Market Microstructure and Algorithmic Trading

Mathematical Institute
Oxford-Man Institute

University of Oxford

Lecture notes – HT 2024

Fayçal Drissi

If you have questions or issues related to the course, please contact me at faycal.drissi@gmail.com.

Contents

1 Introduction **3**
- 1.1 Limit order books 3
- 1.2 Market impact 6
- 1.3 Over-the-counter markets 9
- 1.4 Optimal execution 10
- 1.5 Mathematical tools 11
  - 1.5.1 Convex analysis 11
  - 1.5.2 Stochastic optimal control 13

2 Optimal routing **17**
- 2.1 Optimal routing of agressive orders 17
- 2.2 Smart routing of limit orders 19

3 The Almgren-Chriss model in discrete time **20**
- 3.1 Modelling framework 21
- 3.2 Performance criterion 23
- 3.3 Solution 23
- 3.4 Discussion 26

1 Introduction

Recent regulatory changes due to Reg NMS and MiFID along with the automation of algorithmic trading shifted the focus of researchers and practitioners towards liquidity and market microstructure problems. New strands of the mathematical finance literature now focus on new topics such as order execution, market making, systemic risk, and counterparty risk from a microstructure perspective.

Electronic markets are mainly organized around Limit Order Books (LOBs) and Over-The-Counter (OTC) markets to exchange securities between market participants. This course introduces the literature on optimal trading in high frequency markets with a focus on optimal order execution in LOBs and optimal market making in OTC markets. Optimal trading problems share a common structure that will be followed throughout this course. In each section, we (i) identify a decision problem motivated by practical situations faced by market operators, we (ii) propose a parsimonious model which summarises the environment in key variables that must be considered, we (iii) frame the decision problem as an optimisation problem which can be addressed using classical mathematical tools, and finally we (iv) obtain a solution (often in closed-form) which we study through simulations and discussions.

Next, we describe the mechanisms of LOBs and OTC markets. LOBs allow traders to submit different types of orders that indicate the price, the volume, and the intention to buy or sell a security, and they rely on financial institutions and trusted third-parties that facilitate trading by collecting orders and matching buyers and sellers. OTC markets are based on a network of financial institutions and market makers that constantly stream quotes at which they are ready to buy and sell securities.

1.1 Limit order books

Limit order books concentrate most of the trading activity in the stock market. Here, we recall the main mechanisms behind LOBs which motivate many of our modelling assumptions in the following sections. The LOB allows to match buyers with sellers. Both can send two types of orders; *limit orders* (LOs) and *market orders* (MOs). Here, we focus on price-time-priority LOBs, which are the most popular.

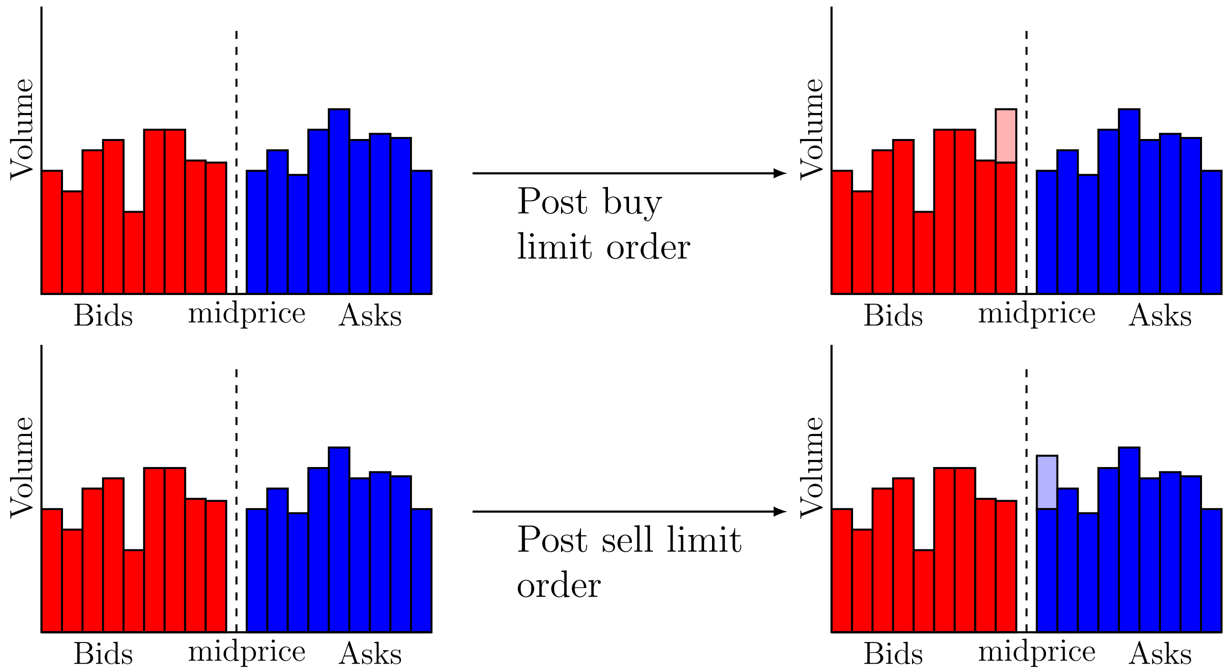

Figure 1: Changes in the LOB after a buy LO.

An LO is composed of a price level, a volume, and an indicator to buy or to sell the asset. Once an LO is *posted* by a trader, it *rests* (or sits) in the LOB until it is matched against an MO of the opposite side, or it is cancelled by the agent that posted it. If the LO is a buy LO, then it sits on the *bid* side of the LOB, and if it is a sell LO, then it sits on the *ask* side; see Figure 1 which represents an LOB before and after a buy or sell LO is posted. Limit orders are called *passive* orders because they do not consume liquidity immediately. The cancellation or filling of an LO that is resting in the LOB can be either full (for the entire posted quantity) or partial; see Figure 2.

Traders can post multiple LOs at different price levels (see Figure 3). When two LOs are posted on the same side of the book and at the same price level, then the order of execution in an LOB with price-time priority depends on their position in the queue. Time priority in LOBs means that the queues are First-In-First-Out (FIFO) queues. More precisely, when two LOs are sitting on the same side of the LOB at the same price level, then the LO that was posted first is executed before. Price priority in LOBs means that if two buy (sell) LOs are sitting on the bid (ask) side of the LOB at different price levels, then the LO with the highest (lowest) price is executed first.

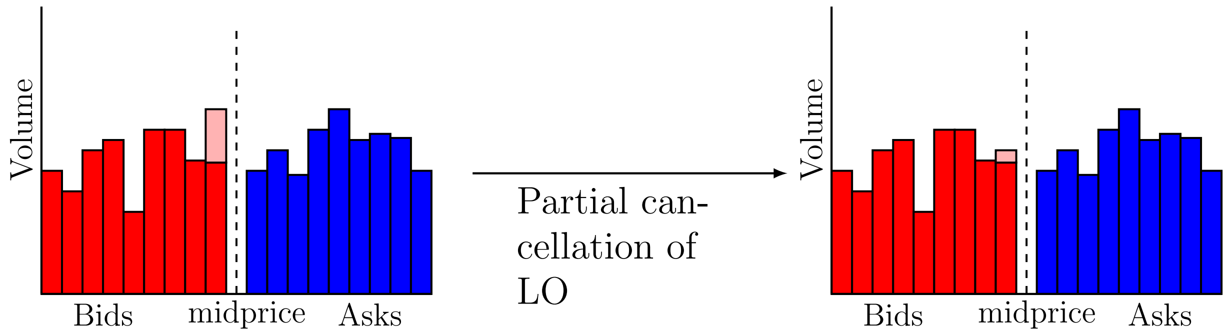

Figure 2: Changes in the LOB after a cancellation.

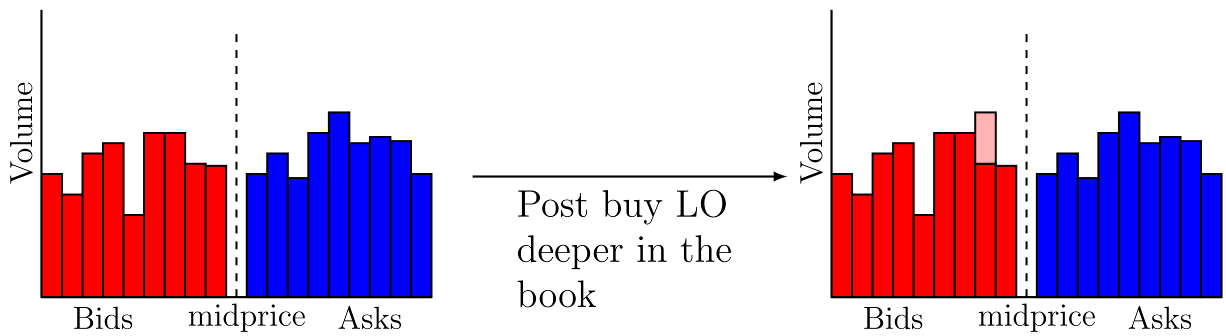

Figure 3: Changes in the LOB after a buy LO at the second best level.

