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Assignment 1

Problem 1.

(a) N is the number of unordered pairs of distinct keys in $[U]$. This means $N = \binom{U}{2}$. Next, we want to find the expected number of unordered pairs of distinct keys x and y such that $h(x) = h(y)$ when we pick a random hash function h from $\mathcal{H}$ (Given that $\mathcal{H}$ is c-universal).

From the definition of c-universal, we know that $\underset{h \in \mathcal{H}}{Pr}[h(x) = h(y)] \leq \frac{c}{m}$.

Let I_n be an indicator random variable that is 1 when $h(x) = h(y)$ and 0 otherwise for any given unordered pair of distinct keys x and y.

From this, we can determine $\mathbb{E}[I_n] \leq \frac{c}{m}$.

Let I be the number of unordered pairs of distinct keys x and y where $h(x) = h(y)$.

We can now express $\mathbb{E}[I]$:

$$
\begin{aligned}
I &= \sum_{x \neq y} I_n \\
\mathbb{E}[I] &= \sum_{x \neq y} \mathbb{E}[I_n] \\
&\leq N \cdot \frac{c}{m} \\
&\leq \binom{U}{2} \cdot \frac{c}{m} \\
&\leq \frac{U \cdot (U-1) \cdot c}{2 \cdot m}
\end{aligned}
$$

(b) Let's consider one hash function $h \in \mathcal{H}$. Consider that h maps s_1 number of distinct keys to value 1, and s_2 number of distinct keys to value 2, and so on, all the way to s_m. This means that for any value of s_i, there are exactly $\binom{s_i}{2}$ possible pairings of keys such that $h(x) = h(y)$.

This let's us express I differently:

$$
\begin{aligned}
I &= \sum_{i=1}^{m} \binom{s_i}{2} \\
&= \sum_{i=1}^{m} \frac{(s_i) \cdot (s_i - 1)}{2}
\end{aligned}
$$

(c) Now let's figure out s_i. For any given s_i, we know that $\mathbb{E}[s_i] \geq \frac{U}{m}$. This is because we are trying to fit U keys into m spaces. We can now use the equation in part b to determine the following:

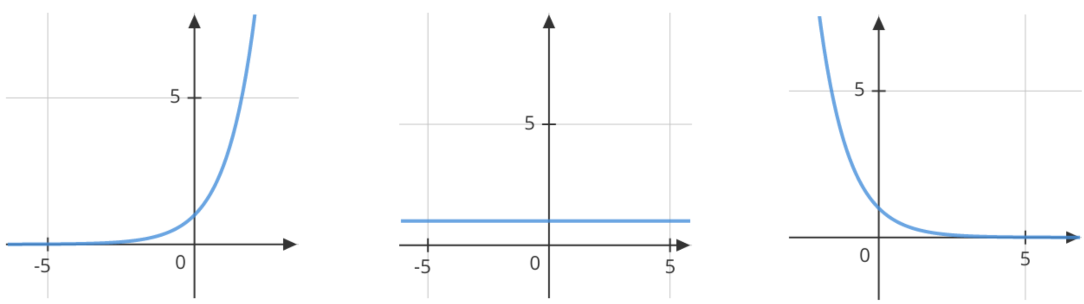
$$
\begin{aligned}
I &= \sum_{i=1}^{m} \frac{s_i \cdot (s_i - 1)}{2} \\
\mathbb{E}[I] &= \sum_{i=1}^{m} \frac{\mathbb{E}[s_i] \cdot (\mathbb{E}[s_i] - 1)}{2} \\
&\geq m \cdot \frac{\frac{U}{m} \cdot (\frac{U}{m} - 1)}{2} \\
&\geq \frac{U \cdot (\frac{U}{m} - 1)}{2}
\end{aligned}
$$

(d) By combining the inequalities of part A and part C, we can define a bound for the expected number of unordered pairs of distinct keys x and y such that $h(x) = h(y)$ when we pick a random hash function $h \in \mathcal{H}$.

$$\frac{U \cdot (\frac{U}{m} - 1)}{2} \leq \mathbb{E}[I] \leq \frac{U \cdot (U-1) \cdot c}{2 \cdot m}$$

Problem 2.

(a) We want to show that $f(x) = e^{tx}$ is convex for $t > 0$. By definition, a convex function is also described such that all points along any line between two points on the function has a value greater than or equal to the function value underneath the points. $f(x) = e^{tx}$ is convex over all t by observation:



(b) Let Z be a random variable with probability density function g in the interval [0,1]. $p = \mathbb{E}[Z]$. Let's also define a Bernoulli random variable such that $Pr[X = 1] = p$. We want to show that for any convex function f, the following is true.

$$\mathbb{E}[f(Z)] \leq \mathbb{E}[f(X)]$$

Let's start with the left hand side (LHS). We know that $\mathbb{E}[f(Z)] = \int_0^1 f(t)g(t)dt$ and $\mathbb{E}[Z] = \int_0^1 tg(t)dt$. We can also express t as $(t)(1) + (1-t)(0)$. With these in mind, we can start simplifying the LHS:

$$
\begin{aligned}
LHS &= \mathbb{E}[f(Z)] \\
&= \int_0^1 f(t)g(t)dt \\
&= \int_0^1 f((t)(1)+(1-t)(0))g(t)dt
\end{aligned}
$$

