# LINEARNI VREMENSKI NEPROMJENJIVI SUSTAVI (LTI) U PROSTORU STANJA

→ kontinuirano

$$\boxed{\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}}$$

$x \in \mathbb{R}^n$ n-dimenzija prostora stanja

$u \in \mathbb{R}^p$ - ulaz

$y \in \mathbb{R}^q$ - izlaz

$$\dot{x}(t) \equiv \begin{bmatrix} \frac{dx_1(t)}{dt} \\ \vdots \\ \frac{dx_n(t)}{dt} \end{bmatrix} \quad u(t) \equiv \begin{bmatrix} u_1(t) \\ \vdots \\ u_p(t) \end{bmatrix} \quad y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_q(t) \end{bmatrix}$$

LINEARNOST

$$\boxed{f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2)}$$

→ diskretno

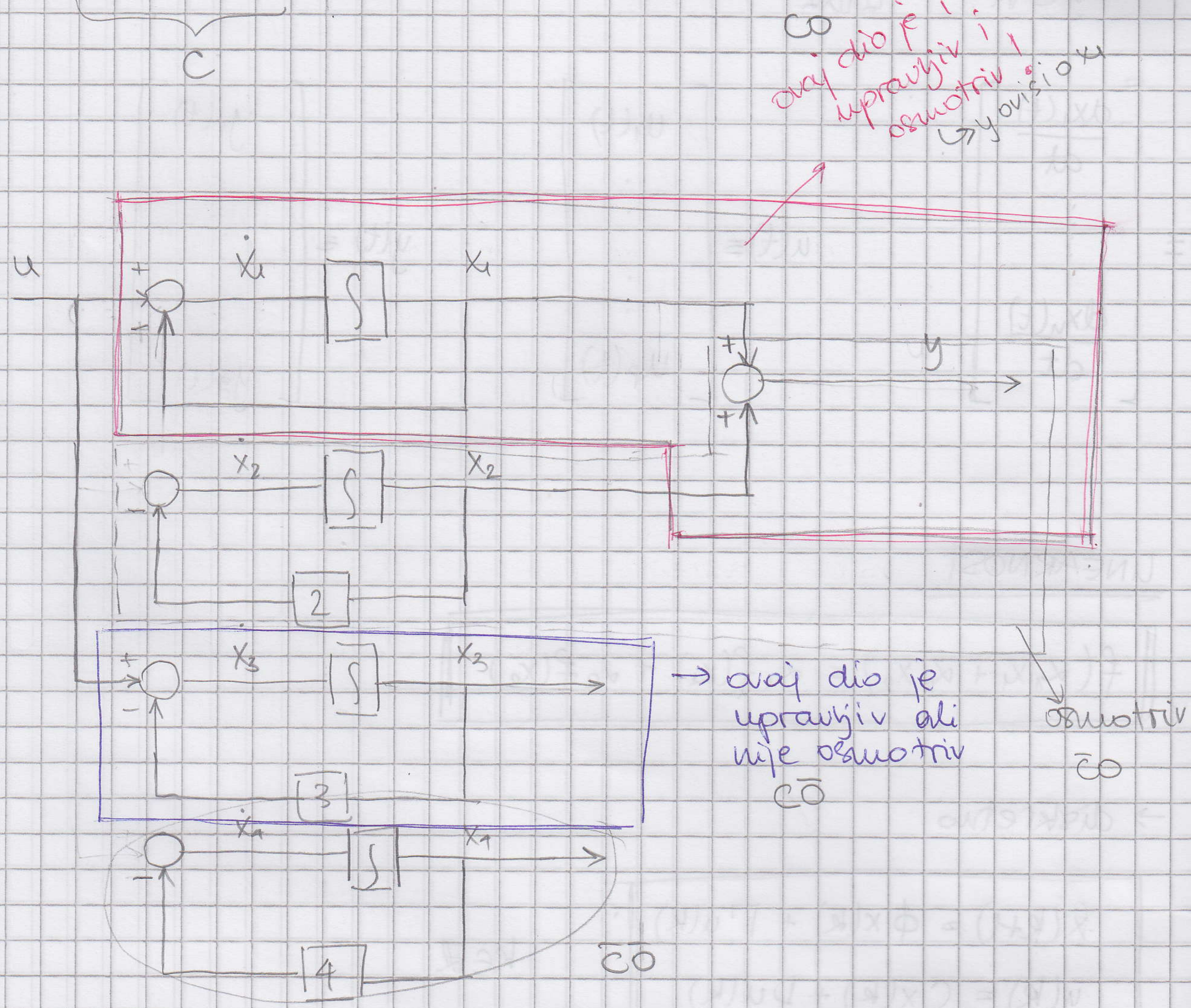
$$\boxed{\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}} \quad k \in \mathbb{Z}$$

