Homework 5

(due Tuesday, October 20)

Instructions: This homework is to be submitted on GradeScope as a *single* pdf (not in parts) by 11:59 pm on the due date. You may either type your solutions in a word processor and print to a pdf, or write them by hand and submit a scanned copy. Do write and submit your answers as if they were a professional report. There will be point deductions if the submission isn't neat (is disordered, difficult to read, scanned upside down, etc....).

Begin by reviewing your class notes, the slides, and the textbook. Then do the exercises below. Show your work. An unjustified answer may receive little or no credit.

Read: 2.2 (for Thursday-Friday), 2.3 (for Tuesday)

1. [8 Points] Let $G = (\{S, A, B\}, \{a, b\}, R, S)$ be the context-free grammar whose set R of productions consists of:

$$\begin{aligned} S &\rightarrow ABS \mid AB \\ A &\rightarrow aA \mid a \\ B &\rightarrow bA \end{aligned}$$

For each of the following strings, state whether it is in $L(G)$ or not. Provide derivations for those strings that are in $L(G)$ and reasons for those that are not.

(a) $aabaab$

(b) $aaaaba$

(c) $aabbaa$

(d) $abaaba$

2. [10 Points] Consider the context-free grammar for arithmetic expressions that was discussed in class. Its rules were

$$\begin{aligned} E &\rightarrow E + T \mid T \\ T &\rightarrow T \times F \mid F \\ F &\rightarrow (E) \mid \mathbf{a} \end{aligned}$$

Give parse trees and derivations for the following strings

(a) $\mathbf{a}$

(b) $\mathbf{a} + \mathbf{a} + \mathbf{a}$

(c) $((\mathbf{a}))$

3. [10 Points] Consider the context-free grammar $G = (V, \Sigma, R, S)$ where $V = \{S, A, B\}$, $\Sigma = \{a, b\}$ and R has the rules

$$\begin{aligned} S &\to bA \mid aB \mid \varepsilon \\ A &\to aS \mid bAA \\ B &\to bS \mid aBB \end{aligned}$$

Intuitively, A generates strings with one more a than b's, B generates strings with one more b than a's, and S generates the strings with equal numbers of a's and b's.

(a) Give a leftmost derivation of the string $a\,a\,b\,b$. (3 points)

(b) Give a rightmost derivation of the string $a\,a\,b\,a\,b\,b$ (4 points)

(c) Draw a parse tree for the string $a\,b\,b\,a$ (3 points)

4. [18 Points] Give context-free grammars that generate the following languages over the alphabet $\Sigma = \{a, b\}$. In each case, try to minimize the number of variables and specify all components of the quadruple (V, Σ, R, S).

(a) The empty language.

(b) $\{w \mid \text{the length of } w \text{ is odd}\}$. (Optimal solution has only one variable).

(c) $\{a^m b^n \mid 0 \leq m \leq n\}$ (Optimal solution has only one variable).

(d) $\{w \mid w \text{ contains at least four } a\text{'s}\}$.

(e) $\{w \mid w \neq \varepsilon \text{ and } w \text{ begins and ends with the same symbol}\}$.

5. [10 Points] Context-free Grammars
Construct context-free grammars that generate the languages below. For each grammar specify all components of the quadruple (V, Σ, R, S).

(a) $L_1 = \{a^i\, b^j\, c^k \mid j = i + k \text{ and } i, j, k \geq 0\}$

(b) $L_2 = \{a^i\, b^j\, c^k \mid i = j + k \text{ and } i, j, k \geq 0\}$

6. [6 Points] Construct a right-regular grammar for the language

$$L = \big\{w \in \{0, 1\}^* \mid w \text{ is a binary number divisible by } 3\big\}$$

(We accept the empty string and strings that have leading zeros: ε, 110 and 00110 all belong to L.) Hint: Problem 10f in homework 2.

7. [12 Points] Use the method from class (lecture 10a) to convert each of the following grammars to Chomsky normal form. Note that not all passes are always needed (for example, pass 2 is not needed if there is no ε-rule).

   (a) $S \rightarrow\ a\,S\,a\,b \mid B$
   $B \rightarrow\ b\,b\,C \mid b\,b$
   $C \rightarrow\ \varepsilon \mid c\,C$

   (b) $S \rightarrow\ A\,B$
   $A \rightarrow a \mid B$
   $B \rightarrow b \mid A \mid \varepsilon$

8. [10 Points] For each of the following languages over the alphabet $\{a, b\}$, draw the state diagram of a pushdown automaton that accepts this language. For full credit, your automaton should have as few states as possible. (Below, assume that $m, n \geq 0$).

   (a) $\{a^n b^m \mid n \leq m\}$.

   (b) $\{a^n b^m \mid n \geq m\}$.

9. [8 Points] Construct a pushdown automaton P such that (assume $m, n \geq 0$):

$$L(P) = \{a^m b^n \mid n = 2m\}$$

   Specify the components of your automaton and draw a state-diagram. For full credit, your automaton should have as few states as possible.

10. [8 Points] Construct a pushdown automaton P such that (assume $m, n \geq 0$):

$$L(P) = \{a^m b^n \mid m \leq n \leq 2m\}$$

   Specify the components of your automaton and draw a state-diagram. For full credit, your automaton should have as few states as possible.