Domaća zadaća 6.

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Zadatak 1

Odrediti jednadžbu ravnine π koja prolazi točkama $S(2,1,0)$ i $T(-1,0,2)$, a paralelna je s osi y.

↳ Ravnina je paralelna s osi y ⇒ normala ravnine je okomita s osi y

⇒ $\vec{n} \perp \vec{j}$

↳ $\vec{ST} = (-1-2)\vec{i} + (0-1)\vec{j} + (2-0)\vec{k} \quad -3\vec{i} - \vec{j} + 2\vec{k}$

↳ $\vec{n} = \vec{j} \times \vec{ST} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ -3 & -1 & 2 \end{vmatrix} = \vec{i}(2-0) - \cancel{\vec{j}(0)} + \vec{k}(0-(-3)) = 2\vec{i} + 3\vec{k}$

$\pi \ldots \; 2(x-2) + \cancel{0(y-1)} + 3\cdot(z-0) = 0$

$2x - 4 + 3z = 0$

$\boxed{2x + 3z - 4 = 0}$

Zadatak 2.

Odrediti jedn. pravca kroz $T(3,2,1)$, paralelnog s ravninama $\pi_1 \ldots x+y-z+2=0$

$\pi_2 \ldots -x+2y+z-5=0$

$p \parallel \pi_1$, $p \parallel \pi_2$ } $\vec{c_p} = \vec{n_{\pi_1}} \times \vec{n_{\pi_2}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{vmatrix}$

vektor smjera

↳ $n_{\pi_1} = \vec{i} + \vec{j} - \vec{k}$

$n_{\pi_2} = -\vec{i} + 2\vec{j} + \vec{k}$

↳ $\vec{c_p} = \vec{i}(1-(-2)) - \cancel{\vec{j}\cdot(1-1)} + \vec{k}(2-(-1))$

$\boxed{\vec{c_p} = 3\vec{i} + 3\vec{k}}$

↳ može se zapisati kao $3(\vec{i}+\vec{k})$ jer je svaki $\vec{c_p}$ pomnožen s skalarom isto $\vec{c_p}$

↳ kanonska jednadžba : uvrstimo T i $\vec{c_p}$

$\boxed{\frac{x-3}{3} = \frac{y-2}{0} = \frac{z-1}{3}}$

ili

$\boxed{\frac{x-3}{1} = \frac{y-2}{0} = \frac{z-1}{1}}$

Zadatak 3.

Odrediti točku na osi Oz, a leži točka u ravnini π koja prolazi točkama:

$A(1,0,-1)$, $B(2,1,1)$, $C(0,3,2)$

↳ Jednadžba ravnine kroz 3 točke:

$\begin{vmatrix} x-1 & y-0 & z-(-1) \\ 2-1 & 1-0 & 1-(-1) \\ 0-1 & 3-0 & 2-(-1) \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-1 & y & z+1 \\ 1 & 1 & 2 \\ -1 & 3 & 3 \end{vmatrix} = 0$

ili pošto A leži na y osi, a C na x osi normala mora biti okomita na vektore $\vec{AB}$ i $\vec{BC}$ pa se dobi jednadžba normale i ubacivanjem bilo koje od točki dobije se jednadžba ravnine

↳ $(x-1)(\overset{-3}{3-6}) - y\cdot(\overset{5}{3-(-2)}) + (z+1)(\overset{4}{3-(-1)}) = 0$

$-3x+3-5y+4z+4=0 \Rightarrow \boxed{\pi \ldots -3x-5y+4z+7=0}$

↳ ubacimo $T(0,0,t_z) \Rightarrow 4t_z = -7$

$\boxed{t_z = \frac{-7}{4}} \Rightarrow \boxed{T(0,0,-\frac{7}{4})}$

Zadatak 4.

Odrediti jednadžbu ravnine paralelnu s pravcem $\frac{x}{1} = \frac{y}{-3} = \frac{z}{2}$ i koja odsjeca na pozitivnim dijelovima koordinatnih osi Ox i Oy segmente duljine 1 i 2.

↳ $\vec{c_p} = \vec{i} - 3\vec{j} + 2\vec{k}$

↳ točka koju ravnina odsjeca ⇒ $T(1,2,t_z)$

↳ segmentni oblik jednadžb: $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1$

$\frac{x}{1} + \frac{y}{2} + \frac{z}{t_z} = 1 \;/\cdot 2t_z$

$2t_z x + y t_z + 2z = 2t_z$

↳ $\vec{n_\pi} \perp \vec{c_p} \Rightarrow (2t_z\vec{i} + t_z\vec{j} + 2\vec{k})\cdot(\vec{i} - 3\vec{j} + 2\vec{k}) = 0$

⇒ $2t_z - t_z + 4 = 0 \Rightarrow \boxed{t_z = 4}$

$8x + 4y + 2z = 8 \;/:2$

$\boxed{4x + 2y + z = 4}$

Zadatak 5.