The state of the LOB at a specific time is described by the size of the LO queues and their price levels. There are two price levels which correspond to the highest price across buy LOs and the lowest price across sell LOs. These are referred to as the *best bid* and the *best ask*, respectively. The average of the best bid and best ask is called the *midprice*; See Figure 1.

An MO consists of a volume and an indicator to buy and sell the security. When an MO is submitted by a trader, it is matched against opposite LOs that are resting in the LOB; see Figure 4 which shows the state of the LOB before and after a buy MO has been submitted.

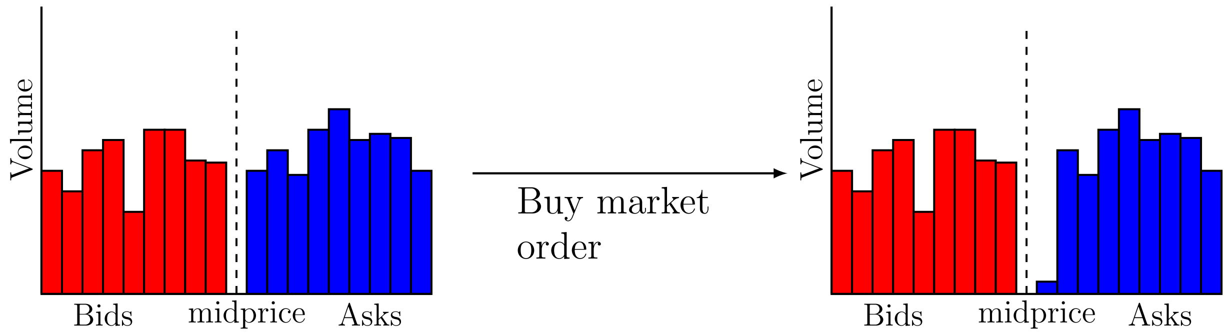

Figure 4: Changes in the LOB after a buy MO.

1.2 Market impact

The effect that MOs have on prices can be described in several ways. The literature on market impact generally splits the impact of MOs on prices into a **temporary** and a **permanent** impact. However, these two components of market impact are not completely dissociated, and more realistic interpretations can be studied and modelled; see Section **??** for an example.

Temporary market impact – Execution costs. When an MO is submitted, it is matched against resting LOs on the opposite side of the book. If the size of the MO is larger than the size of the queue at the best available price, then the first queue is depleted, and the MO is matched with LOs resting on the second best available price. This continues until the entire MO is filled. Figure 5 depicts how large MOs *walk the book* further when compared to smaller MOs.

The volume of an MO and the volumes and prices of resting LOs determine the overall transaction price of the MO. We refer to the difference between the midprice[1] and the execution price (per share) obtained by the agent as the *execution costs*, or the *instantaneous market impact*, or the *temporary market impact*.

[1] One may choose another reference price like the best ask or best bid.

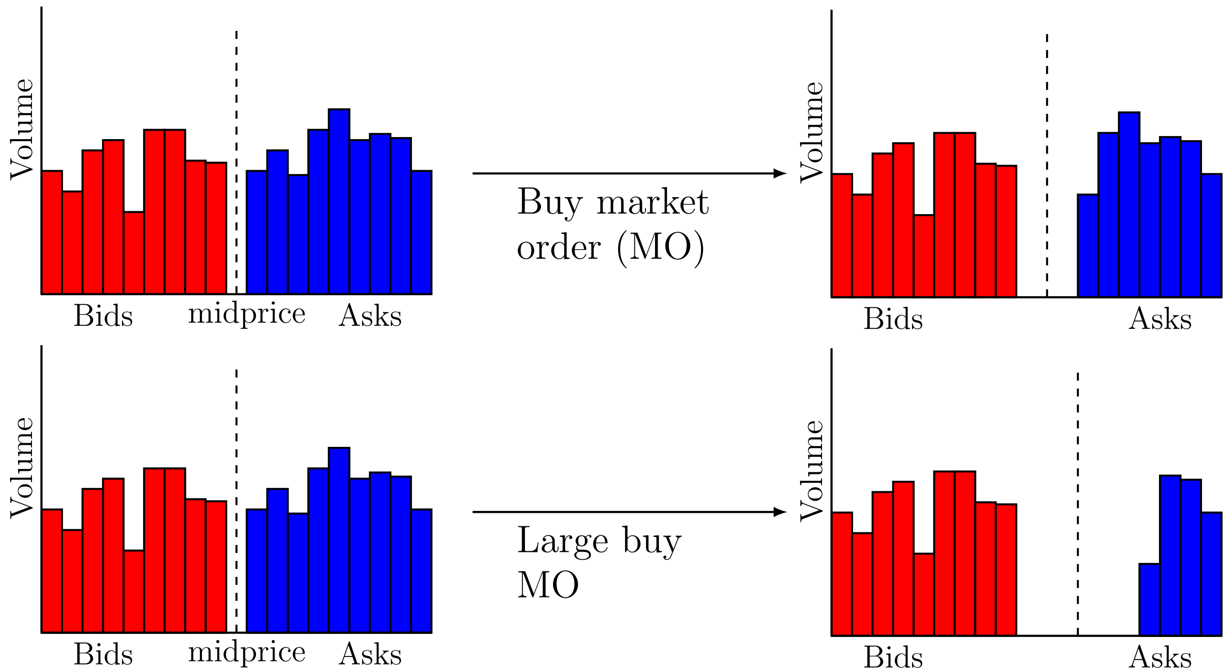

Figure 5: Limit order book.

Provided with a snapshot of the LOB, i.e., the state of the LOB at some given time, traders can compute the execution price per share for various volumes of MOs as they walk the book. Thus, they can obtain an estimate of the execution costs as a function of the volume of an MO. Figure 6 shows the execution costs per share as a function of the volume at different times throughout a trading day for the share MSFT quoted on NYSE.

First, note that the minimum execution cost for small volumes is \$ 0.005 which corresponds to half the *tick size*. The tick size corresponds to the smallest price increment between two successive price levels in the LOB. Thus, the smallest *bid-ask spread*, i.e., the difference between the best bid and the best ask, is equal to 1 tick, which corresponds to \$ 0.01 for MSFT. In many cases throughout the trading day we consider in Figure 6, the bid-ask spread is larger than 1 tick. Thus, even for MOs with small sizes that do not deplete the best opposite queue, the trader must *cross the spread* and incur execution costs that correspond to half the bid-ask spread.

Also, note that the difference between the average execution price is approximately a linear function of the volume. Denote the midprice by S and the volume of a an MO by v. We can approximate the execution price per

