Γ

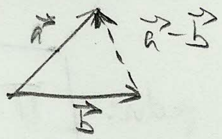
MATEMATIKA 2

SAŽETAK BY Cope

- prema predavanjima prof. Marangunića 2014./2015.
- svaki ciklus je posebno numeriran (2. ciklus kreće od 8. poglavlja)
- na kraju dodani dokazi iz 2. ciklusa te tablica Mc-Laurinovih redova

MAT 2

- oduzimanje vektora:



1. VEKTORI

- ort vektor: $\boxed{\vec{a}_0 = \dfrac{\vec{a}}{|\vec{a}|}}$
- baza u $V^2 = L(\vec{a}_1, \vec{a}_2) = \{\vec{a} = \lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2 \mid \lambda_1, \lambda_2 \in \mathbb{R}\}$
- konveksna kombinacija: $\vec{OC} = s\vec{OA} + (1-s)\vec{OB}$
- polovište dužine: $\boxed{\vec{r}_C = \dfrac{\vec{r}_A + \vec{r}_B}{2}}$
- težište tetraedra: $\vec{r}_T = \dfrac{\vec{r}_A + \vec{r}_B + \vec{r}_C + \vec{r}_D}{4}$
- težište trokuta: $\vec{r}_T = \dfrac{\vec{r}_A + \vec{r}_B + \vec{r}_C}{3}$
- kanonska baza koord. sustava: $\mathbb{R}^3 \Rightarrow (O; \hat{\imath}, \hat{\jmath}, \hat{k})$

$$\vec{OT} = \vec{r}_T = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$|\vec{r}_T| = \sqrt{x^2+y^2+z^2}$$

- opći vektor: $\boxed{\vec{AB} = (x_B - x_A)\hat{\imath} + (y_B - y_A)\hat{\jmath} + (z_B - z_A)\hat{k}}$

* skalarni produkt: $\boxed{\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\varphi}$ $\quad \boxed{\cos\varphi = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}}$

$\Rightarrow \vec{a} \cdot \vec{a} = \vec{a}^2$

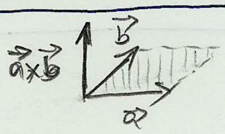
$\Rightarrow \vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0$

$\boxed{\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z}$

- skalarna projekcija: $a_b = \vec{a} \cdot \vec{b}_0$
- vektorska projekcija: $\vec{a}_b = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2} \cdot \vec{b}$

* vektorski produkt:

- veličina: $\boxed{|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\varphi = P_{\square}}$
- $(\vec{a} \times \vec{b}) \perp \vec{a}$
- $\quad$ -II- $\quad \perp \vec{b}$



- $P_\Delta = |\vec{a}| \cdot \dfrac{v_a}{2}$

$\vec{a} \backslash \vec{b}$	$\hat{\imath}$	$\hat{\jmath}$	k
$\hat{\imath}$	$\vec{0}$	$\hat{k}$	$-\hat{\jmath}$
$\hat{\jmath}$	$-\hat{k}$	$\vec{0}$	$\hat{\imath}$
$\hat{k}$	$\hat{\jmath}$	$-\hat{\imath}$	$\vec{0}$

$$\vec{a} \times \vec{b} = \begin{array}{c} \\ \end{array}\begin{vmatrix} \overset{+}{\hat{\imath}} & \overset{-}{\hat{\jmath}} & \overset{+}{\hat{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

(Laplaceov razvoj)

* mješoviti produkt (3 vektora)

$$\boxed{\begin{aligned}(\vec{a} \times \vec{b}) \cdot \vec{c} &= |\vec{a} \times \vec{b}| \cdot |\vec{c}| \cdot \cos\varphi = \\ &= |\vec{a} \times \vec{b}| \cdot v_{AB} = V_{\square}\end{aligned}}$$

$$\boxed{(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}}$$

$\Rightarrow (\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{c} \times \vec{a}) \cdot \vec{b} = (\vec{b} \times \vec{c}) \cdot \vec{a}$

$\rightarrow$ CIKLIČKE PERMUTACIJE

$\Rightarrow (\vec{a} \times \vec{b}) \cdot \vec{c} = -(\vec{b} \times \vec{a}) \cdot \vec{c}$

$$V_{\text{TETRAEDRA}} = \underset{\text{piramida}}{\frac{1}{3}} \cdot \underset{\text{baza } \Delta \text{ a ne } \square}{\frac{1}{2}} V_{\text{PARALELEPIPEDA}}$$

2. PRAVAC I RAVNINA

* ravnina:

- opći oblik jedn.:

$$\boxed{\begin{aligned}&\Pi \equiv A(x-x_0)+B(y-y_0)+C(z-z_0)=0\\&\vec{n}=A\vec{i}+B\vec{j}+C\vec{k}\\&T_0(x_0,y_0,z_0)\end{aligned}}$$

$$Ax+By+Cz+D=0$$

↳ određuje se T

$$\Rightarrow \overrightarrow{T_0T}\cdot\vec{n}=0 \Rightarrow \perp$$

- jedn. Π kroz 3 točke:

$$\boxed{\Pi \equiv \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0}$$

- segmentni oblik:

$$Ax+By+Cz=-D \;/:(-D) \Rightarrow \boxed{\Pi=\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=1}$$

$$\frac{x}{\underbrace{-\frac{D}{A}}_{=p}}+\frac{y}{\underbrace{-\frac{D}{B}}_{=q}}+\frac{z}{\underbrace{-\frac{D}{C}}_{=r}}=1$$

→ p, q, r → odsječci na koord. osima

→ ako je neki = 0, onda je ravnina paralelna s tom osi

- udaljenost točke od ravnine:

$$n_0=\frac{A\vec{i}+B\vec{j}+C\vec{k}}{\sqrt{A^2+B^2+C^2}} \qquad \boxed{d=\overrightarrow{T'T}\cdot\vec{n}_0=\frac{|Ax_T+By_T+Cz_T+D|}{\sqrt{A^2+B^2+C^2}}}$$

- Hesseov oblik ravnine: $\boxed{\dfrac{Ax+By+Cz+D}{\sqrt{A^2+B^2+C^2}}}$

- simetralna ravnina:

$$\boxed{(\Pi_s)_{1,2}=\frac{|A_1x+B_1x+C_1x+D_1|}{\sqrt{A_1^2+B_1^2+C_1^2}}=\pm\frac{|A_2x+B_2x+C_2x+D_2|}{\sqrt{A_2^2+B_2^2+C_2^2}}}$$

- kut među ravninama:

$$\boxed{\cos\varphi=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}}$$

- pramen ravnina: ⇒ sve prolaze kroz presječni pravac

$$\boxed{A_1x+B_1y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0}$$

* pravac: određen sa $T_0(x_0,y_0,z_0)$ i vekt. smjera: $\boxed{\vec{s}=l\vec{i}+m\vec{j}+n\vec{k}}$

- parametarska jedn.:

$$\boxed{p\begin{cases}x=x_0+\lambda l\\y=y_0+\lambda m\\z=z_0+\lambda n\end{cases}}$$

- kanonska jedn.:

$$\boxed{p\equiv\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}}$$

- kroz 2 točke:

$$\boxed{p\equiv\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}}$$

- pravac kao presjek 2 ravnine: $\boxed{\begin{array}{l}\vec{s}=\vec{n}_1\times\vec{n}_2\\ T_0\in(\Pi_1\wedge\Pi_2)\end{array}\Big\}\text{ kanonske jedn.}}$ *

* međusobni položaj p i Π
- sjecište: $\boxed{\left.\begin{array}{l}\Pi = Ax+By+Cz+D=0\\ p=\begin{cases}x=x_0+\lambda l\\ y=y_0+\lambda m\\ z=z_0+\lambda n\end{cases}\end{array}\right\}\text{ sustav}}$
  - iz sustava dobijemo λ i onda uvrstimo u jedn. p
- kut između p i Π: $\boxed{\sin\varphi=\cos(90^\circ-\varphi)=\frac{|\vec{s}\cdot\vec{n}|}{|\vec{s}|\cdot|\vec{n}|}}$
- udaljenost točke od pravca: $\boxed{d=\frac{|\overrightarrow{T_pT}\times\vec{s}|}{|\vec{s}|}}$ *

3. FUNKCIJE VIŠE VARIJABLI

* Euklidski prostor
- $(\vec{x},\vec{y})=\vec{x}\cdot\vec{y}=x_1y_1+x_2y_2+x_3y_3\dots+x_ny_n$
- duljina vektora $\vec{x}\Rightarrow\|\vec{x}\|=\sqrt{\vec{x}^2}=\sqrt{x_1^2+x_2^2+\dots+x_n^2}$

$\boxed{\begin{array}{l}\to(\vec{x},\vec{x})\geq 0\\ \to(\vec{x}+\vec{y},\vec{z})=(\vec{x},\vec{z})+(\vec{y},\vec{z})\\ \to(\vec{x},\vec{y})=(\vec{y},\vec{x})\end{array}}$

- nejednakost Cauchy-Schwartz-Bunikovskog
$$\boxed{(\vec{x},\vec{y})^2\leq\|\vec{x}\|^2\cdot\|\vec{y}\|^2\iff|(\vec{x}\cdot\vec{y})|\leq\|\vec{x}\|\cdot\|\vec{y}\|}$$ *
- norma: $\boxed{\|\vec{x}\|=\sqrt{(\vec{x},\vec{x})}=\sqrt{\vec{x}^2}=\sqrt{x_1^2+x_2^2+\dots+x_n^2}}$
- metrika $\boxed{d(\vec{x},\vec{y})=\|\vec{y}-\vec{x}\|}$
- otvorena kugla u R^n: $K_\varepsilon(S)=\{T\in R^3\mid d(\underbrace{S}_{\text{središte}},T)<\underbrace{\varepsilon}_{\text{radijus}}\}$
- skup $\Omega\subseteq R^n$ otvoren ako za svaku njegovu točku postoji $K_\varepsilon(\vec{x})\subseteq\Omega$

