volume, and the intention to buy or sell a security, and they rely on financial institutions and trusted third-parties that facilitate trading by collecting orders and matching buyers and sellers. OTC markets are based on a network of financial institutions and market makers that constantly stream quotes at which they are ready to buy and sell securities.

1.1 Limit order books

Limit order books concentrate most of the trading activity in the stock market. Here, we recall the main mechanisms behind LOBs which motivate many of our modelling assumptions in the following sections. The LOB allows to match buyers with sellers. Both can send two types of orders; *limit orders* (LOs) and *market orders* (MOs). Here, we focus on price-time-priority LOBs, which are the most popular.

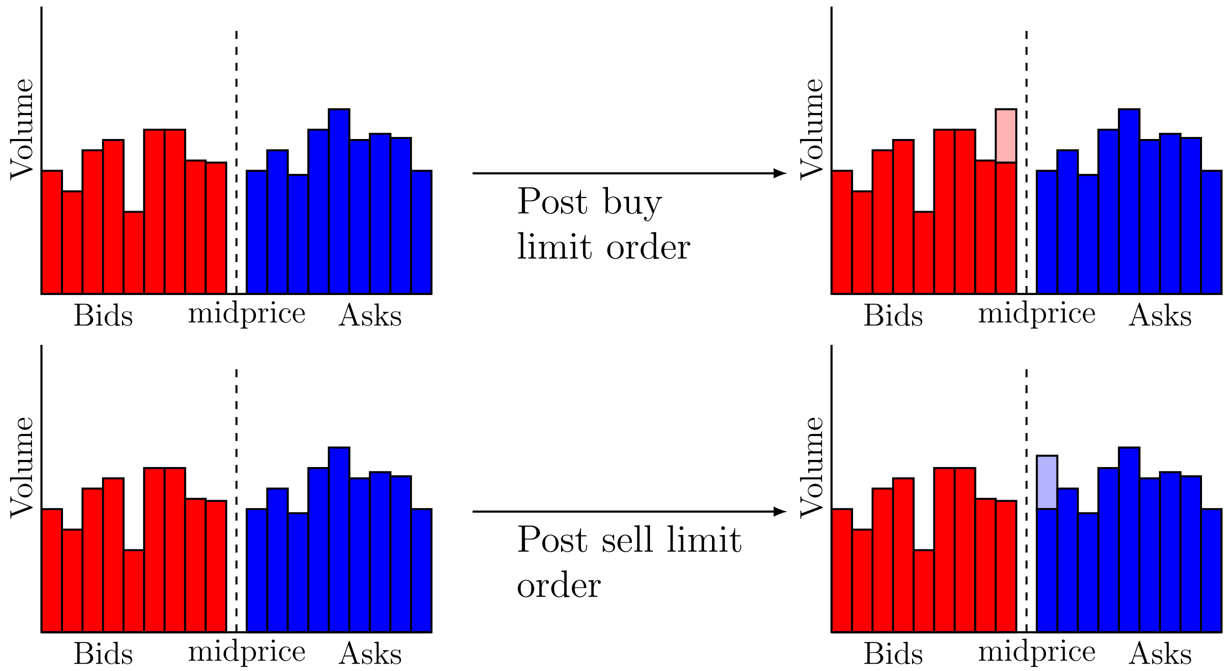

Figure 1: Changes in the LOB after a buy / sell LO.

An LO is composed of a price level, a volume, and an indicator to buy or to sell the asset. Once an LO is *posted* by a trader, it *rests* (or sits) in the LOB until it is matched against an MO of the opposite side, or it is cancelled by the agent that posted it. If the LO is a buy LO, then it sits on the *bid* side of the LOB, and if it is a sell LO, then it sits on the *ask* side; see Figure 1 which represents an LOB before and after a buy or sell LO is posted. Limit orders are called *passive* orders because they do not consume liquidity immediately. The cancellation or filling of an LO that is resting in the LOB can be either full (for the entire posted quantity) or partial; see Figure 2.

Traders can post multiple LOs at different price levels (see Figure 3). When two LOs are posted on the same side of the book and at the same price level, then the order of execution in an LOB with price-time priority depends on their position in the queue. Time priority in LOBs means that the queues are First-In-First-Out (FIFO) queues. More precisely, when two LOs are sitting on the same side of the LOB at the same price level, then the LO that was posted first is executed before. Price priority in LOBs means that if two buy (sell) LOs are sitting on the bid (ask) side of the LOB at different price levels, then the LO with the highest (lowest) price is executed first.

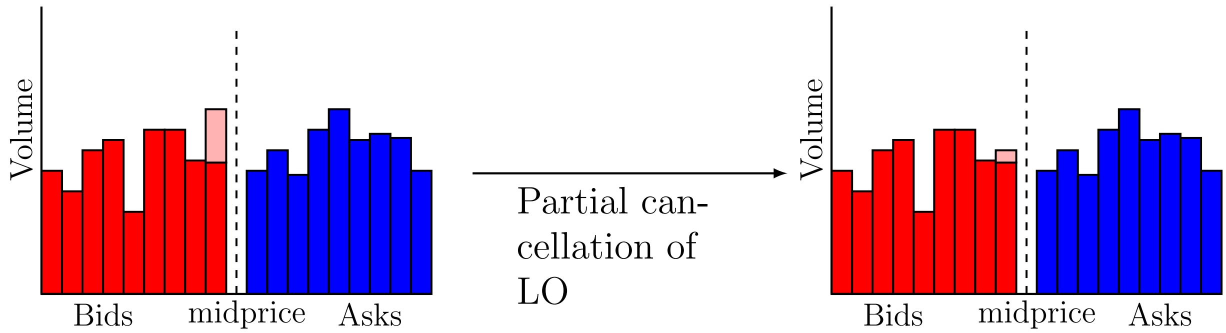

Figure 2: Changes in the LOB after a cancellation.

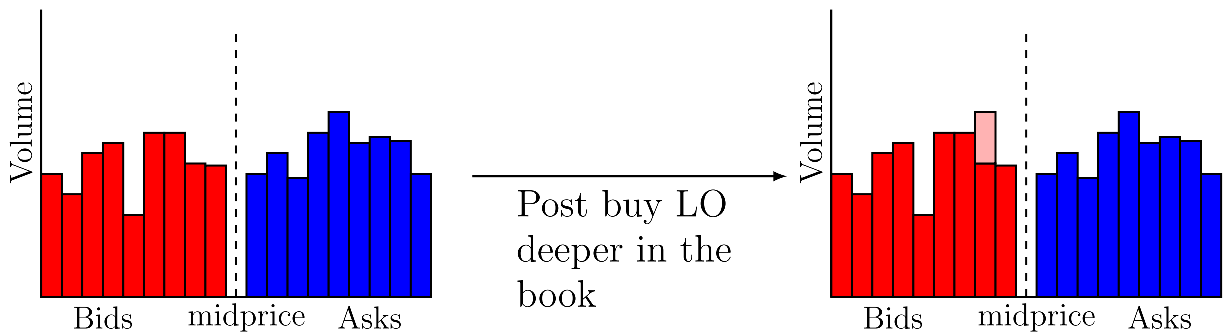

Figure 3: Changes in the LOB after a buy LO at the second best level.

The state of the LOB at a specific time is described by the size of the LO queues and their price levels. There are two price levels which correspond to the highest price across buy LOs and the lowest price across sell LOs. These are referred to as the *best bid* and the *best ask*, respectively. The average of the best bid and best ask is called the *midprice*; See Figure 1.

An MO consists of a volume and an indicator to buy and sell the security. When an MO is submitted by a trader, it is matched against opposite LOs that are resting in the LOB; see Figure 4 which shows the state of the LOB before and after a buy MO has been submitted.

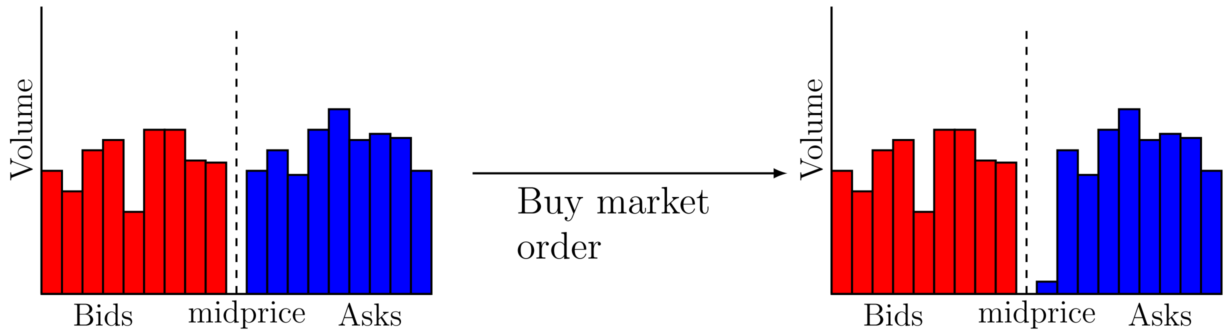

Figure 4: Changes in the LOB after a buy MO.

1.2 Market impact

The effect that MOs have on prices can be described in several ways. The literature on market impact generally splits the impact of MOs on prices into a **temporary** and a **permanent** impact. However, these two components of market impact are not completely dissociated, and more realistic interpretations can be studied and modelled; see Section 8 for an example.

Temporary market impact – Execution costs. When an MO is submitted, it is matched against resting LOs on the opposite side of the book. If the size of the MO is larger than the size of the queue at the best available price, then the first queue is depleted, and the MO is matched with LOs resting on the second best available price. This continues until the entire MO is filled. Figure 5 depicts how large MOs *walk the book* further when compared to smaller MOs.

The volume of an MO and the volumes and prices of resting LOs determine the overall transaction price of the MO. We refer to the difference between the midprice[1] and the execution price (per share) obtained by the agent as the *execution costs*, or the *instantaneous market impact*, or the *temporary market impact.*

[1]One may choose another reference price like the best ask or best bid.

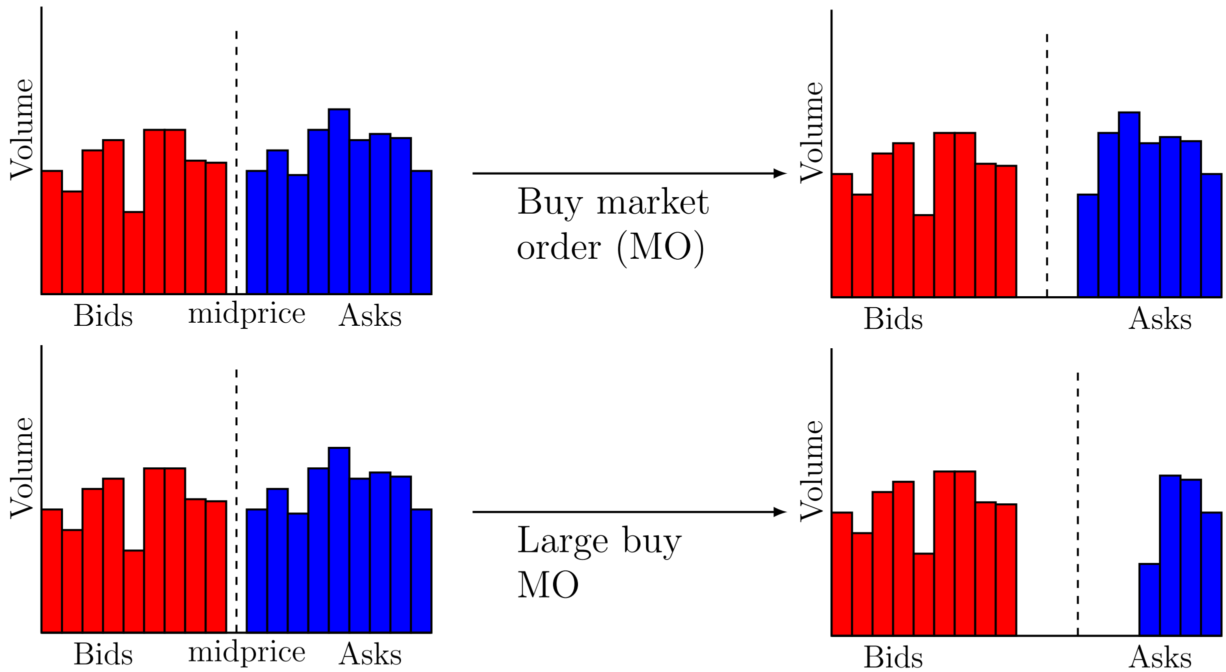

Figure 5: Impact of small and large MOs on the limit order book.

Provided with a snapshot of the LOB, i.e., the state of the LOB at some given time, traders can compute the execution price per share for various volumes of MOs as they walk the book. Thus, they can obtain an estimate of the execution costs as a function of the volume of an MO. Figure 6 shows the execution costs per share as a function of the volume at different times throughout a trading day for the share MSFT quoted on NYSE.

First, note that the minimum execution cost for small volumes is \$ 0.005 which corresponds to half the *tick size*. The tick size corresponds to the smallest price increment between two successive price levels in the LOB. Thus, the smallest *bid-ask spread*, i.e., the difference between the best bid and the best ask, is equal to 1 tick, which corresponds to \$ 0.01 for MSFT. In many cases throughout the trading day we consider in Figure 6, the bid-ask spread is larger than 1 tick. Thus, even for MOs with small sizes that do not deplete the best opposite queue, the trader must *cross the spread* and incur execution costs that correspond to half the bid-ask spread.

Also, note that the difference between the average execution price is approximately a linear function of the volume. Denote the midprice by S and the volume of a an MO by v. We can approximate the execution price per

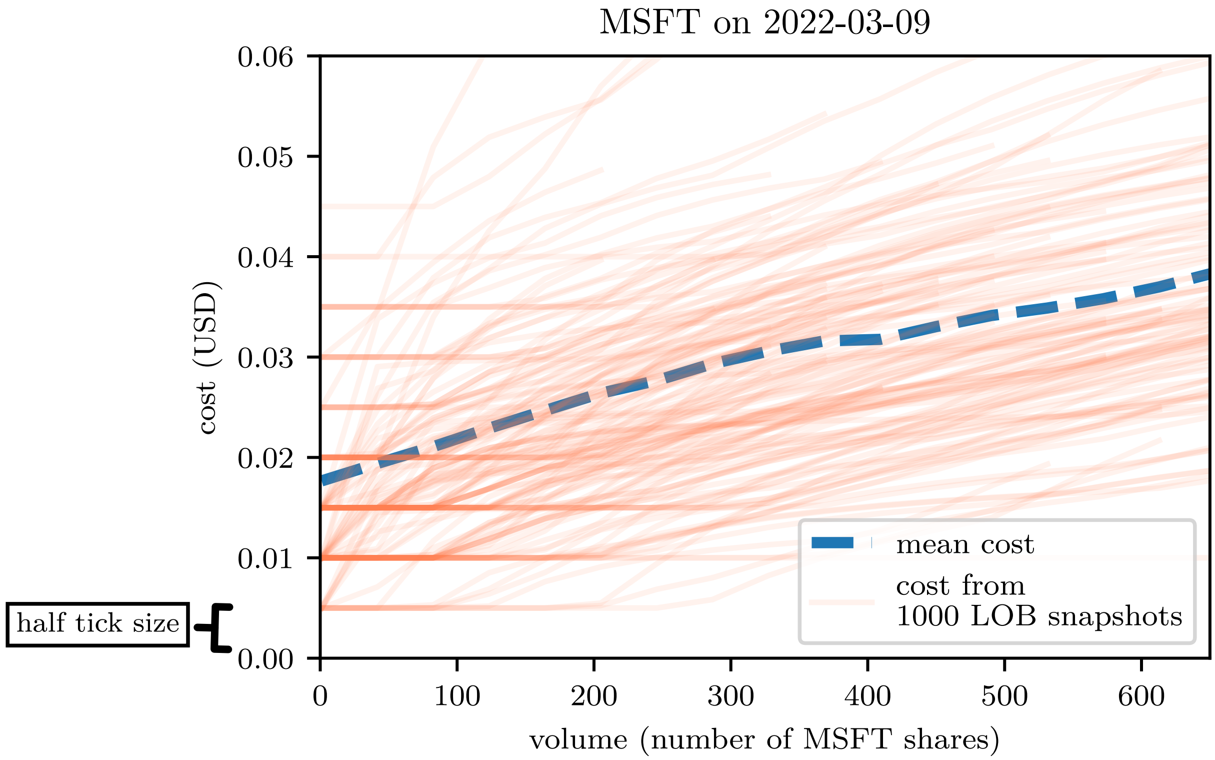

Figure 6: Execution costs defined as a function of trading volume for multiple snapshots of the LOB of MSFT quoted on Nasdaq. The execution costs are defined as the difference between the execution price per share for a given volume of share, and the midprice.

share with the linear form $S+\eta\, v$. We will make this modelling assumption in many of the models presented in the following sections. Part of the literature prefers to approximate execution costs with a concave power law, and we study this assumption in Section 6.

Permanent market impact. Permanent price impact refers to the relationship between the volume of an MO and the midprice at future times after the execution. Large MOs leave a lasting and long-term effect on the midprice. Large buy MOs move the price upward and large sell MOs move the price downward; see Figure 7. An interpretation of this modelling assumption is that market participants are trading based on information on the fundamental value of the firm, so their trading activity should reflect permanent change in the value of the share. The seminal work in Gatheral (2010) shows with simple arguments that a model with permanent impact that is not linear in the size of the MO leads to dynamic arbitrage, i.e., leads to strategies with zero risk and positive profits, which is an undesirable feature in an optimal trading model because these strategies do not exist in practice; see Section 4.4 for more details.

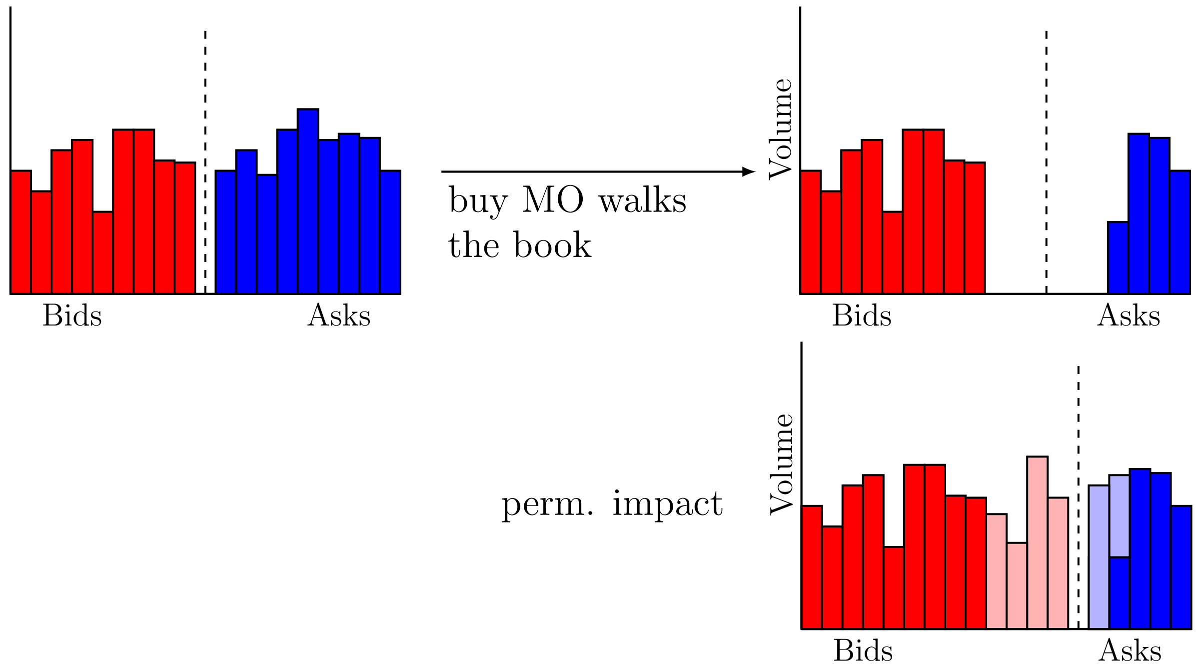

Figure 7: In the first two panels, an MO walks the book so the next midprice exhibits the temporary price impact. Immediately after the MO, market participants replenish the LOB. The difference between the midprice in the last panel and that of the first panel is the permanent impact.

1.3 Over-the-counter markets

There are two types of market participants in any trading venue; liquidity takers and liquidity providers. Liquidity takers are traders that need to buy or sell an asset and look for a counterparty to match their trade. Liquidity providers or market makers are counterparty to both buyers and sellers and profit from roundtrip trades (in LOBs, they earn the spread).

Most electronic exchanges clear the demand and supply of liquidity in LOBs, which are essentially trading venues for stocks. Alternatively, OTC markets are off-exchange "quote-driven" (or dealer-driven) markets that are based on a network of market makers that set prices at which liquidity takers can trade. These markets are the main exchanges for FX and Corporate Bonds securities.

OTC market makers faces a complex problem. They provide bid and ask quotes for various assets that exhibit complex joint dynamics without seeing the full depth of price and clients. Consequently, it is key to properly account for risks at the portfolio level. However, a large proportion of multi-asset market making models in the literature only consider correlated Brownian dynamics. Additionally, multi-asset market making is challenging due to high dimensionality and the resulting numerical challenges to obtain the optimal quotes.

1.4 Optimal execution

Market operators with large orders regularly come to the market for different reasons, e.g., financial firms that shifted from individual trading to centralised execution in dealing desks, banks that manage their liquidity risk through central risk books (CRB) at the firm level, agency brokers who act on behalf of pension funds, hedge funds, mutual funds, or sovereign funds, etc.

When the orders to be executed represent a significant portion of the overall traded volume, market operators cannot place all the desired quantity in a single order because either the available liquidity is insufficient, or the trade would adversely impact the price and be excessively costly (e.g., by walking the book). The market operator (agent) must slice the parent order into smaller (child) orders which are executed over a given period of time.

Slow execution of child orders exposes the agent to adverse price fluctuations, i.e., the price might increase when the agent needs to buy or the price might decrease when agent needs to sell. However, fast execution of child orders exposes the agent to higher execution costs due to crossing the spread and walking the book. It also impacts the price adversely due to the agent's own trading activity (permanent impact). A balance must be struck between trading fast to minimise market risk and trading slowly to minimise trading costs. This classical problem gave rise to an extensive literature which focuses on the problem of optimal execution and scheduling of large orders.

The agent must formulate a model to decide how to optimally execute a large order. The models presented in this course, much like those in the literature, are solved for different modelling assumptions on the agent's aversion to risk, their aversion to holding inventory, the execution costs incurred when sending orders, the permanent impact of their trading activity, the type of orders they use, the dynamics of the midprice, etc.

1.5 Mathematical tools

This section provides a very brief overview of the mathematical tools needed in several of the algorithmic trading problems studied in the following sections.

1.5.1 Convex analysis

Sections 3 and 4 rely on results of convex analysis and convex duality. In this section, we recall the main results that are used. The proofs and further analysis can be found in the classical monographs on convex analysis; see e.g., Rockafellar (1997).

First, we introduce the central tool of Legendre-Fenchel transform in convex analysis.

Definition 1. *The Legendre-Fenchel transform of a convex function f is the function f^* defined by*

$$\tilde{f} : p \in \mathbb{R}^d \mapsto \sup_{x \in \mathbb{R}^d} p \cdot x - f(x).$$

The Legendre-Fenchel transform $\tilde{f}$ is a mapping from the graph of a function to the set of its tangents. $\tilde{f}$ is well defined when f is asymptotically super-linear[2]. It is also a convex function, and the Legendre-Fenchel of $\tilde{f}$ is f. Finally, when f is strictly convex, then $\tilde{f}$ is continuously differentiable.

We focus on a specific problem that arises when addressing optimal trading problems with convex trading costs, called Bolza problems; see Guéant (2016) for numerous optimal trading problems solved using this framework. A particular type of Bolza problems focuses on finding the minimisers of the

[2]A convex function $f : \mathbb{R}^d \mapsto \mathbb{R}$ is asymptotically super-linear when

$$\lim_{|x| \to \infty} \frac{f(x)}{|x|} = \infty.$$

function J:

$$J : x \in W^{1,1}((0,T), \mathbb{R}^d) \mapsto \int_0^T (f(x(t)) + g(x'(t)))\, dt, \tag{1}$$

over the set

$$\mathcal{C} = \left\{x \in W^{1,1}((0,T), \mathbb{R}^d), x(0) = a, x(T) = b\right\},$$

where $W^{1,1}(U)$ is the set of real-valued absolutely continuous functions on the open set U, and $d \geq 1$ is an integer.

We assume that $f(\cdot)$ and $g(\cdot)$ are classical convex functions that are measurable and bounded from below. The problem (1) is called the primal problem and we introduce the dual optimisation problem

$$I : p \in W^{1,1}((0,T), \mathbb{R}^d) \mapsto \int_0^T \left(\tilde{f}_t(p'(t)) + \tilde{g}_t(p(t))\right) dt + a \cdot p(0) - b \cdot p(T),$$

over the set of absolutely continuous functions.

Theorem 1. *If f and g are continuous convex functions and they are asymptotically super-linear, then there exists a minimiser x^* of J and a minimiser p^* of I.*

If f is differentiable, and g is strictly convex, then the minimiser can be characterised by the following Hamiltonian system of equations:

$$\begin{cases} {p^*}'(t) & = f'(x^*(t)), \\ {x^*}'(t) & = \tilde{g}'(p^*(t)), \\ x^*(0) & = a, \\ x^*(T) & = b. \end{cases}$$

Discrete time. In discrete time, we consider the functional $J : \mathbb{R}^{d\,N+d} \mapsto \mathbb{R}$ given by

$$J(x_0, \ldots, x_N) = \sum_{n=1}^{N-1} f\left(x_n\right) + \sum_{n=0}^{N-1} g\left(x_{n+1} - x_n\right),$$

where f and g are convex functions. Our goal is to find the minimisers of J over

$$\mathcal{C} = \left\{(x_0, \ldots, x_N) \in \mathbb{R}^{d(N+1)}, x_0 = a, x_N = b\right\},$$

where a and b are fixed. The dual problem is

$$I(p_0,\dots,p_{N-1})=\sum_{n=1}^{N-1}\tilde{f}_n\left(p_n-p_{n-1}\right)+\sum_{n=0}^{N-1}\tilde{g}_n\left(p_n\right)+a\cdot p_0-b\cdot p_{N-1}.$$

The discrete-time counterpart of Theorem 1 is:

Theorem 2. *If $x^*=(x_0^*,\dots,x_N^*)$ minimises J and $p^*=(p_0^*,\dots,p_{N-1}^*)$ minimises I, then they can be characterised by the following Hamiltonian system of equations:*

$$\begin{cases} p_n^*-p_{n-1}^*=f^{'}\left(x_n^*\right) & \forall n\in\{1,\dots,N-1\}\,,\\ x_{n+1}^*-x_n^*=\tilde{g}^{'}\left(p_n^*\right) & \forall n\in\{0,\dots,N-1\}\,. \end{cases}$$

1.5.2 Stochastic optimal control

This section recalls the dynamic programming principle for jump-diffusion processes; see the monograph Pham (2009) for more details on stochastic optimal control and the book Cartea et al. (2015) for various optimal trading models solved using the tools of stochastic optimal control.

In dynamic optimisation problems, an agent seeks to maximise a reward over some time window $[0,T]$. The agent takes actions $\mathbf{u}$ that affect the dynamics of some underlying system $\mathbf{X}^{\mathbf{u}}$; the superscript indicates that $\mathbf{X}$ is affected (controlled) by $\mathbf{u}$. Often, the actions of the agent incur costs (or rewards) over the time window, which can also depend on the time and the state of the system, so they must be accounted for. At each time t, the cumulative past actions of the agent affect the future dynamics of the system and future potential costs.

We consider a filtered probability space $\left(\Omega,\mathcal{F},\mathbb{P};\mathbb{F}=(\mathcal{F}_t)_{t\in[0,T]}\right)$ satisfying the usual conditions and which supports all the processes that we introduce, where $T>0$ is a fixed time horizon.

Consider an agent who is faced with a control problem of a system whose dynamics contain a diffusive and a jump component. Let $(\mathbf{N}_t^{\mathbf{u}})_{t\in[0,T]}$ denote a $p-$dimensional counting process with controlled intensity $(\lambda_t^{\mathbf{u}})_{t\in[0,T]}$ where $\lambda_t^{\mathbf{u}}=\lambda\left(t,\mathbf{X}_t^{\mathbf{u}},\mathbf{u}_t\right)$ and $(\mathbf{u}_t)_{t\in[0,T]}$ is an $m-$dimensional control process.[3] Also, let $(\mathbf{W}_t)_{t\in[0,T]}$ denote an $m-$dimensional standard Brownian motion.

[3] $\mathbf{N}^{\mathbf{u}}$ is called a doubly stochastic Poisson process because its intensity is itself stochastic, and in our case controlled. Recall that $\mathbf{N}_t^{\mathbf{u}}-\int_0^t\lambda_s^{\mathbf{u}}\,ds$ is a martingale.

Let $(\mathbf{X}^{\mathbf{u}}_t)_{t\in[0,T]}$ denote a controlled $m-$dimensional system with dynamics

$$d\mathbf{X}^{\mathbf{u}}_t = \mu\left(t,\mathbf{X}^{\mathbf{u}}_t,\mathbf{u}_t\right)\,dt + \mathbf{V}\left(t,\mathbf{X}^{\mathbf{u}}_t,\mathbf{u}_t\right)d\mathbf{W}_t + \gamma\left(t,\mathbf{X}^{\mathbf{u}}_t,\mathbf{u}_t\right)d\mathbf{N}^{\mathbf{u}}_t,\;\; \mathbf{X}^{\mathbf{u}}_0 = X_0, \tag{2}$$

where the $m-$dimensional vector of drifts μ, the $m\times m$ variance matrix $\mathbf{V}$, and the $m\times p-$dimensional jump matrix γ are Lipschitz continuous and X_0 is known.

The agent has a *performance criterion* they wish to maximise which takes the form

$$\boxed{\mathbb{E}\left[G(\mathbf{X}^{\mathbf{u}}_T) + \int_0^T F(s,\boldsymbol{X}^{\boldsymbol{u}}_s,\boldsymbol{u}_s)\,ds\right]}. \tag{3}$$

where $G:\mathbb{R}^m\mapsto\mathbb{R}$ is the *terminal reward* function and $F:[0,T]\times\mathbb{R}^m\times\mathbb{R}^p\mapsto\mathbb{R}$ is the *running reward* function. The functions G and F are assumed to be uniformly bounded.

The agent seeks to maximise the performance criterion (3), so we define their *value function* as:

$$\boxed{H(\mathbf{X}_0) = \sup_{\boldsymbol{u}\in\mathcal{A}}\mathbb{E}\left[G(\mathbf{X}^{\mathbf{u}}_T) + \int_0^T F(s,\boldsymbol{X}^{\boldsymbol{u}}_s,\boldsymbol{u}_s)\,ds\right],} \tag{4}$$

where $\mathcal{A}$ is the *admissible set* of strategies that the agent may use:

$$\mathcal{A} = \Big\{(\mathbf{u}_s)_{s\in[0,T]},\ \mathbb{R}^m\text{-valued},\ \mathbb{F}\text{-predictable, such that} \tag{5}$$
$$(\mathbf{X}^{\mathbf{u}}_s)_{s\in[0,T]}\text{ admits a strong solution}\Big\}.$$

The predictability assumption in (5) is necessary in financial decision problems because it ensures that the agent can only use strategies that do not use future information. Often in the following sections, we require the admissible set to include additional constraints to ensure that the problem is well defined and that a solution exists.

Instead of optimising H in (4), it is more convenient to consider a time-indexed succession of optimisation problems that explicitly take into account the feedback effect between the actions of the agent and their impact on the future dynamics and costs. More precisely, we embed the problem (4) into a class of problems indexed by time t:

$$\boxed{H(t,\mathbf{x}) = \sup_{\boldsymbol{u}\in\mathcal{A}_t} \mathbb{E}\left[G(\mathbf{X}_T^{\mathbf{x},\mathbf{u}}) + \int_t^T F(s, \boldsymbol{X}_s^{\mathbf{x},\boldsymbol{u}}, \boldsymbol{u}_s)\, ds\right],} \tag{6}$$

where the process $(\mathbf{X}_s^{\mathbf{x},\mathbf{u}})_{s\in[t,T]}$ follows the same dynamics as in (2) but starts with initial value $\mathbf{X}_t^{\mathbf{x},\mathbf{u}} = \mathbf{x}$, and where we define for all $t \in [0,T]$:

$$\mathcal{A}_t = \Big\{ (\mathbf{u}_s)_{s\in[t,T]},\ \mathbb{R}^m\text{-valued},\ \mathbb{F}\text{-predictable, such that}$$
$$(\mathbf{X}_s^{\mathbf{u}})_{s\in[t,T]} \text{ admits a strong solution} \Big\},$$

and we set $\mathcal{A} = \mathcal{A}_0$, so $H(0,\mathbf{x}) = H(\mathbf{x})$.

Often in this course, we will drop the notation $\mathbf{X}^{\mathbf{x},\mathbf{u}}$, and write (6) as

$$\boxed{H(t,\mathbf{x}) = \sup_{\boldsymbol{u}\in\mathcal{A}_t} \mathbb{E}_{t,\mathbf{x}}\left[G(\mathbf{X}_T^{\mathbf{u}}) + \int_t^T F(s, \boldsymbol{X}_s^{\boldsymbol{u}}, \boldsymbol{u}_s)\, ds\right],}$$

where $\mathbb{E}_{t,x}$ represents the expectation conditional on $\mathbf{X}_t^{\mathbf{u}} = \mathbf{x}$.

The following classical result (see Pham (2009)) shows that the value function satisfies the Dynamic Programming Principle (DPP).

Theorem 1. *The value function (6) satisfies the DPP*

$$\boxed{H(t,\boldsymbol{x}) = \sup_{\boldsymbol{u}\in\mathcal{A}} \mathbb{E}\left[H(T, \boldsymbol{X}_T^{\mathbf{x},\boldsymbol{u}}) + \int_t^T F(s, \boldsymbol{X}_s^{\mathbf{x},\boldsymbol{u}}, \boldsymbol{u}_s)\, ds\right],}$$

for all $t \in [0,T]$ *and* $\mathbf{x} \in \mathbb{R}^m$.

The DPP connects the value function to its future expected value, regularised by the expected reward or penalty F. In its infinitesimal version, the DPP gives the Dynamic Programming Equation (DPE), or the Hamilton-Jacobi-Bellman (HJB) equation

$$\boxed{\begin{array}{c} \partial_t H(t,\boldsymbol{x}) + \sup_{\boldsymbol{u}\in\mathcal{A}}(\mathcal{L}_t^{\boldsymbol{u}} H(t,\boldsymbol{x}) + F(t,\boldsymbol{x},\boldsymbol{u})) = 0 \\[2ex] \text{subject to the terminal condition} \quad H(T,\boldsymbol{x}) = G(\boldsymbol{x}), \end{array}}$$

where $\mathcal{L}_t^{\mathbf{u}}$ is the infinitesimal generator of the process $\mathbf{X}_t^{\mathbf{x},\mathbf{u}}$.

Sketch of the proof The dynamic programming principle gives

$$H(t,\boldsymbol{x}) = \sup_{u} \left\{ \mathbb{E}\left[H(t+h, \boldsymbol{X}^{u}_{t+h}) + \int_{t}^{t+h} F(s, \boldsymbol{X}^{u}_{s}, \boldsymbol{u}_s)\, ds \,\middle|\, \boldsymbol{X}_t = \boldsymbol{x} \right] \right\} , \tag{7}$$

for any $h \in (0, T-t)$.

By Itô's formula we have

$$\begin{aligned} & H\left(t+h, \boldsymbol{X}^{\boldsymbol{u}}_{t+h}\right) - H(t,\boldsymbol{x}) \\ = & \int_{t}^{t+h} \left(\partial_t + \mathcal{L}^{\boldsymbol{u}}_t\right)[H](s, \boldsymbol{X}^{\boldsymbol{u}}_{s}) ds + \int_{t}^{t+h} \sigma(s, \boldsymbol{X}^{u}_{s}, \boldsymbol{u}_s) \nabla_{\boldsymbol{x}} H(s, \boldsymbol{X}^{u}_{s})\, dW_s \,, \end{aligned}$$

where $\mathcal{L}^{\boldsymbol{u}}_t$ is the infinitesimal generator of $\boldsymbol{X}^{u}$.