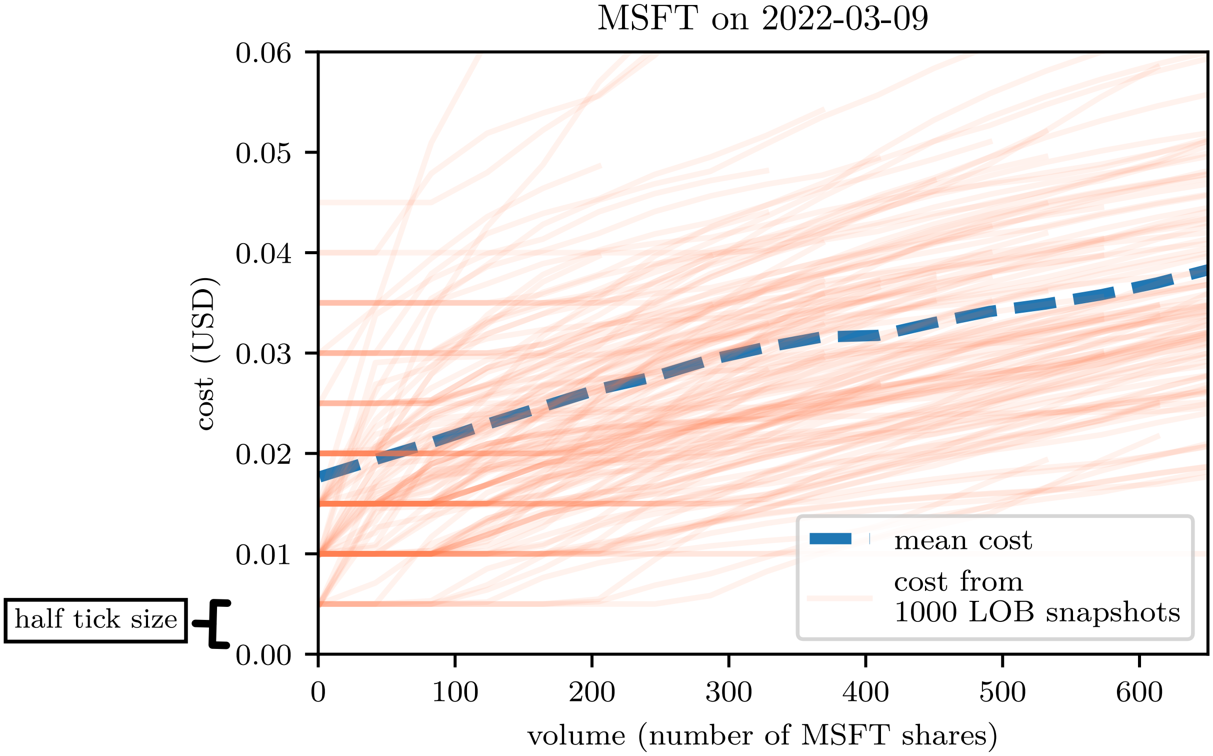

Figure 6: Execution costs defined as a function of trading volume for multiple snapshots of the LOB of MSFT quoted on Nasdaq. The execution costs are defined as the difference between the execution price per share for a given volume of share, and the midprice.

share with the linear form $S + \eta\, v$. We will make this modelling assumption in many of the models presented in the following sections. Part of the literature prefers to approximate execution costs with a concave power law, and we study this assumption in Section **??**.

Permanent market impact. Permanent price impact refers to the relationship between the volume of an MO and the midprice at future times after the execution. Large MOs leave a lasting and long-term effect on the midprice. Large buy MOs move the price upward and large sell MOs move the price downward; see Figure 7. An interpretation of this modelling assumption is that market participants are trading based on information on the fundamental value of the firm, so their trading activity should reflect permanent change in the value of the share. The seminal work in Gatheral (2010) shows with simple arguments that a model with permanent impact that is not linear in the size of the MO leads to dynamic arbitrage, i.e., leads to strategies with zero risk and positive profits, which is an undesirable feature in an optimal trading model because these strategies do not exist in practice; see Section **??** for more details.

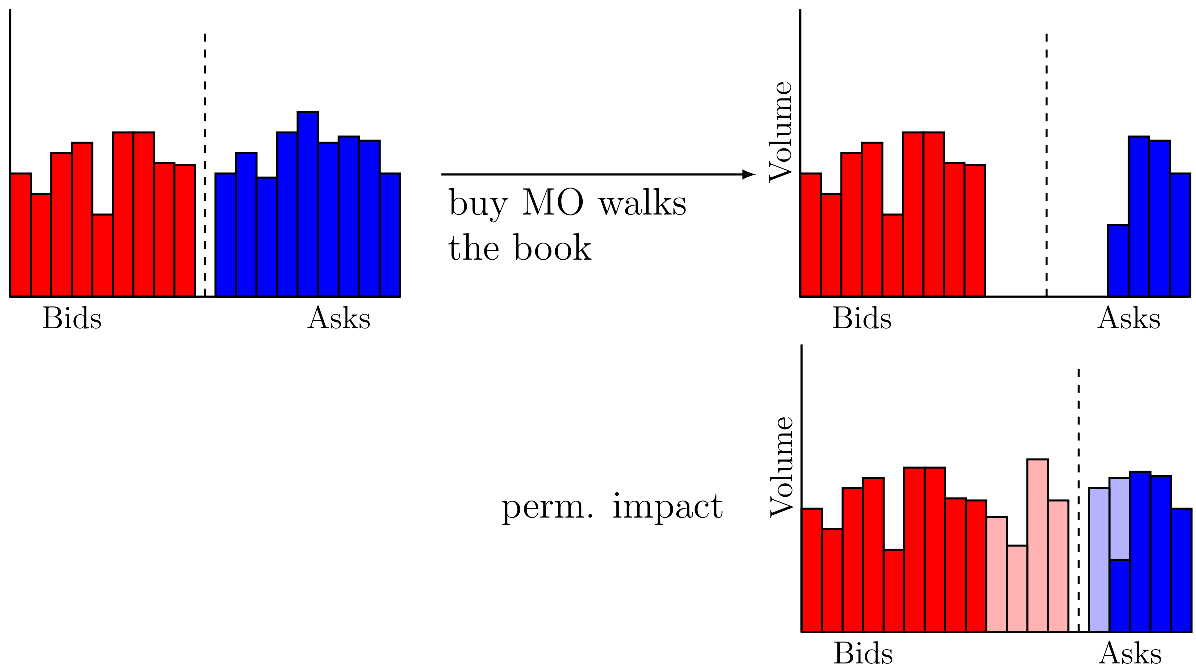

Figure 7: In the first two panels, an MO walks the book so the next midprice exhibits the temporary price impact. Immediately after the MO, market participants replenish the LOB. The difference between the midprice in the last panel and that of the first panel is the permanent impact.

1.3 Over-the-counter markets

There are two types of market participants in any trading venue; liquidity takers and liquidity providers. Liquidity takers are traders that need to buy or sell an asset and look for a counterparty to match their trade. Liquidity providers or market makers are counterparty to both buyers and sellers and profit from roundtrip trades (in LOBs, they earn the spread).

Most electronic exchanges clear the demand and supply of liquidity in LOBs, which are essentially trading venues for stocks. Alternatively, OTC markets are off-exchange "quote-driven" (or dealer-driven) markets that are based on a network of market makers that set prices at which liquidity takers can trade. These markets are the main exchanges for FX and Corporate Bonds securities.

OTC market makers faces a complex problem. They provide bid and ask quotes for various assets that exhibit complex joint dynamics without seeing the full depth of price and clients. Consequently, it is key to properly account for risks at the portfolio level. However, a large proportion of multi-asset market making models in the literature only consider correlated Brownian dynamics. Additionally, multi-asset market making is challenging due to high dimensionality and the resulting numerical challenges to obtain the optimal quotes.

1.4 Optimal execution

Market operators with large orders regularly come to the market for different reasons, e.g., financial firms that shifted from individual trading to centralised execution in dealing desks, banks that manage their liquidity risk through central risk books (CRB) at the firm level, agency brokers who act on behalf of pension funds, hedge funds, mutual funds, or sovereign funds, etc.

When the orders to be executed represent a significant portion of the overall traded volume, market operators cannot place all the desired quantity in a single order because either the available liquidity is insufficient, or the trade would adversely impact the price and be excessively costly (e.g., by walking the book). The market operator (agent) must slice the parent order into smaller (child) orders which are executed over a given period of time.

Slow execution of child orders exposes the agent to adverse price fluctuations, i.e., the price might increase when the agent needs to buy or the price might decrease when agent needs to sell. However, fast execution of child orders exposes the agent to higher execution costs due to crossing the spread and walking the book. It also impacts the price adversely due to the agent's own trading activity (permanent impact). A balance must be struck between trading fast to minimise market risk and trading slowly to minimise trading costs. This classical problem gave rise to an extensive literature which focuses on the problem of optimal execution and scheduling of large orders.

The agent must formulate a model to decide how to optimally execute a large order. The models presented in this course, much like those in the literature, are solved for different modelling assumptions on the agent's aversion to risk, their aversion to holding inventory, the execution costs incurred when sending orders, the permanent impact of their trading activity, the type of orders they use, the dynamics of the midprice, etc.

1.5 Mathematical tools

This section provides a very brief overview of the mathematical tools needed in several of the algorithmic trading problems studied in the following sections.

1.5.1 Convex analysis

Sections 3 and **??** rely on results of convex analysis and convex duality. In this section, we recall the main results that are used. The proofs and further analysis can be found in the classical monographs on convex analysis; see e.g., Rockafellar (1997).

First, we introduce the central tool of Legendre-Fenchel transform in convex analysis.