# UPRAVLJIVOST i OSMOTRIVOST (controlability & observability)

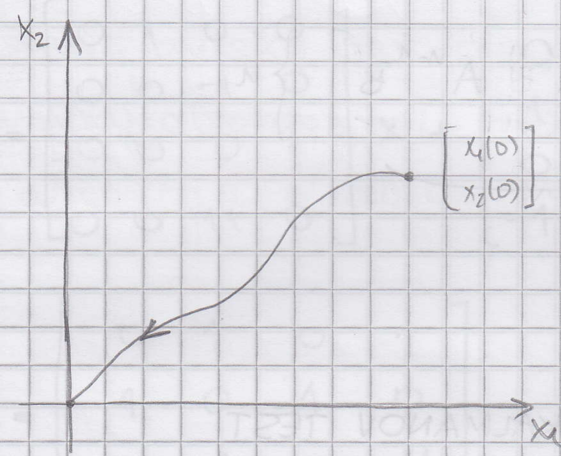
SISO

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}}_{A} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{B} \cdot u$$

$$y = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}}_{C} \cdot x$$



## ⇒ UPRAVLJIVOST

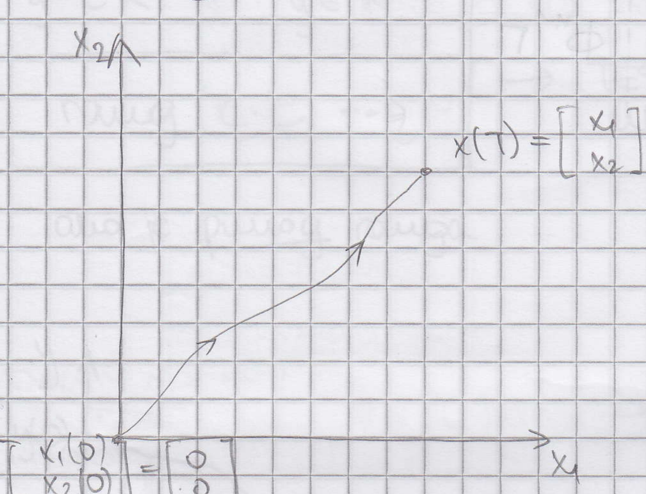



LINEARNI SUSTAVI
→ ako je sust. upr. onda je i dohv. i obrnuto
↳ kod kont.
kod diskr. dohv. → upr.

$$\begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix} \rightarrow \quad x(T)=\begin{bmatrix}0\\0\end{bmatrix}$$

→ u konačnom vremenu $T>0$ možemo dovesti sustav iz nekog $\overbrace{\text{stanja}}^{\text{(početnog)}}$ u ishodište koord. sustava (to radimo pomoću upravljačkog signala)

## ⇒ DOHVATLJIVOST (inverz upravljivosti)




→ u konačnom vremenu stignemo do neke točke $x(T)$

sust. je dohvatljiv ako postoji upr. signal koji u konačnom vrem. $T$ tjera signal u konačno stanje $x(T)$

KONTINUIRANO

$\mathcal{C}$ – matrica upravljivosti (Control)

$$\mathcal{C} = \underbrace{\left[ \underbrace{B}_{n\times p} \;\vdots\; \underbrace{AB}_{n\times n\times n\times p} \;\vdots\; A^2B \;\vdots\; \dots \;\vdots\; A^{n-1}B \right]}_{n\times p} \Big\} n$$

→ sustav je dohvatljiv ako $\mathcal{C}$:

$$\boxed{\mathrm{rang}(\mathcal{C}) = n}$$ → KALMANOV TEST DOHVATLJIVOSTI

rang matrice – koliko ima linearno nezavisnih redaka u matrici

$$\mathcal{C} = \Sigma^T \cdot S \cdot \Sigma$$

DISKRETNO

$$\mathcal{C} = \left[ \Gamma \;\vdots\; \Phi\Gamma \;\vdots\; \Phi^2\Gamma \;\vdots\; \dots \;\vdots\; \Phi^{n-1}\Gamma \right]$$

Primjer

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 11 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot u$$

$n=4$

$A^2 = [-1\ 0\ -1\ 11\ 11]$

$A^3 = [0\ -1\ 120\ 11]$

$$\mathcal{C} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -11 \\ -1 & 0 & -11 & 0 \end{bmatrix}$$

