Wilson-Fisher RG flow

The Renormalization Group (RG), tells us, fundamentally, how a theory changes as the energy scale with which we probe it changes. One of the best ways to understand the qualitative behaviour of the RGE of a given theory is to plot the renormalization flow trajectories.

Let us explore one of the more classical examples where interesting behaviour the Wilson-Fisher

Defining the RG equations

The equations for the renormalization group evolution (RGE) of the coupling and anomalous dimension of the Wilson-Fisher in dimension 4-ϵ model are defined as follows (to first non-vanishing

$$\mu \frac{d}{d\mu} m_R^2 = \frac{\lambda_R}{16\pi^2} m_R^2 + O\left(\lambda_R^2\right)$$

$$\mu \frac{d}{d\mu} \lambda_R = -\epsilon\, \lambda_R + \frac{3\lambda_R^2}{16\pi^2} + O\left(\lambda_R^3\right)$$

It’s useful to define dimensionless version of ${m_R}^2$ as $\tilde{m}_R = \frac{1}{\mu} m_R$, such that the equations now become

$$\mu \frac{d}{d\mu} \tilde{m}_R^2 = \left(-2 + \frac{\lambda_R}{16\pi^2}\right) \tilde{m}_R^2 + O\left(\lambda_R^2\right)$$

$$\mu \frac{d}{d\mu} \lambda_R = -\epsilon\, \lambda_R + \frac{3\lambda_R^2}{16\pi^2} + O\left(\lambda_R^3\right)$$

To visualize the flow caused by these equations, it’s best to define the RHS as a set of functions

In[94]:=
```
RG1[λ_, m2_] := (-2 + λ / (16 π ^ 2)) * m2;

RG2[ϵ_, λ_, m2_] := (-ϵ * λ + 3 λ ^ 2 / (16 π ^ 2));
```

Next, consider the doublet (λ_R, ${m_R}^2$), at any specific point in the space defined by the set of all possible such doublets (which we henceforth, define as the *coupling space*) . Hence, to visualize the trajectories in the coupling space of our theory, we identify these functions as tangent vectors of possible trajectories at every point in the coupling space. Then, we can use a function like StreamPlot, to visualize the set of tangent vectors, which, together, will form visible trajectories.

Plotting as a function of ϵ

In order to make the manipulation of the plot is smooth it's useful to precompute a list of plots which will then be displayed in the manipulate tool.

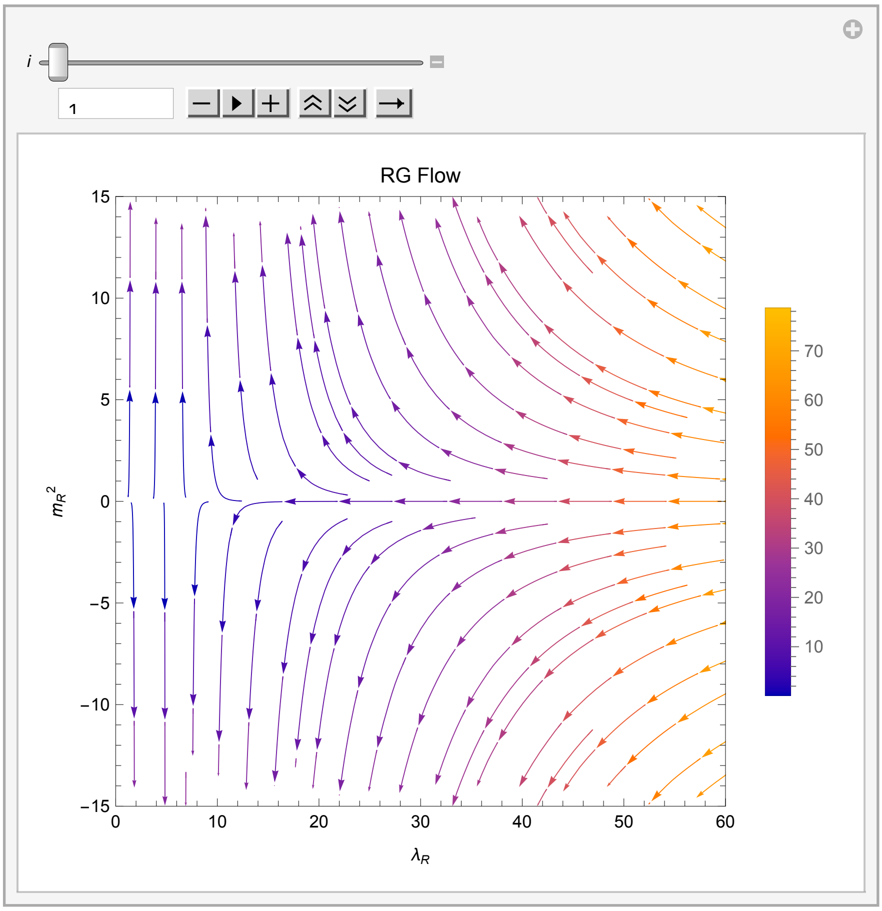
In[97]:=
```
ListofPlots =
  Table[StreamPlot[{-RG2[ϵ, λ, m2], -RG1[λ, m2]}, {λ, 0, 20 π^2 / 3}, {m2, -15, 15},
    PlotLabel → "RG Flow", FrameLabel → {"λR", "mR^2"}, PlotLegends → Automatic,
    StreamPoints → 150, PlotRange → {{0, 60}, {-15, 15}}], {ϵ, 0.1, 1, 0.01}];
```

