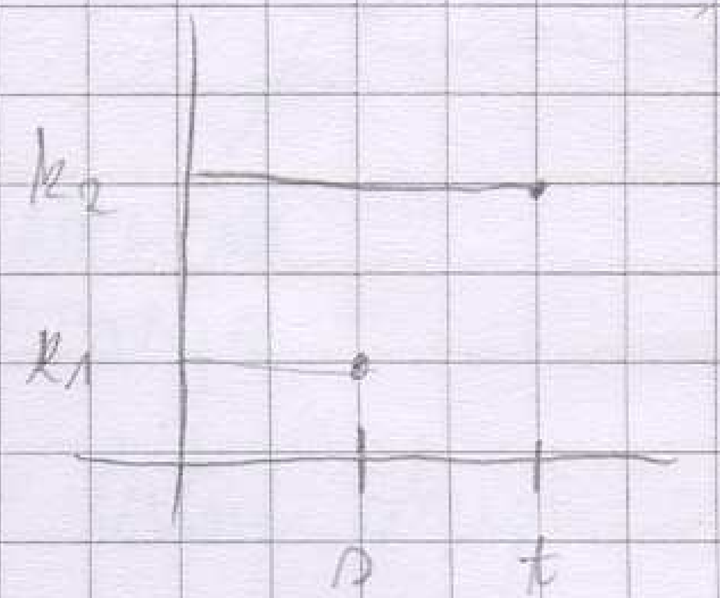
FALI JEDNO PREDAVANJE

Poissonov proces - nastavak

Poissonov proces N_t je okarakteriziran sa

1) $N_0 = 0$
2) N ima nezavisne priraste
3) razdioba $N_t - N_s$ je Poissonova s parametrom $\lambda(t-s)$, $0 \le s < t$

$$P(N_t - N_s = k) = \frac{[\lambda(t-s)]^k}{k!} e^{-\lambda(t-s)}$$



$$P\big((N_s, N_t) = (k_1, k_2)\big) = P(N_s = k_1, N_t = k_2), \quad k_1 \le k_2, \quad s < t$$

$$= \cancel{P}(N_t = k_2, N_t - N_s = k_2 - k_1) = P(N_s = k_1, N_t - N_s = k_2 - k_1) =$$

$$= P(N_s = k_1) \cdot P(N_t - N_s = k_2 - k_1) = \frac{(\lambda s)^{k_1}}{k_1!} e^{-\lambda s} \cdot \frac{[\lambda(t-s)]^{k_2 - k_1}}{(k_2 - k_1)!} \cdot e^{-\lambda(t-s)}$$

$$= \frac{\lambda^{k_2} s^{k_1} (t-s)^{k_2 - k_1}}{k_1! (k_2 - k_1)!} \cdot e^{-\lambda t}$$

PRIMJER:

broj poziva

očekivani broj poziva u min = 1,2

$N_2 = 2$, $N_4 = 3$ - vjerojatnost? $\Rightarrow k_1 = 2, k_2 = 3$

$s = 2$, $t = 4$

$E(N_1) = \overset{\lambda \cdot t}{\lambda \cdot 1} = 1{,}2 \Rightarrow \lambda = 1{,}2$

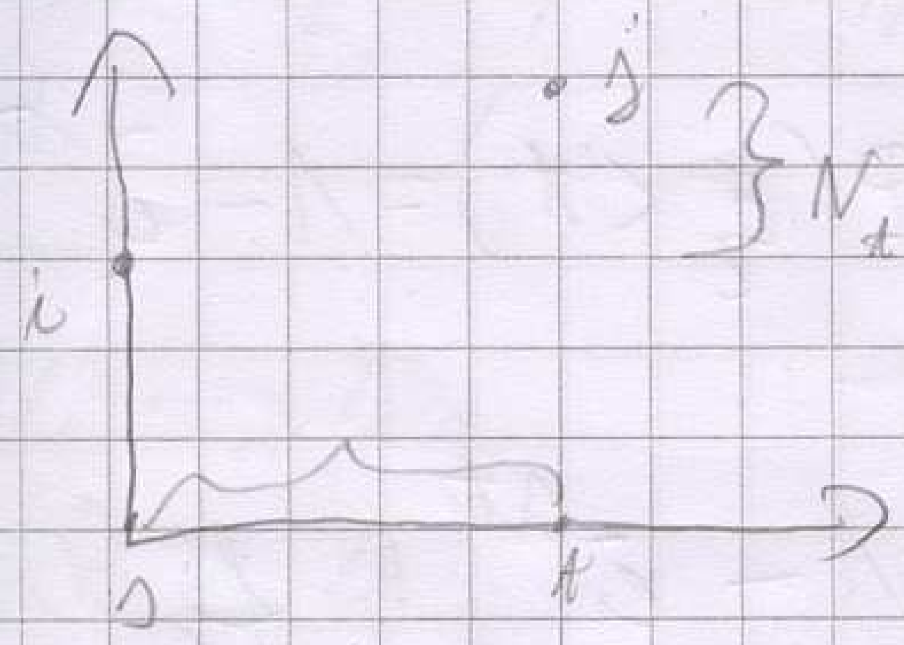
$P(N_2 = 2, N_4 = 3) =$

$\dots = \dfrac{1{,}2^3 \quad 2^2 \quad 2^1}{2! \cdot 1!} \cdot e^{-1{,}2 \cdot 4} = 0{,}0569$

$P(N_s = i, N_t = j) = P(N_s = i) \cdot P(N_t = j \mid N_s = i)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad P(N_t - N_s = j - i)$



$\Big\} N_t - N_s = j - i$

$P(N_t = j \mid N_s = i) = \dfrac{[\lambda \cdot (t-s)]^{j-i}}{(j-i)!} e^{-\lambda(t-s)}$

$t_1 < t_2 < \dots < t_n$

$0 \le k_1 \le k_2 \le \dots \le k_n$

$P(N_{t_1} = k_1, N_{t_2} = k_2, \dots, N_{t_n} = k_n) =$

$= P(N_{t_1} = k_1, N_{t_2} - N_{t_1} = k_2 - k_1, \dots, N_{t_n} - N_{t_{n-1}} = k_n - k_{n-1}) =$

$= \dfrac{\lambda^{k_n} \cdot t_1^{k_1} \cdot (t_2 - t_1)^{k_1} \cdots (t_n - t_{n-1})^{k_n - k_{n-1}}}{k_1! \, (k_2 - k_1)! \cdots (k_n - k_{n-1})!} e^{-\lambda t_n} = \dots$

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a) $\lambda = 2 \qquad P(N_4 = 0) = ?$

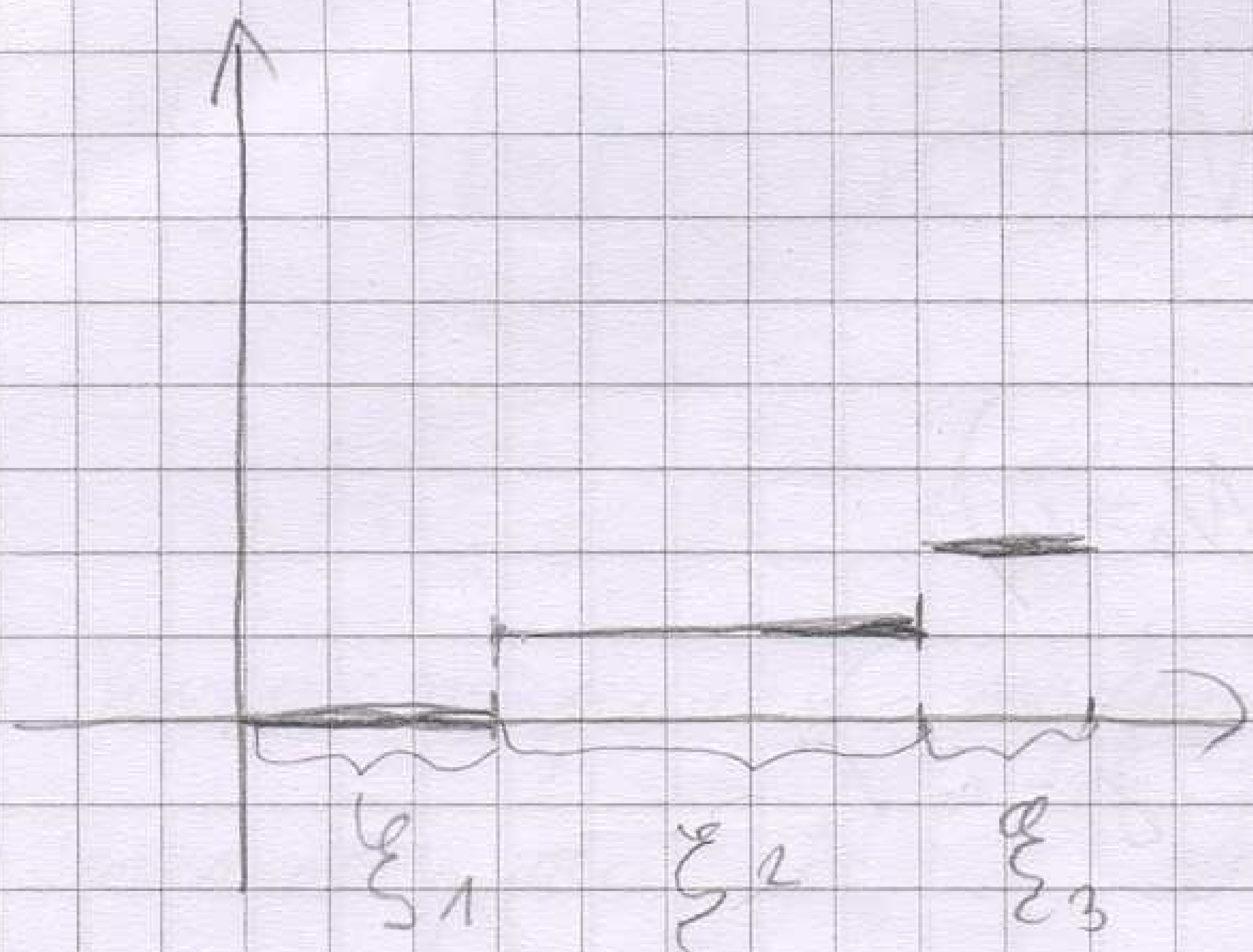
$P(N_4 = 0) = \dfrac{(\lambda t)^k}{k!} e^{-\lambda t} = 1 \cdot e^{-8} = e^{-8}$