$|\mathcal{C}| = 100 \neq 0$

$\longrightarrow$ rang$(\mathcal{C}) = n$ ako je $|\mathcal{C}| \neq 0$

↳ u slučaju ako je $\mathcal{C}$ kvadratna matrica

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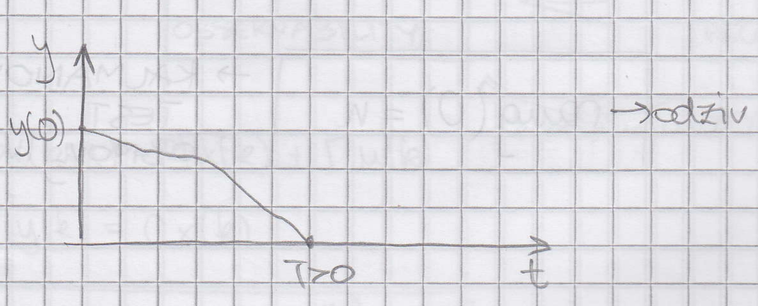
IZLAZNA UPRAVLJIVOST

$y = C \cdot x \qquad y \in R^2$

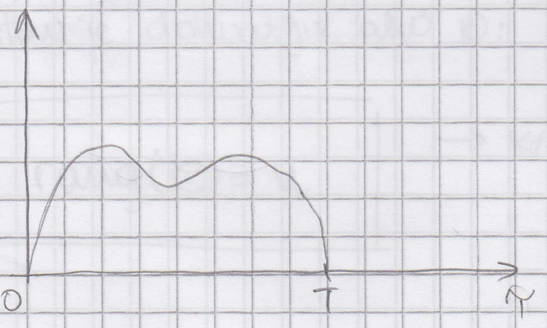
$\boxed{\text{rang}(C \cdot \mathcal{C}) = 2}$ $\rightarrow$ TEST IZLAZNA UPRAVLJIVOST

ako je punog ranga



## ⇒ OSMOTRIVOST

→ sustav je osmotriv ako za bilo koje početno stanje $x_0 = x(0)$ postoji konačno vrijeme $T > 0$ takvo da poznavanje $y(\tau)$ $(0 < \tau < T)$ je dovoljno da bi se odredilo $x(0)$




## OBNOVLJIVOST

→ sustav je obnovljiv ako za bilo koje konačno stanje $x(T) = x_f$ postoji konačan $T$ takvo da poznavanje $y(\tau)$ $(0 < \tau < T)$ je dovoljno da bi odredili $x_f$

KONTINUIRANO

$O$ - matrica osmotrivosti

$$O = \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{n} \Big\} q \cdot n$$

$$\boxed{\text{rang}(O) = n}$$ → KALMANOV TEST OBNOVLJIVOSTI

DISKRETNO

$$\mathcal{O} = \begin{bmatrix} C \\ C\Phi \\ C\Phi^2 \\ \vdots \\ C\Phi^{n-1} \end{bmatrix}$$

→ test je isti!

<u>Primjer</u>

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$n = 4$

$\det(\mathcal{O}) \neq 0$ → sust. je obnovljiv!

$$\begin{tikzcd}
\boxed{\substack{\text{UPRAVLJIVOST} \\ \text{CONTROLLABILITY}}} \arrow[r, bend left, "\Phi \text{ nesingularna}"] & \boxed{\substack{\text{DOHVATLJIVOST} \\ \text{REACHABILITY}}} \arrow[l, Rightarrow] \\
\boxed{\substack{\text{OSMOTRIVOST} \\ \text{OBSERVABILITY}}} \arrow[r, bend right, "\Phi \text{ nesingularna}"'] & \boxed{\substack{\text{OBNOVLJIVOST} \\ \text{RECONSTRUCTABILITY}}} \arrow[l, Rightarrow]
\end{tikzcd}$$

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

$$y(k) = C x(k)$$

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$$x(n) = \Phi^n x(0) + \sum_{i=0}^{n-1} \Phi^{n-1-i} \cdot \Gamma \cdot u(i) = \Phi^n x(0) + \Phi^{n-1} \cdot \Gamma \cdot u(0) + \Phi^{n-2} \Gamma \cdot u(1) + \ldots + \Gamma \cdot u(n-1)$$

$$= \Phi^n \cdot x(0) + \mathcal{C} \cdot \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix}$$

Primjer

$$x(k+1) = \underbrace{\begin{bmatrix} 1 & 1 \\ -0.25 & 0 \end{bmatrix}}_{\Phi} x(k) + \underbrace{\begin{bmatrix} 1 \\ -0.5 \end{bmatrix}}_{\Gamma} u(k)$$

$$x(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

a) $x(2) = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$ dal je dohvatljivo stanje $x(2)$?

$$x(2) = \Phi^2 x(0) + \Phi\Gamma u(0) + \Gamma \cdot u(1)$$

$$\Phi^2 = \begin{bmatrix} 3/4 & 1 \\ -1/4 & -1/4 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} 1 & 0.5 \\ -0.5 & -0.25 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ -1/4 & -1/4 \end{bmatrix} \rightarrow \det(\mathcal{C}) = 0$$

