6) A) $S_n = \sum_{n=1}^{\infty} n^n z^n$, $R = ?$

$c_n = n^n$

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty} \sqrt[\cancel{n}]{n^{\cancel{n}}} = \lim_{n\to\infty} n = \infty \Rightarrow R = 0$$

B) $S_n = \sum_{n=1}^{\infty} \cos(in) z^n$, $R = ?$

$$c_n = \cos(in) = \operatorname{ch} n = \frac{e^n + e^{-n}}{2}$$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \frac{e^{n+1} + e^{-(n+1)}}{e^n + e^{-n}} = \lim_{n\to\infty} \frac{e^{-(n+1)}}{e^{-n}} \cdot \frac{e^{2(n+1)} + 1}{e^{2n} + 1} =$$

$$= \frac{\cancel{e^{n}} \cdot e^{-1}}{\cancel{e^{n}}} \cdot \frac{\frac{\cancel{e^{2n}} \cdot e^2}{\cancel{e^{2n}}} + \frac{1}{e^{2n}} \nearrow 0}{\frac{\cancel{e^{2n}}}{\cancel{e^{2n}}} + \frac{1}{e^{2n}} \nearrow 0} = e^{-1} \cdot e^2 = e \Rightarrow R = e^{-1} = \frac{1}{e}$$

7) A) $S_n = \sum_{n=1}^{\infty} \frac{n}{2^n} z^n$, $R = ?$

$c_n = \frac{n}{2^n}$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{\frac{n+1}{\cancel{2^n} \cdot 2}}{\frac{n}{\cancel{2^n}}} = \frac{1}{2} \lim_{n\to\infty} \frac{n+1}{n} =$$

$$= \frac{1}{2} \frac{\frac{\cancel{n}}{\cancel{n}} + \frac{1}{n} \nearrow 0}{\frac{\cancel{n}}{\cancel{n}}} = \frac{1}{2} \Rightarrow R = 2$$

B) $S_n = \sum_{n=1}^{\infty} \operatorname{ch} \frac{i}{n} z^n$, $R = ?$

$c_n = \operatorname{ch} \frac{i}{n} = \cos \frac{1}{n}$

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty} \sqrt[n]{\cos \frac{1}{n}} = \lim_{n\to\infty} \left(\cos \frac{1}{n}\right)^{\frac{1}{n}} = \left(\cos \frac{1}{\infty}\right)^{\frac{1}{\infty}} =$$

$$= (\cos 0)^0 = 1 \Rightarrow R = 1$$

8) A) $S_n = \sum_{n=1}^{\infty} n! z^n$, $R = ?$

$c_n = n!$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} \frac{\cancel{n!}(n+1)}{\cancel{n!}} = \infty \Rightarrow R = 0$$

B) $S_n = \sum_{n=1}^{\infty} \left(\frac{z}{in}\right)^n, \quad R = ?$

$$c_n = \frac{1}{(in)^n}$$

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty} \sqrt[\cancel{n}]{\frac{1}{|in|^{\cancel{n}}}} = \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0 \Rightarrow R = \infty$$

9) A) $S_n = \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} z^n, \quad R = ?$

$$c_n = \frac{(2n)!}{(n!)^2}$$

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \frac{\frac{(2(n+1))!}{((n+1)!)^2}}{\frac{(2n)!}{(n!)^2}} = \lim_{n\to\infty} \frac{\frac{\cancel{(2n)!}(2n+1)(2n+2)}{\cancel{(n!)^2}(n+1)^2}}{\frac{\cancel{(2n)!}}{\cancel{(n!)^2}}} =$$

$$= \lim_{n\to\infty} \frac{(2n+1)\,2\cancel{(n+1)}}{(n+1)^{\cancel{2}}} = 2 \lim_{n\to\infty} \frac{2n+1}{n+1} = 2 \cdot \frac{\frac{2n}{\cancel{n}} + \frac{1}{n} \to 0}{\frac{\cancel{n}}{\cancel{n}} + \frac{1}{n} \to 0} = 4 \Rightarrow R = \frac{1}{4}$$

B) $S_n = \sum_{n=1}^{\infty} e^{i\frac{\pi}{n}} z^n, \quad R = ?$

$$c_n = e^{i\frac{\pi}{n}}$$

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty} \sqrt[n]{\left|e^{i\frac{\pi}{n}}\right|} = \lim_{n\to\infty} \left|e^{i\pi}\right|^{\frac{1}{n^2}} = \left|e^{i\pi}\right|^{\frac{1}{\infty}} = \left|e^{i\pi}\right| =$$

$$= |\cos\pi + i\sin\pi| = |-1| = 1$$

10) A) $S_n = \sum_{n=1}^{\infty} \frac{n^n}{(2n)!} z^n, \quad R = ?$

$$c_n = \frac{n^n}{(2n)!}$$

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(2(n+1))!}}{\frac{n^n}{(2n)!}} = \lim_{n\to\infty} \frac{\frac{(n+1)^n (n+1)}{\cancel{(2n)!}(2n+1)(2n+2)}}{\frac{n^n}{\cancel{(2n)!}}} =$$

$$= \lim_{n\to\infty} \frac{\frac{(n+1)^n \cancel{(n+1)}}{(2n+1)\,2\cancel{(n+1)}}}{n^n} = \frac{1}{2} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \cdot \lim_{n\to\infty} \frac{1}{2n+1} = \frac{1}{2} \cdot e \cdot 0 = 0$$

$$\Rightarrow R = \infty$$

B) $S_n = \sum_{n=1}^{\infty} e^{in} z^n, \quad R = ?$

$c_n = e^{in}$

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|c_n|} = \sqrt[n]{|e^{in}|} = |e^{i}| = |\cos 1 + i \sin 1| = \sqrt{\cos^2 1 + \sin^2 1}$$

$$= 1 \Rightarrow R = 1$$

11) A) $f(z) = \frac{1}{(1+z^3)^2}, \quad z_0 = 0$

$$f(z) = (1+z^3)^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} z^{3n} = \sum_{n=0}^{\infty} \frac{-2(-2-1)\cdot\ldots\cdot(-2-n+1)}{1\cdot 2\cdot 3\cdot\ldots\cdot n} z^{3n} =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1\cdot 2\cdot 3\cdot\ldots\cdot(n+1)}{1\cdot 2\cdot 3\cdot\ldots\cdot n} z^{3n} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)!}{n!} z^{3n} =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\cancel{n!}(n+1)}{\cancel{n!}} z^{3n} = \sum_{n=0}^{\infty} (-1)^n (n+1) z^{3n}$$

B) $f(z) = \sin(3z-1), \quad z_0 = -1$

$$f(z) = \sin(3(z+1-1)-1) = \sin(3(z+1)-4) =$$

$$= \sin(3(z+1))\cos 4 - \cos(3(z+1))\sin 4 =$$

$$= \cos 4 \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1}(z+1)^{2n+1}}{(2n+1)!} - \sin 4 \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n}(z+1)^{2n}}{(2n)!}$$

12) A) $f(z) = \frac{1}{(z+1)(z-2)}, \quad z_0 = 0$

$$\frac{1}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2}$$

$1 = A(z-2) + B(z+1)$ $\qquad A+B=0 \Rightarrow B=-A \Rightarrow\Rightarrow B = \frac{1}{3}$

$1 = Az - 2A + Bz + B$ $\qquad -2A+B = 1 \Rightarrow -2A - A = 1 \Rightarrow$

$1 = (A+B)z + (-2A+B)$ $\qquad \Rightarrow -3A = 1 \Rightarrow A = -\frac{1}{3}$

$$f(z) = -\frac{1}{3}\frac{1}{z+1} + \frac{1}{3}\frac{1}{z-2} = -\frac{1}{3}\frac{1}{1+z} + \frac{1}{3}\frac{1}{-2\left(1-\frac{z}{2}\right)} =$$

$$= -\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{6}\sum_{n=0}^{\infty}\frac{z^n}{2^n} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^{n+1}z^n - \frac{1}{3}\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} =$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left[(-1)^{n+1} - 2^{-(n+1)}\right] z^n$$

B) $f(z) = \cos z, \quad z_0 = \frac{\pi}{4}$

$$f(z) = \cos\left(z - \frac{\pi}{4} + \frac{\pi}{4}\right) = \cos\left(z - \frac{\pi}{4}\right)\cos\frac{\pi}{4} - \sin\left(z-\frac{\pi}{4}\right)\sin\frac{\pi}{4} =$$

$$= \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}(-1)^n \frac{\left(z-\frac{\pi}{4}\right)^{2n}}{(2n)!} - \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}(-1)^n \frac{\left(z-\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}$$

13) A) $f(z) = \operatorname{arsh} z, \quad z_0 = 0$

$$f'(z) = \frac{1}{\sqrt{1+z^2}} = (1+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n} z^{2n} \Big/ \int dz$$

$$f(z) = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\frac{z^{2n+1}}{2n+1}$$

B) $f(z) = \ln(3-z), \quad z_0 = 1$

$$f(z) = \ln(3-(z+1-1)) = \ln(4-(z+1)) = \ln\left(4\left(1-\frac{z+1}{4}\right)\right) =$$

$$= \ln 4 + \ln\left(1 - \frac{z+1}{4}\right) = \ln 4 + \sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n}\left(-\frac{z+1}{4}\right)^n =$$

$$= \ln 4 + \sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n}\cdot(-1)^n\left(\frac{z+1}{4}\right)^n = \ln 4 + \sum_{n=1}^{\infty}(-1)^{2n-1}\cdot\frac{(z+1)^n}{4^n\cdot n} =$$

$$= \ln 4 - \sum_{n=1}^{\infty}\frac{(z+1)^n}{4^n\cdot n}$$

14) A) $f(z) = \ln(z^2 - 3z + 2)$

$$f(z) = \ln(z^2 - z - 2z + 2) = \ln(z(z-1) - 2(z-1)) = \ln(z-1)(z-2)$$

$$= \ln(z-1) + \ln(z-2)$$

$$f'(z) = \frac{1}{z-1} + \frac{1}{z-2} = (-1)\frac{1}{1-z} - \frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \Big/ \int$$

$$f(z) = C - \sum_{n=0}^{\infty}\frac{z^{n+1}}{n+1} + C - \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\frac{z^{n+1}}{n+1} = C - \sum_{n=1}^{\infty}\frac{z^n}{n} + C - \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{z^n}{n}$$

$$= C - \sum_{n=1}^{\infty}(1+2^{-n})\frac{z^n}{n} \Rightarrow f(0) = C \quad \Big\} \; C = \ln 2$$

$$f(0) = \ln(0 - 0 + 2) = \ln 2$$

$$f(z) = \ln 2 - \sum_{n=1}^{\infty}(2^{-n}+1)\frac{z^n}{n}$$

B) $f(z) = \sin^2 z$

$$f(z) = \frac{1-\cos(2z)}{2} = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{4^n z^{2n}}{(2n)!}$$