Replace $H\left(t+h, \boldsymbol{X}^{\boldsymbol{u}}_{t+h}\right)$ in (7), divide by h on both sides of (7), and send h to 0, to obtain the HJB

$$\partial_t H(t,\boldsymbol{x}) + \sup_{u}(\mathcal{L}^{u}_t H(t,\boldsymbol{x}) + F(t,\boldsymbol{x},\boldsymbol{u})) = 0.$$

□

For the diffusion-jump process in (2), the infinitesimal generator acts on functions H as follows:

$$\boxed{\begin{aligned} \mathcal{L}^{\boldsymbol{u}}_t H(t,\boldsymbol{x}) \;=\; & \boldsymbol{\mu}(t,\boldsymbol{x},\boldsymbol{u}) \cdot \nabla_x H(t,\boldsymbol{x}) + \tfrac{1}{2}\mathrm{Tr}\left(\boldsymbol{\Sigma}(t,\boldsymbol{x},\boldsymbol{u}) D^2_{xx} H(t,\boldsymbol{x})\right) \\ & + \textstyle\sum_{j=1}^{p} \lambda_j(t,\boldsymbol{x},\boldsymbol{u}) \left[H(t,\boldsymbol{x} + \boldsymbol{\gamma}_{:,j}(t,\boldsymbol{x},\boldsymbol{u})) - H(t,\boldsymbol{x})\right], \end{aligned}} \tag{8}$$

where $\boldsymbol{\Sigma}(t,\boldsymbol{x},\boldsymbol{u}) = \boldsymbol{V}(t,\boldsymbol{x},\boldsymbol{u})\boldsymbol{V}(t,\boldsymbol{x},\boldsymbol{u})^{\intercal}$ is the covariance matrix, $D^2_{xx}H(t,\boldsymbol{x})$ is the Hessian matrix of H, and $\boldsymbol{\gamma}(t,\boldsymbol{x},\boldsymbol{u})_{:,j}$ is the jth column of the $m \times p$ matrix $\boldsymbol{\gamma}(t,\boldsymbol{x},\boldsymbol{u})$.

The first term in (8) represents the change in the value function due to the drift of $\mathbf{X}$, the second term represents the diffusion volatility, and the third term corresponds to the arrival of a jump in each component of $\mathbf{N}$. When a jump of the jth component of $\mathbf{N}$ occurs, the components of the system $\mathbf{X}$ only jump according to the jth column of $\boldsymbol{\gamma}(t,\boldsymbol{x},\boldsymbol{u})$.

Solving for the supremum term in the HJB provides the optimal control in *feedback form*, i.e., as a function of the value function. One usually solves the HJB equation, which is a nonlinear PDE. The solution the the PDE is only a *candidate solution* and one needs to prove that it is in fact the solution to the original control problem through a verification argument; see Pham (2009). When a classical solution to the HJB equation exists, i.e., it is $\mathcal{C}^1$ in time and $\mathcal{C}^2$ in x, and if the control is admissible, i.e., it is in $\mathcal{A}$, then by standard results, the solution to the HJB is indeed the value function we seek and the resulting control is an optimal Markov control. Finally, the results and discussions above hold for any terminal stopping time $\tau \leq T$.

2 Optimal routing

We start this course with a simple problem that illustrates the methodology to adopt when addressing an optimal trading problem. More precisely, we (i) identify a decision problem, we (ii) propose a parsimonious model of the environment through variables that describe the key quantities to consider, we (iii) frame the decision problem as an optimisation problem that can be solved with classical mathematical tools, and finally we (iv) obtain a solution that we can study and implement in practice.

2.1 Optimal routing of agressive orders

Identifying the decision problem. Liquidity in trading venues is limited. Often, operators need to split a large order over N available trading venues to obtain better execution prices and to reduce the costs of their trading activity.

Here, we study the problem of optimal routing of "*marketable*" orders in multiple trading venues. A marketable order is a buy (resp. sell) order at a price higher than the best ask (resp. lower than the best bid). A buy (sell) marketable order is defined by two variables $Q^\star$ and $P^\star$. $Q^\star$ is the quantity of the order and $P^\star$ is the maximum (minimum) price that the trader is willing to accept.

Modelling framework. Let $(Q^\star, P^\star)$ be a marketable buy order; the analysis for sell orders is identical. Let $Q_n(p)$ be the visible quantity that is available at price p in the $n-$th trading venue when $n \in \{1, \ldots, N\}$.

Optimisation problem. The agent splits the parent order $(Q^\star, P^\star)$ into N child orders $(Q_n(p_n), p_n)$ to send to the N trading venues. The agent wishes to choose $(p_1, \cdots, p_n)$ to minimise the overall execution price

$$\sum_{n=1}^{N} p_n \cdot Q_n(p_n),$$

and ensure that the parent order is fully executed, i.e., we require that

$$Q^\star = \sum_{n=1}^{N} Q_n(p_n).$$

Recall that $P^\star$ is the maximum price that the agent is willing to pay, so we need the additional constraint that $P^\star \geq p_n$ for all $n \in \{1, \cdots, N\}$.

Solution. The above problem is a classical constrained optimisation problem. We look for the stationary points of the Lagrangian function and we write

$$Q_n(p_n) + p_n\, Q_n^{'}(p_n) = \lambda\, Q_n^{'}(p_n), \qquad \text{for } n \in \{1, \cdots, N\},$$

where λ is the Lagrange multiplier.

Next, we assume the linear form

$$Q_n(p) = q_n + c_n \cdot p.$$

Thus we obtain

$$(q_n + c_n \cdot p_n^\star) + p_n^\star\, c_n = \lambda\, c_n \implies \qquad\qquad p_n^\star = \frac{\lambda}{2} - \frac{q_n}{2\, c_n}\,.$$

Substitute λ in the constraint $Q^\star = \sum_{n=1}^{N} Q_n(p_n)$ to obtain

$$Q^\star = \sum_{n=1}^{N} \{q_n + c_n \cdot p_n^\star\} = \sum_{n=1}^{N} q_n/2 + c\,\lambda/2, \quad c = \sum_n c_n.$$

Finally, the optimal prices to target in each trading venue are given by

$$\boxed{p_n^\star = \frac{Q^\star}{c} - \frac{q_n}{2\, c_n}\left(1 + \frac{c_n}{q_n} \cdot \frac{\overline{q}}{\overline{c}}\right), \quad \overline{c} = \frac{1}{N}\sum_n c, \;\; \overline{q} = \frac{1}{N}\sum_n q}$$

The problem above is simple, yet it contains all the ingredients of optimal trading. In practice, agents take into account more practical issues of financial environments such as latency in high frequency markets, cancellations in limit order books (randomness of $Q_n(p)$), time periods throughout which they spread their trading activity, hidden and Iceberg orders, etc. Each issue can be modelled with additional variables and assumptions and complicates the model and the solution. For example, the next section consider the optimal routing of a large limit order instead of a marketable order.

2.2 Optimal routing of limit orders

Identifying the decision problem. Crossing the spread is the first source of execution costs. Thus, agents mostly trade with limit orders. However, there is structural uncertainty when using limit orders because of price and time priority; waiting on a bad queue generates **opportunity costs**. Here, we consider the problem of an agent that can trade in N different LOBs and wishes to split a large buy limit order.

Modelling framework. Let Q_n denote the size of the best bid queue in the $n-$th LOB. If the agent posts a buy limit order at the best bid price, then they must wait for the Q_n previous LOs to be filled before their LO is filled. The agent assumes that the best bid queue is consumed according to a Poisson P^n_t with intensity λ_n. This intensity can be estimated using historical data.

Optimisation problem. The objective of the agent is to split the parent order into N child LOs with quantities $(q_1, \cdots, q_N)$. The agent wishes to minimize, *on average*, the time $t^\star$ that they need to execute the quantity $Q^\star = \sum_n q_n$.

Solution. In each trading venue $n \in \{1, \ldots, N\}$, after the LO of size q_n is posted, the size of the best bid queue is $Q_n + q_n$. We denote by t_n the time needed for the queue to be totally consumed. We write

$$\int_0^{t^n} dP^n_t = q_n + Q_n \implies \mathbb{E}\left[P^n_{t^n}\right] = t_n\, \lambda_n = q_n + Q_n .$$

The agent seeks to minimise the maximum of $\{t^1, \ldots, t^N\}$, so we necessarily have that $t^\star = t^n$ for all $n \in \{1, \cdots, N\}$.

In particular, we write

$$t^\star = t_n = Q^\star / \sum_n \lambda_n + \sum_n Q_n / \sum_n \lambda_n \implies \boxed{q^\star_n = \rho_n \frac{Q^\star}{N} + \left(\rho_n\, \overline{Q} - Q\right),}$$

where

$$\rho_n = \lambda_n / \overline{\lambda}, \quad \overline{\lambda} = \frac{1}{N} \sum_n \lambda_n, \quad \text{and} \quad \overline{Q} = \frac{1}{N} \sum_n Q_n \,.$$

3 The Almgren-Chriss model in discrete time

Investors and market operators such as institutional traders, mutual funds, and brokers, regularly interact in financial markets to buy or sell large amounts of assets. If the quantity of an order represents a significant portion of the trading volume, then executing the order in one single trade is costly and often impossible. As a result, the optimal split of large blocks of assets (meta orders) into smaller trades (child orders) has become a classical problem in the quantitative finance literature and in practice.

The optimal execution literature formulates models to help agents control their overall trading costs. The results of these models are trading schedules that optimally balance between (i) trading slowly to minimise execution costs (measured as the difference between a reference price and the average price obtained for a trade) and adverse price movements (which are a consequence of their trading activity), and (ii) trading rapidly to minimise inventory and price risk.

The pioneer works in the optimal execution literature are Almgren and Chriss (1999) and Almgren and Chriss (2000). The original Almgren-Chriss model is a discrete-time model where an agent liquidates an initial inventory by posting market orders and maximises a mean-variance objective function. At present, almost all practitioners slice their large orders into child orders according to optimised trading schedules.

Various extensions and models for the optimal execution of large orders have been proposed in the last two decades; see Section 14. All these models share a common structure; first one formulates (and motivates) a dynamic model of the financial environment where the agent operates, second, one defines the control variables of the agent (the space of decisions), and finally, the agent solves an optimisation problem which results in an optimal behaviour (the optimal trading strategy).

The following sections introduce the different ingredients necessary to formulate the problem of optimal trading in the original Almgren-Chriss framework; namely inventory, execution costs, market impact, price dynamics, and performance criterion. The derivations are based on Almgren and Chriss (2000) and Guéant (2016).

3.1 Modelling framework

An agent holds Q_0 units of a single stock at time $t = 0$. The goal of the agent is to unwind (or liquidate) their **initial inventory** by time $T > 0$. We divide the time window (or trading window) $[0, T]$ into N slices of length Δt and we denote the subdivisions by

$$t_0 = 0 < \cdots < t_n = n\,\Delta t < \cdots < t_N = N\,\Delta t = T.$$

Inventory. At the start of each time interval $[t_n, t_{n+1}]$, the agent chooses the number of shares they buy or sell over the interval. We denote by $\nu_{n+1}\,\Delta t$ the amount of shares that the agent buys or sells over $[t_n, t_{n+1}]$. If $\nu_{n+1} \leq 0$ then the agent sells shares, if $\nu_{n+1} \geq 0$, then the agent buys shares.

Let Q_n denote the number shares (or **inventory**) in the agent's portfolio at time t_n. Thus, the agent's inventory evolves as

$$\boxed{Q_{n+1} = Q_n + \nu_{n+1}\,\Delta t, \quad \text{for } 0 \leq n < N.} \tag{9}$$

Execution costs. Let S_n denote the midprice of the stock at time t_n. For each trade of size $\nu_{n+1}\,\Delta t$ throughout $[t_n, t_{n+1}]$, the agent pays costs like the bid-ask spread and the cost of walking the book. Thus, the agent's trades are not executed at the mid-price S_n but at a less favourable price $\tilde{S}_n$. The difference between the midprice and the execution price is called the execution cost, or the temporary impact, or the instantaneous impact.

In the Almgren-Chriss model, the execution costs depend on the agent's trade size $\nu_n\,\Delta t$ and on trading activity of other agents. We introduce the deterministic **market volume** V_{n+1}, which is the volume traded by other agents throughout $[t_n, t_{n+1}]$. In practice, the market volume V_{n+1} is random. However, market activity depends on the time of day. On average, it is deterministic and has a characteristic U-shape; see Figure 8.

To model execution costs, we assume that the execution price per share received by the agent takes the linear form

$$\tilde{S}_{n+1} = S_n + \eta\,\nu_{n+1}/V_{n+1}.$$

The cost parameter η is positive so the agent buys (sells) at prices higher (lower) than the mid-price S_n. Thus, the amount paid (received) for $\nu_{n+1}\,\Delta t$ shares bought (sold) between t_n and t_{n+1} is

$$\nu_{n+1}\,\tilde{S}_{n+1}\,\Delta t = \nu_{n+1}\,(S_n + \eta\,\nu_{n+1}/V_{n+1})\,\Delta t.$$

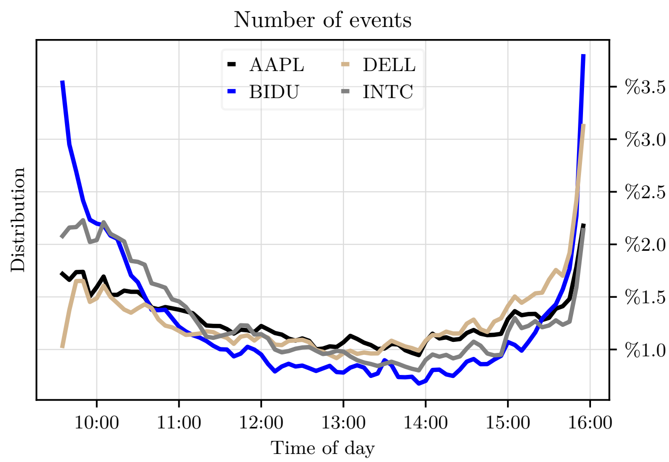

Figure 8: Distribution of trading volume throughout a trading day, measured in portion of LOB events, averaged through trading days between October and December 2022 for multiple shares quoted on Nasdaq. Source: Cartea et al. (2023).

Note that the same comments on the linear shape of the execution costs curve as a function of trading volume in Section 1.2 can be made on the execution costs curve as a function of participation rate ν_{n+1}/V_{n+1}; see Figure **??**

Let X_n denote the amount of cash on the agent's cash account at time t_n and let X_0 be the initial cash of the agent. The dynamics of the agent's cash account are

$$\boxed{X_{n+1} = X_n - \nu_{n+1}\, S_n\, \Delta t - \eta\, \frac{\nu_{n+1}^2}{V_{n+1}}\, \Delta t, \quad \text{for } 0 \le n < N.} \tag{10}$$

Permanent impact. We assume that the agent's trading activity has a permanent impact on the midprice that is relative to the size $\nu_{n+1}\, \Delta t$ of their trades. In particular, we assume that this impact is linear in the trading size and we assume that the dynamics of the midprice are

$$\boxed{S_{n+1} = S_n + \underbrace{\sigma\, \sqrt{\Delta t}\, \epsilon_{n+1}}_{\text{market risk}} + \underbrace{k\, \nu_{n+1}\, \Delta t}_{\text{linear perm. impact}}\,, \quad \text{for } 0 \le n < N,} \tag{11}$$

where $\{\epsilon_n\}$ are independent and identically distributed (i.i.d.) $\mathcal{N}(0,1)$ variables, $\sigma > 0$ is the arithmetic volatility of the midprice, and $k > 0$ scales the magnitude of the linear permanent impact. Later in Section 4.4, we show

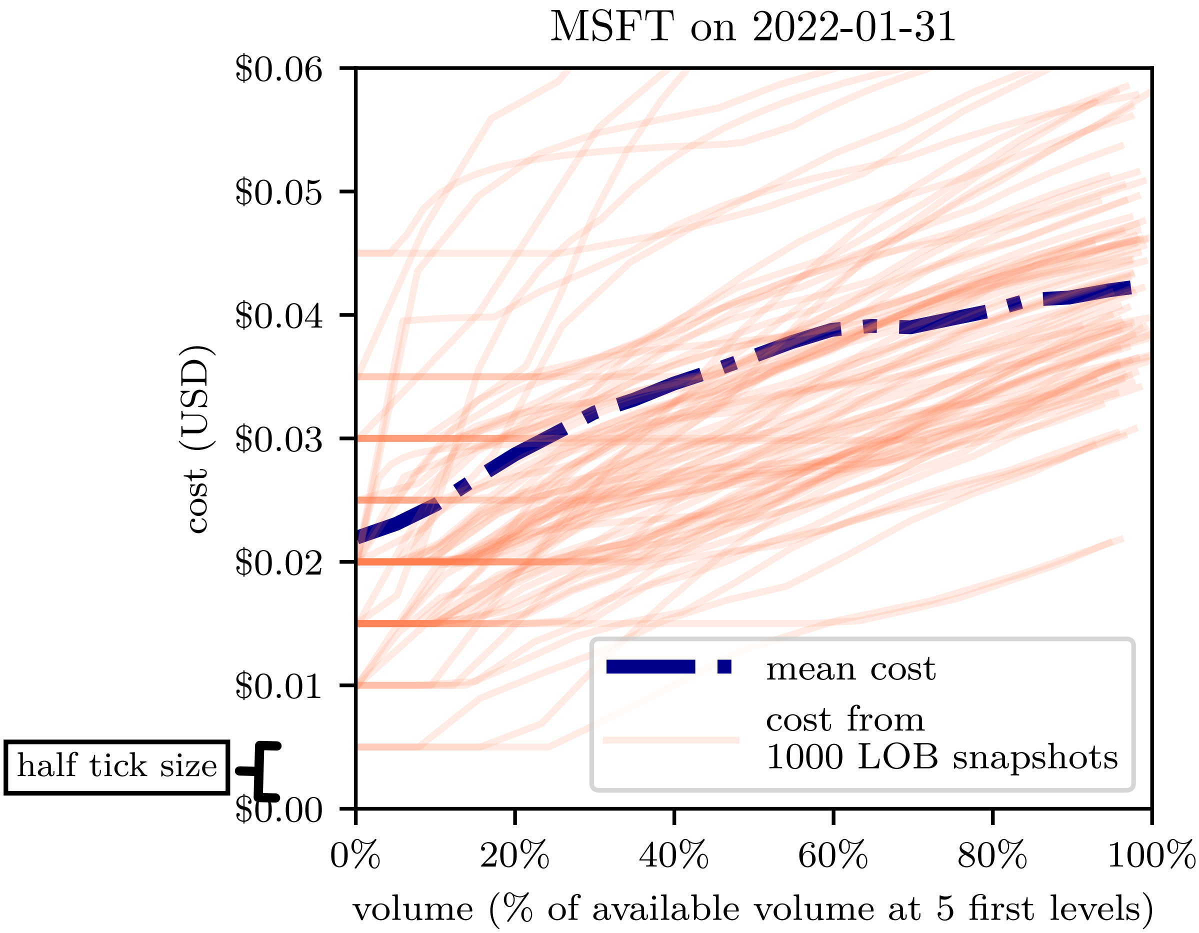

Figure 9: Execution costs defined as a function of participation rate for multiple snapshots of the LOB of MSFT quoted on Nasdaq. The total trade volume is approximated by the total available liquidity. The execution costs are defined as the difference between the execution price per share for a given volume of share, and the midprice.

that the permanent impact of the agent's trading activity must be linear to prevent *dynamic arbitrage*.

3.2 Performance criterion

In the previous section, we have modelled the dynamics of the key variables of the problem. Now, the agent must choose a performance criterion to optimise. Recall that the goal of the agent is to find the optimal **trading schedule** $\{\nu_1, \ldots, \nu_N\}$, so their performance criterion defines what optimality means.

This section follows the steps of Almgren and Chriss (2000) (see also Guéant (2016)). We seek a *liquidation strategy* $v = (\nu_1, \ldots, \nu_n)$ maximising the mean-variance objective function

$$\boxed{\mathbb{E}[X_N] - \frac{\gamma}{2}\,\mathbb{V}[X_N],} \tag{12}$$

where $\gamma > 0$ is the risk aversion parameter and it scales the magnitude of risk aversion in the performance of the agent. The higher the value of γ, the more the agent penalises market risk, i.e., holding the risky asset.

We focus on deterministic strategies that are **admissible**, i.e., they are in the set of admissible strategies $\mathcal{A}^{\text{det}}$:

$$\boxed{(\nu_n)_n \in \mathcal{A}^{\text{det}} = \left\{(\nu_1, \ldots, \nu_N) \in \mathbb{R}^n,\ \ \sum_{n=0}^{N-1} \nu_{n+1}\,\Delta t = -Q_0\right\}}.$$

A deterministic strategy is a strategy that only depends on model parameters and time. Thus, it can be computed at the beginning of the execution process, in particular, it does not depend on the evolution of the price.

3.3 Solution

To solve our problem, we first compute the terminal wealth X_N of the agent, and then compute the performance criterion (12). The following result provides a formula for the terminal wealth of the agent and their performance criterion.

Proposition 1. *The terminal wealth of the agent is*

$$\begin{aligned} X_N =& X_0 + Q_0\,S_0 - \frac{k}{2} Q_0^2 + \sigma\,\sqrt{\Delta t}\sum_{n=0}^{N-1} Q_{n+1}\,\epsilon_{n+1} \\ &- \sum_{n=0}^{N-1} \nu_{n+1}^2 \left(\frac{\eta - \frac{k}{2}\,V_{n+1}\,\Delta t}{V_{n+1}}\right)\Delta t\,, \end{aligned}$$

So the mean and variance of the terminal wealth are

$$\boxed{\begin{cases} \mathbb{E}\left[X_N\right] & = X_0 + Q_0\, S_0 - \frac{k}{2} Q_0^2 - \sum_{n=0}^{N-1} \nu_{n+1}^2 \, \left(\frac{\eta - \frac{k}{2}\, V_{n+1}\, \Delta t}{V_{n+1}}\right) \Delta t, \\ & \\ \mathbb{V}\left[X_N\right] & = \sigma^2\, \Delta t \, \sum_{n=0}^{N-1} Q_{n+1}^2 \, . \end{cases}} \tag{13}$$

Proof. Use the dynamics of the inventory in (9), those of the midprice in (11), and those of the cash in (10) to write

$$\begin{aligned} X_N =& X_0 - \sum_{n=0}^{N-1} \left(Q_{n+1} - Q_n\right) S_n - \eta \sum_{n=0}^{N-1} \frac{\nu_{n+1}^2}{V_{n+1}} \Delta t \\ =& X_0 - \sum_{n=0}^{N-1} Q_{n+1} S_n + \sum_{n=0}^{N-1} Q_n S_n - \eta \sum_{n=0}^{N-1} \frac{\nu_{n+1}^2}{V_{n+1}} \Delta t \\ =& X_0 - \sum_{n=0}^{N-1} Q_{n+1} \left(S_{n+1} - \sigma \sqrt{\Delta t} \epsilon_{n+1} - k \nu_{n+1} \Delta t\right) \\ & + \sum_{n=0}^{N-1} Q_n S_n - \eta \sum_{n=0}^{N-1} \frac{\nu_{n+1}^2}{V_{n+1}} \Delta t \\ =& X_0 + Q_0 S_0 + \sigma \sqrt{\Delta t} \sum_{n=0}^{N-1} Q_{n+1} \epsilon_{n+1} \\ & + k \sum_{n=0}^{N-1} Q_{n+1} \nu_{n+1} \Delta t - \eta \sum_{n=0}^{N-1} \frac{\nu_{n+1}^2}{V_{n+1}} \Delta t. \end{aligned}$$

To obtain the desired result, write the term $k \sum_{n=0}^{N-1} Q_{n+1} \nu_{n+1} \Delta t$ as

$$\begin{aligned} k \sum_{n=0}^{N-1} Q_{n+1} \nu_{n+1} \Delta t &= k \sum_{n=0}^{N-1} Q_{n+1} \left(Q_{n+1} - Q_n\right) \\ &= k \sum_{n=0}^{N-1} \left(\frac{Q_{n+1} + Q_n}{2} + \frac{Q_{n+1} - Q_n}{2}\right) \left(Q_{n+1} - Q_n\right) \\ &= \frac{k}{2} \sum_{n=0}^{N-1} \left(Q_{n+1}^2 - Q_n^2\right) + \frac{k}{2} \sum_{n=0}^{N-1} \left(Q_{n+1} - Q_n\right)^2 \end{aligned}$$

$$= -\frac{k}{2} Q_0^2 + \frac{k}{2} \sum_{n=0}^{N-1} \nu_{n+1}^2 \Delta t^2.$$

We only consider deterministic controls ν_n, so the final value of the cash process is normally distributed with mean and variance in (13). $\square$

The first term $X_0 + Q_0 \, S_0$ in (13) is the marked-to-market (MtM) value of the agent's wealth at time $t = 0$. The second term corresponds to costs (or to a discount term) originating from the permanent impact. This term does not depend on the specific liquidation strategy followed by the agent, so these costs are unavoidable. The last term corresponds to running execution costs throughout the trading window. In contrast to permanent impact costs, the running execution costs depend on the liquidation strategy.

To obtain an analytical solution, we simplify our model to consider a flat market volume curve and we write $V_n = V$ for all n.[4] In that case, minimising the mean-variance objective (12) reduces to minimising the function $\mathfrak{J}$ over $\mathcal{A}^{\text{det}}$ where

$$\mathfrak{J} : \nu \in \mathbb{R}^n \mapsto \tilde{\eta} \sum_{n=0}^{N-1} \frac{\nu_{n+1}^2}{V} \Delta t + \frac{\gamma}{2} \sigma^2 \, \Delta t \sum_{n=0}^{N-1} Q_{n+1}^2 \, ,$$

and

$$\tilde{\eta} = \eta - \frac{k}{2} V \, \Delta t.$$

The above problem minimises $\mathfrak{J}$ over the liquidation strategies $\nu \in \mathcal{A}^{\text{det}}$. Observe that this problem is equivalent to minimising the functional J over the **trading curves** $Q \in \mathcal{C}^{\text{det}}$ where

$$\boxed{J : q \in \mathbb{R}^{N+1} \mapsto \tilde{\eta} \sum_{n=0}^{N-1} \frac{\left(Q_{n+1} - Q_n\right)^2}{V \, \Delta t} + \frac{\gamma}{2} \sigma^2 \, \Delta t \sum_{n=0}^{N-1} Q_{n+1}^2,}$$

and

$$\mathcal{C}^{\text{det}} = \left\{ Q = \left(Q_0, \ldots, Q_N\right), \; Q_0 = Q_0, \; Q_N = 0 \right\}.$$

[4]In many instances in algorithmic trading models, one needs simplifying assumptions to obtain analytical and interpretable formulae.

The optimal liquidation strategy $\nu^\star \in \mathcal{A}^{\text{det}}$ is given by

$$\nu^\star = \frac{Q_n^\star - Q_{n-1}^\star}{\Delta t}, \quad \text{for } 0 < n \leq N,$$

where $Q^\star \in \mathcal{C}^{\text{det}}$ is the optimal trading curve.

Assumption 1. *We assume that $\tilde{\eta} > 0 \implies \eta > \frac{k}{2} V \, \Delta t$.*

Assumption 1 ensures that the problem is convex. It is not restrictive because $k\,\Delta t$ is small enough in practice and the terms in Δt^2 are generally dropped.

The functional J is strictly convex and $\mathcal{C}^{\text{det}}$ is convex, thus by standard results there exists a unique minimiser to J over $\mathcal{C}^{\text{det}}$. Let $Q^\star$ be this minimiser. The optimal trading curve $Q^\star$ is uniquely characterised by the following Hamiltonian system (see Theorem 2):

$$\begin{cases} p_{n+1} &= p_n + \gamma\,\sigma^2\,\Delta t\, Q^\star_{n+1}\,, \quad & 0 \leq n < N-1\,, \\ Q^\star_{n+1} &= Q^\star_n + \frac{V}{2\,\tilde{\eta}}\,\Delta t\, p_n\,, & 0 \leq n < N\,, \end{cases}$$

with the boundary conditions $Q_0^\star = Q_0$ and $Q_N^\star = 0$.

Thus, $Q^\star$ is the solution the second-order recursive equation

$$Q^\star_{n+2} - \left(2 + \frac{\gamma\,\sigma^2\,V}{2\,\tilde{\eta}}\Delta t^2\right) Q^\star_{n+1} + Q^\star_n = 0\,,$$

which admits the solution

$$\boxed{Q_n^\star = Q_0\,\frac{\sinh\left(\alpha\,(T - t_n)\right)}{\sinh(\alpha T)}} \tag{14}$$

where α uniquely solves

$$2\,\cosh\left(\alpha\,\Delta t\right) = \frac{\gamma\,\sigma^2\,V}{2\,\tilde{\eta}}\Delta t^2\,.$$

3.4 Discussion

Here, we consider an example where the midprice has an arithmetic volatility $\sigma = 1\$ \cdot \text{day}^{-1/2}$ (approx. 32% annualized vol) and we set $S_0 = 100$. Assume the flat market volume is $V = 4,000,000$ shares per day, and we set

$\eta = 0.1\$ \cdot \text{share}^{-1}$. An agent holds an initial inventory $Q_0 = 200,000$ shares to liquidate, which corresponds to 5% participation rate. Figure 10 shows the optimal trading curve $Q^\star$ for multiple values of the risk aversion parameter γ, the volatility σ, the execution costs η, and the market trading volume V.

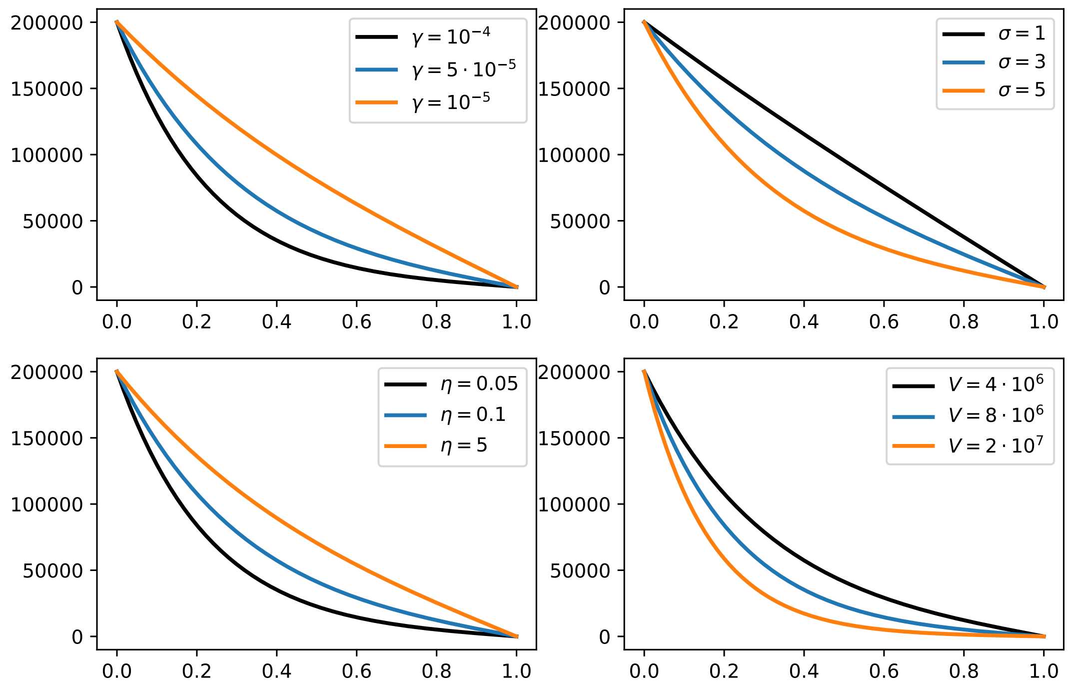

Figure 10: Optimal trading curves and impact of parameters.

Liquidity parameters. η (or $\tilde{\eta}$) and V are scaling factors for the execution costs $\eta\,\nu_{n+1}/V$ paid by the trader at time t_n. When execution costs are high, the agent trades slowly. The larger the value of η, the larger the execution costs; η is a proxy for the depth of the LOB and can be estimated with a linear regression using snapshots of the LOB; see Figure 6. The smaller the value of V, the larger the execution costs; the price of liquidity depends on the participation rate v_n/V. In particular, we can show that

$$\frac{dQ^\star}{d\eta}/Q_0 \geq 0 \quad \text{and} \quad \frac{dQ^\star}{dV}/Q_0 \leq 0.$$

In practice, in order to set the value of the execution cost / temporary market impact parameter η, one supposes that the additional cost incurred per share when trading a given volume $\nu_n \Delta t$ is proportional to the participation rate to the market (in practice we consider a flat volume curve that matches the average daily volume); see the discussions in Almgren and Chriss (2001). For example, for each $\mathfrak{p}\%$ of participation rate, one assumes a cost corresponding to half the bid-ask spread is incurred.[5] The value of $\mathfrak{p}$ can be obtained from historical data.

Risk parameters. The volatility σ measures price risk. The larger its value, the faster the agent needs to liquidate their inventory to reduce the exposure to price risk. In particular one shows that

$$\frac{dQ^\star}{d\sigma}/Q_0 \leq 0.$$

The risk aversion parameter γ determines the balance between maximising wealth (or equivalently minimising execution costs) and minimising price risk. The larger its value, the more the agent is sensitive to price risk, thus the agent trades faster; see Figure 10. Also, one shows that

$$\frac{dQ^\star}{d\gamma}/Q_0 \leq 0.$$

Finally, observe that when γ is very small, the optimal trading curve is a straight line:

$$\lim_{\gamma \to 0} Q^\star(t) = Q_0 \left(1 - t/T\right).$$

[5]The average bid-ask spread is close to the tick value when the asset is very liquid; see Figure 6.

4 The Almgren-Chriss model in continuous time

4.1 Modelling framework

We work on a filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}\right)$ satisfying the usual conditions, where $T > 0$. We assume this probability space to be large enough to support all the processes introduced in this section.

Inventory. This section introduces the continuous-time counterpart of the model of Section 3. At time $t = 0$, an agent holds an initial number Q_0 of shares that they wish to unwind by the terminal time $T > 0$. The trader's position over the trading window $[0, T]$ is modeled by the process $(Q_t)_{t\in[0,T]}$. The dynamics of this process are given by

$$\boxed{dQ_t = \nu_t \, dt,}$$

where $(\nu_t)_{t\in[0,T]}$ is a progressively measurable control process satisfying the unwind constraint

$$\int_0^T \nu_t \, dt = -Q_0.$$

The value ν_t at time t stands for the trading speed, i.e., the instantaneous trading volume of the agent.

Execution costs. The midprice of the stock is modelled by the (controlled) process $(S_t)_{t\in[0,T]}$. At time t, the agent trades a quantity $\nu_t \, dt$ of shares, and the price obtained by the agent for each share is $\tilde{S}_t = S_t + \eta \, \nu_t/V$ where V is the (constant) trading speed of other agents. Similar to the previous section, we consider a flat volume of trading in the market throughout the trading day. This assumption can be relaxed; see Section 7.5. We model the cash account of the agent by the process $(X_t)_{t\in[0,T]}$ with dynamics

$$\boxed{dX_t = -\nu_t \, \tilde{S}_t \, dt = -\nu_t \, (S_t + \eta \, \nu_t/V) \, dt.}$$

Market impact. The agent's trading activity permanently impacts the price. We assume this impact is linear; Section 4.4 shows that linear impact is necessary to prevent dynamic arbitrage. We write the dynamics of the midprice as

$$\boxed{dS_t = \sigma\, dW_t + k\, \nu_t\, dt,}$$

where $(W_t)_{t\in[0,T]}$ is a standard Brownian motion, $\sigma > 0$ is the arithmetic volatility parameter, and $k \geq 0$ scales the magnitude of the permanent market impact.

4.2 Performance criterion

Similar to the previous section, the goal of the agent is to find an optimal liquidation strategy $\nu \in \mathcal{A}$ that maximises the mean-variance criterion

$$\mathbb{E}[X_T] - \frac{\gamma}{2}\mathbb{V}[X_T],$$

where $\gamma > 0$ is the risk aversion parameter.

We denote by $\mathcal{A}$ the set of admissible strategies (or controls) and we write[6]

$$\boxed{\mathcal{A} = \left\{ (\nu_t)_{t\in[0,T]} \in \mathbb{H}^0\left(\mathbb{R}, (\mathcal{F}_t)_t\right), \int_0^T \nu_t\, dt = -Q_0, \int_0^T |\nu_t|\, dt \in L^\infty(\Omega) \right\}.}$$

For a strategy $\nu \in \mathcal{A}$, the terminal wealth of the agent writes

$$\begin{aligned} X_T =& X_0 - \int_0^T \nu_t\, S_t\, dt - \eta \int_0^T \frac{\nu_t^2}{V}\, dt \\ =& X_0 + Q_0\, S_0 + \int_0^T k\, \nu_t\, Q_t\, dt + \sigma \int_0^T Q_t\, dW_t - \eta \int_0^T \frac{\nu_t^2}{V}\, dt, \\ =& X_0 + Q_0\, S_0 - \frac{k}{2}Q_0^2 + \sigma \int_0^T Q_t\, dW_t - \eta \int_0^T \frac{\nu_t^2}{V}\, dt. \end{aligned}$$

4.3 Solution

To solve the problem, we first restrict the optimisation to the set of deterministic trading strategies that we denote $\mathcal{A}^{\text{det}}$. Thus, similar to the previous

[6] $L^\infty(\Omega)$ is the set of bounded processes. $\mathbb{H}^0\left(\mathbb{R}, (\mathcal{F}_t)_t\right)$ is the set of real-valued progressively measurable processes.

section, the terminal wealth X_T is normally distributed with mean and variance

$$\begin{cases} \mathbb{E}\left[X_T\right] & = \underbrace{X_0 + Q_0\, S_0}_{\text{MtM}} - \underbrace{\frac{k}{2} Q_0^2}_{\text{perm. impact.}} - \underbrace{\eta \int_0^T \frac{\nu_t^2}{V\, dt}}_{\text{execution costs}} \\ \mathbb{V}\left[X_T\right] & = \sigma^2 \int_0^T Q_t^2\, dt. \end{cases}$$

Thus, the problem of maximising a mean-variance criterion in $\mathcal{A}^{\text{det}}$ reduces to minimising the functional

$$\eta \int_0^T \frac{\nu_t^2}{V}\, dt + \frac{\gamma}{2}\sigma^2 \int_0^T Q_t^2\, dt.$$

Similar to the previous section, the above problem is equivalent to minimising the functional J over the set $\mathcal{C}^{\text{det}}$ of deterministic absolutely continuous functions:

$$\boxed{J\left(Q\right) = \int_0^T \left(\eta\, \frac{Q'\left(t\right)^2}{V} + \frac{\gamma}{2}\sigma^2 \int_0^T Q_t^2\right)\, dt\,,}$$

with constraints $Q(0) = Q_0$ and $Q(T) = 0$.

