

Market Microstructure and Algorithmic Trading

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Lecture notes – HT 2024

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1 Introduction

Recent regulatory changes due to Reg NMS and MiFID along with the automation of algorithmic trading shifted the focus of researchers and practitioners towards liquidity and market microstructure problems. New strands of the mathematical finance literature now focus on new topics such as order execution, market making, systemic risk, and counterparty risk from a microstructure perspective.

Electronic markets are mainly organized around Limit Order Books (LOBs) and Over-The-Counter (OTC) markets to exchange securities between market participants. This course introduces the literature on optimal trading in high frequency markets with a focus on optimal order execution in LOBs and optimal market making in OTC markets. Optimal trading problems share a common structure that will be followed throughout this course. In each section, we (i) identify a decision problem motivated by practical situations faced by market operators, we (ii) propose a parsimonious model which summarises the environment in key variables that must be considered, we (iii) frame the decision problem as an optimisation problem which can be addressed using classical mathematical tools, and finally we (iv) obtain a solution (often in closed-form) which we study through simulations and discussions.

Next, we describe the mechanisms of LOBs and OTC markets. LOBs allow traders to submit different types of orders that indicate the price, the volume, and the intention to buy or sell a security, and they rely on financial institutions and trusted third-parties that facilitate trading by collecting orders and matching buyers and sellers. OTC markets are based on a network of financial institutions and market makers that constantly stream quotes at which they are ready to buy and sell securities.

1.1 Limit order books

Limit order books concentrate most of the trading activity in the stock market. Here, we recall the main mechanisms behind LOBs which motivate many of our modelling assumptions in the following sections. The LOB allows to match buyers with sellers. Both can send two types of orders; *limit orders* (LOs) and *market orders* (MOs). Here, we focus on price-time-priority LOBs, which are the most popular.

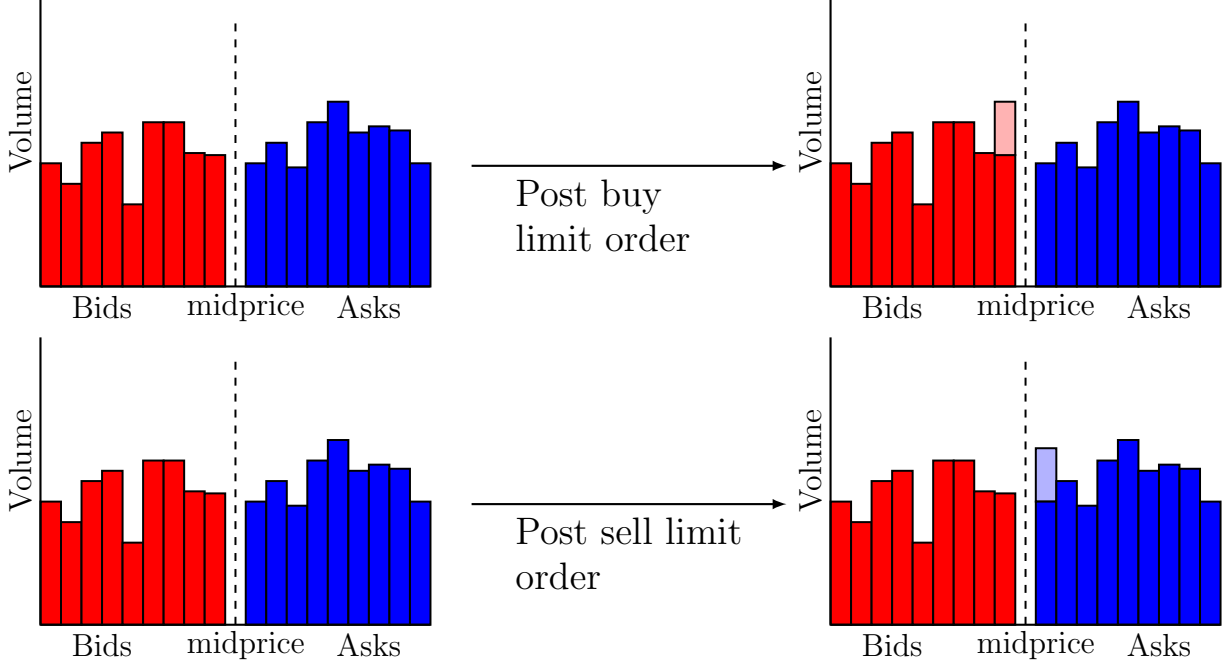


Figure 1: Changes in the LOB after a buy / sell LO.

An LO is composed of a price level, a volume, and an indicator to buy or to sell the asset. Once an LO is *posted* by a trader, it *rests* (or *sits*) in the LOB until it is matched against an MO of the opposite side, or it is cancelled by the agent that posted it. If the LO is a buy LO, then it sits on the *bid* side of the LOB, and if it is a sell LO, then it sits on the *ask* side; see Figure 1 which represents an LOB before and after a buy or sell LO is posted. Limit orders are called *passive* orders because they do not consume liquidity immediately. The cancellation or filling of an LO that is resting in the LOB can be either full (for the entire posted quantity) or partial; see Figure 2.

Traders can post multiple LOs at different price levels (see Figure 3). When two LOs are posted on the same side of the book and at the same price level, then the order of execution in an LOB with price-time priority depends on their position in the queue. Time priority in LOBs means that the queues are First-In-First-Out (FIFO) queues. More precisely, when two LOs are sitting on the same side of the LOB at the same price level, then the LO that was posted first is executed before. Price priority in LOBs means that if two buy (sell) LOs are sitting on the bid (ask) side of the LOB at different price levels, then the LO with the highest (lowest) price is executed first.