b) $P(N_1 = 0, N_4 = 0) = \dfrac{1 \cdot 1 \cdot 3^0}{0! \cdot 0!} \cdot e^{-4 \cdot 2} = e^{-8}$

c) $P(N_4 = 1) = 8 e^{-8}$

$k_1 = 0 \quad \dfrac{1}{1} e^{-2} \cdot \dfrac{6}{1} \cdot e^{-6} = 6 e^{-8}$

$k_1 = 1$



$\xi_1, \xi_2, \dots$ – nezavisne
– jednako distribuirane

$$F_\xi(t) = P(\xi < t) = 1 - e^{-\lambda t}$$

$$F_\xi = P(\xi < t) = P(N_t \geq 1) = 1 - P(N_t = 0) = 1 - e^{-\lambda t}, \quad f_\xi(t) = \lambda$$

$$\{N_t = k\} = \{\xi_1 + \xi_2 + \dots + \xi_k < t, \ \xi_1 + \xi_2 + \dots + \xi_{k+1} > t\}$$

$$W_n = \xi_1 + \xi_2 + \dots + \xi_n$$

$$f_{W_n}(t) = (f_{\xi_1} * f_{\xi_2} * \dots * f_{\xi_n})(t)$$

$$\hat{f}_{W_n}(s) = \left[\hat{f}_\xi(s)\right]^n = \left(\frac{\lambda}{s-\lambda}\right)^n = \frac{\lambda^n}{(s-\lambda)^n} \bullet\!\!-\!\!\circ \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} =$$

$$= \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}$$

GAMA RAZDIOBA, PARAMETRI: n, λ

$$\{N_t < n\} = \{\xi_1 + \xi_2 + \xi_n > t\} = \{W_n > t\}$$

$$F_{N_t}(n) = 1 - F_{W_n}(t) =$$

$$= 1 - \int_0^t f_{W_n}(t)\,dt =$$

$$= \int_t^\infty f_{W_n}(t)\,dt = \frac{1}{(n-1)!} \int_t^\infty \lambda^n x^{n-1} e^{-\overset{y}{\lambda x}}\,dx =$$

$$= \frac{1}{(n-1)!} \int_{\lambda t}^\infty y^{n-1} e^{-y}\,dy = \frac{1}{(n-1)!} \left[-e^{-y} y^{n-1} \Big|_{\lambda t}^{\infty} - \int_{\lambda t}^\infty (n-1) y^{n-2} e^{-y} \right.$$

$$= \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t} + \frac{1}{(n-2)!} \int_{\lambda t}^\infty y^{n-2} e^{-y}\,dy$$

$$= \sum_{j=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^j}{j!}$$

$$P(N_t = n) = F_{N_t}(n+1) - F_{N_t}(n) = \sum_{j=0}^{n} \dots - \sum_{j=0}^{n-1} \dots = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$N_1 \to \lambda_1$

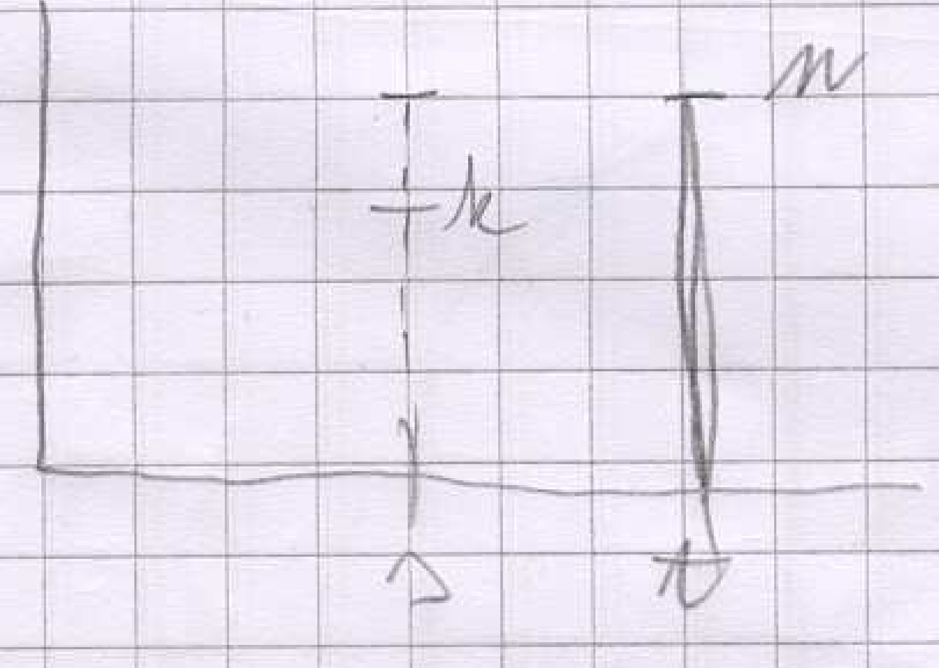
$N_2 \to \lambda_2$

$N = N_1 + N_2 \to \lambda_1 + \lambda_2$

$$P(N(t) = n) = P(N_1(t) + N_2(t) = n) = \sum_{r=0}^{n} P(N_1(t) = r, N_2(t) = n-r)$$

$$= \sum_{r=0}^{n} \frac{(\lambda_1 t)^r}{r!} e^{-\lambda_1 t} \cdot \frac{(\lambda_2 t)^{n-r}}{(n-r)!} e^{-\lambda_2 t}$$

$$= \frac{1}{n!} \left[\underbrace{\sum_{r=0}^{n} \binom{n}{r} \lambda_1^r \lambda_2^{n-r}}_{(\lambda_1+\lambda_2)^n} \right] \cdot t^n e^{-(\lambda_1+\lambda_2)t} \quad //$$



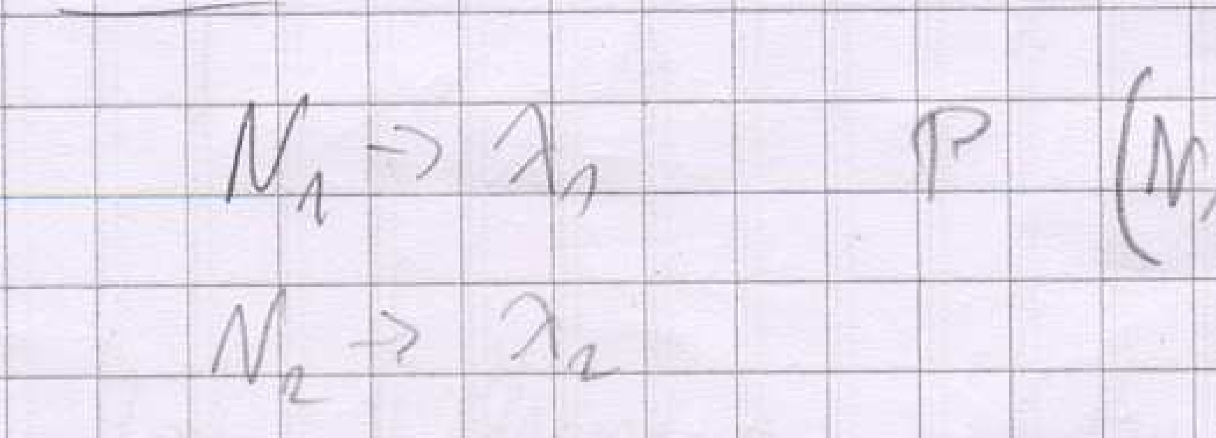
$$P(N_s = k \mid N_t = n) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$p = \frac{s}{t}$$

$N_1 \to \lambda_1$ $\qquad P(N_1(t) = k \mid N_1(t) + N_2(t) = n) = \binom{n}{k} p^k (1-p)^{n-k}$

$N_2 \to \lambda_2$

$$p = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$



$$P(A) = \sum_k P(H_k)\, P(A \mid H_k) = (\infty \text{ hipoteze}$$

$$= \int_{-\infty}^{\infty} P(A \mid \xi = t)\, f(t)\, dt$$

- za fiksni interval: $P(N=k) = \frac{(at)^k}{k!} e^{-at}$

- za slučajni interval:

$$P(N=k) = \int_0^\infty P(N=k \mid \xi = t)\, f_\xi(t)\,dt =$$

$$= \int_0^\infty \frac{(at)^k}{k!} e^{-at} \cdot b e^{-bt}\,dt =$$

$$= \frac{b a^k}{k!} \int_0^\infty t^k e^{-(a+b)t}\,dt = \begin{vmatrix} u = (a+b)t \\ du = (a+b)dt \end{vmatrix} =$$

$$= \frac{b a^k}{k!} \int_0^\infty \frac{u^k}{(a+b)^k} \cdot e^{-u} \frac{du}{a+b} =$$

$$= \frac{b a^k}{(a+b)^{k+1}} \cdot \frac{1}{k!} \cdot \underbrace{\int_0^\infty u^k e^{-u}\,du}_{\Gamma(k+1) = k!} = \frac{b a^k}{(a+b)^{k+1}} =$$

