Teorem srednje vrijednosti

$\forall x, y \in U_\varepsilon(x^*) \quad \exists \xi \in U_\varepsilon(x^*)$ td $\quad f(x) - f(y) = f'(\xi)(x-y)$

dobijemo

$$|f(x) - f(y)| = |f'(\xi)(x-y)| = |f'(\xi)|\,|x-y| \leq L\,|x-y| < |x-y|$$

T:

$x_0 \in U_\varepsilon(x^*) \to x_n \in U_\varepsilon(x^*)$, $\forall n$ $\quad \to$ niz x_n konvergira prema x^*

(i) $\quad x_0 \in U_\varepsilon(x^*) \Rightarrow x_1 \in U_\varepsilon(x^*)$

$$|x_1 - x^*| = |f(x_0) - f(x^*)| < |x_0 - x^*| < \varepsilon$$

(ii) pretp. $x_0 \in U_\varepsilon(x^*) \to x_n \in U_\varepsilon(x^*)$

(iii)

$$|x_{n+1} - x^*| = |f(x_n) - f(x^*)| < |x_n - x^*| < \varepsilon$$

brzina konvergencije

$\boxed{\text{definicija}}$

Niz realnih brojeva x_n nekako konvergira linearno prema $x^* \in \mathbb{R}$ ako postoji $L \in \langle 0,1\rangle$ i $m_0 \in N$ t.d. $\forall m \geq m_0$ vrijedi:

$$|x_{n+1} - x^*| \leq L\,|x_n - x^*|$$

linearna konvergencija prespora

Pr.

$$x_{n+1} - x^* = 0.99(x_0 - x^*) \qquad L = 0.99$$

za manji L imamo brža konvergencija

$|x_0 - x^*| = 1$

$|x_0 - x^*| = 0.99\,|x_{n-1} - x^*| = \ldots = 0.99\,|x_0 - x^*|$

$|x_n - x^*| = 0.99^n \qquad 0.99^n \leq \varepsilon$

$\varepsilon = 10^{-10} \qquad m = 2291$

def. $(x_n)_{n\in\mathbb{N}_0}$ konvergentan niz s $\lim x_n = x^*$

Kažemo da niz $(x_n)_{n\in\mathbb{N}}$ konvergira kvadratično prema x^* ako $\exists L>0$ i $n_0 \in \mathbb{N}$ t.d. $\forall n \geq n_0$

$$|x_{n+1}-x^*| \leq L|x_n-x^*|^2$$

def x_n, $n\in\mathbb{N}_0$ konvergentan niz t.d. $\lim x_n = x^*$ kažemo da je konv. niza x_n prema x^* reda p ako postoji $L>0$ i $n_0\in\mathbb{N}$ t.d. $\forall n\geq n_0$

$$|x_{n+1}-x^*| \leq L|x_n-x^*|^p$$

p ne mora biti cijeli broj

Lema Dovoljni uvjeti za lokalnu konvergenciju proizvoljnog reda (Metode jednostavne iteracije)

$$f:\mathbb{R}\to\mathbb{R} \qquad f(x^*)=x^* \qquad f\in C^p(U_\varepsilon(x^*))$$

$$f'(x^*) = f''(x^*) = \dots = f^{(p-1)}(x^*) = 0$$

$$f^{(p)}(x^*) \neq 0$$

tada MJI konvergira lokalno reda p prema x^*.

dokaz

Taylorov razvoj za f oko x^*:

$$f(x) = \underbrace{f(x^*)}_{x^*} + \sum_{e=1}^{p-1} \frac{\overset{=0}{f^{(e)}(x^*)}}{e!}(x-x^*)^e + \frac{f^{(p)}(\xi)}{p!}\cdot(x-x^*)^p$$

$\boxed{x=x_n}$

$f(x_n) = x_{n+1}$

$$x_{n+1}-x^* = \frac{f^{(p)}(\xi)}{p!}\cdot(x_n-x^*)^p$$

f-ja neprekidna po max. postoji

$$|x_{n+1}-x^*| = \underbrace{\left|\frac{f^{(p)}(\xi)}{p!}\right|}_{L \text{ (konst.)}}|x_n-x^*|^p \leq \frac{\max|f^{(p)}(\xi)|}{p!}|x_n-x^*|^p$$

$\boxed{\text{primjer.}}$ odredite MJI kojom bi izračunali vrijednost $\sqrt{a}$, $a>0$

dokažite konv. te metode i odredite red konv.

