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Analytic K-Homology

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PREFACE

Analytic K-homology draws together ideas from algebraic topology, functional analysis and geometry. It is a tool — a means of conveying information among these three subjects — and it has been used with spectacular success to prove and indeed discover remarkable theorems across a wide span of mathematics. These include results in operator theory which make no mention of topology or geometry at all, and results in topology and geometry which are apparently far removed from functional analysis.

The purpose of this book is to acquaint the reader with the essential ideas of analytic K-homology and develop some of its applications. We shall begin this introduction with a brief account of the origins of the subject, followed by a first look at the central notion of 'abstract elliptic operator' which ties most of the different aspects of analytic K-homology together.

The roots of analytic K-homology

The subject of analytic K-homology had two separate beginnings, one in operator theory and one in the index theory of Atiyah and Singer, but we shall start this story in a third place, in the realm of algebraic topology.

The abstract machinery of algebraic topology assigns to each generalized *cohomology* theory a dual *homology* theory. From this point of view, K-homology is nothing more than the homology theory dual to Atiyah–Hirzebruch K-theory. Now cohomology and homology theories stand in relation to one another more or less as vector spaces stand in relation to their duals. Thus an essential feature of any dual pair of homology/cohomology theories is the bilinear pairing between the homology and cohomology groups of each space, the exemplar of which is the pairing between ordinary homology and de Rham cohomology given by integration of differential forms over simplices. An important step in the investigation of any homology/cohomology theory is to find a concrete realization of this pairing, just as the analysis of a linear space is often centered (especially in the case of topological linear spaces) around a concrete identification of its dual.

Unfortunately the abstract approach to dual theories does not provide a concrete and geometric account of this pairing. The first purpose of analytic K-homology is to remedy this shortcoming in the case of K-theory: at its core is a very simple pairing based on Fredholm index theory. But what makes analytic K-homology especially remarkable, and much more than an appendix to Atiyah–Hirzebruch K-theory, is the extraordinary volume of mathematics which travels in both directions between topology and functional analysis through this basic Fredholm index pairing.

The Fredholm index of a linear operator T is the integer quantity

$$\text{Index}\,(\mathsf{T}) = \text{Dim}(\text{Kernel}(\mathsf{T})) - \text{Dim}(\text{Cokernel}(\mathsf{T})),$$

which is defined whenever the kernel and cokernel of T are finite-dimensional. The index is stable under finite-rank perturbations of the operator T (in finite dimensions this is the well-known Rank-Nullity Theorem from college linear algebra). In the context of Hilbert space, where this book is located, the index is also stable under small-norm perturbations of the bounded operator T. Since every compact Hilbert space operator is a norm-limit of finite rank operators, it follows that the Fredholm index is stable under compact operator perturbations of T. In fact a lovely theorem of Atkinson asserts that a bounded Hilbert space operator has finite-dimensional kernel and cokernel if and only if it is invertible modulo compact operators. (The connection between index theory and compact operators is already made evident by the famous Fredholm Alternative: if K is a compact operator then the linear equation $\mathsf{Kf} + \mathsf{f} = \mathsf{g}$ has a solution for every g if and only if the homogeneous equation $\mathsf{Kf} + \mathsf{f} = 0$ has no non-trivial solution.)

Operator theory has long considered the problem of classifying Hilbert space operators 'modulo compact operators'. Weyl and von Neumann showed that two selfadjoint operators are unitarily equivalent modulo compact operators if and only if they have the same spectrum apart from isolated eigenvalues of finite multiplicity. In the 1960's Brown, Douglas and Fillmore began an investigation of *essentially normal* operators, meaning those for which $\mathsf{T}^*\mathsf{T}$ and $\mathsf{T}\mathsf{T}^*$ are equal modulo compact operators, by asking themselves the following question: is the unilateral shift operator on the Hilbert space $\ell^2(\mathbb{N})$ unitarily equivalent modulo compact operators to the bilateral shift operator on $\ell^2(\mathbb{Z})$? The essential spectrum, meaning the part of the spectrum which is stable under compact perturbations, is for both operators the unit circle S^1 in the complex plane. According to the Weyl–von Neumann Theorem the essential spectrum is a complete classification invariant for selfadjoint operators. But in the present case a new invariant emerges, namely the Fredholm index. Indeed the index of the unilateral shift is -1, whereas the index of the bilateral shift is 0, while the stability properties of the index show that it is an invariant for unitary equivalence modulo compact operators.

Using simple operator theory techniques it is not hard to show that two essentially normal operators with essential spectrum S^1 are unitarily equivalent modulo compact operators if and only if they have the same Fredholm index. But the situation for other essential spectra $X \subseteq \mathbb{C}$ (for example, the closed unit disk) is considerably more complicated. Brown, Douglas and Fillmore introduced the classifying structure Ext (X) to help attack the problem, and then they proved two very unexpected things: first, Ext (X) is actually an abelian group, and second, Ext (X) is the degree-one K-homology group of X. This is the Brown–Douglas–Fillmore Theorem. The determination of Ext (X), which is to say the classification of essentially normal operators, was thereby carried out by reducing the classification problem to a computational problem in algebraic topology.

The group Ext (X) also classifies C*-algebra extensions of the form

$$0 \longrightarrow \mathfrak{K}(H) \longrightarrow E \longrightarrow C(X) \longrightarrow 0\,,$$

where $\mathfrak{K}(H)$ is the C*-algebra of compact operators on a Hilbert space H and C(X) is the commutative C*-algebra of continuous complex-valued functions on X (which can now be any compact metric space). Thinking of Ext (X) from this point of view, we can give a very simple account of the crucial Fredholm index pairing with K-theory. Indeed the K-theory group $K^{-1}(X)$ is generated by homotopy classes of maps from X to the group of invertible complex matrices. Given an extension, as above, an invertible, matrix-valued function on X lifts to a matrix over the algebra E which is invertible modulo compact operators. This matrix has a Fredholm index, and thanks to the stability properties of the index this procedure for combining an extension and an invertible, matrix-valued function defines a bilinear pairing between Ext (X) and $K^{-1}(X)$. From this point of view, the Brown–Douglas–Fillmore Theorem is the assertion that if $X \subseteq \mathbb{C}$ then the homomorphism

$$\text{Index} : \operatorname{Ext}(X) \longrightarrow \operatorname{Hom}(K^{-1}(X), \mathbb{Z})$$

is an isomorphism of abelian groups.

The index theory of Atiyah and Singer presents a second view of the Fredholm index pairing between K-theory and K-homology. Suppose that X is a compact manifold and that D is an linear elliptic operator on X. Then D has a Fredholm index. But in addition if V is a vector bundle on X then a standard construction in index theory (essentially a tensor product) produces a new linear elliptic operator D_V, 'with coefficients in V', and the assignment $V \mapsto \operatorname{Index} D_V$ determines a homomorphism

$$\operatorname{Index}_D : K^0(X) \longrightarrow \mathbb{Z}$$

In order to extend this discussion to spaces other than manifolds, Atiyah identified the key functional-analytic properties of an elliptic operator on a manifold and so developed an abstract notion of elliptic operator, now called a *Fredholm module*. Kasparov developed Atiyah's idea and showed that the abelian group generated by homotopy classes of Fredholm modules is another analytic model for K-homology, this time for the degree-zero K-homology group of X.

Kasparov's K-homology has proved to be an extremely powerful and flexible tool in index theory. For example the proof of the Atiyah–Singer Index Theorem itself can be presented very simply and conceptually using the product structure on K-homology. Moreover Kasparov's work has allowed a considerable strengthening of the index theory of Atiyah and Singer. Kasparov developed his theory as a tool in differential topology, and indeed some of the most powerful theorems in the topological theory of manifolds (pertaining particularly to the Novikov conjecture) rely very heavily on Kasparov's machinery. In several cases no proofs of these theorems are known which do not employ functional analysis to a very considerable extent. Thus, thanks to Kasparov's discoveries, functional analysis has repaid to topology the debt incurred by Brown, Douglas and Fillmore!

Abstract Elliptic Operators

Let us describe in a little more detail the Fredholm modules that Atiyah invented. In an effort to make the main idea as plain as possible we shall take a look at Fredholm modules which are associated not to a space X but to a group G. One of the virtues of the functional-analytic approach is that these two very different types of objects — spaces and groups — can be placed on more or less the same footing. Both give rise to C^*-algebras: the commutative C^*-algebras $C(X)$ in the first case and the group C^*-algebras $C^*(G)$ in the second.

The *unitary representation ring* $R(G)$ of G is the abelian group whose generators are finite-dimensional unitary representations V of G, with the relations

$$[V \oplus W] = [V] + [W].$$

An element of $R(G)$ can always be represented by a formal difference $[V_0] - [V_1]$ of finite-dimensional unitary representations. Note that if we were to omit the finite-dimensionality restriction then $R(G)$ would collapse to zero, thanks to the calculation

$$[V] + [V_\infty] = [V \oplus V_\infty] = [V_\infty],$$

where $V_\infty = \bigoplus^\infty V$. Nevertheless, one *can* make sense of a formal difference $[V_0] - [V_1]$ of infinite-dimensional representations of G, if along with the two representations V_0 and V_1 themselves there is given an 'approximate isomorphism' between them. This is what a Fredholm module provides. To be precise, a *Fredholm module* over G consists of a pair of Hilbert space representations of G, V_0 and V_1, together with an operator $U\colon V_0 \to V_1$ which is unitary modulo compact operators and also an intertwiner modulo compact operators (meaning that $Ug - gU$ is compact for each $g \in G$). From these objects one can construct a *Fredholm representation ring*[1] $R_{\mathrm{Fred}}(G)$ in more or less the same way that $R(G)$ is constructed. It is plain that there is a natural map $R(G) \to R_{\mathrm{Fred}}(G)$,

[1]Both $R(G)$ and $R_{\mathrm{Fred}}(G)$ are commutative rings; the former by tensor product of representations and the latter by a much more elaborate construction — the so-called Kasparov product.

but it is typically far from an isomorphism (although it *is* an isomorphism if G is finite or compact).

Kasparov's group $K_0(X)$ is defined using Fredholm modules over X, which are pairs (V_0, V_1) of Hilbert space representations of the C^*-algebra $C(X)$ equipped with a linear operator $U: V_0 \to V_1$ which is an 'approximate isomorphism' in the same sense as above. What Atiyah observed is that each order zero elliptic pseudodifferential operator on a closed manifold X gives rise to a Fredholm module, and so Kasparov's theory in some sense incorporates the index theory of elliptic operators on X. What became evident after the Brown–Douglas–Fillmore work came to the attention of Atiyah and Singer is that each C^*-algebra extension of the type we considered earlier corresponds to a *selfadjoint* Fredholm module — which is an approximate self-symmetry of a *single* Hilbert space representation. So in retrospect the single notion of Fredholm module very elegantly connects together the two realizations of analytic K-homology.

About this book

We turn now to a description of this book. The first part (Chapters 1–7) leads toward a proof of the Brown–Douglas–Fillmore Theorem. Chapter 1 introduces the basic concepts of C^*-algebra theory and operator theory. In Chapter 2 we develop the theory of the Fredholm index, which, as we have indicated, is fundamental to analytic K-homology. The chapter also provides the basic definitions of C^*-algebra extension theory. Chapter 3 is devoted to a number of technical results in C^*-algebra theory: Stinespring's Theorem, nuclearity and Voiculescu's Theorem are some of the topics covered here. The fourth chapter is devoted to K-theory for C^*-algebras. The pace here is brisk, as many readers will already be familiar with K-theory, but the chapter is self-contained and could also serve as a rapid introduction to the subject.

In Chapter 5 we begin the study of K-homology proper. We define the K-homology of a C^*-algebra A in terms of the K-theory of a suitable dual algebra $\mathfrak{D}(A)$. We use the technical results of Chapter 3 to develop the analysis of these dual algebras, and we obtain the long exact sequence of K-homology associated to a C^*-algebra extension. Chapter 6 makes the connection between K-homology and *coarse geometry* — the study of geometric spaces from the point of view of their large-scale structure. The connection was originally explored from the perspective of index theory on non-compact manifolds — we shall return to this subject in the second half of the book — but our immediate business is to use coarse geometry to prove the *homotopy invariance* of K-homology, and thereby show that K-homology is a generalized homology theory in the sense of algebraic topology. By this stage we shall have assembled nearly all the tools we require to prove the Brown–Douglas–Fillmore Theorem. In Chapter 7 we complete the proof, assuming a certain naturality property of the index pairing between K-theory and K-homology. This naturality property is given a direct proof in Chapter 8, but for the reader who wants to see the BDF Theorem proved without getting involved in the second half of the book, we indicate a more circuitous way around the problem in the exercises to Chapter 7. We also prove the Universal Coefficient Theorem for K-homology, which can be thought of as a generalization of the BDF Theorem to higher-dimensional spaces X.

The second half of the book is centered around index theory. In Chapter 8 we introduce Kasparov's definition of K-homology in terms of Fredholm modules, and we show that his definition is equivalent to the duality-based one of Chapter 5. Then we carry out some key computations involving boundary maps and the index pairing. In Chapter 9 we describe the product structure on K-homology. This complicated but powerful construction is Kasparov's major contribution to the theory: it allows very simple proofs of the main properties of K-homology, and as we shall see it also connects in a very beautiful way with the theory of elliptic operators. That theory is the subject of Chapter 10; after reviewing the basic results of elliptic operator theory on manifolds, we show how an elliptic operator gives rise to a Fredholm module and therefore to a K-homology class, and how the Kasparov product on K-homology corresponds to the external product defined by Atiyah on the class of elliptic operators. In Chapter 11 we apply K-homology to prove the Atiyah–Singer Index Theorem, at least in an illustrative special case. The proof could be put onto one line — an indication of the power of K-homology theory. We also use K-homology to prove some related index theorems: the Toeplitz Index Theorem for strongly pseudoconvex domains in $\mathbb{C}^n$, and (in an exercise) the Callias Index Theorem for operators of 'Dirac–Schrödinger' type. All these results are rather simple consequences of basic calculations of K-homology theory.

Finally, in Chapter 12, we introduce the topic of higher index theory. Here one contemplates an 'index' which is no longer an integer but an element of a C*-algebra K-theory group. Higher index theory turns out to be critically important to a number of geometric problems, of which we have selected the positive scalar curvature problem — which manifolds carry positive scalar curvature metrics? — as an illustrative and important example. Central to higher index theory are the *Baum–Connes conjectures.* Using our work on coarse geometry we shall be able to state the conjectures, prove them in some special cases, and describe their relationship to the positive scalar curvature problem.

Some topics which are not covered in the book

The plan of this book changed several times over several years of writing. But right from the beginning we decided to avoid any serious discussion of Kasparov's bivariant KK-theory. We did so because we wanted to minimize the C*-algebra background demanded of the reader, and because we wanted to emphasize, at least at the beginning of the book, the Brown–Douglas–Fillmore theory, which is not really illuminated by Kasparov's two-variable theory. But by avoiding KK-theory we have occasionally been forced to develop circuitous arguments at points where the bivariant theory provides a direct approach. In addition, KK-theory is more or less essential for the further development of the ideas introduced in Chapter 12. Other topics, such as the foliation index theorem of Connes and Skandalis, are also out of convenient reach.

In recent years other definitions of analytic K-homology have been developed. Perhaps most notable among them is a version of K-homology for C^*-algebras which is based on a notion of *asymptotic morphism* between C^*-algebras. While this version of K-homology has an appealing simplicity, and while it is quite well adapted to Atiyah–Singer index theory, it is less well suited to the Brown–Douglas–Fillmore theory. Moreover the theory of asymptotic morphisms is best developed in the context of a two-variable theory, like Kasparov's KK-theory, and since we are concentrating on the one-variable theory it did not seem altogether appropriate to include an account of the theory of asymptotic morphisms in this book.

After some hesitation, we decided not to include any extended discussion of *real* C^*-algebras and their K-theory and K-homology groups in the main text of the book. The connections between K-homology and manifold theory attain their most precise form if one works in the real case; but the underlying linear algebra is more complicated, and in places the more versatile and powerful bivariant KK-theory of Kasparov is called for. This means that the real case is perhaps more suited to a second look at the subject. We have however attempted to write a book which, while not real, can be 'realized' without excessive difficulty. Following a well-known precedent, we have included in Appendix B an outline of how this is to be achieved.

Advice to the reader

To read the first part of this book you will need to have completed a basic course in functional analysis, and have had some exposure to 'homological' mathematics. The latter might come from studying classical algebraic topology, or homological algebra, or the K-theory of C^*-algebras. Chapters 1–7 then constitute a somewhat discursive introduction to K-homology, culminating in the proof of the Brown–Douglas–Fillmore Theorem. The pace is not too hurried and the arguments are mostly rather detailed. These chapters could be the core of an introductory graduate course on K-homology.

The pace quickens in the second half of the book with the introduction of Kasparov theory, and from Chapter 8 on a bit more mathematical sophistication is demanded of you. For example, in many arguments you will need to supply at least some of the details yourself. Moreover, since the central examples of

Fredholm modules come from elliptic operator theory, some familiarity with the machinery of smooth manifolds, vector bundles, and so on will be required. Finally several sections require an acquaintance with specialized topics in Riemannian geometry, complex function theory, and other subjects.

Although the second part of the book does depend on the first, you could jump right in at Chapter 8 if you are well-prepared and want to learn about the connection to index theory as soon as possible. You would need to refer back to earlier chapters as necessary, especially to Chapter 6. Appendix A contains various elementary results about graded algebras and modules. You should probably review it before embarking on Chapter 8.

Each chapter ends with a series of exercises. Quite a number of the exercises develop ideas not fully presented in the text; several require familiarity with other subjects (for example, geometry or representation theory); some are quite difficult. We hope that at least some of the problems are challenging enough and interesting enough to repay careful thought, since it goes without saying that working through exercises is by far the best way to learn this or any other subject.

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This book is dedicated to our families:

Yvonne and Julia,
Liane, Nathan, and Miriam.

CONTENTS

1

C*-ALGEBRAS AND OPERATOR THEORY

This chapter gives a rapid introduction to operators on Hilbert space and the elements of C^*-algebra theory. It is intended for those readers who have sufficient mathematical background to benefit from a compressed account of the subject. For a more complete introduction the reader should consult one of the many available books on C^*-algebras (see the notes at the end of the chapter). Rather than start at the very beginning, we have taken for granted those aspects of Banach algebra theory which are nearly always presented in undergraduate and elementary graduate courses in functional analysis. The theory of Banach algebras involves an interaction of algebraic notions (spectrum, spectral radius) and topological ones (open sets, norms); C^*-algebras are those Banach algebras for which this interaction attains its most perfect form.

1.1 Bounded Operators and Functional Calculus

We begin by studying operators on Hilbert space. Throughout the book the scalar field $\mathbb{C}$ will be used, so that 'Hilbert space' means 'complex Hilbert space'. By convention, the Hilbert space inner product $\langle v, w\rangle$ will be taken to be linear in the variable w and conjugate-linear in v.

Let H be a Hilbert space. A *bounded linear operator* on H is a linear map $T\colon H \to H$ such that the *operator norm* of T, defined by

$$\|T\| = \sup\{\, \|Tv\| : \|v\| \leqslant 1 \,\},$$

is finite. We shall often abbreviate 'bounded linear operator' to 'bounded operator', or just 'operator'.

We can of course equally well discuss bounded linear maps from one Hilbert space to another, but the operators on H have the advantage that they can be composed. Denote by $\mathfrak{B}(H)$ the algebra of all bounded operators on H. Equipped with the above norm it is a Banach algebra. The *adjoint* of a bounded linear operator T is the operator T^* characterized by the identity

$$\langle T^*v, w\rangle = \langle v, Tw\rangle \qquad \forall v, w \in H.$$

The operation $T \mapsto T^*$ is a conjugate-linear *involution* on the algebra $\mathfrak{B}(H)$: that is, $T^{**} = T$ and $(ST)^* = T^*S^*$. If T is a bounded operator on H then the Cauchy–Schwarz inequality implies that

$$\|T\|^2 = \sup_{\|v\|\leqslant 1} \langle Tv, Tv\rangle = \sup_{\|v\|\leqslant 1} \langle T^*Tv, v\rangle \leqslant \sup_{\|v\|\leqslant 1} \|T^*Tv\| = \|T^*T\|.$$

Thus we obtain the inequalities

$$\|T\|^2 \leqslant \|T^*T\| \leqslant \|T^*\|\|T\| \tag{1.1.1}$$

which imply that $\|T\| \leqslant \|T^*\|$. By symmetry $\|T\| = \|T^*\|$, and so the involution on $\mathfrak{B}(H)$ is isometric. A Banach algebra such as $\mathfrak{B}(H)$ which is equipped with an isometric involution is called a *Banach $*$-algebra.*

1.1.2 EXAMPLE If $H = \mathbb{C}^n$ then $\mathfrak{B}(H)$ identifies with the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices over $\mathbb{C}$, and the adjoint corresponds to the matrix operation of conjugate transpose.

The inequalities (1.1.1) together with the equality $\|T\| = \|T^*\|$ further imply that

$$\|T^*T\| = \|T\|^2.$$

This is called the *C^*-identity* for operators on Hilbert space.

There is some standard terminology for various kinds of operators on Hilbert space, summarized in the table below.

Operator	Description	Formula
Selfadjoint	$\langle Tu, v\rangle = \langle u, Tv\rangle$	$T = T^*$
Unitary	Isometric isomorphism of H to itself	$U^*U = UU^* = I$
Normal	Commutes with its adjoint	$T^*T = TT^*$
Isometry	Isometric map of H into H	$V^*V = I$
Projection	Orthogonal projection onto closed subspace	$P = P^* = P^2$
Partial Isometry	Isometry from orthogonal complement of kernel into H	$V^*V =$ projection

1.1.3 REMARK It makes perfect sense to speak of unitary, isometric and partially isometric operators from one Hilbert space to another, rather than considering only operators mapping H to itself.

1.1.4 EXAMPLE Every selfadjoint operator is normal, as is every unitary operator.

1.1.5 LEMMA *If T is a normal operator then* $\|T^2\| = \|T\|^2$.

PROOF From the C^*-identity and the definition of normality,

$$\|T^2\|^2 = \|T^{2*}T^2\| = \|(T^*T)^*(T^*T)\| = \|T^*T\|^2 = \|T\|^4. \quad \square$$

1.1.6 DEFINITION Suppose that T is a bounded operator on H. Denote by $C^*(T)$ the Banach subalgebra of $\mathfrak{B}(H)$ generated by T, its adjoint, and the identity operator I.

The notation is meant to suggest that $C^*(T)$ is the C^*-algebra generated by T. C^*-algebras will be introduced in Section 1.3 below.

If T is a normal operator then $C^*(T)$ is a *commutative* Banach algebra and we can apply to it the following well-known techniques from elementary functional analysis, due mainly to Gelfand.

The *dual* of a commutative Banach algebra A is the set $\widehat{A}$ of non-zero, continuous algebra homomorphisms from A into $\mathbb{C}$. It is a locally compact Hausdorff space in the topology of pointwise convergence and we denote by $C_0(\widehat{A})$ the algebra of continuous, complex-valued functions on $\widehat{A}$ which vanish at infinity. Equipped with the supremum norm it is a Banach algebra. The homomorphism

$$A \to C_0(\widehat{A})$$

which maps $a \in A$ to the function $\widehat{a}(\alpha) = \alpha(a)$ is called the *Gelfand transform*.

If A has a multiplicative unit then $\widehat{A}$ is a compact Hausdorff space. In this case we shall write $C(\widehat{A})$ in place of $C_0(\widehat{A})$: the subscript 0 indicates that we consider only functions which converge to 0 at infinity, and this condition is superfluous in the compact case. When A has a unit a simple but important result of Gelfand asserts that *an element* $a \in A$ *is invertible if and only if its Gelfand transform* $\widehat{a}$ *is invertible*. If we recall that the *spectrum* of $a \in A$ is the non-empty compact set

$$\text{Spectrum}_A(a) = \{\lambda \in \mathbb{C} \mid \lambda 1 - a \text{ is not invertible in } A\}$$

then Gelfand's result may be reformulated as follows:

1.1.7 THEOREM *If A is a commutative Banach algebra with unit then*

$$\text{Spectrum}_A(a) = \text{Spectrum}_{C(\widehat{A})}(\widehat{a}). \quad \square$$

1.1.8 REMARK The algebra $C(\widehat{A})$ is a Banach $*$-algebra; the involution maps a function $f(x)$ to its complex conjugate $\overline{f(x)}$. If A is also a Banach $*$-algebra, we may ask whether the Gelfand transform takes the involution on A to the involution on $C(\widehat{A})$. A homomorphism of Banach $*$-algebras which respects the involutions in this way is called a *$*$-homomorphism*. Equivalently, a homomorphism of Banach $*$-algebras is a $*$-homomorphism if and only if it maps selfadjoint elements to selfadjoint elements. But the selfadjoint elements of the $*$-algebra $C(\text{Spectrum}(T))$ are precisely the real-valued continuous functions. Thus we obtain from Theorem 1.1.7:

1.1.9 LEMMA *The Gelfand transform for a commutative unital Banach $*$-algebra A is a $*$-homomorphism if and only if every selfadjoint element of A has real spectrum.* □

There are examples of Banach $*$-algebras for which this condition is not satisfied. Exercise 1.9.2 asks the reader to construct one.

1.1.10 LEMMA *Let A be a commutative Banach algebra with unit. The Gelfand transform for A is an isometric homomorphism of Banach algebras if and only if $\|a^2\| = \|a\|^2$, for every $a \in A$.*

PROOF The *spectral radius* of an element $a \in A$ is the quantity

$$r_A(a) = \sup\{\,|\lambda| : \lambda \in \mathrm{Spectrum}_A(a)\,\}.$$

From Gelfand's Theorem 1.1.7 and the definition of the norm on $C(\widehat{A})$ it follows that $r_A(a) = \|\widehat{a}\|$. Now the *spectral radius formula,* proved in any first account of Banach algebra theory, asserts that

$$r_A(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}.$$

If $\|a^2\| = \|a\|^2$ then it is clear from the spectral radius formula that $r_A(a) = \|a\|$, and so $\|\widehat{a}\| = \|a\|$. Conversely, if the Gelfand transform is isometric then

$$\|a^2\| = \|\widehat{a}^2\| = \|\widehat{a}\|^2 = \|a\|^2. \quad \square$$

It follows from Lemmas 1.1.5 and 1.1.10 that if T is a normal operator then the Gelfand transform for $C^*(T)$ is isometric. We are going to prove the following much more precise result:

1.1.11 SPECTRAL THEOREM FOR NORMAL OPERATORS *If T is a normal Hilbert space operator then the map $\alpha \mapsto \alpha(T)$ is a homeomorphism from the dual of $C^*(T)$ onto* Spectrum(T), *and the Gelfand transform*

$$C^*(T) \to C(\mathrm{Spectrum}(T))$$

is an isometric $*$-isomorphism.

REMARK By Spectrum(T) we mean here the spectrum of T viewed as an element of the Banach algebra $\mathfrak{B}(H)$.

In view of Lemma 1.1.9, the following calculation will be an important step in our proof of Theorem 1.1.11:

1.1.12 LEMMA *If T is a selfadjoint Hilbert space operator then* Spectrum(T) *is a subset of $\mathbb{R}$.*

PROOF Let λ and μ be real numbers. The identity

$$\|(T - (\lambda + i\mu)I)v\|^2 = \|(T - \lambda I)v\|^2 + \mu^2\|v\|^2$$

is proved by expanding the norms as inner products. It shows that if $\mu \neq 0$ then both $T - (\lambda + i\mu)I$ and its adjoint $T - (\lambda - i\mu)I$ are bounded below. But it is easily checked that an operator is invertible if and only if both it and its adjoint are bounded below. □

PROOF OF THE SPECTRAL THEOREM Denote by $D^*(T)$ the smallest Banach $*$-algebra of operators on H which contains $C^*(T)$ and has the property that if $S \in D^*(T)$ and S is invertible in $\mathfrak{B}(H)$ then $S^{-1} \in D^*(T)$. It is easy to see that $D^*(T)$ is commutative and is composed entirely of normal operators. It has the property that if $S \in D^*(T)$ then the spectrum of S considered as an element of $D^*(T)$ is the same as the spectrum of S considered as an element of $\mathfrak{B}(H)$. In particular, the spectrum in $D^*(T)$ of every selfadjoint element is real, and so the Gelfand transform for $D^*(T)$ is a $*$-homomorphism.

A Banach algebra homomorphism $\alpha: D^*(T) \to \mathbb{C}$ (which, by the above, is automatically a $*$-homomorphism) is determined by its value on T, in the sense that if $\alpha_1(T) = \alpha_2(T)$ then $\alpha_1 = \alpha_2$. Indeed, the Banach $*$-algebra of operators on which α_1 and α_2 agree contains $C^*(T)$ and has the property that if an invertible operator S belongs to it then so does S^{-1}. The dual of $D^*(T)$ therefore identifies with a closed subset of Spectrum(T) via $\alpha \mapsto \alpha(T)$. It follows from Gelfand's Theorem 1.1.7 that this map is a homeomorphism onto Spectrum(T). By Lemmas 1.1.5 and 1.1.10 the Gelfand transform is isometric.

Let us consider now the range of the Gelfand transform. It is a closed, $*$-subalgebra of $C(\text{Spectrum}(T))$ and it contains all polynomial functions (because $\widehat{T}$ is the polynomial function $z(\lambda) = \lambda$). By the Stone–Weierstrass Theorem then, the range is all of $C(\text{Spectrum}(T))$.

We have now shown that the Gelfand transform for the commutative Banach $*$-algebra $D^*(T)$ is an isometric $*$-isomorphism. To complete our proof of the Spectral Theorem let us show that $C^*(T) = D^*(T)$. Since the Gelfand transform of $T \in D^*(T)$ is the polynomial function $z(\lambda) = \lambda$, and since z generates $C(\text{Spectrum}(T))$ as a Banach $*$-algebra with unit, it follows that T generates $D^*(T)$ as a Banach $*$-algebra with unit. But the Banach $*$-algebra with unit generated by T is precisely $C^*(T)$. □

1.1.13 DEFINITION Let T be a normal operator on a Hilbert space H and let $f \in C(\text{Spectrum}(T))$. Denote by $f(T)$ the corresponding element of $C^*(T)$. The $*$-homomorphism $C(\text{Spectrum}(T)) \to \mathfrak{B}(H)$ defined by $f \mapsto f(T)$ (the inverse of the Gelfand transform) is called the *functional calculus* for T.

1.1.14 REMARKS If $f(\lambda) = a_0 + a_1\lambda + \cdots + a_n\lambda^n$ then of course

$$f(T) = a_0 + a_1 T + \cdots + a_n T^n.$$

If $f(\lambda) = \lambda^\alpha$ then we will write T^α in place of $f(T)$, and we will also use similar notation for other functions. Notice that, since the function λ^α is typically defined only for $\lambda \geqslant 0$, whenever we form T^α we shall tacitly assume that the spectrum of T is contained in the positive reals. We shall say more about this assumption in the next section.

The spectrum of $f(T)$ is the image under f of the spectrum of T:

$$\mathrm{Spectrum}(f(T)) = f[\mathrm{Spectrum}(T)].$$

To see this, we return to the proof of the Spectral Theorem: the identity is a consequence of Gelfand's Theorem 1.1.7, applied to $D^*(T)$, together with the fact that the spectrum of $f(T)$ in $D^*(T)$ is equal to the spectrum of $f(T)$ in $\mathfrak{B}(H)$.

Various other useful properties of the functional calculus are mentioned in the exercises.

1.2 Positive Operators and the Strong Operator Topology

The most natural topology on $\mathfrak{B}(H)$ is the *norm topology*, in which a sequence $\{T_n\}$ converges to T if and only if $\|T_n - T\| \to 0$. However, it is frequently useful to consider other topologies on $\mathfrak{B}(H)$. Let us just mention one out of the several in regular use: the *strong operator topology* has the property that a sequence (or a net) $\{T_n\}$ converges to T in the strong operator topology if and only if $T_n v \to Tv$, for all $v \in H$. The terminology, which is standard, is a bit unfortunate since the strong topology is weaker (that is, it has fewer open sets) than the norm topology.

The space $\mathfrak{B}(H)$ has a useful completeness property in the strong topology, analogous to the fact that that every bounded monotone sequence in $\mathbb{R}$ has a limit. To state it we need the following:

1.2.1 DEFINITION AND LEMMA *Let T be a bounded selfadjoint operator. The following are equivalent:*

(a) *the spectrum of T is non-negative;*
(b) *T is of the form $T = S^*S$, for some S;*
(c) *$\langle v, Tv\rangle \geqslant 0$, for all $v \in H$.*

If any one of these holds then we say that T is positive, *and write $T \geqslant 0$.*

PROOF To prove (a) $\Rightarrow$ (b) we use the functional calculus to define $S = T^{\frac{1}{2}}$. To prove (b) $\Rightarrow$ (c) we simply note that

$$\langle v, S^*Sv\rangle = \langle Sv, Sv\rangle = \|Sv\|^2.$$

To prove that (c) $\Rightarrow$ (a) we use the functional calculus to write $T = T_+ - T_-$, where

$$\begin{aligned} T_+ &= f(T) \quad \text{for} \quad f(x) = \max\{x, 0\}, \\ T_- &= f(T) \quad \text{for} \quad f(x) = \max\{-x, 0\}. \end{aligned}$$

The selfadjoint operators $T_\pm$ have non-negative spectrum and $T_+T_- = 0$. If (c) holds then the formula

$$\|T_-^{\frac{3}{2}} w\|^2 = \langle T_-^{\frac{3}{2}} w, T_-^{\frac{3}{2}} w\rangle = \langle T_-^2 w, T_- w\rangle = -\langle TT_- w, T_- w\rangle \leqslant 0$$

shows that $T_-^{\frac{3}{2}} = 0$ and hence $T_- = 0$. Thus $T = T_+$, and so T has non-negative spectrum. □

1.2.2 Remark It is apparent from (c) above that the sum of two positive operators is positive, and that the product of a positive scalar and a positive operator is positive; moreover, if T is positive then so is S^*TS, for any S. We may define an ordering on the selfadjoint operators by saying that $T_1 \leqslant T_2$ if and only if $T_2 - T_1$ is positive. This ordering is a useful tool but it must be handled with care; see Exercise 1.9.12 for some interesting examples.

1.2.3 Proposition *If $\{T_n\}_{n=1}^\infty$ is a sequence of positive operators and if*

$$\sup_n \|T_1 + \cdots + T_n\| < \infty$$

then the infinite sum $\sum_j T_j$ converges in the strong operator topology.

Proof If $v \in H$ then the series $\sum_j \langle T_j v, v\rangle$ has non-negative terms and uniformly bounded partial sums, and is therefore convergent. The inequality

$$\|Tv\|^2 \leqslant \|T^{\frac{1}{2}}\|^2 \|T^{\frac{1}{2}} v\|^2 = \|T\| \langle Tv, v\rangle,$$

applied to $T = \sum_{j=M}^N T_j$, shows that the partial sums of $\sum_j T_j v$ form a Cauchy sequence in H. □

1.2.4 Example Here is a simpler instance of strong convergence. Let $\{T_1, T_2, \dots\}$ be a sequence of bounded operators on Hilbert spaces $\{H_1, H_2, \dots\}$, and suppose that $\sup_n \|T_n\| < \infty$. Then the formula

$$T(v_1, v_2, \dots) = (T_1 v_1, T_2 v_2, \dots)$$

defines a bounded operator on the Hilbert space direct sum $\oplus_{n=1}^\infty H_n$, called the *direct sum* $\oplus_{n=1}^\infty T_n$ of the operators T_n. The finite direct sums $\oplus_{n=1}^N T_n$, viewed as operators on the infinite direct sum $\oplus_{n=1}^\infty H_n$ in the obvious way, converge in the strong operator topology to $\oplus_{n=1}^\infty T_n$, as $N \to \infty$. There are similar definitions and assertions for *uncountable* families of operators $\{T_\alpha\}$.

1.3 C*-Algebras

1.3.1 DEFINITION A *C*-algebra* is a Banach $*$-algebra which is isometrically $*$-isomorphic to a norm-closed $*$-subalgebra of $\mathfrak{B}(H)$, for some Hilbert space H.

The C*-algebras we have just defined are sometimes referred to as 'concrete' C*-algebras, meaning that they can be concretely realized as algebras of operators on Hilbert space. There is also a notion of 'abstract' C*-algebra, in which there is no explicit reference to Hilbert space; we shall consider this in a little while. The main theorem of elementary C*-algebra theory asserts that the two notions coincide. We shall prove this later in the chapter.

1.3.2 EXAMPLE Let X be a compact Hausdorff space. Denote by $C(X)$ the collection of all complex-valued continuous functions on X, which is a Banach $*$-algebra with the supremum norm and complex-conjugation involution. If μ is any measure on X then $C(X)$ acts by multiplication operators on $H = L^2(X, \mu)$, and this representation gives a $*$-homomorphism $C(X) \to \mathfrak{B}(H)$, which is isometric provided that μ assigns non-zero measure to each non-empty open set. Thus, $C(X)$ is a C*-algebra. If X is only *locally* compact, then an analogous construction shows that the algebra $C_0(X)$ of continuous functions which tend to zero at infinity is a C*-algebra. Notice that $C_0(X)$ does not have a unit if X is not compact.

1.3.3 REMARK The same argument shows that $C_b(X)$, the algebra of *bounded* continuous functions on a locally compact Hausdorff space X, is a unital C*-algebra. This algebra is generally less useful than $C_0(X)$ (but see Exercise 1.9.9).

1.3.4 EXAMPLE Let H be a Hilbert space. A *rank-one operator* on H is an operator T of the form

$$T(\nu) = \langle \eta, \nu \rangle \xi$$

for some fixed vectors $\xi, \eta \in H$. The rank-one operators generate a subalgebra of $\mathfrak{B}(H)$ called the algebra of *finite-rank* operators, and the norm closure of the algebra of finite-rank operators is an important C*-algebra called the algebra of *compact* operators on H. It is denoted by $\mathfrak{K}(H)$. It is a fact that an operator T belongs to $\mathfrak{K}(H)$ if and only if it maps bounded subsets of H to precompact ones (that is, if and only if T is compact in the usual sense of functional analysis), but the characterization in terms of finite-rank operators will be more important for our purposes.

1.3.5 EXAMPLE Let G be a discrete group, and let $\ell^2(G)$ be the Hilbert space of square-summable complex functions on G. Each $g \in G$ defines a unitary operator U_g on $\ell^2(G)$, by way of the left translation action of G on itself; and these operators U_g generate a subalgebra of $\mathfrak{B}(\ell^2(G))$ isomorphic to the complex group algebra $\mathbb{C}[G]$. The norm closure of this copy of $\mathbb{C}[G]$ is a C*-subalgebra of $\mathfrak{B}(\ell^2(G))$ which is called the *reduced group* C*-*algebra* of G and is denoted by $C_r^*(G)$. We shall carry out a more detailed study of this algebra, and its connections with the harmonic analysis of G, in Chapter 3.

1.3.6 EXAMPLE If A is a C*-algebra of operators on a Hilbert space H, then $M_n(A)$, the algebra of $n \times n$ matrices over A, is a C*-algebra of operators on the direct sum of n copies of H.

C*-algebras comprise a very special class within the realm of all Banach algebras, in which algebra and functional analysis are particularly well integrated. The following result is a good illustration of this.

1.3.7 PROPOSITION *If A and B are C*-algebras, and if $\alpha: A \to B$ is a $*$-homomorphism, then α is continuous and in fact $\|\alpha\| \leqslant 1$. In particular, if $\alpha: A \to B$ is an algebraic $*$-isomorphism between two C*-algebras then α is automatically isometric.*

PROOF The C*-identity $\|a_1^* a_1\| = \|a_1\|^2$ is valid for any element a_1 in a C*-algebra (since it is valid for any Hilbert space operator). If we can show that $\|\alpha(a)\| \leqslant \|a\|$, for every selfadjoint element a, then since every element of the form $a = a_1^* a_1$ is selfadjoint we shall be able to conclude that

$$\|a_1\|^2 = \|a_1^* a_1\| \geqslant \|\alpha(a_1^* a_1)\| = \|\alpha(a_1)\|^2,$$

and we shall be done.

We may assume that A and B are C*-algebras of operators on two Hilbert spaces, and we may certainly represent A and B as algebras of operators in such a way that neither contains the identity operator. Denote by $\widetilde{A}$ the C*-algebra generated by A and the identity operator, and define $\widetilde{B}$ similarly. The C*-algebra $\widetilde{A}$ contains A as a two-sided ideal of linear codimension 1, and the same for $\widetilde{B}$, and any $*$-homomorphism $\alpha: A \to B$ extends in a unique way to a $*$-homomorphism $\alpha: \widetilde{A} \to \widetilde{B}$ which maps identity operator to identity operator. It is clear on purely algebraic grounds that if $a \in \widetilde{A}$ then the spectrum of $\alpha(a) \in \widetilde{B}$ is a subset of the spectrum of $a \in \widetilde{A}$. Hence the spectral radius of $\alpha(a)$ is bounded by the spectral radius of a. If a is selfadjoint then $\|a\|^2 = \|a^* a\| = \|a^2\|$, and so it follows from the proof of Lemma 1.1.10 that norm of a is equal to its spectral radius, and the same for $\alpha(a)$. Hence

$$\|a\| = r(a) \geqslant r(\alpha(a)) = \|\alpha(a)\|.$$

This completes the proof. $\square$

It follows from the proposition that the C*-algebra $\widetilde{A}$ constructed in the course of the proof is unique up to isometric $*$-isomorphism. Since the same device will be used a number of times we make the following formal definition.

1.3.8 DEFINITION If A is a C*-algebra then the *unitalization* of A is the unique (up to canonical isometric $*$-isomorphism) C*-algebra $\widetilde{A}$ with multiplicative unit which contains A as a closed, two-sided ideal of linear codimension one ($\widetilde{A}$ is thus spanned by A and the multiplicative unit of $\widetilde{A}$).

We shall use the abbreviated term *unital* to refer to C*-algebras with multiplicative units, and $*$-homomorphisms which preserve the units. We note[2] that even if A is unital, $A \neq \widetilde{A}$ and the unit of A is *not* the unit of $\widetilde{A}$.

1.3.9 PROPOSITION *Let* A *be a unital* C*-*subalgebra of* $\mathfrak{B}(H)$. *If* T *is any element in* A *then*

$$\text{Spectrum}_A(T) = \text{Spectrum}_{\mathfrak{B}(H)}(T).$$

PROOF Note first that $T - \lambda I$ is invertible (either in A or in $\mathfrak{B}(H)$) if and only if both of $(T-\lambda I)^*(T-\lambda I)$ and $(T-\lambda I)(T-\lambda I)^*$ are invertible. So it suffices to show that if a selfadjoint operator S is invertible in $\mathfrak{B}(H)$ then its inverse lies in A. But it follows from the Spectral Theorem that S^{-1}, if it exists, is a norm limit of polynomials in S. □

1.3.10 COROLLARY *If* $a \in A \subseteq B$, *where* A *and* B *are* C*-*algebras with the same unit, then* $\text{Spectrum}_A(a) = \text{Spectrum}_B(a)$. □

Because of the corollary, from here on we shall drop the subscript and just write Spectrum(a). Notice that this kind of 'spectral permanence' definitely does not hold for general Banach algebras; see Exercise 1.9.18.

Here is another immediate consequence of Proposition 1.3.9:

1.3.11 COROLLARY *The spectrum of a selfadjoint element in a unital* C*-*algebra is real.* □

Let us turn now to a characterization of *commutative* C*-algebras. From 1.3.2 we are familiar with some examples, namely the algebras $C_0(X)$ associated to locally compact Hausdorff spaces X. A celebrated theorem of Gelfand and Naimark shows that all commutative C*-algebras are of this kind:

1.3.12 THEOREM *If* A *is a commutative* C*-*algebra then the Gelfand transform is an isometric* $*$-*isomorphism from* A *onto* $C_0(\widehat{A})$.

PROOF By introducing the unitalization $\widetilde{A}$, if necessary, we may assume that A is unital. Every element of a commutative C*-algebra is normal (meaning, as before, that $a^*a = aa^*$). The argument in the proof of Lemma 1.1.5 therefore shows that $\|a^2\| = \|a\|^2$, and so it follows from Lemma 1.1.10 that the Gelfand transform is isometric. It follows from Corollary 1.3.11 that the Gelfand transform is a $*$-homomorphism, and then it follows from the Stone–Weierstrass Theorem that its image is all of $C_0(\widehat{A})$. □

[2]Some authors employ a different convention.

Let us concentrate for a moment on commutative and *unital* C*-algebras. If $\alpha\colon A \to B$ is a unital $*$-homomorphism between commutative and unital C*-algebras then there is an induced continuous map $\widehat{\alpha}\colon \widehat{B} \to \widehat{A}$ (given by composing algebra homomorphisms from B to $\mathbb{C}$ with α). Conversely, a continuous map from $\widehat{B}$ to $\widehat{A}$ obviously induces a $*$-homomorphism from $C(\widehat{A})$ to $C(\widehat{B})$. From here it is a short step to the following result, which indicates a close relation between C*-algebra theory and the topology of compact spaces:

1.3.13 THEOREM *The category of commutative, unital C*-algebras (and unital $*$-homomorphisms) is equivalent to the opposite of the category of compact Hausdorff spaces and continuous maps.* □

This theorem is the basis for the idea that C*-algebra theory might make useful contributions to topology, which is a theme throughout this book.

Using the device of unitalization, it is a simple matter to prove a version of Theorem 1.3.13 for non-unital C*-algebras. If X is a locally compact space and if $A = C_0(X)$ then $\widetilde{A} \cong C(\widetilde{X})$, where $\widetilde{X}$ denotes the one-point compactification of X (our convention is that if X is already compact then $\widetilde{X}$ is the disjoint union of X and an additional 'point at infinity'). If $B = C_0(Y)$ then $*$-homomorphisms from A to B correspond to unital $*$-homomorphisms from $\widetilde{A}$ to $\widetilde{B}$, which in turn correspond to basepoint-preserving continuous maps from $\widetilde{Y}$ to $\widetilde{X}$. If we define the category of locally compact Hausdorff topological spaces in such a way that the morphisms from X to Y are precisely the basepoint-preserving maps from $\widetilde{X}$ to $\widetilde{Y}$ then we obtain immediately the following result:

1.3.14 THEOREM *The category of all commutative C*-algebras and $*$-homomorphisms is equivalent to the opposite of the category of locally compact, Hausdorff topological spaces.* □

1.3.15 REMARK A morphism from X to Y, in our category of locally compact spaces, is a continuous and proper map from an open subset U of X to Y — the corresponding map $\widetilde{X} \to \widetilde{Y}$ sends the points of $X \setminus U$ to the point at infinity in $\widetilde{Y}$.

We conclude this section with a useful fact about non-commutative C*-algebras. It is a simple consequence of the observations we have just made in the commutative case.

1.3.16 LEMMA *If $\alpha\colon A \to B$ is an injective unital $*$-homomorphism between unital C*-algebras then α is isometric (and so, in particular, α has closed range).*

PROOF Thanks to the C*-identity, it suffices to prove that $\|\alpha(a)\| = \|a\|$ for every selfadjoint $a \in A$. By replacing A and B with $C^*(a)$ and $C^*(\alpha(a))$, if necessary, it suffices to prove the lemma in the case where A and B are commutative. But then $A \cong C(X)$ and $B \cong C(Y)$, and α is induced from a continuous and surjective map from Y to X. A simple direct calculation in this case completes the proof. □

1.4 The GNS Construction

1.4.1 DEFINITION A *representation* of a C*-algebra A is a $*$-homomorphism $\pi\colon A \to \mathfrak{B}(H)$. The Hilbert space H need not be separable (although usually — after this section — we shall assume that it is). We say that π is *non-degenerate* if $\pi[A]H$ is dense in H[3] (if A is unital, this amounts to saying that π is a unital $*$-homomorphism). We say that the representation is *faithful* if π is injective.

It should be clear what is meant by the direct sum of a (finite or infinite[4]) family of representations of A. Two representations $\pi\colon A \to \mathfrak{B}(H)$ and $\pi'\colon A \to \mathfrak{B}(H')$ are *unitarily equivalent* if there is a unitary operator $U\colon H \to H'$ such that $U\pi(a)U^* = \pi'(a)$, for all $a \in A$.

We shall need only a passing familiarity with representation theory, limited to the following important construction of representations from certain linear functionals.

1.4.2 DEFINITION A *state* on a unital C*-algebra A is a linear functional $\sigma\colon A \to \mathbb{C}$ such that $\sigma(1) = 1$ and $\sigma(a^*a) \geqslant 0$, for every $a \in A$.

There is also a definition for non-unital algebras, but let us concentrate here on the unital case.

1.4.3 EXAMPLE Let $\pi\colon A \to \mathfrak{B}(H)$ be a non-degenerate representation, and let $v \in H$ be a unit vector. Then the functional $a \mapsto \langle v, \pi(a)v\rangle$ is a state, called the *vector state* associated to the pair (π, v).

In fact, every state is a vector state:

1.4.4 THEOREM *Let σ be a state on a unital C*-algebra A. There is a representation $\pi\colon A \to \mathfrak{B}(H)$ and a unit vector $v \in H$ such that $\sigma(a) = \langle v, \pi(a)v\rangle$ for all $a \in A$, and such that the subspace $\pi[A]v$ is dense in H. The pair (π, v) is unique, up to unitary equivalence.*

PROOF We shall construct the representation $\pi\colon A \to \mathfrak{B}(H)$ by manufacturing the Hilbert space H from the vector space underlying A. The sesquilinear form

$$\langle a, b\rangle = \sigma(a^*b)$$

on A satisfies all the axioms for an inner product, except that $\langle a, a\rangle = 0$ need not imply $a = 0$. The remaining axioms are enough to prove the Cauchy–Schwarz inequality

[3] In fact, one can show that $\pi[A]H$ is always a *closed* subspace of H; this is a form of the Cohen Factorization Theorem (Exercise 1.9.17). Thus, in fact, π is non-degenerate iff $\pi(A)H = H$.

[4] For infinite direct sums, see our discussion in Example 1.2.4.

$$|\langle a, b\rangle|^2 \leqslant \langle a, a\rangle\langle b, b\rangle,$$

and it follows that the set

$$N = \{\, a \in A \mid \langle a, a\rangle = 0\,\},$$

is a vector subspace of A. The formula

$$\langle a + N, b + N\rangle = \langle a, b\rangle = \sigma(a^*b)$$

defines an inner product on the quotient space A/N and we denote by H the Hilbert space completion of A/N. Observe that N is not only a vector subspace of A but in fact a left ideal of A, thanks to the inequality[5] $b^*a^*ab \leqslant \|a\|^2 b^*b$, which implies that

$$\langle ab, ab\rangle \leqslant \|a\|^2\langle b, b\rangle.$$

So we can define, for $a \in A$, an operator $\pi(a)$ on A/N by $\pi(a)(b+N) = ab+N$. The operator $\pi(a)$ extends by continuity to a bounded operator on H, and we obtain a representation $\pi\colon A \to \mathfrak{B}(H)$. If $v = 1 + N$ then $\langle v, \pi(a)v\rangle = \sigma(a)$, as required by the existence part of the theorem.

If $\pi'\colon A \to \mathfrak{B}(H')$ is a representation and $v' \in H'$ is a unit vector such that $\langle v', \pi'(a)v'\rangle = \sigma(a)$ then the formula $\pi(a)v \mapsto \pi'(a)v'$ defines an isometry from the dense subspace $\pi[A]v \subset H$ into H'. This follows from the calculation

$$\|\pi(a)v\|^2 = \langle v, \pi(a^*a)v\rangle = \sigma(a^*a) = \langle v', \pi'(a^*a)v'\rangle = \|\pi'(a)v'\|^2.$$

If $\pi'[A]H'$ is dense in H' then the isometry extends to a unitary isomorphism $U\colon H \to H'$ such that $U\pi(a)U^* = \pi'(a)$, and $Uv = v'$; in other words U is a unitary equivalence from the pair (π, v) to (π', v'). $\square$

A pair consisting of a representation $\pi\colon A \to \mathfrak{B}(H)$ and a unit vector v such that $\pi[A]v$ is dense in H is called a *cyclic representation*, and v itself is called a *cyclic vector*. The result above, called the *GNS construction* (after Gelfand, Naimark and Segal), shows that there is a one-to-one correspondence between states of A and unitary equivalence classes of cyclic representations of A. This allows us to analyze *all* representations in terms of states, because we have the following

1.4.5 PROPOSITION *Every representation of a unital C^*-algebra is a (possibly infinite) direct sum of cyclic representations.*

The proof is an easy transfinite argument, for instance using Zorn's Lemma, which we leave to the reader. $\square$

[5] Here we are making use of the ordering on selfadjoint operators defined in 1.2.2. It is easy to see that $a^*a \leqslant \|a\|^2 1$, so

$$\|a\|^2 b^*b - b^*a^*ab = ((\|a\|^2 1 - a^*a)^{1/2} b)^*((\|a\|^2 1 - a^*a)^{1/2} b).$$

Hence $\|a\|^2 b^*b - b^*a^*ab \geqslant 0$, as required.

1.5 Representations of Commutative C*-Algebras

In this section we shall use Proposition 1.4.5 to learn a little about the representation theory of commutative C*-algebras.

Suppose that X is a compact metrizable space (the hypothesis of metrizability is made to avoid some technical issues in measure theory). By the Riesz representation theorem of elementary functional analysis, every state of the C*-algebra C(X) is of the form

$$\sigma(f) = \int_X f d\mu,$$

where μ is a Borel probability measure on X. If we apply the GNS construction to this state, then we obtain the Hilbert space $H = L^2(X, \mu)$ together with the representation of C(X) on it by pointwise multiplication. By Proposition 1.4.5 then, every representation of C(X) is unitarily equivalent to a direct sum of pointwise multiplication representations on Hilbert spaces $L^2(X, \mu)$.

When is the representation of C(X) on $L^2(X, \mu)$ unitarily equivalent to the representation of C(X) on $L^2(X, \mu')$? If $U: L^2(X, \mu) \to L^2(X, \mu')$ is a unitary equivalence, and if φ is the image under U of the constant function 1, then for every $f \in C(X)$,

$$\int_X f(x)\, d\mu = \int_X f(x)|\varphi(x)|^2\, d\mu'.$$

It follows that μ is absolutely continuous with respect to μ' (in other words, every μ'-null set is μ-null). Applying the same argument to U^{-1} we conclude that in addition μ' is absolutely continuous with respect to μ. Conversely, if μ and μ' are mutually absolutely continuous measures on X then pointwise multiplication by the square root of the Radon–Nikodym derivative is a unitary isomorphism between $L^2(X, \mu)$ and $L^2(X, \mu')$.

1.5.1 Example Let X be a compact subset of $\mathbb{C}$, and let π be a faithful and non-degenerate representation of C(X). Let $z \in C(X)$ denote the identity function $z(\lambda) = \lambda$. Then z is a normal element in C(X) with spectrum X, and so the operator $\pi(z) = T$ is normal with spectrum X. Conversely, if T is a normal operator with spectrum X then the unital C*-subalgebra of $\mathfrak{B}(H)$ generated by T is isomorphic to C(X). In short, a faithful and non-degenerate representation of C(X) is the same thing as a normal operator with spectrum X.

1.5.2 Definition Two bounded operators T and T′ on Hilbert spaces H and H′ are *unitarily equivalent* if there is a unitary operator $U: H' \to H$ such that $T = UT'U^*$.

This definition is so arranged that two normal operators are unitarily equivalent if and only if they have the same spectrum, say X, and the corresponding representations of C(X) are unitarily equivalent in our earlier sense.

1.5.3 Example On a finite-dimensional Hilbert space, two normal operators are unitarily equivalent if and only if they have the same eigenvalues, counting multiplicities. Thus every normal operator is unitarily equivalent to a more or less unique diagonal operator.

From Proposition 1.4.5 we conclude that:

1.5.4 Proposition *Every normal Hilbert space operator is unitarily equivalent to a direct sum of multiplication operators* M *of the form* $M\varphi(\lambda) = \lambda\varphi(\lambda)$ *on Hilbert spaces* $L^2(X, \mu)$, $X \subseteq \mathbb{C}$. □

Let us say that a normal operator with spectrum X is *cyclic* if the corresponding representation of $C(X)$ is cyclic; so that then the operator is unitarily equivalent to a single multiplication operator $M\varphi(\lambda) = \lambda\varphi(\lambda)$ on some $X \subseteq \mathbb{C}$. It follows that the classification, up to unitary equivalence, of cyclic normal operators with spectrum X is the same as the classification of fully supported probability measures on X up to the equivalence relation of mutual absolute continuity. Thus the classification problem for normal operators involves, at its core, some rather subtle issues in measure theory. We mention this because in the next chapter we shall take a look at an apparently similar classification problem which turns out to have a totally different solution, rooted in algebraic topology.

The measure-theoretic description of representations of $C(X)$ allows us to extend the functional calculus defined in 1.1.13 in the following important way:

1.5.5 Definition If X is a locally compact separable metrizable space then denote by $B(X)$ the C^*-algebra of bounded Borel functions on X (with the supremum norm).

1.5.6 Proposition *Let* T *be a normal Hilbert space operator. The functional calculus homomorphism* $f \mapsto f(T)$, *from* $C(\mathrm{Spectrum}(T))$ *to* $\mathfrak{B}(H)$, *extends to a* C^**-homomorphism from* $B(\mathrm{Spectrum}(T))$ *to* $\mathfrak{B}(H)$.

Proof Assume without loss of generality that T is a multiplication operator by x on $L^2(X, \mu)$, where $X = \mathrm{Spectrum}(T)$. Then for any bounded Borel function f we may define $f(T)$ to be the operator of multiplication by $f(x)$. It is clear that this produces an operator with the required properties. □

1.5.7 Remark The same argument shows that any representation of a commutative C^*-algebra $C_0(X)$ (X being locally compact Hausdorff and second countable) extends to a representation of $B(X)$.

The above *Borel functional calculus* is the unique extension of the continuous functional calculus which has the property that if the sequence[6] $\{f_n\}_{n=1}^{\infty}$ of functions converges pointwise to f, and if $\{f_n\}$ is uniformly bounded, then

[6] It is important here that we work with *sequences* rather than more general nets; compare the Dominated Convergence Theorem.

$f_n(T) \to f(T)$ in the strong operator topology. This continuity property is a simple consequence of the Dominated Convergence Theorem.

1.6 Abstract C*-Algebras

As we have already mentioned, there is a way to define C*-algebras which makes no mention of Hilbert spaces.

1.6.1 DEFINITION An *abstract* C*-*algebra* is a Banach $*$-algebra A such that $\|a^*a\| = \|a\|^2$, for every $a \in A$.

Since the C*-identity $\|T^*T\| = \|T\|^2$ holds for Hilbert space operators, it is clear that every C*-algebra (as we defined the term) is an abstract C*-algebra. The main theorem in elementary C*-algebra theory is the converse of this:

1.6.2 GELFAND–NAIMARK REPRESENTATION THEOREM *Every abstract* C*-*algebra is isometrically* $*$-*isomorphic to a* C*-*subalgebra of* $\mathfrak{B}(H)$, *for some Hilbert space* H.

Here is a sketch of the proof. If A is a non-unital abstract C*-algebra then it is a simple matter to construct a unitalization $\widetilde{A}$ which is an abstract C*-algebra.[7] So there is no harm in restricting to the unital case. The first step in the proof is to repeat all of the arguments in Sections 1.3 and 1.4, but for abstract C*-algebras. The GNS construction will then provide a means of representing abstract C*-algebras as operators on a Hilbert space. The only new idea needed here is a proof that selfadjoint elements in an abstract C*-algebra have real spectrum: our argument in Section 1.3 depended on the corresponding result for selfadjoint Hilbert space operators which we proved in Section 1.1. A fairly simple proof which works for abstract C*-algebras is sketched in the exercises.

The next step in the proof of Theorem 1.6.2 is to show that an abstract C*-algebra A has 'sufficiently many' states, from which a large enough direct sum of GNS representations may be constructed to represent A faithfully on a Hilbert space. Here we encounter a more difficult problem:

1.6.3 LEMMA *If* A *is an abstract* C*-*algebra then every element in* A *of the form* a^*a *is positive (that is, its spectrum is non-negative). Moreover, the sum of two positive elements is positive.* □

The proof is surprisingly delicate, and we shall defer it to the exercises. Note that the lemma is certainly a simple consequence of Theorem 1.6.2, so it seems that we cannot avoid proving it. However, the positivity of a^*a is easily verified directly in all the abstract C*-algebras the reader will encounter in this book (or

[7] To construct $\widetilde{A}$, embed A as an algebra of left-multiplication operators on the Banach space A. One can then define $\widetilde{A}$ as the Banach algebra of operators generated by A and the identity. This gives the norm on $\widetilde{A}$ — verification of the C*-identity is left to the reader.

elsewhere), so it is not unreasonable to adopt it as an extra *axiom* for abstract C^*-algebras. The early researchers in this area followed this course, before a proof of Lemma 1.6.3 was discovered, and the reader will come to no harm by doing the same.

One way or the other, let us grant ourselves the lemma and proceed.

1.6.4 LEMMA *If* a *is a selfadjoint element in a unital abstract* C^**-algebra* A *then*

$$\|a\| = \sup\{ |\sigma(a)| : \sigma \text{ is a state of } A\}.$$

Every linear functional on A *is a linear combination of states.*

PROOF If A is *commutative* then $A \cong C(X)$, for some compact space X. The identity $\|a\| = \sup_\sigma\{\sigma(a)\}$ is thus clear, since for each $a \in C(X)$ there is an evaluation homomorphism $\sigma\colon C(X) \to \mathbb{C}$ such that $\sigma(a) = \|a\|$, and every such homomorphism is a state. In the non-commutative case, given a selfadjoint element $a \in A$, there is a state σ of the commutative C^*-algebra $C^*(a)$ such that $\sigma(a) = \|a\|$. Note that σ, chosen as above, has norm one (indeed, it is easy to show that every state has norm one). Extend σ to a linear functional on A, still of norm one (this is possible thanks to the Hahn–Banach Theorem). Replace $\sigma(a)$ with $\frac{1}{2}(\sigma(a) + \overline{\sigma(a^*)})$, if necessary, to obtain a linear functional which is real-valued on selfadjoint elements. We claim that it is a state on A. Suppose that $\|a\| \leqslant 1$. Then $\|a^*a\| \leqslant 1$ and $\|1 - a^*a\| \leqslant 1$ (it is here that we use the difficult result $a^*a \geqslant 0$) so

$$|1 - \sigma(a^*a)| = |\sigma(1 - a^*a)| \leqslant 1.$$

It follows that $\sigma(a^*a) \geqslant 0$.

The proof that every bounded linear functional is a linear combination of states is another application of the Hahn–Banach Theorem, and is omitted. □

To complete the proof of the Gelfand–Naimark Theorem, we proceed as follows. For each $a \in A$ let σ_a be a state such that $\sigma_a(a^*a) = \|a^*a\|$. Let $\pi_a\colon A \to \mathfrak{B}(H_a)$ be the corresponding GNS representation and form the (very large) direct sum representation $\pi = \bigoplus_{a\in A} \pi_a$ on the (very large) Hilbert space $H = \bigoplus_{a\in A} H_a$. If $b \in A$ then the norm of $\pi(b^*b)$ is $\sup_{a\in A} \|\pi_a(b^*b)\|$, and so

$$\|\pi(b)\|^2 = \|\pi(b^*b)\| = \sup_{a\in A} \|\pi_a(b^*b)\| \geqslant \|\pi_{b^*b}(b^*b)\| = \|b^*b\| = \|b\|^2.$$

Thus π is an isometric $*$-homomorphism from A into $\mathfrak{B}(H)$, as required.

1.7 Ideals and Quotients

By an *ideal* in a C^*-algebra we shall always mean a closed, two-sided ideal J such that[8] $a \in J \Leftrightarrow a^* \in J$. Observe that J is, in particular, a C^*-algebra in its

[8] Actually this last condition is implied by the preceding ones.

own right.

It is unlikely that an ideal in a C*-algebra will have a unit, and it is often awkward to replace J by the C*-algebra $\widetilde{J}$ obtained by adding a unit (since $\widetilde{J}$ will no longer be an ideal), so we often must cope with the absence of a unit in a different way:

1.7.1 DEFINITION An *approximate unit* for a C*-algebra A is a net $\{u_\alpha\}$ of positive elements in A, each of norm less than or equal to one, such that

(a) $u_\alpha \geqslant u_\beta$ if $\alpha \geqslant \beta$, and

(b) for every $a \in A$, $\lim_{\alpha\to\infty} \|au_\alpha - a\| = 0$.

The following result is proved in the exercises:

1.7.2 PROPOSITION *Every C*-algebra possesses an approximate unit. If A is separable then there is a sequential approximate unit* $\{u_n\}_{n=1}^{\infty}$. □

In almost any concrete example, it is easy to write down an approximate unit, without recourse to the proposition. Let us give one instance of this.

1.7.3 EXAMPLE If $A = C_0(X)$ (and X is second countable, for simplicity) then we can form an approximate unit by starting with open sets $U_1, U_2, \ldots$, such that

(a) $\bigcup U_n = X$, and

(b) $\overline{U_n}$ is compact and contained in U_{n+1}.

Take u_n to be a continuous, $[0,1]$-valued function which is supported in U_{n+1} and identically 1 on U_n.

1.7.4 THEOREM *If J is a closed ideal in a C*-algebra then the *-algebra A/J is a C*-algebra in the quotient norm.*

PROOF Let $\pi\colon A \to A/J$ be the canonical projection and let $\{u_\alpha\}$ be an approximate unit for J. We shall first obtain the formula

$$\|\pi(a)\| = \inf_\alpha \|a(1-u_\alpha)\|. \tag{1.7.5}$$

If $a \in A$ and $b \in J$ then we note that

$$\inf_\alpha \|a(1-u_\alpha)\| = \inf_\alpha \|(a+b)(1-u_\alpha)\|,$$

since $b(1-u_\alpha) \to 0$. It follows that

$$\inf_\alpha \|a(1-u_\alpha)\| = \inf_\alpha \|(a+b)(1-u_\alpha)\| \leqslant \inf_\alpha \|a+b\|\|1-u_\alpha\| \leqslant \|a+b\|,$$

for every $b \in J$. Taking the infimum over $b \in J$ we obtain

$$\inf_\alpha \|a(1-u_\alpha)\| \leqslant \|\pi(a)\|.$$

Conversely, we may write $a(1-u_\alpha) = a+b$, where $b = -au_\alpha$, and so

$$\|a(1-u_\alpha)\| \geqslant \inf_{b\in J} \|a+b\| = \|\pi(a)\|.$$

Hence $\inf_\alpha \|a(1-u_\alpha)\| \geqslant \|\pi(a)\|$, and the formula (1.7.5) follows.

Now (1.7.5) implies that the quotient norm is a C^*-norm:

$$\|\pi(a^*)\pi(a)\| = \inf_\alpha \|a^*a(1-u_\alpha)\| \geqslant \inf_\alpha \|(1-u_\alpha)a^*a(1-u_\alpha)\| = \|\pi(a)\|^2.$$

This implies that A/J is an (abstract) C^*-algebra. □

1.7.6 REMARK Note that if we assume the 'axiom' $a^*a \geqslant 0$ for A then the same property for the abstract C^*-algebra A/J is an immediate consequence (see the remarks following Lemma 1.6.3).

1.7.7 COROLLARY *The range of any $*$-homomorphism $\alpha\colon A \to B$ is norm-closed.*

PROOF By adjoining units, if necessary, we may assume that α is a unital $*$-homomorphism between unital C^*-algebras. Now let J be the kernel of α, factor α through the quotient C^*-algebra A/J, and apply Lemma 1.3.16. □

1.8 Unbounded Operators

The results of Section 1.5 tell us that every bounded, selfadjoint operator is unitarily equivalent to a direct sum of multiplication operators $M\varphi(\lambda) = \lambda\varphi(\lambda)$ on Hilbert spaces $L^2(X,\mu)$, where X is a compact subset of $\mathbb{R}$.

In this book we shall apply the techniques of operator theory to the study of certain differential operators arising from geometry. It is notorious, however, that differential operators are not bounded, or even fully defined, on L^2 spaces. For instance, if f is an L^2 function on the real line, then the function df/dx is not defined in general; and even if f is differentiable, one cannot estimate the L^2 norm of df/dx in terms of the L^2 norm of f. Despite these problems, the classical theory of the Fourier transform tells us that the operator d/dx *is* unitarily equivalent to a multiplication operator — namely, the multiplication operator by the unbounded function $\lambda(\xi) = -i\xi$ on $L^2(\mathbb{R},\mu)$, where μ is a multiple of Lebesgue measure. In this section we shall investigate conditions under which a general unbounded operator is similar to an 'unbounded multiplication' of this sort. The techniques of this section will not be needed until Chapter 10 of the book.

Let H be a Hilbert space. An *unbounded operator* on H is a linear map T from a subspace $\mathrm{Domain}(T) \subseteq H$ (the *domain* of T) to H; usually we require $\mathrm{Domain}(T)$ to be dense, in which case T is said to be *densely defined.* If T and T' are unbounded operators for which $\mathrm{Domain}(T) \subseteq \mathrm{Domain}(T')$ and $Tx = T'x$

for $x \in \text{Domain}(T)$, then one writes $T \subseteq T'$ and says that T' is an *extension* of T. The *graph* of T is the subspace $\{(v, Tv) : v \in \text{Domain}(T)\}$ of $H \times H$; the operator T is *closed* if its graph is closed, and *closable* if it has a closed extension. If T is closed and has domain all of H, then it is bounded, by the Closed Graph Theorem of elementary functional analysis.

Any closable operator T has a *closure* $\overline{T} \supseteq T$, which is the operator whose graph is the closure of the graph of T. Usually we shall omit the overline, unless some crucial issue turns on the distinction between an operator and its closure; the jargon is that we have extended T ('by continuity') to its *minimal domain*:

1.8.1 LEMMA *Let T be a closable operator on a Hilbert space H. Then $u \in H$ belongs to the domain of the closure of T (that is, to the minimal domain of T) if and only if there is a sequence $\{u_j\}_{j=1}^{\infty}$ in the domain of T such that $u_j \to u$ while $\|Tu_j\|$ remains bounded.*

PROOF Suppose that there is such a sequence. Since $\{Tu_j\}$ is bounded, we may use a diagonal procedure to extract a subsequence $\{u_{j_k}\}$ for which $\{Tu_{j_k}\}$ is weakly convergent; that is, $\langle Tu_{j_k}, v\rangle$ is convergent for any v. Now it is a simple consequence of the Hahn–Banach theorem that the weak and norm closures of any *convex* subset of H are the same; hence, by replacing the u_{j_k} by suitable convex combinations w_k, we may find a sequence such that $\{Tw_k\}$ converges in norm; and of course $w_k \to u$. Now the sequence of points (w_k, Tw_k) belongs to the graph of T and converges to a point whose first coordinate is u; so u belongs to the minimal domain. $\square$

1.8.2 DEFINITION Let T be an unbounded, densely defined operator. The *adjoint* of T is the unbounded operator T^* defined by the formula

$$\langle T^*u, v\rangle = \langle u, Tv\rangle.$$

To be precise, u belongs to $\text{Domain}(T^*)$ if there is a constant C such that $|\langle u, Tv\rangle| \leqslant C\|v\|$ for all $v \in \text{Domain}(T)$. Since $\text{Domain}(T)$ is dense this implies that $v \mapsto \langle u, Tv\rangle$ extends uniquely to a continuous linear functional φ on H, and by a standard result on Hilbert space there must be a unique $w \in H$ such that $\varphi(v) = \langle w, v\rangle$ for all u. This w is, by definition, the element T^*u.

It is easy to see that the operator T^* is always closed.

1.8.3 DEFINITION An operator T is *symmetric* if $T \subseteq T^*$, that is $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in \text{Domain}(T)$. The operator T is *selfadjoint* if $T = T^*$. In other words, T is selfadjoint if it is symmetric and if, in addition, $\text{Domain}(T) = \text{Domain}(T^*)$.

If T is a symmetric operator then it is closable and $T \subseteq \overline{T} \subseteq T^*$. The domain of T^* is sometimes called the *maximal domain* of T, and if it coincides with the minimal domain (the domain of $\overline{T}$) we say that T is *essentially selfadjoint*. In other words, a symmetric operator is essentially selfadjoint if it has one and only one selfadjoint extension.

Let T be a closed operator. We shall equip Domain(T) with the *graph norm*, that is, the norm $\|\cdot\|_T$ defined by

$$\|u\|_T^2 = \|u\|^2 + \|Tu\|^2.$$

Notice that Domain(T) is complete in this norm (it is isometric to the graph of T, which is closed in $H \times H$). We define the *spectrum* of a closed operator T to be the complement of the set of $\lambda \in \mathbb{C}$ such that the bounded linear map $(T-\lambda I)\colon \text{Domain}(T) \to H$ has a two-sided inverse.[9] The next result explains our interest in selfadjoint, rather than just symmetric, operators:

1.8.4 LEMMA *The spectrum of a selfadjoint operator is real.*

PROOF We can use much the same argument as in the bounded case (Lemma 1.1.12) but it is worth checking to see where the full force of the selfadjointness condition is required. The identity

$$\|(T-(\lambda+i\mu)I)v\|^2 = \|(T-\lambda I)v\|^2 + \mu^2\|v\|^2 \tag{1.8.5}$$

still holds for all $v \in \text{Domain}(T)$: expand the left side as

$$\langle (T-\lambda I)v, (T-\lambda I)v\rangle + \langle i\mu v, i\mu v\rangle - \langle (T-\lambda I)v, i\mu v\rangle - \langle i\mu v, (T-\lambda I)v\rangle$$

and use selfadjointness to cancel the last two terms. Equation 1.8.5 shows that $S = T - (\lambda + \mu i)I$ is injective and has closed range whenever $\mu \neq 0$. Now suppose that $v \in H$ is orthogonal to the range of S. Then, by definition of the adjoint, v belongs to the domain of S^* and $S^*v = 0$. But since T is selfadjoint, $S^* = T-(\lambda-\mu i)I$ is also injective, and it follows that S is onto. (If T were merely symmetric, we could not make this step.) We have shown S is a bijection from Domain(T) to H, so it has a set-theoretic inverse S^{-1}. Finally, Equation 1.8.5 shows that S^{-1} is a bounded operator, with norm at most $1/|\mu|$. □

Now let T be a selfadjoint operator. The identity

$$\|Tv + iv\|^2 = \|Tv\|^2 + \|v\|^2 = \|Tv - iv\|^2$$

(which is a special case of 1.8.5 above) shows that the invertible operators $T \pm i$ are actually isometries of Domain(T) (with its graph norm) onto H. Therefore

$$U = (T - iI)(T + iI)^{-1}$$

is an invertible isometric map — a unitary — from H onto itself. It is called the *Cayley transform* of T. By construction, the operator $I - U = 2i(T + iI)^{-1}$ is

[9] By the Closed Graph Theorem such an inverse is automatically a bounded operator from H to Domain(T), and so in particular is a bounded operator on H.

injective and has dense range; conversely, any unitary U with this property is the Cayley transform of a unique unbounded selfadjoint operator T, which may be recovered from U by

$$T = i(I+U)(I-U)^{-1}, \quad \text{Domain}(T) = \text{Image}(I-U).$$

The main theorem on unbounded selfadjoint operators is now a simple deduction:

1.8.6 PROPOSITION *An unbounded, selfadjoint operator T is unitarily equivalent to a direct sum of multiplication operators, each of which is of the form $M\varphi(\lambda) = \lambda\varphi(\lambda)$ on $L^2(\mathbb{R}, \mu)$ for some measure μ on $\mathbb{R}$.*

PROOF By Proposition 1.5.4 the Cayley transform U of T is unitarily equivalent to a direct sum of multiplication operators $N\psi(z) = z\psi(z)$ on $L^2(S^1, \nu)$, for some measure ν on S^1. Suppose for simplicity that U is equivalent to a single such operator. Since $\ker(U-I) = 0$, the set $\{1\}$ has ν-measure zero; so we may regard N as acting on $L^2(S^1 \setminus \{1\}, \nu)$. The homeomorphism $g\colon S^1 \setminus \{1\} \to \mathbb{R}$ given by $g(z) = i(1+z)/(1-z)$ then identifies $L^2(S^1, \nu)$ with $L^2(\mathbb{R}, \mu)$, where $|g'(z)|^{\frac{1}{2}} d\mu(g(z)) = d\nu(z)$. Thus the operator $T = i(I+U)(I-U)^{-1}$ can be thought of as acting on $L^2(\mathbb{R}, \mu)$ as a multiplication operator, and it is trivial to check that in fact it acts by multiplication by λ. □

As before we can use this to define a functional calculus: for an operator T unitarily equivalent to multiplication by λ on $L^2(\mathbb{R}, \mu)$, and for a bounded Borel function f on $\text{Spectrum}(T) = \text{Support}(\mu)$, we define $f(T)$ to be the operator that is unitarily equivalent to multiplication by $f(\lambda)$. The mapping $f \mapsto f(T)$ is a $*$-homomorphism from $B(\text{Spectrum}(T))$ to $\mathfrak{B}(H)$, which agrees with the 'natural' definition when f is a rational function whose poles are not on the real axis. Moreover, if $\{f_n\}$ is a uniformly bounded sequence of Borel functions and $f_n \to f$ pointwise, then $f_n(T) \to f(T)$ in the strong operator topology. The proof follows from the Dominated Convergence Theorem, just as before.

1.9 Exercises

1.9.1 Let A be a unital Banach algebra. Show that if $a \in A$ and $\|a\| < 1$, then $1 + a$ is invertible (use power series). Deduce that the group $G(A)$ of invertible elements of A is an open subset of A.

Show also that if Ω is an open subset of $\mathbb{C}$ then the set of $a \in A$ whose spectrum lies within Ω is an open subset of A.

1.9.2 Let A be the *disk algebra*, that is the Banach algebra (with pointwise operations and supremum norm) of continuous functions on the closed unit disk

$\overline{\mathbb{D}}$ in the complex plane which are holomorphic in the interior of $\mathbb{D}$. Show that the operation

$$f^*(z) = \overline{f(\bar{z})}$$

defines an involution on A which makes it into a commutative Banach $*$-algebra in which not every selfadjoint element has real spectrum.

1.9.3 Let f be a continuous function on $\mathbb{C}$. Show that if $\{T_n\}$ is a norm-convergent sequence of normal operators on a Hilbert space then $\lim_{n\to\infty} f(T_n)$ is equal to $f(\lim_{n\to\infty} T_n)$. Does the analogous statement hold good if we replace the norm topology by the strong topology?

1.9.4 Let T be a normal operator and let $f\colon \text{Spectrum}(T) \to \mathbb{C}$ be a continuous function. Show that if f is $\{0,1\}$-valued then $f(T)$ is a projection. Similarly show that if f is real-valued, then $f(T)$ is selfadjoint; if f has absolute value 1, then $f(T)$ is unitary; and if $f \geqslant 0$, then $f(T)$ is positive.

1.9.5 Show that if λ is an isolated point in the spectrum of a normal operator T then λ is an eigenvalue of T. Is this true if T is not normal?

1.9.6 Show that the operation of composition, $\mathfrak{B}(H) \times \mathfrak{B}(H) \to \mathfrak{B}(H)$, is *not* continuous in the strong operator topology; but show that its restriction to bounded subsets of $\mathfrak{B}(H)$ *is* continuous in this topology. Are the sets of invertible or unitary operators on a Hilbert space topological groups in the strong topology?

1.9.7 Let H be a Hilbert space. Recall that the *weak topology* on H is the weakest topology which makes all the maps $\varphi_w\colon H \to \mathbb{C}$, $v \mapsto \langle w, v\rangle$, continuous. It is a standard result of functional analysis that the unit ball of H is *compact* in the weak topology.

Let $T \in \mathfrak{B}(H)$ be a compact selfadjoint operator. Show that the quadratic function $v \mapsto \langle Tv, v\rangle$ is weakly continuous on the unit ball of H, and that it attains an extremal value at an eigenvector of H. Hence prove that H has an orthonormal basis consisting of eigenvectors for H (this is the *spectral theorem* for compact selfadjoint operators).

Deduce that if $A \subseteq \mathfrak{B}(H)$ is a commutative C^*-algebra of compact operators, then there is an orthonormal basis of H with respect to which all the elements of A can be simultaneously diagonalized.

1.9.8 Let X be a locally compact Hausdorff space. Prove that X is second countable if and only if $C_0(X)$ is separable. Conclude that the category of separable and commutative C^*-algebras is equivalent to the opposite of the category of locally compact, second countable, Hausdorff topological spaces.

1.9.9 Let X be a locally compact Hausdorff space, and let $C_b(X)$ denote the algebra of bounded, continuous, complex-valued functions on X. Show that $C_b(X)$

is a commutative C*-algebra. Deduce that $C_b(X) = C(Z)$, where Z is a compact Hausdorff space containing X as a dense open subset (a *compactification* of X). Prove moreover that Z is the universal compactification, in the sense that if Z' is another compactification of X then there is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Z \arrow[d] \\ & Z' \end{tikzcd}$$

where the vertical arrow is a uniquely determined surjection. (The space Z is called the *Stone–Čech compactification* of X.)

1.9.10 Let $A = C(X)$ be a commutative and unital C*-algebra. If Z is a closed subset of X then the set of those functions in A whose restriction to Z is zero constitutes an ideal in A. Prove that every ideal arises in this way.

1.9.11 The following argument proves that selfadjoint elements in abstract C*-algebras have real spectrum. Let a be a selfadjoint element of a unital abstract C*-algebra A. Using the C*-identity, prove that

$$\|a \pm i\nu\|^2 \leqslant \|a\|^2 + \nu^2$$

for any $\nu > 0$. Deduce that the spectrum of a is contained in the set

$$S_\nu := \{z \in \mathbb{C} : |z \pm i\nu|^2 \leqslant \|a\|^2 + \nu^2\}.$$

Now show that $\bigcap_\nu S_\nu$ is the interval $[-\|a\|, \|a\|]$ of the real axis, and deduce that a has real spectrum.

1.9.12 Let x and y be positive elements of a unital C*-algebra A.

(a) Prove that if x, y are invertible and $x \leqslant y$, then $y^{-1} \leqslant x^{-1}$.

(b) Prove that if $x \leqslant y$ then $x^{\frac{1}{2}} \leqslant y^{\frac{1}{2}}$.

(c) Is it true that, whenever x and y are positive with $x \leqslant y$, we also have $x^2 \leqslant y^2$? (Hint: consider 2×2 matrices.)

(In fact, the condition (c) above holds for all positive elements $x, y \in A$ only if A is commutative. See [104, Section 1.3] for more information.)

1.9.13 Let B be a unital C*-algebra, and let $T \in M_n(B)$ be selfadjoint. Prove that T is positive if and only if

$$b \cdot T \cdot b^*$$

is a positive element of B for every $1 \times n$ row vector b of elements of B.

1.9.14 What is wrong with the following simple 'proof' of the fact that in an abstract C*-algebra the spectrum of a^*a is a subset of $[0, \infty)$? Form the C*-algebra of 2×2-matrices with entries in A. The matrix

$$\begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$$

is a selfadjoint element in this C*-algebra, hence its spectrum is contained in $\mathbb{R}$. But the square of this matrix is

$$\begin{pmatrix} aa^* & 0 \\ 0 & a^*a \end{pmatrix},$$

and, being the square of a selfadjoint element, its spectrum is contained in $[0, \infty)$. It follows that the spectrum of a^*a (and of aa^*) is contained in $[0, \infty)$.

1.9.15 Here is a correct way to prove that $a^*a \geqslant 0$ in an abstract C*-algebra.

(a) The set of positive elements is closed under addition. (Hint: a selfadjoint element x is positive if and only if $\| \|x\| 1 - x\| \leqslant \|x\|$)

(b) Let $b = (a^*a)_-$ be the negative part of the selfadjoint element a^*a: our objective is to show that $b = 0$. Write $e = ba^* = c + id$, where c and d are selfadjoint, and calculate:

$$-e^*e = b^3 \geqslant 0 \quad \text{and} \quad ee^* = 2c^2 + 2d^2 - e^*e \geqslant 0.$$

(c) In any unital algebra, $\mathrm{Spectrum}(xy) = \mathrm{Spectrum}(yx)$, with the possible exception that 0 may be in one spectrum but not the other. Hence the spectrum of e^*e is $\{0\}$, and so $e^*e = b^3 = 0$, and $b = 0$.

1.9.16 Show that every C*-algebra has an approximate unit, as follows. First show that if $a \geqslant b \geqslant 0$ then

$$\lim_{n \to \infty} b \left(b + \tfrac{1}{n}\right)^{-1} a = a.$$

Now, for simplicity, assume that A is separable, and let $a_1, a_2, \ldots$ be a sequence of positive elements of norm 1 or less whose linear span is dense in A. Let $u = \sum_j 2^{-j} a_j$ and let $u_n = u(u + \frac{1}{n})^{-1}$. Show that $\{u_n\}$ is an approximate unit.

1.9.17 Let A be a C*-algebra.

(a) Let $x, a \in A$ with $a \geqslant xx^* \geqslant 0$. Show that the elements

$$\left[\tfrac{1}{j} + a\right]^{-\frac{1}{2}} a^{\frac{1}{4}} x, \qquad j = 1, 2, \ldots$$

form a Cauchy sequence in A which converges to an element $b \in A$ with $x = a^{\frac{1}{4}} b$ and $\|a\|^{\frac{1}{4}} \geqslant \|b\|$.

Now let $\pi\colon A \to \mathfrak{B}(H)$ be a representation.

(b) Show that for any $v \in \overline{\pi[A]H}$ and any $\varepsilon > 0$, one can find x in the unit ball of A such that $\|v - \pi(x)v\| < \varepsilon$. (Use an approximate unit.)

(c) Given $v \in \overline{\pi[A]H}$, use part (b) above to define by induction a sequence $\{x_n\}$ of elements in the unit ball of A and a sequence $\{v_n\}$ of elements of $\overline{\pi[A]H}$ such that $v_1 = v$ and

$$v_n = v_{n-1} - \pi(x_{n-1})v_{n-1}, \quad \|v_n\| \leqslant 4^{-n}.$$

for $n > 1$.

(d) Show that $\sum \pi(x_n)v_n$ converges to v.

(e) Let $a = \sum 4^{-n} x_n x_n^*$. Using part (a) above write $2^{-n}x_n = a^{\frac{1}{4}} b_n$ with $\|b_n\|$ bounded. Show that

$$v = \pi(a^{\frac{1}{4}}) \sum \pi(b_n)\,(2^n v_n)$$

where the series converges in norm, and deduce that $v \in \pi[A]H$.

Thus $\pi[A]H$ is always closed in H. In particular, if π is a non-degenerate representation, then $\pi[A]H = H$.

1.9.18 Let $A \subseteq B$ be an inclusion of unital Banach algebras.

(a) Consider the set $G(A)$ of invertible elements of A as a subset of A. (Recall from 1.9.1 that $G(A)$ is open.) Prove that no boundary point of $G(A)$ is invertible in B.

(b) Deduce that, for $a \in A$, $\mathrm{Spectrum}_A(a)$ is the union of $\mathrm{Spectrum}_B(x)$ and certain connected components of the complement of $\mathrm{Spectrum}_B(x)$.

(c) Take $B = C(S^1)$ and let A be the norm closure of the polynomials in z (where $z\colon S^1 \to \mathbb{C}$ is the usual embedding of the unit circle). Compute the spectrum of the element z in the algebra B and in the algebra A.

1.9.19 Let A be a unital C*-algebra.

(a) Let $a \in A$ be normal, and suppose that $a' \in A$ commutes with a. Show that a' commutes with $f(a)$ for every $f \in C(\mathrm{Spectrum}(a))$.

(b) Suppose now that $a \in A$ is selfadjoint and that $a' \in A$ *anticommutes* with a (meaning that $aa' + a'a = 0$). Show that a' anticommutes with $f(a)$ if f is an odd function, and commutes with $f(a)$ if f is an even function.

(c) Let $b \in A$ be arbitrary, and let $f \in C(\mathrm{Spectrum}(b^*b) \cup \mathrm{Spectrum}(bb^*))$. Show that $bf(b^*b) = f(bb^*)b$.

1.9.20 Prove that a closed symmetric operator is selfadjoint if and only if it has real spectrum (compare Lemma 1.8.4).

1.9.21 Let T be an unbounded selfadjoint operator. For a complex number $\lambda \notin \mathrm{Spectrum}(T)$, denote by $R_\lambda(T)$ the *resolvent* $R_\lambda(T) = (T - \lambda I)^{-1}$. Let S be a bounded selfadjoint operator. Show that $S + T$ is selfadjoint. Suppose in addition that $\lambda \notin \mathrm{Spectrum}(S + T)$; show that we have the identity

$$R_\lambda(S + T) = R_\lambda(T) - R_\lambda(T)\, S\, R_\lambda(S + T).$$

By iteration obtain the series expansion

$$R_\lambda(S + T) = R_\lambda(T) - R_\lambda(T) S R_\lambda(T) + R_\lambda(T) S R_\lambda(T) S R_\lambda(T) - \cdots,$$

valid when $\|S\|$ is sufficiently small. (We shall need this *perturbation series expansion* at one point in Chapter 10.)

1.10 Notes

We have assumed that the reader has some knowledge of elementary functional analysis. The book of Rudin [117] will provide ample background.

The standard reference works on C*-algebras are the books of Dixmier [48] and Pedersen [104]. Both books contain an extensive bibliography of the subject. Two recent books by Fillmore [56] and Davidson [45] are more modest in scope and for this reason might be more useful than the standard tomes.

The strong operator topology is most directly associated with the theory of von Neumann algebras. Although sometimes confused, C*-algebras and von Neumann algebras are quite different in character, the former being associated to topology (via the characterization of commutative C*-algebras given in the text) and the latter to measure theory. Much of von Neumann algebra theory is organized around the brilliant research of Alain Connes, and the reader is referred to the amazing book [41] for Connes' own perspective on this work.

The elements of C*-algebra representation theory are developed in Arveson's book [4]. A fuller treatment can be found in [48] and [104].

The theory of unbounded operators is extensively discussed in the books of Dunford and Schwarz [52] and Kato [85]. A shorter exposition can be found in Rudin's book mentioned above.

2

INDEX THEORY AND EXTENSIONS

We are ready now to look at some of the specialized topics to which this book is devoted, which involve the interaction of operator theory and algebraic topology. The most fundamental example is the theory of the *Fredholm index*; we study this first.

2.1 Fredholm Operators and the Calkin Algebra

Recall from Chapter 1 that a bounded linear operator on a Hilbert space H is *compact* if it is a limit, in the norm topology, of finite-rank operators. The set of all compact operators on H is denoted $\mathfrak{K}(\mathrm{H})$; it is an ideal in the C^*-algebra $\mathfrak{B}(\mathrm{H})$ (Exercise 2.9.1).

As far as this book is concerned, compact operators will play the role of 'small perturbations' of the zero operator. Thus if K is compact and T is any operator we shall regard $\mathrm{T}+\mathrm{K}$ as a small perturbation of T, so that T and $\mathrm{T}+\mathrm{K}$ are 'essentially' the same operator. As we shall see later, we are led to this point of view quite naturally by examples from analysis and geometry.

In keeping with this perspective we introduce the following C^*-algebra.

2.1.1 Definition The quotient C^*-algebra $\mathfrak{B}(\mathrm{H})/\mathfrak{K}(\mathrm{H})$ is called the *Calkin algebra,* and is denoted $\mathfrak{Q}(\mathrm{H})$. If T is a bounded operator on H then we shall denote by $\pi(\mathrm{T})$ its image in $\mathfrak{Q}(\mathrm{H})$.

2.1.2 Definition A bounded operator $\mathrm{T}\colon \mathrm{H}_1 \to \mathrm{H}_2$ is a *Fredholm operator* if its kernel and cokernel are finite-dimensional vector spaces. The *Fredholm index* of a Fredholm operator T is the integer quantity

$$\operatorname{Index}(\mathrm{T}) = \operatorname{Dim}(\operatorname{Kernel}(\mathrm{T})) - \operatorname{Dim}(\operatorname{Cokernel}(\mathrm{T})).$$

2.1.3 Remark An extra condition is sometimes included in the definition of a Fredholm operator: that $\operatorname{Image}(\mathrm{T})$ be a *closed* subspace of H_2. This condition is superfluous, however, as the following argument shows. The restriction of T to the orthogonal complement of its kernel is an injective linear map $\mathrm{T}\colon \operatorname{Kernel}(\mathrm{T})^{\perp} \to \mathrm{H}_2$, and since $\operatorname{Image}(\mathrm{T})$ has finite codimension this restriction can be extended to a bijective, continuous linear map $\widetilde{\mathrm{T}}\colon \operatorname{Kernel}(\mathrm{T})^{\perp} \oplus \mathbb{C}^n \to \mathrm{H}_2$. By the Closed Graph Theorem, $\widetilde{\mathrm{T}}$ is a homeomorphism, and it follows that $\operatorname{Image}(\mathrm{T}) = \widetilde{\mathrm{T}}(\operatorname{Kernel}(\mathrm{T})^{\perp})$ is closed.

The relevance of the Fredholm index to the study of the Calkin algebra is as follows:

2.1.4 ATKINSON'S THEOREM *A bounded operator* $T \in \mathfrak{B}(H)$ *is Fredholm if and only if its image in the Calkin algebra is invertible.*

In the proof we shall need to recall the elementary fact (Exercise 1.9.1) that any element a of a unital Banach algebra A such that $\|1 - a\| < 1$ is invertible in A.

PROOF OF ATKINSON'S THEOREM Let T be a Fredholm operator on H. Then $\mathrm{Image}(T)$ is a closed subspace of H and the restricted operator $T\colon \mathrm{Kernel}(T)^{\perp} \to \mathrm{Image}(T)$ is invertible. If S is its (bounded) inverse, and if we extend the domain of S to all of H by defining S to be zero on $\mathrm{Image}(T)^{\perp}$ then we obtain the formulas

$$I - ST = \text{orthogonal projection onto } \mathrm{Kernel}(T)$$

and

$$I - TS = \text{orthogonal projection onto } \mathrm{Image}(T)^{\perp}.$$

Since both $\mathrm{Kernel}(T)$ and $\mathrm{Image}(T)^{\perp}$ are finite-dimensional, the orthogonal projections are compact. Therefore, $\pi(S)$ is inverse to $\pi(T)$ in $\mathfrak{Q}(H)$.

Conversely, if $\pi(S)$ is inverse to $\pi(T)$ in $\mathfrak{Q}(H)$ then we may write $ST = I + K_1$ and $TS = I + K_2$, where K_1 and K_2 are compact operators. There are finite-rank operators F_1 and F_2 such that $\|F_1 - K_1\| < 1$ and $\|F_2 - K_2\| < 1$. But we noted above that if X is any operator such that $\|X\| < 1$ then $I - X$ is invertible, and therefore $I - F_1 + K_1$ is an invertible operator. The formula

$$\begin{aligned}(I - F_1 + K_1)^{-1} ST &= (I - F_1 + K_1)^{-1}(I + K_1) \\ &= I + (I - F_1 + K_1)^{-1} F_1 = I + \text{finite-rank operator}\end{aligned}$$

shows that T is left-invertible modulo finite-rank operators, and hence has finite-dimensional kernel. A similar calculation using K_2 and F_2 shows that T is right-invertible modulo finite-rank operators, and hence has finite-dimensional cokernel. Thus T is a Fredholm operator. □

2.1.5 PROPOSITION *Let* S *and* T *be Fredholm operators. Then* ST *is also a Fredholm operator and*

$$\mathrm{Index}\,(ST) = \mathrm{Index}\,(S) + \mathrm{Index}\,(T).$$

In addition, if T *is Fredholm then so is* T^* *and*

$$\mathrm{Index}\,(T^*) = -\mathrm{Index}\,(T).$$

PROOF Elementary linear algebra provides us with the exact sequence

$$0 \to \operatorname{Kernel} T \to \operatorname{Kernel} ST \to \operatorname{Kernel} S \to$$
$$\to \operatorname{Cokernel} T \to \operatorname{Cokernel} ST \to \operatorname{Cokernel} S \to 0$$

of vector spaces. Since S and T have finite-dimensional kernel and cokernel, this exact sequence tells us that ST does too. The asserted formula for Index (ST) follows from the principle that the alternating sum of dimensions in a finite exact sequence of finite-dimensional vector spaces is zero. To prove the second part of the proposition, we note that the range of an operator T is closed if and only if the range of its adjoint is closed (the proof is left to the reader), in which case Cokernel(T) is isomorphic to Kernel(T^*). Hence if T is Fredholm then so is T^* and

$$\operatorname{Index}(T) = \operatorname{Dim}(\operatorname{Kernel}(T)) - \operatorname{Dim}(\operatorname{Kernel}(T^*)).$$

The formula $\operatorname{Index}(T^*) = -\operatorname{Index}(T)$ obviously follows from this. □

2.1.6 PROPOSITION *The Fredholm index has the following stability properties:*

(a) *The set of all Fredholm operators is an open subset of* $\mathfrak{B}(H)$, *and the Fredholm index is a continuous (and therefore locally constant) function on this set.*
(b) *If* T *is Fredholm and* K *is compact then* $T + K$ *is Fredholm, and* $\operatorname{Index}(T) = \operatorname{Index}(T + K)$.

PROOF Atkinson's Theorem implies that the set of Fredholm operators is an open subset of $\mathfrak{B}(H)$ which is stable under compact perturbations. To prove continuity of the index, fix a Fredholm operator T. Let $J\colon H_1 \to H$ be the inclusion of $H_1 = \operatorname{Kernel}(T)^{\perp}$ into H, and let $Q\colon H \to H_2$ be the orthogonal projection of H onto $H_2 = \operatorname{Image}(T)$. Both Q and J are Fredholm operators, whose indices are Dim(Cokernel(T)) and minus Dim(Kernel(T)) respectively, so that

$$\operatorname{Index}(T) + \operatorname{Index}(J) + \operatorname{Index}(Q) = 0.$$

Indeed, the operator $QTJ\colon H_1 \to H_2$ is invertible. Since the set of invertible operators is open, the operator

$$QT'J\colon H_1 \to H_2$$

is invertible for all $T'\colon H \to H$ sufficiently close in norm to T. Hence the index of $QT'J$ is zero. But then by 2.1.5,

$$\operatorname{Index}(Q) + \operatorname{Index}(T') + \operatorname{Index}(J) = 0,$$

and this implies that $\operatorname{Index}(T') = \operatorname{Index}(T)$.

The second part of the proposition follows from the first: the linear path from T to $T + K$ lies entirely within the Fredholm operators, and the index function is continuous and therefore constant along it. □

2.2 The Essential Spectrum

In this section we undertake an algebraic analysis of elements in the Calkin algebra. As with the elements in any C^*-algebra, of primary importance is their spectrum:

2.2.1 DEFINITION Let T be a bounded operator on a Hilbert space H. The *essential spectrum* of T is the spectrum of its image $\pi(T)$ in the Calkin algebra $\mathfrak{Q}(H)$.

The essential spectrum of a bounded operator T is a non-empty, compact subset of the spectrum of T. By Atkinson's theorem, the essential spectrum of T may be described as the set of those complex numbers λ for which $T - \lambda I$ fails to be a Fredholm operator.

The following result describes the essential spectrum of a selfadjoint operator.

2.2.2 PROPOSITION *The essential spectrum of a selfadjoint operator T is made up of the limit points of the spectrum of T together with the isolated eigenvalues of T with infinite multiplicity. Thus a complex number belongs to the spectrum of T, but not the essential spectrum, if and only if it is an isolated eigenvalue of T with finite multiplicity.*

PROOF If λ is an isolated point of the spectrum of T then the operator $T - \lambda I$ is invertible on the orthogonal complement of the λ-eigenspace of T (where it is zero). So $T - \lambda I$ is a Fredholm operator if and only if the eigenspace is finite-dimensional. To complete the proof we must show that if λ is a limit point of the spectrum of T then λ belongs to the essential spectrum. If μ is a real number then the selfadjoint operator $T - \mu I$ can fail to be invertible only if there is a sequence $\{v_n\}$ of unit vectors such that $\|Tv_n - \mu v_n\| \to 0$. If $\mu' \neq \mu$ and if $\{v'_n\}$ is a sequence of unit vectors such that $\|Tv'_n - \mu' v'_n\| \to 0$ then $\langle v_n, v'_n\rangle \to 0$ (this generalizes the well-known fact that eigenvectors associated to distinct eigenvalues of a selfadjoint operator are orthogonal). By a diagonal argument, followed by Gram–Schmidt orthogonalization, if λ is a limit point of the spectrum of T then there is an *orthonormal* sequence $\{w_n\}$ such that $\|Tw_n - \lambda w_n\| \to 0$. In fact such a sequence may be constructed which is orthogonal to any given finite-dimensional subspace of H. Thus $T - \lambda I$ is not bounded below on the orthogonal complement of any finite-dimensional subspace and so it is not a Fredholm operator. □

There is a remarkable classification theorem involving the essential spectrum of selfadjoint operators. To formulate it we need the following notion:

2.2.3 DEFINITION Two bounded operators T_1 and T_2 on Hilbert spaces H_1 and H_2 are *essentially unitarily equivalent* if there is a unitary isomorphism of Hilbert spaces $U\colon H_1 \to H_2$ such that $T_1 - U^*T_2U$ is a compact operator.

Essential unitary equivalence is an equivalence relation, and we aim to classify selfadjoint operators up to essential unitary equivalence. It is clear that essentially unitarily equivalent operators have the same essential spectrum. Conversely:

2.2.4 THEOREM *If* T_1 *and* T_2 *are selfadjoint operators on separable Hilbert spaces, and if they have the same essential spectrum, then they are essentially unitarily equivalent.*

The proof of this theorem is a fairly simple consequence of the following famous result:

2.2.5 WEYL–VON NEUMANN THEOREM *Every bounded selfadjoint operator on a separable Hilbert space* H *is an arbitrarily small compact perturbation of a diagonal operator.*

By definition, an operator D is *diagonal* if there exists an orthonormal basis for H consisting of eigenvectors for D. In detail, the theorem states that given any bounded selfadjoint operator T, and any $\varepsilon > 0$, there exists a diagonal operator D with $T - D$ compact and $\|T - D\| \leqslant \varepsilon$.

PROOF OF THE WEYL–VON NEUMANN THEOREM We know that T is unitarily equivalent to a (countable) direct sum of operators $M\varphi(\lambda) = \lambda\varphi(\lambda)$ on $L^2(X, \mu)$, where X is the spectrum of T and μ is a Borel measure on X. To prove the theorem it suffices to consider one model operator M. For each $n = 1, 2, \ldots$, partition X into finitely many Borel sets of diameter $\varepsilon 2^{-n-3}$ or less, in such a way that the $(n+1)$st partition is a refinement of the nth. Denote by H_n the finite-dimensional subspace of $H = L^2(X, \mu)$ consisting of functions which are constant on the members of the nth partition and denote by P_n the orthogonal projection onto H_n. It is easy to calculate that

$$\|P_n T - TP_n\| \leqslant \varepsilon 2^{-n-2}.$$

For instance, T is within $\varepsilon 2^{-n-2}$ of a pointwise multiplication operator T_n associated to a function which is constant on each member of the nth partition, and such an operator T_n commutes with P_n. If we define $Q_n = P_n - P_{n-1}$ (we set $P_0 = 0$) then Q_n is also an orthogonal projection and

$$\|Q_n T - TQ_n\| \leqslant \varepsilon 2^{-n}.$$

The series $\sum_n Q_n$ converges strongly to I, and so

$$T = \sum TQ_n = \sum Q_n T Q_n - \sum (Q_n T - TQ_n) Q_n,$$

with convergence in the strong topology. In fact, however, our estimates show that the last series converges in the norm topology. Since each term in this last series is a finite-rank operator, the sum is compact and (by our estimate) has norm $\leqslant \varepsilon$. As for the first series, it is a direct sum of selfadjoint operators on the finite-dimensional spaces $Q_n H$. Each summand is a diagonal operator, and hence the first series sums to a diagonal operator. □

PROOF OF THEOREM 2.2.4 Both T_1 and T_2 are compact perturbations of diagonal operators, and of course these diagonal operators have the same essential spectrum. But the essential spectrum of a diagonal operator is just the set of limit points of its eigenvalue sequence, and so by permuting the eigenbasis of one of our diagonal operators it is easy to arrange a unitary equivalence, modulo compact operators, with the other (see Exercise 2.9.4 for the details). The same unitary will implement a unitary equivalence, modulo compact operators, between T_1 and T_2. □

2.3 The Toeplitz Extension

The remarkable stability properties of the Fredholm index bring operator theory into close contact with topology. To illustrate, consider the following important example.

2.3.1 DEFINITION Let S^1 denote the unit circle in $\mathbb{C}$. The *Hardy space* $H^2(S^1)$ is the closed subspace of $L^2(S^1)$ spanned by the functions z^n, for $n \geqslant 0$. A *Toeplitz operator* on $H^2(S^1)$ is a bounded operator T_g of the form

$$T_g(f) = P(gf) \qquad (f \in H^2(S^1)),$$

where $g \in L^\infty(S^1)$ and P is the orthogonal projection from $L^2(S^1)$ onto $H^2(S^1)$. The function g is called the *symbol* of T_g.

Recall that the *winding number* $\mathrm{Winding}(g) \in \mathbb{Z}$ of a continuous function $g\colon S^1 \to \mathbb{C}\setminus\{0\}$ is its class in the fundamental group $\pi_1(\mathbb{C}\setminus\{0\})$, which we identify with $\mathbb{Z}$ in such a way that the winding number of the function $g(z) = z$ is $+1$.

2.3.2 TOEPLITZ INDEX THEOREM *If T_g is a Toeplitz operator on $H^2(S^1)$ whose symbol is a continuous and nowhere vanishing function on S^1 then T_g is a Fredholm operator and*

$$\mathrm{Index}\,(T_g) = -\mathrm{Winding}(g).$$

PROOF Denote by M_g the operator of pointwise multiplication by g on $L^2(S^1)$. The set of all continuous g for which the commutator $PM_g - M_gP$ is compact is a C^*-subalgebra of $C(S^1)$. A direct calculation shows that $g(z) = z$ is in this C^*-subalgebra, for in this case the commutator is a rank-one operator.

Since the function $g(z) = z$ generates $C(S^1)$ as a C^*-algebra (by the Stone–Weierstrass Theorem) it follows that $PM_g - M_gP$ is compact for every continuous g. Identifying T_g with PM_g, we see that

$$\begin{aligned} T_{g_1}T_{g_2} &= PM_{g_1}PM_{g_2} = PPM_{g_1}M_{g_2} + \text{compact operator} \\ &= PM_{g_1g_2} + \text{compact operator} \\ &= T_{g_1g_2} + \text{compact operator,} \end{aligned}$$

for all continuous g_1 and g_2. In particular, if g is continuous and nowhere zero then $T_{g^{-1}}$ is inverse to T_g, modulo compact operators, and so by Atkinson's Theorem, T_g is Fredholm. Now the continuity of the Fredholm index shows that Index (T_g) only depends on the homotopy class of the continuous function $g\colon S^1 \to \mathbb{C} \setminus \{0\}$. So it suffices to check the theorem on a representative of each homotopy class; say the functions $g(z) = z^n$, for $n \in \mathbb{Z}$. This is an easy computation, working in the orthogonal basis $\{z^n \mid n \geqslant 0\}$ for $H^2(S^1)$. □

2.3.3 PROPOSITION *The map* $\alpha\colon g \mapsto \pi(T_g)$ *is an injective* $*$*-homomorphism from* $C(S^1)$ *to the Calkin algebra* $\mathfrak{Q}(H^2(S^1))$.

PROOF Our calculations in the proof of 2.3.2 above show that $T_{g_1g_2} = T_{g_1}T_{g_2}$ modulo compact operators, and it is easy to check that $T_g^* = T_{\bar{g}}$. Hence α is indeed a $*$-homomorphism. The kernel of α is an ideal in $C(S^1)$, and every such ideal consists of all those functions which vanish on some closed subset $X \subseteq S^1$. By symmetry, X must be rotationally invariant; so either $X = \emptyset$ or else $X = S^1$. But $X = \emptyset$ is impossible, since not every Toeplitz operator is compact; so $X = S^1$ and α is injective. □

Since injective $*$-homomorphisms preserve spectrum we deduce:

2.3.4 COROLLARY *Let* $g \in C(S^1)$. *The essential spectrum of the Toeplitz operator* T_g *is equal to the range of* g.

We may rephrase Proposition 2.3.3 by constructing a certain short exact sequence of C^*-algebras. Let H be the Hardy space $H^2(S^1)$ and let $\mathfrak{T}$ be the C^*-subalgebra of $\mathfrak{B}(H)$ generated by the compact operators together with the Toeplitz operators T_g for all $g \in C(S^1)$. The image $\pi(\mathfrak{T})$ of $\mathfrak{T}$ in the Calkin algebra is isomorphic to $C(S^1)$ by 2.3.3 above, and so we get a short exact sequence

$$0 \longrightarrow \mathfrak{K}(H) \longrightarrow \mathfrak{T} \longrightarrow C(S^1) \longrightarrow 0 \tag{2.3.5}$$

called the *Toeplitz extension.*

REMARK Our Fredholm index calculations show that invertibles in $C(S^1)$ need not lift to invertibles in $\mathfrak{T}$. Thus, the Toeplitz extension does not *split* — there is no $*$-homomorphism from $C(S^1)$ to $\mathfrak{T}$ which is a left inverse to $\pi\colon \mathfrak{T} \to C(S^1)$.

2.4 Essentially Normal Operators

All Hilbert spaces considered in this section will be separable and infinite-dimensional.

2.4.1 DEFINITION A bounded operator T on a Hilbert space is *essentially normal* if the operator $TT^* - T^*T$ is compact. It is *essentially selfadjoint* if the operator $T - T^*$ is compact.

2.4.2 EXAMPLE The Toeplitz operators T_g with continuous symbol g are essentially normal, because $T_g^* = T_{\bar{g}}$, and we have already remarked that Toeplitz operators with continuous symbols commute with one another modulo compact operators.

Every compact perturbation of a selfadjoint operator is essentially selfadjoint. Conversely, if T is essentially selfadjoint then it is a compact perturbation of a selfadjoint operator (namely $\frac{1}{2}(T+T^*)$). It follows from 2.2.4 that two essentially selfadjoint operators are essentially unitarily equivalent if and only if they have the same essential spectrum. Thus the classification of essentially selfadjoint operators up to essential unitary equivalence is quite straightforward.[10]

The classification of essentially *normal* operators is more complicated. For instance, the functions $g_1(z) = z$ and $g_2(z) = z^2$ have the same range, and so the Toeplitz operators T_{g_1} and T_{g_2} have the same essential spectrum. Nevertheless they cannot be essentially unitarily equivalent, since the first has Fredholm index -1, while the second has index -2, and it is clear from Proposition 2.1.6 that essentially unitarily equivalent operators have the same index. Furthermore neither operator is a compact perturbation of a normal operator, since every normal Fredholm operator has index zero.

To bring the classification problem for essentially normal operators into sharper focus we introduce the following algebraic object:

2.4.3 DEFINITION If X is a non-empty compact subset of $\mathbb{C}$ then denote by $\operatorname{Ext}(X)$ the set[11] of all essential unitary equivalence classes of essentially normal operators with essential spectrum X.

The term 'Ext' is short for 'Extension', and will be further explained in the next section.

[10]Indeed, it is much *more* straightforward than the classification of selfadjoint operators up to unitary equivalence, which as we saw in the last chapter involves the classification of Borel probability measures on the spectrum.

[11]In slightly more detail: let $\mathfrak{C}$ denote the collection of pairs (T, H), where H is a separable, infinite-dimensional Hilbert space and T is an essentially normal operator on it with essential spectrum X. Essential unitary equivalence defines an equivalence relation $\sim$ on $\mathfrak{C}$. We set $\operatorname{Ext}(X) = \mathfrak{C}/\sim$. Since $\mathfrak{C}$ is not a set in Zermelo–Fraenkel set theory, some readers may feel nervous here. Exercise 2.9.5 should reassure them.

There is an interesting addition operation on Ext (X), given by

$$[T_1] + [T_2] = [T_1 \oplus T_2],$$

which makes Ext (X) into a commutative semigroup (meaning that the addition operation is commutative and associative: no claim is made — yet — about the existence of a zero element).

2.4.4 EXAMPLE It follows from Theorem 2.2.4 that if $X \subseteq \mathbb{R}$ then all selfadjoint operators with essential spectrum X determine the same element of Ext (X). Moreover, every essentially normal operator with essential spectrum X is a compact perturbation of a selfadjoint operator. Hence Ext (X) = 0, the trivial semigroup.

2.4.5 EXAMPLE Let X be the unit circle S^1. As we have already noted, Fredholm index theory shows that Ext $(S^1) \neq 0$. In fact there is a surjective homomorphism of semigroups

$$\text{Index} : \text{Ext}(S^1) \to \mathbb{Z},$$

defined by mapping [T] to Index (T).

Brown, Douglas and Fillmore examined this example in more detail, and were able to show:

2.4.6 PROPOSITION *The homomorphism* Index : Ext $(S^1) \to \mathbb{Z}$ *is an isomorphism.*

A proof is outlined in the exercises. With this in hand, they were led to consider Ext (X) for general X, an unexpectedly complicated project which required a far richer interplay between topology and analysis. Their final result was 2.4.8 below. To state it, it is convenient to introduce one extra piece of terminology.

2.4.7 DEFINITION Let X be a non-empty and compact subset of $\mathbb{C}$. An *index function* for X is an integer-valued function on the set of bounded components of the complement of X in $\mathbb{C}$. If T is an essentially normal operator with essential spectrum X then the *index function associated to* T is the map $\lambda \mapsto \text{Index}(T - \lambda I)$.

Observe that the function $\lambda \mapsto \text{Index}(T - \lambda I)$ is indeed defined on the complement of the essential spectrum. Since it is a locally constant function which vanishes for λ sufficiently large, it is an index function in the sense of the above definition.

2.4.8 BROWN–DOUGLAS–FILLMORE THEOREM *If* X *is any non-empty and compact subset of* $\mathbb{C}$ *then the semigroup* Ext (X) *is a group. The map which associates to* $[T] \in \text{Ext}(X)$ *the index function of* T *is an isomorphism from* Ext (X) *to the group of all index functions for* X.

In short, Fredholm index theory is the only obstruction which prevents two essentially normal operators with the same essential spectrum from being essentially unitarily equivalent.

It may be shown that if X is a compact subset of the plane then the free abelian group on the set of bounded components of $\mathbb{C} \setminus X$ is isomorphic to the group $\pi^1(X)$ which consists of homotopy classes of continuous maps from X to the unit circle. Thus Theorem 2.4.8 asserts that there is a natural isomorphism

$$\mathrm{Ext}\,(X) \cong \mathrm{Hom}(\pi^1(X), \mathbb{Z}).$$

This introduction of algebraic topology into the *statement* of Theorem 2.4.8 is actually an important component of its *proof*. Although we shall not follow the most direct route, we shall eventually obtain this proof in Chapter 7.

2.5 C*-Algebra Extensions

Let T be an essentially normal operator on a separable Hilbert space H, with essential spectrum $X \subseteq \mathbb{C}$. Denote by E_T the C*-algebra generated by T, the identity operator, and all compact operators on H. Then there is a short exact sequence of C*-algebras

$$0 \longrightarrow \mathfrak{K}(H) \longrightarrow E_T \longrightarrow C(X) \longrightarrow 0\,,$$

where the map $\mathfrak{K}(H) \to E_T$ is inclusion, and the map $E_T \to C(X)$ sends $T \in E_T$ to the function $z \mapsto z$ in $C(X)$. Indeed, the quotient C*-algebra $E_T/\mathfrak{K}(H)$ is commutative and generated by $\pi(T)$. Hence by the functional calculus there is an isomorphism

$$E_T/\mathfrak{K}(H) \cong C(\mathrm{Spectrum}(\pi(T))) = C(X),$$

under which $f(\pi(T))$ corresponds to $f \in C(X)$.

Suppose that T and T' are essentially normal operators which are essentially unitarily equivalent by a unitary $U\colon H \to H'$. Then there is a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{K}(H) \arrow[r] \arrow[d, "\mathrm{Ad}_U"] & E_T \arrow[r] \arrow[d, "\mathrm{Ad}_U"] & C(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{K}(H') \arrow[r] & E_{T'} \arrow[r] & C(X) \arrow[r] & 0
\end{tikzcd}$$

in which the vertical arrows are given by conjugation with U, and this shows that the short exact sequences associated to T and T' are in a natural sense isomorphic. Conversely, if the short exact sequences are isomorphic in this sense, then T and T' are essentially unitarily equivalent. The semigroup $\mathrm{Ext}\,(X)$ of the previous section may therefore be thought of as made up of isomorphism classes of *extensions* of $C(X)$ by the compact operators, in the sense of the following definitions.

2.5.1 Definition Let A be a C*-algebra and let $\mathfrak{K}(H)$ be the C*-algebra of compact operators on a separable and infinite-dimensional Hilbert space H. An *extension of* A *by* $\mathfrak{K}(H)$ is a short exact sequence of C*-algebras and $*$-homomorphisms, of the form

$$0 \longrightarrow \mathfrak{K}(H) \longrightarrow E \longrightarrow A \longrightarrow 0 .$$

To say it as plainly as possible, an extension is a one-to-one $*$-homomorphism of $\mathfrak{K}(H)$ onto a closed, two-sided ideal of a C*-algebra E, and a $*$-homomorphism of E onto A whose kernel is this ideal. It is important to point out that the $*$-homomorphisms, and not just the C*-algebras, are part of the data comprising an extension. In particular, describing E, up to isomorphism, does not fully describe the extension.

2.5.2 Definition Two extensions of A by the compact operators, written as

$$0 \longrightarrow \mathfrak{K}(H) \longrightarrow E \longrightarrow A \longrightarrow 0$$

and

$$0 \longrightarrow \mathfrak{K}(H') \longrightarrow E' \longrightarrow A \longrightarrow 0 ,$$

are *isomorphic* if there are $*$-isomorphisms $\alpha\colon \mathfrak{K}(H) \to \mathfrak{K}(H')$ and $\beta\colon E \to E'$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{K}(H) \arrow[r] \arrow[d, "\alpha"] & E \arrow[r] \arrow[d, "\beta"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{K}(H') \arrow[r] & E' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

is commutative.

2.5.3 Remark Every $*$-isomorphism $\alpha\colon \mathfrak{K}(H) \to \mathfrak{K}(H')$ is conjugation by some unitary isomorphism of Hilbert spaces.[12] Furthermore, if an isomorphism $\alpha = \mathrm{Ad}_U\colon \mathfrak{K}(H) \to \mathfrak{K}(H')$ extends to an isomorphism $\beta\colon E \to E'$ in such a way that the diagram above commutes, then this extension is unique. Thus an isomorphism of extensions is completely determined by a unitary $U\colon H \to H'$, and one can think of isomorphism of extensions as a kind of 'unitary equivalence.' In particular, two essentially normal operators with essential spectrum X are essentially unitarily equivalent if and only if the extensions of $C(X)$ that they determine are isomorphic.

[12]To build the unitary $U\colon H \to H'$ from α, pick a unit vector v in H and another unit vector v' in H' such that $\alpha(p_v) = p_{v'}$, where p_v and $p_{v'}$ are the orthogonal projections onto the span of v and v' respectively. Then define $Ukv = \alpha(k)v'$.

2.6 Extensions and the Calkin Algebra

The following lemma presents a convenient way of constructing extensions.

2.6.1 LEMMA *Suppose that φ is a $*$-homomorphism from A to the Calkin algebra $\mathfrak{Q}(H)$. There is, up to isomorphism, a unique extension of A by $\mathfrak{K}(H)$ which fits into a commutative diagram of the following sort:*

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{K}(H) \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & \mathfrak{K}(H) \arrow[r] & \mathfrak{B}(H) \arrow[r, "\pi"] & \mathfrak{Q}(H) \arrow[r] & 0. \end{tikzcd}$$

PROOF To construct the extension, define E to be the 'pull-back'

$$E = \{T \oplus a \in \mathfrak{B}(H) \oplus A : \pi(T) = \varphi(a)\}.$$

There are obvious maps from $\mathfrak{K}(H)$ into E, and from E to A and $\mathfrak{B}(H)$, and we obtain a commutative diagram of extensions of the required sort. To prove uniqueness, suppose that $0 \to \mathfrak{K}(H) \to E' \xrightarrow{\pi'} A \to 0$ also fits into the above sort of diagram. If we denote by ρ' the map $E' \to \mathfrak{B}(H)$ then we get a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{K}(H) \arrow[r] \arrow[d, equal] & E' \arrow[r, "\pi'"] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathfrak{K}(H) \arrow[r] & E \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

by mapping $e' \in E'$ to the element $\rho'(e') \oplus \pi'(e') \in E$. □

2.6.2 DEFINITION Two $*$-homomorphisms, $\varphi\colon A \to \mathfrak{Q}(H)$ and $\varphi'\colon A \to \mathfrak{Q}(H')$, are *unitarily equivalent* if there is a unitary isomorphism $U\colon H \to H'$ such that

$$\varphi'(a) = \mathrm{Ad}_U\, \varphi(a)$$

for all $a \in A$, where $\mathrm{Ad}_U\colon \mathfrak{Q}(H) \to \mathfrak{Q}(H')$ is the isomorphism induced by conjugating with the unitary $U\colon H \to H'$.

We leave it to the reader to check that unitarily equivalent $*$-homomorphisms from A to $\mathfrak{Q}(H)$ give rise to isomorphic extensions.

Let us now show that *every* extension of A by $\mathfrak{K}(H)$ arises from some $*$-homomorphism $\varphi\colon A \to \mathfrak{Q}(H)$. Suppose we are given an extension

$$0 \longrightarrow \mathfrak{K}(H) \longrightarrow E \xrightarrow{\pi} A \longrightarrow 0.$$

Let J be the ideal in E which is identified isomorphically with $\mathfrak{K}(H)$. The identification gives a representation ρ of J on the Hilbert space H. It extends to a representation of E on H (which we shall also call ρ) by the formula

$$\rho(e)(\rho(j)v) = \rho(ej)v \qquad (e \in E, j \in J, v \in H).$$[13]

We may now define the required $*$-homomorphism $\varphi\colon A \to \mathfrak{Q}(H)$ by the formula

$$\varphi(\pi(e)) = \pi(\rho(e)) \qquad (e \in E).$$

Using 2.5.3 we see that isomorphic extensions give rise to unitarily equivalent $*$-homomorphisms from A to $\mathfrak{Q}(H)$, and we thus arrive at the following result:

2.6.3 PROPOSITION *There is a one-to-one correspondence between isomorphism classes of extensions of A by $\mathfrak{K}(H)$ and unitary equivalence classes of $*$-homomorphisms from A into the Calkin algebra, in which an extension and a $*$-homomorphism correspond if there is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{K}(H) \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & \mathfrak{K}(H) \arrow[r] & \mathfrak{B}(H) \arrow[r, "\pi"] & \mathfrak{Q}(H) \arrow[r] & 0.
\end{tikzcd}$$

From now on we shall use the term 'extension of A by $\mathfrak{K}(H)$' to refer either to a short exact sequence $0 \to \mathfrak{K}(H) \to E \to A \to 0$ or to a $*$-homomorphism $\varphi\colon A \to \mathfrak{Q}(H)$. We shall use the term 'unitary equivalence' to refer either to isomorphism of short exact sequences or to unitary equivalence of $*$-homomorphisms into the Calkin algebra.

2.7 The Extension Semigroup

All Hilbert spaces considered in this section will be separable and infinite-dimensional.

How may we classify extensions of A up to unitary equivalence? Some simple algebraic invariants of extensions $\varphi\colon A \to \mathfrak{Q}(H)$ immediately present themselves:

(a) the kernel of the $*$-homomorphism φ, and

(b) for unital algebras A, the unitality or otherwise of φ.

But the heart of the matter is the classification of extensions $\varphi\colon A \to \mathfrak{Q}(H)$ of unital C^*-algebras A for which φ is unital and injective. For example, the extensions constructed from essentially normal operators are of this sort.

[13]Since $\rho[J]H = \mathfrak{K}(H)H = H$ there is at most one extension of ρ to E which satisfies this formula. To give a more official definition of ρ, select an approximate unit u_n for J and define $\rho(e)v = \lim_{n\to\infty} \rho(eu_n)v$. A simple calculation shows the limit exists and defines a representation of E as required.

2.7.1 DEFINITION Let A be a separable C^*-algebra with unit. Denote by $\operatorname{Ext}(A)$ the set of unitary equivalence classes[14] of unital, injective extensions $\varphi\colon A \to \mathfrak{Q}(H)$.

The set $\operatorname{Ext}(A)$ may be equipped with a natural addition operation: the obvious inclusion

$$\mathfrak{Q}(H) \oplus \mathfrak{Q}(H') \subseteq \mathfrak{Q}(H \oplus H'),$$

defined by $\pi(T) \oplus \pi(T') \mapsto \pi(T \oplus T')$, allows us to form the unital, injective extension

$$\varphi \oplus \varphi'\colon A \to \mathfrak{Q}(H) \oplus \mathfrak{Q}(H') \subseteq \mathfrak{Q}(H \oplus H')$$

from two unital injective extensions $\varphi\colon A \to \mathfrak{Q}(H)$ and $\varphi'\colon A \to \mathfrak{Q}(H')$. The unitary equivalence class of $\varphi \oplus \varphi'$ depends only on the unitary equivalence classes of φ and φ'.

2.7.2 PROPOSITION *With the addition defined above,* $\operatorname{Ext}(A)$ *is an abelian semigroup.* $\square$

In other words, the addition operation is commutative and associative.

2.7.3 PROPOSITION *Suppose that* X *is a compact subset of* $\mathbb{C}$. *Then the semigroup* $\operatorname{Ext}(X)$ *introduced in Definition* 2.4.3 *is the same the semigroup* $\operatorname{Ext}(C(X))$ *introduced in Definition* 2.7.1. $\square$

This follows from Remark 2.5.3 and our discussion in this section.

Is the semigroup $\operatorname{Ext}(A)$ a group? The question has two parts:

(a) Does $\operatorname{Ext}(A)$ have a zero element — a class $0 \in \operatorname{Ext}(A)$ for which $0+[\varphi] = [\varphi]$ for all φ? Of course, by abstract nonsense, there is at most one zero element.

(b) Does every class $[\varphi]$ have an additive inverse — a class $[\varphi']$ for which $[\varphi] + [\varphi'] = 0$?

2.7.4 DEFINITION A unital extension $\varphi\colon A \to \mathfrak{Q}(H)$ is *split* if there is a *unital* $*$-homomorphism $\widetilde{\varphi}\colon A \to \mathfrak{B}(H)$ such that $\varphi = \pi \circ \widetilde{\varphi}$, where π is the quotient map from $\mathfrak{B}(H)$ to $\mathfrak{Q}(H)$. We shall call $\widetilde{\varphi}$ a *multiplicative lifting*[15] of φ.

In terms of short exact sequences $\mathfrak{K}(H) \to E \xrightarrow{q} A$, a unital extension is split if and only if there is a unital $*$-homomorphism $\sigma\colon A \to E$ that is a left inverse to q. It is important that the lifting $\widetilde{\varphi}$ (or the splitting σ) be *unital*. Extensions which are lifted by non-unital $\widetilde{\varphi}$ can behave quite differently from extensions which are lifted by unital $\widetilde{\varphi}$.

[14] Once again we consider extensions $A \to \mathfrak{Q}(H)$ for *all* choices of Hilbert space H. Compare Definition 2.4.3 and the footnote thereto.

[15] In the next chapter we shall need to consider linear 'liftings' which are no longer $*$-homomorphisms; hence the terminology.

