Problem: Striking behavior of circular orbits

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Prelude

When thinking of features of general relativity entirely absent from Newton's gravity, what comes to mind most often, for good reason, are black holes. They are certainly the most extreme feature of GR, but there are many more, less jarring, exotic behaviors for which there are no Newtonian analogs. The idea of this problem is to explore an example of this, while dispelling a common misconception that outside of the event horizon, the Schwarzschild solution is not qualitatively very different from Newtonian Gravity.

Problem

Let's set the scene first. We have an uncharged and static star, with mass M and radius $R < 3M$ and an object with mass $m \ll M$ orbiting around it at some r with constant angular velocity.

The Newtonian subsection ahead is likely trivial for someone looking for problems in GR. Its purpose is just emphasize the stark contrast between the Newtonian and general relativity situations, feel free to skip it if you feel comfortable with what's stated.

The Newtonian Situation

Show that, for Newtonian gravity, for any star mass M and orbiting radius r, there exists an angular velocity $\dot{\phi}$ such that a circular orbit can be achieved by say, a rocket, without the need of any additional force to sustain it beyond gravity.

General relativity

Now in the framework of general relativity, prove (and find it) that there exists a radius r^* where the acceleration of the rocket in a circular orbit at constant angular velocity is entirely independent of $\dot{\phi}$. Interpret this result, can there ever be an orbit like in the Newtonian situation at this radius?

Next, suppose you are on a rocket trying to sustain an orbit like this in the region $R < r < r^*$, and suppose you want to minimize the force necessary to hold this orbit, what angular velocity should you aim for? How does this compare with Newtonian intuition?

References

[1] J. P. L. M. A. Abramowicz, "A note on a paradoxical property of the schwarzschild solution," *Acta Physica Polonica, Series B*, vol. 5, pp. 327–329, 1974.