Signali i sustavi

Pismeni ispit – 7. rujna 2016.

1. (20 bodova) Zadan je vremenski diskretan signal $f(n) = \cos(\frac{4\pi}{3}n) - \sin(\frac{4\pi}{3}n)$.

a) **(6 bodova)** Izračunajte snagu signala postupkom u vremenskoj domeni.

b) **(7 bodova)** Odredite DTFS zadanog signala.

c) **(7 bodova)** Izračunajte snagu signala postupkom u frekvencijskoj domeni.

2. (20 bodova) Zadani su vremenski kontinuirani signali $f_1(t) = \cos(4t)$ i $f_2(t) = e^{-4|t|}$ te neka je $f_3(t) = f_1(t)f_2(t)$.

a) **(6 bodova)** Odredite CTFT (u generaliziranom smislu) kontinuiranog signala f_1.

b) **(7 bodova)** Odredite CTFT kontinuiranog signala f_2.

c) **(7 bodova)** Odredite CTFT zadanog signala f_3 koristeći teorem o konvoluciji signala u frekvencijskoj domeni.

3. (20 bodova) Vremenski diskretan kauzalan LTI sustav zadan je jednadžbom diferencija $y(n)+\frac{1}{4}y(n-1) = 2u(n)+u(n-1)$ te početnim uvjetom $y(-1) = 2$.

a) **(10 bodova)** Odredite odziv sustava na pobudu $u(n) = 4(\frac{1}{2})^n\mu(n)$ pomoću Z transformacije.

b) **(10 bodova)** Odredite odziv sustava na pobudu $u(n) = 4(\frac{1}{2})^{n-1}\mu(n-1)$ pomoću Z transformacije.

4. (20 bodova) Vremenski kontinuiran kauzalan LTI sustav zadan je diferencijalnom jednadžbom $y'(t)+3y(t) = 3u'(t)+u(t)$.

a) **(6 bodova)** Odredite prijenosnu funkciju sustava te ispitajte stabilnost sustava.

b) **(7 bodova)** Odredite i skicirajte amplitudno frekvencijsku karakteristiku sustava.

c) **(7 bodova)** Odredite prisilni odziv sustava na svevremensku pobudu $u(t) = 2\cos(4t) + 2\sin(4t)$.

5. (20 bodova) Vremenski kontinuiran kauzalan LTI sustav bez nula zadan je odzivom nepobuđenog sustava $y_{np}(t) = 2t+1$ i odzivom mirnog sustava $y_m(t) = 4t^2\mu(t)$ na kauzalnu pobudu $u(t) = \mu(t)$.

a) **(6 bodova)** Odredite diferencijalnu jednadžbu sustava (minimalni red).

b) **(7 bodova)** Odredite impulsni odziv sustava.

c) **(7 bodova)** Odredite totalni odziv sustava na kauzalnu pobudu $u(t) = e^{-2t}\mu(t)$.

1. $f(n) = \cos\left(\frac{4\pi}{3}n\right) - \sin\left(\frac{4\pi}{3}n\right)$

$\omega_0 = \frac{4\pi}{3}$ $\quad N_0 = \frac{2\pi}{\frac{4\pi}{3}} = \frac{6}{4} = \frac{3}{2}$

a)

$$P = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2$$

N_0 mora biti cijeli broj

$\boxed{N_0 = 3}$

$$P = \frac{1}{3}\sum_{n=0}^{2} |x(n)|^2 = \frac{1}{3}\left(|\cos(0) - \sin(0)|^2 + \left|\cos\left(\frac{4\pi}{3}\right) - \sin\left(\frac{4\pi}{3}\right)\right|^2 + \left|\cos\left(\frac{8\pi}{3}\right) - \sin\left(\frac{8\pi}{3}\right)\right|^2\right)$$

$$= \frac{1}{3}\cdot\left(|1-0|^2 + \left|-\frac{1}{2} + \frac{\sqrt{3}}{2}\right|^2 + \left|-\frac{1}{2} - \frac{\sqrt{3}}{2}\right|^2\right)$$

$$= \frac{1}{3}\left(1 + \left(\frac{1}{4} - \frac{\sqrt{3}}{3} + \frac{3}{4}\right) + \left(\frac{1}{4} + \frac{\sqrt{3}}{2} + \frac{3}{4}\right)\right)$$