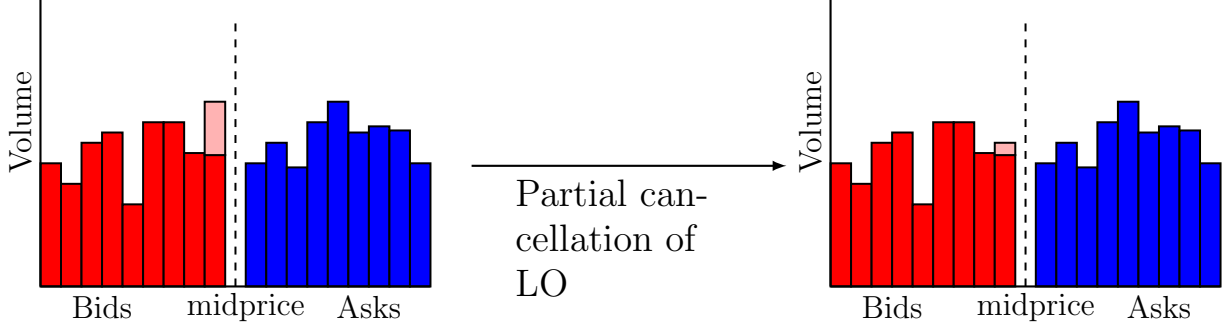


Figure 2: Changes in the LOB after a cancellation.

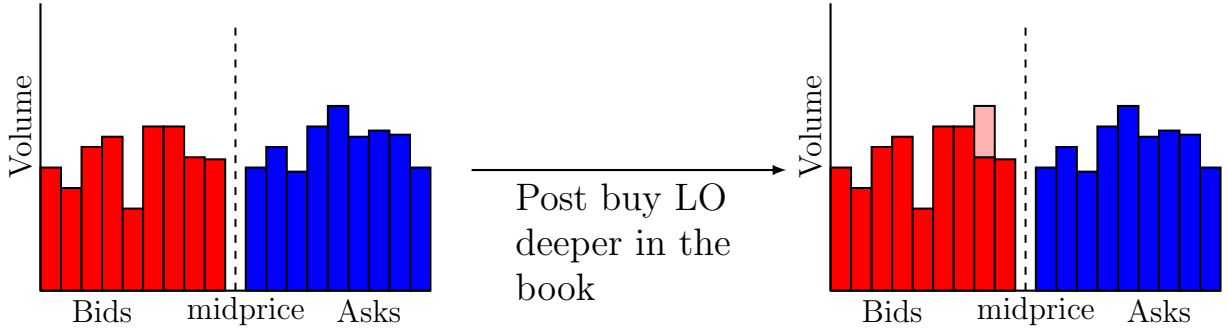


Figure 3: Changes in the LOB after a buy LO at the second best level.

The state of the LOB at a specific time is described by the size of the LO queues and their price levels. There are two price levels which correspond to the highest price across buy LOs and the lowest price across sell LOs. These are referred to as the *best bid* and the *best ask*, respectively. The average of the best bid and best ask is called the *midprice*; See Figure 1.

An MO consists of a volume and an indicator to buy and sell the security. When an MO is submitted by a trader, it is matched against opposite LOs that are resting in the LOB; see Figure 4 which shows the state of the LOB before and after a buy MO has been submitted.

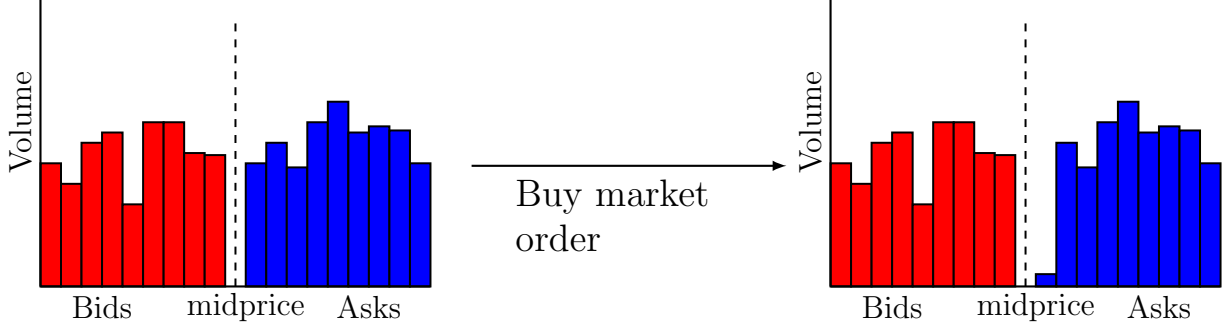


Figure 4: Changes in the LOB after a buy MO.

1.2 Market impact

The effect that MOs have on prices can be described in several ways. The literature on market impact generally splits the impact of MOs on prices into a **temporary** and a **permanent** impact. However, these two components of market impact are not completely dissociated, and more realistic interpretations can be studied and modelled; see Section 10 for an example.

Temporary market impact – Execution costs. When an MO is submitted, it is matched against resting LOs on the opposite side of the book. If the size of the MO is larger than the size of the queue at the best available price, then the first queue is depleted, and the MO is matched with LOs resting on the second best available price. This continues until the entire MO is filled. Figure 5 depicts how large MOs *walk the book* further when compared to smaller MOs.

The volume of an MO and the volumes and prices of resting LOs determine the overall transaction price of the MO. We refer to the difference between the midprice¹ and the execution price (per share) obtained by the agent as the *execution costs*, or the *instantaneous market impact*, or the *temporary market impact*.