$\text{rang}(\mathcal{C}) < 2$

↳ sustav nije dohv.

ali stanje $x(2)$ možda je?

$$\underbrace{\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}}_{x(2)} = \begin{bmatrix} 5/2 \\ -1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -1/4 \end{bmatrix} u(0) + \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} u(1)$$

$$-1/2 = 5/2 + 1/2\, u(0) + u(1)$$
$$1 = -1 - 1/4\, u(0) - 1/2\, u(1)$$

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$$-3 = 1/2\, u(0) + u(1) \quad \rightarrow \quad u(1) = -3 - 1/2\, u(0) = -3 - \frac{1}{3} = -10/3$$

$$2 = -1/4\, u(0) - 1/2\, u(1)$$

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$$2 = -1/4\, u(0) + 3/2 + u(0)$$
$$1/2 = 3/4\, u(0) \quad \rightarrow \quad u(0) = 2/3$$

→ stanje je dohvatljivo

# REGULATOR PO VARIJABLAMA STANJA

$\dot{x} = Ax + Bu$

$y = Cx$

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- $x(k+1) = \Phi x(k) + Bu(k)$

  $y(k) = Cx(k)$

$M : \exists M^{-1}$ → matrica transformacije

$\Phi \to \tilde{\Phi} = M \Phi M^{-1}$

$\Gamma \to \tilde{\Gamma} = M \cdot \Gamma$

$\tilde{\mathcal{C}} = [\tilde{\Gamma} \mid \tilde{\Phi}\tilde{\Gamma} \mid \dots \mid \tilde{\Phi}^{n-1}\tilde{\Gamma}]$

$\tilde{\mathcal{C}} = [M \cdot \Gamma \mid M\Phi \cancel{M^{-1}} \cancel{M}\Gamma \mid M\Phi^2 \cancel{M^{-1}} \cancel{M}\Gamma \mid \dots \mid M\Phi^{n-1}\Gamma]$

$= M[\Gamma \mid \Phi\Gamma \mid \dots \mid \Phi^{n-1}\Gamma] = M\mathcal{C}$

transformacija:

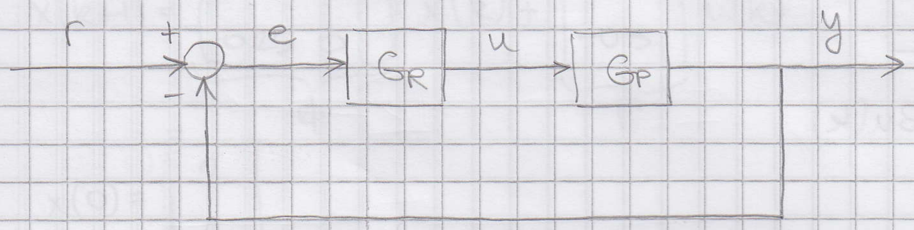
$z = T \cdot x \; / \; T^{-1}$ $\quad T \in R^{n \times n}$ (nije sing.)

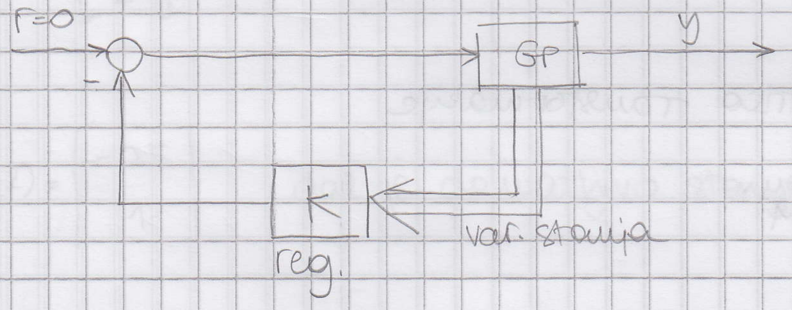
$T^{-1} z = x$

$T^{-1} z(k+1) = \Phi T^{-1} z(k) + \Gamma u(k) \mid T$

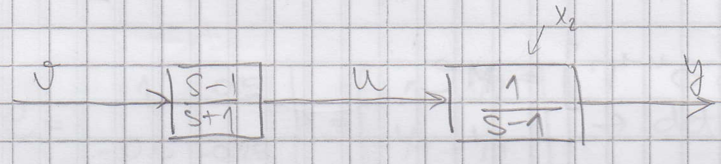
$z(k+1) = \underbrace{T \Phi T^{-1}}_{\tilde{\Phi}} z(k) + T\Gamma u(k)$