$$f(x)=0 \quad \rightarrow \quad x=\varphi(x)$$

problem fiksne točke

$$x-\sqrt{a}=0$$

$$f(x)=x^2-a=0$$

$$x^2=a \quad /+x^2$$

$$2x^2=a+x^2 \quad /:2$$

$$x^2=\frac{1}{2}(a+x^2) \quad /:x$$

$$x=\frac{1}{2}\left(x+\frac{a}{x}\right) \rightarrow \varphi(x)=\frac{1}{2}\left(x+\frac{a}{x}\right)$$

konvergentan niz $\Rightarrow$ monoton i ograničen

padajući + ogr. odozdo

rastući + ogr. odozgo

iterativni postupak

$$x_{n+1}=\varphi(x_n)=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)$$

$\Rightarrow$ ispitivanje konvergencije

1. način: T: x_n ograničen odozdo + padajući

tvrdnja

$$x_{n+1}-\sqrt{a}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)-\sqrt{a}=\frac{1}{2}\frac{(x_n-\sqrt{a})^2}{x_n}$$

$$x_0>0 \qquad x_1-\sqrt{a}=\frac{1}{2}\frac{(x_0-\sqrt{a})^2}{x_0}>0 \rightarrow x_1>\sqrt{a}$$

T1: $x_n>\sqrt{a}\ \ \forall n$ ograničen odozdo

(i) $x_0>0 \rightarrow x_1\sqrt{a}$

(ii) $x_n>\sqrt{a} \rightarrow x_{n+1}>\sqrt{a}$

T2: x_n monotono padajući

$$x_{n+1}-x_n=\frac{1}{2}\left(\frac{a}{x_n}-x_n\right)=\frac{a-x_n^2}{2x_n}<0 \rightarrow x_{n+1}\leq x_n \ \ \forall n$$

x_n konvergira

red konvergencije

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)$$

$$\lim x_{n+1}=\frac{1}{2}\lim x_n+\lim\frac{a}{x_n}$$

$$x^*=\frac{1}{2}\left(x^*+\frac{a}{x^*}\right)$$

$$x^*=\sqrt{a}$$

brzina konvergencije

$|x_{n+1} - x^*|$ greška u n+1 koraku

$x^* = \sqrt{a}$

$$|x_{n+1} - \sqrt{a}| = |x_n - \sqrt{a}|^2 \cdot \frac{1}{|2x_n|} \le \frac{1}{2\sqrt{a}} |x_n - \sqrt{a}|^2$$

kvadratična konvergencija

$$\left|\frac{1}{2x_n}\right| = \frac{1}{2|x_n|} < \frac{1}{2\sqrt{a}}$$

$x_n > \sqrt{a}$

$\frac{1}{x_n} < \frac{1}{\sqrt{a}}$

2. način:

$$f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right), \quad f(\sqrt{a}) = \sqrt{a}$$

dovoljan uvjet: $|f'(x^*)| < 1$ konvg.

$$f'(x) = \frac{1}{2} - \frac{a}{2x^2} = \frac{1}{2}\left(1 - \frac{a}{x^2}\right)$$

$$f'(x^*) = \frac{1}{2}\left(1 - \frac{a}{a}\right) = 0 < 1$$

$$f''(x) = \frac{a}{x^3} \qquad f''(x^*) \neq 0$$

kvadratična konvergencija

Napomena

ako niz (x_n) konvergira kvadratično prema x^*, tada svaka sljedeća iteracija ima približno 2x više točnih decimala

x^* = točno, n-ti x_n ima p_n točnih decimala

$$\frac{|x_n - x^*|}{|x^*|} \approx 10^{-p_n} \quad \Big/ \log_{10}$$

$$-\log_{10} \frac{|x_n - x^*|}{|x^*|} \approx p_n \rightarrow \log_{10} \frac{|x_n - x^*|}{|x^*|} \approx -p_n$$

ovo vrijedi zbog kvadratične konv.

$$\frac{|x_{n+1} - x^*|}{x^*} \le \frac{L\,|x_n - x^*|^2}{|x^*|} = L|x^*| \left(\frac{|x_n - x^*|}{|x^*|}\right)^2 \quad \Big/ \log_{10}$$

$$\underbrace{\log_{10} \frac{|x_{n+1} - x^*|}{|x^*|}}_{-p_{n+1}} \le \log_{10} L|x^*| + \log_{10} \left|\frac{x_n - x^*}{x^*}\right|^2$$