¹One may choose another reference price like the best ask or best bid.

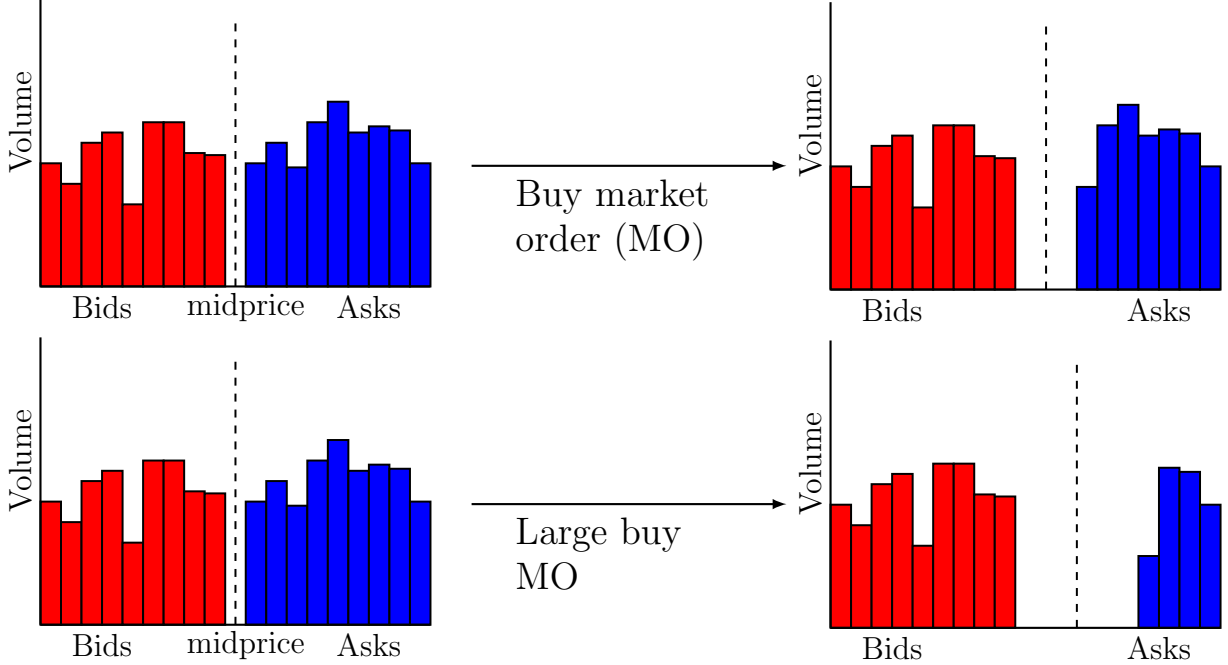


Figure 5: Impact of small and large MOs on the limit order book.

Provided with a snapshot of the LOB, i.e., the state of the LOB at some given time, traders can compute the execution price per share for various volumes of MOs as they walk the book. Thus, they can obtain an estimate of the execution costs as a function of the volume of an MO. Figure 6 shows the execution costs per share as a function of the volume at different times throughout a trading day for the share MSFT quoted on NYSE.

First, note that the minimum execution cost for small volumes is \$0.005 which corresponds to half the *tick size*. The tick size corresponds to the smallest price increment between two successive price levels in the LOB. Thus, the smallest *bid-ask spread*, i.e., the difference between the best bid and the best ask, is equal to 1 tick, which corresponds to \$0.01 for MSFT. In many cases throughout the trading day we consider in Figure 6, the bid-ask spread is larger than 1 tick. Thus, even for MOs with small sizes that do not deplete the best opposite queue, the trader must *cross the spread* and incur execution costs that correspond to half the bid-ask spread.

Also, note that the difference between the average execution price is approximately a linear function of the volume. Denote the midprice by S and the volume of a an MO by v . We can approximate the execution price per