15) A) $f(z) = \sqrt{z+i}$

$$f(z) = \sqrt{i}\sqrt{1+\frac{z}{i}} = \sqrt{i}\left(1+\frac{z}{i}\right)^{\frac{1}{2}} = \sqrt{i}\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-i)^n z^n =$$

$$= \sqrt{i}\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-1)^n i^n z^n$$

B) $f(z) = e^z \sin z$

$$f(z) = e^z \frac{e^{zi}-e^{-zi}}{2i} = \frac{1}{2i}\left(e^{z(1+i)} - e^{z(1-i)}\right) =$$

$$= \frac{1}{2i}\sum_{n=0}^{\infty}\left(\frac{(1+i)^n}{n!} - \frac{(1-i)^n}{n!}\right)z^n = \frac{1}{2i}\sum_{n=0}^{\infty}\frac{1}{n!}\left((1+i)^n-(1-i)^n\right)z^n$$

16) A) $f(z) = \frac{z}{(z^2+1)(z^2-4)}$

$$\frac{z}{(z^2+1)(z^2-4)} = \frac{Az+B}{z^2+1} + \frac{Cz+D}{z^2-4}$$

$$z = (Az+B)(z^2-4)+(Cz+D)(z^2+1)$$

$$z = Az^3+Bz^2-4Az-4B+Cz^3+Dz^2+Cz+D$$

$$z = (A+C)z^3+(B+D)z^2+(-4A+C)z-4B+D$$

$A+C=0 \Rightarrow C=-A \Rightarrow\Rightarrow C = 1/5$

$-4A+C=1 \Rightarrow -4A-A=1 \Rightarrow -5A=1 \Rightarrow A=-1/5$

$B+D=0 \Rightarrow D=-B \Rightarrow D=0$

$-4B+D=0 \Rightarrow -4B-B=0 \Rightarrow -5B=0 \Rightarrow B=0$

$$f(z) = -\frac{1}{5}\frac{z}{z^2+1} + \frac{1}{5}\frac{z}{z^2-4} = -\frac{z}{5}\frac{1}{1+z^2} + \frac{z}{5}\frac{1}{-4\left(1-\frac{z^2}{4}\right)} =$$

$$= -\frac{z}{5}\sum_{n=0}^{\infty}(-1)^n z^{2n} - \frac{z}{20}\sum_{n=0}^{\infty}\frac{z^{2n}}{4^n} = \frac{1}{5}\sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1} - \frac{1}{5}\sum_{n=0}^{\infty}\frac{z^{2n+1}}{4^{n+1}} =$$

$$= \frac{1}{5}\sum_{n=0}^{\infty}\left[(-1)^{n+1} - 4^{-(n+1)}\right]z^{2n+1}$$

B) $f(z) = \mathrm{ch}^2 z$

$$f(z) = \frac{1+\mathrm{ch}(2z)}{2} = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{4^n z^{2n}}{(2n)!}$$

17) A) $f(z) = z^6 + 9z^4$

$z^6 + 9z^4 = 0$ $\qquad$ $z_1 = 0$, kratnost 4

$z^4(z^2+9) = 0$ $\qquad$ $z_{2,3} = \sqrt{-9} = \pm 3i$, kratnosti 1

B) $f(z) = (z-i)\,\text{sh} z$

$(z-i)\,\text{sh} z = 0$

$z_0 = i$ kratnost 1 $\qquad$ $\text{sh} z = 0 \Rightarrow z_k = \text{Arsh}\, 0 = \text{Ln}(0 + \sqrt{0^2+1}) =$

$= \ln 1 + i(\arg \frac{0}{1} + 2k\pi) = 2k\pi i$

$z_{1,2\ldots n} = (2k\pi + 1)i, \; k \in \mathbb{Z}$

kratnost 1

18) A) $f(z) = z^3 \sin z$

$z^3 \sin z = 0$ $\qquad$ $\sin z = 0 \Rightarrow z_k = \arcsin 0 = -i\,\text{Ln}(i\cdot 0 + \sqrt{1-0^2})$

$z_0 = 0$ kratnost 4 $\qquad$ $= -i(\ln 1 + i(\arg \frac{0}{1} + 2k\pi)) = k\pi$

kratnost 1 $\quad k \in \mathbb{Z} \setminus \{0\}$

B) $f(z) = \dfrac{1 - \text{ch} z}{z}$

$1 - \text{ch} z = 0 \Rightarrow \text{ch} z = 1 \Rightarrow z = \text{Arch}\, 1 = \text{Ln}(1 + \sqrt{1^2 - 1}) =$

$= \ln 1 + i(\arg \frac{0}{1} + 2k\pi) = 2k\pi i, \; k \in \mathbb{Z} \setminus \{0\}$

ispitivanje kratnosti:

$g(z) = 1 - \text{ch} z \Rightarrow g'(z) = -\text{sh} z \Rightarrow g''(z) = -\text{ch} z$

$g'(2k\pi i) = -\text{sh}(2k\pi i) = -i\sin(2k\pi) = 0$ $\qquad$ kratnost 2

$g''(2k\pi i) = -\text{ch}(2k\pi i) = -\cos(2k\pi) = -1$

$z_0 = 0$ kratnost 1

19) A) $f(z) = 2(\text{ch} z - 1) - z^2$ $\qquad$ $f(0) = 0$ $\qquad$ $z = 0$

$f'(z) = 2\,\text{sh} z - 2z$ $\qquad$ $f'(0) = 0$

$f''(z) = 2\,\text{ch} z - 2$ $\qquad$ $f''(0) = 0$

$f'''(z) = 2\,\text{sh} z$ $\qquad$ $f'''(0) = 0$

$f^{IV}(z) = 2\,\text{ch} z$ $\qquad$ $f^{IV}(0) = 2$ $\qquad$ kratnost 4 $\quad z = 0$

B) $f(z) = e^{\sin z} - e^{\text{tg} z}$ $\qquad$ $f(0) = 1 - 1 = 0$ $\qquad$ $z = 0$

$f'(z) = e^{\sin z}\cos z - e^{\text{tg} z}\dfrac{1}{\cos^2 z}$ $\qquad$ $f'(0) = 1 - 1 = 0$ $\qquad$ $\Big\}$ $f''(0) = 1 - 0 - 1 - 0 = 0$

$f''(z) = e^{\sin z}\cos^2 z - e^{\sin z}\sin z - e^{\text{tg} z}\dfrac{1}{\cos^4 z} - e^{\text{tg} z}\cdot(-2)\dfrac{1}{\cos^3 z}(-\sin z)$

$f'''(z) = e^{\sin z}\cos^3 z + e^{\sin z}\cdot 2\cos z \sin z - e^{\sin z}\cos z \sin z - e^{\sin z}\cos z -$

$- e^{\text{tg} z}\dfrac{1}{\cos^6 z} - e^{\text{tg} z}\cdot(-4)\dfrac{1}{\cos^5 z}(-\sin z) - 2e^{\text{tg} z}\dfrac{\text{tg} z}{\cos^4 z} - 2e^{\text{tg} z}\dfrac{1}{\cos^4 z} -$

$- 2e^{\text{tg} z}\cdot(-2)\dfrac{\text{tg} z}{\cos^3 z}(-\sin z)$ $\qquad$ $f'''(0) = \cancel{1} - 0 - 0 - \cancel{1} - 1 - 0 - 0 - 2 - 0 = -3$

kratnost 3 $\quad z = 0$

20) A) $f(z) = z^2(e^{z^3}-1) \quad z=0$

$$f(z) = z^2\left(\cancel{1} + z^3 + \frac{z^6}{2} + \frac{z^9}{6} + \ldots \cancel{-1}\right) = z^5\left(1 + \frac{z^3}{2} + \frac{z^6}{6} + \ldots\right) =$$

$$= z^5\varphi(z), \quad \varphi(0) = 1 \qquad \text{kratnost } 5 \quad z=0$$

B) $f(z) = \dfrac{sh^2 z}{z}$ 2008 2MI

$$f(z) = \frac{(e^z - e^{-z})^2}{2^2 z} = \frac{e^{2z} - 2 + e^{-2z}}{4z} = \quad 1 + 2z + \frac{4z^2}{2} - 2 + 1$$

$$= \frac{\cancel{1+2z} + \frac{4z^2}{2} \cancel{+ \frac{8z^3}{6}} + \frac{16z^4}{24} + \ldots \cancel{-2+1-2z} + \frac{4z^2}{2} \cancel{- \frac{8z^3}{6}} + \frac{16z^4}{24} + \ldots}{4z}$$

$$= \frac{4z^2 + \frac{16z^4}{12} + \ldots}{4z} = \frac{\cancel{4}z^{\cancel{2}}\left(1 + \frac{4z^2}{12_3} + \ldots\right)}{\cancel{4z}} = z\left(1 + \frac{z^2}{3} + \ldots\right)$$

$$= z\varphi(z), \quad \varphi(0) = 1 \qquad \text{kratnost } 1 \quad z=0$$

$sh z = 0 \quad z = Arsh\,0 = Ln(0 \pm \sqrt{0^2+1}) = Ln(\pm 1)$

$\left.\begin{array}{l} z_1 = i2k\pi \\ z_2 = i(\pi + 2k\pi) \end{array}\right\} z = ik\pi$

