Solution - Renormalizability of Massive Vector Fields

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1 Solution

1.1 Dimensional Analysis

The action of our theory is given by

$$S_{\mathrm{MQED}} = \int \mathrm{d}^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu + \bar{\psi}(i\not{\partial} - M)\psi + g\bar{\psi}\not{A}\psi \right)$$

Since we're in spacetime dimension $d = 4$, the requirement that the action is dimensionless implies that the kinetic term must have energy dimension 4, therefore the kinetic term of the spin-1 field implies that

$$4 = 2 + 2[A_\mu] \quad \to \quad [A_\mu] = 1$$

Whereas for the fermion field

$$4 = 1 + 2[\psi] \quad \Rightarrow \quad [\psi] = \frac{3}{2}$$

Therefore, the implication from the interaction term is

$$4 = 1 + 2 \times \frac{3}{2} + [g] \quad \Rightarrow \quad [g] = 0$$

That is, g is dimensionless, and standard wisdom would imply that this theory is renormalizable.

1.2 Renormalizable?

1.2.1 The UV behavior of the propagator

The goal is to find the propagator for a free massive spin-1 field. The path integral prescription teaches us that the propagator for a free theory is always simply the inverse of the quadratic form in the Lagrangian, since recall that given a quadratic operator $O_{\mu\nu}$

$$Z[J] = \int \mathcal{D}A\, e^{i\int \mathrm{d}^4x A^\mu O_{\mu\nu} A^\nu + J_\mu A^\mu} = N e^{i\int \mathrm{d}^4x \mathrm{d}^4y J^\mu(x) O^{-1}_{\mu\nu}(x-y) J^\nu(y)}$$

Such that

$$\langle A_\mu(x) A_\nu(y) \rangle = (-i)^2 \frac{\delta^2 Z[J]}{\delta J^\mu(x) \delta J^\nu(y)} = i O^{-1}_{\mu\nu}(x-y)^{[1]}$$

To find the quadratic operator in the free Lagrangian, we need to manipulate it slightly. Initially, it reads

$$-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} m^2 A_\mu^2 = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{2} m^2 A_\mu^2$$

Expanding

$$-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\nu A_\mu \partial^\mu A^\nu + \frac{1}{2} m^2 A_\mu A^\mu$$

[1]Note how this is different from what the canonical quantization procedure requires, we would need to canonically quantize A_μ then compute $\langle A_\mu(x) A_\nu(y) \rangle$ explicitly, this is massively more complicated, particularly in theories with constraints, as is the case for a massive spin-1 field.

Integrating by parts allows to rewrite this as

$$\frac{1}{2}A^\mu[(\Box+m^2)g_{\mu\nu}-\partial_\mu\partial_\nu]A^\nu \quad \overset{\Leftrightarrow}{\text{Momentum Space}} \quad \frac{1}{2}A^\mu\underbrace{[-(p^2-m^2)g_{\mu\nu}+p_\mu p_\nu]}_{\equiv K_{\mu\nu}}A^\nu$$

So $K_{\mu\nu}$ is the quadratic form we were looking for, and finding the propagator $\Pi^{\mu\nu}$ is really equivalent to looking for its inverse. Lorentz covariance requires that $\Pi^{\mu\nu}$ must be built out of the available tensor $g^{\mu\nu}$ and $p^\mu p^\nu$, such that

$$\Pi^{\mu\nu}=Ag^{\mu\nu}+\frac{B}{m^2}p^\mu p^\nu$$

where the $1/m^2$ factor was added to make $[A]=[B]$. Next, note that the requirement that $\Pi_{\mu\nu}=K^{-1}_{\mu\nu}$ imposes, by definition, that $K_{\mu\alpha}\Pi^{\alpha\nu}=\delta^\nu_\mu$. Algebraically, this is to say that

$$[-(p^2-m^2)g_{\mu\alpha}+p_\mu p_\alpha]\left[Ag^{\alpha\nu}+\frac{B}{m^2}p^\alpha p^\nu\right]=\delta^\nu_\mu$$

Expanding the LHS

$$-A(p^2-m^2)\delta^\nu_\mu+Ap_\mu p^\nu-B\frac{p^2-m^2}{m^2}p_\mu p^\nu+B\frac{p^2}{m^2}p_\mu p^\nu=A(m^2-p^2)\delta^\nu_\mu+(A-B)p_\mu p^\nu=\delta^\nu_\mu$$