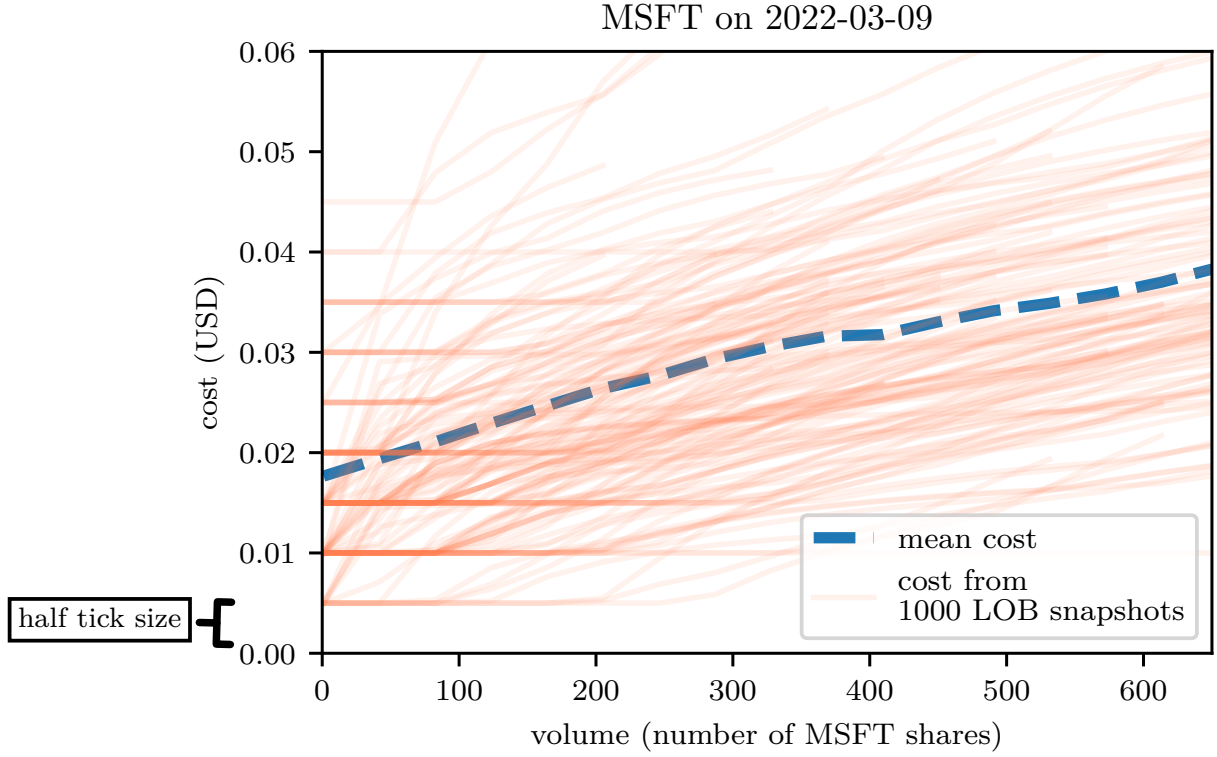


Figure 6: Execution costs defined as a function of trading volume for multiple snapshots of the LOB of MSFT quoted on Nasdaq. The execution costs are defined as the difference between the execution price per share for a given volume of share, and the midprice.

share with the linear form $S + \eta v$. We will make this modelling assumption in many of the models presented in the following sections. Part of the literature prefers to approximate execution costs with a concave power law, and we study this assumption in Section 6.

Permanent market impact. Permanent price impact refers to the relationship between the volume of an MO and the midprice at future times after the execution. Large MOs leave a lasting and long-term effect on the midprice. Large buy MOs move the price upward and large sell MOs move the price downward; see Figure 7. An interpretation of this modelling assumption is that market participants are trading based on information on the fundamental value of the firm, so their trading activity should reflect permanent change in the value of the share. The seminal work in Gatheral (2010) shows with simple arguments that a model with permanent impact that is not linear

in the size of the MO leads to dynamic arbitrage, i.e., leads to strategies with zero risk and positive profits, which is an undesirable feature in an optimal trading model because these strategies do not exist in practice; see Section 4.4 for more details.

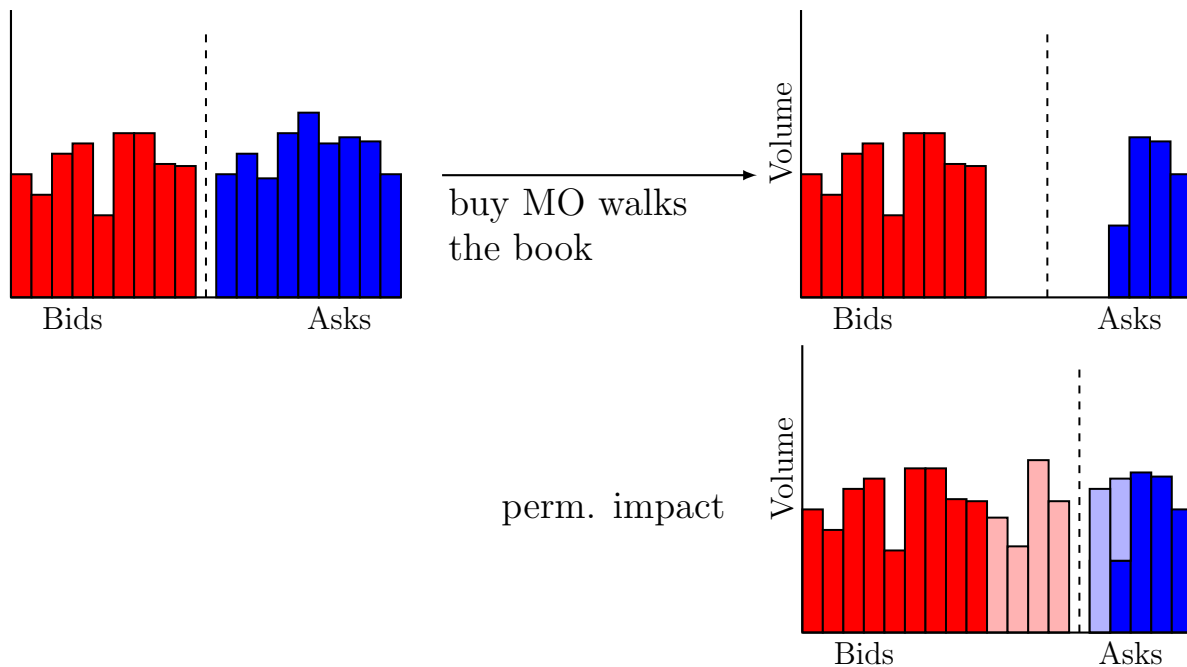


Figure 7: In the first two panels, an MO walks the book so the next midprice exhibits the temporary price impact. Immediately after the MO, market participants replenish the LOB. The difference between the midprice in the last panel and that of the first panel is the permanent impact.

1.3 Over-the-counter markets

There are two types of market participants in any trading venue; liquidity takers and liquidity providers. Liquidity takers are traders that need to buy or sell an asset and look for a counterparty to match their trade. Liquidity providers or market makers are counterparty to both buyers and sellers and profit from roundtrip trades (in LOBs, they earn the spread).