Definition 1. *The Legendre-Fenchel transform of a convex function f is the function f^* defined by*

$$f^* : p \in \mathbb{R}^d \mapsto \sup_{x \in \mathbb{R}^d} p \cdot x - f(x).$$

The Legendre-Fenchel transform f^* is a mapping from the graph of a function to the set of its tangents. f^* is well defined when f is asymptotically super-linear[2]. It is also a convex function, and the Legendre-Fenchel of f^* is f. Finally, when f is strictly convex, then f^* is continuously differentiable.

We focus on a specific problem that arises when addressing optimal trading problems with convex trading costs, called Bolza problems; see Guéant (2016) for numerous optimal trading problems solved using this framework. A particular type of Bolza problems focuses on finding the minimisers of the

[2]A convex function $f : \mathbb{R}^d \mapsto \mathbb{R}$ is asymptotically super-linear when

$$\lim_{|x| \to \infty} \frac{f(x)}{|x|} = \infty.$$

function J:

$$J : x \in W^{1,1}((0,T), \mathbb{R}^d) \mapsto \int_0^T (f(x(t)) + g(x'(t)))\, dt, \tag{1}$$

over the set

$$\mathcal{C} = \{x \in W^{1,1}((0,T), \mathbb{R}^d), x(0) = a, x(T) = b\},$$

where $W^{1,1}(U)$ is the set of real-valued absolutely continuous functions on the open set U, and $d \geq 1$ is an integer.

We assume that $f(\cdot)$ and $g(\cdot)$ are classical convex functions that are measurable and bounded from below. The problem (1) is called the primal problem and we introduce the dual optimisation problem

$$I : p \in W^{1,1}((0,T), \mathbb{R}^d) \mapsto \int_0^T (f_t^*(p'(t)) + g_t^*(p(t)))\, dt + a \cdot p(0) - b \cdot p(T),$$

over the set of absolutely continuous functions.

Theorem 1. *If f and g are continuous convex functions and they are asymptotically super-linear, then there exists a minimiser x^* of J and a minimiser p^* of I.*

If f is differentiable, and g is strictly convex, then the minimiser can be characterised by the following Hamiltonian system of equations:

$$\begin{cases} p^{*'}(t) & = f^{'}(x^*(t)), \\ x^{*'}(t) & = g^{*'}(p^*(t)), \\ x^*(0) & = a, \\ x^*(T) & = b. \end{cases}$$

Discrete time. In discrete time, we consider the functional $J : \mathbb{R}^{dN+d} \mapsto \mathbb{R}$ given by

$$J(x_0, \ldots, x_N) = \sum_{n=1}^{N-1} f(x_n) + \sum_{n=0}^{N-1} g(x_{n+1} - x_n),$$

where f and g are convex functions. Our goal is to find the minimisers of J over

$$\mathcal{C} = \left\{(x_0, \ldots, x_N) \in \mathbb{R}^{d(N+1)}, x_0 = a, x_N = b\right\},$$

where a and b are fixed. The dual problem is

$$I(p_0, \dots, p_{N-1}) = \sum_{n=1}^{N-1} f_n^* (p_n - p_{n-1}) + \sum_{n=0}^{N-1} g_n^* (p_n) + a \cdot p_0 - b \cdot p_{N-1}.$$

The discrete-time counterpart of Theorem 1 is:

Theorem 2. *If $x^* = (x_0^*, \dots, x_N^*)$ minimises J and $p^* = (p_0^*, \dots, p_{N-1}^*)$ minimises I, then they can be characterised by the following Hamiltonian system of equations:*

$$\begin{cases} p_n^* - p_{n-1}^* = f^{'} (x_n^*) & \forall n \in \{1, \dots, N-1\}, \\ x_{n+1}^* - x_n^* = g^{*'} (p_n^*) & \forall n \in \{0, \dots, N-1\}. \end{cases}$$

1.5.2 Stochastic optimal control

This section recalls the dynamic programming principle for jump-diffusion processes; see the monograph Pham (2009) for more details on stochastic optimal control and the book Cartea et al. (2015) for various optimal trading models solved using the tools of stochastic optimal control.

In dynamic optimisation problems, an agent seeks to maximise a reward over some time window $[0, T]$. The agent takes actions $\mathbf{u}$ that affect the dynamics of some underlying system $\mathbf{X}^{\mathbf{u}}$; the superscript indicates that $\mathbf{X}$ is affected (controlled) by $\mathbf{u}$. Often, the actions of the agent incur costs (or rewards) over the time window, which can also depend on the time and the state of the system, so they must be accounted for. At each time t, the cumulative past actions of the agent affect the future dynamics of the system and future potential costs.

We consider a filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}\right)$ satisfying the usual conditions and which supports all the processes that we introduce, where $T > 0$ is a fixed time horizon.

Consider an agent who is faced with a control problem of a system whose dynamics contain a diffusive and a jump component. Let $(\mathbf{N}_t^{\mathbf{u}})_{t \in [0,T]}$ denote a $p-$dimensional counting process with controlled intensity $(\lambda_t^{\mathbf{u}})_{t \in [0,T]}$ where $\lambda_t^{\mathbf{u}} = \lambda\left(t, \mathbf{X}_t^{\mathbf{u}}, \mathbf{u}_t\right)$ and $(\mathbf{u}_t)_{t \in [0,T]}$ is an $m-$dimensional control process.[3] Also, let $(\mathbf{W}_t)_{t \in [0,T]}$ denote an $m-$dimensional standard Brownian motion.

[3] $\mathbf{N}^{\mathbf{u}}$ is called a doubly stochastic Poisson process because its intensity is itself stochastic, and in our case controlled. Recall that $\mathbf{N}_t^{\mathbf{u}} - \int_0^t \lambda_s^{\mathbf{u}} \, ds$ is a martingale.

Let $(\mathbf{X}_t^{\mathbf{u}})_{t\in[0,T]}$ denote a controlled $m-$dimensional system with dynamics

$$d\mathbf{X}_t^{\mathbf{u}} = \mu\left(t, \mathbf{X}_t^{\mathbf{u}}, \mathbf{u}_t\right)\, dt + \mathbf{V}\left(t, \mathbf{X}_t^{\mathbf{u}}, \mathbf{u}_t\right) d\mathbf{W}_t + \gamma\left(t, \mathbf{X}_t^{\mathbf{u}}, \mathbf{u}_t\right) d\mathbf{N}_t^{\mathbf{u}}, \quad \mathbf{X}_0^{\mathbf{u}} = X_0, \tag{2}$$

where the $m-$dimensional vector of drifts μ, the $m \times m$ variance matrix $\mathbf{V}$, and the $m \times p-$dimensional jump matrix γ are Lipschitz continuous and X_0 is known.

The agent has a *performance criterion* they wish to maximise which takes the form

$$\boxed{\mathbb{E}\left[G(\mathbf{X}_T^{\mathbf{u}}) + \int_0^T F(s, \boldsymbol{X}_s^{\boldsymbol{u}}, \boldsymbol{u}_s)\, ds\right]}. \tag{3}$$

where $G : \mathbb{R}^m \mapsto \mathbb{R}$ is the *terminal reward* function and $F : [0,T] \times \mathbb{R}^m \times \mathbb{R}^p \mapsto \mathbb{R}$ is the *running reward* function. The functions G and F are assumed to be uniformly bounded.

The agent seeks to maximise the performance criterion (3), so we define their *value function* as:

$$\boxed{H(\mathbf{X}_0) = \sup_{\boldsymbol{u}\in\mathcal{A}} \mathbb{E}\left[G(\mathbf{X}_T^{\mathbf{u}}) + \int_0^T F(s, \boldsymbol{X}_s^{\boldsymbol{u}}, \boldsymbol{u}_s)\, ds\right],} \tag{4}$$

where $\mathcal{A}$ is the *admissible set* of strategies that the agent may use:

$$\begin{aligned} \mathcal{A} = \Big\{ &(\mathbf{u}_s)_{s\in[0,T]},\ \mathbb{R}^m\text{-valued},\ \mathbb{F}\text{-predictable, such that} \\ &\qquad\qquad (\mathbf{X}_s^{\mathbf{u}})_{s\in[0,T]} \text{ admits a strong solution} \Big\}. \end{aligned} \tag{5}$$

The predictability assumption in (5) is necessary in financial decision problems because it ensures that the agent can only use strategies that do not use future information. Often in the following sections, we require the admissible set to include additional constraints to ensure that the problem is well defined and that a solution exists.