Because the function J is strictly convex and the set of admissible trading curves is convex, the functional J admits a unique minimiser $Q^\star$. To characterise this minimiser, observe that the Legendre-Fenchel transform of the function $x \mapsto \eta\, x^2$ is $p \mapsto p^2/4\,\eta$, so $Q^\star$ is the unique solution to the Hamiltonian system (see Theorem 1):

$$\begin{cases} p'\left(t\right) & = \gamma\, \sigma^2\, Q^\star\left(t\right), \\ Q^{\star'}\left(t\right) & = V\, p(t)/2\,\eta, \\ Q^\star\left(0\right) & = Q_0, \\ Q^\star\left(T\right) & = 0. \end{cases}$$

The above system leads to the ODE

$$Q^{\star''}(t) = V\, \frac{\gamma\, \sigma^2}{2\,\eta}\, Q^\star\left(t\right)\,,$$

with boundary conditions $Q^\star(0) = Q_0$ and $Q^\star(T) = 0$. The solution to the

ODE is the optimal trading curve $Q^\star \in \mathcal{C}^{\mathrm{det}}$:

$$\boxed{Q^\star(t) = Q_0 \frac{\sinh\left((T-t)\sqrt{\frac{\gamma\, V\, \sigma^2}{2\,\eta}}\right)}{\sinh\left(T\sqrt{V\, \frac{\gamma\, \sigma^2}{2\,\eta}}\right)},} \tag{15}$$

which leads to the optimal liquidation strategy $\nu^\star \in \mathcal{A}^{\mathrm{det}}$:

$$\boxed{\nu^\star(t) = -Q_0\sqrt{\frac{\gamma\, V\, \sigma^2}{2\,\eta}} \frac{\cosh\left((T-t)\sqrt{\frac{\gamma\, V\, \sigma^2}{2\,\eta}}\right)}{\sinh\left(T\sqrt{V\, \frac{\gamma\, \sigma^2}{2\,\eta}}\right)}.}$$

Recall the optimal strategy (14) in the discrete-time Almgren-Chriss framework, and notice that when $\Delta t \to 0$, the strategy is exactly that of the continuous-time counterpart:

$$\begin{cases} Q^\star_n & = \frac{Q_0\,\sinh(\alpha(T-t_n))}{\sinh(\alpha T)} \\ 2\,\cosh\left(\alpha\,\Delta t\right) & = \frac{\gamma\,\sigma^2\,V}{2\,\tilde{\eta}}\Delta t^2 \end{cases} \xrightarrow[\Delta t \to 0]{\text{Taylor expansion}} \begin{cases} Q^\star_t & = \frac{Q_0\,\sinh(\alpha(T-t))}{\sinh(\alpha T)} \\ \alpha & = \sqrt{\frac{\gamma\,\sigma^2\,V}{2\,\tilde{\eta}}}. \end{cases}$$

Similar to Section 3, we restricted the space of admissible strategies to that of deterministic ones. A deterministic optimal strategy is interesting in practice because it can be computed before the liquidation starts, and does not depend on the price path. For many years, market operators used execution algorithms that pre-compute trading schedules before the start of the trading window. In practice, execution algorithms are split into **two layers**: the first is strategic and defines the optimal trading curve to implement. The second is **tactical** (or sometimes speculative) and tracks the optimal trading curve using different types of orders, different trading venues, etc; see Figure 11.

It can be shown in the case of the mean-variance performance criterion used by Almgren and Chriss, that there exist stochastic strategies that outperform the best deterministic one. However, it has been shown in Schied et al. (2010) that when the agent uses the CARA utility (Constant Absolute Risk Aversion), then the best strategy over both stochastic and deterministic admissible strategies is the same as the best deterministic strategy.

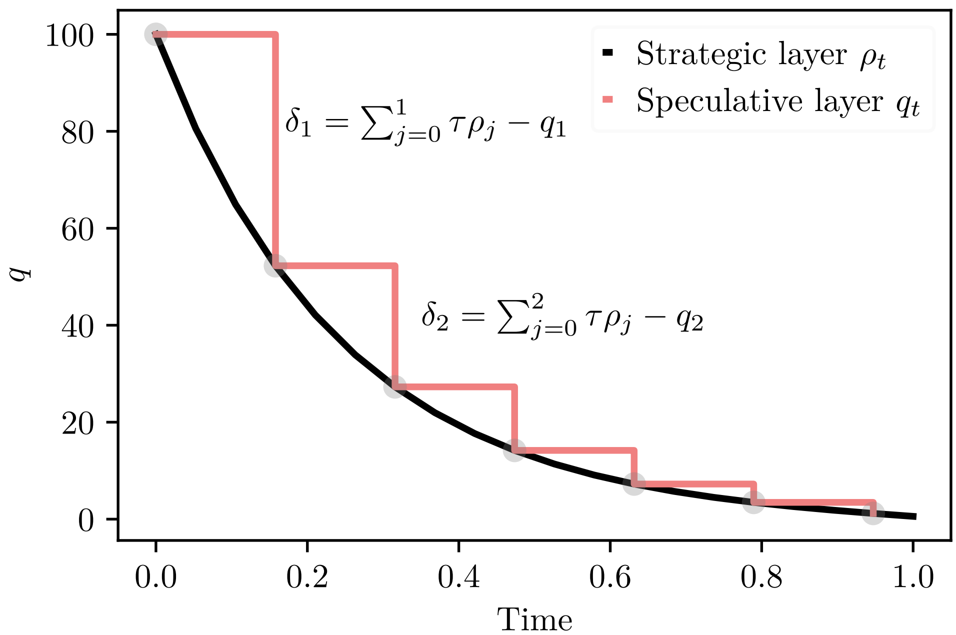

Figure 11: Execution algorithms in two layers.

4.4 Market impact must be linear

The discussion of this section is out of the scope of this course, but provides an interesting insight on the reason for the modelling assumptions concerning the permanent impact of trading. The discussion is based on the pioneer work in Gatheral (2010) and the work in Guéant (2016).

In our model of Section 4.1, we assume that the permanent impact of the agent's trading activity is linear. Here, we show that this key assumption guarantees the absence of *dynamic arbitrage* which is introduced in Gatheral (2010). A dynamic arbitrage strategy corresponds to a roundtrip strategy that is profitable on average. More precisely, there is a dynamic arbitrage if there exist $0 \leq t_1 < t_2 \leq T$, and an admissible strategy ν such that (i) it is a roundtrip strategy, i.e., $\int_{t_1}^{t_2} \nu_t \, dt = 0$, and (ii) it is profitable on average, i.e., $\mathbb{E}\left[X_{t_2} \mid \mathcal{F}_{t_1}\right] > X_{t_1}$.

Let $\kappa(\cdot)$ be a general permanent impact function and suppose there are no execution costs so $\eta = 0$. Then the dynamics of the processes in our problem (q_t, S_t, X_t) are

$$\begin{cases} dq_t = \nu_t \, dt, \\ dS_t = \sigma \, dW_t + \kappa\left(\nu_t\right) \, dt, \\ dX_t = -\nu_t \, S_t \, dt. \end{cases}$$

Next, we show that $\kappa(\cdot)$ must be linear for absence of dynamic arbitrage to hold. Consider an interval $[t_1, t_2]$ and the following roundtrip strategy:

$$v_t = \begin{cases} \alpha & \text{if } t \in [t_1, \tau(\alpha, \beta)] \\ -\beta & \text{if } t \in [\tau(\alpha, \beta), t_2] \,, \end{cases}$$

where α and β have the same sign and $\tau(\alpha, \beta) = \frac{\alpha\, t_1 + \beta\, t_2}{\alpha + \beta}$.

An agent which follows the strategy ν wealth X_{t_2} at time t_2 given by

$$X_{t_2} = X_{t_1} - \int_{t_1}^{t_2} v_t S_t dt = X_{t_1} + \int_{t_1}^{t_2} (q_t - q_{t_1})\, \sigma dW_t + \int_{t_1}^{t_2} (q_t - q_{t_1})\, \kappa\,(v_t)\, dt.$$

The roundtrip strategy implies that $q_{t_1} = q_{t_2}$ and a short calculation shows that

$$\begin{aligned}
\mathbb{E}\left[X_{t_2} \mid \mathcal{F}_{t_1}\right] =& X_{t_1} + \int_{t_1}^{t_2} (q_t - q_{t_1})\, \kappa\,(v_t)\, dt \\
=& X_{t_1} + \int_{t_1}^{\tau(a,b)} (q_t - q_{t_1})\, \kappa\,(v_t)\, dt + \int_{\tau(a,b)}^{t_2} (q_t - q_{t_2})\, \kappa\,(v_t)\, dt \\
=& X_{t_1} + \int_{t_1}^{\tau(a,b)} \alpha\,(t - t_1)\, \kappa(\alpha) dt + \int_{\tau(a,b)}^{t_2} \beta\,(t_2 - t)\, \kappa(-\beta) dt \\
=& X_{t_1} + \frac{1}{2}\alpha \left(\frac{\beta}{\alpha + \beta}\right)^2 (t_2 - t_1)^2\, \kappa(\alpha) \\
&+ \frac{1}{2}\beta \left(\frac{\alpha}{\alpha + \beta}\right)^2 (t_2 - t_1)^2\, \kappa(-\beta) \\
=& X_{t_1} + \frac{1}{2}\frac{\alpha\beta}{(\alpha + \beta)^2}\, (t_2 - t_1)^2\, (\beta\kappa(\alpha) + \alpha\kappa(-\beta)).
\end{aligned}$$

To guarantee absence of dynamic arbitrage, we require that $\mathbb{E}\left[X_{t_2} \mid \mathcal{F}_{t_1}\right] \leq X_{t_1}$ for any choice of α and β. Thus, we require that

$$\forall \alpha, \beta \in \mathbb{R}, \quad \alpha\, \beta > 0 \Rightarrow \beta\, \kappa(\alpha) + \alpha\, \kappa(-\beta) \leq 0.$$

Now, replace α with $-\beta$, and β with $-\alpha$ above to obtain

$$\forall \alpha, \beta \in \mathbb{R}, \quad \alpha\, \beta > 0 \Rightarrow \alpha\, \kappa(-\beta) + \beta\, \kappa(\alpha) \geq 0.$$

Therefore,

$$\forall \alpha, \beta \in \mathbb{R}, \quad \alpha\, \beta > 0 \Rightarrow \beta\, \kappa(\alpha) = -\alpha\, \kappa(-\beta).$$

In particular, set $\alpha = \beta$ to find that $\kappa(\cdot)$ is an odd function on $\mathbb{R}^*$. If $\alpha \neq 0$ and $\beta = \operatorname{sign}(\alpha)$, we obtain

$$\forall \alpha \in \mathbb{R}^*, \quad \kappa(\alpha) = -\alpha \operatorname{sign}(\alpha)\, \kappa(-\operatorname{sign}(\alpha)) = \alpha \kappa(1).$$

We also need to prove $\kappa(0) = 0$ and we use another specific roundtrip strategy. Assume $\kappa(0) \neq 0$ and consider the following roundtrip strategy:

$$v_t = \begin{cases} \kappa(0) & \text{if } t \in \left[t_1, t_1 + \frac{\tau}{3}\right], \\ 0 & \text{if } t \in \left[t_1 + \frac{\tau}{3}, t_1 + \frac{2\tau}{3}\right], \\ -\kappa(0) & \text{if } t \in \left[t_1 + \frac{2\tau}{3}, t_2\right], \end{cases}$$

where $\tau = t_2 - t_1$. Similar calculations as above show that

$$\begin{aligned} \mathbb{E}\left[X_{t_2} \middle| \mathcal{F}_{t_1}\right] &= X_{t_1} + \int_{t_1}^{t_2} (q_t - q_{t_1})\, \kappa(\nu_t)\, dt \\ &= X_{t_1} + \int_{t_1}^{t_1+\tau/3} \kappa(0)(t - t_1)\, \kappa(\kappa(0))\, dt + \int_{t_1+\tau/3}^{t_1+2\,\tau/3} \frac{\tau}{3} \kappa(0)^2\, dt \\ &\quad + \int_{t_1+2\,\tau/3}^{t_2} \kappa(0)(t_2 - t)\, \kappa(-\kappa(0))\, dt \\ &= X_{t_1} + \kappa\left(0\right)^2 \frac{\left(t_2 - t_1\right)^2}{9} > X_{t_1}. \end{aligned}$$

Thus, we require that $\kappa(0) = 0$ to guarantee absence of dynamic arbitrage. The conclusion is

$$\forall \alpha \in \mathbb{R}, \quad \kappa(\alpha) = \alpha\, \kappa(1).$$

Finally, if one assumes the form $\kappa(\nu) = k\, \nu$, then it is straightforward to see that for any strategy ν satisfying the roundtrip condition we have

$$\mathbb{E}[X_{t_2} | \mathcal{F}_{t_1}] = X_{t_1} + \int_{t_1}^{t_2} k\, (q_t - q_{t_1})\, \nu_t\, dt = X_{t_1} + \frac{k}{2} \left(q_{t_2} - q_{t_1}\right)^2 = X_{t_1},$$

so there is no dynamic arbitrage iif the permanent impact is linear.

5 Optimal execution in the Cartea-Jaimungal framework

The models of Section 3 and 4 rely on a mean-variance objective and one key constraint; a **fuel constraint** which stipulates that the sum of trades executed by the agent should sum up to minus the initial inventory. This constraint is also called a **hard constraint** because it stipulates that the terminal inventory Q_T must be zero. When addressing trading problems in a dynamic setup, the hard constraint leads to singular boundary condition that makes optimal trading problems difficult to address mathematically. Álvaro Cartea, Sebastian Jaimungal, and their co-authors proposed a framework with penalisation terms that (i) relax the hard constraint, (ii) are interpretable, (iii) can be estimated from market data, and (iv) most importantly enable one to use the classical tools of stochastic optimal control. This framework is now the standard approach to solve optimal trading problems of all kinds in the mathematical finance literature. Many of the models discussed in the course are inspired by the book Cartea et al. (2015).

The model of this section also deals with the classical problem of how an agent can buy or sell a large amount of shares with minimum costs while also minimising the risk of adverse price movements.

5.1 Modelling framework

Similar to Section 4, the model of this section requires to describe the dynamics of the number of shares that the agent is holding (inventory), the dynamics of the midprice of the asset, the execution costs of the agent's trading activity, and the impact of the agent's orders on the midprice.

We work on a filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}\right)$ satisfying the usual conditions, where $T > 0$. We assume this probability space to be large enough to support all the processes introduced in this section.

Inventory. At time $t = 0$, the agent holds an initial inventory composed of Q_0 shares. The agent wishes to liquidate this position by some terminal time $T > 0$. The trader's inventory over $[0, T]$ is modelled by the process $(Q^\nu_t)_{t\in[0,T]}$ whose dynamics are

$$\boxed{dQ^\nu_t = \nu_t \, dt,} \tag{16}$$

where $(\nu_t)_{t\in[0,T]}$ is the control process, i.e., the variable that the agent controls in the optimisation problem. Similar to Section 4, the value of ν_t at time t stands for the trading speed, i.e., the instantaneous trading volume of the agent. Here, we do not restrict the sign of the agent's control ν. When $\nu_t > 0$, the agent is buying shares, and when $\nu_t < 0$, the agent is selling them. Note that when a process has the control ν as a superscript, then it means that the process is (indirectly) controlled by the agent's decisions.

Midprice. The midprice process is modelled by the process $(S_t^\nu)_{t\in[0,T]}$, and it is affected by the speed of trading ν. We assume this impact is linear and we write the dynamics of the midprice as

$$\boxed{dS_t^\nu = \sigma\, dW_t + k\, \nu_t\, dt, \quad S_0 \in \mathbb{R}_+^\star \text{ is known,}} \tag{17}$$

where W is a standard Brownian motion, $\sigma > 0$ is the volatility parameter, and $k \geq 0$ scales the magnitude of the permanent market impact.

Execution costs. At any one time t, the number of shares displayed (or equivalently the number of LOs resting in the book) at each price level is limited. A large MO will walk the book, so the average price per share will be worse than the current midprice; see the discussions in Section 1.2. More precisely, the execution price per share received by the agent is $\tilde{S}_t^\nu = S_t^\nu + \eta\, \nu_t$.[7] So the agent's cash process $(X_t^\nu)_{t\in[0,T]}$ satisfies the SDE

$$\boxed{dX_t^\nu = -\nu_t\, \tilde{S}_t\, dt = -\nu_t\, (S_t + \eta\, \nu_t)\, dt, \quad X_0 \in \mathbb{R} \text{ is known.}} \tag{18}$$

5.2 Performance criterion

In the CJ framework, although the agent's real objective is to complete the liquidation of her initial inventory, the model allows to fall short of this target so $Q_T \neq 0$; the fuel constraints are thus dropped. However, at the end of the trading window, the agent must execute a buy or a sell MO for the remaining amount so they incur additional costs that this trade might cause. To model this aspect, the CJ framework introduces a terminal inventory penalty parameterised by a **terminal penalty parameter** $\alpha \geq 0$. This parameter includes any costs from walking the book and any other penalties

[7] We drop the market volume V and assume it is incorporated in the estimation of η.

that the agent incurs. In practice, the agent might (and usually does) artificially inflate the value of α to enforce total liquidation by the terminal time T.

The CJ framework also introduces another penalty term in the model; a **running inventory penalty** of the form $\phi \int_0^T (Q_t^\nu)^2$ where $\phi \geq 0$ is the **urgency** parameter. The running inventory penalty does not impact the wealth of the agent; recall the dynamics of the cash in (18). This term must be considered as a means for the agent to incorporate an artificial cost for holding inventory throughout the trading window. This parameter allows the agent to model their urgency to get rid of inventory. Higher values of ϕ lead to quicker execution because it (artificially) increases the cost of holding shares.

We denote by $\mathcal{A}_t$ the set of admissible strategies (or controls) that are $\mathcal{F}-$predictable and square integrable, i.e., $\mathbb{E}\left[\int_t^T g_s^2\, ds\right] < \infty$, and we denote by $\mathcal{A} = \mathcal{A}_0$. The agent's performance criterion in the CJ framework is composed of three components, the terminal wealth, the terminal penalty, and the running inventory penalty, and we write

$$\boxed{\begin{aligned} H^\nu(t,x,S,q) &= \mathbb{E}_{t,x,S,q}\left[\underbrace{X_T^\nu}_{\text{Terminal Cash}} + \underbrace{Q_T^\nu\,(S_T^\nu - \alpha Q_T^\nu)}_{\text{Terminal Execution}} - \underbrace{\phi \int_t^T (Q_u^\nu)^2}_{\text{Inventory Penalty}}\right] \\ &= \mathbb{E}_{t,x,S,q}\left[\underbrace{X_T^\nu + Q_T^\nu\, S_T^\nu}_{\text{Terminal Wealth}} - \underbrace{(\alpha Q_T^\nu)^2}_{\text{Terminal Penalty}} - \underbrace{\phi \int_t^T (Q_u^\nu)^2}_{\text{Inventory Penalty}}\right], \end{aligned}}$$

where $\mathbb{E}_{t,x,S,q}$ denotes the expectation conditional (with a slight abuse of notation) on $X_t^\nu = x$, $S_t^\nu = S$, and $Q_t^\nu = q$. The value function is

$$H(t,x,S,q) = \sup_{\nu\in\mathcal{A}} H^\nu(t,x,S,q).$$

5.3 Solution

To solve this optimal trading problem, we use the tools of stochastic optimal control; see Section 1.5.2. The dynamic programming principle (DPP) suggests that the value function H satisfies the HJB equation

$$\partial_t H + \sup_{\nu \in \mathcal{A}_t} \left(\mathcal{L}_t^\nu H - \phi\, q^2 \right) = 0,$$

subject to the terminal condition $H(T, x, S, q) = x + q\, S - \alpha\, q^2$, where $\mathcal{L}^\nu$ is the infinitesimal generator of the processes of the problem X, S, and Q, i.e.,

$$\mathcal{L}_t^\nu = \mathcal{L}^X + \mathcal{L}^Q + \mathcal{L}^S$$

and it acts on the value function H. Recall the SDEs in (16)-(17)-(18), thus $\mathcal{L}^S = \frac{1}{2}\, \sigma^2\, \partial_{SS} + k\, \nu\, \partial_S$, $\mathcal{L}^Q = \nu\, \partial_q$, and $\mathcal{L}^X = -\nu\, (S + \eta \nu)\, \partial_x$.

By substituting in the HJB above, and keeping only the terms in ν in the supremum term, we obtain the HJB

$$\boxed{\begin{aligned} 0 &= \left(\partial_t + \frac{1}{2} \sigma^2\, \partial_{SS} \right) H - \phi\, q^2 \\ &\quad + \sup_{\nu} \left\{ -\nu\, \left(S + \eta\, \nu \right) \partial_x H + k\, \nu\, \partial_S H + \nu\, \partial_q H \right\}, \end{aligned}} \tag{20a}$$

with terminal condition

$$H(T, x, S, q) = x + S\, q - \alpha\, q^2, \quad \forall\, (x, S, q) \in \mathbb{R}^3.$$

The first-order condition allows us to obtain the optimal speed in feedback form, i.e., the optimal trading speed as a function of the value function (and state variables):

$$\boxed{\nu^\star = \frac{1}{2\, \eta}\, \frac{-S\, \partial_x H + k\, \partial_S H + \partial_q H}{\partial_x H}.} \tag{21}$$

Upon substituting the optimal feedback control (21) into the HJB (20a), it reduces to

$$\boxed{\begin{aligned} 0 &= \left(\partial_t + \frac{1}{2} \sigma^2\, \partial_{SS} \right) H - \phi\, q^2 \\ &\quad + \frac{1}{4\, \eta}\, \frac{\left(-S\, \partial_x H + k\, \partial_S H + \partial_q H \right)^2}{\partial_x H} \end{aligned}} \tag{22a}$$

To further study our optimal trading problem, we must solve the HJB (22a) which takes the form of a nonlinear PDE in H. The usual approach is to *guess* a functional form for H in order to reduce the dimensionality of the problem and obtain PDEs that can be solved in closed-form.[8] To propose a guess for a functional form (an ansatz) for the solution to the HJB , it is helpful to look at the boundary or terminal conditions to get an idea of which variables are relevant in the value function.

The terminal condition above suggests the ansatz

$$\boxed{H(t,x,S,q) = x + q\,S + h(t,S,q)\,,} \tag{23}$$

where $h(t,S,q)$ is still to be determined. The ansatz (23) has a simple interpretation. The first term is the accumulated cash, the second term is the book value of the inventory marked-to-market (i.e. the value of the shares at the current midprice), and the last term is the added value from optimally liquidating the remaining shares.

Substitute (23) into (22a) to obtain the new PDE in $h(t,q)$:

$$0 = \left(\partial_t + \frac{1}{2}\sigma^2\,\partial_{SS}\right) h - \phi\,q^2 + \frac{1}{4\,\eta}\,\left(k\,(q + \partial_S h) + \partial_q h\right)^2\,,$$

subject to the terminal condition

$$h(T,S,q) = -\alpha\,q^2.$$

The above PDE contains no explicit dependence on the state variable S and the terminal condition is independent of S as well. It follows that h does not depend on S so $\partial_S h(t,S,q) = 0$ and we write $h(t,S,q) = h(t,q)$. Thus, we obtain the PDE for h :

$$\boxed{0 = \partial_t h - \phi\,q^2 + \frac{1}{4\,\eta}\,\left(k\,q + \partial_q h\right)^2\,,} \tag{24}$$

and the optimal feedback control in (21) simplifies to

$$\boxed{\nu^\star = \frac{1}{2\,\eta}\,\left(k\,q + \partial_q h\right).} \tag{25}$$

In the form (24) of the PDE, it appears that the solution admits a separation of variables and takes the form of second-degree polynomial in q; this will

[8]Note that closed-form solutions of non-linear PDEs are very rarely obtained.

be the case in all the optimal trading models of this course because we are purposefully in the classical and tractable linear-quadratic-Gaussian (LQG) control framework.

$$\boxed{h(t,q) = h_0(t) + h_1(t)\, q + h_2(t)\, q^2\,.} \tag{26}$$

Upon substituting (26) in the PDE (24), we obtain that for all $q \in \mathbb{R}$,

$$\begin{aligned}0 = &\left\{\partial_t h_2\,(t) - \phi + \frac{1}{4\,\eta}\,\left(k\, + 2\,h_2\,(t)\right)^2\right\} q^2 \\ &+ \left\{\partial_t h_1\,(t) + \frac{1}{2\,\eta} h_1\,(t)\,(k\, + 2\,h_2\,(t))\right\} q + \left\{\partial_t h_0 - \frac{1}{4\,\eta}\,h_1\,(t)^2\right\}.\end{aligned}$$

Clearly, if the equality above is verified for all q, then each coefficient of the polynomial must be equal to zero, so we obtain the system of ODEs

$$\boxed{\begin{cases} 0 = & \partial_t h_2\,(t) - \phi + \frac{1}{4\,\eta}\,\left(k\, + 2\,h_2\,(t)\right)^2 \\ 0 = & \partial_t h_1\,(t) + \frac{1}{2\,\eta} h_1\,(t)\,(k\, + 2\,h_2\,(t)) \\ 0 = & \partial_t h_0 - \frac{1}{4\,\eta}\,h_1\,(t)^2\;, \end{cases}}$$

subject to the terminal conditions $h_2(T) = -\alpha,\;\; h_1(T) = 0,\;$ and $h_0(T) = 0$.

Clearly, the solution to the ODE in h_1 is $h_1(t) = 0$, so the solution to the ODE in h_0 is also zero. The non-linear ODE in h_2 is of **Riccati** type and can be integrated exactly; Riccati ODEs frequently arise in linear-quadratic-Gaussian control problems and there is an extensive literature on the study of existence, solutions, and numerical approximations of these equations (see e.g., Abou-Kandil et al. (2012)).

First, let $h_2(t) = -\frac{k}{2} + \mathfrak{h}_2(t)$, so the ODE in h_2 can be solved by solving the ODE in $\mathfrak{h}_2$:

$$\frac{\partial_t \mathfrak{h}_2}{\eta\,\phi - \mathfrak{h}_2^2} = \frac{1}{\eta},$$

subject to the terminal condition $\mathfrak{h}_2(T) = \frac{k}{2} - \alpha$. Integrating both sides of the above over the *remaining* trading window $[t, T]$ yields

$$\log\frac{\sqrt{\eta\phi} + \mathfrak{h}_2(T)}{\sqrt{\eta\phi} - \mathfrak{h}_2(T)} - \log\frac{\sqrt{\eta\phi} + \mathfrak{h}_2(t)}{\sqrt{\eta\phi} - \mathfrak{h}_2(t)} = 2\gamma\,(T - t)\;,$$

so that

$$\boxed{\mathfrak{h}_2(t) = \sqrt{\eta\phi}\frac{1+\zeta e^{2\gamma(T-t)}}{1-\zeta e^{2\gamma(T-t)}}\,,}$$

where

$$\gamma = \sqrt{\frac{\phi}{\eta}} \quad \text{and} \quad \zeta = \frac{\alpha - \frac{1}{2}k + \sqrt{\eta\phi}}{\alpha - \frac{1}{2}k - \sqrt{\eta\phi}}.$$

We have fully determined the solution to the HJB and the optimal trading strategy $\nu^\star$ can be obtained explicitly from the feedback form (25) and we write

$$\boxed{\nu_t^* = -\gamma \frac{\zeta\ e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta\ e^{\gamma(T-t)} - e^{-\gamma(T-t)}} Q_t^{\nu^*}.} \tag{27}$$

The optimal speed (27) is proportional to the investor's current inventory $Q_t^{\nu^\star}$ ($Q_t^{\nu^\star}$ is the agent's inventory when they trade with the strategy $\nu^\star$), and the proportional factor depends non-linearly on time. From the formula of the optimal speed (27), we can obtain the agent's inventory. First use (25) and note that

$$dQ_t^{\nu^\star} = \nu_t^\star\, dt = \frac{1}{\eta}\mathfrak{h}_2(t) Q_t^{\nu^\star}\, dt$$

so that

$$Q_t^{\nu^\star} = Q_0 \ \exp\left\{\int_0^t \frac{\mathfrak{h}_2(s)}{\eta} ds\right\}.$$

Finally, to obtain the explicit optimal trading curve $Q^{\nu^\star}$, we compute the integral as

$$\begin{aligned}
\int_0^t \frac{\chi(s)}{\eta} ds &= \frac{1}{k}\int_0^t \sqrt{\eta\,\phi}\frac{1+\zeta e^{2\gamma(T-s)}}{1-\zeta e^{2\gamma(T-s)}} ds \\
&= \gamma \int_0^t \frac{e^{-2\gamma(T-s)}}{e^{-2\gamma(T-s)} - \zeta} ds + \gamma \int_0^t \frac{\zeta e^{2\gamma(T-s)}}{1-\zeta e^{2\gamma(T-s)}} ds \\
&= \log\left(e^{-\gamma(T-s)} - \zeta e^{\gamma(T-s)}\right) |_0^t \\
&= \log \frac{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{\zeta e^{\gamma T} - e^{-\gamma T}},
\end{aligned}$$

hence

$$\boxed{Q_t^{\nu^*} = \frac{\zeta\ e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{\zeta\ e^{\gamma T} - e^{-\gamma T}} Q_0.} \tag{28}$$

5.4 Discussion

Deterministic strategy. Observe that the optimal strategy is deterministic even though we did not restrict the space of admissible strategies to that of deterministic ones. Thus, the optimal trading curves and model parameters can be estimated prior to the trading window and market operators can focus on the second layer. The CJ framework has been widely adopted in the literature and the industry because it is tractable.

Infinite penalty. In the limit when the quadratic liquidation penalty goes to infinity, i.e., when $\alpha \to \infty$, we obtain $\zeta \underset{\alpha\to\infty}{\longrightarrow} 1$ and the optimal trading speed and trading curve simplify to

$$Q_t^{\nu^*} \xrightarrow[\alpha\to+\infty]{} \frac{\sinh\left(\gamma(T-t)\right)}{\sinh\left(\gamma T\right)} Q_0,$$

and

$$\nu_t^* \xrightarrow[\alpha\to+\infty]{} \gamma \frac{\cosh\left(\gamma(T-t)\right)}{\sinh\left(\gamma T\right)} Q_0 \,.$$

Both equations do no depend on the permanent impact k and resemble those obtained in the classical Almgren-Chriss model with hard constraint; see equations (14) and (15). However, when $\alpha < \infty$, the strategy does depend on the permanent impact k.

Next, we study the optimal trading curve given by the model as a function of model parameters. Figure 12 shows the optimal trading curve (28) for different values of model parameters.

Terminal penalty. Figure 12 shows that the terminal penalty mainly plays a role at the end of the trading window. For fixed execution costs η, the agent can leave some terminal penalty to execute at the end of the trading window if the relative terminal cost α is small. Note that when $\alpha = 0.1$, the liquidation is completed by the terminal time T.

Running inventory penalty. As the running penalty increases, the trading curves become more convex and the optimal strategy aims to sell more assets sooner in the trading window. This is intuitive because ϕ is the agent's urgency and penalises holding inventory. When ϕ approaches zero, the optimal curve resembles a straight line, this is because

$$Q_t^{\nu^\star} \underset{\phi \to 0}{\longrightarrow} \frac{t}{T + k/\alpha}.$$

Execution costs. As the execution costs driven by η increase, the agent trades slowly. Also, when the execution costs are significantly lower than the terminal penalty, the agent will choose to liquidate with a block order at the terminal time and avoid trading throughout the trading window.

Permanent impact. The permanent impact plays a less significant role than other model parameters; recall that it played no role in the Almgren-Chriss framework with hard constraints, and it also does not play a role when we consider an infinite terminal penalty $\alpha \to \infty$.

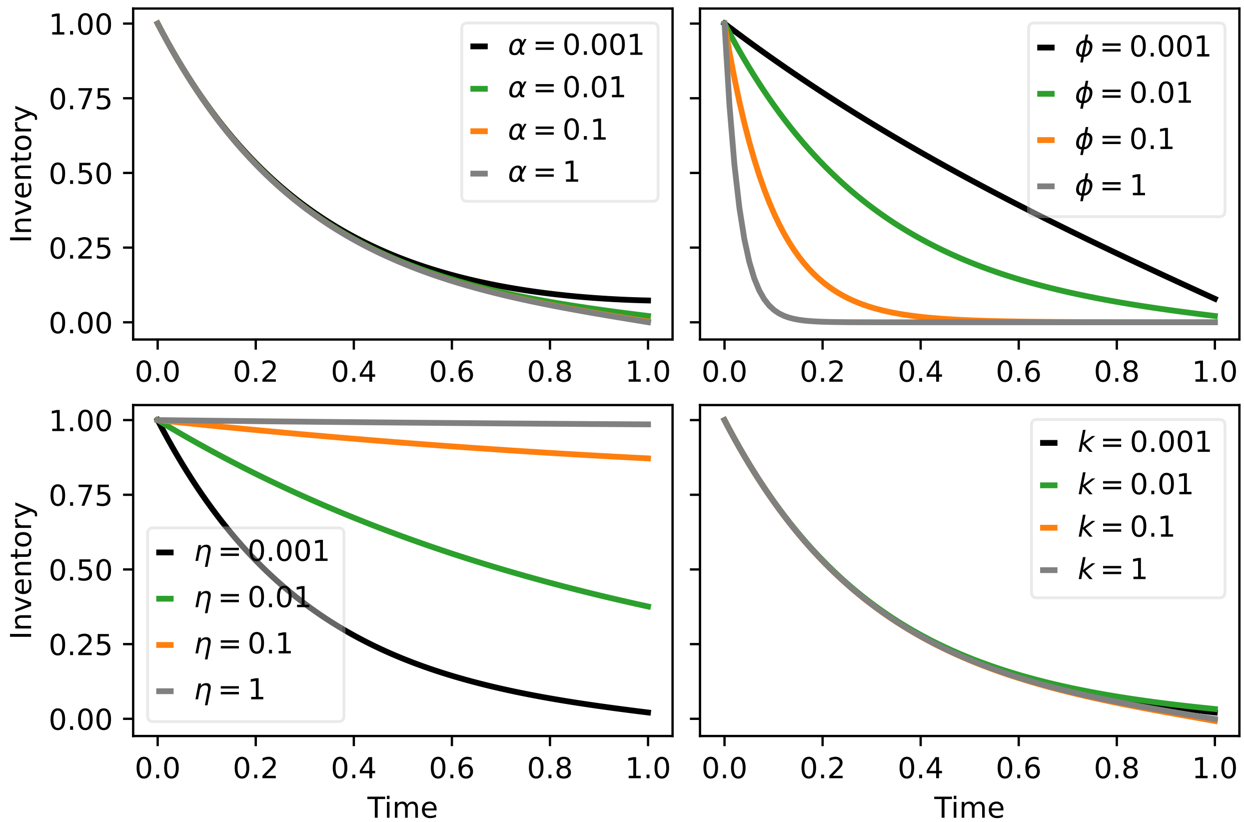

Figure 12: Optimal trading curve (28) for multiple values of the parameters: terminal penalty α, running inventory penalty ϕ, execution costs η, and permanent impact k. The default parameter values are $\alpha = 0.01$, $\phi = 0.01$, $k = 0.001$, and $\eta = 0.001$. The other parameters are $T = 1$ and $q_0 = 1$.

6 Optimal execution with nonlinear impact

Section 5 assumed that the instantaneous impact (execution costs) of the agent's trading activity in (18) is linear in the speed of trading. This assumption is key for solving the problem because we are able to frame the optimisation problem in a linear-quadratic-Gaussian framework which lead to Riccati equations that can be either solved explicitly or efficiently solved with numerical approximation techniques. In Section 1.2, we showed that a linear model is a good approximation for this impact. However, the recent literature argues that a power law function $f : \nu \mapsto |\nu|^a$ with power $a > 0$ less than one fits the observed data better. Here, we study the problem of optimal execution when the instantaneous costs are driven by a general nonlinear function f of the trading speed ν.