# REGULATOR PO VARIJABLAMA STANJA (regulator nultog reda)

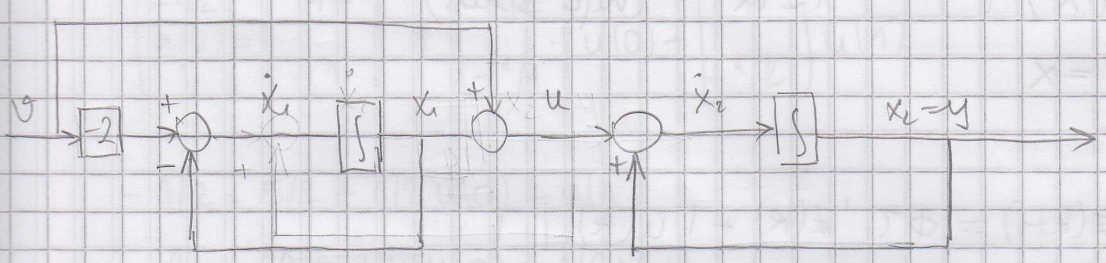



Primjer - STABILIZACIJA NESTABILNOG PROCESA




$$\frac{s-1}{s+1} = \frac{s+1-2}{s+1} = 1 - \frac{2}{s+1}$$

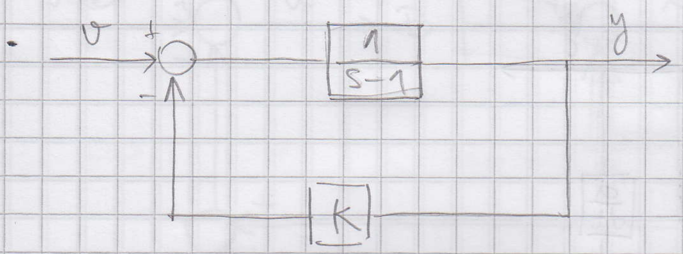



$$\left.\begin{array}{l} v(t) = 0 \\ u(t) = 0 \\ x_1(0) = 0 \end{array}\right\} x_1(t) = 0$$

$x_2(0) > 0 \rightarrow x_2(t) \rightarrow \infty$

$x_2(0) < 0 \rightarrow x_2(t) \rightarrow -\infty$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} v \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

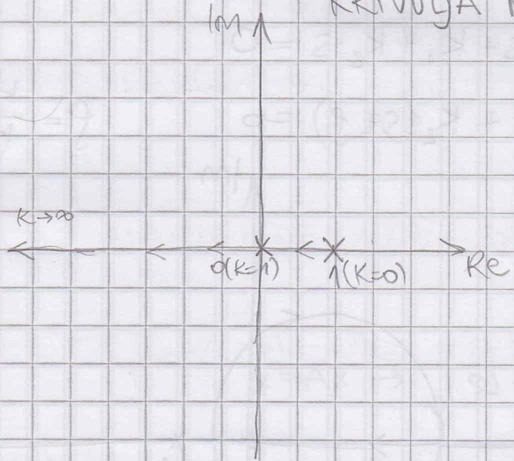



$$G_z(s) = \frac{\frac{1}{s-1}}{1 + K\frac{1}{s-1}} = \frac{1}{s-1+K}$$

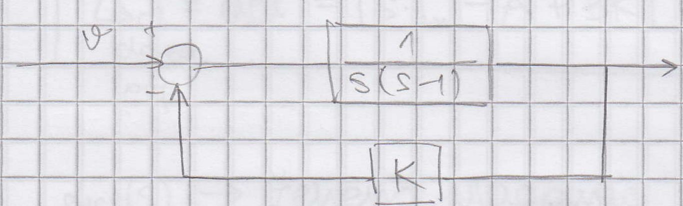
KRIVULJA MJESTA KORIJENA (KMK)

