INTEGRALNI RAČUN

11.1. NEODREĐENI INTEGRALI

ZZV-5.) Nađi funkciju $f(x)$ za koju vrijedi $F'(x) = e^{-x}$ uz uvijet $F(0)=0$.

→ DIFERENCIJALNE JEDNADŽBE → $F(x) = ?$

↳ cijela FIZIKA 2 ....

$F(x) = -e^{-x} + C$ → SKUP svih funkcija različitih za constant

$F(0) = -e^{0} + C = 0$

$\boxed{C = 1}$ → $\boxed{F(x) = -e^{-x} + 1}$

ČESTO NA ISPITIMA

def.

Funkcija $F(x)$ se zove PRIMITIVNA funkcija od $f(x)$ na intervalu $[a,b]$ ako za svaki $x \in (a,b)$ vrijedi $\boxed{F'(x) = f(x)}$.

npr.

ako je $f(x) = \sin x$ → $F_1(x) = -\cos x + 314$

$F_2(x) = -\cos x + \pi^{\pi^{\pi^{\cdots}}}$ u mat.

⋮

TM Neka su F_1 i F_2 dvije primitivne funkcije od $f(x)$. Tada se one razlikuju za konstantu, tj. $F_2(x) - F_1(x) = C \in \mathbb{R}$.

Dokaz:

$F_1'(x) = F_2'(x) = f(x)$

→ po korolaru La Grangeova TM srednje vrijednosti odmah slijedi da se razlikuju za konstantu ∎

Skup svih primitivnih funkcija od f na (a,b) nazivamo NEODREĐENI INTEGRAL od f i označavamo ga s

$$\int f(x)\,dx = \{F(x)+C,\ c\in\mathbb{R}\}$$

Funkcija za koju postoji neodređeni integral na (a,b) se zove INTEGRABILNA funkcija.

$$\boxed{\int f(x)\,dx = F(x)+C \iff F'(x)=f(x)}$$

podintegralna funkcija

DIFERENCIJAL VARIJABLE PO KOJOJ INTEGRIRAMO

$$\boxed{\int x^{\alpha}\cdot dx = \frac{x^{\alpha+1}}{\alpha+1}+C}$$

$\alpha \neq -1$

sve suprotno od deriviranja

podjelimo i povećamo za 1 ← pomnožimo i smanjimo za 1

$\int dx = x + C$ → phh trivić.... (za ostale fakultete)

za $\alpha = -1$: $\int \frac{dx}{x} = \ln x + C \to D(f)=\mathbb{R}^+$

$D(f)=\mathbb{R}\setminus\{0\}$

→ po def., NISU NA ISTOM intervalu, nemaju istu $D(f)$ pa $\ln x$ NIJE PRIMITIVNA FUNKCIJA OD $\frac{1}{x}$!!!

→ zato definiramo:

$$\boxed{\int \frac{dx}{x} = \ln|x|+C}$$

→ tablica integrala → sve naopako

PRIMJEDBA:

$$\boxed{\int \frac{dx}{x^2-a} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C}$$ na $(-\infty, -a)$ ili $(-a, a)$ ili $(a, +\infty)$

↓

PAMTI NAPAMET!

$$\boxed{\int \frac{dx}{\cos^2 x} = tg\,x + C}$$ **ALI** na intervalima $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$....

↳ BITNO KASNIJE ZA ODREĐENE!

→ NEELEMENTARNI INTEGRALI:

npr.) $\int \frac{\sin x}{x}\,dx$ ili $\int e^{x^2}\,dx$ → NE ZNAMO !! (sve do 4. godine...)

• NEPOSREDNO (DIREKTNO) INTEGRIRANJE •

Pr1.) $\int (2x^2 - 4e^x + 5)\,dx = 2\int x^2\,dx - 4\int e^x\,dx + 5\int dx =$

$$= 2\frac{x^3}{3} - 4e^x + 5x + C$$

Zad.) $\int \frac{(x+1)^2}{\sqrt{x}}\,dx = \int \frac{x^2+2x+1}{\sqrt{x}}\,dx = \int \left(\frac{x^2}{\sqrt{x}} + \frac{2x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right)dx =$

$$= \int \left(x^{\frac{3}{2}} + 2\cdot x^{1/2} + x^{-\frac{1}{2}}\right)dx = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{2\cdot x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$

TABLICA INTEGRALA

1) $\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + c$

2) $\int a^x dx = \frac{a^x}{\ln a} + c$

3) $\int \sin x \, dx = -\cos x + c$

4) $\int \cos x dx = \sin x + c$

5) $\int \frac{dx}{\sin^2 x} = -\mathrm{ctg} x + c$

6) $\int \frac{dx}{\cos^2 x} = \mathrm{tg} x + c$

7) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + c$

8) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \mathrm{arctg}\left(\frac{x}{a}\right) + c$

9) $\int \mathrm{sh} x dx = \mathrm{ch} x + c$, $\int \mathrm{ch} x dx = \mathrm{sh} x + c$

10) $\int \frac{dx}{\mathrm{sh}^2 x} = -\mathrm{cth} x + c$

11) $\int \frac{dx}{\mathrm{ch}^2 x} = \mathrm{th} x + c$

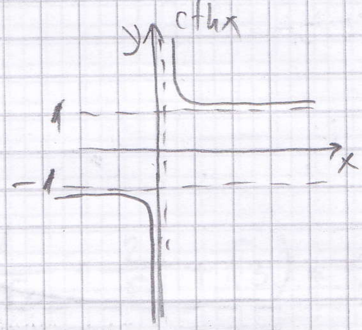
12) $\int \frac{dx}{\sqrt{x^2 + a}} = \ln\left| x + \sqrt{x^2 + a} \right| + c$

13) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left| \frac{x-a}{x+a} \right| + c$

AUDITORNE VJEŽBE ⟵

Ispitaj sve i nacrtaj $f(x) = cth\left(\frac{x^2}{1-x}\right)$.

