An asymptotically exact a posteriori error estimator for non-selfadjoint Steklov eigenvalue problem *

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Abstract

This paper aims to introduce an asymptotically exact a posteriori error estimator for non-selfadjoint Steklov eigenvalue problem arising from inverse scattering by using the complementary technique, which provides an asymptotically exact estimate for eigenpair of non-selfadjoint Steklov eigenvalue problem. Besides, as its applications, we design a novel cascadic adaptive method for non-selfadjoint Steklov eigenvalue problem based on the asymptotically exact estimate. In our novel algorithm, we will transform the non-selfadjoint Steklov eigenvalue problem into some boundary value problems on the adaptive space sequence and some non-selfadjoint Steklov eigenvalue problem on a low dimensional finite element space. The involved boundary value problems are solved by executing some smoothing steps which is the key point of cascadic algorithm. The mesh refinement strategy and the number of smoothing steps for the cascadic adaptive method will be controlled by the proposed asymptotically exact a posteriori error estimator.

Keywords. Non-selfadjoint Steklov eigenvalue problem, asymptotically exact a posteriori error estimator, cascadic multigrid method, adaptive method, complementary method.

AMS subject classifications. 65N30, 35J61, 65M55, 65B99.

1 Introduction

Steklov eigenvalue problem is a vital class of mathematical problem, which is widely used in engineering and physics. For example, in the analysis of pendulum vibrations (see [1]), in the study of the vibration models in contact with incompressible fluid (see [7]), and in the study of stability of mechanical oscillators immersed in viscous fluid (see [20]). Thus we pay attention to Steklov eigenvalue problem in this paper and the strategy we adopt is finite element method. The theoretical analysis of finite element method for Steklov eigenvalue problem can be referred to [2, 12]. In addition, other numerical results of Steklov eigenvalue problem can be found in [3, 8, 9, 17, 28, 33, 42, 45, 53] and the literature cited therein. But so far, almost all of the above work only consider the self-adjoint situation. Few studies have discussed the finite element method for non-selfadjoint Steklov eigenvalue problem (see,e.g.,[16]). The non-selfadjoint Steklov eigenvalue problem will lead to a non-Hermitian matrices, and the computation of complex eigenvalues of such

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non-Hermitian matrices are quite challenging. Thus in this paper, we study the error estimate of non-selfadjoint Steklov eigenvalue problem and construct some efficient finite element algorithms.

In this paper, an asymptotically exact a posteriori error estimator for a class of non-selfadjoint Steklov eigenvalue problems arising from inverse scattering is introduced, and its applications in adaptive finite element method are also studied. The reason why we study an asymptotically exact a posteriori error estimator lies in the fact that although the approximate solutions of partial differential equations can be obtained through many numerical methods such as finite difference method and finite element method, we do not know the exact solution of the equation in most case, so we can not get the accurate error estimate for the approximations. This problem widely exists in numerical simulations. It is generally believed that only numerical solutions can not meet the practical applications and we also need to get some information of errors. Although some a priori error estimates can be deduced by theoretical analysis of finite element method, these estimates are limited to verify the asymptotic convergence rate. Therefore, we need some accurate a posteriori error estimates. Based on the complementary energy method [40, 46], a posteriori error estimator for the non-selfadjoint Steklov eigenvalue problem is designed. When the mesh size is small enough, we can prove that the proposed error estimator is asymptotically exact.

As applications of the proposed a posteriori error estimator, a novel cascadic adaptive finite element method for solving non-selfadjoint Steklov eigenvalue problems arising from inverse scattering is designed. As we know, multigrid and multilevel methods [13, 50] own the optimal error estimate with linear computation complexity. At present, many multigrid methods have been developed, such as full multigrid method and cascadic multigrid method. In view of the simplicity and efficiency of cascadic multigrid method, this algorithm is adopted in our paper. Cascadic multigrid method is proposed in [11] by P. Deuflhard. We know that when the smoothing process of multigrid method reaches finer mesh, the error inherited from coarser mesh is low frequency, and it is difficult to eliminate such low frequency error with the traditional smoothers. So the correction steps are necessary. The main difference between cascadic multigrid method and other multigrid methods lies in whether the correction step is reserved. In each finite element space of cascadic multigrid method, only the smoothing steps are needed. If the number of smoothing steps and degrees of freedom have an inverse relationship, we can derive the optimal error estimate as that of the standard finite element method. In addition, due to the lower dimension of coarser level, we can still get the linear computation complexity of cascadic multigrid method. For more information on cascadic multigrid method, see [11, 43] and the references cited therein.

In [6], Babuška is the first to propose the concept of adaptive finite element method (AFEM). The core idea of AFEM is average error distribution. So far, AFEM has been widely used to solve singular partial differential equations, and its efficiency has been proved theoretically and numerically. The theoretical analysis of AFEM is very mature. For example, Dörfler [24] introduce Dörfler's marking strategy and prove strict energy error reduction for Laplace problem under initial grid conditions. Mekchay and Nochetto [37] introduced similar results for second order elliptic operators by introducing total error. Now the most commonly used form is propose in [18] by Cascon etc. For eigenvalue problems, AFEM is also a competitive algorithm. Similar theoretical and numerical results can be found in [21, 25, 31]. For more results on AFEM, see [8, 18, 39, 44] and the references cited. The key point of AFEM lies in choosing the marked elements appropriately. By using the asymptotically exact error estimator, we can obtain a high quality mesh.

In this paper, as the applications of the proposed asymptotically exact a posteriori error estimator, a novel cascadic adaptive method is established by combining the latest research results of AFEM, cascadic multigrid method and multilevel correction method (see [30, 47]). The main idea of the novel algorithm is to transform the non-selfadjoint Steklov eigenvalue problem into a series of boundary value problems on adaptive spaces. Then we further improve the accuracy of approximate eigenfunctions by solving some small scale non-selfadjoint Steklov eigenvalue problems. Similarly to the standard cascadic multigrid method, we only do some smoothing steps for the involved boundary value problems. Since the dimension of the non-selfadjoint Steklov eigenvalue problem in the second step is relatively small with mesh refinement, the computation complexity is determined by the smoothing steps in adaptive spaces. The number of smoothing steps in our adaptive algorithm will depend on the error of eigenfunctions which will be replaced by the asymptotically exact a posteriori error estimator. Overall, the computation complexity of the novel cascadic adaptive algorithm will not be much higher than that of the standard adaptive finite element method.

The outline of the paper is organized as follows. Section 2 introduces the finite element method for the non-selfadjoint Steklov eigenvalue problem. In Section 3, an asymptotically exact a posteriori error estimator is proposed based on complementary method. Section 4 presents a novel cascadic adaptive method for solving non-selfadjoint Steklov eigenvalue problems. In Section 5, some numerical experiments are presented to verify the theoretical results in this paper. Finally, some concluding comments are given.

2 Finite element method for non-selfadjoint Steklov eigenvalue problem

In this paper, we will use the letter C to denote a mesh-independent constant, and use $x_1 \lesssim y_1$, $x_2 \gtrsim y_2$, $x_3 \approx y_3$ to denote $x_1 \leq C_1 y_1$, $x_2 \geq c_2 y_2$, $c_3 x_3 \leq y_3 \leq C_3 x_3$. The standard notations for Sobolev spaces $W^{s,p}(\Omega)$ are used. For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$, $V = H^1(\Omega)$.

Here, we consider the following non-selfadjoint Steklov eigenvalue problem:

$$\left\{ \begin{array}{rcll} \Delta u + k^2 n(x) u &=& 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \lambda u &=& 0, & \text{on } \partial\Omega, \end{array} \right. \tag{2.1}$$

where $\Omega \subset \mathcal{R}^2$ is a bounded polygonal domain and $\frac{\partial}{\partial \nu}$ is the outward normal derivative on $\partial\Omega$. k is the wavenumber and $n(x)$ is the index of refraction. We assume $n(x)$ denotes a bounded complex value function given by:

$$n(x) = n_1(x) + i\frac{n_2(x)}{k}, \tag{2.2}$$

where $i = \sqrt{-1}, n_1(x) \geq \delta > 0$ and $n_2(x) \geq 0$ are bounded and piecewise smooth functions.

In order to give the weak form of (2.1), the following bilinear forms are defined:

$$a(u,v) = (\nabla u, \nabla v) - (k^2 n(x) u, v), \ (u,v) = \int_\Omega u\bar{v}dx \quad \text{and} \quad b(u,v) = \int_{\partial\Omega} u\bar{v}ds. \tag{2.3}$$

Then the weak form of (2.1) has following form: Find $(\lambda, u) \in \mathcal{C} \times V$ such that $b(u,u) = 1$ and

$$a(u,v) = -\lambda b(u,v), \quad \forall v \in V. \tag{2.4}$$

Now, we introduce the finite element method for non-selfadjoint Steklov eigenvalue problem (2.4). First we generate a shape regular decomposition of the computing domain Ω into triangles or rectangles (cf. [14, 19]). Then we consider the finite element discretization on the conforming mesh $\mathcal{T}_k$ over Ω. Let $h_k := \max\{h_T : T \in \mathcal{T}_k\}$, where h_T denotes the diameter of each $T \in \mathcal{T}_k$.

Let $V_k \subset V$ be the finite element space of continuous piecewise polynomials over $\mathcal{T}_k$, and be equipped with the same norm $\|\cdot\|_{1,\Omega}$ as that of space V. The standard finite element scheme for non-selfadjoint Steklov eigenvalue problem (2.4) is: Find $(\lambda_k, u_k) \in \mathcal{C} \times V_k$ such that $b(u_k, u_k) = 1$ and

$$a(u_k, v_k) = -\lambda_k b(u_k, v_k), \quad \forall v_k \in V_k. \tag{2.5}$$

For the non-selfadjoint Steklov eigenvalue problem (2.4), there exists the corresponding adjoint eigenvalue problem: Find $(\lambda^*, u^*) \in \mathcal{C} \times V$ such that $b(u^*, u^*) = 1$ and

$$a(v, u^*) = -\bar{\lambda}^* b(v, u^*), \quad \forall v \in V. \tag{2.6}$$

Note that the primal and dual eigenvalues are connected via $\lambda = \bar{\lambda}^*$.