Most electronic exchanges clear the demand and supply of liquidity in LOBs, which are essentially trading venues for stocks. Alternatively, OTC markets are off-exchange “quote-driven” (or dealer-driven) markets that are based on a network of market makers that set prices at which liquidity takers

can trade. These markets are the main exchanges for FX and Corporate Bonds securities.

OTC market makers faces a complex problem. They provide bid and ask quotes for various assets that exhibit complex joint dynamics without seeing the full depth of price and clients. Consequently, it is key to properly account for risks at the portfolio level. However, a large proportion of multi-asset market making models in the literature only consider correlated Brownian dynamics. Additionally, multi-asset market making is challenging due to high dimensionality and the resulting numerical challenges to obtain the optimal quotes.

1.4 Optimal execution

Market operators with large orders regularly come to the market for different reasons, e.g., financial firms that shifted from individual trading to centralised execution in dealing desks, banks that manage their liquidity risk through central risk books (CRB) at the firm level, agency brokers who act on behalf of pension funds, hedge funds, mutual funds, or sovereign funds, etc.

When the orders to be executed represent a significant portion of the overall traded volume, market operators cannot place all the desired quantity in a single order because either the available liquidity is insufficient, or the trade would adversely impact the price and be excessively costly (e.g., by walking the book). The market operator (agent) must slice the parent order into smaller (child) orders which are executed over a given period of time.

Slow execution of child orders exposes the agent to adverse price fluctuations, i.e., the price might increase when the agent needs to buy or the price might decrease when agent needs to sell. However, fast execution of child orders exposes the agent to higher execution costs due to crossing the spread and walking the book. It also impacts the price adversely due to the agent's own trading activity (permanent impact). A balance must be struck between trading fast to minimise market risk and trading slowly to minimise trading costs. This classical problem gave rise to an extensive literature which focuses on the problem of optimal execution and scheduling of large orders.

The agent must formulate a model to decide how to optimally execute a large order. The models presented in this course, much like those in the literature, are solved for different modelling assumptions on the agent's aversion

to risk, their aversion to holding inventory, the execution costs incurred when sending orders, the permanent impact of their trading activity, the type of orders they use, the dynamics of the midprice, etc.

1.5 Mathematical tools

This section provides a very brief overview of the mathematical tools needed in several of the algorithmic trading problems studied in the following sections.

1.5.1 Convex analysis

Sections 3 and 4 rely on results of convex analysis and convex duality. In this section, we recall the main results that are used. The proofs and further analysis can be found in the classical monographs on convex analysis; see e.g., [Rockafellar \(1997\)](#).

First, we introduce the central tool of Legendre-Fenchel transform in convex analysis.

Definition 1. *The Legendre-Fenchel transform of a convex function f is the function f^* defined by*

$$f^* : p \in \mathbb{R}^d \mapsto \sup_{x \in \mathbb{R}^d} p \cdot x - f(x).$$

The Legendre-Fenchel transform f^* is a mapping from the graph of a function to the set of its tangents. f^* is well defined when f is asymptotically super-linear². It is also a convex function, and the Legendre-Fenchel of f^* is f . Finally, when f is strictly convex, then f^* is continuously differentiable.

We focus on a specific problem that arises when addressing optimal trading problems with convex trading costs, called Bolza problems; see [Guéant \(2016\)](#) for numerous optimal trading problems solved using this framework. A particular type of Bolza problems focuses on finding the minimisers of the

²A convex function $f : \mathbb{R}^d \mapsto \mathbb{R}$ is asymptotically super-linear when

$$\lim_{|x| \rightarrow \infty} \frac{f(x)}{|x|} = \infty.$$

function J :

$$J : x \in W^{1,1}((0, T), \mathbb{R}^d) \mapsto \int_0^T (f(x(t)) + g(x'(t))) dt, \quad (1)$$

over the set

$$\mathcal{C} = \{x \in W^{1,1}((0, T), \mathbb{R}^d), x(0) = a, x(T) = b\},$$

where $W^{1,1}(U)$ is the set of real-valued absolutely continuous functions on the open set U , and $d \geq 1$ is an integer.

We assume that $f(\cdot)$ and $g(\cdot)$ are classical convex functions that are measurable and bounded from below. The problem (1) is called the primal problem and we introduce the dual optimisation problem

$$I : p \in W^{1,1}((0, T), \mathbb{R}^d) \mapsto \int_0^T (f_t^*(p'(t)) + g_t^*(p(t))) dt + a \cdot p(0) - b \cdot p(T),$$

over the set of absolutely continuous functions.

Theorem 1. *If f and g are continuous convex functions and they are asymptotically super-linear, then there exists a minimiser x^* of J and a minimiser p^* of I .*

If f is differentiable, and g is strictly convex, then the minimiser can be characterised by the following Hamiltonian system of equations:

$$\begin{cases} p^{*'}(t) &= f'(x^*(t)), \\ x^{*'}(t) &= g^{*'}(p^*(t)), \\ x^*(0) &= a, \\ x^*(T) &= b. \end{cases}$$

Discrete time. In discrete time, we consider the functional $J : \mathbb{R}^{dN+d} \mapsto \mathbb{R}$ given by

$$J(x_0, \dots, x_N) = \sum_{n=1}^{N-1} f(x_n) + \sum_{n=0}^{N-1} g(x_{n+1} - x_n),$$

where f and g are convex functions. Our goal is to find the minimisers of J over

$$\mathcal{C} = \{(x_0, \dots, x_N) \in \mathbb{R}^{d(N+1)}, x_0 = a, x_N = b\},$$

where a and b are fixed. The dual problem is

$$I(p_0, \dots, p_{N-1}) = \sum_{n=1}^{N-1} f_n^*(p_n - p_{n-1}) + \sum_{n=0}^{N-1} g_n^*(p_n) + a \cdot p_0 - b \cdot p_{N-1}.$$