1) $D(f) = R \setminus \{0,1\}$



2)

1° V.A.:

$$\lim_{x\to 0^+} cth\frac{x^2}{1-x} = cth\left(\lim_{x\to 0^+} \frac{x^2}{1-x}\right) = +\infty$$ ISTO

V.A. $\Rightarrow \boxed{x=0}$

$$\lim_{x\to 1^-} cth\frac{x^2}{1-x} = cth\left(\lim_{x\to 1^-} \frac{x^2}{1-x}\right) = cth(+\infty) = 1$$ NE

$$\lim_{x\to 1^+} cth\frac{x^2}{1-x} = cth\left(\lim_{x\to 1^+} \frac{x^2}{1-x}\right) = cth(-\infty) = -1$$ NE

2° H.A.:

$$\lim_{x\to +\infty} cth\frac{x^2}{1-x} = cth\left(\lim_{x\to +\infty}\frac{x^2}{1-x}\right) = cth\left(\lim_{x\to +\infty} \frac{1}{\frac{1}{x^2}-\frac{1}{x}}\right) =$$

$$= cth(-\infty) = -1 \quad \boxed{y=-1} \quad \text{D.H.A}$$

$$\lim_{x\to -\infty} cth\frac{x^2}{1-x} = 1 \quad \boxed{y=1} \quad \text{L.H.A}$$

3) ekstremi

$$f'(x) = \frac{-1}{sh^2\left(\frac{x^2}{1-x}\right)} \cdot \frac{2x(1-x)+x^2}{(1-x)^2}$$

$$f'(x) = 0 \iff 2x - 2x^2 + x^2 = 0$$

$$2x - x^2 = 0$$

$$x(2-x) = 0$$

NE $x=0$ → NIJE U D(f)

$\boxed{x=2}$

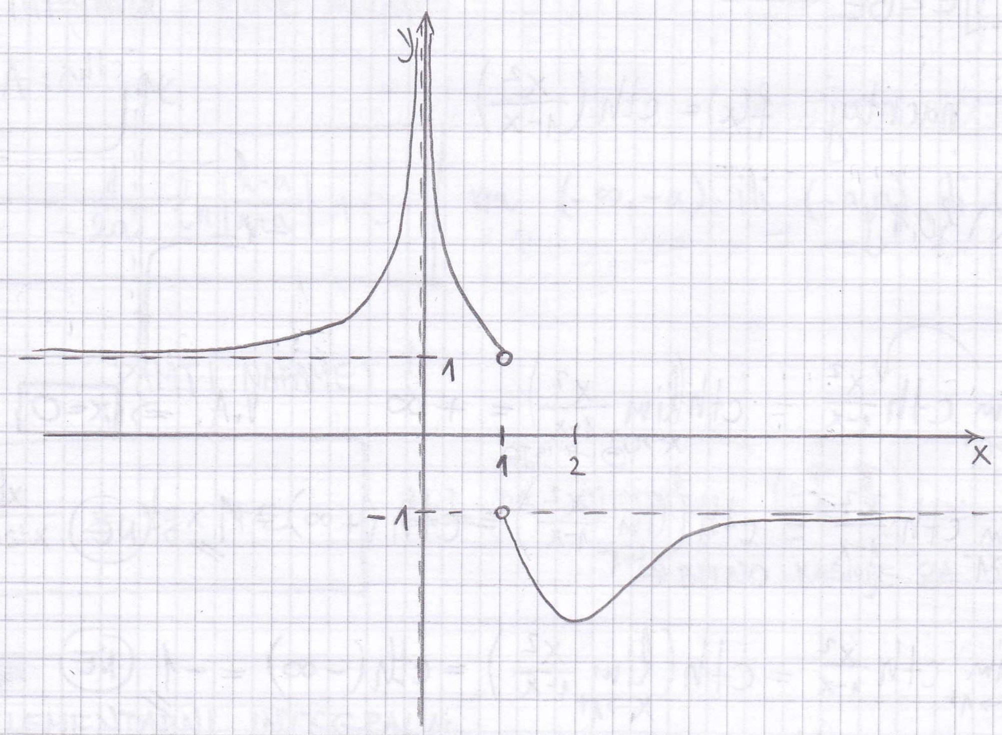
$f(2) = cth(-4)$

$f(2) < -1$

$T_m(2, <-1)$

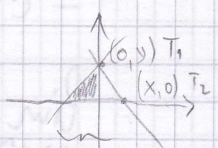
	$-\infty$	0	1	2	$+\infty$
$f'(x)$	+	−	−	+	
$f(x)$	↗	↘	↘	↗	

PAZI! NIJE MAX (at 0) MIN (at 2)



zzv-E-11) Iz točke na osi ordinata povučene su tangente na $y=1-x^2$ tako da s osi apscisa zatvaraju trokut minimalne površine. Odredite ga.

$$y'=-2x$$



2 trokutića čine veliki trok.

$T_1(0,y)$

$$t_1 \dots \; y-(1-x_0^2) = -2x_0 \cdot (0-x_0)$$

$$y = x_0^2+1$$

$$t_2 \dots \; 0-(1-x_0)^2 = -2x_0(x-x_0) \rightarrow T_2(x,0)$$

$$x = \frac{1}{2x_0}+\frac{1}{2}x_0$$

$$P = \frac{1}{2}\cdot(x_0^2+1)\left(\frac{1}{2x_0}+\frac{1}{2}x_0\right)\cdot 2$$

$$= \frac{1}{2}\left(x_0^3+2x_0+\frac{1}{x_0}\right)$$

$$P' = \frac{1}{2}\left(3x^2+2-\frac{1}{x_0^2}\right)$$

$$P'=0 \iff \left(3x_0^2+2-\frac{1}{x_0^2}\right)=0$$

$$3x_0^4+2x_0-1=0$$

$$x_{1,2}^2 = \frac{-2\pm4}{6} \begin{cases} \nearrow -1 \quad \text{NE} \\ \searrow \frac{1}{3} \Rightarrow x=\pm\frac{1}{\sqrt{3}} \end{cases}$$

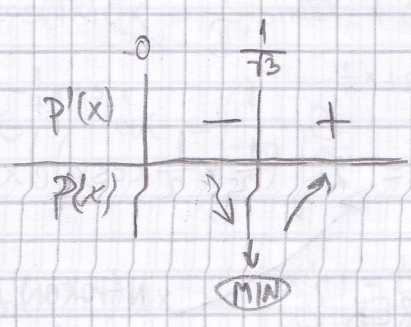
čin:

1) derivirate P'

2) $P''\left(\frac{1}{\sqrt{3}}\right) > 0$

lok. min $= \frac{1}{\sqrt{3}}$

I način:



$$P = \frac{1}{2}\left(\frac{1}{3\sqrt{3}} + \frac{2}{\sqrt{3}} + \sqrt{3}\right) = \frac{8}{3\sqrt{3}}$$

* PREDAVANJA *

„SVOJSTVA NEODREĐENIH INTEGRALA"

a) $\frac{d}{dx}\int f(x)\,dx = \left(\int f(x)\,dx\right)' = f(x)$

b) $\int f'(x)\,dx = f(x) + C$ Dokaz: očito! (po definiciji)

c) $\int a\,f(x)\,dx = a\int f(x)\,dx$

d) $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$!!