Suppose now that $\varphi\colon A \to \mathfrak{Q}(H)$ is split. Then we can form the 'infinite direct sum' $\varphi \oplus \varphi \oplus \cdots \in \operatorname{Ext}(A)$ as follows: take a lift $\widetilde{\varphi}\colon A \to \mathfrak{B}(H)$, form the infinite direct sum $\widetilde{\varphi} \oplus \widetilde{\varphi} \oplus \ldots$, which is a well-defined $*$-homomorphism from A to $\mathfrak{B}(H \oplus H \oplus \ldots)$, and then project down again to the Calkin algebra $\mathfrak{Q}(H \oplus H \oplus \ldots)$. It is clear that

$$[\varphi] + [\varphi \oplus \varphi \oplus \ldots] = [\varphi \oplus \varphi \oplus \ldots],$$

in $\operatorname{Ext}(A)$, and so it is tempting to 'cancel off' the terms $[\varphi \oplus \varphi \oplus \ldots]$ in this equation to get $[\varphi] = 0$. Of course, cancellation is not always valid in semigroups, but our calculations allow us to say this:

2.7.5 Proposition *If A is a unital C^*-algebra for which $\operatorname{Ext}(A)$ is a group then the class of any split extension is the zero element in $\operatorname{Ext}(A)$.* □

We shall show in the next chapter that if A is separable then the class of any split extension is always the zero element, whether or not $\operatorname{Ext}(A)$ is a group, and in particular, any two split extensions are equivalent. This is a remarkable and very important theorem of Voiculescu.

Let us turn now to the question of additive inverses.

2.7.6 Definition A unital, injective extension $\varphi\colon A \to \mathfrak{Q}(H)$ is *semisplit* if there is another unital and injective extension $\varphi'\colon A \to \mathfrak{Q}(H')$ such that $\varphi \oplus \varphi'$ is a split extension.

Thus if we assume Voiculescu's theorem, then a unital, injective extension of a separable unital C^*-algebra is semisplit if and only if its class in $\operatorname{Ext}(A)$ has an additive inverse.

2.7.7 Definition Let A be a C^*-algebra with unit and suppose that $\rho\colon A \to \mathfrak{B}(H)$ is a non-degenerate representation of A on a separable Hilbert space H. Let $P \in \mathfrak{B}(H)$ be a projection which commutes with the action of A modulo compact operators. In other words, assume that

$$\rho(a)P - P\rho(a) \in \mathfrak{K}(H), \tag{2.7.8}$$

for all $a \in A$. The *abstract Toeplitz operator* $T_a \in \mathfrak{B}(PH)$ with symbol $a \in A$, associated to the pair (ρ, P), is the operator

$$PH \xrightarrow{\text{inclusion}} H \xrightarrow{\rho(a)} H \xrightarrow{\text{projection}} PH$$

The *abstract Toeplitz extension* associated to the pair (ρ, P) is the homomorphism

$$\varphi_P\colon A \to \mathfrak{Q}(PH)$$

defined by formula $\varphi_P(a) = \pi(T_a)$.

2.7.9 Remark Condition 2.7.8 is used to show that φ_P is a $*$-homomorphism. It abstracts the critical property of the Hardy space projection $L^2(S^1) \to H^2(S^1)$ which was used in the proof of Theorem 2.3.2. Thus the Toeplitz extension of 2.3.5 is indeed an abstract Toeplitz extension in the sense of our definition.

We use the notation φ_P, rather than the more correct $\varphi_{(\rho,P)}$, because we shall typically fix a representation ρ and vary P to obtain different extensions φ_P. The extensions φ_P are unital, but need not be injective.

2.7.10 Proposition *A unital, injective extension $\varphi\colon A \to \mathfrak{Q}(H)$ is semisplit if and only if it is unitarily equivalent to an abstract Toeplitz extension.*

Proof Suppose, on the one hand, that φ is semisplit and that $\varphi \oplus \varphi'$ lifts to a $*$-homomorphism $\rho\colon A \to \mathfrak{B}(H \oplus H')$. If P denotes the orthogonal projection from $H \oplus H'$ onto $H \oplus 0$ then $P(H \oplus H') \cong H$, and the abstract Toeplitz extension $\varphi_P\colon A \to \mathfrak{Q}(P(H \oplus H'))$ is unitarily equivalent to φ. Suppose, on the other hand, that φ_P is a Toeplitz extension which is injective (so that we can study its class in Ext (A)), and denote by $\rho\colon A \to \mathfrak{B}(H)$ the non-degenerate representation from which it is constructed. We may form the abstract Toeplitz extension $\varphi_{P^\perp}$ using the projection $P^\perp$ complementary to P, which also satisfies the condition 2.7.8. The direct sum $\varphi_P \oplus \varphi_{P^\perp}$ is split. Indeed the representation ρ is a lifting; this is because $P\rho(a)P + P^\perp\rho(a)P^\perp = \rho(a)$ modulo $\mathfrak{K}(H)$, thanks to 2.7.8. If $\varphi_{P^\perp}$ is injective (and so defines a class in Ext (A)) then we are done; otherwise, just form the direct sum of ρ with a suitable non-degenerate representation and repeat the argument to obtain an injective $\varphi_{P^\perp}$. $\square$

2.8 Geometric Examples of Extensions

So far we have seen only two examples of extensions: those associated to essentially normal operators (Section 2.5) and the Toeplitz extension (Section 2.3).[16] This section will outline three somewhat more elaborate constructions of extensions of $C(X)$ by $\mathfrak{K}(H)$ by means of global analysis. As with the examples we have already considered, the Fredholm index theory associated to these extensions is important and very interesting.

We shall proceed at a rather rapid pace, but the reader can at this stage safely disregard the details of what follows since we shall not return to these examples until much later in the book.

2.8(a) *Higher-Dimensional Toeplitz Operators*

Toeplitz operators on the circle were defined in terms of the Hardy space $H^2(S^1)$, which is the closed linear subspace of $L^2(S^1)$ generated by the functions $z \mapsto z^n$,

[16] In fact one can easily show that the Toeplitz extension is associated to an essentially normal operator, so we have in reality seen only *one* example.

for $n \geqslant 0$. Elementary complex analysis shows that $H^2(S^1)$ can also be characterized as the closure of the space of *boundary values* of those holomorphic functions on the open unit disk that admit a continuous extension to the closed unit disk. This suggests a higher-dimensional generalization: consider an appropriate bounded open subset $\Omega \subseteq \mathbb{C}^n$, with smooth boundary $\partial\Omega$, and define $H^2(\partial\Omega)$ to be the closure in $L^2(\partial\Omega)$ of the boundary values of the holomorphic functions that extend continuously to $\overline{\Omega}$.

2.8.1 DEFINITION If f is a bounded measurable function on $\partial\Omega$ then the *Toeplitz operator with symbol* f is the operator T_f on $H^2(\partial\Omega)$ given as the composition

$$H^2(\partial\Omega) \xrightarrow{\text{inclusion}} L^2(\partial\Omega) \xrightarrow{M_f} L^2(\partial\Omega) \xrightarrow{\text{projection}} H^2(\partial\Omega),$$

where M_f denotes pointwise multiplication by f.

Following the case of the unit disk, we should like to assert that if f and f' are continuous functions then the composition of Toeplitz operators $T_f T_{f'}$ is equal to the Toeplitz operator $T_{ff'}$, modulo compact operators. Unfortunately this is not usually true (the simplest example is the product of the unit disk with itself in $\mathbb{C}^2$). However it *is* true if Ω is say the unit ball in $\mathbb{C}^n$. A more general statement is this:

2.8.2 PROPOSITION *Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^n$. If f is a continuous function on $\partial\Omega$ then the multiplication operator M_f on $L^2(\partial\Omega)$ commutes, modulo compact operators, with the orthogonal projection from $L^2(\partial\Omega)$ onto the space $H^2(\partial\Omega)$. Therefore if f and f' are two continuous functions on $\partial\Omega$ then the product $T_f T_{f'}$ is equal to $T_{ff'}$, modulo compact operators.*

Pseudoconvexity is a notion of convexity which is invariant under complex-analytic changes of coordinates. All smooth, bounded domains in $\mathbb{C}$ are strongly pseudoconvex. An open ball in $\mathbb{C}^n$ is strongly pseudoconvex but the product of two or more balls is not.

2.8.3 PROPOSITION *Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^n$. If f is a continuous function on $\partial\Omega$ then the Toeplitz operator T_f is compact if and only if f is identically zero.*

The two propositions allow us to associate a unital, injective extension of C^*-algebras to the strongly pseudoconvex domain Ω, as follows.

2.8.4 DEFINITION The *Toeplitz algebra* $\mathfrak{T}(\partial\Omega)$ is the C^*-algebra generated by the Toeplitz operators T_f, where $f \in C(\partial\Omega)$, together with all the compact operators on $H^2(\partial\Omega)$. The *Toeplitz extension* for $\partial\Omega$ is the short exact sequence

$$0 \longrightarrow \mathfrak{K}(H^2(\partial\Omega)) \longrightarrow \mathfrak{T}(\partial\Omega) \longrightarrow C(\partial\Omega) \longrightarrow 0\,,$$

where the quotient map sends $T_f \in \mathfrak{T}(\partial\Omega)$ to $f \in C(\partial\Omega)$.

If f is a continuous, complex-valued function on $\partial\Omega$ which is nowhere zero then the Toeplitz operator T_f is Fredholm. The proof is the same as the one in Section 2.3. However, for higher-dimensional domains the Fredholm index theory of T_f may not be very interesting. For instance if Ω is a ball in $\mathbb{C}^n$, and if $n > 1$, then the index of T_f is zero. This is because every f is homotopic, through nowhere-zero continuous functions, to a constant map on $\partial\Omega$, while the Fredholm index of T_f depends only on the homotopy class of f.

To obtain interesting Fredholm index problems in this case we must consider not single Toeplitz operators but Toeplitz *systems*. Thus, suppose that $F\colon \partial\Omega \to M_N(\mathbb{C})$ is a continuous matrix-valued function on $\partial\Omega$. Since this is the same thing as an $N \times N$ matrix of continuous, complex-valued functions on $\partial\Omega$ we may associate to F an $N \times N$ matrix T_F of Toeplitz operators. As before, $T_{F_1} T_{F_2} = T_{F_1 F_2}$ modulo matrices of compact operators. Therefore, if F is nowhere singular (that is, everywhere invertible), then T_F, viewed as an operator on an N-fold direct sum of Hardy spaces, is Fredholm. There is no domain Ω for which the indices of all these Toeplitz systems are zero. In particular, the Toeplitz extension is never split.

What is the index of the Toeplitz system T_F? The question directs us toward some very substantial issues in analysis and topology, and for the time being we can only sketch an answer. The index of T_F is a homotopy invariant of the map F from $\partial\Omega$ to the group $GL(N, \mathbb{C})$ of invertible $N \times N$ complex matrices, and so first of all we seek a description of the group of homotopy classes of such maps. If Ω is an open ball in $\mathbb{C}^n$ and if $N \geqslant n$ then the celebrated Bott Periodicity Theorem asserts that this homotopy group is infinite cyclic, so that to each map $F\colon \partial\Omega \to GL(N, \mathbb{C})$ there is associated an integer 'degree' which completely classifies F up to homotopy. An explicit calculation of $\operatorname{Index}(T_F)$ for a single degree-one map F leads to the index formula

$$\operatorname{Index}(T_F) = -\operatorname{degree}(F),$$

by an argument which generalizes our proof of the Toeplitz Index Theorem 2.3.2. For other domains Ω the index problem is solved by a combination of topological ideas which extend Bott periodicity and analytic ideas which, in effect, reduce the index problem to the case where Ω is a ball.

2.8(b) *The Pseudodifferential Operator Extension*

Let M be a smooth manifold without boundary (but not necessarily compact). Denote by S^*M the *cosphere bundle* of M, obtained from the cotangent bundle of M by deleting the zero cotangent vectors and identifying non-zero cotangent vectors which differ only by multiplication by a positive scalar (if M is equipped with a Riemannian metric then S^*M identifies with the space of unit length cotangent vectors; hence the name). We are going to sketch the construction of the *pseudodifferential operator extension*

$$0 \longrightarrow \mathfrak{K}(L^2(M)) \longrightarrow \mathfrak{P}(M) \longrightarrow C_0(S^*M) \longrightarrow 0 \tag{2.8.5}$$

for the manifold M.

The first case to consider is that for which M is an open subset of $\mathbb{R}^n$. Suppose that σ is a smooth complex-valued function on T^*M which has the following homogeneity property:

$$\sigma(x, t\xi) = \sigma(x, \xi) \quad \text{for } t \geqslant 1 \text{ and } |\xi| \geqslant 1.$$

Assume further that σ is 'compactly supported in the M-direction', which is to say that $\sigma(x, \xi)$ vanishes for x outside some compact subset of M. Then the linear map $D_\sigma \colon C_c^\infty(M) \to C_c^\infty(M)$ given by the integral formula

$$D_\sigma f(x) = \frac{1}{(2\pi)^n} \int \sigma(x, \xi)\widehat{f}(\xi)e^{i\langle x, \xi\rangle}\, d\xi, \tag{2.8.6}$$

where $\widehat{f}$ denotes the Fourier transform of the function f, is an example of a *pseudodifferential operator* on M. Thanks to its homogeneity, the function σ defines a function σ_0 on the cosphere bundle S^*M; we shall refer to this as the *symbol* of the operator D_σ. The theory of pseudodifferential operators implies that:

2.8.7 PROPOSITION *The operator D_σ extends by continuity to a bounded linear operator on $L^2(M)$. The map which associates to each D_σ its symbol σ_0 extends to a $*$-homomorphism from the C^*-algebra $\mathfrak{P}(M)$ generated by all the D_σ onto the C^*-algebra $C_0(S^*M)$.* □

Let us call the above $*$-homomorphism from $\mathfrak{P}(M)$ to $C_0(S^*M)$ the *symbol map* for $\mathfrak{P}(M)$.

2.8.8 PROPOSITION *An operator $T \in \mathfrak{P}(M)$ is compact if and only if its symbol is zero. Every compact operator belongs to $\mathfrak{P}(M)$.* □

If M is an open subset of $\mathbb{R}^n$ then the two propositions combine to determine the injective extension

$$0 \longrightarrow \mathfrak{K}(L^2(M)) \longrightarrow \mathfrak{P}(M) \longrightarrow C_0(S^*M) \longrightarrow 0\,,$$

that we are seeking. To construct the extension for a general manifold we need two facts, the first very elementary but the second quite substantial. The simple fact is that if M is an open subset of $\mathbb{R}^n$, and if $g \in C_c^\infty(M)$, then the multiplication operator M_g is a pseudodifferential operator — namely the one associated to the function $\sigma(x, \xi) = g(x)$. So M_g is an element of $\mathfrak{P}(M)$. The second fact relates to the invariance of pseudodifferential operators under smooth changes of coordinates. To state it, observe that a diffeomorphism $\Phi: M \to M'$ of smooth manifolds induces an isomorphism $\Phi^*: T^*M' \to T^*M$ and thus a $*$-isomorphism $\Phi_*: C_0(S^*M) \to C_0(S^*M')$.

2.8.9 PROPOSITION *If $\Phi: M \to M'$ is a diffeomorphism of open subsets in $\mathbb{R}^n$ then the transform*[17] *under Φ of an operator in $\mathfrak{P}(M)$ with symbol σ_0 is an operator in $\mathfrak{P}(M')$ with symbol $\Phi_*\sigma_0$.*

This invariance property, combined with a partition of unity argument, allows us to define $\mathfrak{P}(M)$ for any smooth manifold as the C^*-algebra of those bounded operators T on $L^2(M)$ such that

(a) $\lim \|TM_{g_n} - T\| = 0 = \lim \|M_{g_n}T - T\|$ for some approximate unit g_n for $C_0(M)$,
(b) T commutes with $C_0(M)$ modulo compact operators, and
(c) for each coordinate chart U and each $g \in C_0(U)$, the operator M_gTM_g belongs to $\mathfrak{P}(U)$.

The symbol of T is well-defined as a continuous function on the cosphere bundle S^*M, vanishing at infinity, and now Propositions 2.8.7 and 2.8.8 extend to general M. This completes the construction of the pseudodifferential operator extension for general M.

The index theory of the pseudodifferential operator extension associated to a *closed* manifold is rather similar to that of the higher-dimensional Toeplitz extensions. The operator D_σ is Fredholm if and only if its symbol σ_0 is nowhere zero. For many manifolds M the indices of these individual pseudodifferential operators D_σ are all zero, but in all cases there are pseudodifferential *systems* which are non-trivial from the point of view of index theory. As in the Toeplitz case, the index of certain pseudodifferential systems may be calculated by using Bott periodicity. A complete analysis of the index problem for pseudodifferential operators was carried out by Atiyah and Singer, who were able to reduce the general problem to the cases covered by Bott periodicity. The Index Theorem of Atiyah and Singer will be discussed in Chapter 11.

2.8(c) *Spectral Projections*

Let M be a smooth, closed manifold. A variety of supplementary topological or geometric hypotheses on M give rise to natural symmetric, first-order, elliptic partial differential operators on M. For example, if M is given a Riemannian metric and an orientation then there is an associated *signature operator*, while

[17] The diffeomorphism $\Phi: M \to M'$ induces a map $C_c^\infty(M') \to C_c^\infty(M)$ by composition. We obtain a unitary $U: L^2(M') \to L^2(M)$ by multiplying by the square root of the Jacobian of f. By 'the transform' of an operator $T \in \mathfrak{B}(L^2(M))$ we mean the operator $U^*TU \in \mathfrak{B}(L^2(M'))$.

to a Spin or Spin^c structure on M there corresponds the Dirac operator. The reader unfamiliar with these matters can proceed for now with a more elementary example: the operator $-i d/d\theta$ on the unit circle S^1 (as it happens, this is both the signature operator and the Dirac operator for the manifold S^1; needless to say the situation for higher-dimensional manifolds is more complicated).

Every symmetric, first-order, partial differential operator D on a smooth closed manifold M is essentially selfadjoint.[18] As a result, if χ is any bounded Borel function on $\mathbb{R}$ then the functional calculus associates to χ and D a bounded operator $\chi(D)$. We are interested here in the characteristic function of the half-line $[0, \infty)$. In this case the operator $P = \chi(D)$ is a projection. If D is, in addition, elliptic then it may be shown that P is a pseudodifferential operator of essentially the type considered in the previous paragraph. Hence:

2.8.10 PROPOSITION *If D is elliptic then for every continuous function $g \in C(M)$ the commutator $M_g P - P M_g$ is a compact operator.* □

We therefore obtain from the projection P an abstract Toeplitz extension $\varphi_P \colon C(M) \to \mathfrak{Q}(H)$ by the formula

$$\varphi_P(g) = \pi(P M_g P),$$

where as usual π is the projection from the bounded operators to the Calkin algebra.

The index-theoretic aspects of the extension φ_P are again governed by the theorem of Atiyah and Singer; they will be taken up in Chapter 11. It is a remarkable fact that up to unitary equivalence *every* unital, injective extension of $C(M)$ by $\mathfrak{K}(H)$ is of the form φ_P. Clearly this indicates a close connection between extension theory and geometry.

2.9 Exercises

2.9.1 Show that if S is a rank-one operator and T is any operator, then ST and TS are rank-one operators. Deduce that, as asserted in the text, the compact operators $\mathfrak{K}(H)$ on a Hilbert space H form a closed ideal in $\mathfrak{B}(H)$.

2.9.2 Let $\mathfrak{F}$ be the vector space of finite-rank operators from H to H', where H and H' are Hilbert spaces. Show that an inner product may be defined on $\mathfrak{F}$ by

$$\langle S, T \rangle_{HS} = \mathrm{Trace}(S^* T)$$

where the trace of a finite-rank operator is the sum of its diagonal matrix entries relative to an orthonormal basis. This inner product is called the *Hilbert–Schmidt* inner product.

[18] This and various other aspects of the present discussion will be taken up in much more detail in Chapter 10.

Let $\|\cdot\|_{HS}$ denote the corresponding norm. Prove that

$$\|T\| \leqslant \|T\|_{HS}$$

for any $T \in \mathfrak{F}$. Deduce that the completion of $\mathfrak{F}$ with respect to the Hilbert–Schmidt inner product may be identified with a certain space of bounded operators from H to H'. These are called the *Hilbert–Schmidt operators*.

Prove that every Hilbert–Schmidt operator is compact, and give an example of a compact operator which is not Hilbert–Schmidt. If either H or H' is of finite dimension n, prove that every bounded operator is Hilbert–Schmidt and that $\|T\|_{HS} \leqslant n^{\frac{1}{2}}\|T\|$.

2.9.3 An operator $T \in \mathfrak{B}(H)$ is said to be of *trace class* if it can be written $T = T_1T_2$, where T_1 and T_2 are Hilbert–Schmidt operators. Show that if this is so, then the number

$$\mathrm{Trace}(T) = \langle T_1^*, T_2\rangle_{HS}$$

depends only on T, and that

$$\mathrm{Trace}(ST) = \mathrm{Trace}(TS)$$

for all bounded operators S.

Suppose now that $S, T \in \mathfrak{B}(H)$ and that $K_1 = ST - I$ and $K_2 = TS - I$ are of trace class. Show that T is Fredholm and that

$$\mathrm{Index}\,(T) = \mathrm{Trace}(K_2) - \mathrm{Trace}(K_1).$$

2.9.4 Let $\{\lambda_i\}_{i=1}^{\infty}$ and $\{\mu_j\}_{j=1}^{\infty}$ be two bounded sequences of real numbers, and suppose that they have the same limit points. Show that there is a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $|\lambda_i - \mu_{\sigma(i)}| \to 0$ as $i \to \infty$. Use this fact to show that two selfadjoint diagonal operators having the same essential spectrum are essentially unitarily equivalent.

2.9.5 Let H be a fixed separable infinite-dimensional Hilbert space. Show that an essentially normal operator T' on any other such Hilbert space H' is essentially unitarily equivalent to an essentially normal operator T on H, and that T is determined by this condition up to essential unitary equivalence. Deduce that $\mathrm{Ext}\,(X)$ may be defined as the collection of equivalence classes of essentially normal operators (with essential spectrum X) on the *fixed* Hilbert space H. Note that this latter definition is not subject to set-theoretic criticism.

How is the addition operation on $\mathrm{Ext}\,(X)$ represented in this new formulation?

2.9.6 Use the Weyl–von Neumann Theorem and the functional calculus to show that any two unitary operators with the same essential spectrum are essentially unitarily equivalent.

2.9.7 Let H be the Hardy space $H^2(S^1)$. Show that the C^*-subalgebra of $\mathfrak{B}(H)$ generated by the Toeplitz operators T_g, $g \in C(S^1)$, contains all the compact operators on H. Thus we did not need to add the compact operators as generators when defining the Toeplitz extension $\mathfrak{T}$.

2.9.8 The Toeplitz operator T_z with symbol z is called the *unilateral shift*. It is a proper isometry (that is, an isometry which is not a unitary).

(a) Prove the *Wold decomposition theorem*: every isometry V is unitarily equivalent to a direct sum

$$U \oplus (\oplus^n T_z) = U \oplus T_z \oplus T_z \oplus \cdots ,$$

where U is unitary and the number n of copies of T_z can be finite or infinite. (Hint: Let $E = \text{Kernel}(V^*)$ and let $n = \text{Dim}(E)$. Show that $V^j(E)$ is orthogonal to $V^k(E)$ for $j \neq k$.)

(b) Deduce that if v is an isometry in a C^*-algebra A, then there is a unique $*$-homomorphism $\mathfrak{T} \to A$ which maps T_z to v.

2.9.9 Prove that, as asserted in the text, every normal Fredholm operator has index zero.

2.9.10 Let A be a unital C^*-algebra.

(a) Show that an extension of A with kernel J can be identified with an extension of A/J.

(b) Show that a possibly non-unital extension of A can be identified with a unital extension of the unitalization $\widetilde{A}$.

(c) Classify *all* extensions of $A = C[0, 1]$, up to unitary equivalence.

2.9.11 In terms of the description of extensions as short exact sequences

$$0 \longrightarrow \mathfrak{K}(H) \longrightarrow E \longrightarrow A \longrightarrow 0 ,$$

what does it mean for an extension to be unital? What does it mean for an extension to be injective? How is the addition of extensions described in this picture?

2.9.12 Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. The *Bergman space* $H^2(\mathbb{D})$ is the space of holomorphic functions on D which are square-integrable with respect to Lebesgue measure. It is a closed subspace of $L^2(\mathbb{D})$. Show that the orthogonal projection $P\colon L^2(\mathbb{D}) \to H^2(\mathbb{D})$ commutes, modulo compact operators, with multiplication by every continuous, complex-valued f function on the closed unit disk $\overline{\mathbb{D}}$.

Denote by T_f the Toeplitz operator on $H^2(\mathbb{D})$ obtained by compressing pointwise multiplication by f to $H^2(\mathbb{D})$. If g is the restriction of f to $S^1 = \partial D$ then prove directly (without recourse to the Brown–Douglas–Fillmore Theorem or the next exercise) that the operators $T_f\colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ and $T_g\colon H^2(S^1) \to H^2(S^1)$ are essentially unitarily equivalent.

2.9.13 The purpose of this exercise is to show that $\mathrm{Ext}\,(S^1) \cong \mathbb{Z}$. For the first part of the argument we can replace S^1 with any compact set $X \subseteq \mathbb{C}$; show that if T is essentially normal, with essential spectrum X, and if $\lambda \in X$, then there is an orthonormal sequence $\{v_n\}$ such that

$$\|Tv_n - \lambda v_n\| \to 0 \quad \text{and} \quad \|T^* v_n - \bar{\lambda} v_n\| \to 0.$$

Conclude that T is essentially unitarily equivalent to a direct sum $T' \oplus D'$, where T' is essentially normal and D' is diagonal, with essential spectrum X. Using the fact that any two diagonal operators with the same essential spectrum are essentially unitarily equivalent, conclude that, if D is any diagonal operator with essential spectrum X, then

$$T \oplus D \approx T' \oplus D' \oplus D \approx T' \oplus D' \approx T,$$

where '$\approx$' denotes essential unitary equivalence. This shows that the diagonal operators give a zero element in $\mathrm{Ext}\,(X)$. Now specialize to $X = S^1$. Show that if $\mathrm{Index}\,(T) = 0$ then T is a compact perturbation of a unitary. (Hint: perturb T to make it invertible, then form $U = T(T^*T)^{-\frac{1}{2}}$.) Conclude from the preceding discussion that T gives the zero element in $\mathrm{Ext}\,(S^1)$. Deduce that the homomorphism $\mathrm{Index}: \mathrm{Ext}\,(S^1) \to \mathbb{Z}$ is an isomorphism. (See [46].)

2.9.14 Show that $\mathrm{Ext}\,(M_n(\mathbb{C})) \cong \mathbb{Z}/n\mathbb{Z}$.

2.9.15 Two extensions $\varphi_0, \varphi_1\colon A \to \mathfrak{Q}(H)$ are *weakly equivalent* if there is a unitary $u \in \mathfrak{Q}(H)$ such that $\varphi_0 = \mathrm{Ad}_u\,\varphi_1$. The weak equivalence classes of unital, injective extensions form a commutative semigroup $\mathrm{Ext}_w(A)$. Assume for the rest of the exercise that A is a C^*-algebra for which $\mathrm{Ext}\,(A)$ is a group. Show that $\mathrm{Ext}_w(A)$ is also a group and that there is an exact sequence

$$\mathbb{Z} \to \mathrm{Ext}\,(A) \to \mathrm{Ext}_w(A) \to 0.$$

Calculate this sequence for $A = M_n(\mathbb{C})$. Show also that if there is a unital $*$-homomorphism $A \to \mathbb{C}$ (for instance, if A is commutative) then the natural map $\mathrm{Ext}\,(A) \to \mathrm{Ext}_w(A)$ is an isomorphism.

2.9.16 The CAR (= canonical anticommutation relations) algebra is the C^*-algebra obtained by completing the increasing union of matrix algebras

$$M_2(\mathbb{C}) \subseteq M_4(\mathbb{C}) \subseteq M_8(\mathbb{C}) \subseteq M_{16}(\mathbb{C}) \subseteq \dots,$$

where the inclusion maps are given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Show that Ext (CAR) is isomorphic to the additive group of the ring of 2-adic integers. (See [105].)

2.9.17 Among the only other C^*-algebras for which a direct computation of Ext (A) is possible are the *Cuntz algebras* $\mathcal{O}_n$. These are the C^*-algebras generated by n isometries $V_1, \dots, V_n$ such that $V_1V_1^* + \cdots + V_nV_n^* = I$. One can show that $\mathcal{O}_n$ does not depend on the choice of V_i (the C^*-algebras generated by any two such systems of n isometries are canonically isomorphic). Using this fact, and assuming that Ext $(\mathcal{O}_n)$ is a group, show that Ext $(\mathcal{O}_n) \cong \mathbb{Z}$ and $\text{Ext}_w(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$. (See [103].)

2.9.18 Denote by $SU(2)$ the group of determinant-one 2×2 unitary matrices and denote by Ω the open unit ball in $\mathbb{C}^2$. If v is a unit vector in $\mathbb{C}^2$ then there is a unique matrix U_v in $SU(2)$ whose first column is v, and the map $F\colon v \mapsto U_v$ is a homeomorphism from $\partial\Omega$ onto $SU(2)$. Calculate the index of the 2×2 Toeplitz system T_F. Hint: This is best done using the representation theory of the compact group $SU(2)$. Construct representations of $SU(2)$ on the domain and range of the operator T_F — as Hilbert spaces they are both $H^2(\partial\Omega)\oplus H^2(\partial\Omega)$ — in such a way that T_F becomes an intertwining operator. Decompose both representations into finite-dimensional, isotypical subspaces and use Schur's Lemma to deduce that T_F decomposes into a corresponding direct sum of finite-dimensional 'isotypical operators' $T_F^\pi\colon V^\pi \to W^\pi$. Then

$$\text{Index}\,(T_F) = \sum_\pi \text{Index}\,(T_F^\pi) = \sum_\pi (\text{Dim}(V^\pi) - \text{Dim}(W^\pi)),$$

from which Index (T_f) may be computed using explicit representation theory calculations. For related information on the connection between Fredholm indices and representation theory see [28].

2.9.19 As in the previous exercise, denote by Ω the unit ball in $\mathbb{C}^2$. Show that the Fredholm index of 2×2 Toeplitz systems on $\partial\Omega$ determines a homomorphism from the group $\pi_3(GL(2,\mathbb{C}))$ to $\mathbb{Z}$. Show that the inclusion $SU(2) \subset GL(2,\mathbb{C})$ induces an isomorphism $\pi_3(SU(2)) \cong \pi_3(GL(2,\mathbb{C}))$. Use the fact that $\pi_3(S^3) \cong \mathbb{Z}$ to deduce that $\pi_3(GL(2,\mathbb{C}))$ is infinite cyclic, and then use the result of the previous exercise to precisely formulate and prove an index theorem for 2×2 Toeplitz systems on $\partial\Omega$. Can you extend your result to $N \times N$ systems?

2.9.20 If f is a smooth and compactly supported function on $\mathbb{R}$ then the *Hilbert transform* of f is the function

$$Hf(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{x-t}\,dt$$

(the integral is regarded as a principal value). Show that $Hf \in L^2(\mathbb{R})$ and that H extends by continuity to a bounded operator on $L^2(\mathbb{R})$. Show that if $g \in C_0(\mathbb{R})$ then H commutes with the multiplication operator M_g modulo compact operators. Show that H and $C_0(\mathbb{R})$ generate the pseudodifferential operator extension for the manifold $\mathbb{R}$.

2.9.21 Give the precise relation between the Toeplitz extension for S^1, the pseudodifferential operator extension for S^1, and the extension associated to the operator $-i\,d/d\theta$ on S^1.

2.10 Notes

An alternate account of Fredholm operator theory is presented by Douglas in [49], along with an extensive introduction to Toeplitz operator theory. Douglas wrote his book just prior to his work with Brown and Fillmore, and it is interesting to look at the exercises to Chapter 5 of [49] (Exercise 5.16 in particular) in the light of what came later. Atkinson's Theorem was proved in [19].

The theory of $\operatorname{Ext}(A)$, initially for commutative A, was developed by Brown, Douglas, and Fillmore in their celebrated papers [33, 34]. For an inspiring survey of this work see [50]. The relationship between extensions and homomorphisms to the Calkin algebra was observed earlier by Busby [36], and the homomorphism corresponding to a given extension is sometimes called its *Busby invariant*.

A comprehensive reference for the theory of Toeplitz operators in several complex variables is [30]. The relationship between their index theory and the Atiyah–Singer Index Theorem was worked out in the context of K-homology by Baum, Douglas, and Taylor [25, 24]. For a shorter account, see [64]. A survey article of Atiyah [8] gives a beautiful introduction to the Bott Periodicity Theorem from the point of view of index theory. Atiyah writes that 'this theorem of Bott must rank as one of the real achievements of topology and it is certainly something of which everybody should be aware'.

The theory of pseudodifferential operators can be found in [126, 127]. It is typically the case that the applications of analysis to topology do not require the most sophisticated versions of pseudodifferential theory. The reader may also wish to consult Section 5 of [16]. Our sign and normalization conventions for the Fourier transform agree with those of [16, 127] and disagree with those of [117, 126].

3

COMPLETELY POSITIVE MAPS

The main result of the chapter is Voiculescu's Theorem, that if A is a separable and unital C^*-algebra then the semigroup Ext (A) has a zero element. But we shall begin by exploring in more depth the question of invertibility of extensions. It is a happy coincidence that substantially the same techniques can be applied to both matters. Using the same tools we will also prove a fundamental technical result of Kasparov which will be central to our later development of K-homology theory.

3.1 Completely Positive Maps

In Definition 2.7.6 we introduced the notion of semisplit extension. The following definition and theorem give a very simple and useful characterization of this concept.

3.1.1 Definition A bounded linear map $\sigma\colon A \to B$ between two unital C^*-algebras is *completely positive* if $\sigma(1) = 1$ and

$$\sum_{i,j} b_i^* \sigma(a_i^* a_j) b_j \geqslant 0$$

for all n, all $a_1, \ldots, a_n \in A$, and all $b_1, \ldots, b_n \in B$.

3.1.2 Remarks It is easy to check that the other conditions on a completely positive map automatically imply that σ is bounded, so we could have left boundedness out of the definition. We could also have given a 'non-unital' definition, simply by disregarding the condition $\sigma(1) = 1$, but we shall not need to consider this more general sort of completely positive map.

3.1.3 Stinespring's Theorem *Let* A *be a unital* C^**-algebra. A unital linear map* $\sigma\colon A \to \mathfrak{B}(H)$ *is completely positive if and only if there are*

(a) *an isometry* $V\colon H \to H_1$, *and*

(b) *a non-degenerate representation* $\rho\colon A \to \mathfrak{B}(H_1)$,

such that $\sigma(a) = V^*\rho(a)V$ *for all* $a \in A$.

Proof The map $\sigma(a) = V^*\rho(a)V$ is completely positive since

$$\sum_{i,j} b_i^* \sigma(a_i^* a_j) b_j = \left(\sum_i \rho(a_i) V b_i\right)^* \left(\sum_i \rho(a_i) V b_i\right) \geqslant 0.$$

The proof that every completely positive map is of this form uses a device similar to the GNS construction of Theorem 1.4.4. Define a sesquilinear form on the algebraic tensor product $A \otimes H$ (taken over $\mathbb{C}$) by

$$\left\langle \sum_i a_i \otimes \xi_i, \sum_j a_j' \otimes \xi_j' \right\rangle = \sum_{i,j} \langle \xi_i, \sigma(a_i^* a_j')\xi_j' \rangle.$$

This is positive-semidefinite because σ is a completely positive map. As in the GNS construction, take the quotient of $A \otimes H$ by the subspace N comprising vectors with length zero, to obtain a genuine inner product space, and then complete this quotient space to obtain a Hilbert space H_1. The natural representation of A on $A \otimes H$ by left multiplication descends to a representation $\rho\colon A \to \mathfrak{B}(H_1)$, and the map $V\colon \xi \mapsto 1 \otimes \xi + N$ has the stated properties. □

3.1.4 Remark It is natural to ask whether complete positivity is equivalent to positivity (meaning that $\sigma(a^*a) \geqslant 0$ for every $a \in A$). In fact this is not so, although counterexamples are not so easy to find. For instance, if A or B is commutative then every positive map $\varphi\colon A \to B$ is automatically completely positive. See Exercises 3.9.1 and 3.9.2.

Here is the application of complete positivity to semisplit extensions.

3.1.5 Theorem *An extension $\varphi\colon A \to \mathfrak{Q}(H)$ is semisplit if and only if there is a completely positive map $\sigma\colon A \to \mathfrak{B}(H)$ such that $\varphi(a) = \pi(\sigma(a))$ for all $a \in A$.*

In short, $\varphi\colon A \to \mathfrak{Q}(H)$ is semisplit if and only if it has a *completely positive lifting* to $\mathfrak{B}(H)$. We isolate one component of the proof as a lemma: it will have other applications later.

3.1.6 Lemma *Let A be a C^*-algebra, and suppose that $\rho\colon A \to \mathfrak{B}(H \oplus H')$ is a representation of A on a Hilbert space which is decomposed as an orthogonal direct sum. Let*

$$\rho(a) = \begin{pmatrix} \rho_{11}(a) & \rho_{12}(a) \\ \rho_{21}(a) & \rho_{22}(a) \end{pmatrix}$$

be the matrix representation of ρ relative to this direct sum decomposition. Suppose further that ρ_{11} is a $*$-homomorphism modulo the compact operators: that is, the composite map

$$A \xrightarrow{\rho_{11}} \mathfrak{B}(H) \xrightarrow{\pi} \mathfrak{Q}(H)$$

is a $*$-homomorphism. Then $\rho_{12}(a)$ and $\rho_{21}(a)$ are compact for all $a \in A$, and ρ_{22} is also a $*$-homomorphism modulo the compact operators.

PROOF Since ρ is a $*$-homomorphism, matrix multiplication gives

$$\rho_{11}(aa^*) = \rho_{11}(a)\rho_{11}(a)^* + \rho_{12}(a)\rho_{12}(a)^*.$$

Since ρ_{11} is a $*$-homomorphism modulo the compact operators, it follows that $\rho_{12}(a)\rho_{12}(a)^*$ is compact for each a, and as a result[19] $\rho_{12}(a)$ and $\rho_{21}(a) = \rho_{12}(a^*)^*$ are compact. From this it follows that ρ_{22} is a $*$-homomorphism modulo the compact operators. □

PROOF OF THEOREM 3.1.5 If φ is semisplit then there is some φ' such that $\varphi \oplus \varphi'$ lifts to a unital $*$-homomorphism $\rho\colon A \to \mathfrak{B}(H \oplus H')$. If $V\colon H \to H \oplus H'$ is the obvious inclusion then the map $\sigma(a) = V\rho(a)V^*$ is a lifting of φ which is completely positive and unital. Conversely, suppose that $\sigma\colon A \to \mathfrak{B}(H)$ is a completely positive lifting of φ. Stinespring's Theorem provides a non-degenerate representation $\rho\colon A \to \mathfrak{B}(H_1)$ and an isometry $V\colon H \to H_1$ such that $\sigma(a) = V^*\rho(a)V$ for all $a \in A$. We use this isometry to identify H with a subspace of H_1. Applying Lemma 3.1.6 we deduce that $\rho(a)$ commutes, modulo compact operators, with the orthogonal projection $P = VV^*\colon H_1 \to H$. It follows that φ is unitarily equivalent to the abstract Toeplitz extension $\varphi_P(a) = \pi(P\rho(a)P)$. By Proposition 2.7.10, φ is semisplit. □

3.2 Quasicentral Approximate Units

Every $*$-homomorphism is a completely positive map. But for many purposes completely positive maps are much more flexible than $*$-homomorphisms. For instance, if σ_1 and σ_2 are completely positive then so is any 'average' of the form

$$\sigma(a) = x_1^*\sigma_1(a)x_1 + x_2^*\sigma_2(a)x_2,$$

provided that $x_1^*x_1 + x_2^*x_2 = 1$.

As well as such averages we shall also need to discuss limits of completely positive maps. Such limits will always be taken in the *point-norm topology*. By definition, a sequence $\sigma_n\colon A \to B$ of maps between C^*-algebras converges in this topology to a map σ if, for every $a \in A$, the sequence $\sigma_n(a)$ converges (in the norm of B) to $\sigma(a)$. Clearly, a point-norm limit of completely positive maps is itself completely positive.

A much deeper closure property of completely positive maps is this:

3.2.1 PROPOSITION *Let A be a separable and unital C^*-algebra, and let J be a separable ideal in a unital C^*-algebra B. The set of all those completely positive maps $A \to B/J$ which lift to completely positive maps into B is closed in the point-norm topology.*

[19] It follows easily from the C^*-identity for the Calkin algebra that if T is a bounded operator such that TT^* is compact, then T itself is compact.

If $\pi\colon B \to B/J$ denotes the quotient map then we shall say in brief that a completely positive map $\sigma\colon A \to B/J$ is *liftable* if there is a completely positive map $\tilde{\sigma}\colon A \to B$ such that $\sigma = \pi \circ \tilde{\sigma}$. Thus the proposition asserts that the set of liftable completely positive maps into B/J is point-norm closed.

Before proving Proposition 3.2.1 let us use the result to prove a lifting theorem for completely positive maps from *commutative* C*-algebras.

3.2.2 THEOREM *Let* J *be a separable ideal in a unital* C*-*algebra* B. *Let* A *be a unital, separable and commutative* C*-*algebra. Then every completely positive map* $\sigma\colon A \to B/J$ *is liftable.*

3.2.3 COROLLARY *If* A *is unital, separable and commutative then every unital extension* $\varphi\colon A \to \mathfrak{Q}(H)$ *is semisplit.* □

PROOF OF THEOREM 3.2.2 Write $A = C(X)$, where X is a compact metric space, and for each n choose a finite cover of X by open sets of diameter $1/n$ or less. Choose a partition of unity $\{f_1, f_2, \ldots, f_{k_n}\}$ subordinate to this cover, and also choose points $\{x_1, x_2, \ldots, x_{k_n}\}$, one from each member of the open cover. Suppose we are given a completely positive map $\sigma\colon C(X) \to B/J$. Define completely positive maps

$$\sigma_n\colon C(X) \to B/J$$

by

$$\sigma_n(f) = \sigma\left(\sum_{i=1}^{k_n} f(x_i) f_i\right).$$

As $n \to \infty$ the mesh of the associated cover tends to zero, and so $\sigma_n(f) \to \sigma(f)$. Thus σ is the point-norm limit of the completely positive maps σ_n. But each of the maps σ_n is liftable: simply choose[20] positive operators F_i such that $\pi(F_i) = \sigma(f_i)$ and $\sum F_i = I$, then lift σ_n to $\tilde{\sigma}_n(f) = \sum f(x_i) F_i$. Proposition 3.2.1 now completes the proof. □

The proof of Proposition 3.2.1 requires the following concept, which is fundamental to a number of the constructions in this book.

3.2.4 DEFINITION Let J be an ideal in a C*-algebra B. An approximate unit $\{u_\alpha\}$ for J is *quasicentral* for B if $\|bu_\alpha - u_\alpha b\| \to 0$, for every $b \in B$.

3.2.5 REMARK As was the case with ordinary approximate units, if B is separable we may require our quasicentral approximate unit to be a sequence, while if B is non-separable we must settle for a net.

It is typically straightforward to exhibit quasicentral approximate units for concrete J and B. In fact they always exist, although for simplicity we shall give the proof in the separable case only.

[20] How? See Exercise 3.9.4.

3.2.6 THEOREM *If* J *is an ideal in a separable* C^**-algebra* B *then there is an approximate unit for* J *which is quasicentral for* B. *In fact, if* $\{u_n\}$ *is any approximate unit for* J *then there is a quasicentral approximate unit* $\{v_n\}$ *for which each* v_n *is a finite convex combination of the elements* $\{u_n, u_{n+1}, \ldots\}$.

PROOF If J is an ideal in B, and if $\{u_1, u_2, \ldots\}$ is an approximate unit for J then $\varphi(u_n b - b u_n) \to 0$ for every state φ of B and every $b \in B$. To see this, use the GNS construction to represent φ as a vector state $\varphi(b) = \langle \pi(b)v, v \rangle$, and then use the fact that the operators $\pi(u_n)$ converge strongly to the orthogonal projection onto the closed B-invariant subspace $\overline{\pi(J)H}$; any orthogonal projection onto a closed B-invariant subspace must commute with B. Since every bounded linear functional on B is a linear combination of states, it follows that for every $b \in B$ the sequence $x_n = b u_n - u_n b$ converges to zero in the weak topology of B. By a standard consequence of the Hahn–Banach Theorem, some sequence y_n of convex combinations of the x_n converges to zero in the norm topology; we may assume that y_n is a convex combination of $\{x_n, x_{n+1}, \ldots\}$. Now $y_n = b w_n - w_n b$, where w_n is a convex combination of the elements $\{u_n, u_{n+1}, \ldots\}$. Furthermore by a simple diagonal argument we can arrange that $\|b w_n - w_n b\| \to 0$, not just for a single b, but for all b in a finite subset of B. Now let $\{b_1, b_2, \ldots\}$ be a countable dense set in B and take v_n to be a convex combination of the elements $\{u_n, u_{n+1}, \ldots\}$ (namely one of the w_k above) such that $\|v_n b_j - b_j v_n\| < 1/n$ for all $j \leqslant n$. □

PROOF OF PROPOSITION 3.2.1 Let $\sigma_n \colon A \to B/J$ be a sequence of completely positive maps, and let $\rho_n \colon A \to B$ be completely positive liftings. Suppose that the sequence $\{\sigma_n\}$ converges to σ; we must show that $\sigma \colon A \to B/J$ lifts to a unital completely positive map $\rho \colon A \to B$. There is no loss of generality in assuming that B is separable (if not, replace it by the C^*-subalgebra generated by the images of all the maps ρ_n).

Let $\{a_1, a_2, \ldots\}$ be a dense set in A. By passing to a subsequence of $\{\sigma_n\}$, if necessary, we can assume that the following convergence condition holds:

$$j < N \quad \Rightarrow \quad \|\sigma_N(a_j) - \sigma_{N-1}(a_j)\| < 2^{-N}.$$

We shall now alter, one at a time, the liftings ρ_n to liftings ρ'_n, for which

$$j < N \quad \Rightarrow \quad \|\rho'_N(a_j) - \rho'_{N-1}(a_j)\| < 2^{-N}.$$

This will prove the lemma, since then we can define $\rho \colon A \to B$ to be the limit of the ρ'_n, which will lift $\sigma \colon A \to B/J$. We set $\rho'_1 = \rho_1$ and, supposing that $\rho'_1, \ldots, \rho'_N$ have been defined, we define ρ'_{N+1} as follows. Let $u_1, u_2, \ldots$ be a quasicentral approximate unit for $J \subseteq B$ and set

$$\rho'_{N+1}(a) = (1 - u_k)^{\frac{1}{2}} \rho_{N+1}(a)(1 - u_k)^{\frac{1}{2}} + u_k^{\frac{1}{2}} \rho'_N(a) u_k^{\frac{1}{2}},$$

for sufficiently large k. It follows from the formula (1.7.5) for the quotient C*-algebra norm in B/J that if k is large enough then

$$j < N+1 \quad \Rightarrow \quad \|\rho'_{N+1}(a_j) - \rho'_N(a_j)\| < 2^{-N-1},$$

as required. □

A slight strengthening of Theorem 3.2.6 will be useful to us later. First recall a well-known definition:

3.2.7 Definition If a and b are elements of a C*-algebra A, their *commutator* $[a,b]$ is the element $ab - ba$ of A. If $X, Y \subseteq A$ and $[x,y] \in Y$ for all $x \in X$, $y \in Y$ we shall say that X *derives* Y.

3.2.8 Proposition *Let* E *be a separable* C*-*subalgebra of* $\mathfrak{B}(H)$, *and let* Δ *be a separable linear subspace of* $\mathfrak{B}(H)$ *that derives* E. *Then there exists an approximate unit* $\{u_n\}$ *for* E *which is* quasicentral relative to Δ, *in the sense that* $\|u_n x - x u_n\| \to 0$ *for all* $x \in \Delta$.

Proof Let A denote the C*-algebra generated by Δ and E. Let J be the closed linear subspace $\overline{AE + E}$ of A; since Δ derives E, J is in fact an ideal in A. Let $\{v_n\}$ be an approximate unit for E. Then it is an approximate unit for J, and according to Theorem 3.2.6 there is a quasicentral approximate unit $\{u_n\}$ for J in A made up of convex combinations of the $\{v_n\}$. Then $\{u_n\}$ is also an approximate unit for E and has the required properties. □

3.3 Nuclearity

By introducing the following class of C*-algebras we can generalize the results of the previous section.

3.3.1 Definition A separable unital C*-algebra A is *nuclear* if the identity map $A \to A$ is the point-norm limit of a sequence of compositions of completely positive maps $A \to A_n \to A$, where each A_n is a finite-dimensional C*-algebra.

3.3.2 Remarks We could have given a definition for non-unital algebras, and also non-separable ones; but the above is adequate for our purposes. It is sometimes useful to note that the finite-dimensional algebras A_n can be replaced by matrix algebras $M_{k_n}(\mathbb{C})$. This is because every A_n can be realized as a block-diagonal subalgebra of a matrix algebra (Exercise 3.9.5), and the operation of compression to the block diagonal gives a completely positive 'retraction' onto A_n.

3.3.3 Example The C^*-algebra $C(X)$ is nuclear. Using the notation in the proof of Theorem 3.2.2, for each finite open cover there are completely positive maps

$$C(X) \to \mathbb{C} \oplus \cdots \oplus \mathbb{C} \to C(X),$$

the first being $f \mapsto (f(x_1), f(x_2), \dots, f(x_{k_n}))$ and the second being

$$(\lambda_1, \lambda_2, \dots, \lambda_{k_n}) \mapsto \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_{k_n} f_{k_n}.$$

As the mesh of the cover goes to zero, the composition converges to the identity.

The following example is far richer, and gives more substance to the notion of nuclearity.

3.3.4 Example Let G be a (countable) discrete group. Recall from 1.3.5 that $C_r^*(G)$ denotes the C^*-algebra of operators on $\ell^2(G)$ generated by the unitary operators U_g of left translation by $g \in G$. We shall argue that $C_r^*(G)$ is nuclear if G is an *amenable* group.

Recall that G is amenable if there is a bounded linear functional $\Lambda \colon \ell^\infty(G) \to \mathbb{C}$ such that

(a) $\Lambda(1) = 1$, and

(b) Λ is invariant under the left translation action of G on $\ell^\infty(G)$.

The functional Λ is called an *invariant mean* on G. The main theorem in amenable group theory says that G is amenable if and only if there exists a *Følner sequence* $\{G_n\}$ of finite subsets of G such that

$$\frac{|gG_n \triangle G_n|}{|G_n|} \to 0,$$

for each $g \in G$. (The notation $\triangle$ stands for the symmetric difference.)

Suppose now that G is amenable and let $\{G_n\}$ be a Følner sequence. Let P_n denote the orthogonal projection of $\ell^2(G)$ onto $\ell^2(G_n)$. Then for each n we may define completely positive maps

$$C_r^*(G) \to \mathfrak{B}(\ell^2(G_n)) \to C_r^*(G)$$

by

$$T \mapsto P_n T P_n \mapsto \frac{1}{|G_n|} \sum_{g \in G} g^{-1} P_n T P_n g,$$

where the sum converges in the strong topology. By the Følner condition, the sequence of maps $C_r^*(G) \to C_r^*(G)$ so defined converges pointwise to the identity, and thus $C_r^*(G)$ is nuclear.

3.3.5 REMARK The converse is also true: if $C^*_r(G)$ is nuclear, then G is amenable. A proof is outlined in Exercise 3.9.7. It follows that there exist non-nuclear C^*-algebras, for example the reduced C^*-algebra of a non-abelian free group.

The main result of this section generalizes Theorem 3.2.2.

3.3.6 THEOREM *Let* J *be a separable ideal in a unital* C^**-algebra* B. *Let* A *be a unital, separable and nuclear* C^**-algebra. Then every completely positive map* $\sigma\colon A \to B/J$ *is liftable.*

In brief, nuclear C^*-algebras have the 'completely positive lifting property'. The key to the proof is to show that the matrix algebras $M_k(\mathbb{C})$ have the completely positive lifting property, which is Proposition 3.3.8 below. For then, by nuclearity, σ is a point-norm limit of completely positive maps σ_n which factor through matrix algebras, and hence are liftable. Proposition 3.2.1 now shows that σ itself is liftable, exactly as in the proof of Theorem 3.2.2.

We consider then the lifting of completely positive maps from matrix algebras. Denote by $e_{\alpha\beta} \in M_k(\mathbb{C})$ the matrix with entry 1 in the (α, β) position and 0 elsewhere. Note that

$$e_{\alpha\beta}e_{\beta\gamma} = e_{\alpha\gamma} \quad \text{and} \quad {e_{\alpha\beta}}^* = e_{\beta\alpha}.$$

3.3.7 LEMMA *Let* B *be a unital* C^**-algebra. A unital linear map* $\sigma\colon M_k(\mathbb{C}) \to B$ *is completely positive if and only if the* $k\times k$ *matrix* $\boldsymbol{\sigma} = [\sigma(e_{\alpha\beta})]$ *is positive in* $M_k(B)$.

PROOF Consider the expression

$$S = \sum b_i^* \sigma(a_i^* a_j) b_j$$

which appears in Definition 3.1.1. If we write each a_i as a linear combination of the $e_{\alpha\beta}$ then S becomes a sum of expressions of the form

$$S' = \sum c_\alpha \sigma(e_{\alpha\beta}) c_\beta^*.$$

Thus σ is completely positive if and only if all the expressions S' are positive. But S' may be written in matrix notation as $c \cdot \boldsymbol{\sigma} \cdot c^*$, where c is the row vector with entries c_β, and so (using Exercise 1.9.13) σ is completely positive if and only if the matrix $\boldsymbol{\sigma}$ is positive. □

3.3.8 PROPOSITION *Every completely positive map* $\sigma\colon M_k(\mathbb{C}) \to B/J$ *is liftable.*

PROOF By Lemma 3.3.7, the completely positive map σ corresponds to a positive matrix $\boldsymbol{\sigma} = [\sigma(e_{\alpha\beta})]$ with entries in B/J. We may lift $\boldsymbol{\sigma}$ to a positive matrix $\tilde{\boldsymbol{\sigma}}$ with entries in B, and $\tilde{\boldsymbol{\sigma}}$ corresponds to a map $\tilde{\sigma}\colon M_k(\mathbb{C}) \to B$. We now apply Lemma 3.3.7 to show that $\tilde{\sigma}$ is a completely positive map. There is only one detail which must be attended to: we must ensure that the lifted map is unital. In terms of the matrix $\tilde{\sigma}$ the unitality of $\tilde{\sigma}$ corresponds to the fact that the sum of the diagonal entries — the 'trace' of $\tilde{\sigma}$ — is 1. Now the trace s is certainly 1 modulo J, and by adding a positive element of J to the $(1,1)$ entry of $\tilde{\sigma}$ we can ensure that s is also invertible. Multiplying $\tilde{\sigma}$ by the 'scalar' $s^{-\frac{1}{2}}$ on both right and left we obtain a new lifting matrix whose trace is exactly 1. □

As was remarked above, this completes the proof of Theorem 3.3.6.

3.4 Voiculescu's Theorem

The purpose of this and the next two sections is to prove Voiculescu's Theorem, that if A is a separable and unital C^*-algebra then the semigroup $\operatorname{Ext}(A)$ has a zero element. We shall begin by formulating the theorem using terms which do not refer explicitly to extensions, but which are convenient for the proof. In the next section we shall reduce the proof to a (highly non-trivial) finite-dimensional calculation, and in the last of the three sections we shall complete that calculation.

3.4.1 Definition If T and T' are bounded operators on the same Hilbert space H, we shall write

$$T \sim T'$$

if T and T' differ by a compact operator on H.

Let us adapt an earlier definition, using this notation:

3.4.2 Definition Let $\rho\colon A \to \mathfrak{B}(H)$ and $\rho'\colon A \to \mathfrak{B}(H')$ be two maps. We shall write

(a) $\rho' \approx \rho$ if there is a unitary $U\colon H' \to H$ such that $\rho'(a) \sim U^*\rho(a)U$, for all $a \in A$, and

(b) $\rho' \lesssim \rho$ if there is an isometry $V\colon H' \to H$ such that $\rho'(a) \sim V^*\rho(a)V$, for all $a \in A$.

The relation $\approx$ is an equivalence relation: to borrow from Chapter 2 we might call it essential unitary equivalence. The relation $\lesssim$ is transitive, but it is not a partial order: $\rho \lesssim \rho'$ and $\rho' \lesssim \rho$ do not together imply $\rho \approx \rho'$.

Here is our formulation of Voiculescu's Theorem:

3.4.3 Theorem *Let H be a separable Hilbert space, let E be a unital separable C^*-algebra, and let $\rho\colon E \to \mathfrak{B}(H)$ be a non-degenerate representation. Let L be a separable Hilbert space and let $\sigma\colon E \to \mathfrak{B}(L)$ be a completely positive map. If σ has the property that*

$$\rho(e) \in \mathfrak{K}(H) \quad \Rightarrow \quad \sigma(e) = 0$$

for every $e \in E$, then $\sigma \lesssim \rho$.

3.4.4 REMARK It follows from Stinespring's Theorem that σ is a compact perturbation of a completely positive map if and only if $\sigma \lesssim \rho$ for some representation ρ.

Why does Theorem 3.4.3 imply that Ext (A) has a zero element? To explain the connection we need Theorem 3.1.5 and Lemma 3.1.6. In our present language these results may be reformulated as follows.

3.4.5 LEMMA *If* $\rho\colon \mathrm{A} \to \mathfrak{B}(\mathrm{H})$ *and* $\rho'\colon \mathrm{A} \to \mathfrak{B}(\mathrm{H}')$ *are representations of* A *and if* $\rho' \lesssim \rho$ *then there is a completely positive map* $\rho''\colon \mathrm{A} \to \mathfrak{B}(\mathrm{H}'')$ *such that* $\rho \approx \rho' \oplus \rho''$. □

Before using Theorem 3.4.3 to prove that Ext (A) has a zero element we shall present a slightly simpler consequence:

3.4.6 THEOREM *Let* $\rho\colon \mathrm{A} \to \mathfrak{B}(\mathrm{H})$ *and* $\rho'\colon \mathrm{A} \to \mathfrak{B}(\mathrm{H}')$ *be non-degenerate representations of a separable, unital* C^**-algebra on separable Hilbert spaces. Suppose that* $\rho[\mathrm{A}] \cap \mathfrak{K}(\mathrm{H}) = 0$. *Then* $\rho' \oplus \rho \approx \rho$. *Thus if, in addition,* $\rho'[\mathrm{A}] \cap \mathfrak{K}(\mathrm{H}') = 0$, *then* $\rho \approx \rho'$.

Another way to state the hypothesis on ρ is this: $\pi \circ \rho\colon \mathrm{A} \to \mathfrak{Q}(\mathrm{H})$ must be injective. Thus the theorem implies that any two unital, injective and *split* extensions of A are unitarily equivalent.

PROOF Let L be the direct sum of countably many copies of the Hilbert space H' and let $\sigma\colon \mathrm{A} \to \mathfrak{B}(\mathrm{L})$ be the direct sum of countably many copies of the representation ρ'. It follows from Theorem 3.4.3 that $\sigma \lesssim \rho$. It therefore follows from Lemma 3.4.5 that $\rho \approx \sigma \oplus \rho''$ for some completely positive map ρ''. But it is clear that $\rho' \oplus \sigma \approx \sigma$ and hence

$$\rho \approx \sigma \oplus \rho'' \approx \rho' \oplus \sigma \oplus \rho'' \approx \rho' \oplus \rho$$

which is what we wanted to prove. □

The proof that Ext (A) has a zero element (represented by the class of any split extension) is an elaboration of the above argument:

3.4.7 THEOREM *Let* A *be a separable unital* C^**-algebra. If* φ *is a unital injective extension of* A, *and if* φ' *is a split unital injective extension of* A, *then* $\varphi' \oplus \varphi$ *is unitarily equivalent to* φ. *Hence the class of any split unital injective extension is a zero element for* Ext (A).

PROOF Let $\mathrm{E} \subseteq \mathfrak{B}(\mathrm{H})$ be the inverse image of $\varphi[\mathrm{A}] \subseteq \mathfrak{Q}(\mathrm{H})$ under the map $\pi\colon \mathfrak{B}(\mathrm{H}) \to \mathfrak{Q}(\mathrm{H})$. Let $\rho\colon \mathrm{E} \to \mathfrak{B}(\mathrm{H})$ be the identity representation. Let $\rho'\colon \mathrm{A} \to \mathfrak{B}(\mathrm{H}')$ be a multiplicative lifting of the split extension φ'. Regard it as a representation of E, via the natural map $\mathrm{E} \to \varphi[\mathrm{A}] \cong \mathrm{A}$ and note that ρ' vanishes on $\mathrm{E} \cap \mathfrak{K}(\mathrm{H})$. We now proceed as in the proof of Theorem 3.4.6. Let L be the direct sum of countably many copies of the Hilbert space H' and let

$\sigma\colon E \to \mathfrak{B}(L)$ be the direct sum of countably many copies of the representation ρ'. Then $\sigma \lesssim \rho$ by Theorem 3.4.3 and so $\rho \approx \sigma \oplus \rho''$ for some completely positive map ρ''. Since $\rho' \oplus \sigma \approx \sigma$ as before, we once again obtain the equivalences

$$\rho \approx \sigma \oplus \rho'' \approx \rho' \oplus \sigma \oplus \rho'' \approx \rho' \oplus \rho.$$

Passing to the Calkin algebra we conclude that φ and $\varphi' \oplus \varphi$ are unitarily equivalent, as required. □

Combining Theorem 3.4.7 with our earlier results on completely positive liftings we get:

3.4.8 THEOREM *If* A *is a unital, separable and nuclear* C^**-algebra then* $\mathrm{Ext}(A)$ *is a group.* □

3.5 Block-Diagonal Maps

Our strategy for proving Voiculescu's Theorem is this: we are going to show that $\sigma \lesssim \rho$ by breaking σ into an infinite number of small, finite-dimensional pieces and absorbing the pieces into ρ, one at a time. To make this approach workable we shall need to use a stronger (but regrettably more technical) version of the relation $\lesssim$, one which is better adapted to arguments involving infinite constructions.

3.5.1 DEFINITION Let $\rho\colon A \to \mathfrak{B}(H)$ and $\rho'\colon A \to \mathfrak{B}(H')$ be two maps. We shall write $\rho' \lessapprox \rho$ if for every finite-dimensional subspace F of H there is a sequence of isometries $V_n\colon H' \to H$ such that

(a) the range of each V_n is orthogonal to F,
(b) for each $a \in A$ the operators $\rho'(a) - V_n{}^*\rho(a)V_n$ are compact, and
(c) for each $a \in A$ the operators $\rho'(a) - V_n{}^*\rho(a)V_n$ also tend to zero in norm as $n \to \infty$.

Obviously if $\sigma \lessapprox \rho$ then $\sigma \lesssim \rho$. The virtue of the new definition is this:

3.5.2 LEMMA *Let* E *be a separable, unital* C^**-algebra and let* $\rho\colon E \to \mathfrak{B}(H)$ *be a completely positive map. Let* $\sigma_n\colon E \to \mathfrak{B}(H_n)$ *be a sequence of completely positive maps, where each Hilbert space* H_n *is finite-dimensional. If* $\sigma_n \lessapprox \rho$ *for all* n *then* $\sigma = \bigoplus \sigma_n \lesssim \rho$.

PROOF Choose a dense sequence $\{e_j\}$ in E. Construct inductively a sequence $V_n\colon H_n \to H$ of isometries such that

(a) V_nH_n is orthogonal to $\{V_mH_m, e_jV_mH_m, e_j^*V_mH_m : j, m < n\}$, and
(b) $\|\sigma_n(e_j) - V_n^*e_jV_n\| < 2^{-n}$ for all $j \leqslant n$.

Then let $V = \bigoplus V_n$, which is an isometry from $\bigoplus H_n$ into H. It follows from condition (a) that

$$V^*e_jV = (V_1 + \cdots + V_j)^*e_j(V_1 + \cdots + V_j) + V_{j+1}^*e_jV_{j+1} + V_{j+2}^*e_jV_{j+2} + \cdots.$$

Hence for every j

$$\sigma(e_j) - V^*e_jV = \Big(\sigma_1(e_j) + \cdots + \sigma_j(e_j) - (V_1 + \cdots + V_j)^*e_j(V_1 + \cdots + V_j)\Big) + \sum_{n>j}\big[\sigma_n(e_j) - V_n^*e_jV_n\big].$$

The terms on the right-hand side of the top line in the display are of finite rank; the nth term in the bottom line is of finite rank and has norm less than 2^{-n}. Thus $\sigma(e_j) - V^*a_jV$ is a norm limit of finite-rank operators, and is therefore compact. □

3.5.3 Remark A slight strengthening of the above argument produces the conclusion $\bigoplus \sigma_n \lessapprox \rho$.

3.5.4 Definition Let A be a unital C*-algebra. A map $\sigma\colon A \to \mathfrak{B}(H)$ is *block-diagonal* if there is a direct sum decomposition $H = \bigoplus H_n$, where each H_n is finite-dimensional, and a corresponding decomposition $\sigma = \bigoplus \sigma_n$.

Note that if σ is completely positive then the maps σ_n appearing in the definition must themselves be completely positive.

3.5.5 Theorem *Let A be a separable, unital C*-algebra and let $\sigma\colon A \to \mathfrak{B}(H)$ be a completely positive map. Then there is a block-diagonal completely positive map $\sigma'\colon A \to \mathfrak{B}(H')$ such that $\sigma \lesssim \sigma'$.*

As the reader will see, complete positivity is not really needed: a suitable continuity hypothesis will suffice (see the last line of the proof). However, our only application of the theorem is to completely positive maps.

Proof Let $u_1, u_2, \ldots$ be an approximate unit for $\mathfrak{K}(H)$ which is made up of finite-rank operators and is quasicentral relative to $\sigma[A]$; Proposition 3.2.8 assures us that such an approximate unit exists. Let $\{a_i\}$ be a countable dense subset of A. For fixed k we may arrange (refining our approximate unit if necessary) that the norms $\|u_n\sigma(a_i) - \sigma(a_i)u_n\|$ are small enough[21] to ensure that

$$\|d_n\sigma(a_i) - \sigma(a_i)d_n\| < 2^{-n} \qquad (i = 1, \ldots k, \quad n = 1, 2, \ldots)$$

where $d_n = (u_n - u_{n-1})^{\frac{1}{2}}$ (and we set $u_0 = 0$). Using a diagonal procedure we may further refine the approximate unit so that for every k,

[21] See Exercise 3.9.6.

$$(3.5.6)\quad \|d_n\sigma(a_i) - \sigma(a_i)d_n\| < 2^{-n-k} \qquad (i = 1, \dots k, \quad n = k+1, k+2, \dots).$$

Let $H_n = d_nH$, which is a finite-dimensional subspace of H, let $H' = \bigoplus H_n$, and define $V\colon H \to H'$ by $Vv = (d_1v, d_2v, \dots)$. This is an isometry since

$$\|Vv\|^2 = \sum_{n=1}^{\infty} \|d_nv\|^2 = \sum_{n=1}^{\infty} \langle (u_n - u_{n-1})v, v\rangle = \lim_{n\to\infty} \langle u_nv, v\rangle = \langle v, v\rangle.$$

Let P_n be the orthogonal projection of H onto H_n, let $\sigma_n(a) = P_n\sigma(a)P_n$, and define $\sigma'(a) = \bigoplus \sigma_n(a) \in \mathfrak{B}(H')$. Then

$$\sigma(a) - V^*\sigma'(a)V = \sigma(a) - \sum_n d_n\sigma(a)d_n = \sum_n (\sigma(a)d_n - d_n\sigma(a))d_n.$$

By 3.5.6, the sum on the right hand side is norm convergent whenever a belongs to the dense set $\{a_i\}$. Since the individual terms are compact, so is their sum. Thus $\sigma(a) - V^*\sigma'(a)V$ is compact for all a belonging to a dense set; by an approximation argument using the continuity of σ and σ', it follows that $\sigma(a) - V^*\sigma'(a)V$ is compact for all $a \in A$. □

3.5.7 Remark It is a simple matter to strengthen the theorem: there is a sequence of pairs (σ'_k, V_k) as in the theorem for which the compact operators

$$\sigma(a) - V_k^*\sigma'_k(a)V_k$$

converge to zero in norm for each $a \in A$ (but this is not quite the same as asserting that $\sigma \precapprox \sigma'$). We shall not however use this observation.

The following result now reduces Voiculescu's Theorem to the case of finite-dimensional, completely positive maps:

3.5.8 Theorem *Let E be a separable, unital C^*-algebra and let $\rho\colon E \to \mathfrak{B}(H)$ be a representation. Let J be the ideal in E comprising those elements e for which $\rho(e)$ is a compact operator. Suppose that $\sigma \precapprox \rho$ for every finite-dimensional Hilbert space L and every completely positive map $\sigma\colon E \to \mathfrak{B}(L)$ which vanishes on J. Then $\sigma \precsim \rho$ for every separable Hilbert space L and every completely positive map $\sigma\colon E \to \mathfrak{B}(L)$ which vanishes on J.*

Proof Lemma 3.5.2 and the hypotheses of the theorem imply that $\sigma' \precsim \rho$ for every block-diagonal completely positive map which vanishes on J. Theorem 3.5.5 implies that for every completely positive map which vanishes on J there is a block-diagonal map σ' vanishing on J such that $\sigma \precsim \sigma'$. Hence

$$\sigma \precsim \sigma' \precsim \rho,$$

as required. □

3.6 Proof of Voiculescu's Theorem

Thanks to Theorem 3.5.8, to prove Theorem 3.4.3 it suffices to consider the special case where σ is finite-dimensional, as long as we obtain in this case the strengthened conclusion $\sigma \lessapprox \rho$. This is what we shall now do.

To prove Theorem 3.4.3, we may assume that E is a unital C^*-subalgebra of $\mathfrak{B}(H)$ and that ρ is the identity representation. In addition we may assume that $\mathfrak{K}(H) \subseteq E$ since we can extend any completely positive map $\sigma\colon E \to \mathfrak{B}(L)$ which vanishes on $E \cap \mathfrak{K}(H)$ to $E + \mathfrak{K}(H)$ by defining σ to be zero on any compact operator not already in E. The one-dimensional case of Voiculescu's Theorem is thus implied by the following result in C^*-algebra theory:

3.6.1 Glimm's Lemma *Let E be a separable unital C^*-subalgebra of $\mathfrak{B}(H)$ which contains $\mathfrak{K}(H)$ and let $\rho\colon E \to \mathfrak{B}(H)$ be the identity representation. If σ is any state on E which vanishes on $\mathfrak{K}(H)$ then $\sigma \lessapprox \rho$. In other words, for every finite-dimensional subspace F of H there is a sequence of unit vectors $\{v_k\}$ in H orthogonal to F such that*

$$\sigma(e) = \lim_{k\to\infty} \langle v_k, e v_k\rangle,$$

for every $e \in E$.

Let $\mathcal{S}$ be the set of all states for which Glimm's Lemma is true. In other words, let $\mathcal{S}$ be the set of all states σ on E which, for every finite-dimensional $F \subseteq H$, can be written as a limit of vector states

$$\sigma(e) = \lim_{k\to\infty} \langle v_k, e v_k\rangle,$$

where all the v_k are orthogonal to F. It is easy to see that every such state must vanish on $\mathfrak{K}(H)$, and our job is to show that in fact $\mathcal{S}$ is the set of all states vanishing on $\mathfrak{K}(H)$.

3.6.2 Lemma *A state $\sigma\colon E \to \mathbb{C}$ belongs to $\mathcal{S}$ if and only if the following holds: for every $\varepsilon > 0$, every finite set $e_1, \ldots, e_n$ in E, and every finite-dimensional subspace $F \subseteq H$, there is a unit vector orthogonal to F such that*

$$|\sigma(e_i) - \langle v, e_i v\rangle| < \varepsilon,$$

for $i = 1, \ldots, n$.

Proof This is straightforward. □

3.6.3 Lemma *The set $\mathcal{S}$ is weak*-closed and convex.*

Proof The weak* topology is the topology of pointwise convergence of states, and it is therefore clear from the preceding lemma that $\mathcal{S}$ is weak*-closed. Suppose then that σ_0 and σ_1 are states in $\mathcal{S}$ and that $0 \leqslant t \leqslant 1$. We want to show that the state $\sigma_t = (1-t)\sigma_0 + t\sigma_1$ belongs to $\mathcal{S}$. Let $\varepsilon > 0$, let $e_1, \ldots, e_n \in E$, and let F be a finite-dimensional subspace of H. Let v_0 be a unit vector orthogonal to F such that

$$|\sigma_0(e_i) - \langle v_0, e_i v_0\rangle| < \varepsilon,$$

for $i = 1, \ldots, n$. Let v_1 be a unit vector orthogonal to F, and also to the vectors v_0, $e_i v_0$ and $e_i^* v_0$ for $i = 1, \ldots, n$, such that

$$|\sigma_1(e_i) - \langle v_1, e_i v_1\rangle| < \varepsilon,$$

for $i = 1, \ldots, n$. Let $v_t = (1-t)^{\frac{1}{2}} v_0 + t^{\frac{1}{2}} v_1$. Then v_t is orthogonal to F and

$$\begin{aligned} |\sigma_t(e_i) - \langle v_t, e_i v_t\rangle| \leqslant (1-t)|\sigma_0(e_i) - \langle v_0, e_i v_0\rangle| + t|\sigma_1(e_i) - \langle v_1, e_i v_1\rangle| \\ + (t(1-t))^{\frac{1}{2}} \left(|\langle v_0, e_i v_1\rangle| + |\langle v_1, e_i v_0\rangle|\right). \end{aligned}$$

The first and second terms on the right hand side are bounded by $(1-t)\varepsilon$ and $t\varepsilon$, respectively. The last two terms are zero because v_1 is orthogonal to $e_i v_0$ and $e_i^* v_0$. It follows that

$$|\sigma_t(e_i) - \langle v_t, e_i v_t\rangle| < \varepsilon,$$

for $i = 1, \ldots, n$, and this proves the lemma. □

We need one final fact before beginning the proof of Glimm's Lemma.

3.6.4 LEMMA *Let E and $\mathcal{S}$ be as above. Denote by $\pi\colon \mathfrak{B}(H) \to \mathfrak{Q}(H)$ the quotient map. If e is a positive element of E then*

$$\|\pi(e)\| = \sup_{\psi \in \mathcal{S}} \psi(e).$$

PROOF According to the Weyl–von Neumann Theorem, every selfadjoint operator e is a compact perturbation of a diagonal operator. Since the states ψ in $\mathcal{S}$ vanish on $\mathfrak{K}(H)$ it suffices to prove the lemma for a diagonal operator e. If λ is an element of the essential spectrum of e then λ is a limit of a sequence of eigenvalues for e. If ψ is a weak*-limit point of the corresponding sequence of eigenvector states then $\psi \in \mathcal{S}$ and $\psi(e) = \lambda$. Since the norm of $\pi(e)$ is the largest element in the essential spectrum of e it follows that $\sup \psi(e) \geqslant \|\pi(e)\|$. Since every state $\psi \in \mathcal{S}$ may be regarded as a state on $E/\mathfrak{K}(H)$, and since every state has norm one, the reverse inequality $\sup \psi(e) \leqslant \|\pi(e)\|$ is immediate. □

PROOF OF GLIMM'S LEMMA Suppose that there is some state $\sigma\colon E \to \mathbb{C}$ which vanishes on $\mathfrak{K}(H)$ yet does not belong to the weak*-closed, convex set $\mathcal{S}$. Denote by $\mathfrak{X}$ the real topological vector space of real-valued, bounded linear functionals on the selfadjoint elements of E. We equip $\mathfrak{X}$ with the weak* topology and note that the continuous linear functionals on $\mathfrak{X}$ are precisely the evaluation functionals $\varphi \mapsto \varphi(e)$. By the Hahn–Banach Theorem, there is an element $e \in E = \mathfrak{X}^*$ and a scalar $\alpha \in \mathbb{R}$ such that $\sigma(e) > \alpha > \psi(e)$, for all $\psi \in \mathcal{S}$. Replacing e by $e + \|e\| I$ if necessary, we may assume e is positive. It follows from Lemma 3.6.4 that $\sigma(e) > \|\pi(e)\|$. But this is impossible since σ may be regarded as a state on $E/\mathfrak{K}(H)$. $\square$

The proof of the one-dimensional case of Voiculescu's Theorem is now complete. The general finite-dimensional case reduces quickly to the one-dimensional case.

3.6.5 LEMMA *Let A be a unital C^*-algebra and let $\sigma\colon A \to M_n(\mathbb{C})$ be a linear map. Define a linear functional $\bar{\sigma}\colon M_n(A) \to \mathbb{C}$ by*

$$\bar{\sigma}([a_{ij}]) = \frac{1}{n} \sum_{i,j=1}^{n} \langle \varepsilon_i, \sigma(a_{ij})\varepsilon_j \rangle,$$

where $\varepsilon_1, \ldots, \varepsilon_n$ is the standard orthonormal basis for $\mathbb{C}^n$. The linear map σ is completely positive if and only if the linear functional $\bar{\sigma}$ is a state.

PROOF The linear map σ is completely positive if and only if the quantity

$$\sum_{i,j=1}^{N} \langle \nu_i, \sigma(a_i{}^* a_j)\nu_j \rangle$$

is non-negative, for every N, every $a_1, \ldots, a_N \in A$ and every $\nu_1, \ldots, \nu_N \in \mathbb{C}^n$. Every positive $n \times n$ matrix over A is a sum of matrices of the form $[a_i^* a_j]$, where $a_1, \ldots, a_n \in A$. So by setting $N = n$ and specializing $\nu_1, \ldots, \nu_N$ to the standard basis for $\mathbb{C}^n$ we see from the formula for $\bar{\sigma}$ that the complete positivity of σ implies the positivity of $\bar{\sigma}$, as required. Conversely if $\bar{\sigma}$ is a state then from a basis expansion $\nu_i = \sum_{k=1}^{n} \alpha_{ik}\varepsilon_k$ we get

$$\sum_{i,j=1}^{N} \langle \nu_i, \sigma(a_i{}^* a_j)\nu_j \rangle = \sum_{k,l=1}^{n} \langle \varepsilon_k, \sigma(c_k{}^* c_l)\varepsilon_l \rangle = n \cdot \bar{\sigma}([c_k{}^* c_l]),$$

where $c_k = \sum_{i=1}^{N} \alpha_{ik} a_i$. So the positivity of $\bar{\sigma}$ implies the complete positivity of σ. $\square$

3.6.6 REMARK Since every state of a C^*-algebra B extends to any C^*-algebra D which contains B (see the proof of Lemma 1.6.4), it follows from Lemma 3.6.5 that any completely positive map from B into $M_n(\mathbb{C})$ extends to a completely positive map from D into $M_n(\mathbb{C})$. It is interesting to note that positive maps do not necessarily extend: *complete* positivity is crucial here.

3.6.7 PROPOSITION *Let* E *be a separable unital* C^**-subalgebra of* $\mathfrak{B}(H)$ *which contains* $\mathfrak{K}(H)$ *and let* $\rho\colon E \to \mathfrak{B}(H)$ *be the identity representation. If* $\sigma\colon E \to \mathfrak{B}(\mathbb{C}^n)$ *is any completely positive map which vanishes on* $\mathfrak{K}(H)$ *then* $\sigma \lesssim \rho$.

PROOF View $M_n(E)$ as a C^*-subalgebra

$$M_n(E) \subseteq \mathfrak{B}(H \oplus \cdots \oplus H).$$

Given a completely positive map $\sigma\colon E \to \mathfrak{B}(\mathbb{C}^n)$ which is zero on the compacts, the state $\bar{\sigma}\colon M_n(E) \to \mathbb{C}$ constructed in Lemma 3.6.5 is also zero on the compacts. It follows from Glimm's Lemma that for every finite-dimensional subspace of $H \oplus \cdots \oplus H$, say of the form

$$F \oplus \cdots \oplus F \subseteq H \oplus \cdots \oplus H,$$

there are vector states

$$[e_{ij}] \mapsto \langle v, [e_{ij}]v\rangle$$

of $M_n(E)$ which converge pointwise to $\bar{\sigma}$, where $v = (v_1, \ldots, v_n)$ is orthogonal to our given finite-dimensional subspace. If $V\colon \mathbb{C}^n \to H$ is the operator mapping ε_i to $\sqrt{n}v_i$ then we can rewrite the above vector state as

$$[e_{ij}] \mapsto \frac{1}{n}\sum_{i,j}\langle \varepsilon_i, V^*e_{ij}V\varepsilon_j\rangle.$$

Comparing this with the formula

$$\bar{\sigma}\colon [e_{ij}] \mapsto \frac{1}{n}\sum_{i,j}\langle \varepsilon_i, \sigma(e_{ij})\varepsilon_j\rangle,$$

we see that the matrix-valued functions $e \mapsto V^*eV$, mapping E to $\mathfrak{B}(\mathbb{C}^n)$, converge entrywise, and hence in norm, to $\sigma\colon E \to \mathfrak{B}(\mathbb{C}^n)$. Since the range of V is orthogonal to F, this finishes the proof, except for the fact that V may not be an isometry. However, by looking at $1 \in E$ we see that V^*V converges to I, so it is eventually invertible, and we can replace V with $V(V^*V)^{-\frac{1}{2}}$ to get isometries with all the desired properties. □

We have now completed the proof of Voiculescu's Theorem 3.4.3.

3.7 Property T and Ext

The purpose of this section is to present an example of a separable C^*-algebra A for which $\operatorname{Ext}(A)$ is not a group. Obviously A must be a non-nuclear C^*-algebra, and since the only examples of these we have met so far are the reduced C^*-algebras of non-amenable groups, it will not be a surprise that our example is based on the representation theory of certain non-amenable groups.

To keep our presentation fairly brief, we shall assume that the reader has some familiarity with unitary representation theory.

3.7.1 Definition A (discrete) group G has Kazhdan's *property T* if

(a) it is generated by a finite set $S \subseteq G$, and

(b) there is a constant $\varepsilon_S > 0$ with the following property: if π is any unitary representation of G on a Hilbert space H , and if there is a unit vector v in H for which $\|\pi(g)v - v\| < \varepsilon_S$ for every $g \in S$, then there is a unit vector $w \in H$ such that $\pi(g)w = w$, for all $g \in G$.

In brief, a finitely generated group has property T if and only if every unitary representation which 'almost has fixed vectors' actually *does* have fixed vectors.

3.7.2 Example No infinite amenable group can have property T, since the regular representation of an infinite amenable group almost has fixed vectors (consider the normalized characteristic functions of the sets making up a Følner sequence), but does not have any genuine fixed vectors.

3.7.3 Example Every finite group has property T.

It is far from easy to exhibit even a single infinite property T group. However it is a fact that $G = SL(3, \mathbb{Z})$ has property T. We shall eventually base our example on this group.

Non-amenable groups have the property that some of their unitary representations are 'separated from' the regular representation. For a non-amenable group G it is therefore convenient to study another group C*-algebra, larger than the algebra $C^*_r(G)$ of Definition 1.3.5 and less closely connected to the regular representation. To define it, note first that the group algebra $\mathbb{C}[G]$ is a $*$-algebra under the involution $[g]^* = [g^{-1}]$, and that each unitary representation of G on a Hilbert space extends to a non-degenerate representation of $\mathbb{C}[G]$. Conversely, each non-degenerate representation of the $*$-algebra $\mathbb{C}[G]$ determines a unitary representation of G.

3.7.4 Definition The *full* C^**-algebra* of a discrete group G, denoted $C^*(G)$, is the completion of the group algebra $\mathbb{C}[G]$ in the norm $\|f\| = \sup_\pi \|\pi(f)\|$, where the supremum is over all unitary representations of G on Hilbert space.

By construction, non-degenerate representations of $C^*(G)$ correspond one-to-one with unitary representations of G. In contrast, non-degenerate representations of $C^*_r(G)$ correspond one-to-one with those unitary representations of G which 'contribute' to the regular representation (for an example, see Exercise 3.9.16).

3.7.5 Structure of $C^*(G)$ for Finite Groups *If* G *is finite then associated to each unitary equivalence class* $[\pi]$ *of irreducible unitary representations of* G *there is a central projection* $P_\pi \in C^*(G)$, *such that*

(a) *in any unitary representation* $\rho\colon G \to U(H)$, P_π *acts as the orthogonal projection onto the* π*-isotypical subspace of* H,

(b) *the projections* P_π *associated to inequivalent irreducible representations are orthogonal to one another,*

(c) *the* C^**-subalgebra* $P_\pi C^*(G) P_\pi$ *is isomorphic to* $\mathfrak{B}(H_\pi)$*, where* H_π *denotes the Hilbert space of the representation* π*, and*

(d) *the sum of all the* P_π *is* 1. □

3.7.6 STRUCTURE OF $C^*(G)$ FOR PROPERTY T GROUPS *If* G *is a property T group then associated to each unitary equivalence class* $[\pi]$ *of irreducible* finite-dimensional *unitary representations of* G *there is a central projection* $P_\pi \in C^*(G)$*, such that*

(a) *in any unitary representation* $\rho\colon G \to U(H)$*,* P_π *acts as the orthogonal projection onto the* π*-isotypical subspace of* H*,*

(b) *the projections* P_π *associated to inequivalent finite-dimensional irreducible representations are orthogonal to one another, and*

(c) *the* C^**-subalgebra* $P_\pi C^*(G) P_\pi$ *is isomorphic to* $\mathfrak{B}(H_\pi)$*, where* H_π *denotes the Hilbert space of the representation* π. □

The proofs of these two results are rather different. Theorem 3.7.5 is a matter of pure algebra: for a finite group, $C^*(G)$ is just the group algebra $\mathbb{C}[G]$, and the result is standard fare in any introductory course on representation theory. The main idea in the proof is Schur's Lemma (3.7.7 below). By contrast, Theorem 3.7.6 has substantial analytical content. It illustrates the distinction between $C^*(G)$ and $C^*_r(G)$ for a property T group, since the projections[22] P_π are invisible from the standpoint of the regular representation. A proof of Theorem 3.7.6 is outlined in Exercise 3.9.17; it depends on an 'approximate' version of Schur's Lemma for property T groups, which we shall prove in 3.7.8. If G is an infinite property T group then the projections P_π will not add up to 1: this results from the fact that G has infinite-dimensional irreducible unitary representations along with its finite-dimensional ones.

REMARK There is no similar structure theorem for groups which do not have property T: for instance $C^*(\mathbb{Z})$ has no non-trivial projections, even though all its irreducible unitary representations are one-dimensional.

Let us now develop the version of Schur's Lemma appropriate to property T groups. Recall first the usual statement of Schur's Lemma:

[22] Indeed, the 'Kazhdan projections' P_π are rather mysterious objects. One might expect that elements of $C^*(G)$ could be regarded as infinite series $\sum \lambda_g g$ of group elements, convergent in some suitable sense. But the coefficients of such an infinite series can be recovered from its action on the regular representation, and we would be forced to conclude that the 'coefficients' of the projection P_π are all zero, even though the projection itself is non-zero. There is, in fact, no such series expansion for the Kazhdan projections.

3.7.7 SCHUR'S LEMMA *Let* G *be a group. Let* π *be an irreducible unitary representation of* G *on a finite-dimensional Hilbert space* H_π, *and let* ρ *be an arbitrary unitary representation of* G *on a Hilbert space* H_ρ. *If there is a non-zero operator* $W\colon H_\pi \to H_\rho$ *such that* $W\pi(g) = \rho(g)W$, *for all* $g \in G$, *then* π *is unitarily equivalent to a subrepresentation of* ρ. □

This is proved by arguing that W^*W is a scalar multiple of the identity (for its eigenspaces are subrepresentations of the irreducible representation π), and so, for a suitable $\lambda > 0$, the operator $V = \lambda W$ is an isometry implementing a unitary equivalence between π and a subrepresentation of ρ. It is not really necessary to assume H_π is finite-dimensional, but the infinite-dimensional case introduces technicalities which are not relevant here.

3.7.8 SCHUR'S LEMMA FOR PROPERTY T GROUPS *Let* G *be a property T group, and let* S *and* ε_S *be as in Definition* 3.7.1. *Then there is a second constant* $\varepsilon'_S > 0$ *with the following property: if*

(a) π *is an irreducible unitary representation of* G *on a finite-dimensional Hilbert space* H_π, *and*
(b) ρ *is a unitary representation of* G *on a Hilbert space* H_ρ, *and*
(c) *there is an isometry* $V\colon H_\pi \to H_\rho$ *such that* $\|\pi(g) - V^*\rho(g)V\| < \varepsilon'_S$ *for all* $g \in S$,

then π *is unitarily equivalent to a subrepresentation of* ρ.

PROOF Set $\varepsilon'_S = \frac{1}{2}\varepsilon_S^2$, and suppose that π, ρ, and V satisfy conditions (a)–(c) above.

Let $n = \operatorname{Dim} H_\pi$. Let H_σ denote the Hilbert space of Hilbert–Schmidt operators (Exercise 2.9.2) from H_π to H_ρ. The operator V belongs to H_σ (in fact, since H_π is finite-dimensional, every bounded linear operator from H_π to H_ρ belongs to H_σ) and has Hilbert–Schmidt[23] norm $\|V\|_{HS} = n^{\frac{1}{2}}$. Moreover, a unitary representation σ of G on H_σ is defined by

$$\sigma(g)(T) = \rho(g) \cdot T \cdot \pi(g)^*.$$

We shall now argue that V is almost fixed by the representation σ. The identity

$$(V\pi(g) - \rho(g)V)^* (V\pi(g) - \rho(g)V) = \pi(g)^* (V^*\rho(g)V - \pi(g)) - (V^*\rho(g)V - \pi(g))^* \pi(g)$$

shows that $\|V\pi(g) - \rho(g)V\|^2 \leqslant 2\|V^*\rho(g)V - \pi(g)\|$. Therefore,

$$\|V - \sigma(g)V\| = \|V\pi(g) - \rho(g)V\| < \varepsilon_S.$$

[23]It is important in this proof to distinguish between the operator norm (denoted $\|\cdot\|$) and the Hilbert–Schmidt norm (denoted $\|\cdot\|_{HS}$).

Since the domain of V is n-dimensional,

$$\|V - \sigma(g)V\|_{HS} \leqslant n^{\frac{1}{2}} \|V - \sigma(g)V\| < n^{\frac{1}{2}} \varepsilon_S,$$

(see Exercise 2.9.2). Thus the unit vector $n^{-\frac{1}{2}} V$ in H_σ satisfies the conditions of Definition 3.7.1, and therefore there is a non-zero fixed vector $W \in H_\sigma$ for σ. To say that W is a fixed vector is to say that, considered as an operator from H_π to H_ρ, it satisfies the identity $W\pi(g) = \rho(g)W$. The conditions of Lemma 3.7.7 are therefore fulfilled, and so π is unitarily equivalent to a subrepresentation of ρ. □

Now let $G = SL(3, \mathbb{Z})$. As we have already noted, this is a property T group. In addition, G has an infinite family of pairwise inequivalent finite-dimensional irreducible unitary representations $\pi_n \colon G \to U(H_n)$ (obtained from the finite quotients $SL(3, \mathbb{Z}/k\mathbb{Z})$).

Let J be the ideal in $C^*(G)$ generated by the projections P_{π_n} supplied by Theorem 3.7.6. If we form the direct sum representation

$$\rho = \bigoplus \pi_n \colon C^*(G) \to \mathfrak{B}(H), \tag{3.7.9}$$

where $H = \bigoplus H_n$, then ρ maps J to $\mathfrak{K}(H)$, and we obtain an extension

$$\varphi \colon C^*(G)/J \to \mathfrak{Q}(H).$$

3.7.10 Theorem *The above extension has no completely positive lifting. Consequently,* Ext $(C^*(G)/J)$ *is not a group.*

Proof Suppose, for the sake of a contradiction, that there is a completely positive lifting $\sigma \colon C^*(G)/J \to \mathfrak{B}(H)$. Think of σ as a completely positive map from $C^*(G)$ into $\mathfrak{B}(H)$ which is zero on J, and note that σ, so viewed, is a compact perturbation of the representation ρ. Stinespring's Theorem provides a Hilbert space H' containing H, and a non-degenerate representation

$$\rho' \colon C^*(G) \to C^*(G)/J \to \mathfrak{B}(H'),$$

such that

$$\rho(a) \sim Q\rho'(a)Q \qquad \forall a \in C^*(G). \tag{3.7.11}$$

where Q denotes the orthogonal projection of H' onto H. If Q_n denotes the orthogonal projection from H' onto $H_n \subseteq H \subseteq H'$ then (3.7.11) implies that

$$\lim_{n\to\infty} \|\pi_n(a) - Q_n\rho'(a)Q_n\| = \lim_{n\to\infty} \|Q_n(\rho(a) - Q\rho'(a)Q)Q_n\| = 0.$$

It follows from Lemma 3.7.8 that for large enough n, say $n \geqslant N$, each π_n occurs as a subrepresentation of ρ', and so

$$n \geqslant N \quad \Rightarrow \quad \|\rho'(a)\| \geqslant \|\pi_n(a)\|.$$

But the central projection $P_{\pi_N} \in C^*(G)$ lies in J, and so $\rho'(P_{\pi_N}) = 0$, whereas $\pi_N(P_{\pi_N}) = Q_N$. Contradiction. □

3.8 Kasparov's Technical Theorem

The result of this section is another application of the existence of quasicentral approximate units. It will turn out to yield several key constructions in the theory of Ext (A) and the more general K-homology groups $K^*(A)$. We shall give one preliminary application of the Technical Theorem at the end of the section.

3.8.1 KASPAROV TECHNICAL THEOREM *Let* H *be a separable Hilbert space and let* E_1 *and* E_2 *be separable* C^**-subalgebras of* $\mathfrak{B}(H)$ *such that* $E_1 \cdot E_2 \subseteq \mathfrak{K}(H)$. *Let* Δ *be a separable linear subspace of* $\mathfrak{B}(H)$ *which derives* E_1 *(that is,* $[\Delta, E_1] \subseteq E_1$*). Then there is a selfadjoint operator* $X \in \mathfrak{B}(H)$*, with* $0 \leqslant X \leqslant 1$*, such that*

(a) $(1 - X) \cdot E_1 \subseteq \mathfrak{K}(H)$,
(b) $X \cdot E_2 \subseteq \mathfrak{K}(H)$, *and*
(c) $[X, \Delta] \subseteq \mathfrak{K}(H)$.

3.8.2 REMARK The conclusion may be expressed by saying that X 'essentially separates' E_1 and E_2 and 'essentially commutes' with Δ. In later applications it will be important to note that the space of operators X satisfying the conditions of the theorem is connected (in fact, convex).

PROOF Choose dense sequences $\{x_m\}$, $\{y_m\}$, $\{z_m\}$ in the unit balls of E_1, E_2, and Δ respectively. Using Proposition 3.2.8, choose an approximate unit $\{u_n\}$ for E_1 such that

$$\|u_n x_m - x_m\| < 2^{-n} \quad \forall m \leqslant n \tag{3.8.3}$$

$$\|[u_n, z_m]\| < 2^{-n} \quad \forall m \leqslant n \tag{3.8.4}$$

Choose also an approximate unit w_n for $\mathfrak{K}(H)$ such that if we define $d_n = (w_n - w_{n-1})^{\frac{1}{2}}$ then[24]

$$\|d_n u_{m_1} y_{m_2}\| < 2^{-n} \quad \forall m_1, m_2 \leqslant n \tag{3.8.5}$$

$$\|[d_n, x_m]\| < 2^{-n} \quad \forall m \leqslant n \tag{3.8.6}$$

$$\|[d_n, y_m]\| < 2^{-n} \quad \forall m \leqslant n \tag{3.8.7}$$

$$\|[d_n, z_m]\| < 2^{-n} \quad \forall m \leqslant n \tag{3.8.8}$$

Now consider the infinite series $\sum d_n u_n d_n$. The partial sums of this series form an increasing sequence of positive operators, which is bounded above by 1 since

$$\sum_{n=1}^{N} d_n u_n d_n \leqslant \sum_{n=1}^{N} d_n^2 = w_N \leqslant 1;$$

[24] See the proof of Theorem 3.5.5 for this sort of construction.

so by Proposition 1.2.3 the series converges in the strong operator topology to an operator X. We shall prove that X has the desired properties.

(a) We may write $1 - X = \sum d_n(1 - u_n)d_n$, a strongly convergent series. It is enough to prove that $(1 - X)x_m$ is compact for all m. The series

$$(1 - X)x_m = \sum d_n(1 - u_n)d_n x_m$$

converges strongly (because multiplication by a fixed operator is strongly continuous) and all its terms are compact; it is enough, therefore, to prove that it converges in norm. But

$$d_n(1 - u_n)d_n x_m = d_n((1 - u_n)x_m)d_n + d_n(1 - u_n)[d_n, x_m]$$

and by 3.8.3 and 3.8.6 this has norm at most $2 \cdot 2^{-n}$ as soon as $n > m$.

(b) Similarly, it is enough to prove the norm convergence of

$$Xy_m = \sum d_n u_n d_n y_m.$$

Writing

$$d_n u_n d_n y_m = d_n u_n y_m d_n + d_n u_n [d_n, y_m]$$

and using 3.8.5 and 3.8.7, we may argue as before.

(c) Finally, it is enough to prove the norm convergence of

$$[X, z_m] = \sum [d_n u_n d_n, z_m].$$

Here we write

$$[d_n u_n d_n, z_m] = [d_n, z_m]u_n d_n + d_n[u_n, z_m]d_n + d_n u_n [d_n, z_m]$$

and use 3.8.4 and 3.8.8 to get a similar estimate. □

Here is an example which illustrates the rôle of the Technical Theorem in providing what we might call a 'non-commutative partition of unity'.

3.8.9 PROPOSITION *Let A be a separable, unital C^*-algebra and let*

$$0 \longrightarrow J \xrightarrow{\alpha} A \xrightarrow{\beta} B \longrightarrow 0$$

be a short exact sequence of C^*-algebras which is split by a unital $*$-homomorphism $\sigma\colon B \to A$. Let H be a separable Hilbert space and let $\varphi\colon A \to \mathfrak{Q}(H)$ be a unital and injective extension. If the extensions $\varphi\alpha\colon J \to \mathfrak{Q}(H)$ and $\varphi\sigma\colon B \to \mathfrak{Q}(H)$ are both split[25] *then the extension φ is itself split.*

[25] Since J will typically not be unital we do not require here that the extension $\varphi\alpha$ lift to a *unital* $*$-homomorphism into $\mathfrak{B}(H)$, as we would in the case of a unital C^*-algebra.

PROOF Assume that the extension $\varphi\alpha$ is split and denote by $\rho\colon J \to \mathfrak{B}(H)$ a multiplicative lifting. Since J is an ideal in A, the representation ρ of J extends to a representation of A. We do not assert that this extension will be non-degenerate; in fact the possible degeneracy of the extension will introduce a complication which we shall have to confront at the end of the proof. But at any rate, let us proceed by composing with the quotient map into $\mathfrak{Q}(H)$ to obtain a split (but not necessarily unital) extension $\psi\colon A \to \mathfrak{Q}(H)$ with the property that $\psi\alpha = \varphi\alpha$. We have now constructed the following commutative diagram of maps and $*$-homomorphisms

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, "\alpha"] \arrow[d, "\rho"] & A \arrow[r, bend left, "\beta"] \arrow[d, bend right, "\psi"'] \arrow[d, bend left, "\varphi"] & B \arrow[l, bend left, "\sigma"] \arrow[r] & 0 \\
& \mathfrak{B}(H) \arrow[r, "\pi"'] & \mathfrak{Q}(H) & &
\end{tikzcd}$$

Using the Technical Theorem, we shall prove below the following:

3.8.10 CLAIM *The extensions* $\psi \oplus \varphi\sigma\beta$ *and* $\psi\sigma\beta \oplus \varphi$ *are unitarily equivalent.*

Grant this for a moment. If the extension ψ is in fact unital then the proof is completed as follows: the hypothesis that $\varphi\sigma$ splits implies that $\psi \oplus \varphi\sigma\beta$ splits, and therefore, by the above unitary equivalence, that $\psi\sigma\beta \oplus \varphi$ splits. But the latter is the direct sum of φ and a split extension. By Voiculescu's Theorem the direct sum is unitarily equivalent to φ and hence φ splits as required. If the extension ψ is non-unital then the representation of A on H from which ψ is obtained must map the identity of A to a projection operator P on H with infinite codimension. By adding to our representation a non-degenerate representation of A on $P^{\perp}H$ which factors through B, we obtain a new representation of A which *is* non-degenerate, and which is still an extension of the representation ρ of J. The proof is completed by using this new representation to define an extension ψ, and the proceeding as before. $\square$

PROOF OF THE CLAIM Let $E_1 \subseteq \mathfrak{B}(H)$ be the C^*-subalgebra $\rho(J) + \mathfrak{K}$, let E_2 be the C^*-subalgebra generated by $\{T \in \mathfrak{B}(H) : \pi(T) = \varphi(a) - \psi(a),\ a \in A\}$, and let $\Delta = \pi^{-1}(\varphi[A])$. As J is an ideal in A, and as the $*$-homomorphisms φ and ψ agree on J, we see that $E_1 \cdot E_2 \subseteq \mathfrak{K}(H)$ and $[\Delta, E_1] \subseteq E_1$. Thus by the Technical Theorem there is a selfadjoint operator X such that

(a) $0 \leqslant X \leqslant 1$,

(b) $j \in J \quad\Rightarrow\quad \pi(1 - X)\varphi(\alpha(j)) = 0$,

(c) $a \in A \quad\Rightarrow\quad \pi(X)(\varphi(a) - \psi(a)) = 0$, and

(d) $[\pi(X), \varphi[A]] = 0$.

Now the operator

$$U = \begin{pmatrix} -(1-X)^{\frac{1}{2}} & X^{\frac{1}{2}} \\ X^{\frac{1}{2}} & (1-X)^{\frac{1}{2}} \end{pmatrix}$$

is unitary and implements the required equivalence. □

3.9 Exercises

3.9.1 Let $A = C(X)$, where X is compact Hausdorff, and let $\{x_i\}$ be a finite subset of X. Let $\sigma\colon A \to B$ be a map of the form

$$\sigma(f) = \sum_i f(x_i)P_i,$$

where the P_i are positive elements of the C^*-algebra B and $\sum P_i = 1$. Show that σ is completely positive. If $\sigma'\colon A \to B$ is a positive, unital linear map, show that σ' is a point-norm limit of maps of the form σ above, and deduce that σ' is completely positive.

3.9.2 Let $\tau\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be the transposition map

$$\tau\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Show that τ is a positive linear map which is *not* completely positive.

3.9.3 Let A be the commutative C^*-algebra of continuous functions on the closed unit ball in $\mathbb{R}^n$, and let I be the ideal consisting of those functions vanishing on the boundary sphere, so that A/I is the algebra of continuous functions on the $(n-1)$-sphere.

(a) Show that there is no $*$-homomorphism $A/I \to A$ that splits the quotient map $A \to A/I$.

(b) Write down an explicit completely positive map $A/I \to A$ that splits $A \to A/I$.

(c) Can the completely positive map you wrote down in (b) appear as the top left entry in a $*$-homomorphism $A/I \to M_k(A)$?

3.9.4 Let B be a unital C^*-algebra and J an ideal in it.

(a) Prove the fact, used in the text, that a positive element x of B/J can be lifted to a positive element $\tilde{x}$ of B. (Take a selfadjoint lift and then adjust it by means of the functional calculus.)

(b) Prove further that if $x_1, \ldots, x_n$ are positive elements of B/J whose sum is 1, then one can find positive lifts $\tilde{x}_1, \ldots, \tilde{x}_n$ whose sum is 1 also. (Take positive lifts as in (a), let s be their sum; arrange that s be invertible, and then replace each $\tilde{x}$ by $s^{-\frac{1}{2}}\tilde{x}s^{-\frac{1}{2}}$.)

3.9.5 Show that every finite-dimensional C*-algebra is isomorphic to a direct sum of matrix algebras over $\mathbb{C}$.

3.9.6 Let A be a unital C*-algebra and let φ be a continuous function from $[0,1]$ to $\mathbb{C}$. Show that for every $\varepsilon > 0$ there is $\delta > 0$ with the following property: for all a, a' in the unit ball of A, with $a \geqslant 0$, we have

$$\|aa' - a'a\| \leqslant \delta \Rightarrow \|\varphi(a)a' - a'\varphi(a)\| \leqslant \varepsilon.$$

(Consider first the case of polynomial φ, then use the Stone–Weierstrass Theorem.)

3.9.7 Let G be a countable discrete group. Suppose that $C_r^*(G)$ is nuclear, with a sequence of approximate factorizations

$$C_r^*(G) \to \mathfrak{B}(H_n) \to C_r^*(G),$$

where the H_n are finite-dimensional Hilbert spaces. Show that the completely positive maps $C_r^*(G) \to \mathfrak{B}(H_n)$ extend to completely positive maps $\mathfrak{B}(\ell^2(G)) \to \mathfrak{B}(H_n)$ (see Lemma 3.6.5 and the subsequent remark). Using the Banach–Alaoglu Theorem from elementary functional analysis, show that the states

$$\mathfrak{B}(\ell^2(G)) \to \mathfrak{B}(H_n) \to C_r^*(G) \to \mathbb{C},$$

where $C_r^*(G) \to \mathbb{C}$ is the 'trace' $T \mapsto \langle [e], T[e] \rangle$, have a limit point: a state $\mathfrak{B}(\ell^2(G)) \to \mathbb{C}$. Finally use the approximate factorization property to show that this state restricts to an invariant mean on $\ell^\infty(G) \subseteq \mathfrak{B}(\ell^2(G))$.

3.9.8 The tensor product of two Hilbert spaces, H_1 and H_2, is the completion of the algebraic tensor product of H_1 and H_2 in the inner product norm determined by the formula

$$\langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle = \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle.$$

If $T_1 \in \mathfrak{B}(H_1)$ and $T_2 \in \mathfrak{B}(H_2)$ then the formula

$$(T_1 \otimes T_2)(v_1 \otimes v_2) = T_1 v_1 \otimes T_2 v_2$$

determines a bounded operator on the Hilbert space tensor product $H_1 \otimes H_2$. If A_1 and A_2 are C*-algebras of operators on H_1 and H_2 respectively, then the C*-algebra tensor product of A_1 and A_2, denoted $A_1 \otimes A_2$, is the completion of the algebraic tensor product of A_1 and A_2 in the operator norm of $H_1 \otimes H_2$.

If A_1 and A_2 are any two C*-algebras then we define the C*-algebra tensor product $A_1 \otimes A_2$ by first representing A_1 and A_2 faithfully on Hilbert spaces H_1 and H_2, then proceeding as above. The purpose of this exercise is to show that $A_1 \otimes A_2$ does not depend on the choices of the representations for A_1 and A_2. The following line of reasoning works for separable C*-algebras, to which the general case is easily reduced.

(a) Show that replacing H_1 by $H_1 \oplus H_1 \oplus \ldots$ and H_2 by $H_2 \oplus H_2 \oplus \ldots$ does not change $A_1 \otimes A_2$.
(b) Having replaced H_1 and H_2 as above, show that the map from $A_1 \otimes A_2$ into the Calkin algebra $\mathfrak{Q}(H_1 \otimes H_2)$ is isometric.
(c) Use Theorem 3.4.3 to show that if H_1' and H_2' are other choices of representing Hilbert spaces then the norm on the C*-algebra $\pi[A_1 \otimes A_2] \subseteq \mathfrak{Q}(H_1' \otimes H_2')$ is less than or equal to the norm on $\pi[A_1 \otimes A_2] \subseteq \mathfrak{Q}(H_1 \otimes H_2)$.
(d) Use the argument of (a) and (b) to show that the tensor product norm computed from $H_1' \otimes H_2'$ is less than or equal to the norm computed from $H_1 \otimes H_2$. By symmetry, the two norms must be equal.

3.9.9 REMARKS The tensor product we have constructed is given the name *spatial*, or *minimal*, tensor product, since it is an awkward, complicating feature of C*-algebra theory that other norms on the algebraic tensor product exist.[26] Our choice is generally the best behaved. For instance it is functorial, and if $A_1 \to A_1'$ is an inclusion then so is $A_1 \otimes A_2 \to A_1' \otimes A_2$. This is not generally so of other tensor products.

3.9.10 Let $A = C(X)$ be a commutative, unital C*-algebra, and let B be another C*-algebra. Let $C(X; B)$ denote the C*-algebra of continuous B-valued functions on X, with supremum norm and pointwise involution. Prove that the map

$$f \otimes b \mapsto fb,$$

from the algebraic tensor product of A and B to $C(X; B)$, extends to a $*$-isomorphism of $C(X) \otimes B$ with $C(X; B)$.

3.9.11 Prove that if $\sigma\colon A_1 \to A_1'$ is a completely positive map then $\sigma \otimes 1$, defined on the algebraic tensor product in the obvious way, extends to a completely positive map $\sigma \otimes 1\colon A_1 \otimes A_2 \to A_1' \otimes A_2$ (use Stinespring's Theorem).

3.9.12 Use the previous exercise to show that the tensor product of a short exact sequence

$$0 \to J_1 \to A_1 \to A_1/J_1 \to 0$$

with a C*-algebra A_2, that is,

$$0 \to J_1 \otimes A_2 \to A_1 \otimes A_2 \to A_1/J_1 \otimes A_2 \to 0,$$

remains an exact sequence, if either

(a) the surjection $A_1 \to A_1/J_1$ has a completely positive section, or
(b) the C*-algebra A_2 is nuclear.

[26] In fact, if A_2 is any non-nuclear C*-algebra then there is an A_1 such that different completions of the algebraic tensor product exist.

3.9.13 REMARKS A C*-algebra A_2 is *exact* if its tensor product with any short exact sequence, as above, remains exact. Thus nuclear C*-algebras are exact. It is not hard to show that C*-subalgebras of exact C*-algebras are exact. So are quotients, but this is much harder. If F_2 is the free group on two generators then $C^*_r(F_2)$ is an example of a C*-algebra which is exact but not nuclear. The full C*-algebra $C^*(F_2)$ is not even exact. (The same remarks apply with F_2 replaced by $G = SL(3, \mathbb{Z})$.)

3.9.14 Show that if $A \to B$ is a unital inclusion of nuclear C*-algebras then there is a completely positive map $B \to A$ such that $A \to B \to A$ is the identity (use the remark following Lemma 3.6.5). Conclude that an increasing union of nuclear C*-algebras is nuclear. Thus the CAR algebra from Exercise 2.9.16 is nuclear.

REMARK One can show that the Cuntz algebras $\mathcal{O}_n$ from Exercise 2.9.17 are also nuclear.

3.9.15 Define a non-unital C*-algebra A to be nuclear if and only if $\widetilde{A}$ (the C*-algebra with a unit adjoined) is nuclear. Check that the previous exercises carry over to the non-unital case.

3.9.16 Let G be a group and denote by $\tau\colon \mathbb{C}[G] \to \mathbb{C}$ the $*$-homomorphism associated to the trivial representation of G. Suppose that τ extends to a $*$-homomorphism from $C^*_r(G)$ to $\mathbb{C}$. Show that if τ is extended to a state on $\mathfrak{B}(\ell^2(G))$ using the Hahn–Banach Theorem (see the proof of Lemma 1.6.4), and if this state is then restricted to $\ell^\infty(G)$, then what is obtained is an invariant mean for G. Conclude that the following are equivalent:

(a) the trivial representation of G extends to a $*$-homomorphism on $C^*_r(G)$;
(b) the regular representation of $C^*(G)$ is injective;
(c) the group G is amenable.

3.9.17 Here is one way to prove the structure theorem for the full C*-algebra of a property T group. It is convenient to represent $C^*(G)$ as a C*-algebra of operators on a Hilbert space H. If π is a finite-dimensional, irreducible unitary representation of G then we can assume that the π-isotypical subspace of H is a single copy, H_π, of the representation space for π.

(a) The first step in the argument is to show that there is *some* element of $C^*(G)$ which is non-zero on H_π but zero on the orthogonal complement $H_\pi^\perp$. Suppose, for the sake of a contradiction, that no such element exists. Then $C^*(G)$ is represented faithfully on $H_\pi^\perp$ and we can assume that the representation contains no non-zero compact operator (if not, replace $H_\pi^\perp$ with $H_\pi^\perp \oplus H_\pi^\perp \oplus \ldots$; note that the infinite direct sum does not contain a copy of the representation π either). Apply Glimm's Lemma for matrices to the completely positive map $\pi\colon C^*(G) \to \mathfrak{B}(H_\pi)$ to conclude that the representation π is 'approximately' a subrepresentation in $H_\pi^\perp$; then apply Schur's Lemma for property T groups to conclude that π is actually a subrepresentation in $H_\pi^\perp$. This contradicts the definition of $H_\pi^\perp$.

(b) Show that the set of all elements of $C^*(G)$ which are zero on $H_\pi^\perp$ is a subalgebra of $\mathfrak{B}(H_\pi)$ whose commutant is trivial (since π is irreducible), and hence is all of $\mathfrak{B}(H_\pi)$.

(c) Complete the argument by observing that if $P_\pi \in C^*(G)$ denotes the identity in $\mathfrak{B}(H_\pi)$ then P_π has all the properties described in 3.7.6.

3.10 Notes

Stinespring's Theorem can be found in [123]. Our proof of Voiculescu's Theorem, and the lifting properties which imply that Ext is a group, is based on the influential exposition of Arveson [5]. The original proof that Ext is a group was quite different; see [33], and compare [34].

Nuclear C^*-algebras were first studied in connection with the theory of C^*-algebra tensor products, and the original definition referred to a uniqueness property for tensor products. The reader is referred to the survey article of Lance [92] for more information. The class of *exact* C^*-algebras is larger than that of nuclear C^*-algebras; for more information about them, and related approximation properties, see the lectures of Wassermann [133].

Voiculescu's Theorem appears in [131]. As the title of Voiculescu's paper suggests, his result includes the Weyl–von Neumann Theorem as the special case $A = C(X)$, $X \subseteq \mathbb{R}$. Note that we made use of the Weyl–von Neumann Theorem at a crucial step (see Lemma 3.6.4) in the proof of Voiculescu's Theorem.

For a more extended discussion of amenability and nuclearity of discrete group C^*-algebras, see [45] or [104]. A standard reference for property T is [47]; in particular, the reader will find there a proof that $SL(3, \mathbb{Z})$ has property T. The original example of a C^*-algebra for which Ext is not a group is due to Anderson [2].

Our proof of Kasparov's Technical Theorem comes from [66]; a version of the Theorem first appears in [80].

4

K-THEORY

Let A be a separable, unital and nuclear C*-algebra. In the last two chapters we have organized the algebra extensions

$$0 \longrightarrow \mathfrak{K}(H) \longrightarrow E \longrightarrow A \longrightarrow 0$$

into an abelian group, and we have seen the significance of the class of *split* extensions as the zero element of this group. These constructions are in the spirit of homological algebra, and they suggest that one should develop 'homology' and 'cohomology' theories on the category of not necessarily commutative C*-algebras. (Recall that the category of *commutative* C*-algebras is equivalent to the category of pointed compact Hausdorff spaces, so what we are asking for is a 'non-commutative' extension of classical algebraic topology.) It turns out that, of the various tools of algebraic topology, K-theory is best suited to such a non-commutative development.

Several comprehensive treatments of K-theory for C*-algebras are now available, and this chapter is not intended as a replacement for them. It is a rapid refresher course with an emphasis on certain topics (relative groups, products) which will be needed later and are not always to the fore in the standard texts. However, the chapter is more or less self-contained and could also be used as an introduction to K-theory by a reader who is prepared to fill in some details as we go along.

4.1 The Group $K_0(A)$

As was suggested above, K-theory for C*-algebras is a non-commutative counterpart to K-theory for topological spaces. Let us therefore briefly recall the central definition of topological K-theory. If X is a compact Hausdorff space, then $K^0(X)$ is the abelian group generated by the isomorphism classes of complex vector bundles over X, subject to the relations

$$[E] + [F] = [E \oplus F]$$

for vector bundles E and F. For our purposes a complex vector bundle is most conveniently described by a continuous function $p\colon X \to M_n(\mathbb{C})$ such that $p(x)$ is a *projection* ($p = p^* = p^2$) for each $x \in X$. The ranges of the projections $p(x)$

then fit together to form a vector bundle over X, and a standard classification theorem states that, in the limit as $n \to \infty$, we obtain in this way a one-to-one correspondence between isomorphism classes of vector bundles over X and homotopy classes of projection-valued functions. We can therefore reformulate matters as follows: the abelian group $K^0(X)$ is generated by homotopy classes of maps from X to the space of projections in $M_n(\mathbb{C})$, as n runs over the natural numbers.

We can easily translate this definition to the context of C*-algebras. A projection-valued function from X to $M_n(\mathbb{C})$ is the same thing as a projection in the C*-algebra $M_n(C(X))$ of $n \times n$ matrices over $C(X)$. We are thus led to make the following definition:

4.1.1 DEFINITION Let A be a unital C*-algebra. Denote by $K_0(A)$ the abelian group with one generator, $[p]$, for each projection p in each matrix algebra $M_n(A)$, and the following relations:

(a) if both p and q are projections in $M_n(A)$, for some n, and if p and q are joined by a continuous path of projections in $M_n(A)$, then $[p] = [q]$,

(b) $[0] = 0$, for any size of (square) zero matrix, and

(c) $[p] + [q] = [p \oplus q]$, for any sizes of projection matrices p and q.

If $p \in M_m(A)$ and $q \in M_n(A)$, the notation $p \oplus q$ refers to the projection $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ in $M_{m+n}(A)$. For brevity we shall sometimes refer to a projection in $M_n(A)$ as a projection *over* A. We shall say for short that two projections are *homotopic* if they can be connected by a continuous path of projections.

4.1.2 REMARK Because of relations (b) and (c), every element of $K_0(A)$ is in fact a formal difference $[p] - [q]$ of projections in some $M_n(A)$. Two such formal differences $[p] - [q]$ and $[p'] - [q']$ define the same K-theory class if and only if they are *stably homotopic*, which is to say that there exists a third projection r such that $p \oplus q' \oplus r$ can be joined by a continuous path of projections to $p' \oplus q \oplus r$. To prove this, first observe that the stable homotopy classes of formal differences (p, q) of projections constitute an abelian group G, and then show that $[p] \mapsto (p, 0)$ and $(p, q) \mapsto [p] - [q]$ define mutually inverse homomorphisms $K_0(A) \to G$ and $G \to K_0(A)$.

We note right away that K_0 is a *functor*: if $\alpha\colon A \to B$ is a unital $*$-homomorphism, and p is a projection in $M_n(A)$, then $\alpha(p)$ (defined by applying α element-by-element to the matrix p) is a projection in $M_n(B)$. So α induces a homomorphism $\alpha_*\colon K_0(A) \to K_0(B)$.

Here are some simple examples of K-theory groups.

4.1.3 EXAMPLE Let $A = \mathbb{C}$. Two projections in $M_n(\mathbb{C})$ are connected by a path of projections if and only if they have the same rank. The map $[p] \mapsto \mathrm{Rank}(p)$ induces an isomorphism from $K_0(\mathbb{C})$ to $\mathbb{Z}$.

4.1.4 EXAMPLE Similarly, $K_0(A) \cong \mathbb{Z}$ if A is a matrix algebra $M_m(\mathbb{C})$. In fact, the reader may readily check that if A is any unital algebra then $K_0(A)$ is canonically isomorphic to $K_0(M_m(A))$ (see Lemma 4.2.4).

4.1.5 EXAMPLE If $A = C(X)$, where X is compact Hausdorff, then $K_0(A)$ is the topological K-theory group $K^0(X)$.[27]

We consider next some consequences of the relations which define $K_0(A)$.

4.1.6 EXAMPLE Let $p, q \in M_n(A)$ be projections such that $pq = 0 = qp$. Then $p + q$ is also a projection. Moreover, $[p + q] = [p] + [q]$; for

$$t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} p + \begin{pmatrix} \cos^2(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t) \\ \cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t) & \sin^2(\frac{\pi}{2}t) \end{pmatrix} q$$

is a continuous path of projections connecting $(p + q) \oplus 0$ (for $t = 0$) and $p \oplus q$ (for $t = 1$).

4.1.7 PROPOSITION *Let* p *and* q *be projections in a unital* C^**-algebra* A, *with* $\|p - q\| < 1$. *Then there is a unitary* $u \in A$ *such that* $q = upu^*$.

PROOF Define $x = qp + (1 - q)(1 - p)$. Then $xp = qx$, and a simple calculation shows that

$$x - 1 = 2qp - q - p = (2q - 1)(p - q).$$

Thus $\|x - 1\| < 1$ and so x is invertible. We may define a unitary $u = x(x^*x)^{-\frac{1}{2}}$ by the functional calculus. Now x^*x commutes with p, and so (by Exercise 1.9.19)

$$up = xp(x^*x)^{-\frac{1}{2}} = qx(x^*x)^{-\frac{1}{2}} = qu$$

as required. □

A continuous path of projections can be subdivided into segments to which the proof of Proposition 4.1.7 applies. Thus we obtain:

4.1.8 COROLLARY *Let* $t \mapsto p_t$ *be a continuous path of projections in a unital* C^**-algebra* A. *Then there is a continuous path of unitaries* $t \mapsto u_t$ *in* A *such that* $p_t = u_t p_0 u_t^*$. □

It is also useful to consider some other equivalence relations between projections:

[27] The conventional placement of subscripts and superscripts reflects the facts that $K^0(X)$ is contravariant in X, whereas $K_0(A)$ is covariant in A. Later in the book we shall be concerned with K-*homology* groups, which have the opposite functoriality. To interpret the notation correctly it is therefore necessary to keep carefully in mind whether the argument of a given K-functor is a *topological space* or a C^*-*algebra*.

4.1.9 DEFINITION Let p and q be projections in a unital C^*-algebra A.

(a) We say that p and q are *unitarily equivalent* if $upu^* = q$ for some unitary $u \in A$.

(b) We say that they are *Murray–von Neumann equivalent* if there is $v \in A$ such that $v^*v = p$ and $vv^* = q$.

These are equivalence relations. Moreover, homotopy implies unitary equivalence by Corollary 4.1.8 above, and unitary equivalence implies Murray–von Neumann equivalence. Neither of these implications is reversible (see Exercises 4.10.1 and 4.10.2). But when we pass to matrices, all our notions of equivalence coincide with one another. To see this, we need a simple calculation:

4.1.10 LEMMA *Let v be a partial isometry*[28] *in a unital C^*-algebra. Then the matrix $\begin{pmatrix} v & 1-vv^* \\ v^*v-1 & v^* \end{pmatrix}$ is unitary, and is connected through a path of unitaries to the identity matrix. In particular, if v itself is unitary then the matrix $\begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix}$ is path connected through unitaries to the identity matrix.*

PROOF The path

$$t \mapsto \begin{pmatrix} (\cos(\frac{\pi}{2}t))v & 1-\big(1-\sin(\frac{\pi}{2}t)\big)\,vv^* \\ \big(1-\sin(\frac{\pi}{2}t)\big)\,v^*v-1 & (\cos(\frac{\pi}{2}t))v^* \end{pmatrix}$$

consists of unitaries and connects the given matrix to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which is itself connected by a similar path (take $v = 1$) to the identity matrix. □

We obtain directly from this:

4.1.11 PROPOSITION *Let p and q be Murray–von Neumann equivalent projections in a unital C^*-algebra A. Then $p \oplus 0$ and $q \oplus 0$ are unitarily equivalent via a unitary 2×2 matrix which is path-connected to the identity through unitaries. Hence the projections $p \oplus 0$ and $q \oplus 0$ are homotopic.* □

4.1.12 LEMMA *Let A be a unital C^*-algebra. Projections p and q over A represent the same element of $K_0(A)$ if and only if $p \oplus I_m \oplus 0_n$ is Murray–von Neumann equivalent to $q \oplus I_m \oplus 0_{n'}$ for some $m \times m$ identity matrix I_m and zero matrices 0_n, $0_{n'}$. In particular, Murray–von Neumann equivalent projections represent the same element of $K_0(A)$.*

PROOF By Proposition 4.1.11, Murray–von Neumann equivalent projections become homotopic after stabilization. This yields the 'if' part of the lemma. For the 'only if' part, suppose that p and q define the same K-theory class, and assume by adding suitable zero matrices that both p and q belong to the same matrix algebra over A. Then, by Remark 4.1.2, there is a projection r such that $p \oplus r$ is homotopic to $q \oplus r$, and therefore $p \oplus r \oplus (1-r)$ is homotopic to

[28] Recall that v is a *partial isometry* if vv^* and v^*v are projections, or equivalently if $vv^*v = v$.

$q \oplus r \oplus (1-r)$. But by Example 4.1.6, $r \oplus (1-r)$ is homotopic to $1 \oplus 0$. Thus $p \oplus 1 \oplus 0$ is homotopic to $q \oplus 1 \oplus 0$. But homotopy implies Murray–von Neumann equivalence. □

4.1.13 EXAMPLE Let $A = \mathfrak{B}(H)$ where H is infinite-dimensional. Any two projections in A with infinite-dimensional range are Murray–von Neumann equivalent (Exercise 4.10.1). For any projection p, therefore, the projections $p \oplus 1$ and $0 \oplus 1$ are Murray–von Neumann equivalent. Thus $[p]+[1] = [0]+[1]$ in $K_0(\mathfrak{B}(H))$, and so $[p] = [0] = 0$. Hence $K_0(\mathfrak{B}(H)) = 0$.

4.1.14 REMARK There is an alternative definition of K_0, often preferred by algebraists, in which one takes the generators to be *finite projective modules* over A. The two definitions are linked as follows: the range of a projection $p \in M_n(A)$ is a finite projective A-module, and Murray-von Neumann equivalence of projections corresponds to isomorphism of projective modules. We refer the reader to Appendix A for a further discussion of this approach to K-theory.

Let $\mathcal{A}$ be a normed $*$-algebra, and suppose that its completion $A = \overline{\mathcal{A}}$ is a C*-algebra. Many standard examples of C*-algebras are defined in this way as completions, and it would be helpful in calculation if one could relate the K-theory of A to the projection structure of the uncompleted algebra $\mathcal{A}$. Unfortunately A will in general have many more projections than $\mathcal{A}$ — consider, for example, the situation where A is the algebra of continuous functions on some infinite, compact, disconnected subset of $\mathbb{R}$ and $\mathcal{A}$ is the subalgebra of polynomial functions — but there is one case in which one can say something useful:

4.1.15 PROPOSITION *Let* A *be a unital* C*-*algebra and suppose that there is an increasing sequence*

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A$$

of unital C*-*subalgebras whose union* $\mathcal{A}$ *is dense in* A. *Then the induced map*

$$\varinjlim K_0(A_j) \to K_0(A)$$

is an isomorphism.

PROOF Let $p \in M_n(A)$ be a projection. Then for some j there is a selfadjoint $a \in M_n(A_j)$ with $\|a-p\| < \frac{1}{2}$, so that $\mathrm{Spectrum}(a) \subseteq (-\frac{1}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2})$. Let $f(\lambda) = 0$ for $\lambda < \frac{1}{2}$, $f(\lambda) = 1$ for $\lambda > \frac{1}{2}$; the function f is continuous on Spectrum(a), so $q = f(a)$ is a projection in $M_n(A_j)$, and $\|q-a\| < \frac{1}{2}$. Hence $\|q-p\| < 1$ so, by Proposition 4.1.7, q and p are unitarily equivalent in $M_n(A)$. This proves that p lies in the image of $K_0(A_j) \to K_0(A)$, and thus that the induced map on the direct limit is surjective.

To prove the injectivity of the induced map, suppose that p_0 and p_1 are projections in $M_n(A_j)$ that are connected by a continuous path p_t of projections in $M_n(A)$. There exists a continuous path a_t of selfadjoint elements in $M_n(A_{j'})$, $j' \geqslant j$, with $a_0 = p_0$, $a_1 = p_1$, and $\|a_t - p_t\| < \frac{1}{2}$. Applying the construction of the previous paragraph to each a_t, and using the continuity of the functional calculus (Exercise 1.9.3), we obtain a continuous path q_t of projections in $M_n(A_{j'})$ which connects $p_0 = q_0$ to $p_1 = q_1$. Thus $[p_0] = [p_1]$ in $K_0(A_{j'})$. □

4.1.16 Remark Notice the similarity between the two halves of the proof. To prove injectivity one applies the proof of surjectivity to a path, relative to its endpoints. This kind of proof, summed up in the slogan 'uniqueness is a relative form of existence', occurs frequently in K-theory and elsewhere. We shall usually give only the surjectivity part of similar proofs in future, relying on the reader to fill in the details for the relative version which leads to injectivity.

