Solution: Stable equilibrium in Electrostatics

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Solution

Let's think for a second about what stable equilibrium means physically. First, if there's a point of general equilibrium, it means that there's no resultant force acting on an object placed in that region. But it's more than that, beyond just being in equilibrium, stable equilibrium implies that any perturbation will be met with a restorative force pushing the object back in the direction of equilibrium.

In the context of electrostatics then, the force lines around the region of where of stable equilibrium where a test charge can be placed, must all point towards that point, or in other words [1], the **electric field lines** must all point towards it. Schematically, the electric field must look something like the following image

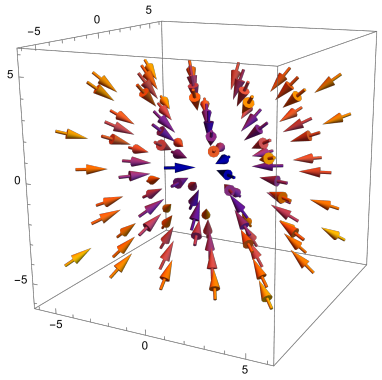

Figure 1: Electric Field lines around a point of stable equilibrium.

But notice the following, if we enclose the region centered around this point with a Gaussian surface, say a sphere for example

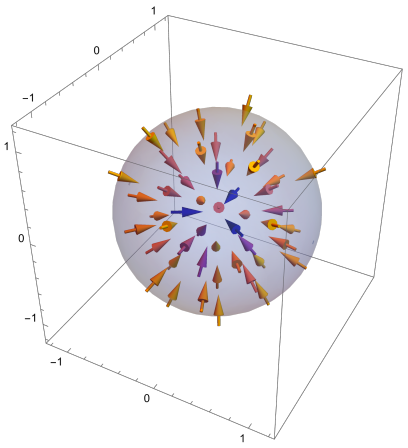

Figure 2: Enclosing the point of stable equilibrium with a Gaussian Surface.

[1] we'll take the point charge to be positive, but the argument would be the same (modulo a few minus signs), if the test charge was negative

Evidently, the flux going through this surface is non-zero

$$\int \vec{E} \cdot \mathrm{d}\vec{a} \neq 0$$

Gauss' Law, would in turn imply there must be some charge inside this region. But there is none! And hence, such a field configuration would violate Gauss' Law, and is then impossible.

Possible Objections

- The test charge is a charge inside the Gaussian Sphere.

  While this is true, it's not an objection to the solution. Importantly, the electric field lines relevant for how the test charge will behave around the point of stable equilibrium exclude the lines produced by itself. After all, a charge cannot apply a resultant force to itself. Hence, the entire argument talks about the field lines that existed prior to us placing the test charge in that point.

- What if, in order to generate the field configuration in Figure 1, we need a charge inside the Gaussian Surface in Figure 2?

  This is also an interesting objection, but it's not really enough to invalidate the solution. Suppose that yes, we needed a charge inside the particular choice of sphere in Figure 2. This would not be a problem, after all, we can just make the ball smaller to exclude it from inside it. Importantly, the stable equilibrium behaviour must remain qualitatively the same arbitrarily close to the point of stability, and hence, we can also make the sphere arbitrarily small so as to exclude any possible charge from inside it.