Solution: Striking behavior of circular orbits

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Solution

Newtonian Situation

Since the main point here is really the GR segment, we'll solve this part somewhat quickly, but let's not be sloppy either. We have a non-rotating star whose mass is much bigger than that of the rocket orbiting it, therefore, we can safely take it to be static and place the origin of our coordinate system at its center. Further, because the only force present is gravity, which is central, angular momentum will be conserved (after all $\vec{r} \times \vec{F} = 0$), which in turn means the orbital plane is conserved as well. This observation allows us to simplify the problem, instead of having to use spherical coordinates, if the plane is always the same, polar coordinates are sufficient. Now since the goal is to eventually use Newton's second law, we need to be able to write acceleration $\ddot{\vec{r}}$ in this choice of coordinates. With that goal in mind, we remember that generally, polar coordinates (r, ϕ) are related to Cartesian coordinates by

$$x = r\cos\phi \qquad y = r\sin\phi$$

In turn, we can write the unit vectors $\hat{r}$ and $\hat{\theta}$, pointing along the direction of positive change of each coordinate as

$$\hat{r} = \frac{\vec{r}}{r} = \frac{r\cos\phi\hat{x} + r\sin\phi\hat{y}}{r} = \hat{x}\cos\phi + \hat{y}\sin\phi$$

Whereas $\hat{\phi}$ will be immediately orthogonal to $\hat{r}$ at every point, and hence

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

Now, we are ready to write the velocity,

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r(-\hat{x}\dot{\phi}\sin\phi + \hat{y}\dot{\phi}\cos\phi) = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

where we've used the fact that unit vectors in Cartesian coordinates are constant. The acceleration then follows suit

$$\ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\dot{\hat{\phi}} = \ddot{r}\hat{r} + 2\dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r(-\hat{x}\dot{\phi}\cos\phi - \hat{y}\dot{\phi}\sin\phi) = \ddot{r}\hat{r} + 2\dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} - r\dot{\phi}^2\hat{r}$$

$$\Rightarrow \quad \boxed{\ddot{\vec{r}} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi}}$$

Now Newton's second law tells us that because gravity is the only force acting on the rocket

$$m\ddot{\vec{r}} = m(\ddot{r} - r\dot{\phi}^2)\hat{r} + m(r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi} = -\frac{Mm}{r^2}\hat{r}$$

Or component-by-component

$$\ddot{r} - r\dot{\phi}^2 = -\frac{M}{r^2} \qquad r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0$$

The second equation tells us what we already argued at the beginning, namely that

$$m(r\ddot{\phi} + 2\dot{r})\dot{\phi} = \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\phi}\right) \equiv \frac{\mathrm{d}L}{\mathrm{d}t} = 0$$

or in plain english, angular momentum is conserved. The second one, on the other hand, is what we're really interested in,

$$\ddot{r} - r\dot{\phi}^2 = -\frac{M}{r^2}$$

This will be respected generally, but if we want the orbit to be circular, we must have $\dot{r}(t) = 0$ and therefore $\ddot{r}(t) = 0$, imposing a particular angular velocity,

$$-r\dot{\phi}^2 = -\frac{M}{r^2} \quad \Rightarrow \quad \boxed{\dot{\phi} = \sqrt{\frac{M}{r^3}}}$$

which is to say, if this is the angular velocity, then we can have a circular orbit sustained by gravity alone.

General Relativity

We start off by setting the stage. A spherically symmetric vacuum solution of Einstein's Field Equations can only be Schwarzschild[1], and hence, in spherical coordinates (t, r, θ, ϕ)

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

Now, let's write down the 4-velocity of the object we're trying to put on a circular orbit with constant angular velocity. With no loss of generality, we can take the orbit to be around $\theta = \pi/2$ such that we need only worry about the variation of ϕ, therefore, the 4-velocity of such an object will be

$$u = \frac{\mathrm{d}}{\mathrm{d}\tau} \quad \Rightarrow \quad u^\mu = (\dot{t}, 0, 0, \dot{\phi})$$

where dotting now, and for the rest of the solution, refers to differentiation with respect to proper time [2]. Next, we can eliminate $\dot{t}$ by imposing the correct normalization of u

$$-1 = u^\mu u_\mu = g_{\mu\nu}u^\mu u^\nu = g_{tt}u^t u^t + g_{\phi\phi}u^\phi u^\phi \underbrace{=}_{\theta=\frac{\pi}{2}} -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + r^2\dot{\phi}^2$$

$$\Rightarrow \quad \dot{t} = \sqrt{\frac{1 + r^2\dot{\phi}^2}{1 - \frac{2M}{r}}}$$

Such that the normalized 4-velocity is

$$u^\mu = \left(\sqrt{\frac{1 + r^2\dot{\phi}^2}{1 - \frac{2M}{r}}}, 0, 0, \dot{\phi}\right) \tag{1}$$

Next, we need to compute the associated 4-acceleration. By definition acceleration is the directional derivative of the velocity along itself, that is

$$a = D_u u \quad \Rightarrow \quad a^\mu = u^\nu\nabla_\nu u^\mu = u^\nu\partial_\nu u^\mu + u^\nu\Gamma^\mu_{\nu\lambda}u^\lambda$$