Evidently then, this implies two things

$$A(m^2-p^2)=1 \quad \Rightarrow \quad A=\frac{-1}{p^2-m^2} \quad \text{and} \quad B=-A=\frac{1}{p^2-m^2}$$

Such that the propagator will be (recovering the $i\epsilon$ prescription as well)

$$\boxed{i\Pi^{\mu\nu}=\frac{i}{p^2-m^2+i\epsilon}\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{m^2}\right)} \tag{1}$$

Crucially, look at the behavior of this propagator as $p\to\infty$[2]

$$\Pi^{\mu\nu}\underbrace{\sim}_{p^2\to\infty}\frac{1}{m^2}$$

This is markedly different from any propagator we've seen so far, as it does not behave as $1/p^2$ for large momentum. Recall that the former is one of the important assumptions we've made when devising the argument of power counting in our diagrams, which means that it will have to be reformulated for our purposes, possibly invalidating the dimensional analysis from the previous problem.

1.2.2 Correcting the Power Counting

The idea of naive power counting is that we are fundamentally analyzing the powers of momentum present in each propagator in the UV. For the propagators of our theory, we have

$$\begin{cases}\sim\frac{1}{m^2} & \text{massive vector}\\ \sim\frac{1}{p} & \text{fermion}\end{cases}$$

So the inverse powers of momentum brought by propagators in any given diagram is entirely determined by the number of internal fermion propagators, which we call P_f. Since every loop contributes with an integration measure d^4p, the superficial degree of divergence D will be given by

$$D=4L-P_f \tag{2}$$

Next, note that the number of loops is given by the number of unfixed momenta in a diagram. Since every vertex imposes one momentum conservation, given P internal propagator, and V vertices, the number of loops will be

$$L=P-(V-1)=P-V+1$$

[2]Note that this argument is really only precise after Wick rotation

where the extra 1 comes from the fact that an overall momentum conservation is guaranteed in any diagram by Poincarè invariance, and is conventionally factored out in the definition of amplitudes. Now, per our definition $P = P_f + P_v$ (P_v being the number of vector propagators), we have

$$L = 1 + P_f + P_v - V \tag{3}$$

Finally, note that every vertex in our theory is attached to two fermion propagators and one vector propagator (interaction is of the form $\bar{\psi}A\psi$), but external lines connect to only one vertex whereas internal propagators connect two, this means that we can write

$$2V = 2P_f + N_f \quad \text{and} \quad V = 2P_v + N_v \quad \Rightarrow \quad P_f = V - \frac{N_f}{2} \quad \text{and} \quad P_v = \frac{1}{2}(V - N_v) \tag{4}$$

where N_v and N_f are the number of vector and fermion external lines respectively. Substituting (3) and (4) into (2), we find

$$D \overset{(3)}{=} 4 + 4P_f + 4P_v - 4V - P_f \overset{(4)}{=} 4 + 4V - 2N_f + 2V - 2N_v - 4V - V + \frac{N_f}{2}$$

$$\Rightarrow \quad \boxed{D = 4 + V - 2N_v - \frac{3}{2}N_f} \tag{5}$$

Crucially, let us take a step back to understand what is going on here. Because of the asymptotic behavior of the vector propagator does not contribute with inverse powers of momentum, there is no P_v contribution in (2), such that in the end, the vertex contribution in (5) does not cancel as we would expect from a theory with $[g] = E^0$. This means that given a fixed number of external legs, we can arbitrarily increase the degree of divergence, the hallmark of a non-renormalizable theory.

1.3 The Stueckelberg mechanism

We begin by integrating in the new scalar field by making the substitution

$$A_\mu(x) \to A_\mu(x) - \frac{1}{m}\partial_\mu\phi(x) \quad \text{and} \quad \psi(x) \to e^{-\frac{i}{m}g\phi(x)}\psi(x)$$