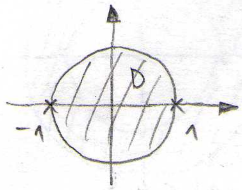
$z=0$ je $k=1$ (brojnik = 2, nazivnik = 1)

$z = ik\pi, \; k \in \mathbb{Z} \setminus \{0\} \; k=2$

1) A) $f(z) = \frac{1}{z^2-1}$, $z_0 = 0$, $D = \{|z| < 1\}$

$z^2 = 1$

$z_{1,2} = \pm 1$



$$f(z) = \frac{1}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}$$

$$1 = A(z+1) + B(z-1) \qquad A+B = 0 \Rightarrow B = -A \Rightarrow B = -\frac{1}{2}$$

$$1 = Az + A + Bz - B \qquad A-B = 1 \Rightarrow A + A = 1 \Rightarrow A = \frac{1}{2}$$

$$1 = (A+B)z + A - B$$

$$f(z) = \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1} = -\frac{1}{2}\frac{1}{1-z} - \frac{1}{2}\frac{1}{1+z} =$$

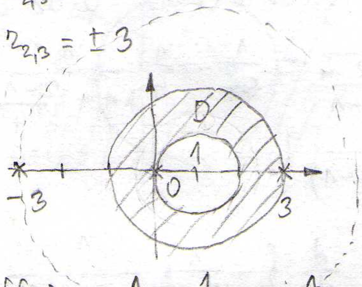
$$= -\frac{1}{2}\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n z^n = -\sum_{n=0}^{\infty} z^{2n}$$

B) $f(z) = \frac{1}{(z^2-9)z^2}$, $z_0 = 1$, $\frac{5}{2} \in D \Rightarrow D = \{1 < |z-1| \leq 2\}$

$z_1 = 0$

$z_{2,3}^2 = 9$

$z_{2,3} = \pm 3$



$$f(z) = \frac{A}{z-3} + \frac{B}{z+3} + \frac{C}{z} + \frac{D}{z^2}$$

$$1 = A(z+3)z^2 + B(z-3)z^2 + Cz(z^2-9) + D(z^2-9)$$

$$z = -3 \qquad 1 = B\cdot(-6)\cdot 9 = -54B \Rightarrow B = -\frac{1}{54}$$

$$z = 3 \qquad 1 = A\cdot 6\cdot 9 = 54A \Rightarrow A = \frac{1}{54}$$

$$z = 0 \qquad 1 = D\cdot(-9) \Rightarrow D = -\frac{1}{9}$$

$$A + B + C = 0 \Rightarrow C = 0$$

$$f(z) = \frac{1}{54}\frac{1}{z-3} - \frac{1}{54}\frac{1}{z+3} - \frac{1}{9}\frac{1}{z^2} = \frac{1}{54}\frac{1}{(z-1)+1-3} - \frac{1}{54}\frac{1}{z-1+1+3} - \frac{1}{9}\frac{d}{dz}\left\{-\frac{1}{z}\right.$$

$$= \frac{1}{54}\frac{1}{-2-(z-1)} - \frac{1}{54}\frac{1}{4+(z-1)} - \frac{1}{9}\frac{d}{dz}\left\{-\frac{1}{(z-1)+1}\right\} =$$

$$= -\frac{1}{108}\frac{1}{1-\frac{z-1}{2}} - \frac{1}{216}\frac{1}{1+\frac{z-1}{4}} - \frac{1}{9}\frac{d}{dz}\left\{\frac{-1}{(z-1)}\cdot\frac{1}{1+\frac{1}{z-1}}\right\} =$$

$$= -\frac{1}{108}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n} - \frac{1}{216}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{4^n} - \frac{1}{9}\frac{d}{dz}\left\{\frac{-1}{z-1}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(z-1)^n}\right\} =$$

$$= -\frac{1}{27}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{27}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{2^{2n+3}} - \frac{1}{9}\frac{d}{dz}\left\{-\sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^{n+1}}\right\} =$$

$$= -\frac{1}{9}\sum_{n=0}^{\infty}(-1)^n\frac{n+1}{(z-1)^{n+2}} - \frac{1}{27}\sum_{n=0}^{\infty}\left(\frac{1}{2^{n+2}} + \frac{(-1)^n}{2^{2n+3}}\right)(z-1)^n$$

$$\left((z-1)^{-n-1}\right)' = -(n+1)(z-1)^{-n-1-1} = -(n+1)(z-1)^{-n-2}$$

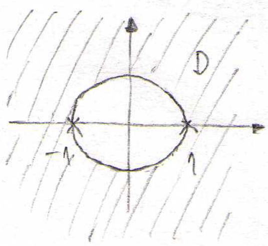
2) A) $f(z) = \frac{1}{z^2-1}$, $z_0 = 0$, $D = \{1 < |z|\}$

$z^2 = 1$

$z_{1,2} = \pm 1$

$$f(z) = \frac{1}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}$$

$$1 = A(z+1) + B(z-1)$$



$$z = 1 \qquad 1 = 2A \Rightarrow A = 1/2$$

$$z = -1 \qquad 1 = -2B \Rightarrow B = -1/2$$

$$f(z) = \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1}$$

$$f(z) = \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1} = \frac{1}{2}\cdot\frac{1}{z}\frac{1}{1-\frac{1}{z}} - \frac{1}{2}\cdot\frac{1}{z}\frac{1}{1+\frac{1}{z}} =$$

$$= \frac{1}{2z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^n} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}} =$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(1-(-1)^n\right)\frac{1}{z^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{z^{2n+2}}$$

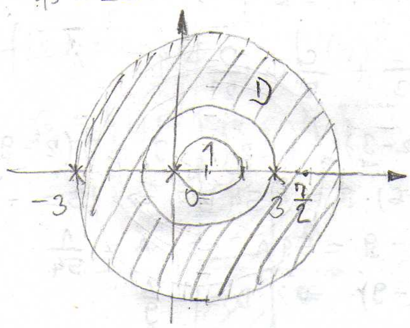
to je različito od nule za $n = 1, 3, 5, 7, 9, \dots$

B) $f(z) = \dfrac{1}{(z^2-9)z^2}$, $z_0 = 1$, $\frac{7}{2} \in D \Rightarrow D = \{2 < |z-1| < 4\}$

$z_1 = 0$

$z_{2,3}^2 = 9$

$z_{2,3} = \pm 3$



$$f(z) = \frac{A}{z-3} + \frac{B}{z+3} + \frac{C}{z} + \frac{D}{z^2} =$$

$$= \frac{1}{54}\frac{1}{z-3} - \frac{1}{54}\frac{1}{z+3} - \frac{1}{9}\frac{1}{z^2}$$

$$= \frac{1}{54}\frac{1}{-2+(z-1)} - \frac{1}{54}\frac{1}{4+(z-1)} - \frac{1}{9}\frac{d}{dz}\left\{-\frac{1}{z}\right\}$$

$$= \frac{1}{54}\frac{1}{z-1}\frac{1}{1-\frac{2}{z-1}} - \frac{1}{54}\cdot\frac{1}{4}\frac{1}{1+\frac{z-1}{4}} -$$

$$- \frac{1}{9}\frac{d}{dz}\left\{-\frac{1}{1+(z-1)}\right\} =$$

$$= \frac{1}{54(z-1)}\sum_{n=0}^{\infty}\frac{2^n}{(z-1)^n} - \frac{1}{216}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{4^n} - \frac{1}{9}\frac{d}{dz}\left\{-\frac{1}{z-1}\frac{1}{1+\frac{1}{z-1}}\right\} =$$

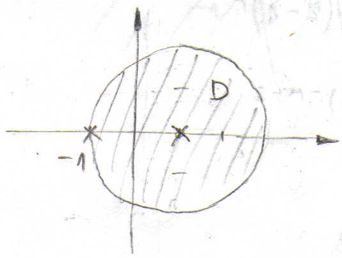
$$= \frac{1}{27}\sum_{n=0}^{\infty}\frac{2^{n-1}}{(z-1)^{n+1}} - \frac{1}{27}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{2^{2n+3}} - \frac{1}{9}\frac{d}{dz}\left\{-\sum_{n=0}^{\infty}(-1)^n\frac{1}{(z-1)^{n+1}}\right\} =$$

$$= -\frac{1}{9}\sum_{n=0}^{\infty}(-1)^n\frac{n+1}{(z-1)^{n+2}} + \frac{1}{27}\sum_{n=0}^{\infty}\frac{2^{n-1}}{(z-1)^{n+1}} - \frac{1}{27}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{2^{2n+3}}$$

3) A) $f(z) = \dfrac{1}{z^2-1}$, $z_0 = 1$, $D = \{0 < |z-1| < 2\}$

$z_{1,2}^2 = 1$

$z_{1,2} = \pm 1$



$$f(z) = \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1} = \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{2+z-1} =$$

$$= \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\cdot\frac{1}{2}\frac{1}{1+\frac{z-1}{2}} =$$

$$= \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{2^{n+1}}$$

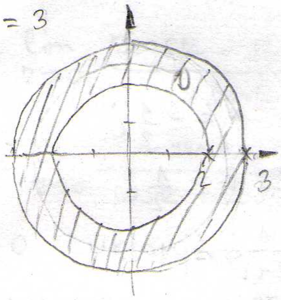
$$= \frac{1}{2}\left[\frac{1}{z-1} - \left(\frac{1}{2} - \frac{1}{4}(z-1) + \frac{1}{8}(z-1)^2 + \dots\right)\right] =$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^{n-1}}{2^n} = \sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^{n-1}}{2^{n+1}}$$

B) $f(z)=\frac{1}{(z-2)(z-3)}$, $z_0=0$, $\frac{5}{2}\in D \Rightarrow D=\{2<|z|<3\}$