4.1.17 Example In Exercise 2.9.16 we introduced the CAR algebra, which is the completion of an increasing union of matrix algebras

$$M_2(\mathbb{C}) \subseteq M_4(\mathbb{C}) \subseteq M_8(\mathbb{C}) \subseteq \cdots$$

where the inclusion maps are given by $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$. Passing to K-theory we find that $K_0(\mathrm{CAR})$ is the direct limit of the sequence $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \ldots$, where each homomorphism is given by multiplication by 2; thus $K_0(\mathrm{CAR}) = \mathbb{Z}[\frac{1}{2}]$, the additive group of the ring of dyadic rationals.

4.1.18 Remark In dealing with non-separable C*-algebras it is occasionally useful to remark that the proposition extends to the case of general direct limits: if $\{A_\alpha\}$ is a direct system of C*-subalgebras of A whose union is dense, then

$$K_0(A) = \varinjlim K_0(A_\alpha).$$

The proof is exactly the same.

4.2 K_0 for Non-Unital Algebras

Although Definition 4.1.1 makes sense even when applied to a non-unital algebra, it turns out that the 'K-theory group' so produced is not especially useful. Here instead is the definition we shall use:

4.2.1 Definition Let J be a non-unital C*-algebra, and let $\widetilde{J}$ be the unitalization of J (Definition 1.3.8), so that there is a short exact sequence

$$0 \longrightarrow J \longrightarrow \widetilde{J} \longrightarrow \mathbb{C} \longrightarrow 0\,.$$

Denote by $K_0(J)$ the kernel of the induced homomorphism $K_0(\widetilde{J}) \to K_0(\mathbb{C}) = \mathbb{Z}$.

4.2.2 REMARK The topologist may find it helpful to think of this definition in the following way. If $J = C_0(X)$, where X is locally compact and Hausdorff, then $\widetilde{J} = C(\widetilde{X})$, where $\widetilde{X}$ is the one-point compactification of X. We are therefore defining the K-theory of X to be the K-theory of $\widetilde{X}$ relative to the point at ∞. This is exactly how K-theory for locally compact spaces is usually defined.

If J happens to have a unit already then the K-theory sequence

$$0 \longrightarrow K_0(J) \longrightarrow K_0(\widetilde{J}) \longrightarrow K_0(\mathbb{C}) \longrightarrow 0$$

is split exact (Exercise 4.10.3), so that the 'new' definition of $K_0(J)$ agrees with the 'old' one. We have therefore extended K_0 to a functor on the category of *all* C^*-algebras and $*$-homomorphisms.

4.2.3 REMARK Many results about the K-theory of non-unital algebras can be deduced from their unital counterparts by simple diagram chasing. As an example we consider Proposition 4.1.15. Let A be a (possibly non-unital) C^*-algebra, and suppose that $\{A_j\}_{j=1}^{\infty}$ is an increasing sequence of C^*-subalgebras whose union is dense in A. Then the union of the $\widetilde{A}_j$ is dense in $\widetilde{A}$. For each j we have a short exact sequence

$$0 \longrightarrow K_0(A_j) \longrightarrow K_0(\widetilde{A}_j) \longrightarrow \mathbb{Z} \longrightarrow 0$$

and since direct limits preserve exactness we obtain in the limit a short exact sequence

$$0 \longrightarrow \varinjlim K_0(A_j) \longrightarrow \varinjlim K_0(\widetilde{A}_j) \longrightarrow \mathbb{Z} \longrightarrow 0 .$$

Applying Proposition 4.1.15 to the middle term and using the definition of $K_0(A)$, this gives us

$$\varinjlim K_0(A_j) \cong K_0(A).$$

In short, Proposition 4.1.15 extends to the non-unital case.

The *stabilization maps* $A \to M_n(A)$, given by $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$, are important examples of non-unital $*$-homomorphisms.

4.2.4 LEMMA *The stabilization maps induce isomorphisms on* K*-theory.*

PROOF A diagram chase as above allows us to assume that A is unital. Since $M_k(M_n(A)) \cong M_{kn}(A)$, we may regard a projection over $M_n(A)$ as a projection over A, and thus we obtain a map $K_0(M_n(A)) \to K_0(A)$. This map is a two-sided inverse to the stabilization map. $\square$

4.2.5 EXAMPLE The C^*-algebra $\mathfrak{K}(H)$ of compact operators is the direct limit of a sequence

$$M_2(\mathbb{C}) \subseteq M_4(\mathbb{C}) \subseteq M_8(\mathbb{C}) \subseteq \cdots$$

where the inclusions are stabilization maps and so induce isomorphisms on K-theory. By Remark 4.2.3 we see that $K_0(\mathfrak{K}(H)) \cong \mathbb{Z}$; the isomorphism sends a finite-rank projection p to $\mathrm{Dim}(\mathrm{Image}(p))$. The same argument shows that the $*$-homomorphism $a \mapsto a \otimes e$, where e is a rank-one projection, induces an isomorphism $K_0(A) \cong K_0(A \otimes \mathfrak{K}(H))$ for any C^*-algebra A. This is called the *stability* property of K-theory.

4.3 Relative K-Theory and Excision

Suppose now that J is an ideal in a unital C^*-algebra A, and let $\pi\colon A \to A/J$ denote the quotient map.

4.3.1 DEFINITION A *relative* K-*cycle* for $(A, A/J)$ is a triple (p, q, x) where p and q are projections in $M_n(A)$, x belongs to $M_n(A)$, and $\pi(x) \in M_n(A/J)$ is a partial isometry implementing a Murray–von Neumann equivalence between $\pi(p)$ and $\pi(q)$. If x itself is a Murray–von Neumann equivalence between p and q, we say that the relative K-cycle (p, q, x) is *degenerate*.

4.3.2 EXAMPLE Let $T \in \mathfrak{B}(H)$ be an operator for which $T^*T - I$ and $TT^* - I$ are compact.[29] Let $A = \mathfrak{B}(H)$ and $J = \mathfrak{K}(H)$, so that A/J is the Calkin algebra $\mathfrak{Q}(H)$. By definition, $\pi(T)$ is unitary in $\mathfrak{Q}(H)$. Thus the triple (I, I, T) is a relative K-cycle for $(A, A/J)$.

4.3.3 DEFINITION The *relative* K-*group* $K_0(A, A/J)$ is defined to be the abelian group with one generator $[p, q, x]$ for each relative K-cycle (p, q, x), and with the following relations:

(a) if (p_0, q_0, x_0) and (p_1, q_1, x_1) are relative K-cycles (belonging to the same matrix algebra) that can be connected by a continuous path (p_t, q_t, x_t) of relative K-cycles, then $[p_0, q_0, x_0] = [p_1, q_1, x_1]$,
(b) if (p, q, x) is degenerate then $[p, q, x] = 0$, and
(c) $[p \oplus p', q \oplus q', x \oplus x'] = [p, q, x] + [p', q', x']$ for all relative K-cycles (p, q, x) and (p', q', x').

4.3.4 REMARK Once again, there is a useful reformulation of this definition in terms of projective A-modules and operators on them, which is discussed in Appendix A.

There is a natural map $K_0(A, A/J) \to K_0(A)$ which sends $[p, q, x]$ to $[p] - [q]$. Using Lemma 4.1.12 we easily obtain:

[29] One sometimes refers to such an operator T as *essentially unitary*.

4.3.5 PROPOSITION *The sequence of groups*

$$K_0(A, A/J) \to K_0(A) \to K_0(A/J)$$

is exact in the middle. □

4.3.6 EXAMPLE Let J be a possibly non-unital C*-algebra, and let $\tilde{J}$ be the unitalization of J. From Proposition 4.3.5, we obtain a homomorphism from the relative group $K_0(\tilde{J}, \mathbb{C})$ onto $K_0(J) = \mathrm{Kernel}[K_0(\tilde{J}) \to K_0(\mathbb{C})]$. In fact, this map is an isomorphism (Exercise 4.10.7). Thus the group $K_0(J)$ is a relative K-group.

It turns out that the group $K_0(A, A/J)$ is *always* isomorphic to $K_0(J)$. In other words, $K_0(A, A/J)$ depends only on the C*-algebraic structure of J, and not at all on the structure of the algebra A in which J is embedded as an ideal. Such results are called *excision* theorems, by analogy with their counterparts in algebraic topology. The Excision Theorem, together with Proposition 4.3.5, will be used to develop the homological properties of K-theory.

4.3.7 DEFINITION If J is an ideal in the unital algebra A, then $\tilde{J}$ may be regarded as a subalgebra of A. The natural forgetful map

$$K_0(J) = K_0(\tilde{J}, \mathbb{C}) \to K_0(A, A/J)$$

is called the *excision map.*

4.3.8 EXCISION THEOREM *The excision map of Definition* 4.3.7 *is always an isomorphism.*

We shall give the proof of this result after we have discussed a couple of examples.

4.3.9 EXAMPLE Let $A = \mathfrak{B}(H)$ and $J = \mathfrak{K}(H)$, and let $T \in A$ be an essentially unitary Fredholm operator. Let $[T] \in K_0(A, A/J)$ be defined by the relative K-cycle (I, I, T) of Example 4.3.2. By the Excision Theorem and Example 4.2.5,

$$K_0(A, A/J) \cong K_0(J) \cong \mathbb{Z}$$

and so $[T]$ corresponds to a certain integer. We claim that this integer is the Fredholm index of the operator T. Indeed, let p denote the orthogonal projection onto Kernel(T) and let q denote the orthogonal projection onto Kernel(T^*). Then the cycle (I, I, T) splits as a direct sum of $(p, q, 0)$ and $(1-p, 1-q, T(1-p))$,[30] and the second of these cycles is homotopic to a degenerate (since T restricts to an invertible Hilbert space operator from Image$(1-p)$ to Image$(1-q)$). The cycle $(p, q, 0)$ already belongs to $K_0(\tilde{J}, \mathbb{C})$ and it corresponds to the integer Dim(Image(p)) − Dim(Image(q)), that is, to the Fredholm index of T.

[30] Here we are using a relative version of Example 4.1.6.

The Excision Theorem does not mean that the introduction of the relative groups served no purpose. Important elements of the groups $K_0(J)$ are often most naturally defined by relative K-cycles. As we have seen, the Fredholm operators furnish such an example. Here is another, more geometric, one:

4.3.10 EXAMPLE Let $\mathbb{D}$ denote the open unit disk in the complex plane. Let $A = C(\overline{\mathbb{D}})$ and let $J = C_0(\mathbb{D})$ be the ideal of continuous functions vanishing on the boundary $\partial\mathbb{D}$, so that $A/J = C(\partial\mathbb{D})$. Let $z\colon \overline{\mathbb{D}} \to \mathbb{C}$ be the inclusion map, which we think of as an element of A. Then the triple[31]

$$(1, 1, \bar{z})$$

defines a relative K-cycle for $(C(\overline{\mathbb{D}}), C(\partial\mathbb{D}))$ and thus by excision a K-theory class in $K_0(C_0(\mathbb{D}))$. Since $\mathbb{D}$ is homeomorphic to the plane $\mathbb{R}^2$, we may also regard the class so defined as an element of $K_0(C_0(\mathbb{R}^2))$. The element of $K_0(C_0(\mathbb{R}^2))$ obtained in this way is called the *Bott generator*; it is the single most important example of a K-theory class. We shall discover later that $K_0(C_0(\mathbb{R}^2))$ is isomorphic to $\mathbb{Z}$ and that the Bott generator corresponds to the integer 1 under this isomorphism.

4.3.11 REMARK We can abstract this construction as follows. Let (X, Y) be a compact Hausdorff pair — that is to say, let X be a compact Hausdorff space and let Y be a closed subspace. Let $x\colon X \to M_n(\mathbb{C})$ be a matrix-valued function whose restriction to Y is unitary. Then (I_n, I_n, x) (where I_n denotes the $n \times n$ identity matrix) defines a relative K-cycle for $(C(X), C(Y))$, and therefore by excision an element of $K_0(C_0(X \setminus Y))$. We say that this element is obtained from x by the *difference construction*. Note the analogy with our construction of the relative cycle associated to a Fredholm operator, in Example 4.3.2.

4.3.12 REMARK The version of the difference construction described in Remark 4.3.11 is not adequate to describe every element of the group $K_0(C_0(X \setminus Y))$ — consider for instance the case $Y = \emptyset$. We need to introduce bundles. Let V and W be Hermitian vector bundles over X, and suppose that there is given a bundle homomorphism $x\colon V \to W$ which becomes a unitary isomorphism when restricted to Y. By the classification theorem for vector bundles, there are projections p and q over $C(X)$ whose ranges are the bundles V and W. It follows from our assumption about x that (p, q, x) is a relative K-cycle for $(C(X), C(Y))$, and thus by excision determines an element of $K_0(C_0(X \setminus Y))$. It is known (Exercise 4.10.6) that every element of $K_0(C_0(X \setminus Y))$ arises in this way from the so-called *difference bundle construction.*

[31] Standard sign conventions require us to choose the complex conjugate $\bar{z}$ rather than z here.

In proving Theorem 4.3.8 we shall make use of some topological properties of surjective $*$-homomorphisms. Let $f\colon E \to B$ be a map of topological spaces. Recall that f has the *path-lifting property* if any commutative diagram of the form

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & E \arrow[d, "f"] \\ {[0,1]} \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

may be extended to a commutative diagram containing a map $[0,1] \to E$ (indicated by the dotted arrow).

4.3.13 LEMMA *Let $\pi\colon A \to B$ be a surjective $*$-homomorphism. Then π has the path-lifting property. The same is true for the restriction of π to a map from the space of selfadjoint elements of A to the space of selfadjoint elements of B.*

PROOF It suffices to show that the $*$-homomorphism

$$C([0,1];A) \to C([0,1];B),$$

induced by π, is surjective. By Exercise 3.9.10 we may identify it with the $*$-homomorphism

$$1 \otimes \pi\colon C[0,1] \otimes A \to C[0,1] \otimes B.$$

The range of $1 \otimes \pi$ certainly contains the *algebraic* tensor product of $C[0,1]$ with B (since π is surjective), and therefore is dense. But the range of any $*$-homomorphism is closed (Corollary 1.7.7), so the proof is completed. $\square$

4.3.14 PROPOSITION *Let $\pi\colon A \to B$ be a surjective, unital $*$-homomorphism. Then*

(a) *the restriction of π to a map from the unitary group of A to the unitary group of B has the path-lifting property, and*

(b) *the restriction of π to a map from the space of projections in A to the space of projections in B has the path-lifting property.*

PROOF Statement (a) means that if u_t is a path of unitaries in B, and U_0 is a unitary in A with $\pi(U_0) = u_0$, then there is a path of unitaries U_t in A which lifts u_t (that is, $\pi(U_t) = u_t$ for all t) and begins with U_0. To prove this, a standard compactness argument shows that it is enough to establish local liftability, and clearly we may assume $U_0 = 1$ without loss of generality. Then for sufficiently small t, the number -1 does not belong to the spectrum of u_t, and so by choosing a continuous branch of the function $\log z$ on $\mathbb{C} \setminus \mathbb{R}^-$ and using the functional calculus we can find a path of selfadjoint elements $x_t \in B$ with $\exp(ix_t) = u_t$ and $x_0 = 0$. By Lemma 4.3.13, x_t can be lifted to a path X_t of selfadjoint elements of A with $X_0 = 0$. Then $U_t = \exp(iX_t)$ defines the desired path of unitaries. This proves the path-lifting property for unitaries. The corresponding property for projections is a consequence, since it follows from Corollary 4.1.8 that every path of projections is the conjugate of a fixed projection by a path of unitaries beginning at the identity. □

PROOF OF THEOREM 4.3.8 We shall outline a proof that the excision map is surjective, and (as threatened in Remark 4.1.16) we shall leave the proof of injectivity to the reader. We must show that any relative K-cycle (p, q, x) for $(A, A/J)$ is equivalent to one for $(\widetilde{J}, \mathbb{C})$.

First step. By forming the direct sum with the degenerate cycle $(1-p, 1-p, 1-p)$, followed by a rotation as in Example 4.1.6, we show that (p, q, x) is equivalent to a K-cycle of the form (p', q', x'), where p' is a matrix over $\widetilde{J}$.

Second step. By Proposition 4.1.11, the matrix

$$\begin{pmatrix} \pi(x') & 1 - \pi(x')\pi(x')^* \\ \pi(x')^*\pi(x') - 1 & \pi(x')^* \end{pmatrix}$$

over A/J is unitary, is connected to the identity by a path of unitaries, and conjugates $\pi(p') \oplus 0$ and $\pi(q') \oplus 0$. Lifting this path of unitaries to a path U_t of unitaries over A we see that the cycle (p', q', x') is equivalent to one of the form (p'', q'', x''), where p'' is a matrix with entries in $\widetilde{J}$, and where $x'' = p''U_1$ with U_1 a unitary matrix over A, connected to the identity by a path U_t.

Third step. Since $t \mapsto U_t$ is a path of unitaries, the path

$$t \mapsto (p'', U_t^* U_1 q'' U_1^* U_t, p'' U_t)$$

consists of relative K-cycles. Thus the cycle (p'', q'', x'') is equivalent to one of the form (p''', q''', x'''), where p''' has entries in $\widetilde{J}$, $x''' = p'''$, and so q''' also has entries in $\widetilde{J}$. So (p''', q''', x''') is a cycle for $(\widetilde{J}, \mathbb{C})$ as required. □

As a consequence of Proposition 4.3.5 and Theorem 4.3.8 (together with the usual diagram chasing to reduce to the unital case) we have the *half-exactness* property of K-theory:

4.3.15 PROPOSITION *Let J be an ideal in a C^*-algebra A. The sequence of K-theory groups*

$$K_0(J) \longrightarrow K_0(A) \longrightarrow K_0(A/J)$$

is exact in the middle. □

4.4 Homotopy

4.4.1 DEFINITION Two $*$-homomorphisms $\alpha_0, \alpha_1 : A \to B$ are *homotopic* if there is a point-norm continuous path of homomorphisms $\alpha_t \colon A \to B$ $(t \in [0,1])$ which includes α_0 and α_1.

4.4.2 EXAMPLE Let X and Y be compact Hausdorff spaces. Then two maps from X to Y are homotopic (in the usual topological sense applicable to such maps) if and only if the induced $*$-homomorphisms from $C(Y)$ to $C(X)$ are homotopic, in the sense of Definition 4.4.1. An analogous statement holds for locally compact spaces X and Y and the corresponding C^*-algebras $C_0(X)$ and $C_0(Y)$, provided that we understand the term 'map from X to Y' to refer to a morphism[32] of the sort indicated in Remark 1.3.15.

4.4.3 PROPOSITION *If α_0 and α_1 are homotopic $*$-homomorphisms from A to B then the induced K-theory maps α_{0*} and α_{1*}, from $K_0(A)$ to $K_0(B)$, are equal.*

This result (called the *homotopy invariance of K-theory*) is an immediate consequence of the definition of the group K_0.

4.4.4 DEFINITION A C^*-algebra A is called *contractible* if the identity homomorphism and the zero homomorphism, from A to A, are homotopic.

4.4.5 EXAMPLE The algebra $C_0(X)$ is contractible if and only if the inclusion of the point at infinity into the one-point compactification of X is a homotopy equivalence (of pointed spaces). The algebra $C_0([0,\infty))$ is contractible, but the algebra $C_0(\mathbb{R})$ is not. Note that if X is compact, then $C(X)$ is *never* contractible by this definition.

By homotopy invariance, a contractible C^*-algebra has zero K-theory.

4.4.6 EXAMPLE A standard example of a contractible C^*-algebra is the *cone* $C(A) = C_0(0,1] \otimes A$ on a C^*-algebra A; the family $\alpha_t f(s) = f(st)$ provides the desired contracting homotopy.

4.4.7 DEFINITION A $*$-homomorphism $\alpha \colon A \to B$ is a *homotopy equivalence* if there exists a $*$-homomorphism $\beta \colon B \to A$ such that the composites $\alpha\beta$ and $\beta\alpha$ are homotopic to the identity $*$-homomorphisms on B and A respectively.

The homomorphisms α and β are said to be *homotopy inverses*. It follows from Proposition 4.4.3 that they induce isomorphisms on K-theory.

4.4.8 EXAMPLE The C^*-algebra A is contractible if and only if the zero homomorphism $0 \to A$ is a homotopy equivalence.

[32] We should also understand the term 'homotopy' to refer to a morphism of this sort from $X \times [0,1]$ to Y.

4.5 Higher K-Theory

In this section we are going to use the half-exactness and homotopy invariance properties of K-theory (Propositions 4.3.15 and 4.4.3) to define 'higher' K-groups which extend the little exact sequence of Proposition 4.3.15 to the left.

4.5.1 REMARK Our discussion will be rather formal, in the sense that no properties of the K-theory functor beyond its half-exactness and homotopy invariance will be used. Therefore, any half-exact and homotopy-invariant functor can be extended in a similar way. This point will be important later. In fact, in Chapter 7 we shall use the same construction to define 'higher K-homology' groups.[33]

4.5.2 DEFINITION Let A be a C^*-algebra and J an ideal in A. The *mapping cone* $C(A, A/J)$ of the surjective $*$-homomorphism $\pi: A \to A/J$ is the algebra consisting of pairs (a, f), where $a \in A$, $f: [0,1] \to A/J$ is continuous, $f(0) = 0$, and $f(1) = \pi(a)$.

For example, the mapping cone of the identity map $A \to A$ is isomorphic to the cone $C(A)$ defined above. The mapping cone $C(A, A/J)$ is a C^*-algebra under the natural pointwise operations. Its K-theory is described by another excision result:

4.5.3 PROPOSITION *The $*$-homomorphism $J \to C(A, A/J)$ given by $a \mapsto (a, 0)$ induces an isomorphism from $K_0(J)$ to $K_0(C(A, A/J))$.*

The proof of this proposition is straightforward. Before giving it, however, we want to show how the proposition is used.

4.5.4 DEFINITION Let A be a C^*-algebra. The *suspension* of A, $S(A)$, is the algebra of continuous functions $f: [0,1] \to A$, with $f(0) = f(1) = 0$.

Using Exercise 3.9.10 we may write

$$S(A) = C_0(0,1) \otimes A = C_0(\mathbb{R}) \otimes A,$$

identifying $(0,1)$ with $\mathbb{R}$ by an orientation-preserving homeomorphism.[34]

4.5.5 DEFINITION Let A be a C^*-algebra. Denote by $K_1(A)$ the group $K_0(S(A))$. More generally, denote by $K_p(A)$ the group $K_0(S^p(A)) = K_0(C_0(\mathbb{R}^p) \otimes A)$.

Notice that the suspension and mapping cone constructions are related by the short exact sequence

$$(4.5.6) \qquad 0 \longrightarrow S(A/J) \longrightarrow C(A, A/J) \longrightarrow A \longrightarrow 0$$

[33]The K-homology functor is *contravariant*, while the K-theory functor is *covariant*. But this does not affect the argument in any essential way.

[34]For definiteness, let us decide to use the homeomorphism $x \mapsto (x - \frac{1}{2})/x(1-x))$.

where the first $*$-homomorphism is given by $f \mapsto (0, f)$. This $*$-homomorphism induces a map

$$K_1(A/J) = K_0(S(A/J)) \longrightarrow K_0(C(A, A/J)) = K_0(J)$$

by Proposition 4.5.3.

4.5.7 DEFINITION The map $\partial\colon K_1(A/J) \to K_0(J)$ so defined is called the *boundary map* (or *connecting map*) associated to the short exact sequence $0 \to J \to A \to A/J \to 0$.

4.5.8 PROPOSITION *Let $0 \to J \to A \to A/J \to 0$ be a short exact sequence of C^*-algebras. The sequence of groups and homomorphisms*

$$K_1(A/J) \xrightarrow{\partial} K_0(J) \longrightarrow K_0(A) \longrightarrow K_0(A/J)$$

is exact both at $K_0(J)$ and at $K_0(A)$.

PROOF Exactness at $K_0(A)$ follows from Proposition 4.3.15. Exactness at $K_0(J)$ follows from Proposition 4.3.15 as well, this time applied to the short exact sequence 4.5.6. $\square$

We can iterate the mapping cone construction indefinitely (Exercise 4.10.15) and obtain the following result:

4.5.9 PROPOSITION *Let $0 \to J \to A \to A/J \to 0$ be a short exact sequence of C^*-algebras. There is a natural semi-infinite exact sequence of abelian groups and homomorphisms*

$$\to K_p(J) \to K_p(A) \to K_p(A/J) \to K_{p-1}(J) \to K_{p-1}(A) \to K_{p-1}(A/J) \to$$

finishing at $K_0(A/J)$. $\square$

Here is a useful consequence of Proposition 4.5.9.

4.5.10 LEMMA *Suppose that the sequence*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

is split by a $*$-homomorphism from A/J to A. Then the associated K-theory sequences

$$0 \longrightarrow K_p(J) \longrightarrow K_p(A) \longrightarrow K_p(A/J) \longrightarrow 0$$

are split exact. $\square$

4.5.11 REMARK To do calculations we need to make our description of the connecting homomorphism $\partial\colon K_1(A/J) \to K_0(J)$ more explicit. First we must find generators for the K_1-group. Notice that for a unital C*-algebra B, we have

$$\widetilde{S(B)} = \{f\colon [0,1] \to B \mid f(0) = f(1) \in \mathbb{C}\}$$

where we identify $\mathbb{C}$ with the subalgebra of B generated by the unit element. Thus $K_0(\widetilde{S(B)})$ is generated by *normalized loops of projections* over B, namely projection-valued maps $p\colon [0,1] \to M_n(B)$ with $p(0) = p(1) \in M_n(\mathbb{C})$; and $K_0(S(B)) = K_1(B)$ is generated by formal differences $[p] - [q]$ of such loops such that, in addition, $p(1) = q(1) \in M_n(\mathbb{C})$.

Now let A be a unital C*-algebra, J an ideal in A, and suppose that p is a normalized loop of projections over a quotient algebra A/J. Since the inclusion of $\mathbb{C}$ in A/J given by the unit element lifts to an inclusion of $\mathbb{C}$ in A, the projection $p(1) \in M_n(\mathbb{C}) \subseteq M_n(A/J)$ can be lifted to a projection

$$P(1) \in M_n(\mathbb{C}) \subseteq M_n(A).$$

By Proposition 4.3.14, there is a path P of projections over A, finishing at $P(1)$, such that $\pi \circ P = p$. The projection $P(0)$ lifts $p(0) \in M_n(\mathbb{C})$, so $P(0)$ is a projection over $\widetilde{J}$. One can check that this projection $P(0)$ is determined by the loop p up to unitary equivalence over $\widetilde{J}$; in particular its K-theory class is well-defined.

4.5.12 DEFINITION With the notation above, the class $[P(0)] - [P(1)] \in K_0(J)$ is called the *twist* of the normalized loop p. We denote it by $\mathrm{Twist}(p)$.

4.5.13 PROPOSITION *Let A be a unital C*-algebra, J an ideal in A. Let p and q be normalized loops of projections over A/J, with $p(1) = q(1)$, so that $[p] - [q]$ defines a class in $K_1(A/J)$. Then*

$$\partial([p] - [q]) = \mathrm{Twist}(p) - \mathrm{Twist}(q)$$

in $K_0(J)$.

PROOF Let P be a lifting of p as in the definition of the twist. Because of Proposition 4.5.3 it suffices to show that the projections

$$(P(1), p) \qquad \text{and} \qquad (P(0), p(0)),$$

over the algebra $\widetilde{C(A, A/J)} = \{(a, f) : a \in A,\ f\colon [0,1] \to A/J,\ f(1) = \pi(a)\}$, are homotopic. The desired homotopy is

$$s \mapsto (P(s), p_s),$$

where by definition $p_s(t) = p(st)$. $\square$

Finally let us return to the proof of the excision result for the mapping cone:

PROOF OF PROPOSITION 4.5.3 The natural map embeds J as an ideal in $C(A, A/J)$, and the quotient $C(A, A/J)/J$ is the algebra $C(A/J)$, which is contractible and so has zero K-theory. Hence, by half-exactness (Proposition 4.3.15), the map $K_0(J) \to K_0(C(A, A/J))$ is surjective. To prove injectivity we introduce another algebra Q, defined to be the algebra of continuous functions $[0, 1] \to A$ with $f(0) \in J$. There are natural homomorphisms $J \to Q$ (as constant functions) and $Q \to J$ (evaluation at 0), and these are homotopy inverses; so $K_0(J)$ is isomorphic to $K_0(Q)$. But there is an obvious surjection $Q \to C(A, A/J)$, whose kernel is the contractible algebra $C_0[0, 1) \otimes J$; so, by half-exactness again, $K_0(J) = K_0(Q) \to K_0(C(A, A/J))$ is an injection. $\square$

4.6 Inner Automorphisms

Let A be a unital C^*-algebra and let $u \in A$ be unitary. Then $\mathrm{Ad}_u(a) = uau^*$ defines an automorphism of A, and it is immediate from the definition that this inner automorphism acts trivially on $K_0(A)$. In this section we shall prove various generalizations of this statement.

4.6.1 LEMMA *Suppose that A is any C^*-algebra (not necessarily unital) and that u is a unitary in a unital C^*-algebra that contains A as an ideal. Then Ad_u induces the identity on $K_p(A)$ for all p.*

PROOF If A is an ideal in the unital C^*-algebra E, then form the C^*-algebra

$$D = \{e_1 \oplus e_2 \in E \oplus E : e_1 - e_2 \in A\},$$

which is sometimes called the *double* of E along A. The C^*-algebra A is included in D as the ideal $A \oplus 0$, and the quotient is clearly isomorphic to E. But the formula $e \mapsto e \oplus e$ gives a splitting $E \to D$ of the quotient map, and so by Lemma 4.5.10 $K_0(A)$ injects into $K_0(D)$. There is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{Ad}_v"'] & D \arrow[d, "\mathrm{Ad}_w"] \\ A \arrow[r] & D \end{tikzcd}$$

where w is the unitary $u \oplus u$ in D. Since the horizontal maps induce injections on K_0 and the map Ad_w induces the identity on $K_0(D)$, the map Ad_u must induce the identity on $K_0(A)$. The result for K_p follows from that for K_0. $\square$

If $v \in A$ is an *isometry* (which means, we recall, that $v^*v = 1$) then the formula $\mathrm{Ad}_v(a) = vav^*$ defines an endomorphism of A. Lemma 4.6.1 extends to such 'inner endomorphisms':

4.6.2 LEMMA *If v is an isometry in A (or in a unital C^*-algebra containing A as an ideal) then the endomorphism $\mathrm{Ad}_v(a) = vav^*$ induces the identity map on K-theory.*

PROOF Consider the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & M_2(A) \\ {\scriptstyle \mathrm{Ad}_v}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}_w} \\ A & \longrightarrow & M_2(A), \end{array}$$

where the horizontal arrows denote the stabilization maps of Lemma 4.2.4 and $w = \begin{pmatrix} v & 1 - vv^* \\ v^*v - 1 & v^* \end{pmatrix}$. Since the matrix w is a unitary in $M_2(A)$ (or in a C^*-algebra containing $M_2(A)$ as an ideal), Lemma 4.6.1 shows that it induces the identity map on K-theory. □

4.6.3 EXAMPLE Let us use these ideas to show that if H is an infinite-dimensional Hilbert space then $K_p(\mathfrak{B}(H)) = 0$ for all p (compare Example 4.1.13). Let H' be the infinite direct sum $H \oplus H \oplus \cdots$, and let $V\colon H \to H'$ be the isometry $v \mapsto (v, 0, 0, \dots)$. Let $\alpha_1 = \mathrm{Ad}_V\colon \mathfrak{B}(H) \to \mathfrak{B}(H')$. By Lemma 4.6.2, the induced map $\alpha_{1*}\colon K_p(\mathfrak{B}(H)) \to K_p(\mathfrak{B}(H'))$ has a two-sided inverse induced by Ad_U, where $U\colon H' \to H$ is a unitary isomorphism. Thus α_{1*} is an isomorphism.

Let $\alpha_2\colon \mathfrak{B}(H) \to \mathfrak{B}(H')$ be the $*$-homomorphism defined by

$$\alpha_2(T)(v_1, v_2, v_3, \dots) = (0, Tv_2, Tv_3, \dots).$$

The $*$-homomorphisms α_1 and α_2 satisfy the conditions of the following 'additivity lemma', whose proof follows from Example 4.1.6:

4.6.4 LEMMA *If $\alpha_1, \alpha_2\colon A \to B$ are $*$-homomorphisms with $\alpha_1[A]\alpha_2[A] = 0$, so that $\alpha_1 + \alpha_2$ is also a $*$-homomorphism, then $\alpha_{1*} + \alpha_{2*} = (\alpha_1 + \alpha_2)_*$ on K-theory.* □

Let W denote the isometry $(v_1, v_2, v_3, \dots) \mapsto (0, v_1, v_2, \dots)$ of H'. Then we have

$$\alpha_2 = \mathrm{Ad}_W \circ (\alpha_1 + \alpha_2).$$

Since Ad_W induces the identity on K-theory, by Lemma 4.6.2, we obtain

$$\alpha_{2*} = (\alpha_1 + \alpha_2)_* = \alpha_{1*} + \alpha_{2*}$$

using Lemma 4.6.4. Hence $\alpha_{1*} = 0$. But we already saw that α_{1*} is an isomorphism from $K_p(\mathfrak{B}(H))$ to $K_p(\mathfrak{B}(H'))$, so the proof that $K_p(\mathfrak{B}(H)) = 0$ is completed.

REMARK This type of argument, which comes down to deducing $1 = 0$ from $\infty + 1 = \infty$, is called an *Eilenberg swindle* — compare the proof of Proposition 2.7.5.

4.7 Products

Let A_1 and A_2 be C*-algebras. In this section we are going to define a bilinear and associative *external product* map

$$\times\colon K_{p_1}(A_1) \times K_{p_2}(A_2) \to K_{p_1+p_2}(A_1 \otimes A_2)$$

on their K-theory groups. The construction of various kinds of products and pairings — on K-theory, on K-homology, and between the two theories — will be a recurrent theme of this book, and the techniques needed for the constructions will become increasingly complicated. But the product that we discuss here has quite a simple definition.

Suppose that A_1 and A_2 are unital. If q_1 is a projection in A_1 and q_2 is a projection in A_2, then $q_1 \otimes q_2$ is a projection in the tensor product $A_1 \otimes A_2$. (The notation $A_1 \otimes A_2$ refers to the minimal tensor product, described in Exercise 3.9.8.) This observation passes to matrices: there are canonical[35] isomorphisms $M_{n_1}(\mathbb{C}) \otimes M_{n_2}(\mathbb{C}) \cong M_{n_1 n_2}(\mathbb{C})$, and using these we find that if q_1 is a projection in $M_{n_1}(A_1)$ and q_2 is a projection in $M_{n_2}(A_2)$ then $q_1 \otimes q_2$ is a projection in $M_{n_1 n_2}(A_1 \otimes A_2)$. This procedure respects the relations used in defining the K-theory groups, and therefore we obtain a product

$$\times\colon K_0(A_1) \times K_0(A_2) \to K_0(A_1 \otimes A_2) \tag{4.7.1}$$

which is associative and bilinear (as a map of abelian groups).

We want to generalize our definition so as to obtain a product on the higher K-theory groups. Recall that the higher K-theory groups $K_p(A)$ are defined in terms of the suspensions $S^p(A) = C_0(\mathbb{R}^p) \otimes A$ of A, and these suspensions are non-unital even if the original A is unital. Thus we must give a definition of the K_0-product which is valid in the non-unital case. The key to doing so is provided by the following lemma:

4.7.2 Lemma *Let A_1 and A_2 be possibly non-unital C^*-algebras, and let $\widetilde{A_1}$ and $\widetilde{A_2}$ be their unitalizations. Let $\pi_j\colon \widetilde{A_j} \to \mathbb{C}$ be the unital $*$-homomorphism with kernel A_j. Then*

$$K_0(A_1 \otimes A_2) = \operatorname{Kernel}\Big(\pi_*\colon K_0(\widetilde{A_1} \otimes \widetilde{A_2}) \to K_0(\widetilde{A_1}) \oplus K_0(\widetilde{A_2})\Big),$$

where $\pi = (1 \otimes \pi_2, \pi_1 \otimes 1)$.

Proof We have split exact sequences

$$0 \longrightarrow A_1 \otimes A_2 \longrightarrow A_1 \otimes \widetilde{A_2} \longrightarrow A_1 \longrightarrow 0$$

[35] Except for questions about the ordering of basis elements. But these only affect matters by an inner automorphism, so the choice makes no difference on the level of K-theory.

and

$$0 \longrightarrow A_1 \otimes \widetilde{A_2} \longrightarrow \widetilde{A_1} \otimes \widetilde{A_2} \longrightarrow \widetilde{A_2} \longrightarrow 0\,.$$

Applying Lemma 4.5.10 to each of these in turn, and recalling that $K_0(A_1)$ injects into $K_0(\widetilde{A_1})$, we obtain the result. □

It follows from the lemma that the product

$$K_0(\widetilde{A_1}) \times K_0(\widetilde{A_2}) \to K_0(\widetilde{A_1} \otimes \widetilde{A_2}) \tag{4.7.3}$$

maps the subgroup $K_0(A_1) \times K_0(A_2)$ of the left-hand side to the subgroup $K_0(A_1 \otimes A_2)$ of the right-hand side. Thus we may make the following definition:

4.7.4 Definition Let A_1 and A_2 be C^*-algebras. The product

$$\times \colon K_0(A_1) \times K_0(A_2) \to K_0(A_1 \otimes A_2)$$

is the restriction to $K_0(A_1) \times K_0(A_2)$ of the product defined by Equation 4.7.3 on the unitalizations of A_1 and A_2. The product

$$\times \colon K_{p_1}(A_1) \times K_{p_2}(A_2) \to K_{p_1+p_2}(A_1 \otimes A_2)$$

is obtained from the product on K_0 by making use of the isomorphism

$$S^{p_1}(A_1) \otimes S^{p_2}(A_2) \cong S^{p_1+p_2}(A_1 \otimes A_2).$$

4.7.5 Remark Notice that the last isomorphism depends on an identification of $C_0(\mathbb{R}^{p_1}) \otimes C_0(\mathbb{R}^{p_2})$ with $C_0(\mathbb{R}^{p_1+p_2})$. We map the coordinates of $\mathbb{R}^{p_1}$ in order to the first p_1 coordinates of $\mathbb{R}^{p_1+p_2}$, and the coordinates of $\mathbb{R}^{p_2}$ to the remaining p_2. It is important to be explicit about this because a permutation σ of the coordinates of $\mathbb{R}^p$ induces the automorphism $(-1)^\sigma$ of $K_p(A)$.

The product so defined is bilinear and associative. It is functorial in the senses described by the next proposition:

4.7.6 Proposition *Let A be a C^*-algebra.*

(a) *If $\alpha\colon A \to A'$ is a $*$-homomorphism, and B is another C^*-algebra, then*

$$\alpha_*(x) \times y = (\alpha \otimes 1)_*(x \times y)$$

for $x \in K_p(A)$ and $y \in K_q(B)$.

(b) *Let*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

be a short exact sequence, and let B *be another* C^**-algebra. Suppose that either* $A \to A/J$ *has a completely positive section, or* B *is nuclear, so that the sequence*

$$0 \longrightarrow J \otimes B \longrightarrow A \otimes B \longrightarrow (A/J) \otimes B \longrightarrow 0$$

is also short exact.[36] *Then*

$$\partial(x \times y) = \partial(x) \times y$$

for $x \in K_p(A/J)$ *and* $y \in K_q(B)$

PROOF The proof of (a) is a simple calculation, and the proof of (b) follows since, by its definition (4.5.7), ∂ is the composite of two maps, one induced by a $*$-homomorphism and the other the inverse of a K-theory isomorphism induced by a $*$-homomorphism. $\square$

It is worth pointing out that the analog of (b) in the second variable involves a possible sign change (arising from permutation of the suspension coordinates, as in Remark 4.7.5).

The reader should beware that in general $K_*(A)$ is *not* a ring. In topological K-theory, an 'internal' product (cup product) $K^*(X) \otimes K^*(X) \to K^*(X)$ can be obtained from the 'external' product (with values in $K^*(X \times X)$) by composition with the diagonal map $X \mapsto X \times X$. The analog of the diagonal map for algebras is $a_1 \otimes a_2 \mapsto a_1 a_2$; but this does not define a $*$-homomorphism $A \otimes A \to A$ unless A is commutative.

Some vestiges of ring structure do survive, however:

4.7.7 PROPOSITION *Let* A *and* B *be* C^**-algebras. Then:*

(a) *The class of the generator* $1 \in K_0(\mathbb{C})$ *is a two-sided unit for the external product. That is*

$$\alpha'_*(1 \times x) = x = \alpha''_*(x \times 1)$$

for all $x \in K_p(A)$*, where* $\alpha' \colon \mathbb{C} \otimes A \to A$ *and* $\alpha'' \colon A \otimes \mathbb{C} \to A$ *are the canonical isomorphisms.*

(b) *The external product is graded commutative in the sense that*

$$y \times x = (-1)^{pq}\theta_*(x \times y)$$

for all $x \in K_p(A)$ *and* $y \in K_q(B)$*, where* $\theta \colon A \otimes B \to B \otimes A$ *is the 'flip' isomorphism.* $\square$

The proofs, which are easy, are left to the reader.

[36] See Exercise 3.9.12.

4.8 Another Description of K_1

By our definitions, $K_1(A)$ is $K_0(SA)$, where S is the suspension operation; see Definition 4.5.4. It is frequently convenient to have a more concrete model of K_1.

4.8.1 DEFINITION Let A be a unital C^*-algebra. Then $K_1^u(A)$ denotes the abelian group with one generator $[u]$ for each unitary matrix in each $M_n(A)$,[37] and the following relations:

(a) if u and v lie in the same $M_n(A)$, and if u and v can be joined by a continuous path of unitaries in $M_n(A)$, then $[u] = [v]$,

(b) $[1] = 0$, and

(c) $[u] + [v] = [u \oplus v]$, for any sizes of unitary matrices u and v.

Let us temporarily use the symbol $\sim$ to denote path-connectedness through unitaries. If u and v are unitaries in A then the following relations hold in $M_2(A)$:

$$u \oplus 1 \sim 1 \oplus u, \qquad u \oplus v \sim uv \oplus 1 \sim vu \oplus 1, \qquad u \oplus u^* \sim 1 \oplus 1.$$

To prove the first relation, use the path $R_t \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} R_t^*$, where R_t is the rotation matrix

$$R_t = \begin{pmatrix} \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \\ -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix}.$$

The other relations follow easily from this. Observe that from the third identity,

$$[u] + [u^*] = [u \oplus u^*] = 0.$$

It follows from this and the other relations above that every element in $K_1^u(A)$ can be written as a single generator $[v]$. Note also that if u and v are matrices of the same size then by the second identity,

$$[u] + [v] = [u \oplus v] = [uv \oplus 1] = [uv]$$

so that addition in the group K_1^u is given by multiplication of matrices.

The notation K_1^u, which is meant to suggest that the definition is made in terms of unitaries, will last only as long as it takes us to prove the following result:

4.8.2 PROPOSITION *Let A be a unital C^*-algebra. The group $K_1^u(A)$ is naturally isomorphic*[38] *to the group $K_1(A) = K_0(S(A))$.*

[37] As we did for projections, we will sometimes refer to a unitary in $M_n(A)$ as a unitary *over* A.

[38] A specific choice of natural isomorphism is provided by the proof below, and we will always use this one.

PROOF By Remark 4.5.11, the group $K_1(A)$ is generated by formal differences of *normalized loops* of projections over A. Such a loop is a projection-valued map $p\colon [0,1] \to M_n(A)$ such that $p(0) = p(1) \in M_n(\mathbb{C})$; we may assume without loss of generality that $p(0) = p(1)$ is a diagonal projection of the form $I_m \oplus 0_{m'}$. By Corollary 4.1.8, there is a unitary-valued map $u\colon [0,1] \to M_n(A)$ such that $p(t) = u(t)p(1)u(t)^*$. The unitary $u(0)$ commutes with $p(1) = p(0)$, so it is of the diagonal form

$$u(0) = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}, \quad \text{where } p(0) = p(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

One can readily check that the class in $K_1^u(A)$ defined by the unitary v depends only on the K_0-class of the normalized loop p. Thus the assignment $[p] \mapsto [v]$ gives a well-defined homomorphism from $K_1(A)$ to $K_1^u(A)$. An inverse to this homomorphism can be defined as follows: given a unitary $v \in M_m(A)$, let $u(t)$ be a path of unitaries in $M_{2m}(A)$ with

$$u(0) = \begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix}, \quad u(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and form the normalized loop of projections $p(t) = u(t)(I_m \oplus 0_m)u(t)^*$ and the constant loop $q(t) = I_m \oplus 0_m$. The assignment $[v] \mapsto [p] - [q]$ gives a homomorphism $K_1^u(A) \to K_1(A)$ inverse to the preceding one. Thus $K_1(A) \cong K_1^u(A)$, as required. $\square$

The description of K_1 by unitaries has the virtue of simplicity. It may be extended[39] to non-unital algebras A by considering those unitary matrices over $\widetilde{A}$ which are equal to the identity matrix modulo A. From now on, we shall not distinguish between K_1 and K_1^u.

Products between K_1 and K_0 also have a simple description in the unitary picture:

4.8.3 PROPOSITION *Let A and B be unital C^*-algebras. Suppose that the unitary u represents a class in $K_1(A)$ and that the projection p represents a class in $K_0(B)$. Then the unitary*

$$u \otimes p + 1 \otimes (1 - p)$$

represents the product $[u] \times [p] \in K_1(A \otimes B)$. $\square$

We leave the verification to the reader (Exercise 4.10.20).

[39] In fact, the long exact sequence shows that $K_1(A) \cong K_1(\widetilde{A})$. It is better not to use this as a definition, however, as the isomorphism is peculiar to *complex* K-theory.

4.8.4 EXAMPLE If $A = \mathbb{C}$ then $K_1(\mathbb{C}) = 0$. This is because every unitary matrix u is path-connected to the identity through unitary matrices, as can be seen by diagonalizing u, then rotating each eigenvalue continuously to 1.

4.8.5 EXAMPLE Let A be any unital C*-algebra. Generalizing the argument of Example 4.8.4, we consider a unitary $u \in A$ and suppose that Spectrum(u) is not the whole of the unit circle. Then there is a continuous branch of the function $z \mapsto \log z$ defined on Spectrum(u), and so by the functional calculus we may define a homotopy $u_t = \exp(t \log u)$ from the identity to u. Thus $[u] = 0$ in $K_1(A)$.

4.8.6 EXAMPLE A similar argument gives another proof that $K_1(\mathfrak{B}(H)) = 0$; each unitary in $\mathfrak{B}(H)$ has a logarithm in $\mathfrak{B}(H)$, defined using the *Borel* functional calculus.

4.8.7 EXAMPLE The *Bott generator* is the class $b \in K_0(C_0(\mathbb{R}^2))$ defined by the relative cycle $[1, 1, \bar{z}]$ for the pair $(\overline{\mathbb{D}}, \partial\mathbb{D})$, where $\mathbb{D}$ is the unit disk in $\mathbb{C}$ (see Example 4.3.10). By definition

$$K_0(C_0(\mathbb{R}^2)) = K_1(C_0(\mathbb{R})) = K_1(C(S^1))$$

where we identify the one-point compactification of $\mathbb{R}$ with the unit circle by way of the Cayley transform $x \mapsto (x-i)/(x+i)$. The Bott generator then corresponds to the K_1-class of the unitary-valued function $z \mapsto \bar{z}$ on $S^1 \subseteq \mathbb{C}$.

4.8.8 EXAMPLE Let us prove that $K_1(\mathfrak{Q}(H)) \cong \mathbb{Z}$. A unitary $u = \pi(T)$ in $\mathfrak{Q}(H)$ lifts to a Fredholm operator $T \in \mathfrak{B}(H)$, and Proposition 2.1.6 shows that the map $T \mapsto \text{Index}(T)$ induces a homomorphism

$$\text{Index} : K_1(\mathfrak{Q}(H)) \to \mathbb{Z}.$$

If $\text{Index}(T) = 0$ then T is a (compact perturbation of) a unitary operator (see Exercise 2.9.13) and it follows from Example 4.8.6 that $[\pi(T)] = 0$. Hence the above 'index homomorphism' is an isomorphism. Notice that we could have carried out this calculation more K-theoretically, using the long exact sequence of K-theory groups associated to the short exact sequence

$$0 \longrightarrow \mathfrak{K}(H) \longrightarrow \mathfrak{B}(H) \longrightarrow \mathfrak{Q}(H) \longrightarrow 0$$

and the fact that $K_p(\mathfrak{B}(H)) = 0$ (see Example 4.6.3).

The example above suggests that the boundary map in the long exact sequence associated to an extension

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

should be thought of as a kind of 'abstract Fredholm index'. Indeed we note that any unitary in A/J has an index in this sense in $K_0(J)$, and if this index is non-zero then the unitary cannot be lifted to a unitary in A. We can describe the boundary map $\partial\colon K_1(A/J) \to K_0(A)$ in a way which makes this connection with index theory more explicit:

4.8.9 LEMMA *Let A be a unital C^*-algebra and let J be an ideal in it; let $\pi\colon A \to A/J$ denote the quotient map. Let u be a unitary element of the quotient algebra A/J. Then*

(a) *there exists $a \in A$ with $\|a\| \leqslant 1$ and $\pi(a) = u$,*

(b) *for any such $a \in A$, the matrix $w \in M_2(A)$ defined by*

$$w = \begin{pmatrix} a & -(1-aa^*)^{\frac{1}{2}} \\ (1-a^*a)^{\frac{1}{2}} & a^* \end{pmatrix}$$

is unitary and satisfies $\pi(w) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$, *and*

(c) *the matrix w is connected to the identity matrix by a path of unitaries.*

PROOF Choose any lift b of u and let $a = b\varphi(b^*b)$, where $\varphi(\lambda) = \min\{1, |\lambda|^{-\frac{1}{2}}\}$. Then $\|a\| \leqslant 1$ and $\pi(a) = u$. To prove that w is unitary is a simple calculation, and it is connected to the identity matrix by a path of unitaries arising from the linear path from a to 1. □

4.8.10 PROPOSITION *Let A be a unital C^*-algebra and let J be an ideal in it; let $\pi\colon A \to A/J$ denote the quotient map. Let u be a unitary element of the quotient algebra A/J, and let $a \in A$ be a lifting of u with $\|a\| = 1$. Then*

(a) *the matrix*

$$P = \begin{pmatrix} aa^* & a(1-a^*a)^{\frac{1}{2}} \\ a^*(1-aa^*)^{\frac{1}{2}} & 1-a^*a \end{pmatrix}$$

is a projection in $M_2(\widetilde{J})$, and is equal modulo J to $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,

(b) *the formal difference $[P] - [Q] \in K_0(J)$ represents the image of $[u] \in K_1(A/J)$ under the boundary map $\partial\colon K_1(A/J) \to K_0(J)$, and*

(c) *in particular, if the lifting a is a partial isometry, so that $1 - a^*a$ and $1 - aa^*$ are projections in J, then their formal difference*

$$[1 - a^*a] - [1 - aa^*] \in K_0(J)$$

represents $\partial[u]$.

Notice that if $A = \mathfrak{B}(H)$, $J = \mathfrak{K}(H)$, and a is a Fredholm partial isometry, then $1 - a^*a$ is the projection onto the kernel of a, and $1 - aa^*$ is the projection onto the kernel of a^*. The formal difference

$$[1 - a^*a] - [1 - aa^*] \in K_0(\mathfrak{K}(H)) = \mathbb{Z}$$

is thus identified with the Fredholm index of a.

PROOF Let w be the unitary from Lemma 4.8.9, and let $w(t)$, $t \in [0, 1]$, be the path of unitaries provided by that lemma from $w(0) = w$ to $w(1) = 1$. The class in $K_1(A/J) = K_0(S(A/J))$ defined by the unitary u is given according to the proof of Proposition 4.8.2 by $[p] - [q]$, where p is the normalized loop of projections

$$p(t) = u(t)p(1)u(t)^*, \quad p(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and q is the constant loop $q(t) = p(1)$. The twist of the loop p (Definition 4.5.12) is

$$\operatorname{Twist}(p) = w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^* = P,$$

and $\operatorname{Twist}(q) = Q$. The result therefore follows from Proposition 4.5.13. □

4.9 Bott Periodicity

Let $b \in K_2(\mathbb{C}) = K_0(C_0(\mathbb{R}^2))$ be the Bott generator. External product with b defines a natural map

$$\beta_A \colon K_0(A) \to K_2(A)$$

for any C^*-algebra A. The main result of K-theory is the following theorem.

4.9.1 BOTT PERIODICITY THEOREM *For any C^*-algebra A, the Bott map*

$$\beta_A \colon K_0(A) \to K_2(A)$$

is an isomorphism.

The proof that we will give is due to Atiyah, and it uses the Toeplitz Index Theorem to construct an inverse to the Bott map. We may assume without loss of generality that A is unital.

4.9.2 LEMMA *The Bott map $\beta \colon K_0(A) \to K_2(A) = K_1(SA)$ may be defined by sending a projection $p \in M_n(A)$ to the unitary loop*

$$p\bar{z} + 1 - p$$

in $M_n(\widetilde{SA})$. Here $z \colon S^1 \to \mathbb{C}$ denotes the identity function on the unit circle $S^1 \subseteq \mathbb{C}$.

PROOF This follows from Example 4.8.7 and Proposition 4.8.3. □

Now we shall define a candidate for the inverse to the Bott map. Let

$$0 \longrightarrow \mathfrak{K}(H) \longrightarrow \mathfrak{T} \longrightarrow C(S^1) \longrightarrow 0$$

be the Toeplitz extension. It has a completely positive section, and therefore by Exercise 3.9.12 the sequence

$$0 \longrightarrow \mathfrak{K}(H) \otimes A \longrightarrow \mathfrak{T} \otimes A \longrightarrow C(S^1) \otimes A \longrightarrow 0$$

is also exact. Associated to this short exact sequence there is a boundary map

$$\partial \colon K_1(C(S^1) \otimes A) \to K_0(\mathfrak{K}(H) \otimes A) \cong K_0(A)$$

and restricting[40] to $C_0(\mathbb{R}) \otimes A \subseteq C(S^1) \otimes A$ we obtain

$$\alpha_A \colon K_2(A) = K_1(C_0(\mathbb{R}) \otimes A) \to K_0(A).$$

We shall show that α_A is inverse to β_A, and thus prove the Bott Periodicity Theorem.

Atiyah's proof depends only on the following formal properties of the homomorphisms α_A (defined for all C^*-algebras A):

(a) for $A = \mathbb{C}$ we have $\alpha_{\mathbb{C}}(b) = 1 \in K_0(\mathbb{C})$, where b is the Bott generator;

(b) for any A and B the diagram

$$\begin{tikzcd} K_2(A) \otimes K_0(B) \arrow[r] \arrow[d, "\alpha_A \otimes 1"] & K_2(A \otimes B) \arrow[d, "\alpha_{A \otimes B}"] \\ K_0(A) \otimes K_0(B) \arrow[r] & K_0(A \otimes B) \end{tikzcd}$$

where the horizontal arrows are given by the K-theory product, is commutative. (In this sense, α is *right-linear* over $K_0(B)$; we may write $\alpha_{A \otimes B}(x \times y) = \alpha_A(x) \times y$.)

It is easy to check that the homomorphism α_A that we have defined has these properties. To prove Property (a), note that if u is a unitary-valued function on S^1, then $\alpha_{\mathbb{C}}$ maps $[u] \in K_1(C(S^1))$ to the index of the Toeplitz operator T_u. The desired result therefore follows from the Toeplitz Index Theorem 2.3.2. Property (b) follows from Proposition 4.7.6.

PROOF OF THE BOTT PERIODICITY THEOREM: First we shall show that α gives a left inverse to the Bott map. For $A = \mathbb{C}$ this is exactly Property (a) above. The general case follows by linearity (Property (b)):

$$\alpha_A(\beta_A(x)) = \alpha_A(b \times x) = \alpha_{\mathbb{C}}(b) \times x = 1 \times x = x,$$

using Proposition 4.7.7(a).

[40]Here we regard S^1 as the one-point compactification of $\mathbb{R}$, by the homeomorphism described in Example 4.8.7.

This shows that the Bott map is injective and to complete the proof we need to show that it is also surjective. In fact, we shall show that every element $y \in K_0(A \otimes C_0(\mathbb{R}^2))$ is of the form $x \times b$ for some $x \in K_0(A)$; by Proposition 4.7.7(b), this will suffice.

The proof will make use of the 'flip' isomorphisms

$$\sigma\colon A \otimes C_0(\mathbb{R}^2) \to C_0(\mathbb{R}^2) \otimes A$$

and

$$\tau\colon C_0(\mathbb{R}^2) \otimes A \otimes C_0(\mathbb{R}^2) \to C_0(\mathbb{R}^2) \otimes A \otimes C_0(\mathbb{R}^2)$$

which interchange the first and last factors of the corresponding tensor products. From the naturality properties of the product,[41]

$$\tau_*(b \times y) = \sigma_*(y) \times b.$$

The key observation is that τ_* is the identity. To see this, notice that τ_* is the homomorphism on K-theory induced by rotating $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ by the 4×4 matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

But this matrix has determinant $+1$ and so is homotopic in $SO(4)$ to the identity. The result follows from the homotopy invariance of K-theory.

Now use the fact that $\alpha_{A \otimes C_0(\mathbb{R}^2)}$ is left inverse to $\beta_{A \otimes C_0(\mathbb{R}^2)}$. We may write

$$y = \alpha_{A \otimes C_0(\mathbb{R}^2)}(b \times y) = \alpha_{A \otimes C_0(\mathbb{R}^2)}(\sigma_*(y) \times b) = \alpha_A(\sigma_*(y)) \times b$$

where the last step uses linearity (Property (b)). This shows that y is of the form $x \times b$, and so completes the proof of the Bott Periodicity Theorem. □

4.9.3 Remark If we identify the groups $K_2(A)$ and $K_0(A)$ by Bott periodicity, then the long exact sequence of K-theory boils down to the *six-term exact*

[41] Specifically, let $\theta\colon B \otimes S \otimes S \to S \otimes B \otimes S$ be the flip corresponding to the cyclic permutation (123) of the factors. Then $\sigma \otimes 1 = \tau \circ \theta$. By Proposition 4.7.7(b) and Proposition 4.7.6(a),

$$\tau_*(b \times y) = \tau_*(\theta_*(y \times b)) = (\sigma \otimes 1)_*(y \times b) = \sigma_*(y) \times b$$

as required.

sequence associated to an extension $0 \to J \to A \to A/J \to 0$. This takes the form

$$\begin{tikzcd} K_1(J) \arrow[r] & K_1(A) \arrow[r] & K_1(A/J) \arrow[d, "\partial_1"] \\ K_0(A/J) \arrow[u, "\partial_0"] & K_0(A) \arrow[l] & K_0(J) \arrow[l] \end{tikzcd}$$

We have described the right-hand vertical map ∂_1 in 4.8.10. The left-hand vertical map ∂_0 can also be described explicitly. Suppose that a class in $K_0(A/J)$ is represented by a difference $[p]-[q]$, where p and q are projections in $M_n(A/J)$. Lift p and q to selfadjoint elements x and y in $M_n(A)$. Observe that $\exp(2\pi i x)$ and $\exp(2\pi i y)$ are unitary elements in $M_n(\widetilde{J})$ which are equal to the identity modulo J, and form

$$\partial_0([p]-[q]) = [\exp(2\pi i x)] - [\exp(2\pi i y)] \in K_1(J).$$

See Exercise 4.10.23 to verify this description, because of which the map ∂_0 is often called the *exponential map*.

4.10 Exercises

4.10.1 Show that two projections $p, q \in \mathfrak{B}(H)$ are Murray–von Neumann equivalent if and only if there is a unitary isomorphism v from the range of p to the range of q. Hence the equivalence class of a projection p can be represented by the cardinal number $\mathrm{Rank}(p) \in \{0, 1, 2, \ldots, \infty\}$ (assuming that H is separable). Show, on the other hand, that p and q are unitarily equivalent if and only if in addition $\mathrm{Rank}(1-p) = \mathrm{Rank}(1-q)$.

4.10.2 Let $X = S^3$ be the 3-sphere, and let $u\colon X \to SU(2)$ be the identity map when $SU(2)$ is identified with S^3 in the standard way (see Exercise 2.9.18). Let

$$p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad p_1 = u p_0 u^*.$$

Show that the unitarily equivalent projections p_0 and p_1 are not homotopic. (Hint: if they are, then u is homotopic to a map factoring through the maximal torus T^2 in $SU(2)$ consisting of diagonal matrices. Using algebraic topology, for instance by calculations with the third homology group, show that this is impossible.)

4.10.3 Show that if A is a unital C^*-algebra and $\widetilde{A}$ nevertheless denotes its unitalization, then $K_0(\widetilde{A}) \cong K_0(A) \oplus \mathbb{Z}$. Show also that if $\alpha\colon A_1 \to A_2$ is a non-unital $*$-homomorphism of unital C^*-algebras, then the induced homomorphism $\alpha_*\colon K_0(A_1) \to K_0(A_2)$ is indeed defined by the 'obvious' formula $\alpha_*[p] = [\alpha(p)]$.

4.10.4 Let $p, q \in M_n(A)$ be projections over a unital C^*-algebra A and suppose that $px = xq$ for some invertible (but perhaps not unitary) $x \in M_n(A)$. Show that $[p] = [q]$ in $K_0(A)$.

4.10.5 Let $\{A_j\}$, $j = 1, 2, \ldots$, be a family of C^*-algebras and let $\alpha_j \colon A_j \to A_{j+1}$ be $*$-homomorphisms. Let $\mathcal{A}$ be the algebraic direct limit of the family $\{A_j, \alpha_j\}$; that is, $\mathcal{A}$ consists of families $\mathbf{a} = \{a_j\}_{j=1}^{\infty}$ with $\alpha_j(a_j) = a_{j+1}$ for sufficiently large j, two such families being identified if they agree for sufficiently large j. Show that the 'norm'

$$\|\mathbf{a}\| = \limsup \|a_j\|$$

satisfies the C^*-identity.

Let A be the C^*-algebra obtained by dividing $\mathcal{A}$ by the ideal $\{\mathbf{a} : \|\mathbf{a}\| = 0\}$ and then subsequently completing relative to the norm $\|\cdot\|$. Show that the induced map

$$\varinjlim K_0(A_j) \to K_0(A)$$

is an isomorphism. Generalize to arbitrary directed systems of C^*-algebras.

4.10.6 Let X be a locally compact Hausdorff space. Use the Excision Theorem to show that every element of $K_0(C_0(X))$ arises from the difference bundle construction.

4.10.7 Let A be a unital C^*-algebra, J an ideal in A, and suppose that the short exact sequence $0 \to J \to A \to A/J \to 0$ is split by a $*$-homomorphism $A/J \to A$. Show (without using the Excision Theorem) that $K_0(A, A/J) \cong \mathrm{Kernel}(K_0(A) \to K_0(A/J))$. In particular, show that the natural map

$$K_0(\widetilde{J}, \mathbb{C}) \to K_0(\widetilde{J})$$

maps $K_0(\widetilde{J}, \mathbb{C})$ isomorphically onto the kernel of the canonical map $K_0(\widetilde{J}) \to \mathbb{Z}$.

4.10.8 Let $f \colon \mathbb{R} \to \mathbb{R}$ be the flip $f(x) = -x$. Show that f induces the inverse automorphism on the abelian group $K_1(A) = K_0(C_0(\mathbb{R}) \otimes A)$, for every C^*-algebra A. Deduce that a permutation σ of the coordinates of $\mathbb{R}^p$ induces the automorphism $(-1)^\sigma$ on $K_0(C_0(\mathbb{R}^p) \otimes A)$ (see Remark 4.7.5).

4.10.9 Let D be a C^*-subalgebra of $\mathfrak{B}(H)$, containing the compact operators. Consider the algebra D' of all operators on $H \oplus H$ which have matrix form

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

where $T_{11} \in D$, T_{12}, T_{21} are compact, and T_{22} is arbitrary. Prove that $K_p(D') \cong K_p(D/\mathfrak{K}(H))$, and that the homomorphism $K_p(D) \to K_p(D')$ induced by the top left corner inclusion corresponds (under your isomorphism) to the homomorphism $K_p(D) \to K_p(D/\mathfrak{K}(H))$ induced by the quotient map.

4.10.10 Suppose that the C^*-algebra J has an approximate unit made up of projections. Show that $K_0(J)$ can be described as the abelian group with one generator for each projection in each matrix algebra over J, and with the same relations as in 4.1.1. (Write J as the direct limit of unital subalgebras $p_n J p_n$, where $\{p_n\}$ is an approximate unit of projections.)

4.10.11 Fix a non-degenerate representation π of $M_n(\mathbb{C})$ on an infinite-dimensional Hilbert space H. Determine the K_0-group of the C^*-algebra of all operators $T \in \mathfrak{B}(H)$ which commute with $\pi[M_n(\mathbb{C})]$ modulo compact operators.

4.10.12 Why doesn't an Eilenberg swindle argument, like the one in Example 4.6.3, show that the K-theory groups of the Calkin algebra $\mathfrak{Q}(H)$ are zero?

4.10.13 Denote by $C^*(F_2)$ the full group C^*-algebra of the free group on two generators (see Section 3.7), and denote by $C(S^1 \vee S^1)$ the commutative C^*-algebra of continuous functions on the 'figure of eight' space. Note that $C(S^1 \vee S^1)$ contains two canonical unitary elements, as does $C^*(F_2)$, and there is a $*$-homomorphism $C^*(F_2) \to C(S^1 \vee S^1)$ mapping generators to generators.

(a) Show that there is $*$-homomorphism $C(S^1 \vee S^1) \to M_2(C^*(F_2))$ which maps the generators u and v to $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}$, respectively.

(b) Use the stability of K-theory, its homotopy invariance, and a 'rotation trick' to show that the two homomorphisms we have defined are inverse to one another in K-theory.

(c) Hence $K_0(C^*(F_2)) \cong \mathbb{Z}$, generated by $[1]$, and $K_1(C^*(F_2)) \cong \mathbb{Z} \oplus \mathbb{Z}$, generated by $[u]$ and $[v]$.

This is one of the few non-commutative C^*-algebras whose K-theory can be determined by an elementary homotopy argument.

4.10.14 Let us look at the K-theory of the Cuntz algebra $\mathcal{O}_n$ which was introduced in Exercise 2.9.17.

(a) Show that if $a \in K_p(\mathcal{O}_n)$ then $(n-1)a = 0$.

(b) Let $\widehat{\mathcal{O}}_n$ be the C^*-algebra generated by n isometries $W_1, \dots, W_n$ on a Hilbert space H whose ranges are orthogonal and add up to a codimension 1 subspace of H. Show that there is an exact sequence

$$0 \to \mathfrak{K}(H) \to \widehat{\mathcal{O}}_n \to \mathcal{O}_n \to 0$$

(use the property of $\mathcal{O}_n$ described in Exercise 2.9.17).

(c) Use the six-term exact sequence to get an exact sequence

$$0 \to K_0(\mathfrak{K}(H)) \to K_0(\widehat{\mathcal{O}}_n) \to K_0(\mathcal{O}_n),$$

and show that the generator $[p] \in K_0(\mathfrak{K}(H))$ maps to the class $(n-1)[1] \in K_0(\widehat{\mathcal{O}}_n)$.

(d) Conclude that the map $k \mapsto k[1]$ is an *injective* homomorphism from $\mathbb{Z}/(n-1)\mathbb{Z}$ to $K_0(\mathcal{O}_n)$.

One can pursue this line of reasoning to conclude that $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ and also $K_1(\mathcal{O}_n) = 0$, but it is not easy [43].

4.10.15 Let J be an ideal in a C*-algebra A. Let $C = C(A, A/J)$ be the mapping cone of the quotient map $A \to A/J$, and let $Z = C(C, A)$ be the mapping cone of the surjective $*$-homomorphism $C \to A$ defined by $(a, f) \mapsto a$. Show that there is a short exact sequence of C*-algebras

$$0 \longrightarrow S(A) \longrightarrow Z \longrightarrow C \longrightarrow 0\,.$$

Hence show that the sequence

$$\cdots \longrightarrow K_1(A) \longrightarrow K_1(A/J) \longrightarrow K_0(J) \longrightarrow \cdots$$

is exact at $K_1(A/J)$.

4.10.16 Let A be a C*-algebra and let $B = A \otimes \mathfrak{K}(H)$. Suppose that B is represented as a subalgebra of $\mathfrak{B}(H')$ for some other Hilbert space H'. Let M be the set of those operators $T \in \mathfrak{B}(H')$ such that left multiplication and right multiplication by T both map B into itself. Prove that M is a C*-algebra. (It is called the *stable multiplier algebra* of A.)

Use an Eilenberg swindle argument to show that $K_p(M) = 0$ for all p, and deduce that $K_0(A) = K_1(M/B)$. (This is, essentially, Kasparov's definition of K-theory [81]; in Kasparov's notation this group would be written $KK(\mathbb{C}, A)$.)

4.10.17 Let

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

be a short exact sequence of C*-algebras, and let T be an element of A which is unitary modulo J. Thus T defines a class $[T] \in K_1(A/J)$, and we call $\partial[T] \in K_0(J)$ the 'index' of T. Give an example to show that a normal T may have a non-zero index. Compare with the result of Exercise 2.9.9.

4.10.18 Let f be a compactly supported, smooth function on $\mathbb{R}$. Define

$$k_x^{(f)}(s,t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} f(x+n) e^{i(s-t)(x+n)}, \quad (s, t \in [0, 2\pi]).$$

(a) Show that $k^{(f)}$ is a smooth function of the variables x, s, and t, periodic in x with period 1.

(b) Fixing f, define a family $T_x^{(f)}$ of operators on $L^2[0, 2\pi]$ parameterized by $x \in S^1 = \mathbb{R}/\mathbb{Z}$ by

$$T_x^{(f)} u(s) = \int_0^{2\pi} k_x^{(f)}(s,t) u(t)\, dt.$$

Show that the assignment $f \mapsto T^{(f)}$ extends to a $*$-homomorphism from $C_0(\mathbb{R})$ to $C(S^1) \otimes \mathfrak{K}(L^2[0, 2\pi])$.

(c) Show that this $*$-homomorphism induces an isomorphism on K_1. (Hint: model the generator for $K_1(C_0(\mathbb{R}))$ by a unitary-valued function f which is equal to 1 outside a small interval; show that $T^{(f)}$ is equal to 'the same' function on S^1 tensored with a certain rank-one projection.)

4.10.19 Let $0 \to J \to A \to A/J \to 0$ be a short exact sequence of C^*-algebras, with A unital, and let u be a unitary over A/J. Show that $\partial[u] \in K_0(J)$ can be represented by the relative K-cycle $[1, 1, a] \in K_0(J, A/J)$, where a is any lift of u to A.

4.10.20 Verify the formula given in Remark 4.8.3 for the external product $K_1(A) \times K_0(B) \to K_1(A \otimes B)$. Is there a similar 'elementary' formula for the product $K_1(A) \times K_1(B) \to K_0(A \otimes B)$?

4.10.21 Let J_0 and J_1 be ideals in a C^*-algebra A, with $J_0 + J_1 = A$. Show that there is a six-term Mayer–Vietoris sequence

$$\begin{tikzcd}
K_1(J_0 \cap J_1) \arrow[r] & K_1(J_0) \oplus K_1(J_1) \arrow[r] & K_1(A) \arrow[d] \\
K_0(A) \arrow[u] & K_0(J_0) \oplus K_0(J_1) \arrow[l] & K_0(J_0 \cap J_1) \arrow[l]
\end{tikzcd}$$

(Consider the six-term exact sequence in K-theory arising from the algebra

$$B = \{f \in C[0,1] \otimes A : f(0) \in J_0, f(1) \in J_1\}$$

and the ideal

$$I = SA = \{f \in C[0,1] \otimes A : f(0) = f(1) = 0\}$$

in B.)

4.10.22 Another Mayer–Vietoris sequence arises from a pullback diagram of C^*-algebras and $*$-homomorphisms, of the form

$$\begin{tikzcd}
B_0 \oplus_A B_1 \arrow[r] \arrow[d] & B_1 \arrow[d, "\alpha_1"] \\
B_0 \arrow[r, "\alpha_0"] & B
\end{tikzcd}$$

where by definition

$$B_0 \oplus_B B_1 = \{(b_0, b_1) \in B_0 \oplus B_1 : \alpha_0(b_0) = \alpha_1(b_1)\}.$$

The Mayer–Vietoris sequence is

$$\begin{tikzcd} K_1(B_0 \oplus_B B_1) \arrow[r] & K_1(B_0) \oplus K_1(B_1) \arrow[r] & K_1(B) \arrow[d] \\ K_0(B) \arrow[u] & K_0(B_0) \oplus K_0(B_1) \arrow[l] & K_0(B_0 \oplus_B B_1) \arrow[l] \end{tikzcd}$$

Show that this sequence is exact if both α_0 and α_1 are surjective, but that it need not be exact if both the maps are zero. How far can you weaken the hypotheses? (Hint: consider the algebra $C = \{(b_0, f, b_1) : b_j \in B_j, f\colon [0,1] \to B, \alpha_j(b_j) = f(j)\}$.)

4.10.23 In this exercise we verify the description of the boundary map ∂_0 in the six-term exact sequence (Remark 4.9.3).

(a) Let $U(A)$ denote the unitary group of a unital C^*-algebra A, and let $U_n(A)$ denote the unitary group of $M_n(A)$. If

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

is a short exact sequence, use the path-lifting property to show that there is an exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_1(U(A)) \longrightarrow \pi_1(U(A/J)) \longrightarrow \pi_0(U(\widetilde{J})) \longrightarrow \pi_0(U(A)) \longrightarrow \cdots.$$

(A loop of unitaries in A/J starting with 1 lifts to a path of unitaries in A starting at 1 and ending at a unitary $u \in \widetilde{J}$. This defines the map $\pi_1(U(A/J)) \to \pi_0(U(\widetilde{J}))$.)

(b) Verify that $\varinjlim \pi_0(U_n(\widetilde{J})) = K_1(J)$.

(c) Use Bott periodicity to show that $\varinjlim \pi_1(U_n(A/J)) = K_0(A/J)$.

(d) Show that the boundary map $K_0(A/J) \to K_1(J)$ obtained from (a), (b), and (c) is the same, up to sign, as the map ∂_0 in the six-term exact sequence.

(e) Let $p \in K_0(A/J)$ be a projection, and let $u\colon t \mapsto e^{-it}p + (1-p)$, $t \in [0, 2\pi]$, be the corresponding loop of unitaries. If $x \in A$ is a selfadjoint lift of p, show that $\tilde{u}\colon t \mapsto e^{-itx}$ is a lift of u to a path of unitaries in A starting with 1.

(f) Deduce that $\partial_0[p] = [e^{2\pi i p}] \in K_1(J)$.

4.10.24 In this exercise we shall develop an alternative proof of the Bott Periodicity Theorem, due to Cuntz [44]. The proof applies to any functor E from the category of C^*-algebras to the category of abelian groups, which satisfies the three properties of stability (Example 4.2.5), half-exactness[42] (Proposition 4.3.15), and homotopy invariance (Proposition 4.4.3).

(a) Suppose that we define $E_n(A) = E(S^n(A))$. Show that the analogs of Proposition 4.5.9 (the long exact sequence), Lemma 4.5.10 (split exactness), and Lemma 4.6.4 (additivity) hold for the functors E_n. (Compare Remark 4.5.1.)

(b) Let $\mathfrak{T}$ denote the Toeplitz algebra. Show that, in order to prove Bott periodicity ($E_2(A) \cong E_0(A)$) it suffices to show that the $*$-homomorphism

$$a \mapsto a \otimes 1, \quad A \to A \otimes \mathfrak{T}$$

induces an isomorphism $E(A) \cong E(A \otimes \mathfrak{T})$ for every A.

(c) Let $\mathfrak{A}$ denote the C^*-subalgebra of $\mathfrak{T} \otimes \mathfrak{T}$ generated by $\mathfrak{K} \otimes \mathfrak{T}$ and $\mathfrak{T} \otimes 1$. Show that there is a short exact sequence

$$0 \longrightarrow \mathfrak{K} \otimes \mathfrak{T} \longrightarrow \mathfrak{A} \longrightarrow C(S^1) \longrightarrow 0\,.$$

(d) Let $V \in \mathfrak{T}$ denote the unilateral shift (the Toeplitz operator with symbol z) and let E denote the rank-one projection $1 - VV^*$ onto the cokernel of V. Show that the elements

$$W_{-1} = V(1-E) \otimes 1 + E \otimes V, \quad W_1 = V(1-E) \otimes 1 + E \otimes 1$$

of $\mathfrak{A}$ are isometries.

(e) Show that there are selfadjoint unitaries $F_{\pm 1}$ in $\mathfrak{A}$ such that $W_{\pm 1} = F_{\pm 1}(V \otimes 1)$.

(f) Use the Spectral Theorem to show that any selfadjoint unitary in a C^*-algebra is connected to the identity by a path of unitaries. Deduce that W_{-1} is connected to W_1 by a path W_t of isometries.

(g) Use the universal property of the Toeplitz algebra (Exercise 2.9.8) to construct a homotopy of $*$-homomorphisms $\alpha_t \colon \mathfrak{T} \to \mathfrak{A}$ such that $\alpha_t(V) = W_t$.

[42] In fact, the proof applies to functors E satisfying only a weakened version of the half-exactness axiom, namely that any short exact sequence $0 \to J \to A \to A/J \to 0$ which admits a completely positive section $A/J \to A$ gives rise to a sequence of E-groups which is exact in the middle. Notice that if the functor E has this property so does the functor $E_B \colon A \mapsto E(A \otimes B)$ for a fixed C^*-algebra B. This is important in the last part of the exercise.

(h) Let $\mathfrak{A}'$ be the C^*-subalgebra of $\mathfrak{A} \oplus \mathfrak{T}$ consisting of those pairs (x, y) such that $\sigma_{\mathfrak{A}}(x) = \sigma_{\mathfrak{T}}(y)$, where

$$\sigma_{\mathfrak{A}} \colon \mathfrak{A} \to \mathfrak{A}/\mathfrak{K} \otimes \mathfrak{T} = C(S^1) \quad \text{and} \quad \sigma_{\mathfrak{T}} \colon \mathfrak{T} \to \mathfrak{T}/\mathfrak{K} = C(S^1)$$

are the 'symbol maps'. Show that there is a split short exact sequence

$$0 \longrightarrow \mathfrak{K} \otimes \mathfrak{T} \longrightarrow \mathfrak{A}' \longrightarrow \mathfrak{T} \longrightarrow 0 .$$

(i) Let $\beta_t \colon \mathfrak{T} \to \mathfrak{A}'$ be given by $\beta_t(x) = (\alpha_t(x), x)$. Use the fact that β_1 and β_{-1} induce the same map on E_*, together with the additivity and stability of the functor E_*, to prove that the $*$-homomorphism $\mathfrak{T} \to \mathfrak{T}$ defined by $V \mapsto 1$ induces the identity map on $E_*(\mathfrak{T})$.

(j) Apply the construction to the functor $E_B(A) = E(A \otimes B)$ to prove that the natural map $B \to B \otimes \mathfrak{T}$ induces an isomorphism on E_*. As we have seen above, this completes the proof of Bott periodicity.

For more details, consult [44].

4.11 Notes

Topological K-theory was invented by Atiyah and Hirzebruch [14], motivated by work of Grothendieck in algebraic geometry. Their key observation was that the Bott Periodicity Theorem, which calculates the homotopy groups of the infinite unitary group, makes it possible to organize the vector bundles over a space into a group with interesting homological properties. Atiyah's introduction [9] to topological K-theory is essential reading. Detailed expositions of C^*-algebra K-theory may be found in the books [27], [109] and [134]; the second and third are an easier read for the beginner.

Bott's theorem was originally proved by means of Morse theory, but, as a consequence of their research into the connections between K-theory and analysis, Atiyah and Bott came upon a different argument, and it was soon realized that this argument could naturally be formulated in the context of Banach algebras (see for example [138]). An important survey paper of Taylor [125] brought K-theory to the attention of functional analysts, and very soon afterwards K-theory established itself as a tool of primary importance in C^*-algebra theory.

The group $K_0(A)$ comes equipped with a natural order structure: the class $[p]$, for p a single projection, is to be thought of as positive. Indeed the idea of comparing and ordering projections long predates K-theory, and originates in the work of Murray and von Neumann on 'rings of operators' [100]. The order structure on K_0 is important for the application of K-theory to the classification of certain types of C^*-algebras, but we do not discuss it here — see [45] for information.

The KK-theory of Kasparov [81, 83, 82] provides a unified framework both for K-theory and for the K-homology theory which is the subject-matter of the remainder of this book. The key construction in this theory is the 'Kasparov product', a very general pairing which includes all the products defined in this book as special cases.

The proof of the Bott Periodicity Theorem given in this book is due to Atiyah [10]. A rather different proof which also involves the Toeplitz extension was given by Cuntz [44]. Cuntz's proof, which is developed in the long Exercise 4.10.24, has the advantage that it applies to *any* functor which shares the half-exactness, homotopy-invariance, and stability properties of K-theory. In fact, these properties come close to characterizing the K-theory functor.

There is a purely *algebraic* K-theory for general rings: a classical reference is [97] and a more modern one is [115]. But algebraic K-theory lacks Bott periodicity, and this gives the subject an entirely different flavor.

The slogan 4.1.16 was invented by Weinberger.

5

DUALITY THEORY

In Chapter 2 we introduced the semigroup Ext (A) of unital, injective extensions $\varphi\colon A \to \mathfrak{Q}(H)$ of a separable and unital C^*-algebra A. In this chapter we shall study Ext (A) as a functor of A. Our main tool is a relation between the Ext groups of separable C^*-algebras and the K-theory groups of certain 'dual' C^*-algebras. Using it we shall derive a number of results analogous to ones proved for K-theory in the last chapter.

5.1 Extension Groups and Dual C*-Algebras

5.1.1 DEFINITION Let A be a separable, unital C^*-algebra and let $\rho\colon A \to \mathfrak{B}(H)$ be a representation of A on a separable Hilbert space. The *dual* algebra of A associated to the representation ρ is the following C^*-subalgebra of $\mathfrak{B}(H)$:

$$\mathfrak{D}_\rho(A) = \{ T \in \mathfrak{B}(H) : [T, \rho(a)] \sim 0 \quad \forall a \in A \}.$$

As in Chapter 3, the notation $T_1 \sim T_2$ means that T_1 and T_2 differ by a compact operator. We shall often drop the representation ρ from our notation and write

$$\mathfrak{D}(A) = \{ T \in \mathfrak{B}(H) : [T, a] \sim 0 \quad \forall a \in A \}$$

if the action of A on the Hilbert space H is understood.

A projection $P \in \mathfrak{D}(A)$ determines a Toeplitz extension $\varphi_P\colon A \to \mathfrak{Q}(PH)$ by the formula

$$\varphi_P(a) = \pi(Pa)$$

(see Definition 2.7.7). Moreover, Murray–von Neumann equivalence of projections corresponds to unitary equivalence of extensions:

5.1.2 LEMMA *Two projections $P_1, P_2 \in \mathfrak{D}(A)$ are Murray–von Neumann equivalent if and only if the associated Toeplitz extensions φ_{P_1} and φ_{P_2} are unitarily equivalent.* □

There is a similar statement for projections in the matrix algebras $M_n(\mathfrak{D}(A))$, which is easily recovered from the above in view of the fact that $M_n(\mathfrak{D}(A))$ identifies with the dual of the n-fold direct sum representation $\rho \oplus \cdots \oplus \rho$ on

$H \oplus \cdots \oplus H$. From here on we shall not concern ourselves with the notational details involved in passing from projections *in* $\mathfrak{D}(A)$ to projections *over* $\mathfrak{D}(A)$.[43]

Since the K_0-group of a C^*-algebra D to some extent classifies projections over D it is natural to suppose that $K_0(\mathfrak{D}(A))$ will play a useful role in the classification of extensions of A by the compact operators. In fact the following definition and proposition point to a very close connection between extension theory and K-theory:

5.1.3 DEFINITION A representation $\rho\colon A \to \mathfrak{B}(H)$ of a C^*-algebra A is *ample* if it is non-degenerate[44] and if no non-zero element in A acts on H as a compact operator. A projection $P \in \mathfrak{D}_\rho(A)$ is *ample* if the operator $P\rho(a)$ is never compact unless $a = 0$.

Observe that a projection $P \in \mathfrak{D}(A)$ is ample if and only if the extension φ_P is injective. Among ample projections, K-theoretic equivalence is the same thing as Murray–von Neumann equivalence:

5.1.4 PROPOSITION *Let A be a separable unital C^*-algebra which is represented amply on a separable Hilbert space. Every element of $K_0(\mathfrak{D}(A))$ is the class $[P]$ of an ample projection over $\mathfrak{D}(A)$ and two ample projections determine the same class in $K_0(\mathfrak{D}(A))$ if and only if they are Murray–von Neumann equivalent.*

PROOF The extension φ_I associated to the identity operator $I \in \mathfrak{D}(A)$ is split, and so by Theorem 3.4.7 the extension $\varphi_I \oplus \varphi_P$ is unitarily equivalent to φ_P, for any ample projection P. It follows that $P \oplus I$ is Murray–von Neumann equivalent to P for any ample projection P over $\mathfrak{D}(A)$. Setting $P = I$ we see that I represents the zero class in K-theory, and of course the same is true for the identity matrix of any size over $\mathfrak{D}(A)$. From the formula $[P] + [I - P] = [I]$ we deduce that $[I - P]$ represents the additive inverse of the class $[P]$. It follows that every class in $K_0(\mathfrak{D}(A))$ is represented by a projection over A, and indeed by an ample projection (since $P \oplus I$ is always ample whether or not P is). But by Lemma 4.1.12 two projections P_1 and P_2 over $\mathfrak{D}(A)$ (or indeed over any C^*-algebra) determine the same K-theory class if and only if there is a diagonal projection Q with entries I and 0 such that $P_1 \oplus Q$ and $P_2 \oplus Q$ are Murray–von Neumann equivalent. Furthermore, Voiculescu's Theorem 3.4.7 implies that if P is any ample projection then P is Murray–von Neumann equivalent to $P \oplus Q$. This completes the proof of the proposition. □

5.1.5 REMARK If $I_n \in M_n(\mathfrak{D}(A))$ denotes the identity matrix then it follows from Theorem 3.4.6 that φ_{I_n} and φ_I are unitarily equivalent. If P_n is an $n \times n$ projection over $\mathfrak{D}(A)$ then any unitary equivalence between σ_{I_n} and σ_I produces

[43] Recall that a projection *over* $\mathfrak{D}(A)$ is a projection in some matrix algebra $M_n(\mathfrak{D}(A))$.

[44] We recall that this means that $\rho[A]H$ is dense in H.

a Murray–von Neumann equivalence between P_n and a projection $P \in \mathfrak{D}(A)$. It follows that classes in $K_0(\mathfrak{D}(A))$ correspond one-to-one with Murray–von Neumann equivalence classes of projections actually *in* $\mathfrak{D}(A)$, as opposed to just *over* $\mathfrak{D}(A)$.

5.1.6 PROPOSITION *Let* A *be a separable, unital and nuclear* C^**-algebra which is represented amply on a separable Hilbert space* H. *There is an isomorphism of abelian groups*

$$K_0(\mathfrak{D}(A)) \cong \mathrm{Ext}\,(A),$$

which associates to a projection $P \in \mathfrak{D}(A)$ *the Toeplitz extension* φ_P.

PROOF This follows almost immediately from the previous proposition. The only thing left to note is that every injective extension φ of A is unitarily equivalent to the Toeplitz extension φ_P associated to some projection $P \in \mathfrak{D}(A)$. But if σ is a completely positive lifting of φ then Voiculescu's Theorem implies that $\sigma \lesssim \sigma_I$. In other words, there is an isometry V from the Hilbert space of σ into H such that $\sigma(a) \sim VaV^*$, for all $a \in A$. If P is the orthogonal projection onto the range of V then Lemma 3.1.6 shows that $P \in \mathfrak{D}(A)$ and that φ_P is unitarily equivalent to φ. $\square$

5.1.7 REMARK If A is not assumed to be nuclear then the same argument provides an isomorphism between $K_0(\mathfrak{D}(A))$ and the group of *invertible* elements in the semigroup $\mathrm{Ext}\,(A)$.

5.2 K-Homology

If ρ_1 and ρ_2 are two ample representations of a separable unital C^*-algebra A on two separable Hilbert spaces H_1 and H_2, then according to Theorem 3.4.6 there is a unitary isomorphism $U\colon H_1 \to H_2$ such that $U\rho_1(a) \sim \rho_2(a)U$ for every $a \in A$. It follows that the $*$-homomorphism Ad_U maps $\mathfrak{D}_{\rho_1}(A)$ isomorphically onto $\mathfrak{D}_{\rho_2}(A)$. Therefore, up to $*$-isomorphism the dual algebra $\mathfrak{D}(A)$ is independent of the choice of ample representation used to define it. Furthermore any two such $*$-isomorphisms Ad_U induce the *same* isomorphism of K-theory groups, because inner automorphisms act trivially on K-theory (see Lemma 4.6.1). These observations allow us to make the following definition:

5.2.1 DEFINITION Let A be a separable and unital C^*-algebra. We define the *reduced analytic* K*-homology groups* of A to be

$$\widetilde{K}^1(A) = K_0(\mathfrak{D}(A)), \quad \widetilde{K}^0(A) = K_1(\mathfrak{D}(A)),$$

where $\mathfrak{D}(A)$ denotes the dual algebra of some arbitrarily chosen[45] ample representation of A.

[45]We are not going to worry about any set-theoretic difficulties attendant to making one such choice for each A.

The reason for the shift in indices (so that $\widetilde{K}^p$ corresponds to K_{1-p}) will become clear in later chapters. For now, let us note that the definition is so arranged that

$$\widetilde{K}^1(A) \cong \mathrm{Ext}\,(A),$$

for every separable, unital and nuclear C*-algebra A.

We shall soon define *unreduced* K-homology for all separable C*-algebras, unital or not, by the simple device of adjoining a unit. But before doing that we are going to make reduced homology into a *functor*. If $\alpha\colon A \to B$ is a unital and *injective* $*$-homomorphism then the composition of α with an ample representation ρ of B produces an ample representation $\rho\alpha$ of A. We obtain an inclusion $\mathfrak{D}_\rho(B) \subseteq \mathfrak{D}_{\rho\alpha}(A)$ and hence a map in reduced K-homology

$$\alpha^*\colon \widetilde{K}^p(B) \to \widetilde{K}^p(A)$$

which does not depend on ρ. This makes $\widetilde{K}^p(A)$ contravariantly functorial on the category of unital and injective $*$-homomorphisms of separable unital C*-algebras. From the perspective of extension theory, we have simply reproduced the obvious functoriality $\mathrm{Ext}\,(B) \to \mathrm{Ext}\,(A)$ which associates to a unital and injective extension φ of B the unital and injective extension $\varphi\alpha$ of A.

There is a way of making Ext, and indeed $\widetilde{K}^p$, contravariantly functorial for arbitrary unital $*$-homomorphisms $\alpha\colon A \to B$. For a unital and injective extension φ of B we first form the unital extension $\varphi\alpha$ of A, and if the result is not injective we replace $\varphi\alpha$ with a direct sum $\varphi\alpha \oplus \psi$, where ψ is any split, unital and injective extension. It follows from Voiculescu's Theorem that the unitary equivalence class of $\varphi\alpha \oplus \psi$ is independent of the choice of ψ, and we obtain a homomorphism $\alpha^*\colon \mathrm{Ext}\,(B) \to \mathrm{Ext}\,(A)$ as required. In K-homology this corresponds to considering a pair of inclusions

$$\mathfrak{D}_\rho(B) \subseteq \mathfrak{D}_{\rho\alpha}(A) \subseteq \mathfrak{D}_{\rho\alpha\oplus\rho'}(A),$$

where ρ' is any ample representation of A (such as a multiplicative lifting of the extension ψ). The induced map on K-theory provides a group homomorphism $\alpha^*\colon \widetilde{K}^p(B) \to \widetilde{K}^p(A)$ which depends only on α.

The following definition and lemma place this construction in a slightly more abstract context:

5.2.2 Definition Let A and B be separable, unital C*-algebras and let $\alpha\colon A \to B$ be a unital $*$-homomorphism. Let ρ_A and ρ_B be ample representations of A and B on separable Hilbert spaces H_A and H_B. An isometry $V\colon H_B \to H_A$ *covers* $\alpha\colon A \to B$ if

$$V^*\rho_A(a)V \sim \rho_B(\alpha(a))$$

for every $a \in A$.

Suppose that V covers α. Then Lemma 3.1.6 shows that the projection VV^* belongs to $\mathfrak{D}(A)$. From this we easily obtain:

5.2.3 LEMMA *If* $\alpha\colon A \to B$ *is a unital* $*$*-homomorphism of separable unital* C^**-algebras, and if* $V\colon H_B \to H_A$ *covers* α*, then the* $*$*-homomorphism* $\mathrm{Ad}_V(T) = VTV^*$ *maps* $\mathfrak{D}(B)$ *into* $\mathfrak{D}(A)$. □

5.2.4 LEMMA *Every unital* $*$*-homomorphism* $\alpha\colon A \to B$ *is covered by some isometry* $V\colon H_B \to H_A$*, and any two such isometries, both covering* α*, induce the same map on* K*-theory:*

$$(\mathrm{Ad}_{V_1})_* = (\mathrm{Ad}_{V_2})_*\colon K_{1-p}(\mathfrak{D}(B)) \to K_{1-p}(\mathfrak{D}(A)).$$

REMARK It is only necessary to assume that H_A is an ample representation; any non-degenerate representation of B on H_B will suffice.

PROOF According to Voiculescu's Theorem 3.4.3, there is an isometry $V\colon H_B \to H_A$ such that $\rho_B(\alpha(a)) \sim V^*\rho_A(a)V$, for all $a \in A$. This takes care of the existence of covering isometries. The uniqueness part of the lemma is proved by an elaboration of the argument used to prove Lemma 4.6.2. Suppose that V_1 and V_2 are two covering isometries. The maps

$$T \mapsto \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T \mapsto \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix},$$

taking $\mathfrak{D}(A)$ into $M_2(\mathfrak{D}(A))$, induce the same isomorphism on K-theory. So it suffices to show that the maps

$$T \mapsto \begin{pmatrix} V_1TV_1^* & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T \mapsto \begin{pmatrix} 0 & 0 \\ 0 & V_2TV_2^* \end{pmatrix},$$

which take $\mathfrak{D}(B)$ to $M_2(\mathfrak{D}(A))$, induce the same map on K-theory. But the second is obtained from the first by conjugating with the unitary

$$\begin{pmatrix} I - V_1V_1^* & V_1V_2^* \\ V_2V_1^* & I - V_2V_2^* \end{pmatrix},$$

which is an element of $M_2(\mathfrak{D}(A))$. So uniqueness follows from the fact that inner automorphisms act trivially on K-theory (Lemma 4.6.1). □

5.2.5 DEFINITION If $\alpha\colon A \to B$ is a unital $*$-homomorphism then we define

$$\alpha^*\colon \widetilde{K}^p(B) \to \widetilde{K}^p(A)$$

to be the map $(\mathrm{Ad}_{V_\alpha})_*\colon K_{1-p}(\mathfrak{D}(B)) \to K_{1-p}(\mathfrak{D}(A))$, where V_α is any isometry which covers α.

5.2.6 LEMMA *The correspondence $\alpha \mapsto \alpha^*$ is a contravariant functor.*

PROOF If $V_\alpha\colon H_B \to H_A$ covers $\alpha\colon A \to B$, and if $V_\beta\colon H_C \to H_B$ covers $\beta\colon B \to C$, then $V_\alpha V_\beta$ covers $\beta\alpha$. $\square$

We are now going to define *unreduced* K-homology groups, which have the advantage of being functorial on the category of all separable C*-algebras and $*$-homomorphisms, unital or not.

5.2.7 DEFINITION Let A be a separable C*-algebra, possibly without unit, and let $\widetilde{A}$ be the C*-algebra with a unit adjoined. Define the (unreduced) K-*homology groups* of A to be

$$K^p(A) = K_{1-p}(\mathfrak{D}(\widetilde{A}))$$

where $\mathfrak{D}(\widetilde{A})$ is the dual of an ample representation of $\widetilde{A}$.

Thus the unreduced K-homology groups of A are the reduced K-homology groups of $\widetilde{A}$. Since any $*$-homomorphism $\alpha\colon A \to B$ gives rise to a unital $*$-homomorphism $\widetilde{\alpha}\colon \widetilde{A} \to \widetilde{B}$, our previous discussion of reduced K-homology makes the unreduced groups contravariantly functorial for arbitrary $*$-homomorphisms.

Suppose that A is a separable and unital C*-algebra. What is the difference between the reduced and unreduced K-homology of A? Let $\rho\colon A \to \mathfrak{B}(H)$ be an ample representation of A. To obtain an ample representation of $\widetilde{A}$ we may form

$$\widetilde{\rho} = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix} : A \to \mathfrak{B}(H \oplus H)$$

and then extend $\widetilde{\rho}$ to a non-degenerate representation of $\widetilde{A}$. The result is certainly an ample representation and, using an obvious matrix notation, the dual of $\widetilde{A}$ assumes a simple, concrete form:

$$\mathfrak{D}(\widetilde{A}) = \begin{pmatrix} \mathfrak{D}(A) & \mathfrak{K}(H) \\ \mathfrak{K}(H) & \mathfrak{B}(H) \end{pmatrix}. \tag{5.2.8}$$

5.2.9 EXAMPLE For $A = \mathbb{C}$ we have

$$\mathfrak{D}(\widetilde{\mathbb{C}}) = \begin{pmatrix} \mathfrak{B}(H) & \mathfrak{K}(H) \\ \mathfrak{K}(H) & \mathfrak{B}(H) \end{pmatrix},$$

from which it follows that

$$K^p(\mathbb{C}) = \begin{cases} \mathbb{Z} & p = 0 \\ 0 & p = 1 \end{cases}$$

In contrast, since the dual of $\mathbb{C}$ is

$$\mathfrak{D}(\mathbb{C}) = \mathfrak{B}(H),$$

it follows that $\widetilde{K}^p(\mathbb{C}) = 0$, for all p.

There is a natural 'augmentation' map

$$j\colon \widetilde{K}^p(A) \to K^p(A),$$

obtained by including $\mathfrak{D}(A)$ as the 'top left corner' of $\mathfrak{D}(\widetilde{A})$ in 5.2.8.

5.2.10 PROPOSITION *For any unital, separable C^*-algebra A there is a natural long exact sequence*

$$\cdots \longrightarrow \widetilde{K}^p(A) \xrightarrow{j} K^p(A) \longrightarrow K^p(\mathbb{C}) \longrightarrow \widetilde{K}^{p+1}(A) \longrightarrow \cdots,$$

in which $j\colon \widetilde{K}^p(A) \to K^p(A)$ is the augmentation map described above, while the map $K^p(A) \to K^p(\mathbb{C})$ arises by functoriality from the unital map $\mathbb{C} \to A$.

PROOF The quotient map

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \mapsto \pi(T_{11})$$

from $\mathfrak{D}(\widetilde{A})$ to $\mathfrak{D}(A)/\mathfrak{K}(H)$ induces an isomorphism in K-theory (the reader was asked to prove this in Exercise 4.10.9). So if we apply the K-theory long exact sequence to the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{K}(H) \arrow[r] \arrow[d, equal] & \mathfrak{D}(A) \arrow[r] \arrow[d] & \mathfrak{D}(A)/\mathfrak{K}(H) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathfrak{K}(H) \arrow[r] & \mathfrak{D}(\mathbb{C}) \arrow[r] & \mathfrak{D}(\mathbb{C})/\mathfrak{K}(H) \arrow[r] & 0 \end{tikzcd}$$

in which the vertical maps come from the inclusion $\mathbb{C} \to A$, then we obtain a diagram

$$\begin{tikzcd} \cdots \arrow[r] & K_{1-p}(\mathfrak{D}(A)) \arrow[r, "j"] \arrow[d] & K_{1-p}(\mathfrak{D}(\widetilde{A})) \arrow[r] \arrow[d] & K_{-p}(\mathfrak{K}(H)) \arrow[r] \arrow[d, equal] & \cdots \\ \cdots \arrow[r] & K_{1-p}(\mathfrak{D}(\mathbb{C})) \arrow[r, "j"'] & K_{1-p}(\mathfrak{D}(\widetilde{\mathbb{C}})) \arrow[r] & K_{-p}(\mathfrak{K}(H)) \arrow[r] & \cdots \end{tikzcd}$$

The proposition now follows from the definitions of reduced and unreduced K-homology, a diagram chase, and the fact that the reduced K-homology of $\mathbb{C}$ is zero. □

5.2.11 COROLLARY *If A is a commutative, separable and unital C^*-algebra then $K^0(A) \cong \widetilde{K}^0(A) \oplus \mathbb{Z}$ and $K^1(A) \cong \widetilde{K}^1(A)$. In particular $K^1(A) \cong \operatorname{Ext}(A)$.*

PROOF If $A = C(X)$ then evaluation at a point of X gives a homomorphism $C(X) \to \mathbb{C}$ splitting the natural map $\mathbb{C} \to C(X)$. So the long exact sequence of the previous proposition splits. □

5.3 Relative K-Homology

Let $0 \to J \to A \to A/J \to 0$ be a short exact sequence of separable C^*-algebras. Our aim in this section is to construct relative K-homology groups $K^*(A, A/J)$ and a functorial six-term exact sequence

(5.3.1)
$$\begin{tikzcd} K^1(A/J) \arrow[r] & K^1(A) \arrow[r] & K^1(A, A/J) \arrow[d] \\ K^0(A, A/J) \arrow[u] & K^0(A) \arrow[l] & K^0(A/J) \arrow[l] \end{tikzcd}$$

which is analogous to the six-term exact sequence in K-theory. (If A is unital there will be a similar sequence in reduced K-homology.) We shall only be able to construct the exact sequence under a technical hypothesis, that the quotient map $A \to A/J$ has a completely positive section. Such a section will always exist if A/J is nuclear, and in particular if A/J is commutative. The technical hypothesis is used in Proposition 5.3.7 below.

Since K-homology is defined in terms of K-theory, it should be no surprise that we shall obtain the six-term exact sequence 5.3.1 from the K-theory six-term exact sequence associated to a certain ideal in the dual algebra $\mathfrak{D}(A)$.

5.3.2 Definition Let $J \subseteq A$ be an ideal in a separable C^*-algebra, and let ρ be a representation of A on a Hilbert space H. We define the *relative dual algebra* $\mathfrak{D}_\rho(A/\!\!/J)$ to be the following ideal in $\mathfrak{D}_\rho(A)$:

$$\mathfrak{D}_\rho(A/\!\!/J) = \{T \in \mathfrak{D}_\rho(A) : T\rho(a) \sim 0 \sim \rho(a)T \quad \forall a \in J\}.$$

As before, we shall often omit mention of the representation ρ, writing

$$\mathfrak{D}(A/\!\!/J) = \{T \in \mathfrak{D}(A) : Ta \sim 0 \sim aT \quad \forall a \in J\}.$$

5.3.3 Remark We shall sometimes say that an operator $T \in \mathfrak{D}(A)$ satisfying the condition appearing in this definition is *locally compact* for J. This terminology is particularly appropriate in the commutative case, to which we shall pay special attention in the next section.

Our focus in this section will be on unreduced homology. Rather than write $\mathfrak{D}(\widetilde{A})$, as in Definition 5.2.7, we shall streamline our notation and write $\mathfrak{D}(A)$, with the understanding that the dual is constructed on a Hilbert space $H_{\widetilde{A}}$ which is equipped with an ample representation of $\widetilde{A}$. Observe that then $\mathfrak{D}(A) = \mathfrak{D}(\widetilde{A})$, since of course every operator on $H_{\widetilde{A}}$ commutes with the identity operator, and so every operator which commutes modulo compacts with A will also commute modulo compacts with $\widetilde{A}$.

If A is unital then to treat the case of reduced K-homology one can simply replace $H_{\widetilde{A}}$ with H_A in the following discussion. If A is non-unital then as far as dual algebras are concerned there is no difference between H_A and $H_{\widetilde{A}}$ since an ample representation of A always extends to an ample representation of $\widetilde{A}$ on the same space.

5.3.4 DEFINITION Let J be an ideal in a separable C*-algebra A. We define the relative K-homology groups[46] of the pair $(A, A/J)$ by representing A on a Hilbert space $H_{\widetilde{A}}$, as above, and applying the formula

$$K^p(A, A/J) = K_{1-p}\left(\mathfrak{D}(A)/\mathfrak{D}(A/\!\!/J)\right).$$

Relative K-homology is contravariantly functorial for $*$-homomorphisms of pairs, that is, $*$-homomorphisms $\alpha\colon A \to A'$ taking J into J'. Indeed, if $V\colon H_{\widetilde{A'}} \to H_{\widetilde{A}}$ is any isometry which covers α then Ad_V maps $\mathfrak{D}(A'/\!\!/J')$ into $\mathfrak{D}(A/\!\!/J)$, and the same argument as the one used to prove Lemma 5.2.4 shows that the induced maps

$$\alpha^*\colon K_{1-p}(\mathfrak{D}(A')/\mathfrak{D}(A'/\!\!/J')) \to K_{1-p}(\mathfrak{D}(A)/\mathfrak{D}(A/\!\!/J))$$

do not depend on the choice of V. The same holds for the maps

$$\alpha^*\colon K_{1-p}(\mathfrak{D}(A'/\!\!/J')) \to K_{1-p}(\mathfrak{D}(A/\!\!/J)).$$

Our aim is to construct the K-homology exact sequence 5.3.1 from the K-theory exact sequence associated to the diagram

$$0 \longrightarrow \mathfrak{D}(A/\!\!/J) \longrightarrow \mathfrak{D}(A) \longrightarrow \mathfrak{D}(A)/\mathfrak{D}(A/\!\!/J). \longrightarrow 0 \tag{5.3.5}$$

Let $V\colon H_{\widetilde{A/J}} \to H_{\widetilde{A}}$ be an isometry which covers the quotient map $\pi\colon A \to A/J$. As we observed in the last section, Ad_V maps $\mathfrak{D}(A/J)$ into $\mathfrak{D}(A)$. In fact Ad_V maps $\mathfrak{D}(A/J)$ into the ideal $\mathfrak{D}(A/\!\!/J)$ (an easy calculation). We want to show that the induced map on K-theory is an isomorphism, and it is here that we need to hypothesize the existence of a completely positive section.

5.3.6 DEFINITION Let us say that the short exact sequence

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

is *semisplit* if the quotient map $\widetilde{A} \to \widetilde{B}$ admits a completely positive section (this is consistent with our terminology in Chapter 2).

5.3.7 PROPOSITION *If the short exact sequence*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

of separable C*-*algebras is semisplit, and if* $V\colon H_{\widetilde{A/J}} \to H_{\widetilde{A}}$ *is any isometry which covers the quotient map* $\pi\colon A \to A/J$, *then the* $*$-*homomorphism*

$$\mathrm{Ad}_V\colon \mathfrak{D}(A/J) \to \mathfrak{D}(A/\!\!/J)$$

induces an isomorphism on K-*theory.*

[46] Beware that in the literature these groups are sometimes denoted $K^p(A, J)$.

PROOF Denote by ρ_A and $\rho_{A/J}$ the given representations of A and A/J on $H_{\widetilde{A}}$ and $H_{\widetilde{A}/J}$, respectively. Let σ be a completely positive section of the quotient map π. By Stinespring's Theorem, there is a separable Hilbert space H and a dilation of the completely positive map $\rho_A\sigma\colon A/J \to B(H_{\widetilde{A}})$ to a representation

$$\rho'_{A/J} = \begin{pmatrix} \rho_A\sigma & * \\ * & * \end{pmatrix} : A/J \to \mathfrak{B}(H_{\widetilde{A}} \oplus H).$$

Now let $H'_{\widetilde{A}/J} = H_{\widetilde{A}} \oplus H$ and let $W\colon H_{\widetilde{A}} \to H'_{\widetilde{A}/J}$ be the obvious inclusion. Then Ad_W maps the ideal $\mathfrak{D}_{\rho_A}(A/\!\!/J)$ into $\mathfrak{D}_{\rho'_{A/J}}(A/J)$. The composition

$$H_{\widetilde{A}/J} \xrightarrow{V} H_{\widetilde{A}} \xrightarrow{W} H'_{\widetilde{A}/J}$$

is an isometry which covers the identity map $\widetilde{A}/J \to \widetilde{A}/J$. It follows from Lemma 5.2.4 that the composition

$$\mathfrak{D}_{\rho_{A/J}}(A/J) \xrightarrow{\mathrm{Ad}_V} \mathfrak{D}_{\rho_A}(A/\!\!/J) \xrightarrow{\mathrm{Ad}_W} \mathfrak{D}_{\rho'_{A/J}}(A/J)$$

induces an isomorphism on K-theory. This shows that Ad_V is injective at the level of K-theory.

To prove surjectivity, define a representation ρ'_A of A on the Hilbert space

$$H'_{\widetilde{A}} = H_{\widetilde{A}} \oplus H'_{\widetilde{A}/J} = H_{\widetilde{A}} \oplus H_{\widetilde{A}} \oplus H$$

by forming the direct sum of ρ_A, acting on the first summand, with $\rho'_{A/J}\pi$, acting on $H'_{\widetilde{A}/J}$. The obvious inclusion $X\colon H'_{\widetilde{A}/J} \to H'_{\widetilde{A}}$ covers $\pi\colon A \to A/J$. We are going to show that the composition

$$\mathfrak{D}_{\rho_A}(A/\!\!/J) \xrightarrow{\mathrm{Ad}\,W} \mathfrak{D}_{\rho'_{A/J}}(A/J) \xrightarrow{\mathrm{Ad}\,X} \mathfrak{D}_{\rho'_A}(A/\!\!/J) \tag{5.3.8}$$

induces an isomorphism on K-theory; this will finish the proof. The composition of isometries

$$H_{\widetilde{A}} \xrightarrow{W} H_{\widetilde{A}} \oplus H \xrightarrow{X} H'_{\widetilde{A}} = H_{\widetilde{A}} \oplus H_{\widetilde{A}} \oplus H$$

includes $H_{\widetilde{A}}$ as the second summand. This isometry does *not* cover the identity map $1_A\colon A \to A$. However, it is homotopic by a rotation to the isometry

$$H_{\widetilde{A}} \xrightarrow{Y} H_{\widetilde{A}} \oplus H_{\widetilde{A}} \oplus H$$

which maps $H_{\widetilde{A}}$ into the first factor of the triple sum, and this isometry *does* cover 1_A. The corresponding homotopy of $*$-homomorphisms

$$T \mapsto \begin{pmatrix} \sin^2(\frac{\pi}{2}t)T & \sin(\frac{\pi}{2}t)\cos(\frac{\pi}{2}t)T & 0 \\ \sin(\frac{\pi}{2}t)\cos(\frac{\pi}{2}t)T & \cos^2(\frac{\pi}{2}t)T & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad t \in [0,1]$$

connects the composition 5.3.8 to the $*$-homomorphism

$$\mathrm{Ad}_V \colon \mathfrak{D}_{\rho_A}(A/\!\!/J) \to \mathfrak{D}_{\rho'_A}(A/\!\!/J),$$

which induces an isomorphism on K-theory by the argument of Lemma 5.2.4. The proof of the proposition is therefore completed by an appeal to the homotopy invariance of K-theory. $\square$

5.3.9 Remark It follows that $K_{1-p}(\mathfrak{D}(A/\!\!/J)) \cong K^p(A/J)$. It is important for the validity of this result that we are working with ample representations of $\widetilde{A}$, not just of A itself — as the reader will perceive on considering the case $A = J = \mathbb{C}$. If A is unital, and if we form the dual algebra by using an ample representation ρ of A (rather than of its unitalization), then we obtain by similar arguments an isomorphism $K_{1-p}(\mathfrak{D}_\rho(A/\!\!/J)) \cong \widetilde{K}^p(A/J)$.

From the six-term exact sequence in K-theory, along with Proposition 5.3.7, we immediately get the following result:

5.3.10 Theorem *Let J be an ideal in a separable C^*-algebra A, and suppose that the extension*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

is semisplit. Then there is a functorial six-term exact sequence

$$\begin{array}{ccccc} K^1(A/J) & \longrightarrow & K^1(A) & \longrightarrow & K^1(A, A/J) \\ \uparrow & & & & \downarrow \\ K^0(A, A/J) & \longleftarrow & K^0(A) & \longleftarrow & K^0(A/J) \end{array} \qquad \square$$

5.4 Excision in K-Homology

In this section we shall prove an excision isomorphism in K-homology:

$$K^p(A, A/J) \cong K^p(J).$$

This holds in full generality: no semisplitting hypothesis is required.

We shall continue to represent our separable C^*-algebras A on Hilbert spaces $H_{\widetilde{A}}$ equipped with ample representations of $\widetilde{A}$.

5.4.1 Lemma *The groups $K_q(\mathfrak{D}(A/\!\!/A))$ are zero for every separable C^*-algebra A and every q.*

Proof This follows from Proposition 5.3.7, which identifies $\mathfrak{D}(A/\!\!/A)$ and $\mathfrak{D}(0)$ at the level of K-theory. $\square$

Exercise 5.6.6 asks the reader to supply a direct proof of Lemma 5.4.1, without recourse to Proposition 5.3.7.

5.4.2 Remark Once again, it is important that in the lemma we work with $H_{\tilde{A}}$ as opposed to H_A: if A is unital then the relative dual $\mathfrak{D}(A/\!/A)$ on the Hilbert space H_A is the C^*-algebra $\mathfrak{K}(H)$ of compact operators, whose K-theory is non-trivial.

It follows from Lemma 5.4.1 and the six-term exact sequence in K-theory that if A is represented on $H_{\tilde{A}}$ then the quotient map

$$\mathfrak{D}(A) \to \mathfrak{D}(A)/\mathfrak{D}(A/\!/A)$$

of associated dual algebras induces an isomorphism

$$K^p(A) \cong K_{1-p}(\mathfrak{D}(A)/\mathfrak{D}(A/\!/A)). \tag{5.4.3}$$

5.4.4 Remark For later purposes we note that, in spite of Remark 5.4.2, if a unital C^*-algebra A is represented amply on a Hilbert space H_A (as opposed to $H_{\tilde{A}}$), and if the dual algebra $\mathfrak{D}(A)$ is formed on this Hilbert space, then the above quotient map still induces an isomorphism 5.4.3. Indeed in this case $\mathfrak{D}(A/\!/A) = \mathfrak{K}(H_A)$ and the isomorphism 5.4.3 was already noted in the proof of Proposition 5.2.10.

With the isomorphism 5.4.3 in hand we are now ready to formulate and prove our excision result:

5.4.5 Excision Theorem *Let A be a separable C^*-algebra and suppose that $\tilde{A}$ is amply represented on a separable Hilbert space $H_{\tilde{A}}$. Let J be an ideal in A and form the C^*-algebras $\mathfrak{D}(A)$ and $\mathfrak{D}(J)$ on $H_{\tilde{A}}$. The inclusion $\mathfrak{D}(A) \subseteq \mathfrak{D}(J)$ maps $\mathfrak{D}(A/\!/J)$ into $\mathfrak{D}(J/\!/J)$ and induces an isomorphism*

$$\mathfrak{D}(A)/\mathfrak{D}(A/\!/J) \xrightarrow[\cong]{} \mathfrak{D}(J)/\mathfrak{D}(J/\!/J)\ .$$

Consequently the diagram of maps

$$\mathfrak{D}(A)/\mathfrak{D}(A/\!/J) \xrightarrow[\cong]{} \mathfrak{D}(J)/\mathfrak{D}(J/\!/J) \longleftarrow \mathfrak{D}(J)$$

induces an excision isomorphism $K^p(A, A/J) \cong K^p(J)$.

Proof To prove that the inclusion of $\mathfrak{D}(J)$ into $\mathfrak{D}(A)$ induces an isomorphism on quotient algebras we must show that

$$\mathfrak{D}(A/\!/J) = \mathfrak{D}(A) \cap \mathfrak{D}(J/\!/J),$$

which implies injectivity of the map, and that

$$\mathfrak{D}(J) = \mathfrak{D}(A) + \mathfrak{D}(J/\!/J),$$

which implies surjectivity. The first equality is an immediate consequence of the definitions. The second is quite non-trivial, and to prove it we shall use

Kasparov's Technical Theorem (3.8.1). Fix $T \in \mathfrak{D}(J)$. Observe then that if $j \in J$ and $a \in A$, then

$$j[T, a] = [T, ja] - [T, j]a \in \mathfrak{K}(H).$$

Thus if we let E_1 be J, E_2 be the algebra generated by all commutators $[a, T]$ for $a \in A$, and Δ be A, then the hypotheses of the Technical Theorem are satisfied. Let X be the operator produced by the theorem. Then

$$j(1 - X)T \sim 0 \qquad \forall j \in J$$

and

$$[XT, a] = X[T, a] + [X, a]T \sim 0 \qquad \forall a \in A.$$

Thus $(1 - X)T \in \mathfrak{D}(J /\!\!/ J)$ and $XT \in \mathfrak{D}(A)$. We have therefore shown that $\mathfrak{D}(J) = \mathfrak{D}(A) + \mathfrak{D}(J /\!\!/ J)$, and the result follows. $\square$

The proof of the Excision Theorem, while short, relies wholly on the quite difficult Kasparov Technical Theorem. We conclude this section with an interesting proof of the theorem in the important special case where A is commutative. Comparing this with the general proof just given is a good way to come to grips with Kasparov's result.

5.4.6 Definition Let X be a compact Hausdorff space and suppose that $C(X)$ acts via a non-degenerate representation on a Hilbert space H. An operator $T \in \mathfrak{B}(H)$ is *pseudolocal* if fTg is a compact operator for every pair of continuous functions f and g on X with disjoint supports.

If $T \in \mathfrak{D}(C(X))$ then T is pseudolocal. Indeed, if f and g have disjoint supports then

$$fTg = f[T, g] \sim 0.$$

The converse is a useful observation of Kasparov:

5.4.7 Kasparov's Lemma *Let X be a compact Hausdorff space and suppose that $C(X)$ acts via a non-degenerate representation on a Hilbert space H. If $T \in \mathfrak{B}(H)$ is pseudolocal then $[T, f] \sim 0$ for every $f \in C(X)$. Thus $\mathfrak{D}(C(X))$ is composed precisely of the pseudolocal operators on H.*

Proof Let us recall that every representation of $C(X)$ extends to a representation of the bounded Borel functions, and observe that if T is pseudolocal then $fTg \sim 0$ for all bounded Borel functions f, g whose supports are disjoint. This is because we can find continuous functions f', g', with disjoint supports, such that $f = ff'$ and $g = gg'$. Hence $fTg = ff'Tg'g \sim 0$.

It suffices to show that if f is a real-valued continuous function on X then $[f, T]$ may be approximated in norm by compact operators. Let $\varepsilon > 0$ and partition the range of f into Borel sets, $U_1, \ldots, U_n$, each of diameter less than ε, in such a way that $\overline{U_i}$ intersects $\overline{U_j}$ if and only if $|i-j| \leqslant 1$ (for instance, one can take the U_i to be a sequence of non-overlapping half-open intervals). Let $f_1, \ldots, f_n$ be the characteristic functions of the Borel sets $f^{-1}[U_1], \ldots, f^{-1}[U_n]$ in X. Observe that

(a) if $|i-j| > 1$ then $f_i T f_j \sim 0$, and
(b) if $\tilde{f} = f(x_1)f_1 + \cdots + f(x_n)f_n$, where $x_1, \ldots, x_n$ are points chosen from $f^{-1}[U_1], \ldots, f^{-1}[U_n]$, then $\|f - \tilde{f}\| < \varepsilon$.

The operator $[f, T]$ is within $2\varepsilon\|T\|$ of $[\tilde{f}, T]$, and since $f_1 + \cdots + f_n = 1$ we have that

$$\begin{aligned} \tilde{f}T - T\tilde{f} &= \sum_{i,j} f(x_i) f_i T f_j - f_i T f(x_j) f_j \\ &\sim \sum_{|i-j|=1} (f(x_i) - f(x_j)) f_i T f_j. \end{aligned}$$

Break the last sum into two parts, one where $i = j+1$ and one where $i = j-1$. The first part is a direct sum of operators $(f(x_{j+1}) - f(x_j)) f_{j+1} T f_j$, from $f_j H$ to $f_{j+1} H$. It follows that the norm of this part is the maximum of the norms of its summands. Therefore since $|f(x_{j+1}) - f(x_j)| < 2\varepsilon$ the norm of the first part is no more than $2\varepsilon\|T\|$. Treating the second part in the same way, we see that the last line of the display is of norm less than $4\varepsilon\|T\|$. It follows that $[f, T]$ is a norm limit of compact operators, as required. □

PROOF OF THEOREM 5.4.5 IN THE COMMUTATIVE CASE Let $A = C_0(X)$ and let $A/J = C_0(Y)$, where Y is a closed subset of X.[47] Place a metric on the one-point compactification $\tilde{X}$, and for each $x \in X \setminus Y$ let U_x be an open metric ball in $\tilde{X}$ of small enough radius that it intersects neither Y nor the point at infinity. The collection of all U_x is an open cover of $X \setminus Y$; choose a locally finite subcover and a partition of unity $\{f_j\}$ subordinate to the subcover. Note that for each $\varepsilon > 0$, the subcover can contain only finitely many balls U_x of diameter bigger than ε. If $T \in \mathfrak{D}(J)$ we form the operator

$$T' = \sum_j f_j^{\frac{1}{2}} T f_j^{\frac{1}{2}}.$$

The sum converges in the strong operator topology. We are going to show that $T' \in \mathfrak{D}(A)$ while $T - T' \in \mathfrak{D}(J/\!\!/J)$. By Kasparov's Lemma 5.4.7, to show that $T' \in \mathfrak{D}(A)$ it suffices to show that if g and h are continuous functions on $\tilde{X}$ and if g and h have disjoint supports then $gT'h \sim 0$. But it follows from the manner in which we constructed our partition of unity that for all but finitely many j either

[47] For the purpose of becoming acquainted with the argument it is useful to consider the case where X is the closed unit interval and Y is the set of its two endpoints.

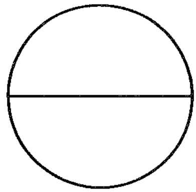

FIG. 5.1. The theta curve

$gf_j = 0$ or $f_j h = 0$. Hence $gT'h$ is a finite sum of terms $gf_j^{\frac{1}{2}}Tf_j^{\frac{1}{2}}h$. Each of these terms is compact since $gf_j^{\frac{1}{2}}Tf_j^{\frac{1}{2}}h \sim Tgf_j^{\frac{1}{2}}f_j^{\frac{1}{2}}h = 0$. We conclude that $fT'g \sim 0$, as required. To prove that $T - T' \in \mathfrak{D}(J/\!/J)$, note that for each compactly supported $f \in J$, almost all the products ff_j are zero. Hence, for a suitable N,

$$fT = \sum_{j=1}^{N} ff_j T \sim \sum_{j=1}^{N} ff_j^{\frac{1}{2}}Tf_j^{\frac{1}{2}} = \sum_{j=1}^{\infty} ff_j^{\frac{1}{2}}Tf_j^{\frac{1}{2}} = fT'.$$

This shows that $T - T' \in \mathfrak{D}(J/\!/J)$, as required, and completes the proof of the theorem. $\square$

5.5 Example: the Theta Curve

In this section we shall classify essentially normal operators whose spectrum is the *theta curve* (Figure 5.1)

$$X = \{z \in \mathbb{C} : |z| = 1 \text{ or } z \in [-1, 1]\}.$$

If T is such an operator then $T \pm \frac{1}{2}i$ are Fredholm operators and the conclusion of our argument will be that T is a compact perturbation of a normal operator if and only if $\operatorname{Index}(T + \frac{1}{2}i) = \operatorname{Index}(T - \frac{1}{2}i) = 0$. Equivalently, we shall show that $\operatorname{Ext}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$, generated by operators with appropriate Fredholm indices. Our argument will use excision and the six-term exact sequence in K-homology, applied in the fairly simple situation of a split short exact sequence of C^*-algebras. We could in fact have obtained the same conclusion somewhat earlier by using Proposition 3.8.9. We note however that the same powerful tool — the Kasparov Technical Theorem — appears in the course of either approach.

Let $Y \subseteq X$ be the top half of the theta curve , $Y = \{z \in X : \operatorname{Im} z \geqslant 0\}$. Then there is a surjective homomorphism of C^*-algebras

$$\alpha \colon C(X) \to C(Y)$$

whose kernel $I = C_0(X \setminus Y)$ consists of functions vanishing on Y. Moreover, α is split by a $*$-homomorphism $\beta \colon C(Y) \to C(X)$ coming from a retraction $X \to Y$.

Form the six-term exact sequence of K-homology associated to the algebra $C(X)$ and the ideal I. Because of the splitting β, this six-term exact sequence breaks up into a pair of split exact sequences, of which the $\mathrm{Ext} = K^1$ part is

$$0 \longrightarrow \mathrm{Ext}\,(Y) \longrightarrow \mathrm{Ext}\,(X) \longrightarrow K^1(I) \longrightarrow 0\,.$$

Now Y is homeomorphic to a circle S^1, so, by Proposition 2.4.6, $\mathrm{Ext}\,(Y) = \mathbb{Z}$. Similarly, by definition, $K^1(I) = \mathrm{Ext}\,(\widetilde{I})$. Since $\widetilde{I} \cong C(S^1)$, we have $K^1(I) \cong \mathbb{Z}$. It follows that $\mathrm{Ext}\,(X) \cong \mathbb{Z} \oplus \mathbb{Z}$, as was required.

We were successful in computing this example because of the splitting of the associated short exact sequence of C^*-algebras. In the next two chapters we shall develop more powerful methods which will allow us to compute Ext for arbitrary closed subsets of the plane.

5.6 Exercises

5.6.1 Compute the reduced and unreduced K-homology groups of the C^*-algebra of $n \times n$ complex matrices (compare Exercise 2.9.14).

5.6.2 Describe the relationship between unreduced K-homology and the 'weak' extension semigroup introduced in Exercise 2.9.15. Compare the exact sequence given there with the one in Proposition 5.2.10.

5.6.3 Show that $K^0(\mathfrak{K}(H)) \cong \mathbb{Z}$.

5.6.4 Generalize the previous exercise by showing that if e is a rank-one projection then the inclusion $a \mapsto a \otimes e$ of A into $A \otimes \mathfrak{K}(H)$ induces an isomorphism in K-homology (see Theorem 9.4.1).

5.6.5 Let A be a separable C^*-algebra and let

$$0 \longrightarrow \mathfrak{K}(H) \longrightarrow E \longrightarrow A \longrightarrow 0$$

be a semisplit extension. By analyzing the associated six-term exact sequence in K-homology, show that the K-homology boundary maps are not always zero. Show that in fact the generator of $K^0(\mathfrak{K}(H))$ maps to the class in $K^1(A)$ determined by the extension.

5.6.6 Give a direct proof of Lemma 5.4.1.

5.6.7 Show that if A is a separable, unital and nuclear C^*-algebra then the group $K^1(C_0(0,1) \otimes A)$ is isomorphic to the group of unitary equivalence classes of those unital, injective extensions of $C(S^1) \otimes A$ whose restrictions to $1 \otimes A \subseteq C(S^1) \otimes A$ are split.

5.6.8 Let $L^2(\mathbb{D})$ be the Hilbert space of square-integrable functions on the open unit disk $\mathbb{D}$ in $\mathbb{C}$. Let A be the C*-algebra of continuous functions on the closed disk and let J be the ideal of functions in A which vanish on the boundary. Represent A on $L^2(\mathbb{D})$ by pointwise multiplication. Show that the projection P onto the Bergman subspace $H^2(\mathbb{D}) \subseteq L^2(\mathbb{D})$ (see Exercise 2.9.12) belongs to the relative dual $\mathfrak{D}(A/\!\!/J)$.

5.6.9 Continuing Exercise 5.6.8, show that the complementary projection $Q = I - P$ is ample. Show that within $\mathfrak{D}(A)$ the projection Q is Murray–von Neumann equivalent to I (this is quite difficult; one approach is to study the unbounded operator $\partial/\partial\bar{z}$ on the disk).