6.1 Modelling framework

The model that we consider here is similar to that of Section 5. We work on a filtered probability space $\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}\big)$ satisfying the usual conditions, where $T > 0$.

Inventory. At time $t = 0$, the agent holds Q_0 shares that she wishes to liquidate by $T > 0$. The agent's inventory is modelled by the process $(Q_t^\nu)_{t\in[0,T]}$ whose dynamics are

$$\boxed{dQ_t^\nu = \nu_t \, dt,}$$

where $(\nu_t)_{t\in[0,T]}$ is the trading speed. When $Q_0 \geq 0$ the execution problem is that of a liquidation, and when $Q_0 \leq 0$ it is an acquisition problem.

Midprice. The midprice process $(S_t^\nu)_{t\in[0,T]}$ is affected by the speed of trading ν and we write

$$\boxed{dS_t^\nu = \sigma \, dW_t + k \, \nu_t \, dt, \quad S_0 \in \mathbb{R}_+^\star \text{ is known,}}$$

where W is a standard Brownian motion, $\sigma > 0$ is the volatility, and $k \geq 0$.

Execution costs. At any one time t, the execution price per share received by the agent is $\tilde{S}_t^\nu = S_t^\nu - f(\nu_t)$ where $f : \mathbb{R} \mapsto \mathbb{R}$ is a nonlinear function. So the agent's cash process $(X_t^\nu)_{t\in[0,T]}$ satisfies the SDE

$$\boxed{dX_t = -\nu_t \, \tilde{S}_t \, dt = -\nu_t \, (S_t + f(\nu_t)) \, dt, \quad X_0 \in \mathbb{R} \text{ is known.}}$$

Note here that $f(\nu_t)$ is the (instantaneous) cost per-share at time t when trading ν_t. Thus, the net cost is $\nu_t\, f(\nu_t)$. For the problem to be well-posed, we define the function

$$F : \nu \mapsto \nu\, f(\nu),$$

and we assume that it is convex. Note that this assumption holds for the linear model of Section 5 where $f : \nu \mapsto \eta\,\nu$ with $\eta > 0$. It also holds for general power-law functions $f : \nu \mapsto \eta\,\nu^a$ where $a > 0$.

6.2 Performance criterion

We consider the same admissible set $\mathcal{A}$ as that of Section 5 and the same performance criterion

$$\boxed{H^\nu(t, x, S, q) = \mathbb{E}_{t,x,S,q}\left[X_T^\nu + Q_T^\nu\, S_T^\nu - \left(\alpha Q_T^\nu\right)^2 - \phi \int_t^T (Q_u^\nu)^2\right],}$$

and the value function is

$$H(t, x, S, q) = \sup_{\nu\in\mathcal{A}} H^\nu(t, x, S, q).$$

6.3 Solution

The dynamic programming principle (DPP) suggests that the value function H should satisfy the HJB

$$0 = \left(\partial_t + \frac{1}{2}\sigma^2\partial_{SS}\right) H - \phi q^2 + \sup_\nu \left\{\left(-\nu\,(S + f(\nu))\,\partial_x - k\,\nu\,\partial_S + \nu\partial_q\right) H\right\},$$

subject to the terminal condition

$$H(T, x, S, q) = x + q\,\left(S - \alpha\, q\right).$$

Here, we follow similar steps as in Section 5. First, we use the usual ansatz

$$H(t, x, S, q) = x + q\,S + h(t, q)\,,$$

which separates the value function into two components: (i) the book value of cash and inventory and (ii) the value of optimally trading the remaining shares.

Substitute the ansatz into the DPE to obtain the following non-linear PDE for h:

$$0 = \partial_t h - \phi q^2 + \sup_\nu \{-\nu\, f(\nu) - (k\, q - \partial_q h)\, \nu\}\,, \tag{30}$$

subject to the terminal condition

$$h(T, q) = -\alpha\, q^2.$$

Recall that $F(\nu) = \nu\, f(\nu)$ is a convex function, so the supremum term can be solved and we write

$$\sup_\nu \{-\nu\, f(\nu) - (k\, q - \partial_q h)\, \nu\} = \tilde{F}\left(-\left(k\, q - \partial_q h\right)\right),$$

where $\tilde{F}$ is the Legendre-Fenchel transform of the function F (See Definition 1).

The case of a power law. Assume for simplicity that $Q_0 < 0$ and that we *restrict the trading speed to be positive to enforce acquisition only.* Assume that the execution costs function is, for $0 < a \leq 1$,

$$f : x \mapsto \eta\, \nu^a$$

so

$$F : x \mapsto \eta\, \nu^{1+a}.$$

Then

$$\tilde{F}(p) = \sup_x \left\{x\, p - \eta\, x^{1+a}\right\}.$$

The above supremum can be found with a first order condition

$$p - \eta\, (1 + a)\, (x^*)^a = 0 \quad \Rightarrow \quad x^* = \left(\frac{p}{(1 + a)\, \eta}\right)^{\frac{1}{a}},$$

so the Legendre-Fenchel transform of F is

$$\tilde{t}F(p) = \xi p^{1+\frac{1}{a}}, \quad \xi = \frac{a\, \eta}{((1 + a)\, \eta)^{1+\frac{1}{a}}}.$$

The nonlinear PDE (30) becomes

$$\boxed{\partial_t h - \phi q^2 + F^*\left(-(k\, q + \partial_q h)\right) = 0, \quad \text{and} \quad h(T, q) = -\alpha q^2.} \tag{31}$$

6.4 Discussion

The PDE (31) can rarely be solved analytically and one usually uses numerical approximation techniques based on grids to obtain a solution that can be implemented. In practice, operators need to obtain optimal trading schedules for a very large number of trades every day and often consider portfolio trades (see Sections 11 and 12). The numerical PDE techniques suffer from the curse of dimensionality because the number of equations to solve grows exponentially with the number of assets and the number of state variables in the model.

Realistic models for multiple assets that contain many variables to describe the environment are not practical. Numerical approximations take too long to compute and implement by the agent in real time. In practice, the profitability of execution and arbitrage strategies relies on computing the strategy within very short periods of time (e.g., milliseconds). Thus, because speed is paramount for market operators, they usually prefer to derive strategies in closed-form or that can be reduced to solving ODEs because they can be deployed in real time. However, this section illustrates how difficult it is to obtain closed-form strategies once the agent considers slightly more complicated models of the financial markets.

Concerning the specific case of linear versus nonlinear execution costs, some researchers argue that, given the extremely low predictive accuracy of market impact models (typically less than 5% R-squared), the cost of increased complexity that arise from moving away from a linear model would outweigh any gains from better describing market impact. Nonetheless, it is worthwhile to investigate the problem in practice to assess if the new model improves performance.

7 Optimal trading with predictive signals

A recent and important stream of the optimal execution literature deals with adding predictive signals of future price changes.[9] Typical examples of these signals include a drift in asset prices, order book imbalances, forecasts of the future order flow of market participants, and other price-based technical indicators. The usual formalism in the literature with predictive signals is to consider Brownian or Black-Scholes dynamics, along with independent mean-reverting Markov signals. The case of Ornstein-Uhlenbeck-type signals is of special interest as it usually leads to closed-form formulas. The model presented below uses the CJ framework introduced in Section 5.

7.1 Modelling framework

Consider a filtered probability space $\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}\big)$, with $T > 0$, satisfying the usual conditions and supporting all the processes we introduce below.

An agent must choose the trading speed at which they send market orders to liquidate $Q_0 \in \mathbb{R}$ shares over a trading window $[0, T]$ where $T > 0$. We denote the trading speed, which is what the investor controls, by $\nu = (\nu_t)_{t\in[0,T]}$ and inventory is $Q = (Q_t)_{t\in[0,T]}$, which is affected by how fast she trades, and satisfies

$$\boxed{dQ^\nu_t = \nu_t\, dt.} \tag{32}$$

We denote by $(S^\nu_t)_{t\in[0,T]}$ the process describing the dynamics of the midprice. The agent uses a price predictor (alpha signal) $(\mu_t)_{t\in[0,T]}$ that drives the stochastic drift of the midprice process. The process μ is assumed Markov, cadlag, and bounded $\mathbb{P}-$a.s.. Also, the investor's trading activity affects the midprice process in two ways. One is permanent (linear) and the other is temporary. The process S^ν satisfies the SDE (stochastic differential equation)

$$\boxed{dS^\nu_t = \mu_t\, dt + k\, \nu_t\, dt + \sigma\, dW_t, \quad S^\nu_0 = S_0,} \tag{33}$$

where $W = (W_t)_{t\in[0,T]}$ is a standard Brownian motion, and μ is assumed independent of W.

[9] This stream of the literature is closely related to the multi-asset optimal execution models below: when trading an asset, the dynamics of another asset within or outside the portfolio can be regarded as a predictive signal that can enhance the execution process.

At any time t, the MO of the agent walks the book, and the average price per share obtained is worse that the current midprice S_t. Here, an MO of order $\nu_t\, dt$ obtains an execution price per share of

$$\tilde{S}_t^\nu = S_t^\nu + \eta\, \nu_t\,,$$

where η is a non-negative constant.

In the dynamics above, the impact of the MO is temporary and only affects the price received by the agent. Moreover, the LOB recovers infinitely fast to its state previous to the arrival of the MO (resilience); See Section 1.2 for further discussions on the modelling assumptions for market impact and Section 8 for a model with transient impact. The investor's cash process $(X_t^\nu)_{t\in[0,T]}$ satisfies the SDE

$$\boxed{dX_t^\nu = -\tilde{S}_t^\nu\, \nu_t\, dt = -\left(S_t^\nu + \eta\, \nu_t\right)\, \nu_t\, dt, \quad X_0^\nu = X_0.} \tag{34}$$

7.2 Performance criterion

The agent's performance criterion is

$$\boxed{H^\nu\left(t, x, s, \mu, q\right) = \mathbb{E}_{t,x,s,\mu,q}\left[X_T + Q_T^\nu\, \left(S_T^\nu - \alpha\, Q_T^\nu\right) - \phi \int_t^T \left(Q_u^\nu\right)^2 du\right]\,,} \tag{35}$$

where $\mathbb{E}_{t,x,s,\mu,q}$ is the expectation conditioned on (with a slight abuse of notation) $X_t = x$, $S_t = S$, $\mu_t = \mu$, and $Q_t = q$. The value function $H : [0,T] \times R^4 \mapsto \mathbb{R}$ of the agent is

$$H\left(t, x, s, \mu, q\right) = \sup_{\nu\in\mathcal{A}} H^\nu\left(t, x, s, \mu, q\right)\,, \tag{36}$$

where $\mathcal{A}$ is the set of admissible strategies consisting of $\mathcal{F}-$predictable processes that satisfy $\int_0^T |\nu_u|\, du < \infty$, $\mathbb{P}-$a.s..

As usual, the right-hand side of the performance criteria (35) is composed of three terms. The first term is the agent's terminal cash from liquidating the shares throughout the trading horizon. The second is the proceeds from liquidating any remaining inventory Q_T^ν at the terminal time T. This terminal inventory is liquidated at the midprice S_T^ν and the agent incurs the costs associated to crossing the spread, liquidity taking fees, and market impact, which is captured by the liquidation penalty parameter $\alpha \geq 0$. Finally, the third term is the running penalty $\phi \int_t^T (Q_u^\nu)^2 \, du$ where $\phi \geq 0$ is the inventory penalty parameter. This penalty is an urgency parameter and does not affect the investor's revenues.[10]

7.3 Solution

The dynamic programming principle for the value function suggests that H in (36) satisfies the HJB (use the dynamics in (32)-(33)-(34)):

$$\boxed{\begin{aligned} 0 = &\left(\partial_t + \tfrac{1}{2}\sigma^2 \, \partial_{SS}\right) H + \mathcal{L}^\mu H - \phi \, q^2 \\ &+ \sup_\nu \left\{ \Big(-\nu \, (S + \eta \, \nu) \, \partial_x + (\mu + k \, \nu) \; \partial_S + \nu \, \partial_q \Big) H \right\}, \end{aligned}}$$

with terminal condition $H\,(T, x, s, \mu, q) = x + q\,s - \alpha\,q^2\,.$

To solve (37), we propose the ansatz

$$\boxed{H(t, x, s, \mu, q) = x + q\,s + h(t, q, \mu)\,,}$$

which has the usual simple interpretation. The first term is the accumulated cash x, the second term is the book value $q\,s$ of the inventory marked-to-market (i.e. the value of the shares at the current midprice), and the last term $h(t, q, \mu)$ is the added value from following an optimal trading strategy using the predictive signal μ up to the terminal date. We find, upon substitution of the ansatz, that the HJB (37) becomes

$$0 = \partial_t h + \mathcal{L}^\mu h - \phi\,q^2 + \mu\,q + \sup_\nu \Big\{ -\eta\,\nu^2 + \nu \; (k\,q + \partial_q h) \Big\}, \tag{38}$$

with terminal condition $h\,(T, q, \mu) = -\alpha\,q^2\,.$

The optimal trading speed in **feedback form** is obtained by solving the first-order condition in (38) and we write

$$\boxed{\nu^\star = \frac{k\,q + \partial_q h}{2\,\eta}.} \tag{39}$$

[10] Including this running inventory penalty is also justified in a setting where the agent considers model uncertainty – i.e. she is ambiguity averse. Cartea et al. (2017) show that including the running penalty is equivalent to the agent considering alternate models with stochastic drifts, but penalizes those models using relative entropy. In that setting, the higher the value of φ, the less confident is the agent about the trend of the midprice.

Substitute (39) in (38) to obtain

$$\boxed{0 = \partial_t h + \mathcal{L}^\mu h - \phi\, q^2 + \mu\, q + \frac{\left(k\, q + \partial_q h\right)^2}{4\,\eta}.} \tag{40}$$

Due to the existence of a linear and a quadratic term in q in (40) and the form of the terminal condition of h, we assume the ansatz

$$\boxed{h(t,\mu,q) = h_0(t,\mu) + q\, h_1(t,\mu) + q^2\, h_2(t,\mu)\,.} \tag{41}$$

Substitute (41) in (38) and collect the terms in q to find the coupled system of PDEs:

$$\boxed{\begin{cases} 0 = & \left(\partial_t + \mathcal{L}^\mu\right) h_0 + \frac{1}{4\,\eta} h_1^2 \\ 0 = & \left(\partial_t + \mathcal{L}^\mu\right) h_1 + \mu + \frac{1}{2\eta}\, h_1\, \left(k + 2h_2\right) \\ 0 = & \left(\partial_t + \mathcal{L}^\mu\right) h_2 - \phi + \frac{1}{4\eta}\, \left(k + 2h_2\right)^2 , \end{cases}} \tag{42}$$

with terminal conditions $h_0\left(T,\mu\right) = h_1\left(T,\mu\right) = 0$ and $h_2(T,\mu) = -\alpha$.

Note that the equation in h_2 in (42) contains no source terms in μ and its terminal condition does not depend on μ, thus the solution must be independent of μ and h_2 is only a function of time. Then h_2 solves the (Riccati) ODE

$$0 = h_2'(t) - \phi + \frac{1}{4\eta}\, \left(\eta + 2\, h_2(t)\right)^2\, ,$$

which can be solved explicitly:

$$\boxed{h_2(t) = \sqrt{\eta\,\phi}\,\frac{1 + \zeta\, e^{2\,\gamma\,(T-t)}}{1 - \zeta\, e^{2\,\gamma\,(T-t)}} - \frac{1}{2}k\,,}$$

where

$$\gamma = \sqrt{\frac{\phi}{\eta}}, \qquad \text{and} \qquad \zeta = \frac{\alpha - \frac{k}{2} + \sqrt{\eta\,\phi}}{\alpha - \frac{k}{2} - \sqrt{\eta\,\phi}}.$$

To solve the equation in h_1 in (42), first note it is a linear PIDE where μ is a source term and where $h_2 + \frac{k}{2}$ is a *discount rate*. The general solution (probabilistic representation) of such an equation can be obtained using the Feynman-Kac theorem and we write

$$h_1(t,\mu) = \mathbb{E}_{t,\mu}\left[\int_t^T \exp\left\{\frac{1}{\eta}\int_t^u \, \left(h_2(s) + \tfrac{1}{2}k\right)\, ds\right\}\, \mu_u\, du\right]\, ,$$

which is simplified to

$$\boxed{h_1(t,\mu) = -\int_t^T \left(\frac{e^{-\gamma(T-u)} - \zeta e^{\gamma(T-u)}}{e^{-\gamma(T-t)} - \zeta e^{\gamma(T-t)}}\right) \mathbb{E}_{t,\mu}\left[\,\mu_u\,\right] du\,.}$$

Finally, we obtain h_0 by a straightforward application of Feynman-Kac:

$$\boxed{h_0(t,\mu) = \frac{1}{4\,\eta}\int_t^T \mathbb{E}_{t,\mu}\left[h_1^2(t,\mu)\right].}$$

The next theorem, **which is beyond the scope of this course**, shows that the candidate solution we obtain in this section is indeed the solution to the optimal control problem.

Theorem 2. *The candidate value function defined in and (41) is indeed the solution to the control problem (36). The trading speed given by*

$$\boxed{\nu_t^* = -\gamma\frac{\zeta e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}}Q_t^{\nu^*} + \frac{1}{2\eta}\int_t^T \left(\frac{\zeta e^{\gamma(T-u)} - e^{-\gamma(T-u)}}{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}}\right) \mathbb{E}\left[\mu_u \mid \mathcal{F}_t^\mu\right] du} \tag{43}$$

is admissible and is optimal. $\mathcal{F}_t^\mu$ is the natural filtration generated by μ.

Proof. We obtained a classical solution ($\mathcal{C}^1$ in time and $\mathcal{C}^2$ is the state variables) so standard results (see Pham (2009)) imply that it suffices to check that the feedback control is admissible. From the feedback form in (39) we obtain the optimal control (43). Since μ is assumed bounded from above a.s., then $\nu^\star$ is bounded from above and below a.s. and is therefore admissible. $\square$

The Almgren-Chriss strategy. The Almgren-Chriss strategy is a special case of the model presented in this Section, and corresponds to the case $\mu \equiv 0$. In that case, the optimal trading speed is

$$\nu_t^{\mathrm{AC}} = -\gamma\frac{\zeta e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}}Q_t^{\nu^*}$$

and corresponds to the classical Almgren-Chriss strategy in Section 4 equation (15).

The optimal strategy. The first term in (43) corresponds to the classical Almgren-Chriss strategy in Section 4 equation (15). The second term in (43) is a speculative component, it adjusts the speed of trading using the weighted average of the future expected value of the signal μ over the remainder of the trading window, i.e., $[t, T]$. In particular, the strategy gives more weight to signal values near t, and the contribution of the expected drift values decreases as it approaches the trading horizon (because of the terminal penalty).

Also, note that when the expected weighted drift value is positive (if the signal predicts an increase of the price on average in the future), then the agent increases the speed of trading (buys the asset), and when the expected weighted drift is negative, the agent decreases the trading speed.

Finally, note that the speculative component in (43) is more significant when the ratio $\frac{k}{\eta}$ is large. Thus, when the permanent impact is large and the execution costs are low, the agent speculates more.

The following section presents some specific uses of the optimal strategy when using LOB-based price predictors or using the trading flow of other agents. Both sections are based on the works Cartea et al. (2023) and Cartea and Jaimungal (2016b), respectively.

7.4 Imbalance and MACD

Here, we explore the use of two popular predictive signals. One is volume imbalance in the LOB, and the second is the Moving Average Convergence Divergence. Many works investigate the predictive power of the volume imbalance on the future midprice; see Bechler and Ludkovski (2015); Cont et al. (2022); Cartea et al. (2023). Volume imbalance has been shown to considerably boost profits of execution and market making strategies; see Cartea et al. (2018a). The MACD is shown to measure the strength of the trend in prices; see Baz et al. (2015).

The volume imbalance signal process $\left(I_t^1\right)_{t\in\mathbb{R}^+}$ is given by

$$I_t^1 = \frac{Q_t^{\mathrm{B}} - Q_t^{\mathrm{A}}}{Q_t^{\mathrm{B}} + Q_t^{\mathrm{A}}} \in [-1, 1] \,, \tag{44}$$

where $\left(Q_t^{\mathrm{A}}\right)_{t\in\mathbb{R}^+}$ and $\left(Q_t^{\mathrm{B}}\right)_{t\in\mathbb{R}^+}$ are the aggregated volume of limit orders resting on the best bid and best ask at event time t, respectively. Volume imbalance captures the difference between the buy and sell pressure; values of I_t^1 close to 1 indicate that the ask queues are relatively short, so it is likely to see a price increase within the next few events. Similarly, when volume imbalance is close to -1, the ask queues are relatively short, so it is likely to see a price decrease within the next few events.

MACD is designed to identify recent trend changes. In discrete event time, the MACD signal $\left(I_t^2\right)_{t\in\mathbb{R}_+}$ is given by

$$\begin{cases} \tilde{S}_t &= \mathrm{E}^{\varepsilon_1}\left(S_t\right) - \mathrm{E}^{\varepsilon_2}\left(S_t\right)\,, \\ I_t^2 &= 10^5\left(\tilde{S}_t - \mathrm{E}^{\varepsilon_3}\left(\tilde{S}_t\right)\right)/S_{t-\varepsilon_2-\varepsilon_3}\,, \end{cases} \tag{45}$$

where $(S_t)_{t\in\mathbb{R}^+}$ is the mid-price process, and $\varepsilon_2 > \varepsilon_1$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are positive integer constants. The exponential moving average $\mathrm{E}^{\varepsilon}(x_t)$ for a discrete-process x observed at event frequency Δt is obtained recursively with

$$\mathrm{E}^{\varepsilon}(x_t) = \varepsilon\, x_t + (1-\varepsilon)\,\mathrm{E}(x_{t-\Delta t})\,,$$

where $\mathrm{E}^{\varepsilon}(x_0) = x_0$. The decay parameter ε is given by $\varepsilon = 1 - \exp\left(-\log 2\,/\,h\right)$, where h is the half-life of the exponential decay.[11]

Ticker	Avg. spread (in ticks)	Avg. queue size best bid	Avg. queue size best ask	Events per minute
AAPL	1.48	532.72	543.38	4472.36
AMZN	1.44	512.35	517.36	3939.42
BIDU	11.37	141.48	152.81	187.09
COST	24.62	73.39	72.33	177.15
DELL	1.60	396.65	390.34	361.98
GOOG	1.43	496.99	514.38	2766.57
INTC	1.16	5247.56	5197.65	1458.23

Table 1: Descriptive statistics of transaction data between 1 October 2022 and 31 December 2022.

[11] The MACD in (45) involves three different exponential filters. The filters $\mathrm{E}^{\varepsilon_1}$ and $\mathrm{E}^{\varepsilon_2}$ are applied to the price S_t to obtain $\tilde{S}_t$. The value of $\tilde{S}$ is an estimate of the rate of change of the midprice, and I^2 is an estimate of the acceleration rate of the midprice; when prices accelerate upward (downward), the value of MACD is large and positive (negative).

We use the data described in Table 1 to compute the two signals. We use the volume at the best bid and best ask quotes to compute the imbalance in (44), and we use $\varepsilon_1 = 12$, $\varepsilon_2 = 26$, and $\varepsilon_3 = 9$ events to compute the MACD in (45). Figure 13 shows the average price innovation conditional on each signal value, 20 and 100 events ahead.

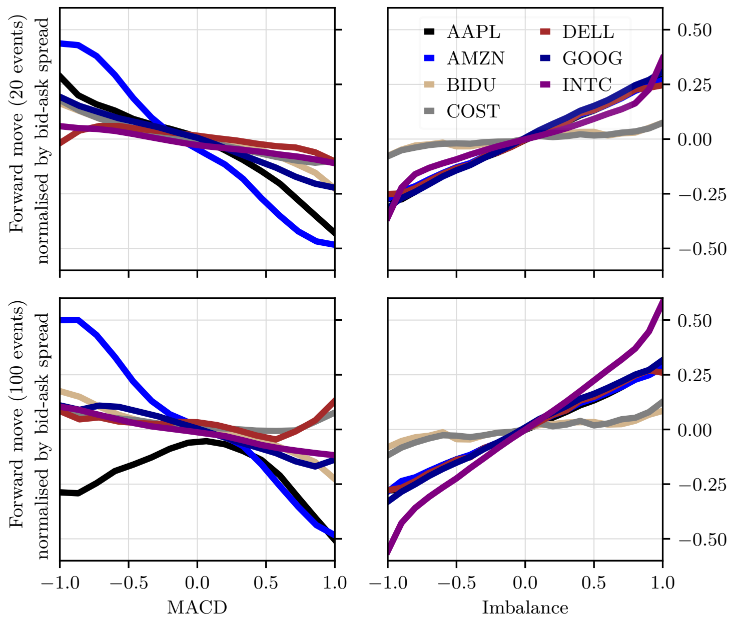

Figure 13: Left: Average price innovation after 20 events (top) and 100 events (bottom) normalised by the average bid-ask spread, as a function of MACD values. Right: Average price innovation after 20 events (top) and 100 events (bottom) normalised by the average bid-ask spread, as a function of imbalance values; data is between October 2022 and December 2022.

Figure 13 shows that both signals have predictive power for the future price innovation 20 and 100 events ahead, which corresponds to time horizons ranging from approximately 1 to 30 seconds in the data we consider. Figure 13 indicates that, over the period that we consider, the imbalance is a short-term momentum signal, while MACD is a mean reversion signal for some securities and a momentum signal for others. Also, Figure 14 shows that both signals are mean reverting.

To apply our findings to optimal execution with volume imbalance or MACD, we consider the following mean-reverting dynamics for the predictive signal μ

$$d\mu_t = -\kappa\,\mu_t + \xi dB_t\,, \tag{46}$$

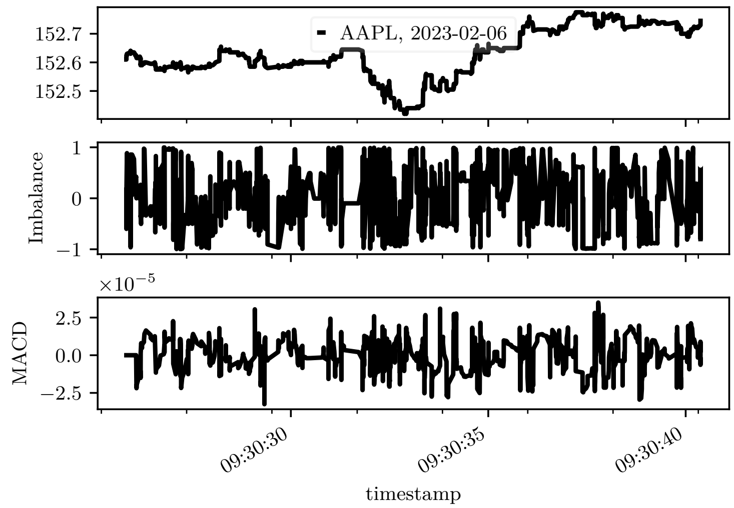

Figure 14: Imbalance and MACD signals for AAPL.

where κ drives the mean reversion speed to 0; see Figure 14, $(B_t)_{t\in[0,T]}$ is a standard Brownian motion independent of all other processes, and $\xi > 0$ drives the dispersion of the signal. Note that all the coefficients of the Ornstein-Uhlenbeck dynamics above can be classically estimated using least squares regression.[12]

Note that the solution to the SDE in (46) for $s \geq t$ is

$$\mu_s = e^{-\kappa(s-t)}\mu_t + \int_t^s e^{-\kappa(s-t)}\gamma \, dB_t,$$

so the expected price drift is

$$\mathbb{E}\left[\mu_s \middle| \mathcal{F}_t^{\mu}\right] = e^{-\kappa(s-t)}\mu_t,$$

and the optimal trading speed in (43) using the volume imbalance or MACD is

$$\boxed{\nu_t^* = -\gamma \frac{\zeta e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}} Q_t^{\nu^*} + \frac{1}{2\eta}\mu_t \int_t^T \frac{\zeta e^{2\gamma(T-u)} - 1}{\zeta e^{2\gamma(T-t)} - 1} e^{(\gamma-\kappa)(u-t)} du.}$$

In particular, the optimal trading speed is linear in the signal.

[12] A time discretisation of an Ornstein-Uhlenbeck model gives rise to an Auto-Regressive model of order 1, or AR(1). The parameters of an AR(1) model are classically estimated by using least squares regression. Conversion of AR(1) coefficients into their continuous-time counterparts is straightforward.

7.5 Order flow

In Cartea and Jaimungal (2016b), the authors consider the following model for μ:

$$\mu_t = k\,\mu^+ - k\,\mu^-\,, \tag{47}$$

where μ^+ is the buying trading flow of other agents, which has a linear permanent impact of magnitude k on the midprice, and μ^- is the selling trading flow of other agents, which has also a linear permanent impact of magnitude k on the midprice.

In particular, they consider the following mean reverting dynamics with jumps for the order flow on both sides of the book

$$d\mu_t^\pm = -\kappa\,\mu_t^\pm + b_{1+N_{t^-}^\pm} dN_t^\pm\,,$$

where $\kappa > 0$ determines the speed of mean reversion, $N_t^\pm$ are two Poisson processes assumed independent of all other processes and with fixed constant intensity λ, and $\{b_1^\pm, b_2^\pm, \dots\}$ are non-negative i.i.d. random variables, with finite first moment, and are also independent of all the other processes in the problem.

The dynamics above imply that the buying and selling orders arrive independently according to Poisson times, and induce an increase in the order flow rate $\mu^\pm$ (the so-called self-exciting nature of the order flow).

The solution to the SDE $\mu^\pm$ for $s \geq t$ is

$$\mu_s^\pm = e^{-\kappa^\pm(s-t)}\mu_t^\pm + \int_t^s e^{-\kappa^\pm(s-u)}\eta_{1+L_{u^-}^\pm} dL_u^\pm,$$

so the expected drift in the price is

$$\mathbb{E}\left[\mu_s^\pm \middle| \mathcal{F}_t^\mu\right] = e^{-\kappa^\pm(s-t)}\left(\mu_t^\pm - \psi^\pm\right) \pm \psi^\pm,$$

where the constants $\psi^\pm$ are given by

$$\psi^\pm = \frac{1}{\kappa^\pm}\lambda^\pm \mathbb{E}[\eta^\pm].$$

For simplicity, we consider the limiting case of the general optimal strategy (43) when the agent requires that all shares are liquidated at the end of the trading horizon and sets $\alpha \to \infty$, so the optimal strategy simplifies to

$$\lim_{\alpha\to\infty} \nu_t^* = -\gamma\frac{\cosh(\gamma(T-t))}{\sinh(\gamma(T-t))}Q_t^{\nu^*} + \frac{1}{2\,\eta}\int_t^T \frac{\sinh(\gamma(T-u))}{\sinh(\gamma(T-t))}\mathbb{E}\left[\mu_u \mid \mathcal{F}_t^\mu\right] du.$$

In this case, the optimal strategy when using the trading flow of other agents and considering the dynamics (47) is

$$
\boxed{
\begin{aligned}
\lim_{\alpha\to\infty} \nu_t^* = &- \gamma\frac{\cosh(\gamma(T-t))}{\sinh(\gamma(T-t))} Q_t^{\nu^*} \\
&+ \frac{1}{2\,\eta}\left[\ell_1^+(t)\left(\mu_t^+ - \psi^+\right) - \ell_1^-(t)\left(\mu_t^- - \psi^-\right) + \ell_0(t)\left(\psi^+ - \psi^-\right)\right],
\end{aligned}
}
$$

where

$$
\ell_0(t) = \frac{1}{\gamma}\frac{\cosh(\gamma(T-t)) - 1}{\sinh(\gamma(T-t))}
$$

and

$$
\ell_1^\pm(t) = \frac{1}{2}\left(\frac{e^{\gamma(T-t)} - e^{-\kappa^\pm(T-t)}}{\kappa^\pm + \gamma} - \frac{e^{-\gamma(T-t)} - e^{-\kappa^\pm(T-t)}}{\kappa^\pm - \gamma}\right) \Bigg/ \sinh(\gamma(T-t)).
$$

7.6 Further readings

Lorenz and Schied (2013) studied the effect of trends in the midprice. The seminal work in the literature on optimal execution with stochastic signals is in Cartea and Jaimungal (2016b) who use a general Markov process that drives the stochastic drift of the asset, and interpret it as the order flow of other market participants. Later, Cartea et al. (2018a) study volume imbalance as a price predictor and considers an optimal trading framework that incorporates order book signals, and Lehalle and Neuman (2019) and Neuman and Voß (2020) extend the CJ framework to incorporate transient market impact. Recently, the following works studied market signals in algorithmic trading strategies: Barger and Lorig (2019); Cartea and Wang (2020); Donnelly and Lorig (2020); Forde et al. (2022); Bergault et al. (2022).

8 Optimal trading with transient impact

The previous sections derived optimal trading strategies that take into account two types of price impact effects; permanent and instantaneous (or temporary). In practice, the nature of price changes, after the occurrence of successive transactions on the same side of the book, involve other dynamical effects.

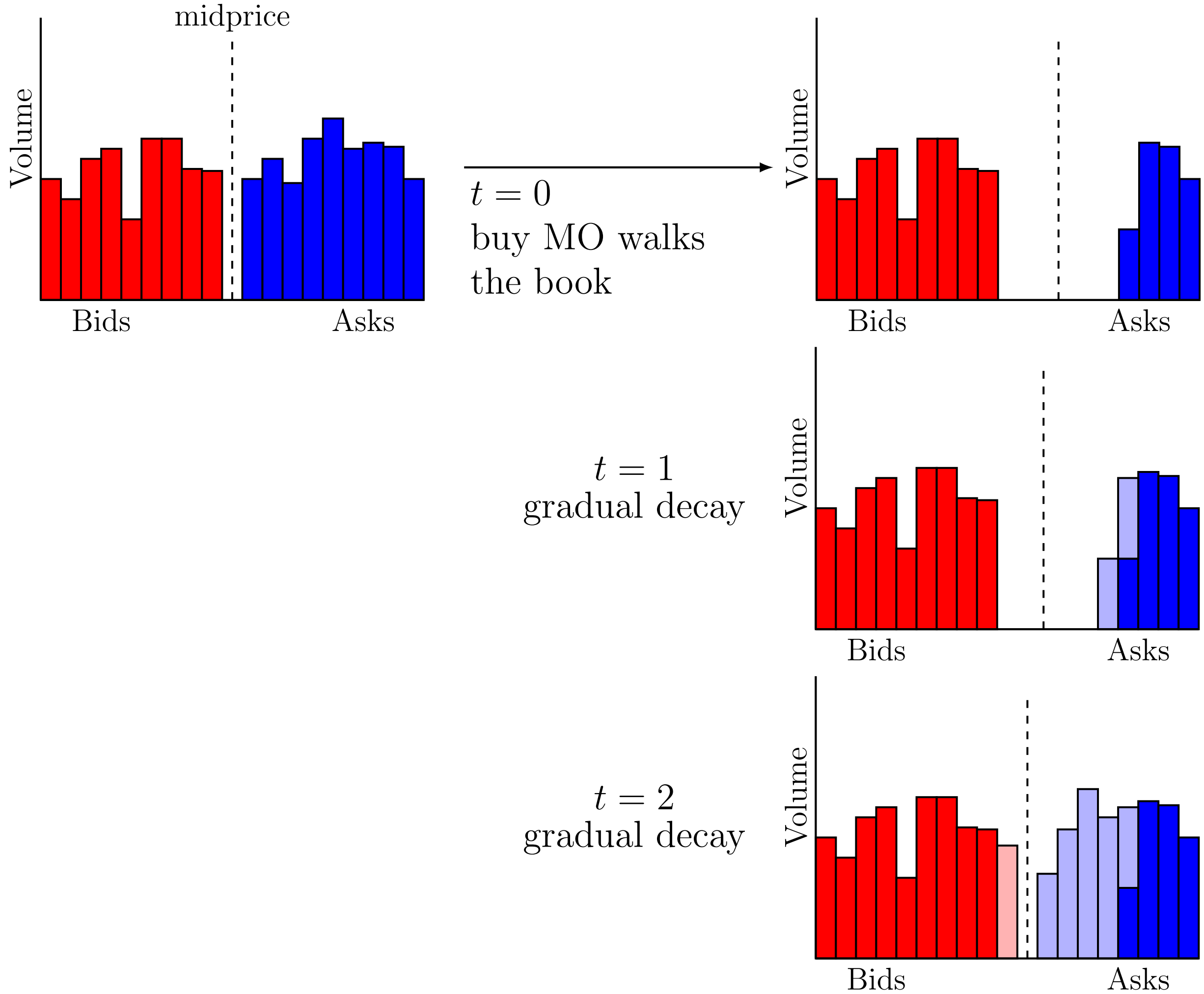

Figure 15: Resilience and transient impact after a large MO. In the first two panels, an MO walks the book so the next midprice exhibits the temporary price impact. Immediately after the MO in the third panel, market participants post new LOs on the ask side so the midprice gradually decays back to its original value. The difference between the midprice in the last panel and that of the first panel is the permanent impact of the MO.