PAZI!

$$\int \frac{(\sqrt{x}+1)^2}{x^2}\,dx = \int\left(\frac{1}{x} + 2x^{-\frac{3}{2}} + \frac{1}{x^2}\right)dx = \ln|x| + 2\cdot\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + \frac{x^{-1}}{-1} + C$$

!!

$$\int \frac{x^2}{x^2+5}\,dx = \int \frac{x^2+5-5}{x^2+5}\,dx = \int 1\,dx - 5\cdot\int\frac{dx}{x^2+5} = x - 5\cdot\frac{1}{\sqrt{5}}\operatorname{arctg}\frac{x}{\sqrt{5}} + C$$

TABLICA:

!

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C}$$

TIPIČNA FINTA!

TIN IKER ŠARIĆ GAY

Pr.) $\int tg^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}dx = \int \frac{1-\cos^2 x}{\cos^2 x}dx = \underbrace{\int \frac{1}{\cos^2 x}dx}_{\text{TABLICA! } tg x + C} - \int dx =$

$= tg x - x + C$

Pr.) $\int \frac{\sin x}{tg \frac{x}{2}}dx = \int \frac{2\cancel{\sin\frac{x}{2}}\cos\frac{x}{2}}{\frac{\cancel{\sin\frac{x}{2}}}{\cos\frac{x}{2}}} = 2\int \cos^2\frac{x}{2}dx = 2\int\left(\frac{1}{2}+\frac{1}{2}\cos x\right)dx =$

S MJESECA SE VIDI ? BUREK ?

NAPOKON KORIST OD PODSJETNIKA !

$= \int dx + \int \cos x\,dx = x + \sin x + C$

Pr.) $\int (3x+14)^{69}dx = ??$

$t \quad dt \; ???$

$\Downarrow$

TM „METODA SUPSTITUCIJE"

Neka je $f(x)$ integrabilna na $[a,b]$, a $\varphi(t)$ neprekinuta diferencijabilna na $[\alpha,\beta]$, $\varphi : [\alpha,\beta] \to [a,b]$ i neka je φ injekcija. Tada uz supstituciju $\boxed{x=\varphi(t)}$ vrijedi:

$$\int f(x)dx = \int f(\varphi(t))\varphi'(t)dt$$

Dokaz:

$\frac{d}{dt}\left[\int f(x)dx\right] = \frac{d}{dt}\left[F(x)+C\right] = \frac{d}{dt}\left[F(\varphi(t))+C\right] = F'(\varphi(t))\cdot\varphi' t =$

$= f(\varphi(t))\cdot\varphi'(t)$

ONO ŠTO JE „NAJRUŽNIJE"

Pr.) $\int (3x+14)^{69}dx = \left| \begin{matrix} 3x+14 = t \\ 3dx = dt \\ dx = \frac{dt}{3} \end{matrix} \right| = \int t^{69}\frac{dt}{3} = \frac{1}{3}\cdot\frac{t^{70}}{70} + C$

derivirati i

!OBAVEZNO dx !!!

$= \frac{1}{3}\cdot\frac{(3x+14)^{70}}{70} + C$

OPĆENITO:

$$\int f(ax+b)\,dx = \left| \begin{matrix} ax+b=t \\ a\,dx=dt \\ dx=\frac{dt}{a} \end{matrix} \right| = \frac{1}{a}\int f(t)\,dt = \frac{1}{a}F(t)+C = \frac{1}{a}F(ax+b)+C$$

!!

Pr.)

$$\int \cos(\pi x)\,dx = \frac{1}{\pi}\sin(\pi x)+C$$

$$\int e^{-\frac{1}{7}x+5}\,dx = \frac{1}{-\frac{1}{7}}e^{-\frac{1}{7}x+5}+C$$

Zd.)

$$\int x^2 \operatorname{ch}(x^3+1)\,dx = \left| \begin{matrix} x^3+1=t \\ 3x^2\,dx=dt \\ dx=\frac{dt}{3x^2} \end{matrix} \right| =$$

AKO OSTANE x sa t-om POD INTEGRALOM ONDA JE SUPSTITUC. KRIVA !!

TAKO MORA BITI

$$= \int \cancel{x^2}\operatorname{ch}t \cdot \frac{dt}{3\cancel{x^2}} =$$

→ NE SMIJE BITI x i t u isto vrijeme !

NE ZABORAVI !

$$= \frac{1}{3}\int \operatorname{ch}t\,dt = \frac{1}{3}\operatorname{sh}(\underline{\underline{x^3+1}})+C$$

Zd.)

$$\int \operatorname{tg}x\,dx = \text{NIJE TABLIČNO !!}$$

$$= \int \frac{\sin x}{\cos x}\underbrace{dx}_{=-dt} = \left| \begin{matrix} \cos x=t \\ -\sin x\,dx=dt \\ dx=\frac{dt}{-\sin x} \end{matrix} \right| = -\int \frac{dt}{t} = -\ln|t|+C = -\ln|\cos x|+C$$

Zd.)

$$\int \frac{dx}{\sin x} \cdot \frac{\sin x}{\sin x} = \int \frac{\sin x}{\sin^2 x}\,dx = \int \frac{\sin x}{1-\cos^2 x}\,dx = \left| \begin{matrix} \cos x=t \\ -\sin x\,dx=dt \\ dx=\frac{dt}{-\sin x} \end{matrix} \right| =$$

$$= \int \frac{dt}{t^2-1} = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C$$

$$= \frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right|+C$$

TIN IKER ŠARIC GAY

11.DZ-4.)