We want to avoid brute-force calculating all of the 64 Christoffel symbols (technically only 40 are independent, but the point stands), so instead, one can note that only the t and ϕ components of u^μ are non-vanishing, as such, the only potentially non-zero terms in the expression above are

$$\begin{aligned}
a^t &= u^t\partial_t u^t + u^\phi\partial_\phi u^t + \Gamma^t_{tt}u^t u^t + \Gamma^t_{t\phi}u^t u^\phi + \Gamma^t_{\phi t}u^\phi u^t + \Gamma^t_{\phi\phi}u^\phi u^\phi \\
a^r &= \Gamma^r_{tt}u^t u^t + \Gamma^r_{t\phi}u^t u^\phi + \Gamma^r_{\phi t}u^\phi u^t + \Gamma^r_{\phi\phi}u^\phi u^\phi \\
a^\theta &= \Gamma^\theta_{tt}u^t u^t + \Gamma^\theta_{t\phi}u^t u^\phi + \Gamma^\theta_{\phi t}u^\phi u^t + \Gamma^\theta_{\phi\phi}u^\phi u^\phi \\
a^\phi &= u^t\partial_t u^\phi + u^\phi\partial_\phi u^\phi + \Gamma^\phi_{tt}u^t u^t + \Gamma^\phi_{t\phi}u^t u^\phi + \Gamma^\phi_{\phi t}u^\phi u^t + \Gamma^\phi_{\phi\phi}u^\phi u^\phi
\end{aligned}$$

[1]This is known as Birkhoff's theorem.

[2]This is not a problem. Generally, differentiation with respect to proper time τ differs from differentiation with respect to coordinate time t by a multiplicative factor of $\frac{\mathrm{d}\tau}{\mathrm{d}t}$, such that none of our conclusions about for example dependence on $\dot{\phi}$ would change if we adopt one or the other.

Unfortunately we need to compute these Christoffel symbols (or you can look them up), either way, we arrive that most of the relevant ones vanish identically

$$\Gamma^t_{tt} = 0 \quad \Gamma^t_{\phi\phi} = 0 \quad \Gamma^\theta_{tt} = 0 \quad \Gamma^\phi_{tt} = 0 \quad \Gamma^\phi_{\phi\phi} = 0 \quad \Gamma^\phi_{t\phi} = \Gamma^\phi_{\phi t} = 0 \quad \Gamma^t_{t\phi} = \Gamma^t_{\phi t} = 0 \quad \Gamma^r_{t\phi} = \Gamma^r_{\phi t} = 0 \quad \Gamma^\theta_{\phi t} = \Gamma^\theta_{t\phi} = 0$$

leaving us having to deal with only

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \underbrace{=}_{\theta=\frac{\pi}{2}} 0 \qquad \Gamma^r_{\phi\phi} = -r\sin^2\theta\left(1-\frac{2M}{r}\right) \underbrace{=}_{\theta=\frac{\pi}{2}} -r\left(1-\frac{2M}{r}\right) \qquad \Gamma^r_{tt} = \frac{M}{r^2}\left(1-\frac{2M}{r}\right)$$

This leaves us with the following acceleration components

$$\begin{aligned}
a^t &= u^t\partial_t u^t + u^\phi\partial_\phi u^t \\
a^r &= u^t\partial_t u^r + u^\phi\partial_\phi u^r + \frac{M}{r^2}\left(1-\frac{2M}{r}\right)u^t u^t - r\left(1-\frac{2M}{r}\right)u^\phi u^\phi \\
a^\theta &= u^t\partial_t u^\theta + u^\phi\partial_\phi u^\theta \\
a^\phi &= u^t\partial_t u^\phi + u^\phi\partial_\phi u^\phi
\end{aligned}$$

Using the expression for the 4-velocity in (1) and remembering that we're aiming for a constant velocity circular orbit, where r and $\dot\phi$ are constant, we find three vanishing components

$$a^t = 0 \quad a^\theta = 0 \quad a^\phi = 0$$

and a non-vanishing one

$$a^r = \frac{1+r^2\dot\phi^2}{1-\frac{2M}{r}}\frac{M}{r^2}\left(1-\frac{2M}{r}\right) - r\dot\phi^2\left(1-\frac{2M}{r}\right) = (1+r^2\dot\phi^2)\frac{M}{r^2} - r\dot\phi^2\left(1-\frac{2M}{r}\right)$$

Which we can simplify further into

$$(1+r^2\dot\phi^2)\frac{M}{r^2} - r\dot\phi^2\left(1-\frac{2M}{r}\right) = \frac{M}{r^2} - (r-3M)\dot\phi^2$$

Hence, our full four-acceleration is

$$\boxed{a^\mu = \left(0, \frac{M}{r^2} - (r-3M)\dot\phi^2, 0, 0\right)} \tag{2}$$