5.6.10 Show that if X is a finite tree then $\operatorname{Ext}(X) = 0$. Can you compute $\operatorname{Ext}(X)$ for every finite graph X?

5.6.11 The purpose of this exercise is to prove that every unitary element in the quotient algebra $\mathfrak{D}(A)/\mathfrak{D}(A/\!\!/J)$ lifts to a partial isometry in $\mathfrak{D}(A)$.

(a) Show that it suffices to consider the case where A is represented on the Hilbert space $H'_{\tilde{A}} = H_{\tilde{A}} \oplus H_{\tilde{A}} \oplus H$ which was used in the second half of the proof of 5.3.7. Operators in $\mathfrak{D}(A)$ can therefore be represented by certain 3×3 matrices.

(b) Show that any element $x \in \mathfrak{D}(A)/\mathfrak{D}(A/\!\!/J)$ can be lifted to a matrix of the form

$$\begin{pmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in $\mathfrak{D}(A)$. Using the functional calculus, show that if $\|x\| \leqslant 1$ then we may assume that $\|T\| \leqslant 1$.

(c) Show that if x is unitary and $\|T\| \leqslant 1$ (where T is as above) then the operator

$$X = \begin{pmatrix} T & 0 & 0 \\ (1 - T^*T)^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a partial isometry in $\mathfrak{D}(A)$ which lifts x.

5.6.12 Prove that every projection in the quotient algebra $\mathfrak{D}(A)/\mathfrak{D}(A/\!\!/J)$ lifts to a projection in $\mathfrak{D}(J)$.

5.7 Notes

The idea of interpreting the BDF groups as K-theory groups of a 'dual' C*-algebra is due to Paschke [102], and the version of the theory described in this chapter was worked out by Higson [68, 70].

Relative K-homology was developed by Baum and Douglas [24, 25] in the course of their study of Toeplitz index theory.

The term 'pseudolocal' which we introduced in Definition 5.4.6 is borrowed from distribution theory, where one says that a linear operator is pseudolocal if its Schwartz kernel is a smooth function on the complement of the diagonal. If a bounded Hilbert space operator T is pseudolocal in this sense, and if f and g are smooth functions with disjoint supports, then fTg is a smoothing operator — and therefore is a compact operator. Every order zero pseudodifferential operator is pseudolocal in the distributional sense and hence in the sense of Definition 5.4.6.

6

COARSE GEOMETRY AND K-HOMOLOGY

Let A and B be C^*-algebras. Recall that two $*$-homomorphisms $\alpha_0, \alpha_1 \colon A \to B$ are *homotopic* if they can be connected by a point-norm continuous family of $*$-homomorphisms $\alpha_t \colon A \to B$, $t \in [0,1]$. It is a simple fact (Proposition 4.4.3) that the K-theory functor is homotopy invariant, which is to say that α_0 and α_1 induce the same homomorphisms $K_p(A) \to K_p(B)$.

It is a deep theorem that the K-homology functor is homotopy invariant in the analogous sense, namely that α_0 and α_1 induce the same homomorphisms $K^p(B) \to K^p(A)$. There are a number of very different proofs of this result, of which the most general and perhaps the most satisfactory is that due to Kasparov, which we shall present in Chapter 8. At the end of this chapter we shall give an alternative proof (for commutative C^*-algebras B), which is based on a remarkable connection between K-homology and the 'coarse' geometry of non-compact spaces.

Coarse geometry is a rich source of interesting C^*-algebras. We begin the chapter with a rather general notion of coarse structure, and then compute the K-theory groups of the C^*-algebras associated to several key examples of such structures. It is these computations which will yield our proof of the homotopy invariance of K-homology. In Chapter 12 we shall touch on another significant application of coarse geometry: the index theory of open manifolds.

6.1 Coarse Structures

Let X be a metric space, with metric d. For a topologist, the significance of the metric lies in the collection of open sets it generates. This passage from the metric to its associated topology loses a good deal of information; in fact only the 'very small scale structure' of the metric is reflected in the topology. For example, the metric

$$d'(x,y) = \min\{d(x,y), 1\}$$

defines the same topology as d.

There is a dual procedure, in which one studies 'very large scale structure'. To formalize this notion let us begin with the following definition.

6.1.1 DEFINITION Let X be a metric space and let S be any set. Two maps $p_1, p_2 \colon S \to X$ are *close* if $\sup_{s \in S} d(p_1(s), p_2(s)) < \infty$.

Closeness is an equivalence relation. If X is of finite diameter then any two maps are close; more generally, closeness depends only on the large-scale properties of the metric. For example, the metric

$$d''(x,y) = \max\{1, d(x,y)\} \quad (\text{if } x \neq y)$$

defines the same closeness relation as d. We shall now abstract the key properties of the closeness relation.

6.1.2 DEFINITION Let X be a set. By a *coarse structure* on X we mean the provision, for each set S, of an equivalence relation (called 'being close') on the set of maps from S to X. These equivalence relations must have the following properties:

(a) if $p_1, p_2 \colon S \to X$ are close and $q \colon S' \to S$ is any map, then $p_1 \circ q$ and $p_2 \circ q$ are close;

(b) if $S = S' \cup S''$ and if $p_1, p_2 \colon S \to X$ are maps whose restrictions to both S' and S'' are close, then p_1 and p_2 are close;

(c) any two constant maps are close to each other.

A set equipped with a coarse structure will be called a *coarse space.*

6.1.3 DEFINITION Let X be a coarse space and let S be a subset of $X \times X$. The set S is called *controlled* if the two coordinate maps

$$\pi_1, \pi_2 \colon S \subseteq X \times X \to X$$

are close. A collection $\mathcal{U}$ of subsets of X is *uniformly bounded* if $\bigcup_{U \in \mathcal{U}} U \times U$ is controlled.

The coarse structure determines the controlled sets. Conversely, the controlled sets determine the coarse structure:

6.1.4 PROPOSITION *Let X be a coarse space. Two maps $p_1, p_2 \colon S \to X$ are close if and only if the image of $p = (p_1, p_2) \colon S \to X \times X$ is controlled.*

PROOF Let $E = \text{Image}(p)$. It is clear from part (a) of Definition 6.1.2 that if E is controlled, then p_1 and p_2 are close. Conversely, suppose that p_1 and p_2 are close. Choose $r \colon E \to S$ such that $p \circ r$ is the identity on E. Then $p_1 \circ r$ and $p_2 \circ r$ are close. But $p_i \circ r = \pi_i \circ p \circ r = \pi_i$. Thus π_1 and π_2 are close on E, so E is controlled. $\square$

6.1.5 DEFINITION A subset $B \subseteq X$ is *bounded* if the one-element family $\{B\}$ is uniformly bounded, that is, if $B \times B$ is controlled.

6.1.6 LEMMA *A non-empty subset* B *of a coarse space* X *is bounded if and only if the inclusion* $B \to X$ *is close to a constant map.*

PROOF Suppose that B is bounded. Then $\pi_1, \pi_2 \colon B \times B \to X$ are close. Choose a 'slice map' $j \colon B \to B \times B$ such that $\pi_1 \circ j$ is the inclusion and $\pi_2 \circ j$ is constant. It follows from part (a) of Definition 6.1.2 that the inclusion is close to a constant map.

Suppose conversely that the inclusion $B \to X$ is close to a constant map. Since $\pi_1 \colon B \times B \to X$ factors through the inclusion $B \to X$, it too is close to a constant map. Similarly, π_2 is close to a constant map. Hence (by part (c) of Definition 6.1.2), π_1 and π_2 are close. □

In our examples, the coarse space X will in addition be equipped with a locally compact topology. It will be natural to require a certain compatibility between the coarse structure and the topology.

6.1.7 DEFINITION Let X be a locally compact space. A coarse structure on X is *proper* if

(a) X has a uniformly bounded open cover, and
(b) every bounded subset of X has compact closure.

These properties imply that the bounded subsets of X are precisely the ones having compact closure. From now on we shall consider only proper coarse structures.

6.1.8 REMARK By our definition, a coarse structure is a huge collection of mutually compatible equivalence relations. The wastefulness of the definition can be reduced by restricting our attention to maps from S to X for some *fixed*, sufficiently large set S. We may then say in general that two maps $p_1, p_2 \colon S' \to X$ are close if the composites $p_1 \circ q$ and $p_2 \circ q$ are close for every map $q \colon S \to S'$ (Exercise 6.7.1). In fact, we can usually take S to be countable; see Lemma 6.1.13 below.

6.1.9 EXAMPLE If X is a metric space then Definition 6.1.1 gives a coarse structure on X, which we call its *metric* coarse structure. This coarse structure is proper if the metric space X is proper, that is, if closed balls in X are compact.

6.1.10 EXAMPLE Suppose that the locally compact Hausdorff space X is equipped with a *metrizable compactification* — that is, a compact metrizable space $\overline{X}$ containing X as an open dense subset. Let $\partial X = \overline{X} \setminus X$. Then we may define a coarse structure as follows: two maps $p_1, p_2 \colon S \to X$ are close if, whenever $\{s_n\}_{n=1}^{\infty}$ is a sequence in S such that one of the sequences $\{p_1(s_n)\}$ or $\{p_2(s_n)\}$ converges to a point of ∂X, both sequences converge to the same point. This is called the *topological* coarse structure associated to the given compactification. It is always proper (Exercise 6.7.2).

According to our definition, the topological coarse structure on $X \subseteq \overline{X}$ is determined by the closeness relation on *sequences* in X. This property is shared by nearly all the coarse structures that we shall consider, in fact by those which are *separable* in the sense of the next definition.

6.1.11 DEFINITION A coarse space X is *separable* if it has a countable uniformly bounded cover.

6.1.12 REMARK Suppose that X is a locally compact metrizable space with a proper coarse structure. If X is separable in the usual topological sense, then it is separable in the coarse sense just defined. For there is a uniformly bounded open cover $\mathcal{U}$; and every open cover of a separable metrizable space has a countable subcover.

If a coarse space X is separable, then every map $S \to X$ is close to one with countable image. Since according to Proposition 6.1.4 the closeness of p_1 and p_2 is a property of $\mathrm{Image}(p_1, p_2) \subseteq X \times X$, we obtain:

6.1.13 LEMMA *Let* X *be a separable coarse space. Two maps* $p_1, p_2 \colon S \to X$ *are close if and only if the sequences* $\{p_1(s_n)\}$ *and* $\{p_2(s_n)\}$ *are close for every sequence* $\{s_n\}$ *in* S. □

6.1.14 DEFINITION Let X_1 and X_2 be coarse spaces. A function $q \colon X_1 \to X_2$ is called a *coarse map* if

(a) whenever p and p′ are close maps into X_1, the composites $q \circ p$ and $q \circ p'$ are close maps into X_2, and
(b) q is proper in the sense that, for every bounded subset $B \subseteq X_2$, $q^{-1}(B)$ is bounded in X_1.

6.1.15 EXAMPLE Suppose that $\mathbb{R}^2$ and $\mathbb{R}$ are given their natural metric coarse structures. Then the absolute value map $v \mapsto |v|$, from $\mathbb{R}^2$ to $\mathbb{R}$, is a coarse map, but the coordinate projections from $\mathbb{R}^2$ to $\mathbb{R}$ are not.

6.1.16 EXAMPLE Let M be a complete, simply connected Riemannian manifold of non-positive curvature. Let $p \in M$ and let V denote the Euclidean space T_pM. By a classical theorem of differential geometry (the Cartan–Hadamard Theorem), the exponential map

$$\exp \colon V \to M$$

is a diffeomorphism, and its inverse $\log \colon M \to V$ decreases all distances. It follows that log is a coarse map for the metric coarse structures. This observation will be important in Chapter 12.

6.1.17 DEFINITION We say that X_1 and X_2 are *coarsely equivalent* if there exist coarse maps $q: X_1 \to X_2$ and $r: X_2 \to X_1$ such that $q \circ r$ and $r \circ q$ are close to the identity maps on X_2 and X_1 respectively. The map q is called a *coarse equivalence*.

6.1.18 EXAMPLE Suppose that $\mathbb{Z}$ and $\mathbb{R}$ are given their natural metric coarse structures. Then the inclusion map $\mathbb{Z} \subseteq \mathbb{R}$ is a coarse equivalence. An inverse up to closeness is the integer part map, which sends each real number x to the greatest integer $\leqslant x$.

6.2 Coarse Geometry of Cones

We shall be interested in coarse structures on *cones*. Let Y be a compact metrizable space. The *closed cone* on Y is defined to be the compact space $\mathcal{C}Y$ obtained from $[0,1] \times Y$ by collapsing $\{0\} \times Y$ to a point. Similarly, the *open cone* on Y is the space $\mathcal{O}Y$ obtained from $[0,1) \times Y$ by collapsing $\{0\} \times Y$ to a point. It is a dense open subset of the compact space $\mathcal{C}Y$. By considering $\mathcal{C}Y$ as a compactification of $\mathcal{O}Y$ we can therefore give $\mathcal{O}Y$ a topological coarse structure. We shall use the notation $\mathcal{O}_c Y$ for the open cone equipped with this coarse structure.

There are also metric coarse structures on $\mathcal{O}Y$. Suppose that Y is a compact subset of the unit sphere in a real Hilbert space E (it is a classical theorem of topology that every compact metrizable space is homeomorphic to such a subset). Let φ be a homeomorphism of $[0,1)$ onto $[0,\infty)$. Then the map

$$\mathcal{O}Y \to E, \qquad (t,y) \mapsto \varphi(t)y$$

identifies $\mathcal{O}Y$ with a certain subset of the metric space E (namely the union of all the rays from the origin in E passing through points of Y) and so defines a metric d_φ on $\mathcal{O}Y$. We use the notation $\mathcal{O}_\varphi Y$ for the open cone equipped with this metric coarse structure, noting explicitly its dependence on the choice of the homeomorphism φ. It is proper since Y is compact.

How are these coarse structures related? Note that there is a partial order on the set of separable coarse structures on a given space: if $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are coarse structures on X we may say that $\mathfrak{C}_1 \preceq \mathfrak{C}_2$, or that $\mathfrak{C}_2$ is *coarser* than $\mathfrak{C}_1$, if the identity map $X_{\mathfrak{C}_1} \to X_{\mathfrak{C}_2}$ is a coarse map. That is to say, $\mathfrak{C}_1 \preceq \mathfrak{C}_2$ if

(a) every $\mathfrak{C}_1$-controlled set is $\mathfrak{C}_2$-controlled, and

(b) every $\mathfrak{C}_2$-bounded set is $\mathfrak{C}_1$-bounded.

For example, if $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are the metric coarse structures associated to proper metrics d_1 and d_2 on a locally compact space X, and if $d_2 \leqslant d_1$, then $\mathfrak{C}_1 \preceq \mathfrak{C}_2$.

6.2.1 PROPOSITION *Let Y be a compact subset of the unit sphere of a Hilbert space E. Then*

(a) *for every homeomorphism* $\varphi\colon [0,1) \to [0,\infty)$, *every controlled set for* $\mathcal{O}_\varphi Y$ *is also controlled for* $\mathcal{O}_c Y$, *and*
(b) *for every controlled set for* $\mathcal{O}_c Y$, *there exists a homeomorphism* $\varphi\colon [0,1) \to [0,\infty)$ *for which that set is controlled for* $\mathcal{O}_\varphi Y$.

Consequently, the coarse structure $\mathcal{O}_c Y$ *is the least upper bound of the coarse structures* $\mathcal{O}_\varphi Y$, *as* φ *runs over the set of all homeomorphisms* $[0,1) \to [0,\infty)$.

PROOF Since all the coarse structures involved are proper they have the same bounded sets, and since they are separable we can compare them by considering only the closeness properties of sequences (Lemma 6.1.13).

First we shall show that $\mathcal{O}_\varphi Y \preceq \mathcal{O}_c Y$ for any φ. Let $\{x_n\}_{n=1}^\infty$ and $\{x'_n\}_{n=1}^\infty$ be sequences in $X = \mathcal{O}Y$ which are close for the coarse structure $\mathcal{O}_\varphi Y$, and suppose that x_n converges to a point $y \in \partial X = Y$. To show that $\{x_n\}_{n=1}^\infty$ and $\{x'_n\}_{n=1}^\infty$ are close for the coarse structure $\mathcal{O}_c Y$, we must show that x'_n converges to the same point y. Let us write $x_n = t_n y_n$, where $t_n \to 1$, $y_n \to y$, and $x'_n = t'_n y'_n$. Since $t_n \to 1$, it follows that $\varphi(t_n) \to \infty$. Since $|\varphi(t'_n) - \varphi(t_n)| \leqslant d_\varphi(x_n, x'_n)$, which is uniformly bounded, it follows that $\varphi(t'_n) \to \infty$ and hence that $t'_n \to 1$. Now the inequality

$$d_Y(y_n, y'_n) \leqslant \frac{d_\varphi(x_n, x'_n)}{\min\{\varphi(t_n), \varphi(t'_n)\}}$$

shows that $d_Y(y_n, y'_n) \to 0$, so that y'_n converges to y and thus x'_n converges to y also.

The first step in proving that $\mathcal{O}_c Y$ is the *least* upper bound of the coarse structures $\mathcal{O}_\varphi Y$ is the following 'calculus lemma', whose proof we leave to the reader:

6.2.2 LEMMA *Let* $\{t_n\}$, $\{t'_n\}$ *be two sequences in* $[0,1)$ *both of which converge to 1. Then there is a homeomorphism* $\psi\colon [0,1) \to [0,\infty)$ *such that* $|\psi(t_n) - \psi(t'_n)| < 1$ *for all* n. □

Returning to the proof of Proposition 6.2.1, it suffices to show that if $\{x_n = t_n y_n\}$ and $\{x'_n = t'_n y'_n\}$ are two sequences in $X = \mathcal{O}Y$ which converge to the same point of $\partial X = Y$ (so that $\{x_n\}$ and $\{x'_n\}$ are close in $\mathcal{O}_c Y$), then there is a φ for which $d_\varphi(x_n, x'_n)$ is bounded (so that $\{x_n\}$ and $\{x'_n\}$ are close in $\mathcal{O}_\varphi Y$).

Apply the calculus lemma to the sequences $\{t_n\}$ and $\{t'_n\}$, and let $\psi\colon [0,1) \to [0,\infty)$ be the homeomorphism so produced. Let $s_n = \min\{t_n, t'_n\}$, so that $\psi(s_n) = \min\{\psi(t_n), \psi(t'_n)\}$, and let $w_n = s_n y_n$, $w'_n = s_n y'_n$. By the characteristic property of ψ, $\{w_n\}$ is close in $\mathcal{O}_\psi Y$ to $\{x_n\}$, and similarly $\{w'_n\}$ is close in $\mathcal{O}_\psi Y$ to $\{x'_n\}$. What is more, since $\{y_n\}$ and $\{y'_n\}$ converge to the same point,

$$\frac{d_\psi(w_n, w'_n)}{\psi(s_n)} \to 0.$$

Choose now a sequence $\{r_j\}$ increasing to infinity, with $|r_j - r_{j-1}| > 1$, such that the following implication is true:

$$\psi(s_n) \geqslant r_j - 1 \quad \Rightarrow \quad \frac{d_\psi(w_n, w_n')}{\psi(s_n)} < \frac{1}{j}.$$

Let $\chi\colon [0,\infty) \to [0,\infty)$ be a piecewise linear homeomorphism taking r_j to j. Then by construction the sequences $\{w_n\}$ and $\{w_n'\}$ are close in $\mathcal{O}_{\chi\psi}Y$. Moreover, the homeomorphism χ has gradient $\leqslant 1$, and therefore $\mathcal{O}_\psi Y \preceq \mathcal{O}_{\chi\psi}Y$. Thus $\{x_n\}$ is close to $\{w_n\}$ in $\mathcal{O}_{\chi\psi}Y$, and similarly $\{x_n'\}$ is close to $\{w_n'\}$. Putting $\varphi = \chi\psi$, we deduce finally that $\{x_n\}$ and $\{x_n'\}$ are close in $\mathcal{O}_\varphi Y$, as was required. □

6.2.3 Remark It will be useful to note that the partially ordered set of coarse structures $\mathcal{O}_\varphi Y$ is *directed* (which means that any two elements have a common upper bound). One can prove this by making use once again of Lemma 6.2.2. Indeed, let $\varphi, \varphi'\colon [0,1) \to [0,\infty)$ be two homeomorphisms. Define sequences $\{t_n\}$ and $\{t_n'\}$ by $\varphi(t_n) = n = \varphi'(t_n')$, and apply the lemma to obtain a homeomorphism $\psi\colon [0,1) \to [0,\infty)$ such that $|\psi(t_n) - \psi(t_n')| \leqslant 1$. Let $\chi\colon [0,\infty) \to [0,\infty)$ be a piecewise linear homeomorphism of gradient $\leqslant 1$ such that $\chi(\psi(t_n)) \leqslant n$. Then

$$\mathcal{O}_\varphi Y \preceq \mathcal{O}_{\chi\psi}Y, \quad \mathcal{O}_{\varphi'}Y \preceq \mathcal{O}_{\chi\psi}Y$$

so $\mathcal{O}_{\chi\psi}Y$ will serve as the required upper bound.

6.3 The C*-Algebra of a Coarse Space

Let X be a locally compact, separable and metrizable space equipped with a proper coarse structure; for brevity we shall refer to such an X simply as a *proper separable coarse space*. In the examples of greatest interest to us, X has a proper metric coarse structure or a topological coarse structure coming from a metrizable compactification. Suppose that $C_0(X)$ is represented non-degenerately on a (separable) Hilbert space H; we shall use the notation $\rho\colon C_0(X) \to \mathfrak{B}(H)$ for the representation, but in accordance with our usual practice we shall omit mention of ρ wherever possible.

6.3.1 Definition Let $v \in H$. The *support* of v is the complement, in X, of the union of all open subsets $U \subseteq X$ such that $\rho(f)v = 0$ for all $f \in C_0(U)$.

We shall construct a C*-algebra of operators on H which reflects the coarse structure of X, by making use of the following two notions.

6.3.2 Definition An operator $T \in \mathfrak{B}(H)$ is *locally compact* on X if fT and Tf are compact operators for all functions $f \in C_0(X)$.

Notice that this is a special case of Remark 5.3.3.

6.3.3 DEFINITION The *support* of an operator $T \in \mathfrak{B}(H)$ is the complement, in $X\times X$, of the union of all open subsets $U\times V \subseteq X\times X$ such that $\rho(f)T\rho(g) = 0$ for all $f \in C_0(U)$ and $g \in C_0(V)$. More generally, if $C_0(X)$ and $C_0(Y)$ are represented on Hilbert spaces H_X and H_Y, then the *support* of a bounded operator $T\colon H_X \to H_Y$ is the complement, in $Y \times X$, of the union of all open sets $U \times V \subseteq Y \times X$ such that $\rho_Y(f)T\rho_X(g) = 0$, for all $f \in C_0(U)$ and $g \in C_0(V)$.

6.3.4 EXAMPLE If T is an integral operator $Tf(x) = \int_X k(x,x')f(x')dx'$, where k is, say, a continuous kernel, and if the measure dx on X is positive on every non-empty open set, then the support of T is the same as the support of k.

6.3.5 DEFINITION Let X be a proper separable coarse space, and suppose that $C_0(X)$ is represented non-degenerately on a Hilbert space H. An operator $T \in \mathfrak{B}(H)$ is *controlled* if $\mathrm{Support}(T)$ is a controlled subset of $X \times X$.

There is a simple calculus of supports. For subsets $A \subseteq Y \times X$ and $B \subseteq X$, denote by $A \circ B$ the subset

$$\{y \in Y : \exists x \in X \text{ such that } (x,y) \in A \text{ and } y \in B\}.$$

Then we have:

6.3.6 LEMMA *For a bounded operator* $T\colon H_X \to H_Y$ *as in Definition* 6.3.3 *above,*

$$\mathrm{Support}(Tv) \subseteq \mathrm{Support}(T) \circ \mathrm{Support}(v)$$

for every compactly supported $v \in H$. *Moreover,* $\mathrm{Support}(T)$ *is the smallest closed subset of* $Y \times X$ *that has this property.*

PROOF Suppose that $y \notin \mathrm{Support}(T) \circ \mathrm{Support}(v)$. Then the closed subsets $\mathrm{Support}(v)$ and $\{x \in X : (y,x) \in \mathrm{Support}(T)\}$ of X are disjoint, so there exists a function $g \in C_0(X)$ equal to 1 on the first subset and equal to 0 on the second. By definition of $\mathrm{Support}(T)$, together with a compactness argument, there exists an open subset $U \subseteq Y$ such that $y \in U$ and $fTg = 0$ for every function $f \in C_0(U)$. But now the identity

$$fTv = fTgv + fT(1-g)v = 0$$

shows that $y \notin \mathrm{Support}(Tv)$. This proves the first statement. As for the second, if $(y,x) \in \mathrm{Support}(T)$, then we can find sequences $u_n \in H_Y$ and $v_n \in H_X$ with $\mathrm{Support}(u_n) \subseteq B(y;\frac{1}{n})$ (with respect to some metric defining the topology), $\mathrm{Support}(v_n) \subseteq B(x;\frac{1}{n})$, and $\langle u_n, Tv_n\rangle \neq 0$; this shows that $(x,y) \in A$ for any closed $A \subseteq Y \times X$ for which $\mathrm{Support}(Tv) \subseteq A \circ \mathrm{Support}(v)$. $\square$

6.3.7 PROPOSITION *Let* X *be a proper separable coarse space, and suppose that* $C_0(X)$ *is represented non-degenerately on a Hilbert space* H. *The collection of controlled operators is a unital $*$-subalgebra of* $\mathfrak{B}(H)$, *and the locally compact and controlled operators form a $*$-ideal in it.*

PROOF A controlled operator T is *proper*, which means that if v is compactly supported, then so is Tv. It then follows from Lemma 6.3.6 that if S and T are controlled operators,

$$\begin{aligned}\mathrm{Support}(ST) &\subseteq \mathrm{Support}(S)\circ\mathrm{Support}(T)\\ &= \{(z,x) : \exists y,\ (z,y)\in\mathrm{Support}(S),\ (y,x)\in\mathrm{Support}(T)\}.\end{aligned}$$

Using the axioms for a coarse structure one sees that this set is controlled. The remainder of the proof is an elementary exercise. □

6.3.8 DEFINITION Let X be a proper separable coarse space and let $\rho\colon C_0(X)\to\mathfrak{B}(H_X)$ be a non-degenerate representation. The C^*-algebra $C^*_\rho(X)$ is the norm closure of the algebra of locally compact, controlled operators on H_X.

Once again we shall usually omit mention of the representation ρ. In fact we shall prove, in Proposition 6.3.12 below, a 'Voiculescu-type' result which implies that the K-theory of the algebra $C^*_\rho(X)$ does not depend on the choice of ρ, provided that ρ is ample. We shall use such a result to make $K_p(C^*(X))$ functorial in X.

6.3.9 DEFINITION Let X and Y be proper coarse spaces and suppose that $C_0(X)$ and $C_0(Y)$ are non-degenerately represented on Hilbert spaces H_X and H_Y. Let $q\colon X\to Y$ be a coarse map. A bounded operator $V\colon H_X\to H_Y$ *covers* q if the maps π_1 and $q\circ\pi_2$, from $\mathrm{Support}(V)\subseteq Y\times X$ to Y, are close.

For example, a controlled operator is one which covers the identity map.

6.3.10 REMARK If S covers q, then it also covers any coarse map close to q.

6.3.11 LEMMA *Let* X *and* Y *be proper separable coarse spaces and suppose that* $C_0(X)$ *and* $C_0(Y)$ *are non-degenerately represented on Hilbert spaces* H_X *and* H_Y. *Let* $q\colon X\to Y$ *be a coarse map. If* $V\colon H_X\to H_Y$ *is an isometry which covers* q, *then the $*$-homomorphism* $\mathrm{Ad}_V(T)=VTV^*$ *maps* $C^*(X)$ *to* $C^*(Y)$.

This is a coarse version of Lemma 5.2.3.

PROOF Let $T\in C^*(X)$. To check that VTV^* is controlled, let $S\subseteq Y\times X\times X\times Y$ be the set of 4-tuples (y,x,x',y') such that $(y,x)\in\mathrm{Support}(V)$, $(x,x')\in\mathrm{Support}(T)$, and $(x',y')\in\mathrm{Support}(V^*)$. Let π_i, $i=1,\ldots,4$, denote the coordinate projections of S. Then, since V covers q, the maps π_1 and $q\circ\pi_2$, from S to Y, are close, and similarly so are the maps $q\circ\pi_3$ and π_4. Since T is controlled the maps π_2 and π_3 are close, and since q is coarse this implies that the maps $q \circ \pi_2$ and $q \circ \pi_3$ are close. Since closeness is transitive, π_1 and π_4 are close on S. But $\mathrm{Support}(VTV^*) \subseteq (\pi_1, \pi_4)(S)$, so VTV^* is controlled.

To show that VTV^* is locally compact, we shall make use several times of the fact that, for a proper coarse structure, the bounded sets are exactly those having compact closure. Let f be a function of compact support on Y; then fV covers q since V does. The map π_1 on $\mathrm{Support}(fV)$ has bounded range, and hence so does $q \circ \pi_2$. Thus $\pi_2(\mathrm{Support}(fV))$ is bounded, by the properness of q, and so there exists a function f' of compact support on X with $fV = fVf'$. But now $fVTV^* = (fV)(f'T)V^*$ is compact since $f'T$ is. Arguing similarly for VTV^*f, we find that VTV^* is locally compact. □

Following the pattern of Chapter 5, we proceed to give a coarse counterpart to Lemma 5.2.4.

6.3.12 PROPOSITION *Let X and Y be proper separable coarse spaces. Suppose that $C_0(X)$ and $C_0(Y)$ are amply represented on Hilbert spaces H_X and H_Y. Then every coarse map $q\colon X \to Y$ is covered by an isometry from H_X to H_Y, and any two such isometries, both covering q, induce the same map on K-theory:*

$$(\mathrm{Ad}_{V_1})_* = (\mathrm{Ad}_{V_2})_* \colon K_p(C^*(X)) \to K_p(C^*(Y)).$$

In particular (taking q to be the identity map) we have:

6.3.13 COROLLARY *Let X be a proper separable coarse space, and let ρ and ρ' be two ample representations of $C_0(X)$. Then the K-theory groups $K_p(C^*_\rho(X))$ and $K_p(C^*_{\rho'}(X))$ are canonically isomorphic.* □

It is in this sense that the K-theory of $C^*(X)$ does not depend on the choice of ample representation.

PROOF OF PROPOSITION 6.3.12 The uniqueness proof is the same as the corresponding part of the proof of Lemma 5.2.4, and is omitted. For the existence proof we must produce a coarse counterpart to Voiculescu's Theorem. We begin by constructing a nice Borel decomposition of Y.

6.3.14 CLAIM *Let Y be a proper separable coarse space. Then Y can be written as the disjoint union of a countable, uniformly bounded collection of Borel subsets each having non-empty interior.*

In principle, the proof of Claim 6.3.14 is simple enough: take a countable uniformly bounded open cover, and inductively thin down its members so as to make them disjoint from their predecessors. But one must be careful not to thin down any set too much, or it may lose all its interior. We defer discussion of the messy details. At any rate, let $\{Y_n\}$ be a decomposition of the sort provided by 6.3.14, and let Q_n be the projection operator on H_Y which corresponds (under the Borel functional calculus 1.5.7) to the characteristic function of Y_n. Since H_Y is ample and each Y_n has non-empty interior, each of the operators Q_n has infinite-dimensional range. Moreover the ranges of the projections Q_n are mutually orthogonal.

Let P_n be the projection on H_X corresponding to the characteristic function of $q^{-1}(Y_n)$. Then the P_n are mutually orthogonal, and since the representation H_X is non-degenerate, the sum $\sum P_n$ converges strongly to I. There exist partial isometries $V_n\colon H_X \to H_Y$ mapping the range of P_n isometrically into the range of Q_n. Put

$$V = \sum V_n = \sum Q_n V_n P_n;$$

this series converges strongly to an isometry $H_X \to H_Y$ which covers q. □

Proof of the Claim Let $\{U_n\}_{n=1}^{\infty}$ be a countable uniformly bounded open cover of Y. We shall need to know that the cover $\{\overline{U}_n\}_{n=1}^{\infty}$ is also uniformly bounded; this is a general fact (the closure of a uniformly bounded cover is uniformly bounded) for which the reader is asked to supply a proof in Exercise 6.7.4. Put $V_1 = U_1$ and

$$V_n = U_n \setminus (U_1 \cup \cdots \cup U_{n-1});$$

then the V_n form a countable uniformly bounded family of disjoint Borel sets covering Y. Some of the sets V_n may however have empty interior. Discard these and let $V_{n_1}, V_{n_2}, \ldots$ be the sets that remain. The closures $\overline{V}_{n_i}$ of the sets V_{n_i} cover Y, since any point y belonging to one of the discarded sets V_n must be the limit of a sequence belonging to some V_m for $m < n$. Finally put $W_1 = \overline{V}_{n_1}$ and

$$W_i = \overline{V}_{n_i} \setminus (\overline{V}_{n_1} \cup \cdots \cup \overline{V}_{n_{i-1}}).$$

Then each W_i has non-empty interior and the disjoint family $\mathcal{W} = \{W_i\}_{i=1}^{\infty}$ covers Y. Moreover $\mathcal{W}$ is uniformly bounded, since $W_i \subseteq \overline{U}_{n_i}$. □

For each proper separable coarse space X let us fix once and for all[48] an ample representation H_X of $C_0(X)$, and let us use this representation in forming the C^*-algebra $C^*(X)$.

6.3.15 Definition If $q\colon X \to Y$ is a coarse map then we define

$$q_*\colon K_p(C^*(X)) \to K_p(C^*(Y))$$

to be the map $(\mathrm{Ad}_{V_q})_*$, where $V_q\colon H_X \to H_Y$ is any isometry that covers q.

[48]We are still not going to worry about any set-theoretic difficulties attendant to making one such choice for each X.

6.3.16 PROPOSITION *If* $q_1, q_2 \colon X \to Y$ *are close, then* $q_{1*} = q_{2*} \colon K_p(C^*(X)) \to K_p(C^*(Y))$. *In particular, a coarse equivalence induces an isomorphism on* K*-theory.*

PROOF If V covers q_1 then it also covers q_2, by Remark 6.3.10. □

Since $V_{q_1} V_{q_2}$ is an isometry that covers $q_1 q_2$, we have:

6.3.17 LEMMA *The correspondence* $q \mapsto q_*$ *is a covariant functor from the category of proper separable coarse spaces and coarse maps to the category of abelian groups and homomorphisms.* □

These facts closely parallel the corresponding discussion for dual algebras and K-homology in Section 5.2. As we shall shortly see, the connection between dual algebras and coarse structures goes deeper than formal similarity.

6.4 K-Theory for Metric Coarse Structures

In this section we are going to compute $K_p(C^*(X))$ for certain metric coarse structures on spaces X, including the standard structure on n-dimensional Euclidean space. In the next section we shall use rather different techniques to compute $K_p(C^*(X))$ for certain topological coarse structures. The first lemma, however, applies in general:

6.4.1 LEMMA *If the proper separable coarse space* X *is also compact then*

$$K_p(C^*(X)) \cong \begin{cases} \mathbb{Z} & \text{if } p = 0, \\ 0 & \text{if } p = 1. \end{cases}$$

PROOF Every operator on X is controlled, so $C^*(X) = \mathfrak{K}(H_X)$ and the result follows. □

Suppose that X is a proper metric space. In particular, it is a proper separable coarse space. The study of metric coarse structures is simplified by the existence of a numerical measure of the size of the support of an operator T on H_X: the *propagation* of T is

$$\operatorname{Prop}(T) = \sup\{d(x,y) : (x,y) \in \operatorname{Support}(T)\}.$$

The metrically controlled operators on H_X are then just those of finite propagation.

Now we shall compute the coarse K-theory for a ray $\mathbb{R}^+ = \{t \in \mathbb{R} : t \geqslant 0\}$.

6.4.2 LEMMA *For all* p *we have*

$$K_p(C^*(\mathbb{R}^+)) = 0,$$

using the Euclidean metric coarse structure.

PROOF Let ρ be the representation of $C_0(\mathbb{R}^+)$ by multiplication operators on $H = L^2(\mathbb{R}^+)$, and let $H' = H \oplus H \oplus H \oplus \cdots$ be the direct sum of infinitely many copies of H, with corresponding representation ρ'. Let V be the inclusion of H as the first summand in H'. Then both ρ and ρ' are ample representations, and V is an isometry covering the identity map. It follows from Proposition 6.3.12 that the $*$-homomorphism $\alpha_1 = \mathrm{Ad}_V$ induces an isomorphism

$$\alpha_{1*}\colon K_p(C^*_\rho(\mathbb{R}^+))) \to K_p(C^*_{\rho'}(\mathbb{R}^+)).$$

We shall show that this induced isomorphism is the zero map. It will follow that $K_p(C^*(\mathbb{R}^+)) = 0$.

Let $U\colon H \to H$ be the right-translation isometry given by

$$Uf(t) = \begin{cases} f(t-1) & \text{if } t \geqslant 1, \\ 0 & \text{if } 0 \leqslant t < 1. \end{cases}$$

Define a $*$-homomorphism $\alpha_2\colon \mathfrak{B}(H) \to \mathfrak{B}(H')$ by the formula

$$\alpha_2(T) = 0 \oplus \mathrm{Ad}_U(T) \oplus \mathrm{Ad}_U^2(T) \oplus \mathrm{Ad}_U^3(T) \oplus \cdots .$$

We make two claims about this $*$-homomorphism:

(a) if T is metrically controlled, then $\alpha_2(T)$ is metrically controlled, and

(b) if T is locally compact, then $\alpha_2(T)$ is locally compact.

Thus α_2 restricts to a $*$-homomorphism $C^*_\rho(\mathbb{R}^+) \to C^*_{\rho'}(\mathbb{R}^+)$.

To prove (a), notice that the support of $\mathrm{Ad}_U(T)$ is just a translate of the support of T, and that therefore T and $\mathrm{Ad}_U(T)$ have the same propagation. The propagation of the direct sum

$$\alpha_2(T) = 0 \oplus \mathrm{Ad}_U(T) \oplus \mathrm{Ad}_U^2(T) \oplus \mathrm{Ad}_U^3(T) \oplus \cdots$$

is the supremum of the propagations of the individual summands, and hence is also equal to the propagation of T.

To prove (b), suppose that T is locally compact. Let f be a compactly supported function on $\mathbb{R}^+$. Then there is an integer N such that $\rho(f)U^n = 0$ for $n > N$. It follows that, in the direct sum

$$\rho'(f)\alpha_2(T) = 0 \oplus \rho(f)\,\mathrm{Ad}_U(T) \oplus \rho(f)\,\mathrm{Ad}_U^2(T) \oplus \cdots ,$$

all the summands $\rho(f)\,\mathrm{Ad}_U^n(T)$ for $n > N$ are zero. Moreover, all the summands without exception are compact, since T is locally compact. It follows that $\rho'(f)\alpha_2(T)$ is compact for every compactly supported function f on $\mathbb{R}^+$. Hence $\alpha_2(T)$ is locally compact.

We have

$$\alpha_2 = \mathrm{Ad}_W \circ (\alpha_1 + \alpha_2)$$

where $W\colon H' \to H'$ is the isometry

$$W(f_1, f_2, \dots) = (0, Uf_1, Uf_2, \dots).$$

The isometry W covers the identity map and so induces the identity on K-theory. Now the proof is completed by an Eilenberg swindle (compare Example 4.6.3): on K-theory we have

$$\alpha_{2*} = (\alpha_1 + \alpha_2)_* = \alpha_{1*} + \alpha_{2*}$$

and thus $\alpha_{1*} = 0$. □

The argument is applicable more widely. In fact, suppose now that Y is a proper metric space and that $X = \mathbb{R}^+ \times Y$ is equipped with the product metric d_X defined by

$$d_X((t, y), (t', y'))^2 = |t - t'|^2 + d_Y(y, y')^2.$$

We can form $H = L^2(\mathbb{R}^+) \otimes H_Y$, and $H' = H \oplus H \oplus \cdots$; these Hilbert spaces carry ample representations of $C_0(X)$. It is convenient to regard H as made up of measurable[49] functions $f\colon \mathbb{R}^+ \to H_Y$ which are square-integrable in the sense that

$$\int_0^\infty \langle f(t), f(t)\rangle_{H_Y} dt < \infty.$$

The definitions of the isometries U, V, and W, and of the $*$-homomorphisms α_1 and α_2, now extend directly to the more general case, and the proof of Lemma 6.4.2 goes through word for word. We obtain:

6.4.3 PROPOSITION *Let Y be a proper metric space and let $X = \mathbb{R}^+ \times Y$, equipped with the product metric and its associated coarse structure. Then* $K_p(C^*(X)) = 0$ *for all* p. □

6.4.4 REMARK This calculation looks very special. However, as we shall explain in a moment, Proposition 6.4.3 will turn out to be the key step in computing $K_p(C^*(X))$ for a variety of more elaborate spaces. In particular, Proposition 6.4.3 is the key to our coarse proof of the homotopy invariance of K-homology.

Let X be a proper metric space and $Y \subseteq X$ a closed subspace. For each $n \in \mathbb{N}$ let Y_n denote the closure of $\{x \in X : d(x, Y) < n\}$. Note that the inclusion map $Y \subseteq Y_n$ is a coarse equivalence, and that Y_n is the closure of its interior.

[49] A Hilbert-space-valued function is *measurable* if its inner product with each fixed vector is measurable.

6.4.5 DEFINITION A metrically controlled subset $S \subseteq X \times X$ is *near* Y if it is contained in $Y_n \times Y_n$ for some n. A metrically controlled operator T is *near* Y if its support is near Y.

The operators near Y form an ideal in the algebra of all metrically controlled operators, and similarly the locally compact operators near Y form an ideal in the algebra of all locally compact metrically controlled operators.

6.4.6 DEFINITION Let X be a proper metric space, and let $Y \subseteq X$ be a closed subspace. The ideal I_Y of $C^*(X)$ *supported near* Y is by definition the norm closure of the set of all locally compact, metrically controlled operators near Y.

REMARK This definition and the subsequent discussion have counterparts in the theory of general coarse structures; see Exercise 6.7.8.

6.4.7 PROPOSITION *Let X be a proper metric space, let $Y \subseteq X$ be a closed subspace, and let I_Y be the ideal of $C^*(X)$ supported near Y. There is an isomorphism*

$$K_p(I_Y) \cong K_p(C^*(Y))$$

between the K-theory of the ideal I_Y and the K-theory of the C^*-algebra associated to Y as a coarse space in its own right.

PROOF Define Y_n as above, and let H_{Y_n} be the subspace of H_X which is the range of the projection operator corresponding to the characteristic function of Y_n under the Borel functional calculus 1.5.7. Since each Y_n is the closure of its interior, the natural representation of $C_0(Y_n)$ on H_{Y_n} is ample. Thus we may identify $C^*(Y_n)$ with the subalgebra of $C^*(X)$ consisting of operators T such that both T and T^* vanish on the orthogonal complement of H_{Y_n}. Moreover, the inclusion $H_{Y_n} \subseteq H_{Y_{n+1}}$ is an isometry which covers the map $Y_n \subseteq Y_{n+1}$ of coarse spaces.

Having made these identifications, we find that $C^*(X)$ contains an increasing sequence of C^*-subalgebras

$$C^*(Y_1) \subseteq C^*(Y_2) \subseteq \cdots,$$

and that the closure of their union is I_Y. By Proposition 4.1.15 and Remark 4.2.3,

$$K_p(I_Y) \cong \varinjlim K_p(C^*(Y_n)).$$

But each of the inclusion maps $Y \to Y_n$ is a coarse equivalence, so by Proposition 6.3.16 the maps in the direct sequence are all isomorphisms, and the result follows. $\square$

6.4.8 REMARK Let $V\colon H_Y \to H_X$ be any isometry covering the inclusion map $Y \subseteq X$. Then Ad_V maps $C^*(Y)$ to $C^*(X)$. In fact, $\mathrm{Ad}_V(C^*(Y)) \subseteq I_Y$ and the proof above shows that the K-theory isomorphism $K_p(C^*(Y)) \to K_p(I_Y)$ is induced by Ad_V.

Suppose now that X is a proper metric space which is written as a union $X = Y \cup Z$ of two closed subspaces. Then $I_Y + I_Z = C^*(X)$. For let $f \in C_0(X)$ be any function for which $f = 1$ on Y and $\mathrm{Support}(f) \subseteq Y_1$. If T is a metrically controlled and locally compact operator, then Tf is near Y and $T(1-f)$ is near Z. Thus $T \in I_Y + I_Z$. Since the sum of two ideals in a C^*-algebra is always closed (Exercise 6.7.7), $I_Y + I_Z = C^*(X)$.

It is always true that $I_{Y\cap Z} \subseteq I_Y \cap I_Z$, but equality does not hold in general (Exercise 6.7.9). It does hold, however, in several important cases. Here is an example:

6.4.9 LEMMA *Let $X = \mathbb{R}^n$, let $Y = \mathbb{R}^- \times \mathbb{R}^{n-1}$, and let $Z = \mathbb{R}^+ \times \mathbb{R}^{n-1}$. Then the associated ideals in $C^*(X)$ satisfy the relation $I_{Y\cap Z} = I_Y \cap I_Z$.*

PROOF We use the notation of the proof of Proposition 6.4.7. The algebra $C^*(X)$ contains an increasing sequence of C^*-subalgebras

$$C^*(Y_1) \cap C^*(Z_1) \subseteq C^*(Y_2) \cap C^*(Z_2) \subseteq \cdots .$$

Moreover, the closure of their union is $I_Y \cap I_Z$.[50] But in the case at hand

$$Y_n \cap Z_n = (Y \cap Z)_n,$$

and so $C^*(Y_n) \cap C^*(Z_n) = C^*((Y\cap Z)_n)$. Since the closure of the union of the subalgebras $C^*((Y\cap Z)_n)$ is the ideal $I_{Y\cap Z}$, the result follows. □

We can now compute the coarse K-theory for Euclidean space.

6.4.10 THEOREM *The groups $K_p(C^*(\mathbb{R}^n))$ are given by*

$$K_p(C^*(\mathbb{R}^n)) = \begin{cases} \mathbb{Z} & \text{if } p \equiv n \pmod 2, \\ 0 & \text{if } p \equiv n+1 \pmod 2. \end{cases}$$

PROOF If $n = 0$ the result follows from Lemma 6.4.1. It suffices then for an inductive proof to establish a 'suspension isomorphism'

$$K_p(C^*(\mathbb{R}^n)) \cong K_{p-1}(C^*(\mathbb{R}^{n-1})).$$

[50] The intersection of two ideals in a C^*-algebra is the same thing as their product, by a functional calculus argument (Exercise 6.7.7). Thus we may write $T = AB$ where $A \in I_Y$ and $B \in I_Z$. Now approximate A by a controlled element A' of $C^*(Y_k)$ and B by a controlled element $B'$of $C^*(Z_\ell)$ for suitably large k, ℓ. The product $A'B'$ belongs to $C^*(Y_n) \cap C^*(Z_n)$ for $n > k + \ell + \mathrm{Prop}(A') + \mathrm{Prop}(B')$.

Let Y and Z be the subspaces described in Lemma 6.4.9 above, and let I_Y, I_Z and $I_{Y\cap Z}$ be the ideals supported near Y, Z and $Y \cap Z$. Using the Mayer–Vietoris sequence of Exercise 4.10.21, there is an exact sequence

$$\begin{tikzcd} K_1(I_{Y\cap Z}) \arrow[r] & K_1(I_Y) \oplus K_1(I_Z) \arrow[r] & K_1(C^*(\mathbb{R}^n)) \arrow[d] \\ K_0(C^*(\mathbb{R}^n)) \arrow[u] & K_0(I_Y) \oplus K_0(I_Z) \arrow[l] & K_0(I_{Y\cap Z}) \arrow[l] \end{tikzcd}$$

The ideals I_Y, I_Z, and $I_{Y\cap Z}$ have the same K-theory as the corresponding C^*-algebras $C^*(Y)$, $C^*(Z)$, and $C^*(Y \cap Z)$. But $Y \cap Z = \mathbb{R}^{n-1}$, while $C^*(Y)$ and $C^*(Z)$ have zero K-theory by Proposition 6.4.3. The suspension isomorphism now follows from the Mayer-Vietoris sequence. $\square$

REMARK By an elaboration of this argument one can compute $K_p(C^*(X))$ whenever $X = \mathcal{O}_\varphi Y$ is an open metric cone on a finite polyhedron Y.

6.5 K-Theory for Topological Coarse Structures

In this section we are going to relate the algebras $C^*(X)$, for topological coarse structures on X, to the relative dual algebras which appeared in Definition 5.3.2. As a result we shall be able to express the groups $K_p(C^*(X))$ in terms of K-homology.

We shall consider the following situation. Let X be a locally compact space, and suppose that X is provided with a metrizable compactification $\overline{X}$. Let $A = C(\overline{X})$ and $J = C_0(X)$, so that $A/J = C(\partial X)$ where $\partial X = \overline{X} \setminus X$. We assume that J is amply represented on a Hilbert space H. The representation of J extends to a representation of A (in this commutative case, we may regard the extension as a consequence of 1.5.7). This extended representation is also ample, since X is dense in $\overline{X}$.

There are two ways of forming a C^*-algebra from the above data:

(a) as in Chapter 5, we may form the relative dual algebra $\mathfrak{D}(A /\!\!/ J)$, which consists of operators on H which are locally compact for the action of J and commute modulo the compact operators with the action of A, or

(b) as in this chapter, we may form the C^*-algebra $C^*(X)$, which is the norm closure of the collection of operators on H that are locally compact for the action of $C_0(X)$ and topologically controlled for the compactification $\overline{X}$.

It turns out that the two constructions yield exactly the same result:

6.5.1 THEOREM *Let X be a locally compact Hausdorff space, equipped with a metrizable compactification $\overline{X}$. Then the algebra $C^*(X)$ associated to the topological coarse structure on X is equal to the relative dual algebra $\mathfrak{D}(A /\!\!/ J)$, where $A = C(\overline{X})$ and $J = C_0(X)$.*

From Proposition 5.3.7 and Remark 5.3.9, we obtain:

6.5.2 COROLLARY *Let* X *be a locally compact space equipped with a metrizable compactification* $\overline{X}$, *and let* $\partial X = \overline{X} \setminus X$. *Then there is an isomorphism*

$$K_p(C^*(X)) \cong \widetilde{K}^{1-p}(C(\partial X)). \quad \square$$

6.5.3 REMARK The Hilbert space H is an ample representation of $A = C(\overline{X})$, not an ample representation of $\widetilde{A}$. This accounts for the appearance of *reduced* K-homology in Corollary 6.5.2.

The proof of Theorem 6.5.1 is given by the next two lemmas. The first shows that $C^*(X) \subseteq \mathfrak{D}(A/\!\!/J)$, and the second shows that $\mathfrak{D}(A/\!\!/J) \subseteq C^*(X)$.

6.5.4 LEMMA *If* $T \in \mathfrak{B}(H)$ *is a locally compact and topologically controlled operator, then* $T \in \mathfrak{D}(A/\!\!/J)$.

PROOF We shall show that if f and g are continuous functions on $\overline{X}$ with disjoint supports then fTg is compact. By Kasparov's Lemma 5.4.7, the result will follow.

Note that

$$\mathrm{Support}(fTg) \subseteq \mathrm{Support}(T) \cap (\mathrm{Support}(f) \times \mathrm{Support}(g)).$$

Let (x_n, x'_n) be a sequence in Support(fTg). If one of x_n or x'_n were to converge to a point of ∂X, then both sequences would have to converge to the same point (by the definition of a topologically controlled set) and this point would have to lie in Support$(f) \cap$ Support(g), contradicting disjointness. We conclude that Support(fTg) contains no sequence either of whose components converges to a point of ∂X, and hence that Support(fTg) is a compact subset of $X \times X$. But a compactly supported, locally compact operator is compact. $\square$

6.5.5 LEMMA *If* T *is an operator belonging to* $\mathfrak{D}(A/\!\!/J)$, *then for every* $\varepsilon > 0$ *there exists a locally compact and topologically controlled operator* T' *with* $\|T - T'\| < \varepsilon$.

The proof begins with a point-set topology calculation, which we leave to the reader:

6.5.6 LEMMA *Let* $\{K_n\}_{n=1}^{\infty}$ *be a sequence of compact subsets of* ∂X *and let* $\{U_n\}_{n=1}^{\infty}$ *be a sequence of open sets in* $\overline{X}$ *such that*

(a) *the closure of* U_n *is disjoint from* K_n,

(b) *for every point* $y \in \partial X$ *there is a subsequence* $\{n_i\}$ *of the natural numbers for which*

$$\{y\} = K_{n_1} \cap K_{n_2} \cap \dots \quad \text{and} \quad X \setminus \{y\} = U_{n_1} \cup U_{n_2} \cup \dots.$$

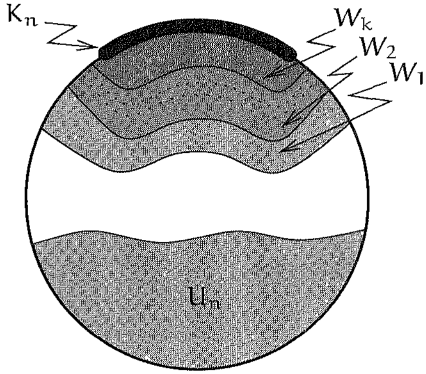

FIG. 6.1. One stage in the proof of Lemma 6.5.5

If $S \subseteq X \times X$ *and if, for every* n*, there is an open neighborhood* V_n *(in* X*) of* K_n *such that*

$$S \cap (U_n \times V_n) = \emptyset \quad \textit{and} \quad S \cap (V_n \times U_n) = \emptyset,$$

then S *is topologically controlled.* □

PROOF OF LEMMA 6.5.5 Choose sequences $\{K_n\}$ and $\{U_n\}$ as in Lemma 6.5.6 above; our assumption that $\overline{X}$ is metrizable implies that such sequences can be found.

Let $T \in \mathfrak{D}(A/\!\!/J)$ and let $\varepsilon > 0$. We shall find an operator $T' \in \mathfrak{D}(A/\!\!/J)$, within ε of T, whose support S satisfies the hypotheses of Lemma 6.5.6. To obtain T', we shall build a sequence $T_0, T_1, T_2, \ldots$ in $\mathfrak{D}(A/\!\!/J)$, starting with $T_0 = T$, and a sequence of open sets $V_1, V_2, \ldots$ in X such that

(a) $K_n \subseteq V_n$,
(b) $\|T_n - T_{n-1}\| < 2^{-n}\varepsilon$,
(c) $\mathrm{Support}(T_n) \subseteq \mathrm{Support}(T_{n-1})$, and
(d) $\mathrm{Support}(T_n) \cap (U_n \times V_n) = \emptyset$ and $\mathrm{Support}(T_n) \cap (V_n \times U_n) = \emptyset$.

Then we shall define $T' = \lim_{n\to\infty} T_n$, which has the required properties.

Supposing that $T_0, \ldots, T_{n-1}$ and $V_1, \ldots, V_{n-1}$ have been constructed, we obtain T_n and V_n as follows. First recall from Remark 1.5.7 that the representation of $A = C(\overline{X})$ on H extends to a representation of the bounded Borel functions. So each Borel subset $B \subseteq \overline{X}$ determines a projection operator P_B on H. Let $W_1, W_2, \ldots,$ be a *decreasing* sequence of neighborhoods of K_n in $\overline{X}$, with intersection K_n (see Figure 6.1). We can assume that the closure of W_1 is disjoint from the closure of U_n, and therefore

$$P_{U_n} T_{n-1} P_{W_1} \sim P_{W_1} T_{n-1} P_{U_n} \sim 0,$$

since T_{n-1} is pseudolocal. Since $K_n = \bigcap W_k$ does not meet X, the projections P_{W_k} converge to zero in the strong operator topology. Since $P_{U_n} T_{n-1} P_{W_1}$ is compact, it follows[51] that

$$\lim_{k\to\infty} \|P_{U_n} T_{n-1} P_{W_k}\| = \lim_{k\to\infty} \|(P_{U_n} T_{n-1} P_{W_1}) P_{W_k}\| = 0,$$

and similarly $\lim_{k\to\infty} \|P_{W_k} T_{n-1} P_{U_n}\| = 0$. So we may define $V_n = W_k$, where k is so large that $\|P_{U_n} T_{n-1} P_{W_k}\| < 2^{-n-1}\varepsilon$ and $\|P_{W_k} T_{n-1} P_{U_n}\| < 2^{-n-1}\varepsilon$, and

$$T_n = T_{n-1} - P_{U_n} T_{n-1} P_{W_k} - P_{W_k} T_{n-1} P_{U_n}.$$

Then items (a) to (d) above hold, as required. □

6.6 The Homotopy Invariance of K-Homology

In this section we shall put coarse geometry to work. We shall give a coarse proof of the homotopy invariance of K-homology. Recall once again the statement of homotopy invariance: if $\alpha_t\colon A \to B$, $t \in [0,1]$, is a point-norm continuous family of $*$-homomorphisms, then α_0 and α_1 induce the same map on K-homology.

A point-norm continuous family of $*$-homomorphisms α_t is the same thing as a single $*$-homomorphism $\alpha\colon A \to C[0,1] \otimes B$. To obtain the general result it therefore suffices to show that the evaluation $*$-homomorphisms

$$\beta_0, \beta_1\colon C[0,1] \otimes B \to B,$$

defined by $\beta_t(f \otimes b) = f(t)b$, induce the same map on K-homology. Each such evaluation $*$-homomorphism is a left inverse to the $*$-homomorphism $\gamma\colon B \to C[0,1] \otimes B$ defined by $\gamma(b) = 1 \otimes b$, so it suffices to show that each of them induces an isomorphism on K-homology. By symmetry, it suffices to show that just one of them, say β_0, induces an isomorphism on K-homology. By the six-term exact sequence of K-homology, the surjective $*$-homomorphism β_0 induces an isomorphism on K-homology if and only if the K-homology of the kernel

$$\mathrm{Kernel}(\beta_0) = C_0(0,1] \otimes B$$

(that is the *cone* on B, in the language of Definition 4.5.4) is zero. It is this statement that we shall prove, in the commutative case:

6.6.1 THEOREM *Let B be a commutative separable C^*-algebra. Then the algebra $C_0(0,1] \otimes B$ has zero K-homology groups.*

Arguing as above, we deduce:

[51] Exercise: If T is a compact operator, and if the sequence $\{S_k\}$ of selfadjoint operators is bounded in norm and tends to zero in the strong topology, then TS_k and $S_k T$ tend to zero in norm.

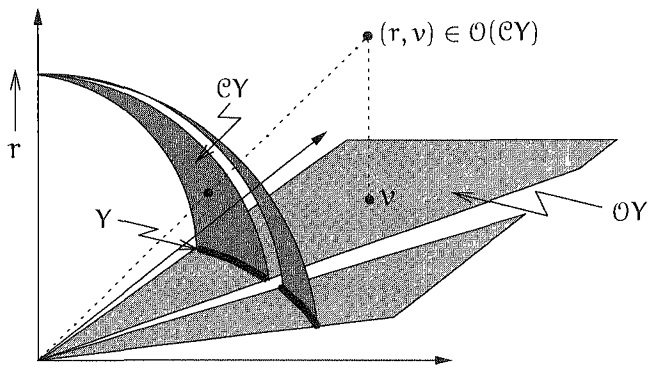

FIG. 6.2. The metric cone on a cone is a product.

6.6.2 COROLLARY *Let* $\alpha_t \colon A \to B$, $t \in [0, 1]$, *be a point-norm continuous family of* $*$*-homomorphisms between separable* C^**-algebras, with* B *commutative. Then* α_0 *and* α_1 *induce the same map on* K*-homology.* □

PROOF OF THEOREM 6.6.1 There is no loss of generality in assuming that B is unital, so that $B = C(Y)$ where Y is a compact metrizable space. Then the unitalization of $C_0(0, 1] \otimes B$ is the algebra of continuous functions on the closed cone $\mathcal{C}Y$. The K-homology of $C_0(0, 1] \otimes B$ is therefore the same thing as the reduced K-homology of $C(\mathcal{C}Y)$.

Let X_c be the open cone on $\mathcal{C}Y$, equipped with its topological coarse structure. Then, by Corollary 6.5.2, the reduced K-homology of $C(\mathcal{C}Y)$ can be identified with the K-theory of the coarse C^*-algebra $C^*(X_c)$. It suffices therefore to show that if $X_c = \mathcal{O}_c(\mathcal{C}Y)$, then $K_p(C^*(X_c)) = 0$.

We shall do this by comparing the topological coarse structure on $X_c = \mathcal{O}_c(\mathcal{C}Y)$ with the metric coarse structures on $X_\varphi = \mathcal{O}_\varphi(\mathcal{C}Y)$ discussed in Section 6.2. Embed Y into the unit sphere of some real Hilbert space E. Then embed $\mathcal{C}Y$ into the unit sphere of the larger Hilbert space $E' = \mathbb{R} \oplus E$ as

$$\mathcal{C}Y = \{(s, ty) : y \in Y,\ 0 \leqslant s, t \leqslant 1,\ s^2 + t^2 = 1\}.$$

Figure 6.2 makes it clear that a point $v' = (r, v) \in E' = \mathbb{R} \oplus E$ lies on a ray through $\mathcal{C}Y$ if and only if $r \geqslant 0$ and v lies on a ray through Y. Consequently, we have an *isometry*

$$\mathcal{O}_\varphi(\mathcal{C}Y) = \mathbb{R}^+ \times \mathcal{O}_\varphi Y,$$

using the product metric on the right-hand side. From Proposition 6.4.3 we deduce that $K_p(C^*(X_\varphi)) = 0$ for each of the metric coarse structures X_φ on X.

The last step of the proof is to relate the topological coarse structure X_c to the metric coarse structures X_φ. By Proposition 6.2.1, the structure X_c is the least upper bound of the structures X_φ. It follows from Proposition 6.2.1 that each $C^*(X_\varphi)$ is a subalgebra of $C^*(X_c)$, and that the union (over all φ) of these subalgebras is dense. Since the homeomorphisms φ form a directed set (Remark 6.2.3), it follows from the continuity of K-theory (Remark 4.1.18) that

$$K_p(C^*(X_c)) = \varinjlim K_p(C^*(X_\varphi)).$$

But all the K-theory groups appearing on the right-hand side are zero. This concludes the proof of Theorem 6.6.1. □

6.7 Exercises

6.7.1 Let X be a coarse space and let S be a set having cardinality greater than or equal to the cardinality of X. Show that two maps $p_1, p_2 \colon S' \to X$ are close if and only if $p_1 \circ q$ and $p_2 \circ q$ are close for every map $q \colon S \to S'$. (Hint: use Proposition 6.1.4.)

6.7.2 Let X be a locally compact Hausdorff space equipped with the topological coarse structure coming from a metrizable compactification $\overline{X}$.

(a) Fix a countable dense set $\{x_n\}$ in X, and let $d_n > 0$ be the distance (in a metric defining the topology of $\overline{X}$) from x_n to ∂X. Show that

$$\mathcal{U} = \{B(x_n; d_n/2)\}$$

is a countable open cover of X which is uniformly bounded for the coarse structure.

(b) Suppose that a subset $Y \subseteq X$ contains a sequence $\{y_k\}_{k=1}^{\infty}$ which converges to a point of ∂X. Show that Y is not bounded for the coarse structure. Deduce that the bounded subsets of X have compact closure.

Hence show that the topological coarse structure is proper and separable.

6.7.3 Let X be a locally compact Hausdorff space and let $\mathcal{E}$ be the family of controlled sets for a proper coarse structure on X. Show that:

(a) The diagonal of $X \times X$ is a member of $\mathcal{E}$.

(b) $\mathcal{E}$ is closed under the formation of subsets and finite unions.

(c) $\mathcal{E}$ is closed under the operations of transposition and product, defined as follows:

  (i) $E^t = \{(y, x) : (x, y) \in E\}$;

  (ii) $E \circ F = \{(x, z) : (x, y) \in E, (y, z) \in F \text{ for some } y \in X\}$.

(d) Every compact subset of $X \times X$ belongs to $\mathcal{E}$.

(e) Every $E \in \mathcal{E}$ is proper, in the sense that

$$E[K] := \{x : \exists y \in K : (x, y) \in E\}$$

has compact closure whenever K does.

Conversely, show that any family $\mathcal{E}$ of subsets of $X \times X$ which satisfies these five properties is the family of controlled sets for a uniquely determined coarse structure on X.

6.7.4 Let $\mathcal{U}$ be a uniformly bounded cover for a proper coarse space. Show that the cover

$$\overline{\mathcal{U}} = \{\overline{U} : U \in \mathcal{U}\}$$

is also uniformly bounded.

6.7.5 Let Γ be a discrete group. Show that there is a unique proper coarse structure on Γ which is translation invariant in the sense that each left translation $L_\gamma \colon \Gamma \to \Gamma$ is close to the identity. Show that if Γ is finitely generated then this structure is a metric coarse structure for a suitable metric on Γ (a 'word length metric').

6.7.6 Generalize Example 6.1.18 to the following situation: if a finitely generated discrete group Γ acts freely and properly discontinuously on a metric space X, with X/Γ compact, then Γ (with its translation-invariant coarse structure) and X are coarsely equivalent (Milnor–Wolf Theorem; see [137]).

6.7.7 Let I and J be (closed) ideals in a C^*-algebra A.

(a) Prove that $I + J$ is closed.
(b) Prove that $IJ = I \cap J$.

(Hints: for (a), use the fact that $(I + J)/J \cong I/(I \cap J)$ is a C^*-algebra; for (b), use the functional calculus to write any selfadjoint element of $I \cap J$ as a product of two such elements.)

6.7.8 Let Y be a subset of the proper separable coarse space X. Say that a controlled subset $S \subseteq X \times X$ is *near* Y if the coordinate maps $\pi_1, \pi_2 \colon S \to X$ are close to a map that factors through the inclusion $Y \to X$. Show that this definition generalizes 6.4.5. Does Proposition 6.4.7 hold in this context?

6.7.9 Let X be a proper metric space and suppose that $X = Y \cup Z$. Let I_Y, I_Z, and $I_{Y \cap Z}$ be the ideals in $C^*(X)$ supported near the given subspaces of X. Give an example to show that the relation

$$I_{Y \cap Z} = I_Y \cap I_Z$$

need *not* hold in general. Prove however that the relation does hold if for every m there is an n such that

$$Y_m \cap Z_m \subseteq (Y \cap Z)_n.$$

(In this case one says that the decomposition $X = Y \cup Z$ is *coarsely excisive.* See [76].)

6.7.10 A metric space X is called *uniformly discrete* if every pair of distinct points of X have distance at least 1.

Let X be a uniformly discrete metric space. A *Ponzi scheme* on X is a function $p \colon X \times X \to \mathbb{R}$ such that

(a) p is bounded,

(b) $\mathrm{Support}(p)$ is controlled,

(c) $p(x, y) = -p(y, x)$ for all $x, y \in X$, and

(d) for each $y \in X$, $\sum_{x \in X} p(x, y) \geqslant 1$.

Show that the existence or otherwise of a Ponzi scheme is a coarse invariant (of uniformly discrete spaces). Show that a finitely generated group, with the coarse structure defined in Exercise 6.7.5, admits a Ponzi scheme if and only if it is not amenable. (See [20].)

6.7.11 A proper metric space X is *flasque* if there is an isometry $T \colon X \to X$ such that $\bigcap_n T^n(X) = \emptyset$. Show that if X is flasque, then $K_p(C^*(X)) = 0$ (use the method of Lemma 6.4.2).

6.7.12 Develop the theory of the algebra $C^*(X)$ for an arbitrary separable coarse space X (without the assumption of an underlying topology on X) as follows.

(a) Let $\mathcal{X}$ be a countable uniformly bounded collection of disjoint subsets of X, with union $\bigcup \mathcal{X} = X$. Define a *coarse module* over X to be a Hilbert space equipped with a representation of the algebra of bounded functions that are constant on each member of $\mathcal{X}$.

(b) Define the notion of *support* for an operator on a coarse module.

(c) Show that the operators with controlled support form a $*$-algebra.

(d) An operator T is *boundedly compact* if fT and Tf are compact whenever the support of f is bounded (f being a function constant on each member of $\mathcal{X}$). Show that the boundedly compact operators form an ideal in the algebra of controlled operators.

(e) Define $C^*(X)$ to be the norm closure of the boundedly compact controlled operators. Reprove all the results of Section 6.3 in this context.

6.8 Notes

The notion of 'large-scale structure' has deep roots in mathematics, and has been particularly influential in the theory of infinite groups. The interested reader should consult the works of M. Gromov [59, 61]. Geometric topologists have developed several theories of *controlled topology* which are closely related to our coarse geometry. See [135] for an extensive discussion of controlled topology, and [108] for an intensive discussion of the underlying algebra. Topologists refer to our metric coarse structures as *boundedly controlled*, and to our topological coarse structures as *continuously controlled.*

The ideas underlying the proof of homotopy invariance in this chapter go back to [1]. The algebra $C^*(X)$ (for a metric coarse structure) was introduced by Roe [110, 112] for index-theoretic purposes which we shall discuss later in the book. The Mayer–Vietoris argument for metric coarse structures appears in [76]. The simple relationship between the algebra $C^*(X)$ for a topological coarse structure and K-homology was worked out in [74], which also contains an axiomatic discussion of coarse structure based on the notion of controlled set (see Exercise 6.7.3).[52] A survey of the C^*-algebraic side of coarse geometry appears in [113].

[52]'Controlled sets' are 'entourages' in the terminology of [74], and our 'coarse structures' are 'unital coarse structures'.

7

THE BROWN–DOUGLAS–FILLMORE THEOREM

Our main objective in this chapter is the Brown–Douglas–Fillmore Theorem. We recall the statement from Chapter 2 (Theorem 2.4.8): if T_1 and T_2 are essentially normal operators with the same essential spectrum X, and if $\mathrm{Index}(T_1 - \lambda I) = \mathrm{Index}(T_2 - \lambda I)$ for every $\lambda \in \mathbb{C} \setminus X$, then T_1 and T_2 are essentially unitarily equivalent; furthermore every locally constant, integer-valued function on $\mathbb{C} \setminus X$ that is zero at infinity arises as the index function $\lambda \mapsto \mathrm{Index}(T - \lambda I)$ of some essentially normal operator T with essential spectrum X.

The proof of the Brown–Douglas–Fillmore Theorem amounts to a calculation of the extension group $\mathrm{Ext}(C(X))$ for compact subsets of the plane, and, as Brown, Douglas and Fillmore found, the proper context for this calculation is algebraic topology. The previous chapters have provided us with most of the tools we need; in the present chapter we shall transport the Ext-functor into the realm of algebraic topology to finish the job.

In Chapter 5 we identified the extension group $\mathrm{Ext}(A)$ of a separable, unital and nuclear C^*-algebra with a K-theory group $K_0(\mathfrak{D}(A))$, which we rewrote as $K^1(A)$ and named K-homology. We studied $K^1(A)$ together with its companion group $K^0(A)$ and developed a six-term exact sequence, along with excision and homotopy invariance results. Taken together, these things show that the functors $A \mapsto K^p(A)$, when specialized to commutative C^*-algebras, determine a *generalized homology theory*. Thanks to its definition, K-homology is of course very closely related to K-theory. There is an integer-valued *index pairing* between the K-theory and the K-homology of any separable C^*-algebra. By specializing to commutative C^*-algebras and applying the machinery of algebraic topology to analyze the index pairing we shall immediately be able to compute $\mathrm{Ext}(C(X))$ in terms of K-theory — at least up to torsion — for any finite simplicial complex X; if X is a subset of the plane then our computation will be exact. Since compact metric spaces, even compact subsets of the plane, can be much more elaborate than finite simplicial complexes, we shall investigate next the 'continuity' properties of K-homology. These will allow us to compute $\mathrm{Ext}(C(X))$ using finite simplicial approximations to X and will supply the final tools needed to prove the Brown–Douglas–Fillmore Theorem.

We shall conclude the chapter with a proof of the *Universal Coefficient Theorem*, which computes the abelian group $\mathrm{Ext}(C(X))$, including its torsion component, in terms of the K-theory of X. Our presentation here will be brisker than the rest of the chapter.

7.1 Generalized Homology Theories

We begin with a notion which will be familiar to the reader from algebraic topology:

7.1.1 DEFINITION A *generalized homology theory* on the category of locally compact and second countable[53] Hausdorff topological spaces is a collection of covariant functors $h_p(X)$, indexed by $p \in \mathbb{Z}$, such that

(a) all the h_p are homotopy functors, and
(b) if $Y \subseteq X$ is a closed subset there is a functorial long exact sequence

$$\cdots \longrightarrow h_p(Y) \longrightarrow h_p(X) \longrightarrow h_p(X \setminus Y) \longrightarrow h_{p-1}(Y) \longrightarrow \cdots .$$

The above definition has a number of variants, in which for instance the category of spaces is broadened or the category of pairs (X, Y) to which axiom (b) applies is narrowed. Our definition requires two items of explanation. First, the morphisms in the category of locally compact and second countable Hausdorff topological spaces are those that we encountered in Chapter 1: a morphism from X to Y is a basepoint-preserving map from the one-point compactification of X to the one-point compactification of Y (see Theorem 1.3.14 and Exercise 1.9.8). Thus, for example, there is a natural morphism from X to $X \setminus Y$, and indeed we are invoking this map in axiom (b) above. Second, the notion of homotopy in the locally compact category is tailored to the notion of morphism: a homotopy of morphisms from X to Y is a morphism from $X \times [0, 1]$ to Y. Thus h_p is a homotopy functor if and only if the projection from $X \times [0, 1]$ to X induces an isomorphism from $h_p(X \times [0, 1])$ to $h_p(X)$, for every X. As we already noted in Chapter 4, the definition of homotopy appropriate to the locally compact category has some potentially confusing aspects. For instance, the correct notion of 'contractible' is 'homotopy equivalent to the empty space'. In particular the one-point space is *not* contractible, whereas for instance a closed half-line $\mathbb{R}^+$ (or indeed any n-dimensional closed half-space) is. However, with a little care, the standard calculations in any first course on algebraic topology carry over to the present context. For example, associated to any generalized homology theory there is a Mayer–Vietoris sequence:

[53]We are concentrating on *second countable* spaces X because we shall require that the C^*-algebra $C_0(X)$ be *separable*.

7.1.2 LEMMA *If* $\{h_p\}_{p\in\mathbb{Z}}$ *is any generalized homology theory, and if* X *is a locally compact and second countable space which is the union of two closed subspaces* X_1 *and* X_2*, then there is a long exact sequence of abelian groups*

$$\cdots \to h_p(X_1 \cap X_2) \to h_p(X_1) \oplus h_p(X_2) \to h_p(X) \to h_{p-1}(X_1 \cap X_2) \to \cdots$$

PROOF This is a diagram chase starting from the long exact sequences in homology (that is, the exact sequences guaranteed by axiom (b) of Definition 7.1.1) associated to the subspaces $X_1 \subseteq X$ and $X_1 \cap X_2 \subseteq X_2$. It is left to the reader as an exercise. □

7.1.3 PROPOSITION *If* $\{h_p\}_{p\in\mathbb{Z}}$ *is any generalized homology theory then, for every* p*,*

$$h_p(\mathbb{R}^n) \cong h_{p-n}(\mathrm{pt}),$$

where pt *denotes the one-point space.*

PROOF Express $\mathbb{R}^n$ as a union of two closed half-spaces whose intersection is $\mathbb{R}^{n-1}$. The half-spaces are contractible, and so have zero homology. Therefore the Mayer–Vietoris sequence of the previous lemma reduces to a collection of isomorphisms

$$h_p(\mathbb{R}^n) \cong h_{p-1}(\mathbb{R}^{n-1}).$$

The result is now clear. □

7.1.4 REMARK The *suspension* of a locally compact space X is the product $SX = \mathbb{R} \times X$. Since the product of any space with a closed half-line is contractible, the same argument provides *suspension isomorphisms*

$$h_p(SX) \cong h_{p-1}(X)$$

for every p.

7.1.5 COROLLARY *Let* S^n *denote the* n*-dimensional sphere. If* $\{h_p\}_{p\in\mathbb{Z}}$ *is any generalized homology theory then*

$$h_p(S^n) \cong h_{p-n}(\mathrm{pt}) \oplus h_p(\mathrm{pt}),$$

for every p.

PROOF Because the subspace $\mathrm{pt} \subseteq S^n$ is a retract, and because $S^n \setminus \mathrm{pt}$ is homeomorphic to $\mathbb{R}^n$, the long exact homology sequence for $\mathrm{pt} \subseteq S^n$ reduces to a family of split short exact sequences

$$0 \longrightarrow h_p(\mathbb{R}^n) \longrightarrow h_p(S^n) \longrightarrow h_p(\mathrm{pt}) \longrightarrow 0\,.$$

The result now follows from Proposition 7.1.3. □

Let us turn now to C^*-algebras and K-homology. We noted in Chapter 1 that the category of locally compact and second countable Hausdorff topological spaces is equivalent to the opposite of the category of separable and commutative C^*-algebras. Under this equivalence, homotopy of morphisms in the category of locally compact spaces corresponds to homotopy of $*$-homomorphisms. Furthermore, associated to every closed subspace $Y \subseteq X$ is a short exact sequence

$$0 \longrightarrow C_0(X \setminus Y) \longrightarrow C_0(X) \longrightarrow C_0(Y) \longrightarrow 0 .$$

So the results that we proved in Chapters 5 and 6 show that the functors

$$h_p(X) = \begin{cases} K^0(C_0(X)) & \text{if p is even,} \\ K^1(C_0(X)) & \text{if p is odd.} \end{cases}$$

constitute a generalized homology theory. Indeed the six-term exact sequence established in Theorem 5.3.10, combined with the Excision Theorem 5.4.5, provide the exact sequences of Definition 7.1.1, axiom (b), while of course we proved the homotopy axiom (a) of Definition 7.1.1 in Section 6.6.

Proposition 7.1.3 and Corollary 7.1.5 make it clear that it is important to determine the value of any homology theory on the one-point space. In the present case we may read from Example 5.2.9 the values

$$h_p(\mathrm{pt}) = \begin{cases} \mathbb{Z} & \text{if p is even,} \\ 0 & \text{if p is odd.} \end{cases}$$

So for example it follows immediately from Corollary 7.1.5 and the relation between K-homology and Ext-theory that:

7.1.6 Corollary *If S^n denotes the n-dimensional sphere then*

$$\operatorname{Ext}(C(S^n)) = \begin{cases} 0 & \text{if n is even,} \\ \mathbb{Z} & \text{if n is odd.} \end{cases} \qquad \square$$

In Chapter 2 we constructed a Toeplitz extension for each odd-dimensional sphere S^{2n-1} using complex function theory on the unit ball of $\mathbb{C}^n$. Using the results of the next section one can show that these extensions determine generators for the groups $\operatorname{Ext}(C(S^{2n-1}))$ (see Exercise 7.7.7 for the case $n = 2$).

7.2 The Index Pairing

The computation of generalized homology groups for spaces other than spheres rapidly becomes complicated. Consider, for example, the real projective plane $\mathbb{RP}^2$, which may be viewed as the space obtained from the closed unit disk $\overline{\mathbb{D}}$ by identifying antipodal points on the boundary of $\overline{\mathbb{D}}$. If we denote by $\mathbb{RP}^1 \subseteq \mathbb{RP}^2$ the subspace corresponding to the boundary of $\overline{\mathbb{D}}$, then the identification map

$\overline{\mathbb{D}} \to \mathbb{R}P^2$ restricts to a two-fold covering map from $\partial\mathbb{D}$ to $\mathbb{R}P^1$, while it is a homeomorphism from $\mathbb{D}$ to $\mathbb{R}P^2 \setminus \mathbb{R}P^1$. Associated to the identification map is a commuting diagram of generalized homology exact sequences (as in axiom (b) of Definition 7.1.1)

$$\begin{tikzcd}
\arrow[r] & h_p(\partial\mathbb{D}) \arrow[r] \arrow[d] & h_p(\overline{\mathbb{D}}) \arrow[r] \arrow[d] & h_p(\mathbb{D}) \arrow[r] \arrow[d] & h_{p-1}(\partial\mathbb{D}) \arrow[r] \arrow[d] & {} \\
\arrow[r] & h_p(\mathbb{R}P^1) \arrow[r] & h_p(\mathbb{R}P^2) \arrow[r] & h_p(\mathbb{R}P^2 \setminus \mathbb{R}P^1) \arrow[r] & h_{p-1}(\mathbb{R}P^1) \arrow[r] & {}
\end{tikzcd}$$

Even though we have already computed all the above homology groups except $h_p(\mathbb{R}P^2)$ in terms of the homology of a point,[54] it is not at all a simple matter to 'solve' for the one 'unknown' $h_p(\mathbb{R}P^2)$ (the reader is asked to do this for the case of K-homology in Exercise 7.7.1).

Rather than confront a series of increasingly complicated problems in algebra as we progress from one space to the next, we shall take an indirect approach to the problem of computing K-homology and reduce it to the problem of computing K-theory groups. The latter is arguably no simpler than the former, but nonetheless by making the reduction we shall in fact have accomplished something. For example we shall soon see that the Brown–Douglas–Fillmore Theorem amounts to a formula for K-homology (for compact subsets of the plane) in terms of K-theory.

To carry out the reduction we are going to combine ideas from Fredholm index theory with some simple techniques from homological algebra. The main tool from algebra is the Five Lemma. Recall this asserts that, given a commuting diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \arrow[r] & \cdots
\end{tikzcd}$$

of abelian groups and group homomorphisms, if the horizontal rows are exact and if each vertical map — except perhaps the middle one — is an isomorphism then in fact *all* of the vertical maps are isomorphisms. The role of index theory is to create diagrams of essentially the above sort. For every separable and commutative[55] C^*-algebra A we are going to construct functorial bilinear *index pairings*

[54] Indeed $\overline{\mathbb{D}}$ is homotopy equivalent to a point while $\partial\mathbb{D}$ and $\mathbb{R}P^1$ are circles — covered by Corollary 7.1.5 — and $\mathbb{D}$ and $\mathbb{R}P^2 \setminus \mathbb{R}P^1$ are planes — covered by Proposition 7.1.3.

[55] The hypothesis of commutativity is not really essential here. We are restricting to the commutative case because so far in the body of the text we have only proved homotopy invariance in K-homology for commutative C^*-algebras.

$$K_p(A) \times K^p(A) \longrightarrow \mathbb{Z}$$

in such a way that if J is an ideal in A then there is a commuting diagram

$$\begin{tikzcd}
\cdots \arrow[r] & K^p(A) \arrow[r] \arrow[d] & K^p(J) \arrow[d] \\
\cdots \arrow[r] & \mathrm{Hom}(K_p(A), \mathbb{Z}) \arrow[r] & \mathrm{Hom}(K_p(J), \mathbb{Z})
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r] & K^{p+1}(A/J) \arrow[r] \arrow[d] & K^{p+1}(A) \arrow[r] \arrow[d] & \cdots \\
\arrow[r] & \mathrm{Hom}(K_{p+1}(A/J), \mathbb{Z}) \arrow[r] & \mathrm{Hom}(K_{p+1}(A), \mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

in which the vertical maps are induced from the index pairing, the top row is the long exact sequence in K-homology associated to $J \subseteq A$, and the bottom row is induced from the long exact sequence in K-theory. Our basic strategy is to deduce from the Five Lemma that if the index pairing induces a vertical isomorphism for J and A/J then it induces an isomorphism for A, thus showing that the index pairing induces an isomorphism

$$K^p(A) \cong \mathrm{Hom}(K_p(A), \mathbb{Z})$$

for larger and larger classes of commutative C*-algebras. Unfortunately there is an algebraic obstacle: the bottom row in the above diagram is not necessarily exact, because the functor $C \mapsto \mathrm{Hom}(C, A)$ (on the category of abelian groups) does not generally preserve exactness. This will impede our progress, and in the present section we shall limit ourselves to situations where this algebraic obstacle can be avoided (a complete analysis of the homological algebra associated to the diagram will be carried out in Section 7.6). However, we shall make enough progress in the current section for the coming application to the Brown–Douglas–Fillmore Theorem.

There are at least two ways to present the index pairing. The first method, which is the most consistent with our treatment of K-homology in Chapter 5 and with our later presentation of Kasparov's theory, combines the definition K-homology given in Chapter 5 with the definition of K-theory in terms of projections and unitaries to provide explicit index-theoretic formulas for the pairings of $K_0(A)$ with $K^0(A)$ and of $K_1(A)$ with $K^1(A)$. This is done in Definitions 7.2.1 and 7.2.3 below. However, a rather involved computation is needed to show that the index pairing so defined is compatible with the boundary maps in the K-homology and K-theory six-term exact sequences. Proposition 7.2.4 presents the relevant formula, but we shall defer its proof until the next chapter. Keeping our present goal in mind — namely the Brown–Douglas–Fillmore Theorem

— a more expedient approach to the index pairing is to introduce 'higher' K-homology groups which are analogous to the higher K-theory groups we studied in Chapter 4. The index pairing between $K_1(A)$ and $K^1(A)$ then automatically determines pairings between the higher K-homology and K-theory groups, and it is immediate that these pairings are compatible with boundary maps. As one might imagine, the approach to the index pairings through higher groups does not altogether avoid the computational problem which arises in the first, more direct approach. One must now show that there are compatible Bott periodicity isomorphisms in higher K-theory and higher K-homology, and this is very closely related to the computation we are trying to avoid (see Exercise 7.7.10). But with a little ingenuity it is possible to obtain the Bott periodicity result as a consequence of calculations presented in earlier chapters (some of them in the exercises), and in this way we can piece together a complete proof of the Brown–Douglas–Fillmore Theorem from the tools we have currently available to us. To summarize, we shall first present the direct approach to the index pairing, leaving the key computation to be finished in the next chapter, and then we shall switch to the expedient approach for the rest of this chapter.

We begin with the pairing between $K_1(A)$ and $K^1(A)$. Let us recall that $K^1(A)$ is the group of Murray–von Neumann equivalence classes of ample projections P in $\mathfrak{D}(A)$.[56] Let P be such a projection and let u be a unitary over $\widetilde{A}$ (giving a generator of $K_1(A)$). Suppose for simplicity that u lies in $\widetilde{A}$ (as opposed to being a matrix over $\widetilde{A}$) and consider the operator PuP. The calculation

$$PuP \cdot Pu^*P \sim Puu^*P = P$$

and the similar calculation

$$Pu^*P \cdot PuP \sim Pu^*uP = P$$

show that PuP is a Fredholm operator on the Hilbert space PH. So we can form $\operatorname{Index}(PuP)$. More generally, if the unitary u belongs, not to $\widetilde{A}$, but to the matrix algebra $M_k(\widetilde{A})$, then we can form $\operatorname{Index}(P_k u P_k)$, where P_k denotes the diagonal $k \times k$ matrix with diagonal entries all P

7.2.1 DEFINITION The *index pairing* between $K_1(A)$ and $K^1(A)$ is the bilinear map $\langle\, ,\, \rangle\colon K_1(A) \times K^1(A) \to \mathbb{Z}$ defined by the formula $\langle [u], [P] \rangle = \operatorname{Index}(P_k u P_k)$, for any projection P in $\mathfrak{D}(A)$ and any $k \times k$ unitary matrix u over $\widetilde{A}$. The *index homomorphism* is the associated group homomorphism

$$\operatorname{Index}\colon K^1(A) \to \operatorname{Hom}(K_1(A), \mathbb{Z}).$$

7.2.2 REMARK Exercise 7.7.2 presents two other formulas for the index pairing.

[56] As in the later parts of Chapter 5, since we are considering here unreduced K-homology, when constructing the dual algebra $\mathfrak{D}(A)$ we use a representation of A which extends to an ample representation of $\widetilde{A}$ on the same separable Hilbert space H.

We leave to the reader the straightforward task of showing that Index $(P_k u P_k)$ only depends on u and P through their K-theory classes and is linear in both u and P (considering these arguments as generators for $K_1(A)$ and $K^1(A)$ respectively). The index pairing is functorial in the sense that if $\alpha\colon A \to B$ is a $*$-homomorphism of separable C^*-algebras then

$$x \in K_1(A), \quad y \in K^1(B) \quad \Rightarrow \quad \langle \alpha_*(x), y\rangle = \langle x, \alpha^*(y)\rangle.$$

To directly define a pairing between $K_0(A)$ and $K^0(A)$ we recall that $K_0(A)$ is generated by formal differences of projection matrices p over $\widetilde{A}$, while $K^0(A)$ is generated by homotopy classes of unitary operators U in $\mathfrak{D}(A)$. If $p \in M_k(\widetilde{A})$, and if U_k denotes the diagonal $k \times k$ matrix with diagonal entries all equal to U, then pU_kp is a Fredholm operator on the Hilbert space pH^k. Thus a second index pairing immediately presents itself:

7.2.3 Definition The *index pairing* between $K_0(A)$ and $K^0(A)$ is the bilinear map

$$\langle\, ,\, \rangle\colon K_0(A) \times K^0(A) \to \mathbb{Z}$$

defined by the formula $\langle [p], [U]\rangle = \operatorname{Index}(pU_kp)$, for any projection p in $M_k(\widetilde{A})$ and any unitary $U \in \mathfrak{D}(A)$. The *index homomorphism* is the associated group homomorphism

$$\operatorname{Index}: K^0(A) \to \operatorname{Hom}(K_0(A), \mathbb{Z}).$$

Once again, we leave the simple details involved in verifying that the pairing is well-defined and functorial to the reader. Less simple is the following computation, which, as we already mentioned, we shall not carry out until the next chapter:

7.2.4 Proposition *Let J be an ideal in a separable C^*-algebra A for which the quotient mapping from A to A/J is semisplit. Denote by ∂ the associated boundary maps in the K-homology and K-theory six-term exact sequences. If $x \in K_0(A/J)$ and $y \in K^1(J)$, then*

$$\langle \partial x, y\rangle = -\langle x, \partial y\rangle.$$

Similarly if $x \in K_1(A/J)$ and $y \in K^0(J)$, then

$$\langle \partial x, y\rangle = \langle x, \partial y\rangle.$$

If Proposition 7.2.4 is granted then it follows immediately that associated to every semisplit short exact sequence of separable C^*-algebras

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

there is a periodic, commuting (up to sign) diagram

$$\begin{array}{ccccc}
\cdots \longrightarrow & K^p(A) & \longrightarrow & K^p(J) & \\
& \downarrow & & \downarrow & \\
\cdots \longrightarrow & \mathrm{Hom}(K_p(A),\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(K_p(J),\mathbb{Z}) &
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & K^{p+1}(A/J) & \longrightarrow & K^{p+1}(A) & \longrightarrow \cdots, \\
& \downarrow & & \downarrow & \\
\longrightarrow & \mathrm{Hom}(K_{p+1}(A/J),\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(K_{p+1}(A),\mathbb{Z}) & \longrightarrow \cdots
\end{array}$$

of precisely the sort we require.[57] Hence:

7.2.5 LEMMA *Let* J *be an ideal in a separable and commutative* C^**-algebra* A. *Suppose that the bottom row in the diagram above is exact. If two out of the three index homomorphisms*

(a) $K^p(J) \to \mathrm{Hom}(K_p(J),\mathbb{Z})$
(b) $K^p(A/J) \to \mathrm{Hom}(K_p(A/J),\mathbb{Z})$
(c) $K^p(A) \to \mathrm{Hom}(K_p(A),J)$

are isomorphisms for $p = 0, 1$, *then the third index homomorphism is also an isomorphism for* $p = 0, 1$. □

We turn now to the alternative treatment of the index pairing, avoiding the as yet unproved Proposition 7.2.4, and based instead on the 'higher' groups

$$K_p(A) = K_1(S^{p-1}(A)) \quad \text{and} \quad K^p(A) = K^1(S^{p-1}(A)),$$

defined for $p \geqslant 1$. Note that the higher K-theory groups are the same as those we defined in Chapter 4. Here, however, we are taking $K_1(A)$, described as equivalence classes of unitary matrices, as our primary object, rather than $K_0(A)$.

From the index pairing between $K_1(A)$ and $K^1(A)$ that we presented in Definition 7.2.1 we immediately obtain functorial pairings

$$K_p(A) \times K^p(A) \to \mathbb{Z}$$

for $p \geqslant 1$, and associated index homomorphisms

$$\mathrm{Index} : K^p(A) \to \mathrm{Hom}(K_p(A),\mathbb{Z}).$$

The higher K-homology groups associated to a C^*-algebra extension can be organized into a long exact sequence similar to the one for their K-theory counterparts. Recall that in Section 4.5 we associated to each short exact sequence of C^*-algebras

[57] The subscripts and superscripts should be taken modulo 2, so that this is a commuting diagram of six-term exact sequences.

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

a long exact sequence of higher K-theory groups

$$\cdots \to K_p(J) \to K_p(A) \to K_p(A/J) \to K_{p-1}(J) \to K_{p-1}(A) \to K_{p-1}(A/J) \to \cdots$$

as follows. We introduced the mapping cone $C(A, A/J)$ in Definition 4.5.2 and defined $*$-homomorphisms

$$S(A/J) \to C(A, A/J) \quad \text{and} \quad J \to C(A, A/J).$$

Then we argued that the second $*$-homomorphism induces an isomorphism in K-theory. This allowed us to form the commuting diagram

$$\begin{tikzcd}
K_{p-1}(S(A/J)) \arrow[r] & K_{p-1}(C(A, A/J)) & K_{p-1}(J) \arrow[l, "\cong"'] \\
K_p(A/J) \arrow[u, equal] \arrow[rr, "\partial"'] & & K_{p-1}(J) \arrow[u, equal]
\end{tikzcd}$$

in which the bottom map was by definition the boundary map in the long exact sequence.

Now the proofs that the $*$-homomorphism $J \to C(A, A/J)$ induces an isomorphism in K-theory, and that the long exact sequence is indeed exact, rested on only two features of K-theory — its homotopy invariance and its half-exactness (see Remark 4.5.1). Therefore, since on the category of separable and commutative C^*-algebras the K-homology functor K^1 has these properties,[58] we obtain immediately a long exact sequence in higher K-homology,

$$K^1(A/J) \longrightarrow K^1(A) \longrightarrow K^1(J) \longrightarrow K^2(A/J) \longrightarrow K^2(A) \longrightarrow \cdots,$$

for each short exact sequence of separable and commutative C^*-algebras.

7.2.6 PROPOSITION *If J is an ideal in a separable and commutative C^*-algebra then the semi-infinite diagram*

[58] In contrast to Section 4.5 here we are only considering the category of separable and commutative C^*-algebras, as opposed to all C^*-algebras. Nonetheless the calculations in Section 4.5 apply since none of the constructions we used there will take us out of the commutative and separable category.

$$
\begin{tikzcd}
K^1(A/J) \arrow[r] \arrow[d] & K^1(A) \arrow[r] \arrow[d] & K^1(J) \arrow[d] \\
\mathrm{Hom}(K_1(A/J),\mathbb{Z}) \arrow[r] & \mathrm{Hom}(K_1(A),\mathbb{Z}) \arrow[r] & \mathrm{Hom}(K_1(J),\mathbb{Z})
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[r] & K^2(A/J) \arrow[r] \arrow[d] & K^2(A) \arrow[r] \arrow[d] & \cdots \\
{} \arrow[r] & \mathrm{Hom}(K_2(A/J),\mathbb{Z}) \arrow[r] & \mathrm{Hom}(K_2(A),\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}
$$

is commutative.

PROOF This follows immediately from the functoriality of the index pairing and the definitions of the boundary maps in K-theory and K-homology. □

Unfortunately Proposition 7.2.6 is not quite adequate for our purposes. This is because the diagram in the proposition is only semi-infinite, so that for example the index homomorphism from $K^1(A)$ to $\mathrm{Hom}(K_1(A),\mathbb{Z})$ is not surrounded on either side by two homomorphisms and we shall never be in a position to apply the Five Lemma to conclude it is an isomorphism.

We shall remedy this shortcoming by using Bott periodicity to extend the diagram in Proposition 7.2.6 infinitely to the left. The higher K-homology groups exhibit the same sort of Bott periodicity as their K-theoretic counterparts:

$$K^p(A) \cong \begin{cases} K^0(A) & \text{if } p \text{ is even,} \\ K^1(A) & \text{if } p \text{ is odd.} \end{cases} \tag{7.2.7}$$

To see this,[59] note first that the suspension $S(A) = C_0(0,1) \otimes A$, the cone $C(A) = C_0(0,1] \otimes A$, and A itself fit together to form a short exact sequence

$$0 \longrightarrow S(A) \longrightarrow C(A) \longrightarrow A \longrightarrow 0\,.$$

The C^*-algebra $C(A)$ is contractible, and so if A is commutative and separable then the K-homology of $C(A)$ vanishes. The six-term K-homology exact sequence associated to the above short exact sequence then reduces to a pair of boundary isomorphisms

$$K^0(S(A)) \cong K^1(A) \quad \text{and} \quad K^1(S(A)) \cong K^0(A).$$

By repeatedly applying these isomorphisms we obtain 7.2.7, as required.

The following result will be obtained in the exercises (it is of course also an immediate consequence of the as yet unproved Proposition 7.2.4).

[59]The argument requires the homotopy invariance of the functors K^0 and K^1; hence for the time being we shall consider only the case where A is commutative.

7.2.8 PROPOSITION *There are Bott periodicity isomorphisms* $K^1(S^2(A)) \cong K^1(A)$ *and* $K_1(S^2(A)) \cong K_1(A)$ *for which the diagram*

$$\begin{tikzcd} K^1(S^2(A)) \arrow[r, "\cong"] \arrow[d] & K^1(A) \arrow[d] \\ \mathrm{Hom}(K_1(S^2(A)), \mathbb{Z}) \arrow[r, "\cong"'] & \mathrm{Hom}(K_1(A), \mathbb{Z}) \end{tikzcd}$$

composed of index homomorphisms and periodicity isomorphisms, is commutative.

Proposition 7.2.8 takes care of the algebraic problem we raised following Proposition 7.2.6: if for example we can show using the Five Lemma that the index pairing induces an isomorphism $K^p(A) \cong \mathrm{Hom}(K_p(A), \mathbb{Z})$ for $p \geqslant 2$ then the same result for $p = 1$ follows immediately from periodicity and the result for $p = 3$. We obtain the following result, which is the counterpart for the higher K-homology groups of Lemma 7.2.5:

7.2.9 LEMMA *Let* J *be an ideal in a separable and commutative* C^**-algebra* A*. Suppose that the bottom row in the diagram of Proposition* 7.2.6 *is exact. If two out of the three index homomorphisms*

(a) $K^p(J) \to \mathrm{Hom}(K_p(J), \mathbb{Z})$
(b) $K^p(A/J) \to \mathrm{Hom}(K_p(A/J), \mathbb{Z})$
(c) $K^p(A) \to \mathrm{Hom}(K_p(A), J)$

are isomorphisms for all $p \geqslant 1$*, then the third index homomorphism is also an isomorphism for* $p \geqslant 1$. □

This is the essential fact about the index pairing that we shall use in what follows. The reader who prefers to follow the first approach to the analysis of the index pairing, relying on Proposition 7.2.4, need only replace all references to Lemma 7.2.9 by references to Lemma 7.2.5 in the proofs which follow, and take all the subscripts and superscripts mod 2. The structure of the argument, by either method, is exactly the same.

Here is the most important application of the lemma:

7.2.10 LEMMA *Let* J *be an ideal in a separable and commutative* C^**-algebra* A*. Suppose that* J*,* A *and* A/J *all have free abelian* K*-theory groups. If two out of the three index homomorphisms*

(a) $K^p(J) \to \mathrm{Hom}(K_p(J), \mathbb{Z})$
(b) $K^p(A/J) \to \mathrm{Hom}(K_p(A/J), \mathbb{Z})$
(c) $K^p(A) \to \mathrm{Hom}(K_p(A), J)$

are isomorphisms for all $p \geqslant 1$*, then the third index homomorphism is also an isomorphism for all* $p \geqslant 1$.

PROOF It is a straightforward exercise in algebra to show that the functor $\mathrm{Hom}(\,\cdot\,,\mathbb{Z})$ transforms exact sequences of *free* abelian groups into exact sequences. Since all the K-theory groups involved in the lemma are free abelian, the bottom row of the commutative diagram in Proposition 7.2.6 is exact. The Five Lemma implies that $K^p(A) \to \mathrm{Hom}(K_p(A),\mathbb{Z})$ is an isomorphism for $p \geqslant 2$ and Bott Periodicity takes care of the case $p = 1$. $\square$

7.2.11 LEMMA *The index pairing induces isomorphisms*

$$K^p(\mathbb{C}) \cong \mathrm{Hom}(K_p(\mathbb{C}),\mathbb{Z})$$

for all $p \geqslant 1$.

PROOF Note that $K^1(\mathbb{C}) = 0$ and $K_1(\mathbb{C}) = 0$. So in view of Proposition 7.2.8 it suffices to consider the case $p = 2$. In other words, it suffices to show that the index pairing induces an isomorphism

$$K^1(C_0(0,1)) \cong \mathrm{Hom}(K_1(C_0(0,1)),\mathbb{Z}).$$

Since both $K^1(C_0(0,1))$ and $K_1(C_0(0,1))$ are isomorphic to $\mathbb{Z}$ it suffices to find a projection $P \in \mathfrak{D}(C_0(0,1))$ and a unitary u such that $\mathrm{Index}(PuP) = \pm 1$. Identifying the unitalization of $C_0(0,1)$ with $C(S^1)$, the projection corresponding to the Toeplitz extension and the unitary function $z \mapsto z$ do the job. $\square$

7.2.12 REMARK We relied on a similar calculation in Chapter 4 in the course of our proof of the Bott Periodicity Theorem.

7.2.13 PROPOSITION *The index homomorphisms*

$$\mathrm{Index} : K^p(C(S^n)) \to \mathrm{Hom}(K_p(C(S^n)),\mathbb{Z})$$

are isomorphisms for all $p \geqslant 1$ *and all spheres* S^n.

PROOF The C^*-algebra $C(S^n)$ may be built in stages by forming extensions, starting from contractible C^*-algebras and $\mathbb{C}$. See the proofs of Proposition 7.1.3 and Corollary 7.1.5. All the C^*-algebras which occur in the construction have free abelian K-theory, so starting from Lemma 7.2.11 and homotopy invariance we can repeatedly apply Lemma 7.2.10 to complete the proof. $\square$

The possible presence of torsion in K-theory limits the extent to which we can extend Proposition 7.2.13 from spheres to other spaces (in the presence of torsion it is difficult to guarantee that the bottom row in the diagram of Proposition 7.2.6 will be exact). However it is possible to 'set aside' the torsion in K-theory and get the following approximate result:

7.2.14 PROPOSITION *If* X *is a finite simplicial complex then for every* $p \geqslant 1$ *the abelian groups* $K^p(C(X))$ *and* $K_p(C(X))$ *are finitely generated, and the map*

$$\mathrm{Index}: K^p(C(X)) \to \mathrm{Hom}(K_p(C(X)), \mathbb{Z})$$

induced from the index pairing has finite kernel and cokernel.

PROOF The first part of the proposition is an easy induction on the dimension of X, using the six-term exact sequence. In view of the first part, the second part is equivalent to the assertion that the 'rationalized' index map

$$\mathrm{Index}_{\mathbb{Q}}: K^p(C(X)) \otimes \mathbb{Q} \to \mathrm{Hom}(K_p(C(X)), \mathbb{Q})$$

is an isomorphism. But this rationalized assertion follows easily from the Five Lemma, since abstract algebra tells us that in the rationalized version of the diagram in Proposition 7.2.6 (where everything is tensored by $\mathbb{Q}$), the bottom row is always exact. □

7.3 Steenrod Homology Theory

The results of the previous two sections allow one to calculate K-homology, at least up to torsion, for finite complexes. But for our analytical purposes we shall need to compute K-homology for spaces more complicated than finite complexes, and to do so we shall show in the present section that K-homology obeys an additional *cluster axiom.*

7.3.1 DEFINITION A *generalized Steenrod homology theory* on the category of locally compact and second countable Hausdorff spaces is a collection of covariant functors $h_p(X)$, indexed by $p \in \mathbb{Z}$, such that

(a) all the h_p are homotopy functors,
(b) if $Y \subseteq X$ is a closed subset there is a functorial long exact sequence

$$\cdots \longrightarrow h_p(Y) \longrightarrow h_p(X) \longrightarrow h_p(X \setminus Y) \longrightarrow h_{p-1}(Y) \longrightarrow \cdots,$$

and
(c) if X is a countable disjoint union of locally compact spaces X_j then the projections $X \to X_j$ induce an isomorphism $h_p(X) \cong \prod_j h_p(X_j)$.

Axioms (a) and (b) are precisely the same as in Definition 7.1.1. Axiom (c) is the cluster axiom, and our first goal in this section is to explain how it allows us to approach the homology groups of a variety of quite complex spaces. The Hawaiian earring H and the Sierpinski gasket G, both pictured in Figure 7.1, are good examples. The Hawaiian earring is actually quite easy to analyze since after removing the basepoint from H the locally compact space which remains

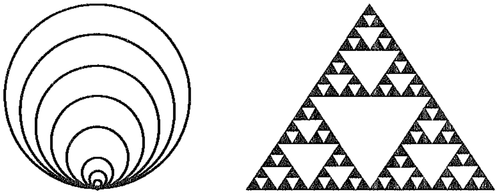

FIG. 7.1. The Hawaiian earring and Sierpinski gasket

is a countable disjoint union of copies of $\mathbb{R}$. So it follows immediately from the cluster axiom that

$$h_p(H) = h_p(\mathrm{pt}) \oplus (\oplus_{j=1}^{\infty} h_{p-1}(\mathrm{pt}))$$

for every p.[60]

The Sierpinski gasket is in an obvious way the intersection of a decreasing sequence of compact spaces each of which is a finite simplicial complex. In fact any compact subset X of the plane or of n-space is such an intersection. To see this, triangulate the ambient space and denote by X_1 the (finite) union of those closed simplices which intersect X; then subdivide the triangulation of the ambient space and denote by X_2 the union of those closed simplices in the subdivision which intersect X; and so on.[61] The analysis presented below, which applies to any compact space X which is the intersection of a decreasing family of compact spaces $\{X_j\}_{j=1}^{\infty}$, relates the homology of X to the homology groups of the spaces X_j.

Let T be the following subset of $[0,1) \times X_1$:

$$T = \{\, (t,x) \in [0,1) \times X_1 \;:\; t \leqslant 1/j \;\Rightarrow\; x \in X_j \,\}.$$

It is a locally compact space, called the *telescope* associated to the sequence of spaces $\{X_j\}_{j=1}^{\infty}$ (to see why, draw a picture in the case where $\{X_j\}_{j=1}^{\infty}$ is a decreasing sequence of concentric disks). The telescope is contractible as a locally compact space. Furthermore T contains a natural copy of X, namely $\{0\} \times X \subseteq T$. In view of the contractibility of T, the long exact sequence for the pair (T, X) reduces to a collection of isomorphisms

$$h_p(T_0) \cong h_{p-1}(X),$$

where

[60]The picture of the Hawaiian earring should also help explain why the cluster axiom is so called.

[61]In fact it is not hard to show that *any* compact metric space is the intersection of a decreasing sequence of compact spaces each of which is homotopy equivalent to a finite simplicial complex.

$$T_0 = T \setminus X = \{ (t,x) \in (0,1) \times X_1 \, : \, t \leqslant 1/j \Rightarrow x \in X_j \}.$$

Now let T_1 be the following closed subspace of T_0:

$$T_1 = \bigcup_j \{1/j\} \times X_j.$$

Observe that

$$T_0 \setminus T_1 = \bigcup_j (1/(j+1), 1/j) \times X_j \cong \bigcup_j SX_j,$$

where SX_j denotes the suspension $\mathbb{R} \times X_j$. Thanks to the cluster axiom the long exact sequence for the pair (T_0, T_1) takes the following form:

$$\longrightarrow \textstyle\prod_j h_p(X_j) \longrightarrow h_p(T_0) \longrightarrow \prod_j h_p(SX_j) \longrightarrow \prod_j h_{p-1}(X_j) \longrightarrow .$$

If we substitute into this sequence the suspension isomorphisms $h_p(SX_j) \cong h_{p-1}(X_j)$ of Remark 7.1.4 and the isomorphisms $h_p(T_0) \cong h_{p-1}(X)$ obtained above we get the following exact sequence relating the homology of X to the homology of the spaces X_j:

(7.3.2)

$$\longrightarrow \textstyle\prod_j h_p(X_j) \longrightarrow h_{p-1}(X) \longrightarrow \prod_j h_{p-1}(X_j) \longrightarrow \prod_j h_{p-1}(X_j) \longrightarrow .$$

Note for later use that all the isomorphisms involved in its construction are connecting maps in the homology long exact sequences for various pairs of spaces.

The exact sequence 7.3.2 is all that we shall use when we turn to K-homology. But it is interesting (and fairly easy) to calculate the map from the direct product $\prod_j h_{p-1}(X_j)$ to itself in 7.3.2, and so make much more explicit the relation between the homology of X and the homology of the spaces X_j. Here is the result:

7.3.3 LEMMA *The map $\prod_j h_{p-1}(X_j) \to \prod_j h_{p-1}(X_j)$ is given, up to sign, by the formula*

$$(a_1, a_2, a_3, \dots) \mapsto (a_1 - \alpha_2(a_2), a_2 - \alpha_3(a_3), a_3 - \alpha_4(a_4), \dots),$$

where $a_j \in h_{p-1}(X_j)$ and $\alpha_j \colon h_{p-1}(X_j) \to h_{p-1}(X_{j-1})$ is the map induced by the inclusion $X_j \subseteq X_{j-1}$. $\square$

Now the kernel of the map in Lemma 7.3.3 is called the *inverse limit* of the system of groups

$$h_{p-1}(X_1) \xleftarrow{\alpha_2} h_{p-1}(X_2) \xleftarrow{\alpha_3} h_{p-1}(X_3) \xleftarrow{\alpha_4} \cdots,$$

and is denoted $\varprojlim h_{p-1}(X_j)$. The cokernel of the map is denoted $\varprojlim^1 h_{p-1}(X_j)$. With this notation, we can reformulate the long exact sequence 7.3.2 as follows:

7.3.4 PROPOSITION *Let* X *be the intersection of a decreasing sequence of compact, metrizable spaces* X_j. *There is a functorial short exact sequence*

$$0 \longrightarrow \varprojlim{}^1 h_{p+1}(X_j) \longrightarrow h_p(X) \longrightarrow \varprojlim h_p(X_j) \longrightarrow 0\,. \quad \square$$

The exact sequence of the proposition is called the '$\varprojlim^1$ sequence'. The map from $h_p(X)$ to $\varprojlim h_p(X_j)$ is of course the one induced by the inclusions $X \subseteq X_j$.

Returning to the Sierpinski gasket, if we write G as a decreasing intersection $G = \cap_j G_j$ in the obvious way then each of the inclusions $G_{j+1} \subseteq G_j$ induces a surjection on homology groups — indeed each inclusion has a right homotopy-inverse. It follows that $\varprojlim^1 h_p(G_j) = 0$, and so from the $\varprojlim^1$ sequence we deduce that $h_p(G) \cong \varprojlim h_p(G_j)$.

7.4 The Cluster Axiom for K-Homology

In this section we shall prove that K-homology is a generalized Steenrod homology theory.

7.4.1 DEFINITION Let $\{A_j\}_{j=1}^{\infty}$ be a sequence of C^*-algebras. The *C^*-algebra direct sum* $\bigoplus_j A_j$ is the C^*-algebra of all sequences $(a_1, a_2, \dots)$ with $a_j \in A_j$ for which $\|a_j\| \to 0$ as $j \to \infty$. The $*$-algebra operations in $\bigoplus_j A_j$ are defined pointwise and the norm is $\sup_j \|a_j\|$.

If X is the disjoint union of locally compact spaces $X_1, X_2, \dots$ then

$$C_0(X) \cong \bigoplus_j C_0(X_j).$$

So to verify the cluster axiom for K-homology it suffices to prove the following result:

7.4.2 PROPOSITION *Let $\{A_j\}_{j=1}^{\infty}$ be a sequence of C^*-algebras. The natural inclusions $A_j \to \bigoplus_j A_j$ induce isomorphisms of abelian groups*

$$K^p(\bigoplus_j A_j) \xrightarrow{\cong} \prod_j K^p(A_j)$$

for every $p \geqslant 1$.

Our proof of the proposition will use the following additional C^*-algebra construction:

7.4.3 DEFINITION Let $\{A_j\}_{j=1}^{\infty}$ be a sequence of C^*-algebras. The *C^*-algebra direct product* $\prod_j A_j$ is the C^*-algebra of all bounded sequences $(a_1, a_2, \dots)$ with $a_j \in A_j$. As with the direct sum, the $*$-algebra operations in $\prod_j A_j$ are defined pointwise and the norm is $\sup_j \|a_j\|$.

PROOF OF PROPOSITION 7.4.2 We shall consider first the case $p = 1$. Let $A = \bigoplus_j A_j$. Suppose that the dual algebras $\mathfrak{D}(A_j)$ are formed using ample representations of $\widetilde{A_j}$ on Hilbert spaces H_j. Then the Hilbert space direct sum $H = \bigoplus_j H_j$ is an ample representation of $\widetilde{A}$, and we shall use this direct sum H to form the dual $\mathfrak{D}(A)$. Corresponding to the inclusions $H_j \subseteq H$ there are $*$-homomorphisms $\mathfrak{D}(A_j) \to \mathfrak{D}(A)$, and these may be organized into a single $*$-homomorphism

$$\prod_j \mathfrak{D}(A_j) \to \mathfrak{D}(A).$$

This is not an isomorphism; its image is the 'diagonal part' of $\mathfrak{D}(A)$, comprising those operators on H which commute with the projection operators P_j onto the spaces H_j. However, the difference between an operator $T \in \mathfrak{D}(A)$ and its 'diagonal part' $\sum_j P_j T P_j$ lies within the ideal $\mathfrak{D}(A /\!\!/ A)$. It follows that there is an induced isomorphism

$$\prod_j \mathfrak{D}(A_j)/\mathfrak{D}(A_j /\!\!/ A_j) \cong \mathfrak{D}(A)/\mathfrak{D}(A /\!\!/ A). \tag{7.4.4}$$

It is tempting now to say that the proof is completed, because of the isomorphism $K^1(A) \cong K_0(\mathfrak{D}(A)/\mathfrak{D}(A /\!\!/ A))$ noted in Section 5.4 and the 'fact' that the K-theory of a direct product of C^*-algebras is the direct product of the K-theory groups. Unfortunately this 'fact' is not true in general (see Exercise 7.7.3 for an example), so we must be more careful.

Since the quotient map from $\mathfrak{D}(A_j)$ to $\mathfrak{D}(A_j)/\mathfrak{D}(A_j /\!\!/ A_j)$ induces an isomorphism in K-theory, and since every K_0-class for the algebra $\mathfrak{D}(A_j)$ is represented by a projection in $\mathfrak{D}(A_j)$ (see Proposition 5.1.4), the map from the group $K_0(\prod_j \mathfrak{D}(A_j)/\mathfrak{D}(A_j /\!\!/ A_j))$ to the product $\prod_j K_0(\mathfrak{D}(A_j)/\mathfrak{D}(A_j /\!\!/ A_j))$ is surjective.

A relative version of the same argument proves injectivity. Thanks to the isomorphism 7.4.4 and thanks to the fact that the quotient map from $\mathfrak{D}(A)$ to $\mathfrak{D}(A)/\mathfrak{D}(A /\!\!/ A)$ induces an isomorphism in K-theory, Proposition 5.1.4 implies that every K_0-class for $\prod_j \mathfrak{D}(A_j)/\mathfrak{D}(A_j /\!\!/ A_j)$ is represented by a projection $p \in \prod_j \mathfrak{D}(A_j)/\mathfrak{D}(A_j /\!\!/ A_j)$, or in other words by a family of projections p_j in the algebras $\mathfrak{D}(A_j)/\mathfrak{D}(A_j /\!\!/ A_j)$. By Exercise 5.6.12 these projections lift to projections P_j in $\mathfrak{D}(A_j)$; we may assume that each P_j is ample in the sense of Definition 5.1.3.[62] Suppose now that the class of the projection p maps to zero in the product $\prod_j K_0(\mathfrak{D}(A_j)/\mathfrak{D}(A_j /\!\!/ A_j))$. Then each p_j determines the zero element of the group $K_0(\mathfrak{D}(A_j)/\mathfrak{D}(A_j /\!\!/ A_j))$, and since the maps from $K_0(\mathfrak{D}(A_j))$ to $K_0(\mathfrak{D}(A_j)/\mathfrak{D}(A_j /\!\!/ A_j))$ are isomorphisms it follows that each P_j determines the zero element in $K_0(\mathfrak{D}(A_j))$. In this case it follows from Proposition 5.1.4 that

[62] If one of the projections P_j is not ample then repeat the entire construction with $p \oplus 1$ in place of p (note that $p \oplus 1$ and p have the same K-theory class).

each P_j is Murray–von Neumann equivalent to I, via some isometry $V_j \in \mathfrak{D}(A_j)$. The family of isometries V_j then determines an isometry in $\prod_j \mathfrak{D}(A_j)/\mathfrak{D}(A_j /\!\!/ A_j)$ which implements a Murray–von Neumann equivalence between the initial projection p and the unit of $\prod_j \mathfrak{D}(A_j)/\mathfrak{D}(A_j /\!\!/ A_j)$. But we noted in the proof of Proposition 5.1.4 that the K-theory class of the unit in $\mathfrak{D}(A)$ is zero. The same is therefore true for the unit in $\prod_j \mathfrak{D}(A_j)/\mathfrak{D}(A_j /\!\!/ A_j) \cong \mathfrak{D}(A)/\mathfrak{D}(A /\!\!/ A)$ and hence the K-theory class of p is trivial, as required. This proves the proposition for the K-homology group K^1.

If $p > 1$ then the result for K^p follows from the result for K^1 and the definition

$$K^p(A) = K^1(S^{p-1}A),$$

since the suspension of a C^*-algebra direct sum is the direct sum of the suspensions of the summands. $\square$

We conclude with a simple consequence which will more or less immediately imply the Brown–Douglas–Fillmore Theorem in the next section.

7.4.5 PROPOSITION *Let X be the intersection of a decreasing sequence of compact metrizable spaces X_j and suppose that the K-theory groups of the spaces X and X_j are all free abelian. If the index homomorphisms*

$$\mathrm{Index} : K^p(C(X_j)) \to \mathrm{Hom}(K_p(C(X_j)), \mathbb{Z})$$

are isomorphisms for all $p \geqslant 1$ and all j then the index homomorphisms

$$\mathrm{Index} : K^p(C(X)) \to \mathrm{Hom}(K_p(C(X)), \mathbb{Z})$$

are isomorphisms for all $p \geqslant 1$ as well.

PROOF It follows from Proposition 4.1.15 and Remark 4.2.3 that the inclusions $C(X_j) \to \oplus_j C(X_j)$ induce isomorphisms

$$\bigoplus_j K_p(C(X_j)) \cong K_p(\bigoplus_j C(X_j)),$$

for every p. These in turn induce isomorphisms

$$\mathrm{Hom}(K_p(\bigoplus_j C(X_j)), \mathbb{Z}) \cong \prod_j \mathrm{Hom}(K_p(C(X_j)), \mathbb{Z}). \tag{7.4.6}$$

With these in hand, the same construction which leads to the exact sequence 7.3.2 produces a commuting diagram

$$
\begin{tikzcd}
\prod_j K^p(C(X_j)) \arrow[r] \arrow[d] & \prod_j K^p(C(X_j)) \arrow[r] \arrow[d] & K^{p-1}(C(X)) \arrow[d] \\
\prod_j \mathrm{Hom}(K_p(C(X_j)),\mathbb{Z}) \arrow[r] & \prod_j \mathrm{Hom}(K_p(C(X_j),\mathbb{Z}) \arrow[r] & \mathrm{Hom}(K_{p-1}(C(X)),\mathbb{Z})
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[r] & \prod_j K^{p-1}(C(X_j)) \arrow[r] \arrow[d] & \prod_j K^{p-1}(C(X_j)) \arrow[r] \arrow[d] & \cdots \\
{} \arrow[r] & \prod_j \mathrm{Hom}(K_{p-1}(C(X_j)),\mathbb{Z}) \arrow[r] & \prod_j \mathrm{Hom}(K_{p-1}(C(X_j)),\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}
$$

in which the horizontal rows are exact (the bottom one thanks to 7.4.6 and our hypothesis that all the K-theory groups are free abelian). Since the vertical maps between the direct products are by hypothesis isomorphisms, it follows from the Five Lemma that for $p \geqslant 2$ the remaining vertical maps are isomorphisms as well. The case $p = 1$ is dealt with by Bott periodicity, as in the discussion preceding Lemma 7.2.9. $\square$

7.5 The Brown–Douglas–Fillmore Theorem

We are now going to apply all of the machinery developed so far to the Brown–Douglas–Fillmore theory of essentially normal operators. We need some elementary results concerning the K-theory of planar sets, whose proofs we shall merely sketch.

7.5.1 Definition Let X be a non-empty compact Hausdorff space.

(a) Denote by $\check{H}^0(X)$ the group of continuous, integer-valued functions on X. Define a *character homomorphism* $K_0(C(X)) \to \check{H}^0(X)$ by assigning to a projection in $M_k(C(X))$ — which is a continuous, projection-valued function on X — its pointwise rank.
(b) Denote by $\check{H}^1(X)$ the group (under pointwise multiplication) of homotopy classes of continuous, nowhere zero, complex-valued functions on X. Define a *character homomorphism* $K_1(C(X)) \to \check{H}^1(X)$ by assigning to a unitary in $M_k(C(X))$ — which is a continuous, unitary-valued function on X — its pointwise determinant.

The groups $\check{H}^0(X)$ and $\check{H}^1(X)$ are familiar to topologists as the lowest two *Čech cohomology groups* of X. The above character homomorphisms are the lowest two components of the *Chern character*.

7.5.2 Proposition *If X is a non-empty, compact subset of the plane then the character homomorphisms $K_p(C(X)) \to \check{H}^p(X)$ are isomorphisms for $p = 0$ and $p = 1$.* $\square$

The proposition is proved in two stages. First, if X is a finite complex then the proof is an induction on the number of simplices in X. Second, a general X may be written as the intersection of a decreasing sequence of complexes X_j (as in Section 7.3). Now we can write the C*-algebra $C(X)$ as the direct limit of the system of C*-algebras

$$C(X_1) \longrightarrow C(X_2) \longrightarrow C(X_3) \longrightarrow \cdots.$$

This is a directed system of the kind that is considered in Exercise 4.10.5, and $K_p(C(X)) = \varinjlim_j K_p(C(X_j))$. Now the groups $\check{H}^p(X)$ satisfy the same 'continuity' condition — namely $\check{H}^p(X) \cong \varinjlim_j \check{H}^p(X_j)$. So the general case follows from the case for complexes.

It follows from Proposition 7.5.2 that the group $K_0(C(X))$ is free abelian (see Exercise 7.7.5). The group $K_1(C(X))$ is also free abelian, although this is perhaps not so clear from the proposition. To clarify matters, here is another description of $K_1(C(X))$.

7.5.3 PROPOSITION *Let X be a non-empty, compact subset of the plane and let $\{\lambda_1, \lambda_2, \ldots\}$ be a sequence of points in $\mathbb{C}\setminus X$, with precisely one λ_j in each bounded component of the complement of X, and none in the unbounded component of the complement. The group $\check{H}^1(X)$ is freely generated by the homotopy classes of the nowhere zero complex functions $z - \lambda_j$ on X.* □

The proof is similar to that of Proposition 7.5.2, namely a reduction to the case of a finite complex, followed by an induction on the number of cells.

From Bott periodicity we obtain the following result:

7.5.4 COROLLARY *If X is a compact subset of the plane then the groups $K_p(C(X))$ are free abelian for all p.* □

7.5.5 THEOREM *If X is a compact subset of the plane then the index homomorphisms*

$$\text{Index} : K^p(C(X)) \to \text{Hom}(K_p(C(X)), \mathbb{Z})$$

are isomorphisms of abelian groups, for all p.

PROOF An induction on the number of simplices, using Lemma 7.2.10, shows that if X is a finite complex then the theorem holds for X. To complete the proof we write a general X as the intersection of a decreasing sequence of finite complexes X_j, and appeal to Proposition 7.4.5. □

The Brown–Douglas–Fillmore Theorem is a simple consequence of this:

7.5.6 BROWN–DOUGLAS–FILLMORE THEOREM *Two essentially normal operators with the same essential spectrum X are essentially unitary equivalent if and only if $\text{Index}(T_1 - \lambda I) = \text{Index}(T_2 - \lambda I)$, for every $\lambda \in \mathbb{C}\setminus X$. Moreover, every locally constant function on $\mathbb{C}\setminus X$ which vanishes at infinity is the index function $\lambda \mapsto \text{Index}(T - \lambda I)$ of some essentially normal operator T with essential spectrum X.*

PROOF The set of essential unitary equivalence classes of essentially normal operators with essential spectrum X is isomorphic to $K^1(C(X))$. The isomorphism is as follows. If $T \in \mathfrak{B}(H)$ is essentially normal with essential spectrum X then in Chapter 3 we showed that there is a normal operator T_1 with spectrum X, acting on a Hilbert space H_1 which contains H, such that if P denotes the orthogonal projection onto H then PT_1P is equal to T modulo compact operators. The functional calculus $f \mapsto f(T_1)$ gives a representation of $C(X)$ on H_1; the projection P essentially commutes with this representation; and to define a K-homology class $[T] \in K^1(A)$ we associate to T the K-theory class of the projection $P \in \mathfrak{D}(A)$. If $[f] \in \check{H}^1(X) \cong K_1(C(X))$ then the index pairing between $[f]$ and $[T]$ is the Fredholm index of the operator $Pf(T_1)P$ on $PH_1 = H$. So if $f(z) = z - \lambda$ then the pairing is

$$[z-\lambda] \otimes [T] \mapsto \mathrm{Index}\,(P(T_1 - \lambda I)P) = \mathrm{Index}\,(T - \lambda I).$$

By Proposition 7.5.3 the group $\check{H}^1(X)$ is generated by the classes of the functions $(z-\lambda)$, where $\lambda \notin X$, while by the injectivity of the index homomorphism in Theorem 7.5.5, two essentially normal operators determine the same K-homology class if and only if their pairings with $\check{H}^1(X)$ are equal. This proves the first part of the theorem. The second part follows from the surjectivity of the index homomorphism. □

7.6 The Universal Coefficient Theorem

The assertion that the index homomorphism is *always* an isomorphism is obviously false, since $\mathrm{Hom}(K_p(A), \mathbb{Z})$ is a torsion-free group, whereas $K^p(A)$ may very well have torsion. In this section we shall show how to describe $K^p(A)$ (for commutative C^*-algebras A) in the presence of torsion.

We shall assume the reader is familiar with elementary homological algebra, since to properly take torsion into account we must introduce the group $\mathrm{Ext}\,(G, G')$ of equivalence classes of abelian group extensions

$$0 \longrightarrow G' \longrightarrow G'' \longrightarrow G \longrightarrow 0\,.$$

We shall, however, remind the reader of the main features of this Ext-functor as we need them.

We are going to prove the following result:

7.6.1 UNIVERSAL COEFFICIENT THEOREM *Let A be a separable and commutative C^*-algebra. For every $p \geqslant 1$ there is a short exact sequence*

$$0 \longrightarrow \mathrm{Ext}\,(K_p(A), \mathbb{Z}) \longrightarrow K^{p+1}(A) \longrightarrow \mathrm{Hom}(K_{p+1}(A), \mathbb{Z}) \longrightarrow 0\,,$$

in which the map from $K^{p+1}(A)$ to $\mathrm{Hom}(K_{p+1}(A), \mathbb{Z})$ is the index homomorphism.

7.6.2 REMARK If $K_p(A)$ is a free abelian group then $\operatorname{Ext}(K_p(A), \mathbb{Z}) = 0$, so the theorem asserts that in this case the index map is an isomorphism. If $K_p(A)$ is a finitely generated abelian group then $\operatorname{Ext}(K_p(A), \mathbb{Z})$ is a finite abelian group, so in this case the theorem implies that the index map is an isomorphism, modulo torsion (compare Proposition 7.2.14).