A) $\int \frac{\ln^3 x}{x} dx = \frac{\ln^4}{4} + C = \ln + C$!!! TAKO OBAVEZNO NA ISPITU !!!

NEMOJ $\ln^3 x = t$ jer je PRESLOŽENA!

$$= \left| \begin{matrix} \ln x = t \\ \frac{1}{x} dx = dt \\ dx = \frac{dt}{\frac{1}{x}} \end{matrix} \right| = \int t^3 dt = \frac{t^4}{4} + C = \frac{\ln^4 x}{4} + C \quad //$$

Z1-09-4.)

$$\int \frac{\operatorname{arctg} x + x}{x^2+1} dx = \underbrace{\int \frac{\operatorname{arctg} x}{x^2+1} dx}_{} + \underbrace{\int \frac{x}{x^2+1} dx}_{}$$

NAJRUŽNIJE

$$\left| \begin{matrix} \operatorname{arctg} x = t \\ \frac{1}{1+x^2} dx = dt \\ dx = \frac{dt}{\frac{1}{1+x^2}} \end{matrix} \right| \qquad \left| \begin{matrix} x^2+1 = k \\ 2x\, dx = dk \\ dx = \frac{dk}{2x} \end{matrix} \right|$$

$$= \int t\, dt + \frac{1}{2} \int \frac{dk}{k} = \frac{t^2}{2} + \frac{1}{2} \ln|k| + C =$$

$$= \frac{(\operatorname{arctg} x)^2}{2} + \frac{1}{2} \ln|x^2+1| + C \quad //$$

11.DZ-4.)

B) $$\int \frac{e^{2x}}{\sqrt{e^x+1}} dx = \left| \begin{matrix} e^x = t \\ e^x dx = dt \\ dx = \frac{dt}{e^x} \end{matrix} \right| = \int \frac{t}{\sqrt{t+1}} dt = \left| \begin{matrix} t+1 = u \\ dt = du \end{matrix} \right| =$$

$\frac{e^{2x}}{\sqrt{t+1}} \cdot \frac{dt}{e^x}$, $e^{2x} \to t$, $t \to u-1$

$$= \int \frac{u-1}{\sqrt{u}} du = \int (u^{\frac{1}{2}} - u^{-\frac{1}{2}}) du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C =$$

$$= \frac{2}{3} (e^x+1)^{\frac{3}{2}} - 2\sqrt{e^x+1} + C \quad //$$

Pr.)

$$\int \frac{dx}{x^2+2x+4} = \int \frac{dx}{(x+1)^2+3} = \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{x+1}{\sqrt{3}} + C \quad //$$

$\to (x+1)^2 - 1$

LINEARNI POMAK NE UTJEČE !

→ TABLICA (prijašnji primjer)

$\int \frac{1\,dx}{x^2+a^2} = \dots$

SMETA → MORAMO SUPST.

-3.)

$$\int \frac{2x-1}{\sqrt{x^2-2x+10}}\,dx = \int \frac{2x-1}{\sqrt{(x-1)^2+9}}\,dx = \left| \begin{array}{l} x-1=t \\ dx=dt \\ x=t+1 \end{array} \right| =$$

$$= \int \frac{2t+1}{\sqrt{t^2+9}}\,dt = \underbrace{2\int \frac{t}{\sqrt{t^2+9}}\,dt}_{\text{NOVA SUPST.}} + \underbrace{\int \frac{dt}{\sqrt{t^2+9}}}_{\text{TABLICA!}} = \int \frac{du}{\underbrace{\sqrt{u}}_{u^{-\frac{1}{2}}}} + \ln\left|t+\sqrt{t^2+9}\right| + C$$

$$\left| \begin{array}{l} t^2+9=u \\ 2t\,dt=du \end{array} \right|$$

$$= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + \ln\left|x-1+\sqrt{(x-1)^2+9}\right| + C =$$

$$= 2\sqrt{(x-1)^2+9} + \ln\left|x-1+\sqrt{(x-1)^2+9}\right| + C \;//$$

-08.-4.)

$$\int \frac{x^9}{x^{10}-2x^5+5}\,dx = \left| \begin{array}{l} x^5=t \\ 5x^4dx=dt \\ x^4dx=\frac{dt}{5} \end{array} \right| = \frac{1}{5}\cdot\int \frac{t}{t^2-2t+5}\,dt =$$

$x^9 \to x^5\cdot x^4$

FALI DETE !!

$$= \frac{1}{5}\int \frac{t}{(t-1)^2+4}\,dt = \left| \begin{array}{l} t-1=u \;\; (t \to u+1) \\ dt=du \end{array} \right| = \frac{1}{5}\int \frac{u+1}{u^2+4}\,du = \frac{1}{5}\int \frac{u}{u^2+4}\,du + \frac{1}{5}\underbrace{\int \frac{du}{u^2+4}}_{\text{tablica}} =$$

$$\left| \begin{array}{l} u^2+4=s \\ 2u\,du=ds \end{array} \right|$$

$$= \frac{1}{10}\ln\left|(x^5-1)^2+4\right| + \frac{1}{10}\,\mathrm{arctg}\,\frac{x^5-1}{10} + C \;//$$

JASNO KO IZ 3.141569CHKE!!!

