# Solution: Flux out of the face of a cube

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## Solution

The idea of solving via both methods is to explicitly feel how dramatically simpler symmetry makes solving our problems. Often one might think, "I'll just solve via brute force, searching for symmetries takes too much time", when in reality, even if you take some time to find it, the time saved afterwards might be worth it. Plus symmetry helps you better understand the physics of the problem, which is always a plus.

### Method 1:

First, let's compute it by integrating directly.

The field of a point charge is given, in SI units, by

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}$$

where $\vec{r}$ is the position vector with the origin situated at the location of the point charge. Now, we take its Cartesian form, after all, they are the natural coordinate choice to deal with a cube, $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$, such that the field is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(x^2+y^2+z^2)^{3/2}}(x\hat{x} + y\hat{y} + z\hat{z})$$

No, we've placed the origin of our coordinate system on top of the charge, with the $\hat{z}$ direction pointing up, $\hat{y}$, to the right, and $\hat{x}$ out of the page. Then, if $L$ is the length of the cube, the face we're interested in will be on the $y = L$ plane, and will have area element $\mathrm{d}\vec{a}$ perpendicular to its surface, given by

$$\mathrm{d}a = \hat{y}\mathrm{d}z\mathrm{d}y^{[1]}$$

Therefore, the flux, will be given by the following integral

$$\phi = \int_F \vec{E}\cdot \mathrm{d}\vec{a} = \int_F \mathrm{d}x\mathrm{d}z\left[\frac{q}{4\pi\varepsilon_0}\frac{y}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{q}{4\pi\varepsilon_0}\int_F \mathrm{d}x\mathrm{d}z\,\frac{y}{(x^2+y^2+z^2)^{3/2}}$$

where $F$ is the face of the cube we're interested in. As we know, the face is defined by the following region in space given our coordinate system

$$x = 0 \to L, \qquad z = 0 \to L \qquad y = L$$

Such that our flux will the be given by

$$\phi = \frac{q}{4\pi\varepsilon_0}\int_0^L \mathrm{d}x\int_0^L \mathrm{d}z\,\frac{L}{(L^2+x^2+z^2)^{3/2}} \tag{1}$$

[1] By convention, the area element points outwards of a closed surface, not inwards.

Integrating this is a bit of a pain, so let's be careful. We start off by integrating with respect to $z$

$$\int_0^L \mathrm{d}z\, \frac{L}{(L^2+x^2+z^2)^{3/2}}$$

here, we can use a trigonometric substitution

$$z = \sqrt{L^2+x^2}\tan u \quad \Rightarrow \quad \mathrm{d}z = \mathrm{d}u\sqrt{L^2+x^2}\sec^2 u$$

such that our integral now becomes

$$\int_{z=0}^{z=L} \mathrm{d}u\, \frac{L\sqrt{L^2+x^2}\sec^2 u}{\left[(L^2+x^2)(1+\tan^2 u)\right]^{3/2}} = \frac{L}{L^2+x^2}\int_{z=0}^{z=L} \mathrm{d}u\, \frac{\sec^2 u}{(1+\tan^2 u)^{3/2}}$$

Now, $\sec^2 u = 1+\tan^2 u$, dramatically simplifying our expression

$$\frac{L}{L^2+x^2}\int_{z=0}^{z=L} \frac{\mathrm{d}u}{\sec u} = \frac{L}{L^2+x^2}\int_{z=0}^{z=L} \mathrm{d}u\cos u$$

which we can just integrate directly, yielding

$$\frac{L}{L^2+x^2}(\sin u)\Big|_{z=0}^{z=L} = \frac{L}{L^2+x^2}\sin\left(\arctan\frac{z}{\sqrt{L^2+x^2}}\right)\Bigg|_{z=0}^{z=L}$$

Finally, in general we know that

$$\sin(\arctan t) = \frac{t}{\sqrt{1+t^2}} \tag{2}$$

and therefore,

$$\frac{L}{L^2+x^2}\sin\left(\arctan\frac{z}{\sqrt{L^2+x^2}}\right)\Bigg|_{z=0}^{z=L} = \frac{L}{L^2+x^2}\left[\frac{z}{\sqrt{L^2+x^2}}\frac{1}{\sqrt{1+\frac{z^2}{L^2+x^2}}}\right]\Bigg|_{z=0}^{z=L} = \frac{L}{L^2+x^2}\left[\frac{z}{\sqrt{L^2+x^2+z^2}}\right]\Bigg|_{z=0}^{z=L}$$

$$= \frac{Lz}{(L^2+x^2)\sqrt{L^2+x^2+z^2}}\Bigg|_{z=0}^{z=L}$$

Finally, reverting back to the initial integration variable and applying the evaluation limits, we find

$$\int_0^L \mathrm{d}z\, \frac{L}{(L^2+x^2+z^2)^{3/2}} = \frac{L^2}{(L^2+x^2)\sqrt{2L^2+x^2}}$$

Substituting this back into the expression we had for the flux in (1), we now have

$$\phi = \frac{q}{4\pi\varepsilon_0}\int_0^L \mathrm{d}x\, \frac{L^2}{(L^2+x^2)\sqrt{2L^2+x^2}}$$