We shall prove the theorem by a somewhat involved reduction to the free abelian case. The following definitions are central to the argument:

7.6.3 DEFINITION A *projective resolution* of a commutative and separable C^*-algebra A is a short exact sequence of commutative and separable C^*-algebras

$$0 \longrightarrow J \longrightarrow B \longrightarrow A \longrightarrow 0$$

such that

(a) $K_p(B)$ is free abelian for every p, and
(b) the induced map $K_p(B) \to K_p(A)$ is surjective for every p.

7.6.4 DEFINITION Let B be a commutative and separable C^*-algebra whose K-theory groups $K_p(B)$ are free abelian, for every p. We shall say that B is *admissible* if the index homomorphisms $\operatorname{Index}: K^p(B) \to \operatorname{Hom}(K_p(B), \mathbb{Z})$ are isomorphisms, for every p. We shall say that a projective resolution

$$0 \longrightarrow J \longrightarrow B \longrightarrow A \longrightarrow 0$$

is admissible if the C^*-algebra B is admissible. (Note that while the K-theory groups $K_p(J)$ are free abelian — since any subgroup of a free abelian group is free — we do *not* require that J be admissible.)

7.6.5 LEMMA *Let* I *be an ideal in a commutative and separable* C^**-algebra* A *and suppose that all of the* K*-theory groups of the* C^**-algebras* I, A *and* A/I *are free abelian. If two of the three* C^**-algebras* I, A, *and* A/I *are admissible, then so is the third.*

PROOF This is proved the same way as Lemma 7.2.10. □

7.6.6 PROPOSITION *Every commutative and separable* C^**-algebra has an admissible projective resolution.*

SKETCH OF THE PROOF This result is essentially due to Atiyah.[63] Suppose first that $A = C(X)$, where X is a finite complex. Atiyah shows that there is an inclusion of X into a finite direct product Y of homogeneous spaces of the type $U(N)$ or $U(N)/U(N-k) \times U(k)$ for which the associated $*$-homomorphism of $B = C(Y)$ onto A is a projective resolution. The admissibility of B follows from

[63] See the paper [6].

the fact that the space Y has a cellular structure for which the skeleta all have free abelian K-theory. So Lemma 7.6.5 applies. Atiyah's argument adapts to the case $A = C(X)$, where X is any compact, metrizable space. Now Y may be an *infinite* direct product, in which case the proof of admissibility for $B = C(Y)$ uses in addition the ideas developed in Section 7.3. The case of a non-unital C^*-algebra A is handled by adjoining a unit. □

7.6.7 DEFINITION Let A be a separable and commutative C^*-algebra. Denote by ${}^{\circ}K^p(A) \subseteq K^p(A)$ the kernel of the index homomorphism $\text{Index}: K^p(A) \to \text{Hom}(K_p(A), \mathbb{Z})$.

7.6.8 LEMMA *The image of the boundary map* $\partial\colon K^{p-1}(J) \to K^p(A)$ *in the* K*-homology long exact sequence associated to an admissible projective resolution*

$$0 \longrightarrow J \longrightarrow B \longrightarrow A \longrightarrow 0\,,$$

is the subgroup ${}^{\circ}K^p(A) \subseteq K^p(A)$.

PROOF Consider the following diagram:

$$\begin{tikzcd} K^{p-1}(J) \arrow[r, "\partial"] & K^p(A) \arrow[r] \arrow[d] & K^p(B) \arrow[d, "\cong"] \\ 0 \arrow[r] & \text{Hom}(K_p(A), \mathbb{Z}) \arrow[r] & \text{Hom}(K_p(B), \mathbb{Z}). \end{tikzcd}$$

The top row is exact at $K^p(A)$ and the bottom horizontal map is injective because the map $K_p(B) \to K_p(A)$ is surjective. The rest of the argument is a diagram chase. □

Let us now fix a projective resolution as in Lemma 7.6.8. Because the maps $K_p(B) \to K_p(A)$ are surjective, the boundary maps in the associated six-term K-theory exact sequence are zero. The six-term sequence therefore reduces to a collection of short exact sequences

$$0 \longrightarrow K_p(J) \longrightarrow K_p(B) \longrightarrow K_p(A) \longrightarrow 0\,.$$

The above is a *free resolution* of the abelian group $K_p(A)$. By basic homological algebra, associated to this free resolution is an exact sequence

$$0 \leftarrow \text{Ext}\,(K_p(A), \mathbb{Z}) \leftarrow \text{Hom}(K_p(J), \mathbb{Z}) \leftarrow \text{Hom}(K_p(B), \mathbb{Z}) \leftarrow \text{Hom}(K_p(A), \mathbb{Z}) \leftarrow 0.$$

The homomorphism from $\text{Hom}(K_p(J), \mathbb{Z})$ to $\text{Ext}\,(K_p(A), \mathbb{Z})$ associates to a group homomorphism $\varphi\colon K_p(A) \to \mathbb{Z}$ the unique equivalence class of abelian group extensions for which there is a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & K_p(J) \arrow[r] \arrow[d, "\varphi"'] & K_p(B) \arrow[r] \arrow[d] & K_p(A) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & G \arrow[r] & K_p(A) \arrow[r] & 0.
\end{tikzcd}$$

At the same time, from Lemma 7.6.8 we obtain an exact sequence

$$0 \longleftarrow {}^{\circ}K^{p+1}(A) \longleftarrow K^p(J) \longleftarrow K^p(B) \longleftarrow K^p(A)/{}^{\circ}K^p(A) \longleftarrow 0$$

It combines with the homological algebra exact sequence to form the following commuting diagram, in which the rows are exact, and the group homomorphism denoted $^{\circ}\mathrm{Index}$ is the one forced by commutativity:

(7.6.9)

$$\begin{tikzcd}
0 & {}^{\circ}K^{p+1}(A) \arrow[l] \arrow[d, "^{\circ}\mathrm{Index}"'] & K^p(J) \arrow[l] \arrow[d, "\mathrm{Index}"'] & K^p(B) \arrow[l] \arrow[d, "\mathrm{Index}", "\cong"'] & K^p(A)/{}^{\circ}K^p(A) \arrow[l] \arrow[d, "\mathrm{Index}"] & 0 \arrow[l] \\
0 & \mathrm{Ext}\,(K_p(A), \mathbb{Z}) \arrow[l] & \mathrm{Hom}(K_p(J), \mathbb{Z}) \arrow[l] & \mathrm{Hom}(K_p(B), \mathbb{Z}) \arrow[l] & \mathrm{Hom}(K_p(A), \mathbb{Z}) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Let us examine in more detail the homomorphism $^{\circ}\mathrm{Index}$. According to its somewhat roundabout definition, if $x \in K^p(J)$ then the map $^{\circ}\mathrm{Index}$ associates to $\partial(x) \in {}^{\circ}K^{p+1}(A)$ the extension associated to the abelian group homomorphism $\mathrm{Index}\,(x)\colon K_p(J) \to \mathbb{Z}$. There is, however, a very simple and direct way of associating an abelian group extension to a class in ${}^{\circ}K^{p+1}(A)$. To each class in $K^{p+1}(A)$ there corresponds an equivalence class of C^*-algebra extensions

$$0 \longrightarrow \mathfrak{K}(H) \longrightarrow E \longrightarrow S^p A \longrightarrow 0\,.$$

A class lies in ${}^{\circ}K^{p+1}(A)$ if and only if the boundary maps in the six-term K-theory sequence associated to the above C^*-algebra extension are zero (see Exercise 7.7.2). So to each class in ${}^{\circ}K^{p+1}(A)$ there is a corresponding abelian group extension

$$0 \longrightarrow K_0(\mathfrak{K}(H)) \longrightarrow K_0(E) \longrightarrow K_0(S^p A) \longrightarrow 0\,.$$

The following lemma shows that $^{\circ}\mathrm{Index}$ is essentially this correspondence.

7.6.10 Lemma *The homomorphism* $^{\circ}\mathrm{Index}\colon {}^{\circ}K^{p+1}(A) \to \mathrm{Ext}\,(K_p(A), \mathbb{Z})$ *does not depend on the choice of the admissible projective resolution used in its definition. It associates to the class of a* C^**-algebra extension*

$$0 \longrightarrow \mathfrak{K}(H) \longrightarrow E \longrightarrow S^p A \longrightarrow 0$$

the additive inverse of the class, in $\operatorname{Ext}(K_p(A), \mathbb{Z})$*, of the abelian group extension*

$$0 \longrightarrow K_0(\mathfrak{K}(H)) \longrightarrow K_0(E) \longrightarrow K_0(S^p A) \longrightarrow 0 .$$

PROOF By replacing A with $S^p A$, if necessary, it suffices to consider the case $p = 1$. Each class $x \in K^1(J)$ is represented by an extension

$$0 \longrightarrow \mathfrak{K}(H) \longrightarrow F \longrightarrow J \longrightarrow 0 .$$

Recall that the boundary map $\partial\colon K^1(J) \to K^2(A)$ passes through the K-homology of the mapping cone $C(B, A)$ associated to our projective resolution. Translating this into the language of extensions we see that the above extension appears as the top row in a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{K}(H) \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{K}(H) \arrow[r] & E \arrow[r] & C(B,A) \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{K}(H) \arrow[r] \arrow[u, equal] & D \arrow[r] \arrow[u] & SA \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

in which the K-homology class of the bottom row is $\partial(x) \in K^1(SA) = K^2(A)$. Now the bottom extension in the above diagram may be placed into a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & SJ \arrow[r] \arrow[d] & SB \arrow[r] \arrow[d] & SA \arrow[r] \arrow[d, "\tau"] & 0 \\
0 \arrow[r] & C(E, C(B,A)) \arrow[r] & \widetilde{C}(E, C(B,A)) \arrow[r] & SA \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{K}(H) \arrow[r] \arrow[u] & D \arrow[r] \arrow[u] & SA \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

whose constituent pieces are as follows:

(a) $C(E, C(B, A))$ is the mapping cone associated to the middle extension in our first diagram, made up of pairs $(e, f) \in E \oplus C(C(B, A))$ such that e maps to $f(0) \in C(B, A)$.

(b) The homomorphism $SJ \to C(E, C(B, A))$ maps $f \in SJ$ to the pair $(0, \tilde{f})$, where $\tilde{f}(t) = (f(t), 0) \in C(B, A)$.

(c) $\widetilde{C}(E, C(B, A))$ is the C*-algebra of pairs (e, f), where $e \in E$ and f is a continuous function from $[0, 1]$ to $C(B, A)$ such that e maps to $f(0) \in C(B, A)$ and such that $f(1) \in SA$. Observe that $\widetilde{C}(E, C(B, A))$ contains $C(E, C(B, A))$ as an ideal with quotient SA, as required.

(d) τ is the 'flip' defined by $\tau(f)(t) = f(1-t)$.

(We leave it to the reader to supply the definitions of the remaining maps in the diagram.) Note that τ induces the map -1 on K-theory. In addition, the composition

$$K_0(SJ) \longrightarrow K_0(C(E, C(B, A))) \xleftarrow{\cong} K_0(\mathfrak{K}(H))$$

gives the map Index (x).[64] It follows that there is a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & K_0(SJ) \arrow[r] \arrow[d, "\text{Index}(x)"'] & K_0(SB) \arrow[r] \arrow[d] & K_0(SA) \arrow[r] \arrow[d, "-1"] & 0 \\
0 \arrow[r] & K_0(\mathfrak{K}(H)) \arrow[r] & K_0(D) \arrow[r] & K_0(SA) \arrow[r] & 0
\end{tikzcd}$$

and in view of the remarks preceding the lemma this completes the proof. □

7.6.11 Definition Let A be a separable and commutative C*-algebra. We shall say that A is *admissible* if the homomorphism Index : $K^p(A) \to \text{Hom}(K_p(A), \mathbb{Z})$ is surjective for every $p \geqslant 1$ and if the homomorphism $^\circ$Index : $^\circ K^{p+1}(A) \to \text{Ext}\,(K_p(A), \mathbb{Z})$ is an isomorphism, for every $p \geqslant 1$.

7.6.12 Remark In Definition 7.6.4 we introduced the notion of admissibility for C*-algebras with free abelian K-theory. Our new definition is clearly consistent with Definition 7.6.4 since if $K_p(A)$ is free abelian then $\text{Ext}\,(K_p(A), \mathbb{Z}) = 0$.

If A is admissible in the sense of Definition 7.6.11 then there is a 'universal coefficient' short exact sequence

$$0 \longrightarrow \text{Ext}\,(K_p(A), \mathbb{Z}) \longrightarrow K^{p+1}(A) \longrightarrow \text{Hom}(K_{p+1}(A), \mathbb{Z}) \longrightarrow 0\,.$$

So to prove the Universal Coefficient Theorem it suffices to show that every separable and commutative C*-algebra is admissible. The first step in this direction is the following:

7.6.13 Admissibility Criterion *If the short exact sequence*

$$0 \longrightarrow J \longrightarrow B \longrightarrow A \longrightarrow 0$$

is an admissible projective resolution then the C*-*algebra* A *is admissible if and only if* J *is admissible.*

Proof Apply the Five Lemma to 7.6.9. □

[64] To see this, use Exercise 7.7.2 and the discussion of the boundary map in K-theory given in Section 4.8.

Roughly speaking, we shall use the admissibility criterion to reduce the proof of admissibility for all C*-algebras to the case of C*-algebras — like the ideals J above — which have free abelian K-theory groups. We saw in Lemma 7.2.10 that, within the class of C*-algebras which have free abelian K-theory, the subclass of admissible C*-algebras is closed under extensions. We are going to prove a similar result without the hypothesis of free abelian K-theory, but we shall relax this hypothesis one step at a time.

7.6.14 LEMMA *Let* I *be an ideal in a separable and commutative* C*-*algebra* A. *If two out of the three* C*-*algebras* I, A *and* A/I *are admissible and have free abelian* K-*theory groups then the third is admissible.*[65]

PROOF Suppose first that I and A are admissible and have free abelian K-theory. Consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & J \arrow[r, equal] \arrow[d] & J \arrow[d] & \\
0 \arrow[r] & I \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \arrow[r] & A \arrow[r] \arrow[d] & A/I \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

in which the right vertical short exact sequence is an admissible projective resolution of A/I and E is the pullback C*-algebra whose elements are pairs (a, b) in $A \oplus B$ such that a and b map to a common element of A/I. The K-theory boundary map for the right vertical sequence is zero. By functoriality of the boundary map the same is true for the left vertical exact sequence. Therefore, there are short exact sequences

$$0 \longrightarrow K_p(J) \longrightarrow K_p(E) \longrightarrow K_p(A) \longrightarrow 0\,,$$

and since both $K_p(J)$ and $K_p(A)$ are free abelian it follows that $K_p(E)$ is free abelian too. Lemma 7.6.5 now shows that E is admissible, and therefore that the left vertical sequence is an admissible projective resolution. So by Criterion 7.6.13 the admissibility of A implies the admissibility of J, which by Criterion 7.6.13 again implies the admissibility of A/I.

[65]But of course the third C*-algebra need not have free abelian K-theory.

The remaining cases of the lemma can be proved by applying the first case to the short exact sequences

$$0 \longrightarrow S(A/I) \longrightarrow C(A, A/I) \longrightarrow A \longrightarrow 0$$

and

$$0 \longrightarrow S(A) \longrightarrow Z(A, A/I) \longrightarrow C(A, A/I) \longrightarrow 0,$$

where $C(A, A/I)$ is the mapping cone for the quotient map $A \to A/I$ and $Z(A, A/I)$ is the following pullback:

$$\begin{tikzcd} Z(A, A/I) \arrow[r] \arrow[d] & C(A, A/I) \arrow[d] \\ C(A) \arrow[r] & A \end{tikzcd}$$

(thus $Z(A, A/I)$ is the mapping cone for the evaluation map $C(A, A/I) \to A$). The inclusion of I into $C(A, A/I)$ induces isomorphisms in K-theory and K-homology, so I is admissible if and only if $C(A, A/I)$ is admissible, and of course if I has free abelian K-theory then so does $C(A, A/I)$. Similarly there is an inclusion of $S(A/I)$ into $Z(A, A/I)$ which induces isomorphisms in K-theory and K-homology, so A/I is admissible if and only if $Z(A, A/I)$ is admissible. □

7.6.15 Lemma *Let* I *be an ideal in a separable and commutative* C^**-algebra* A. *If two of the* C^**-algebras* I, A *and* A/I *are admissible, and if one of these two has free abelian* K*-theory, then all three* C^**-algebras are admissible.*

Proof Using the device employed at the end of the previous lemma to cyclically permute the roles of I, A and A/I, the proof reduces to the situation where A and A/I are admissible and where one of these two algebras has free abelian K-theory. We shall consider here the case where A has free abelian K-theory (which is the only one of the two that we shall use in what follows) and leave the other case to the reader.

Consider the same diagram as in the proof of the previous lemma. A diagram chase shows that E has free abelian K-theory and 7.6.13 implies that J is admissible. Lemma 7.6.14, applied to the left vertical short exact sequence, shows that E is admissible. Lemma 7.6.14, applied to the top horizontal short exact sequence, then shows that I is admissible □

7.6.16 Lemma *Let* I *be an ideal in a separable and commutative* C^**-algebra* A. *If two of the three* C^**-algebras* I, A *and* A/I *are admissible then so is the third.*

Proof By cyclically permuting the roles of I, A and A/I as we did in the proof of Lemma 7.6.14 it suffices to show that if I and A/I are admissible then so is A. Once again we use the diagram in the proof of Lemma 7.6.14. It follows from Lemma 7.6.15, applied to the top horizontal short exact sequence, that E is admissible. The Admissibility Criterion 7.6.13 implies that J is admissible. Finally Lemma 7.6.15, this time applied to the left vertical short exact sequence, implies that A is admissible, as required. □

We are finally in a position to prove the main theorem:

Proof of the Universal Coefficient Theorem The C^*-algebra $C_0(\mathbb{R}^n)$ is admissible. So if X is a finite complex then Lemma 7.6.16 and an induction argument on the dimension of X show that $C(X)$ is admissible. Since it is readily checked from the cluster axiom that a direct sum of admissible C^*-algebras is admissible, it follows from the telescope construction presented in Section 7.3 that $C(X)$ is admissible for every compact metric space X. This proves the Universal Coefficient Theorem for unital commutative C^*-algebras; the non-unital case is handled by adjoining units. □

7.7 Exercises

7.7.1 Complete the discussion which was begun in Section 7.2 and compute the K-homology of the real projective plane.

7.7.2 Let A be a separable and nuclear C^*-algebra and suppose given an extension

$$0 \longrightarrow \mathfrak{K}(H) \longrightarrow E \longrightarrow A \longrightarrow 0\,.$$

Show that if x is the class in $K^1(A)$ associated to this extension then the homomorphism $\operatorname{Index}(x)\colon K_1(A) \to \mathbb{Z}$ may be identified with the boundary map $\partial\colon K_1(A) \to K_0(\mathfrak{K}(H))$ in the K-theory exact sequence for the extension. Show that if $\varphi\colon A \to \mathfrak{Q}(H)$ is the $*$-homomorphism associated to the extension and if u is any unitary element of $\widetilde{A}$ then the index pairing of $x \in K^1(A)$ with $[u] \in K_1(A)$ is the Fredholm index of any operator whose image in the Calkin algebra is $\varphi(u)$.

7.7.3 Show that, if $A_j = \mathbb{C}$ then the K_0-group of the algebra $\prod_{j=1}^{\infty} A_j$ consists of *bounded* sequences of integers, and thus differs from the algebraic direct product of the groups $K_0(A_j)$.

7.7.4 Verify that the telescope of a decreasing sequence of compact spaces is contractible.

7.7.5 Let C denote the Cantor set. Show that the group of continuous $\mathbb{Z}$-valued functions on C is free abelian (exhibit an explicit basis). Deduce that the group of continuous $\mathbb{Z}$-valued functions on *any* compact metric space is free abelian (recall that any compact metric space is a continuous image of the Cantor set).

7.7.6 Show, by an explicit construction, that if X is a compact subset of the plane then the map $\mathrm{Ext}\,(C(X)) \to \mathrm{Hom}(\check{H}^1(X), \mathbb{Z})$ is surjective. (Hint: look at Toeplitz operators on the Bergman space associated to the bounded part of the complement of X. See Exercise 2.9.12.)

7.7.7 Prove that the Toeplitz extension generates the infinite cyclic group $\mathrm{Ext}\,(C(S^3))$. See Exercises 2.9.18 and 2.9.19.

7.7.8 Use the Brown–Douglas–Fillmore Theorem to show that the set of operators in $\mathfrak{B}(H)$ of the form 'normal plus compact' is closed in the norm topology.

7.7.9 The purpose of this exercise is to prove Proposition 7.2.8. The argument begins with a proof that the functor $K^1(A)$ is homotopy invariant on the category of all separable C^*-algebras.

(a) Deduce from the stability of K-homology (Exercise 5.6.4) and its homotopy invariance in the commutative case (Section 6.6) that if $\varepsilon_t \colon C[0,1] \to \mathbb{C}$ is evaluation at $t \in [0,1]$ then

$$\varepsilon_0^* = \varepsilon_1^* \colon K^0(\mathfrak{K}(H)) \to K^0(\mathfrak{K}(H) \otimes C[0,1]).$$

(b) Use the naturality of the boundary map in K-homology and Exercise 5.6.5 to prove that two semisplit extensions of A by $\mathfrak{K}(H)$ which are related by a commuting diagram of semisplit short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{K}(H) \arrow[r] & E_1 \arrow[r] & A \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{K}(H) \otimes C[0,1] \arrow[r] \arrow[u, "\varepsilon_1"] \arrow[d, "\varepsilon_0"'] & E \arrow[r] \arrow[u] \arrow[d] & A \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{K}(H) \arrow[r] & E_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

determine the same class in $K^1(A)$.

(c) Reduce homotopy invariance for $K^1(A)$ to the assertion just proved.

Now in Exercise 4.10.24 we sketched a proof of the Bott Periodicity Theorem which applies to any half-exact, homotopy invariant and stable functor. Check that by using the Bott periodicity isomorphisms in K-theory and K-homology which are provided by Exercise 4.10.24 we obtain a proof of Proposition 7.2.8.

7.7.10 Deduce Proposition 7.2.8 from Proposition 7.2.4.

7.7.11 Use Proposition 7.2.4 to identify the index pairings for the higher K-homology groups with the pairings given in Definitions 7.2.1 and 7.2.3.

7.7.12 Adapt the statement of Theorem 7.3.4 to non-commutative C^*-algebras. Show that the conclusion is consistent with the calculation of Ext (CAR) given in Exercise 2.9.16.

7.7.13 Assume that the Universal Coefficient Theorem is valid for the Cuntz algebras $\mathcal{O}_n$ and the CAR algebra CAR. Show that it is consistent with the calculations of Ext $(\mathcal{O}_n)$ and Ext (CAR) given in Exercises 2.9.17 and 2.9.16, and the K-theory calculations described in Chapter 4.

7.8 Notes

The BDF theorem is one of the high peaks which it is this book's objective to scale, and it is worth pausing to contemplate the way we made our ascent. We began by introducing Ext (X) as a classifying structure for our problem. We deployed significant functional-analytic resources (Voiculescu's and Stinespring's Theorems) in order to show that Ext (X) is a group. Having done this, the same tools sufficed to reveal that Ext (X) is in fact the K-theory group of a certain non-commutative C^*-algebra. Using our knowledge of K-theory we were able to prove excision properties for Ext . The proof that Ext is a homotopy functor rested on interpreting it in a different sense, one suggested by controlled topology and 'coarse geometry'. Finally, we showed that Ext is part of a Steenrod homology theory; and then some techniques of algebraic topology — not quite elementary, since the spaces involved are potentially rather bad — completed the proof of the theorem.

We had the benefit of hindsight in constructing our argument. In BDF's original work, even the fact that Ext (X) is a group did not become apparent until the very end of the argument. The proof of homotopy invariance hinged on the vanishing of Ext (X) when X is a Cantor set — not on the vanishing of Ext (X) when X is a cone. A developed K-theory for C^*-algebras was not available; indeed, BDF theory provided much of the impetus for the development of C^*-algebra K-theory. The moral of the story is that even a quite technical-sounding question may prove to be the key which unlocks a whole new world of mathematical relationships.

The Universal Coefficient Theorem and the relationship which we described in Lemma 7.6.10 between Ext for abelian groups and Ext for C^*-algebras were discovered by Brown [32]. As Rosenberg and Schochet showed (see [116]), it is somewhat easier to prove the theorem by injectively resolving $\mathfrak{K}(H)$ than by projectively resolving A, as we did. However, this can only be done within a more elaborate framework of C^*-algebra extension theory, in which $\mathfrak{K}(H)$ is replaced by a tensor product $D \otimes \mathfrak{K}(H)$.

8

KASPAROV'S K-HOMOLOGY

Our objective in Chapters 5–7 was to analyze the Brown–Douglas–Fillmore groups $\mathrm{Ext}(A)$ for a separable, unital, nuclear C^*-algebra A. To do this we introduced the dual algebra $\mathfrak{D}(A)$, which consists of all those operators which commute modulo the compacts with a suitable representation of A, and we used the theorems of Stinespring and Voiculescu to identify $\mathrm{Ext}(A)$ with $K_1(\mathfrak{D}(A))$. Once this step was made it became natural to consider the other K-group of $\mathfrak{D}(A)$ as well, and so we were led to introduce the K-*homology* groups of A, defined by $K^0(A) = K_1(\mathfrak{D}(A))$ and $K^1(A) = K_0(\mathfrak{D}(A))$.

The definition of K-homology by way of the dual algebra has the disadvantage that it requires us to select one ample Hilbert space representation for each C^*-algebra A. In this chapter we shall reshape the definition into a more flexible form which avoids this problem. The key to this new approach to K-homology is the notion of a *Fredholm module* over a C^*-algebra. A Fredholm operator between Hilbert spaces is a linear isomorphism modulo compact operators. Analogously, a Fredholm module over a C^*-algebra A is an equivalence modulo compact operators between two Hilbert space representations of A. Fredholm modules arise naturally from geometric problems. We shall prove that K-homology groups may be defined by taking Fredholm modules as generators and imposing appropriate relations.

8.1 Fredholm Modules

Let A be a separable[66] C^*-algebra. We do not assume that A is unital.

8.1.1 DEFINITION An (ungraded) *Fredholm module* over A is given by the following data:

(a) a separable Hilbert space H,

(b) a representation $\rho\colon A \to \mathfrak{B}(H)$, and

(c) an operator F on H such that for all $a \in A$,

$$(F^2 - 1)\rho(a) \sim 0, \quad (F - F^*)\rho(a) \sim 0, \quad F\rho(a) \sim \rho(a)F.$$

[66] The definitions in this section make sense for all C^*-algebras. Nevertheless, they are probably appropriate only to separable algebras A, and therefore we shall usually assume that the C^*-algebras for which we want to form K-homology groups are separable.

Recall that the notation $\sim$ denotes equality modulo the compact operators. We note explicitly that ρ is not required to be non-degenerate; indeed the representation ρ, and even the Hilbert space H, are allowed to be zero.

8.1.2 REMARK We shall sometimes refer to (ρ, H, F) as a Fredholm module *over* the representation ρ of A. In accordance with our use of the word 'over' in Chapter 4, we shall also use this terminology for a Fredholm module whose underlying representation is a (finite) multiple of ρ.

It will be important to study Fredholm modules equipped with a *grading*. The reader who is not familiar with $\mathbb{Z}/2$-graded algebra will find a brief introduction in Appendix A. But for the moment it suffices to say that a *graded* Fredholm module is given by the same data as an ungraded one, with the following additional structure:

(a) the Hilbert space H is equipped with a direct sum decomposition $H = H^+ \oplus H^-$,
(b) for each $a \in A$, the operator $\rho(a)$ is *even*. Thus $\rho(a) = \rho^+(a) \oplus \rho^-(a)$, where $\rho^\pm$ are representations of A on $H^\pm$, and
(c) the operator F is *odd*. That is, F has the form

$$F = \begin{pmatrix} 0 & V \\ U & 0 \end{pmatrix}, \tag{8.1.3}$$

where U is an operator from H^+ to H^- and V is an operator from H^- to H^+.

REMARK The operators U and V in (c) above are not independent: V is essentially the adjoint of U. To be precise, the conditions in Definition 8.1.1 translate to the conditions

$$(UV - 1)\rho^-(a) \sim 0,\ \ (VU - 1)\rho^+(a) \sim 0,\ \ (U - V^*)\rho^+(a) \sim 0,\ \ U\rho^+(a) \sim \rho^-(a)U$$

on the operators U and V. So, for example, if A is unital and $\rho^\pm$ are non-degenerate, then U is essentially unitary and $V \sim U^*$.

It is helpful to have in mind a number of examples of Fredholm modules.

8.1.4 EXAMPLE Consider Fredholm modules over the C^*-algebra $\mathbb{C}$, and assume for simplicity that $\rho\colon \mathbb{C} \to \mathfrak{B}(H)$ is the (unique) unital representation. Then an ungraded Fredholm module is given by an essentially selfadjoint Fredholm operator F. A graded Fredholm module is given by an essentially selfadjoint operator F of the form 8.1.3 above, where U and V are Fredholm operators and $V \sim U^*$. When F is a graded Fredholm operator of this sort, we shall use the notation Index (F) to refer to the index of the operator U.

8.1.5 EXAMPLE Let M be a compact manifold. Recall that in Chapter 2 we constructed the *pseudodifferential operator extension* (2.8.5)

$$0 \longrightarrow \mathfrak{K}(L^2(M)) \longrightarrow \mathfrak{P}(M) \longrightarrow C(S^*M) \longrightarrow 0$$

associated to M. Let $D \in M_k(\mathfrak{P}(M))$ be a system of pseudodifferential operators whose symbol is a unitary matrix-valued function[67] on S^*M. Then the Hilbert space and operator

$$H = L^2(M)^k \oplus L^2(M)^k, \qquad F = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix},$$

together with the natural representation ρ of $C(M)$ on H by multiplication operators, define a graded Fredholm module over $C(M)$.

This was the motivating example for the notion of Fredholm module. We shall discuss the connection between elliptic operators and Fredholm modules at much greater length in Chapter 10.

8.1.6 EXAMPLE Let G denote the free group on two generators, and let $A = C_r^*(G)$ be the reduced group C^*-algebra of G (see Example 1.3.5). Let T denote the tree associated to G. This is a one-dimensional simplicial complex with one vertex for each $g \in G$, and an edge joining vertices g and g' if $g^{-1}g'$ is a generator or the inverse of a generator. The group G acts naturally on the tree by left translation. Let H be the graded Hilbert space $\ell^2(T^0) \oplus \ell^2(T^1)$, where T^0 denotes the space of vertices of the tree and T^1 denotes the space of edges. The natural representation of G on H extends to a representation ρ of the reduced group C^*-algebra A, because G acts freely on both T^0 and T^1, with one orbit in the case of T^0 and two orbits in the case of T^1. Thus both $\ell^2(T^0)$ and $\ell^2(T^1)$ are made up of copies of the regular representation of G. Now define an operator $U\colon \ell^2(T^0) \to \ell^2(T^1)$ as follows: for the basis element δ_v corresponding to a vertex v, set $U(\delta_v) = 0$ if $v = e$ is the identity element of G, and otherwise set $U(\delta_v) = \delta_{l(v)}$, where $l(v)$ is the unique edge with one vertex equal to v and the other vertex nearer to e than v is (see Figure 8.1). Put

$$F = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}.$$

We are going to show that (ρ, H, F) is a graded Fredholm module over A. Clearly $F = F^*$, and moreover $F^2 - 1$ is compact (in fact, $F^2 - 1$ is a rank-one projection). It suffices then to show that F commutes modulo compacts with the action of A.

[67] More generally we could consider pseudodifferential systems with unitary symbol, acting from sections of one vector bundle to sections of another.

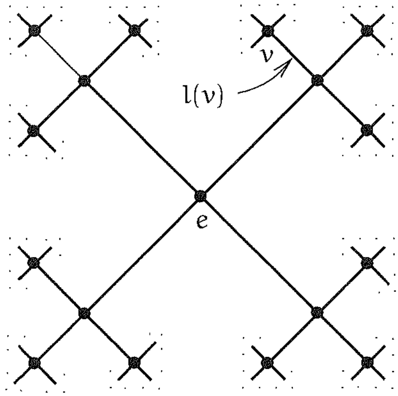

FIG. 8.1. The tree associated to a free group

But, if $g \in G$, then the operator gUg^{-1} is constructed just as U is, except that the special rôle of the vertex e is now played by g. From the geometry of the tree T, the operators U and gUg^{-1} differ only on those basis elements of $\ell^2(T^0)$ corresponding to the finitely many vertices on the unique geodesic path from e to g. Thus $U - gUg^{-1}$ is of finite rank; hence so is the commutator $[U, g]$. Since the group elements g generate the C^*-algebra A, $[U, \rho(a)]$ is compact for every $a \in A$ and we are done.

The next two examples show the close relationship between Fredholm modules and the K-theory of the dual algebras that we defined in Chapter 5. Recall that if $\rho\colon A \to \mathfrak{B}(H)$ is a representation of A, then $\mathfrak{D}_\rho(A)$ denotes the C^*-algebra of all operators T which commute modulo compacts with every $\rho(a)$; and $\mathfrak{D}_\rho(A/\!\!/A)$ denotes the ideal consisting of all T such that $T\rho(a)$ and $\rho(a)T$ are compact for every $a \in A$.

8.1.7 EXAMPLE Let $\rho\colon A \to \mathfrak{B}(H)$ be a representation of A, and let $P \in \mathfrak{B}(H)$ be a projection which commutes, modulo compacts, with every $\rho(a)$. Let $F = 2P - 1$. Then (ρ, H, F) is an ungraded Fredholm module over A.

Note that the hypothesis is simply that P is a projection in $\mathfrak{D}_\rho(A)$. We can assume a little less: if $P \in \mathfrak{D}_\rho(A)$ and if the image of P in the quotient algebra $\mathfrak{D}_\rho(A)/\mathfrak{D}_\rho(A/\!\!/A)$ is a projection, then the same construction will produce a Fredholm module from P. This extra flexibility is sometimes useful.

8.1.8 EXAMPLE Similarly, let $\rho\colon A \to \mathfrak{B}(H)$ be a representation of A and let $U \in \mathfrak{B}(H)$ be a unitary which commutes, modulo compacts, with every $\rho(a)$. Then we may define a graded Fredholm module (ρ', H', F) by

$$H' = H \oplus H, \quad \rho' = \rho \oplus \rho, \quad F = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}.$$

Once again, the hypothesis that U is a unitary in $\mathfrak{D}_\rho(A)$ can be weakened: it is enough to assume that $U \in \mathfrak{D}_\rho(A)$ and that its image in $\mathfrak{D}_\rho(A)/\mathfrak{D}_\rho(A/\!\!/A)$ is unitary.

8.1.9 EXAMPLE Let $\alpha\colon A \to \mathbb{C}$ be a $*$-homomorphism. Define a Fredholm module (ρ, H, F) by

$$H = \mathbb{C} \oplus 0, \quad \rho = \alpha \oplus 0, \quad F = 0.$$

(We are considering H to be a graded Hilbert space whose second graded component is trivial.) The same construction may be applied to a $*$-homomorphism from A to $\mathfrak{K}(H')$, for any (separable) Hilbert space H'.

We defined a graded Fredholm module to be an ungraded module equipped with some extra structure. It is also possible, however, to reverse this process and to identify an *ungraded* Fredholm module with a *graded* one provided with extra structure. To see this, let (ρ, H, F) be an ungraded Fredholm module and form the graded module consisting of

$$H' = H \oplus H, \quad \rho' = \rho \oplus \rho, \quad F' = \begin{pmatrix} 0 & F \\ F & 0 \end{pmatrix}. \tag{8.1.10}$$

The operator

$$\varepsilon = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

is odd, unitary, of square -1, and commutes with each $\rho'(a)$ and with F'. Moreover, the ungraded Fredholm module (ρ, H, F) can be recovered (up to unitary equivalence) from the graded module (ρ', H', F') together with the additional operator ε.

This observation leads us to introduce the notion of *multigraded* Fredholm module, which is a Fredholm module whose underlying Hilbert space is equipped with several anticommuting 'gradings'. In the definition, we use the notions of *multigraded Hilbert space* and *multigraded operator* which are introduced in Appendix A (Definition A.3.1).

8.1.11 DEFINITION Let $p \in \{-1, 0, 1, \dots\}$. A p-*multigraded Fredholm module* for A is given by the following data:

(a) a separable p-multigraded Hilbert space H,
(b) a representation $\rho\colon A \to \mathfrak{B}(H)$ by even multigraded operators, and
(c) an odd multigraded operator F on H such that for all $a \in A$,

$$(F^2 - 1)\rho(a) \sim 0, \quad (F - F^*)\rho(a) \sim 0, \quad F\rho(a) \sim \rho(a)F.$$

Unpacking the definitions of Appendix A, we find that for $p \geqslant 1$, a p-multigraded Fredholm module is a graded Fredholm module (ρ, H, F), together with a list of p odd unitary operators $\varepsilon_1, \ldots, \varepsilon_p$ on H which have square -1, anticommute with one another, commute with F, and commute with $\rho(a)$ for every $a \in A$. Furthermore, a 0-multigraded Fredholm module is just an ordinary graded Fredholm module, and a (-1)-multigraded Fredholm module is an ungraded Fredholm module.

The discussion which preceded Definition 8.1.11 produced a 1-multigraded Fredholm module from a (-1)-multigraded Fredholm module. It can be generalized to show that the p-multigraded Fredholm modules over A are 'periodic' in p, with period 2. In particular, it will turn out that the K-homology groups formed from p-multigraded and $(p+2)$-multigraded Fredholm modules are the same (see Proposition 8.2.13).

REMARK A multigraded Fredholm module may be thought of as one for which the scalar field $\mathbb{C}$ has been replaced by a suitable Clifford algebra. Clifford algebras are ubiquitous in geometry, which means that Fredholm modules arising from geometry are very likely to be multigraded. One can use the periodicity theorem mentioned above to reduce to the ungraded or graded cases, but this reduction sometimes obscures the geometry; furthermore, an analogous reduction is extremely inconvenient in the case of *real* scalars, where the period is 8 rather than 2. For these and other reasons it is important to work with multigraded modules. Nevertheless, on a first reading of this chapter it may be better to concentrate on the ungraded and graded cases. The analytical details are the same, and the reader can postpone consideration of multigradings until Chapter 11.

8.2 The Kasparov Groups

We begin by defining some notions of *equivalence* for Fredholm modules. All the definitions below refer equally to the ungraded, graded, and multigraded cases.

8.2.1 DEFINITION Let (ρ, H, F) be a Fredholm module and let $U \colon H' \to H$ be a unitary isomorphism (preserving the (multi)grading, if there is one). Then $(U^*\rho U, H', U^*FU)$ is also a Fredholm module, and we say that it is *unitarily equivalent* to (ρ, H, F).

8.2.2 DEFINITION Suppose that (ρ, H, F_t) is a family of Fredholm modules parameterized by $t \in [0,1]$, in which the representation and the Hilbert space remain constant but the operator F_t varies with t. If the function $t \mapsto F_t$ is norm continuous, then we say that the family defines an *operator homotopy* between the Fredholm modules (ρ, H, F_0) and (ρ, H, F_1), and that these two Fredholm modules are *operator homotopic*.

8.2.3 DEFINITION Suppose (ρ, H, F) and (ρ, H, F') are Fredholm modules on the same Hilbert space, and that $(F - F')\rho(a)$ is compact for all $a \in A$. We shall then say, by a slight abuse of language, that F is a *compact perturbation* of F'.

Compact perturbation implies operator homotopy; indeed, the linear path from F to F' defines an operator homotopy.

8.2.4 EXAMPLE We can always perform a compact perturbation to make the operator F defining a Fredholm module exactly selfadjoint: simply replace F by $\frac{1}{2}(F + F^*)$.

There is a natural notion of *direct sum* for Fredholm modules: one takes the direct sum of the Hilbert spaces, of the representations, and of the operators F. The *zero module* has zero Hilbert space, zero representation, and zero operator.

Now we can give Kasparov's definition of K-homology.

8.2.5 DEFINITION Let $p \geqslant -1$ be an integer. The *Kasparov* K-*homology group* $KK^{-p}(A)$ is the abelian group with one generator $[x]$ for each unitary equivalence class of p-multigraded Fredholm modules over A and with the relations:

(a) if x_0 and x_1 are operator homotopic p-multigraded Fredholm modules then $[x_0] = [x_1]$ in $KK^{-p}(A)$, and
(b) if x_0 and x_1 are any two p-multigraded Fredholm modules then $[x_0 \oplus x_1] = [x_0] + [x_1]$ in $KK^{-p}(A)$.

8.2.6 REMARK It will be shown in the next section that for $p = -1, 0$ the Kasparov group $KK^{-p}(A)$ just defined is naturally isomorphic to the K-homology group $K^{-p}(A)$ defined by duality theory. Once we have proved this we shall abandon our KK notation, which involves a convenient but non-standard abbreviation: the group that we call $KK^{-p}(A)$ is denoted $KK^{-p}(A, \mathbb{C})$ in the literature.

Notice that the relations imply that the zero module is the zero element of the group. It is easy to see that KK is a *contravariant functor* from the category of C^*-algebras to the category of abelian groups. For if (ρ, H, F) is a Fredholm module over A, and $\alpha\colon A' \to A$ is a $*$-homomorphism, then $(\rho \circ \alpha, H, F)$ is a Fredholm module over A', and this procedure defines a homomorphism

$$\alpha^*\colon KK^{-p}(A) \to KK^{-p}(A').$$

The reader should contrast this natural definition with the somewhat involved construction (Lemma 5.2.4) necessary to obtain the functoriality of K-homology by the 'dual algebra' approach.

We can give a somewhat more explicit description of K-homology by introducing the notion of *degenerate* Fredholm module.

8.2.7 DEFINITION A Fredholm module (ρ, H, F) is *degenerate* if $\rho(a)F = \rho(a)F^*$, $\rho(a)F^2 = \rho(a)$, and $[F, \rho(a)] = 0$ for all $a \in A$. Thus a Fredholm module is degenerate if all the relations in Definition 8.1.1 are satisfied *exactly* (not just modulo the compacts).

8.2.8 PROPOSITION *The class in* $KK^{-p}(A)$ *defined by a degenerate Fredholm module is zero.*

PROOF Let $x = (\rho, H, F)$ be a degenerate Fredholm module for A. Form a new triple $x' = (\rho', H', F')$ in which H' is the direct sum $\bigoplus_{i=1}^{\infty} H$ of infinitely many copies of H, and similarly F' and ρ' are infinite direct sums of copies of F and ρ. In general an infinite direct sum of Fredholm modules is not a Fredholm module, since an infinite direct sum of compact operators need not be compact. But an infinite direct sum of zero operators is still a zero operator! Thus in this case x' *is* a Fredholm module. But clearly $x \oplus x'$ is unitarily equivalent to x', so we have

$$[x] + [x'] = [x']$$

in K-homology, and therefore $[x] = 0$. □

This proof is another example of an Eilenberg swindle (compare Example 4.6.3).

8.2.9 EXAMPLE Let us compute the group $KK^0(\mathbb{C})$. If (ρ, H, F) is a Fredholm module over $\mathbb{C}$, then $\rho(1)$ is a projection $p \in \mathfrak{B}(H)$, and up to compact perturbation (ρ, H, F) is the direct sum of (ρ, pH, pFp) and $(\rho, (1-p)H, (1-p)F(1-p))$. The second module carries the zero action of $\mathbb{C}$ and so is degenerate; the first is determined by the graded Fredholm operator pFp (see Example 8.1.4). The assignment $(\rho, H, F) \mapsto \operatorname{Index} pFp$ gives a homomorphism $K^0(\mathbb{C}) \to \mathbb{Z}$, and using the fact proved in Exercise 2.9.13 that an essentially unitary Fredholm operator with index zero is a compact perturbation of a unitary, we see that this index homomorphism is in fact an isomorphism. Compare Example 5.2.9.

For a multigraded Hilbert space H, let H^{op} denote H with the opposite multigrading. Notice that the identity map $I\colon H \to H^{op}$ then becomes an *odd* unitary isomorphism. If $T \in \mathfrak{B}(H)$ we shall use the notation T^{op} for the same operator considered as an element of $\mathfrak{B}(H^{op})$.

8.2.10 PROPOSITION *The additive inverse in* $KK^{-p}(A)$ *of the* K*-homology class defined by a* p*-multigraded Fredholm module* (ρ, H, F) *is the class defined by* $(\rho^{op}, H^{op}, -F^{op})$.

PROOF We shall show that the direct sum of the two Fredholm modules is operator homotopic to a degenerate. Consider the family F_t of operators on the Hilbert space $H \oplus H^{op}$ defined as follows:

$$F_t = \begin{pmatrix} \cos(\frac{\pi}{2}t)F & \sin(\frac{\pi}{2}t)I \\ \sin(\frac{\pi}{2}t)I & -\cos(\frac{\pi}{2}t)F^{op} \end{pmatrix}.$$

If we equip $H \oplus H^{op}$ with the representation $\rho \oplus \rho^{op}$, then the operators F_t constitute an operator homotopy of Fredholm modules. On the one hand we have $F_0 = F \oplus (-F^{op})$, while on the other hand F_1 is the degenerate operator $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. □

The argument is valid, in particular, for ungraded Fredholm modules (where the distinction between H and H^{op} disappears) and for ordinary graded Fredholm modules.

From the definition, elements of $KK^{-p}(A)$ are formal $\mathbb{Z}$-linear combinations of Fredholm modules. But Proposition 8.2.10 immediately gives:

8.2.11 COROLLARY *Every element of the group* $KK^{-p}(A)$ *can be represented by a single* p*-multigraded Fredholm module.* □

From Proposition 8.2.10 we also obtain:

8.2.12 PROPOSITION *Two* p*-multigraded Fredholm modules* x *and* x' *over* A *give rise to the same element of* $KK^{-p}(A)$ *if and only if they are* stably homotopic, *which is to say that there is a degenerate module* x'' *such that* $x \oplus x''$ *and* $x' \oplus x''$ *are unitarily equivalent to a pair of operator homotopic Fredholm modules.*

PROOF Observe that the collection of equivalence classes of p-multigraded Fredholm modules, under the relation described in the proposition, constitutes an abelian group G. The natural maps $G \to KK^{-p}(A)$ and $KK^{-p}(A) \to G$ define mutually inverse isomorphisms.[68] □

Let (ρ, H, F) be a p-multigraded Fredholm module, with extra multigrading operators $\varepsilon_1, \ldots, \varepsilon_p$. We may define a $(p+2)$-multigraded Fredholm module (ρ', H', F') by setting

$$H' = H \oplus H^{op}, \quad \rho' = \rho \oplus \rho^{op}, \quad F' = F \oplus F^{op}, \quad \varepsilon_i' = \varepsilon_i \oplus \varepsilon_i^{op},$$

and

$$\varepsilon_{p+1} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \varepsilon_{p+2} = \begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix}.$$

This procedure passes to the K-homology groups, giving a *formal periodicity map* which is a homomorphism $KK^{-p}(A) \to KK^{-p-2}(A)$.

8.2.13 PROPOSITION *The formal periodicity map is an isomorphism.* □

The proof is a simple linear algebra calculation, which we leave to the reader (Exercise 8.8.5).

[68] Compare Remark 4.1.2.

8.3 Normalization of Fredholm Modules

The definition of a Fredholm module is a very general and flexible one. For some purposes it is important to know that the group $KK^{-p}(A)$ is not changed if we restrict our attention to Fredholm modules which satisfy some extra normalization conditions. This section collects several results of that sort.

8.3.1 DEFINITION Let (ρ, H, F) be a Fredholm module over A. If $F = F^*$ we shall call (ρ, H, F) a *selfadjoint* Fredholm module. If $\|F\| \leqslant 1$ we shall say that the Fredholm module is *contractive*.

8.3.2 LEMMA *Every Fredholm module is a compact perturbation of a selfadjoint and contractive Fredholm module.*

PROOF The Fredholm modules (ρ, H, F) and $(\rho, H, \frac{1}{2}(F + F^*))$ are equivalent by compact perturbation; this shows we may take F selfadjoint. Having done this, let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be the continuous function whose graph is shown in Figure 8.2, defined by

$$\varphi(\lambda) = \begin{cases} -1 & \text{if } \lambda < -1 \\ \lambda & \text{if } -1 \leqslant \lambda \leqslant 1 \\ +1 & \text{if } \lambda > 1 \end{cases} .$$

Then $\|\varphi(F)\| \leqslant 1$, and a functional calculus argument shows that $\varphi(F)$ is a compact perturbation of F. $\square$

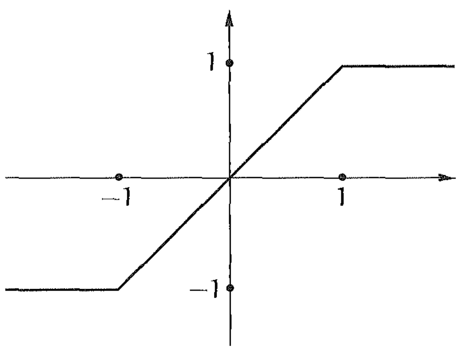

FIG. 8.2. The function $\varphi(\lambda)$ used in the proof of Lemma 8.3.2.

8.3.3 REMARK The argument shows more than the lemma claims. For if F_0 and F_1 are connected by an operator homotopy F_t, then $\varphi(\frac{1}{2}(F_0 + F_0^*))$ and $\varphi(\frac{1}{2}(F_1 + F_1^*))$ are connected by an operator homotopy $\varphi(\frac{1}{2}(F_t + F_t^*))$ which runs through selfadjoint and contractive Fredholm modules. Moreover, if F_0 and F_1 are unitarily equivalent, say by a unitary U with $U^*F_0U = F_1$, then $\varphi(\frac{1}{2}(F_0+F_0^*))$ and

$\varphi(\frac{1}{2}(F_1+F_1^*))$ are unitarily equivalent via U. It follows that if, in the definition of $KK^{-p}(A)$, we restrict attention entirely to selfadjoint and contractive Fredholm modules, and to operator homotopies through these modules, then we obtain the same KK-group as the one provided by Definition 8.2.5. Let us express this by saying that KK may be *normalized* by the requirement that Fredholm modules be selfadjoint and contractive.

8.3.4 DEFINITION We shall say that a selfadjoint and contractive Fredholm module (ρ, H, F) is *involutive* if $F^2 = 1$.

8.3.5 LEMMA *Every class in* $KK^{-p}(A)$ *may be represented by an involutive Fredholm module. In fact* KK *may be normalized by the requirement that Fredholm modules be involutive.*

PROOF Let the given class x be represented by the Fredholm module (ρ, H, F), which we may assume is selfadjoint and contractive. Then x is also represented by $(\rho, H, F) \oplus (0, H, -F)$, since the second module in this sum is degenerate. But $(\rho, H, F) \oplus (0, H, -F)$ is a compact perturbation of

$$\begin{pmatrix} F & (1-F^2)^{\frac{1}{2}} \\ (1-F^2)^{\frac{1}{2}} & -F \end{pmatrix}$$

and this operator is selfadjoint and has square 1. This proves the first statement of the lemma. The argument for normalization is similar (as in Remark 8.3.3), and the details are left to the reader. □

8.3.6 DEFINITION A Fredholm module (ρ, H, F) over a C^*-algebra A is *non-degenerate* if ρ is a non-degenerate representation of A, that is, if $\rho[A]H$ is dense in H. If A is unital, we may also refer to a non-degenerate Fredholm module as a *unital* Fredholm module.

8.3.7 WARNING The terms 'degenerate' (Definition 8.2.7) and 'non-degenerate' (Definition 8.3.6), as applied to Fredholm modules, are not opposites of one another!

8.3.8 LEMMA *Every class in* $KK^{-p}(A)$ *may be represented by a non-degenerate Fredholm module. In fact* KK *may be normalized by the requirement that Fredholm modules be non-degenerate.*

PROOF Let (ρ, H, F) be a general Fredholm module, and let P be the projection onto the subspace $\rho[A]H$ of H.[69] Then (ρ, H, F) is a compact perturbation of the direct sum

$$(\rho, PFP, PH) \oplus (0, (1-P)F(1-P), (1-P)H).$$

[69] This subspace is closed, by the Cohen Factorization Theorem; see Exercise 1.9.17.

The first term is unital, and the second is a compact perturbation of the degenerate module $(0, 0, (1-P)H)$. □

Let A be a C^*-algebra. From the embedding of A in its unitalization $\widetilde{A}$ we obtain a restriction homomorphism $KK^{-p}(\widetilde{A}) \to KK^{-p}(A)$.

8.3.9 LEMMA *The restriction homomorphism* $KK^{-p}(\widetilde{A}) \to KK^{-p}(A)$ *is surjective.*

PROOF Let $x \in KK^*(A)$ be represented by an involutive Fredholm module (ρ, H, F). Define a Fredholm module $(\widetilde{\rho}, \widetilde{H}, \widetilde{F})$ over $\widetilde{A}$ by setting

$$\widetilde{H} = H, \quad \widetilde{F} = F, \quad \widetilde{\rho}(a + \lambda 1) = \rho(a) + \lambda I.$$

The restriction of this module to A is (ρ, H, F), giving the result. □

8.3.10 DEFINITION A graded Fredholm module (ρ, H, F) over a C^*-algebra A is *balanced* if the graded Hilbert space H is the direct sum $H' \oplus H'$ of two copies of a single ungraded Hilbert space H', if H is graded by this direct sum decomposition, and if similarly the graded representation ρ is the direct sum of two copies of a single ungraded representation ρ'.

For example, the graded module of Example 8.1.8 is balanced, but the graded module of Example 8.1.9 is not. By convention, we shall also say that every ungraded Fredholm module is balanced.

8.3.11 REMARK If (ρ, H, F) is a balanced Fredholm module, then we may define a degenerate module (ρ, H, F_0) by putting $F_0 = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. Conversely, it is easy to see that a non-degenerate Fredholm module (ρ, H, F) is unitarily equivalent to a balanced module if there is a degenerate module with the same underlying representation (ρ, H).

There is a natural notion of *balanced equivalence* between balanced Fredholm modules; all homotopies and unitary equivalences used should respect the balanced structure.

8.3.12 PROPOSITION *Every class in* $KK^0(A)$ *may be represented by a balanced Fredholm module. In fact* KK^0 *may be normalized by the requirement that Fredholm modules be balanced, and it may also be normalized by the requirement that Fredholm modules be both balanced and involutive.*

The result also holds for multigraded Fredholm modules $(p = 1, 2, \dots)$ with a suitable definition of 'balanced', but we shall not need this.

PROOF Let (ρ, H, F) be an unbalanced Fredholm module, where $H = H^+ \oplus H^-$ and $F = \left(\begin{smallmatrix} 0 & V \\ U & 0 \end{smallmatrix}\right)$. Let $\widehat{H}$ be the Hilbert space

$$\widehat{H} = \cdots \oplus H^- \oplus H^- \oplus H^+ \oplus H^+ \oplus \cdots$$

(an infinite number of copies of each) and define an operator $\widehat{U}$ on $\widehat{H}$ by the following diagram:

$$
\begin{array}{ccccccccccccc}
 & \cdots\oplus & H^- & \oplus & H^- & \oplus & H^+ & \oplus & H^+ & \oplus & H^+ & \oplus\cdots & \\
\cdots & & \downarrow I & & \downarrow I & & \downarrow U & & \downarrow I & & \downarrow I & & \cdots \\
 & \cdots\oplus & H^- & \oplus & H^- & \oplus & H^- & \oplus & H^+ & \oplus & H^+ & \oplus\cdots &
\end{array}
$$

Define $\widehat{V}$ similarly (with the arrows going up rather than down) and let

$$H' = \widehat{H} \oplus \widehat{H}, \quad F' = \begin{pmatrix} 0 & \widehat{V} \\ \widehat{U} & 0 \end{pmatrix}.$$

Equip H' with the natural representation ρ' of A (an infinite direct sum of copies of ρ^+ and ρ^-). Then (ρ', H', F') is a balanced Fredholm module. Moreover, it is unitarily equivalent to the direct sum of (ρ, H, F) and two degenerate modules whose Hilbert spaces are infinite sums of copies of H^+ and H^- respectively. This proves the first assertion in the statement of the proposition. The second and third assertions are proved by applying the same technique. □

For balanced Fredholm modules there is an interesting variant of the notion of equivalence, which does not have an 'unbalanced' counterpart. Suppose that (ρ_0, H, F) and (ρ_1, H, F) are Fredholm modules which differ only as to the representation ρ. If $\rho_0(a) - \rho_1(a)$ is compact for every A, we might wonder whether the KK-classes of (ρ_0, H, F) and (ρ_1, H, F) are the same. It may be seen by examples (Exercise 8.8.17) that this is not true in general. But the corresponding assertion for balanced modules *is* true. Specifically, we may make the following definition:

8.3.13 Definition Let (ρ_0, H_0, F_0) and (ρ_1, H_1, F_1) be balanced Fredholm modules over a C^*-algebra A. We say that they are *essentially unitarily equivalent* if there is a balanced[70] unitary $U\colon H_0 \to H_1$ such that $UF_0 \sim F_1 U$ and $U\rho_0(a) \sim \rho_1(a)U$ for all $a \in A$.

8.3.14 Proposition *Essentially unitarily equivalent balanced Fredholm modules define the same element of* $KK^0(A)$.

Proof After a unitary equivalence and a compact perturbation, we see that it suffices to consider the special case $H_0 = H_1 = H$, $F_0 = F_1 = F$, and $\rho_0(a) \sim \rho_1(a)$ for all $a \in A$. Consider the Fredholm module $(\rho_0^{op}, H^{op}, -F^{op})$; by Proposition 8.2.10, this Fredholm module represents the inverse of (ρ_0, H, F) in $KK^0(A)$. Moreover, the same homotopy as was used in the proof of Proposition 8.2.10 also shows that the direct sum

$$(\rho_0^{op}, H^{op}, -F^{op}) \oplus (\rho_1, H, F) \tag{8.3.15}$$

[70] That is, a direct sum of two copies of the same unitary operator.

is operator homotopic to the Fredholm module $(\rho_0{}^{op} \oplus \rho_1, H^{op} \oplus H, \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right))$, which is independent of the choice of F. Now if we replace F by a degenerate in the display 8.3.15 — and such a degenerate always exists because of our assumption that (ρ, H, F) is balanced — we obtain a degenerate for the direct sum. We conclude that the display 8.3.15 always represents the zero element of $KK^0(A)$, and therefore that $[\rho_0, H, F] = [\rho_1, H, F]$ in $KK^0(A)$ as required. □

We conclude with another result which is useful in showing that two Fredholm modules are homotopic.

8.3.16 Proposition *Let A be a C^*-algebra, let $\rho\colon A \to \mathfrak{B}(H)$ be a representation of A on a multigraded Hilbert space H, and let F_0 and F_1 be two operators on H which give Fredholm modules for A. Suppose that $\rho(a)(F_0F_1 + F_1F_0)\rho(a^*)$ is positive modulo the compacts*[71] *for every $a \in A$. Then F_0 and F_1 are operator homotopic.*

Proof The operator $T = F_0F_1 + F_1F_0$ belongs to $\mathfrak{D}_\rho(A)$, and it commutes modulo $\mathfrak{D}_\rho(A/\!/A)$ with F_0 and F_1. The hypothesis that $\rho(a)T\rho(a^*)$ is positive modulo compacts, for every $a \in A$, implies, by a functional calculus argument (Exercise 8.8.8), that there is a positive operator S such that $S - T \in \mathfrak{D}_\rho(A/\!/A)$. Now if $t_0^2 + t_1^2 = 1$ then

$$(t_0F_0 + t_1F_1)^2 = 1 + t_0t_1S, \qquad \text{modulo} \quad \mathfrak{D}_\rho(A/\!/A).$$

Therefore if we set $t_0 = \cos(\frac{\pi}{2}t)$ and $t_1 = \sin(\frac{\pi}{2}t)$ then the path

$$F_t = (t_0F_0 + t_1F_1)(1 + t_0t_1S)^{-\frac{1}{2}}, \quad t \in [0, 1]$$

is an operator homotopy from F_0 to F_1,as required. □

8.3.17 Remark One can think of this as a Fredholm module counterpart to Rouché's theorem from complex analysis — a positivity hypothesis yields a homotopy conclusion.

8.4 Kasparov Theory and Duality

In this chapter we have defined Kasparov K-homology groups $KK^{-p}(A)$ when $p \geqslant -1$. In Chapter 5 we used the theory of dual algebras to define different K-homology groups $K^0(A)$ and $K^1(A)$. In this section we shall identify the Kasparov groups $KK^0(A)$ and $KK^1(A)$ with the K-homology groups from Chapter 5.

Recall from Definition 5.1.3 that a representation $\rho\colon A \to \mathfrak{B}(H)$ is *ample* if it is non-degenerate and no non-zero element of A acts on H as a compact operator. Theorem 3.4.6, which is a form of Voiculescu's Theorem, asserts that if ρ is an

[71] That is, its image in the Calkin algebra is positive.

ample representation of a unital C^*-algebra A, and if $\rho' \colon A \to \mathfrak{B}(H')$ is any other non-degenerate representation of A, then there is a unitary $U \colon H \to H' \oplus H$ such that $U\rho(a)U^* - \rho'(a) \oplus \rho(a)$ is compact for all $a \in A$. That is, an ample representation essentially absorbs any non-degenerate representation.

Now let A be a C^*-algebra, perhaps non-unital, and fix once and for all a representation $\rho_A \colon A \to \mathfrak{B}(H_A)$ which is the restriction to A of an ample representation of $\widetilde{A}$. We shall call ρ_A the *universal representation* of A. We shall also need to consider the graded representation $\rho_A \oplus \rho_A$ of A on $H_A \oplus H_A$, and we shall call this the *universal graded representation* of A.

The following two results will be proved at the end of the section.

8.4.1 LEMMA *Let A be a separable C^*-algebra.*

(a) *Every class in $KK^0(A)$ can be defined by an involutive, graded and balanced Fredholm module over the universal graded representation of A.*

(b) *Every class in $KK^1(A)$ can be defined by an involutive, ungraded Fredholm module over the universal representation of A.*

8.4.2 LEMMA *Let A be a separable C^*-algebra.*

(a) *Two involutive, graded and balanced Fredholm modules x and y over the universal representation of A define the same class in $KK^0(A)$ if and only if there is a third such module z such that $x \oplus z$ and $y \oplus z$ can be connected by essential unitary equivalences and (balanced, involutive) operator homotopies.*

(b) *Two involutive, ungraded Fredholm modules x and y over the universal representation of A define the same class in $KK^1(A)$ if and only if there is a third such module z such that $x \oplus z$ and $y \oplus z$ can be connected by essential unitary equivalences and (involutive) operator homotopies.*

Let $\mathfrak{D}(A)$ be the dual algebra (Definition 5.1.1) of A in the universal representation. We shall define group homomorphisms

$$\Phi^1 \colon K^1(A) = K_0(\mathfrak{D}(A)) \to KK^1(A)$$

and

$$\Phi^0 \colon K^0(A) = K_1(\mathfrak{D}(A)) \to KK^0(A)$$

and we shall then use Lemmas 8.4.1 and 8.4.2 to prove that these homomorphisms are isomorphisms. To define Φ^1, let P be a projection in $\mathfrak{D}(A)$ representing a class $[P] \in K_0(\mathfrak{D}(A))$. As in Example 8.1.7, form the ungraded involutive Fredholm module (ρ, H, F) over A, where $F = 2P - I$. The direct sum of projections passes to the direct sum of Fredholm modules, and homotopic projections give rise to operator homotopic Fredholm modules, so we obtain a homomorphism $\Phi^1\colon K_0(\mathfrak{D}(A)) \to KK^1(A)$, as required.[72] Similarly we may define the homomorphism Φ^0 by making use of the construction of Example 8.1.8: given a unitary $U \in \mathfrak{D}(A)$, form the graded balanced involutive Fredholm module $(\rho \oplus \rho, H \oplus H, F)$, where $F = \left(\begin{smallmatrix} 0 & U^* \\ U & 0 \end{smallmatrix}\right)$.

8.4.3 THEOREM *The homomorphisms defined above are isomorphisms between the* K*-homology groups defined in Chapter* 5 *and the Kasparov groups defined in this chapter.*

PROOF An element of the group $KK^0(A)$ or $KK^1(A)$ is in the image of Φ if it can be represented by a (balanced) involutive Fredholm module over the universal representation. So Lemma 8.4.1 immediately implies that the duality homomorphisms are surjective. To prove injectivity, we shall use Lemma 8.4.2. Note the following consequence of our definitions: the Fredholm modules (ρ, H, F) and (ρ, H, F') are essentially unitarily equivalent if and only if there is $U \in \mathfrak{D}(A)$ with $F' = U^*FU$. Suppose now, for example, that P is a projection over $\mathfrak{D}(A)$ such that $\Phi(P) = 0$. By Lemma 8.4.2, there is a projection Q over $\mathfrak{D}(A)$ such that the projection $P \oplus Q$ is connected to $0 \oplus Q$ by a chain of homotopies and unitary equivalences in $\mathfrak{D}(A)$. Hence $[P] = [0]$ in $K_0(\mathfrak{D}(A))$ and we have shown that Φ is injective on $K_0(\mathfrak{D}(A))$. The argument for $K_1(\mathfrak{D}(A))$ is similar. □

PROOF OF LEMMA 8.4.1 By Proposition 8.3.12, KK^0 may be normalized by the requirement that all Fredholm modules be balanced and involutive. Let $x \in KK^0(A)$ be the class of a balanced and involutive Fredholm module (ρ, H, F). Let (ρ_A, H_A, F_A) be the standard degenerate Fredholm module over the universal representation — which is also balanced and involutive — and consider the direct sum $(\rho \oplus \rho_A, H \oplus H_A, F \oplus F_A)$. This sum also represents x. But according to Voiculescu's Theorem, the representation $\rho \oplus \rho_A$ is essentially unitarily equivalent to ρ_A, say by a unitary $U\colon H_A \to H \oplus H_A$. Thus the module $(\rho_A, H_A, U^*(F \oplus F_A)U)$ also represents x, by Proposition 8.3.14. The proof for KK^1 is similar. □

PROOF OF LEMMA 8.4.2 Let x and y be the given Fredholm modules. By Proposition 8.3.12 there is another Fredholm module z' such that $x \oplus z'$ is connected to $y \oplus z'$ by a chain of balanced, involutive operator homotopies and balanced unitary equivalences. Let z'' denote the standard degenerate Fredholm module over the universal representation. According to Voiculescu's Theorem, $z' \oplus z''$ is essentially unitarily equivalent to a Fredholm module z over the universal representation, and now $x \oplus z$ is connected to $y \oplus z$ by a chain of operator homotopies and essential unitary equivalences. Once again, the proof for KK^1 is similar. □

[72] Strictly speaking, we must generalize our construction from projections *in* $\mathfrak{D}(A)$ to projections *over* $\mathfrak{D}(A)$; this is straightforward, and the details are left to the reader.

In view of the results of this section we shall henceforward drop the notation KK, and use the notations $K^{-p}(A)$ for both the 'Fredholm module' and the 'dual algebra' versions of the K-homology groups. In practice, we shall work with the Fredholm module definition; it is much more convenient for calculation.

8.5 Relative K-Homology

8.5.1 DEFINITION Let A be a separable C^*-algebra and let J be an ideal in A. A *relative Fredholm module* for the pair $(A, A/J)$ is given by the following data:

(a) a separable Hilbert space H,
(b) a representation $\rho\colon A \to \mathfrak{B}(H)$, and
(c) an operator F on H such that for all $a \in A$, and all $j \in J$,

$$(F^2 - 1)\rho(j) \sim 0, \quad (F - F^*)\rho(j) \sim 0, \quad F\rho(a) \sim \rho(a)F$$

There are graded and multigraded variants of the definition; we leave these to the reader.

8.5.2 EXAMPLE Suppose that $0 \to J \to A \to A/J \to 0$ is a short exact sequence of C^*-algebras which fits into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\rho'"] & A/J \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathfrak{K}(H') \arrow[r] & \mathfrak{B}(H') \arrow[r] & \mathfrak{Q}(H') \arrow[r] & 0. \end{tikzcd}$$

Then a graded relative Fredholm module can be constructed by taking H to be the graded Hilbert space $H' \oplus 0$, $F = 0$, and $\rho = \rho' \oplus 0$. Compare Example 8.1.9.

In Example 8.1.5 we saw that an elliptic pseudodifferential operator on a closed manifold M gives rise to an (absolute) Fredholm module over $C(M)$. Suppose now that M is a compact manifold with boundary ∂M. It turns out that suitable elliptic pseudodifferential operators on the open manifold $M \setminus \partial M$ give rise to *relative* Fredholm modules for the pair $(C(M), C(\partial M))$. We shall see a quite sophisticated example of this in Chapter 11 when we investigate the index theory of Toeplitz operators.

8.5.3 EXAMPLE The *simplest* example of a compact manifold with boundary is $M = [-1, 1]$ with $\partial M = \{-1, 1\}$. Identify the interior $(-1, 1)$ of M with $\mathbb{R}$ in the standard way. Let $H = L^2(\mathbb{R})$ and let F be the selfadjoint operator on H defined by

$$\widehat{Ff}(\xi) = \frac{\xi}{(1+\xi^2)^{\frac{1}{2}}}\hat{f}(\xi),$$

where the hat denotes the Fourier transformation. Thus F is given by the same integral formula 2.8.6 which defines a pseudodifferential operator, but now the symbol $\sigma(x, \xi) = \xi(1+\xi^2)^{-\frac{1}{2}}$ is no longer compactly supported. Let $\rho\colon C(M) \to \mathfrak{B}(H)$ be the natural representation by multiplication operators. Then (ρ, H, F) is a relative Fredholm module for $(C(M), C(\partial M))$. One can prove this directly using the properties of the Fourier transformation (Exercise 8.8.20), or one can appeal to the general methods of Chapter 10 (see Proposition 10.6.9.)

We can reproduce the entire development of the preceding sections in the relative case. Thus, we can organize the equivalence classes of p-multigraded relative Fredholm modules into a *relative Kasparov group* $KK^{-p}(A, A/J)$.

Let ρ be an ample representation of A. Following the development of Theorem 8.4.3, if $p = 0$ or $p = -1$ then we can identify our relative Kasparov groups with the *relative* K*-homology groups* defined in Chapter 5. Thus

$$KK^0(A, A/J) \cong K^0(A, A/J) = K_1(\mathfrak{D}_\rho(A)/\mathfrak{D}_\rho(A/\!\!/J))$$

and

$$KK^1(A, A/J) \cong K^1(A, A/J) = K_0(\mathfrak{D}_\rho(A)/\mathfrak{D}_\rho(A/\!\!/J)),$$

where

$$\mathfrak{D}_\rho(A/\!\!/J) = \left\{ T \in \mathfrak{D}_\rho(A) \, : \, T\rho(j) \sim 0 \sim \rho(j)T, \quad \forall j \in J \right\}.$$

The details of this are left to the reader. We shall therefore use the notation $K^{-p}(A, A/J)$ for the relative Kasparov group.

The *excision map* for relative K-homology is the natural homomorphism

$$K^{-p}(A, A/J) \longrightarrow K^{-p}(J)$$

which comes from restricting a relative Fredholm module over $(A, A/J)$ to an ordinary Fredholm module over J. Theorem 5.4.5 states that the excision map is an isomorphism.[73]

In order to carry out explicit computations, however, we often need to know that a given element of $K^{-p}(J)$ is represented by a specific relative Fredholm module. For example, let $0 \to J \to A \to A/J \to 0$ be a semisplit short exact

[73] One can think of the excision isomorphism as another normalization statement, like the ones in the Section 8.3. The actual normalization procedure is easily stated in terms of Fredholm modules: an unnormalized module (ρ, H, F) is replaced with the module (ρ, H, XF), where X is a positive operator which commutes with $\rho[A]$ and F modulo compacts, and which has the key properties $(1-X)\cdot\rho[J] \sim 0$ and $X \cdot [F, \rho[A]] \sim 0$. The operator X is supplied by the Kasparov Technical Theorem.

sequence of separable C^*-algebras. Theorem 5.3.10 provides a six-term exact sequence

$$\begin{array}{ccccc} K^1(A/J) & \longrightarrow & K^1(A) & \longrightarrow & K^1(A, A/J) \\ \partial \uparrow & & & & \downarrow \partial \\ K^0(A, A/J) & \longleftarrow & K^0(A) & \longleftarrow & K^0(A/J) \end{array}$$

of K-homology groups. Using relative K-homology, we shall give explicit formulas for the boundary maps. First we need a technical lemma.

8.5.4 LEMMA *Every graded relative Fredholm module for the pair* $(A, A/J)$ *is stably homotopic to one for which* F^2 *is a projection.*

We call such a relative Fredholm module *partially isometric.*[74]

PROOF The lemma follows from the assertion that every unitary element $U \in \mathfrak{D}(A)/\mathfrak{D}(A/\!\!/J)$ lifts to a partial isometry $V \in \mathfrak{D}(A)$, which was proved in Exercise 5.6.11. □

8.5.5 DEFINITION Let

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

be a short exact sequence of separable C^*-algebras which is semisplit by a completely positive map $s\colon \widetilde{A/J} \to \widetilde{A}$ (see Definition 5.3.6). Let $\rho\colon A \to \mathfrak{B}(H)$ be a representation of A on a separable Hilbert space. A *Stinespring dilation* associated to the above data is a $*$-homomorphism

$$\psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} : A/J \to \mathfrak{B}(H \oplus H'),$$

where H' is a separable Hilbert space and $\psi_{11}(x) = \rho(s(x))$.

By Stinespring's Theorem 3.1.3, such dilations always exist. Note that if ρ is a representation on a graded Hilbert space H, then we may produce a *graded* Stinespring dilation; just apply Stinespring's Theorem separately to each of the graded components $\rho^{\pm}$ of ρ.

Here now are the explicit formulas for the K-homology boundary maps.

8.5.6 PROPOSITION *Let*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

be a short exact sequence of separable C^**-algebras, semisplit by a completely positive map* $s\colon \widetilde{A/J} \to \widetilde{A}$. *Let* (ρ, H, F) *be a selfadjoint relative Fredholm*

[74] Some definitions of relative K-homology in the literature explicitly require the partial isometry property as part of the definition.

module (graded or ungraded) for $(A, A/J)$*, and let* $\psi\colon A/J \to \mathfrak{B}(H \oplus H')$ *be a Stinespring dilation. The boundary class* $\partial[\rho, H, F]$ *may be explicitly described as follows:*

(a) *In the case of the map* $\partial\colon K^0(A, A/J) \to K^1(A/J)$*, the module* (ρ, H, F) *is graded; assume that it is also partially isometric. Let* $Q_\pm$ *be the components of the projection* $1 - F^2$ *on the positively and negatively graded parts of* H*. The projections*

$$\begin{pmatrix} Q^\pm & \\ & 0 \end{pmatrix} \in \mathfrak{B}(H \oplus H')$$

commute modulo compacts with $\psi(x)$ *for all* $x \in A/J$ *and thus define ungraded Fredholm modules over* A/J *by the construction of Example 8.1.7. The boundary* $\partial[\rho, H, F]$ *is the difference of the homology classes of these Fredholm modules.*

(b) *In the case of the map* $\partial\colon K^1(A, A/J) \to K^0(A/J)$*, the module* (ρ, H, F) *is ungraded. The operator*

$$\begin{pmatrix} e^{i\pi F} & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{B}(H \oplus H')$$

is a unitary operator on $H \oplus H'$ *commuting modulo compacts with the representation* ψ*; the corresponding balanced Fredholm module defines* $\partial[\rho, H, F]$.

Included in the statement of the proposition is the claim that the homology classes defined in (a) and (b) are independent of the choice of completely positive section and of Stinespring dilation.

PROOF The result follows directly from the explicit formulas 4.8.10 and 4.10.23 for the boundary maps in the six-term K-theory exact sequence associated to the short exact sequence

$$0 \longrightarrow \mathfrak{D}(A /\!\!/ J) \longrightarrow \mathfrak{D}(A) \longrightarrow \mathfrak{D}(A)/\mathfrak{D}(A /\!\!/ J) \longrightarrow 0$$

of dual algebras. □

8.5.7 REMARK In case (a) above it was not really necessary to introduce the notion of Stinespring dilation in order to state the final result. Indeed, the maps $a \mapsto Q^\pm \rho(a)$ are $*$-homomorphisms modulo the compact operators, and they map J to the compacts. Thus they determine $*$-homomorphisms from A/J to the Calkin algebras $\mathfrak{Q}(Q^\pm H)$, and the difference of the K-homology classes of these invertible extensions is the boundary $\partial(\rho, H, F)$.

8.5.8 REMARK The same formula (a) can also be repackaged as follows. Let $Q = 1 - F^2$, let $\widehat{Q} = \left(\begin{smallmatrix} Q & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and let γ be the grading operator on $H \oplus H'$. Then $\partial[\rho, H, F]$ is represented by the ungraded Fredholm module $(\psi, H \oplus H', \gamma(2\widehat{Q}-1))$. Indeed, we see right away from Example 8.1.7 that

$$[\psi, H \oplus H', \gamma(2\widehat{Q} - 1)] = [Q_+] - [Q_-] \in K_1(A/J),$$

and so we recover the formula in (a), as required. While the new formula is a little ungainly, it is quite convenient for some computations.

8.5.9 EXAMPLE Let (ρ, H, F) be the relative Fredholm module defined in Example 8.5.2. Then F is partially isometric, and $\partial(\rho, H, F)$ is the K^1-class defined by the extension $A/J \to \mathfrak{Q}(H)$ induced by ρ. The diligent reader will already have checked this in Exercise 5.6.5.

8.6 Schrödinger Pairs

In this technical section we shall construct certain Fredholm operators whose indices measure the 'interaction' between two bounded selfadjoint operators X and Y. Typically, neither X nor Y will be Fredholm in its own right. The new operators will be used in Section 8.7 to prove the compatibility of the index pairing with boundary maps. We shall also use them in Section 9.3 to prove the homotopy invariance of K-homology, and again in Section 9.6 to prove a compatibility between boundary maps and the Kasparov product.

The position and momentum operators of quantum mechanics provide a fundamental example of a pair (X, Y) to which our construction applies. For this reason we shall use the term *Schrödinger pair* below. See Exercise 11.8.14 for a further application of Schrödinger pairs, this time to index theory on non-compact manifolds.

Recall that a bounded operator X on a Hilbert space H is called *contractive* if $\|X\| \leqslant 1$. If X is a contractive selfadjoint operator then we shall denote by $X^\flat$ the commutative C^*-subalgebra of $\mathfrak{B}(H)$ consisting of all functions $\psi(X)$, where $\psi \in C_0(-1, 1)$.

8.6.1 DEFINITION Let H be a Hilbert space. A pair (X, Y) of selfadjoint, contractive operators on H is an (ungraded) *Schrödinger pair* if

(a) Y commutes with $X^\flat$ modulo $\mathfrak{K}(H)$, and
(b) $X^\flat \cdot Y^\flat \subseteq \mathfrak{K}(H)$.

There is a graded variant of the definition:

8.6.2 DEFINITION Let H be a graded Hilbert space. A pair (X, Y) of odd, selfadjoint, contractive operators on H is a *graded Schrödinger pair* if

(a) Y graded-commutes with $X^\flat$ modulo $\mathfrak{K}(H)$, and

(b) $X^\flat \cdot Y^\flat \subseteq \mathfrak{K}(H)$.

Both definitions may be strengthened, as follows:

8.6.3 DEFINITION If in the preceding two definitions the condition (a) is replaced by the stronger condition that the operator Y commutes (or graded-commutes) modulo $\mathfrak{K}(H)$ with X itself, then we shall call (X, Y) a *strong* Schrödinger pair.

8.6.4 REMARK If (X, Y) is an ungraded Schrödinger pair on H, then the operators

$$X' = \begin{pmatrix} 0 & -iX \\ iX & 0 \end{pmatrix}, \qquad Y' = \begin{pmatrix} 0 & Y \\ Y & 0 \end{pmatrix}$$

constitute a graded Schrödinger pair on $H' = H \oplus H$. This is a 'formal periodicity' similar to that appearing in Equation 8.1.10.

8.6.5 EXAMPLE Let $H = L^2(\mathbb{R})$, let X be the operator of multiplication by the function $x \mapsto x(1+x^2)^{-\frac{1}{2}}$, and let Y be the operator of Example 8.5.3, which multiplies the Fourier transform by the function $\xi \mapsto \xi(1+\xi^2)^{-\frac{1}{2}}$. Then (X, Y) constitute a strong Schrödinger pair. This corresponds to the basic quantum-mechanical example of the one-dimensional position and momentum operators, which was mentioned above.

8.6.6 EXAMPLE Generalizing the previous example, let (ρ, H, F) be an ungraded Fredholm module over a C^*-algebra J. Suppose now that J is an ideal in a C^*-algebra A and that the representation ρ is extended to A. If a is a selfadjoint element of A such that $a^2 - 1 \in J$, then the operators $X = \rho(a)$ and $Y = F$ constitute a Schrödinger pair. If the extension of ρ to A makes (ρ, H, F) into a relative Fredholm module for $(A, A/J)$ then (X, Y) is a strong Schrödinger pair.

8.6.7 DEFINITION Let (X, Y) be a Schrödinger pair. The *Schrödinger operator*[75] associated to (X, Y) is the operator

$$V(X, Y) = \varepsilon X + (1 - X^2)^{\frac{1}{2}} Y,$$

where $\varepsilon = \sqrt{-1}$ in the ungraded case and $\varepsilon = 1$ in the graded case.

These operators are Fredholm:

8.6.8 PROPOSITION *Let (X, Y) be a Schrödinger pair of operators on a Hilbert space H. Then:*

(a) *in the ungraded case, the Schrödinger operator $V(X, Y) = iX + (1 - X^2)^{\frac{1}{2}} Y$ is essentially unitary, and therefore Fredholm, and*

[75] Our terminology is not perfect — the operator of Definition 8.6.7 more closely resembles the *square root* of the physicists' Schrödinger operator than it does that operator itself.

(b) *in the graded case, the Schrödinger operator* $V(X,Y) = X + (1-X^2)^{\frac{1}{2}}Y$ *is essentially selfadjoint, graded and Fredholm.*

PROOF The calculations in the two cases are very similar, and we shall consider only the graded case. It is clear that $V = V(X,Y)$ is odd and essentially selfadjoint. We must show that $V^2 \sim 1$. Since $X^\flat$ graded-commutes with Y modulo compact operators, the operators X and $(1-X)^{\frac{1}{2}}Y$ anticommute modulo compacts. Therefore

$$V^2 - 1 \sim X^2 + (1-X^2)Y^2 - 1 = -(1-X^2)(1-Y^2) \sim 0. \qquad \square$$

8.6.9 EXAMPLE If (X,Y) is a Schrödinger pair for which $X^2 = 1$, then $V(X,Y) = \varepsilon X$ has index zero.

If the Hilbert space H is equipped with a representation of a C^*-algebra B, and if X and Y commute with B modulo compact operators, then of course so does $V(X,Y)$. In the graded case $V(X,Y)$ defines a Fredholm module and hence a K-homology class $[V(X,Y)] \in K^0(B)$. In the ungraded case we also obtain a class $[V(X,Y)] \in K^0(B)$, this time using the balancing construction in Remark 8.6.4. We emphasize that in *both* cases (ungraded and graded) a Schrödinger module defines an element of the *even*[76] K-homology group $K^0(B)$.

8.6.10 PROPOSITION *Let* (X,Y) *be a graded strong Schrödinger pair of operators on a graded Hilbert space* H. *Then*

$$\operatorname{Index} V(X,Y) = \operatorname{Index} V(Y,X).$$

If X *and* Y *commute modulo compact operators with a representation of a* C^**-algebra* B *then there is a similar identity between* K*-homology classes:*

$$[V(X,Y)] = [V(Y,X)] \in K^0(B).$$

PROOF The second statement includes the first. Using the fact that X and Y anticommute modulo the compacts, one calculates that

$$V(X,Y)V(Y,X) + V(Y,X)V(X,Y) \sim 2\left(Y(1-X^2)^{\frac{1}{2}}Y + X(1-Y^2)^{\frac{1}{2}}X\right).$$

The right-hand side of this equation is a positive operator. Therefore by Proposition 8.3.16, $V(X,Y)$ and $V(Y,X)$ define the same K-homology class. $\square$

There is a corresponding ungraded statement, whose proof we leave to the reader:

[76] In Chapter 9 we shall need to contemplate Schrödinger modules equipped with additional multigrading operators. A p-multigraded Schrödinger module then defines an element of $K^{-p}(B)$. But there seems to be no need to burden the reader with a formal discussion of this.

8.6.11 PROPOSITION *Let* (X, Y) *be a strong Schrödinger pair of operators on an ungraded Hilbert space* H. *Then*

$$\operatorname{Index} V(X, Y) = -\operatorname{Index} V(Y, X).$$

If X *and* Y *commute modulo compact operators with a representation of a* C^**-algebra* B *then there is a similar identity between* K*-homology classes:*

$$[V(X, Y)] = -[V(Y, X)] \in K^0(B). \qquad \square$$

We conclude this section with two technical lemmas. Their purpose is to provide a homotopy from $V(X, Y)$ to certain other operators which will arise in the course of our analysis of boundary maps in the next section.

8.6.12 LEMMA *Let* (X, Y) *be a Schrödinger pair of operators on an (ungraded) Hilbert space* H *and let* $P_Y = \frac{1}{2}(1 + Y)$. *Then the operator*

$$W_1(X, Y) = e^{i\pi X} P_Y - (1 - P_Y)$$

is essentially unitary and therefore Fredholm. Moreover,

$$\operatorname{Index} W_1(X, Y) = \operatorname{Index} V(X, Y).$$

PROOF Let $S = \sin(\frac{\pi}{2}X)$, and note that Y commutes modulo compacts with $S^\flat$ and that $(1 - S^2)(1 - Y^2)$ is compact. Now write

$$\begin{aligned} e^{-i\frac{\pi}{2}X} W_1(X, Y) &= \big((1 - S^2)^{\frac{1}{2}} + iS\big)P - \big((1 - S^2)^{\frac{1}{2}} - iS\big)(1 - P) \\ &= iS + (1 - S^2)^{\frac{1}{2}} Y \\ &= V(S, Y). \end{aligned}$$

This proves that $W_1(X, Y)$ is essentially unitary and is homotopic to $V(S, Y)$ through the path $t \mapsto e^{-it\frac{\pi}{2}X} W_1(X, Y)$, $t \in [0, 1]$, so that $[W_1(X, Y)] = [V(S, Y)]$. But now the straight-line path from S to X defines a homotopy $X_t = tX + (1 - t)S$, and for each $t \in [0, 1]$, the operators (X_t, Y) constitute a Schrödinger pair. Thus we obtain a homotopy $V(X_t, Y)$, which shows that $[V(S, Y)] = [V(X, Y)]$. $\square$

8.6.13 LEMMA *Let* (X, Y) *be a graded Schrödinger pair of operators on a graded Hilbert space* H. *Suppose in addition that*

(a) *the operator* $Q_X = 1 - X^2$ *is a projection, and*
(b) *there is an odd selfadjoint involution* Y_0 *on* H *which commutes with* Q_X *modulo the compacts.*

Then the operator

$$W_2(X,Y) = YQ_X + Y_0(1-Q_X)$$

is essentially selfadjoint, graded and Fredholm. Moreover,

$$\operatorname{Index} W_2(X,Y) = \operatorname{Index} V(X,Y).$$

PROOF Notice that Q_X is the projection onto the kernel of X. Let

$$X_t = \cos(\tfrac{\pi}{2}t)X + \sin(\tfrac{\pi}{2}t)Y_0(1-Q_X).$$

Then, for all $t \in [0,1]$, we have $X_t^2 \sim 1 - Q_X$. In addition, $X_t(YQ_X) \sim 0$ and $(YQ_X)X_t \sim 0$. It follows that the path

$$t \mapsto YQ_X + X_t, \quad t \in [0,1]$$

is an operator homotopy from $V(X,Y)$ to $W_2(X,Y)$. □

8.7 The Index Pairing

Recall that our main computational tool in Chapter 7 was the index pairing between K-theory and K-homology. Our objective in this section is to investigate the index pairing from the point of view of Kasparov theory. We begin by reformulating the relevant definitions (namely Definitions 7.2.1 and 7.2.3) in the language of Fredholm modules. Having done so, our main business is to prove that the index pairing is compatible with the boundary maps in K-homology and K-theory. This result (Proposition 8.7.5) is very important, and indeed we have already used it in Chapter 7, deferring the proof.

8.7.1 PROPOSITION *Let A be a unital C^*-algebra, and suppose given*

(a) *an ungraded unital Fredholm module (ρ, H, F) over A, and*
(b) *a unitary u in a matrix algebra $M_k(A)$ over A.*

Denote by P_k the operator $1 \otimes \frac{1}{2}(1+F)$ on the Hilbert space $H^k = \mathbb{C}^k \otimes H$, and denote by U the unitary operator $(1 \otimes \rho)(u)$ on H^k. Then

(a) *the operator*

$$P_kUP_k - (1-P_k)\colon H^k \to H^k$$

is essentially unitary and therefore Fredholm,

(b) *the Fredholm index of $P_kUP_k - (1-P_k)$ depends only on $[u] \in K_1(A)$ and on $[F] \in K^1(A)$, and*
(c) *the pairing $K_1(A) \times K^1(A) \to \mathbb{Z}$ defined by* (b) *above agrees with the index pairing of Definition* 7.2.1.

PROOF Let $W = P_k U P_k - (1 - P_k)$. Since P_k and U commute modulo the compact operators, and since P_k is a projection modulo the compact operators, we have

$$W^*W \sim P_k U^* U P_k + (1 - P_k) \sim 1.$$

Similarly $WW^* \sim 1$, so W is essentially unitary. The assignment

$$(u, (\rho, H, F)) \mapsto \mathrm{Index}\,(W)$$

is additive under direct sums (in either variable), and stable under homotopies and compact perturbations, so it passes to a bilinear pairing $K^1(A) \times K_1(A) \to \mathbb{Z}$. To check the compatibility with Definition 7.2.1, we can suppose that P_k is actually a projection in $\mathfrak{D}_\rho(A)$. Then $W = P_k U P_k \oplus (-I)$, relative to the direct sum decomposition $H^k = \mathrm{Image}(P_k) \oplus \mathrm{Image}(I - P_k)$. The second summand has index zero, and the index of the first summand is exactly the pairing defined in 7.2.1. □

Similarly, in the even case, we have:

8.7.2 PROPOSITION *Let A be a unital C^*-algebra, and suppose given*

(a) *a graded unital Fredholm module (ρ, H, F) over A, and*
(b) *a projection p in a matrix algebra $M_k(A)$ over A.*

Let P be the projection $(1 \otimes \rho)(p)$ on the graded Hilbert space $H^k = \mathbb{C}^k \otimes H$. Write

$$F = \begin{pmatrix} 0 & V \\ U & 0 \end{pmatrix},$$

relative to the graded decomposition $H = H^+ \oplus H^-$. Then

(a) *the operator*

$$P(1 \otimes U)P \colon P(\mathbb{C}^n \otimes H^+) \to P(\mathbb{C}^n \otimes H^-)$$

is essentially unitary, and therefore Fredholm,

(b) *the Fredholm index of $P(1 \otimes U)P$ depends only on the K-theory class $[p] \in K_0(A)$ and the K-homology class $[\rho, H, F] \in K^0(A)$, and*
(c) *the pairing $K_0(A) \times K^0(A) \to \mathbb{Z}$ defined by (b) above agrees with the index pairing of Definition 7.2.3.* □

The index pairings above may be extended to the non-unital case by embedding a possibly non-unital algebra A into its unitalization $\widetilde{A}$ and making use of the excision isomorphisms

$$K_{-p}(\widetilde{A}) \cong K_{-p}(A) \oplus K_{-p}(\mathbb{C}), \qquad K^{-p}(\widetilde{A}) \cong K^{-p}(A) \oplus K^{-p}(\mathbb{C})$$

which follow from the six-term exact sequences in K-theory and K-homology.

8.7.3 REMARK There is an alternative presentation of the even index pairing which makes use of the description of $K_0(A)$ in terms of *Hermitian modules* (see Proposition A.4.5 in Appendix A). Let (ρ, H, F) be a graded Fredholm module over A, representing $y \in K^0(A)$, and let M be a graded Hermitian A-module representing an element x of $K_0(A)$. Then we may form the tensor product $M\hat{\otimes}_\rho H$, which is a graded Hilbert space, and on this Hilbert space we may consider the operator[77] $G = 1\hat{\otimes}F$, which is odd and satisfies $G^2 \sim 1$ and $G \sim G^*$. Thus G is a graded Fredholm operator (in the sense of Example 8.1.4); its index gives the pairing $\langle x, y\rangle$. This is simply a reformulation of Proposition 8.7.2 in the language of Hermitian modules.

8.7.4 EXAMPLE Let $\alpha\colon A \to \mathbb{C}$ be a $*$-homomorphism, and let $[\alpha] \in K^0(A)$ be the K-homology class of Example 8.1.9. Then the index pairing with $[\alpha]$ gives a homomorphism $i_\alpha\colon K_0(A) \to \mathbb{Z}$. By definition, this homomorphism takes a projection p to the index of the zero operator from $\mathrm{Image}(\alpha(p))$ to 0. Thus $i_\alpha(p)$ is the dimension of $\mathrm{Image}(\alpha(p))$, and so i_α is equal to the induced map

$$\alpha_*\colon K_0(A) \to K_0(\mathbb{C}) = \mathbb{Z}$$

on K-theory.

We are now going to use the techniques of Kasparov theory and relative K-homology to show that the index pairings are compatible with the boundary maps in K-theory and K-homology. Recall that this compatibility was a key point in the arguments of Chapter 7 and that we deferred the proof. Now we shall pay the debt by proving Proposition 7.2.4, which for convenience we restate here:

8.7.5 PROPOSITION *Let J be an ideal in a separable C^*-algebra A for which the quotient mapping from A to A/J is semisplit. Denote by ∂ the associated boundary maps in the K-homology and K-theory six-term exact sequences. If $x \in K_0(A/J)$ and $y \in K^1(J)$, then*

$$\langle \partial x, y\rangle = -\langle x, \partial y\rangle. \tag{8.7.6}$$

Similarly if $x \in K_1(A/J)$ and $y \in K^0(J)$, then

$$\langle \partial x, y\rangle = \langle x, \partial y\rangle. \tag{8.7.7}$$

The proofs of these results require extensive but elementary manipulations. We shall treat the two cases separately, but in each case the overall strategy is the same: we shall construct a certain Schrödinger operator V and then we shall

[77] This 'tensor product' is not unique — it is well-defined only modulo the compact operators — but compact perturbations do not affect the index of G. See A.4.8.

show, by separate deformation arguments, first that the index of V is equal to $\langle \partial x, y\rangle$, and second that the index of V is equal to $\langle x, \partial y\rangle$ (up to sign). We shall assume throughout that the algebra A/J is unital; a standard diagram chase shows that this entails no loss of generality.

We begin with Equation 8.7.6.

8.7.8 LEMMA *Suppose given a short exact sequence of separable* C^**-algebras*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0 .$$

Let (ρ, H, F) *be an ungraded Fredholm module for* J*, let* $p \in M_n(A/J)$ *be a projection, and let* $a \in M_n(A)$ *be a lift of* p*. Then the operators*

$$X = (1 \otimes \rho)(2a - 1) \quad \textit{and} \quad Y = 1 \otimes F$$

on the Hilbert space $\mathbb{C}^n \otimes H$ *form a Schrödinger pair. Furthermore, if* (ρ, H, F) *is a* relative *Fredholm module for* $(A, A/J)$ *then* (X, Y) *is a strong Schrödinger pair. The assignment*

$$(p, F) \mapsto \operatorname{Index} V(X, Y)$$

defines bilinear pairings

$$K_0(A/J) \otimes K^1(J) \longrightarrow \mathbb{Z}$$

and

$$K_0(A/J) \otimes K^1(A, A/J) \longrightarrow \mathbb{Z} ,$$

which are compatible with one another via the excision map from $K^1(A, A/J)$ *to* $K^1(J)$.

REMARK To define the operator X above, we extend ρ to a non-degenerate representation of $\widetilde{A}$.

PROOF Example 8.6.6 shows that (X, Y) is a Schrödinger pair, and indeed a strong Schrödinger pair in the case of a relative Fredholm module. It is easy to check that the assignment $(F, p) \mapsto \operatorname{Index} V(X, Y)$ respects the equivalence relations used in defining the K-theory and K-homology groups, and defines a bilinear pairing. $\square$

8.7.9 DEFINITION We shall call the integer $\operatorname{Index} V(X, Y)$ so obtained from $x \in K_0(A/J)$ and $y \in K^1(J)$ (or $y \in K^1(A, A/J)$) the *Schrödinger pairing* of x and y, and denote it $x \diamondsuit y$.

8.7.10 LEMMA *Suppose given a semisplit short exact sequence of separable C^*-algebras*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0 .$$

Let $x \in K_0(A/J)$ *and let* $y \in K^1(A, A/J)$. *Then* $x \lozenge y = -\langle x, \partial y\rangle$.

PROOF Assume that x is represented by a projection p over A/J; to simplify the notation somewhat, let us in fact assume that p actually belongs to A/J. Assume also that y is represented by an ungraded relative Fredholm module (ρ, H, F). The explicit construction of the boundary map in K-homology, given by Proposition 8.5.6, tells us that ∂y is represented by the balanced Fredholm module described by the unitary operator $\left(\begin{smallmatrix} e^{i\pi F} & 0 \\ 0 & -1 \end{smallmatrix}\right)$ acting on the Hilbert space $H \oplus H'$, together with a certain representation $\underline{\psi}\colon A/J \to \mathfrak{B}(H \oplus H')$ which is obtained from a completely positive section $s\colon \widetilde{A/J} \to \widetilde{A}$ by first composing with ρ and then applying Stinespring's Theorem. Let $\widehat{X} = \psi(2p-1) \in \mathfrak{B}(H \oplus H')$. If we write $\widehat{X}$ as an operator matrix, say

$$\widehat{X} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \mathfrak{B}(H \oplus H'),$$

then the top-left entry of $\widehat{X}$ has the form $X_{11} = \rho(2a-1)$, where $a \in A$ is a lift of $p \in A/J$. Thus X_{11} is the operator X which appears in the definition of the Schrödinger pairing. If we write $Y = F$, again as in the definition of the Schrödinger pairing, then the operators X and Y comprise a strong Schrödinger pair and

$$x \lozenge y = \operatorname{Index} V(X, Y).$$

Now the operators

$$\widehat{Y} = \begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \widehat{X} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

comprise a Schrödinger pair $(\widehat{Y}, \widehat{X})$ (although not a strong Schrödinger pair[78]). This is because the off-diagonal terms of $\widehat{X}$ multiply $1 - Y^2$ into the compact operators. Indeed, $1 - Y^2 = 1 - F^2$, while, since $\widehat{X}^2 = 1$,

$$X_{12}X_{21} = 1 - X_{11}^2 = 1 - X^2 \in \rho[J],$$

so the required compactness follows from the axioms for the Fredholm module (ρ, H, F) over J. The same calculation shows that

$$V(\widehat{Y}, \widehat{X}) \sim \begin{pmatrix} V(Y,X) & 0 \\ 0 & 1 \end{pmatrix},$$

[78]This is important: if $(\widehat{Y}, \widehat{X})$ was a strong pair then Proposition 8.6.11 and Example 8.6.9 would imply that $\operatorname{Index} V(\widehat{Y}, \widehat{X}) = 0$, whereas we shall see that $\operatorname{Index} V(\widehat{Y}, \widehat{X}) = \langle x, \partial y\rangle$.

and therefore, thanks to Proposition 8.6.11, that

$$\operatorname{Index} V(\widehat{Y},\widehat{X}) = \operatorname{Index} V(Y,X) = -\operatorname{Index} V(X,Y) = -x \lozenge y.$$

In view of the formula for ∂y provided by Proposition 8.5.6, and in view of the description of the index pairing in Proposition 8.7.1, if we set $P_{\widehat{X}} = \frac{1}{2}(\widehat{X}+1) = \psi(p)$ then $\langle x, \partial y\rangle$ is equal to the index of the operator

$$\begin{pmatrix} e^{i\pi F} & 0 \\ 0 & -1 \end{pmatrix} P_{\widehat{X}} - (1-P_{\widehat{X}}) = e^{i\pi F} P_{\widehat{X}} - (1-P_{\widehat{X}}) = W_1(\widehat{Y},\widehat{X}),$$

where the notation $W_1(\widehat{Y},\widehat{X})$ is defined in Lemma 8.6.12. But Lemma 8.6.12 asserts that $\operatorname{Index} W_1(\widehat{Y},\widehat{X}) = \operatorname{Index} V(\widehat{Y},\widehat{X})$. Therefore

$$\langle x, \partial y\rangle = \operatorname{Index} W_1(\widehat{Y},\widehat{X}) = \operatorname{Index} V(\widehat{Y},\widehat{X}) = \operatorname{Index}(Y,X) = -x \lozenge y,$$

and the proof is complete. □

8.7.11 Lemma *Suppose given a short exact sequence of separable C^*-algebras*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0\,.$$

If $x \in K_0(A/J)$ and $y \in K^1(J)$ then $x \lozenge y = \langle \partial x, y\rangle$.

Proof As in the previous proof, assume that x is represented by a projection $p \in A/J$, lifting to a selfadjoint element $a \in A$, and that y is represented by an ungraded Fredholm module (ρ, H, F). We can assume that the latter is the restriction to J of a unital Fredholm module for $\widetilde{J}$. Put $X = \rho(2a-1)$ and $Y = F$. Now we need the 'exponential' construction of the boundary map on K-theory (Remark 4.9.3). This tells us that ∂x is represented by the unitary $e^{2i\pi a} \in M_k(\widetilde{J})$. By Proposition 8.7.1, $\langle \partial x, y\rangle$ is equal to the index of the operator

$$T = (1\otimes\rho)(e^{2i\pi a})P_Y + (1-P_Y),$$

where $P_Y = \frac{1}{2}(1+Y)$. By the definition of X,

$$T = -e^{i\pi X}P_Y + (1-P_Y) = -W_1(X,Y),$$

and so by Lemma 8.6.12 again,

$$\langle \partial x, y\rangle = \operatorname{Index}(T) = \operatorname{Index} W_1(X,Y) = \operatorname{Index} V(X,Y) = x \lozenge y.$$

The proof is complete. □

Proof of Equation 8.7.6 Combine Lemmas 8.7.10 and 8.7.11. □

Now we shall carry out the corresponding argument for Equation 8.7.7. We leave the following calculation to the reader.

8.7.12 LEMMA *Suppose given a short exact sequence of separable* C^*-*algebras*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0\,.$$

Let (ρ, H, F) *be a graded relative Fredholm module for the pair* $(A, A/J)$. *Let* $u \in M_n(A/J)$ *be a unitary matrix and let* $a \in M_n(A)$ *be a lift of* u *to* A. *Then the operators*

$$X = \begin{pmatrix} 0 & (1 \otimes \rho)(a^*) \\ (1 \otimes \rho)(a) & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 \otimes F & 0 \\ 0 & -1 \otimes F \end{pmatrix}$$

on the graded Hilbert space $\mathbb{C}^n \otimes H \oplus \mathbb{C}^n \otimes H^{op}$ *form a strong Schrödinger pair. Furthermore the assignment*

$$(u, F) \mapsto \operatorname{Index} V(X, Y)$$

defines a bilinear pairing $K_1(A/J) \otimes K^0(A, A/J) \to \mathbb{Z}$. □

The pairing described in Lemma 8.7.12 is clearly a counterpart of the one defined in Lemma 8.7.8. So we shall use again the term *Schrödinger pairing* and the notation $x \lozenge y$, where now $x \in K_1(A/J)$ and $y \in K^0(A, A/J)$. Here is the counterpart of Lemma 8.7.10:

8.7.13 LEMMA *Suppose given a semisplit short exact sequence of separable* C^*-*algebras*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0\,.$$

Let $x \in K_1(A/J)$ *and* $y \in K^0(A, A/J)$. *Then* $x \lozenge y = \langle x, \partial y \rangle$.

PROOF Let y be represented by a graded relative Fredholm module (ρ, H, F) for $(A, A/J)$. We shall use the calculation of the K-homology boundary map given in Proposition 8.5.6. We assume therefore that the relative Fredholm module (ρ, H, F) is partially isometric, or in other words that $1 - F^2$ is a projection Q with graded components $Q^{\pm}$. The boundary ∂y is then the difference $[Q^+] - [Q^-]$ of the homology classes of the extensions of A/J corresponding to the projections $Q^{\pm}$ (see Remark 8.5.7). Let x be represented by a unitary u over A/J; to simplify the notation we shall assume that u actually belongs to A/J. Put

$$X = \begin{pmatrix} 0 & \rho(a^*) \\ \rho(a) & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix},$$

which are operators on the Hilbert space $H \oplus H^{op}$. By definition,

$$x \lozenge y = \operatorname{Index} V(X, Y) = \operatorname{Index} V(Y, X).$$

The operator

$$Q_Y = 1 - Y^2 = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}$$

is a projection, and moreover the operator $X_0 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is an odd selfadjoint involution which commutes modulo the compacts with Y^2. Therefore by Lemma 8.6.13, if

$$W_2(Y, X) = XQ_Y + X_0(1 - Q_Y),$$

then $\operatorname{Index} V(Y, X) = \operatorname{Index} W_2(Y, X)$. The second summand makes no contribution to the index. As for the first summand, the part $(XQ_Y)^+$ which maps the negatively graded component of $Q_Y(H \oplus H^{op}) = QH \oplus QH^{op}$ to the positively graded component is given by

$$(XQ_Y)^+ \sim \begin{pmatrix} 0 & Q^-\rho(a^*)Q^- \\ Q^+\rho(a)Q^+ & 0 \end{pmatrix}.$$

Consequently,

$$\begin{aligned}
\operatorname{Index} W_2(Y, X) &= \operatorname{Index}(Q^+\rho(a)Q^+) + \operatorname{Index}(Q^-\rho(a^*)Q^-) \\
&= \langle x, [Q^+] \rangle - \langle x, [Q^-] \rangle \\
&= \langle x, \partial y \rangle
\end{aligned}$$

as required. $\square$

8.7.14 LEMMA *Suppose given a short exact sequence of separable C^*-algebras*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0\,.$$

If $x \in K_1(A/J)$ and $y \in K^0(A, A/J)$ then $x \lozenge y = \langle \partial x, y \rangle$.

PROOF By adjoining a unit if necessary we may assume that A is unital. Let $y \in K^0(A, A/J)$ be represented by a graded relative Fredholm module (ρ, H, F). Let $x \in K_1(A/J)$ be represented by a unitary matrix over A/J. Once again, to simplify the notation let us assume that $u \in A/J$.

Let us consider first the case where u lifts to a *partial isometry* $a \in A$. As in the previous lemma, let

$$X = \begin{pmatrix} 0 & \rho(a^*) \\ \rho(a) & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix},$$

so that $x \lozenge y = \operatorname{Index} V(X, Y)$. The assumption that a is a partial isometry implies that the operator

$$Q_X = 1 - X^2 = \begin{pmatrix} \rho(1 - a^*a) & 0 \\ 0 & \rho(1 - aa^*) \end{pmatrix}$$

is a projection, in fact the projection onto the kernel of X. Thus

$$V(X,Y) = X + (1 - X^2)Y = (1 - Q_X)X(1 - Q_X) + Q_X Y Q_X,$$

and so Index $V(X,Y) = $ Index $(Q_X Y Q_X)$. It now follows from the formula

$$\partial x = [1 - a^* a] - [1 - aa^*] \in K_0(J),$$

provided by Proposition 4.8.10, together with the definition of the index pairing, that Index $(Q_X Y Q_X) = \langle \partial x, y \rangle$. This completes the proof of the lemma in the special case where u lifts to a partial isometry.

The general case is a reduction to the special case just proved. Every unitary $u \in A/J$ lifts to a contractive element $a \in A$, and if we form the exact sequence

$$0 \longrightarrow M_2(J) \longrightarrow \begin{bmatrix} A & J \\ J & J \end{bmatrix} \longrightarrow A/J \longrightarrow 0\,,$$

then the unitary $u \in A/J$ lifts to the partial isometry $\left(\begin{smallmatrix} a & 0 \\ (1-a^*a)^{1/2} & 0 \end{smallmatrix}\right)$. Since the inclusion of J into $M_2(J)$ as matrices of the type $\left(\begin{smallmatrix} j & 0 \\ 0 & 0 \end{smallmatrix}\right)$ induces an isomorphism in K-theory and K-homology,[79] the general case of the lemma follows from the special case and the functoriality of the boundary maps and of the Schrödinger and index pairings. □

Proof of Equation 8.7.7 Combine Lemmas 8.7.13 and 8.7.14. □

In the next chapter it will be important to note that some of the calculations of this section are also valid in a different context. Specifically, we shall be interested in the semisplit short exact sequence

$$0 \longrightarrow S(A) \longrightarrow C(A) \longrightarrow A \longrightarrow 0$$

relating the suspension $S(A) = C_0(0,1) \otimes A$ and the cone $C(A) = C_0(0,1] \otimes A$ of a unital C^*-algebra A. Let (ρ, H, F) be an ungraded Fredholm module for the $S(A)$ which is a relative module for the inclusion of $S(A)$ as an ideal in $C[0,1] \otimes A$. If $x \in C_0(0,1]$ denotes the identity function $t \mapsto t$ then the operators

$$X = \rho((2x-1) \otimes 1) \quad \text{and} \quad Y = F$$

define a strong Schrödinger pair on H. Moreover, both X and Y commute modulo compacts with the representation $a \mapsto \rho(1 \otimes a)$ of A. Therefore the operator

$$V(X,Y) = iX + (1 - X^2)^{\frac{1}{2}} Y$$

defines a Fredholm module and hence a class in $K^0(B)$.

[79] We shall prove a general matrix stability property for K-homology in the next chapter, but we need only note here that if (ρ, H, F) is a Fredholm module over J then the operator $F \oplus F$ defines a corresponding Fredholm module for $M_2(J)$ which pulls back to (ρ, H, F) along the inclusion $J \to M_2(J)$. This fact — that every module over J comes from a module over $M_2(J)$ — is all that is needed.

8.7.15 LEMMA $\partial[\rho, H, F] = [V(Y,X)] = -[V(X,Y)] \in K^0(B)$.

PROOF We may define a Stinespring dilation $\psi\colon A \to \mathfrak{B}(H \oplus H)$ of ρ by

$$\psi(a) = \begin{pmatrix} \rho(x \otimes a) & \rho(x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} \otimes a) \\ \rho(x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} \otimes a) & \rho((1-x) \otimes a) \end{pmatrix} = U \begin{pmatrix} \rho(1 \otimes a) & 0 \\ 0 & 0 \end{pmatrix} U^*,$$

where U is the (selfadjoint) unitary

$$U = \begin{pmatrix} \rho(x^{\frac{1}{2}} \otimes 1) & \rho((1-x)^{\frac{1}{2}} \otimes 1) \\ \rho((1-x)^{\frac{1}{2}} \otimes 1) & -\rho(x^{\frac{1}{2}} \otimes 1) \end{pmatrix}$$

According to the explicit formula in Proposition 8.5.6 the class $\partial[\rho, H, F]$ is represented by the Fredholm module $(\psi, H \oplus H, \left(\begin{smallmatrix} e^{i\pi F} & 0 \\ 0 & -1 \end{smallmatrix}\right))$. If we conjugate this module by the unitary U above then we obtain (a compact perturbation of) the Fredholm module consisting of the representation $a \mapsto \left(\begin{smallmatrix} \rho(1\otimes a) & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and the operator

$$W_1(Y,X) = \begin{pmatrix} e^{i\pi F}P_X - (1-P_X) & 0 \\ 0 & -1 \end{pmatrix},$$

where $X = \rho((2x-1)\otimes 1)$, $Y = F$ (as above) and $P_X = \frac{1}{2}(X+1)$. But now the same argument that we used to prove Lemma 8.6.12 shows that $W_1(Y,X)$ determines the same K-homology class as $V(Y,X)$. □

Similarly, let (ρ, H, F) be a graded relative Fredholm module for the inclusion of $S(A)$ into $C[0,1] \otimes A$. As above, let $X = \rho(1 \otimes (2a-1))$ and let $Y = F$. Let γ denote the grading operator on H. Then the operator[80]

$$V(X,Y) = \gamma X + (1-X^2)^{\frac{1}{2}} Y$$

defines an ungraded Fredholm module and therefore a class in $K^1(B)$.

8.7.16 LEMMA $\partial[\rho, H, F] = [V(X,Y)] \in K^1(B)$. □

PROOF The proof begins by constructing the same explicit Stinespring dilation ψ that was used in the proof of the previous lemma. Now represent $\partial[\rho, H, F]$ by the Fredholm module $(\psi, H\oplus H', \gamma(2\widehat{Q}-1))$ which was presented in Remark 8.5.8. Conjugating with the unitary U gives (a compact perturbation of) the Fredholm module consisting of the representation $a \mapsto \left(\begin{smallmatrix} \rho(1\otimes a) & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and the operator

$$\begin{pmatrix} \gamma XQ - \gamma(1-Q) & 0 \\ 0 & 0 \end{pmatrix},$$

where $X = \rho((2x-1)\otimes 1)$ and γ now denotes the grading operator on H alone. To complete the proof, a rotation argument like the one used to prove Lemma 8.6.13

[80] Note that thanks to the presence of the factor γ, the operator $V(X,Y)$ defined here is not precisely a Schrödinger operator of the type we have considered up to now.

shows that $\gamma XQ - \gamma(1-Q)$ is operator homotopic to $\gamma XQ + Y$, where $Y = F$ as above. But the anticommutator of $\gamma XQ + Y$ and $V(X,Y) = \gamma X + (1-X^2)^{\frac{1}{2}}Y$ is positive modulo compact operators. So by Proposition 8.3.16 these operators determine the same K-homology class. $\square$

8.8 Exercises

8.8.1 Let (ρ, H, F) be a graded Fredholm module over A. Suppose that we decide to neglect the grading and to regard (ρ, H, F) instead as an ungraded Fredholm module. Show that the class in $K^1(A)$ thereby defined is zero. Formulate and prove a multigraded version of this result.

8.8.2 Let $A = C^*(|\mathbb{R}|)$, the coarse C^*-algebra[81] of Definition 6.3.8 associated to the metric space $\mathbb{R}$. Let $H = L^2(\mathbb{R})$, let $\rho\colon A \to \mathfrak{B}(H)$ be the natural representation, and let $F \in \mathfrak{B}(H)$ be the operator of multiplication by a continuous function $f(x)$ with $f(x) = 1$ for $x > 1$, $f(x) = -1$ for $x < -1$. Prove that (ρ, H, F) is an ungraded Fredholm module over A.

8.8.3 A metric space M is *partitioned* if there is given a proper map φ from M onto $\mathbb{R}$. Use Exercise 8.8.2 and the pairing between K-theory and K-homology to construct a map

$$\varphi_*\colon K_1(C^*(M)) \to \mathbb{Z}$$

associated to a partition φ of a metric space M. (See [111] for the index theory associated to this map.)

8.8.4 Let (ρ, H, F) be a graded Fredholm module over an algebra A, and suppose that ε is an odd unitary operator of square -1, commuting with $\rho(a)$ and F. Show that we must have

$$\varepsilon = \begin{pmatrix} 0 & -V^* \\ V & 0 \end{pmatrix}$$

where $V\colon H^+ \to H^-$ is a unitary operator intertwining ρ^+ and ρ^-. Show further that $VF^+V^* = F^-$ and that (ρ^+, VF^+, H^+) is an ungraded Fredholm module. Show that the original 1-multigraded Fredholm module (ρ, H, F) is equivalent to the 1-multigraded Fredholm module obtained from (ρ^+, VF^+, H^+) by the construction of Equation 8.1.10.

8.8.5 Generalizing the previous exercise, suppose that (ρ, H, F) is a $(p+2)$-multigraded Fredholm module over A. Let $Z = -i\varepsilon_{p+1}\varepsilon_{p+2}$. Show that Z is a selfadjoint, even involution which commutes with F and with ρ, and that the

[81] This algebra is not separable. But this does not affect the calculations we want to do here.

restriction of (ρ, H, F) to the $+1$-eigenspace of Z is a p-multigraded Fredholm module. Show that this procedure gives rise to a homomorphism $K^{-p-2}(A) \to K^{-p}(A)$, and that this homomorphism is inverse to the formal periodicity map $K^{-p}(A) \to K^{-p-2}(A)$. Thus obtain a proof of the formal periodicity theorem (Proposition 8.2.13).

8.8.6 Can K-homology for unital C^*-algebras be normalized by the requirement that Fredholm modules be simultaneously unital and involutive?

8.8.7 Prove that a permutation σ of the p multigrading operators induces the automorphism $(-1)^\sigma$ on $K^{-p}(A)$.

8.8.8 Let $\rho\colon A \to \mathfrak{B}(H)$ be a representation of a C^*-algebra A, and suppose that $P \in \mathfrak{D}_\rho(A)$ is selfadjoint and that $\rho(a)P\rho(a^*)$ is positive modulo compacts for all $a \in A$. Prove that P is positive modulo $\mathfrak{D}_\rho(A/\!\!/A)$. (Use the functional calculus to write $P = P_+ - P_-$, where P_+ and P_- are positive operators with $P_+P_- = P_-P_+ = 0$. Show that $P_- \in \mathfrak{D}_\rho(A/\!\!/A)$.)

8.8.9 This exercise and the next concern a variation on the pairing defined in Lemma 8.7.12. We shall use the notion of relative K_0-cycle, which is discussed in Appendix A. Suppose given a short exact sequence of separable C^*-algebras

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0\,.$$

Let (ρ, H, F) be a graded relative Fredholm module for the pair $(A, A/J)$, and let (M, a) be a relative K_0-cycle for $(A, A/J)$. Show that the operators[82] $X = a\hat{\otimes}1$ and $Y = 1\hat{\otimes}F$ on the graded Hilbert space $M\hat{\otimes}_\rho H$ form a strong Schrödinger pair. Show that the assignment $(F, p) \mapsto \operatorname{Index} V(X, Y)$ defines a bilinear pairing $K_0(A, A/J) \otimes K^0(A, A/J) \to \mathbb{Z}$.

8.8.10 Follow the steps below to show that the index pairing

$$K_0(J) \otimes K^0(J) \longrightarrow \mathbb{Z}$$

agrees with the Schrödinger pairing

$$K_0(A, A/J) \otimes K^0(A, A/J) \longrightarrow \mathbb{Z}$$

of the previous exercise under the excision isomorphisms $K_0(J) \cong K_0(A, A/J)$ (Theorem 4.3.8) and $K^0(J) \cong K^0(A, A/J)$ (Theorem 5.4.5).

(a) First prove the result for the special case where $y \in K^0(A, A/J)$ belongs to the image of the restriction map $\iota^*\colon K^0(A) \to K^0(A, A/J)$.

[82] See Appendix A for the definition of $1\hat{\otimes}F$.

(b) Let $\widetilde{J}$ denote the unitalization of J. Observe that the Schrödinger pairing is compatible with the maps

$$K_0(\widetilde{J}, \mathbb{C}) \to K_0(A, A/J), \quad K^0(A, A/J) \to K^0(\widetilde{J}, \mathbb{C}),$$

in the sense that both possible ways of pairing a class in $K_0(\widetilde{J}, \mathbb{C})$ with a class in $K^0(A, A/J)$ agree. Now use excision to reduce to the case $A = \widetilde{J}$ and apply part (a) above.

8.8.11 Let $x = (\rho, H, F)$ be a Fredholm module over the C*-algebra A, with $F = F^*$ and $F^2 = 1$. Let us say that x is *summable* if $[F, \rho(a)]$ is a trace-class operator for a dense set of $a \in A$. Show that, if x is a summable, graded Fredholm module and if $p \in A$ is a projection, then the index pairing is given by

$$\langle [p], [x] \rangle = \tfrac{1}{2}\operatorname{Trace}(\gamma F[F, \rho(a)]),$$

where $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ denotes the grading operator.

8.8.12 Let A be a C*-algebra, (ρ, H, F) a summable Fredholm module, and let

$$\mathcal{A} = \{\, a \in A \,:\, [F, \rho(a)] \text{ is trace-class} \,\}.$$

Show that $\mathcal{A}$ is a Banach algebra under the norm

$$\|a\|_{\mathcal{A}} = \|a\| + \|[F, \rho(a)]\|_{tr}.$$

Show that if $a \in \mathcal{A}$ has an inverse in the C*-algebra A, then that inverse belongs to the subalgebra $\mathcal{A}$, and deduce that the spectrum of a is the same $\mathcal{A}$ as it is in A. Hence show that any projection $p \in A$ is homotopic to a projection in $\mathcal{A}$.

8.8.13 Let G be the free group on two generators, and let $A = C^*_r(G)$. Let $x = (\rho, H, F)$ be the involutive Fredholm module over A obtained by applying the procedure of Lemma 8.3.5 to the module of Example 8.1.6. Prove that x is summable.

Let $\tau\colon C^*_r(G) \to \mathbb{C}$ be the canonical trace — that is, the linear functional which sends the identity element of G to 1 and every other element of G to 0. (Note that τ is a vector state associated to the regular representation.) Prove that

$$\tau(a) = \tfrac{1}{2}\operatorname{Trace}(\gamma F[F, a]) \tag{8.8.14}$$

for all a belonging to the dense subalgebra $\mathcal{A} \subseteq A$ defined in Exercise 8.8.12.

Using Exercise 8.8.11, deduce that if p is a projection in A, then $\tau(p)$ is an integer. Hence prove that p is either zero or one; in other words, $C^*_r(G)$ has no non-trivial projections. (This is the *Kadison–Kaplansky conjecture* for $C^*_r(G)$.)

8.8.15 Let H be a Hilbert space and let S be a bounded operator on H. Let T be a positive, bounded operator on H such that $ST - TS$ has finite rank. Let P denote the orthogonal projection onto Kernel(T). Show that $SP - PS$ also has finite rank. (Hint: write

$$P = \lim_{\varepsilon \to 0} \varepsilon (T + \varepsilon)^{-1},$$

where the limit exists in the strong operator topology. Prove that the rank is upper semicontinuous in the strong topology.)

8.8.16 Let G be the free group on two generators. Let $W^*(G)$ denote the algebra of all operators on $\ell^2(G)$ which commute with the right action of G. Note that $C^*_r(G) \subseteq W^*(G)$; $W^*(G)$ is the *group von Neumann algebra* of G. Show that the canonical trace τ extends to a strongly continuous positive linear functional on $W^*(G)$, which is faithful in the sense that if $T \in W^*(G)$, $T \geqslant 0$, and $\tau(T) = 0$, then $T = 0$.

Let $a \in \mathbb{C}[G]$ and let L_a denote the operation of left multiplication by a on $\ell^2(G)$. Let Q be the orthogonal projection onto Kernel(L_a). Use Equation 8.8.14 and Exercise 8.8.15 to show that $\tau(Q) \in \mathbb{Z}$. Deduce that L_a is either zero or injective, and hence in particular that $\mathbb{C}[G]$ has no zero-divisors.

8.8.17 Let H be a finite-dimensional graded Hilbert space, and let $\rho \colon \mathbb{C} \to \mathfrak{B}(H)$ be an (even) representation, not necessarily unital. Show that the Fredholm module $(\rho, H, 0)$ over $\mathbb{C}$ defines the integer

$$\operatorname{Dim}(\operatorname{Image} \rho^+(1)) - \operatorname{Dim}(\operatorname{Image} \rho^-(1)) \ \in K^0(\mathbb{C}) = \mathbb{Z}.$$

Deduce that there is no counterpart to Proposition 8.3.14 for unbalanced Fredholm modules. Show, however, that if (ρ, H, F) and (ρ', H', F') are two graded Fredholm modules over the separable algebra A, and if there exists a sequence U_n of even unitaries such that the operators $F'U_n - U_nF$ and $\rho'(a)U_n - U_n\rho(a)$ are compact and tend to zero (in norm) as $n \to \infty$, then $[\rho, H, F] = [\rho', H', F']$ in $K^0(A)$.

8.8.18 Use the explicit formulas for the boundary maps in K-homology given by Proposition 8.5.6 to verify that the boundary maps are zero in the case of an extension which is split by a $*$-homomorphism. (What does a Stinespring dilation amount to in this case?).

8.8.19 Let A be a unital C^*-algebra and let J be an ideal in A. Let $\alpha \colon \mathbb{C} \to A/J$ be the $*$-homomorphism defined by the unit element. Show that, for any $x \in K^1(J)$, we have $\alpha^* \partial(x) = 0$. (Try to find several proofs: using exactness in the K-homology sequence, using properties of the index pairing and exactness for K-theory, computing directly with Proposition 8.5.6.)

8.8.20 Let F be the operator on $L^2(\mathbb{R})$ defined by

$$\widehat{Ff}(\xi) = \frac{\xi}{(1+\xi^2)^{\frac{1}{2}}} \hat{f}(\xi).$$

Show that FM_g and M_gF are compact for each $g \in C_0(\mathbb{R})$, and that $FM_g \sim M_gF$ for the function

$$g(x) = \frac{x}{(1+x^2)^{\frac{1}{2}}}.$$

8.9 Notes

The constructions of this chapter can be traced back to the article [11], in which Atiyah proposed a functional-analytic abstraction of the properties of elliptic pseudodifferential operators. Atiyah showed that his class of 'abstract elliptic operators' would generate K-homology, but did not explicitly describe the relations between them. That was done by Brown Douglas and Fillmore, and by Kasparov [80], who also initiated the consideration of K-homology for non-commutative C^*-algebras. Kasparov was motivated not only by Atiyah's article but also by questions related to the Novikov conjecture from differential topology; we shall take up that story in Chapter 12.

Relative Fredholm modules were introduced by Baum, Douglas and Taylor [24, 25] to provide a topological theory of elliptic operators on compact manifolds with boundary. The connection to duality theory was developed in the unpublished manuscript [68].

The 'Kadison–Kaplansky conjecture' is that for every torsion-free discrete group G, the C^*-algebra $C^*_r(G)$ has no non-trivial projections. The result for the free group is due to Pimsner and Voiculescu [106]. Subsequent work of Cuntz led eventually to the proof using Fredholm modules given in Exercise 8.8.13, which is due to Connes, and was one of the foundational examples for his theory of non-commutative geometry (see Section IV.5 of [41]). Linnell [96] extended the argument to obtain results on zero-divisors in group rings. The 'zero-divisor conjecture' asserts that the group ring of a torsion-free group has no zero-divisors (see Exercise 8.8.16). The best current results on this conjecture use a combination of operator theory and pure algebra.

The construction of the index pairing by way of Schrödinger modules is a special case of the Kasparov product in bivariant KK-theory.

9

THE KASPAROV PRODUCT

In this chapter we shall construct the external product on K-homology. Given two separable C*-algebras A_1 and A_2 this is a $\mathbb{Z}$-bilinear map

$$K^{-p_1}(A_1) \times K^{-p_2}(A_2) \to K^{-p_1-p_2}(A_1 \otimes A_2).$$

Its construction is at the heart of Kasparov's theory, and while the product appears formally similar to the external product in K-theory, the technical details behind it are considerably more subtle.

The fundamental difficulty can already be seen in the simplest case, where we consider the K-homology of the algebra $\mathbb{C}$. Here the issue is to manufacture from two essentially unitary Fredholm operators, U_1 and U_2 a 'tensor product' operator U which is essentially unitary and has

$$\text{Index}(U) = \text{Index}(U_1) \cdot \text{Index}(U_2).$$

As we shall see in Section 9.1, such an operator can be produced as a suitable 'weighted average' of the operators $U_1 \otimes 1$ and $1 \otimes U_2$ and their adjoints. (Actually the exposition runs more smoothly if one works throughout with the corresponding *graded* operators, and this is what we shall do in the main text.) In the special case of the K-homology of $\mathbb{C}$ the weights can be obtained from the functional calculus, but in the general case we shall need the Kasparov Technical Theorem (Theorem 3.8.1) to construct them.

Although it is difficult to construct, the Kasparov product is an exceedingly powerful tool. We shall illustrate this by giving a very short proof of the homotopy invariance of K-homology. In addition, we shall present a proof of Bott periodicity for K-homology and a new construction of the boundary maps in K-homology. Thus we shall organize nearly all of K-homology theory around the Kasparov product. In the last section we shall show that the product is compatible with the index pairing. As we shall see in Chapter 11, this fact is central to the proof of the Atiyah–Singer Index Theorem.

