Derivation EOMs of 2-DOF Pendularm

FENG Chenxi
University of Michigan, Ann Arbor
Email: chenxif@umich.edu

I. Equations of Motion for 2-DOF Pendularm

A double link pendulum is shown in 1. Both pendulum is modeled as simple pendulum, which means the rod is weightless and all the mass is concentrated in the bob. Here, we use the generalized coordinates to represent the position of two bobs.

$$x_1 = l_1 \sin\alpha_1 \tag{1}$$

$$x_2 = l_1 \sin\alpha_1 + l_2 \sin\alpha_2 \tag{2}$$

$$y_1 = -l_1 \cos\alpha_1 \tag{3}$$

$$y_2 = -l_1 \cos\alpha_1 - l_2 \cos\alpha_2 \tag{4}$$

Apply the Lagrangian equations to the system. The kinematics and potential energy are shown as:

$$T_1 = \frac{m_1 v_1^2}{2} = \frac{m_1(\dot{x}_1^2 + \dot{y}_1^2)}{2} \tag{5}$$

$$T_2 = \frac{m_2 v_2^2}{2} = \frac{m_2(\dot{x}_2^2 + \dot{y}_2^2)}{2} \tag{6}$$

$$V_1 = m_1 g y_1 \tag{7}$$

$$V_2 = m_2 g y_2 \tag{8}$$

Then, the Lagrangian is shown as

$$L = T - V = T_1 + T_2 - (V_1 + V_2) \tag{9}$$

By substituting the energy into it

$$L = \frac{m_1(\dot{x}_1^2 + \dot{y}_1^2)}{2} + \frac{m_2(\dot{x}_2^2 + \dot{y}_2^2)}{2} - m_1 g y_1 - m_2 g y_2 \tag{10}$$

o get v_1 and v_2, we calculate the derivatives as,

$$\dot{x}_1 = l_1 \cos\alpha_1 \cdot \dot{\alpha}_1 \tag{11}$$

$$\dot{x}_2 = l_1 \cos\alpha_1 \cdot \dot{\alpha}_1 + l_2 \cos\alpha_2 \cdot \dot{\alpha}_2 \tag{12}$$

$$\dot{y}_1 = -l_1 \sin\alpha_1 \cdot \dot{\alpha}_1 \tag{13}$$

$$\dot{y}_2 = -l_1 \sin\alpha_1 \cdot \dot{\alpha}_1 - l_2 \sin\alpha_2 \cdot \dot{\alpha}_2 \tag{14}$$

Combine Equation 5 to 14,

$$\begin{aligned} L = &\frac{(m_1+m_2)}{2} l_1^2 \dot{\alpha}_1^2 + \frac{m_2}{2} l_2^2 \dot{\alpha}_2^2 + m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \cos(\alpha 1 - \alpha 2) \\ &+ (m_1+m_2) g l_1 \cos\alpha_1 + m_2 g l_2 \cos\alpha_2 \end{aligned} \tag{15}$$

Based on Lagrangian equations of motion shown in Equation 16 and 17

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}_1} - \frac{\partial L}{\partial \alpha_1} = \tau_1 - \tau_2 \tag{16}$$

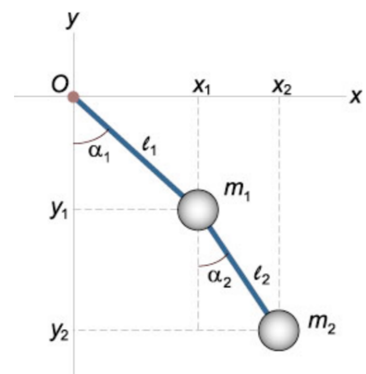

Fig. 1. Model of 2-DOF Pendularm

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}_2} - \frac{\partial L}{\partial \alpha_2} = \tau_2 \tag{17}$$

The next step is to find the derivative of L,

$$\frac{\partial L}{\partial \dot{\alpha}_1} = (m_1+m_2) l_1^2 \dot{\alpha}_1 + m_2 l_1 l_2 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2) \tag{18}$$

$$\frac{\partial L}{\partial \dot{\alpha}_2} = m_2 l_2^2 \dot{\alpha}_2 + m_2 l_1 l_2 \dot{\alpha}_1 \cos(\alpha_1 - \alpha_2) \tag{19}$$

$$\frac{\partial L}{\partial \alpha_1} = -m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) - (m_1+m_2) g l_1 \sin\alpha_1 \tag{20}$$

$$\frac{\partial L}{\partial \alpha_2} = m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) - m_2 g l_2 \sin\alpha_2 \tag{21}$$

Combine Equation 18 to 21 with 16 and 17,we can get the equations of motion for the two pendulum bobs.