broj realizacija ima geometrijsku razdiobu

$$= \frac{b}{a+b} \cdot \left(\frac{a}{a+b}\right)^k$$

$$= p \cdot q^k$$

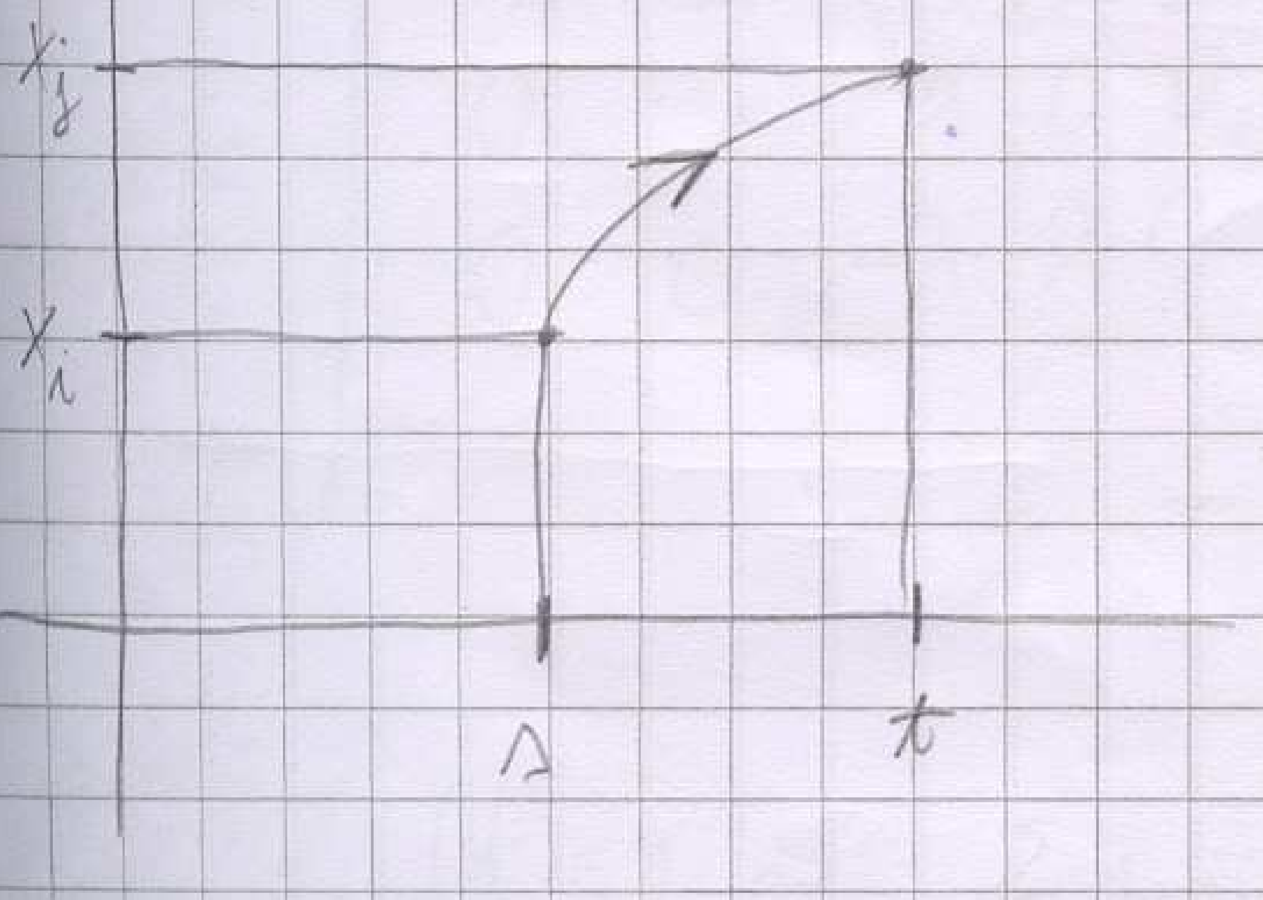
KOLMOGOROVLJEVE JEDNADŽBE

S diskretan

Markovljev proces ako vrijedi

$$P(X_{t_{n+1}} = x_{n+1} \mid X_{t_n} = x_n, \ldots, X_{t_0} = x_0) =$$
$$= P(X_{t_{n+1}} = x_{n+1} \mid X_{t_n} = x_n)$$

$$P(X_{t_{n+1}} = x_{n+1} \mid X_{t_n} = x_n, \ldots, X_{t_0} = x_0) =$$
$$= P(X_{t_{n+1}} - X_{t_n} = x_{n+1} - x_n \mid X_{t_n} - X_{t_{n-1}} = x_n - x_{n-1}, \ldots, X_{t_1} - X_{t_0} = x_1 - x_0, X_{t_0} = x_0)$$
$$= P(X_{t_{n+1}} - X_{t_n} = x_{n+1} - x_n) =$$
$$= P(X_{t_{n+1}} - X_{t_n} = x_{n+1} - x_n \mid X_{t_n} = x_n) =$$
$$= P(X_{t_{n+1}} = x_{n+1} \mid X_{t_n} = x_n)$$



$P(X_t = x_j \mid X_s = x_i)$ - vjerojatnost prijelaza

$\hookrightarrow = p(s, x_i; t, x_j) = p_{ij}^{[s,t]}$

$P^{[s,t]} \equiv (p_{ij}^{[s,t]})$ - matrica prijelaza

$P^{[t,t]} = I$

Tm. (Chapman-Kolmogorov)

$$P^{[s,u]} = P^{[s,t]} \cdot P^{[t,u]}, \quad s \leq t \leq u$$

$$p_{ij}^{[s,u]} = \sum_k p_{ik}^{[s,t]} \cdot p_{kj}^{[t,u]}$$

$$P(N_t = j \mid N_s = i) = \frac{[\lambda(t-s)]^{j-i}}{(j-i)!} e^{-\lambda(t-s)}$$

$$\| $$

$$p_{ij}^{[s,t]}$$

$$P^{[s,t]} = \begin{pmatrix} e^{-\lambda(t-s)} & \lambda(t-s) e^{-\lambda(t-s)} & \frac{\lambda(t-s)^2}{2} e^{-\lambda(t-s)} \\ & e^{-\lambda(t-s)} & \\ & & e^{-\lambda(t-s)} \\ & 0 & \end{pmatrix}$$

Proces je homogen (u vremenu) ako matrica prijelaza $P^{[s,t]}$ ovisi samo o t-s

$$P^{[s,t]} = P^{[s+h,\, t+h]} \quad \forall h$$

$$h = -s: \quad P^{[s,t]} = P^{[0,\, s-t]} =: P(t-s)$$

$$\boxed{P^{[0,t]} = P(t) = P^{[s,\, s+t]}}$$

$$P^{[0,\, t+s]} = P^{[0,t]} \cdot P^{[t,\, t+s]} =$$

$$P(t+s) = P(t) \cdot P(s)$$

$$p_{ij}(t+s) = \sum_k p_{ik}(t)\, p_{kj}(s) \quad \Big/ \frac{\partial}{\partial s}$$

$$p_{ij}'(t+s) = \sum_k p_{ik}(t) \cdot p_{kj}'(s)$$

$$s = 0 \Rightarrow \quad p_{ij}'(t) = \sum_k p_{ik}(t) \cdot \underbrace{p_{kj}'(0)}_{a_{kj}}$$

$P'(t) = P(t) \cdot A \Leftarrow$ KOLMOGOROVLJEVA JEDNADŽBA UNAPRIJED

$P'(t) = A \cdot P(t)$

MATRICA
GUSTOĆA
PRIJELAZA

za Poissonov p:

$$p_{ij}(t) = \frac{(\lambda t)^{j-i}}{(j-i)!} e^{-\lambda t}$$

$$p_{ij}'(t) = \frac{\lambda^{j-i} t^{j-i-1}}{(j-i-1)!} e^{-\lambda t} + \frac{(\lambda t)^{j-i}}{(j-i)!} \cdot e^{-\lambda t} \cdot (-\lambda)$$

$$a_{ij} = p_{ij}'(0)$$

$$p_{ii}(t) = e^{-\lambda t}, \quad p_{ii}'(t) = -\lambda e^{-\lambda t} \Rightarrow a_{ii} = -\lambda$$

$$p_{i,i+1}(t) = \lambda t e^{-\lambda t} \qquad a_{i,i+1} = \lambda$$

$$A = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & \\ 0 & -\lambda & \lambda & 0 & 0 & \cdots \\ 0 & 0 & -\lambda & \lambda & 0 & \\ & & \vdots & & & \end{bmatrix}$$

$$\sum_j p_{ij}(t) = 1 \Rightarrow \sum_j p_{ij}'(0) = 0 \Rightarrow \sum_j a_{ij} = 0$$

$$a_{ij} = \lim_{\Delta t \to 0} \frac{p_{ij}(\Delta t) - p_{ij}(0)}{\Delta t}$$

$i \neq j: \quad a_{ij} = \lim_{\Delta t \to 0} \frac{p_{ij}(\Delta t)}{\Delta t}, \quad p_{ij}(\Delta t) = a_{ij} \Delta t + o(\Delta t)$

$i=j: \; p_{ii}(\Delta t) = 1 + a_{ii}(\Delta t) + o(\Delta t)$

Poisson:

$$P_{ii}(h) = 1 - \lambda h + o(h)$$

$$P_{i,i+1}(h) = \lambda h + o(h)$$

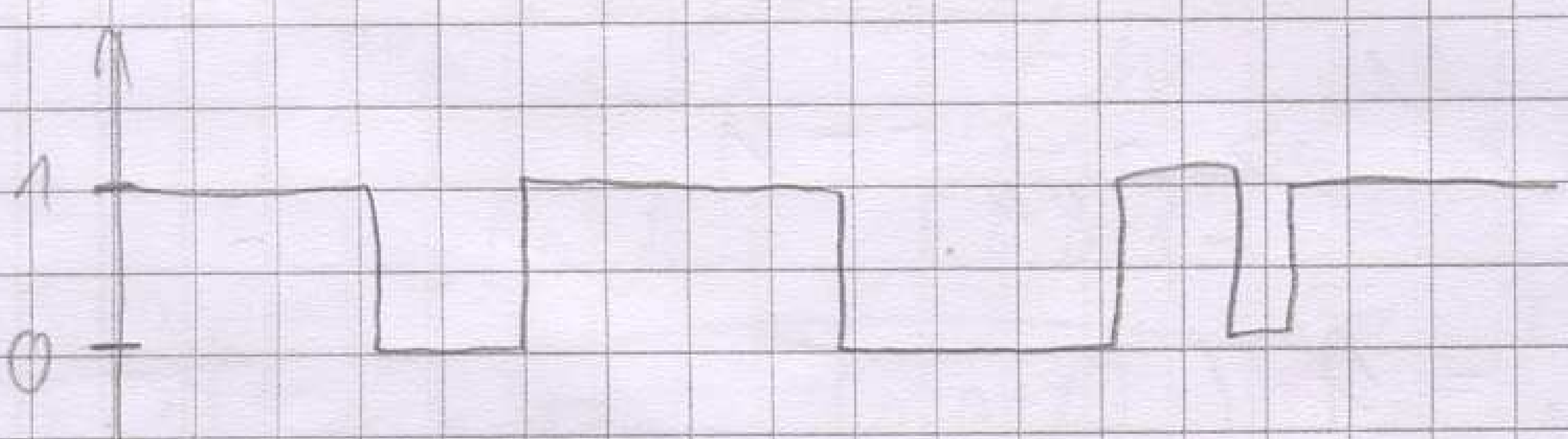
$$p_{i,i+k}(h) = o(h), \; k \geq 2$$

$$a_{i,i+1} = \lambda$$

$$a_{ii} = -\lambda$$

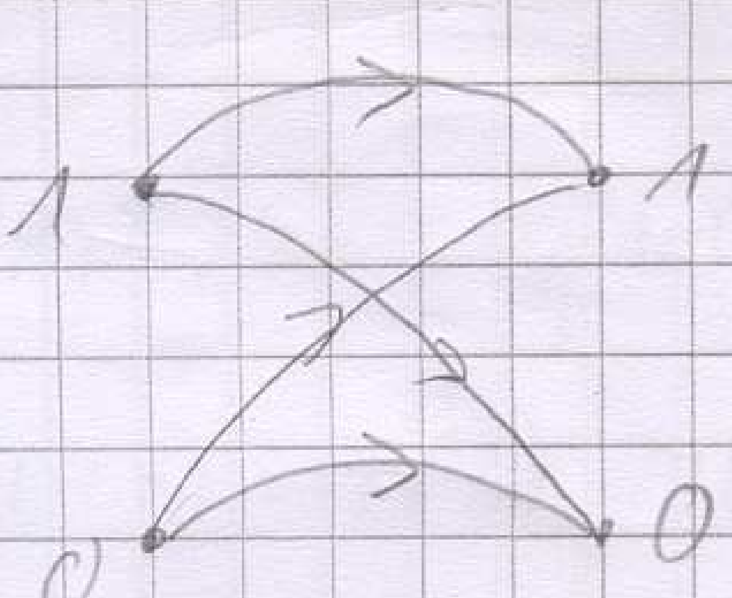
PRIMJER:

1 - ispravno stanje $\sim \mathcal{E}(\mu)$
0 - neispravno stanje $\sim \mathcal{E}(\lambda)$



$$p_{01} = P(X(\Delta t)=1 \mid X(0)=0) = \lambda \Delta t + o(\Delta t)$$

$$p_{10} = P(X(\Delta t)=0 \mid X(0)=1) = \mu \Delta t + o(\Delta t)$$



$$f(t) = \lambda e^{-\lambda t}$$

$$A = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

K. J.

$$\begin{bmatrix} p_{00}'(t) & p_{01}'(t) \\ p_{10}'(t) & p_{11}'(t) \end{bmatrix} = \begin{bmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{bmatrix} \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

$$p_{00}'(t) = -\lambda p_{00}(t) + \mu \overbrace{p_{01}(t)}^{1-p_{00}(t)} = \mu - (\lambda+\mu)\, p_{00}(t)$$

$$p_{11}'(t) = \lambda \underbrace{p_{10}(t)}_{1-p_{11}} - \mu p_{11}(t) = \lambda - (\lambda+\mu)\, p_{11}(t)$$

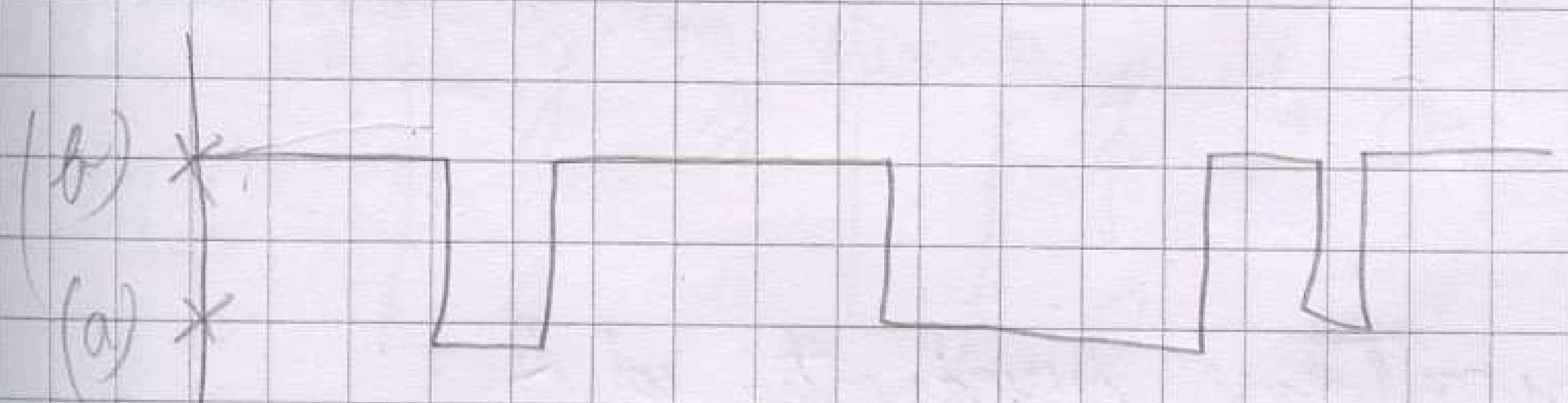
$$p_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

$$p_{00}(0) = 1 \quad , \quad p_{11}(0) = 1$$

$$p_{00}(t) = A + B e^{-(\lambda+\mu)t}$$

$$p_{11}(t) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)\cdot t}$$

$$p_{01}(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$



verjetnosti pod. stanja

$$P(X(t) = 1) = a \cdot P_{01}(t) + b \cdot P_{11}(t) =$$

$$= a\frac{\lambda}{\lambda+\mu} - a\frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} + b\frac{\lambda}{\lambda+\mu} + b\frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} =$$

$$= \frac{\lambda}{\lambda+\mu} + \frac{b\mu - a\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

$$t = 0: \; = \frac{\lambda + b\mu - a\lambda}{\lambda+\mu} = \frac{(1-a)\lambda + b\mu}{\lambda+b}$$

$P'(t) = A\,P(t)$

$P(t) = I$

$P(t) \cdot e^{At}$

$$\boxed{P(t) = e^{At}} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = \sum_{n=0}^{\infty} \frac{A^n}{n!} t^n$$

$A = SDS^{-1}$, $A^2 = SD\cancel{S^{-1}}\cancel{S}DS^{-1} = SD^2S^{-1}$, $A^n = SD^nS^{-1}$

$f(A) := S f(D) S^{-1}$

$$f(D) = \begin{bmatrix} f(d_1) & & 0 \\ & f(d_2) & \\ 0 & & f(d_n) \end{bmatrix}$$

$P(A) = S\,P(D)\,S^{-1}$

POLINOM

ako su različite, mat. je dijagonalna

$A = SDS^{-1} \Rightarrow d_1, d_2 \dots d_n \rightarrow$ svojstvene vrijednosti od A

$\hookrightarrow$ matrica svojstvenih vektora

- preko Laplacea

$P'(t) = AP(t) \Rightarrow s\hat{P}(s) - I = A\hat{P}(s)$

$\hat{P}(s) = [sI - A]^{-1}$

PRIMJER, PRUGI NAČIN:

$A = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$

$P(t) = e^{At}$

$k(x) = \det(xI - A) = \begin{vmatrix} x+\lambda & -\lambda \\ -\mu & x+\mu \end{vmatrix} = x^2 + (\lambda+\mu)x + \dots$

$\Rightarrow$ uvijek jedno rješenje

$x_1 = 0$

$x_2 = -(\lambda+\mu)$ $\Big\rangle \rightarrow$ svojstvene vrijednosti

$$D = \begin{bmatrix} 0 & 0 \\ 0 & -(\lambda+\mu) \end{bmatrix}$$

$$(x I - A)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$x=0$
$$\begin{bmatrix} \lambda & -\lambda \\ -\mu & \mu \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x=0: \; v_1 = v_2 \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x = -(\lambda+\mu)$

$$\begin{bmatrix} -\mu & -\lambda \\ -\mu & -\lambda \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-\mu v_1 - \lambda v_2 = 0$
$\mu v_1 = -\lambda v_2 \qquad \begin{pmatrix} \lambda \\ -\mu \end{pmatrix}$

$$\Rightarrow S = \begin{bmatrix} 1 & \lambda \\ 1 & -\mu \end{bmatrix}, \quad S^{-1} = \frac{1}{-\lambda-\mu}\begin{bmatrix} -\mu & -\lambda \\ -1 & 1 \end{bmatrix} = \frac{1}{\lambda+\mu}\begin{bmatrix} \mu & \lambda \\ 1 & -1 \end{bmatrix}$$

$$P(t) = \begin{bmatrix} 1 & \lambda \\ 1 & -\mu \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & e^{-(\mu+\lambda)t} \end{bmatrix}}_{e^{Dt}} \cdot \frac{1}{\mu+\lambda}\begin{bmatrix} \mu & \lambda \\ 1 & -1 \end{bmatrix}$$

Jordanove blokove (ŽORDANOVI)

$$D = \begin{bmatrix} d_1 & * & & & & \\ & d_1 & d_1 & & & \\ & & & d_2 & * & \\ & & & & d_2 & \\ & & & & & d_3 \; d_3 \; * \\ & & & & & \ddots \end{bmatrix}$$

opći oblik (za konačan broj stanja)

ELEMENTARNA JORDANOVA KLIJETKA

jedan blok:

$$\begin{pmatrix} d & 1 & & & & & \\ & d & 1 & & & & \\ & & d & 0 & & & \\ & & & d & 1 & & \\ & & & & d & 1 & \\ & & & & & d & \\ & & & & & & d \; 1 \\ & & & & & & d \end{pmatrix}$$

do. Taylorovog polinoma

$$f(D) = \begin{bmatrix} f(d) & f'(d) & \frac{1}{2}f''(d) \\ & f(d) & \\ & & f(d) \end{bmatrix}$$