Instead of optimising H in (4), it is more convenient to consider a time-indexed succession of optimisation problems that explicitly take into account the feedback effect between the actions of the agent and their impact on the future dynamics and costs. More precisely, we embed the problem (4) into a class of problems indexed by time t:

$$\boxed{H(t,\mathbf{x}) = \sup_{\boldsymbol{u}\in\mathcal{A}_t} \mathbb{E}\left[G(\mathbf{X}_T^{\mathbf{x},\mathbf{u}}) + \int_t^T F(s,\boldsymbol{X}_s^{\mathbf{x},\boldsymbol{u}},\boldsymbol{u}_s)\,ds\right],} \tag{6}$$

where the process $(\mathbf{X}_s^{\mathbf{x},\mathbf{u}})_{s\in[t,T]}$ follows the same dynamics as in (2) but starts with initial value $\mathbf{X}_t^{\mathbf{x},\mathbf{u}} = \mathbf{x}$, and where we define for all $t \in [0,T]$:

$$\mathcal{A}_t = \Big\{ (\mathbf{u}_s)_{s\in[t,T]},\ \mathbb{R}^m\text{-valued},\ \mathbb{F}\text{-predictable, such that} \\ (\mathbf{X}_s^{\mathbf{u}})_{s\in[t,T]} \text{ admits a strong solution} \Big\},$$

and we set $\mathcal{A} = \mathcal{A}_0$, so $H(0,\mathbf{x}) = H(\mathbf{x})$.

Often in this course, we will drop the notation $\mathbf{X}^{\mathbf{x},\mathbf{u}}$, and write (6) as

$$\boxed{H(t,\mathbf{x}) = \sup_{\boldsymbol{u}\in\mathcal{A}_t} \mathbb{E}_{t,\mathbf{x}}\left[G(\mathbf{X}_T^{\mathbf{u}}) + \int_t^T F(s,\boldsymbol{X}_s^{\boldsymbol{u}},\boldsymbol{u}_s)\,ds\right],}$$

where $\mathbb{E}_{t,x}$ represents the expectation conditional on $\mathbf{X}_t^{\mathbf{u}} = \mathbf{x}$.

The following classical result (see Pham (2009)) shows that the value function satisfies the Dynamic Programming Principle (DPP).

Theorem 1. *The value function (6) satisfies the DPP*

$$\boxed{H(t,\boldsymbol{x}) = \sup_{\boldsymbol{u}\in\mathcal{A}} \mathbb{E}\left[H(T,\boldsymbol{X}_T^{\mathbf{x},\boldsymbol{u}}) + \int_t^T F(s,\boldsymbol{X}_s^{\mathbf{x},\boldsymbol{u}},\boldsymbol{u}_s)\,ds\right],}$$

for all $t \in [0,T]$ *and* $\mathbf{x} \in \mathbb{R}^m$.

The DPP connects the value function to its future expected value, regularised by the expected reward or penalty F. In its infinitesimal version, the DPP gives the Dynamic Programming Equation (DPE), or the Hamilton-Jacobi-Bellman (HJB) equation

$$\boxed{\begin{array}{c}\partial_t H(t,\boldsymbol{x}) + \sup_{\boldsymbol{u}\in\mathcal{A}}(\mathcal{L}_t^{\boldsymbol{u}} H(t,\boldsymbol{x}) + F(t,\boldsymbol{x},\boldsymbol{u})) = 0 \\[1em] \text{subject to the terminal condition} \quad H(T,\boldsymbol{x}) = G(\boldsymbol{x}),\end{array}}$$

where $\mathcal{L}_t^{\mathbf{u}}$ is the infinitesimal generator of the process $\mathbf{X}_t^{\mathbf{x},\mathbf{u}}$.

For the diffusion-jump process in (2), the infinitesimal generator acts on functions H as follows:

$$\boxed{\begin{aligned}\mathcal{L}_t^{\boldsymbol{u}} H(t,\boldsymbol{x}) \;=\;& \boldsymbol{\mu}(t,\boldsymbol{x},\boldsymbol{u})\cdot\nabla_x H(t,\boldsymbol{x}) + \tfrac{1}{2}\mathrm{Tr}\left(\boldsymbol{\Sigma}(t,\boldsymbol{x},\boldsymbol{u}) D^2_{xx} H(t,\boldsymbol{x})\right) \\ &+ \textstyle\sum_{j=1}^{p} \lambda_j(t,\boldsymbol{x},\boldsymbol{u}) \left[H(t,\boldsymbol{x}+\boldsymbol{\gamma}_{:,j}(t,\boldsymbol{x},\boldsymbol{u})) - H(t,\boldsymbol{x})\right],\end{aligned}} \tag{7}$$

where $\boldsymbol{\Sigma}(t,\boldsymbol{x},\boldsymbol{u}) = \boldsymbol{V}(t,\boldsymbol{x},\boldsymbol{u})\boldsymbol{V}(t,\boldsymbol{x},\boldsymbol{u})^\intercal$ is the covariance matrix, $D^2_{xx}H(t,\boldsymbol{x})$ is the Hessian matrix of H, and $\boldsymbol{\gamma}(t,\boldsymbol{x},\boldsymbol{u})_{:,j}$ is the jth column of the $m\times p$ matrix $\boldsymbol{\gamma}(t,\boldsymbol{x},\boldsymbol{u})$.

The first term in (7) represents the change in the value function due to the drift of $\mathbf{X}$, the second term represents the diffusion volatility, and the third term corresponds to the arrival of a jump in each component of $\mathbf{N}$. When a jump of the jth component of $\mathbf{N}$ occurs, the components of the system $\mathbf{X}$ only jump according to the jth column of $\boldsymbol{\gamma}(t,\boldsymbol{x},\boldsymbol{u})$.

Solving for the supremum term in the HJB provides the optimal control in *feedback form*, i.e., as a function of the value function. One usually solves the HJB equation, which is a nonlinear PDE. The solution the the PDE is only a *candidate solution* and one needs to prove that it is in fact the solution to the original control problem through a verification argument; see Pham (2009). When a classical solution to the HJB equation exists, i.e., it is $\mathcal{C}^1$ in time and $\mathcal{C}^2$ in x, and if the control is admissible, i.e., it is in $\mathcal{A}$, then by standard results, the solution to the HJB is indeed the value function we seek and the resulting control is an optimal Markov control. Finally, the results and discussions above hold for any terminal stopping time $\tau \le T$.

2 Optimal routing

We start this course with a simple problem that illustrates the methodology to adopt when addressing an optimal trading problem. More precisely, we (i) identify a decision problem, we (ii) propose a parsimonious model of the environment through variables that describe the key quantities to consider, we (iii) frame the decision problem as an optimisation problem that can be solved with classical mathematical tools, and finally we (iv) obtain a solution that we can study and implement in practice.

2.1 Optimal routing of agressive orders

Identifying the decision problem. Liquidity in trading venues is limited. Often, operators need to split a large order over N available trading venues to obtain better execution prices and to reduce the costs of their trading activity.

Here, we study the problem of optimal routing of "*marketable*" orders in multiple trading venues. A marketable order is a buy (resp. sell) order at a price higher than the best ask (resp. lower than the best bid). A buy (sell) marketable order is defined by two variables $Q^\star$ and $P^\star$. $Q^\star$ is the quantity of the order and $P^\star$ is the maximum (minimum) price that the trader is willing to accept.

Modelling framework. Let $(Q^\star, P^\star)$ be a marketable buy order; the analysis for sell orders is identical. Let $Q_n(p)$ be the visible quantity that is available at price p in the $n-$th trading venue when $n \in \{1, \dots, N\}$.

Optimisation problem. The agent splits the parent order $(Q^\star, P^\star)$ into N child orders $(Q_n(p_n), p_n)$ to send to the N trading venues. The agent wishes to choose $(p_1, \cdots, p_n)$ to minimise the overall execution price

$$\sum_{n=1}^{N} p_n \cdot Q_n(p_n),$$

and ensure that the parent order is fully executed, i.e., we require that

$$Q^\star = \sum_{n=1}^{N} Q_n(p_n).$$

Recall that $P^\star$ is the maximum price that the agent is willing to pay, so we need the additional constraint that $P^\star \geq p_n$ for all $n \in \{1, \cdots, N\}$.

Solution. The above problem is a classical constrained optimisation problem. We look for the stationary points of the Lagrangian function and we write

$$Q_n(p_n) + p_n\, Q_n^{'}(p_n) = \lambda\, Q_n^{'}(p_n), \qquad \text{for } n \in \{1, \cdots, N\},$$

where λ is the Lagrange multiplier.