9.1 The Product of Fredholm Operators

9.1.1 DEFINITION Let A be a C*-algebra and let I be an ideal in A. We shall say that $a \in A$ is *positive modulo* I if the image of a in the quotient C*-algebra A/I is a positive element.

We have already made use of a special case of this notion in Proposition 8.3.16. For a selfadjoint element $a \in A$, let a^+ and a^- be the positive and negative parts of a, defined by

$$a^+ = \tfrac{1}{2}(a + |a|) \quad \text{and} \quad a^- = \tfrac{1}{2}(a - |a|),$$

so that $a^+ \geqslant 0$, $a^- \leqslant 0$, $a^+a^- = a^-a^+ = 0$, and $a = a^+ + a^-$. Then an easy functional calculus argument proves:

9.1.2 LEMMA *The selfadjoint element $a \in A$ is positive modulo I if and only if $a^- \in I$.* □

9.1.3 COROLLARY *Let I and J be two ideals in A. If $a \in A$ is positive modulo I, and also positive modulo J, then it is positive modulo $I \cap J$.* □

9.1.4 DEFINITION Let H be a graded Hilbert space. We shall use the term *graded Fredholm operator* to refer to an odd, selfadjoint Fredholm operator F on H.

Such an operator must have the form

$$F = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix},$$

where U is a Fredholm operator from H^+ to H^-. We shall use the notation Index (F) for the index of the operator U. Equivalently, we may say that Index (F) is equal to the 'graded dimension' of the graded vector space Kernel(F). In other words, Index (F) is the difference

$$\operatorname{Dim}(\operatorname{Kernel}(F)^+) - \operatorname{Dim}(\operatorname{Kernel}(F)^-).$$

Our terminology is consistent with that of Example 8.1.4.

The following result is an immediate consequence of Atkinson's Theorem.

9.1.5 LEMMA *Let H be a graded Hilbert space. An odd, selfadjoint operator F on H is graded Fredholm if and only if there exists some $\varepsilon > 0$ such that $F^2 - \varepsilon 1$ is positive modulo the compact operators.* □

9.1.6 LEMMA *Let F and F' be graded Fredholm operators on a graded Hilbert space H. Suppose that $FF' + F'F$ is positive modulo the compacts. Then* Index $(F) =$ Index (F').

PROOF The criterion provided by Lemma 9.1.5 shows that the map

$$t \mapsto \cos(\tfrac{\pi}{2}t)F + \sin(\tfrac{\pi}{2}t)F'$$

defines a path of graded Fredholm operators connecting F to F'. □

9.1.7 DEFINITION Let F_1 and F_2 be graded Fredholm operators on graded Hilbert spaces H_1 and H_2. A graded Fredholm operator F on the graded tensor product space $H_1 \hat{\otimes} H_2$ is *aligned* with F_1 if

$$F(F_1 \hat{\otimes} 1) + (F_1 \hat{\otimes} 1)F \geqslant 0 \quad \text{modulo compacts.}$$

Similarly F is aligned with F_2 if

$$F(1 \hat{\otimes} F_2) + (1 \hat{\otimes} F_2)F \geqslant 0 \quad \text{modulo compacts.}$$

9.1.8 PROPOSITION *Let F_1 and F_2 be graded Fredholm operators on graded Hilbert spaces H_1 and H_2. There exist graded Fredholm operators on $H_1 \hat{\otimes} H_2$ which are simultaneously aligned with F_1 and F_2. Moreover, any two such operators have the same index.*

PROOF Let $F \in \mathfrak{B}(H_1) \otimes \mathfrak{B}(H_2)$ be defined by $F = F_1 \hat{\otimes} 1 + 1 \hat{\otimes} F_2$. Then

$$F(F_1 \hat{\otimes} 1) + (F_1 \hat{\otimes} 1)F = 2(F_1^2 \hat{\otimes} 1) \geqslant 0$$

and

$$F(1 \hat{\otimes} F_2) + (1 \hat{\otimes} F_2)F = 2(1 \hat{\otimes} F_2^2) \geqslant 0,$$

so that F is aligned with F_1 and F_2. To show that F is Fredholm let $\varepsilon > 0$ be so small that both $F_1^2 - \varepsilon 1$ and $F_2^2 - \varepsilon 1$ are positive modulo the compacts. Then the equation

$$F^2 - \varepsilon 1 = (F_1^2 - \varepsilon 1) \hat{\otimes} 1 + 1 \hat{\otimes} F_2^2$$

shows that $F^2 - \varepsilon 1$ is positive modulo the ideal $I_1 = \mathfrak{K}(H_1) \otimes \mathfrak{B}(H_2)$ in $\mathfrak{B}(H_1) \otimes \mathfrak{B}(H_2)$, and a similar equation (interchanging the rôles of F_1 and F_2) shows that $F^2 - \varepsilon 1$ is positive modulo $I_2 = \mathfrak{B}(H_1) \otimes \mathfrak{K}(H_2)$. It follows from Corollary 9.1.3 that $F^2 - \varepsilon 1$ is positive modulo the ideal[83]

$$I_1 \cap I_2 = I_1 I_2 = \mathfrak{K}(H_1 \hat{\otimes} H_2),$$

which completes the proof that F is Fredholm. Now let F' be another graded Fredholm operator which is aligned both with F_1 and with F_2. Then, by construction, $FF' + F'F$ is positive modulo the compacts. It follows from Lemma 9.1.6 that $\operatorname{Index}(F) = \operatorname{Index}(F')$. □

9.1.9 PROPOSITION *Let F_1 and F_2 be graded Fredholm operators on graded Hilbert spaces H_1 and H_2. If F is a graded Fredholm operator F on $H_1 \hat{\otimes} H_2$ which is aligned with both F_1 and F_2 then*

$$\operatorname{Index}(F) = \operatorname{Index}(F_1) \cdot \operatorname{Index}(F_2).$$

[83] The first equality in the display is proved in Exercise 6.7.7.

PROOF It suffices to check the result for the operator $F = F_1\hat{\otimes}1 + 1\hat{\otimes}F_2$ that we considered in the last proof. By the properties of graded tensor products,

$$F^2 = F_1^2\hat{\otimes}1 + 1\hat{\otimes}F_2^2,$$

since $F_1\hat{\otimes}1$ and $1\hat{\otimes}F_2$ anticommute. The operators on the right are both positive, so

$$\mathrm{Kernel}(F^2) = \mathrm{Kernel}(F_1^2\hat{\otimes}1) \cap \mathrm{Kernel}(1\hat{\otimes}F_2^2) = \mathrm{Kernel}(F_1)\hat{\otimes}\,\mathrm{Kernel}(F_2).$$

The identity $\mathrm{Index}\,(F) = \mathrm{Index}\,(F_1)\,\mathrm{Index}\,(F_2)$ follows from this. □

The operator $F = F_1\hat{\otimes}1 + 1\hat{\otimes}F_2$ will be our first model for the definition of the Kasparov product. However, the discussion so far has not taken into account the requirement, part of the definition of a Fredholm module, that $F^2 - 1$ should be compact. Even if the operators F_1 and F_2 satisfy this condition, the operator F usually will not. For this and other reasons, the simple tensor product construction of F will not do, as it stands, to define an external product in K-homology.

To see what modifications will be needed, let us suppose for a moment that both F_1^2 and F_2^2 are *exactly* equal to 1 (which is admittedly not very interesting from the point of view of Fredholm index theory). Then

$$F^2 = F_1^2\hat{\otimes}1 + 1\hat{\otimes}F_2^2 = 2.$$

Thus if we alter F by introducing 'weights,' defining now

$$F = \frac{1}{\sqrt{2}}(F_1\hat{\otimes}1) + \frac{1}{\sqrt{2}}(1\hat{\otimes}F_2),$$

then (with this new definition) $F^2 = 1$. This construction may be elaborated, as follows:

9.1.10 LEMMA *Let F_1 and F_2 be graded Fredholm operators on graded Hilbert spaces H_1 and H_2. Let N_1 and N_2 be positive operators on $H_1\hat{\otimes}H_2$ which satisfy the following conditions:*

(a) $N_1^2 + N_2^2 = 1$,

(b) N_1 *and* N_2 *commute modulo compact operators with* $F_1\hat{\otimes}1$ *and* $1\hat{\otimes}F_2$, *and*

(c) $N_1 \cdot (F_1\hat{\otimes}1)^2 \sim N_1$ *and* $N_2 \cdot (1\hat{\otimes}F_2)^2 \sim N_2$.

Then the formula

$$F = N_1^{\frac{1}{2}}(F_1\hat{\otimes}1)N_1^{\frac{1}{2}} + N_2^{\frac{1}{2}}(1\hat{\otimes}F_2)N_2^{\frac{1}{2}}$$

defines an odd Fredholm operator which is aligned with F_1 and F_2. Moreover, $F^2 \sim 1$. □

The proof is a direct calculation. Do operators N_1 and N_2 as in the lemma exist? Yes they do, although even in the simple situation we are considering it is not altogether easy to see that this is so. But if for simplicity both F_1 and F_2 are contractive, and if we define a representation of $C[0,1] \otimes C[0,1]$ on $H_1 \hat{\otimes} H_2$ by $f_1 \otimes f_2 \mapsto f_1(F_1^2) \otimes f_2(F_2^2)$, then for example we can take N_1 to be the projection operator on $H_1 \hat{\otimes} H_2$ associated by the Borel functional calculus to the triangle $x_1 > x_2$ in the square $[0,1]^2$, and N_2 to be the complementary projection associated to the triangle $x_1 \leqslant x_2$. We leave the details of this to the reader since we shall prove the existence of N_1 and N_2 in a quite different way in the next section, where we shall use the formula in Lemma 9.1.10 as a model for the general Kasparov product.

Remark The expression

$$N_1(F_1 \hat{\otimes} 1) + N_2(1 \hat{\otimes} F_2)$$

differs from the operator F defined in Lemma 9.1.10 above by a compact operator, and is usually easier to work with; we used the more complicated formula to ensure that F was exactly selfadjoint.

9.2 The Definition of the Kasparov Product

Let (ρ_1, H_1, F_1) and (ρ_2, H_2, F_2) be graded Fredholm modules over separable C^*-algebras A_1 and A_2. Using the ideas of the last section we are going to construct a 'product' Fredholm module over $A_1 \otimes A_2$.

The following definition presents a special case of a concept that was introduced in Definition 3.2.7.

9.2.1 Definition Let H_1 and H_2 be Hilbert spaces. Represent the C^*-algebra $\mathfrak{K}(H_1) \otimes \mathfrak{B}(H_2)$ as the norm-closed linear span of the elementary tensor operators $K_1 \otimes T_2$ on $H_1 \otimes H_2$, where K_1 is compact and T_2 is bounded. A bounded operator F on $H_1 \otimes H_2$ *derives* $\mathfrak{K}(H_1) \otimes \mathfrak{B}(H_2)$ if the commutator $[T, K_1 \otimes T_2]$ belongs to $\mathfrak{K}(H_1) \otimes \mathfrak{B}(H_2)$, for every $K_1 \otimes T_2 \in \mathfrak{K}(H_1) \otimes \mathfrak{B}(H_2)$.

Remark The set of *all* operators which derive $\mathfrak{K}(H_1) \otimes \mathfrak{B}(H_2)$ is a C^*-subalgebra of $\mathfrak{B}(H_1 \otimes H_2)$ which contains — but is considerably larger than — the C^*-algebra $\mathfrak{B}(H_1) \otimes \mathfrak{B}(H_2)$. But $\mathfrak{B}(H_1 \otimes H_2)$ is larger yet.

9.2.2 Definition Let (ρ_1, H_1, F_1) and (ρ_2, H_2, F_2) be graded Fredholm modules over separable C^*-algebras A_1 and A_2. Let ρ be the tensor product representation of $A_1 \otimes A_2$ on the Hilbert space $H = H_1 \hat{\otimes} H_2$. Thus

$$\rho(a_1 \otimes a_2) = \rho_1(a_1) \otimes \rho_2(a_2).$$

We shall say that a graded Fredholm module (ρ, H, F) is *aligned* with (ρ_1, H_1, F_1) and (ρ_2, F_2, H_2) if

$$\rho(a)\Big(F(F_1\hat{\otimes}1) + (F_1\hat{\otimes}1)F\Big)\rho(a^*) \geqslant 0 \quad \text{modulo compacts}$$

and

$$\rho(a)\Big(F(1\hat{\otimes}F_2) + (1\hat{\otimes}F_2)F\Big)\rho(a^*) \geqslant 0 \quad \text{modulo compacts}$$

for every $a \in A_1 \otimes A_2$, and if in addition the operator $\rho(a)F$ derives $\mathfrak{K}(H_1) \otimes \mathfrak{B}(H_2)$ for every $a \in A_1 \otimes A_2$.

This is an adaptation of Definition 9.1.7 to Fredholm modules. If F_1 is p_1-multigraded and F_2 is p_2-multigraded, then $H_1\hat{\otimes}H_2$ carries a natural $(p_1 + p_2)$-multigrading (described explicitly in Appendix A), and we require that an aligned module be $(p_1 + p_2)$-multigraded.

REMARK In the above definition we could replace the technical requirement that $\rho(a)F$ must derive $\mathfrak{K}(H_1) \otimes \mathfrak{B}(H_2)$ with the requirement that $\rho(a)F$ must derive $\mathfrak{B}(H_1) \otimes \mathfrak{K}(H_2)$, without essentially altering what follows. We could even 'symmetrize' the definition by requiring that $\rho(a)F$ must derive *both* C^*-algebras. However, *omitting* the derivation condition entirely would create awkward difficulties.

For brevity, from here on we shall write 'F is aligned with F_1 and F_2' rather than '(ρ, H, F) is aligned with (ρ_1, H_1, F_1) and (ρ_2, H_2, F_2)'. The main result of this section will be as follows:

9.2.3 PROPOSITION *Let (ρ_1, H_1, F_1) and (ρ_2, H_2, F_2) be graded Fredholm modules over separable C^*-algebras A_1 and A_2. There exist Fredholm modules F which are aligned with F_1 and F_2. Moreover, the operator homotopy class of such an F is determined uniquely by the operator homotopy classes of F_1 and F_2.*

The hardest part of the proof is to show the existence of such an F. We shall use the formula given in Lemma 9.1.10, and for this we need the following definition and technical proposition:

9.2.4 DEFINITION Let H be a Hilbert space. A *partition of unity* for H is a collection $\{N_i\}$ of positive operators on H such that $\sum N_i^2 = 1$.[84]

9.2.5 PROPOSITION *Let $H_1 \otimes H_2$ be a tensor product of separable Hilbert spaces, and let Δ be a separable subset of $\mathfrak{B}(H_1 \otimes H_2)$, which derives the C^*-algebra $\mathfrak{K}(H_1)\otimes\mathfrak{B}(H_2)$. There is a partition of unity $\{N_1, N_2\}$ for $H_1 \otimes H_2$ such that*

(a) $N_1 \cdot \big(\mathfrak{K}(H_1) \otimes \mathfrak{B}(H_2)\big) \subseteq \mathfrak{K}(H_1 \otimes H_2)$,

[84] The sum should converge in the strong topology. However, in this section we shall be interested only in *finite* partitions of unity.

(b) $N_2 \cdot (\mathfrak{B}(H_1) \otimes \mathfrak{K}(H_2)) \subseteq \mathfrak{K}(H_1 \otimes H_2)$, *and*

(c) *both* N_1 *and* N_2 *commute modulo* $\mathfrak{K}(H_1 \otimes H_2)$ *with each operator in the set* Δ.

Moreover, if one or both of H_1 *and* H_2 *are graded (or multigraded), and we take graded tensor products, then we may ensure that the operators* N_1 *and* N_2 *preserve the (multi)grading on* $H_1 \hat{\otimes} H_2$.

9.2.6 REMARKS By a 'separable set' in $\mathfrak{B}(H_1 \otimes H_2)$ we just mean a set which has a countable dense subset in the norm topology. Notice that the operators N_1 and N_2 provided by the proposition derive $\mathfrak{K}(H_1) \otimes \mathfrak{B}(H_2)$.

PROOF Let E_1 be the smallest C^*-subalgebra of $\mathfrak{K}(H_1) \otimes \mathfrak{B}(H_2)$ which contains all elementary tensors $K \otimes 1$ and is derived by elements of Δ. Let E_2 be the C^*-algebra of all elementary tensors $1 \otimes K$, where K is again compact. Then both E_1 and E_2 are separable, the product $E_1 \cdot E_2$ is contained within $\mathfrak{K}(H_1 \otimes H_2)$, and $[\Delta, E_1] \subseteq E_1$. Therefore the conditions of the Kasparov Technical Theorem 3.8.1 are satisfied, and so there is an operator X such that $0 \leqslant X \leqslant 1$, $(1 - X) \cdot E_1 \subseteq \mathfrak{K}(H_1 \otimes H_2)$, $X \cdot E_2 \subseteq \mathfrak{K}(H_1 \otimes H_2)$, and such that X commutes modulo compacts with Δ. Take $N_1 = (1 - X)^{\frac{1}{2}}$, $N_2 = X^{\frac{1}{2}}$.

To deal with the graded case of the result, we simply include the grading operators in the space Δ, obtaining an X which commutes modulo compacts with the grading operators. But since conjugation by the grading operators generates a finite group of automorphisms of $\mathfrak{B}(H)$, we may average X over this group (changing it only by a compact perturbation) to make it commute exactly with the gradings. □

9.2.7 DEFINITION We shall say that a partition of unity $\{N_1, N_2\}$ is *adapted* to the given subset Δ if the conditions (a)–(c) of Proposition 9.2.5 are satisfied.

9.2.8 LEMMA *The collection of all partitions of unity adapted to a given separable set* Δ *is non-empty and path-connected.*

PROOF Non-emptiness is simply Proposition 9.2.5 above. To prove the path-connectedness assertion, note that a partition of unity is determined by the operator $X = N_2^2$ which appears in the proof of Proposition 9.2.5. As noted in Remark 3.8.2, the set of all such X is convex and hence path-connected. □

We underline once again that Δ must be separable for this result to apply. For this reason, throughout the rest of this chapter, all algebras whose K-homology we consider shall be taken to be separable, sometimes without explicit comment.

PROOF OF PROPOSITION 9.2.3 Let Δ be the subset of $\mathfrak{B}(H_1) \otimes \mathfrak{B}(H_2)$ consisting of the operators $\rho(a)$ for $a = a_1 \otimes a_2 \in A_1 \otimes A_2$, $F_1 \hat{\otimes} 1$, and $1 \hat{\otimes} F_2$. Let $\{N_1, N_2\}$ be a partition of unity adapted to Δ. Define

$$F = N_1(F_1\hat{\otimes}1) + N_2(1\hat{\otimes}F_2).$$

Then

$$(F^2-1)\rho(a) \sim \Big(N_1^2 \cdot \big((F_1^2-1)\hat{\otimes}1\big) + N_2^2 \cdot \big(1\hat{\otimes}(F_2^2-1)\big)\Big)\rho(a) \sim 0,$$

since (for instance) $(F_1^2-1)\rho_1(a_1)$ is a compact operator on H_1 and therefore $N_1^2\big((F_1^2-1)\rho_1(a_1)\hat{\otimes}\rho_2(a_2)\big)$ is a compact operator on H. By a similar argument,

$$(F^*-F)\rho(a) \sim \Big(N_1 \cdot \big((F_1^*-F_1)\hat{\otimes}1\big) + N_2 \cdot \big(1\hat{\otimes}(F_2^*-F_2)\big)\Big)\rho(a) \sim 0.$$

Finally,

$$[F,\rho(a)] \sim N_1 \cdot \Big([F_1,\rho_1(a_1)]\hat{\otimes}\rho_2(a_2)\Big) + N_2 \cdot \Big(\rho_1(a_1)\hat{\otimes}[F_2,\rho_2(a_2)]\Big) \sim 0.$$

This shows that F defines a Fredholm module. Moreover

$$\rho(a)\Big(F(F_1\hat{\otimes}1) + (F_1\hat{\otimes}1)F\Big)\rho(a^*) \sim 2\rho(a) \cdot N_1 \cdot \rho(a)^* \geqslant 0.$$

This and a similar calculation involving F_2, together with the fact that N_1 and N_2 derive $\mathfrak{K}(H_1) \otimes \mathfrak{B}(H_2)$, show that F is aligned with F_1 and F_2.

Suppose now that F' is aligned with F_1 and F_2. There exists a partition of unity $\{N_1, N_2\}$ adapted to $\Delta \cup \{\rho(a)F' : a \in A_1 \otimes A_2\}$, where Δ is as above. Use it to form

$$F = N_1(F_1\hat{\otimes}1) + N_2(1\hat{\otimes}F_2).$$

Since N_1 and N_2 commute modulo compacts with $\rho(a)F'$, and since F' is aligned with F_1 and F_2, we easily obtain

$$\rho(a)(FF' + F'F)\rho(a^*) \geqslant 0 \quad \text{modulo compacts.}$$

Therefore by Proposition 8.3.16, F' is operator homotopic to F. Thus every aligned module is operator homotopic to one of the sort we constructed in the first part of the proof; and by Lemma 9.2.8 all of those are operator homotopic to one another.

Finally, suppose that one of the operators, say F_1, is varied by an operator homotopy $F_1(t)$. There exists a partition of unity $\{N_1, N_2\}$ adapted to Δ, where now Δ is the (separable) space spanned by the operators $\rho(a)$, $1\hat{\otimes}F_2$, and $F_1(t)\hat{\otimes}1$ (for all $a \in A_1 \otimes A_2$ and all t). Defining

$$F(t) = N_1\big(F_1(t)\hat{\otimes}1\big) + N_2\big(1\hat{\otimes}F_2\big),$$

we now obtain an operator homotopy of Fredholm modules aligned with $F_1(t)$ and F_2. □

9.2.9 DEFINITION A *Kasparov product* of F_1 and F_2 is a Fredholm module which is aligned with F_1 and F_2. We shall use the notation $F_1 \times F_2$ for a Kasparov product of F_1 and F_2.

The above proposition shows that any two graded Fredholm modules admit a Kasparov product, uniquely determined up to operator homotopy, and that the product of Fredholm modules gives rise to a well-defined product on the level of K-homology groups,

$$K^{-p_1}(A_1) \otimes K^{-p_2}(A_2) \to K^{-p_1-p_2}(A_1 \otimes A_2), \tag{9.2.10}$$

for $p_1 \geqslant 0$ and $p_2 \geqslant 0$.[85]

9.2.11 REMARK An examination of the proof shows that Proposition 9.2.3 continues to hold if the two positivity conditions in Definition 9.2.2 are replaced by the weaker requirements that for all $a \in A_1 \otimes A_2$, the operator $\rho(a)(F(F_1 \hat{\otimes} 1) + (F_1 \hat{\otimes} 1)F)\rho(a^*)$ is the sum of a positive operator and a member of $\mathfrak{K}(H_1) \otimes \mathfrak{B}(H_2)$, and that the operator $\rho(a)(F(1 \hat{\otimes} F_2) + (1 \hat{\otimes} F_2)F)\rho(a^*)$ is the sum of a positive operator and a member of $\mathfrak{B}(H_1) \otimes \mathfrak{K}(H_2)$. The derivation condition remains unchanged. In other words, all Fredholm modules F which are 'weakly' aligned with F_1 and F_2 are operator homotopic, and so represent the Kasparov product in K-homology. We will use this observation at one point in Chapter 10.

9.2.12 PROPOSITION *The Kasparov product on* K*-homology is associative.*

PROOF Let (F_j, ρ_j, H_j) be Fredholm modules over separable C^*-algebras A_j, for $j = 1, 2, 3$. Form the iterated Kasparov product $(F_1 \times F_2) \times F_3$ using the construction of the product via partitions of unity, but with the extra requirements that the operators in the second partition of unity commute modulo compact operators with the tensor product operators obtained from the first, and that the second partition operators commute modulo compacts with all three of the operators

$$F_1 \hat{\otimes} 1 \hat{\otimes} 1, \quad 1 \hat{\otimes} F_2 \hat{\otimes} 1, \quad \text{and} \quad 1 \hat{\otimes} 1 \hat{\otimes} F_3.$$

We obtain a formula

$$(F_1 \times F_2) \times F_3 = N_1(F_1 \hat{\otimes} 1 \hat{\otimes} 1) + N_2(1 \hat{\otimes} F_2 \hat{\otimes} 1) + N_3(1 \hat{\otimes} 1 \hat{\otimes} F_3),$$

where

(a) the operators N_1, N_2 and N_3 are positive modulo compact operators,

[85] Some adaptations of the product construction to K^1 are given in the exercises. One can, of course, also appeal to the formal periodicity of K-homology to extend the product to all p_1 and p_2.

(b) $N_1^2 + N_2^2 + N_3^2 = 1$,

(c) $N_1 \cdot (\mathfrak{K}(H_1)\hat{\otimes}\mathfrak{B}(H_2)\hat{\otimes}\mathfrak{B}(H_3)) \subseteq \mathfrak{K}(H_1\hat{\otimes}H_2\hat{\otimes}H_3)$, together with the corresponding conditions for N_2 and N_3, and

(d) N_1, N_2 and N_3 commute modulo compacts with the operators

$$F_1\hat{\otimes}1\hat{\otimes}1, \quad 1\hat{\otimes}F_2\hat{\otimes}1, \quad \text{and} \quad 1\hat{\otimes}1\hat{\otimes}F_3,$$

and with the representation of $A_1 \otimes A_2 \otimes A_3$.

The proof is completed by noting that the other iterated product $F_1 \times (F_2 \times F_3)$ can be represented in the same way, and that any two such weighted averages of $F_1\hat{\otimes}1\hat{\otimes}1$, $1\hat{\otimes}F_2\hat{\otimes}1$ and $1\hat{\otimes}1\hat{\otimes}F_3$ are operator homotopic to one another. □

9.2.13 PROPOSITION *Let $x_1 \in K^{-p_1}(A_1)$ and let $x_2 \in K^{-p_2}(A_2)$. Denote by $\tau\colon A_1 \otimes A_2 \to A_2 \otimes A_1$ the flip map, defined by $\tau(a_1 \otimes a_2) = a_2 \otimes a_1$. Then*

$$x_1 \times x_2 = (-1)^{p_1 p_2}\tau^*(x_2 \times x_1) \in K^{-p_1-p_2}(A_1 \otimes A_2). \qquad \square$$

The proof is straightforward; the sign comes from permuting the multigrading operators (compare Exercise 8.8.7). The next result also has an elementary proof.

9.2.14 PROPOSITION *Let $\alpha\colon B \to A$ be $*$-homomorphism between separable C^*-algebras. Let C be another separable C^*-algebra and let $x \in K^{-q}(C)$. Then the diagram*

$$\begin{tikzcd} K^{-p}(A) \arrow[r, "\alpha^*"] \arrow[d, "\times x"'] & K^{-p}(B) \arrow[d, "\times x"] \\ K^{-p-q}(A \otimes C) \arrow[r, "(\alpha\otimes 1)^*"'] & K^{-p-q}(B \otimes C) \end{tikzcd}$$

commutes, where the rows are induced from the $*$-homomorphisms α and $\alpha \otimes 1$, and the columns are obtained by Kasparov product with x. □

9.3 Index One Operators and Homotopy Invariance

As we saw in Example 8.2.9, the group $K^0(\mathbb{C})$ is isomorphic to $\mathbb{Z}$, and it is generated by the class $\mathbf{1}$ of any graded Fredholm operator of index one, acting on a graded Hilbert space equipped with the non-degenerate representation of $\mathbb{C}$.

9.3.1 PROPOSITION *The generator $\mathbf{1}$ of $K^0(\mathbb{C})$ acts as an identity for the Kasparov product. In other words if A is any separable C^*-algebra and if $x \in K^{-p}(A)$ is any K-homology class then*

$$x \times \mathbf{1} = x = \mathbf{1} \times x$$

in $K^{-p}(A)$.

REMARK We are of course making the identifications $\mathbb{C} \otimes A \cong A \cong A \otimes \mathbb{C}$.

PROOF Let (ρ, H, F) be a Fredholm module over A and represent the class $\mathbf{1} \in K^0(\mathbb{C})$ by the Fredholm module (ρ_0, H_0, F_0), where ρ_0 is the non-degenerate representation of $\mathbb{C}$ on the one-dimensional graded Hilbert space H_0 whose even part is $\mathbb{C}$ and odd part is zero, and where F_0 is the zero operator. The operator $F\hat{\otimes}1$ on $H\hat{\otimes}H_0$ gives a Fredholm module over $A \otimes \mathbb{C}$ which is aligned with F and F_0. Therefore it represents the product. After we identify $A \otimes \mathbb{C}$ with A, and after we identify $H\hat{\otimes}H_0$ with H, $\rho \otimes \rho_0$ with ρ, and $F\hat{\otimes}1$ with F, we get

$$[\rho, H, F] \times [\rho_0, H_0, F_0] = [\rho, H, F],$$

as required. $\square$

9.3.2 REMARK It is worth noting here that if we had taken an infinite-dimensional model for the generator of $K^0(\mathbb{C})$ then the proof of the proposition would have been far less straightforward.

Recall now that two $*$-homomorphisms $\alpha_0, \alpha_1 \colon A \to B$ are *homotopic* if there is a $*$-homomorphism $\alpha\colon A \to C[0,1] \otimes B$ from which α_0 and α_1 may be obtained by evaluation at 0 and 1. In Section 6.6 we proved that homotopic $*$-homomorphisms induce the same map on K-homology, provided that the algebras A and B are *commutative*. We shall now use the Kasparov product to prove homotopy invariance without the commutativity assumption.

9.3.3 THEOREM *Let $\alpha_0, \alpha_1 : A \to B$ be homotopic $*$-homomorphisms between separable C^*-algebras. Then α_0 and α_1 induce the same map on* K*-homology:*

$$\alpha_0^* = \alpha_1^* \colon K^{-p}(B) \longrightarrow K^{-p}(A).$$

PROOF Denote by $\varepsilon_0, \varepsilon_1 : C[0,1] \to \mathbb{C}$ the evaluation maps at 0 and 1. It suffices to prove that

$$(\varepsilon_0 \otimes 1)^* = (\varepsilon_1 \otimes 1)^* \colon K^{-p}(\mathbb{C} \otimes B) \longrightarrow K^{-p}(C[0,1] \otimes B).$$

By Proposition 9.3.1, Kasparov product with $\mathbf{1} \in K^0(\mathbb{C})$ gives an isomorphism from $K^{-p}(B)$ to $K^{-p}(\mathbb{C} \otimes B)$. So it suffices to check that if $x \in K^{-p}(B)$ then

$$(\varepsilon_0 \otimes 1)^*(\mathbf{1} \times x) = (\varepsilon_1 \otimes 1)^*(\mathbf{1} \times x).$$

By the naturality of the product (Proposition 9.2.14),

$$(\varepsilon_0 \otimes 1)^*(\mathbf{1} \times x) = \varepsilon_0^*(\mathbf{1}) \times x \quad \text{and} \quad (\varepsilon_1 \otimes 1)^*(\mathbf{1} \times x) = \varepsilon_1^*(\mathbf{1}) \times x,$$

so it suffices to prove that $\varepsilon_0^*(\mathbf{1}) = \varepsilon_1^*(\mathbf{1})$. But by homotopy invariance in the commutative case (Section 6.6),

$$\varepsilon_0^* = \varepsilon_1^* \colon K^0(\mathbb{C})\colon \longrightarrow K^0(C[0,1]),$$

and so the proof is complete. $\square$

In the above proof it was not necessary to appeal to homotopy invariance in the commutative case. One can manufacture an explicit operator homotopy which shows that $\varepsilon_0^*(\mathbf{1}) = \varepsilon_1^*(\mathbf{1})$, thus obtaining a proof of homotopy invariance which is independent of the constructions of Chapter 6. This is Kasparov's argument, and here are the details.

9.3.4 Definition Let H be the Hilbert space $L^2[-1,1]$, and let $\{e_n\}_{n\in\mathbb{Z}}$ be the orthonormal basis for H comprising the functions

$$e_n(x) = 2^{-\frac{1}{2}} e^{\pi i n x}.$$

Let Y be the bounded operator on H defined by

$$Ye_n = \begin{cases} +e_n & \text{if } n \geqslant 0 \\ -e_n & \text{if } n < 0. \end{cases}$$

Let $g\colon [-1,1] \to [-1,1]$ be a continuous function such that $g(-1) = -1$ and $g(+1) = +1$, and let X_g be the bounded operator on H of pointwise multiplication by g.

9.3.5 Lemma *The operators X_g and Y comprise a Schrödinger pair.*

Proof Note first that $Y^2 - 1 = 0$, and so $Y^\flat = 0$. Hence $X_g^\flat \cdot Y^\flat = 0$. Next, every element of $X_g^\flat$ is multiplication by a continuous function on $[-1,1]$ whose values at ± 1 are equal, and the C^*-algebra of all such functions is generated by the exponential function $e(x) = e^{\pi i x}$. The operator of pointwise multiplication by $e(x)$ acts as a shift operator on H with respect to the basis $\{e_n\}$, and so it commutes modulo compacts with Y (in fact the commutator is a rank-one operator). Therefore every element in the C^*-algebra generated by e gives rise to a multiplication operator which commutes modulo compacts with Y. In particular, $X_g^\flat$ commutes modulo compact operators with Y, as required in the definition of a Schrödinger pair. This completes the proof. □

As we noted in Proposition 8.6.8, the corresponding Schrödinger operator

$$V(X_g, Y) = iX_g + (1 - X_g^2)^{\frac{1}{2}} Y$$

is essentially unitary. Let us compute its index.

9.3.6 Lemma *The Fredholm operator $V(X_g, Y)$ has index one.*

Proof The index of $V(X_g, Y)$ is independent of the choice of g (subject to the condition $g(\pm 1) = \pm 1$ in Definition 9.3.4) because a homotopy between any two choices of g will produce a homotopy of Schrödinger operators $V(X_g, Y)$. So let us take $g(x) = x$. Since $Y^2 = 1$ it follows from Lemma 8.6.12 that the index of $V(X_g, Y)$ is equal to the index of

$$W_1(X_g, Y) = e^{-i\pi X_g} P_Y - (1 - P_Y) \sim P_Y e^{-i\pi X_g} P_Y - (1 - P_Y),$$

where P_Y is the operator $\frac{1}{2}(1 + Y)$. If we denote by Z the rightmost operator in the above display then

$$Ze_n = \begin{cases} e_{n-1} & n > 0 \\ 0 & n = 0 \\ -e_n & n < 0, \end{cases}$$

from which it is clear that Index $(Z) = 1$, as required. □

REMARK The operator $P_Y e^{-i\pi X_g} P_Y$ which appears in the above proof is a Toeplitz operator of the sort considered in Section 2.3. We could have appealed to the Toeplitz Index Theorem 2.3.2 to compute its index.

It will be convenient to work with the following compact perturbation of the operator $V(X_g, Y)$:

9.3.7 DEFINITION With the above notation, the operator $V_g \in \mathfrak{B}(H)$ is defined by

$$V_g = iX_g + (1 - X_g^2)^{\frac{1}{4}} Y (1 - X_g^2)^{\frac{1}{4}}.$$

If we fix a continuous map Φ from $[-1, 1]$ onto $[0, 1]$, then by composing functions in $C[0, 1]$ with Φ we obtain a representation of $C[0, 1]$ on the Hilbert space $H = L^2[-1, 1]$. The operators V_g commute modulo compact operators with this representation and therefore determine Fredholm modules over $C[0, 1]$, and hence classes $[V_g] \in K^0(C[0, 1])$. In fact, having fixed Φ, the classes we obtain from different choices of g are all the same, since a homotopy between two different functions g immediately gives rise to an operator homotopy between the corresponding Fredholm modules. We are going to select Φ carefully and then show that

$$[V_{g_0}] = \varepsilon_0^*(\mathbf{1}) \quad \text{and} \quad [V_{g_1}] = \varepsilon_1^*(\mathbf{1}),$$

for suitable g_0 and g_1. Since $[V_{g_0}] = [V_{g_1}]$, this will show that $\varepsilon_0^*(\mathbf{1}) = \varepsilon_1^*(\mathbf{1})$, as required.

To proceed, fix a function $\Phi\colon [-1, 1] \to [0, 1]$ which assumes the constant value 0 on $[-1, -\frac{1}{2}]$ and the constant value 1 on $[\frac{1}{2}, 1]$.

Let g_0 be a continuous function on $[-1, 1]$ such that $g(-1) = -1$ and $g(x) = +1$ for all $x \geqslant -\frac{1}{2}$. Then V_{g_0} acts as a direct sum of operators on

$$L^2[-1, 1] = L^2[-1, -\tfrac{1}{2}] \oplus L^2[-\tfrac{1}{2}, 1]$$

(it is to obtain this simple direct sum decomposition that we work with V_g rather than $V(X_g, Y)$). On the first summand, V_g acts as an operator of index one, whereas on the second summand it acts as i times the identity operator. So as a Fredholm module, V_{g_0} is a direct sum of an index one module on $L^2[-1, -\frac{1}{2}]$, where the action of $C[0, 1]$ is via the evaluation ε_0, and a degenerate module on $L^2[-\frac{1}{2}, 1]$. Passing to K-homology classes we obtain the formula

$$[V_{g_0}] = \varepsilon_0^*(\mathbf{1}) + 0 = \varepsilon_0^*(\mathbf{1}).$$

Now choose g_1 so that $g(x) = -1$ if $x \leqslant \frac{1}{2}$ and $g(1) = 1$. Then a similar analysis shows that

$$[V_{g_1}] = 0 + \varepsilon_1^*(\mathbf{1}) = \varepsilon_1^*(\mathbf{1}).$$

Kasparov's proof of homotopy invariance is now complete.

9.4 Stability

In Example 4.2.5 we showed that K-*theory* is *stable* in the analytical sense that

$$K_p(A) \cong K_p(A \otimes \mathfrak{K}(H)).$$

To prove this we expressed the algebra $\mathfrak{K}(H)$ as a direct limit of matrix algebras, and then used the continuity of K-theory with respect to direct limits.

It is also true that K-*homology* is stable. But we cannot easily make an analogous argument, since the direct system of matrix algebras will give rise to an *inverse* system of K-homology groups, and our discussion of Steenrod homology in Chapter 7 showed that the behavior of K-homology with respect to *inverse* limits will in general require a much more detailed analysis. Instead, we shall use the Kasparov product to prove the stability of K-homology.

Fix a rank-one projection $e \in \mathfrak{K}(H)$ and let $\alpha\colon \mathbb{C} \to \mathfrak{K}(H)$ be the $*$-homomorphism $\alpha(\lambda) = \lambda e$. We are going to prove that α^* is an isomorphism. In fact, we shall prove a bit more than this, namely:

9.4.1 Theorem *Let* A *be any separable* C^**-algebra. Then the map*

$$(1 \otimes \alpha)^*\colon K^{-p}(A \otimes \mathfrak{K}(H)) \longrightarrow K^{-p}(A \otimes \mathbb{C})$$

is an isomorphism.

Proof Let $k \in K^0(\mathfrak{K}(H))$ be the K-homology class of the Fredholm module consisting of the zero operator on the graded Hilbert space whose even part is H and whose odd part is zero (the representation of $\mathfrak{K}(H)$ on the even space H is the standard one, and of course we must use the zero representation on the odd space 0). Define a homomorphism

$$\beta\colon K^{-p}(A) \longrightarrow K^{-p}(A \otimes \mathfrak{K}(H))$$

by taking the Kasparov product with $k \in K^0(\mathfrak{K}(H))$. The map $\alpha^*\colon K^0(\mathfrak{K}(H)) \to K^0(\mathbb{C})$ takes the class $k \in K^0(\mathfrak{K}(H))$ to the generator $\mathbf{1} \in K^0(\mathbb{C})$. So it follows from the naturality of the Kasparov product with respect to $*$-homomorphisms that

$$(1 \otimes \alpha)^*(\beta(x)) = (1 \otimes \alpha)^*(x \times k) = x \times \alpha^*(k) = x \times \mathbf{1},$$

for every $x \in K^{-p}(A)$. Thus $(1 \otimes \alpha)^*$ is left inverse to β (after $A \otimes \mathbb{C}$ is identified with A) and so α^* is surjective. We need to show that it is also injective. For this purpose let $y \in K^{-p}(A \otimes \mathfrak{K}(H))$ and consider the element

$$k \times (1 \otimes \alpha)^*(y) \in K^{-p}(\mathfrak{K}(H) \otimes A \otimes \mathbb{C}).$$

We are going to show that the transposition isomorphism

$$\tau\colon \mathfrak{K}(H) \otimes A \otimes \mathbb{C} \longrightarrow \mathbb{C} \otimes A \otimes \mathfrak{K}(H),$$

which switches the first and third tensor factors, interchanges the displayed element $k \times (1 \otimes \alpha)^*(y)$ and the element $\mathbf{1} \times y \in K^{-p}(\mathbb{C} \otimes A \otimes \mathfrak{K}(H))$. This will complete the proof. Let

$$\sigma\colon \mathfrak{K}(H) \otimes A \otimes \mathfrak{K}(H) \longrightarrow \mathfrak{K}(H) \otimes A \otimes \mathfrak{K}(H)$$

be the automorphism which switches the first and third tensor factors. Then $\sigma \circ (1 \otimes 1 \otimes \alpha) = (\alpha \otimes 1 \otimes 1) \circ \tau$ and so

$$(1 \otimes 1 \otimes \alpha)^*\sigma^*(k \times y) = \tau^*(\alpha \otimes 1 \otimes 1)^*(k \times y) = \tau^*(\alpha^*(k) \times y) = \tau^*(\mathbf{1} \times y).$$

But the automorphism σ is homotopic to the identity since it is induced from the flip automorphism of the Hilbert space $H \otimes H$ which, like all unitary automorphisms of Hilbert space, is path connected to the identity. Hence

$$(1 \otimes 1 \otimes \alpha)^*\sigma^*(k \times y) = (1 \otimes 1 \otimes \alpha)^*(k \times y) = k \times (1 \otimes \alpha)^*(y),$$

and so $k \times (1 \otimes \alpha)^*(y) = \tau^*(\mathbf{1} \times y)$, as required. □

9.4.2 Remark The 'rotation trick' used here to obtain a right inverse from a left inverse is similar to the one that appears in the proof of the Bott Periodicity Theorem 4.9.1.

9.5 Bott Periodicity

In this section we shall continue our development of K-homology via the Kasparov product by proving a version of the Bott Periodicity Theorem in K-homology. We begin by identifying a particular class which will implement the periodicity isomorphism.

9.5.1 DEFINITION Let H be the Hilbert space $L^2[-1,1]$ and let Y be the selfadjoint operator on H which was introduced in Definition 9.3.4. Thus

$$Ye_n = \begin{cases} +e_n & \text{if } n \geqslant 0 \\ -e_n & \text{if } n < 0. \end{cases}$$

Form the graded Hilbert space $H \oplus H$ and equip it with the 1-multigrading operator $\varepsilon_1 = \left(\begin{smallmatrix} 0 & -1 \\ +1 & 0 \end{smallmatrix}\right)$, and with the standard representation ρ of $C_0(-1,1)$. The *Dirac class* $d \in K^{-1}(C_0(-1,1))$ is the K-homology class of the 1-multigraded Fredholm module determined by the operator $\left(\begin{smallmatrix} 0 & -iY \\ iY & 0 \end{smallmatrix}\right)$. Thus

$$d = [\rho, H \oplus H, \begin{pmatrix} 0 & -iY \\ iY & 0 \end{pmatrix}] \in K^{-1}(C_0(-1,1)).$$

REMARK The reason for the name 'Dirac class' is that, as we shall see in the next two chapters, d is determined by the Dirac operator on $(-1,1)$.

We are going to prove the following result:

9.5.2 THEOREM *The operation of Kasparov product with the Dirac class* $d \in K^{-1}(C_0(-1,1))$ *gives an isomorphism*

$$K^{-p}(A) \xrightarrow[\cong]{d\times} K^{-p-1}(C_0(-1,1) \otimes A).$$

9.5.3 REMARK Using the suspension notation $S(A) \cong C_0(-1,1) \otimes A$ from Chapter 4, and iterating the isomorphism provided by the theorem, we obtain an isomorphism

$$K^{-p}(A) \cong K^{-p-2}(S^2(A)).$$

We noted in the previous chapter that there is a linear-algebraic 'formal' periodicity isomorphism $K^{-p-2} \cong K^{-p}$ (see Proposition 8.2.13),[86] and so we conclude from Theorem 9.5.2 that

$$K^{-p}(A) \cong K^{-p}(S^2(A)),$$

for all $p \geqslant 0$.

We shall prove Theorem 9.5.2 by constructing an inverse to the periodicity map, called the *suspension map*

$$s\colon K^{-p-1}(C_0(-1,1) \otimes A) \longrightarrow K^{-p}(A).$$

To do so it will be convenient to work throughout with Fredholm modules (ρ, H, F) which are *non-degenerate* in the sense of Definition 8.3.6, namely that

[86]The formal periodicity isomorphism can also be obtained from a Kasparov product, this time with a generator of $K^{-2}(\mathbb{C}) \cong \mathbb{Z}$. We leave it to the reader to verify this.

$$\text{(9.5.4)} \qquad \operatorname{Image}(\rho) \cdot H \text{ is dense in } H.$$

We showed in Lemma 8.3.8 that K-homology may be normalized by this non-degeneracy requirement, so that there is no loss of generality in our assumption that Condition 9.5.4 is fulfilled. The reason for requiring it is that a non-degenerate representation of a C^*-algebra J extends in a unique way to a representation of any C^*-algebra $\widehat{J}$ which contains J as an ideal (see the discussion following Definition 2.6.2). If we apply this observation to a Fredholm module (ρ, H, F) over $J = C_0(-1, 1) \otimes A$ then we obtain from ρ a canonical representation of the C^*-algebra of bounded continuous functions on $(-1, 1)$, which allows us to introduce the following operator on H:

9.5.5 DEFINITION Let (ρ, H, F) be a non-degenerate $(p+1)$-multigraded Fredholm module over $C_0(-1, 1) \otimes A$. Denote by $X_0 \colon H \to H$ the image under the representation ρ of the function $x \mapsto x$ on $(-1, 1)$, and denote by $X \colon H \to H$ the odd selfadjoint operator

$$X = \gamma \varepsilon_1 X_0,$$

where γ is the grading operator on H and ε_1 is the first multigrading operator.

REMARK Observe that the operator X *commutes* with the remaining multigrading operators $\varepsilon_2, \ldots, \varepsilon_{p+1}$. Notice also that $X^2 = X_0^2$.

The non-degeneracy assumption also provides H with a representation of A. Using the operator X and this representation we can now define the suspension map.

9.5.6 DEFINITION Let (ρ, H, F) be a non-degenerate $(p+1)$-multigraded Fredholm module over $C_0(-1, 1) \otimes A$. Let X be the operator provided by Definition 9.5.5. The *suspension* of the Fredholm module (ρ, H, F) is the p-multigraded Fredholm module (ρ, H, V) over A, where V is the Schrödinger operator

$$V = X + (1 - X^2)^{\frac{1}{2}} F$$

and where the p-multigrading structure on H is obtained from the operators $\varepsilon_2, \ldots, \varepsilon_{p+1}$ by shifting indices downwards.

REMARK We have taken a slight liberty by referring to V as a Schrödinger operator, since if A is not unital then (X, F) is not precisely a graded Schrödinger pair. Rather, the products in $X^b \cdot F^b$ and the graded commutators of F with elements in X^b are elements not of the compact operators but of the C^*-algebra of 'A-locally compact operators' — those operators T such that $\rho(a) \cdot T$ and $T \cdot \rho(a)$ are compact for every $a \in A$.

By applying the suspension construction to operator homotopies we see right away that suspension determines a group homomorphism

$$s\colon K^{-p-1}(C_0(-1,1)\otimes A) \longrightarrow K^{-p}(A).$$

9.5.7 LEMMA *The suspension map takes the Dirac class* $d \in K^{-1}(C_0(-1,1))$ *to the unit class* $\mathbf{1} \in K^0(\mathbb{C})$.

PROOF The Schrödinger operator is in this case

$$V = \begin{pmatrix} 0 & -X_0 - i(1-X_0^2)^{\frac{1}{2}}Y \\ -X_0 + i(1-X_0^2)^{\frac{1}{2}}Y & 0 \end{pmatrix},$$

where the operator X_0 in the display is pointwise multiplication by the function $g(x) = x$ on the Hilbert space $H = L^2[-1,1]$. Since

$$-X_0 + i(1-X_0^2)^{\frac{1}{2}}Y = i \cdot \left(iX_0 + (1-X_0^2)^{\frac{1}{2}}Y\right)$$

it follows from Lemma 9.3.6 that V has index one. Therefore V determines the unit class in $K^0(\mathbb{C})$, as required. □

9.5.8 LEMMA *If* $y \in K^{-q}(B)$ *is any* K*-homology class then the diagram*

$$\begin{tikzcd}
K^{-p-1}(C_0(-1,1)\otimes A) \arrow[r, "s"] \arrow[d, "\times y"'] & K^{-p}(A) \arrow[d, "\times y"] \\
K^{-p-q-1}(C_0(-1,1)\otimes A\otimes B) \arrow[r, "s"'] & K^{-p-q}(A\otimes B)
\end{tikzcd}$$

in which the horizontal arrows are suspension and the vertical arrows are Kasparov product with y*, is commutative.*

PROOF Let $x \in K^{-p-1}(C_0(-1,1)\otimes A)$, and represent x by a *relative* Fredholm module (ρ_1, H_1, F_1) for the pair $(C[-1,1]\otimes A, C\{-1,1\}\otimes A)$. Thus F_1 commutes modulo compact operators with the operator X_0 of Definition 9.5.5, and anti-commutes with the operator X modulo compacts.[87] Represent $y \in K^{-q}(B)$ by a Fredholm module (ρ_2, H_2, F_2), and form the Kasparov product

$$F = N_1(F_1 \hat{\otimes} 1) + N_2(1 \hat{\otimes} F_2),$$

as in the proof of Proposition 9.2.3 but with the additional requirement on N_1 and N_2 that they commute modulo compact operators with X. Now form the Schrödinger operator

[87] Notice that we can require that (ρ_1, H_1, F_1) be a relative Fredholm module and at the same time that $\rho_1(C_0(-1,1)\otimes A)\cdot H_1$ be dense in H_1. To see this, first represent y by a relative module and then apply the normalization procedure of Lemma 8.3.8, which takes relative modules to relative modules.

$$V = X + (1 - X^2)^{\frac{1}{2}} F$$

on $H_1 \hat{\otimes} H_2$. It follows from the Kasparov Technical Theorem 3.8.1 that there exist even positive operators M_1 and M_2 on $H_1 \hat{\otimes} H_2$ such that

(a) $M_1^2 + M_2^2 = 1$,
(b) M_1 and M_2 commute modulo compact operators with $F_1 \hat{\otimes} 1$, $1 \hat{\otimes} F_2$, X, N_1, N_2, and with the representation of $A \otimes B$ on $H_1 \hat{\otimes} H_2$, and
(c) $M_1 \cdot \rho[C_0(-1,1) \otimes A \otimes B] \sim 0$ and $M_2 \cdot (1 - F^2) \sim 0$.

The operators V and

$$\widetilde{V} = M_1 X + M_2 F$$

then represent the same class in $K^{-p-q}(A \otimes B)$ because

$$\rho(a \otimes b)\big(V\widetilde{V} + \widetilde{V}V\big)\rho(a \otimes b)^* \geqslant 0 \quad \text{modulo compacts.}$$

But now if we form the Schrödinger operator

$$V_1 = X + (1 - X^2)^{\frac{1}{2}} F_1$$

on the Hilbert space H_1, and if we form the Kasparov product $V_1 \times F_2$ using the averaging procedure of Proposition 9.2.3 and weight-operators which commute modulo compacts with X, and with the weights M_1, M_2, N_1 and N_2 already defined, then

$$\rho(a \otimes b)\Big((V_1 \times F_2) \cdot \widetilde{V} + \widetilde{V} \cdot (V_1 \times F_2)\Big)\rho(a \otimes b)^* \geqslant 0 \quad \text{modulo compacts.}$$

This shows that $V_1 \times F_2$ and $\widetilde{V}$ (and hence V too) represent the same class in $K^{-p-q}(A \otimes B)$, and this completes the proof. □

Proof of Theorem 9.5.2 We shall prove that the suspension map is inverse to the operation of Kasparov product with the Dirac class. If $y \in K^{-p}(A)$ then by the preceding two lemmas,

$$s(d \times y) = s(d) \times y = \mathbf{1} \times y = y \in K^{-p}(A),$$

and so suspension is left-inverse to product with d. To prove that it is also right-inverse, we introduce the transposition map

$$\tau\colon A \otimes C_0(-1,1) \longrightarrow C_0(-1,1) \otimes A.$$

Let $x \in K^{-p-1}(SA)$ and use Proposition 9.2.13 to write

$$\begin{aligned} \tau^*\big(d \times s(x)\big) &= (-1)^p s(x) \times d \\ &= (-1)^p s(x \times d). \end{aligned} \tag{9.5.9}$$

Now let

$$\sigma\colon C_0(-1,1)\otimes A\otimes C_0(-1,1) \longrightarrow C_0(-1,1)\otimes A\otimes C_0(-1,1)$$

be the automorphism which switches the first and third tensor product factors. The composition

$$(1\otimes\tau)\circ\sigma\colon C_0(-1,1)\otimes A\otimes C_0(-1,1) \longrightarrow C_0(-1,1)\otimes C_0(-1,1)\otimes A$$

is the map which permutes the rightmost tensor factor to the left. Therefore by Proposition 9.2.13 again,

$$\sigma^*(1\otimes\tau)^*(d\times x) = (-1)^{p+1}x\times d.$$

But σ is homotopic to the automorphism of $C_0(-1,1)\otimes A\otimes C_0(-1,1)$ which simply reverses the orientation of one of the intervals $(-1,1)$, and therefore $\sigma^* = -1$ on K-homology. Hence

$$\sigma^*(1\otimes\tau)^*(d\times x) = -(1\otimes\tau)^*(d\times x) = -d\times\tau^*(x).$$

Therefore $d\times\tau^*(x) = (-1)^p x\times d$, and upon applying the suspension map we get

$$\tau^*(x) = s(d\times\tau^*(x)) = (-1)^p s(x\times d).$$

Combining this with Equation 9.5.9 we conclude that

$$\tau^*(x) = \tau^*(d\times s(x)),$$

and hence $x = d\times s(x)$, as required. □

9.5.10 Remark As with our proof of stability in the previous section, the 'rotation trick' used here to prove Bott periodicity in K-homology is essentially the same as the one used in Chapter 4 to prove Bott periodicity in K-theory.

9.5.11 Remark Since the suspension map s is inverse to the operation of Kasparov product with the Dirac class, it follows from Proposition 9.2.14 that s is *natural*, in the following sense: for any $*$-homomorphism $\alpha\colon A\to B$ the diagram

$$\begin{array}{ccc}
K^{-p-1}(C_0(-1,1)\otimes B) & \xrightarrow{s} & K^{-p}(B) \\
\Big\downarrow{\scriptstyle (1\otimes\alpha)^*} & & \Big\downarrow{\scriptstyle \alpha^*} \\
K^{-p-1}(C_0(-1,1)\otimes A) & \xrightarrow[s]{} & K^{-p}(A),
\end{array}$$

is commutative.

9.6 Boundary Maps and the Kasparov Product

The first purpose of this section is to indicate how the construction of the boundary maps in the K-homology long exact sequence may be organized around the Kasparov product. Of course, we constructed these maps by other means in Chapter 5, and so the second purpose of the section is to show that the product-based construction of the boundary maps comes to the same thing as the construction in Chapter 5.

Granted what we have already proved about the Kasparov theory, the construction of the long exact sequence is actually very simple. Let us begin with a semisplit extension

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0 \tag{9.6.1}$$

of separable C^*-algebras. Our aim is to construct boundary maps

$$\partial\colon K^{-p-1}(J) \longrightarrow K^{-p}(A/J)$$

which fit into a long exact sequence

$$\cdots \longrightarrow K^{-p-1}(A) \longrightarrow K^{-p-1}(J) \xrightarrow{\partial} K^{-p}(A/J) \longrightarrow K^{-p}(A) \longrightarrow \cdots \tag{9.6.2}$$

ending at $K^0(J)$ (the sequence can be extended to the right using formal periodicity[88]). We proved in Chapter 5 that the K-homology functor is half-exact for semisplit extensions, which is to say that the sequence of K-homology groups

$$K^{-p-1}(A/J) \longrightarrow K^{-p-1}(A) \longrightarrow K^{-p-1}(J)$$

is exact at $K^{-p-1}(A)$. Since the Kasparov product does not really illuminate the proof of half-exactness we shall simply accept this result from Chapter 5 and proceed. Recall from Section 4.5 that, starting from from the homotopy invariance and half-exactness of the K-homology functor, one can obtain, by a simple mapping cone construction, a long exact sequence

(9.6.3)

$$K^{-p-1}(A) \longrightarrow K^{-p-1}(J) \xrightarrow{\delta} K^{-p-1}(S(A/J)) \longrightarrow K^{-p-1}(S(A)) \longrightarrow \cdots$$

So to define a boundary map $\partial\colon K^{-p-1}(J) \to K^{-p}(A/J)$ and obtain the long exact sequence 9.6.2 it suffices to use the Bott Periodicity Theorem of the

[88]We noted earlier that the formal periodicity isomorphism is induced by Kasparov product with a generator of $K^{-2}(\mathbb{C})$. We shall see below that the Kasparov boundary map ∂ is compatible with the product, and from this it follows that the entire long exact sequence 9.6.2 is compatible with formal periodicity.

previous section[89] to identify the functor $K^{-p-1}(S(B))$ with $K^{-p}(B)$. Thus if $s_B\colon K^{-p-1}(S(B)) \to K^{-p}(B)$ is the suspension map of Definition 9.5.6, and if $x \in K^{-p-1}(J)$, then we set

$$\partial(x) = s_{A/J}(\delta(x)) \in K^{-p}(A/J).$$

This completes the construction of the long exact sequence in Kasparov's K-homology.

To describe the boundary maps ∂ a little more thoroughly, let us review the construction of the homotopy-theoretic boundary map δ in the exact sequence 9.6.3. Recall that along with the *suspension* $S(A) = C_0(0,1) \otimes A$, we defined the *cone* $C(A) = C_0(0,1] \otimes A$, and that these two C^*-algebras fit into a *suspension short exact sequence*

$$0 \longrightarrow S(A) \longrightarrow C(A) \longrightarrow A \longrightarrow 0\,. \tag{9.6.4}$$

Recall also that we defined the *mapping cone*

$$C(A, A/J) = \big\{\, (a, f) \in A \oplus C([0,1], A/J) \;:\; f(0) = 0 \text{ and } f(1) = \pi(a) \,\big\}.$$

The C^*-algebra $C(A, A/J)$ is an ideal in a C^*-algebra $Q(A, A/J)$, which consists of similar pairs (a, f), except that we omit the condition $f(0) = 0$. There is a commutative diagram of semisplit short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & S(A/J) \arrow[r] \arrow[d] & C(A/J) \arrow[r] \arrow[d] & A/J \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C(A, A/J) \arrow[r] & Q(A, A/J) \arrow[r] & A/J \arrow[r] & 0 \\
0 \arrow[r] & J \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & A/J \arrow[r] \arrow[u, equal] & 0
\end{tikzcd} \tag{9.6.5}$$

The homotopy invariance and half-exactness of K-homology guarantee that the $*$-homomorphism $J \to C(A, A/J)$ induces an isomorphism in K-homology. By definition, the boundary map δ is then the composition

$$K^{-p-1}(J) \xleftarrow[\cong]{} K^{-p-1}(C(A, A/J)) \longrightarrow K^{-p-1}(S(A/J))$$

associated to the leftmost vertical maps in Diagram 9.6.5.

[89]In the previous section we worked with the interval $(-1, 1)$ whereas here we are working with $(0, 1)$. Identify the two via the orientation-preserving map $x \mapsto \frac{1}{2}(x+1)$.

9.6.6 PROPOSITION *Let* $d \in K^{-1}(C_0(0,1))$ *be the Dirac class. The boundary map*

$$\partial\colon K^{-p-1}(C_0(0,1)\otimes A) \longrightarrow K^{-p}(A)$$

associated to the short exact sequence

$$0 \longrightarrow C_0(0,1)\otimes A \longrightarrow C_0[0,1)\otimes A \longrightarrow A \longrightarrow 0$$

has the property that $\partial(d\times x) = x$, *for every* $x \in K^{-p}(A)$.

PROOF Sorting through the constructions, we see that the homotopy-theoretic boundary map

$$\delta\colon K^{-p-1}(S(A)) \to K^{-p-1}(S(A))$$

associated to the short exact sequence in the proposition is the identity. Hence if $y \in K^{-p-1}(S(A))$ then

$$\partial(y) = s(\delta(y)) = s(y).$$

Setting $y = d\times x$ for $x \in K^{-p}(A)$ we conclude that $\partial(d\times x) = s(d\times x)$. But we proved in the previous section that $s(d\times x) = x$. $\square$

Diagram 9.6.5 tells us more:

9.6.7 PROPOSITION *There is a unique way of associating to each semisplit short exact sequence* 9.6.1 *a boundary map* $b\colon K^{-p-1}(J) \to K^{-p}(A/J)$ *in such a way that*

(a) *associated to each commuting diagram of short exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & J_1 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & A_1/J_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J_2 \arrow[r] & A_2 \arrow[r] & A_2/J_2 \arrow[r] & 0
\end{tikzcd}$$

there is a corresponding commuting diagram

$$\begin{tikzcd}
K^{-p-1}(J_1) \arrow[r, "b"] & K^{-p}(A_1/J_1) \\
K^{-p-1}(J_2) \arrow[r, "b"'] \arrow[u] & K^{-p}(A_2/J_2) \arrow[u]
\end{tikzcd}$$

of K*-homology groups,*

(b) *the boundary maps* b_A *for the short exact sequences*

$$0 \longrightarrow S(A) \longrightarrow C(A) \longrightarrow A \longrightarrow 0$$

are compatible with Kasparov product in the sense that

$$b_{A\otimes B}(x \times y) = b_A(x) \times y \in K^{-p-q}(A \otimes B),$$

for every $x \in K^{-p-1}(S(A))$ *and* $y \in K^{-q}(B)$, *and*

(c) *the boundary map for the short exact sequence*

$$0 \longrightarrow C_0(0,1) \longrightarrow C_0[0,1) \longrightarrow \mathbb{C} \longrightarrow 0$$

maps the Dirac class $d \in K^{-1}(C_0(0,1))$ *to the unit class* $\mathbf{1} \in K^0(\mathbb{C})$.

PROOF Remark 9.5.11, Lemma 9.5.8, and Proposition 9.6.6 show that the boundary map ∂ that we constructed earlier in this section has the properties (a)–(c). Conversely, let b have these properties. It follows from functoriality (property (a) above) and Diagram 9.6.5 that the boundary maps for the suspension sequences

$$0 \longrightarrow S(A) \longrightarrow C(A) \longrightarrow A \longrightarrow 0$$

determine all the other boundary maps. It then follows from property (b) that the boundary map for $A = \mathbb{C}$ determines all the others. But since the group $K^{-1}(C_0(0,1))$ is generated by the Dirac class, property (c) determines the boundary map entirely. □

We conclude this section with a discussion of the boundary maps that we constructed in Chapter 5 by means of duality theory. Using the formal periodicity of K-homology (Proposition 8.2.13) we may extend the definitions of those boundary maps to the groups K^{-p}, for $p \leqslant 0$, by requiring one or other of the following diagrams to commute:

(9.6.8)
$$\begin{tikzcd} K^0(J) \arrow[r, "\partial"] \arrow[d, "\cong"'] & K^1(A/J) \arrow[d, "\cong"] \\ K^{-p}(J) \arrow[r, "b"'] & K^{-p+1}(A/J) \end{tikzcd} \qquad \begin{tikzcd} K^1(J) \arrow[r, "\partial"] \arrow[d, "\cong"'] & K^0(A/J) \arrow[d, "\cong"] \\ K^{-p}(J) \arrow[r, "b"'] & K^{-p+1}(A/J) \end{tikzcd}$$

Here the vertical arrows are the periodicity isomorphisms and we must use the left-hand diagram if p is even, and the right-hand diagram if p is odd. Are these boundary maps the same as the ones we have constructed in this section? To answer the question we could apply Proposition 9.6.7, which characterizes the

Kasparov boundary maps. But in fact the formulas we obtained in Chapter 8 provide a more direct approach to the problem.

Since we did so in Chapter 8, we shall assume throughout the rest of this section that A is unital. The problem of identifying the Kasparov boundary maps with the boundary maps 9.6.8 in the non-unital case can easily be handled afterwards by adjoining units.

9.6.9 LEMMA *Let A be a separable, unital C^*-algebra and form the suspension exact sequence*

$$0 \longrightarrow S(A) \longrightarrow C(A) \longrightarrow A \longrightarrow 0 .$$

(a) *Let $x \in K^1(S(A))$ be represented by an ungraded relative Fredholm module (ρ, H, F), for which the representation ρ is extended to $C[0,1]\otimes A$ and commutes modulo compacts with F. Let g denote the function $g(x) = 2x - 1$ on $(0,1)$, and let G be the operator $\rho(g \otimes 1)$. Then the operator*

$$V = iG + (1 - G^2)^{\frac{1}{2}} F \ \in \mathfrak{B}(H),$$

together with the representation $\varphi\colon A \to \mathfrak{B}(H)$ given by $\varphi(a) = \rho(1\otimes a)$, is essentially unitary and defines a Fredholm module over A, and the class in $K^0(A)$ represented by (φ, H, V) is the image under the boundary map $\partial\colon K^1(S(A)) \to K^0(A)$ of the class x.

(b) *Let $x \in K^0(S(A))$ be represented by a graded relative Fredholm module (ρ, H, F) for which, as above, the representation ρ is extended to $C[0,1] \otimes A$ and commutes modulo compacts with F. Let g denote the function $g(x) = 2x-1$ on $(0,1)$, and let G be the operator $\rho(g\otimes 1)$. Let ε denote the grading operator. Then the operator*

$$V = \varepsilon G + (1 - G^2)^{\frac{1}{2}} F \ \in \mathfrak{B}(H),$$

together with the representation $\varphi\colon A \to \mathfrak{B}(H)$ given by $\varphi(a) = \rho(1\otimes a)$, defines an ungraded Fredholm module over A, and the class in $K^1(A)$ represented by this module (φ, H, V) is the image under the boundary map $\partial\colon K^0(S(A)) \to K^1(A)$ of the class x.

PROOF These results are reformulations of Lemmas 8.7.15 and 8.7.16 respectively. □

If we use Diagram 9.6.8 to transfer the above formulas to the groups $K^{-p}(A)$ then we obtain the following:

9.6.10 LEMMA *Let $x \in K^{-p-1}(SA)$ be represented by a $(p+1)$-multigraded relative Fredholm module (ρ, H, F), with grading operator γ and multigrading operators $\varepsilon_1, \ldots, \varepsilon_{p+1}$. Then the operator*

$$V = \gamma \varepsilon_{p+1} G + (1 - G^2)^{\frac{1}{2}} F \in \mathfrak{B}(H),$$

together with the representation $\varphi \colon A \to \mathfrak{B}(H)$ given by $\varphi(a) = \rho(1 \otimes a)$, defines a p-multigraded Fredholm module over A, and the class in $K^{-p}(A)$ represented by (φ, H, V) is the boundary class $b(x) \in K^{-p}(A)$. □

9.6.11 PROPOSITION *The boundary map $b \colon K^{-p-1}(A/J) \to K^{-p}(J)$ defined by Diagram 9.6.8 is the same as the Kasparov boundary map if p is odd, and is minus the Kasparov boundary map if p is even.*

PROOF It follows from Diagram 9.6.5 that we need only prove the proposition for the suspension exact sequences. For these, Lemma 9.6.10 gives an explicit formula for the boundary map b. In addition it follows from Lemma 9.6.6 that the Kasparov boundary map is the suspension homomorphism described by the formula in Definition 9.5.6. Comparing the two formulas, we see that they differ only by a permutation of the multigrading operators $\varepsilon_1, \ldots, \varepsilon_{p+1}$ which interchanges ε_1 and ε_{p+1}. By Exercise 8.8.7 this permutation operation alters the K-homology class of a $(p+1)$ multigraded Fredholm module by a multiplicative factor of $(-1)^{p+1}$. □

9.7 The Kasparov Product and the Index Pairing

The following compatibility relation between the Kasparov product, the K-theory product defined in Section 4.7, and the index pairing will play an important role in our discussion of index theory in Chapter 11.

9.7.1 PROPOSITION *Let A_1 and A_2 be separable C^*-algebras. If $x_1 \in K_0(A_1)$ and $x_2 \in K_0(A_2)$, and if $y_1 \in K^0(A_1)$ and $y_2 \in K^0(A_2)$, then*

$$\langle x_1, y_1 \rangle \langle x_2, y_2 \rangle = \langle x_1 \times x_2, y_1 \times y_2 \rangle.$$

PROOF We may assume that A_1 and A_2 are unital, since the non-unital case follows easily by adjoining units. Let x_1 and x_2 be represented by projections over A_1 and A_2; in fact to simplify the notation let us assume that p_1 and p_2 are projections within A_1 and A_2. Let y_1 and y_2 be represented by selfadjoint Fredholm modules (ρ_1, H_1, F_1) and (ρ_2, H_2, F_2). Then $\langle x_1, y_1 \rangle$ is the index of the graded Fredholm operator $p_1 F_1 p_1$ on $p_1 H_1$, while $\langle x_2, y_2 \rangle$ is the index of the graded Fredholm operator $p_2 F_2 p_2$ on $p_2 H_2$ (to streamline the notation further we have omitted mention of the representations ρ_1 and ρ_2). Construct the product using the procedure described in Proposition 9.2.3, but then symmetrize the result as follows, so as to obtain a selfadjoint operator:

$$F_1 \times F_2 = N_1^{\frac{1}{2}}(F_1 \hat{\otimes} 1)N_1^{\frac{1}{2}} + N_2^{\frac{1}{2}}(1 \hat{\otimes} F_2)N_2^{\frac{1}{2}}.$$

The quantity $\langle x_1 \times x_2, y_1 \times y_2 \rangle$ is then the index of the graded Fredholm operator $(p_1 \otimes p_2)(F_1 \times F_2)(p_1 \times p_2)$ on $p_1 H_1 \hat{\otimes} p_2 H_2$. But

$$(p_1 \otimes p_2)(F_1 \times F_2)(p_1 \times p_2) \sim N_1^{\frac{1}{2}}(p_1 F_1 p_1 \hat{\otimes} p_2)N_1^{\frac{1}{2}} + N_2^{\frac{1}{2}}(p_1 \hat{\otimes} p_2 F_2 p_2)N_2^{\frac{1}{2}}$$

So it follows from Lemma 9.1.10 and Proposition 9.1.9 that

$$\operatorname{Index}(p_1 \otimes p_2)(F_1 \times F_2)(p_1 \times p_2) = \operatorname{Index} p_1 F_1 p_1 \cdot \operatorname{Index} p_2 F_2 p_2,$$

as required. □

9.7.2 Remark An analogous result also holds for the odd-dimensional index pairings — see Exercise 9.8.4.

9.8 Exercises

The purpose of the first two exercises is to explicitly define a Kasparov product pairing

$$K^{p_1}(A_1) \otimes K^{p_2}(A_2) \longrightarrow K^{p_1+p_2}(A_1 \otimes A_2),$$

where $p_1, p_2 \in \{0, 1\}$ and the sum $p_1 + p_2$ is computed modulo 2.

9.8.1 Let A_1 and A_2 be separable C^*-algebras. Let (ρ_1, H_1, F_1) be an *ungraded* Fredholm module over A_1 and let (ρ_2, H_2, F_2) be a graded Fredholm module over A_2. Let $H = H_1 \otimes H_2$ and let $\rho = \rho_1 \otimes \rho_2$. Show that if we denote by γ the grading operator on H_2 then the operator

$$F = N_1(F_1 \otimes \gamma) + N_2(1 \otimes F_2),$$

where $\{N_1, N_2\}$ is a suitable partition of unity, defines an ungraded Fredholm module (ρ, H, F). Define in this way a pairing

$$K^1(A_1) \otimes K^0(A_2) \longrightarrow K^1(A_1 \otimes A_2).$$

9.8.2 Let A_1 and A_2 be separable C^*-algebras. Let (ρ_1, H_1, F_1) be an ungraded Fredholm module over A_1 and let (ρ_2, H_2, F_2) be an ungraded Fredholm module over A_2. Let $H = H_1 \otimes H_2$ and let $\rho = \rho_1 \otimes \rho_2$. Show that the operator

$$F = \begin{pmatrix} 0 & N_1(F_1 \otimes 1) - N_2(1 \otimes iF_2) \\ N_1(F_1 \otimes 1) + N_2(1 \otimes iF_2) & 0 \end{pmatrix}$$

where $\{N_1, N_2\}$ is a suitable partition of unity, defines a (balanced) graded Fredholm module (ρ, H, F). Define in this way a pairing

$$K^1(A_1) \otimes K^1(A_2) \longrightarrow K^0(A_1 \otimes A_2) .$$

9.8.3 Use periodicity to extend the Kasparov product to a pairing

$$K^{p_1}(A_1) \otimes K^{p_2}(A_2) \longrightarrow K^{p_1+p_2}(A_1 \otimes A_2) ,$$

which is defined for all $p_1, p_2 \in \mathbb{Z}$ (define the higher K-homology groups using suspension, as in Chapter 7). Describe its associativity and commutativity properties, and its compatibility with the pairings defined in the previous two exercises.

9.8.4 Let A_1 and A_2 be separable C^*-algebras. Show that if $x_1 \in K_{p_1}(A_1)$ and $x_2 \in K_{p_2}(A_2)$, and if $y_1 \in K^{p_1}(A_1)$ and $y_2 \in K^{p_2}(A_2)$, then

$$\langle x_1, y_1\rangle\langle x_2, y_2\rangle = (-1)^{p_1 p_2}\langle x_1 \times x_2, y_1 \times y_2\rangle,$$

for all $p_1 \geqslant 0$ and $p_2 \geqslant 0$. Here the K-homology products are defined using Exercise 9.8.3.

9.8.5 Let A_1 and A_2 be C^*-algebras and let J_1 and J_2 be ideals in them. Represent the relative K-theory groups $K_0(A_i, A_i/J_i)$ as in Proposition A.5.2: that is, by cycles made up of a graded Hermitian module M_i over A_i together with an odd $a_i \in \mathrm{End}(M_i)$ such that $a_i - a_i^*$ and $a_i^2 - 1$ are carried by J_i. Show that the formula

$$a = N_1(a_1 \hat{\otimes} 1) + N_2(1 \hat{\otimes} a_2),$$

suitably interpreted, defines an external product

$$K_0(A_1, A_1/J_1) \times K_0(A_2, A_2/J_2) \to K_0(A_1 \otimes A_2, A_1 \otimes A_2/J_1 \otimes J_2)$$

which agrees under the excision isomorphisms with the product of Definition 4.7.4.

9.8.6 Let G be a discrete group and let $C^*(G)$ be its full C^*-algebra (Definition 3.7.4). The diagonal map $G \to G \times G$ gives rise to a $*$-homomorphism $\Delta\colon C^*(G) \to C^*(G) \otimes C^*(G)$. Use Δ to give $K^*(C^*(G))$ the structure of a graded-commutative ring. This is Kasparov's *representation ring* for G. Show that if G is finite then Kasparov's ring is the usual representation ring of G.

9.8.7 Let G be a discrete group. The trivial representation of G determines a $*$-homomorphism from the full C^*-algebra $C^*(G)$ to $\mathbb{C}$, which defines a homology class $\delta \in K^0(C^*(G))$. We say that the group G is K-*amenable* if δ lies in the image of the map

$$K^0(C_r^*(G)) \to K^0(C^*(G))$$

induced from the regular representation of G. Show that if G is K-amenable then the map $K^0(C_r^*(G)) \to K^0(C^*(G))$ is an isomorphism. (The corresponding statement is also true in K-theory, but to prove it one needs various products between K-theory and K-homology which have not been developed in the text.)

9.8.8 Use the Fredholm module of Example 8.1.6 to prove that the free group on two generators is K-amenable.

9.8.9 Let A and B be unital C^*-algebras. Show that the formula $(a, T) \mapsto \pi[\rho(a \otimes 1)T]$ defines a $*$-homomorphism

$$A \otimes \mathfrak{D}_\rho(A \otimes B) \to \mathfrak{D}_\sigma(B)/\mathfrak{K}(H),$$

where $\sigma(b) = \rho(1 \otimes b)$. Use this to define a 'slant product'

$$\backslash \colon K_p(A) \otimes K^q(A \otimes B) \to K^{q-p}(B),$$

and, if A is commutative, a 'cap product'

$$\frown \colon K_p(A) \otimes K^q(A) \to K^{q-p}(A).$$

Investigate the compatibility of these various products with the Kasparov product.

9.9 Notes

The Kasparov product was introduced in [80]. The construction was motivated by the 'product' of elliptic pseudodifferential operators used by Atiyah and Singer in their proof of the Index Theorem [16]; we shall deal with that connection in the next two chapters. Among many other remarkable results, Kasparov's paper [80] contains the proof of the homotopy invariance of K-homology (Theorem 9.3.3) that we have discussed in this chapter. This is the most general argument for homotopy invariance, and it long preceded the coarse approach set out earlier in this book.

Our characterization of the product in terms of 'alignment' conditions derives from the work of Connes and Skandalis [42, 121].

There is a lengthy list of products involving K-homology and K-theory groups, and a lengthy list of compatibility relations between them. The products are unified by Kasparov's *bivariant* KK-theory [81]. This theory, which we shall not discuss in detail in this book, associates abelian groups $KK^{-p}(A, B)$ to every pair (A, B) of separable C^*-algebras in such a way that

(a) the group $KK^{-p}(A, \mathbb{C})$ is the K-homology group $K^{-p}(A)$,
(b) the group $KK^{-p}(\mathbb{C}, B)$ is the K-theory group $K_p(B)$, and
(c) there are compatible and associative 'Kasparov' products

$$KK^{-p}(A, B) \times KK^{-q}(B, C) \to KK^{-p-q}(A, C)$$

and

$$KK^{-p}(A_1, B_1) \times KK^{-q}(A_2, B_2) \to KK^{-p-q}(A_1 \otimes A_2, B_1 \otimes B_2).$$

Moreover, a semisplit short exact sequence

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

gives rise to an element of $KK^1(A/J, J)$, and the associated boundary maps (in both K-homology and K-theory) are given by Kasparov product with this element. The various compatibilities that we have proved in this chapter and the last — between the K-theory and K-homology boundary maps, the index pairing, and the products in K-theory and K-homology — are all manifestations of the associativity of the Kasparov product in KK-theory.

10

ELLIPTIC DIFFERENTIAL OPERATORS

K-homology for C^*-algebras arose in response to a number of developments in functional analysis and topology. We have already treated in detail the Brown–Douglas–Fillmore theory of C^*-algebra extensions. But equally important to K-homology was the topological theory of elliptic operators on manifolds. The index theorem of Atiyah and Singer showed that elliptic operator theory deserved the careful attention of topologists; Atiyah then abstracted the functional-analytic properties of elliptic operators, and Kasparov organized these 'abstract elliptic operators' into a fully functioning homology theory. In this chapter we shall review some of the theory of (first-order) linear elliptic differential operators, and then we shall explain how an elliptic operator on a manifold determines a Fredholm module, and therefore a K-homology class. We shall prove that the Kasparov product on K-homology corresponds to the natural 'external product' of elliptic operators. This was Kasparov's original motivation for the introduction of his product.

We shall assume that the reader has some familiarity with Fourier analysis and the language of smooth manifold theory. For more details, he or she should consult one of the references mentioned in the notes at the end of the chapter.

10.1 First-Order Differential Operators

10.1.1 DEFINITION Let M be a smooth manifold and let S be a smooth complex vector bundle over M. Let $C^\infty(M;S)$ be the space of smooth sections of S. A *first-order linear differential operator* on S is a complex-linear map

$$D\colon C^\infty(M;S) \to C^\infty(M;S)$$

which has the following properties:

(a) if u_1 and u_2 are smooth sections of S which agree on an open set $U \subseteq M$ then Du_1 and Du_2 agree on U, and

(b) for each coordinate patch $U \subseteq M$, if we choose coordinates x_j in U and a trivialization for the bundle S over U, then D can be represented in local coordinates by a formula

$$Du = \sum_j A_j \frac{\partial u}{\partial x_j} + Bu,$$

where the A_j and B are smooth, matrix-valued functions on U.

10.1.2 REMARKS Since we shall not have occasion to consider differential operators of higher order, let us make the convention that the term 'differential operator' will always mean 'first-order linear differential operator'. We shall also use the term 'differential operator on M' as an abbreviation of the term 'differential operator on some complex vector bundle over M'. Finally, we emphasize that throughout this chapter we shall be considering only manifolds *without* boundary.

Notice that item (a) in Definition 10.1.1 gives sense to the local description of D in item (b). In fact a differential operator on M may be *restricted* to any open subset of M. The operation of restriction, in its turn, allows us to speak of the *support* of a differential operator D: it is the complement of the largest open set U for which the restriction of D to U is zero.

The functions A_j and B which appear in Definition 10.1.1 depend on the particular coordinate system and trivialization chosen. But if $\xi = \sum_j \xi_j \, dx_j$ is a cotangent vector at a point $x \in M$, and if we form the expression

$$\sigma_D(x, \xi) = \sum_j A_j \xi_j,$$

then $\sigma_D(x, \xi)$, interpreted as an endomorphism of the vector space S_x, is independent of the choice of coordinates. Indeed, a calculation shows that if g is any smooth function, and if $\rho(g)$ denotes the operator on $C^\infty(M; S)$ of multiplication by g, then

$$\sigma_D(x, dg)u(x) = \big([D, \rho(g)]u\big)(x)$$

for any smooth section u of S.

10.1.3 DEFINITION Let $D\colon C^\infty(M; S) \to C^\infty(M; S)$ be a differential operator. The *symbol* of D is the vector bundle morphism[90]

$$\sigma_D \colon T^*M \to \mathrm{End}(S)$$

which is defined by the formula above.

We shall study differential operators using C^*-algebraic and Hilbert space techniques, and to do so we need to introduce appropriate inner products. For this we need the following items:

(a) a Hermitian metric $(\ ,\ )$ on the vector bundle S, and
(b) a smooth measure μ on M (this measure will frequently be the canonical one arising from a Riemannian metric on M, but other choices are possible).

[90]Note that the cotangent bundle T^*M is a *real* vector bundle over M, whereas $\mathrm{End}(S)$ is *complex*. For the purposes of this definition we use only the underlying real vector bundle structure on $\mathrm{End}(S)$.

We shall often assume without explicit mention that our manifolds and bundles are provided with these additional structures. Granted them, we define an inner product on the space $C_c^\infty(M; S)$ of compactly supported, smooth sections of S by the formula

$$\langle u, v\rangle = \int_M (u(x), v(x))\, d\mu(x).$$

We form a Hilbert space $L^2(M; S)$ by taking the completion of $C_c^\infty(M; S)$ relative to this inner product.

10.1.4 PROPOSITION *Let* $D\colon C^\infty(M; S) \to C^\infty(M; S)$ *be a differential operator. There is a unique differential operator* $D^\dagger\colon C^\infty(M; S) \to C^\infty(M; S)$ *such that*

$$\langle Du, v\rangle = \langle u, D^\dagger v\rangle,$$

for all $u, v \in C_c^\infty(M; S)$. *The symbols of* D *and* $D^\dagger$ *are related by the formula*

$$\sigma_{D^\dagger}(x, \xi) = -\sigma_D(x, \xi)^*.$$

10.1.5 DEFINITION The operator $D^\dagger$ is called the *formal adjoint* of D.

PROOF OF THE PROPOSITION To prove existence, a partition of unity argument shows that it suffices to consider the case where D is supported in a single coordinate patch, so that D may be described by the formula in Definition 10.1.1. The main idea is then to integrate by parts, as in the calculation

$$\int \frac{du}{dx} v\, dx = -\int u \frac{dv}{dx}\, dx,$$

which shows that the formal adjoint of the operator $\frac{d}{dx}$ on $\mathbb{R}$ (with the canonical metric and measure) is the operator $-\frac{d}{dx}$. Further details of the construction are left to the reader. To prove uniqueness, note that the range of the difference, say E, of two formal adjoints is orthogonal to every $u \in C_c^\infty(M; S)$:

$$v \in C_c^\infty(M; S) \quad \Rightarrow \quad \langle u, Ev\rangle = 0.$$

This implies that $E = 0$, as required. $\square$

10.2 Symmetric and Selfadjoint Differential Operators

We shall now develop some of the Hilbert space theory of differential operators. We need to proceed with care, because differential operators are not bounded (the L^2-norm of the derivative of u cannot be controlled in terms of the L^2-norm of u). However, the theory of unbounded operators on Hilbert space which we introduced in Chapter 1 applies to this situation. We shall therefore think of a differential operator D as an unbounded operator whose domain is the space $C_c^\infty(M;S) \subseteq L^2(M;S)$.

Recall that an unbounded Hilbert space operator T is *closable* if the norm-closure of its graph is the graph of another unbounded operator, called the *closure* of T and denoted $\overline{T}$.

10.2.1 LEMMA *Every differential operator* D *is closable.*

PROOF All we need to know is that the closure of the graph of D is again a graph, in other words that the closure of the graph intersects each 'vertical' in just one point. By linearity, it suffices to prove that if $\{v_j\}$ is a sequence in $C_c^\infty(M;S)$ with $v_j \to 0$ in H, and if $Dv_j \to w$ in H, then $w = 0$. If u is any member of $C_c^\infty(M;S)$ then

$$\langle u, w\rangle = \lim\langle u, Dv_j\rangle = \lim\langle D^\dagger u, v_j\rangle = 0.$$

Thus w is orthogonal to the dense subspace $C_c^\infty(M;S) \subseteq L^2(M;S)$, and therefore $w = 0$. $\square$

As in Chapter 1 we shall omit the overline when no crucial issue turns on the distinction between D and $\overline{D}$, or when (as with the functional calculus) the use of the closure $\overline{D}$ is implied.

Recall now that an unbounded operator T is *symmetric* if $\langle Tu, v\rangle = \langle u, Tv\rangle$, for all u and v in the domain of T. For differential operators this amounts to the condition that D is equal to its formal adjoint $D^\dagger$. We shall primarily be concerned with symmetric operators in this book, and we shall therefore tailor the following definition to the symmetric case:

10.2.2 DEFINITION Let $D\colon C_c^\infty(M;S) \to C_c^\infty(M;S)$ be a symmetric differential operator. The *minimal domain* of D is the domain of the closure $\overline{D}$. The *maximal domain* of D is the domain of the adjoint operator D^*.

10.2.3 REMARK The maximal domain of D can be described very simply in the language of distribution theory. It is the set of those $u \in L^2(M;S)$ such that the derivative Du, which is defined as a distribution, is equal to a square-integrable section of S.

If D is symmetric then its Hilbert space adjoint D^* is an extension of its closure $\overline{D}$, which is in turn an extension of D. All three operators can be different from one another.

10.2.4 EXAMPLE A classic example where the minimal and maximal domains differ is provided by the operator $D = i\frac{d}{dx}$ defined on the non-compact manifold $M = (0,1)$ with the usual Riemannian metric (so that the Hilbert space H is equal to $L^2[0,1]$). Here the identity function $v(x) = x$ belongs to the maximal domain of D, because its (distributional) derivative is an L^2 function. But v does not belong to the minimal domain of D, since a continuity argument shows that $\langle Du, 1\rangle = 0$ for all u belonging to the minimal domain, while $\langle Dv, 1\rangle \neq 0$.

The above example suggests that non-selfadjointness of symmetric operators has something to do with 'boundary conditions' on a non-compact manifold M. The following lemma makes the same point.

10.2.5 LEMMA *Let D be a symmetric differential operator and suppose that u is a* compactly supported *element of $L^2(M;S)$. Then u belongs to the minimal domain of D if and only if it belongs to the maximal domain.*

PROOF The proof makes use of *Friedrichs' mollifiers* on M. Let K be a compact subset of M. Then there exist, for all sufficiently small $t > 0$, operators $F_t\colon L^2(K;S) \to L^2(M;S)$ such that

(a) $\|F_t\| \leqslant 1$,
(b) for each $u \in L^2(K;S)$, $F_t u \to u$ in $L^2(M;S)$ as $t \to 0$,
(c) for each $u \in L^2(K;S)$, $F_t u$ is smooth, with compact support, and
(d) the commutator $[D, F_t]$ extends to a bounded operator from $L^2(K;S)$ to $L^2(M;S)$, whose norm is bounded independent of t.

The construction of these mollifiers is described in Exercise 10.9.1. Now suppose that u belongs to the maximal domain of D and is supported within K. Then $F_t u$ is smooth and compactly supported, and tends to u as $t \to 0$. Moreover we have

$$DF_t u = F_t Du + [D, F_t]u.$$

Therefore $DF_t u$ is uniformly bounded in norm, and so Lemma 1.8.1 completes the proof. □

Notice the similarity between the notion of Friedrichs' mollifier and that of quasicentral approximate unit (see Definition 3.2.4).

10.2.6 COROLLARY *Every symmetric differential operator on a compact manifold without boundary is essentially selfadjoint. More generally, every compactly supported, symmetric differential operator on an open manifold is essentially selfadjoint.* □

As Example 10.2.4 shows, the selfadjointness of general differential operators on open manifolds is a more delicate matter. To get a result in this situation we shall use a crucial fact about first-order operators: if D is such an operator, and if $\rho(g) \in \mathfrak{B}(H)$ is the multiplication operator by a smooth function g, then the commutator $[D, \rho(g)]$ is also a 'multiplication' operator (more precisely, a smooth endomorphism of the bundle on which D acts). Indeed, we have already noted the formula

$$\sigma_D(x, dg)u(x) = \big([D, \rho(g)]u\big)(x). \tag{10.2.7}$$

Note in particular that if g is smooth and compactly supported, then the operator $[D, \rho(g)]$ is bounded.

10.2.8 Definition Let D be a differential operator on a manifold M. We shall say that M is *complete* for D if there is a smooth, proper function $g\colon M \to \mathbb{R}$ such $[D, \rho(g)]$ is a bounded Hilbert space operator.

10.2.9 Example If $M = \mathbb{R}$ and $D = \frac{d}{dx}$ then M is complete for D: take $g(x) = x$.

10.2.10 Proposition *If D is a symmetric differential operator on M, and if M is complete for D, then D is essentially selfadjoint.*

Proof Using completeness one can produce a sequence of compactly supported functions $g_n\colon M \to [0, 1]$ such that $g_n \to 1$ uniformly on compact sets, and $[D, \rho(g_n)] \to 0$ in norm. Let u belong to the maximal domain of D. Then $\rho(g_n)u$ also belongs to the maximal domain of D, and hence to the minimal domain (by Lemma 10.2.5). But $\rho(g_n)u \to u$ in norm as $n \to \infty$, while

$$D\rho(g_n)u = \rho(g_n)Du + [D, \rho(g_n)]u \to Du,$$

as $n \to \infty$. Thus u belongs to the minimal domain. $\square$

10.2.11 Proposition *Let M be a complete Riemannian manifold and let D be a symmetric differential operator on M. If the* propagation speed

$$c_D = \sup\{\, \|\sigma_D(x, \xi)\| \,:\, x \in M,\ \xi \in T^*_x M,\ \|\xi\| = 1\,\}$$

is finite, then D is essentially selfadjoint.

Proof We may assume that M is connected. Fix a point $x_0 \in M$. We would like to take $g(x)$ to be the Riemannian distance $d(x, x_0)$ from x to x_0. This is a Lipschitz function but unfortunately it need not be smooth. However we can always find a smooth approximation g to the Riemannian distance, for instance such that $|g(x) - d(x, x_0)| \leqslant 1$ and $\|dg(x)\| \leqslant 2$ for all x. The completeness of M implies that g is a proper function. The hypothesis on σ_D implies that $[D, \rho(g)]$ is a bounded operator. Therefore M is complete for D, and hence D is essentially selfadjoint. $\square$

10.3 Wave Operators

Suppose now that the differential operator D *is* essentially selfadjoint. Then we may form various functions of D using spectral theory, as in Section 1.8.

Of particular interest are the unitary operators e^{isD}, which are the solution operators for the 'wave equation'

$$\frac{\partial u}{\partial s} = iDu.$$

The following proposition describes a standard feature of partial differential equations of this type (which are called symmetric hyperbolic systems). The method of proof — by 'energy estimates' — is also standard.

10.3.1 PROPOSITION *Let* D *be an essentially selfadjoint differential operator on* M. *Let* K *be a compact subset of* M *and let* W *be an open neighborhood of* K. *There exists* $\varepsilon > 0$ *such that if* u *is supported within* K, *and if* $|s| < \varepsilon$, *then* $e^{isD}u$ *is supported within* W.

10.3.2 EXAMPLE To illustrate the proposition, let $M = \mathbb{R}$ and let $D = i\frac{d}{dx}$. Then e^{isD} is the operator on $L^2(\mathbb{R})$ of translation by s, and therefore certainly has the required 'finite propagation' property.

PROOF OF THE PROPOSITION By treating first D and then $-D$, it suffices to consider nonnegative s. Let $g\colon M \to \mathbb{R}$ be a smooth, compactly supported function such that

$$g(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } x \notin W. \end{cases}$$

Let $f\colon \mathbb{R} \to [0,1]$ be a smooth, *non-decreasing* function such that

$$\begin{cases} f(t) < 1 & \text{if } t < 1 \\ f(t) = 1 & \text{if } t \geqslant 1. \end{cases}$$

Choose a constant c such that

$$c > \|[D, \rho(g)]\|.$$

Having selected these things, define for each $s \geqslant 0$ a smooth function h_s on M by the formula

$$h_s(x) = f(g(x) + cs)\,.$$

Then $\{h_s\}_{s\geqslant 0}$ is a family of functions whose level sets $L_s = \{x : h_s(x) = 1\}$ increase in size as s increases. Moreover

$$s < \frac{1}{c} \quad \Rightarrow \quad K \subseteq L_0 \subseteq L_s \subseteq W.$$

The family $\{\rho(h_s)\}_{s\geqslant 0}$ of bounded operators is norm-differentiable with respect to s, with derivative $\rho(\dot{h}_s)$, where we use the dot to denote partial differentiation with respect to s. Now note that

$$\dot{h}_s(x) = cf'(g(x) + cs) \geqslant 0.$$

We may also calculate the exterior derivative of h_s with respect to the x-variable:

$$dh_s(x) = f'(g(x) + cs)\, dg(x) = \frac{1}{c}\dot{h}_s(x) dg(x).$$

Using this and Formula 10.2.7 for the commutator of D with a smooth function we obtain

$$[D, \rho(h_s)] = \frac{1}{c}\, \rho(\dot{h}_s)[D, \rho(g)].$$

Bearing in mind that $\rho(\dot{h}_s)$ is positive and commutes with the bounded, selfadjoint operator $i[D, \rho(g)]$, and that $c > \|[D, \rho(g)]\|$, we see that

$$\rho(\dot{h}_s) - i[D, \rho(h_s)] = \frac{1}{c}\rho(\dot{h}_s)\big(c - i[D, \rho(g)]\big) \geqslant 0\,.$$

Suppose now that u is a smooth section which is supported within K, and let $u_s = e^{isD}u$. Then

$$\frac{\partial}{\partial s}\big\langle \rho(h_s)u_s, u_s\big\rangle = \big\langle \rho(\dot{h}_s)u_s, u_s\big\rangle - i\big\langle [D, \rho(h_s)]u_s u_s\big\rangle,$$

which in view of the preceding inequality is non-negative. Therefore

$$s \geqslant 0 \quad \Rightarrow \quad \langle \rho(h_s)u_s, u_s\rangle \geqslant \langle \rho(h_0)u, u\rangle\ .$$

But

$$\langle \rho(h_0)u, u\rangle = \langle u, u\rangle = \langle u_s, u_s\rangle,$$

firstly because u is supported in K and secondly because the operator e^{isD} is unitary. Hence

$$s \geqslant 0 \quad \Rightarrow \quad \langle \rho(h_s)u_s, u_s\rangle \geqslant \langle u_s, u_s\rangle.$$

Since $1 \geqslant h_s \geqslant 0$, this implies that $\rho(h_s)u_s = u_s$, so that u_s is supported in the level set $L_s = \{x : h_s(x) = 1\}$. In particular, if $s < 1/c$ then u_s is supported within W, as required. □

10.3.3 COROLLARY *Let* D *be an essentially selfadjoint differential operator on* M. *Let* f *and* g *be bounded functions on* M. *If* Support(g) *is compact and if* Support(f) *is disjoint from* Support(g) *then there exists* $\varepsilon > 0$ *such that* $\rho(f)e^{isD}\rho(g) = 0$ *whenever* $|s| < \varepsilon$. □

10.3.4 COROLLARY *Let* K *be a compact subset of a manifold* M. *Let* D_1 *and* D_2 *be essentially selfadjoint differential operators on* M *which are equal in a neighborhood of* K. *There exists* $\varepsilon > 0$ *such that*

$$e^{isD_1}\rho(g) = e^{isD_2}\rho(g)$$

for all $|s| < \varepsilon$ *and all* g *supported in* K.

PROOF It suffices to show that there exists $\varepsilon > 0$ such that if u is supported within K then $e^{isD_1}u = e^{isD_2}u$, for all $|s| < \varepsilon$. Let

$$u_{1,s} = e^{isD_1}u, \quad \text{and} \quad u_{2,s}(s) = e^{isD_2}u.$$

It follows from Proposition 10.3.1, together with the fact that $D_1 = D_2$ near K, that

$$D_1 u_{1,s} = D_2 u_{1,s} \quad \text{and} \quad D_1 u_{2,s} = D_2 u_{2,s},$$

for all small s. Since $\dot{u}_{1,s} = iD_1 u_{1,s}$ and $\dot{u}_{2,s} = iD_2 u_{2,s}$ it follows from an expansion of the norm as an inner product that

$$\frac{d}{ds} \|u_{1,s} - u_{2,s}\|^2 = 0,$$

for small s. This gives the result. □

Having analyzed the wave operators e^{isD} we can now use Fourier analysis to examine more general operators $\varphi(D)$. For the statement and proof of the following proposition we shall borrow some ideas from distribution theory.

10.3.5 PROPOSITION *Let D be an essentially selfadjoint differential operator on M. If φ is a bounded Borel function on $\mathbb{R}$ whose Fourier transform is compactly supported then*

$$\langle \varphi(D)u, v\rangle = \frac{1}{2\pi}\int \langle e^{isD}u, v\rangle\, \widehat{\varphi}(s)\, ds,$$

for all $u, v \in C_c^\infty(M; S)$.

The integral is to be interpreted in the sense of distribution theory: it represents the pairing of the compactly supported distribution $\widehat{\varphi}$ with the smooth function $s \mapsto \langle e^{isD}u, v\rangle$. Although the proposition supposes only that φ is a bounded *Borel* function, which is all that is needed to define $\varphi(D)$ using the functional calculus, the hypothesis that $\widehat{\varphi}$ is compactly supported actually implies that φ is *smooth*.

PROOF If φ is a Schwartz class function then the proof follows from the inversion formula

$$\varphi(x) = \frac{1}{2\pi}\int e^{isx}\, \widehat{\varphi}(s)\, ds,$$

where the right-hand side is a Riemann integral. To prove the general case, choose a smooth, compactly supported function ψ on $\mathbb{R}$ with integral 1 and let $\psi_\varepsilon(x) = \frac{1}{\varepsilon}\psi(\frac{1}{\varepsilon}x)$. The inverse Fourier transform $\check{\psi}_\varepsilon$ converges to 1, uniformly on compact sets, as $\varepsilon \to 0$, while the convolution $\widehat{\varphi} \star \psi_\varepsilon$ converges to $\widehat{\varphi}$ in the sense of tempered distributions. The special case of the proposition already proved gives

$$\langle \varphi(D)\check{\psi}_\varepsilon(D)u, v\rangle = \frac{1}{2\pi}\int \langle e^{isD}u, v\rangle\, \widehat{\varphi} \star \psi_\varepsilon(s)\, ds,$$

and now the general case follows from a continuity argument. □

We conclude with a technical computation which uses similar ideas and which will be of use later.

10.3.6 LEMMA *Suppose that* T_1 *and* T_2 *are essentially selfadjoint operators defined on the same domain and that* $T_1 - T_2$ *is bounded. Then*

$$\|e^{isT_1} - e^{isT_2}\| \leqslant |s| \cdot \|T_1 - T_2\|,$$

for all $s \in \mathbb{R}$.

PROOF It follows from the Fundamental Theorem of Calculus that

$$\Big\langle \big(e^{isT_1} - e^{isT_2}\big)u, v\Big\rangle = i\int_0^s \Big\langle \big(e^{itT_1}(\overline{T}_1 - \overline{T}_2)e^{i(s-t)T_2}\big)u, v\Big\rangle \, dt,$$

for u in the common domain of T_1 and T_2. Hence

$$\Big|\langle \big(e^{isT_1} - e^{isT_2}\big)u, v\rangle\Big| \leqslant s \cdot \|T_1 - T_2\| \cdot \|u\| \cdot \|v\|,$$

which proves the lemma. □

10.3.7 PROPOSITION *Let* ψ *be a bounded Borel function whose distributional Fourier transform* $\widehat{\psi}$ *is compactly supported. Suppose in addition that the product* $s\widehat{\psi}(s)$ *is a smooth function, and let*

$$C_\psi = \frac{1}{2\pi}\int |s\widehat{\psi}(s)| \, ds.$$

If T_1 *and* T_2 *are essentially selfadjoint operators which share a common invariant domain, and if* $T_1 - T_2$ *is bounded, then*

$$\|\psi(T_1) - \psi(T_2)\| \leqslant C_\psi \cdot \|T_1 - T_2\|.$$

By an *invariant domain* we mean a domain for an unbounded operator T which is mapped into itself by T (so that the powers T^k are all defined on the domain).

PROOF Represent $\psi(T_1)$ and $\psi(T_2)$ by Fourier integrals, following the proof of Proposition 10.3.5. If u is an element in the domain of T_1 and T_2, and if $v \in H$, then

$$\Big\langle \big(\psi(T_1) - \psi(T_2)\big)u, v\Big\rangle = \frac{1}{2\pi}\int \Big\langle \big(e^{isT_1} - e^{isT_2}\big)u, v\Big\rangle \widehat{\psi}(s) \, ds.$$

It follows from the lemma just proved that the inner product in the integral equals s times a smooth function which is pointwise bounded by $\|T_1 - T_2\| \cdot \|u\| \cdot \|v\|$. Hence

$$\Big|\Big\langle \big(\psi(T_1) - \psi(T_2)\big)u, v\Big\rangle\Big| \leqslant \|T_1 - T_2\| \cdot \|u\| \cdot \|v\| \cdot C_\psi$$

as required. □

10.4 Ellipticity

10.4.1 Definition Let M be a smooth manifold and let S a smooth complex vector bundle over M. A differential operator $D\colon C^\infty(M;S) \to C^\infty(M;S)$ is *elliptic* if its symbol $\sigma_D(x,\xi)$ is an invertible endomorphism of S_x, for all non-zero $\xi \in T^*_x M$. If U is an open subset of M then the operator D is *elliptic over* U if the restriction of D to U is elliptic.

It is the task of elliptic analysis to translate the invertibility of the symbol of an elliptic operator into an approximate invertibility of the differential operator itself. In this section we shall review the key steps in this analysis, but in several places we shall make the simplifying assumption that our manifold M is a torus $\mathbb{T}^n$ and that S is a trivial line bundle. To proceed from this special case to the general case is a matter of grafting the proof for the torus onto a general manifold using partitions of unity: the details may be pursued in the references listed at the end of the chapter.

10.4.2 Definition Let S be a smooth vector bundle over M and let K be a compact subset of M. The *Sobolev space* $L^2_1(K;S)$ consists of all elements of $L^2(M;S)$ which are supported within K and which belong to the maximal domain of all (first-order) differential operators acting on the bundle S over M.

In view of the local description of differential operators which is provided by Definition 10.1.1, it is clear that if K is contained within a single coordinate chart, and if the bundle S is trivialized over this chart, then a section u belongs to $L^2_1(K;S)$ if and only if it is supported within K and each distributional partial derivative $\partial u/\partial x_j$ is square-integrable. Taken together, the L^2-norms of these partial derivatives, together with the L^2-norm of u itself, provide $L^2_1(K;S)$ with the structure of a Hilbert space. Different choices of coordinates will produce different norms on $L^2_1(K;S)$, but the underlying topological vector space will be the same. So we can apply to $L^2_1(K;S)$ many of the ordinary notions of operator theory such as bounded operator, compact operator, and so on, although not metric-dependent notions such as isometry or selfadjoint operator. If K is a general compact set then by means of a partition of unity argument we can equip $L^2_1(K;S)$ with a similar 'Hilbertian' structure.

In the case of the torus $\mathbb{T}^n$ and the trivial bundle over it, the Sobolev space may be described very simply in terms of Fourier series. Since $\mathbb{T}^n$ is itself compact, let us take $K = \mathbb{T}^n$. Expand a function $u \in L^2(\mathbb{T}^n)$ as a Fourier series

$$u(z) = \frac{1}{(2\pi)^n} \sum_{k\in\mathbb{Z}^n} \widehat{u}(k) z^k,$$

where $z = (z_1, \ldots, z_n) \in \mathbb{T}^k$, where $z^k = z_1^{k_1} \cdots z_n^{k_n}$, and where

$$\widehat{u}(k) = \int_{\mathbb{T}^n} u(z)z^{-k}\,dz.$$

The Sobolev space $L^2_1(\mathbb{T}^n)$ consists of those L^2-functions for which the weighted sum

$$\|u\|_1^2 = \sum_k (1+|k|^2)|\widehat{u}(k)|^2$$

is finite (here $|k| = |k_1| + \cdots + |k_n|$). This is because partial differentiation in the jth coordinate direction on $\mathbb{T}^n$ corresponds to the operation $\widehat{u}(k) \mapsto k_j\widehat{u}(k)$ on Fourier coefficients (up to a multiplicative factor).

10.4.3 Rellich Lemma *Let* S *be a smooth vector bundle over* M *and let* K *be a compact subset of* M*. The inclusion* $L^2_1(K;S) \to L^2(M;S)$ *is a compact operator.*

Proof Using the 'grafting' technique indicated in the introduction to this section, the proof is reduced to the case of the torus and the inclusion $L^2_1(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$. Here the inclusion is the norm limit of the sequence $\{P_N\}$ of *finite-rank* operators which send a function u to the N'th partial sum $P_N u = (2\pi)^{-n}\sum_{|k|\leqslant N}\widehat{u}(k)z^k$ of its Fourier series. □

It is immediate from the definition of the Sobolev space that every (first order) differential operator D is *bounded* when considered as an operator from $L^2_1(K;S)$ to $L^2(M;S)$. The main result in elliptic analysis is the following partial converse to this statement:

10.4.4 Gårding's Inequality *Let* D *be a first-order differential operator on* M *and let* K *be a compact subset of* M*. If* D *is elliptic over a neighborhood of* K *then there is a constant* $c > 0$ *such that*

$$\|u\| + \|Du\| \geqslant c \cdot \|u\|_1,$$

for all $u \in L^2_1(K;S)$*, where* $\|\ \|$ *denotes the norm in* $L^2(M;S)$ *and* $\|\ \|_1$ *denotes the norm in the Sobolev space* $L^2_1(K;S)$.

Note that in view of the presence of the constant c, the inequality only really involves the equivalence class of the norm $\|\ \|_1$ (which is all that is intrinsic to $L^2_1(K;S)$).

Proof We shall first prove the theorem for a constant-coefficient elliptic operator on a torus. Let $v = Du$. It is sufficient to show that there are constants $\alpha > 0$ and $\beta \geqslant 0$ (independent of u) such that

$$|\widehat{v}(k)| \geqslant \alpha \cdot |k| \cdot |\widehat{u}(k)| - \beta \cdot |\widehat{u}(k)|.$$

Now to prove this inequality, write $D = \sum A_j \frac{\partial}{\partial x_j} + B$, where the A_j and B are constant matrices. For a multi-index k, let $A \cdot k$ denote $\sum_j A_j k_j$. Ellipticity implies that $A \cdot k$ is invertible for $k \neq 0$ and indeed that $|(A \cdot k)| \geqslant \alpha|k|$, for some positive constant α. Now we compute (integrating by parts) that

$$\begin{aligned} \widehat{v}(k) &= \int (A \cdot k)\, u(z)\, z^k \, dz + B \int u(z)\, z^k \, dz. \\ &= (A \cdot k)\widehat{u}(k) + B\widehat{u}(k). \end{aligned}$$

It follows that $|\widehat{v}(k)| \geqslant \alpha|k| \cdot |\widehat{u}(k)| - \beta|\widehat{u}(k)|$, as required.

We shall now prove the result for general operators on the torus which act on sections of a trivial bundle. (Once again, the result for arbitrary D on arbitrary M is obtained by 'grafting' onto the torus, as suggested in the introduction to this section.) Suppose then that D is elliptic over $U \subset \mathbb{T}^n$. By comparing D with the constant-coefficient operator D_x obtained by freezing the coefficients of D at a point $x \in K$, we see that around each $x \in K$ there is a compact neighborhood K_x such that

$$c_x \cdot \|u\|_1 \leqslant \|u\| + \|Du\|,$$

for some $c_x > 0$ and all $u \in L^2_1(K_x; S)$. Take a finite family $\{K_{x_j}\}$ of such neighborhoods which covers K and let $\{\varphi_{x_j}\}$ be a smooth partition of unity which is subordinate to this finite cover. If $u \in L^2_1(K; S)$ then as long as $c \leqslant c_{x_j}$, for all j, we have that

$$\begin{aligned} c\|u\|_1 &= c\|\sum_{j=1}^{N} \varphi_{x_j} u\|_1 \\ &\leqslant \sum_{j=1}^{N} c_j \|\varphi_{x_j} u\|_1 \\ &\leqslant \sum_{j=1}^{N} \|\varphi_{x_j} u\| + \sum_{j=1}^{N} \|D\varphi_{x_j} u\|. \end{aligned}$$

But now of course $\|\varphi_{x_j} u\| \leqslant \|u\|$, while

$$\begin{aligned} \|D\varphi_{x_j} u\| &\leqslant \|\varphi_{x_j} Du\| + \|[D, \varphi_{x_j}]u\| \\ &\leqslant \|Du\| + K_j\|u\|, \end{aligned}$$

where $K_j = \|[D, \varphi_{x_j}]\|$ (remember that the commutator $[D, \varphi_{x_j}]$ is a bounded operator). Putting everything together, if $K \geqslant K_j$ for all j then

$$c\|u\|_1 \leqslant N\Big((K+1)\|u\| + \|Du\|\Big),$$

and adjusting the constant c appropriately we get the required estimate. □

To illustrate Gårding's inequality, let us suppose that M is compact and that D is a symmetric elliptic operator on M. The minimal domain of D (the domain of the closure of D) is then precisely the Sobolev space $L^2_1(M;S)$. Indeed we know from the definition of the Sobolev space that $L^2_1(M;S)$ is contained within the maximal domain of D, and by Lemma 10.2.5 the maximal domain is equal to the minimal domain. But if $\{u_j\}$ is a sequence of smooth sections, and if $\{u_j\}$ and $\{Du_j\}$ are Cauchy sequences in the L^2-norm, so that $\lim u_j$ is an element of the minimal domain, then Gårding's inequality implies that $\{u_j\}$ is Cauchy in the L^2_1-norm, so that $\lim u_j$ is an element of $L^2_1(M;S)$.

It also follows from Gårding's inequality that the resolvent operator $(i+\overline{D})^{-1}$ maps $L^2(M;S)$ continuously into $L^2_1(M;S)$. Indeed, since $\overline{D}$ is selfadjoint, every element u of $L^2(M;S)$ is of the form $(i+\overline{D})v$, for some v in the domain of $\overline{D}$, which is to say, some v in the Sobolev space $L^2_1(M;S)$. But then

$$\|(i+\overline{D})^{-1}u\|_1 = \|v\|_1 \leqslant \tfrac{1}{c}(\|v\| + \|\overline{D}v\|) = \tfrac{1}{c}\|(i+\overline{D})v\| = \tfrac{1}{c}\|u\|,$$

as required.[91] If we combine this estimate with the Rellich Lemma we can conclude that $(i+\overline{D})^{-1}$ is *compact*, when considered as an operator from $L^2(M;S)$ to itself. Since the resolvent function $(i+x)^{-1}$ generates all of $C_0(\mathbb{R})$ we conclude:

10.4.5 PROPOSITION *Let D be a symmetric first-order elliptic operator on a compact manifold M. If $\varphi \in C_0(\mathbb{R})$ the operator $\varphi(D)\colon L^2(M;S) \to L^2(M;S)$ is compact.* □

The information contained in this proposition is often expressed in another way, with which the reader may be more familiar. The set $\{\varphi(D) : \varphi \in C_0(\mathbb{R})\}$ is a commutative C^*-algebra of compact operators on $L^2(M;S)$. According to the spectral theorem for compact operators (Exercise 1.9.7), all the elements of this C^*-algebra can be simultaneously diagonalized; in other words, there exists an orthonormal basis $\{v_j\}$ of $L^2(M;S)$ whose members are eigenvectors for every operator $\varphi(D)$. It follows that each $v_j \in L^2_1(M;S)$ and that

$$\overline{D}v_j = \lambda_j v_j.$$

for some real λ_j. In other words, there is an orthonormal basis for $L^2(M;S)$ consisting of eigenvectors for $\overline{D}$. Furthermore, since the operator e^{-D^2} (say) is compact it follows that $e^{-\lambda_j^2} \to 0$ as $j \to \infty$, and therefore $|\lambda_j| \to \infty$. To summarize:

10.4.6 PROPOSITION *Let D be a symmetric elliptic operator on a compact manifold M. There is a sequence $\{v_j\}$ in the Sobolev space $L^2_1(M;S)$ and a sequence $\{\lambda_j\}$ of real numbers such that*

[91] We could also have proved this using the Closed Graph Theorem.

(a) $\{v_j\}$ *is an orthonormal basis for* $L^2(M;S)$,
(b) $\overline{D}v_j = \lambda_j v_j$, *for all* j, *and*
(c) $\lim_{j\to\infty} |\lambda_j| = \infty$. □

10.4.7 COROLLARY *Let* D *be a symmetric elliptic operator on a compact manifold* M. *Then* $\overline{D}$ *is an unbounded Fredholm operator: its kernel is finite-dimensional and its range is closed.* □

A modest further development of elliptic analysis shows that each eigenvector v_j in Proposition 10.4.6 is actually *smooth*. This is a consequence of the following:

10.4.8 ELLIPTIC REGULARITY PRINCIPLE *If* D *is elliptic over* U, *and if* u *is a distribution such that* Du *is smooth over* u, *then in fact* u *itself is smooth over* U. □

This may be proved by first introducing the 'higher' Sobolev spaces $L^2_k(K;S)$, of L^2-sections all of whose k'th order distributional partial derivatives are L^2-sections, and then extending the proof of Gårding's Inequality to obtain the *basic elliptic estimate*

$$\|u\|_k + \|Du\|_k \geqslant c_k \cdot \|u\|_{k+1}.$$

Notice that the basic elliptic estimate implies, for example, that the eigenvectors of elliptic operators on compact manifolds belong to every Sobolev space $L^2_k(M;S)$. The last main step toward the proof of the elliptic regularity principle is the *Sobolev Embedding Theorem*, which asserts that that $\cap_k L^2_k(K;S)$ is made up entirely of smooth sections of S. This is proved by reducing to the torus, where the result corresponds to the assertion that the smooth functions on a torus are precisely those L^2-functions whose Fourier coefficients tend to zero more rapidly than the reciprocal of any polynomial.

When we develop index theory in the next chapter we shall be computing the Fredholm indices of elliptic differential operators

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix},$$

on compact manifolds, and our functional-analytic approach will lead us to consider not D itself but rather its operator-theoretic closure. Thus we shall actually be computing the quantity

$$\operatorname{Index}(\overline{D}) = \operatorname{Dim}(\operatorname{Kernel}((\overline{D})^+)) - \operatorname{Dim}(\operatorname{Kernel}((\overline{D})^-))$$

Elliptic regularity guarantees that this agrees with the more natural index of D computed from its action on smooth sections only.

10.5 Elliptic Operators on Open Manifolds

We noted in the last section that if D is a symmetric elliptic operator on a compact manifold then $\varphi(D)$ is a compact operator, for every $\varphi \in C_0(\mathbb{R})$. The purpose of this section is to present various versions of this result for operators on non-compact manifolds, beginning with the following simple computation:

10.5.1 PROPOSITION *Let M be a manifold (not necessarily compact), and let D be an essentially selfadjoint differential operator on M. If D is elliptic over an open subset $U \subseteq M$ then for every $\varphi \in C_0(\mathbb{R})$ and every $g \in C_0(U)$ the operator $\rho(g)\varphi(D)\colon L^2(M;S) \to L^2(M;S)$ is compact.*

PROOF It suffices to prove this for $\varphi(x) = (i+x)^{-1}$, which generates $C_0(\mathbb{R})$. Moreover we may assume that g is smooth and supported in some compact set K. If u is an element in the domain of $\overline{D}$, and if g is smooth and compactly supported in $K \subseteq M$, then the compactly supported section $\rho(g)u$ lies in the maximal domain of D, and hence in the Sobolev space $L^2_1(K;S)$. By Gårding's inequality, the product $\rho(g)\varphi(D)$ maps $L^2(M;S)$ continuously into $L^2_1(K;S)$. The proposition now follows from the Rellich Lemma. $\square$

We may express this result in the language of Definition 6.3.2 as follows:

10.5.2 PROPOSITION *Let D be an essentially selfadjoint elliptic differential operator on an open manifold M. If $\varphi \in C_0(\mathbb{R})$ then the operator $\varphi(D)\colon L^2(M;S) \to L^2(M;S)$ is locally compact.* $\square$

This result can be refined in an important way which incorporates more of the coarse-geometric language that we introduced in Chapter 6. We begin by improving a little the analysis of the wave equation that we presented in Section 10.3. We shall assume that the manifold M is equipped with a Riemannian metric and that the operator D has the following property:

10.5.3 DEFINITION Let M be a Riemannian manifold and let D be a differential operator on M. We define the *propagation speed* of D to be the quantity

$$c_D = \sup\{\, \|\sigma_D(x,\xi)\| \,:\, x \in M,\ \xi \in T^*_x M,\ \|\xi\| = 1\,\}.$$

We shall say that D has *finite propagation speed* if $c_D < \infty$.

REMARK We already encountered the quantity c_D in Section 10.2, where we showed that if M is complete, and if $c_D < \infty$, then D is essentially selfadjoint.

10.5.4 PROPOSITION *Let D be a elliptic differential operator on a connected complete Riemannian manifold M, and suppose that D has finite propagation speed. Let C be a closed subset of M and let U be an open neighborhood of C. Suppose that*

$$\inf\{\, d(x,y) \,:\, x \in C,\ y \notin U\,\} > \varepsilon > 0,$$

where $d(x,y)$ *is the distance function*[92] *on* M *associated to the Riemannian metric. If* u *is supported within* C*, and if* $|s| < \varepsilon c_D^{-1}$*, then* $e^{isD}u$ *is supported within* U.

PROOF There is a smooth function g on M such that

$$g(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin U, \end{cases}$$

and such that the gradient of g is everywhere less than $1/\varepsilon$. The operator $\rho(g)$ preserves the maximal, and hence also the minimal, domain of D. Since $\|[D, \rho(g)]\| = \|[\overline{D}, \rho(g)]\| < \varepsilon^{-1}c_D$, the proof of Proposition 10.3.1 applies to the present situation. □

Proposition 10.5.4 provides an important link between differential operator theory and the coarse geometry of Chapter 6. Let M be a connected, complete Riemannian manifold. Equip M with the coarse structure associated to its Riemannian metric and recall from Definition 6.3.5 that an operator $T \in \mathfrak{B}(L^2(M;S))$ is said to be *controlled* for this coarse structure if there is a constant $R > 0$ such that $\rho(f)T\rho(g) = 0$ whenever $\operatorname{Support}(f)$ and $\operatorname{Support}(g)$ contain no points within R of one another.

10.5.5 LEMMA *Let* M *be a connected, complete Riemannian manifold and let* D *be a symmetric, finite propagation speed elliptic operator on* M. *Let* φ *be a bounded Borel function on* $\mathbb{R}$ *and suppose that* $\operatorname{Support}(\widehat{\varphi}) \subseteq (-t,t)$. *If* f *and* g *are bounded functions on* M *such that* $\operatorname{Support}(f)$ *and* $\operatorname{Support}(g)$ *contain no points within* tc_D *of one another, then* $\rho(f)T\rho(g) = 0$. *In particular,* $\varphi(D)$ *is a controlled operator for the metric coarse structure on* M.

PROOF If $\varphi(x) = e^{isx}$ then the lemma follows immediately from Proposition 10.5.4 above. The general case follows from the Fourier transform description of $\varphi(D)$ provided by Proposition 10.3.5. □

Recall from Definition 6.3.8 that the *coarse* C^**-algebra* $C^*(M)$ consists of all those $T \in \mathfrak{B}(L^2(M;S))$ which are norm limits of locally compact and controlled operators.

10.5.6 PROPOSITION *Let* M *be a connected, complete Riemannian manifold and let* D *be a finite propagation speed, symmetric elliptic operator on* M. *If* $\varphi \in C_0(\mathbb{R})$ *then* $\varphi(D) \in C^*(M)$.

PROOF If $\varphi \in C_0(\mathbb{R})$ then Proposition 10.5.2 tells us that $\varphi(D)$ is locally compact. If φ has compactly supported Fourier transform then Lemma 10.5.5

[92]Since we require that M be connected, the Riemannian metric defines a distance function $d(x,y)$ on all of M, rather than a distance function on each path component.

tells us that $\varphi(D)$ is controlled. But the functions $\varphi \in C_0(\mathbb{R})$ which have compactly supported Fourier transform comprise a dense subset of $C_0(\mathbb{R})$, and so $\varphi(D) \in C^*(M)$ for every $\varphi \in C_0(\mathbb{R})$, as required. □

10.6 The Homology Class of a Selfadjoint Operator

The goal of this chapter is to prove that every symmetric elliptic differential operator defines a K-homology class. The purpose of this section is to carry out the construction for the special case of operators on complete manifolds.

As we turn to K-homology we need to incorporate into our discussion of differential operators the notions of grading and multigrading. So we shall assume in this section that D is a symmetric first-order elliptic operator, acting on sections of a p-multigraded smooth vector bundle S over a smooth manifold M. We shall assume that D anticommutes with the grading operator on S and that it commutes with the multigrading operators on S. Thus we shall assume that D is an *odd* and *multigraded* differential operator.

As mentioned above, we shall also assume that M is complete. To be precise, we shall assume that M is complete for D, in the sense of Definition 10.2.8, so that D is essentially selfadjoint. Our goal is to obtain from D a p-multigraded Fredholm module (ρ, H, F) over the C^*-algebra $C_0(M)$, and therefore a homology class $[D] \in K^{-p}(C_0(M))$.

We shall take $H = L^2(M; S)$, and we shall equip H with the natural representation ρ of $C_0(M)$ by multiplication operators. It remains to construct F. Since D is essentially selfadjoint, for each bounded Borel function φ on $\mathbb{R}$ we may form the operator $\varphi(D)$ by applying the functional calculus to the closure of D. The operator F will be obtained in this way, by choosing a suitable function on $\mathbb{R}$:

10.6.1 Definition A smooth function $\chi\colon \mathbb{R} \to [-1, 1]$ is a *normalizing function* (Figure 10.1) if

(a) χ is odd (that is, $\chi(-\lambda) = -\chi(\lambda)$ for all λ),
(b) $\chi(\lambda) > 0$ when $\lambda > 0$, and
(c) $\chi(\lambda) \to \pm 1$ as $\lambda \to \pm\infty$.

The rôle of normalizing functions in the context of K-homology is to convert the unbounded operator D to a bounded one, while preserving the essential index-theoretic features of D. We observe that

(a) any two normalizing functions differ by an element of $C_0(\mathbb{R})$, and
(b) for every $t > 0$ there is a normalizing function χ for which the support of the distributional Fourier transform $\widehat{\chi}$ is contained in $(-t, t)$, and for which $s\widehat{\chi}(s)$ is a smooth function.

See Exercise 10.9.3 for the proof of (b). The operator $\chi(D)$ shares the grading properties of D because of the following simple computation:

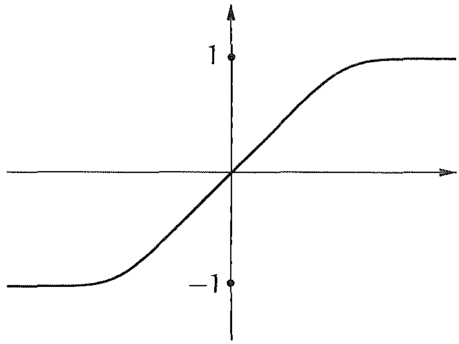

FIG. 10.1. Normalizing function

10.6.2 LEMMA *Let* D *be an unbounded essentially selfadjoint operator on a Hilbert space* H, *and let* T *be a bounded operator which preserves the domain of* D. *Suppose that* $TD = -DT$ *on the domain of* D. *Let* φ *be a bounded Borel function on* $\mathbb{R}$. *Then* T *commutes with* $\varphi(D)$ *if* φ *is even and* T *anticommutes with* $\varphi(D)$ *if* φ *is odd.*

PROOF Let us grade the C^*-algebra $C_0(\mathbb{R})$ by even and odd functions. The computation $T(i \pm D)^{-1} = (i \mp D)^{-1}T$ shows that the operator T graded-commutes with the resolvent operators $(i \pm D)^{-1}$. Since the resolvent functions $(i \pm x)^{-1}$ generate the C^*-algebra $C_0(\mathbb{R})$, it follows that T graded-commutes with $\varphi(D)$, for every $\varphi \in C_0(\mathbb{R})$. The same result for all bounded Borel functions follows by an approximation argument, using the strong topology. □

Rather more substantial are the following results. For both we assume that D is symmetric and that M is complete for D.

10.6.3 LEMMA *Suppose that* D *is elliptic over an open subset* U *of* M. *If* χ_1 *and* χ_2 *are normalizing functions, and if* $g \in C_0(U)$, *then the operators* $\chi_1(D)\rho(g)$ *and* $\chi_2(D)\rho(g)$ *differ by a compact operator.*

PROOF Write

$$\chi_1(D)\rho(g) - \chi_2(D)\rho(g) = \varphi(D)\rho(g) = \big(\rho(\bar{g})\varphi(D)\big)^*,$$

where $\varphi = \chi_1 - \chi_2$. Since $\varphi \in C_0(\mathbb{R})$ it follows from Proposition 10.5.1 that $\rho(\bar{g})\varphi(D)$ is compact, as required. □

10.6.4 LEMMA *Suppose that* D *is elliptic over an open subset* U *of* M. *If* χ *is a normalizing function, and if* $g \in C_0(U)$, *then the commutator* $[\chi(D), \rho(g)]$ *is compact.*

PROOF Let U^+ be the one-point compactification of U. The representation ρ extends to a non-degenerate representation of $C(U^+)$ on $H = L^2(M; S)$. According to Kasparov's Lemma 5.4.7, it suffices to show that if f and g are continuous functions on U^+ with disjoint supports then $\rho(f)\chi(D)\rho(g)$ is a compact operator. One of the functions f or g is compactly supported within U, and without loss of generality we may assume it is g (if not then replace the expression $\rho(f)\chi(D)\rho(g)$ with its adjoint). According to Corollary 10.3.3 the product $\rho(f)e^{isD}\rho(g)$ is then zero for all sufficiently small s. So if the support of $\widehat{\chi}$ is sufficiently small, as it is for a suitable choice of χ, then it follows from Proposition 10.3.5 that $\rho(f)\chi(D)\rho(g) = 0$. It therefore follows from Lemma 10.6.3 that the operator $\rho(f)\chi(D)\rho(g)$ is at least compact for every choice of χ. $\square$

We can now state and prove the main result of this section:

10.6.5 THEOREM *Let D be a symmetric, p-multigraded elliptic differential operator on a smooth manifold M, and suppose that M is complete for D. Let $H = L^2(M;S)$ and let ρ be the representation of $C_0(M)$ on H by multiplication operators. If χ is a normalizing function, and if F is the operator $\chi(D)$, then the triple (ρ, H, F) is a Fredholm module. The K-homology class $[\rho, H, F] \in K^{-p}(C_0(M))$ does not depend on the choice of χ.*

PROOF Clearly F is selfadjoint. Furthermore it is odd and multigraded because of Lemma 10.6.2. It follows from Proposition 10.5.2 that since

$$F^2 - 1 = \chi^2(D) - 1,$$

and since $\chi^2 - 1 \in C_0(\mathbb{R})$, the operator $(F^2 - 1)\rho(g)$ is compact for every $g \in C_0(M)$. By the previous lemma, the commutator $[F, \rho(g)]$ is compact for every $g \in C_0(M)$, and so all the axioms for a Fredholm module have now been verified. Lemma 10.6.3 shows that the Fredholm modules associated to two different normalizing functions are compact perturbations of one another, in the sense of Definition 8.2.3. Hence they determine the same K-homology class, as required. $\square$

10.6.6 DEFINITION Let D be a symmetric elliptic differential operator on a p-multigraded bundle S over M, and suppose that M is complete for D. The homology class $[D] \in K^{-p}(C_0(M))$ is, by definition, the class of the Fredholm module (ρ, H, F), where F is the operator $\chi(D)$ obtained from some normalizing function χ, and ρ and H are as above.

