目录

讨论班设置

我们以 Morita 的 Geometry of Differential Forms 为主要参考书，这本书主要讲了微分形式，Hodge 理论和 Chern-Weil 理论（几何示性类）。

通过 Hodge 理论，我们将对 de Rham 上同调群有更深刻的认识。我们也会见识一些利用调和形式简化证明的例子。

我们将从曲率矩阵构造整体的（闭）微分形式，进而得到一个 de Rham 同调类。这样我们便从几何结构得到了一个拓扑量。

从代数的角度，示性类可以看作从流形上的向量丛集合到其上同调群的映射，借此可以区分不同构的向量丛。从这个角度可以自然过渡到拓扑 K 理论。

我们会从较基本的内容谈起，特别我们会较详细地介绍最基本的黎曼几何，因此学过流形和拓扑的同学即可参与。

在讨论班中我们也将给出高维 Gauss-Bonnet 定理的内蕴证明（Chern）。

最后会根据进度决定是否讨论主丛上的示性类。

大致安排如下：

第一周：张量、微分形式、黎曼度量（袁榕含报告）、Hodge 星算子。

第二周：Laplacian 算子、Hodge 定理（季雨衡报告证明思路）、应用。

第三周：向量丛（周铭杨报告）、外蕴观点下的协变导数。

第五周：测地线、联络、曲率、微分形式。

第六周：不变多项式、示性类、Chern-Weil 定理的多种证明。

第七周：复流形、Chern 类、和 Pontrjagin 类的关系、Chern 类的多种定义。

第八周：Euler 类和 Gauss-Bonnet 公式。

第九周：主丛相关。

第十周：主丛上的联络和曲率。

第十一周：主丛上的示性类。

第十二周：拓展，Morita 上的一些专题讨论。

第十二周：拓展。

1 第一周，黎曼度量和星算子

粗略地说，黎曼流形是一个向量长度被定义的流形。

具体地来说，我们仅仅假设我们有向量长度的概念，结果就是我们会得到体积的概念和这个流形的形状，包括其是怎么弯曲的。

我们会在一个 de Rham 上同调的代表元集合中，找到一个最好的闭形式来表示这个上同调。

这个闭形式叫做全纯形式，我们可以用 Laplacian 算子来区分它。

1.1 黎曼度量

M 是 C^∞ 流形，如果对于任意点 $p \in M$，切平面 T_pM 上被赋予了正定内积结构

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

使得 g_p 关于 p 是光滑的，我们就说 $g = \{g_p : p \in M\}$ 是 M 上的一个黎曼度量，并称 M 是黎曼流形。

强调：正定、g_p 的光滑性怎么定义。Einstein 求和约定。

$$g_{ij}(p) = g_p(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$$

$$g_{ij}(p) = \frac{\partial y_k}{\partial x_i}(p)\frac{\partial y_l}{\partial x_j}(p)h_{kl}(p)$$

二阶对称张量、矩阵表示。

$$ds^2 = g = g_{ij}dx^i \otimes dx^j$$

联络与度量相容：$\nabla g = 0$.

例子一，n 维欧氏空间。

例子二，上半平面 $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ 上赋予黎曼度量

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

成为双曲平面。以后会研究其上的测地线。

例子三，黎曼流形的子流形有诱导黎曼度量。

长度定义，$X \in T_pM$，

$$\|X\| = \sqrt{g(X, X)} = \sqrt{\langle X, X\rangle}.$$

$\|X\| = 0$ 当且仅当 $X = 0$.

对于光滑曲线 $c:[a,b]\to M$, 定义长度为

$$L(C)=\int_a^b \|\dot{c}(t)\|dt.$$

即将速度积分得到长度。容易验证积分与坐标选取无关（留作习题）。

强调与坐标选取无关的重要性。

一个微分流形上的黎曼度量有很大的自由度。

命题：任何一个光滑流形上存在黎曼度量。P147、148。单位一分解。强调 Hermite 度量不能全纯。

1.2 微分形式

定义 $\hat{g_p}(X)$：$\hat{g_p}(X)(Y)=g_p(X,Y)$，则得到线形映射

$$\hat{g_p}:T_pM\to T_p^*M.$$

容易证明这个线形映射是单射，由于维数，我们知道 $\hat{g_p}$ 给出了从 T_pM 到 T_p^*M 的同构。

我们可以通过这个方式，将流形上的光滑向量场全体 $\mathscr{X}(M)$ 和一阶微分形式全体 $\mathcal{A}^1(M)$ 等同。从分量的角度，这个可以由指标升降给出

$$dx^i=g^{ij}\frac{\partial}{\partial x_j},\qquad \frac{\partial}{\partial x_j}=g_{ij}dx^i$$

其中矩阵 $\{g^{ij}\}$ 是矩阵 $\{g_{ij}\}$ 的逆矩阵。从此我们不再区分 $\mathscr{X}(M)$ 和 $\mathcal{A}^1(M)$。

对于一个光滑函数 f,df 是一阶微分形式，于是可以对应一个向量场，我们记为 $\operatorname{grad} f$，称作 f 的梯度。

$$g(\operatorname{grad} f,X)=df(X)=Xf,$$

对于任意 $X\in\mathscr{X}(M)$。具体地，对于局部自然坐标 $\{x^i\}$，$df=\frac{\partial f}{\partial x^i}dx^i$，于是

$$\operatorname{grad} f=\frac{\partial f}{\partial x^i}g^{ij}\frac{\partial}{\partial x^j}.$$

如果 $\operatorname{grad} f$ 无零点，则我们有“水平超曲面”的概念。而且 $g(\operatorname{grad} f,X)=Xf$ 给出了 $\operatorname{grad} f$ 垂直于水平超曲面。

由于我们将 $\mathscr{X}(M)$ 与 $\mathcal{A}^1(M)$ 等同，这个同构给出了在 $\mathcal{A}^1(M)$ 上的内积

$$\langle\omega,\eta\rangle=\omega_i\eta_j g^{ij}.$$

我们推广到 k 形式的情况:

$$\langle\alpha_1\wedge\cdots\wedge\alpha_k,\beta_1\wedge\cdots\wedge\beta_k\rangle=\det(\langle\alpha_i,\beta_j\rangle).$$

补充定义退化的情况。

如果 $e_1,\cdots,e_n$ 是 V 的标准正交基，其对偶基记为 $\theta^1,\cdots,\theta^n$，则

$$\theta^{i_1}\wedge\cdots\wedge\theta^{i_k},\ 1\le i_i<\cdots<i_k\le n,$$

构成 $\bigwedge^k V^*$ 的标准正交基。

1.3 Hodge 星算子

设 M 是可定向 n 维黎曼流形。为什么需要可定向。

$\bigwedge^k T_p^*M$ 和 $\bigwedge^{n-k} T_p^*M$ 维数相同，因此同构。特别地，若 $\theta^1, \cdots, \theta^n$ 是一组标准正交基，则可定义线形算子

$$* : \wedge^k T_p^*M \to \wedge^{n-k} T_p^*M$$

通过

$$*(\theta^1 \wedge \cdots \wedge \theta^k) = \theta^{k+1} \wedge \cdots \wedge \theta^n.$$

特别地，

$$*1 = \theta^1 \wedge \cdots \wedge \theta^n,\ *(\theta^1 \wedge \cdots \wedge \theta^n) = 1.$$

通过变换 $p \in M$，我们得到线形映射

$$* : \mathcal{A}^k(M) \to \mathcal{A}^{n-k}(M).$$

需要注意的问题是，若 $\omega \in \mathcal{A}^k(M)$，我们需要证明 $*\omega$ 是光滑的，为此我们考虑局部坐标。

考虑局部的标准正交基 $\theta^1, \cdots, \theta^n$，这可以由 Gram-Schmidt 正交过程得到。设

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k} \theta^{i_1} \wedge \cdots \wedge \theta^{i_k},$$