$$= \frac{1}{3}\cdot 3 = 1$$

b) DTFS

$$f(n) = \cos\left(\frac{4\pi}{3}n\right) - \sin\left(\frac{4\pi}{3}n\right) = \sum_{k=0}^{N-1} X[k]e^{2\pi jkn/N}$$

$$= \frac{e^{j\frac{4\pi}{3}n} + e^{-j\frac{4\pi}{3}n}}{2} - \frac{e^{j\frac{4\pi}{3}n} - e^{-j\frac{4\pi}{3}n}}{2j}$$

$$= \frac{1}{2}\left(e^{\frac{2\pi jn\cdot(2)}{N}} + e^{\frac{2\pi jn(-2)}{N}}\right) + \frac{1}{2}e^{j\pi}\left(e^{\frac{2\pi jn(2)}{N}} - e^{\frac{2\pi jn(-2)}{N}}\right)$$

$X_2 = \frac{1}{2} + \frac{1}{2}e^{j\frac{\pi}{2}}$ $\qquad X_{-2} = X_{-2+N} = X_1 = \frac{1}{2} - \frac{1}{2}e^{j\frac{\pi}{2}}$

$$= \frac{1}{2} + \frac{1}{2}e^{-j\frac{\pi}{2}}$$

c) $P = \sum_{k=0}^{N-1} |X_k|^2 - \sum_{k=0}^{2} |X_k|^2 = \left|\frac{1}{2} + \frac{1}{2} e^{j\frac{\pi}{2}}\right|^2 + \left|\frac{1}{2} + \frac{1}{2} e^{-j\frac{\pi}{2}}\right|^2$

$= \left|\frac{1}{2} + \frac{1}{2} j\right|^2 + \left|\frac{1}{2} - \frac{1}{2} j\right|^2 = \sqrt{\frac{1}{4} + \frac{1}{4}}^{\,2} + \sqrt{\frac{1}{4} + \frac{1}{4}}^{\,2}$

$= \underline{\underline{1}}$

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2. $f_1(t) = \cos(4t) \qquad f_2(t) = e^{-4|t|} \qquad f_3(t) = f_1(t) f_2(t)$

a) službene formule str. 3

$\cos(\omega_0 t) \circ\!\!-\!\!\bullet\ \pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$

$\omega_0 = 4$

$\cos(4t) \circ\!\!-\!\!\bullet\ \pi\left[\delta(\omega - 4) + \delta(\omega + 4)\right]$

b) $F_2(\omega) = \int_{-\infty}^{\infty} e^{-4|t|} e^{-j\omega t} dt = \int_0^{\infty} e^{-4t} e^{-j\omega t} dt + \int_{-\infty}^{0} e^{4t} e^{-j\omega t} dt$

$= \int_0^{\infty} e^{(-4-j\omega)t} dt + \int_{-\infty}^{0} e^{(4-j\omega)t} dt$

$= \frac{-1}{4+j\omega} e^{(-4-j\omega)t}\Big|_0^{\infty} + \frac{1}{4-j\omega} e^{(4-j\omega)t}\Big|_{-\infty}^{0}$

$= \frac{-1}{4+j\omega}(0-1) + \frac{1}{4-j\omega}(1-0) = \frac{1}{4+j\omega} + \frac{1}{4-j\omega}$

$= \frac{4-j\omega+4+j\omega}{(4-j\omega)(4+j\omega)} = \underline{\underline{\frac{8}{16+\omega^2}}} = F_2(\omega)$