-7.)

$$\int \sqrt{a^2-x^2}\,dx = \left| \begin{array}{l} x=a\sin t \\ dx=a\cos t\,dt \\ t=\arcsin\frac{x}{a} \end{array} \right| = \int \sqrt{\underbrace{a^2-a^2\sin^2 t}_{a^2(1-\sin^2 t)}}\cdot a\cos t\,dt =$$

DOBRO PAMTI!!

$$= \int \sqrt{a^2\cos^2 t}\cdot a\cos t\,dt = a^2\int \cos^2 t\,dt =$$

$|\cos t|$, $\cos^2 t$

$$\boxed{\begin{array}{l} \cos^2 x = \frac{1+\cos 2x}{2} \\ \sin^2 x = \frac{1-\cos 2x}{2} \end{array}}$$

$$= a^2\int \frac{1+\cos 2t}{2}\,dt = \frac{a^2}{2}\left(t+\frac{1}{2}\sin 2t\right) + C =$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{a^2}{4}\sin\left(2\overset{k}{\arcsin\frac{x}{a}}\right) + C =$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{a^2}{\cancel{4}}\cancel{2}\sin\left(\arcsin\frac{x}{a}\right)\cos\left(\arcsin\frac{x}{a}\right) + C = \longrightarrow$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{\cancel{a}\cdot x}{2\cdot\cancel{a}}\cdot\sqrt{1-\left(\frac{x}{a}\right)^2} + C =$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{ax}{2}\cdot\sqrt{1-\left(\frac{x}{a}\right)^2} + C$$

11.DZ-11.)

$$\int\frac{\cos^3 x}{\sqrt{\sin x}}\,dx = \left|\begin{array}{l} t=\sin x \\ \cos x\,dx = dt \\ dx = \frac{dt}{\cos x}\end{array}\right| =$$

$$= \int\frac{1-t^2}{\underbrace{\sqrt{t}}_{t^{\frac{1}{2}}}}\,dt = \int\left(t^{-\frac{1}{2}} - t^{\frac{3}{2}}\right)dt = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} - \frac{t^{\frac{5}{2}}}{\frac{5}{2}} + C =$$

$$= 2\sqrt{\sin x} - \frac{2}{5}(\sin x)^{\frac{5}{2}} + C$$

Pr.) $\int x^2 \sin x\,dx = \{\text{NIJEDNA SUPSTITUCIJA}\} = \cancel{\frac{x^3}{3}\cdot(-\cos x) + C}$ NE !

→ INTEGRIRANJE NE VRIJEDI ZA UMNOŽAK I KVOCIJENT !

$\int a\cdot b\,dx \neq \int a\,dx\cdot\int b\,dx$!!

TM „PARCIJALNA INTEGRACIJA"

Neka su f i g diferencijabilne funkcije. Ako postoje sljedeći integrali, tada vrijedi:

$$\int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\,g(x)\,dx$$

Dokaz: → ČESTO NA ISPITIMA

$$\left[f(x)\,g(x) - \int f'(x)\,g(x)\,dx\right]' = \cancel{f'(x)g(x)} + f(x)g'(x) - \cancel{f'(x)g(x)} =$$

$$= f(x)\,g'(x)$$

TRIVIJALNO

→ sada pišemo:

$$\boxed{\int u\cdot dv = u\cdot v - \int v\,du}$$

2)

POANTA: DOBITI JEDNOSTAVNIJI INTEGRAL

$$\int \underbrace{x}_{u}\,\underbrace{e^x dx}_{dv} = \left| \begin{array}{ll} \overset{\text{DERIVIRAMO}}{u=x} & \overset{\text{INTEGRIRAMO}}{dv=e^x dx} \\ du = 1\,dx & v=e^x \end{array} \right| = x\cdot e^x - \int e^x dx = x\cdot e^x - e^x + C$$

„SVE" ZNAMO DERIVIRATI!

UVIJEK ODABERI ONO ŠTO ZNAŠ INTEGRIRATI!

→ dakle, ne mora uvijek prvo biti u, a drugo dv!!

2.)

$$\int \underbrace{x^2}_{u}\,\underbrace{\ln x\, dx}_{dv} =$$

dv → NEMAMO $\int \ln x\, dx$ NE ZNAMO!!!

zato:

$$\left| \begin{array}{ll} u=\ln x & dv = x^2 dx \\ du=\frac{1}{x}dx & v=\frac{x^3}{3} \end{array} \right|$$

$$= \frac{x^3}{3}\ln x - \int \frac{x^{\cancel{3}2}}{3}\cdot\frac{1}{\cancel{x}}dx = \frac{x^3}{3}\ln x - \frac{1}{3}\cdot\frac{x^3}{3} + C$$

$$\int \underbrace{\arcsin x}_{u}\,\underbrace{dx}_{dv}$$

u → JER TO ZNAMO DERIVIRATI!!

$$= \left| \begin{array}{ll} u=\arcsin x & dv=dx \\ du=\frac{1}{\sqrt{1-x^2}}dx & v=x \end{array} \right| =$$

$$= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx = \left| \begin{array}{l} 1-x^2=t \\ -2x\,dx = dt \\ -x\,dx = \frac{dt}{2} \end{array} \right| =$$

$$= x\arcsin x + \frac{1}{2}\int \frac{dt}{\sqrt{t}} = x\arcsin x + \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C =$$

$$= x\cdot\arcsin x + \sqrt{1-x^2} + C$$

→ NAJČEŠĆE NA ISPITU! SVE NAS ISPITAJU!

Zad.)

$$\int \sqrt{x}\,\sin(\sqrt{x})\,dx = \text{PRVO SUPSTITUCIJA S MJESECA} = \left| \begin{array}{l} \sqrt{x} = t \\ \frac{1}{2\sqrt{x}}\,dx = dt \\ dx = 2\sqrt{x}\,dt \end{array} \right| =$$

$\sqrt{x} = t$ → SVE PREKO t

$$= 2\int t^2 \sin t\,dt = \text{S MJESECA PARCIJALNA} = \left| \begin{array}{ll} u = t^2 & dv = \sin t \\ du = 2t\,dt & v = -\cos t \end{array} \right| =$$

SMANJUJEMO POTENCIJU I OLAKŠAVAMO SI

$$= -2t^2\cos t + 4\underbrace{\int t\cos t\,dt}_{\text{parcijalno opet!}} = \left| \begin{array}{ll} u = t & dv = \cos t\,dt \\ du = dt & v = \sin t \end{array} \right| =$$

$$= -2t^2\cos t + 4\cdot\left(t\sin t - \int \sin t\,dt\right) =$$

$$= -2t^2\cos t + 4t\sin t + 4\cos t + C =$$

$$= -2x\cos(\sqrt{x}) + 4\sqrt{x}\cdot\sin(\sqrt{x}) + 4\cos(\sqrt{x}) + C$$ → SIGURNO TAKO NEŠTO NA ISPITU

Z1-08'-3.)