$$-p_{n+1} \approx \log_{10} L|x^*| - 2p_n \qquad \boxed{p_{n+1} \approx 2p_n}$$

primjer:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \quad x_0 = 2$$

vidi skriptu

n	x_n	$e_n = x_n - \sqrt{2}$
0	2	10^{-2}
1		

greške se duplaju → imamo duplo više točnih decimala

200. MJI rješavamo jedbu $xe^x - 1 = 0$

$x^* = 0.56714329 \qquad x_0 = 0.5$

(i) $xe^x = 1 \quad /: e^x$

$x = e^{-x}$

$\boxed{f_1(x) = e^{-x}}$

(ii) $xe^x = 1$

$xe^x + x = 1 + x$

$x(e^x + 1) = 1 + x$

$x = \frac{1+x}{1+e^x} \qquad \boxed{f_2(x) = \frac{1+x}{1+e^x}}$

(iii) $xe^x - 1 = 0$

$xe^x + x - 1 = x \qquad \boxed{f_3(x) = xe^x + x - 1}$

ispitivanje konvergencije

(i) $|f_1'(x)| = |-e^{-x}| = \frac{1}{e^x} \Rightarrow |f_1'(x^*)| < 1$ konvergira linearno i sporo

(ii) $|f_2'(x)| = \frac{1+e^x - e^x(1+x)}{(1+e^x)^2} = \frac{|1 - xe^x|}{(1+e^x)^2}$ konvergira brže

$f'(x^*) = 0$

(iii) $|f_3'(x)| = |1 - e^x - xe^x| \qquad |f_3'(x^*)| = |-e^{x^*}| = |e^{x^*}| > 1$

divergira

NEWTONOVA METODA

pretp.

f diferencijabilna

$f'(x) \neq 0$ na nekoj okolini nultočke (ja u der. pojavljuje u nazivniku)

izvod newtonove metode

x_0 = poč. vrijed.

taylorov razvoj f do x_0 — zanemarujemo

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi)}{2}(x - x_0)^2 \quad \xi \in \langle x_0, x \rangle$$

$$f(x_1) \simeq f(x_0) + f'(x_0) \cdot (x_1 - x_0)$$

x_1 = nultočka od f $f(x_1) = 0$

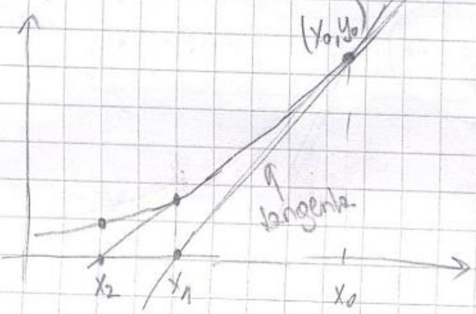
$$0 \approx f(x_0) + f'(x_0)(x_1 - x_0)$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

Newtonova metoda

$$\boxed{x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}}$$

grafički opis metode



$x_0 \in \mathbb{R}$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

presjek tangente s osi x

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

$\boxed{\text{brzina konvergencije}}$

$$\boxed{|x_{n+1} - x^*| \leq L\,|x_n - x^*|^2}$$

1. način

Taylorov razvoj oko x_n, $x = x^*$

$$f(x) = f(x_n) + f'(x_n)(x - x_n) + \frac{f''(\xi)(x-x_n)}{2}$$

$x = x^*$

$$0 = f(x^*) = f(x_n) + f'(x_n)\cdot(x^* - x_n) + \frac{f''(\xi)(x^*-x_n)}{2} \Big/ : f'(x_n)$$

$$\underbrace{-\frac{f(x_n)}{f'(x_n)} + x_n}_{x_{n+1}} - x^* = \frac{f''(\xi_n)}{2f'(x_n)}(x^* - x_n)^2$$

$$x_{n+1} - x^* = \frac{f''(\xi_n)}{2f'(x_n)}\cdot(x^* - x_n)^2$$

može biti ograničeno nekim brojem

2. način

$$f(x) = 0 \longrightarrow x = g(x)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = g(x_n) \Rightarrow g(x) = x - \frac{f(x)}{f'(x)}$$

sad derivirai i gledai koja $\frac{d}{dx} g(x) \neq 0$

kvadratična konv.

Kriterij zaustavljanja

1. ocjena reziduala

$$|f(x_n)| \le \varepsilon |f(x_0)|$$

2. ocjena duljine koraka $\quad |x_{n+1} - x_n| \le \varepsilon |x_0|$

pr. za računanje pomoću Newtona

$f(x) = x^2 - a = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

kvadrat. konv.