$z_1=2$

$z_2=3$



$$f(z)=\frac{A}{z-2}+\frac{B}{z-3}$$

$$1=A(z-3)+B(z-2)$$

$z=2 \quad 1=-A \Rightarrow A=-1$

$z=3 \quad 1=B \Rightarrow B=1$

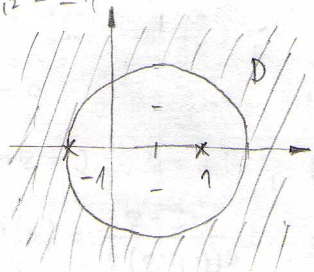
$$f(z)=-\frac{1}{z-2}+\frac{1}{z-3}=-\frac{1}{z}\frac{1}{1-\frac{2}{z}}-\frac{1}{3}\frac{1}{1-\frac{z}{3}}=$$

$$=-\frac{1}{z}\sum_{n=0}^{\infty}\frac{2^n}{z^n}-\frac{1}{3}\sum_{n=0}^{\infty}\frac{z^n}{3^n}=-\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}-\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}$$

4) A) $f(z)=\frac{1}{z^2-1}$, $z_0=1$, $D=\{2<|z-1|\}$

$z_{1,2}^2=1$

$z_{1,2}=\pm 1$



$$f(z)=\frac{1}{2}\frac{1}{z-1}-\frac{1}{2}\frac{1}{z+1}=\frac{1}{2}\frac{1}{z-1}-\frac{1}{2}\frac{1}{2+z-1}$$

$$=\frac{1}{2}\frac{1}{z-1}-\frac{1}{2}\frac{1}{z-1}\frac{1}{1+\frac{2}{z-1}}=\frac{1}{2}\frac{1}{z-1}-$$

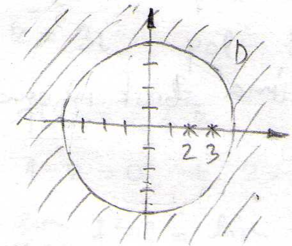
$$-\frac{1}{2}\frac{1}{z-1}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{(z-1)^n}=\frac{1}{2}\left[\frac{1}{z-1}-\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{(z-1)^{n+1}}\right]$$

$$=\frac{1}{2}\left[\cancel{\frac{1}{z-1}}-\left(\cancel{\frac{1}{z-1}}-\frac{2}{(z-1)^2}+\frac{4}{(z-1)^3}-\frac{8}{(z-1)^4}+\dots\right)\right]$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{2^{n+1}}{(z-1)^{n+2}}=\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{(z-1)^{n+2}}$$

B) $f(z)=\frac{1}{(z-2)(z-3)}$, $z_0=0$, $4\in D$

$z_1=2$ $\quad z_2=3$



$$f(z)=-\frac{1}{z-2}+\frac{1}{z-3}=-\frac{1}{z}\frac{1}{1-\frac{2}{z}}+\frac{1}{z}\frac{1}{1-\frac{3}{z}}=$$

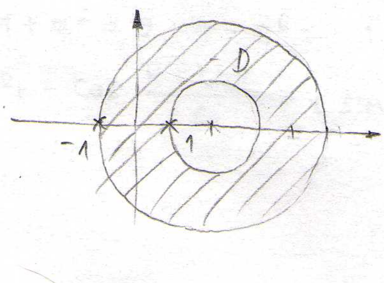
$$=-\frac{1}{z}\sum_{n=0}^{\infty}\frac{2^n}{z^n}+\frac{1}{z}\sum_{n=0}^{\infty}\frac{3^n}{z^n}=-\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}+\sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}}=$$

$$=\sum_{n=0}^{\infty}\frac{3^n-2^n}{z^{n+1}}$$

5) A) $f(z)=\frac{1}{z^2-1}$, $z_0=2$, $D=\{1<|z-2|<3\}$

$z_{1,2}^2=1$

$z_{1,2}=\pm 1$



$$f(z)=\frac{1}{2}\frac{1}{z-1}-\frac{1}{2}\frac{1}{z+1}=\frac{1}{2}\frac{1}{1+(z-2)}-\frac{1}{2}\frac{1}{3+(z-2)}$$

$$=\frac{1}{2}\frac{1}{z-2}\frac{1}{1+\frac{1}{z-2}}-\frac{1}{2}\cdot\frac{1}{3}\frac{1}{1+\frac{z-2}{3}}=$$

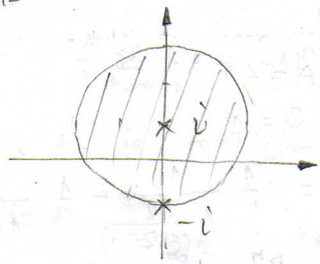
$$=\frac{1}{2}\frac{1}{z-2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(z-2)^n}-\frac{1}{2}\cdot\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\frac{(z+3)^n}{3^n}=$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(z-2)^{n+1}}-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(z+3)^n}{3^{n+1}}$$

B) $f(z) = \frac{1}{z^2+1}$, $z_0 = i$, $0 \in D$

$z_{1,2}^2 = -1$

$z_{1,2} = \pm i$



$f(z) = \frac{A}{z-i} + \frac{B}{z+i}$

$1 = A(z+i) + B(z-i)$

$z = i \quad 1 = 2iA \rightarrow A = -\frac{1}{2}i$

$z = -i \quad 1 = -2iB \rightarrow B = \frac{1}{2}i$

$f(z) = -\frac{1}{2}i \frac{1}{z-i} + \frac{1}{2}i \frac{1}{z+i} =$

$= -\frac{1}{2}i \frac{1}{z-i} + \frac{1}{2}i \frac{1}{2i+z-i} =$

$= -\frac{1}{2}i \frac{1}{z-i} + \frac{1}{2}i \frac{1}{2i} \frac{1}{1+\frac{z-i}{2i}} = -\frac{1}{2}i \frac{1}{z-i} + \frac{1}{2}i \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(2i)^{n+1}} =$

$= \frac{1}{2}i \left[-\frac{1}{z-i} + \frac{1}{2i} + \frac{z-i}{4} - \frac{(z-i)^2}{8i} - \frac{(z-i)^3}{16} + \dots \right] =$

$= -\frac{1}{2i} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(z-i)^{n-1}}{(2i)^{n+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^{n-1}}{(2i)^{n+1}}$

6) A) $f(z) = \frac{1}{z(z^2+1)^2}$ sing = ?

$f(z) = \frac{1}{z(z-i)^2(z+i)^2}$, $z(z-i)^2(z+i)^2 = 0$

$z = 0$ pol prvog reda

$z = i$, $z = -i$ polovi drugog reda

B) $f(z) = \sin\frac{1}{z} + \frac{1}{z^2}$ sing = ?

$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \frac{1}{z^2}$ – to je red koji ima beskonačno dug glavni dio Laurentovog reda

$z = 0$ bitni singularitet

7) A) $f(z) = \frac{z^2+z}{z^5+2z^4+z^3}$ sing = ?

$f(z) = \frac{z(z+1)}{z^3(z^2+2z+1)} = \frac{\cancel{z(z+1)}}{z^{3\,2}(z+1)^2} = \frac{1}{z^2(z+1)}$

$z = 0$ pol drugog reda

$z = -1$ pol prvog reda

B) $f(z) = \mathrm{ctg}\, z - \frac{1}{z}$ sing = ?

$f(z) = \frac{\cos z}{\sin z} - \frac{1}{z} = \frac{z\cos z - \sin z}{z \sin z}$ problematična točka je $z=0$

$c_0 = \lim_{z\to 0} \frac{z\cos z - \sin z}{z\sin z} = L'H = \lim_{z\to 0} \frac{\cancel{\cos z} - z\sin z - \cancel{\cos z}}{\sin z + z\cos z} = L'H =$

$= \lim_{z\to 0} \frac{-\sin z - z\cos z}{\cos z + \cos z - z\sin z} = \frac{0}{2} = 0$

$z=0$ uklonjivi singularitet

$z\sin z = 0$, $z = k\pi$ $k\in \mathbb{Z}\setminus\{0\}$ polovi prvog reda

8) A) $f(z) = z^2 \mathrm{ctg}\, z$ sing = ?

$f(z) = z^2 \frac{\cos z}{\sin z}$

$c_0 = \lim_{z\to 0} z^2 \frac{\cos z}{\sin z} = L'H = \lim_{z\to 0} \frac{2z\cos z - z^2 \sin z}{\cos z} = \frac{0}{1} = 0$

$z=0$ uklonjivi singularitet $\qquad \lim_{z\to k\pi} \frac{k^2\pi^2(\pm 1)}{0} = \infty$

$\sin z = 0$, $z = k\pi$, $k\in\mathbb{Z}\setminus\{0\}$ polovi prvog reda

B) $f(z) = \frac{z+3i}{z(z^2+9)^3}$ sing = ?

$f(z) = \frac{\cancel{z+3i}}{z(z-3i)^3(z+3i)^{\cancel{3}2}} = \frac{1}{z(z+3i)^2(z-3i)^3}$

$z=0$ pol prvog reda

$z=-3i$ pol drugog reda

$z = 3i$ pol trećeg reda

9) A) $f(z) = e^{\frac{z}{1-z}}$

$1-z=0 \Rightarrow z=1$

$\lim_{z\to 1} f(z) = \lim_{z\to 1} e^{\frac{1}{1-z}} = e^{\frac{1}{0}} = e^{\infty} = \infty$

$z=1$ bitni singularitet

B) $f(z) = \frac{z^3}{1+z^6}$

$1+z^6 = 0 \Rightarrow z^6 = -1$

$z_k = \cos\frac{\pi + 2k\pi}{6} + i\sin\frac{\pi+2k\pi}{6}$, $k = 0,1,2,3,4,5$ polovi prvog reda

10) A) $f(z)=\frac{1}{z^3+z}$ $\quad$ Res = ?

$$f(z)=\frac{1}{z(z^2+1)}=\frac{1}{z(z-i)(z+i)}$$

$$\operatorname{Res}(f(z),0)=\lim_{z\to 0}\cancel{z}\,\frac{1}{\cancel{z}(z-i)(z+i)}=\frac{1}{-i\cdot i}=1$$

$$\operatorname{Res}(f(z),i)=\lim_{z\to i}\cancel{(z-i)}\,\frac{1}{z\cancel{(z-i)}(z+i)}=\frac{1}{i\cdot 2i}=-\frac{1}{2}$$

$$\operatorname{Res}(f(z),-i)=\lim_{z\to -i}\cancel{(z+i)}\,\frac{1}{z(z-i)\cancel{(z+i)}}=\frac{1}{-i\cdot(-2i)}=-\frac{1}{2}.$$

B) $f(z)=\frac{1-\cos z}{z^3(z-1)}$ $\quad$ Res = ?