2.c) $f_1(t) f_2(t) \circ\!\!-\!\!\bullet \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} F_1(x) F_2(\omega - x)\, dx$

$$F_1(\omega) = \pi \left[\delta(\omega - 4) + \delta(\omega + 4) \right]$$

$$F_2(\omega) = \frac{8}{16 + \omega^2}$$

$$F_3(\omega) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \pi \left[\delta(x-4) + \delta(x+4) \right] \cdot \frac{8}{16 + (\omega - x)^2}\, dx$$

$$= \frac{1}{2} \left[\int\limits_{-\infty}^{\infty} \delta(x-4) \frac{8}{16 + (\omega - x)^2}\, dx + \int\limits_{-\infty}^{\infty} \delta(x+4) \frac{8}{16 + (\omega - x)^2}\, dx \right]$$

$$= \frac{1}{2} \left[\frac{8}{16 + (\omega - 4)^2} + \frac{8}{16 + (\omega + 4)^2} \right] = \frac{4}{16 + \omega^2 - 8\omega + 16} + \frac{4}{16 + \omega^2 + 8\omega + 16}$$

$$= \frac{4}{\omega^2 - 8\omega + 32} + \frac{4}{\omega^2 + 8\omega + 32} = 4 \cdot \frac{\omega^2 + 8\omega + 32 + \omega^2 - 8\omega + 32}{(\omega^2 - 8\omega + 32)(\omega^2 + 8\omega + 32)}$$

$$= 4 \cdot \frac{2(\omega^2 + 32)}{\omega^4 + \cancel{8\omega^3} + \cancel{32\omega^2} - \cancel{8\omega^3} - \cancel{64\omega^2} - \cancel{256\omega} + \cancel{32\omega^2} + \cancel{256\omega} + 1024}$$

$$= \frac{8 \cdot (\omega^2 + 32)}{\omega^4 + 1024} = F_3(\omega)$$

3. $y(n) + \frac{1}{4}y(n-1) = 2u(n) + u(n-1)$ uvjet $y(-1) = 2$

a) $u(n) = 4\cdot\left(\frac{1}{2}\right)^n \mu(n)$ odziv pomoću z - transform.

$$Y(z) + \frac{1}{4}\left(\frac{1}{z}Y(z) + y(-1)\right) = 2U(z) + U(z)\frac{1}{z}$$

$$Y(z)\left(\frac{1}{4z} + 1\right) + \frac{1}{2} = U(z)\left(2 + \frac{1}{z}\right)$$

$$Y(z)\ \frac{4z+1}{4z} = U(z)\ \frac{2z+1}{z} - \frac{1}{2}$$

$$\boxed{U(z) = 4\cdot\frac{z}{z-\frac{1}{2}} \quad \text{formule!}}$$

$$Y(z)\ \frac{4z+1}{4z} = 4\cdot\frac{z}{z-\frac{1}{2}}\cdot 2\cdot\frac{z+\frac{1}{2}}{z} - \frac{1}{2}$$

$$= 8\cdot\frac{z+\frac{1}{2}}{z-\frac{1}{2}} - \frac{1}{2} = 8\cdot\left(1 + \frac{1}{z-\frac{1}{2}}\right) - \frac{1}{2} = \frac{15}{2} + \frac{8}{z-\frac{1}{2}}$$

$$Y(z) = \frac{15}{2}\cdot\frac{4z}{4z+1} + \frac{8\cdot 4z}{(4z+1)\left(z-\frac{1}{2}\right)} = \frac{15}{2}\cdot\frac{z}{z+\frac{1}{4}} + 8\cdot\frac{z}{\left(z+\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}$$

$$Az - \frac{1}{2}A + Bz + \frac{1}{4}B = z \Rightarrow \begin{matrix} A + B = 1 \\ -\frac{1}{2}A + \frac{1}{4}B = 0 \end{matrix}$$

$$A = \frac{1}{3} \qquad B = \frac{2}{3}$$

$$= \frac{15}{2}\cdot\frac{z}{z+\frac{1}{4}} + 8\cdot\left(\frac{A}{z+\frac{1}{4}} + \frac{B}{z-\frac{1}{2}}\right) = \frac{3}{2}\cdot\frac{z}{z+\frac{1}{4}} + 8\cdot\left(\frac{\frac{1}{3}}{z+\frac{1}{4}} + \frac{\frac{2}{3}}{z-\frac{1}{2}}\right)$$

$$= \frac{15}{2}\ \frac{z}{z+\frac{1}{4}} - \frac{32}{3}\ \frac{-\frac{1}{4}}{z+\frac{1}{4}} + \frac{32}{3}\ \frac{-\frac{1}{2}}{z-\frac{1}{2}} = Y(z)$$