We can now use the definition of a convex function, which is that for $0 \le \lambda \le 1$, and convex function f, the following is true: $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$. We can plug this into what we have so far:

$$
\begin{aligned}
LHS &= \int_0^1 f((t)(1)+(1-t)(0))g(t)dt \\
&\le \int_0^1 tf(1)+(1-t)f(0)g(t)dt \\
&\le \int_0^1 tf(1)g(t)dt + \int_0^1 (1-t)f(0)g(t)dt \\
&\le f(1)\int_0^1 tg(t)dt + f(0)[\int_0^1 g(t)dt - \int_0^1 tg(t)dt] \\
&\le f(1)\mathbb{E}[Z] + f(0)[1-\mathbb{E}[Z]] \\
&\le f(1)p + f(0)(1-p)
\end{aligned}
$$

Now let's start examining the right hand side (RHS). Because X is a Bernoulli random variable (aka discreet values), we can express $\mathbb{E}[f(X)]$ as a computed expression:

$$
\begin{aligned}
RHS &= f(0)Pr[X=0] + f(1)Pr[X=1] \\
&= f(0)(1-p) + f(1)p
\end{aligned}
$$

Combine the expressions we have and we get:

$$
\begin{aligned}
LHS &\le RHS \\
\mathbb{E}[f(Z)] &\le \mathbb{E}[f(X)]
\end{aligned}
$$

(c) Let $Y_1, \ldots, Y_n$ be independent identical distributed random variables over [0,1]. Let $Y = \sum_i Y_i$. We want to show that for $\delta \le 1$, $Pr[Y - E[Y] > \delta] \le exp(-\delta^2/2n)$.

We can use Hoeffding's equality here. In particular, if we let $\epsilon = \delta/n$, we get the above expression. The reasoning behind this is that we can treat each Y_i as an identical experiment. The variance of the experiment being conducted n times should decrease, but $E[Y]$ stays the same. Therefore, we can divide δ by n.

$$Pr[X - E[X] \geq \epsilon n] \leq exp(\frac{-\epsilon^2 n}{2})$$
$$\epsilon = \delta/n$$
$$Pr[Y - E[Y] \geq (\delta/n)n] \leq exp(\frac{-(\delta/n)^2 n}{2})$$
$$Pr[Y - E[Y] \geq \delta] \leq exp(\frac{-\delta^2}{2n})$$

Problem 3.

(a) We want to show that the hash functions in $\mathcal{H}$ have the following property: for any key $x \in [U]$ and $v \in [m]$, we have

$$\Pr_{h \in \mathcal{H}}[h(x) = v] = \frac{1}{m}$$

To show this, we can break down v in terms of the hash function definition. More specifically, we can relate the two together:

$$\begin{aligned} h(x) = & H_0(x_0) \quad \oplus \quad H_1(x_1) \quad \oplus \quad \dots \quad \oplus \quad H_{c-1}(x_{c-1}) && (1) \\ v = & \quad v_0 \quad \oplus \quad v_1 \quad \oplus \quad \dots \quad \oplus \quad v_{c-1} && (2) \end{aligned}$$

We can also express $Pr[h(x) = v]$ as the probability that each character is the correct hash:

$$\Pr_{h \in \mathcal{H}}[h(x) = v] = \prod_{i=0}^{c-1} Pr[H_i[x_i] = v_i]$$

Individually, $Pr[h_i[x_i] = v_i]$ is $\frac{1}{m^{1/c}}$. We know this because the size of each H_i is $m^{1/c}$, and we are trying to select an individual entry. Overall, then, we can compute the above expression to arrive at our answer

$$\begin{aligned} \Pr_{h \in \mathcal{H}}[h(x) = v] &= \prod_{i=0}^{c-1} Pr[H_i[x_i] = v_i] \\ &= (\frac{1}{m^{1/c}})^c \\ &= \frac{1}{m} \end{aligned}$$

(b) We want to show that for two different keys $x, y \in [U]$ and $u, v \in [m]$, the following property is true:

$$\Pr_{h \in \mathcal{H}}[h(x) = u \text{ and } h(y) = v] = \frac{1}{m^2}$$

Intuitively, because x and y are distinct keys, we can regard the above expression as the product of two independent events.

$$\Pr_{h \in \mathcal{H}}[h(x) = u \text{ and } h(y) = v] = \Pr_{h \in \mathcal{H}}[h(x) = u] \times \Pr_{h \in \mathcal{H}}[h(y) = v]$$

The probability of each event happening independently together is the same as the probability we computed above:

$$\begin{aligned} \Pr_{h \in \mathcal{H}}[h(x) = u] &= \frac{1}{m} \\ \Pr_{h \in \mathcal{H}}[h(y) = v] &= \frac{1}{m} \\ \Pr_{h \in \mathcal{H}}[h(x) = u \text{ and } h(y) = v] &= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m^2} \end{aligned}$$