The field strength is evidently invariant,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu\left(A_\nu - \frac{1}{m}\partial_\nu\phi\right) - \partial_\nu\left(A_\mu - \frac{1}{m}\partial_\mu\phi\right) = F_{\mu\nu} + \frac{1}{m}[\partial_\mu, \partial_\nu]\phi = F_{\mu\nu}$$

and so is the fermionic sector

$$\begin{aligned}\bar{\psi}(i\partial\!\!\!/ - M)\psi + g\bar{\psi}A\!\!\!/\psi \quad &\to \quad e^{\frac{i}{m}g\phi(x)}\bar{\psi}(i\partial\!\!\!/ - M)e^{-\frac{i}{m}g\phi(x)}\psi + ge^{\frac{i}{m}g\phi(x)}\bar{\psi}\left(A\!\!\!/ - \frac{1}{m}\partial_\mu\phi\right)e^{-\frac{i}{m}g\phi(x)} \\ &= \bar{\psi}(i\partial\!\!\!/ - M)\psi + g\bar{\psi}A\!\!\!/\psi + \frac{g}{m}\bar{\psi}\partial\!\!\!/\phi\psi - \frac{g}{m}\bar{\psi}\partial\!\!\!/\phi\psi = \bar{\psi}(i\partial\!\!\!/ - M)\psi + g\bar{\psi}A\!\!\!/\psi\end{aligned}$$

But the mass term is not

$$\frac{1}{2}m^2(A_\mu - \frac{1}{m}\partial_\mu\phi)(A^\mu - \frac{1}{m}\partial^\mu\phi) \quad \to \quad \frac{1}{2}m^2 A_\mu A^\mu + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - mA^\mu\partial_\mu\phi$$

Such that our new Lagrangian, known as a Stueckelberg Lagrangian, is given by

$$\boxed{\mathcal{L}_{SB} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A_\mu^2 + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - mA^\mu\partial_\mu\phi + \bar{\psi}(i\partial\!\!\!/ - M)\psi + g\bar{\psi}A\!\!\!/\psi}$$

Note that the dynamics of this Lagrangian are not immediately obvious, particularly because of the way the vector field and the scalar are coupled in a bilinear.

Now, we prove that this is invariant under gauge transformations of the form

$$A_\mu \to A_\mu + \partial_\mu\alpha \quad \phi \to \phi + m\alpha \quad \text{and} \quad \psi \to e^{ig\alpha}\psi$$

$F_{\mu\nu}$ is manifestly invariant. The next three terms conspire to cancel the extra contributions:

$$\frac{1}{2}m^2A_\mu^2+\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-mA^\mu\partial_\mu\phi\to\frac{1}{2}m^2(A_\mu+\partial_\mu\alpha)^2+\frac{1}{2}\partial_\mu(\phi+m\alpha)\partial^\mu(\phi+m\alpha)-m(A_\mu+\partial_\mu\alpha)\partial_\mu(\phi+m\alpha)$$
$$=\frac{1}{2}m^2A_\mu^2+m^2A^\mu\partial_\mu\alpha+m^2\partial_\mu\alpha\partial^\mu\alpha+\frac{1}{2}\partial_\mu\phi\partial^\mu\phi+m\partial_\mu\phi\partial^\mu\alpha-mA_\mu\partial^\mu\phi-m^2\partial_\mu\alpha\partial^\mu\alpha-m\partial_\mu\phi\partial^\mu\alpha-m^2A_\mu\partial^\mu\alpha$$
$$=\frac{1}{2}m^2A_\mu^2+\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-mA^\mu\partial_\mu\phi$$

Whereas, for the fermion and interacting sector

$$\bar\psi(i\not\partial-m)\psi+g\bar\psi\not A\psi\to\bar\psi(i\not\partial-m)\psi-g\bar\psi\not\partial\alpha\psi+g\bar\psi\not A\psi+g\bar\psi\not\partial\alpha\psi$$
$$=\bar\psi(i\not\partial-m)\psi+g\bar\psi\not A\psi$$

Therefore, the entire Lagrangian is invariant under the gauge transformation.

Next, note that the gauge transformation of the $\phi(x)$ field is markedly different from usual gauge transformations. The fact that it's a simple additive function means **any choice** of function for $\phi(x)$ is a possible gauge fixing condition. With this in mind then, we could simply choose $\phi(x)=0$ (this is known as the unitary gauge)[3], in this case the Lagrangian reduces to

$$\mathcal{L}_{SB}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}m^2A_\mu^2+\bar\psi(i\not\partial-M)\psi+g\bar\psi\not A\psi$$

Such that the generating functional

$$Z[0]=\int\mathcal{D}\phi\mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar\psi e^{iS[\phi,A,\psi,\bar\psi]}\underbrace{\to}_{\phi=0}\left(\int\mathcal{D}\phi\right)\int\mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar\psi e^{iS[A,\psi,\bar\psi]}\propto\int\mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar\psi e^{iS_{MQED}[A,\psi,\bar\psi]}$$

Which is really just the generating functional for the original massive QED theory, that is, massive QED is a particular gauge choice of this more general gauge theory. This is an "accident" if you'd like, it's not something we required from the start, but it is true nonetheless.