The discrete weak form associated with (2.6) is given by: Find $(\lambda_k^*, u_k^*) \in \mathcal{C} \times V_k$ such that $b(u_k^*, u_k^*) = 1$ and

$$a(v, u_k^*) = -\bar{\lambda}_k^* b(v, u_k^*), \quad \forall v \in V_k. \tag{2.7}$$

Note that the discrete primal and dual eigenvalues are also connected via $\lambda_k = \bar{\lambda}_k^*$. Hereafter,we use (λ_k, u_k) and (λ_k, u_k^*) to denote the solutions of (2.5) and dual problem (2.7), respectively.

Define $\eta(V_k)$ as

$$\eta(V_k) = \sup_{f \in H^{1/2}(\partial\Omega), \|f\|_{1/2,\partial\Omega}=1} \inf_{v \in V_k} \|Tf - v\|_1, \tag{2.8}$$

where the operator $T : H^{-1/2}(\partial\Omega) \to V$ is defined by

$$a(Tf, v) = b(f, v), \quad \forall f \in H^{-1/2}(\partial\Omega) \text{ and } \forall v \in V. \tag{2.9}$$

Similarly, define $\eta^*(V_k)$ as

$$\eta^*(V_k) = \sup_{f \in H^{1/2}(\partial\Omega), \|f\|_{1/2,\partial\Omega}=1} \inf_{v \in V_k} \|T^* f - v\|_{1,\Omega}, \tag{2.10}$$

where the operator $T^* : H^{-1/2}(\partial\Omega) \to V$ is defined by

$$a(v, T^* f) = b(v, f), \quad \forall f \in H^{-1/2}(\partial\Omega) \text{ and } \forall v \in V. \tag{2.11}$$

Let $M(\lambda)$ and $M^*(\lambda)$ denote two eigenspaces corresponding to the eigenvalue λ of (2.4) and (2.6), respectively,

$$\begin{aligned} M(\lambda) &= \{u \in V : u \text{ is an eigenfunction of (2.4) corresponding to } \lambda\}, \\ M^*(\lambda) &= \{u^* \in V : u^* \text{ is an eigenfunction of (2.6) corresponding to } \lambda\}. \end{aligned}$$

Then we introduce the following notations for error estimates

$$\begin{aligned} \delta_k(\lambda) &:= \sup_{u \in M(\lambda), \|u\|_{0,\partial\Omega}=1} \inf_{v_k \in V_k} \|u - v_k\|_{1,\Omega}, \\ \delta_k^*(\lambda) &:= \sup_{u^* \in M^*(\lambda), \|u^*\|_{0,\partial\Omega}=1} \inf_{v_k \in V_k} \|u^* - v_k\|_{1,\Omega}. \end{aligned}$$

For simplicity, we only consider the nondefective eigenvalues (the ascent equals to 1) of the non-selfadjoint Steklov eigenvalue problem in this paper. Then the algebraic multiplicity equals to the geometric multiplicity and the generalized eigenspace is the same as the eigenspace. We have following error estimates for the non-selfadjoint Steklov eigenvalue problem.

Theorem 2.1 ([36, 54]). *When the mesh size h is small enough, for any finite element approximate eigenpairs $(\lambda_k, u_k) \in \mathcal{C} \times V_k$ and $(\lambda_k, u_k^*) \in \mathcal{C} \times V_k$, there exist the exact eigenpairs (λ, u) and (λ, u^*), such that the following error estimates hold*

$$\|u - u_k\|_{1,\Omega} \lesssim \delta_k(\lambda), \tag{2.12}$$

$$\|u^* - u_k^*\|_{1,\Omega} \lesssim \delta_k^*(\lambda), \tag{2.13}$$

$$\begin{aligned}
\|u-u_k\|_{0,\Omega} &\lesssim \eta^*(V_k)\|u-u_k\|_{1,\Omega}, &(2.14)\\
\|u^*-u_k^*\|_{0,\Omega} &\lesssim \eta(V_k)\|u^*-u_k^*\|_{1,\Omega}, &(2.15)\\
\|u-u_k\|_{0,\partial\Omega}^2 &\lesssim \eta^*(V_k)\|u-u_k\|_{1,\Omega}^2, &(2.16)\\
\|u^*-u_k^*\|_{0,\partial\Omega}^2 &\lesssim \eta(V_k)\|u^*-u_k^*\|_{1,\Omega}^2, &(2.17)\\
|\lambda-\lambda_k| &\leq \|u-u_k\|_{1,\Omega}\|u^*-u_k^*\|_{1,\Omega}. &(2.18)
\end{aligned}$$

Lemma 2.1 ([34]). *Assume $(\lambda,u)\in\mathcal{C}\times V$ and $(\lambda,u^*)\in\mathcal{C}\times V$ satisfy (2.4) and (2.6), respectively, and suppose $\psi,\ \psi^*\in V$ such that $b(\psi,\psi^*)\neq 0$. Set*

$$\widehat{\lambda}=-\frac{a(\psi,\psi^*)}{b(\psi,\psi^*)}.$$

Then we have following expansion

$$\widehat{\lambda}-\lambda \;=\; \frac{-a(\psi-u,\psi^*-u^*)-\lambda b(\psi-u,\psi^*-u^*)}{b(\psi,\psi^*)}.$$

3 An asymptotically exact a posteriori error estimator

This section is devoted to designing and analyzing the asymptotically exact a posteriori error estimator for the non-selfadjoint Steklov eigenvalue problem and its adjoint problem.

Firstly, let us look back upon the following Green's formula.

Lemma 3.1. *Let $\Omega\subset\mathcal{R}^2$ be a bounded domain with Lipschitz boundary condition, and the unit outward normal is denoted by ν. Then the following Green's formula holds*

$$(\mathrm{div}\mathbf{p},v)+(\mathbf{p},\nabla v)=b(\mathbf{p}\cdot\nu,v),\quad \forall v\in H^1(\Omega)\text{ and }\forall\mathbf{p}\in\mathbf{W}, \tag{3.1}$$

where $\mathbf{W}:=\{\mathbf{p}\in(L^2(\Omega))^d:\mathrm{div}\mathbf{p}\in L^2(\Omega),\mathbf{p}\cdot\nu\in L^2(\partial\Omega)\}$.

Based the Green's formula and finite element error analysis, we can derive the following error estimates for the approximate eigenpairs (λ_k,u_k) and (λ_k,u_k^*).

Theorem 3.1. *When the mesh size h_k is small enough, the approximate eigenfunctions u_k and u_k^* satisfy following estimates*

$$\|u-u_k\|_{1,\Omega}\leq\frac{1+C\sqrt{\eta^*(V_k)}}{1-C\eta^*(V_k)}\min_{\mathbf{p}\in\mathbf{W}}\eta(\lambda_k,u_k,\mathbf{p}), \tag{3.2}$$

$$\|u^*-u_k^*\|_{1,\Omega}\leq\frac{1+C\sqrt{\eta(V_k)}}{1-C\eta(V_k)}\min_{\mathbf{p}\in\mathbf{W}}\eta^*(\lambda_k,u_k^*,\mathbf{p}), \tag{3.3}$$

where the a posteriori error estimators $\eta(\lambda_k,u_k,\mathbf{p})$ and $\eta^*(\lambda_k,u_k^*,\mathbf{p})$ are defined by

$$\eta(\lambda_k,u_k,\mathbf{p}):=\big(\|\mathrm{div}\mathbf{p}+k^2n(x)u_k\|_0^2+\|\mathbf{p}-\nabla u_k\|_0^2+\|\lambda_k u_k+\mathbf{p}\cdot\nu\|_{0,\partial\Omega}^2\big)^{\frac{1}{2}}, \tag{3.4}$$

$$\eta^*(\lambda_k,u_k^*,\mathbf{p}):=\big(\|\mathrm{div}\mathbf{p}+\bar{k}^2\bar{n}(x)u_k^*\|_0^2+\|\mathbf{p}-\nabla u_k^*\|_0^2+\|\bar{\lambda}_k u_k^*+\mathbf{p}\cdot\nu\|_{0,\partial\Omega}^2\big)^{\frac{1}{2}}. \tag{3.5}$$

Proof. From (2.4)-(2.5), (3.1) and Lemma 2.1, set $w=u-u_k$ there holds

$$\begin{aligned}
&\|u-u_k\|_{1,\Omega}^2\\
=\;& a(u-u_k,w)+(k^2n(x)(u-u_k),w)+(u-u_k,w)\\
=\;& -\lambda b(u,w)-(\nabla u_k,\nabla w)+(k^2n(x)u_k,w)
\end{aligned}$$

$$
\begin{aligned}
& +(k^2n(x)(u-u_k),w)+(u-u_k,w) \\
= & -\lambda_k b(u_k,w)-(\nabla u_k,\nabla w)+(k^2n(x)u_k,w) \\
& +(k^2n(x)(u-u_k),w)+(u-u_k,w)-b(\lambda u-\lambda_k u_k,w) \\
& +(\mathrm{div}\mathbf{p},w)+(\mathbf{p},\nabla w)-b(\mathbf{p}\cdot\nu,w) \\
= & (\mathrm{div}\mathbf{p}+k^2n(x)u_k,w)+(\mathbf{p}-\nabla u_k,\nabla w)-b(\lambda_k u_k+\mathbf{p}\cdot\nu,w) \\
& +(k^2n(x)(u-u_k),w)+(u-u_k,w)-b(\lambda u-\lambda_k u_k,w) \\
\leq & \|\mathrm{div}\mathbf{p}+k^2n(x)u_k\|_{0,\Omega}\|w\|_{0,\Omega}+\|\mathbf{p}-\nabla u_k\|_{0,\Omega}\|\nabla w\|_{0,\Omega} \\
& +\|\lambda_k u_k+\mathbf{p}\cdot\nu\|_{0,\partial\Omega}\|w\|_{0,\partial\Omega}+C\eta^*(V_k)\|u-u_k\|_{1,\Omega}\|w\|_{1,\Omega} \\
\leq & \left(\|\mathrm{div}\mathbf{p}+k^2n(x)u_k\|_{0,\Omega}^2+\|\mathbf{p}-\nabla u_k\|_{0,\Omega}^2+\|\lambda_k u_k+\mathbf{p}\cdot\nu\|_{0,\partial\Omega}^2\right)^{1/2} \\
& (\|w\|_{0,\Omega}^2+\|\nabla w\|_{0,\Omega}^2+\|w\|_{0,\partial\Omega}^2)^{1/2}+C\eta^*(V_k)\|u-u_k\|_{1,\Omega}\|w\|_{1,\Omega}, \qquad (3.6)
\end{aligned}
$$

where the Hölder inequality is used.