Poissonov proces s gomilanjem

$$P(N_t = k) \sim e^{\lambda t(z-1)}$$

X_i – broj unesrećenih u jednoj nesreći, $X_i \sim \begin{pmatrix} 0 & 1 & 2 & 3 & \\ p_0 & p_1 & p_2 & p_3 & \dots \end{pmatrix}$

broj unesrećenih

$$M_t \sim e^{\lambda t(\psi_X(z) - 1)}$$

PRIMJER

Kupci ulaze (1 ili 2) jednako p. Vrijeme ulaska $\sim \mathcal{P}(\lambda)$. Odredi f-ju izvodnicu broja kupaca.

$$X_i \sim \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad \psi_X(s) = \frac{1}{2}s + \frac{1}{2}s^2$$

$$\psi_M(s) = e^{\lambda t\left(\frac{1}{2}s + \frac{1}{2}s^2 - 1\right)}$$

$$\psi_M'(1) = \lambda \cdot t \cdot \left(\frac{1}{2} + s\right) \cdot e^{(\quad)} = \frac{3}{2}\lambda t$$

$\lambda = \frac{1}{2}$

$t = 4$

$$\psi_M(s) = e^{s+s^2-2} = e^{-2} \cdot e^{s+s^2} = e^{-2} \cdot \left[1 + s + s^2 + \frac{1}{2}(s+s^2)^2 + \frac{1}{6}(s+s^2)^3 + \frac{1}{24}(s+s^2)^4 + \dots\right]$$

$$M_4 \sim \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & \dots \\ 1 & 1 & \frac{3}{2} & \frac{7}{6} & \frac{25}{24} & \dots \end{pmatrix}$$

e^{-2} *

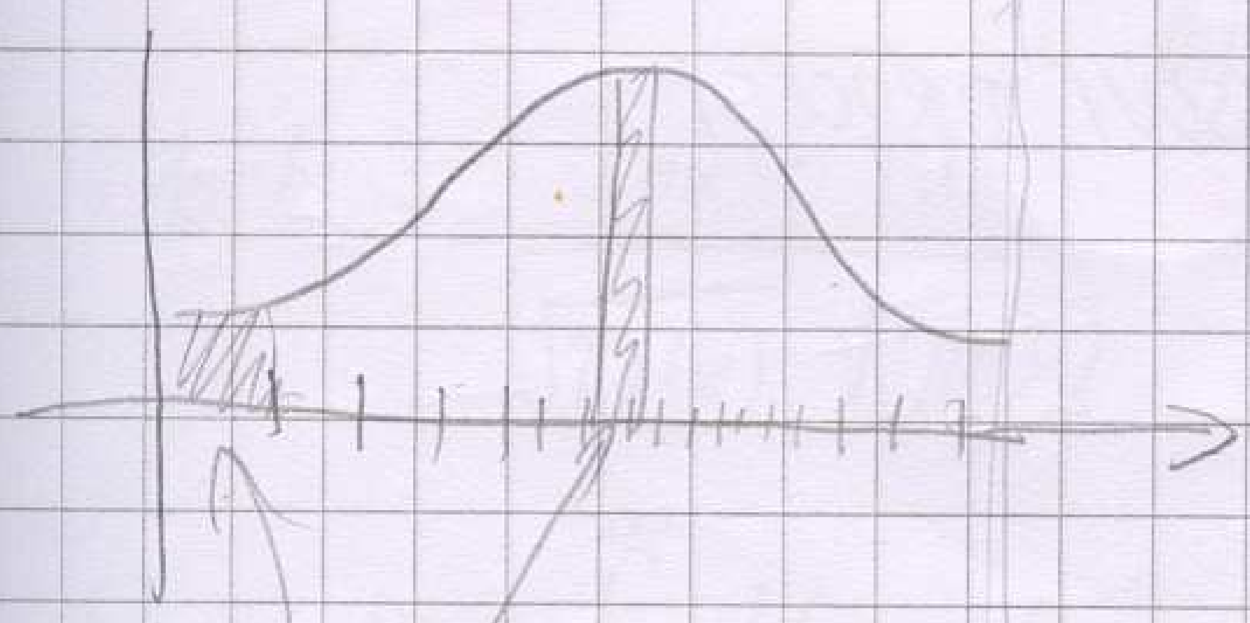
Promjenjiv intenzitet

$$\psi_{N_t}(s) = e^{\lambda t(s-1)}$$

nastao integriranjem

$\lambda \to \lambda(t)$

$$\to e^{\int_0^t \lambda(u)\,du \cdot (s-1)}$$



← intenzitet konstantan, a vrijeme ubrzava / usporava

ista površina

$$\overline{\lambda} = \frac{1}{t}\int_0^t \lambda(u)\,du$$

$$\rightarrow e^{\overline{\lambda}\cdot t}$$

λ ovisi o stanju

$$p_{n,n+k}(h) = \begin{cases} \lambda h + o(h), & k=1 \\ o(h), & k \geq 2 \\ 1-\lambda h + o(h), & k=0 \end{cases}$$

$\lambda = \lambda_n$

$$A = \begin{bmatrix} -\lambda_0 & \lambda_0 & & & \\ 0 & -\lambda_1 & \lambda_1 & & \\ & & -\lambda_2 & \lambda_2 & \\ & & & \ddots & \ddots \end{bmatrix}$$

$$P'(t) = A \cdot P(t)$$

$$p_n'(t) = -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t)$$

$$p_0'(t) = -\lambda_0 p_0(t)$$

Proces rađanja i umiranja

$$p_{ij}(h) = \begin{cases} \lambda_i h + o(h), & j = i+1 \\ \mu_i h + o(h), & j = i-1 \\ o(h), & |j-i| \geq 2 \\ 1-(\lambda_i+\mu_i)h + o(h) \end{cases}$$

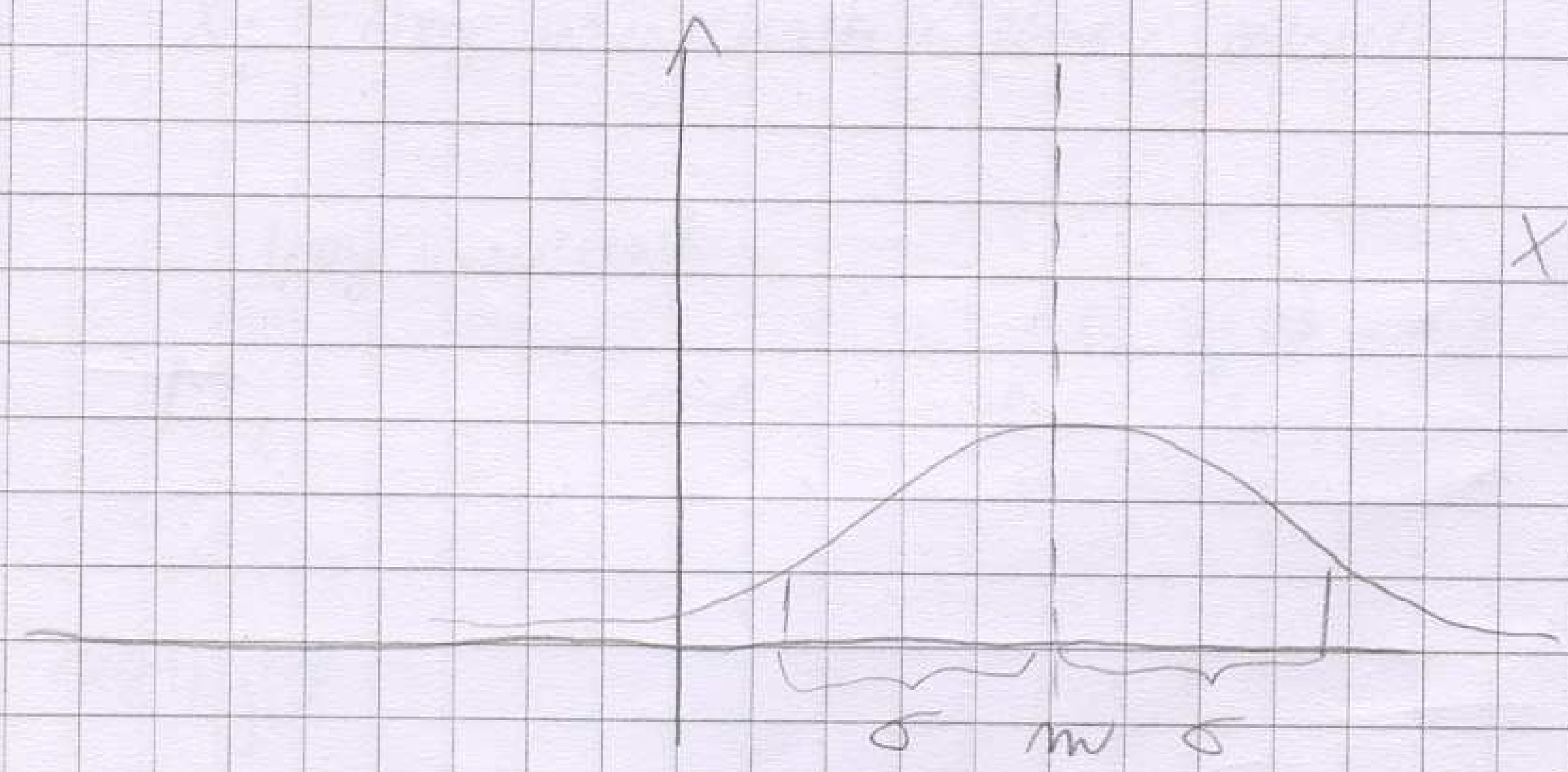
INTENZITET UMIRANJA

INTENZITET RAĐANJA

$$A = \begin{bmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -\lambda_1-\mu_1 & \lambda_1 & & \\ & \mu_2 & -\lambda_2-\mu_2 & \lambda_2 & \\ & & \mu_3 & -\lambda_3-\mu_3 & \lambda_3 \end{bmatrix}$$