$$\int \underbrace{e^{-x}}_{u}\,\underbrace{\sin x\,dx}_{dv} = \text{PARCIJALNA S MARSA BUREK ??} = \left| \begin{array}{ll} u = e^{-x} & dv = \sin x \\ du = -e^{-x}\,dx & v = -\cos x \end{array} \right|$$

$$= -e^{-x}\cos x - \int e^{-x}\cos x\,dx = \left| \begin{array}{ll} u = e^{-x} & dv = \cos x\,dx \\ du = -e^{-x}\,dx & v = \sin x\,dx \end{array} \right| =$$ CIKLIČKI $\int$ → ILKO IH JAKO VOLI ILKO <3

$$= -e^{-x}\cos x - \left[e^{-x}\sin x + \underbrace{\int e^{-x}\sin x\,dx}_{\text{POČETNI INTEGRAL}}\right]$$

$$\boxed{I = \int e^{-x}\sin x\,dx}$$!!

$$I = -e^{-x}\cos x - e^{-x}\sin x - I$$

$$2I = -e^{-x}(\cos x - \sin x) \quad |:2$$

$$\boxed{I = -\frac{1}{2}e^{-x}(\cos x - \sin x) + C}$$

→ CIKLIČKI INTEGRAL

SIGURNO NA Z1 !!

$$\int \frac{x\cos x}{\sin^2 x}dx = \left| \begin{matrix} u = x & dv = \frac{\cos}{\sin^2 x}dx \\ du = dx & v = -\frac{1}{\sin x} \end{matrix} \right| =$$

INTEGRIRATI

$$\int \frac{\cos x}{\sin^2 x}dx = \left| \begin{matrix} \sin x = t \\ \cos x dx = dt \end{matrix} \right| =$$

UVIJEK DONJU ZA SUPST.

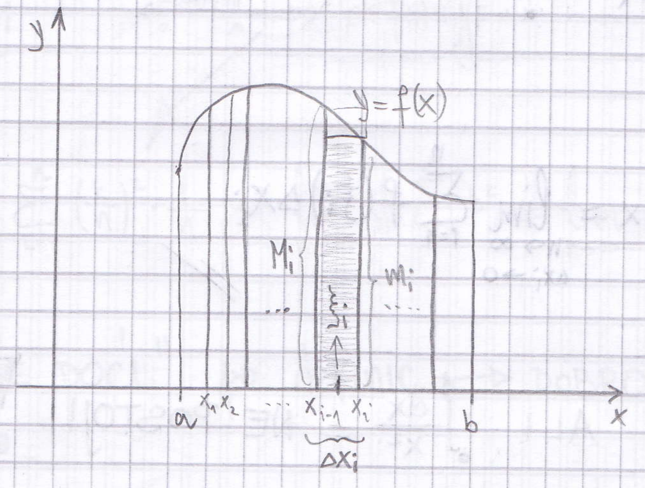
$$= \int \frac{dt}{t^2} = \frac{t^{-1}}{-1} = -\frac{1}{\sin x} + c$$

$$= \frac{-x}{\sin x} + \int \frac{dx}{\sin x} =$$

prošlo predavanje $\frac{\cdot \sin x}{\cdot \sin x}$

$$= -\frac{x}{\sin x} - \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + c$$

11.2. ODREĐENI INTEGRAL



$$a = x_0 < x_1 < x_2 < \dots < x_{i-1} < x_i < \dots x_n < b$$

→ SUBDIVIZIJA INTERVALA $[a,b]$

$\Delta x_i = x_i - x_{i-1}$

$m_i = \inf f(x) \rightarrow$ najmanja vrijednost $f(x)$ na $[x_{i-1}, x_i]$

$M_i = \sup f(x) \rightarrow$ najveća –||– na $[x_{i-1}, x_i]$

$s_\Delta = \sum_{i=1}^{n} m_i \cdot \Delta x_i$ → DONJA INTEGRALNA SUMA

$S_\Delta = \sum_{i=1}^{n} M_i \Delta x_i$ → GORNJA INT. SUMA

$$\sigma_\Delta = \sum_{i=1}^{n} f(\xi_i) \Delta x_i$$

$m(b-a) \leq s_\Delta \leq \sigma_\Delta \leq S_\Delta \leq M(b-a)$ (*) (očito sa slike)

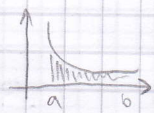
$m = \inf(f(x))$ na $[a,b]$

$M = \sup(f(x))$ na $[a,b]$

$$\left.\begin{array}{ll} I_* = \sup\limits_{\Delta} s_\Delta & \text{donji Riemannov integral} \\ I^* = \inf\limits_{\Delta} S_\Delta & \text{gornji Riemanov integral} \end{array}\right\} I_* \leq I^* \quad \text{(očito)}$$

def. Kažemo da je funkcija $f(x)$ integrabilna (u Riemannovom smislu) na $[a,b]$ ako je $I_* = I^* = I$ i tada taj I nazivamo ODREĐENI INTEGRAL funkcije $f(x)$ na $[a,b]$ i označavamo ga sa

$$\int_a^b f(x)\,dx .$$



analogna def.:

$$\int_a^b f(x)\,dx = \lim_{\substack{n \to \infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i$$

Pr.) $\int \frac{dx}{x^2} = -\frac{1}{x} + C$ ALI $\int_0^1 \frac{dx}{x^2}$ = NE POSTOJI !!

TM

Omeđena funkcija $f(x)$ na $[a,b]$ je integrabilna ako i samo ako za svaki $\varepsilon > 0$ postoji razdioba Δ tako da je $S_\Delta - s_\Delta \leq \varepsilon$.

Dokaz: PMF !!

TM

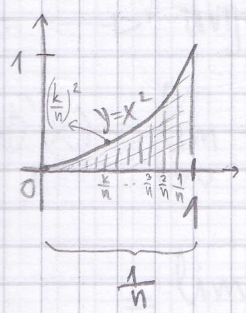
Ako je $f(x)$ neprekinuta na $[a,b]$ onda je integrabilna na $[a,b]$.

VAŽNO^2

Ako je $f(x)$ omeđena na $[a,b]$ i ima samo konačno mnogo točki prekida, tada je integrabilna na $[a,b]$.