Odrediti sve točke na pravcu zadanom kao presjek $\begin{cases} 5x+3y-1=0 \\ 2x+3z+5=0 \end{cases}$

koje su jednako udaljene od ravnina $3x+3y-2=0$ i $4x+y+z+4=0$

↳ moramo izraziti y i z preko zajedničkog parametra u zadanim jednadžbama.

↳ x je slobodni parametar

↳ $x=t, \; t\in\mathbb{R}$

$\begin{cases} x = t \\ y = \frac{1}{3} - \frac{5}{3}t \\ z = -\frac{5}{3} - \frac{2}{3}t \end{cases} \Rightarrow \vec{c_p} = t\vec{i} + (\frac{1}{3} - \frac{5}{3}t)\vec{j} + (-\frac{5}{3} - \frac{2}{3}t)\vec{k}$

↳ udaljenost točaka na pravcu je jednaka od obje ravnine

↳ $d(M_1,\pi) = d(M_2,\pi) \Rightarrow \frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$ je formula

$\frac{|3t+1-5t-2|}{\cancel{\sqrt{18}}} = \frac{|4t+\frac{1}{3}-\frac{5}{3}t-\frac{5}{3}-\frac{2}{3}t+4|}{\cancel{\sqrt{18}}} \Rightarrow 2t+1 = \pm\left(\frac{5}{3}t + \frac{8}{3}\right)$

↳ $2t+1 = \frac{5}{3}t + \frac{8}{3} \Rightarrow \frac{1}{3}t = \frac{5}{3} \Rightarrow t_1 = 5 \Rightarrow M_1(5,-8,-5)$

$2t+1 = -\frac{5}{3}t - \frac{8}{3} \Rightarrow \frac{11}{3}t = -\frac{11}{3} \Rightarrow t_2 = -1 \Rightarrow M_2(-1,2,-1)$

Zadatak 6.

Naći jednadžbe ravnina na kojim leže sve točke jednako udaljene od ravnina

$x+2y+2z=2$ i $4x-4y+7z=1$

$d = \frac{|Ax+By+Cz+D|}{\sqrt{A^2+B^2+C^2}}$

$d_1 = d_2 \Rightarrow \frac{|x+2y+2z-2|}{\sqrt{9}} = \frac{|4x-4y+7z-1|}{\sqrt{81}}$

$\Rightarrow 3x+6y+6z-6 = \pm(4x-4y+7z-1)$

⇒ $3x+6y+6z-6 = 4x-4y+7z-1$

$\boxed{x-10y+z+5=0 \ldots \pi_1}$

⇒ $3x+6y+6z-6 = -4x+4y-7z+1$

$\boxed{7x+2y+13z-7=0 \ldots \pi_2}$

Zadatak 7.

Odredi presjek pravca i ravnine i u kojem su međusobnom položaju?

$p \ldots \frac{x-1}{3} = \frac{y+1}{2} = \frac{z-2}{-2}$

$\pi \ldots 2x-y+2z+2=0$

↳ $\vec{c_p} = 3\vec{i} + 2\vec{j} - 2\vec{k}$, $\vec{n_\pi} = 2\vec{i} - \vec{j} + 2\vec{k}$

$\vec{c_p}\cdot\vec{n_\pi} = 6-2-4 = 0$

↳ $\vec{c_p} \perp \vec{n_\pi} \Rightarrow$ $\boxed{\text{pravac i ravnina su paralelni, nema presjeka}}$

Zadatak 8.

Izračunati kut koji zatvaraju pravac i ravnina.

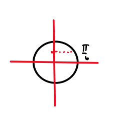
$p \ldots \frac{x-5}{1} = \frac{y+1}{0} = \frac{z-8}{1}$

$\pi \ldots 2x-2y+3=0$

↳ $\vec{c_p} = \vec{i} + \vec{k}$

↳ $\vec{n_\pi} = 2\vec{i} - 2\vec{j}$

$\sin\varphi = \frac{|\vec{c_p}\cdot\vec{n_\pi}|}{|\vec{c_p}|\cdot|\vec{n_\pi}|} = \frac{|2|}{\sqrt{2}\cdot\sqrt{8}} = \frac{2}{\sqrt{2}\cdot 2\sqrt{2}} = \frac{2}{4} = \frac{1}{2}$



$\Rightarrow \sin\varphi = \frac{1}{2} \Rightarrow \boxed{\varphi = \frac{\pi}{6}}$

Zadatak 9.