则

$$*\omega = \sum_{i_1 < \cdots < i_k} \mathrm{sgn}(I, J) f_{i_1 \cdots i_k} \theta^{j_1} \wedge \cdots \wedge \theta^{j_{n-k}}.$$

解释 $\mathrm{sgn}(I, J)$ 的定义。

定义 $v_M = *1$ 是体积形式（或体积元），它的局部标准坐标表示为

$$v_M = \theta^1 \wedge \cdots \wedge \theta^n.$$

如果用自然坐标，则

$$v_M = \det(a^i_j) dx^1 \wedge \cdots \wedge dx^n = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n.$$

对于区域 $D \subset M$，其体积为 $\int_D v_M$。若 M 是紧致的，则 M 的体积为 $\int_M v_M$。

命题 1.1. 关于 Hodge$*$ 算子的一些性质：

1. $*(f\omega + g\eta) = f * \omega + g * \eta.$
2. $* * \omega = (-1)^{k(n-k)} \omega.$
3. $\omega \wedge *\eta = \eta \wedge *\omega = \langle \omega, \eta \rangle v_M.$
4. $*(\omega \wedge *\eta) = *(\eta \wedge *\omega) = \langle \omega, \eta \rangle.$

5. $\langle *\omega, *\eta\rangle = \langle \omega, \eta\rangle$.

对于光滑向量场 $X \in \mathscr{X}(M)$，用 ω_X 表示 X 对应的一阶微分形式，定义 X 的散度为

$$divX = *d*\omega_X.$$

设 X 有局部标准坐标表示 $X = X^i e_i$，则 $\omega_X = \sum_i X^i\theta^i$.

$$\begin{aligned}
divX &= \sum_i *d*(X^i\theta^i) \\
&= \sum_i *d(X^i(-1)^{i-1}\theta^1\wedge\hat{\theta^i}\wedge\theta^n) \\
&= \sum_i *e_i(X^i)\theta^1\wedge\cdots\wedge\theta^n \\
&= e_i(X^i).
\end{aligned}$$

定理 1.1. M 是可定向紧黎曼流形。如果 X 是 M 上的向量场，则我们有等式

$$\int_M divXv_M = \int_{\partial M}\langle X, n\rangle v_{\partial M},$$

其中 n 是边界 ∂M 上的方向向外的单位法向量。特别地，若 M 是闭流形，

$$\int_M divXv_M = 0.$$

证明留作习题，答案见香蕉空间《微分形式与代数拓扑》板块。

2 第二周，Laplacian 和 Hodge 定理

在前两节中，如无特别强调，我们规定 (M, g) 是闭的（紧致无边）、可定向黎曼流形。紧致性只在涉及积分时是必须的。

2.1 Laplacian

我们用记号 $\langle \omega_p, \eta_p \rangle$ 来表示两个 k 阶微分形式在点 p 的内积，将其积分得到

$$(\omega, \eta) := \int_M \langle \omega, \eta \rangle v_M,$$

容易验证其具有线性性、对称性和正定性，因此是 $\mathcal{A}^k(M)$ 上的内积。我们再定义不同阶的微分形式之间的内积是 0，于是我们得到了 $\mathcal{A}^*(M)$ 上的内积。

我们定义 ω 的长度 $\|\omega\| = \sqrt{(\omega, \omega)}$。

由性质1.1的第三条，

$$(\omega, \eta) = \int_M \langle \omega, \eta \rangle v_M = \int_M \omega \wedge *\eta = \int_M \eta \wedge *\omega.$$

且由性质1.1的第五条，

$$(*\omega, *\eta) = (\omega, \eta),$$

因此 Hodge $*$ 算子给出了 $\mathcal{A}^k(M) \to \mathcal{A}^{n-k}(M)$ 的一个等距映射。

考虑以下交换图表

$$\begin{tikzcd} \mathcal{A}^k(M) \arrow[r, "*"] \arrow[d, dashed, "\delta"] & \mathcal{A}^{n-k}(M) \arrow[d, "d"] \\ \mathcal{A}^{k-1}(M) \arrow[r, "(-1)^k *"] & \mathcal{A}^{n-k+1}(M) \end{tikzcd}.$$

其中系数 $(-1)^k$ 是为了使得 δ 成为 d 的伴随算子，定义

$$\delta := (-1)^k *^{-1} d* = (-1)^{n(k+1)+1} * d * .$$

容易验证，

$$*\delta = (-1)^k d*, \qquad \delta* = (-1)^{k+1} * d, \qquad \delta \circ \delta = 0.$$

命题 2.1. 关于 $\mathcal{A}^*(M)$ 上的内积 (,)，δ 是外微分算子的伴随算子，即

$$(d\omega, \eta) = (\omega, \delta\eta).$$

当然，d 也是 δ 的伴随算子。

定义 2.1. 对于黎曼流形 M，我们定义 Laplacian 算子（或 Laplace-Beltrami 算子）如下

$$\Delta := d\delta + \delta d : \mathcal{A}^k(M) \to \mathcal{A}^k(M).$$

若 $\omega \in \mathcal{A}^*(M)$ 满足 $\Delta\omega = 0$，则我们称 ω 是调和形式。特别地，若 $\Delta f = 0$ 成立，则称函数 f 为调和函数。

注意到 $\Delta = d\delta + \delta d = (d+\delta)^2$，我们定义一阶微分算子

$$P = d + \delta,$$

则有 $\Delta = P^2$。

通过计算欧式空间 $\mathbb{R}^n$ 上的简单微分形式 $\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_k}$，

$$\Delta\omega = -\sum_{s=1}^{n} f_{ss} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

我们能够看见黎曼流形上的 Laplacian 和 Laplace 算子的联系。

注 2.1. 注意到我们定义的 Laplacian 和通常的 Laplace 算子不一致，这里是为了保证 Δ 的正算子性质（即 Δ 的特征值皆非负），假设已知 P 的自伴随性，

$$(\Delta f, f) = (P^2 f, f) = (Pf, Pf) \geq 0.$$

命题 2.2. Laplacian 算子 Δ 的一些性质：

1. $*\Delta = \Delta *$，则若 ω 调和，则 $*\omega$ 调和。且 $*: \mathbb{H}^k \to \mathbb{H}^{n-k}$ 给出了同构。

2. Δ, P 自伴随。

3. $\Delta\omega = 0$ 等价于 $d\omega = 0$ 且 $\delta\omega = 0$。

推论 2.1. 若 f 是可定向闭黎曼流形 M 上的光滑函数，则

$$\int_M \Delta f v_M = 0.$$

证明.

$$\Delta f v_M = (\delta df) v_M = *(\delta df) = -d(*df),$$

则

$$\int_M \Delta f v_M = -\int_M d(*df) = 0.$$

推论 2.2. M^n 是连通可定向闭黎曼流形，则 M 上的调和函数是常值，n 阶的调和形式是一个常数乘以体积元 v_M。

由此 0 阶和最高阶调和形式构成的向量空间都同构于 $\mathbb{R}$，而注意到 $H^0_{dR}(M) = H^n_{dR}(M) = \mathbb{R}$。Hodge 定理说明了对于一般的 k，de Rham 上同调类被唯一的调和形式表示。

2.2 Hodge 定理

定义 M 上的 k 阶调和形式全体

$$\mathbb{H}^k(M) := \{\omega \in \mathcal{A}^k(M) : \Delta\omega = 0\}.$$

由于调和形式都是闭的，所以通过取 de Rham 上同调类，我们得到自然的线性映射

$$\mathbb{H}^k(M) \to H^k_{dR}(M).$$

引理 2.1. 这个映射是单的。

由 de Rham 定理，$H^k_{dR}(M) \cong H^k(M;\mathbb{R})$ 是有限维的，再结合以上引理，得 $\mathbb{H}^k(M)$ 也是有限维的。