We can now impose what we wanted, namely that the acceleration is independent of $\dot\phi$:

$$\partial_{\dot\phi} a^\mu = 0 \quad \forall\ \mu$$

Evidently, we need only worry about a^r, for which we find

$$\partial_{\dot\phi} a^r = (r-3M)\dot\phi \overset{!}{=} 0 \quad \Rightarrow \quad \boxed{r = 3M}$$

Or we could've simply noted that for $r = 3M$ the second term, which is the only one that depends on $\dot\phi$, vanishes entirely. Now, to find the four-acceleration for such a radius, we need only evaluate (2) at $r = 3M$, yielding

$$\boxed{a^\mu = \left(0, \frac{1}{9M}, 0, 0\right)}$$

This should be quite the striking result. For starters, the fact that this four-acceleration is non-zero means there are no geodesic orbits at this radius, that is, in stark contrast to the Newtonian situation, gravity alone cannot sustain this orbit regardless of its angular velocity[3]. But perhaps more perplexing is the fact that this acceleration

[3] There's a subtlety here that should be appreciated. Note that in the Newtonian situation, gravity is an explicit force, hence, to be in an orbit sustained by only gravity means that the object must be accelerated by gravity alone. In GR though, a circular orbit sustained only by gravity means looking for a a circular geodesic (and hence zero acceleration) in whatever curved spacetime we're in, which is why this can never be achieved at $r = 3M$.

does not depend on angular velocity, regardless of how fast the object spins around the star, the acceleration required to keep it in this orbit is the same, it's as if, in its own frame, centrifugal (pseudo)force is entirely absent. Indeed, this leads us to the final part of the problem.

Let's now look at the region $R < r < 3M$, note that for such a situation, looking at (2), a^r is not only strictly positive, but monotonically increasing with $\dot{\phi}$. The behavior should be clear, but this is an example plot for $a^r(\dot{\phi})$ for $r = 2.5M$

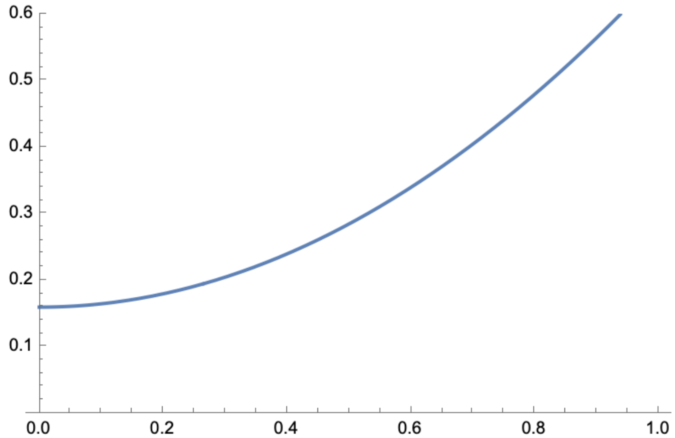

Figure 1: Radial component of four-acceleration as a function of $\dot{\phi}$ for $r = 2.5M$

What this tells us is that if the goal is to minimize the acceleration a rocket needs to exert in order to hold this orbit, they should just have no angular velocity at all[4]. This is incredibly weird if you try to compatibilize it with the Newtonian understanding. For the sake of concreteness let's go back to it for a moment; we had shown previously that for a stable orbit, the radial acceleration is given by

$$\ddot{r} = r\dot{\phi}^2 - \frac{GM}{r^2}$$

Here the behavior is very different, in particular, for a given r, regardless of its value, if the goal is to minimize this quantity, one can simply take

$$\dot{\phi} = \sqrt{\frac{GM}{r^3}} \equiv \dot{\phi}^*$$

for which $\ddot{r} = 0$. If $\dot{\phi} < \dot{\phi}^*$ the gravitational force takes over, and one needs to exert outward force to hold the orbit. Conversely, for $\dot{\phi} > \phi^*$, one would need an inward force because gravity alone cannot hold the orbit.

There is no situation, **ever**, in Newtonian gravity, where the force necessary to hold this orbit is minimized if the rocket is standing still. Note that qualitatively, however, GR's behavior would match up were the sign of the $mr\dot{\phi}^2$ term flipped, then, with the centripetal term and gravity having the same sign, no matter the angular velocity, one would need an outward force to hold this orbit, with a minimum at $\dot{\phi} = 0$. This is the phenomenon of "inversion of the centrifugal force" that literature often say happens to in this region of spacetime. And while this picture does make this situation a bit more palpable, one should be careful with such statements because they are evidently frame dependent, if we're not in the frame of the rocket itself [5], then there's no such thing as centrifugal pseudo-force. Regardless, the phenomenon is very interesting and clearly shows that there's a lot of interesting and unexpected physics in Schwarzschild spacetime beyond simply black holes.

[4]Though to be fair, calling this an orbit is a bit of stretch, but the point stands, angular velocity **increases** always increases the acceleration.

[5]which emphatically in our GR treatment we're not, after all the Schwarzschild metric only takes the form we've used for observers that see the star/black hole as spherically symmetric and static, the latter of which clearly would not hold were the observer spinning around it