* vektorska funkcija $\mathcal{C}$: $\boxed{\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}}$
* derivacija vektorske funkcije:
$$\boxed{\vec{r}\,'(t)=\lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t}=\lim_{\Delta t\to 0}\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}}\qquad\boxed{\vec{r}\,'(t)=x'(t)\vec{i}+y'(t)\vec{j}+z'(t)\vec{k}}$$
- tangenta na $\mathcal{C}$: $\boxed{t=\frac{x-x_0}{x'(t_0)}=\frac{y-y_0}{y'(t_0)}=\frac{z-z_0}{z'(t_0)}}$
- $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$
- parametrizacija $\mathcal{C}$:
  - → jedna varijabla → parametar t
  - → ostale → $f(t)$

* najvažnije plohe u prostoru

(A) RAVNINA $\pi = A(x-x_0)+B(y-y_0)+C(z-z_0)=0$

(B) CILINDRIČNE (VALJKASTE) PLOHE

- $F(x,y)=0$ -> fali jedna varijabla
- $\mathcal{C}$ - izvodbena krivulja

-> $x^2+y^2=R^2$ => KRUŽNI CILINDAR

$z=y^2$ => PARABOLIČNI CILINDAR

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ => ELIPTIČNI CILINDAR

(C) STOŽASTE (KONUSNE) PLOHE

- svi pravci kroz izvodbenu krivulju $\mathcal{C}$ i vrh V

$$\boxed{\mathcal{C} \dots \begin{cases} F(x,y)=0 \\ z=C \end{cases}} \quad , \quad V(x_V, y_V, z_V)$$

- izvornice: $i = \frac{x-x_V}{x_0-x_V} = \frac{y-y_V}{y_0-y_V} = \frac{z-z_V}{z_0-z_V}$ $\quad x_0 = C\cdot\frac{x}{z}$, $\quad y_0 = C\cdot\frac{y}{z}$

- u $\mathcal{C}$ se naprave zamjene: $\boxed{F\left(C\cdot\frac{x}{z}, C\cdot\frac{y}{z}\right)}$

-> $z^2=\frac{1}{R^2}(x^2+y^2)$ => KRUŽNI STOŽAC

$z^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ => ELIPTIČNI STOŽAC

(D) ROTACIJSKE (ROTACIONE) PLOHE

ZADANA KRIVULJA	OS OKO KOJE ROTIRA	JEDNADŽBA ROT. PLOHE
$y=f(x)$	O_x	$\sqrt{y^2+z^2}=f(x)$
	O_y	$y=f\left(\sqrt{x^2+z^2}\right)$

-> $x^2+z^2+y^2=R^2$ => SFERA

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ => TROOSNI ELIPSOID

$y=x^2+z^2$ => PARABOLOID (KRUŽNI)

$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ => ELIPTIČKI PARABOLOID

-> rotacija hiperbole $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ do:

osi x => $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{b^2}=1$ => DVOPLOŠNI HIPERBOLOID

osi y => $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{a^2}=1$ => JEDNOPLOŠNI HIPERBOLOID

-> sedlasta ploha: $\boxed{z=\frac{x^2}{a^2}-\frac{y^2}{b^2}}$ => HIPERBOLIČKI PARABOLOID

4. DIFF. RAČUN FJA. VIŠE VARIJABLI

* Limes i neprekinutost

– limes niza: $\boxed{\vec{x}_0 = \lim_{k\to\infty} \vec{x}_k \iff (\forall \varepsilon > 0)(\exists k_0 \in N)\left[k > k_0 \Rightarrow \|\vec{x}_k - \vec{x}_0\| < \varepsilon\right]}$

$$d(\vec{x}_k, \vec{x}_0) = \|\vec{x}_k - \vec{x}_0\| < \varepsilon$$

– limes fje. više var. (Heineova definicija)

*

$$\boxed{\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = L \iff \forall(\vec{x}_k)\left[\underbrace{\{\vec{x}_n\}}_{\text{vrijednost niza}} \subseteq D_{(f)} \setminus \{\vec{x}_0\} \wedge \lim \vec{x}_k = \vec{x}_0 \Rightarrow \lim_{k\to\infty} f(\vec{x}_k) = L\right]}$$

– neprekinutost (Cauchy):

$$\boxed{(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)\left(\|x - x_0\| < \delta\right) \Rightarrow \left(|f(x) - f(x_0)| < \varepsilon\right)}$$

– $f(\vec{x})$ neprekinuta u $\vec{x}_0 \iff \lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$

– limes je neovisan o smjeru približavanja granične točke

* parcijalne derivacije

$z = f(x,y)$, $T_0(x_0, y_0) \in M$

$$\left(\frac{\partial f}{\partial x}\right)_0 = f'_x(x_0, y_0) = \varphi'_1(x_0) = \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$\left(\frac{\partial f}{\partial y}\right)_0 = f'_y(x_0 y_0) = \varphi'_2(y_0) = \lim_{\Delta y\to 0} \frac{f(x_0, y_0+\Delta y) - f(x_0, y_0)}{\Delta y}$$

– tangencijalna ravnina u točki $T(x_0, y_0, z(x,y))$

*

$$\Pi \equiv z - z_0 = \left(\frac{\partial z}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial z}{\partial y}\right)_0 (y - y_0)$$

$$n = \frac{x - x_0}{\left(\frac{\partial z}{\partial x}\right)_0} = \frac{y - y_0}{\left(\frac{\partial z}{\partial y}\right)_0} = \frac{z - z_0}{-1}$$

→ kod implicitno zadane

$F(x,y,z) = 0$,

$T_0(x_0, y_0, z_0)$

$n = \frac{z - z_0}{\left(\frac{\partial F}{\partial z}\right)_0} \ldots$

– gradijent fje:

$$\boxed{\begin{array}{l} \operatorname{grad} f = \frac{\partial f}{\partial x_1}\vec{e}_1 + \frac{\partial f}{\partial x_2}\vec{e}_2 + \ldots + \frac{\partial f}{\partial x_n}\vec{e}_n \\ \operatorname{grad} f = \nabla f \end{array}}$$

$$\boxed{\nabla = \frac{\partial}{\partial x_n}\vec{e}_1 + \frac{\partial}{\partial x_2}\vec{e}_2 + \ldots + \frac{\partial}{\partial x_n}\vec{e}_n}$$

$\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ = kanonska baza prostora $(\vec{i}, \vec{j}, \vec{k} \ldots)$

- Niveau plohe:

$u = f(x,y,z) \to$ fiksiramo $u = konst. = C \to \boxed{f(x,y,z) = C} \to$ nivo ploha

- smjer $\overrightarrow{\text{grad}} f$ se poklapa sa smjerom $\vec{n}$ na nivo-plohu

↳ smjer najbrže promjene fje u danoj točki

* diferencijabilnost i diferencijal

- fja diferencijabilna ako u toj točki postoji $\text{grad} f(\vec{x})$ te ako je za svaki $\vec{h} = (h_1, h_2 \ldots h_n)$ promjena fje. Δf moguće prikazati u obliku:

$$\boxed{f(\vec{x}+\vec{h}) - f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{h} + o(\vec{h}) \quad , \|\vec{h}\| \ll} \quad *$$

$$- \boxed{d f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{h}} = \frac{\partial f}{\partial x_1}(x_1) h_1 + \frac{\partial f}{\partial x_2}(x_2) h_2 + \ldots + \frac{\partial f}{\partial x_n}(x_n) h_n$$

- nalaženje df:

$$u = u(x_1, x_2, \ldots x_n) \quad \boxed{du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \ldots + \frac{\partial u}{\partial x_n} dx_n}$$

* Izračun približne vrijednosti fje:

$$\boxed{f(\vec{x}+\vec{h}) \approx f(\vec{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \quad , \|\vec{h}\| \ll}$$

→ posebice: $z(x+\Delta x, y+\Delta y) \approx z(x,y) + \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y \quad , \Delta x \ll , \Delta y \ll$

→ relativna pogreška produkta 2 faktora ≈ zbroju rel. pogrešaka tih faktora

$$\frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{f(x,y)} \approx \frac{\Delta x}{x} + \frac{\Delta y}{y}$$

5. DIFER. RAČUN FJA. VIŠE VAR. II

* derivacija složene fje.

- LANČANO PRAVILO: $\boxed{\frac{dy}{dt} = (f \circ g)'(t) = f'(g(t)) \cdot g'(t)}$

- više var:

$$v = f(u_1, \ldots, u_m)$$

$$u_i = \varphi_i(x_1, \ldots, x_n)$$

$$v = f[\varphi_1(x_1, \ldots, x_n), \varphi_2(x_1, \ldots, x_n), \ldots, \varphi_m(x_1, \ldots, x_n)] = f_1(x_1, x_2, \ldots, x_n)$$

- vrijedi: $\frac{\partial v}{\partial x_i} = \frac{\partial v}{\partial u_1}\frac{\partial u_1}{\partial x_i} + \frac{\partial v}{\partial u_2}\frac{\partial u_2}{\partial x_i} + \ldots + \frac{\partial v}{\partial u_m}\frac{\partial u_m}{\partial x_i} \Rightarrow \boxed{\frac{\partial v}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial v}{\partial u_j}\frac{\partial u_j}{\partial x_i}}$

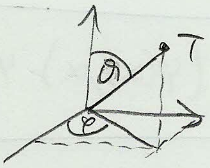
- npr. $z = f(u,v)$, $u = f(x,y)$, $v = f(x,y)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{du}{dx} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dx} = f'_u(u,v)\cdot u'_x(x,y) + f'_v(u,v)\cdot v'_x(x,y)$$

→ analogno $\frac{\partial z}{\partial y}$

- sferne koordinate:

$x = r\cdot\cos\varphi\cdot\sin\vartheta$
$y = r\cdot\sin\varphi\cdot\cos\vartheta$
$z = r\cdot\cos\vartheta$