1.4 Renormalizable

1.4.1 Gauge-Fixing

Now we need to gauge fix, to do so, let us use the Fadeev-Popov (FP) prescription, but first, briefly recall what it entails. The idea is that gauge invariance implies that given any field configuration, there is a set of other field configurations, called its **gauge orbit**, which are all physically equivalent. This means that if we have any hope of having a path integral that makes sense, we cannot indiscriminately sum over all field configurations, we need to sum over only over **physical configurations**, or one member of each gauge orbit. This selection is what we call an ideal gauge fixing[4], and may be expressed as a functional equation

$$F[\phi,A](x)=0 \tag{6}$$

such that only one field **per gauge orbit**, can satisfy it. Next, we decompose a general field configuration A_μ^α, ϕ^α (these are the fields being integrated over if we do not gauge fix at all), as a sum between a field configuration satisfying (6), which we'll call A_μ, ϕ, and a gauge transformation:

$$A_\mu^\alpha=A_\mu+\partial_\mu\alpha\qquad\phi^\alpha=\phi+m\alpha.$$

Here, note the functional equivalent of the known identity for delta functions:

$$1=\frac{\mathrm{d}f}{\mathrm{d}x}\bigg|_{f=0}\int\mathrm{d}x\,\delta(f(x))\quad\Rightarrow\quad1=\underbrace{\det\left(\frac{\delta F[A^\alpha,\phi^\alpha](x)}{\delta\alpha(y)}\right)\bigg|_{F=0}}_{\equiv\Delta_F[A,\phi]}\int\mathcal{D}\alpha\,\delta[F[A^\alpha,\phi^\alpha]]$$

[3]Note how this is not possible for A_μ for example, not every vector field can be expressed as the divergence of a scalar field.

[4]Strictly speaking it's possible to have partial gauge fixings, which do not cut every gauge orbit only once, an example is the Lorenz gauge $\partial_\mu A^\mu=0$, this choice leaves a residual gauge freedom in the form of $A_\mu\to A_\mu+\partial_\mu f$ so long as $\Box f=0$. For abelian theories this is not a problem, these copies factor out of the path integral and never contribute, so we shall treat every gauge fixing here as ideal. But they can be a problem in non-Abelian theories, giving rise to Gribov copies. For more details see "An Introduction to QFT", Anthony G. Williams chapters 6.4 and 9.2.

This allows us to write the generating functional as

$$Z[0] = 1\int \mathcal{D}A^\alpha\mathcal{D}\phi^\alpha\mathcal{D}\psi\mathcal{D}\bar{\psi}e^{iS_{SB}} = \int \mathcal{D}\alpha\mathcal{D}A^\alpha\mathcal{D}\phi^\alpha\mathcal{D}\psi\mathcal{D}\bar{\psi}\Delta_F[A^\alpha,\phi^\alpha]\delta[F[A^\alpha,\phi^\alpha]]e^{iS_{SB}}$$

Now, we apply the inverse gauge transformation to our fields, $A^\alpha_\mu \to A_\mu$ and $\phi^\alpha \to \phi$. The measure is invariant[5], and so is the action, such that we arrive at

$$= \int \mathcal{D}\alpha\mathcal{D}A\mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\bar{\psi}\Delta_F[A,\phi]\delta[F[A,\phi]]e^{iS_{SB}} = \left(\int \mathcal{D}\alpha\right)\mathcal{D}A\mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\bar{\psi}\Delta_F[A,\phi]\delta[F[A,\phi]]e^{iS_{SB}}$$

$$\Rightarrow \boxed{Z[0] \propto \mathcal{D}A\mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\bar{\psi}\Delta_F[A,\phi]\delta[F[A,\phi]]\exp\left(i\int \mathrm{d}^4x\mathcal{L}_{SB}\right)} \tag{7}$$

Now integration happens only over the physical configurations of the field and we are hence safe from over-counting.