定理 2.1. Hodge 定理。

可定向紧黎曼流形上的任意 de Rham 上同调类可以被调和形式唯一表示。换句话说，映射 $\mathbb{H}^k(M) \to H^k_{dR}(M)$ 是同构。

唯一性已由上条引理给出，难点在于调和形式的存在性。

引理 2.2. $\mathcal{A}^k(M)$ 中的 $\mathbb{H}^k, d\mathcal{A}^{k-1}, \delta\mathcal{A}^{k+1}$ 相互正交。因此我们有直和

$$\mathbb{H}^k \oplus d\mathcal{A}^{k-1} \oplus \delta\mathcal{A}^{k+1} \subset \mathcal{A}^k(M),$$

且 $d\mathcal{A}^{k-1} \oplus \delta\mathcal{A}^{k+1}$ 含于 $\operatorname{Ker} H$。更进一步，$\mathcal{A}^k$ 中正交于以上直和的元素一定为 0。

注 2.2. 最后一句话并不能证明 $\mathbb{H}^k \oplus d\mathcal{A}^{k-1} \oplus \delta\mathcal{A}^{k+1} = \mathcal{A}^k(M)$，因为 $\mathcal{A}^k(M)$ 是无限维的。但 Kodaira 和 de Rham 证明了以下的 Hodge 分解。

定理 2.2. Hodge 分解。

可定向紧黎曼流形上的任意 k 阶微分形式可以被唯一地写成调和形式、恰当形式和对偶恰当形式的和；即

$$\mathcal{A}^k(M) = \mathbb{H}^k \oplus d\mathcal{A}^{k-1} \oplus \delta\mathcal{A}^{k+1}.$$

利用 Hodge 分解可以很容易给出 Hodge 定理的证明。

Hodge 分解的证明思路 Danny Ji

In this note, we will assume two theorems as proved and head towards the Hodge Theorem.

Of course we should get familiar with our goal first. Given a compact oriented Riemannian manifold M, there's a nature map $H^k(M) \to H^k_{dR}(M)$. It is not difficult to prove it's injective, but may take a while to figure out the exciting fact that it's actually bijective.

定理 2.3 (The Hodge Theorem). Given a compact oriented Riemannian manifold M, each de Rham cohomology class on it contains a unique harmonic form. In other words, the nature map $H^k(M) \to H^k_{dR}(M)$ is a linear isomorphism.

proof of uniqueness in Theorem 2.3. Assume that there exists $\omega \in \mathcal{A}^k(M)$ such that $d\omega$ is a harmonic form, which leads to the fact that $\delta d\omega = 0$. Therefore we have

$$0 = (\delta d\omega, \omega) = (d\omega, d\omega) = \|d\omega\|^2.$$

So we can come to the conclusion that $d\omega = 0$, or in other words, any form that is both exact and harmonic must be 0. □

The proof of existence requires solving the differential equation $\Delta\omega = \alpha$, with $\alpha \in \mathcal{A}^k(M)$ a known closed form. Assume that we've found $\omega \in \mathcal{A}^k(M)$ satisfying the equation above, then $(\omega, \Delta\beta) = (\Delta\Omega, \beta) = (\alpha, \beta)$ for any $\beta \in \mathcal{A}^k(M)$. Thus ω determines a bounded linear functional l on $\mathcal{A}^k(M)$ such that $l(\beta) = (\omega, \beta)$ and $l(\Delta\beta) = (\alpha, \beta)$ for any $\beta \in \mathcal{A}^k(M)$. l is called a **weak solution** to the partial differential equation $\Delta\omega = \alpha$, and ω is an **ordinary solution**.

As we've seen, an ordinary solution can determine a weak solution, and it remains to answer whether the converse is true. The following theorem gives out a positive consequence, and turns the problem into finding a weak solution of the equation.

定理 2.4 (Regularity Theorem). Given a compact oriented Riemannian manifold M and $\alpha \in A^k(M)$, assume that l is a weak solution of $\Delta\omega = \alpha$. Then there exists $\omega \in \mathcal{A}^k(M)$ such that

$$l(\beta) = (\omega, \beta) \text{ for all } \beta \in \mathcal{A}^k(M).$$

Moreover, we also need some technique of analysis to find the weak solution, which is shown in the following theorem.

定理 2.5. Given a compact oriented Riemannian manifold M and $\{\alpha_n\}$ a sequence of smooth p-forms on M such that both $\{\|\alpha_n\|\}$ and $\{\|\Delta\alpha_n\|\}$ are bounded, then there exists a subsequence of $\{\alpha_n\}$ which is a Cauchy sequence of $\mathcal{A}^k(M)$.

We will omit the proof of Theorem 2.4 and 2.5 as they requires more knowledge regarding Sobolev spaces and elliptic operators, which is far from the main topic of the seminar. Anyone interested in these topic can take chapter 4 of GTM 65 or chapter 6 of GTM 94 for reference. To show how they work, we prove a lemma first:

引理 2.3. Given a compact oriented Riemannian manifold M, there exists an orthogonal direct sum decomposition $\mathcal{A}^k(M) = (H^k(M))^\perp \oplus H^k(M)$ and a constant $c > 0$ such that

$$\|\beta\| \leq c\|\Delta\beta\| \text{ for all } \beta \in (H^k(M))^\perp.$$

证明. Suppose the contrary, then there exists a sequence $\{\beta_j\}$ in $(H^k(M))^\perp$ such that $\|\beta_j\| = 1$ and $\lim_{j\to\infty}\|\Delta\beta_j\| = 0$. Theorem 2.5 implies that it contains a Cauchy subsequence, which can be assumed to be $\{\beta_j\}$ itself. Thus $\lim_{j\to\infty}(\beta_j, \alpha)$ exists for all $\alpha \in \mathcal{A}^k(M)$, and we can define a bounded linear functional l by

$$l(\alpha) = \lim_{j\to\infty}(\beta_j, \alpha) \text{ for all } \alpha \in \mathcal{A}^k(M).$$

And for any $\alpha \in \mathcal{A}^k(M)$, as $\lim_{j\to\infty}\|\Delta\beta_j\| = 0$,

$$l(\Delta\alpha) = \lim_{j\to\infty}(\beta_j, \Delta\alpha) = \lim_{j\to\infty}(\delta\beta_j, \alpha) = 0.$$

So l is a weak solution of $\Delta\beta = 0$, and by Theorem 2.4 there exists $\beta \in \mathcal{A}^k(M)$ such that $l(\alpha) = (\alpha, \beta)$ for all $\alpha \in \mathcal{A}^k(M)$, leading to the fact that $\lim_{j\to\infty}\beta_j = \beta$. Since $\|\beta_j\| = 1$ and $\beta_j \in (H^k(M))^\perp$ which is a closed set, we have $\|\beta\| = 1 \neq 0$ and $\beta \in (H^k(M))^\perp$. However, it contradicts to the fact that $\Delta\beta = 0$. □

Then comes the most important step in the proof of the Hodge theorem.

定理 2.6 (The Hodge Decomposition Theorem). Given a compact oriented Riemannian manifold M, we have the following orthogonal direct sum decomposition of $\mathcal{A}^k(M)$

$$\begin{aligned}
\mathcal{A}^k(M) &= \Delta(\mathcal{A}^k(M)) \oplus H^k(M) \\
&= d\delta(\mathcal{A}^k(M)) \oplus \delta d(\mathcal{A}^k(M)) \oplus H^k(M) \\
&= d(\mathcal{A}^{k-1}(M)) \oplus \delta(\mathcal{A}^{k+1}(M)) \oplus H^k(M).
\end{aligned}$$

证明. It's obvious that we just need to verify the first line. First we claim that the dimension of $H^k(M)$ is finite. Otherwise, for an infinite orthonormal sequence in $H^k(M)$, Theorem 2.5 implies that it must contains a Cauchy sequence, causing a contradiction. Let $\omega_1, \cdots, \omega_m$ be an orthonormal basis of $H^k(M)$. Considering orthogonal decomposition $\mathcal{A}^k(M) = (H^k(M))^\perp \oplus H^k(M)$, it's sufficient to prove that $(H^k(M))^\perp = \Delta(\mathcal{A}^k(M))$.