The discrete-time counterpart of Theorem 1 is:

Theorem 2. *If $x^* = (x_0^*, \dots, x_N^*)$ minimises J and $p^* = (p_0^*, \dots, p_{N-1}^*)$ minimises I , then they can be characterised by the following Hamiltonian system of equations:*

$$\begin{cases} p_n^* - p_{n-1}^* = f'(x_n^*) & \forall n \in \{1, \dots, N-1\}, \\ x_{n+1}^* - x_n^* = g^{*'}(p_n^*) & \forall n \in \{0, \dots, N-1\}. \end{cases}$$

1.5.2 Stochastic optimal control

This section recalls the dynamic programming principle for jump-diffusion processes; see the monograph [Pham \(2009\)](#) for more details on stochastic optimal control and the book [Cartea et al. \(2015\)](#) for various optimal trading models solved using the tools of stochastic optimal control.

In dynamic optimisation problems, an agent seeks to maximise a reward over some time window $[0, T]$. The agent takes actions \mathbf{u} that affect the dynamics of some underlying system $\mathbf{X}^{\mathbf{u}}$; the superscript indicates that \mathbf{X} is affected (controlled) by \mathbf{u} . Often, the actions of the agent incur costs (or rewards) over the time window, which can also depend on the time and the state of the system, so they must be accounted for. At each time t , the cumulative past actions of the agent affect the future dynamics of the system and future potential costs.

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]})$ satisfying the usual conditions and which supports all the processes that we introduce, where $T > 0$ is a fixed time horizon.

Consider an agent who is faced with a control problem of a system whose dynamics contain a diffusive and a jump component. Let $(\mathbf{N}_t^{\mathbf{u}})_{t \in [0, T]}$ denote a p -dimensional counting process with controlled intensity $(\lambda_t^{\mathbf{u}})_{t \in [0, T]}$ where $\lambda_t^{\mathbf{u}} = \lambda(t, \mathbf{X}_t^{\mathbf{u}}, \mathbf{u}_t)$ and $(\mathbf{u}_t)_{t \in [0, T]}$ is an m -dimensional control process.³ Also, let $(\mathbf{W}_t)_{t \in [0, T]}$ denote an m -dimensional standard Brownian motion.

³ $\mathbf{N}^{\mathbf{u}}$ is called a doubly stochastic Poisson process because its intensity is itself stochastic, and in our case controlled. Recall that $\mathbf{N}_t^{\mathbf{u}} - \int_0^t \lambda_s^{\mathbf{u}} ds$ is a martingale.

Let $(\mathbf{X}_t^{\mathbf{u}})_{t \in [0, T]}$ denote a controlled m -dimensional system with dynamics

$$d\mathbf{X}_t^{\mathbf{u}} = \mu(t, \mathbf{X}_t^{\mathbf{u}}, \mathbf{u}_t) dt + \mathbf{V}(t, \mathbf{X}_t^{\mathbf{u}}, \mathbf{u}_t) d\mathbf{W}_t + \gamma(t, \mathbf{X}_t^{\mathbf{u}}, \mathbf{u}_t) d\mathbf{N}_t^{\mathbf{u}}, \quad \mathbf{X}_0^{\mathbf{u}} = X_0, \quad (2)$$

where the m -dimensional vector of drifts μ , the $m \times m$ variance matrix \mathbf{V} , and the $m \times p$ -dimensional jump matrix γ are Lipschitz continuous and X_0 is known.

The agent has a *performance criterion* they wish to maximise which takes the form

$$\mathbb{E} \left[G(\mathbf{X}_T^{\mathbf{u}}) + \int_0^T F(s, \mathbf{X}_s^{\mathbf{u}}, \mathbf{u}_s) ds \right]. \quad (3)$$

where $G : \mathbb{R}^m \mapsto \mathbb{R}$ is the *terminal reward* function and $F : [0, T] \times \mathbb{R}^m \times \mathbb{R}^p \mapsto \mathbb{R}$ is the *running reward* function. The functions G and F are assumed to be uniformly bounded.

The agent seeks to maximise the performance criterion (3), so we define their *value function* as:

$$H(\mathbf{X}_0) = \sup_{\mathbf{u} \in \mathcal{A}} \mathbb{E} \left[G(\mathbf{X}_T^{\mathbf{u}}) + \int_0^T F(s, \mathbf{X}_s^{\mathbf{u}}, \mathbf{u}_s) ds \right], \quad (4)$$

where \mathcal{A} is the *admissible set* of strategies that the agent may use:

$$\mathcal{A} = \left\{ (\mathbf{u}_s)_{s \in [0, T]}, \mathbb{R}^m\text{-valued, } \mathbb{F}\text{-predictable, such that} \right. \quad (5)$$

$$\left. (\mathbf{X}_s^{\mathbf{u}})_{s \in [0, T]} \text{ admits a strong solution} \right\}.$$

The predictability assumption in (5) is necessary in financial decision problems because it ensures that the agent can only use strategies that do not use future information. Often in the following sections, we require the admissible set to include additional constraints to ensure that the problem is well defined and that a solution exists.