Figure 15 shows that following the impact of a large transaction in the LOB after an MO walks the book, there is a resulting gap which is not instantaneously replenished. The dynamics used for the midprice and cash in the previous models suggest that this process is instantaneous. Empirical evidence from multiple studies show that it takes time for LOs to be submitted in place of those that were filled. The rate at which these orders are replenished is called resilience, and thus we say that the models of Sections 3, 4, 5, 6, 7, and 12 consider *infinite resilience*.

Resilience in the LOB is important when considering the execution costs of trading activity. Figure 15 shows an idealised situation where sell orders are gradually replenishing the LOB without any further transaction (MO). If a subsequent buy MO is sent to the LOB after only a short time following the first MO, then it would be subjected to higher prices than if it had occurred after a longer time. This difference in transaction price depends on the amount of time between trades and is called the transient price impact. The transient price impact was not considered in the previous models which make the implicit assumption that the agent waits long enough between consecutive trades so the transient impact is negligible. This assumption is realistic in very liquid markets (e.g., large cap equities) and when the agent does not trade frequently.

A general formulation of transient price impact can be incorporated in the dynamics of the midprice. This section describes a model that takes into account the transient impact of successive market orders. In particular, we introduce an **impact function** that describes the instantaneous response of the LOB to a transaction, and a **decay kernel** that models the resilience of the LOB.

8.1 Modelling framework

Consider a filtered probability space $\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}\big)$, with $T > 0$, satisfying the usual conditions and supporting all the processes we introduce below.

An agent must choose the trading speed at which they send market orders to liquidate $Q_0 \in \mathbb{R}$ shares over a trading window $[0, T]$ where $T > 0$. The trading speed is $\nu = (\nu_t)_{t\in[0,T]}$ and the inventory is $Q = (Q_t)_{t\in[0,T]}$ and it satisfies

$$\boxed{dQ^\nu_t = \nu_t \, dt.} \tag{49}$$

The investor's trading activity affects the midprice process in two ways. One is permanent (linear) and the other is transient. We denote by $(S_t^\nu)_{t\in[0,T]}$ the process describing the dynamics of the midprice. A general formulation of transient price impact can be incorporated into the dynamics of S^ν by introducing a strictly increasing impact function h and a non-increasing decay kernel function G. The process S^ν follows the dynamics

$$\boxed{S_t^\nu = S_0 + \underbrace{\sigma W_t}_{\substack{\text{market}\\ \text{risk}}} + \underbrace{k\int_0^t \nu_s ds}_{\text{permanent impact}} + \underbrace{\int_0^t h\left(\nu_s\right)\, G\left(t-s\right)\, ds}_{\text{transient impact}},} \tag{50}$$

where $W=(W_t)_{t\in[0,T]}$ is a standard Brownian motion, $\sigma>0$ is the volatility, and $k\geq 0$ is the permanent price impact parameter.

The function h determines the magnitude of transient impact and the decay kernel G controls how quickly the impact of trading decays through time; see Figure 15.

At any time t, the average price per share obtained is $\tilde{S}_t^\nu$ and we write

$$\tilde{S}_t^\nu = S_t^\nu + \eta\,\nu_t\,,$$

where η is a non-negative constant that scales the linear execution costs. The investor's cash process $X^\nu = (X_t)_{t\in[0,T]}$ satisfies the SDE

$$\boxed{dX_t^\nu = -\tilde{S}_t^\nu\,\nu_t\,dt = -\left(S_t^\nu + \eta\,\nu_t\right)\,\nu_t\,dt, \quad X_0^\nu = X_0.} \tag{51}$$

Performance criterion. The agent's performance criterion is

$$\boxed{H^\nu\left(t,x,s,q\right) = \mathbb{E}_{t,x,s,q}\left[X_T + Q_T^\nu\,\left(S_T^\nu - \alpha\,Q_T^\nu\right) - \phi\int_t^T \left(Q_u^\nu\right)^2 du\right]\,,} \tag{52}$$

where $\mathbb{E}_{t,x,s,q}$ is the expectation conditioned on (with a slight abuse of notation) $X_t=x$, $S_t=S$, and $Q_t=q$. The first term on the right-hand side of the performance criteria (52) is the agent's terminal cash, the second represents the proceeds from liquidating any remaining inventory Q_T^ν at the terminal time, the third term is a penalty term that captures the cost of liquidation the terminal penalty, and finally, the last term is the running inventory penalty and models the urgency of execution.

8.2 The case of exponential decay

Here, we consider that the decay of the transient impact is exponential and we write $G(t-s) = \exp(-\beta(t-s))$ for $s \leq t$. The parameter ρ represents the rate of decay.

Using similar arguments as in Section 1.2 where we proved that permanent impact must be linear to avoid dynamic arbitrage, Gatheral and his co-authors also obtained an important and similar result for transient impact in Gatheral (2010). Assuming a particular price process, the authors of Gatheral et al. (2012) demonstrate that a model that combines a nonlinear impact function h with exponential decay admits price manipulation, which is an undesirable feature that disqualifies the model.

Thus, we consider a linear instantaneous impact function

$$h : \nu \mapsto \lambda \, \nu.$$

The coefficient λ determines the distance between the midprices before and after an MO arrives in the LOB.

To further study our problem, note that the transient component of the midprice dynamics in (50) is now

$$\int_0^t \lambda \, \nu_s \, e^{-\beta \, (t-s)} \, ds.$$

We define a new state variable $(I_t^\nu)_{t\in[0,T]}$, where $I_0 = 0$, that quantifies the accumulated transient impact and we write

$$I_t^\nu = \int_0^t \lambda \, \nu_s \, e^{-\beta \, (t-s)} \, ds.$$

The dynamics above solve the following SDE:

$$\boxed{dI_t^\nu = (\lambda \, \nu_t - \beta \, I_t^\nu) \, dt, \quad I_0 = 0,}$$

so the dynamics of the midprice are

$$\boxed{dS_t^\nu = \sigma \, dW_t + \left(\tilde{\lambda} \, \nu_t - \beta \, I_t^\nu\right) \, dt, \quad S_0^\nu = S_0 \text{ known.}} \tag{53}$$

The performance criterion of the agent becomes

$$H^\nu \, (t, x, s, I, q) = \mathbb{E}_{t,x,s,I,q} \left[X_T + Q_T^\nu \, (S_T^\nu - \alpha \, Q_T^\nu) - \phi \int_t^T (Q_u^\nu)^2 \, du \right],$$

where $\mathbb{E}_{t,x,s,I,q}$ is the expectation conditioned on $X_t = x$, $S_t = S$, $I_t = I$, and $Q_t = q$. The value function $H : [0,T] \times R^4 \mapsto \mathbb{R}$ of the agent is

$$H\left(t, x, s, I, q\right) = \sup_{\nu \in \mathcal{A}} H^{\nu}\left(t, x, s, I, q\right) \,, \tag{54}$$

where $\mathcal{A}$ is the set of admissible strategies consisting of $\mathcal{F}-$predictable processes that satisfy $\int_0^T |\nu_u| \, du < \infty$, $\mathbb{P}-$a.s..

Use the dynamics in (49)-(51)-(53) to write the HJB equation associated with the problem (54) as

$$\begin{aligned} 0 =& \partial_t H - \phi\, q^2 + \frac{1}{2}\sigma^2\, \partial_{SS} H - \beta\, I\, \partial_S H - \beta\, I\, \partial_I H \\ &+ \sup_{\nu} \left\{ \tilde{\lambda}\, \nu\, \partial_S H + \lambda\, \nu\, \partial_I H + \nu\, \partial_q H - (S + \eta\, \nu)\, \nu\, \partial_x H \right\}, \end{aligned}$$

subject to the terminal condition

$$H\left(T, x, S, I, q\right) = x + q\, S - \alpha\, q^2.$$

As usual, we use the ansatz

$$H\left(t, x, S, I, q\right) = x + q\, S + h\left(t, q, I\right) \,,$$

and find the following nonlinear PDE for $h(t, q, I)$:

$$0 = \partial_t h - \phi\, q^2 - \beta\, I\, q - \beta\, I\, \partial_I h + \sup_{u} \left\{ \nu \left(\tilde{\lambda}\, q + \lambda\, \partial_I h + \partial_q h \right) - \eta\, \nu^2 \right\}, \tag{55}$$

subject to the terminal condition

$$h\left(T, I, q\right) = -\alpha\, q^2.$$

The supremum term in (55) can be solved with a first order condition, so the optimal feedback trading speed is

$$\boxed{\nu^{\star} = \frac{\tilde{\lambda}\, q + \lambda\, \partial_I h + \partial_q h}{2\, \eta},} \tag{56}$$

and the PDE simplifies to

$$0 = \partial_t h - \phi\, q^2 - \beta\, I\, q - \beta\, I\, \partial_I h + \frac{\left(\tilde{\lambda}\, q + \lambda\, \partial_I h + \partial_q h \right)^2}{4\, \eta}, \tag{57}$$

subject to the same terminal condition.

The terminal condition $h\,(T, I, q) = -\alpha\, q^2$ suggests that h is a quadratic polynomial in q, and the nonlinear PDE (57) suggests that the dependence on I is also quadratic. We introduce a second ansatz and look for a solution $h(T, q, I)$ of the following form:

$$h(t, q, I) = A(t)q^2 + B(t)\, q\, I + C(t)\, I^2 + D(t)\, q + E(t)\, I + F(t)\,, \qquad (58)$$

for all $(t, q, I) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, or equivalently

$$\boxed{h(t, q, I) = \begin{pmatrix} q \\ I \end{pmatrix}^\intercal P(t) \begin{pmatrix} q \\ I \end{pmatrix} + \begin{pmatrix} D(t) \\ E(t) \end{pmatrix}^\intercal \begin{pmatrix} q \\ I \end{pmatrix} + F(t)\,,}$$

where $P : [0, T] \to \mathcal{S}_2(\mathbb{R})$ is defined as

$$\boxed{P(t) = \begin{pmatrix} A(t) & \frac{1}{2}B(t) \\ \frac{1}{2}B(t)^\intercal & C(t) \end{pmatrix}.} \qquad (59)$$

The following result shows that the ansatz (58) leads to a matrix Riccati equation.

Proposition 2. *Assume there exist $A \in C^1([0, T], \mathbb{R})$, $B \in C^1([0, T], \mathbb{R})$, and $C \in C^1([0, T], \mathbb{R})$ such that P in (59) satisfies the matrix Riccati equation*

$$\boxed{0 = P'(t) + Q + Y^\intercal P(t) + P(t)Y + P(t)UP(t),} \qquad (60)$$

with terminal condition

$$P(T) = \begin{pmatrix} -\alpha & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$\boxed{Q = \begin{pmatrix} -\phi + \frac{\tilde{\lambda}^2}{4\,\eta} & -\frac{\beta}{2} \\ -\frac{\beta}{2} & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} \frac{\tilde{\lambda}}{2\,\eta} & 0 \\ \frac{\tilde{\lambda}\,\lambda}{2\,\eta} & -\beta \end{pmatrix}, \quad \text{and} \quad U = \frac{1}{\eta}\begin{pmatrix} 1 & \lambda \\ \lambda & \lambda^2 \end{pmatrix},}$$

then h defined in (58) with $D \equiv E \equiv F \equiv 0$ solves the PDE (57).

Proof. First, replace the ansatz (58) in the PDE (57). The term stemming from the supremum in (55) writes

$$\left(\tilde{\lambda}\, q + \lambda\, \partial_I h + \partial_q h\right)^2 = \left(\tilde{\lambda} + \lambda\, B(t) + 2\, A(t)\right)^2 q^2 + (2\, \lambda\, C(t) + B(t))^2\, I^2$$

$$
\begin{aligned}
&+ 2\left(\tilde{\lambda} + \lambda\, B(t) + 2\, A(t)\right)(2\,\lambda\, C(t) + B(t))\, q\, I \\
&+ 2\,(\lambda\, E(t) + D(t))\left(\tilde{\lambda} + \lambda\, B(t) + 2\, A(t)\right) q \\
&+ 2\,(\lambda\, E(t) + D(t))\,(2\,\lambda\, C(t) + B(t))\, I \\
&+ (\lambda\, E(t) + D(t))^2\,,
\end{aligned}
$$

so we find that for all q and I that the following polynomial is always zero:

$$
\begin{aligned}
0 = &\left(A'(t) - \phi + \frac{1}{4\,\eta}\left(\tilde{\lambda} + \lambda\, B(t) + 2\, A(t)\right)^2\right) q^2 \\
&+ \left(B'(t) - \beta - \beta\, B(t) + \frac{1}{2\,\eta}\left(\tilde{\lambda} + \lambda\, B(t) + 2\, A(t)\right)(2\,\lambda\, C(t) + B(t))\right) q\, I \\
&+ \left(C'(t) - 2\,\beta\, C(t) + \frac{1}{4\,\eta}\,(2\,\lambda\, C(t) + B(t))^2\right) I^2 \\
&+ \left(D'(t) + \frac{1}{2\,\eta}\,(\lambda\, E(t) + D(t))\left(\tilde{\lambda} + \lambda\, B(t) + 2\, A(t)\right)\right) q \\
&+ \left(E'(t) - \beta\, E(t) + \frac{1}{2\,\eta}\,(\lambda\, E(t) + D(t))\,(2\,\lambda\, C(t) + B(t))\right) I \\
&+ F'(t) + (\lambda\, E(t) + D(t))^2\,.
\end{aligned}
$$

Thus every coefficient of the above polynomial is alway zero. Thus we obtain the following system of ODEs

$$
\begin{cases}
0 = & A'(t) - \phi + \frac{1}{4\,\eta}\left(\tilde{\lambda} + \lambda\, B(t) + 2\, A(t)\right)^2 \\
0 = & B'(t) - \beta - \beta\, B(t) + \frac{1}{2\,\eta}\left(\tilde{\lambda} + \lambda\, B(t) + 2\, A(t)\right)(2\,\lambda\, C(t) + B(t)) \\
0 = & C'(t) - 2\,\beta\, C(t) + \frac{1}{4\,\eta}\,(2\,\lambda\, C(t) + B(t))^2 \\
0 = & D'(t) + \frac{1}{2\,\eta}\,(\lambda\, E(t) + D(t))\left(\tilde{\lambda} + \lambda\, B(t) + 2\, A(t)\right) \\
0 = & E'(t) - \beta\, E(t) + \frac{1}{2\,\eta}\,(\lambda\, E(t) + D(t))\,(2\,\lambda\, C(t) + B(t)) \\
0 = & F'(t) + (\lambda\, E(t) + D(t))^2\,.
\end{cases}
$$

This system of ODEs can be decomposed into three groups of equations; the first three ODEs for A, B and C are independent of the others and can be solved independently, the linear ODEs for D and E, and finally F. Note that the ODEs for D and E admit as a (unique) solution $D \equiv E \equiv 0$ so the solution for F is also zero. Thus it remains to study the first group of ODEs. Defining P as in (59) and the matrices Q, U, and Y as in Proposition 2, we obtain that P solves indeed the matrix Riccati equation (60). □

Using the ansatz (58) in the optimal feedback control in (56) simplifies to

$$\boxed{\nu^\star = \frac{\tilde{\lambda}\, q + \lambda\, \left(B(t)\, q + 2\, C(t)\, I\right) + 2\, A(t) q + B(t)\, I}{2\, \eta}.} \tag{61}$$

8.3 Simulation results

Note that because I^ν is deterministic, the strategy also is deterministic. The Riccati equation (60) cannot be solved in closed-form. However, because it is an ODE, there exists very efficient approximation techniques to obtain the solution and study its behaviour.[13] Thus, the strategy can be implemented in practice.

We solve numerically for the solution P to the matrix Riccati equation (60) and obtain the optimal strategy (61). We use the model parameters $T = 1$ second, $\eta = 0.01$, $\alpha = 10$, and $Q_0 = 10$. We also set a zero permanent impact ($k = 0$) and zero urgency ($\phi = 0$) to focus on the effect of transient impact on the optimal strategy, and we set arbitrary large value of the terminal penalty to ensure full liquidation.

Figure 16 shows the optimal trading curve for various values of λ. When $\lambda = 0$, the optimal strategy is a straight line, which is a TWAP strategy. This is expected because when transient impact is zero, and so is permanent impact and inventory penalty, then the strategy is TWAP. When $\lambda \neq 0$, the behaviour corresponds to trading at a faster pace closer to the beginning and the end of the trading period $[0, T]$, and trading at a slower pace throughout the trading period. The interpretation is that the agent wishes to minimise the time when they are trading with high transient price impact. The effect of transient market impact is larger when there are successive large orders; a single MO will only suffer execution costs (η) and it is subsequent trades that incur transient impact costs. By trading quickly early, the agent will spend more time trading slowly while waiting for the exponential resilience to drive the price back to a profitable level. When the transient impact has decayed significantly by the end of the trading window, the agent places again a large

[13] See the Jupyter notebook https://github.com/FDR0903/HFT_course/blob/main/transient_impact.ipynb.

trade to complete the liquidation; see Chen et al. (2019) for more details and discussions for a simpler yet similar strategy.

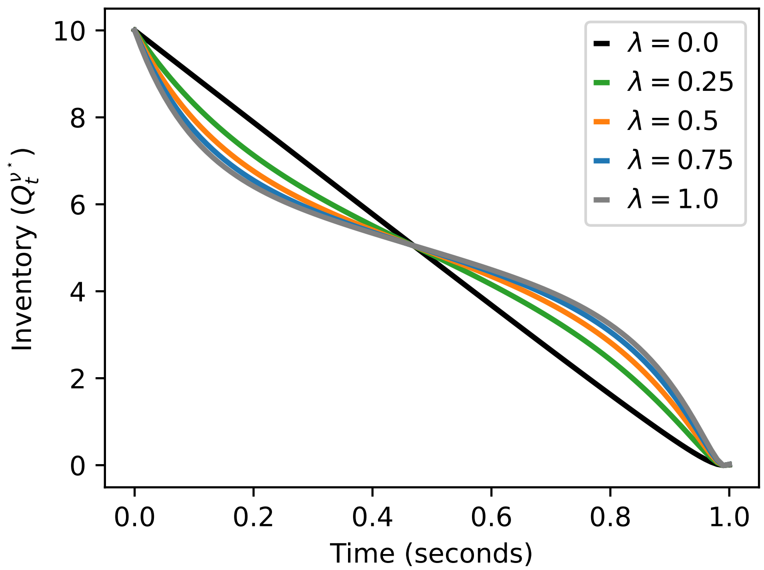

Figure 16: Optimal trading curve with transient impact price impact for various values of the transient impact parameter λ. We use the model parameters $k = \phi = 0$, $T = 1$ second, $\eta = 0.01$, $\alpha = 10$, $Q_0 = 10$, and $\beta = 1$.

Figure 17 shows the optimal trading curve for various values of β. When $\beta = 0$, we also obtain a TWAP strategy. This is expected because when there is no resilience, the transient impact is simply a permanent impact. Also, as β becomes larger ($\beta = 20$ in the figure), the decay becomes instantaneous and this is close to having zero transient impact. In this case, the strategy is also a TWAP as discussed above. When $\beta \neq 0$ and $\beta << \infty$, the behaviour of the strategy is the same as that described for Figure 16, i.e., faster trading at the start and end of the trading window to profit from the resilience of the LOB.

8.4 Further readings

Lorenz and Schied (2013) investigate the stability of optimal strategies in the presence of transient price impact with exponential decay and general dynamics of a drift in the underlying price process accounting for the price impact of other agents. Obizhaeva and Wang (2013) and Alfonsi et al. (2008) propose a single-asset market impact model where price dynamics are derived from a dynamic LOB model with resilience. Alfonsi and Schied (2010) derive

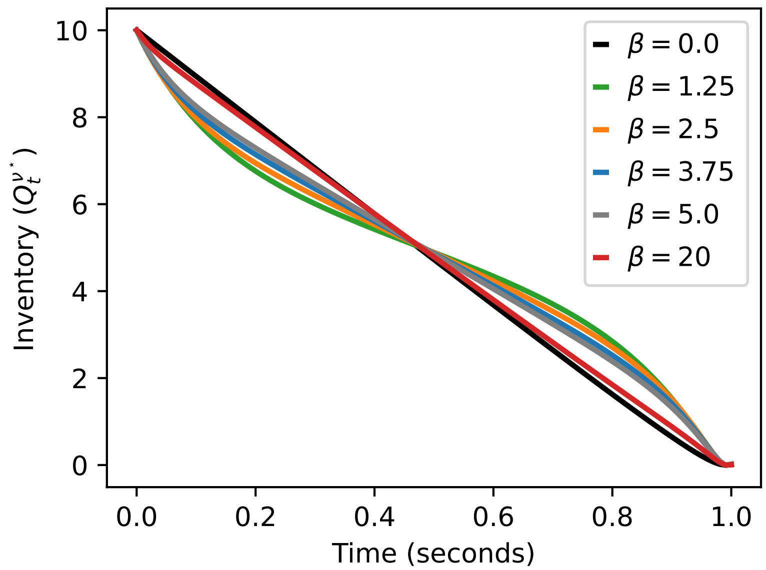

Figure 17: Optimal trading curve with transient impact price impact for various values of the decay coefficient β. We use the model parameters $k = \phi = 0$, $T = 1$ second, $\eta = 0.01$, $\alpha = 10$, $Q_0 = 10$, and $\lambda = 0.5$.

explicit optimal execution strategies in a discrete-time LOB model with general shape functions and an exponentially decaying price impact. Gatheral (2010) uses the no-dynamic-arbitrage principle to address the viability of market impact models. Gatheral et al. (2012) obtain explicit optimal strategies with a transient market impact in an expected cost minimisation setup. Chen et al. (2019) obtain closed-form strategies with transient impact with exponential decay when the terminal penalty is infinite.

9 Optimal trading with limit orders

The previous sections mainly discussed optimal execution strategies that use market orders (MOs) only. In LOBs, sending MOs guarantees execution. However, they incur execution costs in the form of the bid-ask spread and the cost of walking the book due to limited liquidity. In practice, market operators mainly use limit orders (LOs) that do not cross the spread in an attempt to reduce execution costs and not reveal their actions by adversely impacting the midprice. However, LOs are not guaranteed to be executed due to **price priority**, i.e., there is no guarantee that an opposing matching order will arrive at the desired price level. Also, limit orders that are posted on a level of the LOB are in a queue of previously posted LOs that have **time priority**.

9.1 Fill probability

Let S_t denote the midprice of an asset at time t. An agent that wishes to buy a quantity ΔQ posts a buy LO – the analysis for sell LOs is identical – with a desired price level $S_t - \delta^{\text{bid}}$. We call δ the *depth* of the LO and if the LO is executed, the depth measures the price improvement. The larger the value of δ, the lower the probability that the order will get executed, because time and price priority reduce the probability that an order arrives and walks the LOBs up to the posted depth.

The probability that an order gets filled at a given depth is called the **fill probability**, and it naturally decreases with δ. The fill probability depends on the current state of the LOB. Consider the top panels of Figure 18. The probability to be filled when posting at the best bid depends on the size of buy LOs already posted at the best bid, and on the selling pressure on the other side. Consider now the bottom panels of Figure 18. The probability to be filled when posting at the second best bid depends on the size of buy LOs already posted at the best bid and the second best bid, and on the selling pressure on the other side. Thus, the deeper the LO is posted, the less likely it is that opposing MOs large enough to walk the LOB will arrive.

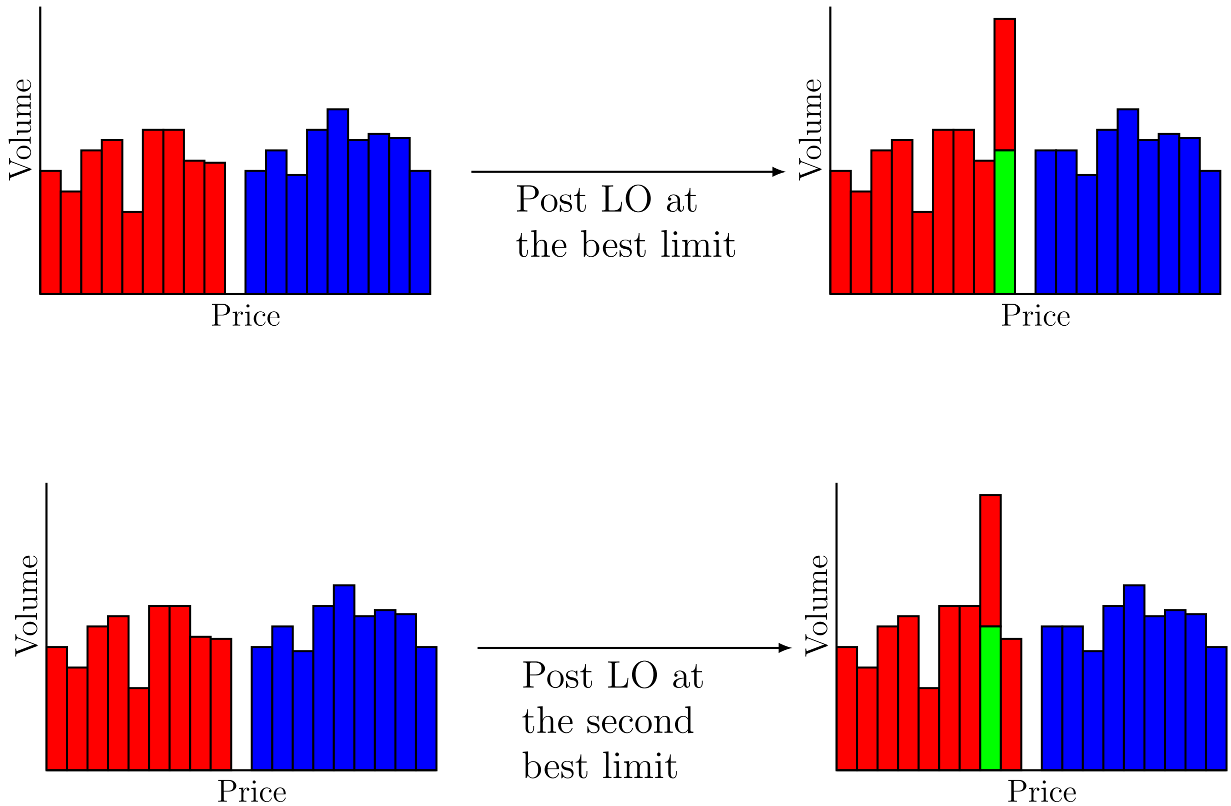

Figure 18: Limit order book.

One can use historical data to estimate the fill probability from LOB data as a function of the depth of the LO. Figure 19 shows the estimated fill probability of LOs (buy and sell) as a function of its depth for multiple shares quoted on Nasdaq. The figure shows that fill probability in LOBs has an power law type of decay, and can be approximated with an exponential decay.

If one assumes that the volume of individual MOs is exponentially distributed with mean volume $\overline{v}$, and that the LOB, on average, is block shaped with a fixed height A, i.e., the posted volume at each price level is equal to a constant A, then one recovers the exponential decay of the fill probability. More precisely, conditional on the arrival of a sell MO of size v, the probability that the buy LO is executed is given by

$$\mathbb{P}[\text{execution of LO of depth } \delta] = \mathbb{P}[v > A\,\delta] = \exp\left\{-\frac{A}{\overline{v}}\,\delta\right\}. \tag{62}$$

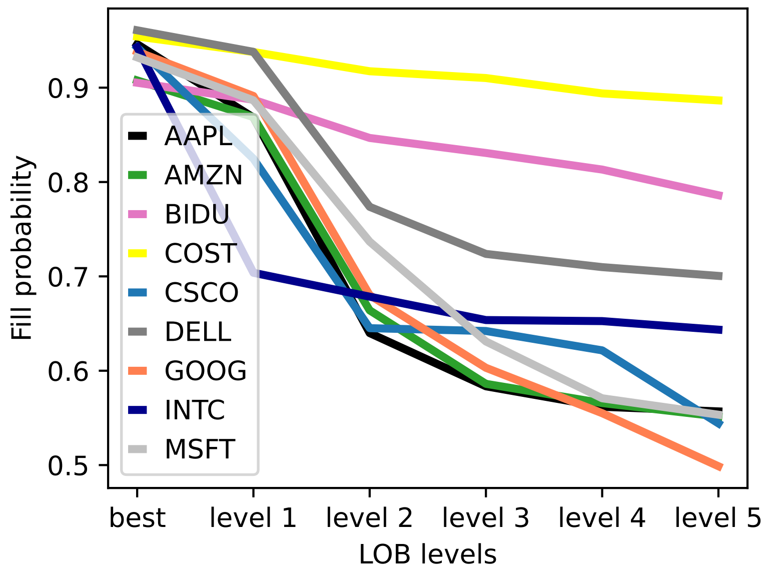

Figure 19: Fill probability at different levels of the LOB for multiple shares quoted on Nasdaq. Source: Arroyo et al. (2023).

We will use the assumption (62) of exponential decay of the fill probability for our model of optimal trading with LOs but also for the optimal market making model of Section 10. Although we do not take this into account in the models of this course, it must be noted that in practice, the fill probability depends significantly on the time the LO spends in the LOB before being cancelled or executed. The work in Arroyo et al. (2023) shows that this dependence can be modelled and estimated; see Figure 20.

9.2 Modelling framework

We work on a filtered probability space $\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}\big)$, with $T > 0$, satisfying the usual conditions. The space supports all the processes we introduce below.

An agent must liquidate (sell) $Q_0 > 0$ shares over a trading window $[0, T]$ where $T > 0$. We denote by $(S_t)_{t\in[0,T]}$ the asset's midprice which follows the dynamics

$$\boxed{dS_t = \sigma\, dW_t, \quad S_0 \in \mathbb{R}_+ \text{ is known,}}$$

where $\sigma > 0$ is the volatility parameter and W is a standard Brownian motion.

The strategy used by the agent relies on continuous post-and-cancel of sell LOs. At every instant in the trading window $[0, T]$, the agent reassesses

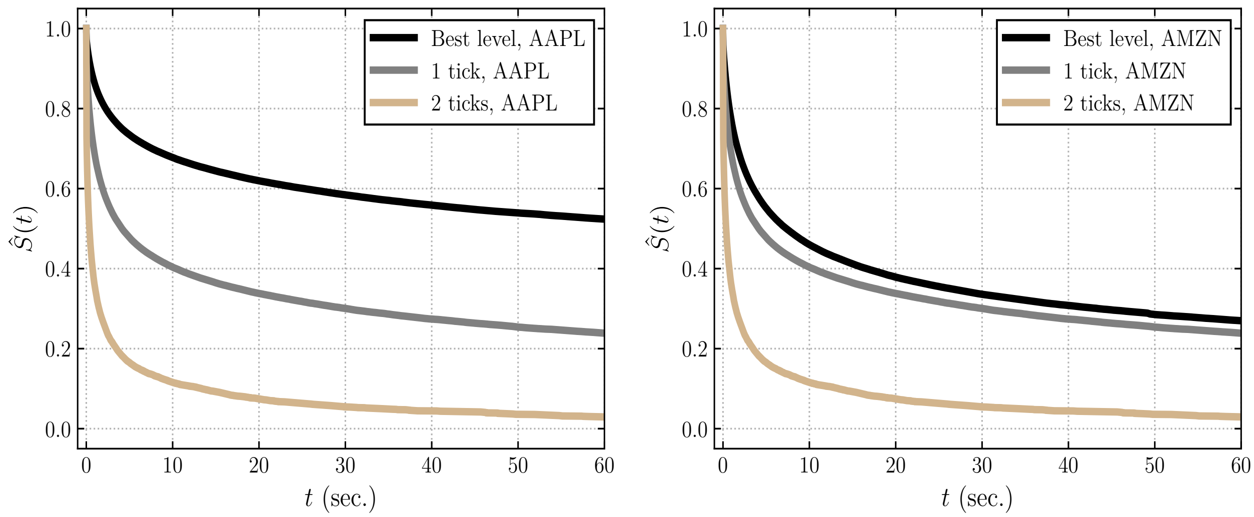

Figure 20: Estimate of the survival function when placing LOs at different depths of the LOB. **Left:** AAPL, **right:** AMZN. Source: Arroyo et al. (2023).

market conditions and the inventory level, cancels any LO resting in the book, and posts a new sell LO at the optimal depth δ. In particular, the agent tracks not only their inventory, but also the arrival of MOs from other traders that will possibly match the agent's posted LOs. Thus, the process $(\delta)_{t\in[0,T]}$ denotes the depth at which the agent continuously posts limit sell orders, i.e., the agent posts LOs at the price $S_t + \delta_t$ at time t. δ is an $\mathcal{F}-$predictable process that is left-continuous with right limits (cádlág).

As discussed in the previous section, the arrival of buy market orders that are large enough to lift the agent's sell LO depends on the depth of the LO. We model the arrival of orders that can reach the depth δ with a (controlled) counting process $\left(N_t^\delta\right)_{t\in[0,T]}$ that counts the number of market buy orders which reached the agent's LOs, i.e., it counts MOs which walk the sell side (ask) of the book to a price greater than or equal $S_t + \delta_t$. We assume that this counting process has an intensity $\Lambda(\delta)$ that depends on δ and we write

$$\boxed{\Lambda(\delta) = \lambda \exp(-\kappa\,\delta)\,,}$$

where $\kappa > 0$ is the exponential decay parameter. The constant λ represents a baseline intensity of buy MO orders that arrive in the LOB. Higher values of the decay parameter κ means that for a fixed depth, the fill probability $e^{-\kappa\,\delta}$ decreases, i.e., the number of orders that walk the book to the price $S_t + \delta_t$

decreases.

The agent's cash process $(X_t)_{t\in[0,T]}$ satisfies the SDE

$$\boxed{dX_t^\delta = (S_t + \delta_t)\ dN_t^\delta, \quad X_0 \in \mathbb{R} \text{ is known,}}$$

and the agent's inventory $(Q_t)_{t\in[0,T]}$ which remains to be liquidated follows the dynamics

$$\boxed{Q_t = Q_0 - N_t^\delta.}$$

We assume here that Q_0 is an integer (number of shares), thus the inventory Q of the agent can only take the values $\{Q_0, Q_0 - 1, \ldots, 0\}$

9.3 The performance criterion

The agent wishes to maximise the profit from liquidating the initial inventory Q_0 and requires that most of the shares are sold by the terminal time T. We define the stopping time

$$\tau = T \wedge \min\{t : \ Q_t^\delta = 0\}$$

corresponding to the minimum of T and the first hitting time of zero by the inventory process $(Q_t)_{t\in[0,T]}$. If the agent holds any remaining inventory at the end of the trading horizon, they liquidate it using one single MO at the price $S_\tau - \alpha Q_\tau$, so the agent receives $Q_\tau\,(S_\tau - \alpha\, Q_\tau)$, where $\alpha \geq 0$ is the liquidation penalty (linear impact function).

We consider the admissible space $\mathcal{A}$ of strategies δ that are bounded from below. The agent's optimisation problem is to find

$$\boxed{H(x,S) = \sup_{\delta\in\mathcal{A}} \mathbb{E}\left[X_\tau^\delta + Q_\tau^\delta\,(\,S_\tau - \alpha\,Q_\tau^\delta)\mid X_{0^-}^\delta = x, S_0 = S, Q_{0^-}^\delta = Q_0\right],}$$

and the corresponding value function is

$$H(t,x,S,q) = \sup_{\delta\in\mathcal{A}} \mathbb{E}_{t,x,S,q}\left[X_\tau^\delta + Q_\tau^\delta(S_\tau - \alpha Q_\tau^\delta)\right], \tag{63}$$

where $\mathbb{E}_{t,x,S,q}[\cdot]$ is the expectation conditional on $X_{t-}^\delta = x$, $S_t = S$, and $Q_{t-}^\delta = q$.

In contrast to Sections 5 and 7, the agent here does not have any urgency. It is straightforward to extend the present model to one where the agent penalises inventory that is different from 0.

9.4 Solution

The dynamic programming principle (DPP) suggests that the value function (63) solves the following dynamic programming equation (DPE):

$$\boxed{\begin{aligned}&\partial_t H + \frac{1}{2}\sigma^2\partial_{SS}H\\ &\qquad + \sup_{\delta}\left\{\lambda e^{-\kappa\delta}\left[H(t,x+(S+\delta),S,q-1) - H(t,x,S,q)\right]\right\} = 0,\end{aligned}}$$

with boundary conditions

$$\begin{cases} H(t,x,S,0) & = x,\\ H(T,x,S,Q) & = x + q\,(S-\alpha\,q)\,. \end{cases}$$

In contrast to previous sections, the optimal trading problem where the state variables jump result in a non-linear partial integral differential equation (PIDE) rather than a non-linear PDE. The operator ∂_{SS} corresponds to the generator of the Brownian motion that drive the midprice, the term

$$\Lambda(\delta)\,(H(\text{state variable after jump}) - H(\text{state variable before jump}))$$

is the generator of the counting process N that drives the cash and the inventory of the agent; the difference $H(t,x+(S+\delta),S,q-1) - H(t,x,S,q)$ is the change in the agent's value function when an MO fills the agent's LO (increase by $S+\delta$ in cash and decrease by 1 in inventory) and $\lambda\,e^{-\kappa\,\delta}$ is the rate of arrival of other market participants' buy MOs which fill the agent's sell LO.