$s-1+K=0$

$s=1-K$



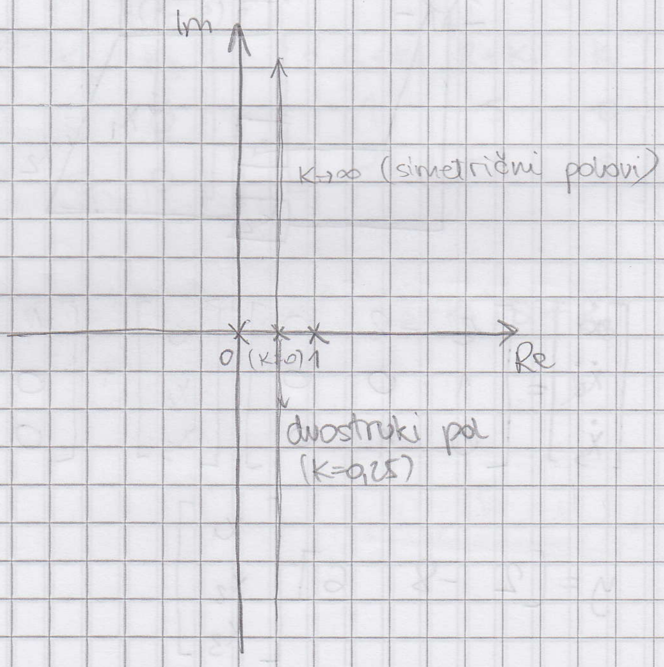

MATLAB → rlocus

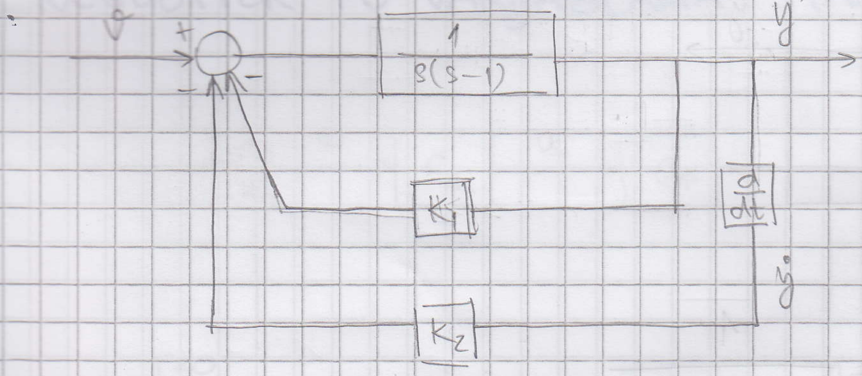



$$G_z(s) = \frac{1}{s^2-s+K}$$

$s^2-s+K=0$

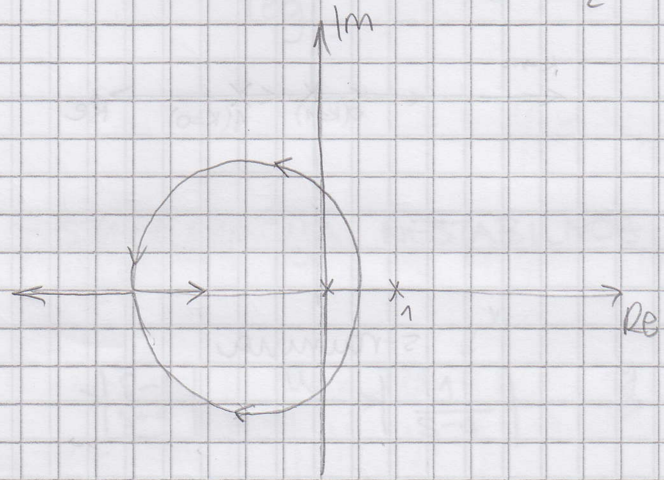
$$s_{12} = \frac{1 \pm \sqrt{1-4K}}{2}$$

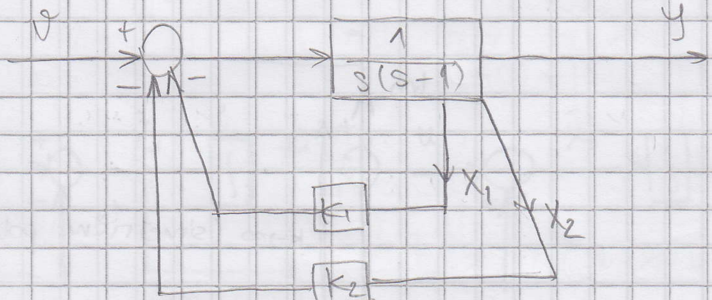





$$s^2 - s + K_1 + K_2 \cdot s = 0$$

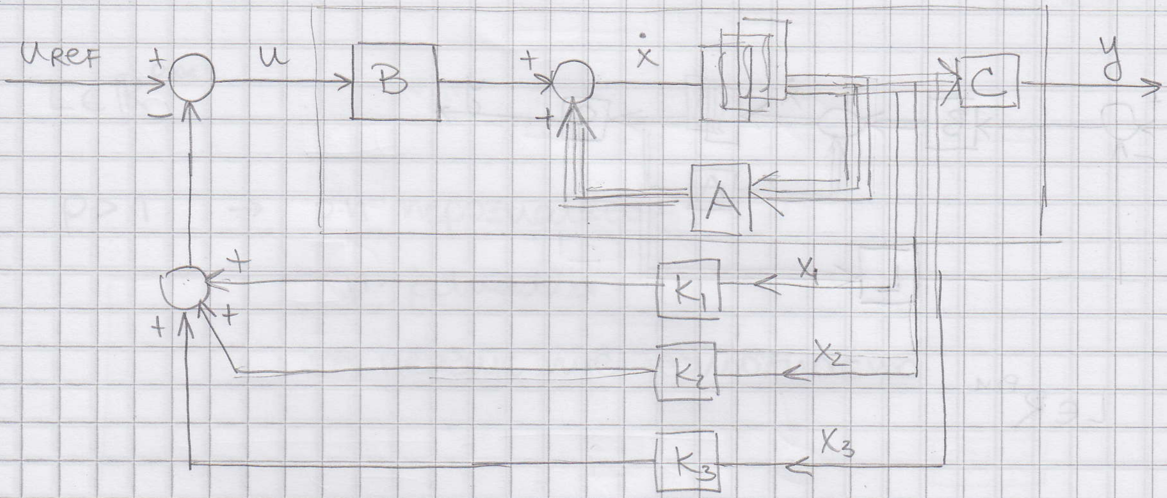
$$s^2 - s + K_2 (s + \rho) = 0 \qquad \rho = \frac{K_1}{K_2} = 1$$