For any $\alpha, \beta \in \mathcal{A}^k(M)$ with $\Delta\beta = 0$, we have

$$(\Delta\alpha, \beta) = (\alpha, \Delta\beta) = 0.$$

Thus $(H^k(M))^\perp \supset \Delta(\mathcal{A}^k(M))$.

Conversely, for $\alpha \in (H^k(M))^\perp$, we define a linear functional l on $\Delta(\mathcal{A}^k(M))$ by

$$l(\Delta\alpha) = (\alpha, \beta) \text{ for all } \beta \in \mathcal{A}^k(M).$$

For $\beta \in \mathcal{A}^k(M)$, if $\Delta\beta = 0$, then $(\alpha, \beta) = 0$, so l is well-defined. We use H to denote the projection from $\mathcal{A}^k(M)$ to $H^k(M)$, then we can define $\eta = \beta - H\beta$. Lemma 2.3 implies that for any $\beta \in \mathcal{A}^k(M)$, there exists $c > 0$ such that

$$|l(\Delta\beta)| = |l(\Delta\eta)| = |(\alpha, \eta)| \leq \|\alpha\|\|\eta\| \leq c\|\alpha\|\|\Delta\eta\| \leq \|\alpha\|\|\Delta\beta\|.$$

So l is a bounded linear functional, which can be extended to be a bounded linear functional on $\mathcal{A}^k(M)$. We use the same notation, and find that l is a weak solution of $\Delta\omega = \alpha$, so by Theorem 2.4, there exists $\omega \in \mathcal{A}^k(M)$ such that $\Delta\omega = \alpha$, hence $(H^k(M))^\perp \subset \Delta(\mathcal{A}^k(M))$.

Above all, we have $(H^k(M))^\perp = \Delta(\mathcal{A}^k(M))$. □

注 2.3. The operator G defined in the way that $G(\alpha)$ is the unique solution of $\Delta\omega = \alpha - H(\alpha)$, and is called the **Green Operator**. We can easily verify that G is a bounded self-adjoint linear operator and commutes with any linear operator which commutes with Δ. Also, for any $\alpha \in \mathcal{A}^k(M)$, $\alpha = \Delta G\alpha + H\alpha$.

For those who are not familiar with functional analysis, the last part of the proof actually is the application of Hahn-Banach Theorem. Here we give a description of one version of the theorem: assume that L_0 is a subspace of a normed linear space L and f_0 is a bounded linear functional on L_0, then there exists a bounded linear functional f on L satisfying $f|_{L_0} = f_0$ and $\|f\| = \|f_0\|$.

Finally we can finish the proof of the Hodge theorem.

proof of existence in Theorem 2.3. For $[\omega]_{dR} \in H^k_{dR}(M)$, we pick a representative ω of the de Rham cohomology class, which satisfies $d\omega = 0$. Theorem 2.6 implies that

$$\begin{aligned} \omega &= \Delta G\omega + H\omega \\ &= d\delta G\omega + \delta G d\omega + H\omega \\ &= d(\delta G\omega) + H\omega. \end{aligned}$$

Thus $[\omega]_{dR} = [H\omega]_{dR}$ and $H\omega \in H^k(M)$, which is a harmonic form contained in the de Rham cohomology class. □

2.3 Hodge 定理的应用

Poincaré 对偶定理

定理 2.7. Poincaré 对偶定理。

对于连通、可定向的闭 n 维流形，双线性映射

$$\begin{aligned} H^k_{dR}(M) \times H^k_{dR}(M) &\to \mathbb{R} \\ ([\omega],[\eta]) &\mapsto \int_M \omega \wedge \eta \end{aligned}$$

是非退化的，因此诱导了同构

$$H^{n-k}_{dR}(M) \cong H^k_{dR}(M)^*.$$

流形和 Euler 数

定义 Betti 数 $\beta_i = \dim H^i_{dR}(M)$，定义 Euler-Poincaré 示性数

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim H^i_{dR}(M).$$

定理 2.8. 奇数维闭流形的 Euler-Poincaré 示性数是 0。

相交数和流形的指标

M 是连通、可定向 n 维闭流形，Poincaré 对偶有另一种表达

$$H_k(M;\mathbb{R}) \cong H^{n-k}_{dR}(M).$$

考虑 k 维可定向闭子流形 $N \subset M$，其 k 阶的基本类 $[N] \in H_k(N;\mathbb{Z}) \subset H_k(M;\mathbb{Z}) \subset H_k(M;\mathbb{R})$ 决定了 $[N]^* \in H^{n-k}_{dR}(M)$。可以证明，对于 N 的任意小的开邻域 U，我们可以取 $[N]^*$ 中的代表元 $\omega \in \mathcal{A}^{n-k}(M)$ 使得其支集含于 U。事实上，对于任意 $\alpha \in \mathcal{A}^k(M)$，有

$$\int_N \alpha|_N = \int_M \alpha \wedge \omega.$$

详细内容可以参考

Bott, R. and Tu, W., Differential Forms in Algebraic Topology, Springer, 1982

这是一本很好的书。

对于两个可定向闭子流形 N_1^k 和 N_2^{n-k}，我们定义其相交数为

$$[N_1] \cdot [N_2] = [N_1]^* \wedge [N_2]^* \in H^n_{dR}(M) \cong \mathbb{R}.$$

事实上这个数是一个整数，是 N_1 和 N_2 的交点个数（带符号）。于是我们可以定义流形的指标

定义 2.2. 流形的指标。

考虑 $4k$ 维的连通可定向闭流形 M，映射

$$H_{2k}(M;\mathbb{R}) \times H_{2k}(M;\mathbb{R}) \to \mathbb{R}$$

是 $H_{2k}(M;\mathbb{R})$ 上的二次型，我们称其为 M 上的相交型（乱翻译的）。我们考虑这个二次型的特征值，正的个数减去负的个数，即被称为 M 的指标 $\operatorname{sign} M$。

定理 2.9. Hirzebruch 指标定理。

定义 L 形式

$$L(TM,\nabla) = \det\left(\left(\frac{\frac{\sqrt{-1}}{2\pi}R}{\tanh(\frac{\sqrt{-1}}{2\pi}R)}\right)^{\frac{1}{2}}\right) \in \mathcal{A}^*(M),$$

则

$$L(M) := \langle L(TM),[M]\rangle = \int_M L(TM,\nabla),$$

等于 M 的指标。特别地 $L(M)$ 是整数。

3 第三周，向量丛和协变导数

示性类是研究向量丛的重要工具，它们用上同调的语言刻画了丛的弯曲情况。

3.1 向量丛

Vector Bundle

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Contents

1 Vector Bundles

1.1 Preliminaries on Manifolds

Definition 1.1. An n-dimensional smooth manifold is a second countable topological space M and a coordinate open covering $\{U_\alpha\}$ such that

1. For each U_α, there is an homeomorphism $\varphi_\alpha : U_\alpha \to \varphi(U_\alpha) \subseteq \mathbb{R}^n$.

2. For each α, β with $U_\alpha \cap U_\beta \neq \varnothing$, the **transition function** $\varphi_{\alpha\beta} := \varphi_\alpha \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is smooth.

Here $(U_\alpha, \varphi_\alpha)$ is called a **coordinate chart** of M, and $\{(U_\alpha, \varphi_\alpha)\}$ is called an **atlas** of M.