The terminal condition at $t = T$ represents the cash the agent has acquired plus the value from liquidating the remaining shares at $S - \alpha\,q$ per share. The boundary condition when $q = 0$ represents the cash that the agent holds at the stopping time τ.

We use the following ansatz for the value function (see Section 5 for more details):

$$\boxed{H(t,x,S,q) = x + q\,S + h(t,q)\,.} \tag{65}$$

Substitute (65) into the DPE (64) and find that $h(t,q)$ satisfies the coupled system of non-linear ODEs:

$$\boxed{\begin{array}{rcl}\partial_t h + \sup_\delta\left\{\lambda e^{-\kappa\delta}\left[\delta + h(t,q-1) - h(t,q)\right]\right\} & = & 0,\\ h(t,0) & = & 0,\\ h(T,q) & = & -\alpha q^2.\end{array}} \tag{66}$$

The optimal depth in feedback form can be obtained by solving the supremum term in (66), which can easily be done with the first order condition:

$$\begin{aligned} 0 &= \partial_\delta \left\{ \lambda e^{-\kappa\delta} \left[\delta + h(t, q-1) - h(t,q) \right] \right\} \\ &= \lambda \left(-\kappa e^{-\kappa\delta} \left[\delta + h(t, q-1) - h(t,q) \right] + e^{-\kappa\delta} \right) \\ &= \lambda e^{-\kappa\delta} \left(-\kappa \left[\delta + h(t, q-1) - h(t,q) \right] + 1 \right), \end{aligned}$$

so the optimal strategy $\delta^\star$ in feedback form is given by

$$\boxed{\delta^*(t,q) = \frac{1}{\kappa} + [h(t,q) - h(t,q-1)].} \tag{67}$$

The firm term in (67) stems from optimising the instantaneous expected profits from selling one share. Indeed, observe that the expected revenue of selling one share at the price $S+\delta$, minus the cost S, is $(S+\delta-\delta)\,\Lambda(\delta) = \delta\, e^{-\kappa\,\delta}$, whose maximum is reached for $\delta = 1/\kappa$. The second term $h(t,q) - h(t,q-1)$ can be interpreted as a correction to the static depth $1/\kappa$. This term is a **reservation price** and it is equal to the price p such that $H(t, x+p, S, q-1) = H(t,x,S,q)$, so it is an additional wealth that the agent demands for selling the asset such that their value function remains unchanged.

Substitute the optimal depth (67) in feedback form into (66) to obtain the coupled system of ODEs for $h(t,q)$

$$\boxed{\partial_t h + \frac{\tilde{\lambda}}{\kappa} \exp\left\{ -\kappa \left[h(t,q) - h(t,q-1) \right] \right\} = 0\,,}$$

where $\tilde{\lambda} = \lambda\, e^{-1}$. Finally, consider the ansatz

$$\boxed{h(t,q) = \frac{1}{\kappa} \log \omega(t,q)\,,} \tag{68}$$

so the new equation for $\omega(t,q)$ is

$$\begin{aligned} 0 &= \partial_t h + \frac{\tilde{\lambda}}{\kappa} \exp\left\{ -\kappa \left[h(t,q) - h(t,q-1) \right] \right\} \\ &= \frac{1}{\kappa} \frac{\partial_t \omega(t,q)}{\omega(t,q)} + \frac{\tilde{\lambda}}{\kappa} \frac{\omega(t,q-1)}{\omega(t,q)}, \end{aligned}$$

which simplifies to

$$\boxed{\partial_t \omega(t,q) + \tilde{\lambda}\,\omega(t,q-1) = 0\,,} \tag{69}$$

with terminal and boundary conditions

$$\omega(T,q) = e^{-\kappa\,\alpha\,q^2}\,, \qquad \text{and} \qquad \omega(t,0) = 1\,.$$

The coupled system of ODEs (9.4) can be solved explicitly by completing the following steps:

(i) Compute $\omega(t,q)$ for $q = 1\text{share}, 2\text{shares}, \ldots$ by integrating (9.4).

(ii) Notice that the solution for each q is a polynomial in $(T-t)$ which increase in order as q increases. Thus, use the ansatz

$$\omega(t,q) = \sum_{n=0}^{q} a_n^q (T-t)^n$$

in the solution and find that the coefficients a_n^q satisfy the recursion equations

$$a_n^q = \frac{\tilde{\lambda}}{n} a_{n-1}^{q-1}$$

for $n = 1, \ldots, q, \quad q = 1, 2, \ldots,$ and $a_0^q = e^{-\kappa\,\alpha\,q^2}$.

(iii) Show by induction that the above form is indeed correct.

(iv) Show that

$$a_n^q = \frac{\tilde{\lambda}^n}{n!} e^{-\kappa\,\alpha\,(q-n)^2},$$

for $n = 1, \ldots, q$, and $q = 1, 2, \ldots$.

Thus, the solution to system of ODEs us

$$\boxed{\omega(t,q) = \sum_{n=0}^{q} \frac{\tilde{\lambda}^n}{n!} e^{-\kappa\alpha(q-n)^2}\,(T-t)^n\,.}$$

The solution provides the function $h(t,q)$ in (68) and the value function $H(t,x,S,q)$ in (65). Also, the formula for $h(t,q)$ can be substituted in the feedback formula (67) to obtain the optimal depth

$$\boxed{\delta^*(t,q) = \frac{1}{\kappa}\left[1 + \log \frac{\sum_{n=0}^{q} \frac{\tilde{\lambda}^n}{n!} e^{-\kappa\alpha(q-n)^2}(T-t)^n}{\sum_{n=0}^{q-1} \frac{\tilde{\lambda}^n}{n!} e^{-\kappa\alpha(q-1-n)^2}(T-t)^n}\right]\quad,} \tag{70}$$

for $q > 0$.

9.5 Simulations

Here we study the dependence of the quotes over the model parameters, time, and the agent's inventory. Figure 21 shows the optimal quotes as a function of time, inventory, and penalty parameter. The optimal depth decreases in the inventory of the agent. The larger the inventory, to more the agent is willing to accept a lower premium δ to increase the probability that the sell LO order is filled, which ensures the liquidation of the shares by the end of the trading horizon and avoids crossing the spread at the terminal time.

For a fixed inventory level, the optimal depth decreases with time. This is because the agent becomes more averse to inventory as time passes due to the terminal penalty α. Also, Figure 21 shows that for a fixed time t and inventory q, the optimal depth decreases when the penalty α decreases. This is because increasing the penalty forces the agent to liquidate the position faster, so the depth at which the LOs ar posted is smaller to increase the fill probability. Note that near the terminal time, the terminal penalty plays a more significant role in the optimal depth compared with the start of the trading window.

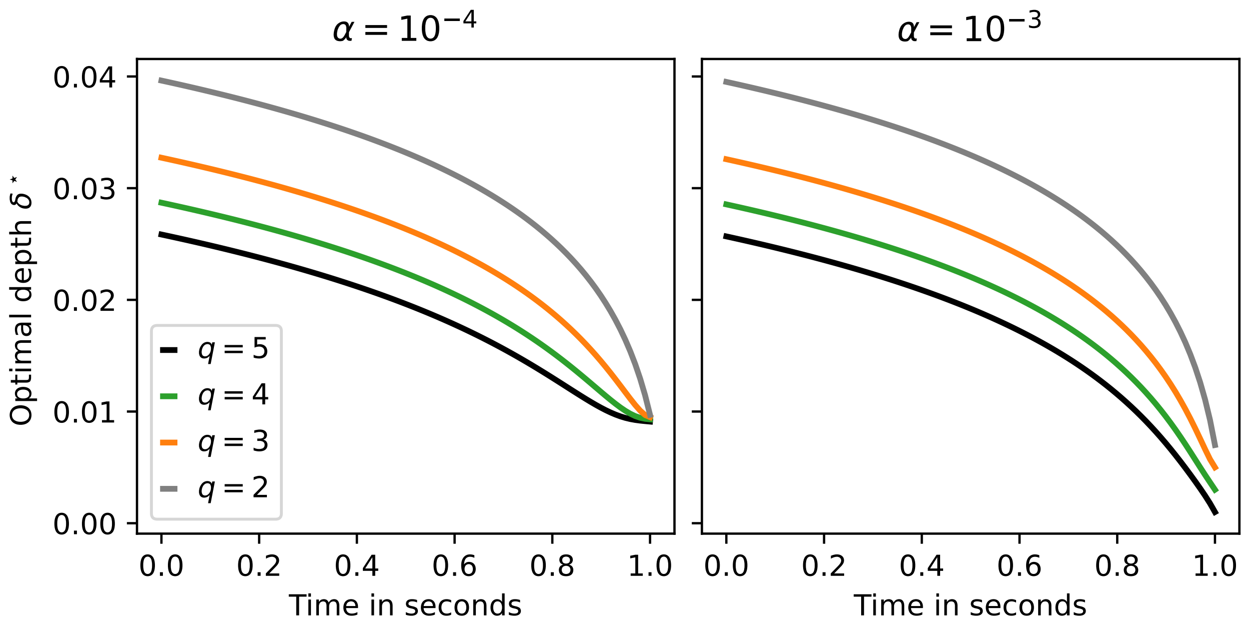

Figure 21: Optimal depth $\delta^\star$ in (70) of the agent's sell LOs as a function of time for different values of the inventory. The parameters are $\lambda = 50$, $T = 1$ minute, $\kappa = 100$, and $Q_0 = 5$. On the left panel, the terminal penalty parameter is $\alpha = 10^{-4}$ and on the right panel, the terminal penalty parameter is $\alpha = 10^{-3}$.

Finally, Figure 22 shows how the quotes react to different values of the intensity of buy MO arrival κ and the exponential decay parameter κ. Clearly, larger values of λ imply more buying pressure so the intensity of MOs that can reach the agent's LO increase. Thus, the agent exploits this by posting the LO deeper in the book. On the other hand, higher values of κ imply a smaller probability that the agent's LO will be filled when a buy MO arrives. So the agent decreases the depth at which the LO is posted.

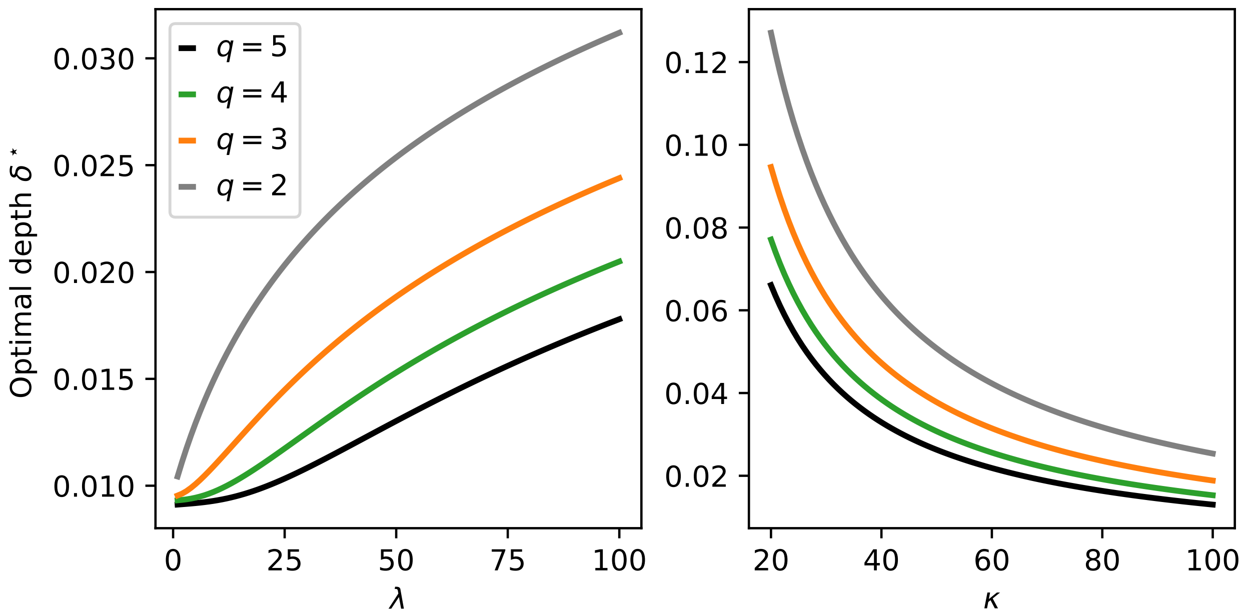

Figure 22: Optimal depth $\delta^\star$ in (70) of the agent's sell LOs as a function of time for different values of the inventory. The parameters of the left panel are $t = 0.8$ minutes, $T = 1$ minute, $\kappa = 100$, $\alpha = 10^{-4}$, and $Q_0 = 5$. The parameters of the right panel are $t = 0.8$ minutes, $T = 1$ minute, $\lambda = 50$, $\alpha = 10^{-4}$, and $Q_0 = 5$.

10 Optimal market making

Most electronic exchanges clear the demand and supply of liquidity in LOBs, where a large panel of market participants such as institutional or individual investors, dealers, brokers, and arbitrageurs can interact. Alternatively, OTC markets are off-exchange "quote-driven" markets that are based on a network of market makers that set prices at which liquidity takers can trade. To set prices, liquidity providers in OTC markets constantly stream bid and ask quotes at which they are ready to immediately trade, or they respond to requests made by their clients; the so-called RFQs (Request For Quote).[14]

OTC market makers faces a complex problem. They provide bid and ask quotes for various assets that exhibit complex joint dynamics without seeing the full depth of price and clients. Consequently, it is key to properly account for risks at the portfolio level. However, a large proportion of multi-asset market making models in the literature only consider correlated Brownian dynamics. Additionally, multi-asset market making is challenging due to high dimensionality and the resulting numerical challenges to obtain the optimal quotes.

To derive the optimal bid and ask quotes that they provide to market participants, market makers solve utility maximisation problem. The first works to tackle this problem mathematically are in Ho and Stoll (1981) and Ho and Stoll (1983) where the authors derive a quoting strategy in a multi-period sequential bidding setup. Shortly after, the work in Glosten and Milgrom (1985) analyses the market maker's decision problem within a model with informed and uninformed traders, giving rise to numerous models that oppose insiders and liquidity consumers who interact through a market maker that sets prices. The literature has later been revived after the seminal work in Avellaneda and Stoikov (2008) who considered a market with a reference price in the form of a Brownian motion, and modelled the number of orders at the bid and the ask as a counting process whose intensity depends on the distance of the quotes from the reference price. Although the work in Avellaneda and Stoikov (2008) and its extensions were intended to solve optimal market making within an LOB, the approach is better suited for an OTC market maker that receives RFQs, unless the quoted spreads in the LOB are

[14]For a better understanding of how the system of RFQs operates, we refer the interested reader to Fermanian et al. (2016).

large in terms of the tick size.

This section deals with the optimal trading problem faced by a market maker who proposes liquidity for an asset in an OTC market by continuously streaming bid and ask quotes at which they are ready to buy or sell the asset.[15]

10.1 Modelling framework

Consider a filtered probability space $\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}\big)$, with $T > 0$, satisfying the usual conditions and supporting all the processes we introduce below.

A market maker operates in an OTC market for a trading period $[0, T]$ (usually one trading day) and is in charge of a single asset whose midprice is modelled by the process $(S_t)_{t\in[0,T]}$ which follows the dynamics

$$\boxed{dSt = \sigma\, dW_t\,, \quad S_0 \in \mathbb{R}_+ \text{ is known,}}$$

where W is a standard Brownian motion.

The market maker chooses the bid price $S^b_t = S_t - \delta^b_t$ and the ask price $S^a_t = S_t + \delta^a_t$ that they stream to their clients and at which they are ready to immediately fill a trade. The control variables of the agent are the distances δ^b and δ^a which determine the price of liquidity; the larger these values, the more expensive it is for the market maker's clients to buy and sell the security.

To model the arrival of buy orders at the ask and sell orders at the bid, we use two counting (controlled) processes $\left(N^{b,\delta}_t\right)$ and $\left(N^{a,\delta}_t\right)$ to model the number of sell orders and buy orders, respectively, that the market maker fills and which arrive at Poisson times with intensities $\Lambda^b(\delta^b)$ and $\Lambda^a(\delta^a)$, respectively. The intensity functions Λ^b and Λ^a are decreasing in δ^b and δ^a, respectively; the more the liquidity proposed by the market maker is expensive, the fewer orders they will receive.

The inventory of the market maker is modelled by the process $\left(Q^\delta_t\right)$. When the market maker receives a buy (sell) order at the proposed ask price (bid price), the inventory decreases (increases) by one unit of the security. Thus,

[15] A slight modification of the model that we study here can also be used for a market maker who posts LOs on both sides of the of the midprice in a limit order book.

the dynamics of the inventory are

$$\boxed{dQ_t^\delta = dN_t^{\delta,b} - dN_t^{\delta,a}, \quad Q_0^\delta = 0.}$$

The cash of the market maker is modelled by the process $\left(X_t^\delta\right)$. When the market maker receives a buy order at the ask price, the cash increases by $S_t + \delta_t^a$ per share sold by the market maker. When the market maker receives a sell order at the bid price, the cash decreases by $S_t - \delta_t^b$ per share bought by the market maker. Thus, the dynamics of the market maker's cash are

$$\boxed{dX_t^\delta = -\left(S_t - \delta_t^b\right)\, dN_t^{\delta,b} + \left(S_t + \delta_t^b\right)\, dN_t^{\delta,a}, \quad X_0^\delta = X_0 \in \mathbb{R} \text{ know.}} \tag{71}$$

In this section and similar to Section 9, we consider the following exponential intensity functions :

$$\begin{cases} \Lambda^b(\delta^b) = & \lambda^b\, e^{-\kappa^b\, \delta^b} \\ \Lambda^a(\delta^a) = & \lambda^a\, e^{-\kappa^a\, \delta^a}\,, \end{cases}$$

where (λ^b, λ^a) represent a baseline intensity of order arrival when the price of liquidity is zero, and (κ^b, κ^a) are the rates of decay of the sell and buy pressure, respectively, as a function of the price of liquidity.

Finally, we also assume that the market maker manages inventory risk by choosing caps on how long or short their position is in the security. Thus the market maker chooses boundaries $\underline{q} < 0$ and $\overline{q} > 0$ such that they stop filling trades whenever the inventory satisfies $Q \notin [\underline{q}, \overline{q}]$; the inventory and the boundaries are integers that represent a number of shares. To model this aspect, we modify the intensity functions above and we set

$$\boxed{\begin{cases} \Lambda^b(\delta^b) = & \lambda^b\, e^{-\kappa^b\, \delta^b}\, \mathbb{1}_{q<\overline{q}} \\ \Lambda^a(\delta^a) = & \lambda^a\, e^{-\kappa^a\, \delta^a}\, \mathbb{1}_{q>\underline{q}}\,, \end{cases}} \tag{72}$$

where $\mathbb{1}$ is the indicator function. Mathematically, the intensity functions (72) imply that orders (artificially) stop coming once the inventory of the market maker hits one of the boundaries. In practice, this can be achieved by quoting a very large price at the bid or the ask to discourage clients from trading.

10.2 Performance criterion

The market makers seeks a strategy $\delta = \left(\delta^b, \delta^a\right)$ that maximises the terminal wealth. We also assume that by the terminal time T, the market maker wishes to get rid of any remaining inventory Q_T^δ by using an MI at a price which is worse that the midprice S_T. Finally, the market maker also includes a running inventory penalty that penalises holding large positions throughout the trading period.

The performance criterion is

$$\boxed{H^\delta(t,x,S,q) = \mathbb{E}_{t,\boldsymbol{x},q,S}\left[X_T + Q_T^\delta(S_T^\delta - \alpha Q_T^\delta) - \phi\int_{\boldsymbol{t}}^T (Q_u)^2 du\right]},$$

where $\alpha \geq 0$ models the terminal penalty from liquidating the remaining inventory, $\phi \geq 0$ scales the quadratic running inventory penalty. The value function of the market maker is

$$H(t,x,S,q) = \sup_{\delta\in\mathcal{A}} H^\delta(t,x,S,q),$$

where $\mathcal{A}$ is the set of admissible $\mathcal{F}-$predictable processes that are bounded from below.

10.3 Solution with no inventory penalty

Here we consider the simple case when the agent does not penalise holding inventory, so they set the penalty parameters $\phi = \alpha = 0$ and $\overline{q} = -\underline{q} = \infty$. To solve our problem, we note that the dynamic programming principle suggests that the value function solves the DPE

$$\begin{aligned}
0 = {} & \partial_t H(t,x,q,S) + \frac{1}{2}\sigma^2\,\partial_{SS}H(t,x,q,S) \\
& + \lambda^a \sup_{\delta^a}\left\{e^{-\kappa^a\,\delta^a}\left(H(t,x+(S+\delta^a),q-1,S) - H(t,x,q,S)\right)\right\}\mathbb{1}_{q>\underline{q}} \\
& + \lambda^b \sup_{\delta^b}\left\{e^{-\kappa^b\,\delta^b}\left(H\left(t,x-(S-\delta^b),q+1,S\right) - H(t,x,q,S)\right)\right\}\mathbb{1}_{q<\overline{q}},
\end{aligned} \tag{73}$$

subject to the terminal condition

$$H(T,x,S,q) = x + q\,S,$$

The terms in the DPE (73) represent various components in the value of the agent's utility: (i) the change in the value function due to the arrival of orders that are filled by the market maker, and (ii) the change in the value function due to the diffusion of the asset price.

To solve the DPE (73), we proceed as in the previous sections, i.e., we propose ansatzs that simplify the equation or reduce its dimensionality, based on the form of the equation and its terminal condition. The first and usual ansatz is based on the form of the terminal condition, and allows to reduce the dimensionality from 4 to 2 and we write

$$H(t, x, q, S) = x + q\,S + h(t, q), \tag{74}$$

which represent the accumulated cash x, the marked-to-market value of the agent's inventory $q\,S$, and the added value from following the optimal market making strategy $h(t, q)$.

Substitute (74) into (73) to obtain the following equation in $h(t, q)$:

$$\begin{aligned} 0 = \partial_t h(t, q) &+ \lambda^a \sup_{\delta^a} \left\{ e^{-\kappa^a \delta^a} \left(\delta^a + h(t, q-1) - h(t, q) \right) \right\} \\ &+ \lambda^b \sup_{\delta^b} \left\{ e^{-\kappa^b \delta^b} \left(\delta^b + h(t, q+1) - h(t, q) \right) \right\}, \end{aligned} \tag{75}$$

subject to the terminal condition

$$h(T, q) = 0.$$

Now, notice that there are no terms in (75) that depend on the inventory q and neither does the terminal condition. Thus, h does not depend on the inventory and we write $h(t, q) = h(t)$. This aspect is expected because the market maker does not penalise holding inventory and only maximises wealth, so the value function only depends on the inventory through the marked-to-market value of the wealth $x + q\,S$. Equation (75) simplifies to

$$0 = \partial_t h(t, q) + \lambda^a \sup_{\delta^a} \left\{ \delta^a\, e^{-\kappa^a \delta^a} \right\} + \lambda^b \sup_{\delta^b} \left\{ \delta^b\, e^{-\kappa^b \delta^b} \right\}, \tag{76}$$

subject to the terminal condition

$$h(T, q) = 0.$$

Solving the supremum terms in (76) allows one to obtain the optimal feedback distances $\delta^{b,\star}$ and $\delta^{a,\star}$:

$$\boxed{\delta^{b,*} = \frac{1}{\kappa^b} \quad \text{and} \quad \delta^{a,*} = \frac{1}{\kappa^a}.}$$

The optimal strategy corresponds to proposing a constant bid and an ask quotes. The values $(1/\kappa^b,\, 1/\kappa^a)$ optimise the instantaneous expected profit from a roundtrip trade; i.e., a simultaneous order filled at the bid and the ask by the market maker. To see this, note that the expected profit (see the dynamics in (71)) from a roundtrip trade is $\delta^b\,\Lambda^b(\delta^b) + \delta^a\,\Lambda^a(\delta^a)$ which is maximal for $\delta^b = 1/\kappa$ and $\delta^a = 1/\kappa$, where $\Lambda^b(\,\cdot\,)$ and $\Lambda^a(\,\cdot\,)$ are given by (72).

The models of optimal market making in the literature usually lead to strategies that attempt to improve on the baseline strategy (77) by taking into account additional aspects of the problem as in the next subsection which incorporates inventory penalisation.

10.4 Solution to the general problem

To solve our problem, we note that the dynamic programming principle gives the HJB

$$\boxed{\begin{aligned} 0 &= \partial_t H(t,x,q,S) + \frac{1}{2}\sigma^2\,\partial_{SS} H(t,x,q,S) - \phi\, q^2 \\ &+ \lambda^a \sup_{\delta^a} \left\{ e^{-\kappa^a\,\delta^a} \left(H(t, x + (S+\delta^a), q-1, S) - H(t,x,q,S) \right) \right\} \mathbb{1}_{q>\underline{q}} \\ &+ \lambda^b \sup_{\delta^b} \left\{ e^{-\kappa^b\,\delta^b} \left(H\left(t, x - (S-\delta^b), q+1, S\right) - H(t,x,q,S) \right) \right\} \mathbb{1}_{q<\bar{q}}, \end{aligned}} \tag{78a}$$

subject to the terminal condition

$$H(T,x,S,q) = x + q\,S - \alpha\, q^2.$$

The terms in the DPE (78a) represent (i) the change in the value function due to the arrival of orders that are filled by the market maker if the inventory is within the range $[\underline{q}, \overline{q}]$, (ii) the change in the value function due to the diffusion of the asset price, and (iii) finally the change in the value function due to the effect of penalising deviations of inventories from zero along the path of the trading strategy δ.

To solve the DPE (78a), we propose the usual ansatz

$$\boxed{H(t, x, q, S) = x + q\, S + h(t, q),} \tag{79}$$

which represent the accumulated cash x, the marked-to-market value of the agent's inventory $q\, S$, and the added value from following the optimal market making strategy $h(t, q)$.

Substitute (79) into (78a) to obtain the following equation in $h(t, q)$:

$$\boxed{\begin{aligned} 0 = \partial_t h(t, q) - \phi q^2 + \lambda^a \sup_{\delta^a} \left\{ e^{-\kappa^a \delta^a} \left(\delta^a + h(t, q-1) - h(t, q) \right) \right\} \mathbb{1}_{q > \underline{q}} \\ + \lambda^b \sup_{\delta^b} \left\{ e^{-\kappa^b \delta^b} \left(\delta^b + h(t, q+1) - h(t, q) \right) \right\} \mathbb{1}_{q < \overline{q}}, \end{aligned}}$$

subject to the terminal condition

$$h(T, q) = -\alpha\, q^2.$$

Solving the supremum terms in (80) allows one to obtain the optimal feedback distances $\delta^{b,\star}$ and $\delta^{a,\star}$:

$$\boxed{\begin{aligned} \delta^{b,*}(t, q) &= \frac{1}{\kappa^b} - h(t, q+1) + h(t, q), \quad q < \overline{q}, \\ \delta^{a,*}(t, q) &= \frac{1}{\kappa^a} - h(t, q-1) + h(t, q), \quad q > \underline{q}. \end{aligned}}$$

The optimal feedback distances can be decomposed into a first component $1/\kappa^{a,b}$ and a second component related to a change in the value function due to an inventory change after a trade is filled. The first component is the optimal strategy of a market maker who does not penalise inventory; see Subsection 10.3. The second term in both $\delta^{b,*}(t, q)$ and $\delta^{a,*}(t, q)$ corresponds to inventory through time. We study how this component impacts the optimal strategy in Subsection 10.5.

Next, substitute the optimal feedback strategy (81) into (80) and obtain the DPE

$$\boxed{\begin{aligned}\phi q^2 = \partial_t h(t,q) + \frac{e^{-1}\lambda^a}{\kappa^a}\exp\{-\kappa^a(-h(t,q-1)+h(t,q)\}\mathbb{1}_{q>\underline{q}} \\ + \frac{e^{-1}\lambda^b}{\kappa^b}\exp\{-\kappa^b(-h(t,q+1)+h(t,q))\}\mathbb{1}_{q<\overline{q}},\end{aligned}} \tag{82a}$$

subject to the same terminal condition.

The work in Avellaneda and Stoikov (2008) was the first to find an analytical solution to the DPE (82a) when the intensity functions Λ^b and Λ^a decay at the same rate $\kappa = \kappa^a = \kappa^b$. In that case, we propose the following ansatz

$$\boxed{h(t,q) = \frac{1}{\kappa}\log\omega(t,q),}$$

which simplifies the DPE to

$$\begin{aligned}0 =& \partial_t\omega(t,q) - \phi\,\kappa\,q^2\,\omega(t,q) \\ &+ e^{-1}\lambda^a\,\omega\,(t,q-1)\,\mathbb{1}_{q>\underline{q}} + e^{-1}\lambda^b\,\omega\,(t,q+1)\,\mathbb{1}_{q<\overline{q}},\end{aligned} \tag{83}$$

subject to the terminal condition

$$\omega(T,q) = \exp(-\kappa\,\alpha\,q^2).$$

Equation (83) corresponds to a system of ODEs. To see this, recall that the inventory q can only take the finitely many values $\{\underline{q}, \underline{q}+1, \ldots, \overline{q}-1, \overline{q}\}$, thus for a fixed time t, $\omega(t,q)$ can only take the finitely many value $\{\omega(t,\underline{q}), \omega(t,\underline{q}+1), \ldots, \omega(t,\overline{q}-1), \omega(t,\overline{q})\}$.

For each time t, we define the vector

$$\mathbf{w}(t) = \big(\omega(t,\underline{q}), \omega(t,\underline{q}+1), \ldots, \omega(t,\overline{q}-1), \omega(t,\overline{q})\big)^\intercal,$$

and note that the DPE (83) implies that $\mathbf{w}$ solves the ODE

$$\boxed{0 = \partial_t\mathbf{w}(t) + \mathbf{A}\,\omega(t),} \tag{84}$$

where $\mathbf{A}$ is an $\overline{q}-\underline{q}+1-$square matrix, i.e., $\mathbf{A} \in \mathcal{M}_{\overline{q}-\underline{q}+1}(\mathbb{R})$ matrix, and we write

$$\mathbf{A}_{q,i} = \begin{cases} -\phi\,\kappa\,q^2, & i = q, \\ \lambda^a\,e^{-1}, & i = q-1, \\ \lambda^b\,e^{-1}, & i = q+1, \\ 0, & \text{otherwise,} \end{cases}$$

The solution to the first-order homogeneous matrix ODE (84) is straightforward and we obtain

$$\boxed{\mathbf{w}(t) = \exp\left(\mathbf{A}\,(T-t)\right)\,\mathbf{z},}$$

where the vector $\mathbf{z}$ is $(\overline{q}-\underline{q}+1)-$dimensional is defined as

$$\mathbf{z} = \left(e^{-\alpha\,\kappa\,\underline{q}^2}, \ldots, e^{-\alpha\,\kappa\,\overline{q}^2}\right)^{\intercal}.$$

10.5 Simulations

Here, we study how the quotes depend on model parameters. Figure 23 shows the optimal quotes at time $t = 0$ for various values of the parameters. Each column of panels in Figure 23 fixes all parameters but one to some default values and we set $T = 30$ minutes and $\alpha = 10^{-4}$. Each couple of panels of each column in Figure 23 shows the optimal bid and sell distances, respectively; recall that the quoted price for sell orders is $S_t - \delta^{b,\star}$ and the quoted price for buy orders is $S_t + \delta^{b,\star}$.

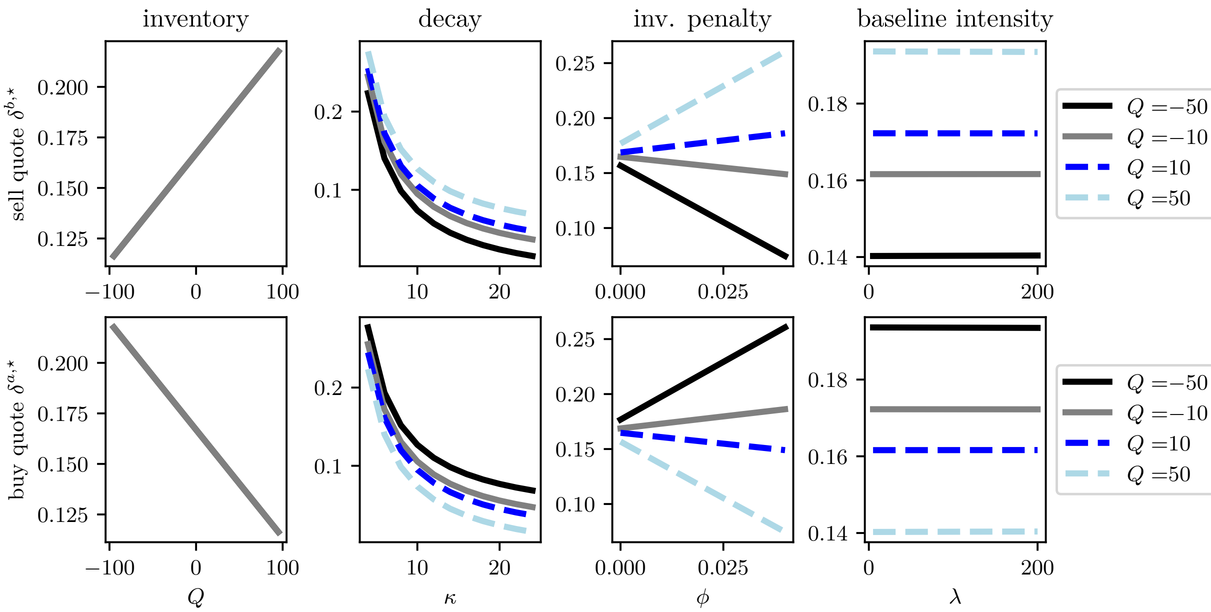

Figure 23: Optimal distances $\delta^{b,\star}$ and $\delta^{a,\star}$ in (81) as a function of model parameters for different values of the inventory. The default parameters are $\lambda = 50$, $T = 1$ minute, $\phi = 2 \times 10^{-3}$, $\kappa = 100$, and $Q_0 = 5$. The terminal time is $T = 30$ minutes and the terminal penalty is $\alpha = 10^{-4}$.

The first column in Figure 23 shows that the market maker skews the price of liquidity as a function of the inventory. When the position is long ($Q > 0$), the sell price increases and the buy price decreases. This is to attract buyers and discourage sellers so the inventory is pushed back to zero. Similarly, when the market maker is short the security ($Q < 0$), the sell price decreases and the buy price in crease. This is to attract sellers and discourage buyers.

The second column shows that the price of both sell and buy orders decreases when the decay parameter κ decreases. Recall that κ represents the decay in probability of receiving orders when the price of liquidity increases. Thus, κ quantifies the sensitivity of traders to the price of liquidity. The more sensitive the traders, the less expensive the quotes.

The third columns studies the quotes as a function of the magnitude of the running inventory penalty ϕ. When the agent is long the security ($Q > 0$), increasing the inventory penalty forces the agent to revert their inventory faster to zero by attracting buyers and and discouraging sellers (increasing sell price and decreasing buy price). When the agent is short the security ($Q < 0$), increasing the inventory penalty forces the agent to revert their inventory faster to zero by attracting sellers and and discouraging buyers. This effect is stronger for larger long or short positions. Finally, the last columns shows that the intensity of order arrival plays a second-order role in the optimal strategy.

10.6 Further readings

The model in Avellaneda and Stoikov (2008) gave rise to an extensive literature on optimal market making. Guéant et al. (2013) provide a rigorous analysis of the problem, for which they consider the exponential intensity functions suggested in Avellaneda and Stoikov (2008). The authors use the tools of stochastic optimal control to prove that the problem reduces to a system of linear ODEs under inventory constraints. They also analyse the asymptotic regime of the optimal quotes when the time horizon T tends to $+\infty$, and propose closed-form approximations using results from spectral analysis. In addition, Cartea and his co-authors also study optimal market making models. Cartea et al. (2015) use stochastic control tools to rigorously characterise the optimal market making strategy that uses an objective function given by the expectation of the P&L minus a running penalty. Cartea et al. (2017) study the ambiguity in the specification of fill probabilities and the dynamics of intensities and prices. Finally, Cartea and Wang (2020) incorporate signals in the market making strategy.

More recently, other features have also been considered in the literature, such as general intensity shapes depending on unobservable factors in Campi and Zabaljauregui (2020), persistence in the order flow in Jusselin (2021), variability in the request sizes in Bergault and Guéant (2021), and client tiering and access to liquidity pools in Barzykin et al. (2021). The case of options market making is studied in the literature starting with Stoikov and Sağlam (2009) where the authors use a mean-variance framework. In El Aoud and Abergel (2015), the authors extend the problem by considering a stochastic volatility model for prices. Finally, Guilbaud and Pham (2013) propose a framework for studying optimal market making policies in a limit order book where the bid-ask spread follows a Markov chain with finitely many values and an agent sends market and limit orders to maximise their expected utility. The problem is solved using a mixed regime switching regular / impulse control problem.