Next, we assume the linear form

$$Q_n(p) = q_n + c_n \cdot p.$$

Thus we obtain

$$(q_n + c_n \cdot p_n^\star) + p_n^\star\, c_n = \lambda\, c_n \implies \qquad p_n^\star = \frac{\lambda}{2} - \frac{q_n}{2\, c_n}\,.$$

Substitute λ in the constraint $Q^\star = \sum_{n=1}^N Q_n(p_n)$ to obtain

$$Q^\star = \sum_{n=1}^{N} \{q_n + c_n \cdot p_n^\star\} = \sum_{n=1}^{N} q_n/2 + c\,\lambda/2, \quad c = \sum_n c_n.$$

Finally, the optimal prices to target in each trading venue are given by

$$\boxed{p_n^\star = \frac{Q^\star}{c} - \frac{q_n}{2\, c_n}\left(1 + \frac{c_n}{q_n} \cdot \frac{\overline{q}}{\overline{c}}\right), \quad \overline{c} = \frac{1}{N}\sum_n c, \ \ \overline{q} = \frac{1}{N}\sum_n q}$$

The problem above is simple, yet it contains all the ingredients of optimal trading. In practice, agents take into account more practical issues of financial environments such as latency in high frequency markets, cancellations in limit order books (randomness of $Q_n(p)$), time periods throughout which they spread their trading activity, hidden and Iceberg orders, etc. Each issue can be modelled with additional variables and assumptions and complicates the model and the solution. For example, the next section consider the optimal routing of a large limit order instead of a marketable order.

2.2 Smart routing of limit orders

Identifying the decision problem. Crossing the spread is the first source of execution costs. Thus, agents mostly trade with limit orders. However, there is structural uncertainty when using limit orders because of price and time priority; waiting on a bad queue generates **opportunity costs**. Here, we consider the problem of an agent that can trade in N different LOBs and wishes to split a large buy limit order.

Modelling framework. Let Q_n denote the size of the best bid queue in the $n-$th LOB. If the agent posts a buy limit order at the best bid price, then they must wait for the Q_n previous LOs to be filled before their LO is filled. The agent assumes that the best bid queue is consumed according to a Poisson P_t^n with intensity λ_n. This intensity can be estimated using historical data.

Optimisation problem. The objective of the agent is to split the parent order into N child LOs with quantities $(q_1, \cdots, q_N)$. The agent wishes to minimize, *on average*, the time $t^\star$ that they need to execute the quantity $Q^\star = \sum_n q_n$.

Solution. In each trading venue $n \in \{1, \ldots, N\}$, after the LO of size q_n is posted, the size of the best bid queue is $Q_n + q_n$. We denote by t_n the time needed for the queue to be totally consumed. We write

$$\int_0^{t^n} dP_t^n = q_n + Q_n \implies \mathbb{E}\left[P_{t^n}^n\right] = t_n\,\lambda_n = q_n + Q_n.$$

The agent seeks to minimise the maximum of $\{t^1, \ldots, t^N\}$, so we necessarily have that $t^\star = t^n$ for all $n \in \{1, \cdots, N\}$.

In particular, we write

$$t^\star = t_n = Q^\star / \sum_n \lambda_n + \sum_n Q_n / \sum_n \lambda_n \implies \boxed{q_n^\star = \rho_n \frac{Q^\star}{N} + \left(\rho_n\, \overline{Q} - Q\right),}$$

where

$$\rho_n = \lambda_n / \overline{\lambda}, \quad \overline{\lambda} = \frac{1}{N} \sum_n \lambda_n, \quad \text{and} \quad \overline{Q} = \frac{1}{N} \sum_n Q_n \,.$$

3 The Almgren-Chriss model in discrete time

Investors and market operators such as institutional traders, mutual funds, and brokers, regularly interact in financial markets to buy or sell large amounts of assets. If the quantity of an order represents a significant portion of the trading volume, then executing the order in one single trade is costly and often impossible. As a result, the optimal split of large blocks of assets (meta orders) into smaller trades (child orders) has become a classical problem in the quantitative finance literature and in practice.

The optimal execution literature formulates models to help agents control their overall trading costs. The results of these models are trading schedules that optimally balance between (i) trading slowly to minimise execution costs (measured as the difference between a reference price and the average price obtained for a trade) and adverse price movements (which are a consequence of their trading activity), and (ii) trading rapidly to minimise inventory and price risk.

The pioneer works in the optimal execution literature are Almgren and Chriss (1999) and Almgren and Chriss (2000). The original Almgren-Chriss model is a discrete-time model where an agent liquidates an initial inventory by posting market orders and maximises a mean-variance objective function. At present, almost all practitioners slice their large orders into child orders according to optimised trading schedules.

Various extensions and models for the optimal execution of large orders have been proposed in the last two decades; see Section **??**. All these models share a common structure; first one formulates (and motivates) a dynamic model of the financial environment where the agent operates, second, one defines the control variables of the agent (the space of decisions), and finally, the agent solves an optimisation problem which results in an optimal behaviour (the optimal trading strategy).

The following sections introduce the different ingredients necessary to formulate the problem of optimal trading in the original Almgren-Chriss framework; namely inventory, execution costs, market impact, price dynamics, and performance criterion. The derivations are based on Almgren and Chriss (2000) and Guéant (2016).

3.1 Modelling framework

An agent holds Q_0 units of a single stock at time $t = 0$. The goal of the agent is to unwind (or liquidate) their **initial inventory** by time $T > 0$. We divide the time window (or trading window) $[0, T]$ into N slices of length Δt and we denote the subdivisions by

$$t_0 = 0 < \cdots < t_n = n\,\Delta t < \cdots < t_N = N\,\Delta t = T.$$

Inventory. At the start of each time interval $[t_n, t_{n+1}]$, the agent chooses the number of shares they buy or sell over the interval. We denote by $\nu_{n+1}\,\Delta t$ the amount of shares that the agent buys or sells over $[t_n, t_{n+1}]$. If $\nu_{n+1} \leq 0$ then the agent sells shares, if $\nu_{n+1} \geq 0$, then the agent buys shares.

Let Q_n denote the number shares (or **inventory**) in the agent's portfolio at time t_n. Thus, the agent's inventory evolves as

$$\boxed{Q_{n+1} = Q_n + \nu_{n+1}\,\Delta t, \quad \text{for } 0 \leq n < N.} \tag{8}$$

Execution costs. Let S_n denote the midprice of the stock at time t_n. For each trade of size $\nu_{n+1}\,\Delta t$ throughout $[t_n, t_{n+1}]$, the agent pays costs like the bid-ask spread and the cost of walking the book. Thus, the agent's trades are not executed at the mid-price S_n but at a less favourable price $\tilde{S}_n$. The difference between the midprice and the execution price is called the execution cost, or the temporary impact, or the instantaneous impact.

In the Almgren-Chriss model, the execution costs depend on the agent's trade size $\nu_n\,\Delta t$ and on trading activity of other agents. We introduce the deterministic **market volume** V_{n+1}, which is the volume traded by other agents throughout $[t_n, t_{n+1}]$. In practice, the market volume V_{n+1} is random. However, market activity depends on the time of day. On average, it is deterministic and has a characteristic U-shape; see Figure 8. To model execution costs, we assume that the execution price per share received by the agent takes the linear form

$$\tilde{S}_{n+1} = S_n + \eta\,\nu_{n+1}/V_{n+1}.$$

The cost parameter η is positive so the agent buys (sells) at prices higher (lower) than the mid-price S_n. Thus, the amount paid (received) for $\nu_{n+1}\,\Delta t$ shares bought (sold) between t_n and t_{n+1} is

$$\nu_{n+1}\,\tilde{S}_{n+1}\,\Delta t = \nu_{n+1}\,(S_n + \eta\,\nu_{n+1}/V_{n+1})\,\Delta t.$$

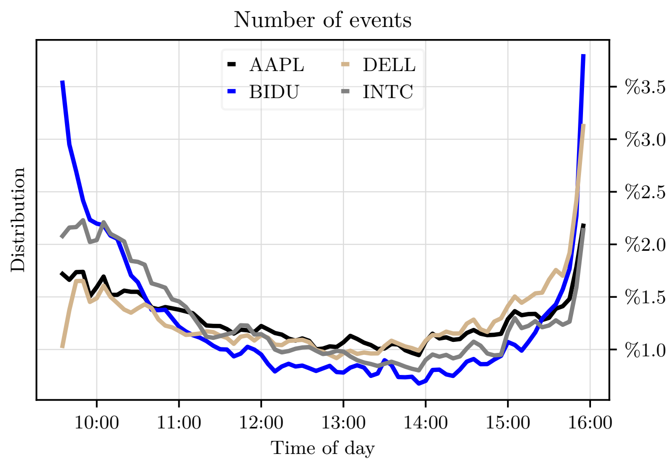

Figure 8: Distribution of trading volume throughout a trading day, measured in portion of LOB events, averaged through trading days between October and December 2022 for multiple shares quoted on Nasdaq. Source: Cartea et al. (2023).