Definition 1.2. A **smooth map** $f : N \to M$ between two smooth manifold $(M, \{(U_\alpha, \varphi_\alpha)\})$ and $(N, \{(V_\beta, \psi_\beta)\})$ is a continuous map, such that for each α, β with $f(U_\alpha) \cap V_\beta \neq \varnothing$, the map $\psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \to \psi_\beta(f(U_\alpha) \cap V_\beta)$ is smooth.

Definition 1.3. The **differential** of a smooth map $f : N \to M$ around $p \in N$ is the differential of the map $\psi_{f(p)} \circ f \circ \varphi_p^{-1}$, where $(U_p, \varphi_p), (V_{f(p)}, \psi_{f(p)})$ are coordinate charts around p and $f(p)$, respectively.

Remark. One needs to verify that this definition is invariant under different choices of charts.

Definition 1.4. M is a smooth manifold. The tangent space T_pM over $p \in M$ is

$$T_pM = \left\{ \frac{d\gamma(0)}{dt} : \gamma \in C^\infty(\mathbb{R}, M), \gamma(0) = p \right\}$$

where the differential is taken in the sence of 1.3.

1.2 Tangent Bundle of a Manifold

Let M be a C^∞ manifold and consider the set of all tangent spaces at all points of M:

$$TM = \bigcup_{p \in M} T_pM,$$

the projection map $\pi : TM \to M$ sending $X \in T_pM$ to $\pi(X) = p$.

TM admits a natural structure of C^∞ manifold. Suppose $\mathcal{S}$ is an atlas on M, $(U, \varphi) \in \mathcal{S}$. For any tangent vector $v \in T_pU$, its image $\varphi_*(v)$ can be represented in local coordinates

$$\varphi_*(v) = a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n}.$$

Now consider

$$\tilde{\varphi} : \pi^{-1} \to \varphi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n},$$

$$\tilde{\varphi}(v) = (\varphi(p), a_1, \ldots, a_n) \in \varphi(U) \times \mathbb{R}^n,$$

then

$$\tilde{\mathcal{S}} = \{(\pi^{-1}(U), \tilde{\varphi} : (U, \varphi) \in \mathcal{S})\}$$

is an atlas for TM.

1.3 Vector Bundles

Definition 1.5. Let M be a C^∞ manifold. By an n-dimensional real(resp. complex) **vector bundle** $\xi = (E, \pi, M)$ over M we mean that $\pi : E \to M$ is a C^∞ map from a smooth manifold E onto M such that

1. for each $p \in M$, $\pi^{-1}(p)$ has the structure of an n-dimensional real(resp. complex) vector space;
2. **local triviality**: for each $p \in M$ there are an open neighborhood U and a diffeomorphism $\varphi_U : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^n$.

Suppose there are two overlapping trivialization neighborhood U_α and U_β, with trivializations φ_α and φ_β. Then the composite map

$$\varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

can be written in the form

$$\varphi_\alpha \circ \varphi_\beta^{-1}(p, v) = (p, g_{\alpha\beta}(p)v)$$

where the smooth map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n; \mathbb{R})$ is called **transition function**.

Proposition 1.6. *The transition functions $\{g_{\alpha\beta}\}$ of a trivialization $\{U_\alpha, \varphi_\alpha\}$ of vector bundle E satisfiy the* ***cocycle condition****:*

1. $g_{\alpha\beta}(x) = g_{\beta\alpha}^{-1}$, $\forall x \in U_{\alpha\beta}$,

2. $g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$, $\forall x \in U_{\alpha\beta\gamma}$.

Remark. Transition function relies on the choice of local trivialization. However, if $\{U_\alpha, \varphi'_\alpha\}$ is another local trivialization of ξ, then there are smooth maps $\lambda_\alpha : U_\alpha \to \mathrm{GL}(n; \mathbb{R})$(Actually, $\lambda_\alpha = \varphi_\alpha(\varphi'_\alpha)^{-1}$) such that for arbitrary $x \in U_{\alpha\beta}$,

$$g_{\alpha\beta}(x) = \lambda_\alpha(x) g'_{\alpha\beta}(x) \lambda_\beta^{-1}(x).$$

If one can choose a appropriate local trivialization such that $g_{\alpha\beta} \in \mathrm{GL}^+(n; \mathbb{R})$, the vector bundle is called **orientable**.

The cocycle condition though, can be used to introduce another definition of vector bundles, which, after generalizing transition function to auto-diffeomorphisms of fibres, leads to the notion of fibre bundle.

Proposition 1.7. *Given a coordinate open covering $\{U_\alpha\}$ of a C^∞ manifold B and a family of differentiable functions $g_{\alpha\beta}$ satisfying the cocycle condition, there is a vector bundle $\xi = (E, \pi, B)$ with B the base, and with $g_{\alpha\beta}$ as the transition functions.*

Proof. Consider $E := \coprod_\alpha U_\alpha \times \mathbb{R}^n / \sim$, where the equivalence condition is

$$(\alpha, b, u) \sim (\beta, c, v) \Leftrightarrow b = c, u = g_{\alpha\beta}(b)(v).$$

□

Definition 1.8 (bundle map)**.** A bundle map between vector bundles $\pi : E \to M$ and $\pi' : F \to N$ is a C^∞ map $\tilde{f} : E \to F$ such that the diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & F \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes, and $\tilde{f}$ restricted to each fibre is a linear map.

Definition 1.9. Two vector bundles over M is said to be **isomorphic** if there is a bundle map, which restricts to identity on M, and restrics to linear isomorphism on each fibre.

Definition 1.10. For a given vector bundle $\pi : E \to M$, a C^∞ map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$ is called a **section**. We denote by $\Gamma(E)$ the set of sections of a vector bundle E.

The section set $\Gamma(E)$ naturally form a module over $C^\infty(M)$. Conversely, every projective module over $C^\infty(M)$ corresponds to a vector bundle over M.

Theorem 1.11 (Serre-Swan)**.** *There is a one-to-one correspondence between isomorphic classes of vector bundles over M with projective modules over the smooth function ring of M.*

Giving a trivialization $\varphi : \pi^{-1}(U) \simeq U \times \mathbb{R}^n$ on an open subset U is equivalent to choosing sections $s_i : U \to E, i = 1, \ldots, n$, over U such that $\forall p \in U$, $s_1(p), \ldots, s_n(p)$ form a basis of E_p. Whether one can take out a global **non-vanishing** section of a vector bundle reflects a lot information of the vector bundle. For example,

Proposition 1.12. *A n-dimensional vector bundle is isomorphic to a trivial bundle if and only if there exists n linear independent global sections.*

Theorem 1.13 (Hopf Index)**.** *On a compact orientable smooth manifold M, the sum of index of singular points of a smooth vector field over M equals to its Euler charateristic number $\chi(M)$.*

2 Several Construction of Vector Bundles

Whitney Sum

Given two vector bundles $\pi : E \to B$ and $\pi' : E' \to B$ over the same base space, their **direct sum** or **whitney sum** is defined to be

$$E \oplus E' := \{((b, v), (c, u)) \in E \times E' : b = c\}.$$

Complexification

Given any n-dimensional real vector bundle E we can construct an n-dimensional complex vector bundle $E \otimes \mathbb{C}$ by complexifying each fibre E_p.

Tensor and Hom

Given two vector bundles $\pi : E \to B$ and $\pi' : E' \to B$, the vector bundle $E \otimes E'$ over B generated by transition functions $\{g_{\alpha\beta} \otimes g'_{\alpha\beta}\}$ is called the **tensor product** of E and E'.