$$\begin{aligned} &l_2 \ddot{\alpha}_2 + l_1 \ddot{\alpha}_1 \cos(\alpha_1 - \alpha_2) - l_1 \dot{\alpha}_1^2 \sin(\alpha_1 - \alpha_2) \\ &+ g \sin\alpha_2 - \frac{\tau_2}{m_2 l_2} = 0 \end{aligned} \tag{22}$$

$$\begin{aligned} &(m_1+m_2) l_1 \ddot{\alpha}_1 + m_2 l_2 \ddot{\alpha}_2 \cos(\alpha_1 - \alpha_2) \\ &+ m_2 l_2 \dot{\alpha}_2^2 \sin(\alpha_1 - \alpha_2) \\ &+ (m_1+m_2) g \sin\alpha_1 - \frac{\tau_1 - \tau_2}{l_1} = 0 \end{aligned} \tag{23}$$

From Equation 22, we can get the angular acceleration of the second pendulum.

$$\begin{aligned} \ddot{\alpha}_2 = &\frac{l_1}{l_2} \dot{\alpha}_1^2 \sin(\alpha_1 - \alpha_2) - \frac{l_1}{l_2} \ddot{\alpha}_1 \cos(\alpha_1 - \alpha_2) \\ &- \frac{g}{l_2} \sin(\alpha_2) + \frac{\tau_2}{m_2 l_2^2} \end{aligned} \tag{24}$$

Combine Equation 23 with Equation 24 and extract $\ddot{\alpha}_1$,

$$\begin{aligned}
&[(m_1+m_2)l_1 - m_2 l_1 \cos{(\alpha_1-\alpha_2)}^2]\ddot{\alpha}_1 \\
&\quad = (\frac{\tau_1-\tau_2}{l_1} - \frac{\tau_2}{l_2}\cos{(\alpha_1-\alpha_2)}) \\
&\qquad - m_2 l_1 \dot{\alpha}_1^2 \sin{(\alpha_1-\alpha_2)}\cos{(\alpha_1-\alpha_2)} \\
&\qquad + m_2 g \sin\alpha_2 \cos{(\alpha_1-\alpha_2)} \\
&\qquad - m_2 l_2 \dot{\alpha}_2^2 \sin{(\alpha_1-\alpha_2)} - (m_1+m_2) g \sin\alpha_1
\end{aligned} \tag{25}$$

$$\begin{aligned}
&\Rightarrow \ddot{\alpha}_1 \\
&= (\frac{\tau_1-\tau_2}{l_1} - \frac{\tau_2}{l_2}\cos{(\alpha_1-\alpha_2)}) \\
&\quad - m_2 l_1 \dot{\alpha}_1^2 \sin{(\alpha_1-\alpha_2)}\cos{(\alpha_1-\alpha_2)} \\
&\quad + m_2 g \sin\alpha_2 \cos{(\alpha_1-\alpha_2)} - m_2 l_2 \dot{\alpha}_2^2 \sin{(\alpha_1-\alpha_2)} \\
&\quad - (m_1+m_2) g \sin\alpha_1 / [(m_1+m_2)l_1 - m_2 l_1 \cos{(\alpha_1-\alpha_2)}^2]
\end{aligned} \tag{26}$$

Putting the $\ddot{\alpha}_1$ into 23 , we can get $\ddot{\alpha}_2$

$$\begin{aligned}
&l_2[m_1 + m_2 \sin{(\alpha_1-\alpha_2)}^2]\ddot{\alpha}_2 \\
&\quad = -\frac{\tau_1-\tau_2}{l_1}\cos{(\alpha_1-\alpha_2)} \\
&\qquad + (m_1+m_2) g \sin\alpha_1 \cos{(\alpha_1-\alpha_2)} \\
&\qquad + m_2 l_2 \dot{\alpha}_2^2 \sin{(\alpha_1-\alpha_2)}\cos{(\alpha_1-\alpha_2)} + \frac{(m_1+m_2)\tau_2}{m_2 l_2} \\
&\qquad - (m_1+m_2) g \sin\alpha_2 + (m_1+m_2) l_1 \dot{\alpha}_1^2 \sin{(\alpha_1-\alpha_2)}
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\Rightarrow \ddot{\alpha}_2 \\
&\quad = -\frac{\tau_1-\tau_2}{l_1}\cos{(\alpha_1-\alpha_2)} \\
&\qquad + (m_1+m_2) g \sin\alpha_1 \cos{(\alpha_1-\alpha_2)} \\
&\qquad + m_2 l_2 \dot{\alpha}_2^2 \sin{(\alpha_1-\alpha_2)}\cos{(\alpha_1-\alpha_2)} \\
&\qquad + \frac{(m_1+m_2)\tau_2}{m_2 l_2} - (m_1+m_2) g \sin\alpha_2 + (m_1 \\
&\qquad\quad + m_2) l_1 \dot{\alpha}_1^2 \sin{(\alpha_1-\alpha_2)} / l_2[m_1 + m_2 \sin{(\alpha_1-\alpha_2)}^2]
\end{aligned} \tag{28}$$