Ako je $f(x)$ omeđena i monotona na $[a,b]$ tada je integrabilna na $[a,b]$. → ANALOGIJA SA KONVERGENCIJOM

$$\int_0^1 x^2\,dx = ?$$



po definiciji

$$\int_0^1 x^2\,dx = \lim_{x\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2\cdot\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}k^2 = \lim_{n\to\infty}\frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\begin{matrix}\sim 2n^3\\ \sim 6n^3\end{matrix} = \frac{1}{3}$$

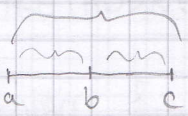
„GOVORI SE DA ĆE DOĆI" po definic. → DOBRO PROUČITI !!!

SVOJSTVA.

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



$$\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$$

DOKAZE NE TREBAMO, IAKO SU NARAVNO TRIVIJ.

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

TM Ako je $f(x) \geq g(x)$ na $[a,b]$ tada je $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

TM „TEOREM SREDNJE VRIJEDNOSTI INTEGRALNOG RAČUNA"

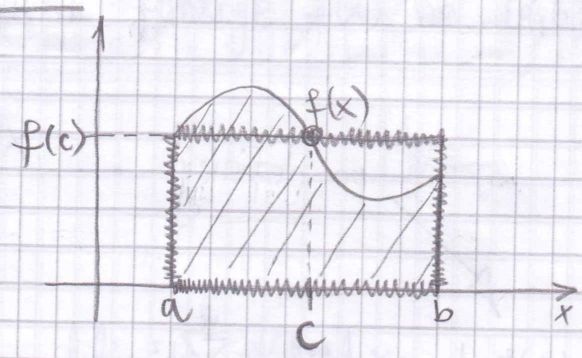
Neka je f neprekinuta funkcija na $[a,b]$. Tada postoji $c \in (a,b)$ tako da

$$\boxed{\int_a^b f(x)\,dx = f(c)\cdot(b-a)}.$$

TAKOĐER:

→ geom. interpretacija:

ISPIT



→ površina ispod krivulje $f(x)$ od a do b jednaka je površini pravokutnika

→ NIJE BILO NITI JEDNOG ZI DO SADA BEZ OVOG TM (ILI ZADATKA S NJIME

Dokaz: (*)

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \quad |:(b-a)$$

$$m \leq \frac{\int_a^b f(x)\,dx}{b-a} \leq M \Rightarrow f(c) = \frac{\int_a^b f(x)\,dx}{b-a}$$

„OSNOVNI TEOREM INTEGRALNOG RAČUNA"

Neka je f neprekinuta funkcija na $[a,b]$ i neka je $x \in [a,b]$.
Tada je $\phi(x) = \int_a^x f(x)\,dx$ diferencijalna i vrijedi:

$$\phi'(x) = \frac{d}{dx}\int_a^x f(x)\,dx = f(x).$$

ILKO MJENJA OZNAKE! Ö

Dokaz: $\xrightarrow{\text{GOTOVO}}$ UVIJEK DOĐE DOKAZ OVOG TEOREMA NA ISPIT!!

$$\phi'(x) = \lim_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h} = \lim_{h\to 0}\frac{\int_a^{x+h} f(x)\,dx - \int_a^x f(x)\,dx}{h} =$$

$$= \lim_{h\to 0}\frac{\int_x^{x+h} f(x)\,dx}{h} \overset{\text{TEOR. SREDNJE VRIJEDN. I.R.}}{=\!=} \lim_{h\to 0}\frac{f(c_h)\cdot\cancel{h}}{\cancel{h}} \overset{f \text{ NEPREKINUTA}}{=}$$

(postoji $c_h \in [x, x+h]$) → prethodni TM

SMANJUJE SE

$$= f(x)$$

„NEWTON-LEIBNIZOVA FORMULA"

Neka je $f(x)$ neprekinuta funkcija na $[a,b]$, te neka je $F(x)$ bilokoja primitivna funkcija od $f(x)$. Tada vrijedi

$$\boxed{\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)}$$

Dokaz: (VOLE GA STAVITI KAO I ONE PRETHODNE) ! ZADNJI DOKAZ U MAT 1 !

$\phi(x)$ iz prethodnog $\boxed{\text{TM}}$ je primitivna fja od $f(x)$ pa je bilokoja druga primitivna fja $F(x) = \phi(x) + C$

$$F(b) - F(a) = \phi(b) - \phi(a) = \int_a^b f(x)\,dx - \underbrace{\int_a^a f(x)\,dx}_{0} = \int_a^b f(x)\,dx$$

Pr.) $\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$ //

GRRR

$\boxed{\text{TM}}$ "METODA SUPSTITUCIJE"

$$\int_a^b f(x)\,dx = \left| \begin{array}{l} x=\varphi(t) \\ \varphi(\alpha)=a \\ \varphi(\beta)=b \end{array} \right| = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

$\boxed{\text{TM}}$ "PARCIJALNA INTEGRACIJA"

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

11. DZ. - 7.)

NAJRUŽNIJI

$x^2=t-1$

$$\int_0^{\sqrt{3}} x^5\sqrt{x^2+1}\,dx = \left| \begin{array}{l} x^2+1=t \\ 2x\,dx=dt \\ dx=\frac{dt}{2x} \\ x=0 \to \alpha=1 \\ x=1 \to \beta=4 \end{array} \right| = \int_1^4 x^{\cancel{5}\,4}\sqrt{t}\,\frac{dt}{2\cancel{x}} = \frac{1}{2}\int_1^4 \underbrace{(t-1)^2\sqrt{t}}_{(t^2-2t+1)\,t^{\frac{1}{2}}}\,dt =$$

PAZI!

$$= \frac{1}{2}\int_1^4 \left(t^{\frac{5}{4}} - 2t^{\frac{3}{2}} + t^{\frac{1}{2}}\right)dt = \frac{1}{2}\left[\frac{t^{\frac{7}{2}}}{\frac{7}{2}} - 2\cdot\frac{t^{\frac{5}{2}}}{\frac{5}{2}} + \frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]\Bigg|_1^4 = \dots = \frac{848}{105}$$ //

!! PAZI → SADA SE NE VRAĆA x u t !!

GRANICE PROMIJENJENE $\Big|_1^4$!!!