Therefore, we get

$$
\|u-u_k\|_{1,\Omega}^2 \leq \eta(\lambda_k,u_k,\mathbf{p})(1+C\sqrt{\eta^*(V_k)}\|u-u_k\|_{1,\Omega}+C\eta^*(V_k)\|u-u_k\|_{1,\Omega}^2.
$$

Then (3.2) follows from the arbitrariness of $\mathbf{p}\in\mathbf{W}$.

Similarly, for the adjoint problem, set $w=u^*-u_k^*$ we have

$$
\begin{aligned}
& \|u^*-u_k^*\|_{1,\Omega}^2 \\
= & a(w,u^*-u_k^*)+(w,\bar{k}^2\bar{n}(x)(u^*-u_k^*))+(w,u^*-u_k^*) \\
= & -\lambda b(w,u^*)-(\nabla w,\nabla u_k^*)+(w,\bar{k}^2\bar{n}(x)u_k^*) \\
& +(w,\bar{k}^2\bar{n}(x)(u^*-u_k^*))+(w,u^*-u_k^*) \\
= & -\lambda_k b(w,u_k^*)-(\nabla w,\nabla u_k^*)+(w,\bar{k}^2\bar{n}(x)u_k^*) \\
& +(w,\bar{k}^2\bar{n}(x)(u^*-u_k^*))+(w,u^*-u_k^*)-b(w,\bar{\lambda}u^*-\bar{\lambda}_k u_k^*) \\
& +(w,\mathrm{div}\mathbf{p})+(\nabla w,\mathbf{p})-b(w,\mathbf{p}\cdot\nu) \\
= & (w,\mathrm{div}\mathbf{p}+\bar{k}^2\bar{n}(x)u_k^*)+(\nabla w,\mathbf{p}-\nabla u_k^*)-b(w,\bar{\lambda}_k u_k^*+\mathbf{p}\cdot\nu) \\
& +(w,\bar{k}^2\bar{n}(x)(u^*-u_k^*))+(w,u^*-u_k^*)-b(w,\bar{\lambda}u^*-\bar{\lambda}_k u_k^*) \\
\leq & \|\mathrm{div}\mathbf{p}+\bar{k}^2\bar{n}(x)u_k^*\|_{0,\Omega}\|w\|_{0,\Omega}+\|\mathbf{p}-\nabla u_k^*\|_{0,\Omega}\|\nabla w\|_{0,\Omega} \\
& +\|\bar{\lambda}_k u_k^*+\mathbf{p}\cdot\nu\|_{0,\partial\Omega}\|w\|_{0,\partial\Omega}+C\eta(V_k)\|u^*-u_k^*\|_{1,\Omega}\|w\|_{1,\Omega} \\
\leq & \left(\|\mathrm{div}\mathbf{p}+\bar{k}^2\bar{n}(x)u_k^*\|_{0,\Omega}^2+\|\mathbf{p}-\nabla u_k^*\|_{0,\Omega}^2+\|\bar{\lambda}_k u_k^*+\mathbf{p}\cdot\nu\|_{0,\partial\Omega}^2\right)^{1/2} \\
& (\|w\|_{0,\Omega}^2+\|\nabla w\|_{0,\Omega}^2+\|w\|_{0,\partial\Omega}^2)^{1/2}+C\eta(V_k)\|u^*-u_k^*\|_{1,\Omega}\|w\|_{1,\Omega}. \qquad (3.7)
\end{aligned}
$$

That is

$$
\|u^*-u_k^*\|_{1,\Omega}^2 \leq \eta^*(\lambda_k,u_k^*,\mathbf{p})(1+C\sqrt{\eta(V_k)})\|u^*-u_k^*\|_{1,\Omega}+C\eta(V_k)\|u^*-u_k^*\|_{1,\Omega}^2.
$$

Then (3.3) follows from the arbitrariness of $\mathbf{p}\in\mathbf{W}$. □

Now, it's time for us to concentrate on numerical algorithm for the optimization problem involved in (3.2): Find $\widehat{\mathbf{p}}\in\mathbf{W}$ such that

$$
\eta(\lambda_k,u_k,\widehat{\mathbf{p}})=\min_{\mathbf{p}\in\mathbf{W}}\eta(\lambda_k,u_k,\mathbf{p}) \qquad (3.8)
$$

and the optimization problem involved in (3.3): Find $\widehat{\mathbf{p}}^*\in\mathbf{W}$ such that

$$
\eta^*(\lambda_k,u_k^*,\widehat{\mathbf{p}}^*)=\min_{\mathbf{p}\in\mathbf{W}}\eta^*(\lambda_k,u_k^*,\mathbf{p}). \qquad (3.9)
$$

Actually, we only need to solve two boundary value problems which are equivalent to the optimization problems (3.8) and (3.9).

Theorem 3.2. *The following boundary value problem is equivalent to the optimization problem (3.8): Find* $\widehat{\mathbf{p}} \in \mathbf{W}$ *such that*

$$\widehat{a}(\widehat{\mathbf{p}}, \mathbf{q}) = F(\mathbf{q}), \qquad \forall \mathbf{q} \in \mathbf{W}, \tag{3.10}$$

and following boundary value problem is equivalent to the optimization problem (3.9): Find $\widehat{\mathbf{p}}^* \in \mathbf{W}$ *such that*

$$\widehat{a}(\widehat{\mathbf{p}}^*, \mathbf{q}) = F^*(\mathbf{q}), \qquad \forall \mathbf{q} \in \mathbf{W}, \tag{3.11}$$

where

$$\begin{aligned}
&\widehat{a}(\mathbf{p}, \mathbf{q}) := (\mathrm{div}\mathbf{p}, \mathrm{div}\mathbf{q}) + (\mathbf{p}, \mathbf{q}) + b(\mathbf{p}\cdot\nu, \mathbf{q}\cdot\nu),\\
&F(\mathbf{q}) := \big(\nabla u_k, \mathbf{q}\big) - (k^2 n(x) u_k, \mathrm{div}\mathbf{q}\big) - \lambda_k b(u_k, \mathbf{q}\cdot\nu),\\
&F^*(\mathbf{q}) := \big(\nabla u_k^*, \mathbf{q}\big) - (\bar{k}^2 \bar{n}(x) u_k^*, \mathrm{div}\mathbf{q}\big) - \bar{\lambda}_k b(u_k^*, \mathbf{q}\cdot\nu).
\end{aligned}$$

Proof. Suppose $\widehat{\mathbf{p}} \in \mathbf{W}$ is the solution of (3.8), then for any $t \in \mathcal{R}$ and $\mathbf{q} \in \mathbf{W}$, we have

$$\eta^2(\lambda_k, u_k, \widehat{\mathbf{p}} + t\mathbf{q}) \geq \eta^2(\lambda_k, u_k, \widehat{\mathbf{p}}). \tag{3.12}$$

According to the definition presented in (3.4), (3.12) can be simplified into the following equation

$$At^2 + Bt \geq 0, \tag{3.13}$$

where

$$A = \|\mathrm{div}\mathbf{q}\|_0^2 + \|\mathbf{q}\|_0^2 + \|\mathbf{q}\cdot\nu\|_{0,\partial\Omega}^2$$

and

$$\begin{aligned}
B &= (\mathrm{div}\mathbf{p} + k^2 n(x) u_k, \mathrm{div}\mathbf{q}) + (\mathrm{div}\mathbf{q}, \mathrm{div}\mathbf{p} + k^2 n(x) u_k)\\
&\quad + (\mathbf{p} - \nabla u_k, \mathbf{q}) + (\mathbf{q}, \mathbf{p} - \nabla u_k)\\
&\quad + b(\mathbf{p}\cdot\nu + \lambda_k u_k, \mathbf{q}\cdot\nu) + b(\mathbf{q}\cdot\nu, \mathbf{p}\cdot\nu + \lambda_k u_k)\\
&= 2Re\big((\mathrm{div}\mathbf{p} + k^2 n(x) u_k, \mathrm{div}\mathbf{q}) + (\mathbf{p} - \nabla u_k, \mathbf{q}) + b(\mathbf{p}\cdot\nu + \lambda_k u_k, \mathbf{q}\cdot\nu)\big).
\end{aligned} \tag{3.14}$$

The equation (3.13) is equivalent to $B = 0$. Thus

$$Re\big((\mathrm{div}\mathbf{p} + k^2 n(x) u_k, \mathrm{div}\mathbf{q}) + (\mathbf{p} - \nabla u_k, \mathbf{q}) + b(\mathbf{p}\cdot\nu + \lambda_k u_k, \mathbf{q}\cdot\nu)\big) = 0. \tag{3.15}$$

Further, we can replace $\mathbf{q}$ in (3.14) with $i\mathbf{q}$ to obtain

$$Im\big((\mathrm{div}\mathbf{p} + k^2 n(x) u_k, \mathrm{div}\mathbf{q}) + (\mathbf{p} - \nabla u_k, \mathbf{q}) + b(\mathbf{p}\cdot\nu + \lambda_k u_k, \mathbf{q}\cdot\nu)\big) = 0. \tag{3.16}$$

Combining (3.15) and (3.16) leads to the desired result (3.10), and (3.11) can be proved similarly. □

Remark 3.1. *There have developed many special techniques for the div problem as (3.10) and (3.11). For assemble of stiffness matrix, we refer to [41]. For the solving process, the preconditioners in [4] and optimal multilevel method in [51] are good candidates.*

From (3.4), (3.5) and Theorem 3.2, the following identities hold.