Next, the idea of the R_ξ gauge is that, instead of using a hard delta condition, it's more manageable to use a smoother gaussian average of different gauge fixings. So let us instead consider a family of gauge fixings with the form suggested in the problem's statement

$$F[A,\phi,\alpha] = \partial_\mu A^\mu_\alpha + m\phi_\alpha - f = 0$$

For this particular choice,

$$\frac{\delta F}{\delta\alpha(y)} = \partial_\mu\frac{\delta A^\mu_\alpha}{\delta\alpha(y)} + m\frac{\delta\phi_\alpha(x)}{\delta\alpha(y)} = (\Box_x + m)\delta^4(x-y)$$

Crucially, this determinant is field-independent and hence we can discard it from our path integral as a normalizing constant. This is where non-Abelian gauge theories are different, discarding this determinant is not generally possible because it is field dependent, and we deal with it by introducing the famous FP ghosts.

Then, for this choice, up to a multiplicative constant, we can also introduce an extra integral to the generating functional in (7), which averages over different choices of f

$$Z[0] \propto \int \mathcal{D}f\exp\left(-\frac{i}{2\xi}\int \mathrm{d}^4x f^2(x)\right)\int \mathcal{D}\alpha\mathcal{D}A\mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\bar{\psi}\delta[\partial_\mu A^\mu + m\phi - f]\exp\left(i\int \mathrm{d}^4x\mathcal{L}_{SB}\right)$$
$$= \int \mathcal{D}\alpha\mathcal{D}A\mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}f\,\delta[\partial_\mu A^\mu + m\phi - f]\exp\left\{i\int \mathrm{d}^4x\left[\mathcal{L}_{SB} - \frac{1}{2\xi}f^2(x)\right]\right\}$$

performing the integral over f, fixes $f = \partial_\mu A^\mu + m\phi$ and we find the R_ξ gauge-fixed form of the Stueckelberg lagrangian

$$\boxed{Z[0] \propto \int \mathcal{D}\alpha\mathcal{D}A\mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left\{i\int \mathrm{d}^4x\left[\mathcal{L}_{SB} - \frac{1}{2\xi}(\partial_\mu A^\mu + m\phi)^2\right]\right\}} \tag{8}$$

note that $\xi \to 0$ results in a recovery of the delta hard gauge-fixing.

1.4.2 Decoupling Gauge

With (8) built, let us look at the kinetic portion of the scalar and vector sector more closely

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A_\mu^2 - \frac{1}{2\xi}(\partial_\mu A^\mu + m\phi)^2 + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - mA^\mu\partial_\mu\phi$$
$$= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{1}{2}m^2A_\mu^2 + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2\xi}m^2\phi^2 - m\left(A^\mu\partial_\mu\phi + \frac{1}{\xi}\partial_\mu A^\mu\phi\right)$$
$$= \frac{1}{2}A^\mu\left[(\Box + m^2)g_{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu\right]A^\nu + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2\xi}m^2\phi^2 - \left(1 - \frac{1}{\xi}\right)mA^\mu\partial_\mu\phi$$

where in the last equality we integrated by parts several of the terms in the second line. Something remarkable happens when $\xi = 1$, particularly with the last term, the two fields decouple! In this choice of gauge fixing, the kinetic portion we were inspecting becomes

$$\frac{1}{2}A^\mu[(\Box + m^2)g_{\mu\nu}]A^\nu + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2$$

[5] There is a mathematical proof of this if you're worried, for any compact group the Haar measure is gauge invariant, see Hamermesh 1962, p. 313, for example.

Note, additionally, how this choice of gauge also eliminates the $\partial_\mu \partial_\nu$ term in the quadratic form for the vector field. This means that the propagator in this gauge will lack the $\frac{p_\mu p_\nu}{m^2}$ term present in (1), it will be simply

$$\boxed{i\Pi^{\mu\nu}_{GF} = \frac{-ig^{\mu\nu}}{p^2 - m^2 + i\epsilon}}$$

In this sense, this gauge is analogous to the Feynman gauge in QED (but you could've guessed that from $\xi = 1$ and the presence of the $\partial_\mu A^\mu$ term). Remarkably then, the propagator now goes as $1/p^2$ in the limit where $p \to \infty$. This means that we have successfully constructed a gauge where the usual analysis that $[g] = E^0$ implies naive renormalizability holds!

Importantly, the conclusion here should **not** be that $[g] = E^0$ always implies renormalizability in some non-trivial way, even when it seems like it does not, this is just one example where it does. Matter of factly, this entire idea does not work for non-Abelian theories, after all if it did, the Higgs would not be necessary. The conclusion should really be that proof of renormalizability is far more complicated than it might seem naively.