$z=0$ pol trećeg reda

$$\operatorname{Res}(f(z),0)=\frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left(\cancel{z^3}\,\frac{1-\cos z}{\cancel{z^3}(z-1)}\right)=\frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left(\frac{\sin z(z-1)-(1-\cos z)}{(z-1)^2}\right)$$

$$=\frac{1}{2}\lim_{z\to 0}\left(\frac{(\cos z(z-1)\cancel{+\sin z-\sin z})(z-1)^2-(\sin z(z-1)-(1-\cos z))2(z-1)}{(z-1)^4}\right)$$

$$=\frac{1}{2}\lim_{z\to 0}\left(\frac{\cos z(z-1)^2-2\sin z(z-1)+2(1-\cos z)}{(z-1)^3}\right)=$$

$$=\frac{1}{2}\cdot\frac{1-0+0}{-1}=-\frac{1}{2}$$

$z=1$ pol prvog reda

$$\operatorname{Res}(f(z),1)=\lim_{z\to 1}\cancel{(z-1)}\,\frac{1-\cos z}{z^3\cancel{(z-1)}}=1-\cos 1$$

11) A) $f(z)=\frac{1}{z-z^3}$ $\quad$ Res = ?

$$f(z)=\frac{1}{z(1-z^2)}=\frac{1}{z(1-z)(1+z)}=\frac{-1}{z(z-1)(z+1)}$$

$$\operatorname{Res}(f(z),0)=\lim_{z\to 0}\cancel{z}\,\frac{-1}{\cancel{z}(z-1)(z+1)}=1$$

$$\operatorname{Res}(f(z),1)=\lim_{z\to 1}\cancel{(z-1)}\,\frac{-1}{z\cancel{(z-1)}(z+1)}=-\frac{1}{2}$$

$$\operatorname{Res}(f(z),-1)=\lim_{z\to -1}\cancel{(z+1)}\,\frac{-1}{z(z-1)\cancel{(z+1)}}=-\frac{1}{2}$$

B) $f(z)=\frac{\sin\pi z}{(z-1)^3}$

$z=1$ pol drugog reda

$$\operatorname{Res}(f(z),1)=\frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left(\cancel{(z-1)^2}\,\frac{\sin\pi z}{(z-1)^{\cancel{3}}}\right)=\lim_{z\to 1}\frac{\cos\pi z\cdot\pi(z-1)-\sin\pi z}{(z-1)^2}=$$

$$=\frac{0-0}{0}=0$$

12) A) $f(z)=\frac{z^2}{z^4-1}$, Res = ?

$$f(z)=\frac{z^2}{(z^2-1)(z^2+1)}=\frac{z^2}{(z-1)(z+1)(z-i)(z+i)}$$

$$\text{Res}(f(z),1)=\lim_{z\to 1}\cancel{(z-1)}\frac{z^2}{\cancel{(z-1)}(z+1)(z-i)(z+i)}=\frac{1}{2(1-i)(1+i)}=\frac{1}{2\cdot 2}=\frac{1}{4}$$

$$\text{Res}(f(z),-1)=\lim_{z\to -1}\cancel{(z+1)}\frac{z^2}{(z-1)\cancel{(z+1)}(z-i)(z+i)}=\frac{1}{-2\cdot 2}=-\frac{1}{4}$$

$$\text{Res}(f(z),i)=\lim_{z\to i}\cancel{(z-i)}\frac{z^2}{(z-1)(z+1)\cancel{(z-i)}(z+i)}=\frac{-1}{-2\cdot 2i}=\frac{1}{4i}=-\frac{1}{4}i$$

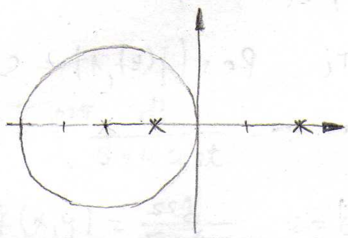
$$\text{Res}(f(z),-i)=\lim_{z\to -i}\cancel{(z+i)}\frac{z^2}{(z-1)(z+1)(z-i)\cancel{(z+i)}}=\frac{-1}{-2\cdot(-2i)}=-\frac{1}{4i}=\frac{1}{4}i$$

B) $f(z)=\frac{e^{3z}-1}{z^3}$, Res = ?

$z=0$ pol drugog reda

$$\text{Res}(f(z),0)=\lim_{z\to 0}\frac{e^{3z}-1}{z^3}=L'H=\lim_{z\to 0}\frac{\cancel{3}e^{3z}}{\cancel{3}z^2}=\lim_{z\to 0}\frac{3e^{3z}}{2z}=\lim_{z\to 0}\frac{9e^{3z}}{2}=\frac{9}{2}$$

13) A) $\int_\Gamma \frac{z\,dz}{(z-2)(z+1)^2}=?$ $\quad \Gamma \ldots |z+2|=2$



$z_1=2$ pol prvog reda

$z_2=-1$ pol drugog reda

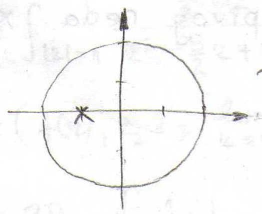
$$\text{Res}(f(z),2)=\lim_{z\to 2}\cancel{(z-2)}\frac{z}{\cancel{(z-2)}(z+1)^2}=\frac{2}{9} \quad \times$$

$$\text{Res}(f(z),-1)=\lim_{z\to -1}\frac{d}{dz}\left(\cancel{(z+1)^2}\frac{z}{(z-2)\cancel{(z+1)^2}}\right)=$$

$$=\lim_{z\to -1}\frac{\cancel{z}-2-\cancel{z}}{(z-2)^2}=-\frac{2}{9} \quad \checkmark$$

$$\oint_\Gamma \frac{z\,dz}{(z-2)(z+1)^2}=2\pi i\cdot\left(-\frac{2}{9}\right)=-\frac{4\pi i}{9}$$

B) $\int_\Gamma \sin\frac{z}{z+1}\,dz=?$ $\quad \Gamma\ldots |z|=2$



$z=-1$ bitni singularitet

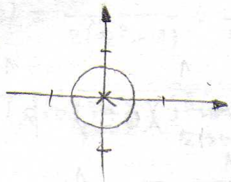
$$f(z)=\sin\frac{z}{z+1}=\sin\left(\frac{z+1-1}{z+1}\right)=\sin\left(1-\frac{1}{z+1}\right)=$$

$$=\sin 1\cos\frac{1}{z+1}-\cos 1\sin\frac{1}{z+1}=$$

$$=\sin 1\left(1-\frac{1}{(z+1)^2 2!}+\frac{1}{(z+1)^4 4!}+\ldots\right)-\cos 1\left(\frac{1}{z+1}-\frac{1}{(z+1)^3 3!}+\ldots\right)$$

$$\oint_\Gamma \sin\frac{z}{z+1}\,dz=2\pi i\cdot(-\cos 1)=-2\pi i\cos 1 \qquad \text{Res}(f,-1)=-\cos 1$$

14) A) $\int_\Gamma \frac{e^z\,dz}{z^4+z^3}$, $\Gamma \ldots |z| = \frac{1}{2}$



$$f(z) = \frac{e^z}{z^3(z+1)} \qquad z=-1 \text{ pol prvog reda } \times$$

$z = 0$ pol trećeg reda ✓

$$\mathrm{Res}(f(z),0) = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left(\cancel{z^3}\frac{e^z}{\cancel{z^3}(z+1)}\right) =$$

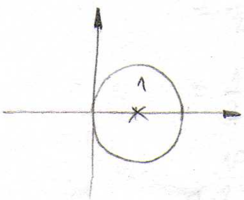
$$= \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left(\frac{e^z(z+1)-e^z}{(z+1)^2}\right) = \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left(\frac{ze^z}{(z+1)^2}\right) =$$

$$= \frac{1}{2}\lim_{z\to 0}\left(\frac{(e^z+ze^z)(z+1)^2 - ze^z\cdot 2(z+1)}{(z+1)^4}\right) = \frac{1}{2}\lim_{z\to 0}\frac{e^z(z+1)^2-2ze^z}{(z+1)^3}$$

$$= \frac{1}{2}\cdot 1 = \frac{1}{2}$$

$$\oint_\Gamma \frac{e^z\,dz}{z^4+z^3} = \cancel{2}\pi i\cdot\frac{1}{\cancel{2}} = \pi i$$

B) $\int_\Gamma \sin\frac{1}{z-1}\,dz$, $\Gamma \ldots |z-1| = 1$

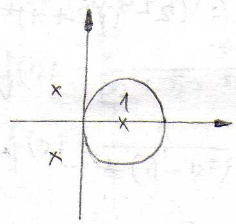


$z = 1$ bitni singularitet

$$f(z) = \sin\frac{1}{z-1} = \left(\frac{1}{z-1} - \frac{1}{(z-1)^3 3!} + \ldots\right)$$

$$\oint_\Gamma \sin\frac{1}{z-1} = 2\pi i\, 1 = 2\pi i \qquad \mathrm{Res}(f(z),1) = c_{-1} = 1$$