Theorem 3.3. *Assume* $\widehat{\mathbf{p}}$ *and* $\widehat{\mathbf{p}}^*$ *be the solutions of boundary value problems (3.10) and (3.11), respectively, we can derive*

$$\eta^2(\lambda_k, u_k, \mathbf{p}) = \eta^2(\lambda_k, u_k, \widehat{\mathbf{p}}) + \|\widehat{\mathbf{p}} - \mathbf{p}\|_{\mathbf{W}}^2 + \|(\mathbf{p} - \widehat{\mathbf{p}})\cdot\nu\|_{0,\partial\Omega}^2, \quad \forall \mathbf{p} \in W, \tag{3.17}$$

$$\eta^{*2}(\lambda_k, u_k^*, \mathbf{p}) = \eta^{*2}(\lambda_k, u_k^*, \widehat{\mathbf{p}}^*) + \|\widehat{\mathbf{p}}^* - \mathbf{p}\|_{\mathbf{W}}^2 + \|(\mathbf{p} - \widehat{\mathbf{p}}^*)\cdot\nu\|_{0,\partial\Omega}^2, \quad \forall \mathbf{p} \in W. \tag{3.18}$$

Proof. Using the definition in (3.4), we have

$$\begin{aligned} & \eta^2(\lambda_k, u_k, \mathbf{p}) = \eta^2(\lambda_k, u_k, \widehat{\mathbf{p}} + \mathbf{p} - \widehat{\mathbf{p}}) \\ = \ & \eta^2(\lambda_k, u_k, \widehat{\mathbf{p}}) + \|\mathrm{div}(\mathbf{p} - \widehat{\mathbf{p}})\|_0^2 + \|\mathbf{p} - \widehat{\mathbf{p}}\|_0^2 + \|(\mathbf{p} - \widehat{\mathbf{p}}) \cdot \nu\|_{0,\partial\Omega}^2 \\ & + 2Re(\mathrm{div}\widehat{\mathbf{p}} + k^2 n(x) u_k, \mathrm{div}(\mathbf{p} - \widehat{\mathbf{p}})) + 2Re(\widehat{\mathbf{p}} - \nabla u_k, \mathbf{p} - \widehat{\mathbf{p}}) \\ & + 2Re\, b(\widehat{\mathbf{p}} \cdot \nu + \lambda_k u_k, (\mathbf{p} - \widehat{\mathbf{p}}) \cdot \nu). \end{aligned} \tag{3.19}$$

From (3.10), there holds

$$(\mathrm{div}\widehat{\mathbf{p}} + k^2 n(x) u_k, \mathrm{div}(\mathbf{p} - \widehat{\mathbf{p}})) + (\widehat{\mathbf{p}} - \nabla u_k, \mathbf{p} - \widehat{\mathbf{p}}) + b(\widehat{\mathbf{p}} \cdot \nu + \lambda_k u_k, (\mathbf{p} - \widehat{\mathbf{p}}) \cdot \nu) = 0. \tag{3.20}$$

Combining (3.19) and (3.20) leads to the result (3.17), and (3.18) can be derived similarly. □

Before we use the a posteriori error estimators (3.4) and (3.5), we still need to prove the efficiency and reliability in the following theorem.

Theorem 3.4. *Suppose (λ_k, u_k) and (λ_k, u_k^*) are approximate eigenpairs of (2.5) and its adjoint problem (2.7). Let $\widehat{\mathbf{p}}$ and $\widehat{\mathbf{p}}^*$ be the solutions of (3.10) and (3.11), then we can derive the following efficiency and reliability*

$$\theta_1 \|u - u_k\|_{1,\Omega} \le \eta(\lambda_k, u_k, \widehat{\mathbf{p}}) \le \theta_2 \|u - u_k\|_{1,\Omega}, \tag{3.21}$$

$$\theta_1^* \|u^* - u_k^*\|_{1,\Omega} \le \eta^*(\lambda_k, u_k^*, \widehat{\mathbf{p}}^*) \le \theta_2^* \|u^* - u_k^*\|_{1,\Omega}, \tag{3.22}$$

where

$$\theta_1 := \frac{1 - C\eta^*(V_k)}{1 + C\sqrt{\eta^*(V_k)}}, \ \theta_1^* := \frac{1 - C\eta(V_k)}{1 + C\sqrt{\eta(V_k)}}, \ \theta_2 := \sqrt{1 + C\eta^*(V_k)}, \ \theta_2^* := \sqrt{1 + C\eta(V_k)}.$$

Furthermore, we have the following asymptotic accuracy

$$\lim_{h_k \to 0} \frac{\eta(\lambda_k, u_k, \widehat{\mathbf{p}})}{\|u - u_k\|_{1,\Omega}} = 1 \qquad \text{and} \qquad \lim_{h_k \to 0} \frac{\eta^*(\lambda_k, u_k^*, \widehat{\mathbf{p}}^*)}{\|u^* - u_k^*\|_{1,\Omega}} = 1. \tag{3.23}$$

Proof. The left sides of (3.21) and (3.22) can be derived by using (3.2) and (3.3) directly . Next we only need to prove the right sides of (3.21) and (3.22). Since $\nabla u \in \mathbf{W}$, we can obtain the following identity based on (2.1) and (3.4)

$$\begin{aligned} \eta^2(\lambda_k, u_k, \nabla u) &= \|k^2 n(x) u_k + \Delta u\|_0^2 + \|\nabla(u - u_k)\|_0^2 + \|\lambda_k u_k + \nabla u \cdot \nu\|_{0,\partial\Omega}^2. \\ &= \|k^2 n(x)(u_k - u)\|_0^2 + \|\nabla(u - u_k)\|_0^2 + \|\lambda_k u_k - \lambda u\|_{0,\partial\Omega}^2. \end{aligned}$$

Thus (3.8) implies

$$\begin{aligned} \eta^2(\lambda_k, u_k, \widehat{\mathbf{p}}) &\le \eta^2(\lambda_k, u_k, \nabla u) \\ &\le \|u - u_k\|_{1,\Omega}^2 + C\eta^*(V_k)\|u - u_k\|_{1,\Omega}^2. \end{aligned}$$

Now we have proved the right side of (3.21), and the right side of (3.22) can be derived similarly. Further, (3.23) can be proved easily based on the fact that $\eta(V_k) \to 0$ and $\eta^*(V_k) \to 0$ as $h_k \to 0$. □

Though we have transformed the optimal problems into two boundary value problems (3.10) and (3.11), we do not know the exact solutions $\widehat{\mathbf{p}}$ and $\widehat{\mathbf{p}}^*$. Thus we need to solve these two boundary value problems numerically to derive some approximate solutions, which can be used to construct the computable error estimators.

Corollary 3.1. *Let $\mathbf{p}_k \in \mathbf{W}$ and $\mathbf{p}_k^* \in \mathbf{W}$ be approximate solutions for (3.10) and (3.11), respectively, then we can derive the computable error estimators as follows*

$$\begin{aligned} \theta_1 \|u - u_k\|_{1,\Omega} &\leq \eta(\lambda_k, u_k, \mathbf{p}_k), &(3.24)\\ \theta_1^* \|u^* - u_k^*\|_{1,\Omega} &\leq \eta^*(\lambda_k, u_k^*, \mathbf{p}_k^*). &(3.25)\end{aligned}$$

Corollary 3.2. *If we solve (3.10) and (3.11) by using higher order finite element method to obtain approximations $\mathbf{p}_k$ and $\mathbf{p}_k^*$ which satisfy $\|\widehat{\mathbf{p}} - \mathbf{p}_k\|_{\mathbf{W}} + \|(\mathbf{p}_k - \widehat{\mathbf{p}}) \cdot \nu\|_{0,\partial\Omega} \leq \gamma \|u - u_k\|_{1,\Omega}$, and $\|\widehat{\mathbf{p}}^* - \mathbf{p}_k^*\|_{\mathbf{W}} + \|(\mathbf{p}_k^* - \widehat{\mathbf{p}}^*) \cdot \nu\|_{0,\partial\Omega} \leq \gamma^* \|u^* - u_k^*\|_{1,\Omega}$ for some constants $\gamma > 0$ and $\gamma^* > 0$. Then there holds*

$$\begin{aligned} \eta(\lambda_k, u_k, \mathbf{p}_k) &\leq \sqrt{\theta_2^2 + \gamma^2} \|u - u_k\|_{1,\Omega}, &(3.26)\\ \eta^*(\lambda_k, u_k^*, \mathbf{p}_k^*) &\leq \sqrt{\theta_2^{*2} + \gamma^{*2}} \|u^* - u_k^*\|_{1,\Omega}. &(3.27)\end{aligned}$$

Furthermore, the a posteriori error estimators $\eta(\lambda_k, u_k, \mathbf{p}_k)$ and $\eta^*(\lambda_k, u_k^*, \mathbf{p}_k^*)$ are asymptotically exact if

$$\lim_{h_k \to 0} \frac{\|\widehat{\mathbf{p}} - \mathbf{p}_k\|_{\mathbf{W}} + \|(\mathbf{p}_k - \widehat{\mathbf{p}}) \cdot \nu\|_{0,\partial\Omega}^2}{\|u - u_k\|_{1,\Omega}} = 0, \qquad (3.28)$$

$$\lim_{h_k \to 0} \frac{\|\widehat{\mathbf{p}}^* - \mathbf{p}_k^*\|_{\mathbf{W}} + \|(\mathbf{p}_k^* - \widehat{\mathbf{p}}^*) \cdot \nu\|_{0,\partial\Omega}^2}{\|u^* - u_k^*\|_{1,\Omega}} = 0. \qquad (3.29)$$

Proof. Using (3.17) and (3.21), we can easily derive

$$\begin{aligned} \eta^2(\lambda_k, u_k, \mathbf{p}_k) &= \eta^2(\lambda_k, u_k, \widehat{\mathbf{p}}) + \|\mathbf{p}_k - \widehat{\mathbf{p}}\|_{\mathbf{W}}^2 + \|(\mathbf{p}_k - \widehat{\mathbf{p}}) \cdot \nu\|_{0,\partial\Omega}^2 \\ &\leq \theta_2^2 \|u - u_k\|_{1,\Omega}^2 + \gamma^2 \|u - u_k\|_{1,\Omega}^2 \\ &\leq (\theta_2^2 + \gamma^2) \|u - u_k\|_{1,\Omega}^2, &(3.30)\end{aligned}$$

which is just (3.26) and the asymptotic accuracy of the a posteriori error estimator follows immediately from (3.28). The conclusions for adjoint problem can be derived using (3.18) and (3.22). □

4 Cascadic AFEM for non-selfadjoint Steklov eigenvalue problem

As the applications of the proposed exact a posteriori error estimator in the last section, we design a novel type of cascadic AFEM in this section based on the combination of cascadic multigrid method, adaptive finite element method and multilevel correction method [30, 47].

Firstly, we briefly recall some preliminaries of the standard AFEM. AFEM is composed of four basic components: Solve, Estimate, Mark, Refine. After solving the concerned partial differential equation on current mesh, we need to compute the local error estimator to choose the mesh elements which need to be refined. Through local refinement, we can derive a new mesh with a better quality. Then we can continue a new solving process till we derive the desired accuracy.