where the minus signs come from the fact that generally, when we talk about flow of the RG, we mean how the coupling changes as the scale flows to low energy, hence decreases.

With this table, we can then use a Manipulate function to make this an interactive plot to understand how the RG flow of the theory changes as ϵ changes.

In[85]:=
```
Manipulate[ListofPlots[[i]], {i, 1, Length@ListofPlots, 1}]
```

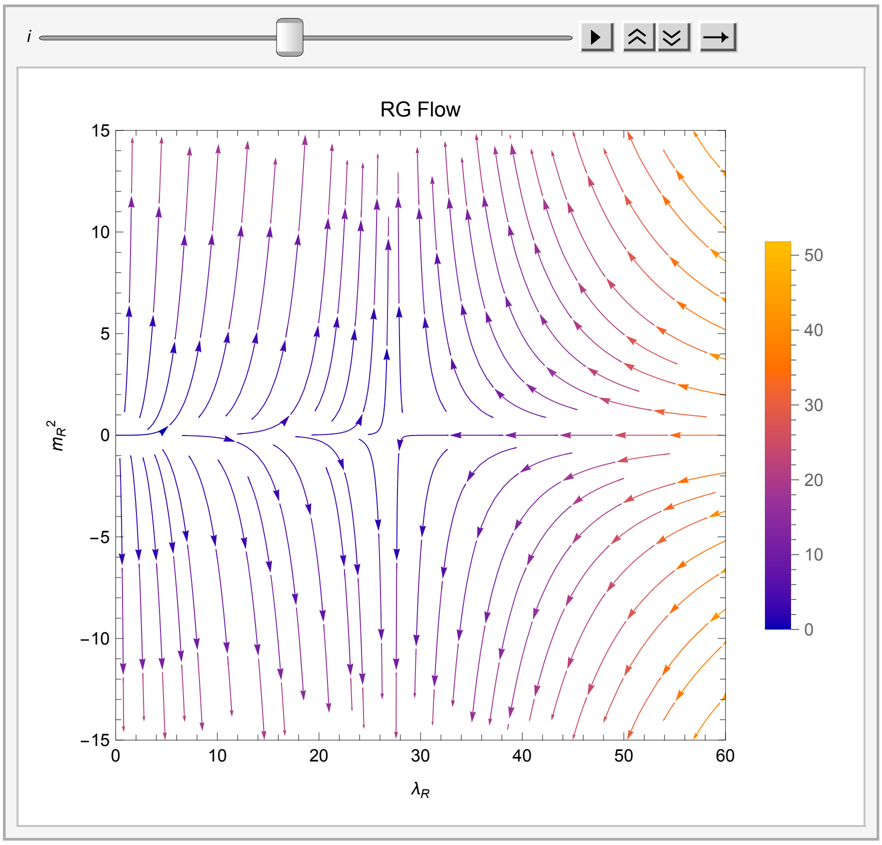
Out[85]=



As well as an animation, using Animate

In[98]:=
```
Animate[ListofPlots⟦i⟧, {i, 1, Length@ListofPlots, 1}]
```

Out[98]=



Note, how, as expected, when d < 4, or equivalently, when $\epsilon > 0$, a new, attractor of trajectories appears, this is a non-trivial fixed point decoupling from the Gaussian fixed point (at $\lambda = 0$, ${m_R}^2 = 0$). This is the famous Wilson-Fisher fixed point, found at

$$\lambda_* = \frac{16\,\pi^2\,\epsilon}{3}$$