Name: Kevin Zhang

**Problem 1.**

Consider convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and a convex set $S$.

(a) Show that the set of minimizers of $f$ over $S$ is convex.

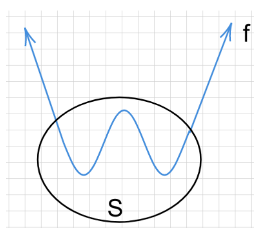
Assume for the sake of contradiction that the set of minimizers $M$ is not convex. What this means is there is a least one point $x_c \notin M$ and two points $x_1, x_2 \in M$ such that $x_c$ is between $x_1$ and $x_2$.

From convexity of $f$, we have that $f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2)$. From lecture, we know that for convex $f$ and convex $S$, any local minimum is also a global minimum. Thus $f(x_1) = f(x_2)$. Combining all of these together (and that $x_c$ is between $x_1$ and $x_2$):

$$
\begin{aligned}
f(\theta x_1 + (1-\theta)x_2) &\leq \theta f(x_1) + (1-\theta) f(x_2) \\
f(\theta x_1 + (1-\theta)x_2) &\leq \theta f(x_1) + (1-\theta) f(x_1) \\
f(x_c) &\leq f(x_1)
\end{aligned}
$$

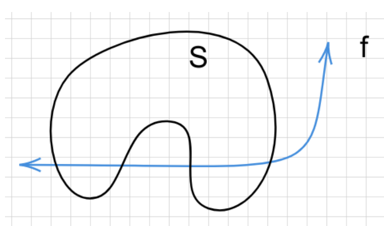
So $x_c$ is a minimizer. But $x_c \notin M$, and $M$ is the set of minimizers. So by contradiction, $M$ must be convex.

(b) Give counterexample if $f$ is not convex.




It's easy to see from the figure that even if two points $x_1, x_2$ are minimizers, not all points between the points would be minimizers. Thus, $M$ cannot be convex.

(c) Give counterexample if $S$ is not convex.




From the figure, the set of minimizers has been split because $S$ is not convex. It is easy to see that there would be some minimizers for $f$ that don't end up in $S$. Thus, $M$ cannot be convex.

**Problem 2.**

Goal is to find $x$ that minimizes $f(x) = \frac{1}{2}\|Ax - b\|^2$.

(a) We want to prove $f$ is convex.

This can be done by showing $f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle$. With $\nabla f(x) = A^T\|Ax - b\|$, we can expand the expression:

$$\begin{aligned}
f(y) - f(x) &\overset{?}{\geq} \langle \nabla f(x), y - x \rangle \\
\frac{1}{2}\|Ay - b\|^2 - \frac{1}{2}\|Ax - b\|^2 &\overset{?}{\geq} \langle A^T\|Ax - b\|, y - x \rangle \\
\frac{1}{2}\left(\|Ay - b\|^2 - \|Ax - b\|^2\right) &\overset{?}{\geq} \langle A^T\|Ax - b\|, y - x \rangle \\
\frac{1}{2}\langle Ay - Ax, (Ay - b) + (Ax - b) \rangle &\overset{?}{\geq} \langle A^T\|Ax - b\|, y - x \rangle \\
\frac{1}{2}A^T\langle y - x, (Ay - b) + (Ax - b) \rangle &\overset{?}{\geq} A^T\langle \|Ax - b\|, y - x \rangle \\
\langle y - x, (Ay - b) + (Ax - b) \rangle &\overset{?}{\geq} 2\langle Ax - b, y - x \rangle
\end{aligned}$$

At this point, we need to do a bit of case analysis to divide by $y - x$. If $y - x < 0$, we need to flip the sign. So, for Case 1 ($y - x > 0$), we have:

$$\begin{aligned}
(Ay - b) + (Ax - b)\rangle &\overset{?}{\geq} 2(Ax - b) \\
(Ay - b)\rangle &\geq (Ax - b)
\end{aligned}$$

And for Case 2($y - x < 0$), we have:

$$\begin{aligned}
(Ay - b) + (Ax - b)\rangle &\overset{?}{\leq} 2(Ax - b) \\
(Ay - b)\rangle &\leq (Ax - b)
\end{aligned}$$

Thus, $f$ is convex.

(b) Prove that $f$ is $\beta$-smooth for as small $\beta$ as you can. We can do this by examining the definition for $\beta$-smoothness:

$$
\begin{aligned}
\|\nabla f(x) - \nabla f(y)\| &\leq \beta \|x - y\| \\
\left\| A^T \|Ax - b\| - A^T \|Ay - b\| \right\| &\leq \beta \|x - y\| \\
\|A^T\| \|Ax - b - (Ay - b)\| &\leq \beta \|x - y\| \\
\|A^T\| \|Ax - Ay\| &\leq \beta \|x - y\| \\
\|A^T A\| \|x - y\| &\leq \beta \|x - y\| \\
\|A^T A\| &\leq \beta
\end{aligned}
$$

We have that $\beta \geq \|A^T A\|_2$, which is the equivalent of $\beta \geq \|A\|_2^2$.