This integral is even worse than the previous one, but we start in the same way, by performing a trigonometric substitution

$$x = \sqrt{2L^2}\tan u \quad \Rightarrow \quad \mathrm{d}x = \mathrm{d}u\,\sqrt{2L^2}\sec^2 u$$

transforming our integral into

$$\frac{q}{4\pi\varepsilon_0}\int_{x=0}^{x=L} \mathrm{d}u\, \frac{L^2\sqrt{2L^2}\sec^2 u}{(L^2+2L^2\tan^2 u)\sqrt{2L^2(1+\tan^2 u)}}$$

once again, using that $\sec^2 u = 1+\tan^2 u$, this simplifies into

$$= \frac{q}{4\pi\varepsilon_0}\int_{x=0}^{x=L} \mathrm{d}u\, \frac{\sec u}{1+2\tan^2 u}$$

Now, notice that the integrand can be rearranged into a more suggestive form

$$\frac{1}{\cos u}\frac{1}{1+2\left(\frac{\sin u}{\cos u}\right)^2} = \frac{1}{\cos u}\frac{\cos^2 u}{\cos^2 u + 2\sin^2 u} = \frac{\cos u}{1+\sin^2 u}$$

Hence, the integral we have to evaluate has now been reduced to

$$\frac{q}{4\pi\varepsilon_0}\int_{x=0}^{x=L} \mathrm{d}u\,\frac{\cos u}{1+\sin^2 u}.$$

Here, the substitution should be very clear, we take

$$v = \sin u \quad\Rightarrow\quad \mathrm{d}v = \mathrm{d}u\cos u$$

such that now we have

$$\phi = \frac{q}{4\pi\varepsilon_0}\int_{x=0}^{x=L} \mathrm{d}v\,\frac{1}{1+v^2}$$

This is a very common integral, that you've likely computed in the past, but for the sake of completeness, let's evaluate it. To do so, we need yet another trigonometric substitution

$$v = \tan w \quad\Rightarrow\quad \mathrm{d}v = \mathrm{d}w\sec^2 w$$

such that

$$\phi = \frac{q}{4\pi\varepsilon_0}\int_{x=0}^{x=L} \mathrm{d}w\frac{\sec^2 w}{1+\sec^2 w}$$

For the last time, $\sec^2 w = \tan^2 + 1$, such that

$$\phi = \frac{q}{4\pi\varepsilon_0}\int_{x=0}^{x=L} \mathrm{d}w = \frac{q}{4\pi\varepsilon_0}(w)\Big|_{x=0}^{x=L}$$

Reverting from $w$ back to $v$, then back to $u$, and then back to $x$

$$\phi = \frac{q}{4\pi\varepsilon_0}(\arctan v)\Big|_{x=0}^{x=L} = \frac{q}{4\pi\varepsilon_0}[\arctan(\sin u)]\Big|_{x=0}^{x=L} = \frac{q}{4\pi\varepsilon_0}\arctan\left[\sin\left(\arctan\left(\frac{x}{\sqrt{2L^2}}\right)\right)\right]\Bigg|_{x=0}^{x=L}$$

Now, using (2) this becomes

$$\phi = \frac{q}{4\pi\varepsilon_0}\arctan\left(\frac{\frac{x}{\sqrt{2L^2}}}{\sqrt{1+\left(\frac{x}{\sqrt{2L^2}}\right)^2}}\right)\Bigg|_0^L$$

Finally, evaluating the limits, we find the flux

$$\phi = \frac{q}{4\pi\varepsilon_0}\left(\arctan\left(\frac{1}{\sqrt{3}}\right) - \arctan 0\right) \quad\Rightarrow\quad \boxed{\phi = \frac{q}{24\varepsilon_0}}$$

What an incredibly boring solution. It works, but surely there is a better way of doing this, which is the second method.

## Method 2:

This method clearly illustrated the power of symmetries, and how dramatically they can simplify anything in Physics.

Instead of thinking about our situation where, a charge is at the vertex of a cube, let's think about another, connected, but different situation. Suppose instead, that out charge is placed at the center of an arrangement of 8 cubes, as in the following diagram

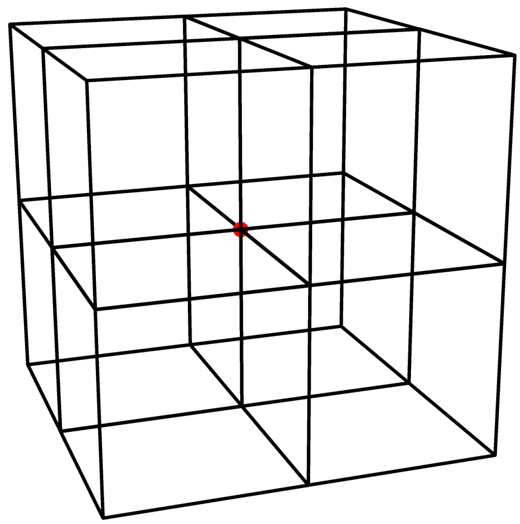

Figure 1: Diagram of the new arrangement, with the charge being at the center of eight cubes.

Gauss' Law tells us that because the outer faces of these cubes close a surface around the charge, the total flux, the sum of the fluxes going through each one of them, is given by

$$\Phi = \frac{q}{\varepsilon_0} \tag{3}$$

Now, notice two things. Firs, this setup has an underlying symmetry, each one of these outer faces is absolutely equivalent from the point of view of the charge, and hence, the flux going through each one of them **must** be the same. Second, the flux going out from each one of these faces is exactly the same as the flux we're interested in computing, after all, if we look at just one of these 8 cubes, we are left with exactly the original setup of this problem. Therefore, we conclude, that the flux we'd like to find is just $\Phi$ divided by the number of external faces in the Figure above, since there are $4 \times 6 = 24$ of them, we have

$$\boxed{\phi = \frac{\Phi}{24} = \frac{q}{24\varepsilon_0}}$$

which is the exact same result, but found through a tremendously faster (and nicer) method.