Za koju vrijednost parametra $\lambda\in\mathbb{R}$ je pravac $p \ldots \frac{x-1}{\lambda} = \frac{y}{2} = \frac{z-1}{-2}$ paralelan s ravninom određenom točkama $A(2,6,0)$, $B(1,2,-1)$, $C(0,2,0)$

$\begin{vmatrix} x-2 & y-6 & z \\ 1-2 & 2-6 & -1 \\ -2 & 2-6 & 0 \end{vmatrix} = 0 \Rightarrow (x-2)\cdot(0-4) - (y-6)(0-2) + z(4-8) = 0$

$\Rightarrow -4x+8+2y-12-4z=0 \Rightarrow -4x+2y-4z-4=0$

$\Rightarrow -2x+y-2z-2=0$

↳ $\vec{n_\pi} = -2\vec{i} + \vec{j} - 2\vec{k}$, $\vec{c_p} = \lambda\vec{i} + 2\vec{j} - 2\vec{k}$ } da pravac i ravnina budu paralelni $\vec{c_p}\cdot\vec{n_\pi}$ mora biti $=0$

↳ $-2\lambda + 2 + 4 = 0 \Rightarrow -2\lambda = -6$

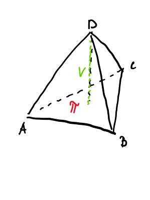
$\boxed{\lambda = 3}$

Zadatak 10

$A(1,0,0)$, $B(1,1,0)$, $C(0,1,2)$, $D(3,3,3)$. Pronaći jednadžbu ravnine u kojoj leži trokut ABC te izračunati duljinu visine iz vrha D piramide ABCD

↳ jednadžba ravnine : $\begin{vmatrix} x-1 & y & z \\ \cancel{1-1}\,0 & 1 & 0 \\ -1 & 1 & 2 \end{vmatrix} = 0 \Rightarrow (x-1)(2) - \cancel{(y)}(0) + z(0-(-1)) = 0$

$\Rightarrow 2x-2+z=0 \Rightarrow \boxed{\pi \ldots 2x-z+2=0}$

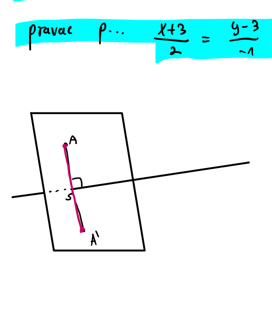


⇒ tražimo udaljenost točke D i ravnine π

$\Rightarrow \boxed{v = d(T,\pi) = \frac{|6+3-2|}{\sqrt{5}} = \frac{7}{\sqrt{5}} = \frac{7\sqrt{5}}{5}}$

Zadatak 11.

Naći točku simetričnu točki $A(2,-1,-1)$ s obzirom na pravac $p \ldots \frac{x+3}{2} = \frac{y-3}{-1} = \frac{z+2}{1}$



⇒ pravac je normala ravnine

↳ $\vec{c_p} = \vec{n_\pi} = 2\vec{i} - \vec{j} + \vec{k}$ ⇒ uvrstimo točku A u jednadžbu da dobijemo jednadžbu ravnine

$2(x-2) - (y+1) + (z+1) = 0$

$2x-4-y-1+z+1=0 \Rightarrow \boxed{\pi \ldots 2x-y+z-4=0}$

↳ $S = p\cap\pi \Rightarrow p \ldots \begin{cases} x = 2t-3 \\ y = -t+3 \\ z = t-2 \end{cases} \Rightarrow$ ubacujemo to u jednadžbu ravnine da dobijemo S

↳ $2(2t-3) - (-t+3) + (t-2) - 4 = 0 \Rightarrow 4t-6+t-3+t-2-4=0 \Rightarrow 6t-15=0 \Rightarrow \boxed{t = \frac{15}{6} = \frac{5}{2}}$

$\boxed{S(2,\frac{1}{2},\frac{1}{2})}$ ⇒ S je polovište A i A'

↳ $2 = \frac{2+x}{2} \Rightarrow x=2$

$\frac{1}{2} = \frac{-1+y}{2} \Rightarrow y=2$

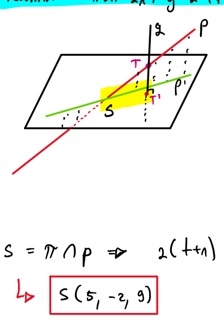
$\frac{1}{2} = \frac{-1+z}{2} \Rightarrow z=2$

$\boxed{A'(2,2,2)}$

Zadatak 12

Pronađi projekciju pravca $p \ldots \frac{x-1}{1} = \frac{y-2}{-1} = \frac{z-1}{2}$ na ravninu $\pi \ldots 2x+y-z+1=0$