(c) Intentionally left blank

(d) Suppose $c = 2$. Imagine we have four distinct keys w, x, y, z, and the hash values for three of them (any three) r, s, t. We want to determine the hash value of the last key u. This is possible if we closely examine the possible values for the keys and hashes:

$$\begin{aligned} h(w) &= H_0[w_0] \quad \oplus \quad H_1[w_1] \\ h(x) &= H_0[x_0] \quad \oplus \quad H_1[x_1] \\ h(y) &= H_0[y_0] \quad \oplus \quad H_1[y_1] \\ h(z) &= H_0[z_0] \quad \oplus \quad H_1[z_1] \end{aligned}$$

At first glance, the characters $w_0, w_1, x_0, x_1, y_0, y_1, z_0, z_1$ have nothing to do with each other. But, because $c = 2$, we can actually constraint the characters:

$$\begin{aligned} w_0, x_0, y_0, z_0 &\in \left\{ c_0^{(0)}, c_1^{(0)} \right\} \\ w_1, x_1, y_1, z_1 &\in \left\{ c_0^{(1)}, c_1^{(1)} \right\} \end{aligned}$$

Therefore:

$$(k_0, k_1) \in \left\{ (c_0^{(0)}, c_0^{(1)}), (c_0^{(0)}, c_1^{(1)}), (c_1^{(0)}, c_0^{(1)}), (c_1^{(0)}, c_1^{(1)}) \right\} \forall k \in \{w, x, y, z\}$$

With these constraints, we can also breakdown the hash values:

$$h_0 = H_0[c_0^{(0)}] \quad \oplus \quad H_1[c_0^{(1)}] \tag{3}$$
$$h_1 = H_0[c_0^{(0)}] \quad \oplus \quad H_1[c_1^{(1)}] \tag{4}$$
$$h_2 = H_0[c_0^{(1)}] \quad \oplus \quad H_1[c_0^{(1)}] \tag{5}$$
$$h_3 = H_0[c_0^{(1)}] \quad \oplus \quad H_1[c_1^{(1)}] \tag{6}$$

$$r, s, t, u \in \{h_0, h_1, h_2, h_3\}$$

With r, s, t, u being distinct, once we have three keys, all we need to do is find the missing pair of characters. This can be achieved with an interesting observation about h_0, h_1, h_2, h_3, which is that they represent the set of all possible hashes. Therefore:

$$h_0 \quad \oplus \quad h_1 \quad \oplus \quad h_2 \quad \oplus \quad h_3 = 0 \tag{7}$$
$$r \quad \oplus \quad s \quad \oplus \quad t \quad \oplus \quad u = 0 \tag{8}$$

(e) We want to show that for any set of d different keys $x^{(1)}, x^{(2)}, \ldots, x^{(d)}$, there exists an index index $i \in [c]$ such that the ith characters of those keys share at least $d^{1/c}$ different values.

We can show this via contradiction. The contradictory argument in this case is that $\forall i \in [c]$, the ith characters of those d keys share less than $d^{1/c}$ different values.

$$h(x^{(1)}) \quad = \quad H_1[x_1^{(1)}] \quad \oplus \quad H_2[x_2^{(1)}] \quad \oplus \quad \ldots \quad \oplus \quad H_{c-1}[x_{c-1}^{(1)}] \tag{9}$$
$$h(x^{(2)}) \quad = \quad H_1[x_1^{(2)}] \quad \oplus \quad H_2[x_2^{(2)}] \quad \oplus \quad \ldots \quad \oplus \quad H_{c-1}[x_{c-1}^{(2)}] \tag{10}$$
$$\vdots \tag{11}$$
$$h(x^{(d)}) \quad = \quad H_1[x_1^{(d)}] \quad \oplus \quad H_2[x_2^{(d)}] \quad \oplus \quad \ldots \quad \oplus \quad H_{c-1}[x_{c-1}^{(d)}] \tag{12}$$

For any vertical slice $x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(d)} \forall i \in [c]$, there are $< d^{1/c}$ distinct letters.

Let Y_i be the # of distinct letters for the ith character. From the contradictory argument above, they must share less than $d^{1/c}$ different values.

$$Y_i < d^{1/c}$$

Let Y be the # of distinct keys. From a tabulation hashing standpoint, a key is composed of its characters. In order for a key to be distinct, at least one of the characters must be distinct. This let's us express Y in terms of Y_i:

$$Y = \prod_{i=1}^{c-1} Y_i$$

Combining the two parts together yields:

$$
\begin{aligned}
Y &= \prod_{i=1}^{c-1} Y_i \\
&< (d^{1/c})^c \\
&< d
\end{aligned}
$$

But, from the original problem statement, the number of distinct keys is d. $Y = d$ and $Y < d$ creates a contradiction.

(f) Intentionally Left Blank

Total Time Taken: 20 Hours.