NORMALNA RAZDIOBA I GAUSSOVI PROCESI

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$



$$X \sim N(m, \sigma^2)$$

2D: $X \sim N(a_1, \sigma_1^2)$
$Y \sim N(a_2, \sigma_2^2)$

X, Y nezavisni

$$(X,Y) \rightarrow f_X(x)\cdot f_Y(y) = \frac{1}{\sigma_1\sigma_2\cdot 2\pi}\cdot e^{-\frac{(x-a_1)^2}{2\sigma_1^2}-\frac{(y-a_2)^2}{2\sigma_2^2}}$$

$a_1 = a_2 = 0$
$\sigma_1^2 = \sigma_2^2 = 1$ $\Rightarrow$ jedinične normalne razdiobe

$$f(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

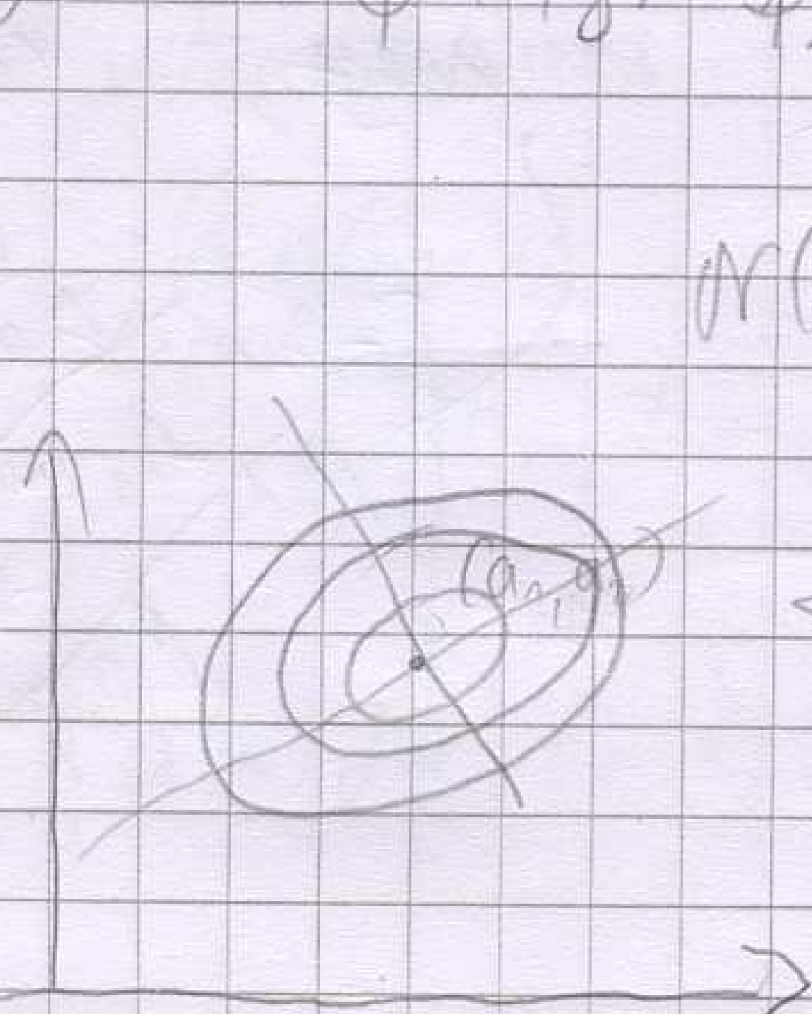
OPĆENITO:

$$(X,Y) \sim N(a_1, a_2; \sigma_1^2, \sigma_2^2, r)$$

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}}\cdot \exp\left\{-\frac{1}{2(1-r^2)}\left[\frac{(x-a_1)^2}{\sigma_1^2} - 2r\frac{(x-a_1)(y-a_2)}{\sigma_1\sigma_2} + \frac{(y-a_2)^2}{\sigma_2^2}\right]\right\}$$

za $r = 0$ $\Rightarrow$ $f(x,y) = f_X(x)\cdot f_Y(y)$ $\Rightarrow$ X, Y su nezavisne normalne varijable

$N(a_1, \sigma_1^2)$ $\quad$ $N(a_2, \sigma_2^2)$



← zakrenuta → postoji zavisnost između X i Y

r – koeficijent relacije

$$k.k. = \frac{cov(X,Y)}{\sqrt{D(X)}\sqrt{D(Y)}} \quad cov = E\left[(X-a_1)(Y-a_2)\right]$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(x-a_1)^2}{2\sigma_1^2}}$$

X, Y nezavisni $\rightarrow r = 0$

$r = 0 \not\Rightarrow X, Y$ nezavisni

} samo kod normalne nezavisnost je isto što i nekoreliranost

$$X \sim \mathcal{N}(0,1) \qquad \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$Y = \sigma X + a$$

$$g_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right| = \frac{1}{2\pi} e^{-\frac{1}{2}\left(\frac{x-a}{2}\right)^2} \sim \mathcal{N}(a, \sigma^2)$$

$y = \sigma x + a$

$x = \frac{y-a}{\sigma}, \quad \frac{dx}{dy} = \frac{1}{\sigma}$

$\vec{X} = (X_1, X_2, \dots, X_n)$

$\vec{a}, R$

$$f(\vec{x}) = \frac{1}{(2\pi)^{n}\sqrt{|R|}} \cdot \exp\left(-\frac{1}{2}(\vec{x}-\vec{a})^T R^{-1} (\vec{x}-\vec{a})\right)$$

$\vec{X} \sim \mathcal{N}(\vec{a}, R)$

BILINEARNA FORMA $(\vec{x} \mid A^T\vec{y})$ $//$

$(A\vec{x} \mid \vec{y}) = \sum_{i,j} x_i y_j$

$(\vec{x} \mid \vec{y}) = \sum x_i y_i$

$\vec{y}^T \vec{x} = (\vec{x} \mid \vec{y})$

$= \vec{y}^T A\vec{x}$

KV. FORMA: $(A\vec{x} \mid \vec{x}) = \sum_{i,j} a_{ij} x_i x_j$

$n=2$: $R = (r_{ij})$

$r_{ij} = cov(X_i, X_j) = E\left[(X_i - a_i)(X_j - a_j)\right]$

$$R = \begin{bmatrix} \sigma_1^2 & r\sigma_1\sigma_2 \\ r\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

$$|R| = \sigma_1^2\sigma_2^2 - r^2\sigma_1^2\sigma_2^2 \Rightarrow \sqrt{|R|} = \sigma_1\sigma_2\sqrt{1-r^2}$$

$\vec{x} = (x_1, x_2, \ldots, x_n)$

$\vec{y} = (y_1, y_2, \ldots, y_n)$

$(\vec{x} \mid \vec{y}) = \vec{y}^T \vec{x} = \sum_{i=1}^{n} x_i y_i$

$(\vec{x}, \vec{x}) = \|\vec{x}\|^2$

$\vec{X} \sim \mathcal{N}(0, I)$

$$f_X(\vec{x}) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\|\vec{x}\|^2} = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_i^2}$$

$\vec{X} = (X_1, X_2, \ldots, X_n)$

$\vec{X} \sim \mathcal{N}(0, I) \Leftrightarrow X_i \sim \mathcal{N}(0,1)$, nezavisne

---

$\vec{Y} = A\vec{X}$

$\vec{X} = A^{-1}\vec{Y}$

$x_1 = b_{11}y_1 + \ldots + b_{1n}y_n$

$\vdots$

$x_n = b_{n1}y_1 + \ldots + b_{nn}y_n$

$A^{-1} = (b_{ij})$

$\frac{\partial x_1}{\partial y_1} = b_{11}$

$\vec{Y} = \psi(\vec{x})$

$g(y_1, y_2, \ldots, y_n) = f(x_1, \ldots, x_n)\,|J|$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \ldots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & & \\ \frac{\partial x_n}{\partial y_1} & \ldots & & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

$$J = \det(A^{-1}) = \frac{1}{\det(A)}$$

$$g(\vec{y}) = \frac{1}{|\det(A)|} \cdot \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left(-\frac{1}{2}\|A^{-1}\vec{y}\|^2\right)$$

$$\|A^{-1}\vec{y}\|^2 = (A^{-1}\vec{y} \mid A^{-1}\vec{y}) = (A^{-1}y)^T (A^{-1}\vec{y}) = \vec{y}^T A^{-1\,T} A^{-1}\vec{y} =$$

$= \vec{y}^T (A^T)^{-1} A^{-1} \vec{y} = \vec{y}^T \underbrace{(AA^T)}_{R}^{-1} \vec{y}$

$\det(R) = \det(AA^T) = \det(A)^2$

$\vec{y} = A\vec{x} + \vec{a} \Rightarrow g(\vec{y}) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\det(R)}} \exp\left\{-\frac{1}{2}(\vec{y}-\vec{a})^T R^{-1} (\vec{y}-\vec{a})\right\}$

$AA^T = R$

R je pozitivno definitna

$(R\vec{x} \mid \vec{x}) \geq 0$

R je simetrična

$R^T = (AA^T)^T = A^{TT} A^T = AA^T = R$

$(R\vec{x} \mid \vec{x}) = (AA^T \vec{x} \mid \vec{x}) = (A^T \vec{x} \mid A^T \vec{x}) = \|A^T \vec{x}\|^2 \geq 0$