(c) Consider matrix $M = I - \gamma A^T A$ for some constant $\gamma$. We want to use gradient descent algorithm with $x^{(t)} \longleftarrow M(x^{(t-1)} - x^*) + x^*$. We want to pick $\gamma$ such that $x^{(t)}$ converges to $x^*$.

We can expand out $x^{(t)}$, since we know it should converge to $x^*$.

$$
\begin{aligned}
&x^{(t)} \longleftarrow M(x^{(t-1)} - x^*) + x^* \\
&x^{(t)} \longleftarrow (I - \gamma(A^T A))(x^{(t-1)} - x^*) + x^* \\
&x^{(t)} \longleftarrow (I - \gamma(A^T A))(x^{(t-1)}) - (I - \gamma(A^T A))(x^*) + x^* \\
&x^{(t)} \longleftarrow x^{(t-1)} - (\gamma(A^T A))(x^{(t-1)}) - x^* + (\gamma(A^T A))(x^*) + x^* \\
&x^{(t)} \longleftarrow x^{(t-1)} - (\gamma(A^T A))(x^{(t-1)}) + (\gamma(A^T A))(x^*)
\end{aligned}
$$

Since we want to convege to $x^*$, $x^{(t-1)} - (\gamma(A^T A))(x^{(t-1)})$ should zero out, and the rest of the term should equal $x^*$. Conveniently, the rest of the term is $(\gamma(A^T A))(x^*)$, so if we let $\gamma = \frac{1}{\|A^T A\|_2} = \frac{1}{\|A\|_2^2}$, we can accomplish both.

TODO: Find the bound

**Problem 3.**

Our goal is to find the first singular vector $v_1$ of a given matrix $A$. This is done by finding a vector $x$ that minimizes $f(x) = -\frac{1}{2}\|Ax\|^2$. We can assume a starting point $x^{(0)}$ such that $\langle x^{(0)}, v_1 \rangle \geq \alpha$.

(a) State a projected gradient descent algorithm with fixed size $\eta$.

The key idea here is to move along with the gradient, and then project back onto $f(x)$. The projection in this case is to normalize the vector, because we're looking for a vector of length 1. The algorithm is the gradient descent algorithm as follows ($\nabla f(x) = -A^T A$):

$y^{(t)} \longleftarrow x^{(t-1)} - \eta \nabla f(x^{(t-1)})$

$x^{(t)} \longleftarrow y^{(t)} / \|y^{(t)}\|$

(b) Let $\eta \geq 1/\sigma_1^2$ and $z^{(t)}$ be the projection of $x^{(t)}$ onto the span of singular vectors with singular values less than $(1-\varepsilon)\sigma_1$. Show that after $t = O(\frac{ln(1/\varepsilon\alpha)}{\varepsilon})$ steps, we have $\|z^{(t)}\| \leq \varepsilon$.

We can express $y^{(t)}$ as $(1 + \eta A^T A)x^{(t-1)}$. If we let $M = (1 + \eta A^T A)$, then after $t$ time steps, $\|z^{(t)}\| = (M^t x^{(0)})/\|M^t x^{(0)}\|$. If we plug in the span of singular vectors for $A$, we get the following (NOTE: I worked with Michael to arrive on this):

$$
\begin{aligned}
\left((M^t x^{(0)})/\|M^t x^{(0)}\|\right)^2 &= \frac{(1 + \eta(1-\varepsilon)\sigma_1^2)^{2t}}{(1 + \eta\sigma_1^2)^{2t}\alpha} \\
\|z^{(t)}\| &= \frac{(1 + (1-\varepsilon))^t}{(2)^t \alpha}
\end{aligned}
$$

We can then solve for $t$:

$$
\begin{aligned}
\|z^{(t)}\| &\leq \varepsilon \\
\frac{(1 + (1-\varepsilon))^t}{(2)^t \alpha} &\leq \varepsilon \\
(2-\varepsilon)^t &\leq \varepsilon\alpha \\
e^{t\varepsilon/2} &\geq (1/\varepsilon\alpha) \\
t &\geq \frac{\ln 2/\varepsilon\alpha}{\varepsilon}
\end{aligned}
$$

Thus, $t$ is $O(\frac{ln(1/\varepsilon\alpha)}{\varepsilon})$.

**Problem 4.**

We have an $\alpha$-strongly convex $f(x)$, over bounded domain $S$. Assume that $\|\nabla f(x)\| \leq G \forall x \in S$. We'll use the projected gradient descent algorithm:

$y^{(t)} \longleftarrow x^{(t-1)} - \eta_t \nabla f(x^{(t-1)})$
$x^{(t)} \longleftarrow \arg\min x \in S\|x - y^{(t)}\|^2/2$

(a) Prove the following:

$\Delta_t = \left(f(x^{(t-1)} - f(x^*)) + \frac{\alpha}{2}\|x^{(t-1)} - x^*\|^2\right) + \frac{1}{2\eta_t}\left(\|x^{(t)} - x^*\| - \|x^{(t-1)} - x^*\|^2\right) \leq \frac{\eta_t G^2}{2}.$

We can do so by examining the two summands separately, and then combining the results.

