Renormalizability of Massive Vector Fields

Francisco Dahab

April 2025

1 Motivation

Naive power-counting of divergences in a theory is precisely just that. If a theory passes the naive test, you're hopeful that it's indeed renormalizable and if it does not, you are less optimistic, but in and of itself, this test is not sufficient to prove renormalizability, not even perturbatively. The idea of this problem is to illustrate this with an example that's not even particularly exotic, a massive spin-1 field.

2 Problem

2.1 Dimensional Analysis

Consider a theory with a massive spin-1 field coupled to a fermion,

$$S = \int \mathrm{d}^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu + \bar{\psi}(i\not{\partial} - M)\psi + g\bar{\psi}\not{A}\psi \right) \tag{1}$$

this is analogous to a version of QED where the photon is massive. What is the dimension of the coupling g? And what would standard wisdom tell you about the renormalizability of this theory given it?

2.2 Renormalizable?

2.2.1 The UV behaviour of the propagator

The conclusion in the previous problem is incorrect, $[g] = E^0$ does not, in this case, indicate renormalizability. To see this, find the free propagator for a spin-1 field. I recommend doing this in the Path Integral prescription, doing it via canonical quantization is immensely more complicated and will lead to nasty non-covariant contributions which cancel non-trivially.

How does this propagator behave in the deep UV, when $p \to \infty$?

2.2.2 Correcting the power counting

Accounting for this, modify the usual power-counting rules and apply it to (1), what is the conclusion about renormalizability now?

2.3 The Stueckelberg Mechanism

As it turns out, this conclusion is also incorrect, and (1) is indeed renormalizable. To see this, consider a different theory, where we integrate in a new scalar field $\phi(x)$[1] via

$$A_\mu(x) \to A_\mu(x) - \frac{1}{m}\partial_\mu \phi(x) \quad \text{and} \quad \psi(x) \to e^{-\frac{i}{m} g\phi(x)}\psi(x)$$

Find the new action and prove that it's now invariant under gauge transformations of the form

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x) \quad \phi(x) \to \phi(x) + m\alpha(x) \quad \text{and} \quad \psi(x) \to e^{ig\alpha(x)}\psi(x)$$

[1] Note that integrating in a field means $\phi(x)$ is a dynamical field with its own path integral.

for any function $\alpha(x)$. This action is called the Stueckelberg Action.

Prove, additionally, that there exists a gauge choice that removes the ϕ dependency from the action, such that the path integral over ϕ factors out, recovering the original theory (1) entirely. This shows that, "accidentally", there exists a superseding gauge theory that reduces to (1) given the correct gauge choice, and hence, the two theories must be physically equivalent as long as we're computing observables.

2.4 Renormalizable

2.5 Gauge-Fixing

Next, using the Faddeev-Popov prescription, gauge fix this in the equivalent of an R_ξ gauge, but instead of constructing it out of the Lorenz gauge $\partial_\mu A^\mu = 0$ as is standard, use $\partial_\mu A^\mu + m\phi$. This is an arbitrary choice, but it's convenient for reasons you will see next.

2.6 The Decoupling Gauge

Now, there should be a choice of ξ that decouples A_μ and ϕ, find it. In this gauge, compute the propagators for both fields, prove that now this theory looks renormalizable by naive power-counting. Once we get to non-Abelian gauge theories, it's worth looking back at this problem and trying to understand why it does not work for massive non-Abelian gauge bosons.

The conclusion here is rather nice. Evidently this is not a full proof of the renormalizability of massive QED (though the full proof does start from the Stueckelberg Action), but it goes to show that not every theory that looks non-renormalizable by power-counting is truly untreatable. In its original form, we cannot naively cancel all infinities at the level of n-point function in massive QED with a finite number of counterterms[2], but we've proven that this is a gauge-dependent statement, there is a better choice of gauge where perturbative renormalization is possible. This is not a completely out of the box idea in physics, in mechanics we often find that one choice of coordinate system is much better suited to deal with a given problem, the situation here is not too dissimilar from this.

[2]One can prove however, that even in this gauge, the contributions arising from the $k^\mu k^\nu/m^2$ contributions do not contribute to the S-matrix rendering it renormalizable, but the construction is quite a bit more complicated than dealing with the theory in a different gauge as we've done here (see for example Itzykson-Zuber chapter 8, section 4).