$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 6 & -8 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & -8 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$




$$u = \overset{=0}{u_{ref}} - \underbrace{\begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}}_{K} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\dot{x} = Ax + Bu \qquad\qquad \dot{x} = Ax \rightarrow |sI - A| = 0$$

$$\dot{x} = (A - B \cdot K)\, x$$

$$\boxed{\Delta_{cl} = \det = |s \cdot I_n - A + BK| = 0} \rightarrow \text{karakteristična jednadžba}$$

closed loop

$\alpha_{cl}(s) \rightarrow$ željeno vladanje sustava

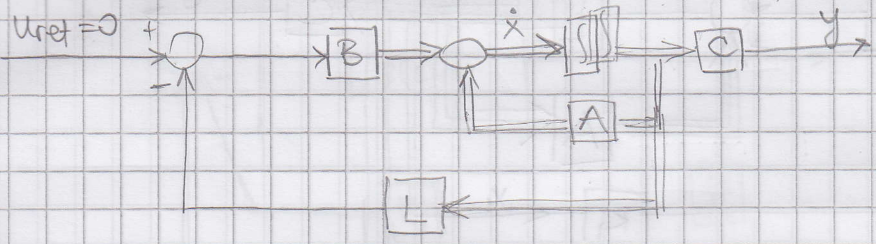
$$s \cdot I - A + BK = \begin{bmatrix} s-6 & 8 & 0 \\ -1 & s & 0 \\ 0 & -1 & s \end{bmatrix} + \begin{bmatrix} K_1 & K_2 & K_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} s-6+K_1 & 8+K_2 & K_3 \\ -1 & s & 0 \\ 0 & -1 & s \end{bmatrix}$$

$$\Delta_{cl} = (s-6+K_1) \cdot s^2 + (8+K_2) \cdot s + K_3 \cdot = s^3 + (K_1 - 6)s^2 + (8 + K_2)s + K_3$$

$$\alpha_{cl}(s) = s^3 + a_2 s^2 + a_1 s + a_0$$

ili

$$\alpha_{cl}(s) = (s - s_1)(s - s_2)(s - s_3)$$




$L \in \mathbb{R}^{p \times n}$

$A_{cl} = A - B \cdot L$ → mat. dinamike zatvorenog kruga

→ karakt. jedn. zat. kruga

$$A_{cl}(s) = |s \cdot I - (A - BL)| = s^n + a_{cl1} s^{n-1} + \ldots + a_{cln} = \alpha_{cl}(s)$$

$$\alpha_{cl}(s) = s^n + \alpha_{cl1} s^{n-1} + \ldots + \alpha_{cln}$$

$$A = \begin{bmatrix} -a_1 & -a_2 & \ldots & -a_n \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & 0 & & \vdots \\ \vdots & & & \vdots \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow BL = \begin{bmatrix} l_1 & l_2 & \ldots & l_n \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{bmatrix}$$

$$A_{cl} = A - B \cdot L = \begin{bmatrix} (-a_1 - l_1) & (-a_2 - l_2) & \ldots & (-a_n - l_n) \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

$$\boxed{l_i = \alpha_i - a_i}$$

$L \in \mathbb{R}^{p \times n}$

$p > 1 \rightarrow p \cdot n$ nepoznanica!