$$y(n) = \frac{15}{2}\left(-\frac{1}{4}\right)^n - \frac{32}{3}\left[\left(-\frac{1}{4}\right)^n - \delta(n)\right] + \frac{32}{3}\left[\left(\frac{1}{2}\right)^n - \delta(n)\right]$$

$$y(n) = \frac{15}{2}\left(-\frac{1}{4}\right)^n - \frac{32}{3}\left(-\frac{1}{4}\right)^n + \cancel{\frac{32}{3}\delta(n)} + \frac{32}{3}\left(\frac{1}{2}\right)^n - \cancel{\frac{32}{3}\delta(n)}$$

$$= -\frac{19}{6}\left(-\frac{1}{4}\right)^n + \frac{32}{3}\left(\frac{1}{2}\right)^n$$

3.b) $u(n) = 4\left(\frac{1}{2}\right)^{n-1} \mu(n-1)$

$$u(n) = u_a(n-1)$$

$$u(z) = \frac{1}{z} \cdot 4 \cdot \frac{z}{z-\frac{1}{2}} - 0 = \frac{4}{z-\frac{1}{2}}$$

$$y(z) \cdot \frac{4z+1}{4z} = \frac{4}{z-\frac{1}{2}} \cdot 2 \cdot \frac{z+\frac{1}{2}}{z} - \frac{1}{2}$$

$$= \frac{8\left(z+\frac{1}{2}\right)}{z\left(z-\frac{1}{2}\right)}$$

$$y(z) = 8 \cdot \frac{z+\frac{1}{2}}{z\left(z-\frac{1}{2}\right)} \cdot \frac{z}{z+\frac{1}{4}} - \frac{1}{2} \cdot \frac{z}{z+\frac{1}{4}}$$

$$= 8 \cdot \frac{z+\frac{1}{2}}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)} - \frac{1}{2} \cdot \frac{z}{z+\frac{1}{4}}$$

$$= 8 \cdot \left(\frac{A}{z-\frac{1}{2}} + \frac{B}{z+\frac{1}{4}}\right) - \frac{1}{2} \cdot \frac{z}{z+\frac{1}{4}}$$

$Az + \frac{1}{4}A + Bz - \frac{1}{2}B = z + \frac{1}{2}$

$A + B = 1$

$\frac{1}{4}A - \frac{1}{2}B = \frac{1}{2}$

$A = \frac{4}{3} \quad B = -\frac{1}{3}$

$$= 8 \cdot \frac{\frac{4}{3}}{z-\frac{1}{2}} + 8 \cdot \frac{-\frac{1}{3}}{z+\frac{1}{4}} - \frac{1}{2} \cdot \frac{z}{z+\frac{1}{4}}$$

$$= \frac{64}{3} \cdot \frac{\frac{1}{2}}{z-\frac{1}{2}} + \frac{32}{3} \frac{-\frac{1}{4}}{z+\frac{1}{4}} - \frac{1}{2} \frac{z}{z+\frac{1}{4}}$$

$$y(n) = \frac{64}{3}\left[\left(\frac{1}{2}\right)^n - \delta(n)\right] + \frac{32}{3}\left[\left(-\frac{1}{4}\right)^n - \delta(n)\right] - \frac{1}{2}\left(-\frac{1}{4}\right)^n$$

$$= \frac{64}{3}\left(\frac{1}{2}\right)^n - \frac{64}{3}\delta(n) + \frac{32}{3}\left(-\frac{1}{4}\right)^n - \frac{32}{3}\delta(n) - \frac{1}{2}\left(-\frac{1}{4}\right)^n$$

$$\boxed{= \frac{61}{6}\left(-\frac{1}{4}\right)^n + \frac{64}{3}\left(\frac{1}{2}\right)^n - 32\,\delta(n) = y(n)}$$