15) A) $\int_\Gamma \frac{e^{2z}\,dz}{z^3-1}$, $\Gamma \ldots |z-1| = 1$



$$f(z) = \frac{e^{2z}}{(z-1)(z^2+z+1)} = \frac{e^{2z}}{(z-1)\left(z+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(z+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}$$

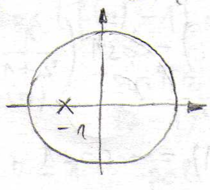
$z = 1$ pol prvog reda ✓

$$\mathrm{Res}(f(z),1) = \lim_{z\to 1}\cancel{(z-1)}\frac{e^{2z}}{\cancel{(z-1)}(z^2+z+1)} = \frac{e^2}{3}$$

$z_{2,3} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ polovi prvog reda $\times$

$$\oint_\Gamma \frac{e^{2z}\,dz}{z^3-1} = 2\pi i\cdot\frac{e^2}{3} = \frac{2}{3}\pi e^2 i$$

B) $\int_\Gamma z\sin\frac{z+1}{z-1}\,dz$, $\Gamma \ldots |z| = 2$



$z = 1$ bitni singularitet

$$f(z) = z\sin\frac{z+1}{z-1} = [(z-1)+1]\sin\frac{z-1+2}{z-1} =$$

$$= (z-1)\sin\left(1+\frac{2}{z-1}\right) + \sin\left(1+\frac{2}{z-1}\right) =$$

$$= (z-1)\sin 1\cos\frac{2}{z-1} + (z-1)\cos 1\sin\frac{2}{z-1} + \sin 1\cos\frac{2}{z-1} + \cos 1\sin\frac{2}{z-1} =$$

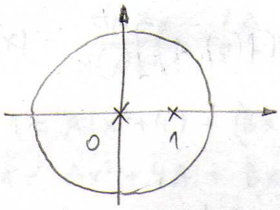
$=$

$=$

$$\oint_\Gamma z\sin\frac{z+1}{z-1}\,dz = 2\pi i\cdot 2(\cos 1 - \sin 1) = 4\pi i(\cos 1 - \sin 1)$$

$$\mathrm{Res}(f(z),1) = -2\sin 1 + 2\cos 1$$

16) A) $\int_\Gamma \frac{\sin z}{z^2-z}$, $\Gamma \ldots |z|=2$



$f(z) = \frac{\sin z}{z(z-1)}$

$z=0$ pol prvog reda

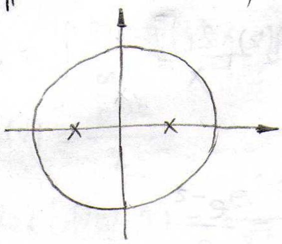
$\mathrm{Res}\,(f(z),0) = \lim_{z\to 0} \cancel{z} \frac{\sin z}{\cancel{z}(z-1)} = 0$

$z=1$ pol prvog reda

$\mathrm{Res}\,(f(z),1) = \lim_{z\to 1} \cancel{(z-1)} \frac{\sin z}{z\cancel{(z-1)}} = \sin 1$

$\oint_\Gamma \frac{\sin z}{z^2-z} = 2\pi i \cdot (0+\sin 1) = 2\pi i \sin 1$

B) $\int_\Gamma \mathrm{tg}(\pi z)\,dz$, $\Gamma \ldots |z| = \pi$



$f(z) = \frac{\sin(\pi z)}{\cos(\pi z)}$

$\cos(\pi z) = 0 \Rightarrow \pi z = \frac{\pi}{2} + k\pi \Rightarrow z_k = \frac{1}{2}(k+1) \quad k \in \mathbb{Z}$

polovi prvog reda

$\mathrm{Res}\,(\mathrm{tg}(\pi z), z_k) = -\frac{1}{\pi}$

$\oint_\Gamma \mathrm{tg}(\pi z)\,dz = 2\cancel{\pi} i \cdot \left(-\frac{1}{\cancel{\pi}}\right) = -2i$

17) A) $I = \int_0^{2\pi} \frac{dt}{5-4\cos t} = ?$

$R(x,y) = \frac{1}{5-4x} \qquad z = e^{it},\ dz = ie^{it}dt = iz\,dt \Rightarrow dt = \frac{dz}{iz}$

$\cos t = \frac{1}{2}(e^{it}+e^{-it}) = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$I = \int_{|z|=1} \frac{1}{5-\cancel{4}^{2}\cdot\frac{1}{\cancel{2}}\left(z+\frac{1}{z}\right)} \cdot \frac{dz}{iz} = \int_{|z|=1} \frac{1}{5-2\left(\frac{z^2+1}{z}\right)} \frac{dz}{iz} = \int_{|z|=1} \frac{\cancel{z}}{5z-2z^2-2} \cdot \frac{dz}{i\cancel{z}} =$

$= \int_{|z|=1} \frac{\frac{i}{2}\,dz}{z^2-\frac{5}{2}z+1} = \int_{|z|=1} \frac{\frac{i}{2}\,dz}{(z-2)(z-\frac{1}{2})}$

$\mathrm{Res}\,(f(z), z_2) = \lim_{z\to z_2} \cancel{(z-z_2)} \frac{\frac{i}{2}}{(z-z_1)\cancel{(z-z_2)}} = \frac{\frac{i}{2}}{z_2-z_1} = \frac{\frac{i}{2}}{\frac{1}{2}-2} = \frac{\frac{i}{\cancel{2}}}{-\frac{3}{\cancel{2}}} = -\frac{1}{3}i$

$I = 2\pi i \cdot \left(-\frac{1}{3}i\right) = \frac{2\pi}{3}$

B) $I = \int_{-\infty}^{\infty} \frac{\sin x}{(x^2+4)(x-1)}\,dx = ?$

$f(x) = \frac{\sin x}{(x^2+4)(x-1)} = \left(\frac{Ax+B}{x^2+4} + \frac{C}{x-1}\right)\sin x$

$1 = (Ax+B)(x-1) + C(x^2+4)$

$1 = Ax^2 - Ax + Bx - B + Cx^2 + 4C$

$1 = (A+C)x^2 + (-A+B)x + (-B+4C)$

$A+C=0 \Rightarrow C=-A \Rightarrow C=\frac{1}{5}$

$-A+B=0 \Rightarrow B=A \Rightarrow B=-\frac{1}{5}$

$-B+4C=1 \Rightarrow -5A=1 \Rightarrow A=-\frac{1}{5}$

$f(x) = -\frac{1}{5}\frac{(x+1)\sin x}{x^2+4} + \frac{1}{5}\frac{\sin x}{x-1} = -\frac{1}{5}\frac{x\sin x}{x^2+4} - \frac{1}{5}\frac{\sin x}{x^2+4} + \frac{1}{5}\frac{\sin x}{x-1}$

$$I = \frac{1}{5}\left[\int_{-\infty}^{\infty} \frac{\sin x}{x-1}\,dx - \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+4}\,dx - \int_{-\infty}^{\infty} \frac{\sin x}{x^2+4}\,dx\right]$$

$$I_1 = \int_{-\infty}^{\infty} \frac{\sin x}{x-1}\,dx = \operatorname{Im} \int_{-\infty}^{\infty} f(x)\,dx = \operatorname{Im}\left[\pi i \operatorname{Res}(f(z),1)\right]$$

$$f(z) = \frac{e^{iz}}{z-1} \qquad \int_{C_R} f(z)\,dz \to 0$$

$$\operatorname{Res}(f(z),1) = \frac{e^{iz}}{(z-1)'}\bigg|_{z=1} = \frac{e^{iz}}{1}\bigg|_{z=1} = e^{i} = \cos 1 + i \sin 1$$

$$I_1 = \operatorname{Im}\left[2\pi i (\cos 1 + i \sin 1\right] = \operatorname{Im}\left[-\pi \sin 1 + \pi i \cos 1\right] =$$

$$= \pi \cos 1$$

$$I_2 = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+4} = \operatorname{Im} \int_{-\infty}^{\infty} f(x)\,dx = \operatorname{Im}\left[2\pi i \operatorname{Res}(f(z), 2i)\right]$$

$$f(x) = \frac{x \cdot e^{ix}}{x^2+1}, \quad f(z) = \frac{z}{z^2+1}$$

$$\operatorname{Res}(f(z),2i) = \frac{z e^{iz}}{(z^2+1)'}\bigg|_{z=2i} = \frac{\cancel{z} e^{iz}}{2\cancel{z}}\bigg|_{z=2i} = \frac{e^{i\cdot 2i}}{2} = \frac{e^{-2}}{2}$$

$$I_2 = \operatorname{Im}\left[\cancel{2}\pi i \cdot \frac{e^{-2}}{\cancel{2}}\right] = \pi e^{-2}$$

$$I_3 = \int_{-\infty}^{\infty} \frac{\sin x}{x^2+4} = \operatorname{Im} \int_{-\infty}^{\infty} f(x)\,dx = \operatorname{Im}\left[2\pi i \operatorname{Res}(f(z), 2i)\right]$$

$$f(x) = \frac{e^{ix}}{x^2+4}, \quad f(z) = \frac{1}{z^2+1}$$

$$\operatorname{Res}(f(z),2i) = \frac{e^{iz}}{(z^2+1)^2}\bigg|_{z=2i} = \frac{e^{iz}}{2z}\bigg|_{z=2i} = \frac{e^{i \cdot 2i}}{2 \cdot 2i} = \frac{e^{-2}}{4i}$$

$$I_3 = \operatorname{Im}\left[2\pi \cancel{i} \cdot \frac{e^{-2}}{4\cancel{i}}\right] = 0$$

$$I = \frac{1}{5}\left[\pi \cos 1 - \pi e^{-2}\right] = \frac{\pi}{5}(\cos 1 - e^{-2})$$