In the remainder of this section (which may be omitted on a first reading) we shall examine the proof of pseudolocality in Lemma 10.6.4 more closely. Suppose that D is a finite propagation speed elliptic operator on a complete Riemannian manifold M. Then the operator $F = \chi(D)$ commutes modulo compact operators not just with all $g \in C_0(M)$ but with rather larger classes of functions. The coarse-geometric language of Chapter 6 provides a convenient way to describe such a class.

10.6.7 DEFINITION Let M be a connected, complete Riemannian manifold. Let us say that a compactification $\overline{M}$ of M is a *coarse compactification* if the topological coarse structure that it induces on M is coarser than the metric coarse structure arising from the Riemannian metric (in the sense that the identity map from the metric coarse structure to the topological coarse structure is a coarse map).

10.6.8 EXAMPLE The compactification of Euclidean n-space by the 'sphere at infinity' is a coarse compactification.

10.6.9 PROPOSITION *Let $\overline{M}$ be a coarse compactification of a connected, complete Riemannian manifold, and let D be a symmetric, finite propagation speed elliptic operator on M. If $g \in C(\overline{M})$ and if χ is any normalizing function then the operator $\chi(D)$ commutes modulo compacts with $\rho(g)$.*

PROOF By Kasparov's Lemma 5.4.7 it suffices to show that if f and g are continuous functions on $\overline{M}$ with disjoint supports then $\rho(f)\chi(D)\rho(g)$ is a compact operator. It follows from the definition of a coarse compactification that

$$\inf\{\, d(x,y) \;:\; x \in \mathrm{Support}(f),\ y \in \mathrm{Support}(g)\,\} > 0.$$

It therefore follows from Proposition 10.5.4 that $\rho(f)\psi(D)\rho(g) = 0$ whenever $\widehat{\psi}$ is supported within a sufficiently small interval about $0 \in \mathbb{R}$. Since every normalizing function χ has the form $\psi+\varphi$, for such a ψ and for some $\varphi \in C_0(\mathbb{R})$, it suffices to show that $\rho(f)\varphi(D)\rho(g) \sim 0$ for every $\varphi \in C_0(\mathbb{R})$. But we have already shown in Proposition 10.5.6 that $\varphi(D)$ belongs to $C^*_{\mathrm{metric}}(M)$, the coarse C^*-algebra associated to the metric on M, and by hypothesis this is a subalgebra of $C^*_{\mathrm{top}}(M)$, the coarse C^*-algebra associated to the compactification $\overline{M}$. Finally, we showed in Theorem 6.5.1 that *every* operator in $C^*_{\mathrm{top}}(M)$ commutes modulo compacts with every continuous function on $\overline{M}$; so $\rho(f)\varphi(D)\rho(g) \sim \rho(f)\rho(g)\varphi(D) = 0$. □

In the circumstances of Proposition 10.6.9, if it should also happen that $\varphi(D)$ is a compact operator for some continuous function φ whose support contains 0 — and in particular if D should be invertible — then we can choose a normalizing function χ so that $\chi(D)^2 - 1$ is compact. Then the operator $F = \chi(D)$ defines a K-homology class for the compactification $\overline{M}$, which restricts to the class $[D]$ of D in $K^{-p}(C_0(M))$ under the map induced by the inclusion $C_0(M) \to C(\overline{M})$. It follows from exactness in the K-homology sequence

$$\cdots \longrightarrow K^{-p}(C(\overline{M})) \longrightarrow K^{-p}(C_0(M)) \xrightarrow{\partial} K^{-p+1}(C(\partial M)) \longrightarrow \cdots$$

that $\partial[D] = 0 \in K^{-p+1}(C(\partial M))$, where $\partial M = \overline{M} \setminus M$. In Chapter 12 we shall see how this argument can be used to produce obstructions to the existence of metrics of positive scalar curvature on some compact aspherical manifolds.

10.7 Elliptic Operators and the Kasparov Product

We are going to define a product operation on elliptic operators which is compatible with the Kasparov product in K-homology. Let D_1 and D_2 be symmetric, graded differential operators on M_1 and M_2, acting on sections of S_1 and S_2, respectively. We define an operator on the graded tensor product bundle[93] $S = S_1 \hat{\otimes} S_2$ over $M = M_1 \times M_2$ by

$$D_1 \times D_2 = D_1 \hat{\otimes} 1 + 1 \hat{\otimes} D_2. \tag{10.7.1}$$

10.7.2 Lemma *Let D_1 and D_2 be symmetric, multigraded elliptic operators. Then the operator $D_1 \times D_2$ is also symmetric, multigraded and elliptic. Moreover, if D_1 and D_2 are elliptic only over open sets U_1 and U_2 then $D_1 \times D_2$ is elliptic over $U_1 \times U_2$.*

Proof We must show that the symbol $\sigma = \sigma_{D_1 \times D_2}$ is invertible when evaluated on each non-zero cotangent vector of $M_1 \times M_2$. Let $x = (x_1, x_2)$ and $\xi = (\xi_1, \xi_2)$. The symbols σ, σ_{D_1}, and σ_{D_2} are related by the identities

$$\sigma = \sigma_{D_1} \hat{\otimes} 1 + 1 \hat{\otimes} \sigma_{D_2}$$

and

$$\sigma^2 = \sigma^2_{D_1} \hat{\otimes} 1 + 1 \hat{\otimes} \sigma^2_{D_2}.$$

If $(\xi_1, \xi_2) \neq 0$ then one of ξ_1 or ξ_2 is non-zero. Thus one of $\sigma^2_{D_1}(x_1, \xi_1)$ or $\sigma^2_{D_2}(x_2, \xi_2)$ is negative definite and the other is negative semidefinite. In either case, σ^2 is negative definite and hence invertible. □

Remark Compare this lemma with Proposition 9.1.8.

Suppose now that D_1 and D_2 are elliptic, and that M_1 and M_2 are complete for D_1 and D_2. Then $M_1 \times M_2$ is complete for $D_1 \times D_2$, and so each of the operators D_1, D_2 and $D_1 \times D_2$ gives rise to a K-homology class. Our objective in this section is to prove the following result:

10.7.3 Theorem *Let D_1 and D_2 be symmetric, multigraded elliptic differential operators on M_1 and M_2, and suppose that M_1 and M_2 are complete for D_1 and D_2. Then:*

$$[D_1 \times D_2] = [D_1] \times [D_2].$$

That is, the K-homology class of $D_1 \times D_2$ is the Kasparov product of the K-homology classes of D_1 and D_2.

The proof follows from a technical computation in operator theory:

[93]If S_1 or S_2 is multigraded, then $S_1 \hat{\otimes} S_2$ is multigraded also. See Definition A.3.3 for our conventions in this regard.

10.7.4 Lemma *Let χ be a normalizing function. Let $F_1 = \chi(D_1)$, let $F_2 = \chi(D_2)$ and let $F = \chi(D_1 \times D_2)$. Then the operators*

$$F(F_1 \hat{\otimes} 1) + (F_1 \hat{\otimes} 1)F \quad \textit{and} \quad F(1 \hat{\otimes} F_2) + (1 \hat{\otimes} F_2)F$$

are positive. Moreover the operator F derives $\mathfrak{K}(H_1) \hat{\otimes} \mathfrak{B}(H_2)$.

In the language of Definition 9.2.2, this lemma shows that the Fredholm module defined by F is *aligned* with F_1 and F_2. By definition of the product (see Definition 9.2.9 and Proposition 9.2.3), it follows that F represents a Kasparov product of F_1 and F_2. Thus Lemma 10.7.4 implies Theorem 10.7.3.

We turn now to the proof of the technical Lemma 10.7.4, which we shall divide into several steps. But let us first introduce the notation

$$T_1 = D_1 \hat{\otimes} 1 \quad \text{and} \quad T_2 = 1 \hat{\otimes} D_2.$$

As we proceed through the following sequence of calculations, the reader should note that we do not need either D_1 or D_2 to be elliptic, only essentially selfadjoint.

10.7.5 Lemma *If φ is any bounded Borel function on $\mathbb{R}$ then*

$$\varphi(T_1) = \varphi(D_1) \hat{\otimes} 1 \quad \textit{and} \quad \varphi(T_2) = 1 \hat{\otimes} \varphi(D_2).$$

Proof An approximation argument shows that if u belongs to the domain of $\overline{D}_1$ and if v is a smooth, compactly supported section on M_2 then the tensor product $u \otimes v$ belongs to the domain of $\overline{T}_1$ and, moreover, $\overline{T}_1(u \otimes v) = \overline{D}_1 u \otimes v$. It follows that

$$(i + \overline{T}_1)^{-1} = (i + \overline{D}_1)^{-1} \hat{\otimes} 1,$$

and since the resolvent function generates $C_0(\mathbb{R})$ as a C^*-algebra it follows that $\varphi(T_1) = \varphi(D_1) \hat{\otimes} 1$ for every $\varphi \in C_0(\mathbb{R})$. The same result for arbitrary bounded φ now follows by taking limits in the strong topology, and the proof of the statement about T_2 is of course similar. □

It follows from the lemma that if φ and ψ are bounded Borel functions then the operators $\varphi(T_1)$ and $\psi(T_2)$ graded-commute with one another, and indeed that $\varphi(T_1)$ graded-commutes with $\overline{T}_2$ (leaving invariant the domain of this unbounded operator), while $\psi(T_2)$ graded-commutes with $\overline{T}_1$.

10.7.6 Lemma *Let $T = T_1 + T_2$. Then*

$$\text{Domain}(\overline{T}) = \text{Domain}(\overline{T}_1) \cap \text{Domain}(\overline{T}_2),$$

and moreover $\overline{T} = \overline{T}_1 + \overline{T}_2$.

Proof The formulas follow from the identity

$$\|T_1 v + T_2 v\|^2 = \|T_1 v\|^2 + \|T_2 v\|^2,$$

which holds for all smooth, compactly supported sections on $M_1 \times M_2$. □

It follows that if φ is any bounded function then $\varphi(T_1)$ and $\varphi(T_2)$ map the domain of $\overline{T}$ into itself.

10.7.7 Lemma *If ψ is a bounded,* even *Borel function, then the operators $\psi(T_1)$ and $\psi(T_2)$ commute with $\overline{T}$.*

Proof This is easily proved directly for $\psi(x) = (1+x^2)^{-1}$ using the identity

$$\frac{1}{1+x^2} = \frac{i}{2}\Big(\frac{1}{i+x} + \frac{1}{i-x}\Big).$$

It then follows by an approximation argument that the lemma holds for arbitrary ψ since $(1+x^2)^{-1}$ generates the C^*-algebra of even functions in $C_0(\mathbb{R})$, and since every bounded even function is a pointwise limit of a bounded sequence of even functions in $C_0(\mathbb{R})$. □

10.7.8 Lemma *If ψ is a bounded,* even *Borel function and φ is any bounded Borel function then the operator $\psi(T)$ commutes with $\varphi(T_1)$ and $\varphi(T_2)$.*

Proof This is first proved directly for $\psi(x) = (1+x^2)^{-1}$ and $\varphi(x) = (i+x)^{-1}$ using the formula presented in the previous proof. The result for general ψ and φ follows by an approximation argument. □

We can now prove the technical lemma:

Proof of Lemma 10.7.4 Let v be a smooth, compactly supported section. If we write $\chi(x) = x\psi(x)$, so that ψ is a positive, even function, then:

$$\begin{aligned}\Big(\chi(T)\chi(T_1) + \chi(T_1)\chi(T)\Big)v &= \Big(\psi(T)\overline{T}\psi(T_1)\overline{T}_1 + \psi(T_1)\overline{T}_1\psi(T)\overline{T}\Big)v \\ &= \psi(T)\psi(T_1)\big(T\,T_1 + T_1T\big)v \\ &= 2\psi(T)\psi(T_1)T_1^2v.\end{aligned}$$

Since $\psi(T)$ and $\psi(T_1)$ are positive and commuting operators, the product operator $\psi(T)\psi(T_1)$ is positive. Moreover

$$\psi(T)\psi(T_1)\overline{T}_1 = \psi(T_1)\overline{T}_1\psi(T) = \overline{T}_1\psi(T_1)\psi(T) = \overline{T}_1\psi(T)\psi(T_1),$$

and so

$$\Big\langle\Big(\chi(T)\chi(T_1) + \chi(T_1)\chi(T)\Big)v, v\Big\rangle = 2\Big\langle\psi(T)\psi(T_1)T_1v, T_1v\Big\rangle,$$

which is positive, as required.

The second positivity assertion is dealt with in a similar way, and so it remains to prove that the operator $F = \chi(T)$ derives the C^*-algebra $\mathfrak{A} =$

$\mathfrak{K}(H_1)\hat{\otimes}\mathfrak{B}(H_2)$. In fact we shall prove that $\chi(T)$ left and right multiplies $\mathfrak{A}$ into itself. Let p_N be the characteristic function of $[-N, N]$ and let

$$P_N = p_N(T_1) = p_N(D_1)\hat{\otimes}1.$$

If $X \in \mathfrak{A}$, then the operators $P_N X$ and XP_N converge in norm to X, as $N \to \infty$. It therefore suffices to show that the operator $P_N\chi(T) = \chi(T)P_N$ belongs to the C^*-algebra $\mathfrak{B} = \mathfrak{B}(H_1)\hat{\otimes}\mathfrak{B}(H_2)$, since $\mathfrak{B}$ certainly left and right multiplies $\mathfrak{A}$ into itself. This is what we shall now do.

The projection operator P_N commutes with $\overline{T}$ and therefore[94]

$$P_N\chi(T) = \chi(P_N T) = \chi(P_N T_1 + P_N T_2).$$

Note that the operator $P_N T_1$ is bounded. The perturbation formula

$$(A+B)^{-1} = A^{-1} - A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} - \dots,$$

(see Exercise 1.9.21), when applied to $A = i + \varepsilon P_N T_2$ and $B = \varepsilon P_N T_1$, shows that if $\varphi(x) = (i+\varepsilon x)^{-1}$, and if ε is so small that the perturbation series converges in norm, then $\varphi(P_N T) \in \mathfrak{B}$. Since $(i+\varepsilon x)^{-1}$ generates $C_0(\mathbb{R})$ it follows that

$$\varphi \in C_0(\mathbb{R}) \quad \Rightarrow \quad \varphi(P_N T) \in \mathfrak{B}. \tag{10.7.9}$$

Now choose a normalizing function ψ for which $s\widehat{\psi}(s)$ is a smooth, compactly supported function. It follows from Proposition 10.3.7 that

$$\lim_{\varepsilon \to 0} \|\psi(\varepsilon P_N T_1 + \varepsilon P_N T_2) - \psi(\varepsilon P_N T_2)\| = 0$$

Since $\psi(\varepsilon P_N T_2) = P_N \psi(\varepsilon T_2) \in \mathfrak{B}$, and since by 10.7.9

$$\psi(\varepsilon P_N T_1 + \varepsilon P_N T_2) - \psi(P_N T_1 + P_N T_2) \in \mathfrak{B},$$

it follows from the above limit that $\psi(P_N T_1 + P_N T_2) \in \mathfrak{B}$. The proof is completed by a second appeal to 10.7.9 since $\psi - \chi \in C_0(\mathbb{R})$. □

10.8 The Homology Class of a Symmetric Operator

In this final section of the chapter we shall remove the hypothesis that M be complete for D, and associate to *every* graded, symmetric elliptic differential operator a homology class $[D] \in K^{-p}(C_0(M))$. We shall prove the natural generalization of the product formula from the last section, and shall also prove a new result, concerning the restriction of elliptic operators and K-homology classes to open subsets of M.

[94]We need to note here and below that $\overline{P_N T} = P_N \overline{T}$, and similarly for T_1 and T_2.

10.8.1 Definition Let S be a smooth, p-multigraded vector bundle over a smooth manifold M and let D be a symmetric, multigraded, elliptic differential operator on S. Let $H = L^2(M; S)$ and let ρ be the representation of $C_0(M)$ on H by pointwise multiplication. We shall say that a p-multigraded Fredholm module (ρ, H, F), defined using the above Hilbert space and representation, is *aligned* with D if, for every open subset $U \subseteq M$, for every essentially selfadjoint differential operator D' which agrees with D on U, and for every $g \in C_0(U)$, we have

$$\rho(g)\Big(F\chi(D') + \chi(D')F\Big)\rho(g)^* \geqslant 0 \text{ modulo } \mathfrak{K}(H),$$

for some (and hence every) normalizing function χ.

It is not apparent that such Fredholm modules exist. But in fact they always do:

10.8.2 Proposition *With data as above,*

(a) *there exist Fredholm modules aligned with* D,
(b) *any two such Fredholm modules are operator homotopic, and*
(c) *if* M *is complete for* D, *then the Fredholm module defined by the operator* $F = \chi(D)$ *of Definition 10.6.6 is aligned with* D.

This allows us to extend Definition 10.6.6 to the non-complete situation:

10.8.3 Definition Let D be a symmetric, multigraded elliptic operator on a p-multigraded bundle S over M. The homology class $[D] \in K^{-p}(C_0(M))$ is the homology class of any Fredholm module which is aligned with D.

The main tool for proving Proposition 10.8.2 is the following computation:

10.8.4 Lemma *Let* D_1 *and* D_2 *be essentially selfadjoint differential operators on* M *and suppose that they restrict to the* same *elliptic operator on some open subset* U *of* M. *Let* χ *be a normalizing function and let* $g \in C_0(U)$. *Then the operators* $\chi(D_1)\rho(g)$ *and* $\chi(D_2)\rho(g)$ *are equal to one another, modulo compact operators.*

Proof It suffices to prove the lemma for smooth functions g with compact support. Furthermore by Proposition 10.5.1, having fixed g it suffices to prove the lemma for any one normalizing function χ. Let us choose χ so that if s belongs to the support of $\widehat{\chi}$ then $e^{isD_1}\rho(g) = e^{isD_2}\rho(g)$ (compare Corollary 10.3.4). It then follows from Proposition 10.3.5 that $\chi(D_1)\rho(g) = \chi(D_2)\rho(g)$. □

Proof of Proposition 10.8.2 We begin by constructing one Fredholm module which is aligned with D. Fix a normalizing function χ. Choose a cover $\{U_j\}$ of M by relatively compact open sets, and let $\{g_j\}$ be a partition of unity subordinate to $\{U_j\}$. Let D_j be an essentially selfadjoint operator which coincides with D on the set U_j (for example, we could obtain D_j by multiplying D on the left and right by a suitable cutoff function, since by Corollary 10.2.6 every compactly supported operator is essentially selfadjoint). Now write

$$F = \sum_j \rho(g_j)^{\frac{1}{2}} \chi(D_j) \rho(g_j)^{\frac{1}{2}}.$$

The partial sums are bounded in norm and the series converges in the strong operator topology. The operator F is selfadjoint. If g is a compactly supported function on M then the product $\rho(g)F$ is represented by a *finite* sum, and it follows from Lemma 10.8.4 that

$$\rho(g)F \sim \rho(g)\chi(D_g), \tag{10.8.5}$$

where D_g is any essentially selfadjoint operator which agrees with D in a neighborhood of the support of g. It follows from this and Lemma 10.6.4 that $\rho(g)$ commutes with F, modulo compact operators. It follows from Proposition 10.5.1 that

$$\rho(g)(F^2 - 1) \sim \rho(g)(\chi(D_g)^2 - 1) \sim 0,$$

and so we have proved that F defines a Fredholm module. Equation 10.8.5 and Lemma 10.8.4 show that this Fredholm module is aligned with D. Moreover, if D itself is essentially selfadjoint then we may take $D = D_g$ in Equation 10.8.5 and we see that F is a locally compact perturbation of the module defined using $\chi(D)$ of Definition 10.6.6. In particular, that module is aligned with D.

To complete the proof, suppose that F_1 defines another Fredholm module which is aligned with D. Then by what we have already shown,

$$\rho(g)(F_1 F + F F_1)\rho(g)^* \sim \rho(g)(F_1 \chi(D_g) - F_1 \chi(D_g))\rho(g)^* \geqslant 0 \quad \text{modulo compacts.}$$

It follows from Proposition 8.3.16 that F and F_1 are operator homotopic. □

10.8.6 REMARK It is apparent from the proof that in order to check that a Fredholm module F is aligned with D, we need not verify the condition of Definition 10.8.1 for *all* choices of open set U and operator D'; it is enough to check it for open sets U covering M and a single choice of D' for each U.

Let us turn now to the generalization of Theorem 10.7.3.

10.8.7 THEOREM *Let D_1 and D_2 be symmetric, multigraded elliptic differential operators on M_1 and M_2. Then:*

$$[D_1 \times D_2] = [D_1] \times [D_2].$$

That is, the K-homology class of $D = D_1 \times D_2$ is the Kasparov product of the K-homology classes of D_1 and D_2.

PROOF The proof is nearly identical to the proof of Theorem 10.7.3. Let F, F_1, and F_2 be operators which are constructed as in the proof of Proposition 10.8.2 so as to be aligned with D, D_1, and D_2. Now the Fredholm module defined by F is aligned with F_1 and F_2 in the extended sense of Remark 9.2.11. Indeed, let $g = g_1 \otimes g_2$ be compactly supported, and let D_g, D_{g_1}, and D_{g_2} be essentially selfadjoint operators which agree with D, D_1, and D_2 in neighborhoods of the supports of g, g_1, and g_2 respectively. According to Equation 10.8.5 from the preceding proof,

$$\rho(g)F \sim \rho(g)\chi(D_g), \quad \rho(g_1)F_1 \sim \rho(g_1)\chi(D_{g_1}), \quad \rho(g_2)F_2 \sim \rho(g_2)\chi(D_{g_2}).$$

Thus $\rho(g)(F(F_1\hat{\otimes}1) + (F_1\hat{\otimes}1)F)\rho(g)^*$ equals

$$P = \rho(g)\Big(\chi(D_g)(\chi(D_{g_1})\hat{\otimes}1) + (\chi(D_{g_1})\hat{\otimes}1)\chi(D_g)\Big)\rho(g)^*$$

modulo $\mathfrak{K}(H_1) \otimes \mathfrak{B}(H_2)$ (note that we need to use here the result, part of the proof of Lemma 10.7.4, that $\chi(D_g)$ left and right multiplies $\mathfrak{K}(H_1) \otimes \mathfrak{B}(H_2)$ into itself). The operator P is positive, by Lemma 10.7.4, and so

$$\rho(g)\Big(F(F_1\hat{\otimes}1) + (F_1\hat{\otimes}1)F\Big)\rho(g)^* \geqslant 0 \quad \text{modulo } \mathfrak{K}(H_1) \otimes \mathfrak{B}(H_2).$$

By a similar argument

$$\rho(g)\Big(F(1\hat{\otimes}F_2) + (1\hat{\otimes}F_2)F\Big)\rho(g)^* \geqslant 0 \quad \text{modulo } \mathfrak{B}(H_1) \otimes \mathfrak{K}(H_2).$$

Finally, it follows from Lemma 10.7.4 that $\chi(D_g)$ derives $\mathfrak{K}(H_1) \otimes \mathfrak{B}(H_2)$, and therefore so does $F\rho(g) \sim \chi(D_g)\rho(g)$. This completes the proof. □

To conclude this section we consider the behavior of our homology classes under the operation of 'restriction' to an open subset.

10.8.8 PROPOSITION *Let M be a smooth manifold and let D_M be a symmetric, multigraded elliptic operator on M. Let U be an open subset of M and let D_U be the restriction of D_M to U. Then $[D_U] = j^*[D_M]$, where j^* is the map on K-homology induced by the inclusion of C^*-algebras $j\colon C_0(U) \to C_0(M)$.*

PROOF Let F be an operator on $H = L^2(M; S)$ which defines a Fredholm module aligned with D_M. Let P denote the orthogonal projection $L^2(M; S) \to L^2(U; S)$. Then PFP is an operator on $L^2(U; S)$ and defines a Fredholm module over $C_0(U)$, which is clearly aligned with D_U. On the other hand, since $\rho(f)(I - P) = 0$ for $f \in C_0(U)$, the restriction of $[F]$ to a module over $C_0(U)$ is equal to $[PFP]$ plus a degenerate. Thus

$$j^*[D_M] = j^*[F] = [PFP] = [D_U],$$

as required. □

10.9 Exercises

10.9.1 Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth, positive function with compact support and with total mass 1. Define an operator F_t on $L^2(\mathbb{R}^n)$ by

$$F_t u(x) = t^{-n} \int_{\mathbb{R}^n} u(y)\varphi\left(\frac{x-y}{t}\right) dy.$$

Show that $\{F_t\}$ is a family of Friedrichs' mollifiers on $L^2(\mathbb{R}^n)$. Using local coordinates and partitions of unity, graft this family onto an arbitrary manifold M to construct a family of Friedrichs' mollifiers there.

10.9.2 Prove the elliptic regularity principle for constant coefficient elliptic operators on the torus.

10.9.3 Let $g(x)$ be a smooth, even, real-valued function of compact support on $\mathbb{R}$, and let f be the convolution $g \star g$. Suppose that $f(0) = \frac{1}{\pi}$. Let

$$\chi(t) = \int_{-\infty}^{\infty} \frac{e^{itx} - 1}{ix} f(x)\, dx.$$

Prove that χ is a normalizing function whose distributional Fourier transform has compact support. Prove also that $s\widehat{\chi}(s)$ is a smooth function.

10.9.4 In this exercise we will outline an argument of Chernoff [40], which provides an alternative proof that a symmetric differential operator on a complete manifold is essentially selfadjoint (Proposition 10.2.10). The argument also proves that all the powers D^k of D are essentially selfadjoint; we will need this fact (for $k = 2$) in Chapter 11.

(a) Let T be an unbounded, densely defined, symmetric operator on a Hilbert space H. Show that if T is *not* essentially selfadjoint, then there exists a nonzero vector v in the domain of T^{max} such that $T^{max}v = \pm iv$ (for some choice of sign).

(b) Let D be an unbounded, symmetric operator on H whose dense domain $\mathfrak{D}$ is invariant (that is, $D(\mathfrak{D}) \subseteq \mathfrak{D}$), so that the powers D^k are all defined on $\mathfrak{D}$. Suppose that there exists a one-parameter group $t \mapsto U_t$ of unitary operators on H such that

  (i) the operators U_t map $\mathfrak{D}$ into itself, and

  (ii) for every $x \in \mathfrak{D}$, the map from $\mathbb{R}$ to H defined by $t \mapsto x_t = U_t x$ is differentiable, and $\dot{x}_t = iDx_t$.

Show that then D and all its powers are essentially selfadjoint. (Hint: suppose that D^k is not essentially selfadjoint, and let v satisfy $(D^k)^{max}v = \pm iv$. Consider the differential equation satisfied by the inner product $\langle v, x_t\rangle$, for $x \in \mathfrak{D}$.)

(c) Apply (b) to a symmetric first-order differential operator D on a bundle over a manifold complete for D, taking $\mathfrak{D}$ to be the space of compactly supported smooth sections and using Proposition 10.3.1 to prove (i) above. Deduce that all the powers D^k are essentially selfadjoint.

10.9.5 Let D_1 and D_2 be symmetric, first-order elliptic operators acting on the same smooth vector bundle S over a smooth closed manifold M.

(a) Show that D_1 and D_2 have the same symbol if and only if $D_1 - D_2$ is an endomorphism of S (and therefore a bounded operator).
(b) Use Proposition 10.3.7 to show that if D_1 and D_2 have the same symbol then $\lim_{\varepsilon\to 0}\|\chi(\varepsilon D_1) - \chi(\varepsilon D_2)\| = 0$, for a suitably chosen normalizing function χ.
(c) Use Lemma 10.6.3 to deduce that if D_1 and D_2 have the same symbol, and if χ is *any* normalizing function, then $\chi(D_1) \sim \chi(D_2)$. Hence D_1 and D_2 determine the same class in K-homology.
(d) Generalize the above computations and use the results of Section 10.8 to show that if D_1 and D_2 are symmetric, first-order elliptic operators acting on the same smooth vector bundle S over *any* smooth manifold M, and if $\sigma_{D_1} = \sigma_{D_2}$, then D_1 and D_2 determine the same class in K-homology.

10.9.6 Let D be a symmetric elliptic operator on a graded bundle S over a compact manifold M. Write

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} : C^\infty(M;S) \to C^\infty(M;S).$$

Prove that the image of the homology class $[D] \in K^0(C(M))$ under the map $K^0(C(M)) \to K^0(\mathbb{C}) = \mathbb{Z}$ induced by crushing M to a point is the integer

$$\operatorname{Index}(D) = \operatorname{Dim}(\operatorname{Kernel}(D^+)) - \operatorname{Dim}(\operatorname{Kernel}(D^-)).$$

10.9.7 Let D be the symmetric differential operator

$$D = \begin{pmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix}$$

acting on the manifold $\mathbb{R}$, which we consider as 1-multigraded by the operator $\varepsilon_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Prove that the K-homology class of the restriction of D to the interval $(-1,1) \subseteq \mathbb{R}$ is the Dirac class $d \in K^{-1}(C_0(-1,1))$ described in Definition 9.5.1. (Hint: embed $(-1,1)$ in a circle via the map $x \mapsto e^{i\pi x}$, and obtain the K-homology class $[d]$ by restricting to $(-1,1)$ a suitable differential operator on the circle.)

The next several exercises adapt and refine several of the results presented in this chapter to compact manifolds with boundary.

10.9.8 This exercise makes a preliminary calculation about unbounded operators. Let $T\colon H_0 \to H_1$ be a closed, densely defined, unbounded Hilbert space operator.

(a) Prove that the unbounded operator
$$\hat{T} = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$$
on $H = H_0 \oplus H_1$ is selfadjoint (its domain is by definition the direct sum of the domains of T and T^*).

(b) Show that the operators T^*T and TT^* are selfadjoint

(c) Define a bounded Borel function $f\colon \mathbb{R} \to \mathbb{R}$ as follows:
$$f(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$
Let $\hat{U} = f(\hat{T})$ and observe that $\hat{U}\hat{T} = \hat{T}\hat{U}$. Show that $\hat{U}$ maps the orthogonal complement of the kernel of T in H_0 isometrically onto the orthogonal complement of the kernel of T^* in H_1.

(d) Show that on the orthogonal complements of their kernels the operators T^*T and TT^* are unitarily equivalent.

10.9.9 Definition Let $\overline{M}$ be a compact manifold with boundary and let D be an elliptic operator on the interior, M. We shall say that D is *elliptic over* $\overline{M}$ if D is the restriction to M of an elliptic operator on the open manifold obtained from $\overline{M}$ by adjoining an open collar at the boundary.

10.9.10 Let $\widehat{D}$ be a selfadjoint extension of a differential operator on M which is elliptic over $\overline{M}$.

(a) Use the Rellich Lemma to prove that if $g \in C_0(M)$, and if $\varphi \in C_0(\mathbb{R})$, then the operator $\varphi(\widehat{D})\rho(g)$ is compact.

(b) Verify that Proposition 10.3.1 carries over without change to the wave operators $e^{is\widehat{D}}$, and then use the argument in the proof of Lemma 10.6.4 to prove that if $g \in C_0(M)$, and if χ is any normalizing function, then the commutator $[\chi(\widehat{D}), \rho(g)]$ is compact.

(c) Prove that the operator $\chi(\widehat{D})$ defines a class $[\widehat{D}] \in K^{-p}(C_0(M))$.

(d) Prove that $[\widehat{D}]$ is the same as the K-homology class $[D]$ which is constructed in Section 10.8.

10.9.11 DEFINITION Let $\overline{M}$ be a compact manifold with boundary and let D be a differential operator on the interior, M. We shall say that a selfadjoint extension $\widehat{D}$ of D is *local* if the domain of $\widehat{D}$ is invariant under multiplication by smooth functions on $\overline{M}$.

10.9.12 Let D be a symmetric differential operator on M and suppose that D is odd with respect to a grading of the bundle on which it acts:

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}.$$

Define an extension of D by

$$\widehat{D} = \begin{pmatrix} 0 & D_-^{\min} \\ D_+^{\max} & 0 \end{pmatrix},$$

where the decorations indicate that D_- is extended to its minimal domain (the domain of its closure) and D_+ is extended to its maximal domain (all u such that $D_+u \in L^2$, in the sense of distributions).

(a) Prove that $\widehat{D}$ is local, selfadjoint extension of the unbounded operator D.

(b) Prove that if D is elliptic over $\overline{M}$ then the odd-graded part of the domain of $\widehat{D}$ (on which $D_-^{\min}$ acts) is the Sobolev space $L^2(\overline{M}; S)$.

(c) Deduce from the Rellich Lemma that if D is elliptic over $\overline{M}$, and if $\varphi \in C_0(\mathbb{R})$, then the restriction of $\varphi(\widehat{D})$ to the orthogonal complement of the kernel of $\widehat{D}$ is a compact operator. (Use Exercise 10.9.8.)

10.9.13 Let $\widehat{D}$ be a *local* selfadjoint extension of a differential operator on M which is elliptic over $\overline{M}$.

(a) Prove the following extension of Proposition 10.3.1: if K is a compact subset of $\overline{M}$, and if W is an open neighborhood of K in $\overline{M}$, then there exists $\varepsilon > 0$ such that if u is supported within K, and if $|s| < \varepsilon$, then $e^{is\widehat{D}}u$ is supported within W.

Suppose, moreover, that if $\varphi \in C_0(\mathbb{R})$ then the restriction of $\varphi(\widehat{D})$ to the orthogonal complement of the kernel of $\widehat{D}$ is a compact operator.

(b) Prove that if χ is any normalizing function, and if $g \in C(\overline{M})$, then the commutator $[\chi(\widehat{D}), \rho(g)]$ is compact. (Hint: observe that our hypothesis concerning $\varphi(\widehat{D})$ for $\varphi \in C_0(\mathbb{R})$ implies that if $\varphi(0) = 0$ then $\varphi(\widehat{D})$ is compact. With this in hand the compactness argument presented in the text can be easily adapted to the present situation.)

(c) Conclude that if χ is any normalizing function then the operator $\chi(\widehat{D})$ defines a *relative* K-homology cycle for the pair $(\overline{M}, \partial M)$.

10.9.14 Use the previous exercises and the explicit formula for the boundary map $\partial\colon K^0(J) \to K^1(A/J)$ provided by Proposition 8.5.6 to obtain the following result. Let $\overline{M}$ be a compact manifold with boundary and let D be an odd, symmetric operator on M which is elliptic over $\overline{M}$. Let Q_D be the orthogonal projection onto the space of all $u \in L^2(M; S_+)$ such that $D_+u = 0$, in the sense of distributions. Then for every $g \in C(\overline{M})$ the commutator $[Q_D, \rho(g)]$ is compact, whereas if $g = 0$ on ∂M then $Q_D\rho(g) \sim 0$. The projection Q_D therefore determines an extension

$$0 \longrightarrow \mathfrak{K}(Q_D L^2(M;S)) \longrightarrow E \longrightarrow C(\partial M) \longrightarrow 0\,.$$

Its class in $K^1(C(\partial M))$ is the image of the class $[D] \in K^0(C_0(M))$ under the boundary map in K-homology.

In the following series of exercises we develop an alternative procedure for obtaining a K*-homology class from an elliptic operator. This procedure relies on Cauchy's integral formula, rather than on the Fourier transformation that we used in the main text.*

10.9.15 Let A be a C^*-algebra. An *unbounded Fredholm module* for A is given by the following data:

(a) a graded Hilbert space H,

(b) a degree zero representation $\rho\colon A \to \mathfrak{B}(H)$, and

(c) an odd, selfadjoint, unbounded operator D on H such that:

   (i) for all $a \in A$, and for all $w \in \mathbb{C} \setminus \mathbb{R}$, the operator $(D - w)^{-1}\rho(a)$ is compact, and

   (ii) for a dense set of $a \in A$, the commutator $[D, \rho(a)]$ is densely defined and extends to a bounded operator on H.

Show that an essentially selfadjoint first-order elliptic operator on a manifold M gives rise to an unbounded Fredholm module over $A = C_0(M)$.

10.9.16 Let (ρ, D, H) be an unbounded Fredholm module. Form the bounded operator $F = D(1 + D^2)^{-\frac{1}{2}}$ (note that $F = \chi(D)$ where χ is the normalizing function $x \mapsto x(1 + x^2)^{-\frac{1}{2}}$). Show that $(F^2 - 1)\rho(a)$ is compact for every $a \in A$. In fact, show that $\varphi(D)\rho(a)$ is compact for every $a \in A$ and every $\varphi \in C_0(\mathbb{R})$ (use the Stone–Weierstrass Theorem).

10.9.17 Continue with the notation of the previous exercise. Let $(1 + w^2)^{-\frac{1}{2}}$ be defined as a holomorphic function in the domain Ω which consists of the complex plane $\mathbb{C}$ with the two half-lines $[i, i\infty)$ and $(-i\infty, -i]$ removed; we choose the

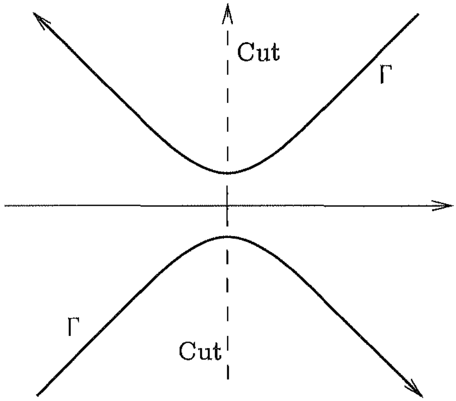

FIG. 10.2. The domain Ω and the contour Γ

branch which is equal to $+1$ for $w = 0$. Let Γ be the contour in Ω shown in the Figure 10.2. Show that

$$(1 + D^2)^{-\frac{1}{2}} = \frac{-1}{2\pi i} \int_\Gamma (1 + w^2)^{-\frac{1}{2}} (D - w)^{-1} \, dw,$$

where the integral converges in norm. Suppose that a belongs to the dense subspace of A such that $[D, \rho(a)]$ is a bounded operator. Show that

$$[F, \rho(a)] = \frac{1}{2\pi i} \int_\Gamma w(1 + w^2)^{-\frac{1}{2}} (D - w)^{-1} [D, \rho(a)] (D - w)^{-1} \, dw,$$

where the integral converges in norm. Deduce that $[F, \rho(a)]\rho(a')$ is compact for all $a, a' \in A$. Using the fact that the products aa' span a dense subspace of A (a consequence of the functional calculus), deduce finally that $[F, \rho(a)]$ is compact for all $a \in A$. Thus (ρ, H, F) is a Fredholm module over A.

10.9.18 Let $\rho \colon J \to \mathfrak{B}(H)$ be a representation of a (non-unital) C^*-algebra J. Let us say that an operator $T \in \mathfrak{B}(H)$ is *carried by* J if there is an approximate unit u_n for J such that the sequences $T\rho(u_n)$ and $\rho(u_n)T$ converge in norm to T. By extending the argument of the previous exercise, prove the following theorem: let J be a non-unital C^*-algebra and let (ρ, D, H) be an unbounded Fredholm module over J. Suppose that J is an ideal in a C^*-algebra A, and that the representation ρ extends to a representation of A. Suppose that, for a dense set of $a \in A$, the commutator $[D, \rho(a)]$ is densely defined and bounded and is carried by J. Then $[F, \rho(a)]$ is compact for all $a \in A$.

10.9.19 Define an appropriate notion of *unbounded relative Fredholm module*, and extend the preceding results to the relative case.

10.9.20 Let D_1 and D_2 be two elliptic operators (on the same bundle S) which have the same symbol. We assume that M is complete for D_1 (and so for D_2), so that both operators are essentially selfadjoint. Use the integral formulas of Exercise 10.9.17 to show that $F_1 = D_1(1 + D_1^2)^{-\frac{1}{2}}$ is a locally compact perturbation of $F_2 = D_2(1 + D_2^2)^{-\frac{1}{2}}$. Deduce that D_1 and D_2 define the same K-homology class (compare Exercise 10.9.5).

10.10 Notes

There are several books in which the reader may find a more detailed development of the analysis of elliptic operators. One which adopts a similar approach to that taken in this chapter is [114]; other possible sources are [26] and [94]. All these books concentrate on the theory of those operators which arise naturally in geometry, called Dirac operators; we shall focus our attention on these in the next chapter. For the distribution theory and Fourier analysis used in this chapter, see [117].

For the idea of combining finite propagation speed and Fourier analysis to obtain estimates for operator functions $\varphi(D)$, see [39]. The application to coarse geometry was made in [110].

The topological significance of Theorem 10.6.5 was noted by Atiyah [11]. Atiyah's approach, however, was based on the theory of elliptic *pseudodifferential* operators, which we have avoided. In fact one can show that the operator $\chi(D)$ is pseudodifferential (at least for a dense set of normalizing functions) and Theorem 10.6.5 then follows from the symbol calculus for pseudodifferential operators. See [126] for the relevant results on functional calculus for differential and pseudodifferential operators. For the product of zeroth-order elliptic pseudodifferential operators see Section 5 of [16]; the devices employed there to work with pseudodifferential (rather than differential) operators can be viewed as precursors to the Kasparov Technical Theorem.

The discussion of K-homology classes for operators on non-complete manifolds is adapted from the unpublished paper [67]. The sequence of exercises on compact manifolds with boundary reproduces work of Baum, Douglas and Taylor [24, 25]. The study of unbounded Fredholm modules was initiated by Baaj and Julg [21].

11

INDEX THEORY

The Atiyah–Singer Index Theorem provides a 'topological' formula for the Fredholm indices of the elliptic operators that we studied in the previous chapter. In this chapter we shall present an introductory treatment of the Index Theorem, including a complete proof of an illustrative and quite useful special case. At least as far as analysis is concerned, our treatment adapts quite easily to more general cases (we shall indicate how).

Our principal aim is to show how the product structures on K-theory and K-homology lead very quickly to the formulation and proof of the Index Theorem, and in keeping with the subject of this book we shall emphasize the analytic aspects of the argument. A separate and very important issue in index theory is to reinterpret the K-theory topological index defined in this chapter in terms of ordinary cohomology, using characteristic classes or other means. Unfortunately a complete treatment of the topological index would take us too far afield and must therefore be omitted.

The chapter ends with an application of the Atiyah–Singer Index Theorem which is very relevant to the theory of C^*-algebra extensions: we shall prove the Toeplitz Index Theorem that we stated in connection with extension theory in Chapter 2. We shall require for the proof some fairly substantial background material in complex analysis, and for this reason our presentation will not be entirely self-contained. But we hope to make the structure of the proof as plain as possible, and thereby convince the reader that the Toeplitz Index Theorem fits very naturally into K-homology theory.

Thus far in the book we have treated K-homology as a contravariant functor on the category of (separable) C^*-algebras. As we now venture toward applications in geometry and topology it will be convenient to shift perspective a little and think of K-homology as a *covariant* functor on the category of locally compact spaces. We therefore introduce the notation

$$K_p(M) = K^{-p}(C_0(M))$$

for the (analytic) K-homology of a locally compact and second countable space M, along with the companion notation

$$K^p(M) = K_{-p}(C_0(M))$$

for its K-theory.

11.1 Dirac Operators

In this section we shall present a very rapid introduction to Dirac operators, which are the most important and most frequently encountered elliptic operators in index theory.

Let M be a Riemannian manifold. The Riemannian metric assigns to each tangent space $T_x M$ an inner product, to which there correspond natural inner products on $T_x^* M$, on the exterior powers $\bigwedge_x^p M$, and so forth.

11.1.1 DEFINITION Let M be a Riemannian manifold and let S be a smooth, graded Hermitian vector bundle over M. A *Dirac operator* on S is an odd, symmetric differential operator on S whose symbol $\sigma(x,\xi)$ has the property that

$$\sigma(x,\xi)^2 u = -\|\xi\|^2 u,$$

for every $x \in M$, every cotangent vector ξ at x, and every $u \in S_x$.

If $\sigma(x,\xi)$ is the symbol of a Dirac operator D then it follows from the symmetry and grading of D that $\sigma(x,\xi)$ is a skew-adjoint, odd endomorphism of S_x, for every x and every ξ. The condition $\sigma(x,\xi)^2 u = -\|\xi\|^2 u$ certainly implies that $\sigma(x,\xi)$ is an invertible endomorphism of S_x whenever $\xi \neq 0$. So every Dirac operator is elliptic.

In some cases, notably those which will be considered in the next section, it is convenient to start not with the operator D but with its symbol $\sigma(x,\xi)$. With this is mind we present a variation of the above definition:

11.1.2 DEFINITION Let M be a Riemannian manifold. A *Dirac bundle* on M is a graded Hermitian vector bundle S over M together with an $\mathbb{R}$-linear morphism of vector bundles

$$T^*M \to \mathrm{End}(S)$$

which associates to each cotangent vector $\xi \in T_x^* M$ a skew-adjoint, odd endomorphism $u \mapsto \xi \cdot u$ of S_x whose square is multiplication by the scalar $-\|\xi\|^2$ on S_x.

The action of a vector $\xi \in T_x^* M$ on S_x will be referred to as *Clifford multiplication* by ξ. This is a reference to the theory of Clifford algebras, for which the relation $\xi^2 = -\|\xi\|^2 1$ is fundamental. We shall say more about Clifford algebras in the next section.

It should be clear that the structure provided by Definition 11.1.2 is nothing more than the symbol of a Dirac operator on S. Thus there is a one-to-one correspondence between Dirac bundle structures on a graded Hermitian bundle S and equivalence classes of Dirac operators on S, two operators being equivalent for this purpose if they have the same symbol.

The following examples show that various important operators from geometry are in fact Dirac operators.

11.1.3 EXAMPLE Recall that on any smooth n-manifold M one has the *de Rham complex*

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M)$$

where $\Omega^p(M)$ is the space of smooth, complex-valued p-forms on M, and d is the exterior derivative. Now suppose that M is Riemannian and let S be the complex exterior algebra bundle $\bigwedge_{\mathbb{C}}^* M$, whose sections are complex-valued differential forms on M. We grade S by dividing the differential forms on M into those of even and odd degree. Let d be the exterior derivative, considered as a first-order partial differential operator on S, and let d^* be its formal adjoint. Then the *de Rham operator* $D = d + d^*$ is a Dirac operator on S. To see this, we use the Leibniz formula

$$d(f\omega) - f d\omega = df \wedge \omega$$

and Equation 10.2.7 to conclude that the symbol of d is the operator E_ξ of exterior multiplication by ξ. Since the symbol of d^* is minus the adjoint of the symbol of d, the symbol of $d + d^*$ is

$$\sigma_D(x, \xi) = E_\xi - E_\xi{}^*.$$

But an algebraic calculation shows that $(E_\xi - E_\xi{}^*)^2 = -\|\xi\|^2 \cdot 1$, and so D is indeed a Dirac operator. In Exercise 11.8.1 the reader is asked to show that the Fredholm index of D, which is defined whenever M is compact, is the Euler characteristic of M.

Let us use the Clifford multiplication notation $\omega \mapsto \xi \cdot \omega$ for the action of the operator $E_\xi - E_\xi{}^*$ on $\bigwedge_x^* M$.

11.1.4 EXAMPLE Suppose now that M is oriented and that $\mathrm{Dim}(M) = 4k$. Let $e_1, \ldots, e_{4k}$ be an oriented local orthonormal frame for T^*M. The operator γ on differential forms defined by the iterated Clifford product

$$\gamma \cdot \omega = (-1)^k e_1 \cdots e_{4k} \cdot \omega$$

is independent of the choice of oriented orthonormal frame. Furthermore, $\gamma^2 = 1$ and $\gamma = \gamma^*$ (see Exercise 11.8.2), so γ defines a grading on $S = \bigwedge_{\mathbb{C}}^* M$. The operator $d + d^*$ anticommutes with γ. Therefore, by the computation in the previous example, $d + d^*$ is a Dirac operator on the γ-graded bundle S. It is called the *signature operator* and its index is the cohomological signature of M. See Exercise 11.8.3.

Note that although the signature operator and the de Rham operator are the same differential operator, the difference in the gradings of the underlying bundle has a very significant effect on index theory — the Fredholm indices of the two operators are not at all the same.

11.1.5 Example Let M be a complex manifold of dimension m. If $z_1, \ldots, z_m$ are complex local coordinates on M then the exterior algebra of complex-valued differential forms on M is generated by the differentials dz_j and $d\bar{z}_j$. The space $\Omega^k(M)$ of complex-valued k-forms on M decomposes as a direct sum of the spaces $\Omega^{p,q}(M)$ that are generated by the elementary forms

$$dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q},$$

where $p + q = k$. These summands do not depend on any choice of coordinates; they correspond to a direct sum decomposition

$$\textstyle\bigwedge_{\mathbb{C}}^* M = \bigoplus_{0 \leqslant p,q \leqslant m} \bigwedge^{p,q} M$$

of the complex exterior algebra bundle of M. The de Rham differential of a form of type (p, q) is a sum

$$d\omega = \partial\omega + \bar{\partial}\omega$$

of forms of types $(p + 1, q)$ and $(p, q + 1)$, respectively. Both ∂ and $\bar{\partial}$ are differentials (that is, $\partial^2 = 0$ and $\bar{\partial}^2 = 0$), and so from the de Rham complex we obtain several new complexes, of which the most important for us is the *Dolbeault complex*

$$\Omega^{0,0}(M) \xrightarrow{\bar{\partial}} \Omega^{0,1}(M) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{0,m}(M). \tag{11.1.6}$$

Since M is a complex manifold, its tangent bundle is equipped with a complex vector bundle structure. A *Hermitian metric* on M is a smooth Hermitian structure on this complex bundle. If we decompose a Hermitian metric h into its real and imaginary parts, say

$$h = g + \sqrt{-1}a,$$

then g is a Riemannian metric on the real manifold underlying M (and a is a real-valued 2-form, about which we shall say more later). Using the Riemannian metric we may form the adjoint of the differential $\bar{\partial}$ and so construct from the Dolbeault complex an operator

$$\mathcal{D} = \bar{\partial} + \bar{\partial}^* \colon \Omega^{0,*}(M) \to \Omega^{0,*}(M)$$

in the same way that the de Rham operator is constructed from the de Rham complex[95]. This is the *Dolbeault operator* and it too is a Dirac operator. Indeed, if $x \in M$ and if f is a smooth function defined near x then

[95] As with the de Rham operator, the grading on the Dirac bundle for the Dolbeault operator $\mathcal{D}$ is given by the degree, modulo 2, of differential forms.

$$\sigma_{\mathcal{D}}(x, df) = E_{\bar{\partial} f} - E_{\bar{\partial} f}{}^*,$$

where, as in Example 11.1.3, E_ξ is the operator of exterior multiplication by ξ (although now we are considering complexified cotangent vectors). If f is real-valued then $\|df\|^2 = \|\bar{\partial} f\|^2$, and in view of this the same algebra that we cited in Example 11.1.3 proves that $\mathcal{D}$ is a Dirac operator. If M is compact then the index of the Dolbeault operator is the Euler characteristic of the Dolbeault complex. For example, if M is a Riemann surface of genus g then $\operatorname{Index}(\mathcal{D}) = 1 - g$.

Since they are elliptic, the Dirac operators on a Riemannian manifold M determine classes in the K-homology group $K_0(M)$. The computations we carried out in the previous chapter give us two important properties of these Dirac operator classes. To state the first, we remind the reader that associated to the inclusion of an open set U into a locally compact space X there is a 'wrong way' morphism in the category of locally compact spaces, from X to U. It corresponds to the inclusion of $C_0(U)$ into $C_0(X)$, or equivalently to the map from the one-point compactification $\widetilde{X}$ to $\widetilde{U}$ which collapses $X \setminus U$ to the point at infinity in $\widetilde{U}$. We shall denote the induced map on K-homology by

$$\iota^! \colon K_p(X) \to K_p(U)$$

and use the term *restriction map* for it. The exclamation point is to remind the reader that the map is 'going the wrong way'.

11.1.7 PROPOSITION *Let D_M be a Dirac operator on a Riemannian manifold M and let $\iota \colon U \to M$ be the inclusion of an open subset into M. If D_U is the restriction of D_M to U, then $\iota^![D_M] = [D_U]$ in $K_0(U)$.*

PROOF This is just a restatement of Proposition 10.8.8 for Dirac operators. □

Suppose now that D_1 and D_2 are Dirac operators on Riemannian manifolds M_1 and M_2. As in the previous chapter we may form the product

$$D_1 \times D_2 = D_1 \hat{\otimes} 1 + 1 \hat{\otimes} D_2,$$

which is a Dirac operator on $M_1 \times M_2$. From Theorem 10.8.7 we get:

11.1.8 PROPOSITION *Let D_1 and D_2 be Dirac operators on Riemannian manifolds M_1 and M_2. Then*

$$[D_1 \times D_2] = [D_1] \times [D_2] \in K_0(M_1 \times M_2). \qquad \square$$

We conclude this introductory section with a brief mention of connections and curvature, and their relation to Dirac operators.

11.1.9 DEFINITION Let S be a Dirac bundle on M. A *Dirac connection* on S is a complex-linear, affine connection ∇ on S, compatible with the grading of S, such that

(a) if X is any tangent vector field and u, v are sections of S then

$$X \cdot (u, v) = (\nabla_X u, v) + (u, \nabla_X v)$$

(the round brackets denote pointwise inner product, so that (u, v) is a smooth function on M), and

(b) if X is any tangent vector field, if e is any covector field, and u is any section of S then

$$\nabla_X(e \cdot u) = \nabla_X^{LC}(e) \cdot u + e \cdot \nabla_X(u),$$

where ∇^{LC} is the Levi–Civita connection and the products are given by Clifford multiplication.

We shall not require an extensive familiarity with the theory of connections, either in this chapter or elsewhere. But we do want to take note of two important facts. First, if ∇ is a Dirac connection on S then the local formula

$$\text{(11.1.10)} \qquad Du = \sum_i e_i \cdot \nabla_{X_i} u,$$

where $X_1, \ldots, X_n$ is a local orthonormal frame for TM and $e_1, \ldots, e_n$ is the dual orthonormal frame for T^*M (acting by Clifford multiplication), defines a Dirac operator on M which depends only on S and ∇. Secondly, the *square* of this Dirac operator is given by the *Weitzenbock formula*

$$D^2 u = \Delta u + Ru,$$

where $\Delta = \nabla^* \nabla$ is the Laplace operator associated to the connection ∇, and R is the following endomorphism of S:

$$R = \sum_{i<j} e_i e_j (\nabla_{X_i} \nabla_{X_j} - \nabla_{X_j} \nabla_{X_i}) = \sum_{i<j} e_i e_j R(X_i, X_j).$$

Here $R(X_i, X_j)$ is the curvature operator for the connection ∇, familiar from differential geometry. The significance of the Weitzenbock formula is that for operators of the sort described by Equation 11.1.10 we can sometimes derive *analytic* consequences, namely the vanishing of Fredholm indices, from *geometric* hypotheses, namely the positivity of the curvature-based operator R. Indeed, if R is positive-definite, say $R \geqslant \varepsilon I > 0$, then from the Weitzenbock formula we see immediately that

$$\langle Du, Du \rangle = \langle \nabla u, \nabla u \rangle + \langle Ru, u \rangle \geqslant \varepsilon \langle u, u \rangle,$$

and hence that the Dirac operator D has trivial kernel.

The best-known example is the Dirac operator associated to a real spinor bundle on a Riemannian manifold, which we shall discuss in the next section. In this case the operator R turns out to be pointwise multiplication by $\frac{1}{4}$ times the Riemannian scalar curvature function on M. It therefore follows that a real spinor Dirac operator on a closed Riemannian manifold with everywhere positive scalar curvature has zero index. Coupled with the Atiyah–Singer Index Theorem, this observation presents topological obstructions to the construction of positive scalar curvature metrics on some smooth, closed manifolds. We shall develop this point more fully in the next chapter.

11.2 Spin^c-Manifolds

As far as K-theory and K-homology are concerned, the most basic Dirac bundles are the real and complex *spinor bundles*, whose properties we shall explore in this section. A manifold equipped with a spinor bundle has a natural fundamental class in K-homology which is very much analogous to the fundamental class of an oriented manifold in ordinary homology. We shall work out the algebra of these fundamental classes; later in the chapter we shall use this algebra to prove the Atiyah–Singer Index Theorem.

We begin by introducing to Dirac operator theory the same multigradings that we considered in earlier chapters.

11.2.1 DEFINITION A p-*multigraded Dirac operator* is a Dirac operator on a graded Hermitian bundle S together with p odd endomorphisms $\varepsilon_1, \ldots, \varepsilon_p$ of S such that $D\varepsilon_j = \varepsilon_j D$, for all j, and

$$\varepsilon_j = -\varepsilon_j^*, \quad \varepsilon_j^2 = -1, \quad \varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = 0 \quad (i \neq j).$$

Similarly a p-*multigraded Dirac bundle* is a Dirac bundle S equipped with p odd endomorphisms $\varepsilon_1, \ldots, \varepsilon_p$ of S such that

$$\varepsilon_j = -\varepsilon_j^*, \quad \varepsilon_j^2 = -1, \quad \varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = 0 \quad (i \neq j),$$

and such that each ε_j commutes with every Clifford multiplication operator on every fiber S_x.

As in Chapter 10, a p-multigraded Dirac operator D on a Riemannian manifold M defines in a natural way a class $[D] \in K_p(M)$. Propositions 11.1.7 and 11.1.8, dealing with the K-homology classes of Dirac operators in the 0-multigraded case, are easily adapted to p-multigraded operators. Thus if D is a p-multigraded Dirac operator on M, and if U is an open subset of M, then the restriction of the class $[D] \in K_p(M)$ to a class in $K_p(U)$ gives the K-homology class of the restriction of D to U. In addition, if D_1 is a p_1-multigraded Dirac operator on M_1, and if D_2 is a p_2-multigraded Dirac operator on M_2, then the Dirac operator $D_1 \times D_2$ is $(p_1 + p_2)$-multigraded,[96] and

$$[D_1 \times D_2] = [D_1] \times [D_2]$$

in $K_{p_1+p_2}(M_1 \times M_2)$.

The multigrading concept is very closely related to Clifford multiplication. An important consequence of the basic Clifford algebra relation $\xi \cdot \xi \cdot u = -\|\xi\|^2 u$ is that if ξ_1 and ξ_2 are two *orthogonal* cotangent vectors at x then

$$\xi_1 \cdot \xi_2 \cdot u + \xi_2 \cdot \xi_1 \cdot u = 0$$

(to derive this, compare $\|\xi_1 + \xi_2\|^2$ to $\|\xi_1\|^2 + \|\xi_2\|^2$). So from any orthonormal basis of T_x^*M we obtain a collection of Clifford multiplication operators which have the same algebraic structure as the multigrading operators in Definition 11.2.1. This observation suggests the possibility of constructing a natural n-multigraded Dirac operator on a Riemannian n-manifold based on the following concept:

11.2.2 DEFINITION The *complex Clifford algebra* for $\mathbb{R}^n$ is the complex $*$-algebra $\mathbb{C}_n$ generated by elements $e_1, \ldots, e_n$ (corresponding to the standard orthonormal basis of $\mathbb{R}^n$) such that

$$e_j = -e_j^*, \quad e_j^2 = -1 \qquad (j = 1, \ldots, n),$$

and

$$e_i e_j + e_j e_i = 0 \qquad (i \neq j).$$

The algebra $\mathbb{C}_n$ is linearly spanned by the 2^n monomials $e_{j_1} \cdots e_{j_k}$, where $j_1 < \cdots < j_k$ and $0 \leqslant k \leqslant n$. We introduce an inner product on $\mathbb{C}_n$ by deeming these monomials to be orthonormal. The action of the *algebra* $\mathbb{C}_n$ on the *Hilbert space* $\mathbb{C}_n$ by left multiplication is a faithful $*$-representation which gives $\mathbb{C}_n$ the structure of a C^*-algebra. We grade $\mathbb{C}_n$ by assigning to each monomial its degree modulo 2.

Now let M be an n-dimensional Riemannian manifold. If $e_1, \ldots, e_n$ is a local orthonormal frame for T^*M, defined over an open set $U \subseteq M$, then the trivial bundle $U \times \mathbb{C}_n$ over U with fiber $\mathbb{C}_n$ may be given the structure of an n-graded Dirac bundle: Clifford multiplication by an element e_j of the frame is *left* multiplication by the jth generator of $\mathbb{C}_n$, and the n-multigrading operators $\varepsilon_1, \ldots, \varepsilon_n$ for the bundle are *right* multiplication by the same generators.[97]

[96] We refer once again to Definition A.3.3 for our conventions on the multigrading of tensor products.

[97] Notice the important distinction between the left and right actions here: sections of the *bundle* T^*M act by Clifford multiplication on the left (and we need to choose a frame for T^*M to define the action); the *fixed* Clifford generators $\varepsilon_1, \ldots, \varepsilon_n$ act on the right (independent of any choice of frame).

If $f_1, \ldots, f_n$ is a local orthonormal frame on U which is in the same orientation class as $e_1, \ldots, e_n$, and if we construct a new Dirac bundle $U \times \mathbb{C}_n$ using this frame in place of $e_1, \ldots, e_n$, then the new and old Dirac bundles are locally isomorphic. This is because in a neighborhood of any point in U there is a smooth function w, with values in the even part of $\mathbb{C}_n$, such that $ww^* = w^*w = 1$ and such that left multiplication by w intertwines the Clifford multiplication actions associated to the two frames. See Exercise 11.8.6.

On the other hand, if $f_1, \ldots, f_n$ has the opposite orientation then there is no such local isomorphism. Indeed, working on the Dirac bundle associated to the frame $e_1, \ldots, e_n$, the endomorphism

$$u \mapsto (-1)^{n+(n-1)\partial u} f_1 \cdots f_n \cdot \varepsilon_n \cdots \varepsilon_1 u$$

is the identity operator if the frame $f_1, \ldots, f_n$ has the same orientation as $e_1, \ldots, e_n$, and minus the identity if the orientations differ. Therefore we can recover the orientation of frames from the Dirac bundle.

In summary, using Clifford algebras we have obtained a local model for a canonical n-multigraded Dirac bundle on an (oriented) Riemannian n-manifold. We define a complex spinor bundle on M to be a globally defined Dirac bundle which is locally of this model type:

11.2.3 DEFINITION Let M be an n-dimensional Riemannian manifold. A *complex spinor bundle* on M is an n-multigraded Dirac bundle S which is locally isomorphic to the trivial bundle with fiber $\mathbb{C}_n$, the Clifford multiplication being determined from some local orthonormal frame, as above.

11.2.4 REMARK There are parallel notions of *real Dirac bundle* and *real spinor bundle*. The real Clifford algebra $\mathbb{R}_n$ is the $\mathbb{R}$-subalgebra of $\mathbb{C}_n$ generated by the elements $e_1, \ldots, e_n$, and a real spinor bundle on a Riemannian n-manifold is an n-multigraded, real Dirac bundle which is locally isomorphic to $U \times \mathbb{R}_n$. We shall make some use of real spinor bundles in the next chapter. As far as we are concerned in this book, their main advantage over complex spinor bundles is that they may be equipped with a *unique* Dirac connection, for which, as we noted in the previous section, there is a very close connection between the invertibility of the Dirac operator defined by Equation 11.1.10 and the sign of the scalar curvature of the underlying manifold. Of course, if S is a real spinor bundle then its complexification is a complex spinor bundle, and in this way we shall be able to place at least some of the theory of real spinor bundles (in fact, enough of it for our purposes) into the complex setting.

As we noted above, a spinor bundle S determines an orientation for M. In particular, a spinor bundle can only exist if M is orientable. If T^*M admits a globally defined orthonormal frame — in other words, if T^*M is a trivial vector bundle — then there is clearly a corresponding 'trivial' complex spinor bundle. Thus, for example, complex spinor bundles exist on $\mathbb{R}^n$, on tori, on Lie groups, and so on. However, it is not true that every orientable Riemannian manifold admits a complex spinor bundle. Furthermore, while the complexification of a real spinor bundle is a complex spinor bundle, not every complex spinor bundle can be obtained this way. Finally, spinor bundles, either real or complex, are by no means unique. We shall discuss the issue of uniqueness at the end of this section; the problem of constructing spinor bundles (at least in the complex case, and on even-dimensional manifolds) will be taken up in the next section.

From here on, the reader should take the unmodified term 'spinor bundle' to mean 'complex spinor bundle'.

If S_1 is a spinor bundle on M_1 and S_2 is a spinor bundle on M_2 then the product $S_1 \hat{\otimes} S_2$ is a spinor bundle on $M_1 \times M_2$. The case where $M_1 = \mathbb{R}$ is particularly important. We equip the line with the trivial spinor bundle $S_\mathbb{R} = \mathbb{R} \times \mathbb{C}_1$, for which the natural Dirac operator is

$$D_\mathbb{R} = e_1 \cdot \frac{d}{dx}.$$

If we identify $\mathbb{C}_1$ with $\mathbb{C}^2$ via the orthonormal basis $\{1, e_1\}$ for $\mathbb{C}_1$ then $D_\mathbb{R}$ may be written as

$$D_\mathbb{R} = \begin{pmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix},$$

and the multigrading operator ε_1 acts as

$$\varepsilon_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

11.2.5 PROPOSITION *Let S be a Dirac bundle on a Riemannian n-manifold M and let D be a Dirac operator on S. Denote by $s\colon K_{n+1}(\mathbb{R} \times M) \to K_n(M)$ the suspension isomorphism in K-homology. If $D_\mathbb{R}$ is the Dirac operator on the spinor bundle $S_\mathbb{R}$ over $\mathbb{R}$ then*

$$s([D_\mathbb{R} \times D]) = [D]$$

in $K_n(M)$.

PROOF The K-homology class of $D_\mathbb{R} \times D$ is the Kasparov product of the classes of $D_\mathbb{R}$ and D. The naturality of the Kasparov product relative to the suspension isomorphism (Lemma 9.5.8) gives

$$s([D_\mathbb{R} \times D]) = s([D_\mathbb{R}]) \times [D].$$

But it was proved in Exercise 10.9.7 that $[D_\mathbb{R}]$ is the Dirac class $d \in K_1(\mathbb{R})$ described in Definition 9.5.1. According to Lemma 9.5.7, $s(d) = 1$. Therefore

$$s([D_{\mathbb{R}} \times D]) = s([D_{\mathbb{R}}]) \times [D] = \mathbf{1} \times [D] = [D],$$

as required. $\square$

For every spinor bundle S on a Riemannian manifold M there is a corresponding Dirac operator D, and associated to D there is a homology class $[D] \in K_n(M)$. There is generally no canonical Dirac operator on a spinor bundle, but all choices determine the same K-homology class. To see this we can appeal either to Exercise 10.9.5 or to the following argument. Let D and D$'$ be two Dirac operators for the same spinor bundle on M. If we fix two non-empty open intervals in $\mathbb{R}$ with disjoint closures then by means of a partition of unity argument we can construct a Dirac operator $D_{\mathbb{R}\times M}$ on $\mathbb{R} \times M$ which agrees with $D_{\mathbb{R}} \times D$ over the first interval and with $D_{\mathbb{R}} \times D'$ over the second. Since the restriction map from $\mathbb{R}$ to an open interval is a homotopy equivalence, it follows from Proposition 11.1.7 that

$$[D_{\mathbb{R}} \times D] = [D_{\mathbb{R}\times M}] = [D_{\mathbb{R}} \times D'] \in K_{n+1}(\mathbb{R} \times M).$$

It then follows from Proposition 11.2.5 that

$$[D] = [D'] \in K_n(M),$$

as required.

The same argument actually proves a stronger result.

11.2.6 Definition Let M and M$'$ be two Riemannian manifolds with the same underlying smooth manifold, and let S and S$'$ be spinor bundles on M and M$'$, respectively. The pairs (M, S) and (M', S') are *concordant* if there is a pair consisting of a Riemannian metric and a spinor bundle on $\mathbb{R} \times M$ which over some non-empty open interval of $\mathbb{R}$ agrees with $(\mathbb{R} \times M, S_{\mathbb{R}}\hat{\otimes}S)$, and which over some other non-empty open interval of $\mathbb{R}$ agrees with $(\mathbb{R} \times M', S_{\mathbb{R}}\hat{\otimes}S')$.

11.2.7 Proposition *Let* (M, S) *and* (M', S') *be concordant pairs and let* D *and* D$'$ *be Dirac operators for* S *and* S$'$. *Then* $[D] = [D']$ *in* $K_n(M)$. $\square$

11.2.8 Definition A *Spin^c-structure* on a smooth manifold M is a concordance class of Riemannian metrics and complex spinor bundles on M. A *Spin^c-manifold* is a smooth manifold which is provided with a Spin^c-structure.

11.2.9 Remark Similarly, a *Spin-structure* on a smooth manifold M is a concordance class of Riemannian metrics and *real* spinor bundles on M.

11.2.10 Definition Let M be a Spin^c-manifold. The *fundamental class* of M is the K-homology class $[M] \in K_n(M)$ of any Dirac operator on any complex spinor bundle in the concordance class provided by the Spin^c-structure on M.

11.2.11 REMARK If M is a *Spin*-manifold then we take [M] to mean the class obtained by first complexifying the Spin-structure so as to obtain a Spin^c-structure, and then taking the K-homology class of the Spin^c-structure. This ignores whatever finer real structure may be present in the Spin-structure. It is possible, and indeed very simple, to build a more refined 'real' K-homology which properly takes the real structure into account. We shall not do so here, but for the reader who is interested in the real case we note that the material of this and the next two sections carries over to that case without any essential change.

Despite our notation, the class $[M] \in K_n(M)$ *does* in fact depend on the choice of Spin^c-structure, and not on M alone. Our fundamental class notation is taken from ordinary homology theory, where there is a similar ambiguity: the fundamental class [M] in the ordinary homology group $H_n(M)$ depends not only on M but also on the choice of orientation of M. In the context of ordinary homology a connected, orientable manifold has exactly two possible orientations. In K-homology the situation is more complicated, as we shall indicate at the end of this section.

A Spin^c-structure on M restricts to a Spin^c-structure on any open subset of M. In addition, if M_1 and M_2 are Spin^c-manifolds then the product $M_1 \times M_2$ may be equipped with a natural Spin^c-structure by forming the graded tensor product of spinor bundles (the concordance class of the tensor product depends only on the concordance classes of the factors). At the risk of being repetitious, let us specialize to fundamental classes the principal results about Dirac operators from the last section.

11.2.12 PROPOSITION *Let M be a $Spin^c$-manifold and let $[M] \in K_n(M)$ be its fundamental class. If $\iota\colon U \to M$ is the inclusion of an open subset into M, and if $[U] \in K_n(U)$ is the fundamental class of the restricted $Spin^c$-structure, then $\iota^![M] = [U]$.* □

11.2.13 PROPOSITION *If $[M_1]$ and $[M_2]$ are the fundamental classes associated to $Spin^c$-structures on smooth manifolds M_1 and M_2, and if $[M_1 \times M_2]$ is the fundamental class associated to the product $Spin^c$-structure, then*

$$[M_1] \times [M_2] = [M_1 \times M_2]$$

in $K_{n_1+n_2}(M_1 \times M_2)$. □

A third important result concerns the fundamental classes of manifolds with boundary.

11.2.14 DEFINITION Let S be a spinor bundle on an n-dimensional Riemannian manifold with boundary. If e_1 denotes the inward pointing unit normal covector field on the boundary manifold ∂M then the formula

$$X\colon u \mapsto (-1)^{\partial u} e_1 \cdot \varepsilon_1 u$$

defines an automorphism of the restriction of S to ∂M which is even, selfadjoint, and has $X^2 = 1$. The operator X commutes with Clifford multiplication by cotangent vectors orthogonal to e_1, and also with the multigrading operators $\varepsilon_2, \dots, \varepsilon_n$. The -1 eigenbundle for X is a spinor bundle[98] on ∂M whose concordance class depends only on the concordance class of S. It defines the *induced* Spin^c-structure on the boundary of a Spin^c-manifold.

11.2.15 PROPOSITION *If M is the interior of an n-dimensional Spin^c-manifold with boundary, and if we equip the boundary manifold ∂M with the induced Spin^c-structure, then the K-homology boundary map*

$$\partial\colon K_n(M) \to K_{n-1}(\partial M)$$

takes the fundamental class of M to the fundamental class of ∂M:

$$\partial[M] = [\partial M] \in K_{n-1}(\partial M).$$

PROOF Using a collaring neighborhood, the naturality of the boundary map (Proposition 9.6.7(a)), and Proposition 11.2.12, the proof immediately reduces to the case of the manifold $[0,1) \times \partial M$, for which $M = (0,1) \times \partial M$. But in this case every Spin^c-structure on M is isomorphic to the product of the standard structure on $(0,1)$ and the induced structure on ∂M. In addition, according to Proposition 9.6.6 the boundary map $\partial\colon K_n(M) \to K_{n-1}(\partial M)$ is nothing but the suspension isomorphism

$$s\colon K_n((0,1) \times \partial M) \to K_{n-1}(\partial M).$$

So the identity $\partial[M] = [\partial M]$ is a consequence of Proposition 11.2.5. □

We conclude the section by briefly describing the possible variations among Spin^c-structures on a Spin^c-manifold.

Let M be an *oriented* Riemannian manifold and let S be a spinor bundle on M. We shall suppose here and below that the orientation which is provided by S matches the given orientation on M. If L is a complex Hermitian line bundle on M then the tensor product $S \otimes L$ is also a spinor bundle (the Clifford multiplication and multigrading operators act as the identity on the factor L in the tensor product).

If S' is another spinor bundle on M then the complex vector bundle morphisms $S' \to S$ which are compatible with the grading, with the Clifford multiplication, and with the action of the multigrading operators ε_j constitute the

[98] The multigrading operators are obtained from $\varepsilon_2, \dots, \varepsilon_n$ by shifting indices downwards.

sections of a complex Hermitian line bundle L' over M.[99] The spinor bundles S and S' are isomorphic if and only if L' is trivial; more generally S is isomorphic to $S' \otimes L'$. In this way one can show that should one Spin^c-structure exist on a manifold M, the set of all Spin^c-structures on M which determine the same orientation of M is parameterized by the set of isomorphism classes of complex line bundles on M.

One can construct a line bundle invariant for a *single* complex spinor bundle S as follows. The complex conjugate bundle S^* is also a spinor bundle over M and the bundle morphisms $S^* \to S$ which are compatible with the grading, with the Clifford multiplication, and with the action of the multigrading operators ε_j constitute the sections of a complex line bundle $\ell(S)$ over M. The isomorphism class of this line bundle is an invariant of the Spin^c-structure determined by S. Note that $\ell(S \otimes L) = \ell(S) \otimes L^2$, where L^2 denotes the tensor product of L with itself. If the Spin^c-structure on M determined by S is the complexification of a Spin-structure then $\ell(S)$ is a trivial line bundle.

11.3 Even-Dimensional Spin^c-Manifolds

Our presentation of spinor bundles in the previous section is well suited to the multigraded Kasparov groups that were introduced in Chapter 8, and our investment in multigradings pays off handsomely when one comes to consider the case of real (rather than complex) spinor bundles. But in this book we are interested mostly in the complex case, and here a somewhat simpler approach to Spin^c-structures is available. The new approach, which we shall describe for even-dimensional manifolds, is based on the following observation:

11.3.1 LEMMA *Let M be a Riemannian manifold of dimension n. Every n-multigraded Dirac bundle on M of fiber dimension 2^n is a spinor bundle.*

REMARK This is true in both the real and complex cases.

PROOF The issue is to show that every n-multigraded Dirac bundle S of fiber dimension 2^n is locally isomorphic to the model Clifford algebra bundle that we constructed in the last section. To begin, it is convenient to introduce new operators $\tilde{\varepsilon}_j$ from the given multigrading operators ε_j by means of the formulas

$$\tilde{\varepsilon}_j u = (-1)^{\partial u} \varepsilon_j u \qquad (j = 1, \ldots, n).$$

These operators satisfy the relations

$$\tilde{\varepsilon}_j = \tilde{\varepsilon}_j^*, \quad \tilde{\varepsilon}_j^2 = 1 \qquad (j = 1, \ldots, n)$$

and

[99]This is a consequence of the calculations used to prove Lemma 11.3.1 below.

$$\tilde{\varepsilon}_i\tilde{\varepsilon}_j + \tilde{\varepsilon}_j\tilde{\varepsilon}_i = 0 \qquad (i \neq j).$$

Moreover they *anticommute* with the Clifford multiplication operators on our bundle. So if we choose an oriented local orthonormal frame $e_1, \ldots, e_n$ for the cotangent bundle of M then the action of these e_i on any fiber of S, combined with the action of the $\tilde{\varepsilon}_j$, provides a representation of the finite-dimensional algebra with anticommuting generators e_i and $\tilde{\varepsilon}_j$ and the above relations for the ε_j, along with the Clifford relations $e_i^2 = -1$. This algebra, it may be shown, is isomorphic to the full algebra of $2^n \times 2^n$ matrices. It has, up to equivalence, a unique representation of dimension 2^n, and this proves that S is at least fiberwise isomorphic to a spinor bundle. But now bundle theory provides from a fiberwise isomorphism a local isomorphism of bundles around any point of M. $\square$

Let us recall now that we devised in Chapter 8 a means of 'reducing' a p-multigraded (complex) Hilbert space to a $(p-2)$-multigraded Hilbert space, and in this way we obtained an isomorphism from $K^{-p}(A)$ to $K^{-p+2}(A)$. The method adapts immediately to Dirac bundles: if S_n is an n-multigraded Dirac bundle then the operator

$$X = i\varepsilon_{n-1}\varepsilon_n$$

on S_n is even, selfadjoint, and $X^2 = 1$. The $+1$ eigenbundle for X — let us call it S_{n-2} — is a subbundle of S_n which is an $(n-2)$-multigraded Dirac bundle for the structure it inherits from S_n; the dimension of S_{n-2} is exactly half the dimension of S_n.

The bundle S_n may be recovered from S_{n-2} as follows: up to isomorphism it is the direct sum $S_{n-2} \oplus S_{n-2}$, on which the operators ε_{n-1} and ε_n act as the matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

(we use the *reverse* grading on the second summand in the direct sum).

Suppose now that S is a complex spinor bundle on a Riemannian n-manifold, so that the dimension of S is 2^n. If n is even, say $n = 2k$, then after k iterations of the process above we obtain from S a 0-multigraded Dirac bundle S_0 of dimension 2^k.

Reversing the procedure, if S_0 is a Dirac bundle of dimension 2^k on a Riemannian manifold M of dimension $n = 2k$, then after k iterations we obtain from S_0 an n-multigraded Dirac bundle S on M of dimension 2^n, which by Lemma 11.3.1 is a spinor bundle. Hence:

11.3.2 PROPOSITION *Let M be a Riemannian manifold of dimension $n = 2k$. There is a one-to-one correspondence between isomorphism classes of complex spinor bundles on M and isomorphism classes of Dirac bundles of dimension 2^k.* $\square$

11.3.3 DEFINITION Let M be a Riemannian manifold of dimension $n = 2k$. A *reduced complex spinor bundle* on M is a Dirac bundle of dimension 2^k. A Dirac operator on a reduced spinor bundle S_0 defines a *fundamental class* $[M] \in K_0(M)$. It is the image under the isomorphism $K_n(M) \cong K_0(M)$ of the fundamental class $[M] \in K_n(M)$ determined by the spinor bundle associated to S_0.

As was the case with non-reduced spinor bundles, a reduced spinor bundle S_0 on M must of course determine an orientation of M. In fact there is a simple recipe for the orientation: a local orthonormal frame is oriented if and only if the Clifford multiplication operator $i^k e_1 \cdots e_n$ agrees with the grading operator on S_0. This provides the same orientation as the one obtained by constructing from S_0 the associated non-reduced spinor bundle, and then determining an orientation from it.

11.3.4 EXAMPLE The Dolbeault operator on a complex manifold of complex dimension m (and real dimension $n = 2m$) acts on a Dirac bundle of dimension 2^m. Hence the Dolbeault operator determines a Spin^c-structure on M, and the class $[\mathcal{D}] \in K_0(M)$ of the Dolbeault operator corresponds to a fundamental class $[M] \in K_n(M)$ under the periodicity isomorphism $K_n(M) \cong K_0(M)$. The orientation associated to the Dolbeault operator is the standard orientation on a complex manifold.

11.3.5 EXAMPLE The reduced Dirac operator on $\mathbb{R}^2$, corresponding to the canonical Spin^c-structure on the plane, is the operator

$$D = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & 0 \end{pmatrix},$$

with the standard grading on the trivial two-dimensional vector bundle over the plane.

The new reduced fundamental classes $[M] \in K_0(M)$ depend only on the Spin^c-structure associated to S_0 and the analogues of Propositions 11.2.12 and 11.2.13, concerning restrictions to open sets and product manifolds, are valid for them (we shall refrain from formally stating these results yet another time). The invariant $\ell(S)$ may be recovered from the reduced spinor bundle S_0 associated to S: it is the line bundle of morphisms from S_0^* to S_0 which commute with Clifford multiplication.

11.3.6 EXAMPLE In the case of the Spin^c-structure associated to the Dolbeault operator on a complex manifold we have $\ell(S) = \bigwedge^{0,n} M$.

11.4 Index Theory for Hypersurfaces

In this section and the next we shall be concerned with the index theory of Dirac operators on Spin^c-manifolds. Perhaps the most correct way to introduce the subject is as follows. Let S be a complex spinor bundle on a closed Riemannian n-manifold. If E is any complex Hermitian bundle on M then the tensor product $S \otimes E$ is a Dirac bundle. If D_E is a corresponding Dirac operator then we may form the class $[D_E] \in K_n(M)$; if we denote by $c\colon M \to \mathrm{pt}$ the map collapsing M to a point then we can push $[D_E]$ forward by the induced map on K-homology and so obtain an *analytic index* $c_*[D_E] \in K_n(\mathrm{pt})$. It is easy to check that the analytic index depends only on the K-theory class defined by E, and thus we obtain an *analytic index homomorphism*

$$\text{a–Index} : K^0(M) \to K_n(\mathrm{pt}), \qquad E \mapsto c_*(D_E).$$

The *index problem* for the Spin^c-manifold M is to compute the analytic index in terms of the K-theory class of E and topological data associated to the Spin^c-manifold M.

While the above account presents the index problem in its most correct form, and in a way best suited to generalizations and refinements of the index problem (for instance to the case of real spinor bundles, to which we have alluded more than once), there is a more streamlined view of the index problem which is better suited to our present purposes. We note first that if n (the dimension of M) is even then $K_n(\mathrm{pt}) \cong \mathbb{Z}$, while if n is odd then $K_n(\mathrm{pt}) = 0$. So let us right away confine our attention to *even-dimensional* manifolds, and in addition let us avail ourselves of the formal periodicity isomorphism $K_n(M) \cong K_0(M)$ and the notion of reduced spinor bundle to reformulate the index problem, as follows: given a reduced complex spinor bundle S on a closed Spin^c-manifold M, and given a Hermitian vector bundle E over M, compute the Fredholm index

$$\mathrm{Index}\,(D_E) \in \mathbb{Z}$$

of the Dirac operator D_E on $S \otimes E$ in topological terms.

The simplified problem still fits very naturally into the context of K-homology theory, thanks to the following computation:

11.4.1 Lemma *Let M be a closed Riemannian manifold and let D be a Dirac operator associated to some Dirac bundle S on M. If E is a Hermitian vector bundle on M and if D_E is a Dirac operator associated to the tensor product Dirac bundle $S \otimes E$ then*

$$\mathrm{Index}\,(D_E) = \langle [E], [D] \rangle,$$

where the right-hand side is the index pairing between K-theory and K-homology.

Proof Let H be the graded Hilbert space $L^2(S)$, and let χ be a normalizing function. Thus $F = \chi(D)$ gives rise to a Fredholm module (ρ, H, F) defining $[D]$.

Let $P\colon M \to M_n(\mathbb{C})$ be a projection-valued function which determines the vector bundle E. Let D_n be the operator $1 \otimes D$ on the Hilbert space $H_n = \mathbb{C}^n \otimes H$. Then $\operatorname{Index}(D_E)$ is the index of the operator $\chi(PD_nP)$ on PH_n, whereas $\langle [E], [D] \rangle$ is the index of the operator $P\chi(D_n)P$ on PH_n. But it follows from Exercise 10.9.5 that the operators $P\chi(D_n)P$ and $\chi(PD_nP)$ are compact perturbations of one another, and hence they have the same index. □

Returning to the index problem, in this section we are going to concentrate on a quite special situation: we shall suppose that the closed Spin^c-manifold M embeds as a codimension-one submanifold in a Euclidean space. Our strategy for computing $\operatorname{Index}(D_E)$ is to use the embedding to first change D_E to an operator acting on an open neighborhood of M in Euclidean space (without altering the index), and then change the operator again to one which acts on all of Euclidean space (again, without changing the index). The problem of computing $\operatorname{Index}(D_E)$ is now made rather easy, since according to Bott periodicity there is, up to deformation, essentially only one index problem on each Euclidean space. The role of K-homology in carrying out the above program is to provide us with a very flexible notion of 'elliptic operator' — namely the notion of Fredholm module — using which it is very easy to construct the new operators and also show that during the process the Fredholm index is never altered.

We begin the argument with some very simple differential topology.

11.4.2 Lemma *Let M be a closed, even-dimensional Spin^c-manifold and let S be a reduced complex spinor bundle which determines the Spin^c-structure. If M embeds as a codimension-one submanifold of a Euclidean space then the product $M \times \mathbb{R}$ is diffeomorphic, in an orientation-preserving fashion, to an open subset U of Euclidean space $\mathbb{R}^{n+1}$. Moreover, for each such diffeomorphism there is a complex line bundle L on M such that the product of the standard Spin^c-structure on $\mathbb{R}$ and the Spin^c-structure on M associated to $S \otimes L$ is carried via the diffeomorphism to the standard Spin^c-structure on $U \subseteq \mathbb{R}^{n+1}$.*

Proof Since M is a Spin^c-manifold, it is oriented. Therefore for every embedding of M there is a nowhere vanishing normal vector field n. It follows from the Inverse Function Theorem that for a small enough $\varepsilon > 0$ the mapping $(x,t) \mapsto x + tn$ is a diffeomorphism from $M \times (-\varepsilon, \varepsilon)$ onto an open neighborhood of M in $\mathbb{R}^n$. This proves the first part of the lemma. For the second part, we know that the standard Spin^c-structure on U may be obtained from the product Spin^c-structure on $U \cong M \times \mathbb{R}$ by tensor product with some line bundle L on U (indeed any two Spin^c-structures which are compatible with the same orientation are so related). But any line bundle on $U \cong M \times \mathbb{R}$ is isomorphic to a line bundle pulled back from M, and this observation completes the proof. □

Since it is convenient to stay entirely within the world of even-dimensional manifolds, and since if M is even-dimensional the open set U of the Lemma 11.4.2 is necessarily odd-dimensional, let us multiply by an extra factor of $\mathbb{R}$ to get the following:

11.4.3 LEMMA *Let M^n be a closed, even-dimensional Spin^c-manifold and let S be a reduced complex spinor bundle on M which determines the Spin^c-structure. If M embeds as a codimension-one submanifold of a Euclidean space, then the product $M \times \mathbb{R}^2$ is diffeomorphic, in an orientation-preserving fashion, to an open subset U of the Euclidean space $\mathbb{R}^{n+2}$. Moreover, for each such diffeomorphism there is a complex line bundle L on M such that the product of the standard Spin^c-structure on $\mathbb{R}^2$ and the Spin^c-structure on M associated to $S \otimes L$ is carried via the diffeomorphism to the standard Spin^c-structure on $U \subseteq \mathbb{R}^{n+2}$.* $\square$

As we indicated above, the proof of the Index Theorem reduces the index problem for the operator D_E on $M \subseteq \mathbb{R}^n$ (where for simplicity we suppose that n is even) to an integer multiple of a standard index problem on Euclidean space. To describe this standard index problem, recall that according to the Bott Periodicity Theorem 4.9.1 the group $K^0(\mathbb{R}^n)$ is isomorphic to $\mathbb{Z}$, generated by

$$B_n = b \times b \times \cdots \times b,$$

where $b \in K^0(\mathbb{R}^2)$ is the Bott generator[100] (Example 4.3.10).

11.4.4 DEFINITION The class $B_n \in K^0(\mathbb{R}^n)$ described above will be referred to as the *Bott generator* for the even-dimensional Euclidean space $\mathbb{R}^n$.

The calculation that we need for Euclidean space is then:

11.4.5 PROPOSITION *The index pairing between the Bott generator and the fundamental class of an (even-dimensional) Euclidean space is equal to 1; that is, $\langle B_n, [\mathbb{R}^n]\rangle = 1$.*

PROOF Because of the multiplicative properties of the index pairing (Proposition 9.7.1) and of the fundamental class (Proposition 11.2.13), it suffices to consider the case $n = 2$. Moreover, it suffices to compute the pairing of the Dirac operator D on the open unit disk $\mathbb{D}$ with the Bott generator in $K^0(\mathbb{D})$. But

$$b = \partial(x) \in K^0(\mathbb{D}),$$

where $x \in K^{-1}(\partial\mathbb{D})$ is the K-theory class described by the unitary-valued function $z \mapsto \bar{z}$ on $S^1 = \partial\mathbb{D}$, and $\partial\colon K^{-1}(\partial\mathbb{D}) \to K^0(\mathbb{D})$ is the K-theory boundary map associated to the pair $(\overline{\mathbb{D}}, \partial\mathbb{D})$. By Equation 8.7.7,

[100]The Bott generator, as we defined it, lies in $K^0(\mathbb{D})$, where $\mathbb{D}$ is the open unit disk. We obtain a class in $K^0(\mathbb{R}^2)$ including $\mathbb{D}$ into $\mathbb{R}^2$ as an open subset.

$$\langle \partial(x), [D] \rangle = \langle x, \partial[D] \rangle.$$

But $\partial[D] \in K_{-1}(\partial\mathbb{D})$ is the homology class of the Toeplitz extension (there are many ways of proving this: see for example Exercise 10.9.14; one can also use Proposition 11.2.15 and a formal periodicity calculation), so that $\langle x, \partial[D] \rangle$ is equal to the index of the Toeplitz operator $T_{\bar{z}}$, and this is 1 by the Toeplitz Index Theorem. $\square$

11.4.6 Definition Let M^n be an even-dimensional Spin^c-manifold which embeds as a codimension-one submanifold of Euclidean space. The *topological index homomorphism* for M is the map

$$\text{t–Index} : K^0(M) \to \mathbb{Z}$$

given by the formula

$$\text{t–Index}([E]) \cdot B_{n+2} = \iota_! \varphi^*([L^* \otimes E] \times B_2),$$

where $\varphi \colon U \to M \times \mathbb{R}^2$ is an orientation-preserving diffeomorphism from an open subset of $\mathbb{R}^{n+2}$ to $M \times \mathbb{R}^2$, L is the line bundle associated to φ by Lemma 11.4.3, and $\iota \colon U \to \mathbb{R}^{n+2}$ is the inclusion.

Remark We have used the notation $\iota_!$ for the 'wrong way' map in K-theory associated to the inclusion of U into $\mathbb{R}^{n+2}$. The topological index homomorphism is well-defined: one can see this by appealing to a suitable uniqueness theorem for tubular neighborhoods; alternatively, the well-definedness follows from the Index Theorem!

11.4.7 Index Theorem for Hypersurfaces *Let* M *be a smooth, closed, even-dimensional Spin^c-manifold which may be embedded as a codimension-one submanifold of a Euclidean space. If* E *is a smooth vector bundle on* M *and if* $D \otimes E$ *denotes the twisted Dirac operator associated to* E *then*

$$\text{Index}(D_E) = \text{t–Index}(E).$$

In other words, the analytic index is equal to the topological index.

Proof Let S be a reduced spinor bundle on M which is associated to the given Spin^c-structure. Fix an orientation-preserving diffeomorphism φ from an open subset $U \subseteq \mathbb{R}^{n+2}$ to $M \times \mathbb{R}^2$ and denote by L the corresponding line bundle on M, as in Lemma 11.4.2. We shall denote by $[M] \in K_n(M)$ the fundamental class for the Spin^c-structure associated to the reduced spinor bundle $S \otimes L$ (and *not* to S itself), but we shall continue to denote by D the Dirac operator on S. Since the tensor product $L^* \otimes L$ is a trivial line bundle, we have that

$$\mathrm{Index}\,(D_E) = \mathrm{Index}\,(D_{L\otimes L^*\otimes E}),$$

and since the Dirac operator for the Spin^c-structure associated to $S \otimes L$ is D_L it follows that

$$\mathrm{Index}\,(D_{L\otimes L^*\otimes E}) = \big\langle [L^* \otimes E], [D_L] \big\rangle = \big\langle [L^* \otimes E], [M] \big\rangle.$$

We shall now compute the latter index pairing by first passing from M to $M \times \mathbb{R}^2$, then identifying $M \times \mathbb{R}^2$ with the open subset $U \subseteq \mathbb{R}^{n+2}$, and then passing to $\mathbb{R}^{n+2}$ itself, following a sequence of steps just like those used to construct the topological index. First of all, $[M] \times [\mathbb{R}^2] = [M \times \mathbb{R}^2]$, and we have seen that $\big\langle B_2, [\mathbb{R}^2] \big\rangle = 1$. Therefore

$$\begin{aligned} \big\langle [L^* \otimes E], [M] \big\rangle &= \big\langle [L^* \otimes E], [M] \big\rangle \cdot \big\langle B_2, [\mathbb{R}^2] \big\rangle \\ &= \big\langle [L^* \otimes E] \times B_2, [M \times \mathbb{R}^2] \big\rangle. \end{aligned}$$

Next, $\varphi_*[U] = [M \times \mathbb{R}^2]$. Furthermore, if $\iota \colon U \to \mathbb{R}^{n+2}$ is the inclusion map then $\iota^![\mathbb{R}^{n+2}] = [U]$. Therefore $[M \times \mathbb{R}^2] = \varphi_*\iota^![\mathbb{R}^{n+2}]$ and so

$$\begin{aligned} \big\langle [L^* \otimes E] \times B_2, [M \times \mathbb{R}^2] \big\rangle &= \big\langle [L^* \otimes E] \times B_2, \varphi_*\iota^![\mathbb{R}^{n+2}] \big\rangle \\ &= \big\langle \iota_!\varphi^*([L^* \otimes E] \times B_2), [\mathbb{R}^{n+2}] \big\rangle \end{aligned}$$

by the functoriality of the index pairing. Comparing the beginning and end of our computation, we see that we have identified $\mathrm{Index}\,(D_E)$ with t–Index (D_E), as required. $\square$

11.4.8 Remark For the benefit of the reader who is familiar with characteristic class theory, we give a brief account of the reduction of the index theorem for hypersurfaces to a cohomological formula. We make use of the *Chern character* map $\mathrm{ch} \colon K^0(X) \to H_c^{ev}(X; \mathbb{Q})$, and we need to know that

$$\int_{\mathbb{R}^n} \mathrm{ch}(B_n) = 1.$$

Using the properties of the Chern character we may thus obtain

$$\begin{aligned} \text{t–Index}\,(E) &= \int_{\mathbb{R}^n} \mathrm{ch}\,\iota_!\varphi^*([L^* \otimes E] \times B_2) \\ &= \int_{M\times\mathbb{R}^2} \mathrm{ch}([L^* \otimes E] \times B_2) \\ &= \int_M \mathrm{ch}(L^*)\,\mathrm{ch}(E). \end{aligned}$$

The Index Theorem thus takes the form

$$\text{a-Index}(E) = \int_M I(M)\,\mathrm{ch}(E),$$

where the 'index class' $I(M)$ depends only on the geometry of M. The cohomological form of the general index theorem that is described in the next section is in fact the same as this; only the construction of the class $I(M)$ is more complicated.

11.4.9 EXAMPLE Suppose that M is a compact Riemann surface, equipped with the Spin^c-structure determined by the Dolbeault operator (Example 11.3.4). Let E be a *holomorphic line bundle* on M. Then the index of the twisted Dolbeault operator $\mathcal{D}_E$ may be identified, using Hodge theory, with the holomorphic Euler characteristic $\mathrm{Dim}\, H^0(M; \mathcal{O}(E)) - \mathrm{Dim}\, H^1(M; \mathcal{O}(E))$. The index class $I(M)$ must be of the form $1 + x$, where x is a two-dimensional cohomology class, and so the cohomological formula for the topological index is $c_1(E) + x$. Considering the special case of trivial L we find that $x = 1 - g$, where g is the genus, and so finally from the Index Theorem we obtain

$$\mathrm{Dim}\, H^0(M; \mathcal{O}(E)) - \mathrm{Dim}\, H^1(M; \mathcal{O}(E)) = 1 - g + c_1(E),$$

which is the simplest form of the *Riemann–Roch Theorem.*

11.5 The Index Theorem for Spin^c-Manifolds

In this section we shall give a very brief account of the solution of the index problem for general Spin^c-manifolds. Since a proper treatment of the subject requires a Kasparov product which is slightly more general than the one we have developed in this book, we shall present only an outline of the solution. But the interested reader will have no difficulty supplying the missing details, given some familiarity with vector bundle theory.

Let D be the Dirac operator on a closed, even-dimensional Spin^c-manifold and let E be a smooth, Hermitian vector bundle on M. As in the previous section, we are concerned with the problem of computing in topological terms the Fredholm index of the twisted Dirac operator D_E. But now we shall make no further assumption about the manifold M. We begin as we did in the previous section by embedding M into a Euclidean space $\mathbb{R}^K$, but now of some arbitrary even dimension. According to elementary differential topology, if $N(M)$ is the normal bundle for this embedding then there is a canonical, up to isotopy, diffeomorphism from $N(M)$ to an open subset of $\mathbb{R}^K$. In the situation considered in the previous section $N(M)$ was the trivial bundle $M \times \mathbb{R}^2$. Now it may be non-trivial, and our first task is to analyze its structure more closely.

We will need to know that there is a notion of Spin^c-structure on an *arbitrary* real vector bundle V. The definition proceeds in a way which is familiar from our earlier discussion: we define a *Dirac bundle* S for V to be a graded Hermitian bundle S equipped with a Clifford multiplication action $V \to \mathrm{End}(S)$, a *complex spinor bundle* to be a multigraded Dirac bundle of appropriate local type, and a *Spin^c-structure* on V to be a concordance class of complex spinor bundles. There is however an alternative approach to Spin^c-structures, which is more suited to our purposes in this section. This approach makes use of the principal bundles associated to certain Lie groups.

In Exercise 11.8.5 we introduce the compact Lie groups $\mathrm{Spin}^c(n)$, which are constructed from the Clifford algebra on n generators and which belong to an exact sequence

$$0 \longrightarrow S^1 \longrightarrow \mathrm{Spin}^c(n) \longrightarrow SO(n) \longrightarrow 1$$

of groups and homomorphisms. It may be shown that a complex spinor bundle for the vector bundle V determines and is determined by a reduction of the structure group of V from $GL(n,\mathbb{R})$ to $\mathrm{Spin}^c(n)$, along the homomorphism

$$\mathrm{Spin}^c(n) \longrightarrow SO(n) \longrightarrow GL(n,\mathbb{R}).$$

Thus we may regard a Spin^c-structure for V as determined by such a reduction of its structure group.

11.5.1 LEMMA *Any embedding of an n-dimensional Spin^c-manifold M into $\mathbb{R}^K$ gives rise to a Spin^c-structure on the normal bundle $N(M)$.* □

11.5.2 REMARK The lemma is a special case of the general principle that if two out of three vector bundles appearing in a short exact sequence can be equipped with a Spin^c-structure, then the third one can be equipped with a Spin^c-structure also; the corresponding principle for *orientations* will be familiar to the reader.

The next step in the proof of the Index Theorem is to elaborate Kasparov's K-homology theory in the following modest way. Suppose that G is a compact group and that X is a locally compact G-space. We define $K_p^G(X)$ just as we defined $K_p(X)$, from Fredholm modules (ρ, H, F), except we now require that

(a) H is equipped with a continuous unitary representation of G,
(b) the representation $\rho\colon C_0(X) \to \mathfrak{B}(H)$ is G-equivariant, and
(c) the operator F on H is G-equivariant.

Having introduced actions of G everywhere, we now require that all our earlier constructions should be carried out in a manner which is compatible with the new G-structure.

The equivariant Kasparov K-homology theory we obtain has all the features of the theory we have developed over the last several chapters (and indeed were it not for the clutter it would have caused we could have carried G along harmlessly throughout those chapters). But there is, in addition, one new product operation, which combines the old and new K-homology groups. Suppose that Z is the total space of a fiber bundle with structure group G, fiber F and base B. Then there is a Kasparov product

$$K_p(B) \otimes K_q^G(F) \to K_{p+q}(Z).$$

The Kasparov product we have considered up to now is recovered from the more general one by setting $G = \mathrm{pt}$ and $Z = B \times F$.

The new product is mirrored by a similar product operation for Spin^c-structures. If M is a Spin^c-manifold and if F is a smooth G-manifold with a G-equivariant Spin^c-structure[101] then any fiber bundle Z with base M, structure group G and fiber F is provided with a product Spin^c-structure.

Now the Euclidean spaces $\mathbb{R}^n$, considered as $SO(n)$-spaces in the natural way, do not possess equivariant Spin^c-structures. But considered as $\mathrm{Spin}^c(n)$-spaces via the homomorphism $\mathrm{Spin}^c(n) \to SO(n)$ they *do* possess canonical equivariant Spin^c-structures. Setting $G = \mathrm{Spin}^c(n)$ we obtain fundamental classes $[\mathbb{R}^n] \in K_n^G(\mathbb{R}^n)$, as well as 'reduced' classes $[\mathbb{R}^n] \in K_0^G(\mathbb{R}^n)$ whenever n is even.

There is a similar development of equivariant K-*theory*. In particular, there is a product

$$K^p(B) \otimes K_G^q(F) \to K^{p+q}(Z),$$

and there are Bott classes B_n in $K_G^n(\mathbb{R}^n)$, and in $K_G^0(\mathbb{R}^n)$ when n is even.

The formulation and proof of the Atiyah–Singer Index Theorem for Spin^c-manifolds now proceeds almost exactly as in the previous section. In fact there is even a pleasant simplification: given an embedding of M^n into $\mathbb{R}^K$, the topological index is defined by the streamlined formula

$$\text{t–Index}(E) \cdot B_{K-n} = \iota_! \varphi^*([E] \times B_{K-n}).$$

Here φ is the inverse of the canonical, up to isotopy, diffeomorphism from the normal bundle $N(M)$ to an open subset of $\mathbb{R}^K$; ι is the inclusion of that open subset into $\mathbb{R}^K$; and the product $[E] \times B_{K-n} \in K^0(N(M))$ is an instance of the pairing

$$K^0(M) \otimes K^0_{\mathrm{Spin}^c(K-n)}(\mathbb{R}^{K-n}) \to K^0(N(M)).$$

This product, which uses the given Spin^c-structure on $N(M)$ (and hence the Spin^c-structure on M), incorporates the line bundle L which appeared in the topological index of the previous section. By repeating the argument of the

[101] An *equivariant Spin^c-structure* is defined by a complex spinor bundle which is also a G-vector bundle in such a way that Clifford multiplication is a G-map.

previous section, but using the new product (and its naturality properties), we get the Index Theorem

$$\operatorname{Index}(D_E) = \text{t-Index}(E),$$

in the same way as before.

11.6 Toeplitz Index Theorems

In this section we shall formulate and prove an abstract index theorem for Toeplitz operators. In the next section we shall apply the abstract theorem to concrete Toeplitz index problems arising from complex analysis.

The following definition repackages ideas that were developed in Chapter 6 and reprised in Chapter 10 (see in particular the final part of Section 10.6).

11.6.1 DEFINITION Let M be the interior of a compact manifold $\overline{M}$ with boundary ∂M. A Riemannian metric g on M is *controlled* by ∂M if it is complete and if the topological coarse structure associated with ∂M is coarser than the metric coarse structure associated with g. In other words, a complete metric g is controlled by ∂M if and only if $\overline{M}$ is a coarse compactification for the metric coarse structure on M associated with g, in the sense of Definition 10.6.7.

The present change of terminology (from 'coarse compactification' to 'controlled metric') reflects a change in emphasis: instead of tailoring a compactification to a metric we shall now be adapting a metric on M to the pre-existing boundary ∂M.

The following calculation, which is left to the reader, provides the simplest means of constructing controlled metrics on M.

11.6.2 LEMMA *Let M be the interior of a compact manifold $\overline{M}$ with boundary ∂M and let g be a complete Riemannian metric on M. Suppose that for every smooth function f on $\overline{M}$ the gradient of f (with respect to the metric g) vanishes at infinity on M. Then g is controlled by ∂M.* □

11.6.3 EXAMPLE The Poincaré metric on the open unit disk is controlled by the boundary unit circle.

If M is a complete Riemannian manifold and if D is a Dirac operator on M then D is an essentially selfadjoint Hilbert space operator. This follows from Proposition 10.2.11 because every Dirac operator has unit propagation speed. The following proposition reformulates in the current terminology an important functional-analytic consequence of the additional hypothesis that the Riemannian metric on M be controlled by a boundary ∂M:

11.6.4 PROPOSITION *Let M be the interior of a compact manifold $\overline{M}$ with boundary ∂M, and suppose that M is equipped with a Riemannian metric* *which is controlled by* ∂M. *Let* D *be a Dirac operator on* M *and let* $F = \chi(D)$, *where* χ *is a normalizing function. If* f *is any smooth function on* $\overline{M}$ *then the commutator* $[F, f]$ *is a compact operator.*

PROOF See Proposition 10.6.9. $\square$

The proposition furnishes us with a useful means of constructing Schrödinger pairs of the sort considered in Chapter 8:

11.6.5 COROLLARY *With the hypotheses of the previous proposition, if* $|f| \leqslant 1$ *on* M *and* $f^2 = 1$ *on* ∂M *then* F *and* f *comprise a strong Schrödinger pair.*

PROOF It suffices to prove that $(1 - f^2) \cdot (1 - F^2)$ is a compact operator. But since $1 - F^2 = \varphi(D)$ for some $\varphi \in C_0(\mathbb{R})$, the compactness of the product follows from Proposition 10.5.2. $\square$

Exercise 11.8.14 presents an index theorem for Schrödinger pairs of the above sort.

We are now going to consider the consequences of an additional hypothesis, this time concerning the analysis of Dirac operators.

11.6.6 DEFINITION A Dirac operator D on a Dirac bundle S over a complete Riemannian manifold is *partially bounded from below* if there is some $\varepsilon > 0$ such that $\|Du\| \geqslant \varepsilon\|u\|$ for every smooth, compactly supported section of one of the graded components $S^{\pm}$ of the graded bundle S (if we need to specify which component, we shall say that D is partially bounded from below on S^+, or on S^-).

REMARK We shall see in the next section, as a consequence of a more general investigation, that the Dolbeault operator on the Poincaré disk has this property. (In fact we shall derive the analytic property of partial boundedness from a geometric property of M.) But the reader might enjoy verifying the inequality for the Poincaré disk directly.

11.6.7 PROPOSITION *Let* M *be the interior of a compact manifold* $\overline{M}$ *with boundary* ∂M, *and suppose that* M *is equipped with a Riemannian metric which is controlled by* ∂M. *Let* D *be a Dirac operator on* M *and suppose that* D *is partially bounded from below. Then*

(a) *the value* $0 \in \mathbb{R}$ *is an isolated point of the spectrum of* D (*if indeed it appears in the spectrum at all*),

(b) *the orthogonal projection* P *onto the Hilbert space kernel of* D *commutes modulo compact operators with every smooth function on* $\overline{M}$, *and*

(c) *if a smooth function* $f\colon \overline{M} \to \mathbb{C}$ *is identically zero on* ∂M *then the product* Pf *is a compact operator.*

PROOF Assume without loss of generality that D is partially bounded from below on S^-; that is, $\|Du\| \geqslant \varepsilon\|u\|$ for all sections u of S^-. By Exercise 10.9.4, the operator D^2 is essentially selfadjoint; and in view of this, conclusion (a) is equivalent to the assertion that there is some $\varepsilon > 0$ such that

$$\|D^2u\| \geqslant \varepsilon\|Du\|,$$

for every smooth, compactly supported section u. To prove the displayed inequality it suffices to consider sections u of S^+ and S^- separately. If u is a section of S^+ then $v = Du$ is a section of S^- and so $\|Dv\| \geqslant \varepsilon\|v\|$ by hypothesis. If u is a section of S^- then the Cauchy–Schwarz inequality implies that

$$\|D^2u\| \cdot \|u\| \geqslant \|Du\|^2,$$

and combining this with the hypothesized inequality $\|Du\| \geqslant \varepsilon\|u\|$ we get what we want. To prove the remaining parts of the proposition we note that in view of part (a) the projection P may be written in the form $\varphi(D)$, for some $\varphi \in C_0(\mathbb{R})$. From here, the argument is the same as one we used in the proof of Proposition 10.6.9. According to Proposition 10.5.6 the operator $P = \varphi(D)$ lies in the C^*-algebra $C^*_{\text{metric}}(M)$ associated to the metric coarse structure on M provided by the Riemannian metric g. Since g is controlled by ∂M the C^*-algebra $C^*_{\text{metric}}(M)$ is contained within the C^*-algebra $C^*_{\text{top}}(M)$ associated to the topological coarse structure on M provided by ∂M. But Theorem 6.5.1 shows that every operator in $C^*_{\text{top}}(M)$ has the required commutation properties. □

We are now ready to formulate and prove our abstract Toeplitz index theorem.

11.6.8 DEFINITION Let D be a Dirac operator on a complete Riemannian manifold M and denote by P the orthogonal projection onto the kernel of D. If f is any bounded, continuous function on M then the *Dirac–Toeplitz operator* $T_f = PfP$ with symbol f is the compression to Kernel(D) of the operator of pointwise multiplication by f on $L^2(M; S)$.

As in Chapter 2, we can speak of $N \times N$ Dirac–Toeplitz systems, which are $N \times N$ matrices of Dirac–Toeplitz operators $T_{f_{ij}}$ of the above sort. The symbol of the system is the matrix-valued function $F = [f_{ij}]$.

11.6.9 THEOREM *Let M be the interior of a compact Spin^c-manifold $\overline{M}$ with boundary ∂M, and suppose that M is equipped with a Riemannian metric which is controlled by ∂M. Let D be a Dirac operator on M associated to the given Spin^c-structure and suppose that D is partially bounded from below. Let $[f_{ij}]$ be a continuous, matrix-valued function on $\overline{M}$ whose restriction to ∂M is unitary. Then the Dirac–Toeplitz system $T_{[f_{ij}]}$ is Fredholm and*

$$\operatorname{Index}(T_{[f_{ij}]}) = \pm\langle [f_{ij}], \partial[M]\rangle,$$

where $[f_{ij}] \in K^{-1}(\partial M)$ *denotes the* K*-theory class obtained by restricting the matrix-valued function* $[f_{ij}]$ *to* ∂M*, where* $\partial[M] \in K_{-1}(\partial M)$ *is boundary of the fundamental class of* M*, and where the sign is* + *if* D *is partially bounded from below on* S^+ *and is* − *if* D *is partially bounded from below on* S^-.

Remark It is of course possible to identify the class $\partial[M] \in K_{-1}(\partial M)$ explicitly — for example one could use Proposition 11.2.15 and a formal periodicity computation, or one could appeal to Exercise 10.9.14 — but we shall not need to do so below.

Proof It follows from Proposition 11.6.7 that the projection P onto Kernel(D) commutes with continuous functions on $\overline{M}$, modulo compact operators, and moreover $T_{f_1} \sim T_{f_2}$ if f_1 and f_2 are continuous functions on $\overline{M}$ which have a common restriction to ∂M. As a result, if $[g_{ij}]$ is a continuous, matrix-valued function on $\overline{M}$ which is the inverse of $[f_{ij}]$ on ∂M then

$$T_{[f_{ij}]}T_{[g_{ij}]} \sim I \sim T_{[g_{ij}]}T_{[f_{ij}]}$$

and hence $T_{[f_{ij}]}$ is Fredholm. The same reasoning shows that associated to P there is a *Dirac–Toeplitz extension*

$$\varphi_P \colon C(\partial M) \to \mathfrak{Q}(L^2(M;S))$$

which is defined by the formula

$$\varphi_P(f\,|_{\partial M}) = \pi(T_f)$$

for $f \in C(\overline{M})$ (here π is the quotient map from bounded operators to the Calkin algebra). We shall use this extension to compute the index of the Fredholm operator $T_{[f_{ij}]}$. According to part (a) of Proposition 11.6.7, the value 0 is an isolated point in the spectrum of D. If the interval $(-\varepsilon, \varepsilon)$ intersects the spectrum of D only at $0 \in \mathbb{R}$, and if χ is a normalizing function such that $\chi(x)^2 = 1$ whenever $|x| \geqslant \varepsilon$, then the operator $F = \chi(D)$ has the property that $F^2 = 1 - P$. Since it follows from Proposition 11.6.4 that F defines a relative Fredholm module for the pair given by $A = C(\overline{M})$ and $A/J = C(\partial M)$, we can use the formula for $\partial[M]$ provided by Proposition 8.5.6. We conclude that the class $\partial[M] \in K_{-1}(\partial M)$ is represented by the Dirac–Toeplitz extension. Hence

$$\langle [f_{ij}], \partial[M]\rangle = \langle [f_{ij}], [\varphi_P]\rangle.$$

The theorem now follows from the explicit form of the index pairing between an extension and a K-theory class. □

11.7 Index Theory on Strongly Pseudoconvex Domains

Let Ω be a bounded domain in $\mathbb{C}^m$ with smooth boundary. We shall say that a smooth, real-valued function $r\colon \mathbb{C}^m \to \mathbb{R}$ *defines* Ω if

$$\Omega = \{ z \in \mathbb{C}^m \, : \, r(z) > 0 \}$$

and if dr is nowhere vanishing on $\partial\Omega$.

The following notion was introduced in Chapter 2, although not precisely defined there.

11.7.1 DEFINITION A smooth, bounded domain $\Omega \subseteq \mathbb{C}^m$ is *strongly pseudoconvex* if it is defined by a smooth function r with the property that at every point of $\partial\Omega$ and for every $a \in \mathbb{C}^m \setminus \{0\}$,

$$\sum_i a_i \frac{\partial r}{\partial z_i} = 0 \quad \Rightarrow \quad \sum_{i,j} \frac{\partial^2 r}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j < 0.$$

If the condition above is satisfied for one defining function then it is satisfied for any other function which also defines Ω. Furthermore, if Ω is strongly pseudoconvex then it is defined by some function r for which

$$\sum_{i,j} \frac{\partial^2 r}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j < 0$$

at every boundary point and for every $a \in \mathbb{C}^m \setminus \{0\}$. In other words, by choosing r appropriately the tangency condition $\sum_i a_i \frac{\partial r}{\partial z_i} = 0$ which appears in Definition 11.7.1 can be made superfluous.

11.7.2 EXAMPLE The unit ball in $\mathbb{C}^m$ is defined by the function $r(z) = 1 - \|z\|^2$, for which

$$\sum_{i,j} \frac{\partial^2 r}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j = - \sum_i |a_i|^2 < 0,$$

for every $a \in \mathbb{C}^m \setminus \{0\}$. Thus the unit ball is strongly pseudoconvex.

Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^m$. In Chapter 2 we introduced the Toeplitz extension

$$0 \longrightarrow \mathfrak{K}(H^2(\partial\Omega)) \longrightarrow \mathfrak{T}(\partial\Omega) \longrightarrow C(\partial\Omega) \longrightarrow 0\,.$$

Each continuous unitary matrix-valued function on $\partial\Omega$ determines both a K-theory class $[f_{ij}] \in K^1(\partial\Omega)$ and a Fredholm Toeplitz system $T_{[f_{ij}]}$. The *Toeplitz index problem* for Ω is to determine Index $(T_{[f_{ij}]})$ in terms of the K-theory class $[f_{ij}]$. In this section we shall sketch a solution of the Toeplitz index problem and present a complete account of a closely related index problem, based on the following variant of the Toeplitz construction:

11.7.3 DEFINITION Let Ω be an open subset of $\mathbb{C}^m$. The *Bergman space*[102] $A^2(\Omega)$ is the subspace of the Lebesgue space $L^2(\Omega)$ which consists of the square-integrable holomorphic functions on Ω. The *Bergman projection* is the orthogonal projection from $L^2(\Omega)$ to $A^2(\Omega)$. If f is a bounded, measurable function on Ω then the *Bergman–Toeplitz operator* T_f is the compression to the Bergman space of the operator of pointwise multiplication by f on $L^2(\Omega)$.

The orthogonal projection from $L^2(\partial\Omega)$ onto the Hardy space $H^2(\partial\Omega)$ is usually called the *Szegö projection.* In an attempt to avoid confusing the two different sorts of Toeplitz operators (on $H^2(\partial\Omega)$ and on $A^2(\Omega)$) let us from here on refer to Toeplitz operators on the Hardy space $H^2(\partial\Omega)$ as *Szegö–Toeplitz operators*, and the associated extension of $C(\partial\Omega)$ by $\mathfrak{K}(H^2(\partial\Omega))$ as the *Szegö–Toeplitz extension.*

As far as index theory is concerned, the basic result about Bergman–Toeplitz operators is that they determine a *Bergman–Toeplitz extension*

$$0 \longrightarrow \mathfrak{K}(A^2(\Omega)) \longrightarrow \mathfrak{T}(\Omega) \longrightarrow C(\partial\Omega) \longrightarrow 0\,.$$

This follows from the following result, which we shall prove later in this section:

11.7.4 THEOREM *Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^m$ and let $P\colon L^2(\Omega) \to L^2(\Omega)$ be the Bergman projection. If f is a continuous function on $\overline{\Omega}$ then the commutator $[P, f]$ is a compact operator on $L^2(\Omega)$. If in addition f vanishes on $\partial\Omega$ then the product Pf is compact.*

Our treatment of the Szegö–Toeplitz index problem is based on the following result, which we shall *not* prove:

11.7.5 THEOREM *Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^m$. The Bergman–Toeplitz extension for Ω is unitarily equivalent to the Szegö–Toeplitz extension.* □

The idea of the proof is simple enough: restrict a function $\varphi \in A^2(\Omega)$ to the boundary $\partial\Omega$ to obtain a function in $H^2(\partial\Omega)$ (recall that the Hilbert space $H^2(\partial\Omega)$ is generated from boundary values of holomorphic functions on Ω). Unfortunately this simple idea is complicated by the fact that restriction to the boundary is not a bounded Hilbert space operator, and some 'renormalization' must therefore be applied to build a suitable bounded operator from $A^2(\Omega)$ to $H^2(\partial\Omega)$. Exercise 11.8.13 shows how to carry out the renormalization in the case where Ω is the unit ball in $\mathbb{C}^m$. The general case is unfortunately beyond our reach here. So instead of pushing the analysis of Szegö–Toeplitz operators any further we shall instead focus on the Bergman–Toeplitz extension and the associated Bergman-Toeplitz index problem. The reader who is willing to take

[102] Compare Exercise 2.9.12.

Theorem 11.7.5 for granted (or the reader who is willing to look elsewhere for the proof) will immediately obtain an index theorem for Szegö–Toeplitz operators from the index theorem for Bergman–Toeplitz operators which is proved below.

We are going to apply some quite non-trivial geometric ideas to Bergman–Toeplitz operator theory. The key observation which gets us started is this:

11.7.6 LEMMA *Let Ω be any domain in $\mathbb{C}^m$ and let h be any Hermitian metric on Ω. The map*

$$\varphi \mapsto 2^{-\frac{m}{2}} \varphi dz_1 \wedge \cdots \wedge dz_m$$

defines a unitary isomorphism from the Lebesgue space $L^2(\Omega)$ to the Hilbert space of those type $(m, 0)$-forms on Ω which are square-integrable for the metric h.

PROOF Suppose first that h is the restriction to Ω of the standard Hermitian metric on $\mathbb{C}^m$. Then the map is a unitary isomorphism, as required. To deal with the general case we simply note that the inner product on square-integrable $(m, 0)$-forms is given by the integral

$$\langle \varphi dz_1 \wedge \cdots \wedge dz_m, \psi dz_1 \wedge \cdots \wedge dz_m \rangle = \int_\Omega \varphi \bar{\psi} dz_1 \wedge \cdots \wedge dz_m \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_m.$$

Hence the inner product is actually independent of the choice of Hermitian metric h on Ω. □

11.7.7 REMARK It follows from the lemma that the Hilbert space of square-integrable type $(m, 0)$-forms on Ω is invariantly defined and does not depend on any choice of Hermitian metric.[103] We shall use this fact without further comment below.

Thanks to Lemma 11.7.6, Theorem 11.7.4 is equivalent to the following assertion:

11.7.8 THEOREM *Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^m$ and denote by P the orthogonal projection onto the holomorphic forms within the Hilbert space of square-integrable type $(m, 0)$-forms on Ω. If f is a continuous function on $\overline{\Omega}$ then the commutator $[P, f]$ is a compact operator. If in addition f vanishes on $\partial\Omega$ then the product Pf is compact.*

Notice the resemblance between Theorem 11.7.8 and parts (b) and (c) of Proposition 11.6.7 in the previous section. We shall prove Theorem 11.7.8 using Proposition 11.6.7 and the following result:

[103] The reader may be familiar with the related fact that the Hilbert space of middle-dimensional forms on a Riemannian manifold is conformally invariant. For surfaces this result amounts to the same thing as our lemma above.

11.7.9 THEOREM *Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^m$. There is a Hermitian metric* h *on* Ω *such that*

(a) *the Riemannian metric determined by* h *is controlled by* $\partial\Omega$, *and*

(b) *if* $\mathcal{D} = \bar{\partial} + \bar{\partial}^*$ *is the modified Dolbeault operator on* Ω *assembled from the complex*

$$\Omega^{m,0}(M) \xrightarrow{\bar{\partial}} \Omega^{m,1}(M) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{m,m}(M)$$

then the Hilbert space operator $\mathcal{D}$ *is bounded below on all but* $\Omega^{m,0}(M)$.

11.7.10 REMARK The Dolbeault operator $\mathcal{D}$ which appears in the above theorem differs from the one we considered in Section 11.1. The new operator acts on the Dirac bundle $\bigwedge^{m,*} M$, which is the tensor product of the Dirac bundle $\bigwedge^{0,*} M$ considered in Section 11.1, with the line bundle $\bigwedge^{m,0} M$. But in the present situation, where Ω is a domain in $\mathbb{C}^m$, the line bundle $\bigwedge^{m,0} M$ is trivial, and so the new and old Dolbeault operators determine the same K-homology class, namely the fundamental class associated to the Spin^c-structure which Ω inherits from $\mathbb{C}^m$.

It follows from part (b) of Theorem 11.7.9 that the kernel of $\mathcal{D}$ consists exactly of the square-integrable holomorphic forms on Ω of type $(m, 0)$ and that $\mathcal{D}$ is partially bounded from below. So Theorem 11.7.9 and Proposition 11.6.7 together imply Theorems 11.7.4 and 11.7.8.

To prove Theorem 11.7.9 we need to introduce some ideas from Kähler geometry. One says that a Hermitian metric $h = g + \sqrt{-1}a$ on Ω is a *Kähler metric* if $da = 0$, in which case the associated *Kähler form* is $\omega = -\frac{1}{2}a$. If $z_1, \dots, z_m$ are complex local coordinate functions on M, and if we decompose each z_j into its real and imaginary parts $z_j = x_j + \sqrt{-1}y_j$, then the metric h is determined by the quantities $h_{ij} = h(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. One writes

$$h = \sum_{i,j} h_{ij}\, dz_i \otimes d\bar{z}_j,$$

and the Kähler form is then

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j} h_{ij}\, dz_i \wedge \bar{d}z_j.$$

This is interpreted as a real 2-form by writing $dz_i = dx_i + \sqrt{-1}\,dy_i$ and expanding the products in the sum.

The following two propositions will constitute most of the proof of Theorem 11.7.9.

11.7.11 PROPOSITION *Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^m$. There is a Kähler metric on Ω such that*

(a) *the Kähler 2-form ω is the de Rham differential of a uniformly bounded 1-form η, and*
(b) *the underlying Riemannian metric is controlled by $\partial\Omega$.*

REMARK Our argument below will produce a *complex-valued* η such that $d\eta = \omega$. To obtain a real form one can simply take the real part, but in fact complex forms are perfectly adequate for our purposes.

PROOF The domain Ω may be defined by a smooth function r for which the inequality of Definition 11.7.1 is absolute, as in Example 11.7.2. Given such a defining function r, the metric[104]

$$\begin{aligned}\sum_{i,j} h_{ij}\, dz_i \otimes d\bar{z}_j &= -\sum_{i,j} \frac{\partial^2 \log(r)}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j \\ &= \frac{1}{r^2}\sum_{i,j} \frac{\partial r}{\partial z_i}\frac{\partial r}{\partial \bar{z}_j} dz_i \otimes d\bar{z}_j - \frac{1}{r}\sum_{i,j}\frac{\partial^2 r}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j\end{aligned}$$

has the required properties. Note that since r is real-valued the first term in the final display line is positive-semidefinite, while the second term is positive-definite thanks to our choice of r. Thus the formula really does define a metric. To prove (a), note that

$$\omega = -\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log(r) = -\frac{\sqrt{-1}}{2}\sum_{i,j}\frac{\partial^2 \log(r)}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j.$$

Take

$$\eta = -\frac{\sqrt{-1}}{2}\bar{\partial}\log(r) = -\frac{\sqrt{-1}}{2}\sum_j \frac{\partial \log(r)}{\partial \bar{z}_j} d\bar{z}_j$$

and compute. The remaining parts of the lemma are similar computations. □

11.7.12 EXAMPLE If Ω is the unit disk in $\mathbb{C}$ and r is the defining function of Example 11.7.2, then the metric provided by the above proof is the usual Poincaré metric, for which the conclusions of the theorem are well known. If Ω is the unit ball in $\mathbb{C}^m$ then the metric provided by the theorem defines m-dimensional complex hyperbolic space.

If h is any Kähler metric (or indeed any Hermitian metric) on a manifold M then for every $x \in M$ there are coordinates $z_1, \ldots, z_m$ around x such that $h_{ij} = \delta_{ij}$ at x. The Kähler form is then $\omega = \sum_i dx_i \wedge dy_i$ at x (this simple

[104] Here $\frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i})$ and $\frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i})$.

formula explains the factor of $-\frac{1}{2}$ in the definition of ω). Using this one can compute that the operator

$$L\colon \sigma \mapsto \sigma \wedge \omega,$$

which maps k-forms to $(k+2)$-forms, is *injective* as long as $k < m$.[105] We shall use below the fact that, considered as a map on square-integrable forms, L is not only injective but *bounded below*. This follows from the above pointwise computation of ω, which shows that up to unitary equivalence the structure of ω at any two points of M is the same. For the same reason L is also bounded above — that is to say it is a bounded linear operator for the L^2-norm.

11.7.13 PROPOSITION *Let Ω be a complete Kähler manifold of complex dimension m and suppose the Kähler form ω is the de Rham differential of a uniformly bounded 1-form η. If $k \neq m$ then the de Rham Laplacian $D^2 = (d+d^*)^2$ is bounded below on the Hilbert space of square-integrable k-forms.*

PROOF To begin the proof we remark that, since D^2 is essentially selfadjoint (Exercise 10.9.4), it suffices to obtain an inequality of the form

$$\|\sigma\| \leqslant \text{constant} \cdot \|D^2\sigma\|$$

for smooth σ of compact support.