11 Optimal portfolio trading

In practice, operators routinely face the problem of having to execute simultaneously large orders regarding various assets, such as in block trading for funds facing large subscriptions or withdrawals, or when considering multi-asset trades in statistical arbitrage trading strategies. More generally, banks and market makers manage their (il)liquidity and market risk, when it comes to executing trades, in the context of a central risk book; hence the need for multi-asset models. This section considers a generalisation of the model of Section 5 where the agent is in charge of executing large positions in multiple assets simultaneously and they adopt a CARA utility.

11.1 Modelling framework

Consider a filtered probability space $\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}\big)$, with $T > 0$, satisfying the usual conditions. We assume this probability space to be large enough to support all the processes we introduce.

We consider a market with $d \in \mathbb{N}^*$ assets,[16] and an agent that liquidates their holdings in a portfolio over a trading window $[0, T]$. Let $Q_0 \in \mathbb{R}^d$ be the initial holdings of the agent in every asset. The inventory process $(Q_t)_{t\in[0,T]} = \big(Q_t^1, \dots, Q_t^d\big)^\intercal_{t\in[0,T]}$ of the agent evolves with the trading speed $(v_t)_{t\in[0,T]} = (v_t^1, \dots, v_t^d)^\intercal_{t\in[0,T]}$ for each asset, and we write[17]

$$\boxed{dQ_t = v_t dt.} \tag{85}$$

We consider in this section the problem of multi-asset optimal execution in the case where prices are correlated arithmetic Brownian motions. The prices $(S_t)_{t\in[0,T]} = \big(S_t^1, \dots, S_t^d\big)^\intercal_{t\in[0,T]}$ of the d assets follow the dynamics

$$\boxed{dS_t = V dW_t\,,} \tag{86}$$

with $S_0 \in \mathbb{R}^d$ known, the matrix $V \in \mathcal{M}_d(\mathbb{R})$ captures the covariance and we define the covariation matrix $\Sigma = VV^\intercal$, and $(W_t)_{t\in[0,T]} = \big(W_t^1, \dots, W_t^d\big)^\intercal_{t\in[0,T]}$ is a d-dimensional standard Brownian motion with independent coordinates.[18]

[16] We denote by $\mathbb{N}^*$ the set $\mathbb{N}^* := \mathbb{N}\backslash\{0\}$ of positive integers.

[17] The superscript $^\intercal$ designates the transpose operator. It transforms here a line vector into a column vector.

[18] We denote by $\mathcal{M}_{d,k}(\mathbb{R})$ the set of $d \times k$ real matrices and by $\mathcal{M}_d(\mathbb{R}) := \mathcal{M}_{d,d}(\mathbb{R})$ the set of $d \times d$ real square matrices. The set of real symmetric $d \times d$ matrices is denoted by $\mathcal{S}_d(\mathbb{R})$.

Here, we consider that the temporary price impact and/or the execution costs of the agent's trading activity have a positive-definite quadratic form. Thus, the value $(X_t)_{t\in[0,T]}$ of the agent's cash account throughout the trading window follows the dynamics

$$\boxed{dX_t = -v_t^\intercal S_t dt - v_t^\intercal \, \eta \, v_t \, dt,} \tag{87}$$

where the initial cash $X_0 \in \mathbb{R}$ is known, and $\eta \in \mathcal{S}_d^{++}(\mathbb{R})$ is the temporary market impact matrix.[19]

Market impact is also permanentand the impacted price process $(\tilde{S}_t)_{t\in[0,T]} = \left(\tilde{S}_t^1, \dots, \tilde{S}_t^d\right)_{t\in[0,T]}^\intercal$ follows the dynamics:

$$d\tilde{S}_t = dS_t + K v_t dt,$$

where $K \in \mathcal{S}_d(\mathbb{R})$ is the matrix that quantifies the linear permanent impact of the agent's trading activity on the prices. In the case of the CARA utility and Brownian dynamics, Schied et al. (2010) prove that one can reduce the set of admissible strategies to that of absolutely continuous deterministic execution strategies. In particular, the linear permanent price impact does not play a role in the optimisation problem. In the remainder of this section, we do not take permanent price impact into account. We prove this result in Section 12.

11.2 Performance criterion

The terminal wealth of the agent is the sum of the terminal cash X_T and the value of any outstanding terminal inventory Q_T valued at the terminal price S_T. As usual, we add a penalisation term in the objective function. The penalisation term acts as a discount applied to the Marked-to-Market value $Q_T^\intercal S_T$ of the remaining assets and penalises any non-zero terminal position. We assume that the terminal penalty takes a positive-definite quadratic form. Thus, we value the terminal inventory Q_T of the agent at $Q_T^\intercal S_T - Q_T^\intercal \Gamma Q_T$ where $\Gamma \in \mathcal{S}_d^{++}(\mathbb{R})$. The agent maximises the CARA performance criterion

$$\boxed{w^\nu(t,x,q,s) = \mathbb{E}_{t,x,q,s}\left[-\exp\left(-\gamma\left(X_T + Q_T^\intercal \, S_T - Q_T^\intercal \, \Gamma \, Q_T\right)\right)\right],} \tag{88}$$

[19]The subset of positive-definite and positive semi-definite matrices of $\mathcal{S}_d(\mathbb{R})$ are respectively denoted by $\mathcal{S}_d^{++}(\mathbb{R})$ and $\mathcal{S}_d^{+}(\mathbb{R})$.

where $\gamma > 0$ is the absolute risk aversion parameter of the agent. The value function is

$$w(t, x, q, s) = \sup_{\nu \in \mathcal{A}} w^{\nu}(t, x, q, s).$$

11.3 Solution

Use the dynamics in (85)-(86)-(87) to write the HJB equation associated with the problem (88) is

$$\boxed{\begin{aligned} 0 = {} & \partial_t w + \frac{1}{2}\mathrm{Tr}\left(\Sigma\, D^2_{SS} w\right) \\ & + \sup_{v \in \mathbb{R}^d} \left(-(v^\intercal s + v^\intercal \eta v)\partial_x w + v^\intercal \nabla_q w\right), \end{aligned}}$$

for all $(t, x, q, s) \in [0, T) \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ with terminal condition

$$w(T, x, q, s) = -\exp\left(-\gamma\left(x + q^\intercal s - q^\intercal \Gamma q\right)\right) \quad \forall (x, q, s) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d.$$

The terminal condition suggests the ansatz which splits out the accumulated cash, the book value of the shares which are marked-to-market at the midprice, and the added value θ from trading optimally:

$$\boxed{\begin{aligned} & w(t, x, q, s) = -\exp\left(-\gamma\left(x + q^\intercal S + \theta(t, q)\right)\right), \\ & \qquad \forall (t, x, q, s) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d, \end{aligned}}$$

to obtain the new HJB in θ

$$0 = \partial_t \theta(t, q) + \sup_{v \in \mathbb{R}^d} \left(v^\intercal \nabla_q \theta(t, q) - v^\intercal \eta v\right) - \frac{\gamma}{2} q^\intercal \Sigma q, \tag{90}$$

with terminal condition

$$\theta(T, q) = -q^\intercal \Gamma q \quad \forall q \in \mathbb{R}^d.$$

The supremum in (90) is reached with the optimal control in feedback form

$$v^*_s = \frac{1}{2}\eta^{-1} \nabla_q \theta(s, q),$$

for all $s \in [t, T]$. Consequently, we obtain the HJB

$$\boxed{0 = \partial_t \theta(t, q) + \frac{1}{4} \nabla_q \theta(t, q)^\intercal \eta^{-1} \nabla_q \theta(t, q) - \frac{\gamma}{2} q^\intercal \Sigma q,}$$

with terminal condition

$$\theta(T,q) = -q^\intercal \Gamma q, \quad \forall q \in \mathbb{R}^d.$$

We use the second ansatz

$$\boxed{\theta(t,q) = Q^\intercal A(t) q + B(t)^\intercal q + C(t)\,,}$$

for all $(t,q) \in [0,T] \times \mathbb{R}^d$, and find that the problem reduces to finding $A \in C^1([0,T], \mathcal{S}_d(\mathbb{R}))$, $B \in C^1\left([0,T], \mathbb{R}^d\right)$ and $C \in C^1([0,T], \mathbb{R})$ that solve the following ODE system

$$\boxed{\begin{cases} A'(t) = \frac{\gamma}{2}\Sigma - A(t)\eta^{-1}A(t) \\ B'(t) = -A(t)\eta^{-1}B(t) \\ C'(t) = -\frac{1}{4}B(t)^\intercal \eta^{-1} B(t), \end{cases}}$$

with terminal conditions

$$A(T) = -\Gamma, \quad B(T) = C(T) = 0.$$

The solutions for B and C are $B = C = 0$. Therefore, the problem reduces to finding $A \in C^1([0,T], \mathcal{S}_d(\mathbb{R}))$ solution of the following terminal value problem:

$$\begin{cases} A'(t) & = \frac{\gamma}{2}\Sigma - A(t)\eta^{-1}A(t) \\ A(T) & = -\Gamma. \end{cases} \tag{91}$$

We show that when $\Sigma \in S_d^{++}(\mathbb{R})$, A can be found in closed-form. We introduce the change of variables

$$a(t) = \eta^{-\frac{1}{2}} A(t) \eta^{-\frac{1}{2}} \quad \forall t \in [0,T]\,,$$

and notice that (91) is equivalent to the terminal value problem

$$\begin{cases} a'(t) = & \hat{A}^2 - a(t)^2 \\ a(T) = & -C, \end{cases} \tag{92}$$

where

$$\hat{A} = \sqrt{\frac{\gamma}{2}} \left(\eta^{-\frac{1}{2}} \Sigma \eta^{-\frac{1}{2}}\right)^{\frac{1}{2}} \in \mathcal{S}_d^{++}(\mathbb{R})\,,$$

and

$$C = \eta^{-\frac{1}{2}} \Gamma \eta^{-\frac{1}{2}} \in \mathcal{S}_d^+(\mathbb{R}) \,.$$

To solve (92) we use a classical trick for Riccati equations in the next result.

Proposition 3. *Let $\xi : [0,T] \to \mathcal{S}_d(\mathbb{R})$ defined as*

$$\xi\left(t\right) = -\frac{\hat{A}^{-1}}{2} \left(I - e^{-2\hat{A}(T-t)} \right) - e^{-\hat{A}(T-t)} \left(C + \hat{A} \right)^{-1} e^{-\hat{A}(T-t)} \tag{93}$$

be the unique solution of the linear ODE

$$\begin{cases} \xi'(t) = \hat{A}\xi(t) + \xi(t)\hat{A} + I_d \\ \xi(T) = -\left(C + \hat{A} \right)^{-1} . \end{cases} \tag{94}$$

Then $\forall t \in [0,T]$, $\xi(t)$ is invertible and

$$a : t \in [0,T] \to \hat{A} + \xi(t)^{-1} \in \mathcal{S}_d(\mathbb{R})$$

is the unique solution of (92).

Proof. First, we verify that ξ, defined in (93), is the solution of the linear ODE (94). Next, for all $t \in [0,T]$, $-\xi(t)$ is the sum of

$$\frac{\hat{A}^{-1}}{2} \left(I - e^{-2\hat{A}(T-t)} \right) \in \mathcal{S}_d^+(\mathbb{R})$$

and

$$e^{-\hat{A}(T-t)} \left(C + \hat{A} \right)^{-1} e^{-\hat{A}(T-t)} \in \mathcal{S}_d^{++}(\mathbb{R}) \,,$$

so $-\xi(t) \in \mathcal{S}_d^{++}(\mathbb{R})$ and $\xi(t)$ is invertible. We also note that

$$\begin{aligned} a'(t) &= -\xi(t)^{-1}\xi'(t)\xi(t)^{-1} \\ &= -\xi(t)^{-1}\hat{A} - \hat{A}\xi(t)^{-1} - \xi(t)^{-2} \\ &= \hat{A}^2 - \left(\hat{A} + \xi(t)^{-1} \right)^2 \\ &= \hat{A}^2 - a(t)^2 \,, \end{aligned}$$

and $a(T) = -C$, hence the result. $\square$

We deduce the following corollary:

Corollary 1. $\forall t \in [0,T]$,

$$\boxed{A(t) = \eta^{\frac{1}{2}} \left(\hat{A} - \left(\frac{\hat{A}^{-1}}{2} \left(I - e^{-2\hat{A}(T-t)} \right) + e^{-\hat{A}(T-t)} \left(C + \hat{A} \right)^{-1} e^{-\hat{A}(T-t)} \right)^{-1} \right) \eta^{\frac{1}{2}}.} \tag{95}$$

The problem admits a classical solution. The results in Section 12 apply to this problem (when $R = 0$) and rigorously characterise the strategy. In particular,

$$w(t, x, q, s) = -\exp\left(-\gamma \left(x + q^\intercal s + q^\intercal A(t) q\right)\right) ,$$

where A is in (95), is the value function associated with the optimisation problem, and the optimal trading speed is

$$v_s^* = \eta^{-1} q_s^\intercal A\left(s\right) .$$

11.4 Further readings

In contrast to the single-asset case, the existing literature on the joint execution of large orders in multiple assets, or a single asset within a multi-asset portfolio, is limited. The first paper to build multi-asset trading curves in an optimised way is Almgren and Chriss (2001). Almgren and Chriss considered, in an appendix of their seminal paper, a multi-asset extension of their discrete-time model. A few extensions to this model have been proposed in the literature. Lehalle (2009) considers adding an inventory constraint to balance the different portfolio lines during the portfolio execution process. Schied et al. (2010) show that when prices follow Bachelier dynamics, deterministic strategies are optimal for a trader with an exponential utility objective function. Cartea et al. (2015) use stochastic control tools to derive multi-asset optimal execution strategies such as optimal entry/exit times and cointegration-based statistical arbitrage. Bismuth et al. (2019) address optimal portfolio liquidation (along with other problems) by coupling Bayesian learning and stochastic control to derive optimal strategies under uncertainty on model parameters in the Almgren-Chriss framework. Emschwiller et al. (2021) extend optimal trading with Markovian predictors to the multi-asset case, with linear trading costs, using a mean-field approach that reduces the problem to a single-asset one.

12 Optimal trading of cointegrated assets

In practice, market operators such as brokers or mutual funds are in charge of assets from a specific sector or asset class. Thus, the asset prices usually share common stochastic trends and exhibit co-movements. For example, in equity markets, a large number of shares from a specific economic sector have cointegrated dynamics as they share common sources of systemic risk. Often, when trading these assets simultaneously, price movements of a single asset that are adverse when considered independently can become profitable in the portfolio case. Operators need to account for the joint dynamics that assets exhibit.

Most existing papers consider correlated Brownian motions when modelling the joint dynamics of prices. The problem of using single-asset models or unsuitable multivariate models for portfolio trading is that the resulting trading curves of individual assets do not balance well execution costs and price impact with price risk at the portfolio or strategy level.

A classical model for the multivariate dynamics of financial variables that goes beyond that of correlated Brownian motions is the multivariate Ornstein-Uhlenbeck (multi-OU) model. It is especially attractive because it is parsimonious, and yet general enough to cover a wide spectrum of multi-dimensional dynamics; particular cases include correlated Brownian motions but also cointegrated dynamics which are heavily used in statistical arbitrage. Multi-OU dynamics account for the information in the prices of all assets when trading individual assets. This information enhances the trading performance of execution programmes and can serve as a basis to execute statistical arbitrages.[20]

The advantages of multi-OU dynamics for practitioners are numerous. Considering single-asset execution within a portfolio allows to manage risk across a basket of assets. Agents can hold securities on their balance sheets for longer, reducing market impact and execution costs. Moreover, from a regulation point of view, multivariate optimal execution models that naturally offset risks in a portfolio are of great interest. The new FRTB (Fundamental Review of the Trading Book) regulation will lead practitioners to assess liquidity risks within a centralised risk book for capital requirements. In this

[20]When trading an asset, the dynamics of another asset can be regarded as a predictive signal that can enhance the execution process. Thus, the litearture on predictive signals in the optimal execution is closely related to the topic of multi-asset optimal execution with multi-OU dynamics.

context, our model can reduce the liquidity risk of the execution process by taking into account the joint dynamics of the assets.

The work in Cartea et al. (2018c) is the pioneering work in the use of the multi-OU model for the price dynamics in a multi-asset optimal execution problem. The authors proposed a model where the asset prices have multi-OU dynamics and the agent maximises an objective function given by the expectation of the Profit and Loss (PnL) minus a running penalty related to the instantaneous variance of the portfolio.

This section solves a multi-asset optimal execution problem when the agent adopts a CARA utility and the prices follow multi-OU dynamics which account for the presence of cointegration between the assets' prices. We showcase the use of the model to enhance the performance of portfolio execution programmes and to build statistical arbitrage strategies using real data from the Equity and FX markets.

12.1 Modelling framework

We consider a filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}\right)$ satisfying the usual conditions. We assume this probability space to be large enough to support all the processes we introduce.

We consider a market with $d \in \mathbb{N}^\star$ assets, and a trader wishing to liquidate their portfolio over a period of time $[0, T]$, with $T > 0$. their inventory process $(Q_t)_{t\in[0,T]} = \left(Q_t^1, \ldots, Q_t^d\right)^\intercal_{t\in[0,T]}$ evolves as

$$\boxed{dQ_t = v_t dt,} \tag{96}$$

with $Q_0 \in \mathbb{R}^d$ given, where $(v_t)_{t\in[0,T]} = (v_t^1, \ldots, v_t^d)^\intercal_{t\in[0,T]}$ represents the trading rate of the trader for each asset.

The fundamental prices of the d assets are modelled as a d-dimensional Ornstein-Uhlenbeck process $(S_t)_{t\in[0,T]} = \left(S_t^1, \ldots, S_t^d\right)^\intercal_{t\in[0,T]}$ with dynamics

$$\boxed{dS_t = R(\overline{S} - S_t)dt + V dW_t,} \tag{97}$$

and we introduce the market price process $(\tilde{S}_t)_{t\in[0,T]} = \left(\tilde{S}_t^1, \ldots, \tilde{S}_t^d\right)^\intercal_{t\in[0,T]}$ with dynamics:

$$d\tilde{S}_t = dS_t + K\, v_t dt,$$

with $S_0 = \tilde{S}_0 \in \mathbb{R}^d$ given, where $\overline{S} \in \mathbb{R}^d$, $R \in \mathcal{M}_d(\mathbb{R})$, $V \in \mathcal{M}_{d,k}(\mathbb{R})$, $K \in \mathcal{S}_d(\mathbb{R})$, and $(W_t)_{t\in[0,T]} = \left(W_t^1, \ldots, W_t^k\right)^\intercal_{t\in[0,T]}$ is a k-dimensional standard Brownian motion (with independent coordinates) for some $k \in \mathbb{N}^*$. In these dynamics, the matrix R steers the deterministic part of the process, $\overline{S}$ represents the unconditional long-term expectation of $(S_t)_{t\in[0,T]}$, and V drives the dispersion (for what follows, we introduce $\Sigma = VV^\intercal$ the covariation matrix of the process). The matrix K represents the linear permanent impact the agent has on the prices.[21] More precisely, since

$$\begin{aligned} d\tilde{S}_t = dS_t + Kv_t dt &= R(\overline{S} - S_t)dt + Kv_t dt + V dW_t \\ &= R(\overline{S} + K(Q_t - Q_0) - \tilde{S}_t)dt + Kv_t dt + V dW_t, \end{aligned}$$

trading impacts both current market prices and long-term expectations.

Ornstein-Uhlenbeck processes are well suited when prices exhibit mean reversion and/or when there exist one or several linear combinations of asset prices that are stationary. In the latter case, we say that the assets involved in the linear combinations are cointegrated (a situation often encountered in statistical arbitrage). For more details on cointegration in continuous time, we refer to Comte (1999).

Finally, the process $(\tilde{X}_t)_{t\in[0,T]}$ modelling the trader's cash account has the dynamics

$$d\tilde{X}_t = -v_t^\intercal \tilde{S}_t dt - L(v_t)dt,$$

with $\tilde{X}_0 \in \mathbb{R}$ given, where $L : \mathbb{R}^d \to \mathbb{R}_+$ is a function representing the temporary market impact of trades and/or the execution costs incurred by the trader. We assume that L is a positive-definite quadratic form, i.e.[22]

$$\boxed{L(v) = v^\intercal \eta v \quad \text{with} \quad \eta \in \mathcal{S}_d^{++}(\mathbb{R}).}$$

12.2 Performance criterion

The trader seeks to maximise the expected utility of their wealth at the end of the trading window $[0, T]$. This wealth is the sum of the amount $\tilde{X}_T$ on the cash account at time T and the value of the remaining inventory evaluated here at $Q_T^\intercal \tilde{S}_T - \tilde{\ell}(Q_T)$, where the discount term $\tilde{\ell}(Q_T)$ applied to the Mark-to-Market (MtM) value of the remaining assets $(Q_T^\intercal \tilde{S}_T)$ penalises any non-zero

[21] It is assumed symmetric to avoid price manipulation.

[22] The subset of positive-definite and positive semi-definite matrices of $\mathcal{S}_d(\mathbb{R})$ are respectively denoted by $\mathcal{S}_d^{++}(\mathbb{R})$ and $\mathcal{S}_d^{+}(\mathbb{R})$.

terminal position. We assume that $\tilde{\ell}$ is a positive-definite quadratic form, i.e. $\tilde{\ell}(q) = q^\intercal \tilde{\Gamma} q$ with $\tilde{\Gamma} \in \mathcal{S}_d^{++}(\mathbb{R})$ (see below for a stronger assumption on $\tilde{\Gamma}$).

To define the set of admissible controls $\mathcal{A}$, we first introduce a notion of "linear growth" relevant in our context. These technical considerations are beyond the scope of this course, but they are necessary to obtain a rigorous solution to the problem.

Definition 1. *Let $t \in [0,T]$. An $\mathbb{R}^d$-valued, $\mathbb{F}$-adapted process $(\zeta_u)_{u\in[t,T]}$ is said to satisfy a linear growth condition on $[t,T]$ with respect to $(S_u)_{u\in[t,T]}$ if there exists a constant $C_{t,T} > 0$ such that for all $u \in [t,T]$,*

$$\|\zeta_s\| \leq C_{t,T} \left(1 + \sup_{\tau\in[t,u]} \|S_\tau\|\right)$$

almost surely.[23]

We then define for all $t \in [0,T]$:

$$\mathcal{A}_t = \Big\{ (v_u)_{u\in[t,T]},\ \mathbb{R}^d\text{-valued},\ \mathbb{F}\text{-adapted, satisfying} \\ \text{a linear growth condition with respect to } (S_u)_{u\in[t,T]} \Big\},$$

and take $\mathcal{A} := \mathcal{A}_0$.[24]

Mathematically, the trader seeks to solve the dynamic optimisation problem

$$\sup_{v\in\mathcal{A}} \mathbb{E}\left[-\exp\left(-\gamma\left(\tilde{X}_T + Q_T^\intercal \tilde{S}_T - \tilde{\ell}(Q_T)\right)\right)\right], \tag{98}$$

where $\gamma > 0$ is the absolute risk aversion parameter of the trader.

12.3 An equivalent problem

Note that

$$\tilde{X}_T + Q_T^\intercal \tilde{S}_T - \tilde{\ell}(Q_T) = \tilde{X}_0 + Q_0^\intercal \tilde{S}_0 + \int_0^T Q_t^\intercal d\tilde{S}_t - \int_0^T L(v_t)dt - \tilde{\ell}(Q_T)$$

[23]In this section, $\|.\|$ denotes a fixed norm on $\mathbb{R}^d$ (for instance, the Euclidean norm).

[24]We restrict our analysis to linear growth strategies for mathematical convenience, but we expect the candidate control to be optimal among a larger class of processes.

$$
\begin{aligned}
&= X_0 + Q_0^\intercal S_0 + \int_0^T Q_t^\intercal dS_t + \int_0^T Q_t^\intercal K v_t dt \\
&\quad - \int_0^T L(v_t)dt - \tilde{\ell}(Q_T) \\
&= X_T + Q_T^\intercal S_T - \tilde{\ell}(Q_T) + \frac{1}{2}Q_T^\intercal K Q_T - \frac{1}{2}Q_0^\intercal K Q_0,
\end{aligned}
$$

where $X_0 = \tilde{X}_0$ and the process $(X_t)_{t\in[0,T]}$ has dynamics

$$\boxed{dX_t = -v_t^\intercal S_t dt - L(v_t)dt.} \tag{99}$$

Let us now define the penalty function $\ell : \mathbb{R}^d \to \mathbb{R}$ by:

$$\boxed{\ell(q) = \tilde{\ell}(q) - \frac{1}{2}q^\intercal K q = q^\intercal \tilde{\Gamma} q - \frac{1}{2}q^\intercal K q \quad \forall q \in \mathbb{R}^d.}$$

In what follows, we assume that ℓ is a positive semi-definite quadratic form, i.e. $\Gamma = \tilde{\Gamma} - \frac{1}{2}K \in \mathcal{S}_d^+(\mathbb{R})$.

Remark 1. *The assumption on $\tilde{\Gamma}$ is not restrictive as in practice, $\tilde{\Gamma}$ (and therefore Γ) is chosen arbitrarily large to enforce liquidation.*

It is then straightforward to see that Problem (98) is equivalent to the following problem:

$$\boxed{\sup_{v\in\mathcal{A}} \mathbb{E}_{t,x,q,s}\left[-\exp\left(-\gamma\left(X_T + Q_T^\intercal S_T - \ell(Q_T)\right)\right)\right].} \tag{100}$$

The value function of the problem $u : [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is

$$u(t,x,q,s) = \sup_{v\in\mathcal{A}_t} \mathbb{E}_{t,x,q,s}\left[-\exp\left(-\gamma\left(X_T + (Q_T)^\intercal S_T - \ell(Q_T)\right)\right)\right].$$

12.4 Solution

Use the dynamics (96)-(97)-(99) of the problem to write the HJB equation associated with Problem (100) as[25]

$$\boxed{\begin{aligned} 0 = \partial_t w &+ \sup_{v\in\mathbb{R}^d}\left(-(v^\intercal S + L(v))\partial_x w + v^\intercal \nabla_q w\right) \\ &+ (\overline{S} - S)^\intercal R^\intercal \nabla_S w + \frac{1}{2}\mathrm{Tr}\left(\Sigma D_{SS}^2 w\right), \end{aligned}} \tag{101a}$$

[25] u will be solution to that equation, but as we do not know it yet, we write the equation with an unknown function w.

for all $(t, x, q, s) \in [0, T) \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ with the terminal condition

$$w(T, x, q, s) = -\exp\left(-\gamma\left(x + q^\intercal s - \ell(q)\right)\right) \quad \forall (x, q, s) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \tag{102}$$

In order to study (101a), we use the following ansatz:

$$\boxed{\begin{aligned} w(t, x, q, s) = &-\exp\left(-\gamma\left(x + q^\intercal s + \theta(t, q, s)\right)\right), \\ &\forall (t, x, q, s) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{2d}. \end{aligned}} \tag{103a}$$

The interest of this ansatz is based on the following proposition:

Proposition 4. *If there exists $\theta \in C^{1,1,2}([0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ solution to*

$$\begin{aligned} 0 = \partial_t \theta + \sup_{v \in \mathbb{R}^d} \left(v^\intercal \nabla_q \theta - L(v)\right) + \frac{1}{2} Tr\left(\Sigma D^2_{SS} \theta\right) \\ - \frac{\gamma}{2}(q + \nabla_S \theta)^\intercal \Sigma (q + \nabla_S \theta) + (\overline{S} - S)^\intercal R^\intercal (\nabla_S \theta + q) \end{aligned} \tag{104}$$

on $[0, T) \times \mathbb{R}^d \times \mathbb{R}^d$, with terminal condition

$$\theta(T, q, s) = -\ell(q) \quad \forall (q, s) \in \mathbb{R}^d \times \mathbb{R}^d, \tag{105}$$

then the function w defined by the ansatz (103a) is a solution to (101a) on $[0, T) \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ with terminal condition (102).

Proof. Let $\theta \in C^{1,1,2}([0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ be a solution to (104) on $[0, T) \times \mathbb{R}^d \times \mathbb{R}^d$ with terminal condition (105), then we have for all $(t, x, q, s) \in [0, T) \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$:

$$\begin{aligned} &\partial_t w + \sup_{v \in \mathbb{R}^d} \left(-(v^\intercal S + L(v)) \partial_x w + v^\intercal \nabla_q w\right) + (\overline{S} - S)^\intercal R^\intercal \nabla_S w + \frac{1}{2} \mathrm{Tr}\left(\Sigma D^2_{SS} w\right) \\ = &\sup_{v \in \mathbb{R}^d} \left(\gamma (v^\intercal S + L(v)) w - \gamma v^\intercal (\nabla_q \theta + S) w\right) + \frac{\gamma^2}{2} \mathrm{Tr}\left(\Sigma (q + \nabla_S \theta)(q + \nabla_S \theta)^\intercal w\right) \\ &- \gamma \partial_t \theta\, w - \gamma (\overline{S} - S)^\intercal R^\intercal (\nabla_S \theta + q) w - \frac{1}{2} \mathrm{Tr}\left(\gamma \Sigma D^2_{SS} \theta w\right) \\ = &- \gamma w \Bigg(\partial_t \theta + \sup_{v \in \mathbb{R}^d} \left(v^\intercal \nabla_q \theta - L(v)\right) + \frac{1}{2} \mathrm{Tr}\left(\Sigma D^2_{SS} \theta\right) \\ &\qquad - \frac{\gamma}{2}(q + \nabla_S \theta)^\intercal \Sigma (q + \nabla_S \theta) + (\overline{S} - S)^\intercal R^\intercal (\nabla_S \theta + q) \Bigg) \end{aligned}$$

$= 0.$

As it is straightforward to verify that w satisfies the terminal condition (102), the result is proved. $\square$

The above result does not use the quadratic assumptions for L and ℓ. In the quadratic case we consider in this section, θ can be found in almost closed form. To prove this point, the first thing to notice is that the Legendre-Fenchel transform of L writes

$$H : p \in \mathbb{R}^d \mapsto \sup_{v \in \mathbb{R}^d} v^\intercal p - L(v) = \sup_{v \in \mathbb{R}^d} v^\intercal p - v^\intercal \eta v = \frac{1}{4} p^\intercal \eta^{-1} p,$$

as the supremum is reached at $v^* = \frac{1}{2}\eta^{-1}p$.

Consequently, we get the following HJB equation for θ:

$$\boxed{\begin{aligned} 0 =& \partial_t \theta + \frac{1}{4}\nabla_q \theta^\intercal \eta^{-1} \nabla_q \theta + \frac{1}{2}\mathrm{Tr}\left(\Sigma D^2_{SS}\theta\right) \\ &- \frac{\gamma}{2}(q + \nabla_S \theta)^\intercal \Sigma (q + \nabla_S \theta) + (\overline{S} - S)^\intercal R^\intercal (\nabla_S \theta + q), \end{aligned}} \tag{106a}$$

with terminal condition

$$\theta(T, q, s) = -q^\intercal \Gamma q \quad \forall (q, s) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{107}$$

To further study (106a), we introduce a second ansatz and look for a solution θ of the following form:

$$\theta(t, q, s) = q^\intercal A(t) q + q^\intercal B(t) s + s^\intercal C(t) s + D(t)^\intercal q + E(t)^\intercal s + F(t) \tag{108}$$

for all $(t, q, s) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, or equivalently

$$\boxed{\theta(t, q, s) = \begin{pmatrix} q \\ s \end{pmatrix}^\intercal P(t) \begin{pmatrix} q \\ s \end{pmatrix} + \begin{pmatrix} D(t) \\ E(t) \end{pmatrix}^\intercal \begin{pmatrix} q \\ s \end{pmatrix} + F(t)\,,}$$

where $P : [0, T] \to \mathcal{S}_{2d}(\mathbb{R})$ is defined as

$$\boxed{P(t) = \begin{pmatrix} A(t) & \frac{1}{2}B(t) \\ \frac{1}{2}B(t)^\intercal & C(t) \end{pmatrix}.}$$

The interest of this ansatz is stated in the following proposition.

Proposition 5. *Assume there exist* $A \in C^1([0,T], \mathcal{S}_d(\mathbb{R}))$, $B \in C^1([0,T], \mathcal{M}_d(\mathbb{R}))$, $C \in C^1([0,T], \mathcal{S}_d(\mathbb{R}))$, $D \in C^1\left([0,T], \mathbb{R}^d\right)$, $E \in C^1\left([0,T], \mathbb{R}^d\right)$, $F \in C^1([0,T], \mathbb{R})$ *satisfying the system of ODEs*

$$\boxed{\begin{cases} A'(t) = \frac{\gamma}{2}(B(t)+I_d)\Sigma(B(t)^\intercal + I_d) - A(t)\eta^{-1}A(t) \\ B'(t) = (B(t)+I_d)R + 2\gamma(B(t)+I_d)\Sigma C(t) - A(t)\eta^{-1}B(t) \\ C'(t) = R^\intercal C(t) + C(t)R + 2\gamma C(t)\Sigma C(t) - \frac{1}{4}B(t)^\intercal\eta^{-1}B(t) \\ D'(t) = -(B(t)+I_d)R\overline{S} + \gamma(B(t)+I_d)\Sigma E(t) - A(t)\eta^{-1}D(t) \\ E'(t) = -2C(t)R\overline{S} + R^\intercal E(t) + 2\gamma C(t)\Sigma E(t) - \frac{1}{2}B(t)^\intercal\eta^{-1}D(t) \\ F'(t) = -\overline{S}^\intercal R^\intercal E(t) - Tr(\Sigma C(t)) + \frac{\gamma}{2}E(t)^\intercal\Sigma E(t) - \frac{1}{4}D(t)^\intercal\eta^{-1}D(t), \end{cases}} \tag{109}$$

where I_d *denotes the identity matrix in* $\mathcal{M}_d(\mathbb{R})$, *with terminal conditions*

$$A(T) = -\Gamma, \quad B(T) = C(T) = D(T) = E(T) = F(T) = 0. \tag{110}$$

Then θ *defined by* (108) *satisfies* (106a) *on* $[\tau, T) \times \mathbb{R}^d \times \mathbb{R}^d$ *with terminal condition* (107).