$x^2+y^2+z^2=\alpha$

*** usmjerena derivacija**

$$\boxed{\frac{\partial f}{\partial \vec{h}}(\vec{x}) = \lim_{t\to 0}\frac{f(\vec{x}+t\cdot\vec{h}_0)-f(\vec{x})}{t}, \quad \vec{h}_0 = \frac{\vec{h}}{\|\vec{h}\|}}$$

*

- izračun: $\boxed{\frac{\partial f}{\partial \vec{h}}(\vec{x}) = \vec{h}_0\cdot \text{grad} f(\vec{x})}$, $\vec{h}_0 \text{ grad} f = \frac{1}{\sqrt{h_1^2+\dots+h_n^2}}\left[h_1\frac{\partial f}{\partial x_1}+\dots+h_n\frac{\partial f}{\partial x_n}\right]$

- ako $\vec{h} = \vec{i}$ ili $\vec{j}$ ili $\vec{k}$, onda $\frac{\partial f}{\partial \vec{h}}(\vec{x}) = \frac{\partial f}{\partial x_i}(\vec{x})$ $\quad \vec{i}^2=\vec{j}^2=\vec{k}^2=1$
  ↳ x, y, z

- najveći rast f-je, tj. usmj. derivacije — u smjeru vektora $h = \text{grad} f(\vec{x})$

$$\boxed{-\|\text{grad} f(\vec{x})\| \le \frac{\partial f}{\partial h} \le \|\text{grad} f(\vec{x})\|}$$

*** gradijent linearne f-je**

- $D\subseteq \mathbb{R}^n$ omeđen i zatvoren skup, afina f-ja. postiže min i MAX na rubu D
  → konstrukcija nivo krivulja $f=f(x,y) \to f=c \to f(x,y)=c$
  ↓ nivo krivulje
  → grad f
  → min i MAX

6. DIFF. RAČUN FJA. VIŠE VAR III.

*** der. implicitno zadane f-je**

$f(x,y,z)=0$ $\quad \boxed{\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}}$, $\quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} \Rightarrow \frac{\partial f}{\partial z}\neq 0$

→ DOKAZ:

$f(x,y)=0 \Rightarrow df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \;/ \frac{\partial f}{\partial y}\neq 0$

$\Leftarrow$

$dy = -\frac{\frac{\partial f}{\partial x}dx}{\frac{\partial f}{\partial y}} \;/ dx \Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

* derivacija integrala ovisnog o parametru

$$\frac{d}{dx}\int_{x_0}^{x} f(t)\,dt = f(x)$$

$$I(\alpha) = \int_{\varphi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx$$ -> ovisi o parametru

$$\boxed{I'(\alpha) = \frac{d}{d\alpha} I(\alpha) = f(\psi(\alpha),\alpha)\cdot\psi'(\alpha) - f(\varphi(\alpha),\alpha)\cdot\varphi'(\alpha) + \int_{\varphi(\alpha)}^{\psi(\alpha)} \frac{\partial f}{\partial\alpha}(x,\alpha)\,dx}$$

* teorem o srednjoj vrijednosti

- $f: U \mapsto \mathbb{R}$ diff. fja, $U \subseteq \mathbb{R}^n$, $\vec{a}, \vec{b} \in U$, takvi da je spojnica točaka također na U, onda na spojnici postoji $\vec{c}$ takva da je:

$$\boxed{f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c})(\vec{b}-\vec{a})}\ *$$

(1) ako je $\operatorname{grad} f = \vec{0}$, onda je $f = \text{konst.}$

(2) ako $\operatorname{grad} f = \operatorname{grad} g$, onda $\exists C$ takav da $(\forall \vec{x} \in U)(f(\vec{x}) = g(\vec{x}) + C)$

DOKAZ:

(1) $f(\vec{b}) - f(\vec{a}) = \underbrace{\operatorname{grad} f(\vec{c})}_{\vec{0} \text{ po pretp.}}(\vec{b}-\vec{a})$

$f(\vec{b}) - f(\vec{a}) = 0$

$f(\vec{b}) = f(\vec{a}) \rightarrow f = \text{konst. na } U$ Q.E.D.

(2) $\operatorname{grad} f = \operatorname{grad} g$

$\operatorname{grad} f - \operatorname{grad} g = \vec{0}$

$\operatorname{grad}(f-g) = \vec{0} \rightarrow$ iz (1) $\rightarrow f - g = \text{konst.}$ Q.E.D.

* derivacije višeg reda

$$\boxed{z''_{xx} = \frac{\partial^2 z}{\partial x^2}}, \quad \boxed{z''_{xy} = \frac{\partial^2 z}{\partial x \partial y}}$$

- Schwartzov teorem: mješovite derivacije ne ovise o redoslijedu deriviranja

$$\boxed{f''_{xy}(x_0,y_0) = f''_{yx}(x_0,y_0)}$$

* drugi diferencijal i diff. višeg reda

2. $d^2 z = d(dz)$, $\boxed{d^n z = d(d^{n-1} z)}$

$$\boxed{d^2 z = z''_{xx}(dx)^2 + 2z''_{xy}\,dx\,dy + z''_{yy}(dy)^2} \Rightarrow d^2 z = (dz)^{[2]} = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^{[2]} z$$

3. $$\boxed{d^3 z = z'''_{xxx}(dx)^3 + 3z'''_{xxy}(dx)^2 dy + 3z'''_{xyy}\,dx(dy)^2 + z'''_{yyy}(dy)^3}\ *$$

$$\boxed{d^n z = (dz)^{[n]}}$$

$$\boxed{d^2u = u''_{xx}(dx)^2 + u''_{yy}(dy)^2 + u''_{zz}(dz)^2 + 2\left(u''_{xy}\,dx\,dy + u''_{xz}\,dx\,dz + u''_{yz}\,dy\,dz\right)}$$

* Taylorova formula

$$\boxed{f(x,y) = T_n(x,y) + R_n(x,y)}$$

$$\boxed{T_n = f(x_0,y_0) + \sum_{i=1}^{n}\frac{1}{n!}\left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^{[n]} f(x_0,y_0)}$$

, $T_0(x_0,y_0)$ → razvojna točka

$$\left(\text{npr. } \frac{1}{1!}\left[(f'_x)_0(x-x_0) + (f'_y)_0(y-y_0)\right]\right)$$

$$\boxed{R_n = \frac{1}{(n+1)!}\left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^{[n+1]} f(T_c)}$$

, T_c → točka na spojnici T_0T

7. EKSTREMI FJA. VIŠE VARIJABLI

* kvadratna forma

- homogena kv. fja. dviju realnih varijabla

$$\boxed{Q(h,k) = a\cdot h^2 + 2b\cdot hk + c\cdot k^2}, \quad a,b,c \in \mathbb{R}$$

- matrica kv. forme: $\boxed{A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}}$
- glavne minore:

$$\boxed{\begin{aligned} M_1 &= \det[a] = a \\ M_2 &= \det\begin{bmatrix} a & b \\ b & c \end{bmatrix} = ac - b^2 \end{aligned}}$$

- definitnost kv. forme:
  - (a) POZ. DEFINITNA ako $Q(h,k) > 0 \quad \forall (h,k) \neq (0,0)$
  - (b) NEG. DEFINITNA ako $Q(h,k) < 0 \quad \forall (h,k) \neq (0,0)$
  - (c) INDEFINITNA ako $Q(h,k) \lessgtr 0 \quad \forall (h,k) \neq (0,0)$
- Sylvesterov teorem:

M_1	M_2	$Q(h,k)$
⊕	⊕	⊕ def
⊖	⊕	⊖ def
⊕	⊖	indef
⊖	⊖	indef

DOKAZ (za ⊕ def):

$(h,k) \neq (0,0)$, npr. $k \neq 0$

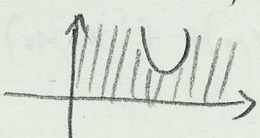
$Q(h,k) = ah^2 + 2bhk + ck^2 =$

izl. k $= \underset{\oplus}{k^2}\underbrace{\left[a\left(\frac{h}{k}\right)^2 + 2b\left(\frac{h}{k}\right) + c\right]}$

→ kv. fja. $\frac{h}{k} = t$

$F(t) = at^2 + 2bt + c$

- diskriminanta $4(b^2 - ac) < 0$ po pretp. $\boxed{M_2 > 0}$
- $a > 0 \Rightarrow Q(h,k) > 0$ $\boxed{M_1 > 0}$

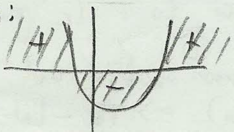


→ analogno za ⊖ def

DOKAZ za indef:

$-M_2 < 0$, tj $ac - b^2 < 0$,

onda $4(b^2 - ac) > 0$ → ima realnih nultočaka:



- $Q(h,k,l)$ kv. homogena fja 3 var.:

$$\boxed{Q(h,k,l) = ah^2 + bk^2 + cl^2 + 2(d\cdot hk + e\cdot hl + f\cdot kl)} \quad *$$

$$\boxed{A = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}}$$

⇒ 2. diferencijal fje. 2 ili 3 varijable je kv. forma s var. dx, dy, dz i matricom A

$z = f(x,y)$ $\quad d^2z = z''_{xx}(dx)^2 + 2z''_{xy}\,dx\,dy + z''_{yy}(dy)^2$

$$\boxed{A = \begin{bmatrix} z''_{xx} & z''_{xy} \\ z''_{xy} & z''_{yy} \end{bmatrix}}$$

- Sylvesterov teorem za fje. 3 var.:

$$\boxed{\begin{aligned} M_1 &= \det[a] = a \\ M_2 &= \det\begin{bmatrix} a & d \\ d & b \end{bmatrix} = ab - d^2 \\ M_3 &= \det[A] \end{aligned}}$$