In order to choose the mesh elements with big errors, we need to use a good a posteriori error estimator. Since we have designed an asymptotically exact error indicator in the last section, we can use it directly in mesh adaptive refinement. In order to simplify the description of our novel algorithm, we will use $\eta_T(\lambda_k, u_k, \mathbf{p}_k)$ and $\eta_T^*(\lambda_k, u_k^*, \mathbf{p}_k^*)$ to denote the restriction of the asymptotically exact a posteriori error estimators $\eta(\lambda_k, u_k, \mathbf{p}_k)$ and $\eta^*(\lambda_k, u_k^*, \mathbf{p}_k^*)$ on each mesh element $T \in \mathcal{T}_k$ in the rest of this paper.

In this paper, we use the well-known Dörfler's marking strategy [24] as follows for mesh adaptive refinement.

Dörfler's Marking Strategy

1. Given a mesh adaptive refinement parameter $\theta \in (0,1)$.
2. Construct a minimal subset $\mathcal{M}_k$ from $\mathcal{T}_k$ by selecting some elements in $\mathcal{T}_k$ such that

$$\sum_{T\in\mathcal{M}_k} (\eta_T^2(\lambda_k, u_k, \mathbf{p}_k) + \eta_T^{*2}(\lambda_k, u_k^*, \mathbf{p}_k^*)) \geq \theta(\eta^2(\lambda_k, u_k, \mathbf{p}_k) + \eta^{*2}(\lambda_k, u_k^*, \mathbf{p}_k^*)).$$

3. Mark all the elements in $\mathcal{M}_k$.

4.1 One correction step

In this subsection, we will introduce a correction step, which will help to simplify the description of our cascadic AFEM.

First, we introduce the basic structure of cascadic multigrid method. The cascadic multigrid method only includes some smoothing steps on each level of finite element space. Through adjusting the number of smoothing steps on different spaces appropriately, cascadic multigrid method can derive the optimal error estimate and optimal computational work. The smoothing operator $S_k : V_k \to V_k$ used in cascadic multigrid method satisfies the following estimates

$$\left\{\begin{array}{rcl} \|S_k^m w_k\|_{1,\Omega} & \leq & \frac{C}{m^\alpha}\frac{1}{h_k}\|w_k\|_0, \\ \|S_k^m w_k\|_{1,\Omega} & \leq & \|w_k\|_{1,\Omega}, \\ \|S_k^m(w_k+v_k)\|_{1,\Omega} & \leq & \|S_k^m w_k\|_{1,\Omega} + \|S_k^m v_k\|_{1,\Omega}, \end{array}\right. \tag{4.1}$$

where m denotes the number of smoothing steps, α depends on the type of smoothing operator. It has been proved in [27] that the symmetric Gauss-Seidel, SSOR, damped Jacobi and Richardson iteration satisfy $\alpha = 1/2$ and the conjugate-gradient iteration satisfies $\alpha = 1$ (cf. [43]).

For simplicity, we use the following symbol

$$w_k = SmoothI(V_k, g, f, \xi_k, m, S_k) \tag{4.2}$$

as deriving an approximate solution w_k by executing m times smoother S_k with initial value ξ_k for the following boundary value problem

$$a_s(u_k, v_k) = b(g, v_k) + (f, v_k), \quad \forall v_k \in V_k, \tag{4.3}$$

where

$$a_s(u_k, v_k) = (\nabla u_k, \nabla v_k) + (u_k, v_k). \tag{4.4}$$

Similarly, we use the following symbol

$$w_k = SmoothII(V_k, g, f, \xi_k, m, S_k) \tag{4.5}$$

as deriving an approximate solution w_k by executing m times smoother S_k with initial value ξ_k for the following boundary value problem

$$a_s(v_k, u_k) = b(v_k, g) + (v_k, f), \quad \forall v_k \in V_k. \tag{4.6}$$

Based on these basic knowledge, we then construct a correction step which is the main component of the whole algorithm. The correction step includes some smoothing process for boundary value problems on the adaptive finite element space and solving process for non-selfadjoint Steklov eigenvalue problem and its adjoint problem on a low dimensional space.

Given approximate eigenpairs for non-selfadjoint Steklov eigenvalue problem and its adjoint problem which were denoted by (λ_k, u_k) and (λ_k, u_k^*). The aim of the correction step is to improve the accuracy of the approximations. The new approximate eigenpairs will be denoted by (λ_{k+1}, u_{k+1}) and $(\lambda_{k+1}, u_{k+1}^*)$. We use V_H to denote a low dimensional space which will be introduced later. Then the correction step is defined in Algorithm 1.

Algorithm 1: One correction step

1. Compute $\eta_T^2(\lambda_k, u_k, \mathbf{p}_k) + \eta_T^{*2}(\lambda_k, u_k^*, \mathbf{p}_k^*)$ on each mesh element $T \in \mathcal{T}_k$ according to the given approximate eigenpairs. Then choose the set $\mathcal{M}_k$ through **Dörfler's Marking Strategy**. Next we refine $\mathcal{T}_k$ to get the new mesh $\mathcal{T}_{k+1}$ and construct corresponding finite element space V_{k+1}.

2. Define two boundary value problems in the new space V_{k+1}: Find $\widehat{u}_{k+1} \in V_{k+1}$ such that
$$a_s(\widehat{u}_{k+1}, v_{k+1}) = -b(\lambda_k u_k, v_{k+1}) + ((1 + k^2 n(x))u_k, v_{k+1}), \quad \forall v_{k+1} \in V_{k+1}. \tag{4.7}$$
Find $\widehat{u}_{k+1}^* \in V_{k+1}$ such that
$$a_s(v_{k+1}, \widehat{u}_{k+1}^*) = -b(v_{k+1}, \bar{\lambda}_k u_k^*) + (v_{k+1}, (1 + \bar{k}^2 \bar{n}(x))u_k^*), \quad \forall v_{k+1} \in V_{k+1}. \tag{4.8}$$
Performing smoothing step (4.2) and (4.5) to equations (4.7) and (4.8), respectively. That is, obtain new approximate solutions $\widetilde{u}_{k+1} \in V_{k+1}$ and $\widetilde{u}_{k+1}^* \in V_{k+1}$ by
$$\widetilde{u}_{k+1} = SmoothI(V_{k+1}, -\lambda_k u_k, (1 + k^2 n(x))u_k, u_k, m_{k+1}, S_{k+1}) \tag{4.9}$$
and
$$\widetilde{u}_{k+1}^* = SmoothII(V_{k+1}, -\bar{\lambda}_k u_k^*, (1 + \bar{k}^2 \bar{n}(x))u_k^*, u_k^*, m_{k+1}^*, S_{k+1}). \tag{4.10}$$

3. Define a new space $V_{H,k+1} = V_H + \text{span}\{\widetilde{u}_{k+1}, \widetilde{u}_{k+1}^*\}$ and solve the following non-selfadjoint Steklov eigenvalue problem: Find $(\lambda_{k+1}, u_{k+1}) \in \mathcal{R} \times V_{H,k+1}$ such that
$$a(u_{k+1}, v_{H,k+1}) = -b(\lambda_{k+1} u_{k+1}, v_{H,k+1}), \quad \forall v_{H,k+1} \in V_{H,k+1}, \tag{4.11}$$
and its adjoint problem: Find $(\lambda_{k+1}, u_{k+1}^*) \in \mathcal{R} \times V_{H,k+1}$ such that
$$a(v_{H,k+1}, u_{k+1}^*) = -b(v_{H,k+1}, \bar{\lambda}_{k+1} u_{k+1}^*), \quad \forall v_{H,k+1} \in V_{H,k+1}. \tag{4.12}$$

For simplicity, we summarize the above two steps by
$$(\lambda_{k+1}, u_{k+1}, u_{k+1}^*) = \text{SmoothCorrection}(V_H, \lambda_k, u_k, u_k^*, m_{k+1}, m_{k+1}^*, S_{k+1}, V_{k+1}).$$

Since the dimension of the non-selfadjoint Steklov eigenvalue problem (4.11) and its adjoint problem (4.12) is small compared to the adaptive space with the refinement of mesh, the computational work of the correction step is mainly occupied by the smoothing steps for the involved boundary value problems. Thus we need to discuss how to decide the number of smoothing steps m_{k+1} and m_{k+1}^*. In the classical cascadic multigrid method, the number of smoothing step is proportional to the degrees of freedom, which can make the solution on the finest mesh has optimal error estimate.

In classical cascadic multigrid method, the number of smoothing step m_k on space V_k satisfies

$$m_k = m_\ell\Big(\frac{n_\ell}{n_k}\Big)^{\frac{1}{d\alpha}}, \tag{4.13}$$

where n_k denotes the number of degrees of freedom for V_k, ℓ denotes the index of the final finite element space.

If the meshes are uniformly refined, then we know each n_k in advance. In this case, (4.13) is an implementable strategy. But if we use the adaptive refinement strategy, the degrees of freedom of the later level is unknown, which means the classical strategy (4.13) is not suitable for AFEM. In order to derive an implementable scheme, we set the final space V_ℓ as the first space that the approximate error is below a tolerance TOL. Thus the following relationship holds

$$\frac{\|u-u_k\|_{1,\Omega}}{\text{TOL}} \approx \Big(\frac{n_\ell}{n_k}\Big)^{\frac{1}{d}}, \tag{4.14}$$

which motivate us to replace (4.13) by

$$m_k = m_\ell\Big(\frac{\|u-u_k\|_{1,\Omega}}{\text{TOL}}\Big)^{\frac{1}{\alpha}}. \tag{4.15}$$

But we still do not know the a priori error estimate $\|u-u_k\|_{1,\Omega}$ in (4.15) since the exact solution u is unknown. Thus, we further approximate $\|u-u_k\|_{1,\Omega}$ by the following equation

$$\|u-u_k\|_{1,\Omega} \approx \|u-u_{k-1}\|_{1,\Omega}\Big(\frac{n_{k-1}}{n_k}\Big)^{\frac{1}{d}}. \tag{4.16}$$

Though we can not derive the exact estimate for $\|u-u_{k-1}\|_{1,\Omega}$, we can approximate it by the asymptotically exact a posteriori error estimator $\eta(\lambda_{k-1}, u_{k-1}, \mathbf{p}_{k-1})$.