4.) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 x \cos^3 x \, dx = \left| \begin{array}{l} \sin x = t \\ \cos x \, dx = dt \\ x = \frac{\pi}{6} \to \alpha = \frac{1}{2} \\ x = \frac{\pi}{2} \to \beta = 1 \end{array} \right| = \int_{\frac{1}{2}}^{1} t^2 \cos^2 x \, dt = \int_{\frac{1}{2}}^{1} t^2 (1 - t^2) \, dt =$

$1 - \sin^2 t = 1 - t^2$

$= \int_{\frac{1}{2}}^{1} (t^2 - t^4) \, dt = \left(\frac{t^3}{3} - \frac{t^5}{5} \right) \Big|_{\frac{1}{2}}^{1} = \left(\frac{1^3}{3} - \frac{1^5}{5} \right) - \left(\frac{\frac{1}{8}}{3} - \frac{\frac{1}{32}}{5} \right)$ OK.

$= \frac{47}{480}$

16.) $\int_0^1 e^{-\sqrt{x}} \, dx = \left| \begin{array}{l} -\sqrt{x} = t \\ -\frac{1}{2\sqrt{x}} dx = dt \\ dx = -2\sqrt{x} \, dt \\ dx = -2t \, dt \\ x = 0 \to \alpha = 0 \\ x = 1 \to \beta = -1 \end{array} \right| = \int_0^{-1} \underbrace{2t}_{u} \cdot \underbrace{e^t dt}_{dv} = \{ \text{PARCIJALNA} \} =$

NAJRUŽNIJE

$= \left| \begin{array}{ll} u = 2t & dv = e^t dt \\ du = 2 \, dt & v = e^t \end{array} \right| = 2te^t \Big|_0^{-1} - \int_0^{-1} 2e^t \, dt =$

$= -2e^{-1} - 0 - 2e^t \Big|_0^{-1} = -2e^{-1} - (2e^{-1} - 2) = -4e^{-1} + 2$

4.) Odredi parametar a tako da vrijedi:

$\int_1^a \frac{\ln^3 x}{x} \, dx = 4$, $a = ?$

$\int_1^a \frac{\ln^3 x}{x} \, dx = \left| \begin{array}{l} \ln x = t \\ \frac{1}{x} dx = dt \\ x = 1 \to \alpha = 0 \\ x = a \to \beta = \ln a \end{array} \right| = \int_0^{\ln a} t^3 \, dt = \frac{t^4}{4} \Big|_0^{\ln a} = \frac{(\ln a)^4}{4} = 4$

najružniji

$(\ln a)^4 = 16 \quad | \sqrt[4]{\ }$

$\boxed{\ln a = \pm 2}$

$\boxed{a = e^{\pm 2}}$

PARNI $\sqrt{\ }$
UVIJEK $\pm$!

ZI-08-7.)

ILKO IH JAKO VOLI !!!

$$\int_0^{\frac{3}{2}} \sqrt{9-x^2}\,dx = \left| \begin{array}{l} x = 3\sin t \\ dx = 3\cos t\,dt \\ x = 0 \to \alpha = 0 \\ x = \frac{3}{2} \to \frac{3}{2} = 3\sin t \\ \quad \sin t = \frac{1}{2} \\ \quad \to \beta = \frac{\pi}{6} \end{array} \right| = \int_0^{\frac{\pi}{6}} 3\cos t \cdot 3\cos t\,dt =$$

$9 - 9\sin^2 t$

$9(1-\sin^2 t) =$

$9\cos^2 t$

$$= 9\int_0^{\frac{\pi}{6}} \cos^2 t\,dt = \frac{9}{2}\int_0^{\frac{\pi}{6}} (1+\cos 2t)\,dt = \frac{9}{2}\left(t + \frac{1}{2}\sin 2t\right)\Big|_0^{\frac{\pi}{6}} =$$

$$= \frac{9}{2}\left(\frac{\pi}{6} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) - \frac{9}{2}\left(0 + \frac{1}{2}\cdot 0\right) = \frac{3\pi}{4} + \frac{9\sqrt{3}}{8}$$

11.ZZV-3.)

$$\int_0^1 x\ln(x^2+1)\,dx = \left| \begin{array}{l} x^2+1 = t \\ 2x\,dx = dt \\ x\,dx = \frac{dt}{2} \\ x = 0 \to \alpha = 1 \\ x = 1 \to \beta = 2 \end{array} \right| = \int_1^2 \ln t\,\frac{dt}{2} = \frac{1}{2}\int_1^2 \ln t\,dt =$$

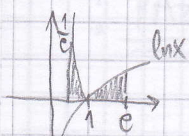
$$= \left| \begin{array}{ll} u = \ln t & dv = dt \\ du = \frac{1}{t}\,dt & v = t \end{array} \right| = \frac{1}{2}\,t\cdot\ln t\Big|_1^2 - \frac{1}{2}\int_1^2 \frac{1}{\cancel{t}}\cancel{t}\,dt = \left(\frac{1}{2}t\ln t - \frac{1}{2}t\right)\Big|_1^2 =$$

$$= (\ln 2 - 1) - \left(0 - \frac{1}{2}\right) = \ln 2 - \frac{1}{2}$$

11.DZ-20.)

prošli primjeri napr

$$\int_{\frac{1}{e}}^{e} |\ln x|\,dx = \int_{\frac{1}{e}}^{1} -\ln x\,dx + \int_1^e \ln x\,dx = -(x\ln x - x)\Big|_{\frac{1}{e}}^{1} + (x\ln x - x)\Big|_1^e =$$



$$= 1 + \left(\frac{1}{e}\ln\frac{1}{e} - \frac{1}{e}\right) + (e\ln e - e) - (0-1) =$$

$$= 2 - \frac{2}{e}$$

BOŽIĆNI POKLONI ZA POD BOR :

1.) $f(x)=\frac{\sqrt{x}}{\ln^2 x}$ ⟶ Odrediti sve i nacrtaj kvalitativni graf !

2.) $\int_1^{\sqrt{3}} \frac{\operatorname{arctg}\frac{1}{x}}{x^2}\,dx = ?$

⟶ SVI SU SE SMIJALI DOK JE OVO PISAO, ALI RAZMISLI ŠTO „POKLONI" ZAPRAVO MOGU ZNAČITI... E MOJ Burek...