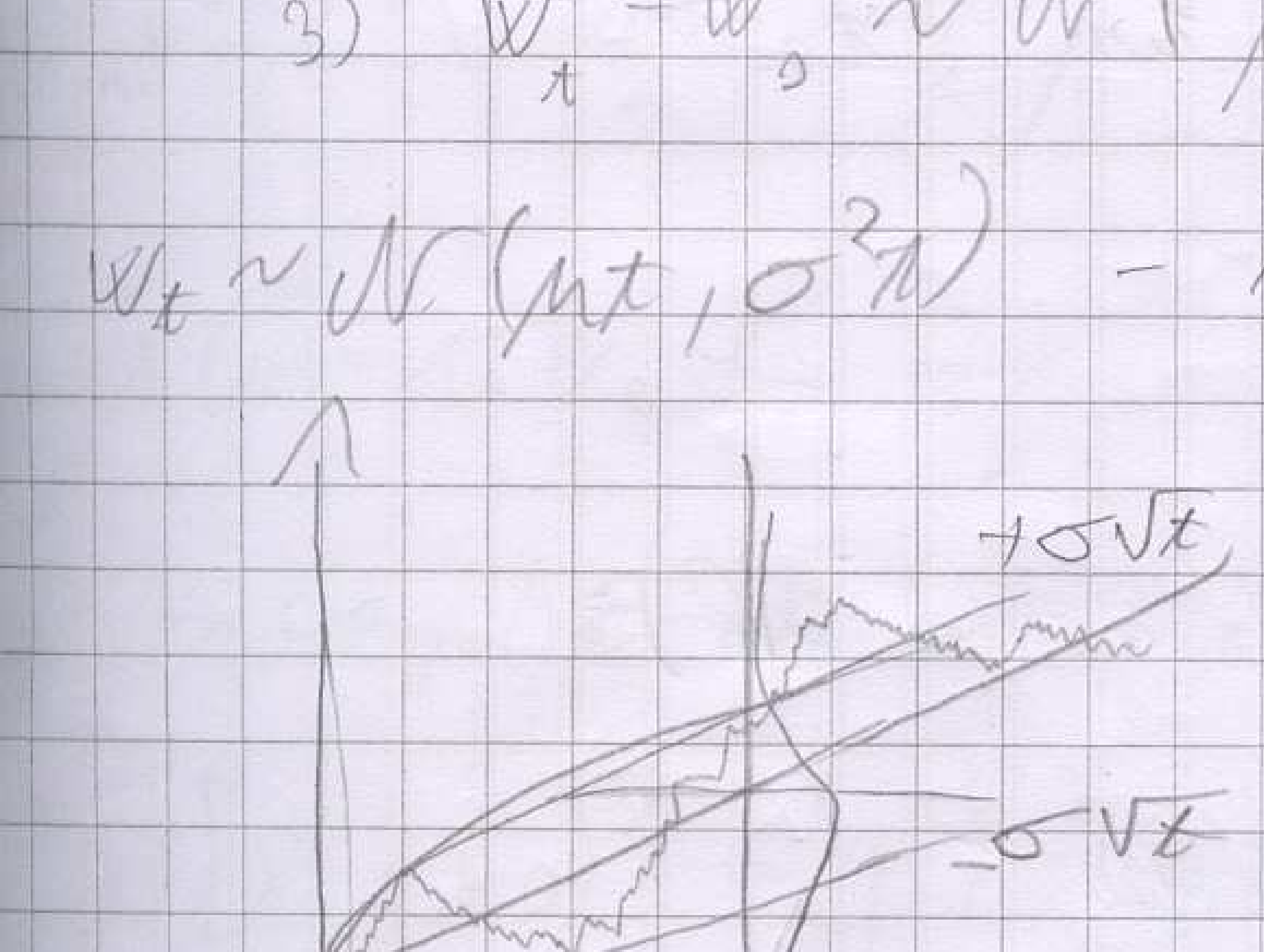
(X_t) je Gaussov ako za sve $t_1 < t_2 < \ldots < t_n$ vektor $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ ima normalnu razdiobu.

BROWNOVO GIBANJE

W_t je Wienerov proces, ako ima sljedeće svojstva

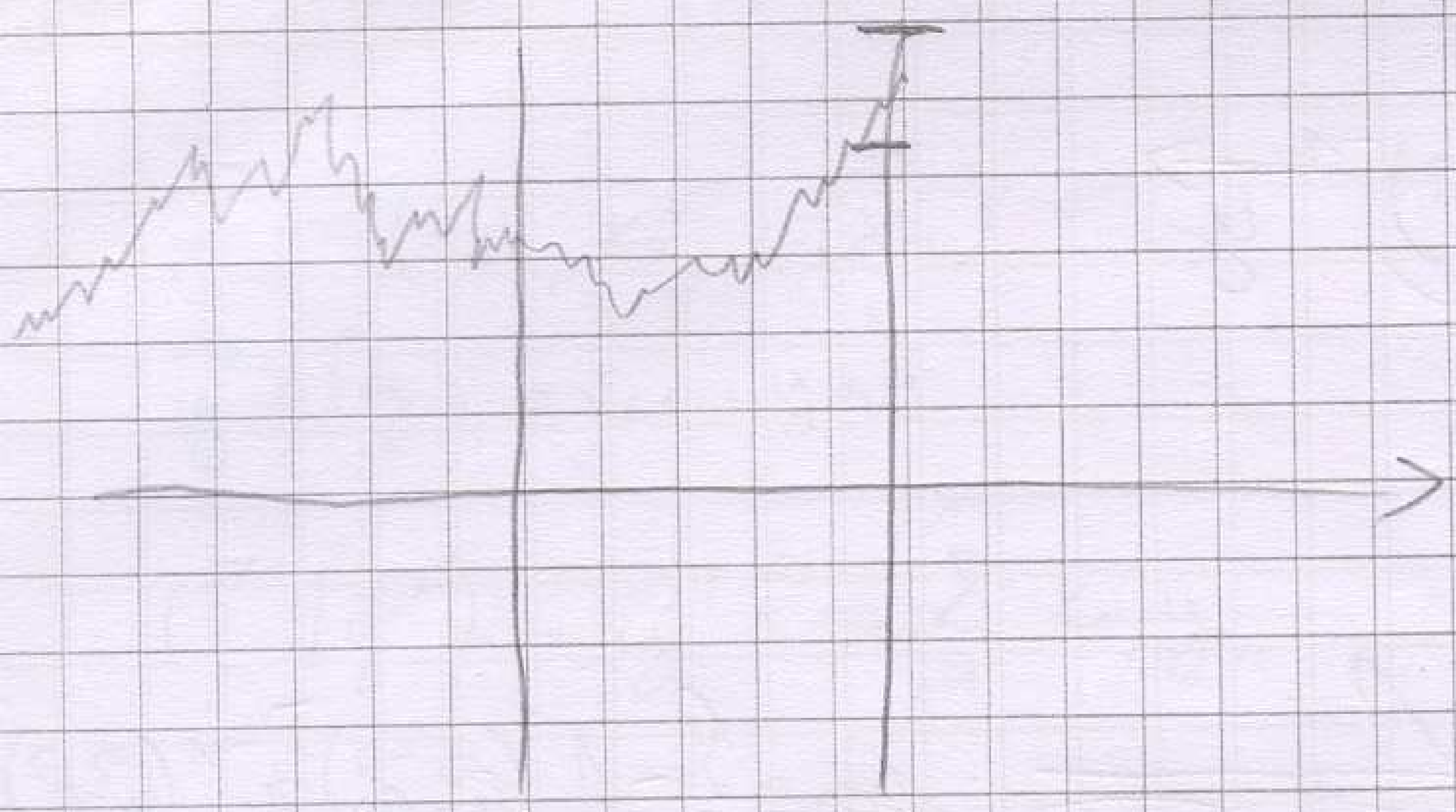
1) $W_0 = 0$
2) W ima nezavisne prirasti
3) $W_t - W_s \sim \mathcal{N}(\mu(t-s), \sigma^2(t-s))$

$W_t \sim \mathcal{N}(\mu t, \sigma^2 t)$ - 1D distribucija



← parabola koja mjeri raspršenje

- trajektorija je kontinuirana, ali nigdje diferencijabilna
- derivacija Brownovog gibanja (u poopćenom smislu) je bijeli šum

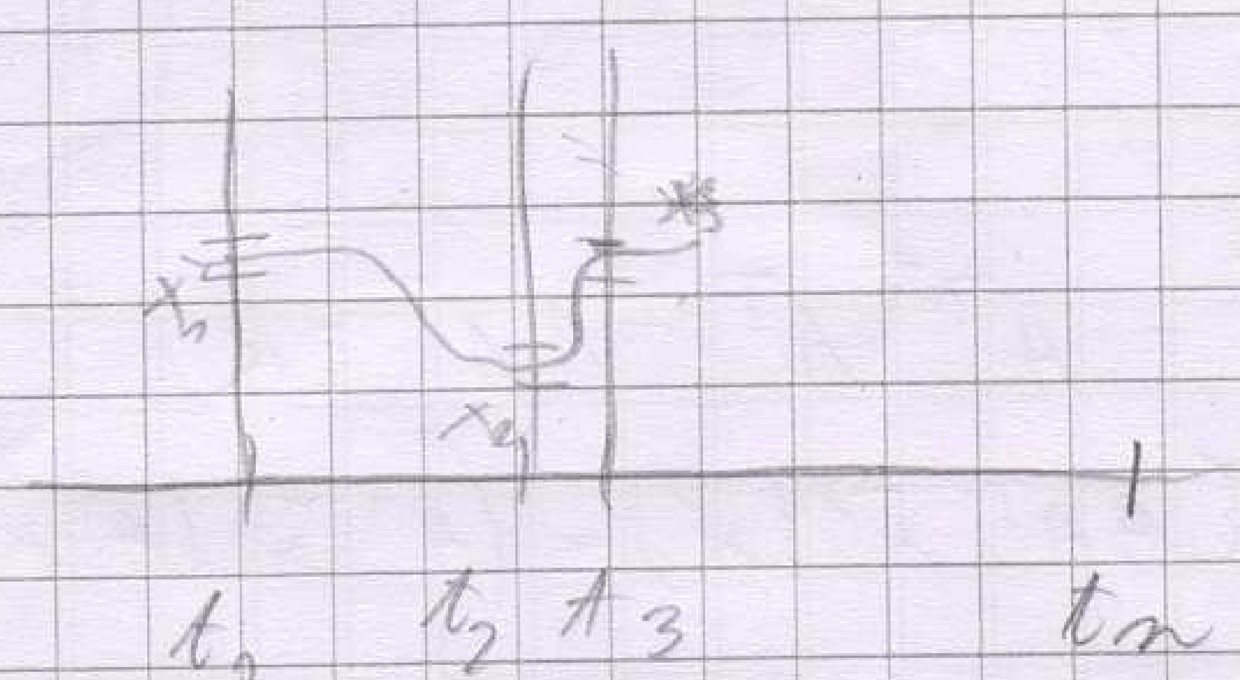


1D gęstość od W_t

$$g_1(t,x) = \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}$$

$$(W_{t_1}, W_{t_2}, \dots, W_{t_n})$$

$$g_{t_1,\dots,t_n}(x_1,\dots,x_n)$$



$$P(x < X < x+\Delta x) \approx \Delta x \cdot g(x) \qquad g_t(x)$$

$$g_{t_1,\dots,t_n}(x_1,\dots,x_n) = g_{t_1}(x_1)\, g_{t_2-t_1}(x_2-x_1) \dots g_{t_n-t_{n-1}}(x_n-x_{n-1}) =$$

$$= \frac{1}{\sigma^n (2\pi)^{\frac{n}{2}} \sqrt{t_1(t_2-t_1)\dots(t_n-t_{n-1})}} \cdot$$

$$\cdot \exp\left\{-\frac{1}{2\sigma^2}\left[\frac{x_1^2}{t_1} + \frac{(x_2-x_1)^2}{t_2-t_1} + \dots + \frac{(x_n-x_{n-1})^2}{t_n-t_{n-1}}\right]\right\}$$