M_1	M_2	M_3	$Q(h,k,l)$
⊕	⊕	⊕	⊕ def
⊖	⊕	⊖	⊖ def
⊕	⊕	⊖	indef.
⊕	⊖	⊕	
⊕	⊖	⊕	
⊖	⊖	⊖	
⊖	⊖	⊕	
⊖	⊕	⊕	

*lokalni ekstremi

- nužni uvjet: $\nabla f(\vec{a}) = \vec{0}$, tj. sve njegove komponente su jednake 0

$$* \quad \boxed{\frac{\partial f}{\partial x_1} = 0, \ \frac{\partial f}{\partial x_2} = 0, \ \dots, \ \frac{\partial f}{\partial x_n} = 0}$$ → rješenja ovog sustava n jedn. su stacionarne točke

- dostatni uvjet:

$d^2f(T_0)$ → poz. def. forma ⇒ LOKALNI MINIMUM

→ neg. def. forma ⇒ LOKALNI MAXIMUM

→ indef. forma ⇒ SEDLASTA TOČKA

→ f - dvaput neprekidno diff. (klase C^2)

→ T_0 - stac. točka

- dokaz za lok. min. (analogno za max)

→ Taylorova formula oko $T_0(x_0,y_0)$

$f(x,y) = T_1(x,y) + R_1(x,y)$

u stac točki $(f'_x)_0 = (f'_y)_0 = 0$

$f(x,y) = f(x_0,y_0) + \frac{1}{2} d^2f(T_c)$

↳ neprekidnost → predznak T_0 i T_c jednaki, $d^2f(T_0) > 0$, onda je u dovoljno malom okolišu od T_0

$f(x,y) > f(x_0,y_0)$ ⇒ T_0 - lok. min

- kad je određivanje predznaka $d^2z(s)$ teško, služimo se Sylvestrovim teoremom
  - → nađemo Hesseovu matricu i minore
  - → ⊕ def = min, ⊖ def = MAX, indef = sedlo
  - → za 2 var → nije primjenjivo ako $M_2 = 0$
  - za 3 var → —||— —||— —||— je bilo koja minora = 0

* uvjetni ekstremi (vezani), LaGrangeova fja,
  - ekstremi na nekom uvjetu, tj. podskupu $S \subseteq D$
  - uvjet: $\boxed{\varphi(\vec{x}) = 0}$ → može ih biti i više
  - metoda LaGrangeovih multiplikanata
    - → fja $z = f(x,y)$ postiže ekstrem u $(x,y) = (x_0, y_0)$ samo ako takav ekstrem u (x_0, y_0) postiže LaGrangeova fja:

$$\boxed{L(x,y;\lambda) = F(x,y) + \underbrace{\lambda}_{\text{LaGrangeov multiplikator}} \cdot \varphi(x,y)}$$

- → određivanje stac. točke i parametra $\lambda = \lambda_0$

$$\boxed{\begin{aligned} L'_x &= F'_x + \lambda \cdot \varphi'_x = 0 \\ L'_y &= F'_y + \lambda \cdot \varphi'_y = 0 \\ L'_\lambda &= \varphi(x,y) = 0 \end{aligned}}$$

→ jedn uvjeta

- → određivanje karaktera:

$$\boxed{\begin{aligned} &\left(\frac{\partial \varphi}{\partial x}\right)_0 dx + \left(\frac{\partial \varphi}{\partial y}\right)_0 dy = 0 \\ &d^2L(s) = L''_{xx}(s)(dx)^2 + 2L''_{xy}(s)\,dx\,dy + L''_{yy}(s)(dy)^2 \end{aligned}}$$

8. REDOVI

- red konvergira prema $S = \lim_{n\to\infty} S_n = \lim \sum_{i=0}^{n} a_i$

- suma geom. reda: $S_n = \frac{q^n - 1}{q-1}$

- geom. redi $\sum_{n=0}^{\infty} q^n$ → KONV. za $|q|<1$ → DIV. za $|q| \geq 1$

- harmonijski red: $\sum_{n=1}^{\infty} \frac{1}{n} \rightarrow$ DIV.

$\frac{1}{a_n} = \frac{1}{2}\left(\frac{1}{a_{n-1}} + \frac{1}{a_{n+1}}\right)$

→ harmonijska sredina

S 1_A: NUŽAN UVJET KONV. – $\boxed{\lim_{n\to\infty} a_n = 0}$

S 1_B: $\lim_{n\to\infty} a_n \neq 0 \rightarrow \sum a_n$ DIV.

REDOVI S POZ. ČLANOVIMA $\sum a_n,\ a_n > 0,\ \forall n$

S2 S_n – rastući niz → red $\sum a_n$ KONV. samo ako je niz (S_n) omeđen odozgo.

Tm Neka su $\sum a_n$ i $\sum b_n$ KONV i $\alpha, \beta \in \mathbb{R}$. Tada i $\sum(\alpha a_n + \beta b_n)$ KONV.

- ako za $n \geq n_0$ članove reda vrijedi $a_n \geq b_n \rightarrow \sum a_n$ MAJORANTA reda $\sum b_n$
- –||– –||– –||– –||– –||– –||– $b_n \leq a_n \rightarrow \sum b_n$ MINORANTA reda $\sum a_n$

S 3_A $\sum b_n$ KONV. ako ima KONV. MAJORANTU

S 3_B $\sum a_n$ DIV. ako ima DIV. MINORANTU

S4 (KRITERIJ USPOREDBE) $\boxed{\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L,\ L\neq 0,\ L\neq\infty \rightarrow \sum a_n \sim \sum b_n}$
↳ ista konvergencija

S5 (INTEGRALNI KRITERIJ)

$a_n = a(n)$, a → poz., neprekidna i padajuća na (n_0, ∞).

Onda $\sum a_n \sim \int_0^{\infty} a(t)\,dt$

$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^r}}$ – KONV. za $r>1$, – DIV. za $r \leq 1$ → DIRICHLETOV RED

$\boxed{\zeta(r) = \sum_{n=1}^{\infty} \frac{1}{n^r}}$ → RIEMANNOVA ZETA FJA.

S6 (D'ALAMBERTOV KRITERIJ) $\boxed{\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q}$ → $\boxed{\begin{array}{l} 0<q<1 \rightarrow \text{KONV.} \\ q>1 \rightarrow \text{DIV.} \\ q=1 \rightarrow \text{nema odluke} \end{array}}$ ← S7 (CAUCHYEV KRITERIJ) $\boxed{\lim_{n\to\infty} \sqrt[n]{a_n} = q}$

S8 Ako je red apsolutno konvergentan, onda je konvergentan.

$\boxed{\sum |a_n| \text{ KONV} \rightarrow \sum a_n \text{ KONV.}}$

S9 (LEIBNIZOV KRITERIJ ZA ALTERNIRANE REDOVE)

$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ KONV. ako vrijedi: (1) (a_n) monotono padajući
(2) $\lim_{n\to\infty} a_n = 0$

LEIBNIZOV RED:

$\boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}}$ → UVJETNO KONVERGIRA PREMA $\ln 2$

9. REDOVI POTENCIJA, TAYLOROV RED

- oko razvojne točke $x_0 \in \mathbb{R}$: $\boxed{\sum_{n=0}^{\infty} C_n (x-x_0)^n = C_0 + C_1(x-x_0) + C_2(x-x_0)^2 + \dots}$

- najčešće: $x_0 = 0 \rightarrow \boxed{\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots}$

- područje konv. → ispitujemo pomoću Cauchya; $\sqrt[n]{a_n} \dots$
→ moramo ispitati posebno na rubovima

S1 Područje KONV. reda potencija je interval sa središtem u x_0.

$\boxed{D = \{x \mid |x-x_0| < R\}}$ R – polumjer konvergencije

Na skupu $\{x \mid |x-x_0| > R\}$ red DIV, na rubovima moramo provjeriti.

S2 (CAUCHY-HADAMARD) – računanje R:

(1) $\boxed{R = \lim_{n\to\infty} \frac{1}{\left|\frac{C_{n+1}}{C_n}\right|} = \lim_{n\to\infty} \left|\frac{C_n}{C_{n+1}}\right|}$ (2) $\boxed{R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|C_n|}}}$

$\sum_{n=0}^{\infty} C_n x^n = R$
$\Downarrow$
$\sum_{n=0}^{\infty} C_n x^{2n} = \sqrt{R}$

TAYLOROV RED

S7 Fja koja na $(x_0-\delta, x_0+\delta)$ ima sve potrebne derivacije može se na tom intervalu prikazati Taylorovim redom oko razvojne točke x_0:

$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}$ samo ako $\boxed{\lim_{n\to\infty} R_n(x) = 0}$

$\boxed{R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\theta x)}$ za svaki $0 < \theta < 1$

za $x_0 = 0 \rightarrow$ McLaurinov red

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

BINOMNI RED

S8 $\boxed{(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n}$ $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{1\cdot 2\cdot 3\cdot \dots \cdot n}$

za $\alpha = \frac{1}{2}$: $\sqrt{1-x} \approx 1 - \frac{1}{2}x \quad |x|<1$

$\sqrt{1+x} \approx 1 + \frac{1}{2}x \quad |x|<1$

DERIVIRANJE I INTEGRIRANJE REDOVA

$f'(x) = \sum_{n=1}^{\infty} n \cdot C_n (x-x_0)^{n-1}$

$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{C_n}{n+1} (x-x_0)^{n+1}$

- kad je područje konv $|x|<1$, možemo:

$x \to -x$
$x \to x^2$
$x \to -x^2$

10. DIF. JEDN. PRVOG REDA

$y = \varphi(x, C_1, \dots, C_n) \rightarrow$ O.R.