Finally, the number of smoothing step is chosen as follows

$$m_k = m_\ell\Big(\frac{\eta(\lambda_{k-1}, u_{k-1}, \mathbf{p}_{k-1})}{\text{TOL}}\big(\frac{n_{k-1}}{n_k}\big)^{\frac{1}{d}}\Big)^{\frac{1}{\alpha}}. \tag{4.17}$$

Similarly, for the adjoint problem, the number of smoothing step is chosen as follows

$$m_k^* = m_\ell^*\Big(\frac{\eta^*(\lambda_{k-1}, u_{k-1}^*, \mathbf{p}_{k-1}^*)}{\text{TOL}}\big(\frac{n_{k-1}}{n_k}\big)^{\frac{1}{d}}\Big)^{\frac{1}{\alpha}}, \tag{4.18}$$

and the good performance of these strategies are examined by a variety of numerical experiments.

4.2 Cascadic adaptive finite element method

In this subsection, we will design a novel cascadic AFEM for the non-selfadjoint Steklov eigenvalue problem based on the correction step define in Algorithm 1. The basic loop of the novel cascadic adaptive algorithm can be described as follows: First we need to solve the non-selfadjoint Steklov eigenvalue problem and its adjoint problem in the initial space. Then based on the approximate solutions, we can execute the correction step constructed in Algorithm 1 till we derive the desired accuracy.

Based on the basic loop described above, and the cascadic type of correction step defined in Algorithm 1, the novel cascadic adaptive algorithm is described in Algorithm 2.

Algorithm 2: Cascadic AFEM for non-selfadjoint Steklov eigenvalue problem

1. Given a refinement parameter $0 < \theta < 1$. Generate a coarse mesh $\mathcal{T}_H$ on the computing domain Ω with mesh size H. Generate the initial mesh $\mathcal{T}_1$ through refining $\mathcal{T}_H$ by some times in the uniform way and construct the initial space V_1. Solve the following non-selfadjoint Steklov eigenvalue problem and its adjoint problem on V_1: Find $(\lambda_1, u_1) \in \mathcal{R} \times V_1$ such that $b(u_1, u_1) = 1$ and
$$a(u_1, v_1) = -\lambda_1 b(u_1, v_1), \quad \forall v_1 \in V_1.$$
Find $(\lambda_1, u_1^*) \in \mathcal{R} \times V_1$ such that $b(u_1^*, u_1^*) = 1$ and
$$a(v_1, u_1^*) = -\lambda_1 b(v_1, u_1^*), \quad \forall v_1 \in V_1.$$
2. Set $k = 1$.
3. Obtain new approximate solutions by Algorithm 1:
$$(\lambda_{k+1}, u_{k+1}, u_{k+1}^*) = \text{SmoothCorrection}(V_H, \lambda_k, u_k, u_k^*, m_{k+1}, m_{k+1}^*, S_{k+1}, V_{k+1}).$$
4. Let $k = k + 1$ and go to step 3.

In the last part of this section, we briefly estimate the computational work involved in Algorithm 2. In order to derive a theoretical result, we have to use additionally, that the sequence of unknowns belong to a geometric progression:
$$n_k < \sigma_0 n_k \leq n_{k+1} < \sigma_1 n_k, \quad k = 1, 2, \cdots. \tag{4.19}$$

Theorem 4.1. *Assume the non-selfadjoint Steklov eigenvalue problem and its adjoint problem in the coarse spaces V_H and V_1 need work M_H and M_1, respectively. If $d > 1/\alpha$, the computational work involved in Algorithm 2 is $O\big(M_1 + M_H \log(n_\ell) + n_\ell\big)$ and furthermore $O\big(n_\ell\big)$ provided M_H and M_1 are small enough. While if $d = 1/\alpha$, the computational work are $O\big(M_1 + M_H \log(n_\ell) + n_\ell \log(n_\ell)\big)$ and $\mathcal{O}\big(n_\ell \log(n_\ell)\big)$, respectively.*

Proof. Let W denote the computational work involved in Algorithm 2 and w_k denote the computational work involved in the k-th level for $k = 1, \cdots, n$. Combining Algorithm 2, (4.13) and (4.19), we have
$$\begin{aligned} W &= \sum_{k=1}^{\ell} w_k = \mathcal{O}\big(M_1 + \sum_{k=2}^{\ell}((m_k + m_k^*)n_k + M_H)\big) \\ &= \mathcal{O}\big(M_1 + \sum_{k=2}^{\ell}(m_k + m_k^*)n_k + M_H(\ell - 1)\big) \\ &= \mathcal{O}\left(M_1 + M_H \log(n_\ell) + (m_\ell + m_\ell^*)n_\ell \sum_{k=2}^{\ell}\Big(\frac{1}{\sigma_0}\Big)^{(\ell-k)(1-\frac{1}{d\alpha})}\right). \end{aligned}$$

So the computational work of Algorithm 2 is $O\big(M_1 + M_H \log(n_\ell) + n_\ell\big)$ when $d > 1/\alpha$ and $O\big(M_1 + M_H \log(n_\ell) + n_\ell \log(n_\ell)\big)$ when $d = 1/\alpha$. Further, the computational work become $\mathcal{O}(n_\ell)$ and $\mathcal{O}(n_\ell \log(n_\ell))$ if M_H and M_1 are small enough. $\square$

5 Numerical experiments

In this section, we use three numerical experiments to verify the theoretical analysis and show the efficiency of our novel cascadic AFEM. In our numerical experiments, we choose conjugate-gradient iteration as the smoother ($\alpha = 1$) with the iteration steps (4.17) and (4.18), $m_\ell = m_\ell^* = 2$.

In order to derive $\eta(\lambda_k, u_k, \mathbf{p}_k)$ and $\eta^*(\lambda_k, u_k^*, \mathbf{p}_k^*)$, we should solve two boundary value problems (3.10) and (3.11) to get approximations $\mathbf{p}_k$ and $\mathbf{p}_k^*$. In our numerical examples, mesh $\mathcal{T}_k$ for non-selfadjoint Steklov eigenvalue problem is used for the boundary value problems (3.10) and (3.11) with finite element space W_k^p defined as follows [15]:

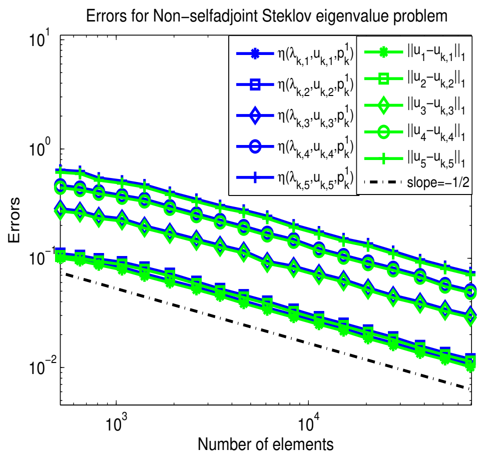
$$W_k^p = \{w \in \mathbf{W} : w|_K \in RT_p, \forall K \in \mathcal{T}_k\}, \tag{5.1}$$

where $RT_p = (\mathcal{P}_p)^d + x\mathcal{P}_p$.

5.1 Example 1

In the first numerical experiment, we aim to test the asymptotic accuracy of the proposed a posteriori error estimators. We solve the non-selfadjoint Steklov eigenvalue problem and its adjoint problem on the square domain $\Omega = (0, 1) \times (0, 1)$ with $k = 1$, $n(x) = 4 + 4i$.

In order to test the asymptotic accuracy of the proposed a posteriori error estimators, we solve the first five eigenpairs of non-selfadjoint Steklov eigenvalue problem and its adjoint problem by Algorithm 2 with the mesh adaptive refinement parameter $\theta = 0.4$. In this numerical experiment, we use the linear finite element space for the non-selfadjoint Steklov eigenvalue problem and its adjoint problem. W_k^0 and W_k^1 are used for the boundary value problems (3.10) and (3.11). Since the exact eigenpairs are unknown, we choose an approximate solution on a sufficiently fine mesh as the exact one. The comparisons between $u - u_k$ and $\eta(\lambda_k, u_k, \mathbf{p}_k)$, $u^* - u_k^*$ and $\eta^*(\lambda_k, u_k^*, \mathbf{p}_k^*)$ are presented in Figures 1 and 2, which show that $\eta(\lambda_k, u_k, \mathbf{p}_k)$ and $\eta^*(\lambda_k, u_k^*, \mathbf{p}_k^*)$ are asymptotically exact when we solve the boundary value problems (3.10) and (3.11) in the space W_k^1.



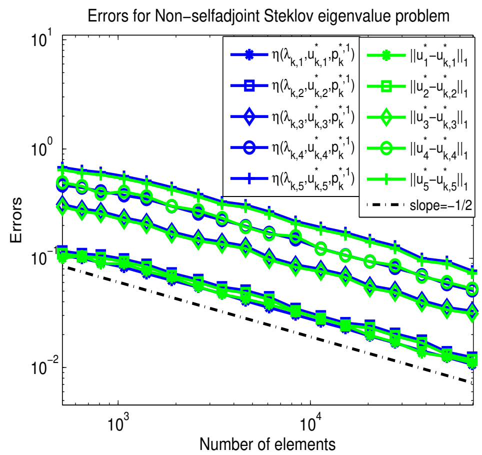


Figure 1: The errors of the first five approximate eigenpairs when the space W_k^1 is adopted

Moreover, we also compare the a posteriori error estimator proposed in our paper and the classical residual type the a posteriori error estimator defined as follows: First, we construct the element residual $\mathcal{R}_T(\lambda_k, u_k)$ and the jump residual $\mathcal{J}_e(\lambda_k, u_k)$ for the approximate eigenpair (λ_k, u_k) as follows:

$$\mathcal{R}_T(\lambda_k, u_k) := \Delta u_k + k^2 n(x) u_k, \qquad \text{in } T \in \mathcal{T}_k, \tag{5.2}$$

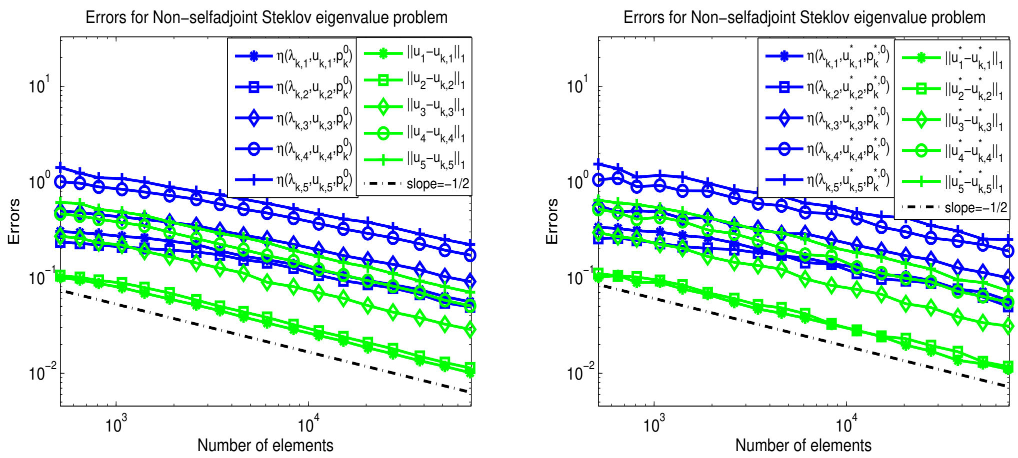


Figure 2: The errors of the first five approximate eigenpairs when the space W_k^0 is adopted