Let X_n denote the amount of cash on the agent's cash account at time t_n and let X_0 be the initial cash of the agent. The dynamics of the agent's cash account are

$$\boxed{X_{n+1} = X_n - \nu_{n+1}\, S_n\, \Delta t - \eta\, \frac{\nu_{n+1}^2}{V_{n+1}}\, \Delta t, \quad \text{for } 0 \leq n < N.} \tag{9}$$

Permanent impact. We assume that the agent's trading activity has a permanent impact on the midprice that is relative to the size $\nu_{n+1}\,\Delta t$ of their trades. In particular, we assume that this impact is linear in the trading size and we assume that the dynamics of the midprice are

$$\boxed{S_{n+1} = S_n + \underbrace{\sigma\,\sqrt{\Delta t}\,\epsilon_{n+1}}_{\text{market risk}} + \underbrace{k\,\nu_{n+1}\,\Delta t}_{\text{linear perm. impact}}, \quad \text{for } 0 \leq n < N,} \tag{10}$$

where $\{\epsilon_n\}$ are independent and identically distributed (i.i.d.) $\mathcal{N}(0, 1)$ variables, $\sigma > 0$ is the arithmetic volatility of the midprice, and $k > 0$ scales the magnitude of the linear permanent impact. Later in Section **??**, we show that the permanent impact of the agent's trading activity must be linear to prevent *dynamic arbitrage*.

3.2 Performance criterion

In the previous section, we have modelled the dynamics of the key variables of the problem. Now, the agent must choose a performance criterion to optimise. Recall that the goal of the agent is to find the optimal **trading schedule** $\{\nu_1, \ldots, \nu_N\}$, so their performance criterion defines what optimality means.

This section follows the steps of Almgren and Chriss (2000) (see also Guéant (2016)). We seek a *liquidation strategy* $v = (\nu_1, \ldots, \nu_n)$ maximising the mean-variance objective function

$$\boxed{\mathbb{E}[X_N] - \frac{\gamma}{2}\,\mathbb{V}[X_N],} \tag{11}$$

where $\gamma > 0$ is the risk aversion parameter and it scales the magnitude of risk aversion in the performance of the agent. The higher the value of γ, the more the agent penalises market risk, i.e., holding the risky asset.

We focus on deterministic strategies that are **admissible**, i.e., they are in the set of admissible strategies $\mathcal{A}^{\text{det}}$:

$$\boxed{(\nu_n)_n \in \mathcal{A}^{\text{det}} = \left\{(\nu_1, \ldots, \nu_N) \in \mathbb{R}^n, \quad \sum_{n=0}^{N-1} \nu_{n+1}\,\Delta t = -Q_0\right\}}.$$

A deterministic strategy is a strategy that only depends on model parameters and time. Thus, it can be computed at the beginning of the execution process, in particular, it does not depend on the evolution of the price.

3.3 Solution

To solve our problem, we first compute the terminal wealth X_N of the agent, and then compute the performance criterion (11). The following result provides a formula for the terminal wealth of the agent and their performance criterion.

Proposition 1. *The terminal wealth of the agent is*

$$X_N = X_0 + Q_0\,S_0 - \frac{k}{2}Q_0^2 + \sigma\,\sqrt{\Delta t}\sum_{n=0}^{N-1} Q_{n+1}\,\epsilon_{n+1}$$

$$
-\sum_{n=0}^{N-1}\nu_{n+1}^2\left(\frac{\eta-\frac{k}{2}\,V_{n+1}\,\Delta t}{V_{n+1}}\right)\Delta t\,,
$$

So the mean and variance of the terminal wealth are

$$
\boxed{\begin{cases}\mathbb{E}\left[X_N\right] & = X_0+Q_0\,S_0-\frac{k}{2}Q_0^2-\sum_{n=0}^{N-1}\nu_{n+1}^2\left(\frac{\eta-\frac{k}{2}\,V_{n+1}\,\Delta t}{V_{n+1}}\right)\Delta t,\\ \\ \mathbb{V}\left[X_N\right] & = \sigma^2\,\Delta t\,\sum_{n=0}^{N-1}Q_{n+1}^2\,.\end{cases}} \tag{12}
$$

Proof. Use the dynamics of the inventory in (8), those of the midprice in (10), and those of the cash in (9) to write

$$
\begin{aligned}
X_N =& X_0-\sum_{n=0}^{N-1}\left(Q_{n+1}-Q_n\right)S_n-\eta\sum_{n=0}^{N-1}\frac{\nu_{n+1}^2}{V_{n+1}}\Delta t\\
=& X_0-\sum_{n=0}^{N-1}Q_{n+1}S_n+\sum_{n=0}^{N-1}Q_nS_n-\eta\sum_{n=0}^{N-1}\frac{\nu_{n+1}^2}{V_{n+1}}\Delta t\\
=& X_0-\sum_{n=0}^{N-1}Q_{n+1}\left(S_{n+1}-\sigma\sqrt{\Delta t}\epsilon_{n+1}-k\nu_{n+1}\Delta t\right)\\
&+\sum_{n=0}^{N-1}Q_nS_n-\eta\sum_{n=0}^{N-1}\frac{\nu_{n+1}^2}{V_{n+1}}\Delta t\\
=& X_0+Q_0S_0+\sigma\sqrt{\Delta t}\sum_{n=0}^{N-1}Q_{n+1}\epsilon_{n+1}\\
&+k\sum_{n=0}^{N-1}Q_{n+1}\nu_{n+1}\Delta t-\eta\sum_{n=0}^{N-1}\frac{\nu_{n+1}^2}{V_{n+1}}\Delta t.
\end{aligned}
$$

To obtain the desired result, write the term $k\sum_{n=0}^{N-1}Q_{n+1}\nu_{n+1}\Delta t$ as

$$
\begin{aligned}
k\sum_{n=0}^{N-1}Q_{n+1}\nu_{n+1}\Delta t &= k\sum_{n=0}^{N-1}Q_{n+1}\left(Q_{n+1}-Q_n\right)\\
&= k\sum_{n=0}^{N-1}\left(\frac{Q_{n+1}+Q_n}{2}+\frac{Q_{n+1}-Q_n}{2}\right)\left(Q_{n+1}-Q_n\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{k}{2}\sum_{n=0}^{N-1}\left(Q_{n+1}^2 - Q_n^2\right) + \frac{k}{2}\sum_{n=0}^{N-1}\left(Q_{n+1} - Q_n\right)^2 \\
&= -\frac{k}{2}Q_0^2 + \frac{k}{2}\sum_{n=0}^{N-1}\nu_{n+1}^2\Delta t^2.
\end{aligned}
$$

We only consider deterministic controls ν_n, so the final value of the cash process is normally distributed with mean and variance in (12). □

The first term $X_0 + Q_0\, S_0$ in (12) is the marked-to-market (MtM) value of the agent's wealth at time $t = 0$. The second term corresponds to costs (or to a discount term) originating from the permanent impact. This term does not depend on the specific liquidation strategy followed by the agent, so these costs are unavoidable. The last term corresponds to running execution costs throughout the trading window. In contrast to permanent impact costs, the running execution costs depend on the liquidation strategy.

To obtain an analytical solution, we simplify our model to consider a flat market volume curve and we write $V_n = V$ for all n.[4] In that case, minimising the mean-variance objective (11) reduces to minimising the function $\mathfrak{J}$ over $\mathcal{A}^{\text{det}}$ where

$$
\mathfrak{J} : \nu \in \mathbb{R}^n \mapsto \tilde{\eta}\sum_{n=0}^{N-1}\frac{\nu_{n+1}^2}{V}\Delta t + \frac{\gamma}{2}\,\sigma^2\,\Delta t\sum_{n=0}^{N-1}Q_{n+1}^2\,,
$$

and

$$
\tilde{\eta} = \eta - \frac{k}{2}\,V\,\Delta t.
$$

The above problem minimises $\mathfrak{J}$ over the liquidation strategies $\nu \in \mathcal{A}^{\text{det}}$. Observe that this problem is equivalent to minimising the functional J over the **trading curves** $Q \in \mathcal{C}^{\text{det}}$ where

$$
\boxed{J : q \in \mathbb{R}^{N+1} \mapsto \tilde{\eta}\sum_{n=0}^{N-1}\frac{\left(Q_{n+1} - Q_n\right)^2}{V\,\Delta t} + \frac{\gamma}{2}\,\sigma^2\,\Delta t\sum_{n=0}^{N-1}Q_{n+1}^2,}
$$

and

$$
\mathcal{C}^{\text{det}} = \left\{Q = \left(Q_0, \ldots, Q_N\right),\; Q_0 = Q_0,\; Q_N = 0\right\}.
$$

[4] In many instances in algorithmic trading models, one needs simplifying assumptions to obtain analytical and interpretable formulae.