Proof. Let $A \in C^1([\tau,T], \mathcal{S}_d(\mathbb{R}))$, $B \in C^1([\tau,T], \mathcal{M}_d(\mathbb{R}))$, $C \in C^1([\tau,T], \mathcal{S}_d(\mathbb{R}))$, $D \in C^1\left([\tau,T], \mathbb{R}^d\right)$, $E \in C^1\left([\tau,T], \mathbb{R}^d\right)$, $F \in C^1([\tau,T], \mathbb{R})$ verify (109) on $[\tau, T)$ with terminal condition (110). Consider $\theta : [\tau,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by (108). Then we obtain for all $(t,q,s) \in [\tau, T) \times \mathbb{R}^d \times \mathbb{R}^d$:

$$\begin{aligned}
& \partial_t\theta + \frac{1}{4}\nabla_q\theta^\intercal\eta^{-1}\nabla_q\theta + \frac{1}{2}\mathrm{Tr}\left(\Sigma D^2_{SS}\theta\right) \\
& - \frac{\gamma}{2}(q + \nabla_S\theta)^\intercal\Sigma(q+\nabla_S\theta) + (\overline{S} - S)^\intercal R^\intercal(\nabla_S\theta + q), \\
= \quad & q^\intercal A'(t)q + q^\intercal B'(t)S + S^\intercal C'(t)S + D'(t)^\intercal q + E'(t)^\intercal S + F'(t) \\
& + q^\intercal A(t)\eta^{-1}A(t)q + q^\intercal A(t)\eta^{-1}B(t)S + \frac{1}{4}S^\intercal B(t)^\intercal\eta^{-1}B(t)S \\
& + D(t)^\intercal\eta^{-1}A(t)q + \frac{1}{2}\left(D(t)\right)^\intercal\eta^{-1}B(t)S + \frac{1}{4}D(t)^\intercal\eta^{-1}D(t) + Tr(\Sigma C(t)) \\
& - \frac{\gamma}{2}\left(q + B(t)^\intercal q + 2C(t)S + E(t)\right)^\intercal\Sigma\left(q + B(t)^\intercal q + 2C(t)S + E(t)\right) \\
& + \overline{S}^\intercal R^\intercal q + \overline{S}^\intercal R^\intercal\left(B(t)^\intercal q + 2C(t)S + E(t)\right) \\
& - S^\intercal R^\intercal q - S^\intercal R^\intercal\left(B(t)^\intercal q + 2C(t)S + E(t)\right) \\
= \quad & q^\intercal\left(A'(t) - \frac{\gamma}{2}(B(t)+I_d)\Sigma(B(t)^\intercal + I_d) + \frac{1}{4}\left(2A(t)\right)\eta^{-1}\left(2A(t)\right)\right)q
\end{aligned}$$

$$
\begin{aligned}
&+ q^\intercal \left(B'(t) - (I_d + B(t))R - 2\gamma(B(t) + I_d)\Sigma C(t) + A(t)\eta^{-1}B(t)\right) S\\
&+ S^\intercal \left(C'(t) - R^\intercal C(t) - C(t)R - 2\gamma C(t)\Sigma C(t) + \frac{1}{4}B(t)^\intercal \eta^{-1}B(t)\right) S\\
&+ \left(D'(t) + (B(t) + I_d)R\overline{S} - \gamma(B(t) + I_d)\Sigma E(t) + A(t)\eta^{-1}D(t)\right)^\intercal q\\
&+ \left(E'(t) + 2C(t)R\overline{S} - R^\intercal E(t) - 2\gamma C(t)\Sigma E(t) + \frac{1}{2}B(t)^\intercal \eta^{-1}D(t)\right)^\intercal S\\
&+ \left(F'(t) + \overline{S}^\intercal R^\intercal E(t) + Tr(\Sigma C(t)) - \frac{\gamma}{2}E(t)^\intercal \Sigma E(t) + \frac{1}{4}D(t)^\intercal \eta^{-1}D(t)\right)\\
= \quad & 0.
\end{aligned}
$$

As it is straightforward to verify that θ satisfies the terminal condition (107), the result is proved. □

Remark 2. *Two remarks can be made on the system of ODEs (109):*

- *This system of ODEs can clearly be decomposed into three groups of equations: the first three ODEs for A, B and C are independent of the others and can be solved as a first step; once we know A, B, and C we can solve the linear ODEs for D and E, and finally F can be obtained with a simple integration;*
- *When $R = 0$ (i.e. in the case where the prices S of the d assets are correlated arithmetic Brownian motions), there is a trivial solution to the last five equations which is $B = C = D = E = F = 0$. The function A can then be found using classical techniques (as shown in Section 11).*

12.5 A matrix Riccati equation

It is noteworthy that the first system, i.e.

$$
\begin{cases}
A'(t) = \frac{\gamma}{2}(B(t) + I_d)\Sigma(B(t)^\intercal + I_d) - A(t)\eta^{-1}A(t)\\
B'(t) = (B(t) + I_d)R + 2\gamma(B(t) + I_d)\Sigma C(t) - A(t)\eta^{-1}B(t)\\
C'(t) = R^\intercal C(t) + C(t)R + 2\gamma C(t)\Sigma C(t) - \frac{1}{4}B(t)^\intercal \eta^{-1}B(t)
\end{cases}
$$

boils down to the following Matrix Riccati ODE in $P = \begin{pmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{pmatrix}$:

$$
\boxed{P'(t) = Q + Y^\intercal P(t) + P(t)Y + P(t)UP(t),} \tag{111}
$$

where

$$Q = \frac{1}{2}\begin{pmatrix} \gamma\Sigma & R \\ R^\intercal & 0 \end{pmatrix} \in \mathcal{S}_{2d}(\mathbb{R}), \qquad Y = \begin{pmatrix} 0 & 0 \\ \gamma\Sigma & R \end{pmatrix} \in \mathcal{M}_{2d}(\mathbb{R}),$$

$$U = \begin{pmatrix} -\eta^{-1} & 0 \\ 0 & 2\gamma\Sigma \end{pmatrix} \in \mathcal{S}_{2d}(\mathbb{R}),$$

and the terminal condition writes

$$P(T) = \begin{pmatrix} -\Gamma & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{S}_{2d}(\mathbb{R}). \tag{112}$$

When compared to the Matrix Riccati ODEs arising in the LQG control literature, the distinctive aspect of our equation is that the matrix U characterizing the quadratic term in the Riccati equation has both positive and negative eigenvalues. In particular, we cannot rely on existing results (see for instance Theorem 3.5 of Freiling (2002)) to prove that there exists a solution to (111) with terminal condition (112). However, the work in Bergault et al. (2022) proves the existence of a solution. we recall these results in the following section.

12.6 Simulations

Here, we show several applications of the strategy. We first use of the optimal strategy for a trader wishing to unwind or execute a statistical arbitrage for a mean-reverting asset. Next, we consider a pair of two cointegrated French stocks (**pairs trading**). We start with a two-asset portfolio liquidation problem and compare the optimal liquidation strategy in our model with that obtained in a multi-asset Almgren-Chriss model.[26]

In the Almgren-Chriss model used for carrying out comparisons, the price dynamics is of the form $dS_t = V_{\mathrm{AC}} dW_t$, where $V_{\mathrm{AC}} \in \mathcal{M}_{d,k}(\mathbb{R})$, i.e. a simple Bachelier dynamics (with correlations). This dynamics differs from that of the OU model we use throughout this section (i.e. $dS_t = R(\overline{S} - S_t)dt + V dW_t$) when $R \neq 0$. In particular, if prices exhibit mean reversion or a cointegrated behavior, as is the case in our examples, the classical Almgren-Chriss model does not properly take the true multivariate dynamics of prices into account, with sometimes important consequences in terms of risk management.

[26]Throughout this section, optimal trading strategies are computed by approximating the solution of the Riccati ODEs using implicit Euler schemes.

12.6.1 Mean reversion

Here, we use data from the FX market, in which asset prices often exhibit mean reversion. We consider a FX futures contract (hereafter CDU1) on the currency pair Canadian Dollar (CAD) / US Dollar (USD) that is exchanged on the Chicago Mercantile Exchange. The contract specifications are given in Table 2.

Underlying asset	Canadian Dollar
Quotation currency	US Dollar
Contract size	CAD 100000
Expiry date	September 14, 2021

Table 2: CDU1 contract specifications

We plot in Figure 24 the mid-price of CDU1,[27] sampled every 60 seconds during the regular trading hours (02:00-16:00 Central Time),[28] over the three following trading days: August 11, August 12, and August 13, 2021.

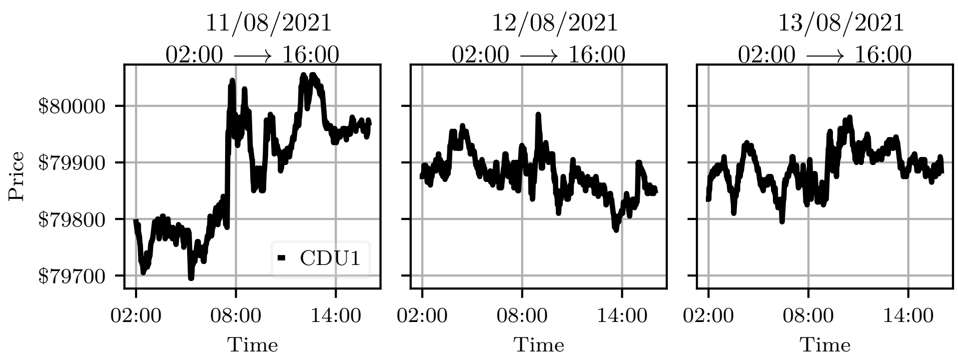

Figure 24: Mid-price of CDU1 sampled every 60 seconds during the regular trading hours (02:00-16:00 CT). Top left: August 11, 2021. Top right: August 12, 2021. Bottom: August 13, 2021.

Liquidation. We consider the case of a trader wishing to unwind a long position in 2250 contracts[29] during the third day, i.e. on August 13, 2021.

[27] CDU1 is usually quoted in USD cents per CAD. However, to use our model, the price must take account of the contract size and be the contract value in USD.

[28] Although CDU1 is quoted continuously with a 60-minute break each day beginning at 16:00 CT, we only consider the trading hours between 02:00 and 16:00 CT because the contract is only liquid during these hours corresponding to European and American market activity.

[29] This represents roughly 5% of the average daily traded volume over the period considered in this example.

We estimate the OU parameters using prices from the two preceding trading days: August 11 and August 12, 2021. Coefficients are classically estimated using least squares regression.[30] In order to set the value of the execution cost / temporary market impact parameter η, we use a similar argument as in Almgren and Chriss (2001): we suppose that the additional cost incurred per contract when trading a given volume is proportional to the participation rate to the market. More precisely, for each percent of participation rate (in practice we consider a flat volume curve that matches the average daily volume), a cost corresponding to half the bid-ask spread[31] is incurred. This results in setting $\eta = 5 \cdot 10^{-3}$ $\cdot$ day. For the terminal penalty parameter Γ, we set a high value to enforce complete liquidation by the end of the trading day. For the risk aversion parameter γ, we choose an intermediate value that does not neutralize any of the financial effects our model could illustrate. The resulting values used to run our algorithms are given in Table 3.[32]

Parameter	Value
T	1 day
Q_0	2250
S_0	\$79835
R	5.1 day^{-1}
$\overline{S}$	\$79887
σ	$243.67\ \$ \cdot \text{day}^{-\frac{1}{2}}$
η	$5 \cdot 10^{-3}\ \$ \cdot \text{day}$
Γ	\$100
γ	$2 \cdot 10^{-5}\ \$^{-1}$

Table 3: Value of the parameters.

We plot in Figure 25 the asset price trajectory $(S_t)_{t\in[0,T]}$ on August 13, 2021 and the inventory process $(Q_t)_{t\in[0,T]}$ corresponding to the use of the optimal strategy.[33] We also plot the inventory process when using a classical

[30] A time discretisation of an Ornstein-Uhlenbeck model gives rise to an Auto-Regressive model of order 1, or AR(1). The parameters of an AR(1) model are classically estimated by using least squares regression. Conversion of AR(1) coefficients into their continuous-time counterparts is straightforward.

[31] The average bid-ask spread is close to the tick value equal to \$5 per contract.

[32] In the one-asset case, $\Sigma = VV^\intercal$ is a scalar. We classically write it as σ^2 and document the value of σ.

[33] In what follows, this strategy is often referred to as ACOU (Almgren-Chriss under Ornstein-Uhlenbeck dynamics) strategy.

Almgren-Chriss (AC) strategy.[34]

The optimal strategy with mean reversion is different from that derived in the Almgren-Chriss model. In particular, the liquidation process is significantly faster in the latter case because unwinding the portfolio appears riskier to a trader who believes that the price evolves as a Brownian rather than a mean-reverting OU. Second, in the case of the ACOU strategy, the trader progressively unwinds their long position over the trading day but also takes advantage of mean reversion. When the price is below $\overline{S}$, the trader reduces the pace of their selling. Symmetrically, when the price is above $\overline{S}$, the trader sells at a faster pace.

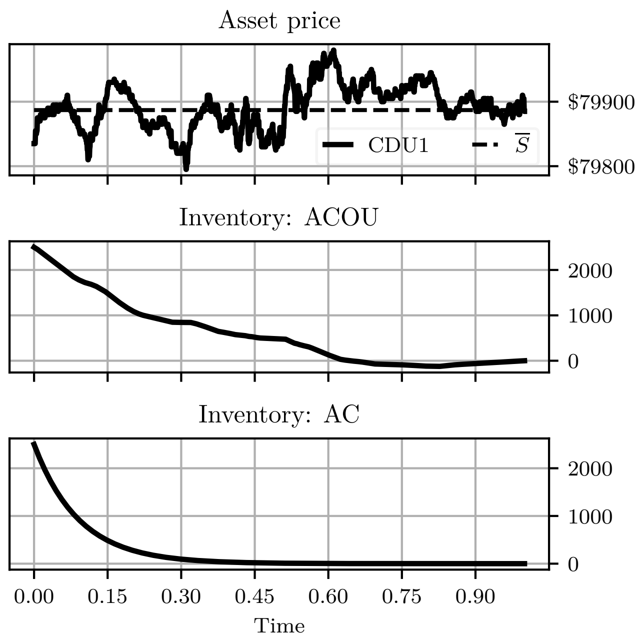

Figure 25: Top: CDU1 price trajectory on August 13, 2021 – $(S_t)_{t\in[0,T]}$. Middle: Trajectory of the inventory when using the optimal strategy corresponding to the estimated Ornstein-Uhlenbeck process – $(Q_t)_{t\in[0,T]}$. Bottom: Trajectory of the inventory when using the optimal strategy corresponding to a Brownian motion (Bachelier) model for the price (classical Almgren-Chriss strategy).

[34]To compute the AC strategy, we estimate the parameter V_{AC} of the Bachelier dynamics. This parameter is a scalar in our one-asset case and we denote it by σ_{AC} instead of V_{AC}. A simple estimation based on price increments leads to $\sigma_{\mathrm{AC}} = 244.02\ \$ \cdot \mathrm{day}^{-\frac{1}{2}}$ which slighlty differs from σ because the drift term in the OU model captures part of the variance.

12.6.2 Pairs trading

Asset prices that exhibit a cointegrated behaviour can be modeled by a multi-OU process. We use data from two French stocks within the banking sector: BNP Paribas (hereafter BNP) and Société Générale (hereafter GLE). Figure 26 shows the mid-prices of BNP and GLE sampled every 60 seconds during the regular trading hours (09:00-17:30) over the week August 09-August 13, 2021. Clearly, the stock prices of the two companies are driven by the same factors and should be cointegrated.

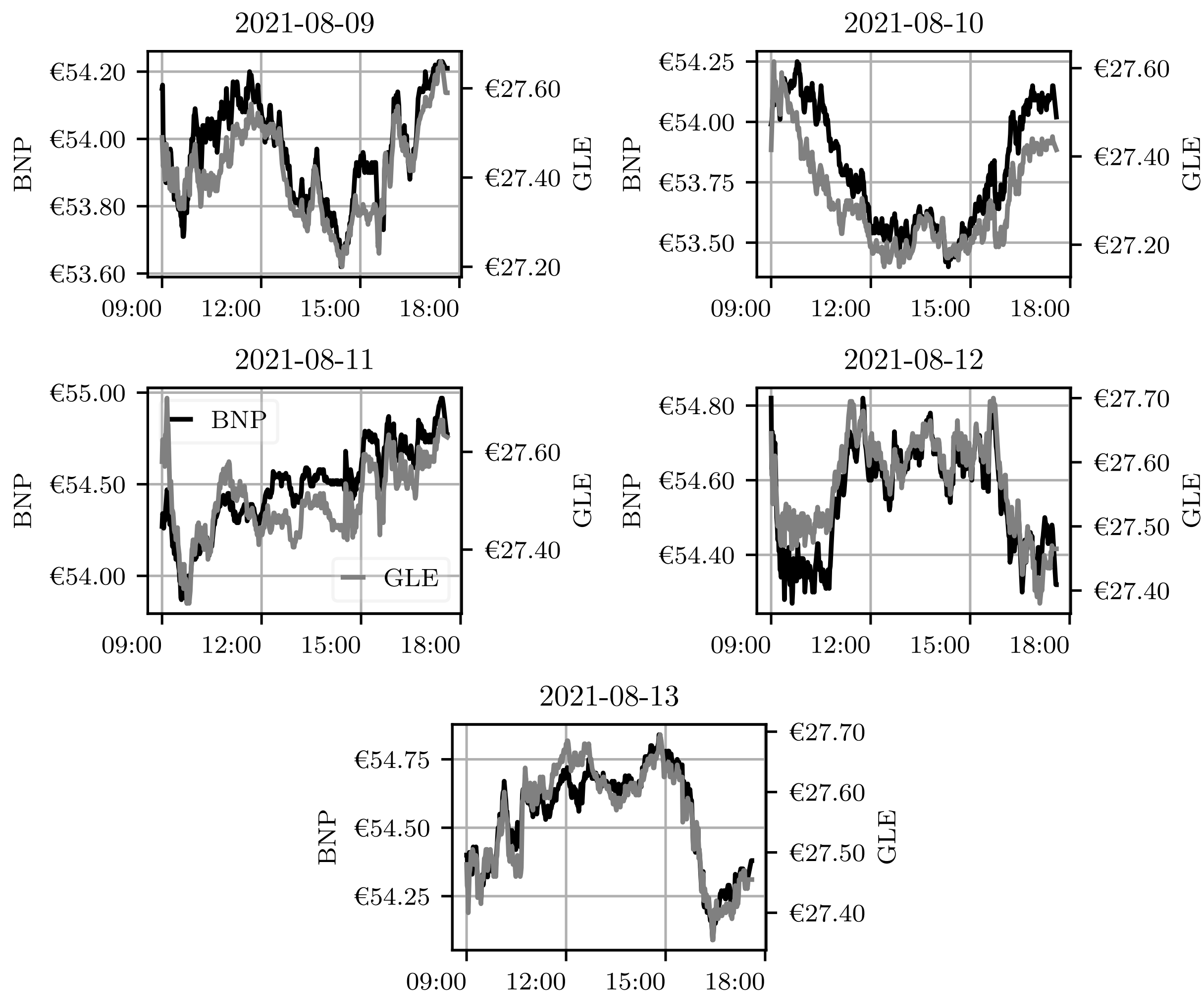

Figure 26: Mid-prices of BNP (left axis) and GLE (right axis) sampled every 60 seconds during the regular trading hours (09:00-17:30) over the week August 09-August 13, 2021.

We consider the case of a trader wishing to execute a statistical arbitrage in the pair BNP/GLE throughout August 13, 2021. The trader estimates the parameters of a multi-OU model using prices from the four preceding trading days, here August 9, 10, 11, and 12, 2021. These parameters are estimated using classical linear regression techniques.[35] Referring to BNP and GLE by respectively using the superscripts 1 and 2, the estimated values of the parameters are given in Table 4. The estimated value of the matrix R suggests that the space of cointegration vectors is spanned by $(1, -3.46)$.

Parameter	Estimate
R	$\begin{pmatrix} 0.33 & 3.95 \\ -2.52 & 10.23 \end{pmatrix}$ day^{-1}
$\overline{S}$	$\left(\overline{S}^1, \overline{S}^2\right) = (€54.23, €27.45)$
Σ	$\begin{pmatrix} 0.47 & 0.20 \\ 0.20 & 0.14 \end{pmatrix}$ $€^2 \cdot$ day^{-1}

Table 4: Multi-OU estimated parameters for the pair (BNP, GLE).

We illustrate the use of our model in the context of pure statistical arbitrage. We consider a trader with no initial inventory who starts trading on August 13, 2021 and wants to maximise the expected utility of their PnL at the end of the day (with no final penalty). To run our algorithm we use the parameters in Tables 4 and 5 with $\gamma = 2 \cdot 10^{-3}$ €$^{-1}$. The results are plotted in Figure 27: the spread process $((S^1_t - \overline{S}^1) - 3.46(S^2_t - \overline{S}^2))_{t\in[0,T]}$, the inventory process $(Q_t)_{t\in[0,T]}$ when using the optimal strategy, and the associated trajectory of the PnL, i.e. the process $(X_t + Q_t^\intercal S_t)_{t\in[0,T]}$.

[35] A time discretisation of a multi-OU model gives rise to a Vector Auto-Regressive model of order 1, or VAR(1). The parameters of a VAR(1) model are classically estimated by using least squares regression. Conversion of VAR(1) coefficients into their continuous-time multi-OU counterparts is straightforward.

Parameter	Value
T	1 day
Q_0	$(0,0)$
S_0	$\left(S_0^1, S_0^2\right) = (€54.4, €27.48)$
η	$\begin{pmatrix} 4 \cdot 10^{-7} & 0 \\ 0 & 2 \cdot 10^{-7} \end{pmatrix}$ € $\cdot$ day
Γ	€$0 \times I_2$
γ	$2 \cdot 10^{-5}$ €$^{-1}$ or $2 \cdot 10^{-3}$ €$^{-1}$

Table 5: Value of the parameters.

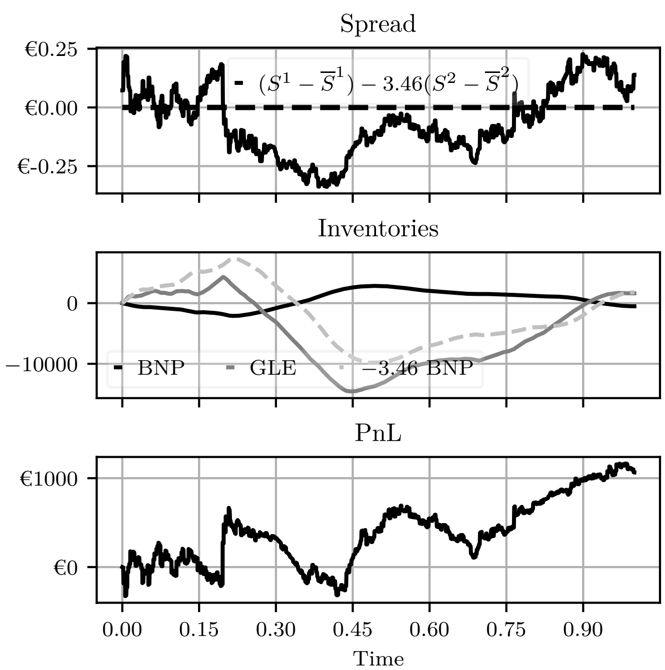

Figure 27: Top: Trajectory of the spread on August 13, 2021 – $((S_t^1 - \overline{S}^1) - 3.46(S_t^2 - \overline{S}^2))_{t\in[0,T]}$. Middle: Trajectory of the inventories when using the optimal strategy corresponding to the estimated multi-OU process with $\gamma = 2 \cdot 10^{-3}$ €$^{-1}$ – $(Q_t)_{t\in[0,T]}$. Bottom: Trajectory of the PnL – $(X_t + Q_t^\intercal S_t)_{t\in[0,T]}$.

We clearly see that the optimal strategy is a long/short strategy. Figure 27 shows the process $(-3.46Q_t^1)_{t\in[0,T]}$ which appears to be in line with $(Q_t^2)_{t\in[0,T]}$. This confirms that the strategy consists mainly in "buying or selling the spread" depending on the sign of the spread process $((S_t^1 - \overline{S}^1) - 3.46(S_t^2 - \overline{S}^2))_{t\in[0,T]}$.

12.7 Conclusions

In this section, we have shown how to account for cross-asset co-movements when executing trades in multiple assets. In our model, the agent has an exponential utility and the prices have multivariate Ornstein-Uhlenbeck dynamics, capturing the complex cross-asset dynamics of prices better than correlated Brownian motions only. The advantage of our approach is twofold: (i) it better accounts for risk at the portfolio level, and (ii) it is versatile and can be used for basket execution and statistical arbitrage.

13 Optimal market making of multiple assets

OTC market makers faces a complex problem. They provide bid and ask quotes for various assets that exhibit complex joint dynamics without seeing the full depth of price and clients. Consequently, it is key to properly account for risks at the portfolio level. However, a large proportion of multi-asset market making models in the literature only consider correlated Brownian dynamics. Additionally, multi-asset market making is challenging due to high dimensionality and the resulting numerical challenges to obtain the optimal quotes.

The models of multi-asset market making suffer from the curse of dimensionality because the number of equations to solve grows exponentially with the number of assets. Multiple approaches are proposed by Guéant and various co-authors to study the issue of the dimensionality in market making problems. Guéant (2017) proposes a general model reconciling the approaches in Guéant et al. (2013) and Cartea et al. (2014) in a multi-asset model, and most importantly obtains interpretable approximations for the optimal quotes. Guéant and Manziuk (2019) use neural networks instead of grids to obtain numerical solutions of the problem in high dimension. They use a model-based actor-critic algorithm for approximating the value function and the optimal strategies within a framework similar to that of Avellaneda and Stoikov (2008). Bergault and Guéant (2021) propose an approach for approximating the optimal quotes by using a low-dimensional factor space on which market risk is projected. Finally, Bergault et al. (2021) propose an approach where the Hamilton-Jacobi equation of the problem is regarded as a perturbation of another simpler quadratic one, whose solution in their case can be obtained in closed form using Linear-Quadratic-Gaussian tools.

Section 13.1 describes a basic multi-asset model which considers correlated Brownian dynamics for the asset prices, and where the market maker adopts a CARA utility. Additional details and proofs of the results in Section 13.1 are in Guéant (2017).

13.1 Modelling framework

Here, a market maker that operates in an OTC market is in charge of a portfolio containing $d \in \mathbb{N}^*$ assets. The joint dynamics of the asset prices

$(S_t)_{t\in[0,T]} = \left(S_t^1, \ldots, S_t^d\right)^\intercal_{t\in[0,T]}$ follow a d-dimensional Brownian process:

$$dS_t = V\, dW_t\,,$$

where the initial values $S_0 \in \mathbb{R}^d$ are known and $(W_t)_{t\in[0,T]}$ is a d-dimensional standard Brownian motion with independent coordinates. We denote by $\Sigma = V^\intercal V$ the variance-covariance matrix associated with S.

Throughout a trading window $[0,T]$, where $T > 0$, the market maker continuously chooses the prices at which she is ready to buy or sell the d assets. We define the processes $(S_t^{i,b})_{t\in[0,T]}$ and $(S_t^{i,a})_{t\in[0,T]}$ as the bid and ask prices of asset i, respectively. We denote by the processes $(\delta_t^{i,b})_{t\in[0,T]}$ and $(\delta_t^{i,a})_{t\in[0,T]}$ the distances of the quotes from the reference price S^i for, respectively, the bid and the ask and we write

$$\delta_t^{i,b} = S_t^i - S_t^{i,b} \quad \text{and} \quad \delta_t^{i,a} = S_t^{i,a} - S_t^i\,, \quad \forall t \in [0,T].$$

For $i \in \{1,\ldots,d\}$, two point processes $\left(N^{i,b}\right)_{t\in[0,T]}$ and $\left(N^{i,a}\right)_{t\in[0,T]}$ model the number of transactions at the bid and ask side for asset i (see Cartea (2013) and Cartea et al. (2018d) for the use of point processes in financial decision problems). We assume that the market maker receives RFQs for a constant size Δ^i for the i-th asset and that she ceases to trade asset i if her (absolute) inventory in asset i exceeds a risk limit Q^i following a trade.

The inventory $(q_t)_{t\in[0,T]} = (q_t^1, \ldots, q_t^N)^\intercal_{t\in[0,T]}$ follows the dynamics

$$\forall i \in \{1,\ldots,d\}, \quad dq_t^i = \Delta^i dN_t^{i,b} - \Delta^i dN_t^{i,a}\,,$$

where the initial inventory q_0^i in the i-th asset is known. The point processes are independent of the Brownian W and have respective intensity processes $\left(\lambda_t^{i,b}\right)_{t\in[0,T]}$ and $\left(\lambda_t^{i,a}\right)_{t\in[0,T]}$ given by

$$\lambda_t^{i,b} = \Lambda^{i,b}\left(\delta_t^{i,b}\right)\mathbb{1}_{q_{t-}^i+\Delta^i<Q^i} \quad \text{and} \quad \lambda_t^{i,a} = \Lambda^{i,a}\left(\delta_t^{i,a}\right)\mathbb{1}_{q_{t-}^i-\Delta^i>-Q^i}\,,$$

for two functions $\Lambda^{i,b}, \Lambda^{i,a} : \mathbb{R} \mapsto \mathbb{R}_+$. The intensity of the arrival of orders at the bid and the ask decreases with the distance of the quotes from the reference price. Thus, we consider that the functions $\Lambda^{i,b}$ and $\Lambda^{i,a}$ are decreasing and go to zero when the quotes go to $+\infty$.

Finally, the cash process $(X_t)_{t\in[0,T]}$ of the market maker evolves as

$$\begin{aligned} dX_t &= \sum_{i=1}^{d} S_t^{i,a}\,\Delta^i\,dN_t^{i,a} - S_t^{i,b}\,\Delta^i\,dN_t^{i,b} \\ &= \sum_{i=1}^{d} \left(\delta_t^{i,b}\,\Delta^i\,dN_t^{i,b} + \delta_t^{i,a}\,\Delta^i\,dN_t^{i,a}\right) - \sum_{i=1}^{d} S_t^i\,dq_t^i. \end{aligned}$$

13.2 Performance criterion

We consider, as in Avellaneda and Stoikov (2008), that the market maker maximises the expected value of a CARA utility function with absolute risk aversion parameter $\gamma > 0$. Thus the market maker maximises the performance criterion

$$\mathbb{E}\left[-\exp\left(-\gamma\left(X_T + q_T^\intercal S_T - q_T^\intercal \Gamma q_T\right)\right)\right]. \tag{113}$$

13.3 Solution

Let $\{e^1, \ldots, e^d\}$ be the canonical basis of $\mathbb{R}^d$, i.e. for all $i \in \{1, \ldots, d\}$, e^i is a vector of zeros except for the ith element which is equal to 1. The HJB equation associated with (113) is

$$\begin{aligned} 0 = {} & \partial_t w(t,x,q,S) + \frac{1}{2}\mathrm{Tr}\left(\Sigma D^2_{SS} w(t,x,q,S)\right) \\ & + \sum_{i=1}^{d} \mathbb{1}_{\{q^i+\Delta^i \le Q^i\}} \sup_{\delta^{i,b}} \Lambda^{i,b}(\delta^{i,b}) \left(w\left(t, x-\Delta^i\left(S^i - \delta^{i,b}\right), q+\Delta^i e^i, S\right) - w\left(t,x,q,S\right)\right) \\ & + \sum_{i=1}^{d} \mathbb{1}_{\{q^i-\Delta^i \ge -Q^i\}} \sup_{\delta^{i,a}} \Lambda^{i,a}(\delta^{i,a}) \left(w\left(t, x+\Delta^i\left(S^i + \delta^{i,a}\right), q-\Delta^i e^i, S\right) - w\left(t,x,q,S\right)\right), \end{aligned} \tag{114}$$

for all $(t,x,q,S) \in [0,T) \times \mathbb{R} \times \mathcal{Q} \times \mathbb{R}^d$, with terminal condition

$$w(T,x,q,S) = -\exp\left(-\gamma\left(x + q^\intercal S - q^\intercal \Gamma q\right)\right), \quad \forall (x,q,S) \in \mathbb{R} \times \mathcal{Q} \times \mathbb{R}^d, \tag{115}$$

where $\mathcal{Q} = \Pi_{i=1}^d \left(\Delta^i\,\mathbb{Z} \cap [-Q^i, Q^i]\right)$.[36]

[36] We denote by $\Delta\,\mathbb{Z}$ the set of multiples of Δ, i.e. $\Delta\mathbb{Z} = \{\ldots, -2\,\Delta, -\Delta, 0, \Delta, 2\,\Delta, \ldots\}$, where $\Delta \in \mathbb{R}_+^*$ is a positive number.

To reduce the dimensionality of the problem, we use the ansatz

$$w(t,x,q,S) = -\exp\left(-\gamma\left(x + q^\intercal S + \theta(t,q)\right)\right), \tag{116}$$

and obtain the following HJB in θ:

$$\begin{aligned}
0 = {} & \partial_t\theta(t,q,S) - \frac{\gamma}{2}q^\intercal\Sigma q \\
& + \sum_{i=1}^{d} \mathbb{1}_{\{q^i+\Delta^i \le Q^i\}}\Delta^i H^{i,b}\left(\frac{\theta(t,q,S) - \theta(t,q+\Delta^i e^i,S)}{\Delta^i}\right) \\
& + \sum_{i=1}^{d} \mathbb{1}_{\{q^i-\Delta^i \ge -Q^i\}}\Delta^i H^{i,a}\left(\frac{\theta(t,q,S) - \theta(t,q-\Delta^i e^i,S)}{\Delta^i}\right),
\end{aligned} \tag{117}$$

on $[0,T)\times\mathcal{Q}$ with terminal condition

$$\theta(T,q) = -q^\intercal\Gamma q, \quad \forall q\in\mathcal{Q}, \tag{118}$$

where the Hamiltonian functions $H^{i,b}$ and $H^{i,a}$ are

$$\begin{aligned}
H^{i,b}(p) &= \sup_{\delta}\frac{\Lambda^{i,b}(\delta)}{\gamma\Delta^i}\left(1 - \exp\left(-\gamma\Delta^i(\delta - p)\right)\right) \\
H^{i,a}(p) &= \sup_{\delta}\frac{\Lambda^{i,a}(\delta)}{\gamma\Delta^i}\left(1 - \exp\left(-\gamma\Delta^i(\delta - p)\right)\right),
\end{aligned}$$

The following results solve the optimal multi-asset market making problem and are proved in Guéant (2017).

Theorem 3. *There exists a unique function $\theta : [0,T]\times\mathcal{Q}\mapsto\mathbb{R}$, $\mathcal{C}^1$ in time, solution of the HJB (117) with terminal condition (118).*

Theorem 4. *Let $\theta : \mathbb{R}^+\times\mathcal{Q}\to\mathbb{R}$ be the unique solution to the HJB (117) with terminal condition (118). Then, the function u defined in (116) is the unique solution to (114) with terminal condition (115). Moreover, for all $i\in\{1,\dots,d\}$, the optimal bid and ask quotes*

$$S_t^{i,b} = S_t^i - \delta_t^{i,b*} \quad \text{and} \quad S_t^{i,a} = S_t^i + \delta_t^{i,a*}$$

are obtained by

$$\delta_t^{i,b*} = \tilde{\delta}^{i,b*}\left(\frac{\theta(t,q_{t-}) - \theta(t,q_{t-}+\Delta^i e^i)}{\Delta^i}\right) \quad \text{when } q_{t-}+\Delta^i e^i \in \Pi_{i=1}^d[-Q^i,Q^i],$$

$$\delta_t^{i,a*} = \tilde{\delta}^{i,a*}\left(\frac{\theta(t,q_{t-}) - \theta(t,q_{t-} - \Delta^i e^i)}{\Delta^i}\right) \quad \textit{when } q_{t-} - \Delta^i e^i \in \Pi_{i=1}^d [-Q^i, Q^i],$$

where the functions $\tilde{\delta}^{i,b*}(.)$ *and* $\tilde{\delta}^{i,a*}(.)$ *are defined as*

$$\tilde{\delta}^{i,b*}(p) = \Lambda^{i,b^{-1}}\left(\gamma \Delta^i H^{i,b}(p) - H^{i,b'}(p)\right),$$
$$\tilde{\delta}^{i,a^*}(p) = \Lambda^{i,a^{-1}}\left(\gamma \Delta^i H^{i,a}(p) - H^{i,a'}(p)\right).$$

14 Extensions and further readings

Some works extend the modeling framework. Forsyth et al. (2012) examine the use of quadratic variation rather than variance in the objective function, Schied and Schoneborny (2009) use stochastic optimal control tools to characterise and find optimal strategies for a Von Neumann–Morgenstern investor, and Guéant (2015) provides results for optimal liquidation within a Von Neumann-Morgenstern expected utility framework with general market impact functions and derives subsequent results for block trade pricing.

Other works extend the assumptions on model parameters. Almgren (2003) studies the case of random execution costs, Almgren (2009, 2012) consider stochastic liquidity and volatility, Lehalle (2008) discusses the statistical aspects of the variability of estimators of the main exogenous variables such as volumes or volatilities in the optimisation phase, Cartea and Jaimungal (2016b) provide a closed-form strategy incorporating order flows from all agents, and Cartea et al. (2018b) analyse how model misspecification of order arrival rates, limit order fill probability, and asset price dynamics affect the agent's optimal trading strategy. Furthermore, Cartea et al. (2021b) and Cartea and Sánchez-Betancourt (2021, 2022b) propose models that incorporate latency to improve trading performance. Finally, Cartea et al. (2022b) and Cartea et al. (2022c,d,a) investigate algorithmic collusion.

Several works consider other types of order and execution strategies than those of the original Almgren-Chriss framework, which focuses on orders of the Implementation Shortfall (IS) type with MOs only. Other execution strategies have been studied in the literature, like Volume-Weighted Average Price (VWAP) orders in Konishi (2002), Frei and Westray (2015), Guéant and Royer (2014), and Cartea and Jaimungal (2016a) but also Target Close (TC) orders and Percentage of Volume (POV) orders, in Guéant (2016). Besides, several models focusing on optimal execution with limit orders have been proposed, as in Bayraktar and Ludkovski (2014), but also in Guéant et al. (2012) and Guéant and Lehalle (2015). The literature also addresses the existence of various trading venues. The case of optimal splitting of orders across different liquidity pools has been addressed in Laruelle et al. (2011) and Cartea et al. (2015).

Finally, a few works in the algorithmic trading literature use model free and data-driven approaches which better scale to complex models of the environment. Guéant and Manziuk (2019) use a model-based deep actor-critic Reinforcement Learning (RL) algorithm to approximate the optimal market making strategy in a high-dimensional setup, Cartea et al. (2021a) use double deep Q network learning to derive statistical arbitrage strategies, and Ning et al. (2021) use a model free Deep Q-Learning algorithm with LOB states as input features to derive execution strategies, Cont et al. (2022, 2023) use data-driven models to characterise cross-impact of order flow; see also Scalzo et al. (2021); Arroyo et al. (2022); Duran-Martin et al. (2022); Borde et al. (2023).

A strand of the literature generalises price dynamics and incorporates signals, uncertainty, and learning in execution. Cartea and Jaimungal (2016b) consider order flow as a driver of asset prices, Cartea et al. (2018a) study volume imbalance as a price predictor and considers an optimal trading framework that incorporates order book signals, and Cartea et al. (2019) use multivariate Ornstein-Uhlenbeck (multi-OU) dynamics for the prices. Laruelle et al. (2013) derive a strategy where the agents learns the parameters of a jump process, Cartea et al. (2017) incorporate model uncertainty in execution, and Casgrain and Jaimungal (2019) derive strategies with learning of latent state distribution upon which prices depend. Bhudisaksang and Cartea (2021a,b) introduce an adaptive control framework which is robust to model misspecification, where the agent continuously learns the drift, and where her uncertainty follows a jump-diffusion. Recently, Cartea and Sánchez-Betancourt (2022a) derive closed-form strategies where a broker provides liquidity to an informed trader and uses the toxic flow to extract the trend signal.

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