18) A) $I = \int_0^{2\pi} \frac{dt}{5+3\cos t} = ?$

$$R(x,y) = \frac{1}{5+3x}, \quad z = e^{it}, \quad dz = ie^{it}\,dt = iz\,dt \Rightarrow dt = \frac{dz}{iz}$$

$$\cos t = \frac{1}{2}(e^{it} + e^{-it}) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$I = \int_{|z|=1} \frac{1}{5 + 3 \cdot \frac{1}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz} = \int_{|z|=1} \frac{1}{5 + \frac{3}{2}\left(\frac{z^2+1}{z}\right)} \frac{dz}{iz} =$$

$$= \int_{|z|=1} \frac{2\cancel{z}}{10z + 3z^2 + 3} \cdot \frac{dz}{\cancel{z}} = \int_{|z|=1} \frac{-\frac{2}{3} i\,dz}{z^2 + \frac{10}{3}z + 1} = \int_{|z|=1} \frac{-\frac{2}{3}i}{(z+\frac{1}{3})(z+3)}$$

$$\operatorname{Res}(f(z),z_1) = \lim_{z\to z_1} \cancel{(z-z_1)} \frac{-\frac{2}{3}i}{\cancel{(z-z_1)}(z-z_2)} = \frac{-\frac{2}{3}i}{z_1 - z_2} = \frac{-\frac{2}{3}i}{-\frac{1}{3}+3} = \frac{-\frac{\cancel{2}}{\cancel{3}}i}{\frac{\cancel{8}\,4}{\cancel{3}}} = -\frac{1}{4}i$$

$$I = \cancel{2}\pi i \cdot \left(-\frac{1}{\cancel{4}_{\,2}} i\right) = \frac{\pi}{2}$$

B) $I = \int_{-\infty}^{\infty} \frac{\sin 3x}{x(x^2+4)}\,dx = ?$

$$f(x) = \frac{\sin 3x}{x(x^2+4)} = \left(\frac{A}{x} + \frac{Bx+C}{x^2+4}\right)\sin 3x$$

$$1 = A(x^2+4) + (Bx+C)x \qquad A+B=0 \Rightarrow B=-A \Rightarrow B = -\frac{1}{4}$$

$$1 = Ax^2 + 4A + Bx^2 + Cx \qquad C=0$$

$$1 = (A+B)x^2 + Cx + 4A \qquad 1 = 4A \Rightarrow A = \frac{1}{4}$$

$$f(x) = \frac{1}{4}\frac{\sin 3x}{x} - \frac{1}{4}\frac{x\sin 3x}{x^2+4}$$

$$I = \frac{1}{4}\left[\int_{-\infty}^{\infty}\frac{\sin 3x}{x}\,dx - \int_{-\infty}^{\infty}\frac{x\sin 3x}{x^2+4}\,dx\right]$$

$$I_1 = \int_{-\infty}^{\infty}\frac{\sin 3x}{x}\,dx = \operatorname{Im}\int_{-\infty}^{\infty} f(x)\,dx = \operatorname{Im}\left[\pi i \operatorname{Res}(f(z),0)\right]$$

$$f(z) = \frac{e^{3iz}}{z}$$

$$\operatorname{Res}(f(z),1) = \left.\frac{e^{3iz}}{z'}\right|_{z=0} = \left.\frac{e^{3iz}}{1}\right|_{z=0} = e^0 = 1$$

$$I_1 = \operatorname{Im}[\pi i \cdot 1] = \pi$$

$$I_2 = \int_{-\infty}^{\infty}\frac{x\sin 3x}{x^2+4}\,dx = \operatorname{Im}\int_{-\infty}^{\infty} f(x)\,dx = \operatorname{Im}\left[2\pi i \operatorname{Res}(f(z),2i)\right]$$

$$f(x) = \frac{xe^{3ix}}{x^2+1}, \quad f(z) = \frac{z}{z^2+1}$$

$$\operatorname{Res}(f(z),2i) = \left.\frac{ze^{3iz}}{(z^2+1)'}\right|_{z=2i} = \left.\frac{\cancel{z}e^{3iz}}{2\cancel{z}}\right|_{z=2i} = \frac{e^{3i\cdot 2i}}{2} = \frac{e^{-6}}{2}$$

$$I_2 = \operatorname{Im}\left[\cancel{2}\pi i \cdot \frac{e^{-6}}{\cancel{2}}\right] = \pi e^{-6}$$

$$I = \frac{1}{4}\left[\pi - \pi e^{-6}\right] = \frac{\pi}{4}(1-e^{-6})$$

19) A) $I = \int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+1)(x^2+4)} = ?$

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$$

$$\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \operatorname{Res}(f(z),i) + 2\pi i \operatorname{Res}(f(z),2i)$$

$$\operatorname{Res}(f(z),i) = \lim_{z\to i}\frac{\cancel{(z-i)}z^2}{\cancel{(z-i)}(z+i)(z^2+4)} = \frac{-1}{2i\cdot 3} = \frac{-1}{6i}$$

$$\operatorname{Res}(f(z),2i) = \lim_{z\to 2i}\frac{\cancel{(z-2i)}z^2}{(z^2+1)\cancel{(z-2i)}(z+2i)} = \frac{-\cancel{4}}{-3\cdot\cancel{4}i} = \frac{1}{3i}$$

$$\int_{-\infty}^{\infty} f(z)\,dz = 2\pi \cancel{i}\cdot\frac{(-1)}{_3\cancel{6}\cancel{i}} + 2\pi\cancel{i}\cdot\frac{1}{3\cancel{i}} = -\frac{\pi}{3} + \frac{2\pi}{3} = \frac{\pi}{3}$$

B) $I=\int_{-\infty}^{\infty}\frac{x\sin x}{x^2-2x+10}=?$

$f(z)=\frac{ze^{iz}}{(z-1-3i)(z-1+3i)}$

$I=\mathrm{Im}\int_{-\infty}^{\infty}f(z)dz=\mathrm{Im}\left[2\pi i\,\mathrm{Res}\left(f(z),1+3i\right)\right]$

$\mathrm{Res}\left(f(z),1+3i\right)=\lim_{z\to1+3i}\frac{\cancel{(z-1-3i)}ze^{iz}}{\cancel{(z-1-3i)}(z-1+3i)}=\frac{(1+3i)e^{i(1+3i)}}{\cancel{1}+3i\cancel{-1}+3i}=$

$=\frac{(1+3i)e^{i-3}}{6i}=e^{-3}(\cos1+i\sin1)\left(\frac{1}{6i}+\frac{1}{2}\right)=$

$=e^{-3}\left(\frac{\cos1}{6i}+\frac{\cos1}{2}+\frac{\sin1}{6}+i\frac{\sin1}{2}\right)$

$I=\mathrm{Im}\left[2\pi i\cdot e^{-3}\left(\frac{\cos1}{\cancel{2}}+\frac{\sin1}{\cancel{6}_3}-i\frac{\cos1}{\cancel{6}_3}+i\frac{\sin1}{\cancel{2}}\right)\right]=$

$=\mathrm{Im}\left[\pi e^{-3}\left(\frac{\cos1}{3}-\sin1+i\cos1+i\frac{\sin1}{3}\right)\right]=\frac{\pi}{3}(3\cos1+\sin1)e^{-3}$

20) A) $I=\int_{-\infty}^{\infty}\frac{x^2+1}{x^4+1}dx=?$

$f(z)=\frac{z^2+1}{z^4+1}$ $\quad z_k=e^{i\frac{\pi+2k\pi}{n}}$ $\quad z_0=e^{i\frac{\pi}{n}}$

$\int_{-\infty}^{\infty}f(z)dz=2\pi i\,\mathrm{Res}\left(f(z),z_0\right)=2\pi i\left.\frac{1}{nz^{n-1}}\right|_{z=z_0}=2\pi i\frac{1}{ne^{i\pi-i\frac{\pi}{n}}}=-2\pi i\frac{e^{i\frac{\pi}{n}}}{n}$

$I=-\frac{2\pi i}{n}\cdot\frac{e^{i\frac{\pi}{n}}}{1-e^{i\frac{2\pi}{n}}}\left(e^{i\frac{2\pi}{n}}+1\right)=-\frac{2\pi i}{n}\cdot\frac{1}{e^{-i\frac{\pi}{n}}-e^{i\frac{\pi}{n}}}\left(e^{i\frac{2\pi}{n}}+1\right)=$

$=\cancel{-}\frac{2\pi\cancel{i}}{n}\cdot\frac{1}{\cancel{-}2\cancel{i}\sin\frac{\pi}{n}}\left(e^{i\frac{2\pi}{n}}+1\right)=\frac{\pi}{n\sin\frac{\pi}{n}}\left(e^{i\frac{2\pi}{n}}+1\right)=[n=4]=$

$=\frac{\pi}{4\sin\frac{\pi}{4}}\left(\left|e^{i\frac{2\pi}{\cancel{4}_2}}\right|+1\right)=\frac{\pi}{{}_2\cancel{4}\cdot\frac{\sqrt2}{\cancel{2}}}\cdot\cancel{2}=\frac{\pi}{\sqrt2}$ $\quad\pi\sqrt2$ $\quad$???