$\mathrm{Hom}(E, E')$ is the set of bundle maps from E to E'. $\mathrm{Hom}(E, E') \simeq E^* \otimes E$. Especially, $E^* := \mathrm{Hom}(E, \mathbb{R})$ is called the **dual** of E. Suppose $\varphi_\alpha : E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$ is a local trivialization of E, then $(\varphi_\alpha^t)^{-1} : E^*|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times (\mathbb{R}^n)^*$ is a local trivialization of E^*, with transition function

$$(\varphi_\alpha^t)^{-1} \circ \varphi_\beta^t : ((\varphi_\alpha \circ \varphi_\beta^{-1})^t)^{-1} = (g_{\alpha\beta}^t)^{-1}.$$

Pullback

Suppose $\pi : E \to M$ is a vector bundle and $f : N \to M$ is a smooth map between two manifolds N and M. It is easy to verify that

$$f^*E := \{(x, v) \in N \times E : f(x) = \pi(v)\}$$

is a vector bundle over N.

Proposition 2.1. *If $\{(U_\alpha, \varphi_\alpha)\}$ is a local trivialization of E and $\{g_{\alpha\beta}\}$ the transition funtion, then $f^*(g_{\alpha\beta})$ are the transition functions of f^*E.*

Proof. The localy trivialization of E induces a local trivialization $\{(f^*(U_\alpha), \psi_\alpha)\}$ of $f^*(E)$, where ψ_α is defined by

$$f^{-1}E|_{f^{-1}(U_\alpha)} \xrightarrow{\psi_\alpha} f^{-1}(U_\alpha) \times \mathbb{R}^n$$
$$(y, e) \mapsto (y, \varphi_\alpha|_{E_{f(y)}}(e)),$$

where $\varphi_\alpha|_{E_{f(y)}}$ is given by $E_{f(y)} \to \{f(y)\} \times \mathbb{R}^n \to \mathbb{R}^n$.

Hence the transition function of f^*E is

$$\begin{aligned} g'_{\alpha\beta}(y) &= \varphi_\alpha|_{E_{f(y)}} \circ (\varphi_\beta|_{E_{f(y)}})^{-1} \\ &= (\varphi_\alpha \circ \varphi_\beta^{-1})|_{f(y)} \\ &= g_{\alpha\beta}(f(y)) \\ &= (f^* g_{\alpha\beta})(y). \end{aligned}$$

□

Quotient Bundle

From a vector bundle $\pi : E \to B$ and its subbundle F one can define their **quotient space** Q, it is a vector bundle for which the following sequence is exact:

$$0 \to F \to E \to Q \to 0,$$

which means for every point $x \in B$, the sequence of vector spaces

$$0 \to F_x \to E_x \to Q_x \to 0$$

is exact. Hence $Q_x = E_x/F_x$ is the quotient space, and $Q = \coprod_{x \in M} Q_x$. Usually one denotes Q by E/F.

By taking frame one can take appropriate local trivialization $\{(U_\alpha, \varphi_\alpha)\}$ of E and $\{U_\alpha, \psi_\alpha\}$ of F such that

$$\psi_\alpha = \varphi_\alpha|_{F_{U_\alpha}}.$$

Under these hypothesis, if the transition function of local trivialization $\{(U_\alpha, \varphi_\alpha)\}$ is $g_{\alpha\beta}$, $\{(U_\alpha, \psi_\alpha)\}$ is $h_{\alpha\beta}$, one have

$$g_{\alpha\beta} = \begin{pmatrix} h_{\alpha\beta} & * \\ 0 & q_{\alpha\beta} \end{pmatrix}$$

whence the transition function of $Q = E/F$ is $q_{\alpha\beta}$, and $\det g_{\alpha\beta} = \det h_{\alpha\beta} q_{\alpha\beta}$. If $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(U_\alpha, \psi_\alpha)\}$ are oriented trivialization, then

$$\det q_{\alpha\beta} > 0,$$

hence Q is orientable, and this orientation is called **direct sum orientation**, in the sense that the natural isomorphism

$$E \simeq F \oplus E/F \tag{2–1}$$

is compatible with these orientations.

Now suppose $i : S \hookrightarrow M$ is a k-dimensional normal submanifold of smooth manifold M of dimension n. There are two vector bundles over S: one is the tangent bundle $TS \to S$, another is the pullback of TM under i.

Definition 2.2. The **normal bundle** of submanifold S in M is the quotient bundle:

$$0 \to TS \to TM|_S \to N_{S/M} \to 0.$$

If manifold M and S is orientable, then the orientation of $TM|_S$ and TS give an orientation of $N_{S/M}$ such that the orientation of $TM|_S$ is the direct sum orientation.

3 Splitting Principle of Complex Bundles

3.1 Hopf Line Bundle

Consider the trivial $(n+1)$-dimensional complex vector bundle $\mathbb{C}P^n \times \mathbb{C}^{n+1}$. An arbitrary point l on $\mathbb{C}P^n$ is a complex line through the origin of $\mathbb{C}^{n+1}$. Now we set

$$L = \{(l, z) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} : z \in l\},$$

then it is a 1-dimensional subbundle of $\mathbb{C}P^n \times \mathbb{C}^{n+1}$ and hence a complex line bundle over $\mathbb{C}P^n$. This is called the **Hopf Line Bundle**, which is a special case of **tautological bundle** over Grassmannian manifold.

Now we study the tangent bundle of the projective space. First the real case. Let L be the Hopf line bundle over $\mathbb{R}P^n$. The usual inner product on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ defines a Riemannian metric on the trivial bundle $\mathbb{R}P^n \times \mathbb{R}^{n+1}$. Since L is a subbundle of this trivial bundle, the orthogonal-copmplement bundle $L^\perp$ is an n-dimensional vector bundle over $\mathbb{R}P^n$. For the Hopf line bundle, $L_l^\perp$ is just the orthogonal complement $l^\perp$. Note that $\mathbb{R}P^n$ is obtained by identification of pairs of points that are symmetric to each other with respect to the origin, any tangent vector $X \in T_l\mathbb{R}P^n$ is expressed by the pair $\{(x, v), (-x, -v)\}$, $x \in S^n$, $v \in l^\perp$. Such pairs induce a linear map

$$f_X : l \mapsto l^\perp$$

by the correspondence

$$l \ni ax \mapsto av \in l^\perp.$$

Conversely, the linear map f_X determines a pair $\{(x, v), (-x, -v)\}$ uniquely. Hence

$$T_l\mathbb{R}P^n = \mathrm{Hom}(l, l^\perp).$$

Proposition 3.1. *Let L be the Hopf line bundle over the real projective space $\mathbb{R}P^n$ and let $L^\perp$ be the orthogonal complement of the subbundle L is the product bundle $\mathbb{R}P^n \times \mathbb{R}^{n+1}$. Then we have a natural bundle isomorphism*

$$T\mathbb{R}P^n \simeq \mathrm{Hom}(L, L^\perp).$$

Corollary 3.2. *Let ϵ be the trivial line bundle over $\mathbb{R}P^n$. Then there is an isomorphism*

$$T\mathbb{R}P^n \oplus \epsilon \simeq L \oplus \cdots \oplus L$$

Proof.

$$T\mathbb{R}P^n \oplus \epsilon \simeq \mathrm{Hom}(L, L^\perp) \oplus \mathrm{Hom}(L, L) \simeq \mathrm{Hom}(L, L^\perp \oplus L),$$

but

$$\mathrm{Hom}(L, L^\perp \oplus L) \simeq \mathrm{Hom}(L, \epsilon^{n+1}) \simeq \mathrm{Hom}(L, \epsilon)^{n+1} \simeq L^{n+1}.$$

□

For complex projective space, one have

Proposition 3.3. *Let L be the Hopf line bundle over $\mathbb{C}P^n$ and ϵ the trivial complex line bundle. Then there is an isomorphism*

$$T\mathbb{C}P^n \oplus \epsilon \simeq L^* \oplus \cdots \oplus L^*.$$

3.2 Projectivization of Vector Bundle

From now on, we deal with complex vector bundle.