$\Phi(x, y, C_1, \dots, C_n) = 0$ tzv. opći integral zadane jednadžbe
= FAMILIJA KRIVULJA

ALGORITAM: za traženje dif. jedn $\rightarrow$ jedn. familije krivulja derivirati n puta i iz sustava eliminirati konstante

$$\boxed{y^2 / \frac{d}{dx} \rightarrow 2yy'}$$

CAUCHYEV PROBLEM PRVOG REDA

$$\boxed{\begin{aligned} y' &= f(x,y) \\ y(x_0) &= y_0 \end{aligned}}$$

$\rightarrow$ odrediti $y = y(x)$ koja zadovoljava

POLJE SMJEROVA / IZOKLINE

$y' = f(x,y)$ — u svakoj točki (x,y) određen je smjer tangente na krivulju

- iscrtamo sve smjerove na izoklinama $\rightarrow$ polje smjerova $\rightarrow$ uočavamo graf integralne krivulje
- izokline — krivulje u čijim se točkama podudaraju vrijednosti
  — $f(x,y) = c$

TIPOVI DIF. JEDN. 1. REDA:

1. SA SEPARIRANIM VARIJABLAMA

$\boxed{f(y)dy = g(x)dx}$ ili $\boxed{y' = \frac{g(x)}{f(x)}}$

ALGORITAM: neposrednim integriranjem

$$\boxed{\int f(y)dy = \int g(x)dx + C}$$

2. OBLIKA $\boxed{y' = f(ax+by+c)}$

ALGORITAM: svodi se na jedn sa separ. var. supstitucijom $\boxed{ax+by+c = z}$

$ax+by+c = z \,/\, \frac{d}{dx}$

$a + by' = z'$ $\rightarrow$ $y' = \frac{z'-a}{b} = f(z)$ $\rightarrow$ $\boxed{\frac{dz}{a+b\cdot f(z)} = dx}$ $\rightarrow$ integriranje $\downarrow$ vratiti supst. $\rightarrow$ ako treba, iz poč. uvj. naći C i partikularno rješenje

3. HOMOGENE

$M(x,y)$ homogena ako vrijedi: $M(tx,ty) = t^{\alpha} M(x,y)$ → stupanj homogenosti

- dif. homogena: $\boxed{y' = f\left(\frac{y}{x}\right)}$

ALGORITAM:

supst. $\boxed{\frac{y}{x} = z}$

$y = x \cdot z \,/\, \frac{d}{dx}$

$y' = z + xz'$

$\boxed{\frac{dy}{dx} = z + x\frac{dz}{dx}}$

uvrstimo u jedn.

$z + x\frac{dz}{dx} = f(z)$

$x\frac{dz}{dx} = f(z) - z$

$\boxed{\frac{dz}{f(z)-z} = \frac{dx}{x}} \,/\int$

$M(x,y)dx + N(x,y)dy = 0 \Leftrightarrow \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$

- ako su M i N homo. jedn istog stupnja, onda je jedn. homogena

$$\int \frac{dx}{tg(x)} = \int \frac{\cos x}{\sin x}dx = \left| \begin{aligned} \sin x &= t \\ dt &= \cos x\, dx \end{aligned} \right| = \int \frac{dt}{t} = \ln|t| = \boxed{\ln|\sin x|}$$

4. KOJE SE SVODE NA HOMOGENE

$$y' = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right)$$

Dva slučaja:

(1) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = 0$ - paralelni pravci

- supst: $a_1x + b_1y = z \,/\, \frac{d}{dx}$

$a_1 + b_1y' = z'$

$$\boxed{\frac{z' - a_1}{b_1} = f\left(\frac{z + c_1}{kz + c_2}\right)}$$

(2) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$ - sijeku se

nove var: $\boxed{\begin{matrix} x = u + x_0 \\ y = v + y_0 \end{matrix}}$ $\begin{matrix} \text{- nezavisna} \\ \text{- zavisna} \end{matrix}$

$$\frac{dy}{dx} = \frac{d(v + x_0)}{d(v + y_0)} = \frac{dv}{du} = f\left(\frac{a_1u + b_1v}{a_2u + b_2v}\right) =$$

$$= f\left(\frac{a_1 + b_1\frac{u}{v}}{a_2 + b_2\frac{u}{v}}\right) = f\left(\frac{u}{v}\right) \to \text{homo.}$$

5. LINEARNE

$$\boxed{y' + f(x) \cdot y = g(x)} \to \text{funkcija smetnje}$$

METODA VARIJACIJE KONSTANTE:

KORAK ① Riješimo skraćenu (homogenu) jedn: $y' + f(x)y = 0$

$$\boxed{y_0 = C \cdot e^{-\int f(x)dx}}$$

KORAK ②: $C = C(x)$

$$\boxed{y = C(x) \cdot e^{-\int f(x)dx}} \to \text{O.R.}$$

- C treba odrediti

$$C(x) = \int g(x) e^{\int f(x)dx} dx + C_1$$

$$\boxed{y = e^{-\int f(x)dx}\left[\int g(x) e^{\int f(x)dx} dx + C_1\right]}$$

$$\boxed{y = \underbrace{C_1 e^{-\int f(x)dx}}_{y_0} + \underbrace{e^{-\int f(x)dx} \cdot \int g(x) e^{\int f(x)dx} \cdot dx}_{y_p \text{ - partikularno rj.}}}$$

$y = y_0 + y_p$

6. BERNOULLIJEVA DIF. JEDN.

$$\boxed{y' + f(x)y = g(x)y^\alpha}, \quad \alpha \neq 0 \; \alpha \neq 1$$

ALGORITAM:

$$\frac{y'}{y^\alpha} + f(x)y^{1-\alpha} = g(x)$$

supst. $y^{1-\alpha} = z \,/\, \frac{d}{dx}$ → nova zavisna

$(1-\alpha)y^{-\alpha} \cdot y' = z'$

$$\boxed{\frac{z'}{1-\alpha} + f(x) \cdot z = g(x)}$$

11. DIF. JEDN. PRVOG REDA (2)

1. EGZAKTNA JEDN.

$$\boxed{P(x,y)dx + Q(x,y)dy = 0}$$

- egzaktna ako postoji $u = u(x,y)$ čiji je totalni diferencijal lijeva strana jednadžbe

tj. $du = Pdx + Qdy \to u(x,y) = C \to$ opće rj. jednadžbe

S1 (UVJET EGZAKTNOSTI)

Da bi $Pdx + Qdy$ bio tot. diferencijal neke fje, nužno je i dostatno da vrijedi:

$$\boxed{\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}}$$

$S1_A$ $\quad Pdx + Qdy = 0, \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

rješenje:

$$\boxed{\begin{gathered}\int_{x_0}^{x} P(x,y)dx + \int_{y_0}^{y} Q(x_0,y)dy = C \\ \text{ili} \\ \int_{x_0}^{x} P(x,y_0)dx + \int_{y_0}^{y} Q(x,y)dy = C\end{gathered}}$$

2. DIF. JEDN. KOJE SE SVODE NA EGZAKTNU, EULEROV MULTIPLIKATOR

Zadana jedn $Pdx + Qdy = 0$, $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$

-> svodi se na egzaktnu množenjem s $\mu(x,y)$ -> Eulerov multip.

$\underbrace{\mu P}_{P} dx + \underbrace{\mu Q}_{Q} dy = 0 \qquad \frac{\partial}{\partial y}[\mu P] = \frac{\partial}{\partial x}[\mu Q] \rightarrow \boxed{\mu_y' P - \mu_x' Q = \mu(Q_x' - P_y')}$

A) $\mu = \mu(x) \rightarrow \mu_y' = 0, \quad \mu_x' = \frac{d\mu}{dx}$ $\qquad \boxed{\ln\mu(x) = \int \frac{P_y' - Q_x'}{Q} dx}$ -> mora biti $f(x)$

B) $\mu = \mu(y) \rightarrow \mu_x' = 0, \quad \mu_y' = \frac{d\mu}{dy}$ $\qquad \boxed{\ln\mu(y) = -\int \frac{P_y' - Q_x'}{P} dy}$ -> mora biti $f(y)$

-> da nađemo μ, podijelimo $P_y' - Q_x'$ sa Q ili P, nađemo μ i pomnožimo s jedn. pa integriramo i dobijemo rješenje

C) $\mu = \mu(xy)$ $\rightarrow \boxed{\ln\mu = \int \frac{Q_x' - P_y'}{xP - yQ} d(xy)}$ $\qquad \mu = \mu(xy) = \mu(u) \rightarrow \mu_y' = \mu_u' \cdot x$, $\mu_x' = \mu_u' \cdot y$

$\frac{d\mu}{du} \rightarrow \mu_u'(xP - yQ) = \mu(Q_x' - P_y')$

$\frac{d\mu}{\mu} = \frac{Q_x' - P_y'}{xP - yQ} du \,/\int$

D) $\mu = \mu(x^2+y^2)$ $\rightarrow \boxed{\ln\mu = \frac{1}{2}\int \frac{Q_x' - P_y'}{yP - xQ} d(x^2+y^2)}$

$\mu = \mu(u) \rightarrow \mu_x' = \mu_u' \cdot 2x$, $\mu_y' = \mu_u' \cdot 2y$ $\rightarrow \mu_u'(2yP - 2xQ) = \mu(Q_x' - P_y') \rightarrow \frac{d\mu}{\mu} = \frac{Q_x' - P_y'}{2yP - 2xQ} \cdot du \,/\int$

3. ORTOGONALNE I IZOGONALNE TRAJEKTORIJE

$F(x,y,y') = 0$ dif. jedn. familije krivulja $\Phi(x,y,C_1) = 0$. Tražimo jedn. familije koje sijeku zadanu familiju pod konst. kutem.

$\boxed{y_1' = \frac{y_2' + \operatorname{tg}\alpha}{1 - y_2' \cdot \operatorname{tg}\alpha}}$ -> dif. jedn. izogon. traj: 1) $F_1\left(x, y, \frac{y' + \operatorname{tg}\alpha}{1 - y' \operatorname{tg}\alpha}\right) = 0$

2) $\alpha \rightarrow -\alpha$

$F_1\left(x, y, \frac{y' - \operatorname{tg}\alpha}{1 + y' \operatorname{tg}\alpha}\right) = 0$

-> $\alpha = 90$ -> ORTOGONALNE TRAJ.