B) $I=\int_{-\infty}^{\infty}\frac{x\cos x}{x^2-2x+10}dx=?$

$f(z)=\frac{ze^{iz}}{(z-1-3i)(1-1+3i)}$

$I=\mathrm{Re}\int_{-\infty}^{\infty}f(z)dz=\mathrm{Re}\left[2\pi i\,\mathrm{Res}\left(f(z),1+3i\right)\right]=$

$=\mathrm{Re}\left[2\pi i\cdot e^{-3}\left(\frac{\cos1}{\cancel{2}}+\frac{\sin1}{\cancel{6}_3}-i\frac{\cos1}{\cancel{6}_3}+i\frac{\sin1}{\cancel{2}}\right)\right]=$

$=\mathrm{Re}\left[\pi\cdot e^{-3}\left(\frac{\cos1}{3}-\sin1+i\cos1+i\frac{\sin1}{3}\right)\right]=\frac{\pi}{3}(\cos1-3\sin1)e^{-3}$

1) A) $\int_0^\infty e^{-x^3}dx = \begin{bmatrix} t=x^3 \Rightarrow x=t^{1/3} & dx=\frac{dt}{3x^2}=\frac{1}{3}t^{-\frac{2}{3}}dt & t_1=0 \\ dt=3x^2dx & & t_2=\infty \end{bmatrix} = \int_0^\infty e^{-t}\cdot\frac{1}{3}t^{-\frac{2}{3}}\,dt =$

$= \frac{1}{3}\int_0^\infty t^{-\frac{2}{3}}e^{-t}dt = \left[x-1=-\frac{2}{3} \Rightarrow x=\frac{1}{3}\right] = \frac{1}{3}\Gamma\left(\frac{1}{3}\right) = \Gamma\left(1+\frac{1}{3}\right) = \Gamma\left(\frac{4}{3}\right)$

B) $\int_{-1}^1 \sqrt{\frac{1+x}{1-x}}\,dx = \int_{-1}^1 (1+x)^{1/2}(1-x)^{-1/2}dx = \begin{bmatrix} 2t=1+x \Rightarrow x=2t-1 & t_1=0 \\ 2dt=dx \quad 1-x=2-2t & t_2=1 \end{bmatrix} =$

$= \int_0^1 \cancel{2^{1/2}}t^{1/2}\cancel{2^{-1/2}}(1-t)^{-1/2}\cdot 2\,dt = 2\int_0^1 t^{1/2}(1-t)^{-1/2}dt \begin{bmatrix} \alpha-1=\frac{1}{2} \Rightarrow \alpha=\frac{3}{2} \\ \beta-1=-\frac{1}{2} \Rightarrow \beta=\frac{1}{2} \end{bmatrix} =$

$= 2\,\frac{\Gamma\left(\frac{3}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = 2\cdot\frac{\Gamma\left(1+\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma(1+1)} = \cancel{2}\,\frac{\frac{1}{\cancel{2}}\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \frac{\sqrt{\pi}\cdot\sqrt{\pi}}{1} = \pi$

2) A) $\int_0^\infty e^{-x^6}dx = \begin{bmatrix} t=x^6 \Rightarrow x=t^{1/6} \Rightarrow dx=\frac{dt}{6x^5}=\frac{1}{6}t^{-\frac{5}{6}}dt & t_1=0 \\ dt=6x^5dx & t_2=\infty \end{bmatrix} = \int_0^\infty e^{-t}\cdot\frac{1}{6}t^{-\frac{5}{6}}\,dt =$

$= \frac{1}{6}\int_0^\infty t^{-\frac{5}{6}}e^{-t}\,dt = \left[x-1=-\frac{5}{6} \Rightarrow x=\frac{1}{6}\right] = \frac{1}{6}\Gamma\left(\frac{1}{6}\right) = \Gamma\left(1+\frac{1}{6}\right) = \Gamma\left(\frac{7}{6}\right)$

B) $\int_0^1 \frac{dx}{\sqrt[3]{1-x^3}} = \int_0^1 (1-x^3)^{-1/3}dx = \begin{bmatrix} t=1-x^3 \Rightarrow x=(1-t)^{1/3} & t_1=1 \quad t_2=0 \\ dt=-3x^2dx \Rightarrow dx=-\frac{1}{3}\frac{dt}{x^2}=-\frac{1}{3}(1-t)^{-2/3}dt \end{bmatrix} =$

$= \int_1^0 t^{-1/3}\cdot\left(-\frac{1}{3}\right)(1-t)^{-2/3}dt = \frac{1}{3}\int_0^1 t^{-1/3}(1-t)^{-2/3}dt = \begin{bmatrix} \alpha-1=-\frac{1}{3} \Rightarrow \alpha=\frac{2}{3} \\ \beta-1=-\frac{2}{3} \Rightarrow \beta=\frac{1}{3} \end{bmatrix} =$

$= \frac{1}{3}\,\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma(1)} = \frac{1}{3}\,\frac{\Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(1-\frac{1}{3}\right)}{\Gamma(1)} = \begin{bmatrix} \text{svojstvo simetrije} \\ \Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z} \end{bmatrix} = \frac{1}{3}\,\frac{\pi}{\sin\frac{\pi}{3}} =$

$= \frac{\pi}{3}\cdot\frac{2}{\sqrt{3}} = \frac{2\pi}{3\sqrt{3}}$

3) A) $\int_0^\infty x^2e^{-x^2}dx = \begin{bmatrix} t=x^2 \Rightarrow x=t^{1/2} & t_1=0,\ t_2=\infty \\ dt=2xdx \Rightarrow dx=\frac{dt}{2x}=\frac{1}{2}t^{-1/2}dt \end{bmatrix} = \int_0^\infty te^{-t}\cdot\frac{1}{2}t^{-1/2}\,dt =$

$= \frac{1}{2}\int_0^\infty t^{1/2}e^{-t}dt = \left[x-1=\frac{1}{2} \Rightarrow x=\frac{3}{2}\right] = \frac{1}{2}\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) =$

$= \frac{1}{4}\Gamma\left(\frac{1}{2}\right) = \frac{1}{4}\sqrt{\pi}$

B) $\int_0^{\pi/2}\frac{dx}{\sqrt{\cos x}} = \int_0^{\pi/2}\cos^{-1/2}x\,dx = \begin{bmatrix} \alpha=-\frac{1}{2} \\ \beta=0 \end{bmatrix} = \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+0}{2}+1\right)} = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}\right)} =$

$= \frac{\sqrt{\pi}}{2}\,\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$

4) A) $\int_0^\infty xe^{-x^3}dx = \begin{bmatrix} t=x^3 \Rightarrow x=t^{1/3} & t_1=0,\ t_2=\infty \\ dt=3x^2dx \Rightarrow xdx=\frac{dt}{3x}=\frac{1}{3}t^{-1/3}dt \end{bmatrix} = \int_0^\infty e^{-t}\cdot\frac{1}{3}t^{-1/3}\,dt =$

$= \frac{1}{3}\int_0^\infty t^{-1/3}e^{-t}\,dt = \left[x-1=-\frac{1}{3} \Rightarrow x=\frac{2}{3}\right] = \frac{1}{3}\Gamma\left(\frac{2}{3}\right)$

B) $\int_0^{\pi/2} \sqrt{tg\, x}\, dx = \int_0^{\pi/2} \sqrt{\frac{\sin x}{\cos x}}\, dx = \int_0^{\pi/2} \cos^{-1/2} x \sin^{1/2} x\, dx = \begin{bmatrix} \alpha = -\frac{1}{2} \\ \beta = \frac{1}{2} \end{bmatrix} =$

$$= \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\cancel{-\frac{1}{2}}+\cancel{\frac{1}{2}}}{2}+1\right)} = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\Gamma(1)} = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)}{2} =$$

$$= \begin{bmatrix} \text{svojstvo simetrije} \\ \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \end{bmatrix} = \frac{1}{2} \frac{\pi}{\sin \frac{\pi}{4}} = \frac{\pi}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

5) A) $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^1 (1-x^4)^{-1/2} dx = \begin{bmatrix} t = 1-x^4 \Rightarrow x = (1-t)^{1/4} \quad t_1 = 1,\ t_2 = 0 \\ dt = -4x^3 dx \Rightarrow dx = \frac{-dt}{4x^3} = -\frac{1}{4}(1-t)^{-3/4} dt \end{bmatrix} =$

$$= \int_1^0 t^{-1/2} \cdot \left(-\frac{1}{4}\right)(1-t)^{-3/4} dt = \frac{1}{4}\int_0^1 t^{-1/2}(1-t)^{-3/4} dt = \begin{bmatrix} \alpha - 1 = -\frac{1}{2} \Rightarrow \alpha = \frac{1}{2} \\ \beta - 1 = -\frac{3}{4} \Rightarrow \beta = \frac{1}{4} \end{bmatrix} =$$

$$= \frac{1}{4} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \sqrt{\pi} \frac{\Gamma\left(1+\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \sqrt{\pi} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

B) $\int_0^\infty \frac{x^2}{1+x^4} dx = \begin{bmatrix} x^2 = tg\,\varphi \Rightarrow x = tg^{1/2}\varphi \quad \varphi_1 = 0,\ \varphi_2 = \pi/2 \\ 2x\,dx = \frac{d\varphi}{\cos^2\varphi} \Rightarrow dx = \frac{d\varphi}{2x\cos^2\varphi} = \frac{tg^{-1/2}\varphi}{2\cos^2\varphi} d\varphi \end{bmatrix} =$

$$= \int_0^{\pi/2} \frac{tg\,\varphi}{1+tg^2\varphi} \cdot \frac{tg^{-1/2}\varphi}{2\cos^2\varphi} d\varphi = \frac{1}{2}\int_0^{\pi/2} \frac{tg^{1/2}\varphi}{1+tg^2\varphi} \frac{d\varphi}{\cos^2\varphi} =$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\frac{\sin^{1/2}\varphi}{\cos^{1/2}\varphi}}{1+\frac{\sin^2\varphi}{\cos^2\varphi}} \cdot \frac{d\varphi}{\cos^2\varphi} = \frac{1}{2}\int_0^{\pi/2} \frac{\frac{\sin^{1/2} d\varphi}{\cos^{1/2} d\varphi}}{\frac{\cos^2\varphi+\sin^2\varphi}{\cancel{\cos^2\varphi}}} \cdot \frac{d\varphi}{\cancel{\cos^2\varphi}} =$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sin^{1/2}\varphi}{\cos^{1/2}\varphi} d\varphi = \frac{1}{2}\int_0^{\pi/2} \cos^{-1/2}\varphi \sin^{1/2}\varphi\, d\varphi = \begin{bmatrix} \alpha = -\frac{1}{2} \\ \beta = \frac{1}{2} \end{bmatrix} =$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\cancel{-\frac{1}{2}}+\cancel{\frac{1}{2}}}{2}+1\right)} = \frac{1}{2} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\Gamma(1)} = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) =$$

$$= \begin{bmatrix} \text{svojstvo simetrije} \\ \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \end{bmatrix} = \frac{1}{4} \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\cancel{4}_2} \cdot \frac{\cancel{2}^1}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$$