One can generalize the Hopf line bundle over $\mathbb{C}P^n$ to any complex vector bundle through a projectivization process. Let $\rho : E \to M$ be a complex vector bundle with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{C})$. The **projectivization** of E, $\pi : P(E) \to M$, is a fibre bundle whose fiber at a point p in M is the projective space $P(E_p)$ and whose transition functions $\bar{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{PGL}(n, \mathbb{C})$ are induced from $g_{\alpha\beta}$. Thus a point of $P(E)$ is a line l_p in the fiber E_p.

Consider the pullback bundle $\pi^{-1}E$, the vector bundle over $P(E)$ whose fiber at l_p is E_p. The **universal subbundle** S over $P(E)$ is defined by

$$S = \{(l_p, v) \in \pi^{-1}E : v \in l_p\},$$

which is also called the tautological line bundle. The **universal quotient bundle** Q is determined by the tautological exact sequence

$$0 \to S \to \pi^{-1}E \to Q \to 0. \tag{3–1}$$

3.3 Chern Class

Before introducing chern class, let us take a look at its real analogy in 1-(complex) dimensional case. Suppose E is an oriented real vector bundle over M of dimension 2, $\{U_\alpha\}$ is a coordinate open cover of M that trivializes E. Using the Riemannian structure of E, over each U_α we can choose an orthonormal frame. This defines on $E^0|U_\alpha$(E^0 is E minus its zero section.) polar coordinates r_α and θ_α; if $x_1, \ldots, x_n$ are coordinates on U_α, the $\pi^*x_1, \ldots, \pi^*x_n, r_\alpha, \theta_\alpha$ are coordinates on $E^0|U_\alpha$. On the overlap $U_\alpha \cap U_\beta$, the radii r_α and r_β are equal but the angular coordinates θ_α and θ_β differ by a rotation, namely

$$\theta_\beta = \theta_\alpha + \pi^*\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}.$$

Since the angle difference is only unique up to an integral multiplication of 2π, all that one can say is

$$\varphi_{\alpha\beta} + \varphi_{\beta\gamma} - \varphi_{\alpha\gamma} \in 2\pi\mathbb{Z},$$

hence $\{\varphi_{\alpha\beta}\}$ does not satisfy the cocycle condition; but if one take $\xi_\alpha := (\frac{1}{2\pi})\sum_\gamma \rho_\gamma d\varphi_{\gamma\alpha}$, where $\{\rho_\gamma\}$ is a partition of unity subordinate to $\{U_\gamma\}$, one will have

$$\frac{1}{2\pi} d\varphi_{\alpha\beta} = \xi_\beta - \xi_\alpha.$$

It follows that $d\xi_\alpha = d\xi_\beta$ on $U_\alpha \cap U_\beta$, hence the $d\xi_\alpha$ piece together to give a global 2-form e on M, which we call the **Euler class** of the oriented vector bundle E (To which step does the orientability condition apply?)

Proposition 3.4. *Suppose $g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(2)$(Why can one always assume the transition functions of a orientable vector bundle fall into $SO(n)$?), then $e(E)$ has extrinsic expression*

$$e(E) = -\frac{1}{2\pi i} \sum_\gamma d(\rho_\gamma d \log g_{\gamma\alpha}) \text{ on } U_\alpha.$$

Definition 3.5. We define the **first Chern class** of a complex line bundle L over a manifold M to be the Euler class of its underlying real bundle $L_\mathbb{R} : c_1(L) = e(L_\mathbb{R}) \in H^2(M)$.

By proposition 3.4, for two complex line bundles L, L' over the same base space, we have

$$c_1(L \otimes L') = c_1(L) + c_1(L').$$

Let L^*be the dual of L. Since the line bundle $L \otimes L^* = \mathrm{Hom}(L, L)$ has a nowhere vanishing section given by the identity map, $L \otimes L^*$ is a trivial bundle. Hence $c_1(L) + c_1(L^*) = c_1(L \otimes L^*) = 0$. Therefore,

Proposition 3.6. *For any complex line bundle L,*

$$c_1(L^*) = -c_1(L).$$

Back to the tautological sequence 3–1. S is a complex line bundle, and we can set $x = c_1(S^*) = e(S^*)$. Then x is a cohomology class in $H^2(P(E))$. Since the restriction of the universal subbundle S on $P(E)$ to a fiber $P(E_p)$ is the universal subbundle $\tilde{S}$ of the projective space $P(E_p)$, by the naturality of Euler class, it follows that $c_1(\tilde{S})$ is the restriction of $-x$ to $P(E_p)$. Hence the cohomology classes $1, x, \ldots, x^{n-1}$ are global classes on $P(E)$ whose restrictions to each fiber $P(E_p)$ freely generate the cohomology of the fiber. By the Leray-Hirsch theorem, the cohomology $H^*(P(E))$ is a free module over $H^*(M)$; these coefficients are by definition the **Chern classes** of the complex vector bundle E:

$$x^n + c_1(E)x^{n-1} + \cdots + c_n(E) = 0, \quad c_i(E) \in H^{2i}(M).$$

In this equation by $c_i(E)$ we really mean $\pi^* c_i(E)$. We call $c_i(E)$ the **ith Chern class** of E and

$$c(E) = 1 + c_1(E) + \cdots + c_n(E) \in H^*(M)$$

its **total Chern class**. With this definition of the Chern classes, we see that the ring structure of the cohomology of $P(E)$ is given by

$$H^*(P(E)) = H^*(M)[x]/(x^n + c_1(E)x^{n-1} + \cdots + c_n(E)). \tag{3–2}$$

Proposition 3.7. *The Euler class definition of first Chern class and the polynomial coefficient definiton coincides.*

3.4 Splitting Principle

GOAL: Given $\tau : E \to M$ a C^∞ complex vector bundle of rank n over a manifold M, construct a space $F(E)$ and a map σ such that:

1. the pullback of E to $F(E)$ splits into a direct sum of line bundles: $\sigma^{-1}E = L_1 \oplus \cdots \oplus L_n$;
2. σ^* embeds $H^*(M)$ in $H^*(F(E))$.

Such a space $F(E)$, which is in fact a manifold by construction, is called a **split manifold** of E.

If E has dimension 1, there is nothing to prove.

If E has rank 2, we can take as a split manifold $F(E)$ the projective bundle $P(E)$, for on $P(E)$ there is the exact sequence

$$0 \to S_E \to \sigma^{-1}E \to Q_E \to 0;$$

by equation 2–1, $\sigma^{-1}E = S_E \oplus Q_E$, which is a direct sum of line bundles.

Now suppose E has rank 3. Over $P(E)$ the line bundle S_E splits off as before. The quotient bundle Q_E over $P(E)$ has rank 2 and so can be split into a direct sum of line bundles when pulled back to $P(Q_E)$. Thus we may take $P(Q_E)$ to be a split manifold $F(E)$. Let $x_1 = \beta^* c_1(S_E^*)$ and $x_2 = c_1(S_{Q_E}^*)$. By equation 3–2,

$$H^*(F(E)) = H^*(M)[x_1, x_2]/(x_1^3 + c_1(E)x_1^2 + c_2(E)x_1 + c_3(E), x_2^2 + c_1(Q_E)x_2 + c_2(Q_E)). \tag{3–3}$$

Generally,

$$\begin{tikzcd} E \arrow[d] & S_1 \oplus Q_1 \arrow[d] & S_1 \oplus S_2 \oplus Q_2 \arrow[d] & \cdots & S_1 \oplus \cdots \oplus S_{n-1} \oplus Q_{n-1} \arrow[d] \\ M & P(E) \arrow[l] & P(Q_1) \arrow[l] & \cdots \arrow[l] & P(Q_{n-2}) = F(E) \arrow[l] \end{tikzcd} \tag{3–4}$$

gives the construction of split manifold. Its cohomology $H^*(F(E))$ is a free $H^*(M)$-module having as a basis all monomials of the form

$$x_1^{a_1} x_2^{a_2} \cdots x_{n-1}^{a_{n-1}},\ 0 \leq a_i \leq n - i, \tag{3–5}$$

where $x_i = c_1(S_i^*)$ in the notation of the diagram.