$\boxed{F_1\left(x, y, -\frac{1}{y'}\right) = 0}$

4. KOJE SE RJ. PARAMETRIZACIJOM

1) $y = f(x, y') \rightarrow y' = p,\ p = p(x) \rightarrow \frac{d}{dx} \rightarrow$ iz dobivene naći $p = p(x) \Rightarrow y = f(x, p)$

2) $x = f(y, y') \rightarrow y' = p,\ p = p(y) \rightarrow \frac{d}{dy} \rightarrow$ iz dobivene naći $p = p(y) \Rightarrow x = f(y, p)$

3) $x = f(y') \rightarrow y' = p,\ p = p(x) \rightarrow \frac{dy}{dx} = p \rightarrow y = \int p\,dx \rightarrow y = \int p \cdot f'(p)\,dp + C$

4) $y = f(y') \rightarrow y' = p,\ p = p(y) \rightarrow \frac{dy}{dx} = p \rightarrow x = \int \frac{1}{p}\,dy \rightarrow x = \int \frac{1}{p} f'(p)\,dp + C$

5. CLAIRAUTOVA JEDN.

$\boxed{y = xy' + \varphi(y')} \rightarrow y = f(x, y')$

ALGORITAM: supst $y' = p,\ p = p(x)$

1. $y = xp + \varphi(p)\ /\frac{d}{dx}$
2. $p = 1 \cdot p + x\frac{dp}{dx} + \varphi'(p)\frac{dp}{dx}$
3. $\frac{dp}{dx}\left(x + \varphi'(p)\right) = 0$

A) $\frac{dp}{dx} = 0 \rightarrow$ naći na O.R.

$\hookrightarrow p = C \rightarrow y = Cx + \varphi(C) \rightarrow$ O.R. $\rightarrow$ familija pravaca

B) $x + \varphi'(p) = 0$

$x = -\varphi'(p)$

$y = -\varphi'(p) \cdot p + \varphi(p)$

$\rightarrow$ singularno rješenje, nije obuhvaćeno općim, predstavlja ANVELOPU (ovojnicu) familije pravaca

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EGZISTENCIJA I JEDNOZNAČNOST RJEŠENJA

S2 (PEANOV TEOREM EGZISTENCIJE)

Ako je $f(x,y)$ neprekinuta u pravokutniku D oko točke (x_0, y_0), tada Cauchyev problem ima bar jedno rješenje u nekom okolišu točke x_0.

S3 (PICARDOV TM. O JEDNOZNAČNOSTI RJ.)

$f(x,y)$ definirana na pravokutniku D oko (x_0, y_0), f neprekinuta na D, $\frac{\partial f}{\partial y}$ omeđena na D.
Onda postoji interval $(x_0 - h, x_0 + h)$ na kojem postoji jednoznačno rj. Cauchyevog problema.

OVOJNICA FAMILIJE KRIVULJA

$\Phi(x, y, C) = 0$ - familija krivulja. Anvelopa $\rightarrow$ u svakoj točki tangenta na anvelopu ujedno je tan. i na nekoj od krivulja od zadane familije.

ALGORITAM TRAŽENJA:

$$\left.\begin{array}{l}(1)\ \Phi(x,y,C) = 0 \\ (2)\ \frac{\partial}{\partial C}\Phi(x,y,C) = 0\end{array}\right\} \text{elim. „}C\text{"}$$

$\rightarrow$ iz (1) i (2) dobije se diskriminantna krivulja

$\Rightarrow$ nađe se dif. jedn. familije i provjeri da li anvelopa zadovoljava tu jedn.

12. DIF. JEDN. VIŠEG REDA

INTEGRIRANJE SNIŽAVANJEM REDA

$F(x, y, y', \dots, y^{(n)}) = 0 \rightarrow$ dif. jedn. n-tog reda

O.R.: $\Phi(x, y, C_1, \dots, C_n) = 0$

1. JEDN. OBLIKA $y^{(n)} = f(x)$

$y^{(n)} = \frac{d}{dx} y^{(n-1)} = f(x)\ /dx\ i \int$

$$\boxed{\begin{array}{l} y^{(n-1)} = \int f(x)\,dx + C_1\ /dx\ i \int \\ y^{(n-2)} = \int\left[\int f(x)\,dx\right]dx + C_1 x + C_2 \end{array}}$$

itd.

2. JEDN. OBLIKA $F(x, y^{(k)}, \dots, y^{(n)}) = 0$

- u jedn. nema: $y, y', y'', \dots, y^{(k-1)}$
- snižavanje reda za $k \rightarrow$ supst. $y^{(k)} = z,\ z = z(x)$

$F(x, z, \dots, z^{(n-k)}) = 0$

$\rightarrow \int tg(x)\,dx$
$\ln|\cos(x)|$

3. JEDN. OBLIKA $F(y, y', \dots, y^{(n)}) = 0$

- nema var. x → snižavanje reda za 1 → supst.

$\boxed{y' = p}$, $p = p(y)$

$\boxed{y'' = p \cdot p'}$, $\boxed{y''' = p^2 p'' + p(p')^2}$ → $\boxed{F_1(y, p_1, p'_1, \dots, p^{(n-1)})}$

4. JEDN. OBLIKA $F(x, y, y', \dots, y^{(n)}) = 0$ HOMOGENE U VAR. $y, y', \dots, y^{(n)}$

- snižavamo red supstitucijom:

$\boxed{y = e^{\int z dx}}$, $z = z(x)$ → $\boxed{y' = y \cdot z}$ → $\boxed{y'' = (z' + z^2) y}$

5. JEDN. OBLIKA $F(x, y, y', \dots, y^{(n)}) = 0$ ZA KOJE VRIJEDI

$$F(x, y, y', \dots, y^{(n)}) = \frac{d}{dx} \Phi(x, y, y', \dots, y^{(n-1)})$$

- ako je ↗ ispunjeno, direktno integriramo

$$F(x, y, \dots, y^{(n)}) = \frac{d}{dx} \Phi(x, y, \dots, y^{(n-1)}) = 0 \;/ \cdot dx \text{ i } \int$$

$$\int d\Phi(x, y, \dots, y^{(n-1)}) = 0$$

$$\boxed{\Phi(x, y, \dots, y^{(n-1)}) = C_1}$$

LINEARNE DIF. JEDN. (LDJ) VIŠEG REDA

(*) $\boxed{A_n(x) y^{(n)} + A_{n-1}(x) y^{(n-1)} + \dots + A_1(x) y' + A_0(x) \cdot y = f(x)}$

→ ako je $f(x) = 0$ → HOMOGENA

→ ako su f-je $A_n, A_{n-1}, \dots, A_0$ konstante, (*) je LDJ s konst. koef.

6. LDJ. 2. REDA

$\boxed{y'' + P(x) y' + Q(x) y = 0}$ → LDJ. 2. reda s nekonst. koef.

S1 (SVOJSTVO LINEARNOSTI SKUPA RJEŠENJA)

(A) ako je y_1 rješenje jednadžbe, onda je i $C \cdot y_1$ rj. jedn, za bilo koju vrijednost C (homogenost)

(B) ako su y_1 i y_2 rješenje, onda je i f-ja $(y_1 + y_2)$ također rješenje (aditivnost)

S2 Ako je y_1 bilo koje rješenje, onda je svako drugo rj. oblika:

$\boxed{y = C_1 y_1(x) + C_2 y_2(x)}$ gdje je $\boxed{y_2(x) = y_1(x) \cdot \int \frac{1}{y_1^2(x)} \cdot e^{-\int P(x) dx} dx}$

7. HOMOGENE LDJ 2. REDA S KONST. KOEF.

S3 $\boxed{y'' + a_1 y' + a_0 y = 0}$ → rješenja u obliku $y = e^{rx}$ → $y'' \rightarrow r^2$, $y' \rightarrow r$, $y \rightarrow 1$

- karakter. jedn. $\boxed{\varphi(r) = r^2 + a_1 r + a_0}$ → r_1 i r_2 - korijeni karakter. jedn.

(1) r_1, r_2 - realni i različiti

$\boxed{y_1 = e^{r_1 x}, \; y_2 = e^{r_2 x}}$

(2) $r_1 = r_2 = r$

$\boxed{y_1 = e^{rx}, \; y_2 = x e^{rx}}$

(3) r_1, r_2 - kompl. → $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$

$\boxed{y_1 = e^{\alpha x} \cos(\beta x), \; y_2 = e^{\alpha x} \sin(\beta x)}$

O.R. u sva 3 slučaja: $y = C_1 y_1 + C_2 y_2$

13. LINEARNE DIF. JEDN. VIŠEG REDA

LINEARNA NEZAVISNOST FJA.

- fje $y_1, y_2, \dots, y_n$ linearno nezavisne ako iz $\alpha_1 y_1(x) + \alpha_2 y_2(x) + \dots + \alpha_n y_n(x) = 0$ slijedi $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ $\Rightarrow$ lin. komb. iščezava samo na trivijalan način.
- fje $y_1, y_2, \dots, y_n$ linearno zavisne ako postoje skalari $\alpha_1, \alpha_2, \dots, \alpha_n$ takvi da barem jedan $\neq 0$, a da vrijedi: $\alpha_1 y_1(x) + \alpha_2 y_2(x) + \dots + \alpha_n y_n(x) = 0$
$\Rightarrow$ Postoji netrivijalna kombinacija koja iščezava.

S1 (LINEAR. NEZAV. FJA I DET. WRONSKOG)

Neka su $y_1(x), y_2(x), \dots, y_n(x) \in \mathcal{C}^{(n-1)}[a, b]$

Wronskijan:

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

$W \neq 0 \rightarrow$ linear. nezav

$= 0 \rightarrow$ linear. zav.