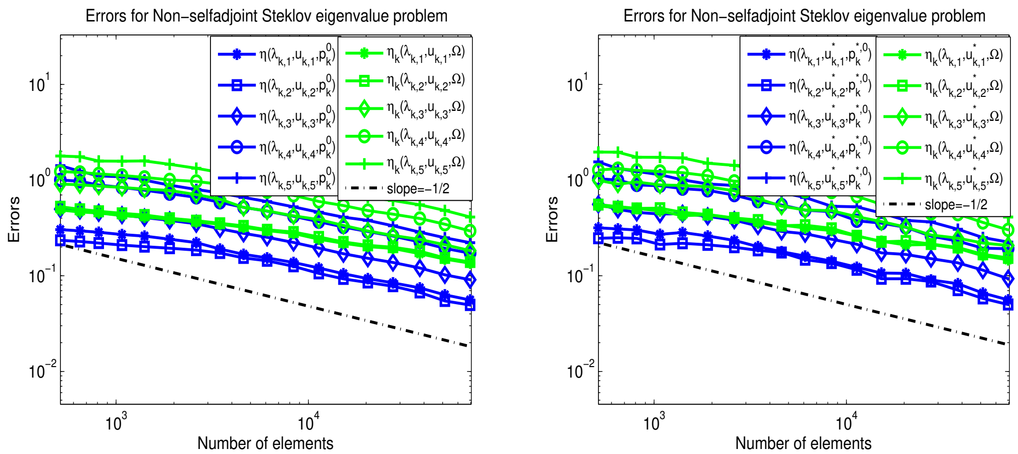


Figure 3: The comparisions between the asymptotically exact a posteriori error estimator and the residual type a posteriori error estimator

$$\mathcal{J}_e(\lambda_k, u_k) := \begin{cases} \frac{1}{2}(\nabla u_k|_{T^+}\cdot\nu^+ + \nabla u_k|_{T^-}\cdot\nu^-), & \text{for} \quad e\in\mathcal{E}_k, \\ \nabla u_k\cdot\nu + \lambda_k u_k, & \text{for} \quad e\in\mathcal{E}_{\partial\Omega}, \end{cases} \tag{5.3}$$

and for adjoint problem, we construct the element residual $\mathcal{R}_T^*(\lambda_k, u_k^*)$ and the jump residual $\mathcal{J}_e^*(\lambda_k, u_k^*)$ for its approximate eigenpair (λ_k, u_k^*) as follows:

$$\mathcal{R}_T^*(\lambda_k, u_k^*) := \Delta u_k^* + \bar{k}^2\bar{n}(x)u_k^*, \qquad \text{in } T\in\mathcal{T}_k, \tag{5.4}$$

$$\mathcal{J}_e^*(\lambda_k, u_k^*) := \begin{cases} \frac{1}{2}(\nabla u_k^*|_{T^+}\cdot\nu^+ + \nabla u_k^*|_{T^-}\cdot\nu^-), & \text{for} \quad e\in\mathcal{E}_k, \\ \nabla u_k^*\cdot\nu + \bar{\lambda}_k u_k^*, & \text{for} \quad e\in\mathcal{E}_{\partial\Omega}, \end{cases} \tag{5.5}$$

where $\mathcal{E}_k$ is the set of interior edges of $\mathcal{T}_k$. For $T\in\mathcal{T}_k$, we define the local error indicators $\eta_k(\lambda_k, u_k, T)$ and $\eta_k^*(\lambda_k, u_k^*, T)$ by

$$\eta_k^2(\lambda_k, u_k, T) := h_T^2\|\mathcal{R}_T(\lambda_k, u_k)\|_{0,T}^2 + \sum_{e\in\mathcal{E}_k, e\subset\partial T} h_e\|\mathcal{J}_e(\lambda_k, u_k)\|_{0,e}^2 \tag{5.6}$$

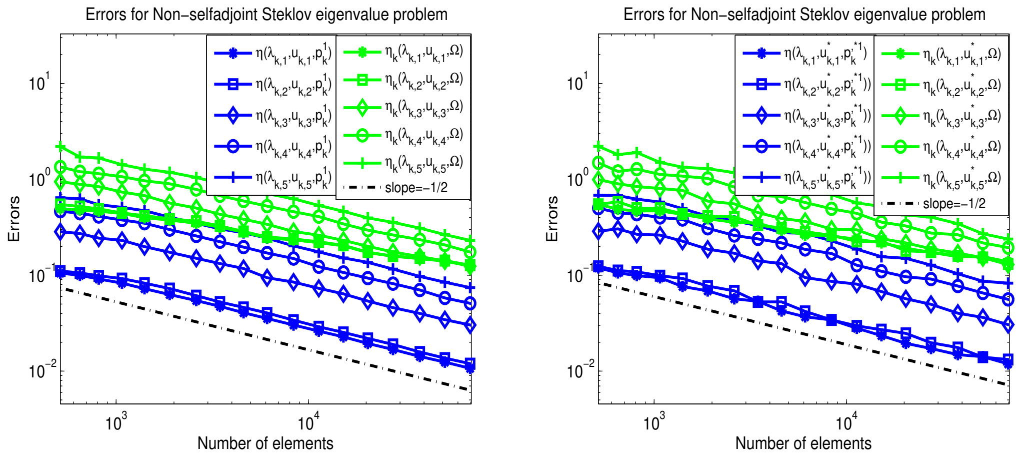


Figure 4: The comparisions between the asymptotically exact a posteriori error estimator and the residual type a posteriori error estimator

and

$$\eta_k^{*2}(\lambda_k, u_k^*, T) := h_T^2 \|\mathcal{R}_T^*(\lambda_k, u_k^*)\|_{0,T}^2 + \sum_{e\in\mathcal{E}_k, e\subset\partial T} h_e \|\mathcal{J}_e^*(\lambda_k, u_k^*)\|_{0,e}^2. \tag{5.7}$$

Given a subset $\omega \subset \Omega$, we define $\eta_k^2(\lambda_k, u_k, \omega)$ and $\eta_k^{*2}(\lambda_k, u_k^*, \omega)$ by

$$\eta_k^2(\lambda_k, u_k, \omega) = \sum_{T\in\mathcal{T}_k, T\subset\omega} \eta_k^2(\lambda_k, u_k, T) \tag{5.8}$$

and

$$\eta_k^{*2}(\lambda_k, u_k^*, \omega) = \sum_{T\in\mathcal{T}_k, T\subset\omega} \eta_k^{*2}(\lambda_k, u_k^*, T). \tag{5.9}$$

The comparisons between our error estimators and the residual type a posteriori error estimators are presented in Figures 3 and 4, which show that our error estimators own a better accuracy.

5.2 Example 2

In the second example, we solve the non-selfadjoint Steklov eigenvalue problem and its adjoint problem on the L-shape domain $\Omega = (-1,1)\times(-1,1)/[0,1]\times[-1,0]$ with $k = 1$ and $n(x) = 1 + i$. We solve the first five eigenvalues of non-selfadjoint Steklov eigenvalue problem by Algorithm 2 with the mesh adaptive refinement parameter $\theta = 0.3$. The number of smoothing steps is chosen according to (4.17) and (4.18). In this numerical experiment, we use the linear finite element space for the non-selfadjoint Steklov eigenvalue problem and its adjoint problem. W_k^1 are used for the boundary value problems (3.10) and (3.11).

The corresponding numerical results are presented in Figures 5 and 6, which shows that Algorithm 2 is able to obtain the optimal error estimate. In order to show the efficiency of Algorithm 2 more clear, we present the computational time of Algorithm 2 and the standard AFEM (i.e. solve the non-selfadjoint Steklov eigenvalue problem and its adjoint problem directly on each adaptive space) in Figure 7. From Figure 7, we can see that Algorithm 2 has a great advantage over the standard AFEM. Besides, the computational work of Algorithm 2 is quasi-linear with respect to the number of elements.

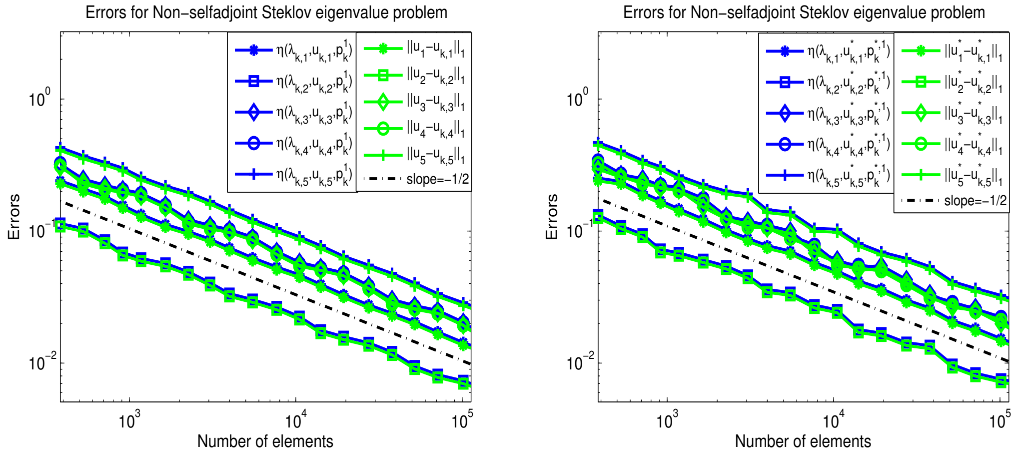


Figure 5: The errors of the Algorithm 2 and direct AFEM for the first five eigenpairs,

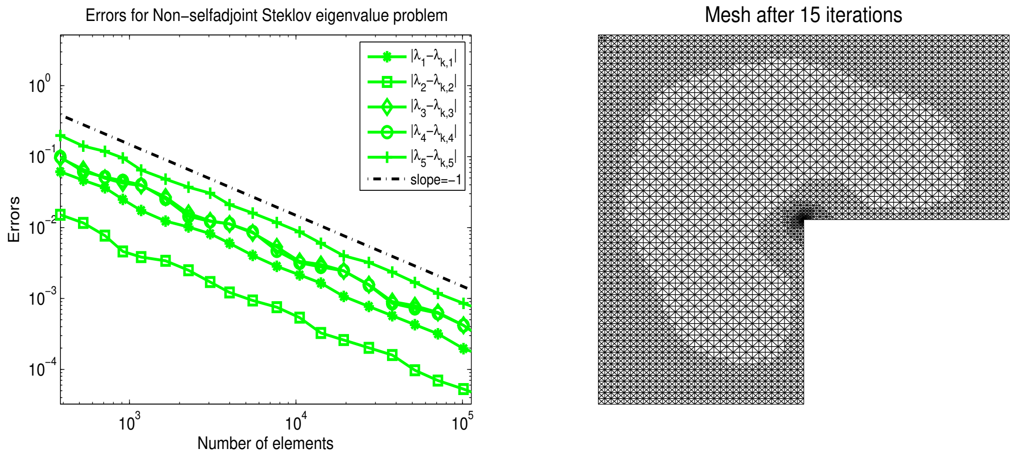


Figure 6: Left: The errors of the Algorithm 2 and direct AFEM for the first five eigenpairs. Right: The mesh after 15 times adaptive refinement.