Theorem 3.8 (Splitting Principle)**.** *To prove a polynomial identity in the Chern classes of complex vector bundles, it suffices to prove it under the assumption that the vector bundles are direct sums of line bundles.*

3.2 外蕴观点下的协变导数

我们先从物理的角度考虑一个运动曲线 $c:(a,b)\to M\subset\mathbb{R}^3$，则我们有速度向量

$$\dot{c}(t)\in T_{c(t)}M.$$

但加速度 $\ddot{c}(t)=d\dot{c}/dt$ 一般来说不在 $T_{c(t)}M$ 内，如果 M 是一个曲面。我们用 N_p 表示 T_pM 在 $T_p\mathbb{R}^3$ 中的正交补，则我们有直和分解 $T_p\mathbb{R}^3=T_pM\oplus N_p$，进而

$$\ddot{c}(t)=(D_h\dot{c})(t)+(D_n\dot{c})(t),$$

其中 $D_h\dot{c}$ 和 $D_n\dot{c}$ 分别表示 $\ddot{c}$ 在 T_p 和 N_p 中的部分（注意此处的 D_h 和 D_n 更像是某种导数）。得益于身处 Euclid 空间，我们可以清楚地写出

$$D_h\dot{c}=\ddot{c}-\langle\ddot{c},n\rangle n,$$

其中 n 是 $c(t)\in M$ 处的单位法向量。

c 是如上的曲线，我们称 c 是测地线，如果其加速度处处垂直于 M 的切平面，即 $(D_h\dot{c})(t)\equiv 0$。

随之而来的推论是因为 $\langle\ddot{c}(t),\dot{c}(t)\rangle=0$，因此 $\langle\dot{c},\dot{c}\rangle$ 是常数，即我们的运动速率不变。我们在此处暂停思考一下，其实要做到速率 $|\dot{c}|$ 不变是十分简单的，我们只需要将该曲线弧长参数化。这点暗示我们测地线的切向量模长不变对其测地线性质并没有太大影响，实际上，我们可以采用如下定义

$$D_h\dot{c}(t)=\lambda\dot{c}(t),\quad \exists\lambda\in\mathbb{R},$$

这样定义的曲线称作仿射测地线。曲线是一个映射，因此不同参数化定义的曲线不是同一条曲线。当我们只考虑曲线的像集时，仿射测地线和测地线是一样的。值得一提的是，仿射测地线的好处是预防我们无法“弧长参数化”的情况，当度量不正定时这有可能发生。

我们摆脱曲线的限制，在 $\mathbb{R}^3$ 中，我们可以定义向量的导数

$$D_XY=(XY_1,XY_2,XY_3)^t,$$

若 $Y=(Y_1,Y_2,Y_3)^t$。可以验证这个定义不依赖于坐标系的选取。

对于 $\mathbb{R}^3$ 中的曲面 S，对于 $p\in S$，$X,Y\in T_pS$，我们有分解

$$D_XY=(D_h)_XY+(D_n)_XY,$$

其中 $(D_h)_XY\in T_pS$ 称作曲面 S 上的协变导数。因为我们只关心切向分量，我们用 ∇_XY 来表示 $(D_h)_XY$。

更一般地，对于流形 M，协变微分是一个映射

$$(X,Y)\in\mathscr{X}(M)\times\mathscr{X}(M)\mapsto\nabla_XY\in\mathscr{X}(M),$$

满足以下条件

$$(1)\ \nabla_{fX_1+gX_2}Y = f\nabla_{X_1}Y + g\nabla_{X_2}Y;$$

$$(2)\ \nabla_X(Y_1+Y_2) = \nabla_X Y_1 + \nabla_X Y_2, \quad \nabla_X(fY) = f\nabla_X Y + (Xf)Y,$$

其中 f 是 M 上的函数，X, Y, X_1, X_2, Y_1, Y_2 是 M 上的向量场。

我们之后会看见，协变微分可以推广到对张量求微分。

命题 3.1. 在 $\mathbb{R}^3$ 中曲面上定义的协变微分满足以上性质。

在流形 M 中，我们称沿着曲线 $c(t)$ 的向量场 Y 是平行的，若 $\nabla_{\frac{\partial}{\partial t}}Y = 0$。称曲线是测地线，若 $\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} = 0$。

接下来我们定义平行移动。对于 $M \in \mathbb{R}^3$ 上任意点 p 和 q，取曲线 $c(t)$ 连接 $t(0) = p$ 和 $t(1) = q$。我们可以定义线性映射 $\tau : T_pM \to T_qM$ 如下。取 $Y_0 \in T_pM$ 为初值，我们试图构造沿着 $c(t)$ 的平行向量场 $Y(t)$，再定义 $\tau(Y_0) = Y(1)$ 即可。我们将平行条件转换为一阶常微分方程，这样平行向量场的存在唯一性即是常微分方程解的存在唯一性。

考虑局部坐标 $c(t) = (u_1(t), u_2(t))$，则 Y 有坐标表示 $Y = Y^1\frac{\partial}{\partial u_1} + Y^2\frac{\partial}{\partial u_2}$，且我们设

$$\nabla_{\frac{\partial}{\partial u_i}}\frac{\partial}{\partial u_j} = \Gamma_{ij}^k\frac{\partial}{\partial u_k},$$

于是

$$\begin{aligned}\nabla_{\frac{\partial}{\partial t}}Y &= \nabla_{\dot{u}_i\frac{\partial}{\partial u_i}}Y^j\frac{\partial}{\partial u_j}\\ &= \dot{u}_i\left(\frac{\partial Y^k}{\partial u_i} + Y^j\Gamma_{ij}^k\right)\frac{\partial}{\partial u_k}\end{aligned}$$

即得到两个常微分方程

$$\frac{dY^k}{dt} + \dot{u}_iY^j\Gamma_{ij}^k, \quad k = 1, 2.$$

由常微分方程理论，我们知道平行向量场 $Y(t)$ 存在且唯一。通过将曲线 $c(t)$ 划分成有限段，我们即可得到 τ。且该方程是线性的，故 τ 是线性映射。

注意到，若 c 是测地线，则

$$\frac{d}{dt}\langle Y(t), \dot{c}(t)\rangle = \langle\nabla_{\frac{\partial}{\partial t}}Y, \dot{c}\rangle + \langle Y, \nabla_{\frac{\partial}{\partial t}}\dot{c}\rangle = 0 + 0 = 0.$$

于是平行向量场与速度方向的夹角保持不变。更一般地，两个平行向量场之间的夹角保持不变。

注意到我们的平行移动是依赖于曲线 $c(t)$ 的选取的，而且这和曲面的曲率性质有关。关于平行移动我们还可以联系和乐群（holonomy group），感兴趣的同学可以参考

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我们在此简单了解和乐群的定义。固定一点 $p \in M$，考虑纤维 E_p 上所有形如 $\tau_\lambda : E_p \to E_p$ 的平行移动，其中 γ 是起终点都在 p 点的分段光滑曲线（此处条件可以放宽）。

容易验证 $\tau_\gamma \cdot \tau_\eta = \tau_{\gamma\cdot\eta}$，其中 $\gamma \cdot \eta$ 表示曲线的复合。容易验证常值曲线给出了恒等线性变换，以及 $\tau_\gamma^{-1} = \tau_{\gamma^{-1}}$。因此所有这样的变换给出了一个群，其是 $GL(E_p)$ 的子群。我们称其为 ∇ 的基点在 p 处的和乐群，常记为 $\mathrm{Hol}_p(\nabla)$。