↳ $T(1,2,1)$



$p \ldots \begin{cases} x = t+1 \\ y = -t+2 \\ z = 2t+1 \end{cases}$ ⇒ da nađemo S ubacujemo u jednadžbu ravnine

$S = \pi\cap p \Rightarrow 2(t+1) - t + 2 - (2t+1) + 1 = 0 \Rightarrow \cancel{2t}+2-t+2-\cancel{2t}-\cancel{1}+\cancel{1}=0$

↳ $\boxed{S(5,-2,9)}$ $\Rightarrow \boxed{t=4}$

ako je pravac g okomit na ravninu (normala) i prolazi točkom T

↳ $\vec{n_\pi} = 2\vec{i} + \vec{j} - \vec{k} = \vec{c_g}$, $T(1,2,1)$

↳ $g \ldots \frac{x-1}{2} = \frac{y-2}{1} = \frac{z-1}{-1} \Rightarrow \begin{cases} x = 2t+1 \\ y = t+2 \\ z = -t+1 \end{cases}$

↳ $T' = g\cap\pi \Rightarrow 2(2t+1) + t+2 - (-t+1) + 1 = 0$

$\Rightarrow 4t+2+t+2+t\cancel{-1}\cancel{+1} = 0 \Rightarrow 6t = -4 \Rightarrow \boxed{t = \frac{-2}{3}}$

↳ $\boxed{T'(-\frac{1}{3},\frac{4}{3},\frac{5}{3})}$

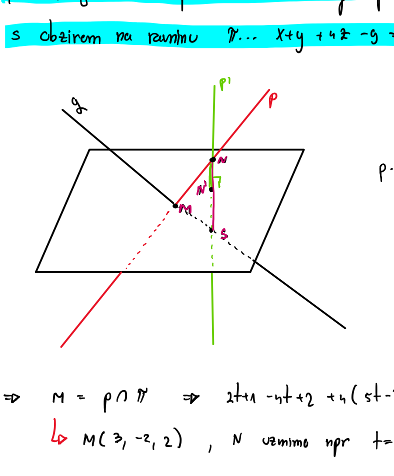
↳ vektor $\vec{ST'}$ leži na pravcu p'

↳ $p' \ldots \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} \Rightarrow \frac{x-5}{\frac{-16}{\cancel{3}}} = \frac{y+2}{\frac{10}{\cancel{3}}} = \frac{z-9}{\frac{-22}{\cancel{3}}} \Rightarrow \frac{x-5}{-16} = \frac{y+2}{10} = \frac{z-9}{-22}$

↳ $\boxed{p' = \frac{x-5}{8} = \frac{y+2}{-5} = \frac{z-9}{11}}$

Zadatak 13.

Pronađi jednadžbu pravca simetričnog pravcu $p \ldots \frac{x-1}{2} = \frac{y-2}{-4} \; \frac{z+3}{5}$ s obzirom na ravninu $\pi \ldots x+y+4z-9=0$



$p \ldots \begin{cases} x = 2t+1 \\ y = -4t+2 \\ z = 5t-3 \end{cases}$

⇒ $M = p\cap\pi \Rightarrow 2t+1-4t+2+4(5t-3)-9=0 \Rightarrow -2t-6+20t-12=0 \Rightarrow 18t = 18 \Rightarrow \boxed{t_M = 1}$

↳ $M(3,-2,2)$, N uzmimo npr $t=0 \Rightarrow N(1,2,-3)$

$p' \ldots \vec{n_\pi}$ i N ga određuju

↳ $\frac{x-1}{1} = \frac{y-2}{1} = \frac{z+3}{4} \Rightarrow p' \ldots \begin{cases} x = t+1 \\ y = t+2 \\ z = 4t-3 \end{cases}$

⇒ $N' = p'\cap\pi \Rightarrow t+1+t+2+4(4t-3)-9=0 \Rightarrow 2t-6+16t-12=0 \Rightarrow \boxed{t_{N'} = 1}$

↳ $N'(2,3,1)$

⇒ N' je polovište od $\overline{SN}$ ⇒ $2 = \frac{x+1}{2} \Rightarrow x=3$

$3 = \frac{y+2}{2} \Rightarrow y=4$

$1 = \frac{z-3}{2} \Rightarrow z=5$

$\boxed{S(3,4,5)}$

⇒ pravac g kroz $M(3,-2,2)$ i $S(3,4,5)$

$g \ldots \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} \Rightarrow \boxed{g \ldots \frac{x-3}{0} = \frac{y+2}{6} = \frac{z-2}{3}}$