$n$ - jednadžbi

$\Rightarrow$ rješenje nije jednoznačno

Primjer

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0{,}627 & 0{,}361 \\ 0{,}0901 & 0{,}853 \end{bmatrix}}_{\Phi} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 0{,}0251 \\ 0{,}115 \end{bmatrix}}_{\Gamma} u$$

$\rightarrow$ Željeni polovi:

$$z_{1,2} = 0{,}5 \pm j0{,}2 \rightarrow \Delta_{cl}(z) = (z - z_1)(z - z_2)$$

$$= (z - 0{,}5 - j0{,}2)(z - 0{,}5 + j0{,}2)$$

$$= z^2 - z + 0{,}29$$

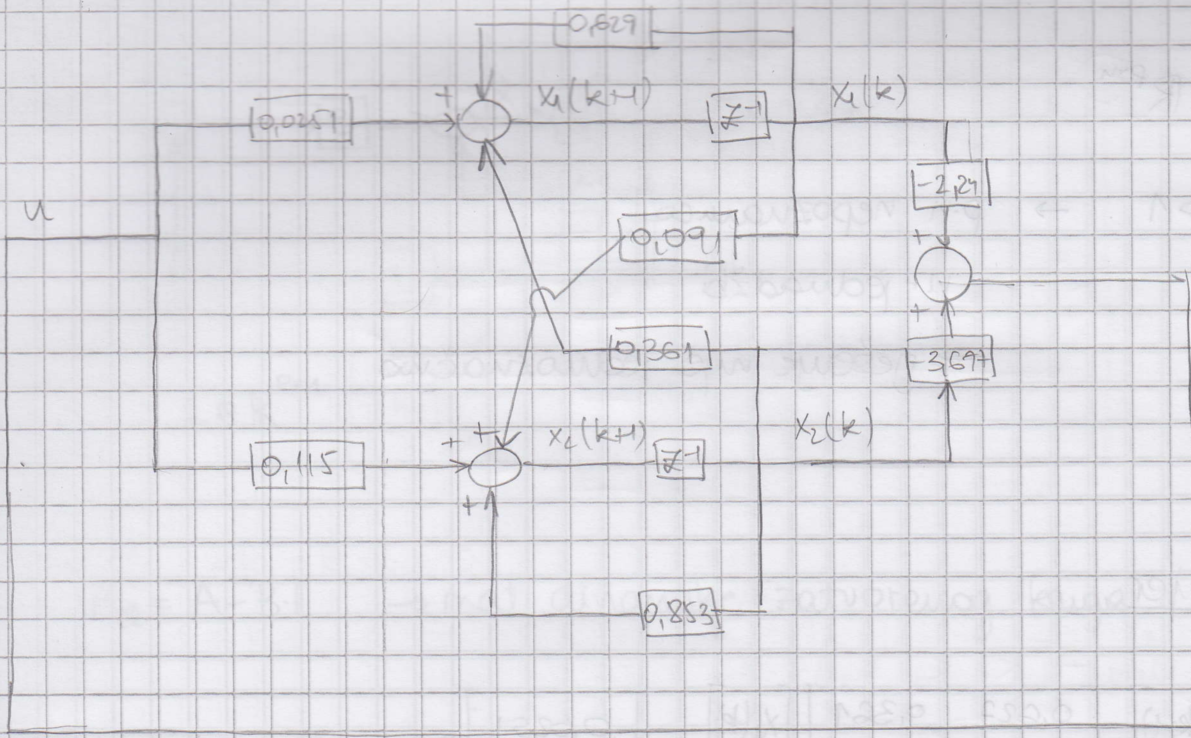
$$\Delta_{cl}(z) = |zI - \Phi + \Gamma \cdot L| = 0$$

$$\Gamma \cdot L = \begin{bmatrix} 0{,}0251 \\ 0{,}115 \end{bmatrix} \begin{bmatrix} l_1 & l_2 \end{bmatrix} = \begin{bmatrix} 0{,}0251 l_1 & 0{,}0251 l_2 \\ 0{,}115 l_1 & 0{,}115 l_2 \end{bmatrix}$$

$$zI - \Phi + \Gamma \cdot L = \begin{bmatrix} z - 0{,}627 + 0{,}0251 l_1 & z - 0{,}361 + 0{,}0251 l_2 \\ z - 0{,}0901 + 0{,}115 l_1 & z - 0{,}853 + 0{,}115 l_2 \end{bmatrix}$$

$$\Delta_{cl}(z) = z^2 - (1{,}48 - 0{,}0251 l_1 - 0{,}115 l_2) z + 0{,}05025 + 0{,}002 l_1 - 0{,}069 l_2$$

$$L = \begin{bmatrix} 2{,}24 & 3{,}677 \end{bmatrix}$$




| POVRATNA VEZA PO VAR. STANJA | POVRATNA VEZA PO IZLAZU |
|---|---|
| • ako i samo ako je sustav potpuno upravljiv za SISO sustav možemo postaviti sve svojstvene vrijednosti zat. kruga po želji | • potrebno je mjeriti samo izlaze |
| • nije potrebno derivirati izlaz | • općenito povećava se red zat. kruga |
| • red zat. kruga jednak je redu procesa | • nule regulatora utječu na zat. krug |