Instead of optimising H in (4), it is more convenient to consider a time-indexed succession of optimisation problems that explicitly take into account the feedback effect between the actions of the agent and their impact on the future dynamics and costs. More precisely, we embed the problem (4) into a

class of problems indexed by time t :

$$H(t, \mathbf{x}) = \sup_{\mathbf{u} \in \mathcal{A}_t} \mathbb{E} \left[G(\mathbf{X}_T^{\mathbf{x}, \mathbf{u}}) + \int_t^T F(s, \mathbf{X}_s^{\mathbf{x}, \mathbf{u}}, \mathbf{u}_s) ds \right], \quad (6)$$

where the process $(\mathbf{X}_s^{\mathbf{x}, \mathbf{u}})_{s \in [t, T]}$ follows the same dynamics as in (2) but starts with initial value $\mathbf{X}_t^{\mathbf{x}, \mathbf{u}} = \mathbf{x}$, and where we define for all $t \in [0, T]$:

$$\mathcal{A}_t = \left\{ (\mathbf{u}_s)_{s \in [t, T]}, \mathbb{R}^m\text{-valued, } \mathbb{F}\text{-predictable, such that } (\mathbf{X}_s^{\mathbf{u}})_{s \in [t, T]} \text{ admits a strong solution} \right\},$$

and we set $\mathcal{A} = \mathcal{A}_0$, so $H(0, \mathbf{x}) = H(\mathbf{x})$.

Often in this course, we will drop the notation $\mathbf{X}^{\mathbf{x}, \mathbf{u}}$, and write (6) as

$$H(t, \mathbf{x}) = \sup_{\mathbf{u} \in \mathcal{A}_t} \mathbb{E}_{t, \mathbf{x}} \left[G(\mathbf{X}_T^{\mathbf{u}}) + \int_t^T F(s, \mathbf{X}_s^{\mathbf{u}}, \mathbf{u}_s) ds \right],$$

where $\mathbb{E}_{t, \mathbf{x}}$ represents the expectation conditional on $\mathbf{X}_t^{\mathbf{u}} = \mathbf{x}$.

The following classical result (see [Pham \(2009\)](#)) shows that the value function satisfies the Dynamic Programming Principle (DPP).

Theorem 1. *The value function (6) satisfies the DPP*

$$H(t, \mathbf{x}) = \sup_{\mathbf{u} \in \mathcal{A}} \mathbb{E} \left[H(T, \mathbf{X}_T^{\mathbf{x}, \mathbf{u}}) + \int_t^T F(s, \mathbf{X}_s^{\mathbf{x}, \mathbf{u}}, \mathbf{u}_s) ds \right],$$

for all $t \in [0, T]$ and $\mathbf{x} \in \mathbb{R}^m$.

The DPP connects the value function to its future expected value, regularised by the expected reward or penalty F . In its infinitesimal version, the DPP gives the Dynamic Programming Equation (DPE), or the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} \partial_t H(t, \mathbf{x}) + \sup_{\mathbf{u} \in \mathcal{A}} (\mathcal{L}_t^{\mathbf{u}} H(t, \mathbf{x}) + F(t, \mathbf{x}, \mathbf{u})) &= 0 \\ \text{subject to the terminal condition } H(T, \mathbf{x}) &= G(\mathbf{x}), \end{aligned}$$

where $\mathcal{L}_t^{\mathbf{u}}$ is the infinitesimal generator of the process $\mathbf{X}_t^{\mathbf{x}, \mathbf{u}}$.

For the diffusion-jump process in (2), the infinitesimal generator acts on functions H as follows:

$$\boxed{\mathcal{L}_t^{\mathbf{u}} H(t, \mathbf{x}) = \boldsymbol{\mu}(t, \mathbf{x}, \mathbf{u}) \cdot \nabla_{\mathbf{x}} H(t, \mathbf{x}) + \frac{1}{2} \text{Tr} \left(\boldsymbol{\Sigma}(t, \mathbf{x}, \mathbf{u}) D_{xx}^2 H(t, \mathbf{x}) \right) + \sum_{j=1}^p \lambda_j(t, \mathbf{x}, \mathbf{u}) \left[H(t, \mathbf{x} + \boldsymbol{\gamma}_{:,j}(t, \mathbf{x}, \mathbf{u})) - H(t, \mathbf{x}) \right],} \quad (7)$$

where $\boldsymbol{\Sigma}(t, \mathbf{x}, \mathbf{u}) = \mathbf{V}(t, \mathbf{x}, \mathbf{u}) \mathbf{V}(t, \mathbf{x}, \mathbf{u})^\top$ is the covariance matrix, $D_{xx}^2 H(t, \mathbf{x})$ is the Hessian matrix of H , and $\boldsymbol{\gamma}(t, \mathbf{x}, \mathbf{u})_{:,j}$ is the j th column of the $m \times p$ matrix $\boldsymbol{\gamma}(t, \mathbf{x}, \mathbf{u})$.

The first term in (7) represents the change in the value function due to the drift of \mathbf{X} , the second term represents the diffusion volatility, and the third term corresponds to the arrival of a jump in each component of \mathbf{N} . When a jump of the j th component of \mathbf{N} occurs, the components of the system \mathbf{X} only jump according to the j th column of $\boldsymbol{\gamma}(t, \mathbf{x}, \mathbf{u})$.

Solving for the supremum term in the HJB provides the optimal control in *feedback form*, i.e., as a function of the value function. One usually solves the HJB equation, which is a nonlinear PDE. The solution to the PDE is only a *candidate solution* and one needs to prove that it is in fact the solution to the original control problem through a verification argument; see [Pham \(2009\)](#). When a classical solution to the HJB equation exists, i.e., it is \mathcal{C}^1 in time and \mathcal{C}^2 in x , and if the control is admissible, i.e., it is in \mathcal{A} , then by standard results, the solution to the HJB is indeed the value function we seek and the resulting control is an optimal Markov control. Finally, the results and discussions above hold for any terminal stopping time $\tau \leq T$.