4. $y'(t)+3y(t)=3u'(t)+u(t)$

a) $sY(s)+3Y(s)=3sU(s)+U(s)$

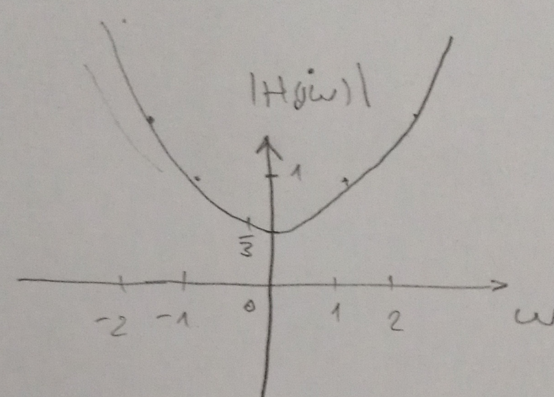
$Y(s)(s+3)=U(s)(3s+1)$

$\frac{Y(s)}{U(s)}=\frac{3s+1}{s+3}=H(s)$ stabilan

$s=-3<1$

b) $H(j\omega)=\frac{3j\omega+1}{j\omega+3}$

$|H(\omega)|=\frac{\sqrt{9\omega^2+1}}{\sqrt{\omega^2+9}}$



c) prisilni na svevremensku pobudu $u(t)=2\cos(4t)+2\sin(4t)$

$|H(4j)|=\sqrt{\frac{9\cdot 4^2+1}{4^2+9}}=\frac{\sqrt{145}}{5}\approx 2,41$

$\measuredangle H(4j)=\text{arctg}\,\frac{3}{1}-\text{arctg}\,\frac{1}{3}=\text{arctg}\,3-\text{arctg}\,1\approx 0,46365$

$y_p(t)=|H(4j)|\,u(t+\measuredangle H(4j))$

$y_p(t)=2,41\cdot\left[2\cdot\cos(4t+0,46365)+2\sin(4t+0,46365)\right]$

5. $y_{np}(t) = 2t+1$ $\quad y_m(t) = 4t^2 \mu(t)$ $\quad u(t) = \mu(t)$

a)

$y(t) = y_m(t) + y_{nep}(t) = 4t^2 + 2t + 1 \longleftarrow$ RED JE 2

NEMA NULA NEMA DERIVACIJE ULAZA

$y''(t) + Ay'(t) + By(t) = C\mu(t)$

$8 + A(8t+2) + B(4t^2+2t+1) = C$

A i B moraju biti ∅ jer s druge strane nema t ni t^2

$\boxed{C = 8}$

$$\boxed{\begin{aligned} &y''(t) = 8u(t) \\ &y(0) = 1 \\ &y'(0) = 2 \end{aligned}}$$

$y(0) = 4\cdot 0^2 + 2\cdot 0 + 1 = \underline{1}$

$y'(t) = 8t+2 \Rightarrow y'(0) = 8\cdot 0 + 2 = 2$

b) $y''(t) = 8u(t)$

$s^2 Y(s) = 8U(s)$

$\frac{Y(s)}{U(s)} = H(s) = \frac{8}{s^2} \circ\!\!-\!\!\bullet \boxed{8t\mu(t) = h(t)}$

c) $y''(t) = 8u(t)$

$u(t) = e^{-2t}\mu(t)$

$s^2Y(s) - sy(0) - y'(0) = 8U(s)$

$U(s) = \frac{1}{s+2}$

$s^2Y(s) - s\cdot 1 - 2 = 8U(s)$

$s^2Y(s) = 8U(s) + s + 2$

$s^2Y(s) = 8\cdot\frac{1}{s+2} + s + 2$

$Y(s) = \frac{8}{s^2(s+2)} + \frac{1}{s} + \frac{2}{s^2} = \frac{A}{s^2} + \frac{Bs}{s^2} + \frac{C}{s+2} + \frac{1}{s} + \frac{2}{s^2}$

$= \frac{4}{s^2} - \frac{2\cancel{s}}{s^{\cancel{2}}} + \frac{2}{s+2} + \frac{2}{s^2} + \frac{1}{s}$

$A(s+2) + Bs(s+2) + Cs^2 = 8$

$As + 2A + Bs^2 + 2Bs + Cs^2 = 8$

$B + C = 0$

$A + 2B = 0$

$2A = 8$

$A = 4 \quad B = -2 \quad C = 2$

$Y(s) = \frac{6}{s^2} - \frac{1}{s} + \frac{2}{s+2}$

$$\boxed{y(t) = (6t - 1 + 2e^{-2t})\mu(t)}$$

1