Solution: Furry's Theorem

By: Matthew D. Schwartz modified by Francisco Dahab

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Solution

The QED Lagrangian is given by

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{D} - m)\psi = \bar{\psi}(i\not{\partial} - m)\psi - ie\bar{\psi}\not{A}\psi$$

Under charge conjugation symmetry (we'll generally refer to it as C-Symmetry from now on), the dynamical fields in our theory change as

$$C : \psi \ \rightarrow \ i\gamma^2\psi^{*}\,[1]$$

Now that we've established that is C-invariant, let's now tackle the problem at hand. Consider then, a photon n point function, which we can express, using Path Integrals, as

$$G_n^{\mu_1\cdots\mu_n}(x_1,\ldots,x_n) = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A\, \prod_{i=1}^{n} A^{\mu_i}(x_i)\, e^{iS_{\text{QED}}[\psi,\bar{\psi},A]}$$

We can then transform the fields under C, because the action itself is is invariant this just becomes[2]

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A\, \prod_{i=1}^{n}(-A^{\mu_i}(x_i))\, e^{iS_{\text{QED}}[\psi,\bar{\psi},A]} = (-1)^n \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A\, \prod_{i=1}^{n} A^{\mu_i}(x_i)\, e^{iS_{\text{QED}}[\psi,\bar{\psi},A]}$$

Now, as long as the field redefinition is well defined such that we are able to integrate over all possible field configurations, it can never change the path integral itself. As such, we find the following equality

$$G_n^{\mu_1\cdots\mu_n}(x_1,\ldots,x_n) = (-1)^n G_n^{\mu_1\cdots\mu_n}(x_1,\ldots,x_n) =$$

Evidently then, if n is odd, we arrive at

$$G_n^{\mu_1\cdots\mu_n}(x_1,\ldots,x_n) = -G_n^{\mu_1\cdots\mu_n}(x_1,\ldots,x_n) \quad \Rightarrow \quad \boxed{G_n^{\mu_1\cdots\mu_n}(x_1,\ldots,x_n) = 0}$$

Which is the statement we were looking for.

Onwards to canonical quantization now. If a theory has C-symmetry and the vacuum respects it, as is the case of QED, then there exists a unitary operator, which we'll call U_C such that

$$U_C\,|\Omega\rangle = |\Omega\rangle$$

where $|\Omega\rangle$ is the vacuum state. That is, the vacuum is invariant under C.

With that established, let us now look at the correlation functions we're interested in, that is

$$G_n^{\mu_1\cdots\mu_n}(x_1,\ldots x_n) = \langle\Omega|T\{A^{\mu_1}(x_1)\ldots A^{\mu_n}(x_n)\}|\Omega\rangle \tag{1}$$

[1] There are other conventions, but they are generally equivalent.

[2] If you're familiar with Anomalies and the Fujikawa approach to them, the fact that the transformation of the fermionic fields is fairly complicated might lead you to worry about non-trivial Jacobians. Rest assured, if you remember for example chiral anomalies, the crux of the issue was the fact that both ψ and $\bar{\psi}$ transformed in the same way under chiral symmetry, but here, they transform in opposite ways, so their contributions cancel each other and we're safe.

We can now populate this expressionw with many factors of an identity operator $\mathbb{1}$ between every object in it, into this without changing its value that is

$$G_n^{\mu_1\cdots\mu_n}(x_1,\dots x_n) = \langle\Omega|\mathbb{1}T\{A^{\mu_1}(x_1)\mathbb{1}\dots\mathbb{1}A^{\mu_n}(x_n)\}\mathbb{1}|\Omega\rangle \tag{2}$$

But because the C-symmetry operator is unitary

$$U_C^\dagger U_C = U_C U_C^\dagger = \mathbb{1}$$

We can write (2) as

$$G_n^{\mu_1\cdots\mu_n}(x_1,\dots x_n) = \langle\Omega|U_C^\dagger U_C T\{A^{\mu_1}(x_1)U_C^\dagger U_C\dots U_C^\dagger U_C A^{\mu_n}(x_n)\}U_C^\dagger U_C|\Omega\rangle$$

We can group factors above suggestively as

$$G_n^{\mu_1\cdots\mu_n}(x_1,\dots x_n) = \langle\Omega|U_C^\dagger T\{\Big(U_C A^{\mu_1}(x_1)U_C^\dagger\Big)U_C\dots U_C^\dagger\Big(U_C A^{\mu_n}(x_n)U_C^\dagger\Big)\}U_C|\Omega\rangle$$

Now, as we've already discussed, the vacuum is invariant under C, such that

$$\langle\Omega|\,U_C^\dagger = \langle\Omega| \quad \text{and} \quad U_C\,|\Omega\rangle = |\Omega\rangle$$

Moreover, $U_C A^{\mu_i}(x_i)U_C^\dagger$ is just the definition of the action of this symmetry on the operators of our theory, and as we discussed

$$C: A^\mu(x) \to -A_\mu(x) \quad \Rightarrow \quad U_C A^{\mu_i}(x_i)U_C^\dagger = -A^{\mu_i}(x_i)$$

Therefore, we accrue a factor of (-1) for every A^μ in our expression,

$$G_n^{\mu_1\cdots\mu_n}(x_1,\dots x_n) = (-1)^n\,\langle\Omega|T\{A^{\mu_1}(x_1)\dots A^{\mu_n}(x_n)\}|\Omega\rangle$$

But note that the expectation value on the RHS of the equation above is just $G_n^{\mu_1\cdots\mu_n}(x_1,\dots x_n)$ again (compare with (1)), therefore we find

$$G_n^{\mu_1\cdots\mu_n}(x_1,\dots x_n) = (-1)^n G_n^{\mu_1\cdots\mu_n}(x_1,\dots x_n)$$

And again, if n is odd

$$G_n^{\mu_1\cdots\mu_n}(x_1,\dots x_n) = -G_n^{\mu_1\cdots\mu_n}(x_1,\dots x_n) \quad \Rightarrow \quad \boxed{G_n^{\mu_1\cdots\mu_n}(x_1,\dots x_n) = 0}$$

This is a situation where both methods are equally simple and nice, but that I find rather instructive nonetheless.

If you're interested in a situation where one of them is way more complicated, see "Guided Problem: Massive Spin 1 Propagator" in QFT → Fundamentals, in my website. Or the "Massive Spin 1 propagator and manifest Lorentz covariance" in the Discussions tab of the website again.