The optimal liquidation strategy $\nu^\star \in \mathcal{A}^{\mathrm{det}}$ is given by

$$\nu^\star = \frac{Q^\star_n - Q^\star_{n-1}}{\Delta t}, \quad \text{for } 0 < n \leq N,$$

where $Q^\star \in \mathcal{C}^{\mathrm{det}}$ is the optimal trading curve.

Assumption 1. *We assume that* $\tilde{\eta} > 0 \implies \eta > \frac{k}{2} V \, \Delta t$.

Assumption 1 ensures that the problem is convex. It is not restrictive because $k \, \Delta t$ is small enough in practice and the terms in Δt^2 are generally dropped.

The functional J is strictly convex and $\mathcal{C}^{\mathrm{det}}$ is convex, thus by standard results there exists a unique minimiser to J over $\mathcal{C}^{\mathrm{det}}$. Let $Q^\star$ be this minimiser. The optimal trading curve $Q^\star$ is uniquely characterised by the following Hamiltonian system (see Theorem 2):

$$\begin{cases} p_{n+1} & = p_n + \gamma \, \sigma^2 \, \Delta t \, Q^\star_{n+1} \, , \quad & 0 \leq n < N-1 \, , \\ Q^\star_{n+1} & = Q^\star_n + \frac{V}{2 \, \tilde{\eta}} \, \Delta t \, p_n \, , & 0 \leq n < N \, , \end{cases}$$

with the boundary conditions $Q^\star_0 = Q_0$ and $Q^\star_N = 0$.

Thus, $Q^\star$ is the solution the second-order recursive equation

$$Q^\star_{n+2} - \left(2 + \frac{\gamma \, \sigma^2 \, V}{2 \, \tilde{\eta}} \Delta t^2 \right) Q^\star_{n+1} + Q^\star_n = 0 \, ,$$

which admits the solution

$$\boxed{Q^\star_n = Q_0 \, \frac{\sinh\left(\alpha \left(T - t_n\right)\right)}{\sinh(\alpha T)}}$$

where α uniquely solves

$$2 \, \cosh\left(\alpha \, \Delta t\right) = \frac{\gamma \, \sigma^2 \, V}{2 \, \tilde{\eta}} \Delta t^2 \, .$$

3.4 Discussion

Here, we consider an example where the midprice has an arithmetic volatility $\sigma = 1\$ \cdot \mathrm{day}^{-1/2}$ (approx. 32% annualized vol) and we set $S_0 = 100$. Assume the flat market volume is $V = 4{,}000{,}000$ shares per day, and we set

$\eta = 0.1\$ \cdot \text{share}^{-1}$. An agent holds an initial inventory $Q_0 = 200,000$ shares to liquidate, which corresponds to 5% participation rate. Figure 9 shows the optimal trading curve $Q^\star$ for multiple values of the risk aversion parameter γ.

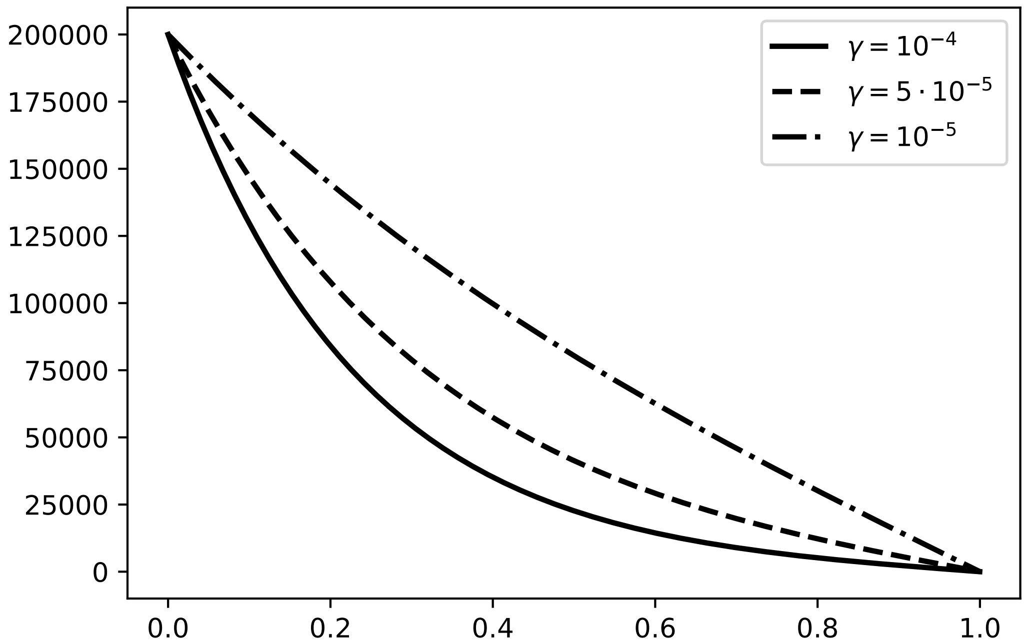

Figure 9: Optimal trading curves.

Liquidity parameters. η (or $\tilde{\eta}$) and V are scaling factors for the execution costs $\eta\, \nu_{n+1}/V$ paid by the trader at time t_n. When execution costs are high, the agent trades slowly. The larger the value of η, the larger the execution costs; η is a proxy for the depth of the LOB and can be estimated with a linear regression using snapshots of the LOB; see Figure 6. The smaller the value of V, the larger the execution costs; the price of liquidity depends on the participation rate v_n/V. In particular, we can show that

$$\frac{dQ^\star}{d\eta}/Q_0 \geq 0 \quad \text{and} \quad \frac{dQ^\star}{dV}/Q_0 \leq 0.$$

In practice, in order to set the value of the execution cost / temporary market impact parameter η, one supposes that the additional cost incurred per share when trading a given volume $\nu_n \Delta t$ is proportional to the participation rate to the market (in practice we consider a flat volume curve that matches the average daily volume); see the discussions in Almgren and Chriss (2001). For example, for each $\mathfrak{p}\%$ of participation rate, one assumes a cost corresponding to half the bid-ask spread is incurred.[5] The value of $\mathfrak{p}$ can be obtained from historical data.

Risk parameters. The volatility σ measures price risk. The larger its value, the faster the agent needs to liquidate their inventory to reduce the exposure to price risk. In particular one shows that

$$\frac{dQ^{\star}}{d\sigma}/Q_0 \leq 0.$$

The risk aversion parameter γ determines the balance between maximising wealth (or equivalently minimising execution costs) and minimising price risk. The larger its value, the more the agent is sensitive to price risk, thus the agent trades faster; see Figure 9. Also, one shows that

$$\frac{dQ^{\star}}{d\gamma}/Q_0 \leq 0.$$

Finally, observe that when γ is very small, the optimal trading curve is a straight line:

$$\lim_{\gamma \to 0} Q^{\star}(t) = Q_0 \left(1 - t/T\right).$$

[5] The average bid-ask spread is close to the tick value when the asset is very liquid; see Figure 6.

References

Almgren, R., Chriss, N., 1999. Value under liquidation. Risk 12, 61–63.

Almgren, R., Chriss, N., 2000. Optimal execution of portfolio transactions. Journal of Risk 3, 5–39.

Almgren, R., Chriss, N., 2001. Optimal execution of portfolio transactions. Journal of Risk 3, 5–40.

Cartea, Á., Drissi, F., Osselin, P., 2023. Bandits for algorithmic trading with signals. Available at SSRN 4484004 .

Cartea, Á., Jaimungal, S., Penalva, J., 2015. Algorithmic and High-Frequency Trading. Cambridge University Press.

Gatheral, J., 2010. No-dynamic-arbitrage and market impact. Quantitative finance 10, 749–759.

Guéant, O., 2016. The Financial Mathematics of Market Liquidity: From Optimal Execution to Market Making. doi:10.1201/b21350.

Pham, H., 2009. Continuous-time stochastic control and optimization with financial applications / Huyen Pham. Springer Berlin.

Rockafellar, R.T., 1997. Convex analysis. volume 11. Princeton university press.