$\mu = 0$

$$R(t,s) = E(X_t X_s) = E(W_t, W_s) = E(W_t(W_s - W_t) + W_t^2) =$$

$$= E(W_t)^2 + E(W_t(W_s-W_t)) =$$

$$= \sigma^2 t + \underbrace{E(W_t)}_{=0} \cdot E(W_s - W_t)$$

$$\boxed{R(t,s) = \sigma^2 \min(t,s)}$$

$(W_{t_1}, \dots, W_{t_n})$

$$r_{ij} = \mathbb{E}(W_{t_i}, W_{t_j}) = \sigma^2 \min(t_i, t_j) = \sigma^2 t_i$$

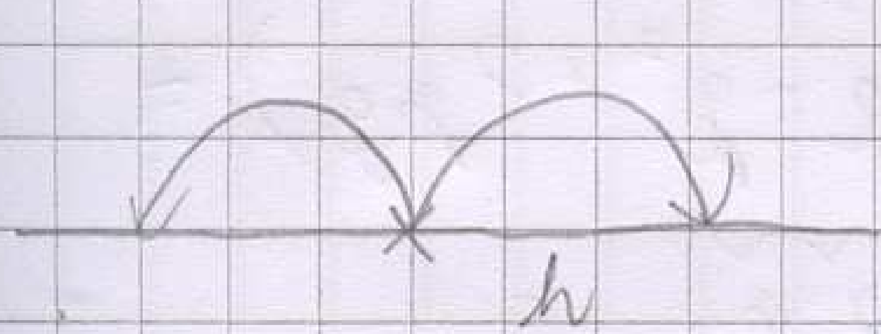
$$R = \sigma^2 \begin{bmatrix} t_1 & t_1 & t_1 & \dots & t_1 \\ t_1 & t_2 & t_2 & \dots & t_2 \\ t_1 & t_2 & \ddots & & \\ & & & \ddots & \\ t_1 & t_2 & \dots & \dots & t_n \end{bmatrix}$$

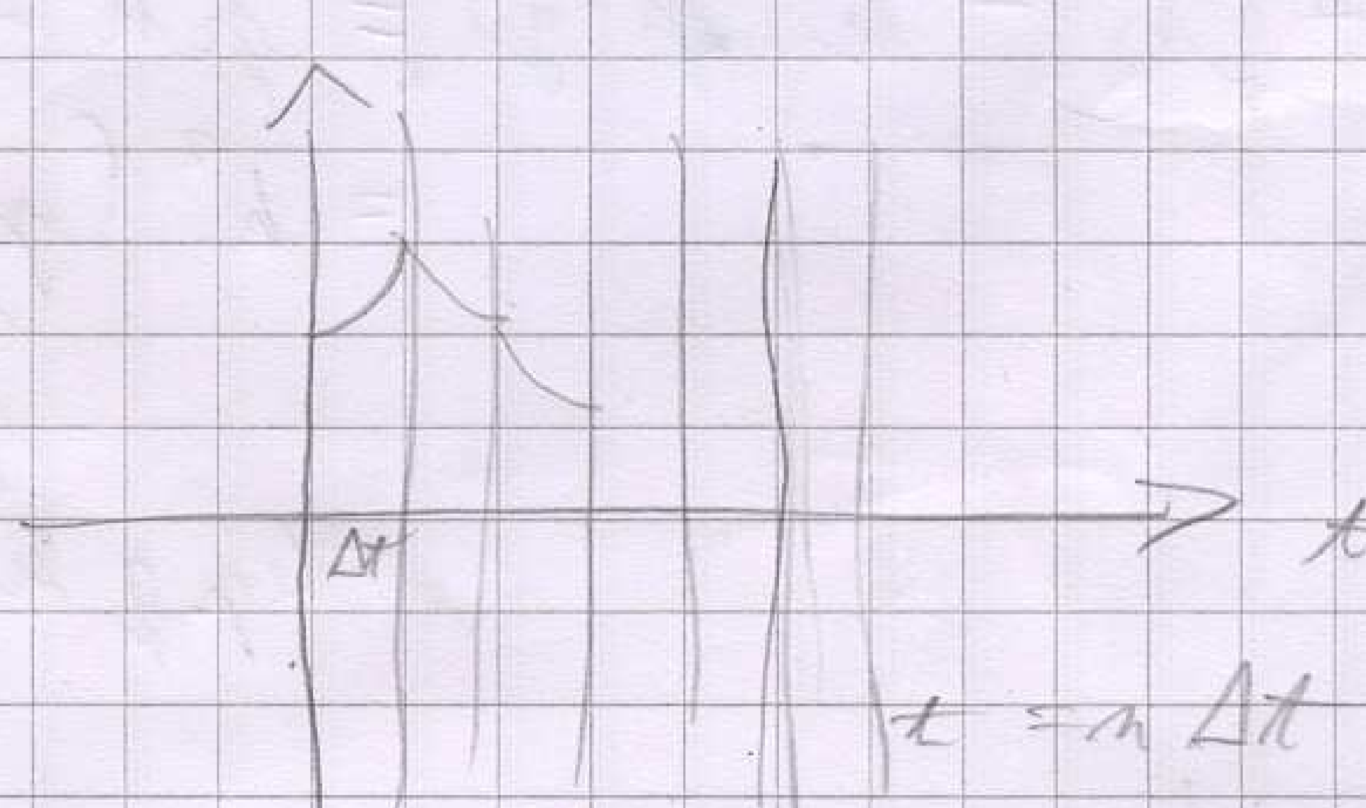
Za $n=2$

$$R = \sigma^2 \begin{bmatrix} t_1 & t_1 \\ t_1 & t_2 \end{bmatrix} \qquad R^{-1} = \frac{1}{\sigma^2} \cdot \frac{1}{t_1(t_2 - t_1)} \begin{bmatrix} t_2 & - \\ -t_1 & \end{bmatrix}$$

$$= \frac{1}{\sigma^2} \begin{bmatrix} \frac{t_2}{t_1(t_2-t_1)} & \frac{-1}{t_2-t_1} \\ \frac{-1}{t_2-t_1} & \frac{1}{t_2-t_1} \end{bmatrix}$$

$$\frac{x_1^2}{t_1} + \frac{(x_2-x_1)^2}{t_2-t_1} = (x_1, x_2)\, R^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$





$$X_k \sim \begin{pmatrix} -h & h \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$X(t) = X_1 + X_2 + \dots + X_n$$

$$n \to \infty,\ \Delta t \to 0,\ h \to 0$$

$$\mathbb{D}[X(t+s)] = \mathbb{D}[X(s) + (X(t+s) - X(s))] =$$

$$= \mathbb{D}[X(s)] + \mathbb{D}[X(t+s) - X(s)] = \mathbb{D}[X(s)] + \mathbb{D}[X(t)]$$

$$\varphi(t+s) = \varphi(t) + \varphi(s) \Rightarrow \varphi(t) = \sigma^2 t$$

$$X_k \sim \begin{pmatrix} -h & h \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$E(X_k) = 0$$
$$D(X_k) = h^2$$

$$\frac{\sum_{k=1}^{n} X_k}{h\sqrt{n}} \sim N(0,1)$$

$$\sum_{k=1}^{n} X_k \sim N(0, h^2 n)$$

$$X(t) \sim N(0, h^2 n) = N\left(0, \frac{h^2}{\Delta t} \cdot t\right) = N(0, \sigma^2 t)$$

$$\sigma^2 = \lim_{\Delta t \to 0} \frac{h^2}{\Delta t}$$

$$X_k \sim \begin{pmatrix} -h & h \\ q & p \end{pmatrix} \qquad E(X_k) = h(p-q)$$

$$D(X_k) = ph^2 + qh^2 - h^2(p-q)^2 = ph^2 + qh^2 - p^2h^2 + 2pqh^2 - q^2h^2 =$$
$$= h^2(\underbrace{p - p^2}_{pq} + \underbrace{q - q^2}_{qp} + 2pq) =$$
$$= 4pq\,h^2$$

$$\frac{\overbrace{\sum_{k=1}^{n} X_k}^{X(t)} - n(p-q)h}{2h\sqrt{npq}} \sim N(0,1)$$

$$X(t) \sim N(n(p-q)h,\ 4h^2npq)$$
$$N\Big(\underbrace{\frac{(p-q)h}{\Delta t}}_{\to \mu} t,\ \underbrace{\frac{4pqh^2}{\Delta t}}_{\to \sigma^2} t\Big)$$

$$X(t) \sim N(\mu t, \sigma^2 t)$$

iz x_0 u x u vremenu t

$p(x_0, x, t)$

$$\frac{\partial p}{\partial t} = \mu \frac{\partial p}{\partial x_0} + \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2}$$

$$\begin{matrix}\mu = 0 \\ \sigma = 1\end{matrix} \Rightarrow \boxed{\frac{\partial p}{\partial t} = \frac{1}{2}\Delta p}$$

LAPLACEOV
OPERATOR

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