1. LDJ n-TOG REDA S KONST. KOEF.

S2 Homo. LDJ n-tog reda s konst koef:

$$(1) \quad \boxed{L(y) \equiv y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0}$$

$y^{(n)} \downarrow r^{(n)}, \; y^{(n-1)} \downarrow r^{n-1}, \; \dots, \; y' \downarrow r, \; y \downarrow 1$

$$(2) \quad \boxed{\varphi(r) \equiv r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0}$$

O.R.: $\boxed{y = C_1 y_1 + C_2 y_2 + \dots + C_n y_n}$ $\quad y_1, y_2, \dots, y_n$ – skup lin. nezav. rješenja (fundam. sustav rješenja)

ALGORITAM:

(1) nađemo sve nultočke: $\varphi(r) = (r - r_1)^{n_1} (r - r_2)^{n_2} \dots (r - r_k)^{n_k}$

(2) svakom realnom korijenu r_i višestrukosti n_i odgovara n_i nezavisnih rješenja

npr. $\underbrace{y_1 = e^{r_1 x}, \; y_2 = e^{r_1 x} \cdot x, \; y_3 = e^{r_1 x} \cdot x^2}_{n=3}, \; \underbrace{y_4 = e^{r_2 x}}_{n=1}$

(3) svakom paru kompl. konj. $r_i = \alpha + i\beta$, $r_{i+1} = \alpha - i\beta$ višestrukosti n_i odgovara $2n_i$ nezavisnih rješenja

$e^{\alpha x} \cos(\beta x), \; e^{\alpha x} x \cos(\beta x), \; e^{\alpha x} x^2 \cos(\beta x), \dots$

$e^{\alpha x} \sin(\beta x), \; e^{\alpha x} x \sin(\beta x), \; e^{\alpha x} x^2 \sin(\beta x), \dots$

(4) O.R. $\rightarrow$ kombinacija svih rješenja

2. NEHOMOGENE LDJ S KONST. KOEF.

$$\boxed{y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = f(x)}$$

O.R.: $\boxed{y = \underbrace{y_h}_{\text{RJ. HOMOGENE}} + y_p}$ – PART. RJ. NEHOMOG. JEDN. UZROKOVANO SMETNJOM $f(x)$

– određivanje y_p: METODA VARIJACIJE KONST.:

npr. $\boxed{y'' + a_1 y' + a_0 y = f(x)}$

$y_h = C_1 y_1(x) + C_2 y_2(x)$

O.R. tražimo u obliku:

$$\boxed{y = C_1(x) y_1(x) + C_2(x) y_2(x)}$$

– uvodimo 2 nepoznate fje $C_1(x)$ i $C_2(x)$

(1) $C_1'(x) y_1(x) + C_2'(x) y_2(x) = 0$

(2) $C_1'(x) y_1'(x) + C_2'(x) y_2'(x) = f(x)$

$\rightarrow$ određujemo $\rightarrow C_1'(x),\ C_2'(x)$

$C_1(x)$ i $C_2(x)$ $\rightarrow C_1(x),\ C_2(x)$

$\rightarrow y = \dots$

ALGORITAM ZA $n=2$

(1) $y'' + a_1 y' + a_0 y = f(x)$

(2) $y = y_h + y_p$ $\{y_1(x), y_2(x)\}$

(3) $y = C_1(x) y_1(x) + C_2(x) y_2(x)$

(4) $C_1(x)$ i $C_2(x)$ $\rightarrow$ iz sustava

(1) $C_1'(x) y_1(x) + C_2'(x) y_2(x) = 0$

(2) $C_1'(x) y_1'(x) + C_2'(x) y_2'(x) = f(x)$

ZA $n=3$

(1) $y''' + a_2 y'' + a_1 y' + a_0 y = f(x)$

(2) $y = y_h + y_p$ $\{y_1(x), y_2(x), y_3(x)\}$

(3) $y = C_1(x) y_1(x) + C_2(x) y_2(x) + C_3(x) y_3(x)$

(4) $C_1(x)$ i $C_2(x)$ i $C_3(x)$ $\rightarrow$ iz sustava

(1) $C_1'(x) y_1(x) + C_2'(x) y_2(x) + C_3'(x) y_3(x) = 0$

(2) $C_1'(x) y_1'(x) + C_2'(x) y_2'(x) + C_3'(x) y_3'(x) = 0$

(3) $C_1'(x) y_1''(x) + C_2'(x) y_2''(x) + C_3'(x) y_3''(x) = f(x)$

ODREĐIVANJE y_p METODOM M.V.K. ZA POSEBNE SLUČAJEVE FJE. SMETNJE $f(x)$

1) $f(x) = P_n(x)$ (POLINOM) $\rightarrow y_p = Q_n(x) \rightarrow$ ako $r=0$ nije korijen karakt. jedn. $\varphi(0) \neq 0$

$\rightarrow y_p = x^\alpha Q_n(x) \rightarrow$ ako je $r=0$ α-struki korijen $\varphi(0) = 0$

2) $f(x) = A \cdot e^{ax}$ (EKSP. FJA) $\rightarrow y_p = h \cdot e^{ax} \rightarrow r=a$ nije korijen $\varphi(a) \neq 0$

$\rightarrow y_p = h \cdot x^\alpha e^{ax} \rightarrow r=a$ je α-struki korijen $\varphi(a) = 0$

3) $f(x) = e^{ax} \cdot P_n(x) \rightarrow y_p = e^{ax} Q_n(x) \rightarrow r=a$ nije korijen $\varphi(a) \neq 0$

$\rightarrow y_p = e^{ax} \cdot x^\alpha \cdot Q_n(x) \rightarrow r=a$ je α-struki korijen $\varphi(a) = 0$

4) $f(x) = A\cos(\omega x) + B\sin(\omega x) \rightarrow y_p = h_1\cos(\omega x) + h_2\sin(\omega x)$, $\rightarrow r = i\cdot\omega$ nije korijen $\varphi(i\omega) \neq 0$

$\rightarrow y_p = x^\alpha(\ )$ $\rightarrow r = i\omega$ α-struki korijen $\varphi(i\omega) = 0$

5) $f(x) = e^{ax}[A\cos(\omega x) + B\sin(\omega x)] \rightarrow y_p = e^{ax}(\ )$ $\rightarrow r = a + i\omega$ nije korijen $\varphi(a+i\omega) \neq 0$

$\rightarrow y_p = x^\alpha e^{ax}(\ )$ $\rightarrow r = a + i\omega$ α-struki korijen $\varphi(a+i\omega) = 0$

RJ. DIF. JEDN. POMOĆU REDOVA

$$f(x)=\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

Tm Dif. jedn. 2. reda: $y''+p(x)y'+q(x)y=v(x)$, p,q i v analitičke. Tada rj. tražimo u obliku $y(x)=\sum_{n=0}^{\infty} a_n (x-x_0)^n$. Računamo $y'(x)$, $y''(x)$ itd.

INTEGRIRANJE

- parcijalno: $\boxed{\int u \cdot dv = uv - \int v \cdot du}$, $v=\int dv$ $= \begin{vmatrix} u & du \\ dv & v \end{vmatrix} = \dots$

- $\ln x$, arc... x, ar... x uvijek u, ostalo dv
- ako nema ovih gore, x je u, ostalo dv

DOKAZI

8. S1$_A$ (NUŽAN UVJET KONV.)

$$\lim_{n\to\infty} a_n = 0$$

DOKAZ: $S_n = S_{n-1} + a_n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[S_n - S_{n-1}\right] = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0 \quad \text{Q.E.D.}$$

S3$_A$ (MAJORANTA)

$\sum a_n \to$ MAJ, KONV

$\sum a_n = M, \ M < \infty$

$S_n = a_1 + a_2 + \dots + a_n \le M$

$b_1 + b_2 + \dots + b_n \le a_1 + a_2 + \dots + a_n \le M$

- niz parc. suma reda $\sum b_n$ ograničen odozgo

$\to \sum b_n$ konv. prema S2

S4 (KRITERIJ USPOREDBE)

$$\lim\left(\frac{a_n}{b_n}\right) = L, \ L \neq 0, \ L \neq \infty$$

DOKAZ: pretp. da $\left(\frac{a_n}{b_n}\right)$ konv. prema L, za velike n vrijedi:

$0 \quad \frac{L}{2} \quad L \quad 2L$

$$\frac{L}{2} < \frac{a_n}{b_n} < 2L$$

dakle: $\boxed{a_n < 2L \cdot b_n}$ za $n \ge n_0$

$$\sum_{n=n_0}^{\infty} a_n < 2L \sum_{n=n_0}^{\infty} b_n$$

MINORANTA $\quad\quad$ → MAJORANTA

- ako $\sum b_n$ KONV, i $\sum a_n$ KONV.
- ako $\sum a_n$ DIV, i $\sum b_n$ DIV

ANALOGNO: $\boxed{\frac{L}{2} b_n < a_n}$ za $n \ge n_0$

$$\frac{L}{2}\sum_{n=n_0}^{\infty} b_n < \sum_{n=n_0}^{\infty} a_n$$

S7 (CAUCHY)

$$q = \lim_{n\to\infty} \sqrt[n]{a_n}$$

a) $q < 1$

$0 \quad q-\varepsilon \quad q \quad q+\varepsilon \quad 1$

odaberemo $\varepsilon > 0$ takav da je $q + \varepsilon < 1$

imamo $(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N})[n > n_0 \Rightarrow \sqrt[n]{a_n} < q + \varepsilon]$

$a_n < (q+\varepsilon)^n$ za $n \ge n_0$

$$\sum_{n=n_0}^{\infty} a_n < \sum_{n=n_0}^{\infty} (q+\varepsilon)^n \to \text{konv. MAJ.} \Rightarrow \text{KONV. i } a_n$$