2 Optimal routing

We start this course with a simple problem that illustrates the methodology to adopt when addressing an optimal trading problem. More precisely, we (i) identify a decision problem, we (ii) propose a parsimonious model of the environment through variables that describe the key quantities to consider, we (iii) frame the decision problem as an optimisation problem that can be solved with classical mathematical tools, and finally we (iv) obtain a solution that we can study and implement in practice.

2.1 Optimal routing of aggressive orders

Identifying the decision problem. Liquidity in trading venues is limited. Often, operators need to split a large order over N available trading venues to obtain better execution prices and to reduce the costs of their trading activity.

Here, we study the problem of optimal routing of “*marketable*” orders in multiple trading venues. A marketable order is a buy (resp. sell) order at a price higher than the best ask (resp. lower than the best bid). A buy (sell) marketable order is defined by two variables Q^* and P^* . Q^* is the quantity of the order and P^* is the maximum (minimum) price that the trader is willing to accept.

Modelling framework. Let (Q^*, P^*) be a marketable buy order; the analysis for sell orders is identical. Let $Q_n(p)$ be the visible quantity that is available at price p in the n -th trading venue when $n \in \{1, \dots, N\}$.

Optimisation problem. The agent splits the parent order (Q^*, P^*) into N child orders $(Q_n(p_n), p_n)$ to send to the N trading venues. The agent wishes to choose (p_1, \dots, p_n) to minimise the overall execution price

$$\sum_{n=1}^N p_n \cdot Q_n(p_n),$$

and ensure that the parent order is fully executed, i.e., we require that

$$Q^* = \sum_{n=1}^N Q_n(p_n).$$

Recall that P^* is the maximum price that the agent is willing to pay, so we need the additional constraint that $P^* \geq p_n$ for all $n \in \{1, \dots, N\}$.

Solution. The above problem is a classical constrained optimisation problem. We look for the stationary points of the Lagrangian function and we write

$$Q_n(p_n) + p_n Q'_n(p_n) = \lambda Q'_n(p_n), \quad \text{for } n \in \{1, \dots, N\},$$

where λ is the Lagrange multiplier.

Next, we assume the linear form

$$Q_n(p) = q_n + c_n \cdot p.$$

Thus we obtain

$$(q_n + c_n \cdot p_n^*) + p_n^* c_n = \lambda c_n \implies p_n^* = \frac{\lambda}{2} - \frac{q_n}{2 c_n}.$$

Substitute λ in the constraint $Q^* = \sum_{n=1}^N Q_n(p_n)$ to obtain

$$Q^* = \sum_{n=1}^N \{q_n + c_n \cdot p_n^*\} = \sum_{n=1}^N q_n/2 + c \lambda/2, \quad c = \sum_n c_n.$$

Finally, the optimal prices to target in each trading venue are given by

$$\boxed{p_n^* = \frac{Q^*}{c} - \frac{q_n}{2 c_n} \left(1 + \frac{c_n}{q_n} \cdot \frac{\bar{q}}{\bar{c}}\right), \quad \bar{c} = \frac{1}{N} \sum_n c, \quad \bar{q} = \frac{1}{N} \sum_n q}$$

The problem above is simple, yet it contains all the ingredients of optimal trading. In practice, agents take into account more practical issues of financial environments such as latency in high frequency markets, cancellations in limit order books (randomness of $Q_n(p)$), time periods throughout which they spread their trading activity, hidden and Iceberg orders, etc. Each issue can be modelled with additional variables and assumptions and complicates the model and the solution. For example, the next section consider the optimal routing of a large limit order instead of a marketable order.

2.2 Optimal routing of limit orders

Identifying the decision problem. Crossing the spread is the first source of execution costs. Thus, agents mostly trade with limit orders. However, there is structural uncertainty when using limit orders because of price and time priority; waiting on a bad queue generates **opportunity costs**. Here, we consider the problem of an agent that can trade in N different LOBs and wishes to split a large buy limit order.

Modelling framework. Let Q_n denote the size of the best bid queue in the n -th LOB. If the agent posts a buy limit order at the best bid price, then they must wait for the Q_n previous LOs to be filled before their LO is filled. The agent assumes that the best bid queue is consumed according to a Poisson P_t^n with intensity λ_n . This intensity can be estimated using historical data.

Optimisation problem. The objective of the agent is to split the parent order into N child LOs with quantities (q_1, \dots, q_N) . The agent wishes to minimize, *on average*, the time t^* that they need to execute the quantity $Q^* = \sum_n q_n$.

Solution. In each trading venue $n \in \{1, \dots, N\}$, after the LO of size q_n is posted, the size of the best bid queue is $Q_n + q_n$. We denote by t_n the time needed for the queue to be totally consumed. We write

$$\int_0^{t_n} dP_t^n = q_n + Q_n \implies \mathbb{E}[P_{t_n}^n] = t_n \lambda_n = q_n + Q_n.$$

The agent seeks to minimise the maximum of $\{t^1, \dots, t^N\}$, so we necessarily have that $t^* = t^n$ for all $n \in \{1, \dots, N\}$.

In particular, we write

$$t^* = t_n = Q^* / \sum_n \lambda_n + \sum_n Q_n / \sum_n \lambda_n \implies q_n^* = \rho_n \frac{Q^*}{N} + (\rho_n \bar{Q} - Q),$$

where

$$\rho_n = \lambda_n / \bar{\lambda}, \quad \bar{\lambda} = \frac{1}{N} \sum_n \lambda_n, \quad \text{and} \quad \bar{Q} = \frac{1}{N} \sum_n Q_n.$$

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