5.3 Example 3

In the last example, we solve the non-selfadjoint Steklov eigenpairs and its adjoint problem on the same computing domain as example 2 with $k = 2$, $n(x) = 2 + 2i$ and $\theta = 0.3$. The number of smoothing steps is chosen according to (4.17) and (4.18). In this numerical experiment, we use the linear finite element space for the non-selfadjoint Steklov eigenvalue problem and its adjoint problem. W_k^1 are used for the boundary value problems (3.10) and (3.11).

We solve the first five eigenvalues of non-selfadjoint Steklov eigenvalue problem by Algorithm 2 and the corresponding numerical results are presented in Figures 8 and 9, which show that Algorithm 2 is able to obtain the optimal error estimate. In order to show the efficiency of Algorithm 2 more clear, we also presented the computational time of Algorithm 2 and the standard AFEM in Figure 10. From Figure 10, we can see that Algorithm 2 has a great advantage over the standard AFEM. Meanwhile, Algorithm 2 has linear computational complexity.

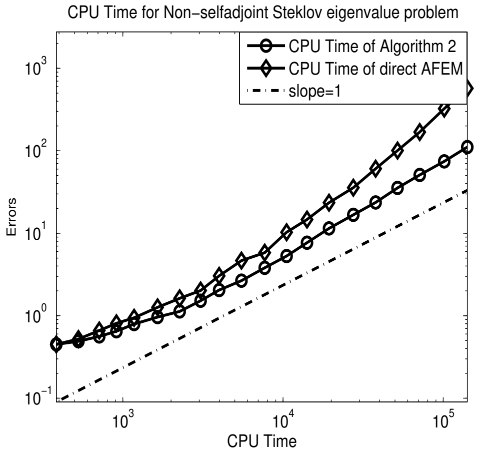


Figure 7: The computational time of the Algorithm 2 and direct AFEM for the first five eigenpairs,

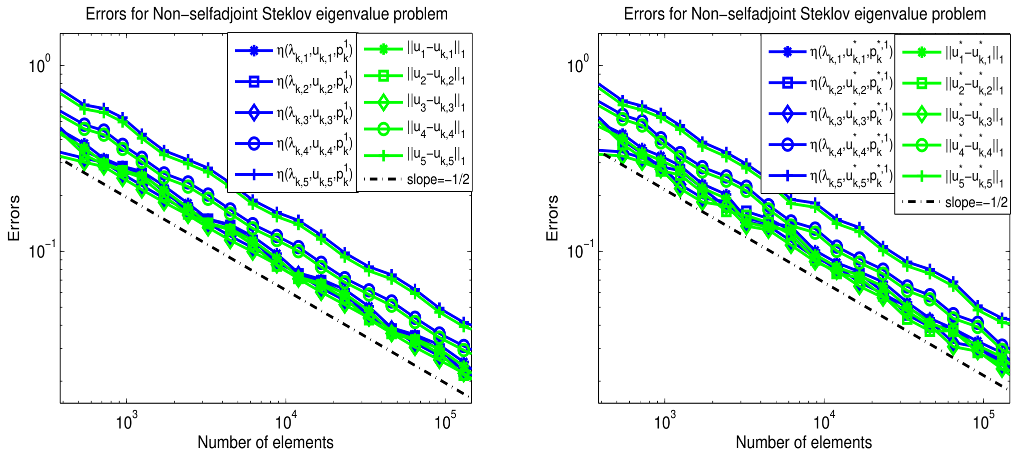


Figure 8: The errors of the Algorithm 2 and direct AFEM for the first five eigenpairs,

6 Concluding remarks

In this paper, we propose an asymptotically exact a posteriori error estimator for the non-selfadjoint Steklov eigenvalue problem and its adjoint problem. Besides, as its applications, we design a novel type of cascadic AFEM, which has a good advantage over the classical AFEM. The new scheme only includes some smoothing steps for involved boundary value problems and solving process for a low dimensional non-selfadjoint Steklov eigenvalue problem and its adjoint problem. Since we do not solve large scale non-selfadjoint Steklov eigenvalue problem directly, the algorithm significantly improves the efficiency.

In the last, we want to mention another application of the proposed a posteriori error estimator. Since the upper bounds for the error estimates of the eigenfunction approximations have been derived in Section 3, we now present a computable error estimate for the eigenvalue approximations of non-selfadjoint Steklov eigenvalue problem. The process can be proved directly by using the Rayleigh quotient expansion for non-selfadjoint Steklov eigenvalue problem.

Theorem 6.1. *Let* $(\lambda_k, u_k) \in \mathcal{C} \times V_k$ *and* $(\lambda_k, u_k^*) \in \mathcal{C} \times V_k$ *be the solutions corresponding to the discrete problems (2.5) and (2.7), respectively. If* $b(u_k, u_k^*) = 1$*, we have the following error*

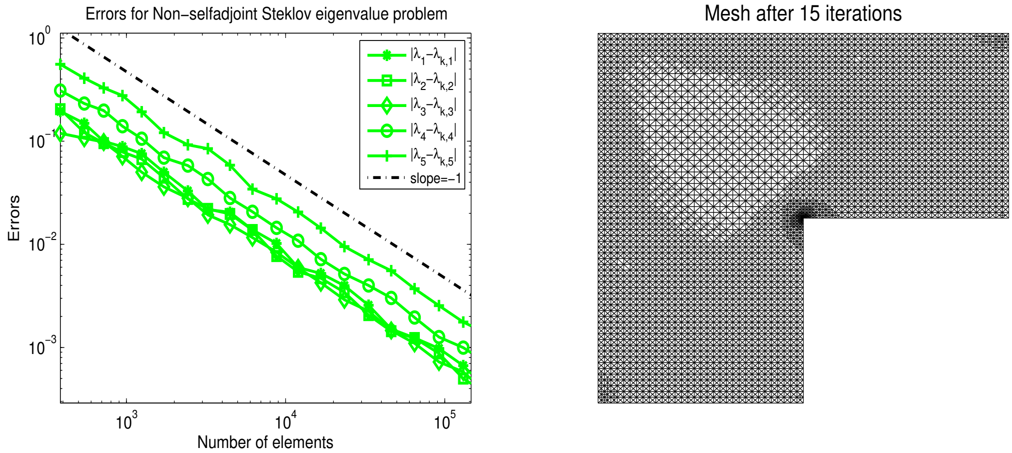


Figure 9: Left: The errors of the Algorithm 2 and direct AFEM for the first five eigenpairs. Right: The mesh after 15 times adaptive refinement.

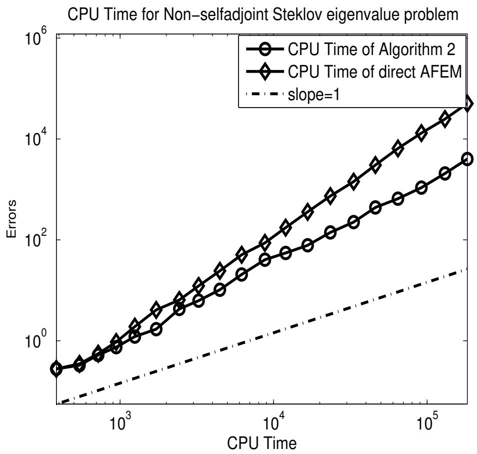


Figure 10: The computational time of the Algorithm 2 and direct AFEM for the first five eigenpairs,

estimate

$$|\lambda_k - \lambda| \leq \theta(h)\eta(\lambda_k, u_k, \mathbf{p}_k)\eta^*(\lambda_k, u_k^*, \mathbf{p}_k^*), \tag{6.1}$$

where $\theta(h) = \frac{\sqrt{(1+C\eta(V_k))(1+C\eta^*(V_k))(1+C\eta(V_k)\eta^*(V_k))}}{(1-C\eta(V_k))(1-C\eta^*(V_k))}$ *and* $\theta(h) \to 1$ *as* $h \to 0$.

Proof. Using Lemma 2.1, we have

$$\begin{aligned} |\lambda_k - \lambda| &= |a(u_k - u, u_k^* - u^*) + \lambda b(u_k - u, u_k^* - u^*)| \\ &= |(\nabla(u_k - u), \nabla(u_k^* - u^*)) - (k^2 n(x)(u_k - u), (u_k^* - u^*)) + \lambda b(u_k - u, u_k^* - u^*)| \\ &\leq \|u_k - u\|_{1,\Omega}\|u_k^* - u^*\|_{1,\Omega} + C\sqrt{\eta(V_k)\eta^*(V_k)}\|u_k - u\|_{1,\Omega}\|u_k^* - u^*\|_{1,\Omega} \\ &\leq \sqrt{1 + C\eta(V_k)\eta^*(V_k)}\|u_k - u\|_{1,\Omega}\|u_k^* - u^*\|_{1,\Omega}. \end{aligned} \tag{6.2}$$

Combining (3.24), (3.25) and (6.2) leads to the desired result. □

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