It suffices to prove the proposition for $k < m$. This is because on any oriented Riemannian n-manifold, the pairing

$$(\omega_1, \omega_2) \mapsto \int_M \omega_1 \wedge \omega_2$$

gives rise to a unitary equivalence between the action of D^2 on k-forms and the action of D^2 on $(n-k)$-forms. Having so restricted k, we make use of the fact that as long as $k < m$ the operator $L\colon \sigma \mapsto \sigma \wedge \omega$ is bounded below on L^2 k-forms. Thus if $k < m$ and if σ is any square integrable k-form then

$$\|\sigma\|^2 \leqslant \text{constant} \cdot \|\sigma \wedge \omega\|^2.$$

Next, if σ is smooth and compactly supported, then

$$\begin{aligned} \|\sigma \wedge \omega\|^2 &= \langle \sigma \wedge \omega, \sigma \wedge \omega \rangle \\ &= \pm \langle \sigma \wedge \omega, d(\sigma \wedge \eta) - d\sigma \wedge \eta \rangle \\ &= \pm \langle \sigma \wedge \omega, d(\sigma \wedge \eta) \rangle \mp \langle \sigma \wedge \omega, d\sigma \wedge \eta \rangle. \end{aligned}$$

[105] In low dimensions it is possible to carry out the computation by brute force; in higher dimensions the computation, which yields a proof of the *Lefschetz Hyperplane Theorem* of algebraic geometry, is surprisingly intricate. But one can, for instance, use the representation theory of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ to organize matters into a manageable form. See [58].

The first of the two inner products which appear on the last line of the display may be estimated as follows:

$$\begin{aligned}|\langle \sigma \wedge \omega, d(\sigma \wedge \eta)\rangle| &= |\langle d^*(\sigma \wedge \omega), \sigma \wedge \eta\rangle| \\ &\leqslant \text{constant} \cdot \|d^*(\sigma \wedge \omega)\| \cdot \|\sigma\|,\end{aligned}$$

where in the last step we applied the Cauchy–Schwarz inequality and the boundedness of η. The second inner product is estimated as follows:

$$\begin{aligned}|\langle \sigma \wedge \omega, d\sigma \wedge \eta\rangle| &\leqslant \text{constant} \cdot \|\sigma \wedge \omega\| \cdot \|d\sigma\| \\ &\leqslant \text{constant} \cdot \|\sigma\| \cdot \|d\sigma\|.\end{aligned}$$

Here the first inequality uses Cauchy–Schwarz and the boundedness of η, and the second uses the boundedness from above of the map $\sigma \mapsto \sigma \wedge \omega$. If we combine both estimates with the identity

$$\|D\varphi\|^2 = \|d\varphi\|^2 + \|d^*\varphi\|^2$$

we obtain

$$\begin{aligned}\|\sigma\|^2 &\leqslant \text{constant} \cdot \|\sigma \wedge \omega\|^2 \\ &\leqslant \text{constant} \cdot (\|d^*(\sigma \wedge \omega)\| + \|d\sigma\|)\|\sigma\| \\ &\leqslant \text{constant} \cdot (\|D(\sigma \wedge \omega)\| + \|D\sigma\|)\|\sigma\|.\end{aligned}$$

Thus $\|\sigma\| \leqslant \text{constant} \cdot (\|D(\sigma \wedge \omega)\| + \|D\sigma\|)$, and an application of the elementary inequality $(a+b)^2 \leqslant 2(a^2+b^2)$ yields

$$\|\sigma\|^2 \leqslant \text{constant} \cdot (\|D(\sigma \wedge \omega)\|^2 + \|D\sigma\|^2).$$

We are almost done, but to complete the argument we must invoke an important symmetry property of Kähler manifolds. We noted above that every Hermitian metric may be reduced to the standard form $h_{ij} = \delta_{ij}$ at any given point. Now it may be shown that a Hermitian metric on M is a Kähler metric if and only if at each point x of M there are complex coordinates $z_1, \ldots, z_m$ so that

$$h_{ij} = \delta_{ij} + O(|z|^2)$$

near x. Working in these coordinates one can calculate that *the operator* $\sigma \mapsto \sigma \wedge \omega$ *commutes with* D^2. Invoking this fact and using also the boundedness of L we conclude that

$$\begin{aligned}\|D(\sigma \wedge \omega)\|^2 &= \langle D(\sigma \wedge \omega), D(\sigma \wedge \omega)\rangle \\ &= \langle D^2(\sigma \wedge \omega), \sigma \wedge \omega\rangle \\ &= \langle D^2\sigma \wedge \omega, \sigma \wedge \omega\rangle \\ &\leqslant \text{constant} \cdot \|D^2\sigma\| \cdot \|\sigma\|.\end{aligned}$$

Since the Cauchy–Schwarz inequality implies that $\|D\sigma\|^2 \leqslant \|D^2\sigma\| \cdot \|\sigma\|$, we conclude that $\|\sigma\| \leqslant \text{constant} \cdot \|D^2\sigma\|$, as required. $\square$

REMARK Since both D and D^2 are essentially selfadjoint, it follows from spectral theory that D is bounded below whenever D^2 is. Thus, if $\deg(\sigma) \neq m$ then

$$\langle D^2 \sigma, \sigma \rangle \geqslant \text{constant} \cdot \langle \sigma, \sigma \rangle,$$

for some non-zero constant.

PROOF OF THEOREM 11.7.9 Let us equip Ω with the metric provided by Proposition 11.7.11, which is controlled by $\partial\Omega$, as required. According to Proposition 11.7.13 the de Rham Laplacian D^2 is bounded below on k-forms as long as $k \neq m$. But a calculation in the local coordinates introduced in the proof of Proposition 11.7.13 shows that $\mathcal{D}^2 = \frac{1}{2} D^2$, and therefore $\mathcal{D}^2$ is similarly bounded below. It follows from this and the remark above that

$$\langle \mathcal{D}^2 \sigma, \sigma \rangle \geqslant \text{constant} \cdot \langle \sigma, \sigma \rangle,$$

if σ is a form of type (m, q) and $q \neq 0$. Therefore the Hilbert space operator $\mathcal{D}$ is bounded below on all but $\Omega^{m,0}(M)$, as required. $\square$

Theorem 11.7.9 and Theorem 11.6.9 provide the following preliminary solution of the Bergman–Toeplitz index problem:

11.7.14 THEOREM *Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^m$, with the Spin^c-structure it inherits as an open subset of a Euclidean space. Equip $\partial\Omega$ with the boundary Spin^c-structure. If $T_{[f_{ij}]}$ is a Bergman–Toeplitz system whose symbol is a continuous, matrix-valued function on $\overline{\Omega}$ which is unitary-valued on $\partial\Omega$ then*

$$\operatorname{Index}(T_{[f_{ij}]}) = \langle [f_{ij}], \partial[\Omega] \rangle,$$

where $[f_{ij}] \in K^{-1}(\partial M)$ is the class obtained by restricting the symbol of the Toeplitz system $T_{[f_{ij}]}$ to ∂M. $\square$

The above index formula can be simplified and made purely topological, as follows. The inclusion $\iota \colon \Omega \to \mathbb{C}^m$ induces a K-theory homomorphism

$$\iota_! \colon K^0(\Omega) \to K^0(\mathbb{C}^m) \cong \mathbb{Z}.$$

By composing with the boundary map $\partial \colon K^{-1}(\partial\Omega) \to K^0(\Omega)$ we obtain a topological index homomorphism

$$\text{t-Index} = \iota_! \circ \partial \colon K^{-1}(\partial\Omega) \to \mathbb{Z},$$

and a corresponding purely topological formula for the index of Toeplitz operators.

11.7.15 THEOREM *With the hypotheses of the previous theorem,*

$$\mathrm{Index}\,(T_{[f_{ij}]}) = \text{t-Index}\,([f_{ij}]).$$

PROOF We proved in Proposition 8.7.5 that $\langle \partial[f_{ij}], [\Omega]\rangle = \langle [f_{ij}], \partial[\Omega]\rangle$, and so it follows from Theorem 11.7.14 that $\mathrm{Index}\,(T_{[f_{ij}]}) = \langle \partial[f_{ij}], [\Omega]\rangle$. But by Proposition 11.2.12 and naturality of the index pairing,

$$\langle \partial[f_{ij}], [\Omega]\rangle = \langle \partial[f_{ij}], \iota^![\mathbb{C}^m]\rangle = \langle \iota_!\partial[f_{ij}], [\mathbb{C}^m]\rangle.$$

The theorem follows, since by Proposition 11.4.5 we identify $K^0(\mathbb{C}^m)$ with $\mathbb{Z}$ by pairing with the fundamental class $[\mathbb{C}^m]$. $\square$

11.8 Exercises

11.8.1 Suppose given a complex

$$H^0 \xrightarrow{d} H^1 \xrightarrow{d} \cdots \xrightarrow{d} H^n$$

of Hilbert spaces and closed, densely defined linear operators (the requirement that $d^2 = 0$ includes the requirement that the range of each differential is included in the domain of the succeeding one).

(a) Show that the operator $D = d + d^*$, whose domain is the intersection of the domains of d and d^*, is a selfadjoint operator on the graded Hilbert space $H = \oplus_j H^j$.

(b) Show that if the odd unbounded operator D is Fredholm then the kernel of D identifies with the cohomology of the given complex and the index of D is the Euler characteristic of the complex.

(c) Conclude that the index of the de Rham operator on a closed manifold is the Euler characteristic of M.

11.8.2 Carry out the computations omitted from Example 11.1.4: show that the operator γ defined there does not depend on the choice of oriented orthonormal frame, and that γ anticommutes with $d + d^*$.

11.8.3 Let M be a compact Riemannian manifold of dimension $4k$ and let D be the Dirac operator constructed in Example 11.1.4, by equipping the Dirac bundle $\bigwedge^*_{\mathbb{C}} M$ with the grading operator $\gamma = (-1)^k e_1 e_2 \ldots e_{4k}$. Show that the index of D is the signature of the symmetric bilinear form

$$H^{2k}(M) \times H^{2k}(M) \to \mathbb{R}$$

on de Rham cohomology which is defined by forming the wedge product of two differential forms and then integrating the resulting $4k$-form over M.

11.8.4 The definition of the *signature operator* given in Example 11.1.4 may be extended to oriented Riemannian manifolds M of arbitrary even dimension $2l$ by equipping the Dirac bundle $\bigwedge_{\mathbb{C}}^* M$ with the grading operator $\gamma = i^l e_1 e_2 \ldots e_{2l}$. Show that if M is closed and if its dimension is not a multiple of 4 then the index of this signature operator is zero. (Hint: use complex conjugation to reverse the grading.)

The next several exercises briefly introduce the groups $\mathrm{Spin}(n)$ and $\mathrm{Spin}^c(n)$, by means of which Spin and Spin^c-structures are defined in other texts.

11.8.5 Identify $\mathbb{R}^n$ with the linear subspace of the Clifford algebra $\mathbb{R}_n$ spanned by the degree-one monomials e_j. Let L be the linear subspace of $\mathbb{R}_n$ spanned by the degree-two monomials $e_i e_j$, where $i \neq j$.

(a) Show that L is a Lie algebra under the usual commutator bracket $[X, Y] = XY - YX$.
(b) Show that if $X \in L$ then $\exp(X)$ is a unitary element of $\mathbb{R}_n$, and that conjugation with $\exp(X)$ maps the subspace $\mathbb{R}^n \subseteq \mathbb{R}_n$ onto itself.
(c) Show that the group $\mathrm{Spin}(n)$ generated by the exponentials $\exp(X)$ is a two-fold covering group of the group $SO(n)$.

11.8.6 Let $a = [a_{ij}]$ be a smooth, $SO(n)$-valued function on an open set in some manifold Riemannian M, and let w be a lifting of a to a $\mathrm{Spin}(n)$-valued function. Show that if two local orthonormal frames on M are related by the equations

$$e_i = \sum_j a_{ij} f_j$$

then left multiplication by w on $\mathbb{R}_n$ or on the complexification $\mathbb{C}_n$ gives an isomorphism between local spinor bundles defined by the two frames.

11.8.7 Let M be an oriented Riemannian manifold and let $E \to M$ be the principal $SO(n)$-bundle of oriented orthonormal frames on M. Let $\tilde{E}$ be a reduction of E to a principal $\mathrm{Spin}(n)$-bundle.

(a) Show that the vector bundle

$$S = \tilde{E} \times_{\mathrm{Spin}(n)} \mathbb{R}_n$$

is a real spinor bundle on M (we let $\mathrm{Spin}(n)$ act on $\mathbb{R}_n$ by left multiplication).
(b) Show that there is a one-to-one correspondence between isomorphism classes of reductions $\tilde{E}$, as above, and isomorphism classes of real spinor bundles on M which are compatible with the given orientation of M.

11.8.8 The group $\mathrm{Spin}^c(n)$ is the group of unitary elements in $\mathbb{C}_n$ generated by $\mathrm{Spin}(n)$ and the complex numbers of modulus one. There are thus group homomorphisms

$$\mathrm{Spin}(n) \longrightarrow \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n).$$

Show that the homomorphism $\mathrm{Spin}^c(n) \to \mathrm{SO}(n)$ appearing above is surjective with kernel S^1. Show that if M is an oriented Riemannian manifold then there is a one-to-one correspondence between Spin^c-structures on M which are compatible with the orientation and reductions to $\mathrm{Spin}^c(n)$ of the principal $\mathrm{SO}(n)$-bundle of oriented orthonormal frames on M.

11.8.9 Show that if two spinor bundles over the same Riemannian manifold are concordant then they are isomorphic.

11.8.10 Let M be a compact, odd-dimensional Spin^c-manifold with boundary ∂M. Prove that the Dirac operator on ∂M has zero index. (Hint: identify the index with the map induced by crushing ∂M to a point, factor this map through the inclusion $\partial M \to M$, and use the exactness of the K-homology sequence.)

This property, called the *cobordism invariance of the index*, was crucial to Atiyah and Singer's first proof of their index theorem. See [101].

11.8.11 Let M be a compact Spin^c manifold, of dimension n. The cap product (Exercise 9.8.9) with the fundamental class $[M]$ defines a homomorphism $P\colon K^p(M) \to K_{n-p}(M)$. Complete the following outline to show that P is an *isomorphism.*

(a) Extend the definition of P to non-compact manifolds M, as a map from $K^p(M)$ to $K^c_{n-p}(M)$, where the compactly supported K-homology group $K^c_r(M)$ is defined to be the direct limit $\varinjlim K_r(L)$ over all compact subsets L of M.

(b) Show that the map P (thus extended) is an isomorphism when $M = \mathbb{R}^n$.

(c) Use a Mayer–Vietoris argument to conclude that P is an isomorphism for all compact manifolds.

This is the K-theory form of Poincaré duality.

11.8.12 This exercise introduces an index-theoretic invariant of real-linear operators which is lost upon complexification. Let H be a *real* Hilbert space and let F be a *skew-adjoint* Fredholm operator on H. Show that the dimension of the kernel of F, modulo 2, is a homotopy invariant of T. Suppose now that H is real and 1-multigraded, so that there is given an isometric, graded operator ε on H such that $\varepsilon^2 = -I$. Show that if F is a selfadjoint odd Fredholm operator on H which commutes with ε then the dimension d of the kernel is divisible by 2 and that the congruence class of d modulo 4 is a homotopy invariant of F. What happens if the condition $\varepsilon^2 = -I$ is replaced by $\varepsilon^2 = +I$? What happens if $\mathbb{R}$ is replaced by $\mathbb{C}$?

11.8.13 Let Ω be the unit ball in $\mathbb{C}^m$. Use monomials $z_1^{k_1} \cdot \cdots \cdot z_m^{k_m}$ to construct orthonormal bases for the Hardy space $H^2(\partial\Omega)$ and the Bergman space $A^2(\Omega)$, and by defining a unitary isomorphism in terms of these bases, verify directly that the Bergman–Toeplitz and Szegö-Toeplitz extensions are unitarily equivalent.

11.8.14 Let M be the interior of a compact manifold $\overline{M}$ with boundary ∂M, and suppose that M is equipped with a Riemannian metric which is controlled by ∂M. Let D be a Dirac operator on a Dirac bundle S over M. Suppose also that W is a graded vector bundle over $\overline{M}$ and that A is an odd, selfadjoint endomorphism of W with $A^2 = 1$ on ∂M, so that (W, A) is a relative cycle defining a class in $K^0(M)$.

(a) Let $\tilde{D} = D\hat{\otimes}1$, $\tilde{A} = 1\hat{\otimes}A$, $\tilde{F} = \tilde{D}(1+\tilde{D}^2)^{-\frac{1}{2}}$. Show that the operators $\tilde{F}$ and $\tilde{A}$, acting on $L^2(M; S\hat{\otimes}W)$, comprise a graded strong Schrödinger pair.
(b) Show that the index of the Schrödinger operator $V(\tilde{F}, \tilde{A}) = \tilde{F} + (1-\tilde{F}^2)^{\frac{1}{2}}\tilde{A}$ is equal to the index pairing of the homology class $[D] \in K_0(M)$ with the K-theory class $[W, A] \in K^0(M)$ (use Lemma 8.7.14).
(c) Show that the 'Dirac–Schrödinger' operator $\tilde{D} + \tilde{A}$ is equal to the product of $V(\tilde{F}, \tilde{A})$ and an invertible positive (unbounded) operator. Thus $\tilde{D} + \tilde{A}$ is Fredholm (in an appropriate sense) and $\operatorname{Index}(\tilde{D} + \tilde{A}) = \operatorname{Index} V(\tilde{F}, \tilde{A})$.
(d) Suppose that D is the Dirac operator associated to a Spin^c-structure, and also that the graded bundle W is trivial, $W = \mathbb{C}^k \oplus \mathbb{C}^k$. Then the restriction of A to ∂M is of the form $\left(\begin{smallmatrix} 0 & u^* \\ u & 0 \end{smallmatrix}\right)$, where $u\colon \partial M \to U(k)$, and u defines a K-theory class $[u] \in K^{-1}(\partial M)$. Show that

$$\operatorname{Index}(\tilde{D} + \tilde{A}) = \text{t–Index}\,[u].$$

11.8.15 Let $e_1, \ldots, e_n$ be an orthonormal cotangent frame defined on an open subset U of a Riemannian manifold M. Use the frame to define the *real* spinor bundle S_U over U with constant fiber $\mathbb{R}_n$, as we did just prior to Definition 11.2.3. Show that if ∇ is a Dirac connection on $S_U \otimes \mathbb{C}$ which additionally compatible with the real structure on S_U (in the sense that the covariant derivatives of a real section are real), and if 1 denotes the section with constant value $1 \in \mathbb{R}_n$, then

$$\nabla_X(1) = -\frac{1}{2}\sum_j e_j \nabla_X^{LC}(e_j).$$

Use this formula to show that if S_M is a real spinor bundle on M then there is a unique connection on S which is compatible with both the Dirac bundle structure on S and its real structure.

11.8.16 Use the above exercise to compute the operator R in the Weitzenbock formula associated to a real spinor bundle. (This requires some Riemannian geometry.)

11.9 Notes

The Atiyah–Singer Index Theorem is a landmark achievement of twentieth century mathematics. The first proof of Atiyah and Singer relied quite heavily on ideas from bordism theory. The proof presented here is essentially the same as their second, K-theoretic proof, which was expounded in the papers [16, 15, 17] of Atiyah, Segal, and Singer. Simpler approaches to various cases of the Index Theorem have been found, but for the most general formulations of the theorem the K-theory approach still has no rivals. The third paper of the series [17] shows how various classical results — in particular, the Riemann–Roch Theorem — may be deduced from the Index Theorem. For more information about the Riemann–Roch Theorem and its applications, one may consult the books [65] (for Riemann surfaces) or [58, 77] (for higher dimensions).

The connection approach to Dirac operators is presented in [94] and [114], among other places. We refer the reader to these texts for a fuller account of the differential geometric calculations we have just hinted at here. The incorporation of Clifford algebras into K-theory and K-homology owes much to a paper of Atiyah, Bott and Shapiro [13]. Our vector bundle approach to Spin^c-structures, particularly the approach via reduced spinor bundles, is based on the discussion given by Baum and Douglas [23].

To formulate and prove the most general sort of index theorem, using the point of view developed in this book, it is best to come to grips with Kasparov's bivariant KK-theory [81]. The generalized products sketched out in Section 11.5 are comfortably contained within KK-theory.

The Toeplitz index problem was formulated by Venugopalkrishna [130], who proved that Toeplitz operators with invertible symbol on a strongly pseudoconvex domain are Fredholm. The Toeplitz Index Theorem was proved by Dynin [53] and Boutet de Monvel [29, 30]. Our general approach through K-homology is due to Baum and Douglas [24]. The specific method of proof, using complete Kähler metrics, is taken from the paper [64], except that we have taken from a celebrated paper of Gromov [60] his lovely proof that the Dolbeault operator in our context is partially bounded.

The 'Dirac–Schrödinger' operators of Exercise 11.8.14 are the subject of an extensive literature, beginning with the paper of Callias [37]. Some other articles on this subject are [3], [35], [89] and [107].

12

HIGHER INDEX THEORY

In this final chapter we shall introduce an important circle of ideas which relates the K-homology of spaces to the K-theory of group C^*-algebras. Consider, for example, a compact manifold M with *finite* fundamental group $G = \pi_1(M)$. Then the universal cover $\widetilde{M}$ of M is also a compact manifold. An elliptic differential operator D on a graded bundle S over M lifts to a similar operator $\widetilde{D}$ over $\widetilde{M}$, and this operator is equivariant with respect to the action of G. Thus the positively and negatively graded parts of $\mathrm{Kernel}(\widetilde{D})$ are finite-dimensional representations of G, and so the index of $\widetilde{D}$ is a formal difference of such representations. In fact the difference is an element of the K-theory of the complex group ring $\mathbb{C}[G]$. So by taking into account the action of the fundamental group, we refined the ordinary index of D, which is an element of $\mathbb{Z} = K_0(\mathbb{C})$, to a 'higher index' which is an element of $K_0(\mathbb{C}[G])$.

This simple construction is only possible when the fundamental group G is finite. However, it turns out that even if G is infinite we can always construct a 'higher index' of D which belongs to $K_0(C^*_r(G))$, the K-theory of the (reduced) group C^*-algebra of G. Moreover the higher index, like the ordinary index, depends only on the K-homology class of the operator D, and in this way we obtain a homomorphism

$$A\colon K_p(M) \to K_p(C^*_r(G))$$

which is called the *assembly map*.[106] Higher index theory and the assembly map are closely related to a range of mathematical issues including representation theory, topology of manifolds, and metrics of positive scalar curvature. We shall use the last of these as an example as we develop the theory.

12.1 Metrics of Positive Scalar Curvature

Let us recall a few notions from Riemannian geometry. If M is a Riemannian 2-manifold (a surface), then its *curvature* is a scalar-valued function on M; it is just the Gauss curvature of classical differential geometry. In higher dimensions the curvature of M is a more complicated object, namely a tensor R^i_{jkl} which

[106] A warning about notation: K_p on the left of the display refers to the K-homology of spaces, whereas K_p on the right of the display refers to the K-theory of C^*-algebras.

assigns a 'sectional' curvature to each two-dimensional surface element embedded in M. To obtain a scalar-valued measure of curvature one must contract the tensor by taking a trace. Specifically, one defines the *scalar curvature* to be

$$\kappa = \sum_{i,j,l} g^{jl} R^{i}_{jil},$$

where g is the metric tensor. Thus the scalar curvature $\kappa(x)$ at a point $x \in M$ is the 'mean value' of the Gaussian curvatures of the two-dimensional surface elements at x.

The Gaussian curvature of a surface completely determines its local isometry type. In contrast, one anticipates from its description as a mean value that in higher dimensions the scalar curvature will have an extremely weak influence on the geometry of M. Indeed, it is known that *any* function on M which is negative somewhere can arise as the scalar curvature of some Riemannian metric on M. It remains only to ask whether every M can also carry a metric whose scalar curvature is everywhere *positive*. The answer is no: there are topological obstructions to positive scalar curvature. One way of realizing these obstructions uses Spin geometry and the Dirac operator.

Suppose that S is a real spinor bundle over the Riemannian manifold M. It may be shown that there is a canonical Dirac connection on S (see Exercise 11.8.15), and we shall denote by D the Dirac operator formed from this Dirac connection according to the procedure of Equation 11.1.10. The square of this Dirac operator satisfies the *Lichnerowicz–Weitzenbock formula*[107]

$$D^2 u = \Delta u + \tfrac{1}{4}\kappa u \tag{12.1.1}$$

where $\Delta = \nabla^* \nabla$ is the Laplace operator associated to the Dirac connection and κ is the scalar curvature function.

If the Riemannian metric on M is complete (and in particular if M is compact), then M is complete for D in the sense of Definition 10.2.8, and so D is essentially selfadjoint. Using the Weitzenbock formula and the positivity of D, we obtain:

12.1.2 LEMMA *Let D denote the Dirac operator on a complete Riemannian Spin-manifold M, and suppose that the scalar curvature of M is bounded below by a strictly positive constant. Then there is an open interval around 0 which does not meet the spectrum of D.* □

This is the only property of metrics of positive scalar curvature that we shall need in the following discussion.

[107] See Exercise 11.8.16, or a textbook on Spin geometry such as [94] or [114].

12.1.3 COROLLARY *Let* M *be a compact, even-dimensional Riemannian Spin-manifold and let* D *be the spinor Dirac operator on* M. *If the Riemannian metric on* M *has positive scalar curvature, then* $\operatorname{Index}(D) = 0$.

PROOF From Lemma 12.1.2, the kernel of D is zero. But the index is the difference of the dimensions of the positively and negatively graded components of the kernel. So the index must be zero too. □

12.1.4 REMARK An alternative proof, while longer, clarifies the connection with K-homology. Recall from Example 10.9.6 that the index of D is equal to the image of the homology class $[D] \in K_0(M)$ under the homomorphism $K_0(M) \to K_0(\mathrm{pt}) \cong \mathbb{Z}$ induced by crushing M to a point. Now the homology class $[D]$ is defined by the operator $F = \chi(D)$, where χ is a normalizing function; and if there is a gap around 0 in the spectrum of D, then we may choose χ in such a way that $F = F^*$ and $F^2 = 1$. These are two of the three conditions for F to define a degenerate Fredholm module. The third condition is not satisfied over the algebra $C(M)$ (because F need not commute exactly with continuous functions on M) but it *is* satisfied over $\mathbb{C} = C(\mathrm{pt})$ (because F certainly commutes with constant functions). Consequently F defines a degenerate Fredholm module over $\mathbb{C}$, and the result follows.

According to the Atiyah–Singer Index Theorem, the index of a Dirac operator on a compact manifold can be calculated in topological terms. For the Dirac operator of a Spin-manifold, the topological expression for the index turns out to be the $\widehat{\mathcal{A}}$-*genus*, obtained by integrating over M a certain polynomial in the Pontrjagin classes. Corollary 12.1.3 and the Index Theorem together thus give rise to a topological obstruction to positive scalar curvature:

12.1.5 THEOREM *A compact, even-dimensional Spin-manifold with non-zero* $\widehat{\mathcal{A}}$-*genus cannot carry a metric of positive scalar curvature.* □

This theorem, however, does not tell the whole story about positive scalar curvature metrics on compact Spin-manifolds. For example, the $\widehat{\mathcal{A}}$-genus vanishes in dimensions congruent to 2 modulo 4. Consider then the case of closed, oriented surfaces: the scalar curvature of such a surface is just its classical Gaussian curvature, and (by the Gauss–Bonnet Theorem) the integral of the Gaussian curvature is equal to the Euler characteristic. Consequently, the only closed, oriented surface that can carry a metric of positive scalar curvature is the 2-sphere. But all closed, oriented surfaces are Spin-manifolds with vanishing $\widehat{\mathcal{A}}$-genus.

All the connected, closed, oriented surfaces other than the 2-sphere are examples of *aspherical* manifolds — that is, their universal covers are contractible. One of the long-standing conjectures in the theory of positive scalar curvature is an ambitious generalization of the phenomenon we have just observed in the two-dimensional case:

12.1.6 CONJECTURE *No compact aspherical manifold carries a metric of positive scalar curvature.*

In the next section we shall prove a special case of this conjecture.

12.2 Non-Positive Sectional Curvature

The following classical theorem in Riemannian geometry provides a practical way of producing examples of aspherical manifolds.

12.2.1 CARTAN–HADAMARD THEOREM *Every complete, simply connected Riemannian manifold of non-positive sectional curvature is contractible. Consequently, a compact connected manifold which admits a metric of non-positive sectional curvature is aspherical.*

OUTLINE OF THE PROOF Let M be a complete and simply connected manifold of non-positive sectional curvature. The curvature condition means that *all* the two-dimensional surface elements in M have non-positive Gaussian curvature, or that geodesics in M diverge *more rapidly* than they do in ordinary Euclidean space. Consequently, any two points of M are joined by a unique geodesic, which is also minimal. It follows that the exponential map

$$\exp\colon T_x M \to M,$$

which associates to a tangent vector $v \in T_x M$ the point at distance $|v|$ from x along the unique geodesic ray setting out in the direction specified by v, is 'expansive' — that is, $d(\exp(v), \exp(v')) \geqslant |v - v'|$ — and is a diffeomorphism of the Euclidean space $T_x M$ onto M. □

We are going to prove that Conjecture 12.1.6 holds for the compact aspherical manifolds produced by the Cartan–Hadamard Theorem. Let M be a manifold which admits *one* metric of non-positive sectional curvature. Our task is to show that *no other* metric on M has positive scalar curvature. We therefore seek structural features of M that are independent of the choice of metric.

12.2.2 PROPOSITION *Let M be a compact connected manifold and let $\widetilde{M}$ be its universal cover. The metric coarse structure on $\widetilde{M}$ is independent of the choice of Riemannian metric on M.*

Each Riemannian metric on M lifts to one on $\widetilde{M}$; the proposition tells us that all these different metrics on $\widetilde{M}$ induce the same metric coarse structure.

PROOF Let d and d' be the distance functions on $\widetilde{M}$ induced by Riemannian metrics g and g' on M. A compactness argument gives us a constant C such that

$$|v|_g \leqslant C|v|_{g'} \quad \text{and} \quad |v|_{g'} \leqslant C|v|_g$$

for all tangent vectors v. Therefore

$$d(x,y) \leqslant Cd'(x,y), \quad d'(x,y) \leqslant Cd(x,y),$$

and the two metrics are coarsely equivalent. □

Suppose now that M is a compact n-manifold admitting a metric of non-positive sectional curvature, and let $W = \widetilde{M}$ be its universal cover. We consider *geodesic rays* $\gamma\colon [0,\infty) \to W$. Two such rays γ_1 and γ_2 are said to be *parallel* if the distance $d(\gamma_1(t), \gamma_2(t))$ remains bounded as $t \to \infty$. The parallelism classes of geodesic rays constitute the *visual boundary* ∂W of W.

It is clear that the visual boundary of Euclidean space $\mathbb{R}^n$ is the sphere S^{n-1}. In general, the exponential map at a point $w_0 \in W$ provides a diffeomorphism from $\mathbb{R}^n \cong T_{w_0}W$ onto W. Each geodesic ray in W is parallel to one and only one such ray starting at w_0, so that the exponential map extends to an identification between $\partial\mathbb{R}^n = S^{n-1}$ and ∂W. We use this identification to give ∂W the structure of a smooth manifold, and we let $\overline{W} = W \cup \partial W$ be the *visual compactification* of W, which is a compact manifold with boundary.

REMARK By our definitions, the topology and the smooth structure on ∂W appear to depend on the choice of basepoint w_0. This is a delicate question; but in any event, the apparent dependence on the choice of w_0 is harmless from the point of view of the argument to be presented below.

12.2.3 PROPOSITION *Let M be a compact manifold of non-positive sectional curvature. Then the visual compactification of $\widetilde{M}$ is a coarse compactification.*

PROOF Recalling the definitions, we must show that the topological coarse structure induced by the visual boundary is coarser than the metric coarse structure, or in other words that if $\{x_n\}$ and $\{y_n\}$ are two sequences in $\widetilde{M}$, with $d(x_n, y_n)$ bounded, and if $\{x_n\}$ converges to a point on the visual boundary, then $\{y_n\}$ converges to that same point. This property is certainly true for Euclidean space; if we use the expansiveness of the exponential map we can therefore infer that it is also true for $\widetilde{M}$. □

We need a result that could have been proved in Chapter 11.

12.2.4 LEMMA *Let N be a Spin (or Spin^c) manifold. The K-homology fundamental class $[N]$ of N is not equal to zero.*

PROOF Proposition 11.4.5 shows that this is so for an even-dimensional Euclidean space, and the odd-dimensional case follows by taking the product with $\mathbb{R}$. For a general N, let $U \subseteq N$ be a coordinate patch, diffeomorphic to $\mathbb{R}^n$. Then $[N]$ restricts to $[U]$, and $[U] \neq 0$. □

12.2.5 THEOREM *Let M be a compact n-manifold which admits a metric of non-positive sectional curvature. Then M has no metric of positive scalar curvature.*

PROOF Suppose that M does admit a metric of positive scalar curvature, and equip its universal covering $W = \widetilde{M}$ with the lifted metric, which also has (uniformly) positive scalar curvature. Since W is contractible its tangent bundle is trivial, and so W admits a real spinor bundle S. Let D denote the Dirac operator for the Spin-connection on S; then, according to Lemma 12.1.2, there is a gap at zero in the spectrum of D. Choose a normalizing function χ which is equal to ± 1 on the spectrum of D, and let $F = \chi(D)$. Then $F^2 = 1$ and $F = F^*$. By Proposition 10.6.9 and Proposition 12.2.3, the operator F commutes modulo compacts with the continuous functions on the visual compactification $\overline{W}$ of W. Thus F defines a Fredholm module and therefore a homology class $[F] \in K_n(\overline{W})$, and, by Definition 10.6.6, $i^![F] = [W] \in K_n(W)$, where i is the inclusion of the open subset W into $\overline{W}$. By exactness in the K-homology sequence

$$K_n(\overline{W}) \xrightarrow{i^!} K_n(W) \xrightarrow{\partial} K_{n-1}(\partial W)$$

we find that $\partial[W] = 0 \in K_{n-1}(\partial W)$. But $\partial[W] = [\partial W]$ by Proposition 11.2.15, and $[\partial W] \neq 0$ by Lemma 12.2.4. This is a contradiction. $\square$

In the next section we shall reformulate the underlying principles of this argument in terms of an assembly map.

12.3 Coarse Geometry and Assembly Maps

Let X be a locally compact space which is equipped with a proper coarse structure. Recall from Chapter 6 that if $\rho\colon C_0(X) \to \mathfrak{B}(H)$ is a non-degenerate representation, then we define $C^*_\rho(X)$ to be the C^*-algebra of operators on H generated by those operators which are locally compact and controlled.

12.3.1 DEFINITION In the above situation, let $D^*_\rho(X)$ denote the C^*-algebra generated by those operators on H which are controlled and which commute modulo compact operators with $\rho(f)$, for every $f \in C_0(X)$.

Extending the terminology of Definition 5.4.6 to the non-compact case, we shall refer to those operators that commute modulo compacts with every $\rho(f)$ as *pseudolocal* on X.

As usual, we shall omit mention of the representation ρ whenever possible. It is clear that $C^*(X)$ is an ideal in the unital C^*-algebra $D^*(X)$. Moreover we can identify the associated quotient algebra in terms which do not involve coarse geometry at all. Let us introduce the following notation: $\mathfrak{D}^*(X)$ will denote the algebra of *all* pseudolocal operators on X, and $\mathfrak{C}^*(X)$ will denote the ideal of *all* locally compact operators. We have met these algebras before; in the notation of Chapter 5 they would be denoted $\mathfrak{D}_\rho(C_0(X))$ and $\mathfrak{D}_\rho(C_0(X)/\!\!/C_0(X))$, respectively.

12.3.2 LEMMA *Let* X *be a locally compact space which is equipped with a proper coarse structure, and suppose that* $C_0(X)$ *is represented non-degenerately on a Hilbert space* H.

(a) *The inclusion of* $D^*(X)$ *into* $\mathfrak{D}^*(X)$ *induces an isomorphism*

$$D^*(X)/C^*(X) \cong \mathfrak{D}^*(X)/\mathfrak{C}^*(X).$$

(b) *Consequently,*[108] *if the representation of* $C_0(X)$ *is ample, then there is an isomorphism*

$$K_{p+1}(D^*(X)/C^*(X)) \cong K_p(X)$$

between the K*-theory of the quotient algebra and the* K*-homology of* X.

PROOF To prove that the inclusion of $D^*(X)$ into $\mathfrak{D}^*(X)$ induces an isomorphism on quotient algebras we must show that

$$C^*(X) = D^*(X) \cap \mathfrak{C}^*(X),$$

which implies the injectivity of the induced map, and

$$\mathfrak{D}^*(X) = D^*(X) + \mathfrak{C}^*(X),$$

which implies surjectivity. The first equality is an immediate consequence of the definitions. To prove the second we shall make use of a truncation procedure which will have other applications.

12.3.3 SUBLEMMA *Let* $\{U_n\}$ *be a locally finite, uniformly bounded open cover of* X, *and let* $\{g_n\}$ *be a subordinate partition of unity. Let* $T \in \mathfrak{B}(H)$. *Then the series*

$$\mathrm{trunc}(T) = \sum_n \rho(g_n)^{\frac{1}{2}} T \rho(g_n)^{\frac{1}{2}}$$

converges strongly to a controlled operator on H. *Moreover, the operation* $\mathrm{trunc}\colon \mathfrak{B}(H) \to \mathfrak{B}(H)$ *is linear and norm continuous.*

[108] See Equation 5.4.3 and Remark 5.4.4.

To apply the sublemma we note that the properness of the coarse structure ensures that there exists a partition of unity of the sort described. If $T \in \mathfrak{D}^*(X)$ and if $T' = \operatorname{trunc}(T)$ then T' is controlled, and moreover we have

$$T' - T = \sum_n \rho(g_n)^{\frac{1}{2}}[T, \rho(g_n)^{\frac{1}{2}}] = -\sum_n [T, \rho(g_n)^{\frac{1}{2}}]\rho(g_n)^{\frac{1}{2}}.$$

Thus for any compactly supported function f on X, both $\rho(f) \cdot (T' - T)$ and $(T' - T) \cdot \rho(f)$ are finite sums of compact operators, and hence are compact. Thus $T' - T$ is locally compact, and the proof of part (a) is complete. Part (b) follows using Equation 5.4.3 and Remark 5.4.4. □

One should compare this proof with that of Theorem 5.4.5. In the language of Remark 8.3.3, it shows that K-homology for X can be normalized by the requirement that Fredholm modules be controlled.

PROOF OF 12.3.3 Suppose that T is positive. Let $T_k = \sum_{n=1}^{k} \rho(g_n)^{\frac{1}{2}} T \rho(g_n)^{\frac{1}{2}}$. Then, for any $v \in H$,

$$\langle T_k v, v\rangle = \sum_{n=1}^{k} \langle T\rho(g_n)^{\frac{1}{2}} v, \rho(g_n)^{\frac{1}{2}} v\rangle \leqslant \|T\| \sum_{n=1}^{k} \|\rho(g_n)^{\frac{1}{2}} v\|^2 \leqslant \|T\|\|v\|^2,$$

so that $\|T_k\| \leqslant \|T\|$ for each k. Now it follows from Proposition 1.2.3 that the series defining $\operatorname{trunc}(T)$ converges strongly to a positive operator and $\|\operatorname{trunc}(T)\| \leqslant \|T\|$. Moreover it is clear that $\operatorname{trunc}(T)$ has support in $\bigcup U_n \times U_n$, so it is controlled. □

We continue to assume that X is a locally compact space equipped with a proper coarse structure, and that $C_0(X)$ is amply represented on a Hilbert space H. Consider now the six-term K-theory exact sequence associated to the short exact sequence of C^*-algebras

$$(12.3.4) \qquad 0 \longrightarrow C^*(X) \longrightarrow D^*(X) \longrightarrow D^*(X)/C^*(X) \longrightarrow 0 .$$

By the lemma above, the boundary map appearing in this short exact sequence can be identified as a homomorphism

$$A\colon K_p(X) \to K_p(C^*(X))$$

from the K-homology of the space X to the K-theory of its coarse C^*-algebra.

12.3.5 DEFINITION The homomorphism $A\colon K_p(X) \to K_p(C^*(X))$ just described is called the *coarse assembly map* associated to the space X. If X is a manifold, and if D is a symmetric elliptic operator over X, then we shall refer to the class $A[D] \in K_p(C^*(X))$ as the *coarse index* of D.

12.3.6 EXAMPLE Suppose that X is *compact*, and so is coarsely equivalent to a point. Then $C^*(X)$ is equal to the algebra $\mathfrak{K}(H)$ of all compact operators on H, and the assembly map $K_0(X) \to K_0(\mathfrak{K}(H)) \cong \mathbb{Z}$ is the same as the index map $K_0(X) \to K_0(\mathrm{pt}) \cong \mathbb{Z}$ obtained by crushing X to a point. The coarse index of an elliptic operator D on a compact X is therefore just its ordinary index.

Our geometric applications of the assembly map will come from the following 'vanishing theorem', which generalizes Corollary 12.1.3.

12.3.7 PROPOSITION *Let X be a complete n-dimensional Riemannian Spin-manifold, with the associated metric coarse structure. Let $[X]$ denote the fundamental class (Definition 11.2.10) of X in $K_n(X)$.*[109] *If X has uniformly positive scalar curvature then $A[X] = 0 \in K_n(C^*(X))$.*

PROOF Since X is complete, the spinor Dirac operator D is essentially selfadjoint, and so the homology class $[X]$ is defined by the Fredholm module consisting of the Hilbert space $H = L^2(S)$ with its natural representation of $C_0(X)$, and the operator $F = \chi(D)$, where χ is a normalizing function.

By the spectral gap assumption (Lemma 12.1.2) we can choose the normalizing function χ such that $\chi(\lambda) = \pm 1$ for all $\lambda \in \mathrm{Spectrum}(D)$. Then $F = F^*$ and $F^2 = 1$, so that the Fredholm module defined by F is involutive.

Now we need to recall the construction of Section 8.4, which sets up the correspondence between the Kasparov definition of K-homology in terms of Fredholm modules and the alternative definition in terms of dual algebras (recall that the latter was used to define the assembly map). One must consider separately the even and odd cases, and we shall focus our attention first on the even one. The fundamental class $[X] \in K_0(X)$ is described by the graded Fredholm module $F = \chi(D)$ on the graded Hilbert space $H = L^2(S)$. This operator is of the form

$$F = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$$

where U is a unitary operator from H^+ to H^-.

In order to represent the associated K-homology class in terms of a dual algebra we need to replace this Fredholm module by a *balanced* Fredholm module (see Proposition 8.3.12). Thus let U' be the unitary operator defined by the diagram

$$\begin{array}{ccccccccccccc} & \cdots\oplus & H^- & \oplus & H^- & \oplus & H^+ & \oplus & H^+ & \oplus & H^+ & \oplus\cdots & \\ \cdots & & \downarrow I & & \downarrow I & & \downarrow U & & \downarrow I & & \downarrow I & & \cdots \\ & \cdots\oplus & H^- & \oplus & H^- & \oplus & H^- & \oplus & H^+ & \oplus & H^+ & \oplus\cdots & \end{array}$$

The top and bottom rows of this diagram define the same Hilbert space, which we use to form the algebras $\mathfrak{D}^*(X)$ and $\mathfrak{C}^*(X)$. Then U' is a unitary in $\mathfrak{D}^*(X)/\mathfrak{C}^*(X)$,

[109] We use the formal periodicity of K-homology to reduce n modulo 2.

so it defines a class in $K_1(\mathfrak{D}^*(X)/\mathfrak{C}^*(X))$, and this represents the K-homology class $[D]$ under the duality isomorphism.

Every normalizing function χ is a uniform limit of normalizing functions whose distributional Fourier transforms are compactly supported. Therefore, by Corollary 10.5.5 and Theorem 10.6.5, the operator F is a norm limit of controlled, pseudolocal operators, and thus U' belongs to $D^*(X)$. So the homology class $[X] \in K_1(\mathfrak{D}^*(X)/\mathfrak{C}^*(X)) = K_0(D^*(X)/C^*(X))$ is the image of a class in $K_1(D^*(X))$, and by exactness in the K-theory sequence

$$K_1(D^*(X)) \longrightarrow K_1(D^*(X)/C^*(X)) \longrightarrow K_0(C^*(X))$$

we conclude that $A[X] = 0$ in $K_0(C^*(X))$.

The proof in the odd case is similar, but easier, since there is no need for 'balancing'. We leave it to the reader. □

12.3.8 Remark The proof of Proposition 12.3.7 was based on the six-term exact sequence in K-theory, whereas the proof of Theorem 12.2.5 was based on the six-term exact sequence in K-homology. The constructions are related as follows. Suppose that the space X is equipped with a topological coarse structure coming from a compactification $\overline{X}$, and $\partial X = \overline{X} \setminus X$. In Corollary 6.5.2 we identified $K_p(C^*(X))$ as the reduced K-homology group $\widetilde{K}_{p-1}(\partial X)$ of the boundary. An elaboration of the argument used to prove this fact also shows that the coarse assembly map $A\colon K_p(X) \to K_p(C^*(X))$ is nothing other than the boundary map

$$\partial\colon K_p(X) \to \widetilde{K}_{p-1}(\partial X)$$

which appears in the six-term exact sequence for the reduced K-homology of the pair $(\overline{X}, \partial X)$. If $\overline{X}$ is a coarse compactification of X (so that the topological coarse structure on X is coarser than the metric one), then we obtain by functoriality a commutative diagram

$$\begin{tikzcd}
K_p(X) \arrow[rr, "A"] \arrow[d, equal] & & K_p(C^*_{\mathrm{metric}}(X)) \arrow[d] \\
K_p(X) \arrow[r, "\partial"'] & \widetilde{K}_{p-1}(\partial X) \arrow[r, equal] & K_p(C^*_{\mathrm{top}}(X))
\end{tikzcd}$$

which allows us to use Proposition 12.3.7 to prove Theorem 12.2.5.

Let M be a compact aspherical n-manifold, and let X be its universal cover (which has a unique Spin-structure since it is contractible). By Lemma 12.2.4, the homology class $[X] \in K_n(X)$ is non-zero. By Proposition 12.3.7, the class $A[D] \in K_n(C^*(X))$ is zero if M has positive scalar curvature. If $A\colon K_n(X) \to K_n(C^*(X))$ was injective, then this would be impossible. Therefore Conjecture 12.1.6 would be proved if we knew that the coarse assembly map for the universal cover of a compact aspherical manifold is always injective. This assertion is part of the

12.3.9 COARSE BAUM–CONNES CONJECTURE *Let* X *be the universal cover of a compact aspherical manifold (or simplicial complex). Then the coarse assembly map* $A\colon K_p(X) \to K_p(C^*(X))$ *is an isomorphism for every* p.

At the time of writing it seems most likely that this conjecture is false in general, but it has nevertheless served well as an organizing principle for the study of index theory and coarse geometry. The next section gives a proof under a negative curvature assumption.

12.4 Scaleable Spaces and the Baum–Connes Conjecture

In this section we describe a class of metric spaces X for which the coarse assembly map is an isomorphism. Our proof will depend on the following simple observation:

12.4.1 LEMMA *The coarse assembly map for a space* X *is an isomorphism if and only if all the* K*-theory groups of the algebra* $D^*(X)$ *are zero.*

PROOF This follows from the exact sequence of K-theory groups associated to the short exact sequence 12.3.4 of C^*-algebras. □

In view of this lemma, we begin our discussion by studying the functorial properties of the K-theory of the algebra $D^*(X)$.

12.4.2 DEFINITION A *uniform map* between proper metric spaces (or, more generally, between locally compact, proper coarse spaces) is a coarse map which is also continuous.

We are going to prove that the correspondence $X \mapsto K_p(D^*(X))$ is a functor on the category of proper metric spaces and uniform maps. To do this we shall follow a well-trodden path: we shall define the notion of *covering isometry* for a uniform map (compare Definitions 5.2.2 and 6.3.9), and we shall show that if V is an isometry covering the uniform map $f\colon X \to Y$, then Ad_V induces a $*$-homomorphism $D^*(X) \to D^*(Y)$ whose action on K-theory is uniquely determined by f. We carried out this procedure in Chapter 5 for K-homology, and in Chapter 6 for the K-theory of $C^*(X)$; now we shall, in effect, carry out both procedures at once.

12.4.3 DEFINITION Let X and Y be proper metric spaces, and let $C_0(X)$ and $C_0(Y)$ be represented non-degenerately on Hilbert spaces H_X and H_Y. Let $f\colon X \to Y$ be a uniform map. An isometry $V\colon H_X \to H_Y$ *uniformly covers* f if V covers f in the sense of Definition 6.3.9 and V covers $f^*\colon C_0(Y) \to C_0(X)$ in the sense of Definition 5.2.2.

We leave to the reader the proof of the following counterpart of Lemma 5.2.3:

12.4.4 LEMMA *Let* $V\colon H_X \to H_Y$ *be an isometry which uniformly covers the uniform map* $f\colon X \to Y$. *Then the* $*$*-homomorphism* $\mathrm{Ad}_V(T) = VTV^*$ *maps*

$D^*(X)$ *into* $D^*(Y)$. *Moreover, the homomorphism induced on* K*-theory by* Ad_V *is independent of the choice of covering isometry* V. □.

To complete our discussion of the functoriality of $K_p(D^*(X))$ we need the *existence* of covering isometries. It is convenient to assume that the representations involved are *very ample* in the following sense.

12.4.5 Definition Let $\rho\colon C_0(X) \to \mathfrak{B}(H_X)$ be a representation. We shall say that ρ is *very ample* if it is unitarily equivalent to a direct sum of countably many copies of some (fixed) ample representation.

Clearly, every very ample representation is ample. The concept allows us to prove:

12.4.6 Lemma *Let* X *and* Y *be proper metric spaces, let* H_X *be equipped with a non-degenerate representation of* $C_0(X)$, *and let* H_Y *be equipped with a very ample representation of* $C_0(Y)$. *Every uniform map* $f\colon X \to Y$ *is uniformly covered by an isometry* $W\colon H_X \to H_Y$.

Proof Suppose that H_Y is the direct sum of countably many copies of an ample representation space L. Let $\{U_j\}$ and $\{V_k\}$ be locally finite open covers of X and Y, by sets of uniformly bounded diameter, having the property that f maps each U_j into some V_{k_j}. Let

$$H_j = \overline{C_0(U_j)H_X}, \qquad L_k = \overline{C_0(V_k)L}.$$

By Voiculescu's Theorem (applied to the C^*-algebra $A = C(\overline{V_{k_j}})$), for each j there is an isometry $W_j\colon H_j \to L_{k_j}$ such that $gW_j - W_j(g \circ f)$ is compact for every $g \in C_0(Y)$. Now let $\{h_j\}$ be a partition of unity subordinate to the cover $\{U_j\}$ and define $W\colon H_X \to H_Y = \bigoplus^\infty L$ by

$$Wv = (W_1 h_1^{\frac{1}{2}} v, W_2 h_2^{\frac{1}{2}} v, \dots).$$

Then W uniformly covers f. □

So long as we always work with very ample representations, we may now define the homomorphism $f_*\colon K_p(D^*(X)) \to K_p(D^*(Y))$ induced by a uniform map $f\colon X \to Y$ to be the homomorphism $(\mathrm{Ad}_W)_*$, where W is any isometry which uniformly covers f.

We are going to prove the Coarse Baum–Connes conjecture for spaces X which admit a 'rescaling map' $r\colon X \to X$, which multiplies all distances by a factor $\leqslant \frac{1}{2}$ and which is 'uniformly homotopic' to the identity map; the paradigm is Euclidean space, with the rescaling map $r(x) = \frac{1}{2}x$. The main difficulty is to identify a suitable notion of 'uniform homotopy'. For example, it is clear that the linear homotopy

$$h(x,t) = \frac{2-t}{2}x$$

does *not* define a uniform map $\mathbb{R}^n \times [0,1] \to \mathbb{R}^n$. To overcome this difficulty we must allow the 'running time' of the homotopy to depend on the starting point x; this 'running time' is T_x in the definition below.

12.4.7 DEFINITION Let X be a complete Riemannian manifold. We shall say that X is *scaleable* if there are a uniform map $r\colon X \to X$, a metric subspace $Z = \{(x,t) : 0 \leqslant t \leqslant T_x\}$ of $X \times \mathbb{R}$, where $x \mapsto T_x$ is a continuous function from X to $[0,\infty)$, and a uniform map $h\colon Z \to X$, such that

(a) $d(r(x), r(x')) \leqslant \frac{1}{2}d(x,x')$ for all $x, x' \in X$,
(b) $h(x,0) = x$, and
(c) $h(x,T_x) = r(x)$.

Parts (b) and (c) of the definition may be expressed by saying that the map r is *uniformly homotopic* to the identity map on X

12.4.8 REMARK The assumption in Definition 12.4.7 that X is a Riemannian manifold is much too strong: but we make it to avoid a few minor technicalities in the presentation below.

The following examples should help clarify this somewhat technical definition.

12.4.9 EXAMPLE Let $X = \mathbb{R}^n$ and let $r(x) = \frac{1}{2}x$. This map is certainly homotopic to the identity, but it takes a little care to produce a homotopy which is a uniform map in the sense required by Definition 12.4.7. To do so we can take $T_x = \frac{1}{2}|x|$ and

$$h(x,t) = \left(1 - \frac{t}{|x|}\right)x.$$

Thus the homotopy h retracts x to $\frac{1}{2}x$ at unit speed.

12.4.10 EXAMPLE Generalizing the previous example, let us show that a complete, simply connected Riemannian manifold of non-positive sectional curvature is scaleable. Let W be such a manifold and choose a basepoint $x_0 \in W$. Let $\exp\colon T_{x_0}W \to W$ be the exponential map, and let $\log\colon W \to T_{x_0}W$ be its inverse. For each $x \in W$ define

$$r(x) = \exp\left(\tfrac{1}{2}\log x\right);$$

that is, $r(x)$ is the point halfway along the geodesic from x_0 to x. Moreover let $T_x = \frac{1}{2}d(x,x_0)$ and

$$h(x,t) = \exp\left\{\left(1 - \frac{t}{d(x,x_0)}\right)\log(x)\right\}$$

Using the geometry of negative curvature, it can be shown that r is a uniform map which decreases distances by a factor of $\frac{1}{2}$ or more, and that h is a uniform homotopy. The geometric details are worked out in Exercise 12.7.4.

We shall see in Proposition 12.4.12 below that the conditions of Definition 12.4.7 imply that $r_* = \mathrm{id}\colon K_p(D^*(X)) \to K_p(D^*(X))$. Let us take this for granted for a moment, and use it to prove the following important result:

12.4.11 THEOREM *Let* X *be a scaleable complete Riemannian manifold. Then the coarse assembly map*

$$A\colon K_p(X) \to K_p(C^*(X))$$

is an isomorphism for every p.

PROOF By Lemma 12.4.1 it suffices to show that $K_p(D^*(X)) = 0$. Let $r\colon X \to X$ be a contraction of X, as in Definition 12.4.7. By the result of Proposition 12.4.12, which for the moment we are taking for granted, it suffices to show that

$$r_* = 0\colon K_p(D^*(X)) \to K_p(D^*(X)).$$

This is what we shall do. For simplicity we shall suppose that r is one-to-one; we shall deal with the slight extra complications of the general case at the end of the proof. We are going to form the induced map $r_*\colon K_p(D^*(X)) \to K_p(D^*(X))$ in an explicit way by making a specific choice of the Hilbert space H_X. Namely, let Σ be a countable dense subset of X which is invariant under r, and let $H = \bigoplus^\infty \ell^2(\Sigma)$, which is a very ample representation of $C_0(X)$. We may define an isometry on $\ell^2(\Sigma)$ by mapping the basis vector corresponding to $x \in \Sigma$ to the basis vector corresponding to $f(x)$ (it is here that we use the assumption that f is one-to-one), and taking the direct sum of infinitely many copies of this isometry we obtain an isometry $V\colon H \to H$ which uniformly covers f.

Further, let $H' = H \oplus H \oplus \cdots$ and let $V_1\colon H \to H'$ be the isometry which includes H as the first summand in H'. Then $\alpha_1 = \mathrm{Ad}_{V_1}$ maps $D^*(X;H)$ into $D^*(X;H')$ and, since the isometry V_1 uniformly covers r, the homomorphism

$$(\alpha_1)_* = (\mathrm{Ad}_{V_1})_*\colon K_p(D^*(X;H)) \to K_p(D^*(X;H'))$$

may be identified with r_*. We are going to use an Eilenberg swindle to show that $(\alpha_1)_*$ induces the zero map on K-theory.

To do this, define another $*$-homomorphism $\alpha_2\colon \mathfrak{B}(H) \to \mathfrak{B}(H')$ by

$$\alpha_2(T) = (0, \mathrm{Ad}_V(T), \mathrm{Ad}_V^2(T), \dots).$$

We shall show that if $T \in D^*(X;H)$, then $\alpha_2(T) \in D^*(X;H')$. Let T be controlled and pseudolocal. Then $\alpha_2(T)$ is controlled, since f decreases all distances.

To prove that the controlled operator $\alpha_2(T)$ is pseudolocal we use Kasparov's Lemma 5.4.7; it suffices to show that $f\alpha_2(T)g \sim 0$ whenever $f, g \in C_0(X)$ have disjoint supports.[110] But since T is controlled, there is a constant $R > 0$ such that if $d(\mathrm{Support}(f), \mathrm{Support}(g)) > R$ then $fTg = 0$. Since r halves all distances, $f\,\mathrm{Ad}_V^k(T)g = 0$ as soon as $2^{-k}R < d(\mathrm{Support}(f), \mathrm{Support}(g))$. Thus the operator

$$f\alpha_2(T)g = (0, f\,\mathrm{Ad}_V(T)g, f\,\mathrm{Ad}_V^2(T)g, \dots)$$

is a *finite* sum of compact operators, and hence is compact.

Having made this observation, we complete the swindle in the usual way (compare the proof of Lemma 6.4.2). We have

$$\alpha_2 = \mathrm{Ad}_W \circ (\alpha_1 + \alpha_2)$$

where W is an isometry which uniformly covers r. Thus on the level of K-theory we have

$$(\alpha_2)_* = (\alpha_1)_* + (\alpha_2)_*$$

(by Lemma 4.6.4), and so $(\alpha_1)_* = 0$.

To finish the proof of the theorem we consider the case where r need not be one-to-one. Now we replace $\ell^2(\Sigma)$ with $\ell^2(\Sigma \times A)$, where A is any countably infinite set, and define an isometry on $\ell^2(\Sigma \times A)$ by mapping the basis vector labeled by (x, a) to a basis vector labeled by some $(r(x), a')$, where a' is chosen so that the map on basis vectors is one-to-one. The rest of the proof proceeds as before. □

Here is the proof of the homotopy result that we used above:

12.4.12 PROPOSITION *Let* X *be a complete Riemannian manifold and let* $r\colon X \to X$ *be a uniform map. Suppose that there are a metric subspace* $Z = \{(x, t) : 0 \leqslant t \leqslant T_x\}$ *of* $X \times \mathbb{R}$, *where* $x \mapsto T_x$ *is a continuous function from* X *to* $[0, \infty)$, *and a uniform map* $h\colon Z \to X$, *such that*

(a) $h(x, 0) = x$, *and*
(b) $h(x, T_x) = r(x)$.

Then $r_* = \mathrm{id}\colon K_p(D^*(X)) \to K_p(D^*(X))$.

PROOF OF PROPOSITION 12.4.12 Think of h as a homotopy between r and the identity. Let Z be the space of the homotopy, as defined above, and let $X_0, X_\infty \subseteq Z$ be defined by

$$X_0 = \{\, (x, 0) : x \in X \,\}, \quad X_\infty = \{\, (x, T_x) : x \in X \,\}.$$

Let i_0 and i_∞ be the inclusions of X_0 and X_∞ into Z, and let $p\colon Z \to X_0$ be the projection onto the first coordinate (Figure 12.1). Note that all these spaces are proper metric spaces, and the maps are uniform maps.

[110] Apply Lemma 5.4.7 to the one-point compactification of X, and use control.

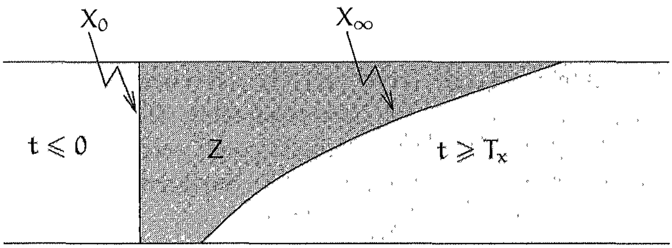

FIG. 12.1. Illustrating the proof of Proposition 12.4.12.

It will suffice to show that i_0, i_∞, and p induce isomorphisms on the K-theory of the algebras D^*. To see this, use the isomorphism induced by $q = p \circ i_\infty$ to identify $K_p(D^*(X_\infty))$ with $K_p(D^*(X_0)) = K_p(D^*(X))$. Having made this identification, note that the maps $K_p(D^*(X)) \to K_p(D^*(Z))$ induced by the inclusions of X_0 and X_∞ into Z agree. But, on composing with $h_*\colon K_p(D^*(Z)) \to K_p(D^*(X))$, the first of these maps gives the identity, and the second gives r_*.

Let us prove that i_0 induces an isomorphism. From the homotopy invariance of ordinary K-homology, i_0 induces an isomorphism from $K_p(X_0)$ to $K_p(Z)$. According to the Five Lemma it therefore suffices to show that i_0 also induces an isomorphism from $K_p(C^*(X_0))$ to $K_p(C^*(Z))$. We describe two decompositions of $X \times \mathbb{R}$ as follows: for the first decomposition put

$$A = \{(x,t) : t \leqslant 0\}, \quad B = \{(x,t) : t \geqslant 0\},$$

and for the second put

$$A' = \{(x,t) : t \leqslant T_x\}, \quad B' = \{(x,t) : t \geqslant 0\}.$$

Notice that $A \cap B = X_0$, while $A' \cap B' = Z$. Moreover, both decompositions are coarsely excisive[111] (Exercise 6.7.9) and so there is a commutative diagram of coarse Mayer–Vietoris sequences

$$\begin{tikzcd}
{} \arrow[r] & K_{p+1}(C^*(X \times \mathbb{R})) \arrow[r] \arrow[d, equal] & K_p(C^*(A)) \oplus K_p(C^*(B)) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & K_{p+1}(C^*(X \times \mathbb{R})) \arrow[r] & K_p(C^*(A')) \oplus K_p(C^*(B')) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
K_p(C^*(A \cap B)) \arrow[r] \arrow[d] & K_p(C^*(X \times \mathbb{R})) \arrow[r] \arrow[d, equal] & {} \\
K_p(C^*(A' \cap B')) \arrow[r] & K_p(C^*(X \times \mathbb{R})) \arrow[r] & {}
\end{tikzcd}$$

[111] It is here that we use the hypothesis that X is a Riemannian manifold — or at least a path metric space.

The spaces A, B, A', and B' are all flasque (Exercise 6.7.11) and so the K-theory of their coarse C^*-algebras is zero. So by the Five Lemma the map $(i_0)_*$ from $K_p(C^*(A \cap B))$ to $K_p(C^*(A' \cap B'))$ is an isomorphism, and it follows that p, which is a one-sided inverse to i_0, also induces an isomorphism on K-theory. Another Mayer–Vietoris argument shows that i_∞ induces an isomorphism on the K-theory of the coarse C^*-algebras. □

12.4.13 REMARK It follows from the discussion above that the universal cover of a compact manifold of non-positive sectional curvature satisfies the coarse Baum–Connes conjecture. A rather more general example that can be treated in the same kind of way is that of the universal cover, W say, of a compact aspherical manifold whose fundamental group is *large-scale hyperbolic* in the sense of Gromov [59]. This hypothesis is equivalent to the statement that there is some $\delta > 0$ for which every geodesic triangle in W (no matter how large) is δ-*thin*: each side lies entirely within distance δ of the union of the other two. It provides a 'coarse' substitute for negative sectional curvature. In particular, while one no longer has a unique geodesic in W between two given points, any two such geodesics are close to each other (within 8δ, say), and we may define a rescaling map $r: W \to W$ by continuously choosing a point $r(x)$ within 20δ of the midpoint of some geodesic segment from x to a fixed basepoint x_0. One concludes as above that the space W satisfies the coarse Baum–Connes conjecture.

12.5 Equivariant Assembly

Let M be a smooth compact manifold with fundamental group G. (In fact all of the discussion in this section and the next could be carried through in a broader context, but the manifold case is the most important one and it allows us to avoid one or two technical issues.) Let $X = \widetilde{M}$ be the universal cover of M.

The space X has a canonical coarse structure generated by any G-invariant Riemannian metric (Lemma 12.2.2). We shall consider the algebras $C^*(X)$, $D^*(X)$, and so on, to be formed from this coarse structure and the very ample representation of $C_0(X)$ by multiplication operators on $H = \bigoplus^\infty L^2(X)$. Notice that G acts on H by translation, compatibly with the representation of $C_0(X)$; thus G acts by automorphisms on the algebras $C^*(X)$ and $D^*(X)$.

12.5.1 DEFINITION Let $C^*_G(X)$ denote the subalgebra of $C^*(X)$ generated by the G-invariant locally compact controlled operators. Similarly let $D^*_G(X)$ denote the subalgebra of $D^*(X)$ generated by the G-invariant pseudolocal controlled operators.

Now consider the short exact sequence of C^*-algebras

$$0 \longrightarrow C^*_G(X) \longrightarrow D^*_G(X) \longrightarrow D^*_G(X)/C^*_G(X) \longrightarrow 0 \tag{12.5.2}$$

This sequence is, in effect, the G-invariant part of the sequence 12.3.4 which was used above to define the coarse assembly map for X.

12.5.3 LEMMA *With the notation above, there is an isomorphism*

$$K_p(C^*_G(X)) \cong K_p(C^*_r(G))$$

from the K*-theory of* $C^*_G(X)$ *to the* K*-theory of the reduced group* C^**-algebra for* G.

12.5.4 LEMMA *With the notation above, there is an isomorphism*

$$K_{p+1}(D^*_G(X)/C^*_G(X)) \cong K_p(M)$$

from the K*-theory of* $D^*_G(X)/C^*_G(X)$ *to the* K*-homology of the compact space* $M = X/G$.

Using these two lemmas, which we shall prove in a moment, we can identify the boundary map appearing in the six-term K-theory sequence associated to the extension 12.5.2 above with a homomorphism

$$A_M : K_p(M) \to K_p(C^*_r(\pi_1(M)))$$

from the K-homology of M to the K-theory of the C^*-algebra of its fundamental group.

12.5.5 DEFINITION We shall call the homomorphism A_M above the (equivariant) *assembly map* for the space M.

PROOF OF LEMMA 12.5.3 Choose a compact fundamental domain D for the action of G on X. Then $X = \bigcup_{g\in G} gD$ and thus

$$L^2(X) = \bigoplus_{g\in G} L^2(gD) = \ell^2(G) \otimes L^2(D),$$

where G acts on $\ell^2(G)$ by the regular representation and on $L^2(D)$ by the trivial representation. Thus $H = \ell^2(G) \otimes L$, where $L = \bigoplus^\infty L^2(D)$. Using the tensor product we may define a $*$-homomorphism $\alpha\colon C^*_r(G) \otimes \mathfrak{K}(L) \to \mathfrak{B}(H)$, and this homomorphism is injective. We claim that its image is precisely the algebra $C^*_G(X)$. Let $\mathbb{C}[G] \odot \mathfrak{K}(L)$ denote the *algebraic* tensor product of the group algebra $\mathbb{C}[G]$ with the compact operators on L. Then $\alpha(\mathbb{C}[G] \odot \mathfrak{K}(L))$ is exactly the algebra of G-invariant, controlled, locally compact operators on H. Thus $\alpha(\mathbb{C}[G] \odot \mathfrak{K}(L))$ is a dense subalgebra of $C^*_G(X)$, and it follows that $\alpha\colon C^*_r(G) \otimes \mathfrak{K}(L) \to C^*_G(X)$ is an isomorphism of C^*-algebras. The stability of K-theory (Example 4.2.5) now gives the result. $\square$

We leave to the reader the task of verifying that the K-theory isomorphism we have constructed is independent of the choice of fundamental domain.

PROOF OF LEMMA 12.5.4 We shall prove that $D^*_G(X)/C^*_G(X)$ is isomorphic to the algebra $D^*(M)/C^*(M)$, whose K-theory represents the K-homology of M by Lemma 12.3.2. Let $\pi\colon X \to M$ be the covering projection. There is some $\delta > 0$ such that if U is any open set of diameter $< \delta$, then $\pi^{-1}(U) = U \times G$. Thus there is a one-to-one correspondence between operators on $L^2(M)$ supported in $U \times U$, and G-invariant operators on $L^2(X)$ supported in $\pi^{-1}(U) \times \pi^{-1}(U)$.

Now any operator on $L^2(M)$ with propagation less than $\delta/2$ can be written (by a straightforward partition of unity argument) as a finite sum of operators supported in sets $U \times U$, where U is of diameter $< \delta$; and each such operator can be lifted to X as above. In this way we obtain a one-to-one correspondence between operators on M of propagation $< \delta/2$ and G-invariant operators on X of propagation $< \delta/2$. Moreover this correspondence preserves the properties of pseudolocality, local compactness, and so on.

Now consider the quotient algebra $D^*_G(X)/C^*_G(X)$. The truncation argument used in the proof of Lemma 12.3.2 shows that each equivalence class $[T]$ in this quotient algebra may be represented by a truncated operator $[\mathrm{trunc}(T)]$, where the truncation is associated to a locally finite partition of unity $\{g_n\}$ on X. Let us form such a partition of unity as follows: first cover M by finitely many open sets $\{U_i\}$ of diameter $< \delta/2$, choose a partition of unity on M subordinate to this cover, and then lift to a G-invariant partition of unity on X by making use of the identifications $\pi^{-1}(U_i) = U_i \times G$. Now, for $T \in D^*_G(X)$, we find that $\mathrm{trunc}(T)$ belongs to $D^*_G(X)$ also and has propagation $< \delta/2$, and the difference $T - \mathrm{trunc}(T)$ belongs to $C^*_G(X)$. Thus any equivalence class in the quotient $D^*_G(X)/C^*_G(X)$ may be represented by an operator of propagation $< \delta/2$. Similarly, each class in $D^*(M)/C^*(M)$ can be represented by an operator of propagation $< \delta/2$. We therefore obtain a one-to-one correspondence between $D^*(M)/C^*(M)$ and $D^*_G(X)/C^*_G(X)$, which is easily seen to be an algebra isomorphism. □

The assembly map $A_M\colon K_p(M) \to K_p(C^*_r(\pi_1(M)))$ is of great importance in topology and is the subject of various conjectures. A crucial example is the *analytic Novikov conjecture*, a special case of which is this:

12.5.6 CONJECTURE *If M is aspherical, then the associated assembly map $A_M\colon K_p(M) \to K_p(C^*_r(\pi_1(M)))$ is injective.*

It is natural to require asphericity here. Indeed the right-hand side of the conjecture depends only on the group $\pi_1(M)$; and the homotopy type of an aspherical space M also depends only on its fundamental group. For this reason one usually refers to Conjecture 12.5.6 as 'the Novikov conjecture for $G = \pi_1(M)$', rather than 'the Novikov conjecture for M'.

12.5.7 REMARK The analytic Novikov conjecture adapts to C^*-algebra theory a well-known conjecture in manifold theory. This original form of the Novikov conjecture has to do with the *homotopy invariance of the higher signatures* for smooth manifolds. Here is a brief account of its connection to the C^*-algebraic assembly map. Let N be an oriented even-dimensional manifold with fundamental group G, and let M be an aspherical space (which for simplicity we assume is also a compact manifold) with the same fundamental group. Homotopy theory provides a *classifying map* $f\colon N \to M$ which pulls back the universal cover of M to the universal cover of N. Let D be the *signature operator* of M, defined in Example 11.1.4. The *higher signature* $\sigma(N)$ of N is the image $f_*[D] \in K_0(M)\otimes\mathbb{Q}$ (classically the term 'higher signature' refers to any one of a family of numerical invariants obtained from σ_N by applying the Chern character and pairing with cohomology, but the information conveyed is the same). Now the Novikov conjecture in its original form states that if N and N' are homotopy equivalent, smooth, oriented manifolds, then $\sigma_N = \sigma_{N'}$ in $K_0(M)\otimes\mathbb{Q}$. But it can be shown that the hypothesis of homotopy equivalence does imply that $A_M(\sigma_N) = A_M(\sigma_{N'})$, so that the homotopy invariance of the higher signatures follows from the injectivity of A_M.

The *Baum–Connes* conjecture is a generalization of the Novikov conjecture. In the simple context of our discussion (that of compact aspherical manifolds) this generalization appears as a surjectivity claim matching the injectivity claim of the analytic Novikov conjecture:

12.5.8 CONJECTURE *If M is aspherical, then the associated assembly map A_M is an isomorphism.*

The Baum–Connes conjecture makes a link between two very different fields of mathematics. The computation of the K-homology of an aspherical G-space M belongs to the realm of classical algebraic topology and homological algebra. By contrast, the computation of the K-theory of $C^*_r(G)$ seems to involve a detailed understanding of the unitary representation theory of the group G. The appearance of the *reduced* C^*-algebra of G is particularly perplexing, as it suggests that only those unitary representations somehow associated to $\ell^2(G)$ should be able to be expressed in topological terms. On grounds of naturality one would have expected the *full* C^*-algebra $C^*(G)$ to appear here; but it is known (thanks to property T, see Exercise 12.7.7) that the counterpart of the Baum–Connes conjecture for the full C^*-algebra is definitely false.

12.5.9 EXAMPLE The Baum–Connes conjecture holds for the group $\mathbb{Z}$. Indeed, let us take $M = S^1$, an aspherical manifold with fundamental group $G = \mathbb{Z}$. Then $K_0(M)$ and $K_1(M)$ are both isomorphic to $\mathbb{Z}$. By Fourier analysis, the group C^*-algebra $C^*_r(\mathbb{Z})$ is isomorphic to the algebra $C(T)$ of continuous functions on another copy of the circle; so $K_0(C^*_r(G))$ and $K_1(C^*_r(G))$ are also both isomorphic to $\mathbb{Z}$. To verify the Baum–Connes conjecture we must check that the generators of the K-homology groups appearing on the left side of the conjecture are mapped to the generators of the K-theory groups appearing on the right side.

We begin with the case of K_0. The generator of $K_0(M)$ can be described by the balanced Fredholm module consisting of a Hilbert space H, an index-one operator U on H whose adjoint is an isometry, and the representation ρ of $C(M)$ on H given by $\rho(f) = f(x_0)$, where x_0 is some fixed point of M; this is a Fredholm module of propagation zero. There is a natural lift to a Fredholm module over the universal cover of M and, if we follow through the construction of the assembly map, we find that the image of the generator is described by the element of $C(T) \otimes \mathfrak{K}(H)$ which is a constant function having value the projection p onto the (one-dimensional) kernel of U. This represents the generator of $K_0(C(T))$.

The one-dimensional case is more interesting. The group $K_1(M)$ is generated by the homology class of the Dirac operator on S^1. Using the finite propagation speed methods of Corollary 10.5.5 one sees that the lift of this homology class to $\widetilde{M} = \mathbb{R}$ provided by Lemma 12.5.4 is equal to the homology class of the Dirac operator on $\mathbb{R}$ (compare Exercise 12.7.6). Since this operator is $\mathbb{R}$-translation invariant, its coarse index lies in the K-theory of the $\mathbb{R}$-translation invariant part of the algebra $C^*(\widetilde{M})$, which may be identified (by Fourier analysis) with the algebra $C_0(\widehat{\mathbb{R}})$ of functions on the dual group $\widehat{\mathbb{R}}$. In fact the index of the Dirac operator on $\mathbb{R}$ is defined by any complex-valued function of absolute value 1 on $\widehat{\mathbb{R}}$ which has winding number equal to 1, that is, the index of $\widetilde{D}$ is the generator of $K_1(C_0(\widehat{\mathbb{R}}))$. The forgetful map which regards $\mathbb{R}$-translation invariant operators as $\mathbb{Z}$-translation invariant operators defines a $*$-homomorphism

$$\alpha\colon C_0(\widehat{\mathbb{R}}) \to C(T) \otimes \mathfrak{K},$$

which by the Poisson summation formula is just the $*$-homomorphism appearing in Exercise 4.10.18. But we showed there that α induces an isomorphism on K_1.

It is beyond the scope of this book to prove the Baum–Connes conjecture in any cases essentially more difficult than the one above. However, in the next section we shall describe a 'descent' argument which proves the analytic *Novikov* conjecture in the case of non-positively curved spaces.

We conclude this section with an observation due to Atiyah.

12.5.10 DEFINITION Let G be a discrete group. Use the regular representation to regard $C_r^*(G)$ as an algebra of operators on $\ell^2(G)$. The *canonical trace* on $C_r^*(G)$ is the positive linear functional

$$\tau(T) = \langle T\delta_e, \delta_e \rangle,$$

where δ_e is the basis vector of $\ell^2(G)$ corresponding to the identity element of G.

REMARK We already used this notion in Exercise 8.8.13.

The term *trace* refers to the fundamental property property $\tau(ST) = \tau(TS)$ for $S, T \in C^*_r(G)$. From this one easily deduces that the assignment $P \mapsto \tau(P)$ yields a homomorphism

$$\tau_* \colon K_0(C^*_r(G)) \to \mathbb{R}.$$

We ask: what is the range of the map τ_*?

12.5.11 PROPOSITION *Let M be a compact manifold with fundamental group G. The following diagram commutes:*

$$\begin{tikzcd} K_0(M) \arrow[r, "A"] \arrow[d, "\text{Index}"'] & K_0(C^*_r(G)) \arrow[d, "\tau_*"] \\ \mathbb{Z} \arrow[r] & \mathbb{R} \end{tikzcd}$$

In particular, if $x \in K_0(C^*_r(G))$ belongs to the range of the assembly map, then $\tau_*(x)$ is an integer.

12.5.12 COROLLARY *If the group G (the fundamental group of a compact aspherical manifold) satisfies the Baum–Connes conjecture, then the range of $\tau_* \colon K_0(C^*_r(G)) \to \mathbb{R}$ consists of integers.* □

This integrality of the canonical trace is one of the more mysterious consequences of the Baum–Connes conjecture.

OUTLINE PROOF OF PROPOSITION 12.5.11 The proof of this result requires some further discussion of the relationship between index theory and traces. Recall that the algebra $\mathfrak{K}(H)$ of compact operators has a 'trace', defined in Exercise 2.9.3. This trace is an *unbounded*, densely defined, positive linear functional on $\mathfrak{K}(H)$, which is invariant in the sense that

$$\mathrm{Trace}(T) = \mathrm{Trace}(UTU^*)$$

for each unitary operator U on H. It turns out that a suitable[112] unbounded trace on a C^*-algebra J still defines a homomorphism from $K_0(J)$ to $\mathbb{R}$, since one can show that every K-theory class can be represented by a projection which is in the domain of the trace. Moreover, for the algebra of compact operators, the

[112] What is needed here is a certain 'semicontinuity' property, analogous to the Monotone Convergence Theorem.

standard trace induces the usual isomorphism $K_0(\mathfrak{K}(H)) \to \mathbb{Z}$. Finally, by using the explicit formula for the boundary map in K-theory, one can show that if

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

is a short exact sequence of C*-algebras, if τ is a (possibly unbounded) trace on J, and if $u \in A$ is an element such that $1 - uu^*$ and $1 - u^*u$ are positive and belong to the domain of τ, then

$$\tau_*(\partial[u]) = \tau(1 - u^*u) - \tau(1 - uu^*), \tag{12.5.13}$$

where $[u] \in K_1(A/J)$ is the K-theory class defined by the element u, which is unitary modulo J. (In the case of the Calkin algebra this is simply the result of Exercise 2.9.3.)

Now compare the short exact sequence

$$0 \longrightarrow C^*(M) \longrightarrow D^*(M) \longrightarrow D^*(M)/C^*(M) \longrightarrow 0$$

with the short exact sequence

$$0 \longrightarrow C^*_G(X) \longrightarrow D^*_G(X) \longrightarrow D^*_G(X)/C^*_G(X) \longrightarrow 0\,.$$

The right-hand algebras in the two sequences are isomorphic, by an isomorphism $\alpha\colon D^*(M)/C^*(M) \to D^*_G(X)/C^*_G(X)$ as described in the proof of Lemma 12.5.4. The left-hand algebras in both short exact sequences have traces: the standard one, denoted Trace, on $C^*(M) = \mathfrak{K}(L^2(M))$ and the tensor product $\tau \otimes \mathrm{Trace}$ on $C^*_G(X) \cong C^*_\tau(G) \otimes \mathfrak{K}$ (here we are using the proof of Lemma 12.5.3). Explicit calculation using formula 12.5.13 shows that

$$\mathrm{Trace}_*(\partial[u]) = (\tau \otimes \mathrm{Trace})_*(\partial[\alpha(u)])$$

and the result follows. □

12.6 The Descent Principle

In this final section we are going to outline a 'descent principle' which will allow us to infer the analytic Novikov conjecture, for a group G, from the coarse Baum–Connes conjecture for the universal cover of a compact aspherical G-space.

We shall allow ourselves to make use of some of the language of G-equivariant homotopy theory, and of homological algebra. In particular we shall assume that the spaces with which we are working are G-*simplicial complexes* which are G-*finite*. A G-simplicial complex X is an ordinary (locally finite) simplicial complex on which a group G acts simplicially, in such a way that the quotient X/G is a simplicial complex. The simplices of X can be organized into orbits, each of which will be called a G-*simplex*. Thus a G-simplex is a product $\Delta \times S$, where Δ is an ordinary simplex and S is a transitive G-set. A G-simplex is *free* if it is of the form $\Delta \times G$, and a G-simplicial complex is *free* if every one of its simplices is free (which is to say that G acts freely on X). Finally, X is G-finite if it has only finitely many G-simplices. The universal cover X of a finite aspherical complex, with fundamental group G, is a G-finite, contractible, free G-simplicial complex; and this information characterizes the space X up to G-equivariant homotopy.

We shall assume that each G-space X that we consider is equipped with a very ample representation of $C_0(X)$ on a Hilbert space H_X. We suppose also that H_X has a unitary G-action which makes the representation of $C_0(X)$ into a G-equivariant map. For example, if X is a triangulated manifold then we might take $H_X = \bigoplus^\infty L^2(X)$.

We begin our investigation by defining some new algebras.

12.6.1 DEFINITION Let X and Y be G-finite G-simplicial complexes. Let $C^*_G(Y, X)$ denote the algebra of continuous, G-equivariant maps T from Y to $C^*(X)$, equipped with pointwise operations and the norm

$$\|T\| = \sup\{\, \|T(y)\| \,:\, y \in Y\}.$$

(Notice that since Y is G-finite, and G acts by unitaries, the supremum is always finite.) We define $D^*_G(Y, X)$ similarly.

12.6.2 REMARK The algebra $C^*_G(\mathrm{pt}, X)$ consists of all G-invariant elements of $C^*(X)$. In particular, there is an inclusion of the algebra $C^*_G(X)$ of Definition 12.5.1 into the algebra $C^*_G(\mathrm{pt}, X)$; this inclusion is an isomorphism if G is amenable (Exercise 12.7.5) and may be an isomorphism in general. There is a similar inclusion for the algebras D^*. However, at least if X is a free G-simplicial complex such as the universal cover of a compact space with fundamental group G, the induced map on the quotient spaces

$$D^*_G(X)/C^*_G(X) \to D^*_G(\mathrm{pt}, X)/C^*_G(\mathrm{pt}, X)$$

is an isomorphism. This follows from the method of proof of Lemma 12.5.4.

We are going to prove:

12.6.3 THEOREM *Let* M *be a finite aspherical complex with fundamental group* G. *Suppose that the universal cover of* M *satisfies the coarse Baum–Connes conjecture. Then the assembly map* $A_M \colon K_p(M) \to K_p(C^*_r(G))$ *is injective, that is,* G *satisfies the analytic Novikov conjecture.*

From Remark 12.4.13 we obtain:

12.6.4 COROLLARY *The fundamental group of a compact non-positively curved Riemannian manifold satisfies the Analytic Novikov Conjecture.* □.

The proof of Theorem 12.6.3 is based on the following diagram, where X is the universal cover of the aspherical complex M:

$$\begin{array}{ccc} K_p(D^*_G(pt, X)) & \longrightarrow & K_p(D^*_G(pt, X)/C^*_G(pt, X)) \\ \downarrow & & \downarrow \\ K_p(D^*_G(X, X)) & \longrightarrow & K_p(D^*_G(X, X)/C^*_G(X, X)) \end{array}$$

We shall outline a proof that

(a) the group $K_p(D^*_G(X, X))$ is equal to zero, and
(b) the right-hand vertical map in the diagram is an isomorphism.

Granted these two statements, Theorem 12.6.3 will follow. For they imply that the first map in the diagram

$$K_p(D^*_G(pt, X)) \longrightarrow K_p(D^*_G(pt, X)/C^*_G(pt, X)) \xrightarrow{\partial} K_{p-1}(C^*_G(pt, X))$$

is zero, and so by exactness that the second map is injective. By Remark 12.6.2 above, the assembly map is injective also.

In proving claims (a) and (b) above we shall make use several times of the following language. A covariant functor L from free G-simplicial complexes to abelian groups will be called *homological* if it is both G-homotopy invariant and *excisive* — meaning that there should be an exact Mayer–Vietoris sequence associated to any decomposition of a space X as the union of two finite G-subcomplexes. A contravariant functor L satisfying the dual conditions will be called *cohomological*. For instance (if G is trivial) then K-theory is cohomological, whereas K-homology is homological.

PROOF OF CLAIM (a) The functor $Y \mapsto K_p(D^*_G(Y, X))$ is cohomological, by the homotopy invariance and excision properties of K-theory. But if $Y = \Delta \times G$ is a single free G-simplex then

$$D^*_G(Y, X) = C(\Delta) \otimes D^*(X).$$

Since Δ is homotopy equivalent to a point, this algebra has the same K-theory as $D^*(X)$, namely zero. But if a cohomological functor is zero on every free G-simplex, then a Five Lemma argument shows it is zero on every finite free G-simplicial complex.

PROOF OF CLAIM (b) For fixed Y, the functor $Z \mapsto K_p(D^*_G(Y,Z)/C^*_G(Y,Z))$ is homological.[113] Consequently, a Five Lemma argument shows that in order to check that the map

$$D^*_G(\mathrm{pt},Z)/C^*_G(\mathrm{pt},Z) \to D^*_G(X,Z)/C^*_G(X,Z)$$

induces an isomorphism on K-theory for all finite free G-simplicial complexes Z, it will suffice to consider the case where Z is a single G-simplex $\Delta \times G$.

Having made this reduction we are now going to fix $Z = W \times G$ and consider $Y \mapsto K_p(D^*_G(Y,Z)/C^*_G(Y,Z))$ as a functor on the category of finite free G-simplicial complexes Y. This functor is cohomological. If Y is a single G-simplex $\Delta' \times G$, with $\operatorname{Dim}\Delta' = q$, then

$$D^*_G(Y,Z)/C^*_G(Y,Z) = C(\Delta') \otimes \prod_G D^*(W)/C^*(W).$$

Consequently, using the proof of the Cluster Axiom for K-homology (Proposition 7.4.2), together with the fact that Δ' is homotopy equivalent to a point, we find that

$$K_p(D^*_G(Y,Z)/C^*_G(Y,Z)) = \prod_G K_p(D^*(W)/C^*(W)) = C^q(Y;K_{p-1}(W)).$$

The group on the right is the group of ordinary (non-equivariant) simplicial q-cochains of Y with values in the K-homology group $K_{p-1}(W)$.

Now imagine assembling the space X from its free G-simplices. By the cohomological nature of the functor $Y \mapsto K_p(D^*_G(Y,Z)/C^*_G(Y,Z))$, we shall obtain a web of interlocking Mayer–Vietoris sequences relating the K-theory of $D^*_G(X,Z)/C^*_G(X,Z)$ to the K-theory of the corresponding algebras for the individual G-simplices. This web of interlocking exact sequences is most conveniently summarized in the form of a *spectral sequence* whose E_1 term is

$$E_1^{pq} = C^q(X;K_{p-1}(W))$$

and which converges to $K_{p+q}(D^*_G(X,Z)/C^*_G(X,Z))$. The E_2 term is

$$E_2^{pq} = H^q(X;K_{p-1}(W)).$$

But X is contractible, so the spectral sequence degenerates at E_2 to give

$$K_p(D^*_G(X,Z)/C^*_G(X,Z)) = K_{p-1}(W) = K_p(D^*_G(\mathrm{pt},Z)/C^*_G(\mathrm{pt},Z)),$$

which is the required result. □

[113] Indeed, if Y and G are trivial this functor is just the K-homology of Z; the homotopy invariance and excision properties in the general case may be proved by following the lines of the argument given in Chapters 5 and 6.

In this final section of the book we have approached the limits of what can by achieved using pure K-homology theory. Just below the surface of the above argument lies a *bivariant* K-theory — homological in one variable, cohomological in the other — and to make further progress with the ideas sketched in this chapter it is necessary to make this theory explicit and to develop its properties. The reader who is eager to learn more should therefore now turn to the papers of Kasparov.

12.7 Exercises

12.7.1 Show that the assembly map for a *finite* group G may be defined in the way sketched in the introduction to this chapter.

12.7.2 Let X be a proper metric space. Let $C_h(X)$ denote the algebra of all bounded functions $f\colon X \to \mathbb{C}$ having the property that, for each fixed $r > 0$, $|f(x) - f(y)| \to 0$ as $x, y \to \infty$ with $d(x, y) < r$.

(a) Show that $C_h(X)$ is a C^*-algebra and that its maximal ideal space can be identified with a certain coarse compactification $\overline{X}$ of X. Show further that $\overline{X}$ is a *universal coarse compactification* of X, that is, any other coarse compactification is a quotient of $\overline{X}$.

(b) Suppose that X is the universal cover of a compact aspherical manifold M. Show that, if every compactly supported (Čech) cohomology class for X maps to zero in the cohomology of $\overline{X}$, then M has no metric of positive scalar curvature.

12.7.3 Let M be a complete Riemannian manifold and let D be a Dirac operator on a graded Dirac bundle S over M. Show that the coarse index of D can be represented by the formal difference of projections

$$\begin{pmatrix} e^{-tD^-D^+} & e^{-\frac{t}{2}D^-D^+}\left(\frac{1-e^{-tD^-D^+}}{D^-D^+}\right)D^- \\ e^{-\frac{t}{2}D^+D^-}\left(\frac{1-e^{-tD^+D^-}}{D^+D^-}\right)D^+ & 1-e^{-tD^+D^-} \end{pmatrix} \ominus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

in the K-theory of the unitalization of $C^*(X)$.

12.7.4 Let x_0, x_1, and y be three points in Euclidean space and let x_t denote the point on the line segment $[x_0, x_1]$ such that $d(x_0, x_t) = td(x_0, x_1)$. Prove that

$$d(x_t, y)^2 = (1-t)d(x_0, y)^2 + td(x_1, y)^2 - t(1-t)d(x_0, x_1)^2.$$

Now suppose that the points x_0, x_1 and y lie not in Euclidean space but in some complete, non-positively curved, simply connected Riemannian manifold

W. Use the expansiveness of the exponential map based at y to obtain the CAT(0)-*inequality*

$$d(x_t, y)^2 \leqslant (1-t)d(x_0, y)^2 + td(x_1, y)^2 - t(1-t)d(x_0, x_1)^2.$$

Fix an origin $O \in W$ and let $r \colon W \to W$ be the map which moves each point halfway along the geodesic segment towards O. Use the CAT(0)-inequality to prove that

$$d(r(x), r(y)) \leqslant \tfrac{1}{2} d(x, y).$$

By further applications of the CAT(0)-inequality, verify also that the map h defined in Example 12.7.4 gives a uniform homotopy from the identity to r.

(See [31] for more about CAT(0)-spaces.)

12.7.5 Show that if G is an *amenable* group then $C^*_G(X)$ is exactly equal to the algebra of G-invariant elements in $C^*(X)$. (It is unknown whether this is true in general.)

12.7.6 Let M be a compact Spin^c-manifold, with fundamental group G, and let X be its universal cover. Let D_M and D_X be the Dirac operators on M and on X respectively. Let χ be a normalizing function. Let $F_M = \chi(D_M)$, which is a pseudolocal operator on M.

(a) Show that the operator $F_X = \chi(D_X)$ belongs to the algebra $D^*_G(X)$.

(b) Show that the isomorphism $D^*_G(X)/C^*_G(X) \cong D^*(M)/C^*(M)$ described by Lemma 12.5.4 takes the equivalence class $[F_X]$ to the equivalence class $[F_M]$.

(The exercise says that 'the lift of Dirac is Dirac'.)

12.7.7 Let M be a compact aspherical manifold with fundamental group G. It is known that the assembly map $A_M \colon K_p(M) \to K_p(C^*_r(G))$ can be factored through the full C^*-algebra of G: there is a commutative diagram

$$\begin{tikzcd} K_p(M) \arrow[r, "A_{\max}"] \arrow[dr, "A_M"'] & K_p(C^*(G)) \arrow[d, "\pi_*"] \\ & K_p(C^*_r(G)) \end{tikzcd}$$

where $\pi \colon C^*(G) \to C^*_r(G)$ is the $*$-homomorphism defined by the regular representation.

(a) Suppose that M is non-positively curved and G has property T. Let $p \in C^*(G)$ be the Kazhdan projection (Theorem 3.7.6) corresponding to the trivial representation of G. Show that $[p] \neq 0 \in K_0(C^*(G))$, whereas p maps to the zero element of $K_0(C^*_r(G))$. Deduce from this, together with the Novikov conjecture for G, that $[p]$ cannot be in the image of $A_{\max}$.

(b) The trivial representation of G induces a homomorphism from $K_0(C^*(G))$ to $\mathbb{Z}$. Use the fact that the composition

$$K_0(M) \to K_0(C^*(G)) \to \mathbb{Z}$$

is the index map to prove again that the Kazhdan projection is not in the range of $A_{\max}$, this time without assuming the Novikov conjecture for G.

12.8 Notes

Theorem 12.1.5 is due to Lichnerowicz [95]. The first proof that the torus admits no metric of positive scalar curvature is due to Schoen and Yau [118, 119]. They use the connection between the scalar curvature and variational problems for minimal hypersurfaces. The first index-theory proofs are due to Gromov and Lawson [62, 63]. The existence of metrics whose scalar curvature is negative somewhere was proved by Kazdan and Warner [86, 87]. For a survey of the positive scalar curvature problem, see [124].

The visual boundary was constructed by Eberlein and O'Neill [54]. For an extensive discussion of non-positive curvature and its applications, see [31].

The coarse Baum–Connes conjecture is stated in [75, 139]. The reader should also consult [140, 141]. For relevant counterexamples, see [51], [140], and [73]. For Gromov's theory of hyperbolic groups see [59, 61]; several expositions of this material are available, see for instance [57] and [31].

The Novikov conjecture was originally formulated in the context of surgery theory, with reference to the L-groups of Wall [132] rather than to C^*-algebra K-theory. In this context, the conjecture provides information about the variety of topological manifold structures on a space X with fundamental group G. A comprehensive survey of the significance and applications of the conjecture (as of 1993) can be found in [55] and the other articles in the proceedings of the 1993 Oberwolfach conference on the Novikov conjecture. The book [98] provides a beautiful introduction to the signature of a manifold and its significance in topology. An account of the homotopy invariance of the C^*-algebraic index of the signature operator may be found in [79].

For background on the Baum–Connes conjecture one should consult [22]. The conjecture has been proved for a class of groups which, although large, is also in some ways rather restricted. The work of Higson and Kasparov [71, 72] proves the conjecture for a class of groups which includes all amenable groups (and no property T groups). The recent work of Lafforgue [90, 91] gives the first example of a property T group which is known to satisfy the conjecture.

Proposition 12.5.11 is known as Atiyah's L^2-index theorem. For more about this result and its connection with von Neumann algebras see [12].

The idea of 'descent' is implicit in Kasparov's work on the Novikov Conjecture. The particular formulation given in this chapter is an adaptation of an argument of Carlsson and Pedersen [38]. For the spectral sequence associated to a filtration see the standard texts on algebraic topology, for instance chapter 9 of [122]. Corollary 12.6.4 is due to Mischencko [99].

The Baum–Connes conjecture (now for *real* K-theory) is of great importance in the study of positive scalar curvature metrics due to Stolz [124]. Under favorable circumstances this study yields necessary and sufficient conditions for the existence of a positive scalar curvature metric on a manifold with given fundamental group.

The universal compactification of Exercise 12.7.2 is sometimes called the *Higson compactification*. For more about its topology see [88].

The reader who wants to proceed further with the study of Kasparov theory might want to look first at [69] and [84] (for an overview). Kasparov's major papers are [81, 83]. Some technical improvements to the theory are in [66] and [121]. There is a book-length exposition in [27]. Finally, Connes' book [41] contains a wealth of ideas about KK-theory as about many other matters.

With the recent (Spring 2000) construction [73] of counterexamples to various generalized forms of the Baum–Connes conjecture, the time may be ripe to ask what exotic geometric phenomena these counterexamples will allow to exist. But that is not a subject for this book.

APPENDIX A

GRADINGS

This appendix summarizes our terminology and conventions regarding various kinds of $\mathbb{Z}/2$-graded objects. We also discuss *Hermitian modules* and their graded counterparts, which may be used to give a rather convenient description of K-theory.

A.1 Graded Vector Spaces and Algebras

A.1.1 DEFINITION Let V be a complex vector space. A *$\mathbb{Z}/2$-grading* of V, or just a *grading* for short, is a decomposition

$$V = V^{+} \oplus V^{-}$$

of V as the direct sum of two subspaces, called the *positive* and *negative* parts of V. A vector space V provided with a grading is called a *graded vector space.* The *grading operator* on a graded vector space V is the involution γ_V whose ± 1-eigenspaces are $V^{\pm}$.

A.1.2 DEFINITION A *graded Hilbert space* is a Hilbert space which is provided with a grading for which the positive and negative parts are closed, orthogonal subspaces (equivalently, the grading operator is a selfadjoint unitary).

One should think of a graded vector space as a formal difference $[V^{+}] - [V^{-}]$ of vector spaces. A *graded vector bundle* is a vector bundle whose fibers are graded vector spaces (and whose transition maps respect the grading).

A.1.3 DEFINITION Let V be a graded vector space. The *opposite* of V is the graded vector space V^{op} which has the same underlying vector space as V, but the opposite grading operator $\gamma_{V^{\mathrm{op}}} = -\gamma_V$. Thus

$$(V^{\mathrm{op}})^{+} = V^{-} \quad \text{and} \quad (V^{\mathrm{op}})^{-} = V^{+}.$$

Let V be a graded vector space. An endomorphism T of V is called *even* if $T(V^{\pm}) \subseteq V^{\pm}$ and *odd* if $T(V^{\pm}) \subseteq V^{\mp}$. In other words, T is even if it commutes with the grading operator γ_V and T is odd if it anticommutes with γ_V. The algebra $\mathrm{End}(V)$ thus becomes an example of a *graded algebra* in the sense of the following definition:

A.1.4 DEFINITION A *graded algebra* is a complex algebra A provided with a direct sum decomposition $A = A^+ \oplus A^-$ such that

$$A^+ \cdot A^+ \subseteq A^+, \quad A^+ \cdot A^- \subseteq A^-, \quad A^- \cdot A^+ \subseteq A^-, \quad A^- \cdot A^- \subseteq A^+.$$

If A is a $*$-algebra, the involution is required to map A^+ to A^+ and A^- to A^-. We call A^+ the subspace of *even* elements of the algebra, and A^- the subspace of *odd* elements. Note that A^- may be zero.

A.1.5 EXAMPLE An important example of a graded $*$-algebra is the *Clifford algebra* $\mathbb{C}_n$, which is the complex unital $*$-algebra generated by n odd operators $e_1, \dots, e_n$ such that

$$e_i e_j + e_j e_i = 0 \ (i \neq j), \quad e_j^2 = -1, \quad e_j = -e_j^*.$$

As a complex vector space, the Clifford algebra has dimension 2^n; the monomials $e_{i_1} \cdots e_{i_k}$, where $i_1 < \cdots < i_k$ and $0 \leqslant k \leqslant n$, comprise a basis. We may introduce an inner product on $\mathbb{C}_n$ by deeming these monomials to be orthonormal. The action of the *algebra* $\mathbb{C}_n$ on the *Hilbert space* $\mathbb{C}_n$ by left multiplication is then a faithful $*$-representation which gives $\mathbb{C}_n$ the structure of a C^*-algebra.

Let A be a graded algebra. An element $a \in A$ is *homogeneous* if it belongs to A^+ or to A^-. The *degree* of a non-zero homogeneous $a \in A$ is

$$\partial a = \begin{cases} 0 & \text{if } a \in A^+, \\ 1 & \text{if } a \in A^-. \end{cases}$$

If a and a' are homogeneous elements of A, then their *graded commutator* is

$$[a, a'] = aa' - (-1)^{\partial a \partial a'} a'a.$$

The definition is extended by linearity to the whole of A.

A.2 Graded Tensor Products

A.2.1 DEFINITION If V_1 and V_2 are graded vector spaces (or Hilbert spaces) their *graded tensor product* $V = V_1 \hat{\otimes} V_2$ is the ordinary tensor product of V_1 and V_2, equipped with the grading operator which is the tensor product of the grading operator of V_1 and the grading operator of V_2.

To put it another way,

$$V^+ = (V_1^+ \otimes V_2^+) \oplus (V_1^- \otimes V_2^-), \qquad V^- = (V_1^- \otimes V_2^+) \oplus (V_1^- \otimes V_2^+).$$

If A_1 and A_2 are graded algebras then the multiplication law

$$(a_1 \hat{\otimes} a_2)(a_1' \hat{\otimes} a_2') = (-1)^{\partial a_2 \partial a_1'} (a_1 a_1') \hat{\otimes} (a_2 a_2')$$

equips their graded vector space tensor product $A = A_1 \hat{\otimes} A_2$ with the structure of a graded algebra. (The multiplication law is extended by linearity from homogeneous elements.) One thinks of the extra sign as arising from a general principle: when two homogeneous objects of degrees d and d' are commuted in a graded algebra, a sign $(-1)^{dd'}$ must always be introduced. Note that this is consistent with the definition of graded commutator given above.

In particular, when $a_1 \in A_1$ and $a_2 \in A_2$ are odd, we have

$$(a_1 \hat{\otimes} 1 + 1 \hat{\otimes} a_2)^2 = a_1^2 \hat{\otimes} 1 + 1 \hat{\otimes} a_2^2.$$

This identity is used in the definition of the products in K-theory and K-homology.

A.2.2 PROPOSITION *Let* V_1 *and* V_2 *be graded finite-dimensional vector spaces. There is a canonical isomorphism*

$$\operatorname{End}(V_1) \hat{\otimes} \operatorname{End}(V_2) \cong \operatorname{End}(V_1 \hat{\otimes} V_2).$$

PROOF If T_1 and T_2 are homogeneous endomorphisms of V_1 and V_2 then we let $T_1 \hat{\otimes} 1$ and $1 \hat{\otimes} T_2$ act on $V_1 \hat{\otimes} V_2$ by

$$(T_1 \hat{\otimes} 1)(v_1 \hat{\otimes} v_2) = T_1 v_1 \hat{\otimes} v_2, \quad (1 \hat{\otimes} T_2)(v_1 \hat{\otimes} v_2) = (-1)^{\partial v_1 \partial T_2} v_1 \hat{\otimes} T_2 v_2.$$

This establishes the required isomorphism. □

A.2.3 PROPOSITION *Let* $\mathbb{C}_m$ *denote the Clifford algebra of Example A.1.5. Then there is a canonical isomorphism* $\mathbb{C}_m \hat{\otimes} \mathbb{C}_n \cong \mathbb{C}_{m+n}$.

PROOF The left-hand side is generated by $e_i \hat{\otimes} 1$, for $1 \leqslant i \leqslant m$, and $1 \hat{\otimes} e_j$, for $1 \leqslant j \leqslant n$. These $m+n$ elements are skew-adjoint, anticommute, and have square -1, so they may be identified with the generators of $\mathbb{C}_{m+n}$. □

A.3 Multigradings

A.3.1 DEFINITION Let $p \in \{0, 1, 2, \dots\}$. A p-*multigraded* Hilbert space is a graded Hilbert space which is equipped with p odd unitary operators $\varepsilon_1, \dots, \varepsilon_p$ such that $\varepsilon_i \varepsilon_j + \varepsilon_j \varepsilon_i = 0$, if $i \neq j$, and $\varepsilon_j^2 = -1$, for every j.

In other words, a p-multigraded Hilbert space H is a graded module over the Clifford algebra $\mathbb{C}_p$; it is useful to think of this as a *right* module structure on H. Notice that a 0-multigraded Hilbert space is just an ordinary graded Hilbert space. It is convenient to make the convention that a (-1)-multigraded Hilbert space is an ordinary *ungraded* Hilbert space.

A.3.2 EXAMPLE If M is a Riemannian manifold then the Hilbert space $H = L^2(M, \mathbb{C}_p)$ of square-integrable functions from M to the finite-dimensional Hilbert space $\mathbb{C}_p$ is p-multigraded by the action of the generators of $\mathbb{C}_p$ on H by right multiplication.

A.3.3 Remark At least if $p_1, p_2 \geqslant 0$, the graded tensor product of a p_1-multigraded Hilbert space and a p_2-multigraded Hilbert space is a (p_1+p_2)-multigraded Hilbert space. Explicitly, if H_1 is p_1-graded and H_2 is p_2-graded then $v_1 \hat{\otimes} v_2$ is (p_1+p_2)-multigraded by making $\varepsilon_1, \ldots, \varepsilon_{p_1}$ act on the tensor product as

$$\varepsilon_j(v_1 \otimes v_2) = (-1)^{\partial v_2} \varepsilon_j(v_1) \otimes v_2,$$

while $\varepsilon_{p_1+1}, \ldots, \varepsilon_{p_1+p_2}$ act as

$$\varepsilon_{p_1+j}(v_1 \otimes v_2) = v_1 \otimes \varepsilon_j(v_2).$$

Notice that the sign conventions used here are compatible with the rule given earlier (Definition A.2.1) on the introduction of signs in graded algebra, provided that we think in terms of the tensor product of *right* modules over the Clifford algebra. They make the natural isomorphism

$$L^2(M_1, \mathbb{C}_{p_1}) \hat{\otimes} L^2(M_2, \mathbb{C}_{p_2}) \cong L^2(M_1 \times M_2, \mathbb{C}_{p_1+p_2}),$$

into an isomorphism of multigraded Hilbert spaces.

If H is a p-multigraded Hilbert space, then a *multigraded operator* on H will mean a bounded operator T that is a right $\mathbb{C}_p$-module map; that is, T commutes with $\varepsilon_1, \ldots, \varepsilon_p$. The multigraded operators on H form a graded C^*-algebra.

A.3.4 Proposition *For* $p \in \{-1, 0, 1, \ldots\}$, *the categories of* p*-multigraded and* $(p+2)$*-multigraded Hilbert spaces are equivalent.*

Proof Assume that $p \geqslant 0$ (we leave the case $p = -1$ to the reader). Let H be a $(p+2)$-multigraded Hilbert space. Let $e = i\varepsilon_{p+1}\varepsilon_{p+2}$; then e is even, selfadjoint and has square 1, so H is decomposed as the direct sum of orthogonal, graded eigenspaces $H_{\pm 1}$ for e. Moreover, e commutes with $\varepsilon_1, \ldots, \varepsilon_p$. Thus H_1 is a p-multigraded Hilbert space.

To reverse the construction, suppose that a p-multigraded Hilbert space H_1 is given and define $H = H_1 \oplus H_1^{op}$. Clearly H is p-multigraded; we make it $(p+2)$-multigraded by defining

$$\varepsilon_{p+1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varepsilon_{p+2} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The two constructions are inverse to one another. □

A.4 Hermitian Modules and K-Theory

Let A be a unital C^*-algebra. We have described the K-theory group, $K_0(A)$, in terms of *projections* in matrix algebras over A. An alternative description, closer to the language of algebra, makes use not of the projections themselves but of their ranges, which are certain modules over A. In this section we shall describe $K_0(A)$ from this alternative point of view. The language of modules is especially useful in connection with products on K-theory and pairings between K-theory and K-homology.

A.4.1 DEFINITION Let A be a unital C^*-algebra. An *inner product module* over A is a right A-module M provided with an 'inner product' $\langle\, ,\rangle\colon M \times M \to A$ such that

(a) for fixed $x \in M$, the map $y \mapsto \langle x, y\rangle$, from M to A, is right A-linear,
(b) $\langle x, y\rangle = \langle y, x\rangle^*$ for all $x, y \in M$, and
(c) the selfadjoint element $\langle x, x\rangle \in A$ is positive for all $x \in M$, and it is zero if and only if $x = 0$.

A *unitary isomorphism* of inner product A-modules is an A-module isomorphism which preserves the inner products.

For example, A is an inner product module over itself, with $\langle x, y\rangle = x^*y$. More generally, A^k is an inner product module: if $x = (x_i)_{i=1}^k$ and $y = (y_i)_{i=1}^k$ belong to A^k, we may define

$$\langle x, y\rangle = \sum_{i=1}^{k} x_i^* y_i.$$

Notice that a matrix $T \in M_k(A)$ acts on A^k (by left multiplication) as an A-module endomorphism; and the adjoint relation $\langle Tx, y\rangle = \langle x, T^*y\rangle$ holds with respect to the A-valued inner product.

A.4.2 DEFINITION Let M be an inner product module over A. We say that M is *Hermitian* if for some k there is a selfadjoint projection $P \in M_k(A)$ such that M is unitarily isomorphic to the submodule $\mathrm{Image}(P) \subseteq A^k$.

A.4.3 REMARK Our Hermitian modules are projective A-modules in the usual sense of algebra. Conversely, a polar decomposition argument shows that any finitely generated projective module in the sense of algebra can be given an inner product with respect to which it is Hermitian. Moreover, this inner product is unique up to isomorphism. Thus the categories of finitely generated projective A-modules and of Hermitian A-modules are equivalent; compare the relationship between the categories of finite-dimensional vector spaces and finite-dimensional inner product spaces.

A.4.4 LEMMA *There is a one-to-one correspondence between*

(a) *unitary isomorphism classes of Hermitian A-modules, and*

(b) *Murray–von Neumann equivalence classes of projections in matrix algebras* $M_k(A)$ *(where we stabilize by identifying* $P \in M_k(A)$ *with* $P \oplus 0 \in M_{k+1}(A)$*).*

PROOF It is clear that if P and Q are Murray–von Neumann equivalent projections then their ranges are unitarily isomorphic Hermitian A-modules. Conversely, suppose that Image(P) and Image(Q) are unitarily isomorphic, by an isomorphism U. By stabilization we may assume $P, Q \in M_k(A)$ for some k. Then $Q^*UP\colon A^k \to A^k$ is an A-module map, so it is given by some $V \in M_k(A)$. This V implements a Murray–von Neumann equivalence between P and Q. □

Because of Lemma A.4.4 we may describe K-theory in terms of Hermitian modules.

A.4.5 PROPOSITION *Let A be a unital* C^**-algebra. The group* $K_0(A)$ *may be described as the abelian group with one generator for each unitary isomorphism class of Hermitian modules over A, and one relation*

$$[M \oplus N] = [M] + [N]$$

for each pair of Hermitian modules M and N. □

A.4.6 PROPOSITION *The algebra* End(M) *of endomorphisms of a Hermitian A-module M is a unital* C^**-algebra.*

PROOF If M is determined by $P \in M_k(A)$, then End(M) is the C^*-algebra $PM_k(A)P$. Notice that the $*$-operation in End(M) obeys the 'adjoint law'

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \in A$$

for all $x, x' \in M$, as we would expect. □

Let M be a Hermitian A-module and let $\rho\colon A \to \mathfrak{B}(H)$ be a Hilbert space representation of A. The vector space $M \otimes H$ carries a positive-semidefinite sesquilinear form defined on elementary tensors by

$$\langle x \otimes v, y \otimes w\rangle = \big\langle v, \rho(\langle x, y\rangle)w\big\rangle.$$

This form is not definite, since, for example, vectors such as $xa \otimes v - x \otimes \rho(a)v$ have length zero. We define $M \otimes_\rho H$ to be the inner product space obtained from $M \otimes H$ after dividing by the subspace of vectors of length zero.

A.4.7 PROPOSITION *The space* $M \otimes_\rho H$ *defined above is a Hilbert space. In fact, if* $P \in M_k(A)$ *is a projection whose range is isomorphic to M, then* $M \otimes_\rho H$ *is isomorphic to the range of the projection* $(1 \otimes \rho)(P)$ *on* $\mathbb{C}^k \otimes H$.

PROOF To simplify the notation we take $k = 1$, so that M is described by a projection P in A itself. There is a surjective linear map

$$\varphi\colon M \otimes H \to \rho(P)H$$

defined by $x \otimes v \mapsto \rho(x)v$. Moreover, we have

$$\left\langle \sum_i x_i \otimes v_i, \sum_i x_i \otimes v_i \right\rangle = \sum_{i,j} \langle v_i, \rho(x_i^* x_j) v_j \rangle = \| \sum_i \rho(x_i) v_i \|^2$$

so that φ takes the semi-inner product on $M \otimes H$ to the inner product on $\rho(P)H$; it therefore induces an isomorphism $M \otimes_\rho H \to \rho(P)H$. □

Given $T \in \mathrm{End}(M)$ we may define an operator $T \otimes 1$ on $M \otimes_\rho H$ by

$$(T \otimes 1)(x \otimes v) = (Tx) \otimes v.$$

The map $T \mapsto T \otimes 1$ gives a $*$-homomorphism $\mathrm{End}(M) \to \mathfrak{B}(M \otimes_\rho H)$. Given $S \in \mathfrak{B}(H)$ we might similarly hope to define an operator $1 \otimes S$ on $M \otimes_\rho H$ by

$$(1 \otimes S)(x \otimes v) = x \otimes (Sv),$$

but this is not well-defined unless S commutes with $\rho(a)$ for every $a \in A$, a condition which is not very often satisfied. However, an 'approximate' version of the construction is available whenever S commutes with $\rho(a)$ *modulo compacts.*

A.4.8 PROPOSITION *Let M be a Hermitian A-module, let $\rho\colon A \to \mathfrak{B}(H)$ be a representation of A, and let S be a bounded operator on H which commutes with $\rho(A)$ modulo compacts. Then there is a class $\mathcal{S}'$ of bounded operators on $M \otimes_\rho H$ which have the following property: for each $S' \in \mathcal{S}'$ and each $x \in M$ the operators*

$$L_x S - S' L_x \colon H \to M \otimes_\rho H$$

are compact, where for each $x \in M$ the operator $L_x\colon H \to M \otimes_\rho H$ maps v to $x \otimes v$. The members of the class $\mathcal{S}'$ differ among themselves by compact operators on $M \otimes_\rho H$.

We shall use the notation $1 \otimes S$ for any member of the class $\mathcal{S}'$ defined above. The 'mapping' $S \mapsto 1 \otimes S$ gives a well-defined $*$-homomorphism from the dual algebra $\mathfrak{D}_\rho(A)$ to the Calkin algebra $\mathfrak{Q}(M \otimes_\rho H)$. Moreover, $1 \otimes S$ commutes modulo the compact operators with $T \otimes 1$, for any $T \in \mathrm{End}(M)$.

PROOF Suppose that M is described by a projection $P \in M_k(A)$, and for simplicity let us once again take $k = 1$. Then $M \otimes_\rho A$ is isomorphic to $\rho(P)H$. We define S': $\rho(P)H \to \rho(P)H$ to be the composite

$$\rho(P)H \xrightarrow{S} H \xrightarrow{\rho(P)} \rho(P)H.$$

Using the fact that S commutes modulo compacts with $\rho(x)$ one easily verifies the desired properties. $\square$

Let J be an ideal in the unital C^*-algebra A. Suppose that M is a Hermitian A-module. We may define an ideal $\mathrm{End}^J(M)$ in the C^*-algebra $\mathrm{End}(M)$ as follows: $T \in \mathrm{End}^J(M)$ if, for every $x \in M$, the image $Tx \in M$ belongs to the linear subspace MJ spanned by elements of the form yj, $y \in Y$, $j \in J$.

A.4.9 DEFINITION We call the members of $\mathrm{End}^J(M)$ the endomorphisms *carried by* J.

If M is represented by a projection $P \in M_k(A)$, then (as we have already observed) $\mathrm{End}(M) = PM_k(A)P$. In this case $\mathrm{End}^J(M) = PM_k(J)P$.

A.4.10 EXAMPLE Let $A = C(X)$, where X is a compact space. Let V be a Hermitian vector bundle over X. Then the sections of A form a Hermitian module M, and $\mathrm{End}(M)$ is just the algebra of sections of the endomorphism bundle $\mathrm{End}(V)$. If Y is a closed subset of X, and J is the ideal in A consisting of those functions which vanish on Y, then $\mathrm{End}^J(M)$ is the space of sections of $\mathrm{End}(V)$ which vanish on Y.

A.5 Graded Hermitian Modules

A.5.1 DEFINITION Let A be a unital C^*-algebra. A *graded Hermitian module* over A is a Hermitian module M (in the sense of the previous section) which is decomposed as the orthogonal direct sum $M = M^+ \oplus M^-$ of two Hermitian submodules.

The endomorphisms of a graded Hermitian module form a graded C^*-algebra. Graded Hermitian modules make possible a very simple definition of the relative K-theory group.

A.5.2 PROPOSITION *Let A be a unital C^*-algebra and let J be an ideal in A. The relative* K*-theory group* $K_0(A, A/J)$ *may be described as the abelian group with the following generators and relations:*

(a) *The generators are homotopy classes of pairs (M, a), where M is a graded Hermitian module over A and $a \in \mathrm{End}(M)$ is an odd endomorphism such that $a - a^*$ and $a^2 - 1$ are carried by J.*

(b) *The direct sum operation on such pairs provides relations*

$$[M' \oplus M'', a' \oplus a''] = [M', a'] + [M'', a''].$$

(c) *In addition, for each degenerate pair, meaning one for which* $a = a^*$ *and* $a^2 = 1$, *there is a relation* $[M, a] = 0$.

PROOF A graded Hermitian module M may be described by a pair (p, q) of projections in $M_n(A)$, p being the projection onto M^+ and q the projection onto M^-. An endomorphism a of the form described above may be written

$$a = \begin{pmatrix} 0 & y \\ x & 0 \end{pmatrix}$$

where $x \in pM_n(A)q$, $y \in qM_n(A)p$, and $x - y^*$, $xy - p$, and $yx - q$ belong to $M_n(J)$. It follows that if $\pi\colon M_n(A) \to M_n(A/J)$ denotes the quotient map, then $\pi(x)$ implements a Murray–von Neumann equivalence between $\pi(p)$ and $\pi(q)$. That is, (p, q, x) is a relative K-cycle according to Definition 4.3.1. The proof is now a simple matter of matching up definitions. □

Our discussion of tensor products between a Hermitian A-module and a representation space of A has a natural extension to the graded case. Let A be a unital C*-algebra, let M be a graded Hermitian A-module, and let $\rho\colon A \to \mathfrak{B}(H)$ be a representation of A on a graded Hilbert space H. The Hilbert space $M \otimes_\rho H$ discussed before is now graded: its even part is $(M^+ \otimes_\rho H^+) \oplus (M^- \otimes_\rho H^-)$ and its odd part is defined in a similar way. We use the notation $M \hat{\otimes}_\rho H$ for $M \otimes_\rho H$ graded in this way.

If T is an endomorphism of the graded module M then we can define an endomorphism $T\hat{\otimes}1$ of $M\hat{\otimes}_\rho H$ in the natural way. If $S \in \mathfrak{B}(H)$ commutes modulo compacts with $\rho(A)$ then the graded analogue of Proposition A.4.8 produces a class $\mathcal{S}'$ of operators $S' = 1\hat{\otimes}S$ with the property that

$$L_x S \pm S' L_x \colon H \to M\hat{\otimes}_\rho H$$

where the sign is negative if one of S, x is even and is positive if they are both odd. Now $1\hat{\otimes}S$ graded-commutes with $T\hat{\otimes}1$. The proofs of these assertions are left to the reader.

A.6 Notes

The use of $\mathbb{Z}/2$-gradings is common in index theory; see [13, 94]. For the connection with the physical theory of supersymmetry see [136].

Our 'Hermitian modules' are actually finite-rank *Hilbert* C*-*modules*. Hilbert modules over C*-algebras are of great importance in Kasparov's KK-theory. The book by Lance [93] gives an excellent introduction.

APPENDIX B

REAL K-HOMOLOGY

B.1 Real C^*-Algebras

B.1.1 DEFINITION A *real* C^**-algebra* is a real, involutive Banach algebra which is isometrically $*$-isomorphic to a norm-closed $*$-algebra of bounded operators on a real Hilbert space.

Obvious examples are the algebras $\mathfrak{B}(H_{\mathbb{R}})$ and $\mathfrak{K}(H_{\mathbb{R}})$ of bounded and compact operators on a real Hilbert space. Equally obvious are the commutative algebras $C_{\mathbb{R}}(X)$ of real-valued continuous functions on compact spaces X. But there are less obvious examples as well: for instance, the quaternion algebra $\mathbb{H}$, and the algebra of all continuous, complex-valued functions on $\mathbb{R}$ which vanish at infinity and which have the symmetry $\overline{f(x)} = f(-x)$ (the latter is isomorphic, by the Fourier transform, to the real C^*-subalgebra of $\mathfrak{B}(L^2(\mathbb{R}))$ generated by convolutions with real-valued, continuous, compactly supported functions on $\mathbb{R}$).

If $A_{\mathbb{R}}$ is a real C^*-algebra then its complexification $A_{\mathbb{C}} = A_{\mathbb{R}} \otimes \mathbb{C}$ is a complex C^*-algebra. The complexification carries a natural 'complex conjugation' operation $a \otimes \lambda \mapsto a \otimes \bar{\lambda}$ from which $A_{\mathbb{R}}$ can be recovered as the fixed points.

B.1.2 DEFINITION A *Real* C^**-algebra* (note the capitalization) is a complex C^*-algebra A which is equipped with a conjugate linear, multiplicative involution $a \mapsto \bar{a}$ which commutes with the adjoint operation.

REMARK Unlike the adjoint operation, the complex conjugation $a \mapsto \bar{a}$ *preserves* the order of products (but otherwise the axioms for it are the same).

If $A_{\mathbb{R}}$ is a real C^*-algebra then its complexification is a Real C^*-algebra, while if $A_{\mathbb{C}}$ is a Real C^*-algebra then the subalgebra of real elements

$$A_{\mathbb{R}} = \{ a \in A_{\mathbb{C}} \ : \ a = \bar{a} \}$$

is a real C^*-algebra.

B.1.3 PROPOSITION *The category of real* C^**-algebras is equivalent to the category of Real* C^**-algebras.* $\square$

It is usually more convenient (especially when considering spectral theory and the functional calculus) to work with Real C^*-algebras rather than real C^*-algebras.

B.2 K-Theory for Real C^*-Algebras

The K_0-group of a Real C^*-algebra is defined using projections $p \in M_k(A)$ such that $\bar{p} = p$ (the complex conjugation operation is extended to matrices in the natural way). Before one reaches the Bott Periodicity Theorem, there is no essential difference between K-theory for Real C^*-algebras and the theory for complex C^*-algebras which is developed in the text. One defines the suspension $S(A) = C_0(0,1) \otimes A$ using the standard complex conjugation on $C_0(0,1)$. The higher K-groups $K_p(A)$ are defined to be $K_0(S^p(A))$, and there is a long exact sequence

$$\to K_p(J) \to K_p(A) \to K_p(A/J) \to K_{p-1}(J) \to K_{p-1}(A) \to K_{p-1}(A/J) \to$$

connecting these higher groups, exactly as in Chapter 4.

The Bott Periodicity Theorem for Real C^*-algebras is an *eightfold* periodicity isomorphism

$$K_p(A) \cong K_p(S^8(A)).$$

There is, however, a vestige of the twofold periodicity which holds for complex C^*-algebras. Let $\widehat{S}$ denote the Real C^*-algebra obtained by equipping $C_0(0,1)$ with the non-standard complex conjugation $\overline{f}(x) = \overline{f(1-x)}$, and let $\widehat{S}(A) = \widehat{S} \otimes A$. Then the arguments presented in Chapter 4 (including the streamlined proof discovered by Cuntz which is presented in the exercises) give an isomorphism

$$K_p(A) \cong K_p(\widehat{S}(S(A))).$$

This exotic type of twofold periodicity may be used to obtain the eightfold periodicity above.

As in the complex case, the group $K_1(A)$ may be defined in terms of unitaries (but now such that $\bar{u} = u$), and the concrete formula that we presented for the boundary map

$$\partial \colon K_1(A/J) \to K_0(J)$$

is valid in the Real context. The other boundary map from complex K-theory, involving the exponential function $e^{i\pi x}$, carries over to the Real case as long as we regard it as a map

$$\partial \colon K_0(A/J) \to K_{-1}(J),$$

where $K_{-1}(J)$ is defined to be $K_0(\widehat{S}(A))$.

The remaining features of K-theory which are presented in Chapter 4 (triviality of inner automorphisms, stability, continuity with respect to direct limits) all carry over to the Real case with no change.

B.3 K-Homology for Real C*-Algebras

We shall pass over the possibility of developing the Brown–Douglas–Fillmore extension theory for real C*-algebras and turn instead to the Kasparov theory, and applications to index theory.

The Kasparov groups $K^{-p}(A)$ are defined exactly as one would expect, using Fredholm modules (ρ, H, F) where ρ is a representation of the Real C*-algebra A on the separable Real p-multigraded Hilbert space H,[114] and F is an operator on H satisfying the conditions presented in Chapter 8, along with the new condition $\bar{F} = F$.

Most of Chapters 8 and 9 carries over without essential change to the Real case. The exceptions are the formal periodicity isomorphisms, which become eightfold periodicities

$$K^{-p}(A) \cong K^{-p+8}(A),$$

and the treatment of the boundary map $\partial\colon K^1(J) \to K^0(A/J)$, which must be reinterpreted as a boundary map

$$\partial\colon K^{-1}(J) \to K^0(A/J)$$

(and this in turn requires a new description of the group $K^{-1}(J)$). Needless to say, the compatibility properties of the boundary map and the index pairing must also be suitably adapted to take into account the difference between K^{-1} and K^1 in the Real case.

Our presentation of the Index Theorem in Chapter 11 was organized around the index pairing, which is integer-valued. This seems to place out of reach the very subtle $\mathbb{Z}/2$-valued generalizations of the Index Theorem which present themselves in the Real case. The remedy is to introduce improved pairings (slant products) between K-theory and K-homology. This could be done at modest cost using the Schrödinger pairings which we introduced in Chapter 8, but the reader who contemplates doing so might be better advised to take up Kasparov's general bivariant theory at this point.

Similar advice might be given to the reader who wishes a fully Real treatment of the assembly maps we introduced in Chapter 12. The process of assembly attempts to put K-theory and K-homology on an equal footing. While we introduced gradings and multigradings into K-homology, we did not do the same in K-theory. The shortcoming is not really noticeable in the complex case, but it now becomes more burdensome. Once again, the best solution is to turn to Kasparov's bivariant theory, which deals all at once with Real C*-algebras and graded C*-algebras, not to mention G-C*-algebras, which play an essential role in the further study of the Novikov and Baum–Connes conjectures.

[114] A Real Hilbert space is of course a complex Hilbert space which is equipped with a complex conjugation.

B.4 Notes

The ideas presented in this section originate in the work of Atiyah and Singer on the real form of the Index Theorem [7, 18]. For a striking application to the positive scalar curvature problem, see the thesis of Hitchin [78].

An elementary approach to K-theory for real and graded C^*-algebras is contained in the papers of van Daele [128, 129]. See also the book [120].

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