First, the left summand. We can use the $\alpha$-convexity to bound this term. From the definition of $\alpha$-convexity, and if we let $y = x^*$ and $x = x^{(t-1)}$, we get the following:

$$
\begin{aligned}
f(y) - f(x) &\geq \langle \nabla f(x), y - x\rangle + \frac{\alpha}{2}\|y - x\|^2 \\
f(x^*) - f(x^{(t-1)}) &\geq \langle \nabla f(x^{(t-1)}), x^* - x^{(t-1)}\rangle + \frac{\alpha}{2}\|x^* - x^{(t-1)}\|^2 \\
f(x^*) - f(x^{(t-1)}) - \frac{\alpha}{2}\|x^* - x^{(t-1)}\|^2 &\geq \langle \nabla f(x^{(t-1)}), x^* - x^{(t-1)}\rangle \\
f(x^{(t-1)}) - f(x^*) + \frac{\alpha}{2}\|x^* - x^{(t-1)}\|^2 &\leq -\langle \nabla f(x^{(t-1)}), x^* - x^{(t-1)}\rangle
\end{aligned}
$$

Next, we can break down the right summand.

$$
\begin{aligned}
RS &= \frac{1}{2\eta_t}\left(\|x^{(t)} - x^*\| - \|x^{(t-1)} - x^*\|^2\right) \\
&= \frac{1}{2\eta_t}\langle x^{(t)} - x^{(t-1)}, x^{(t)} + x^{(t-1)} - 2x^*\rangle \\
&\leq \frac{1}{2\eta_t}\langle y^{(t)} - x^{(t-1)}, y^{(t)} + x^{(t-1)} - 2x^*\rangle \\
&= \frac{1}{2\eta_t}\langle -\eta_t \nabla f(x^{(t-1)}), -\eta_t \nabla f(x^{(t-1)}) + 2x^{(t-1)} - 2x^*\rangle \\
&= \frac{1}{2\eta_t}\left(\eta_t^2(\nabla f(x^{(t-1)}))^2 + 2\eta_t\langle \nabla f(x^{(t-1)}), x^* - x^{(t-1)}\rangle\right) \\
&= \frac{\eta_t}{2}(\nabla f(x^{(t-1)}))^2 + \langle \nabla f(x^{(t-1)}), x^* - x^{(t-1)}\rangle
\end{aligned}
$$

Combining the two terms, and subsituting $(\nabla f(x^{(t-1)}))^2 \leq G$, we get the following:

$$
\begin{gathered}
\left(-\langle \nabla f(x^{(t-1)}), x^* - x^{(t-1)}\rangle\right) + \left(\frac{\eta_t}{2}(\nabla f(x^{(t-1)}))^2 + \langle \nabla f(x^{(t-1)}), x^* - x^{(t-1)}\rangle\right) \\
\frac{\eta_t}{2}(\nabla f(x^{(t-1)}))^2 \\
\frac{\eta_t G^2}{2}
\end{gathered}
$$

(b) We want to design the coefficients of $a_t$ and $\eta_t$ such that for $\sum_{t=1}^{T} a_t \Delta_t$, the coefficents of $\|x^{(t)} - x^*\|^2$ cancel out. We also want to find the bound of the resulting gradient descent algorithm.

If we examine $\Delta_t$ closely over the sum of $a_t$, and attempt to express $a_t$ in terms of $a_{t-1}$, we get following:

$\sum_{t=1}^{T} a_t \Delta_t = ... + a_{t-1}\left(\frac{1}{2\eta_t}\|x^{t-1} - x^*\|^2\right) + ... + a_t\left(\frac{\alpha}{2}\|x^{t-1} - x^*\|^2\right) - a_t\left(\frac{1}{2\eta_t}\|x^{t-1} - x^*\|^2\right) + ...$

$$a_{t-1}(\frac{1}{2\eta_t}) = a_t(\frac{\alpha}{2} - \frac{1}{2\eta_t})$$
$$a_t = a_{t-1}\left(\frac{(\frac{1}{2\eta_t})}{\frac{\alpha}{2} - \frac{1}{2\eta_t}}\right)$$
$$a_t = a_{t-1}\left(\frac{1}{\eta_t \alpha - 1}\right)$$

Next, we can examine the sum to determine $a_0$ and $\eta_t$:

$$\sum_{t=1}^{T} a_t \Delta_t \le \sum_{t=1}^{T} a_t \frac{\eta_t G^2}{2}$$
$$T(f(\bar{x}) - f(x^*)) - \frac{a_T}{2\eta_t}\|x^T - x^*\|^2 + \frac{a_0 \alpha}{2}\|x^0 - x^*\|^2 \le (\frac{1}{\eta_t \alpha - 1})^{(T)(T+1)/2} \frac{T a_0 \eta_t G^2}{2}$$

Note: Not exactly sure how to proceed from here.