(geom red sa $Q = q + \varepsilon < 1$)

b) $q > 1$

$0 \quad 1 \quad q-\varepsilon \quad q \quad q+\varepsilon$

$\varepsilon > 0$ takav da je $q - \varepsilon > 1$

imamo $[\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N})[n > n_0 \Rightarrow \sqrt[n]{a_n} > q - \varepsilon]$

$a_n > (q-\varepsilon)^n$

$$\sum_{n=n_0}^{\infty} a_n > \sum_{n=n_0}^{\infty} (q-\varepsilon)^n \to \text{DIV. MIN.}$$

(geom red sa $Q = q - \varepsilon > 1$)

S6 (D'Alambert)

- Limes niza $\frac{a_{n+1}}{a_n}$ je q $\rightarrow$ odaberemo $\varepsilon>0 \Rightarrow q-\varepsilon < \frac{a_{n+1}}{a_n} < q+\varepsilon$

za $n \geq n_0$

a) $q<1$ – možemo odabrati ε da bude $q+\varepsilon<1$ $\rightarrow$ ispunjen uvjet, red KVG

b) $q>1$ – odaberemo ε takav da bude $q-\varepsilon>1$ $\rightarrow$ red DIV

9. S2$_1$ (Cauchy – Hadamarov)

$$R=\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| \quad \text{znači da postoji} \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{c_{n+1}(x-x_0)^{n+1}}{c_n(x+x_0)^n}\right|=$$

$$=|x-x_0|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|=|x-x_0|\cdot\frac{1}{R}=\frac{|x-x_0|}{R}$$

red KONV za $\frac{|x-x_0|}{R}<1 \Rightarrow |x-x_0|<R$

Q.E.D.

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11. S1 (UVJET EGZAKTNOSTI) Da bi $P(x,y)dx+Q(x,y)dy$ bio tot. dif nekе fje $u=u(x,y)$ nužno je i dostatno da vrijedi: $\boxed{\frac{\partial P(x,y)}{\partial y}=\frac{\partial Q(x,y)}{\partial x}}$

DOKAZ:

(1) NUŽNOST:

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=Pdx+Qdy$$

$$\frac{\partial u}{\partial x}=P \;\Big/\frac{\partial}{\partial y} \qquad \frac{\partial u}{\partial y}=Q \;\Big/\frac{\partial}{\partial x}$$

$$\frac{\partial^2 u}{\partial xy}=\frac{\partial P}{\partial y} \qquad \frac{\partial^2 u}{\partial yx}=\frac{\partial Q}{\partial x} \rightarrow \boxed{\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}}$$

Q.E.D.

po Schwartzovom teoremu jednako

(2) DOSTATNOST

pokažimo da postoji fja koja zadovoljava (1) $u=u(x,y)$

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=Pdx+Qdy$$

$$\frac{\partial u}{\partial x}=P \rightarrow u=u(x,y)=\int_{x_0}^{x}P(x,y)dx+\varphi(y)$$

$$\frac{\partial u}{\partial y}=Q \Rightarrow \int_{x_0}^{x}P'_y(x,y)dx+\varphi'(y)=\int_{x_0}^{x}Q'_x(x,y)dx+\varphi'(y)=$$

$$\cancel{Q(x,y)}=\cancel{Q(x,y)}-Q(x_0,y)+\varphi'(y) \Rightarrow \varphi'(y)=Q(x_0,y)$$

$$\varphi(y)=\int_{y_0}^{y}Q(x_0,y)dy$$

dakle, postoji fja koja zadovoljava (1): $u(x,y)=\int_{x_0}^{x}P(x,y)dx+\int_{y_0}^{y}Q(x_0,y)dy$ Q.E.D.

12. S1 (SVOJSTVO LINEARNOSTI SKUPA RJEŠENJA)

(A) ako je y_1 rješenje, onda je i $C y_1$ rješenje hom. LDJ. 2. reda za svaki C (homogenost)

DOKAZ:

predp. da je $y_1'' + P(x) y_1' + Q(x) y_1 = 0$

dokazati da je $C \cdot y_1$ rješenje

$$(C y_1)'' + P(x)(C y_1)' + Q(x) C y_1 = 0$$

$$C y_1'' + P(x) C y_1' + Q(x) C y_1 = 0$$

$$C \underbrace{(y_1'' + P(x) y_1' + Q(x) y_1)}_{=0 \to \text{predp}} = 0 \quad \to \quad \text{Q.E.D.}$$

(B) ako su y_1 i y_2 dva rj., onda je i fja $(y_1 + y_2)$ rješenje

DOKAZ:

predp: $y_1'' + P(x) y_1' + Q(x) y_1 = 0$

$y_2'' + P(x) y_2' + Q(x) y_2 = 0$

dokazati: $(y_1 + y_2)'' + P(x)(y_1 + y_2)' + Q(x)(y_1 + y_2) = 0$

$$y_1'' + y_2'' + P(x) y_1' + P(x) y_2' + Q(x) y_1 + Q(x) y_2 = 0$$

$$\underbrace{y_1'' + P(x) y_1' + Q(x) y_1}_{=0} + \underbrace{y_2'' + P(x) y_2' + Q(x) y_2}_{=0} = 0$$

po predp

Q.E.D.

13.

S1 (WRONSKIJAN)

Predp. suprotno: $W(y_1, y_2, \dots, y_n) \neq 0$, a da su fje linearno zavisne

Tada postoje skalari $\alpha_1, \alpha_2, \dots, \alpha_n$ od kojih je bar 1 različit od 0, takvi da $\alpha_1 y_1(x) + \alpha_2 y_2(x) + \dots + a_n y_n(x) = 0$. Deriviramo izraz n-1 puta

Dobijemo sustav:

$$\begin{aligned} a_1 y_1 + a_2 y_2 + \dots + a_n y_n &= 0 \\ a_1 y_1' + a_2 y_2' + \dots + a_n y_n' &= 0 \\ &\vdots \\ a_1 y_1^{(n-1)} + a_2 y_2^{(n-1)} + \dots + a_n y_n^{(n-1)} &= 0 \end{aligned}$$

Interpretiramo ovaj sustav kao homogeni s nepoznanicama $\alpha_1, \alpha_2, \dots \alpha_n$, znamo da ima netrivijalno rješenje ako je determinanta sustava jednaka 0. No dat. gornjeg sustava je W, a on po predp. $W \neq 0$, dakle predp. je dovela do proturječja, pa su fje $y_1, y_2, \dots y_n$ lin. nezavisne. Q.E.D.

TABLICA NAJVAŽNIJIH Mc-LAURINOVIH REDOVA

1.	$\sum_0^\infty \frac{x^n}{n!} = e^x \ldots \forall x$ (54)	$\sum_0^\infty (-1)^n \frac{x^n}{n!} = e^{-x} \ldots \forall x$
2.	$\sum_0^\infty \frac{x^{2n+1}}{(2n+1)!} = \mathrm{sh}\, x \ldots \forall x$	(55)
3.	$\sum_0^\infty \frac{x^{2n}}{(2n)!} = \mathrm{ch}\, x \ldots \forall x$	
4.	$\sum_0^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin x \ldots \forall x$	(56)
5.	$\sum_0^\infty (-1)^n \frac{x^{2n}}{(2n)!} = \cos x \ldots \forall x$	
6.	$\sum_1^\infty \frac{x^n}{n} = -\ln(1-x) \ldots \lvert x\rvert<1$	(57) PRIMJEDBA $(x \to -x)$
7.	$\sum_1^\infty (-1)^{n-1} \frac{x^n}{n} = \ln(1+x) \ldots \lvert x\rvert<1$ (57)	za $x=-1$ DIVG za $x=1$ KONV, suma $=\ln 2$
8.	$\sum_0^\infty \frac{x^{2n+1}}{2n+1} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \ldots \lvert x\rvert<1$	
9.	$\sum_0^\infty (-1)^n \cdot \frac{x^{2n+1}}{2n+1} = \mathrm{arctg}\, x \ldots \lvert x\rvert<1$	geom red $x \to -x^2$
10.	BINOMNI RED $\sum_0^\infty \binom{\alpha}{n} x^n = (1+x)^\alpha \ldots \lvert x\rvert<1$ (58)	$\sum_0^\infty (-1)^n \binom{\alpha}{n} x^n = (1-x)^\alpha \ldots \lvert x\rvert<1$
11.	GEOM. RED $\sum_0^\infty x^n = \frac{1}{1-x} \ldots \lvert x\rvert<1$	(58) za $\alpha=-1$
12.	GEOM. RED $\sum_0^\infty (-1)^n \cdot x^n = \frac{1}{1+x} \ldots \lvert x\rvert<1$	
13.	$\sum_1^\infty n \cdot x^n = \frac{x}{(1-x)^2} \ldots \lvert x\rvert<1$	(58) $\alpha=-2$
14.	$\sum_1^\infty 2n \cdot x^{2n} = \frac{2x^2}{(1-x^2)^2} \ldots \lvert x\rvert<1$ geom red; $x \to x^2$ / deriviramo	$\sum_0^\infty (2n+1) x^{2n+1} = \frac{x(1+x^2)}{(1-x^2)^2} \ldots \lvert x\rvert<1$
15.	$\sum_1^\infty \frac{x^{2n}}{2n} = -\frac{1}{2}\ln(1-x^2) \ldots \lvert x\rvert<1$ Integriramo geom red	$\sum_1^\infty (-1)^{n+1} \frac{x^{2n}}{2n} = \frac{1}{2}\ln(1+x^2) \ldots \lvert x\rvert<1$

NAPOMENA: Redovi 8., 9., 14., 15. dobiju se iz geom. redova:

$$1 \underset{(-)}{+} x^2 + x^4 \underset{(-)}{+} x^6 + \ldots = \frac{1}{1 \underset{(+)}{-} x^2}$$

ili

$$x \underset{(-)}{+} x^3 + x^5 \underset{(-)}{+} x^7 + \ldots = \frac{x}{1 \underset{(+)}{-} x^2}$$

njihovim deriviranjem ili integriranjem član po član.