pr. odrediti nultočku od $f(x) = x^6 - x - 1$ koja se nalazi u segmentu $[0,2]$

$f'(x) = 6x^5 - 1 = 0$

$x^5 = 1/6$

$f''(x) > 0$

$\boxed{x^* \in [0,2] \quad f(0)\cdot f(2) < 0}$

$$x_{n+1} = x_n - \frac{x_n^6 - x_n - 1}{6x_n^5 - 1}$$ metodu i x_0 imamo

utječe izbor poč. vrijednosti

pr. $f(x) = \operatorname{arctg} x$

$x^* = 0$

$$x_{n+1} = x_n - \frac{\operatorname{arctg} x_n}{\frac{1}{1+x_n^2}} = x_n - (1+x_n^2)\operatorname{arctg} x_n$$

ako za poč. iter. uzmemo $|x_0| < x_c$
Newt. metoda konvergira u x^*
ako $|x_0| > x_c$ ne konvergira

tražimo još uvjeta kao što slijedi

$x_0 = x_c, \quad x_1 = -x_c, \quad x_2 = x_c, \quad x_3 = -x_c$

povučemo tangentu u točki $(x_c, f(x_c))$ tako da sjecište s osi x bude $-x_c$

$$\operatorname{arctg} x_c = \frac{2x_c}{1+x_c^2} \qquad x_c \in \langle 1.39, 1.40 \rangle$$

$$\boxed{x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}}$$ korak NM

ako korak loš → divergira

Metoda sekante → traži 2 prethodna koraka
-poboljšani red konvergencije

$\boxed{\text{Tm}}$ $f \in C^2$, $f(x^*) = 0$, $f'(x^*) \neq 0$
tada metoda sekante dana formulom konvergira prema x^*

$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

red konvergencije $\frac{1+\sqrt{5}}{2}$

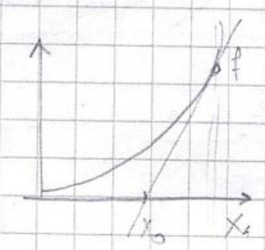
GLOBALNO KONVERGENTNA METODA

Želimo konstruirati metodu koja je uspješna i za početne vrijednosti daleko od x^*

Tvrdnja: Newtonov korak koji vodi k rješenju ima smjer smanjenja vrijednosti fje $|f(x_n)|$

Slučaj

$f(x_0) > 0, f'(x_0) > 0$ fja raste



$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} < x_0$$

Newtonov korak $\Delta = -\frac{f(x_0)}{f'(x_0)}$ je u smjeru pada fje f

analogno za druge slučajeve

U svakom koraku tražimo da vrijedi

$$|f(x_{n+1})| < |f(x_n)|$$

Tada metoda monotono smanjuje apsolutnu vrijednost f-e i $\xrightarrow{k\to\infty}$ ka x^*

Ako je $|f(x_{n+1})| > |f(x_n)|$ onda smatramo da je korak bio big i skraćujemo ga sve dok uvjet ne bude zadovoljen

(Pr.) Newtonova metoda s prilagođenim korakom

$$s_n = -\frac{f(x_n)}{f'(x_n)} \quad \text{velik} \quad |f(x_{n+1})| > f(x_n)| \rightarrow \text{skraćujemo korak}$$

odaberemo $\alpha > 0$ td $|f(x_{n+1})| < |f(x_n)|$ gdje $x_{n+1} = x_n + \alpha s_n$

NEWTONOVA METODA VIŠE DIMENZIJA

$x = [x_1, \dots, x_d]^T \in \mathbb{R}^d$

nelinearni sustav jedbi

$$\begin{cases} f_1(x_1 \dots x_d) = 0 \\ \vdots \\ f_d(x_1, \dots x_d) = 0 \end{cases} \quad \text{ili} \quad F: \mathbb{R}^d \to \mathbb{R}^d \quad F(x) = 0$$

Taylorov razvoj vektorske f-je – do $x^n \to$ n-ta iteracija

x_n – n-ta komponenta vektora

$$F(x) = F(x^n) + DF(x^n)(x - x^n) + \mathcal{O}(\|x - x^n\|^2)$$

$$\left(DF(x^n)\right)_{ij} = \frac{\partial F_i}{\partial x_j}(x^n)$$

biramo x^{n+1} td $F(x^{n+1}) = 0$

$$F(x^{n+1}) = F(x^n) + DF(x^n)(x^{n+1} - x^n)$$

$$\boxed{x^{n+1} = x^n - DF(x^n)^{-1} F(x^n)}$$

računamo ovako $\longrightarrow$

$$X^{n+1} - X^n = -DF(X^n)^{-1} \; F(X^n) \quad / \quad DF(X^n)$$ s L

$$DF(X^n)\left[X^{n+1} - X^n\right] = -F(X^n)$$

P1.

$x_1^2 + x_2^2 = 1$ $\qquad x_0 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$x_1^2 - x_2^2 = -0.5$

zbroji

$2x_1^2 = 0.5$

$$\boxed{\begin{aligned} x_1 &= \pm \frac{1}{2} \\ x_2 &= \pm \frac{\sqrt{3}}{2} \end{aligned}}$$

$F(x) = 0$

$x_1^2 + x_2^2 - 1 = 0$

$x_1^2 - x_2^2 + 0.5 = 0$

$$F(x) = \begin{bmatrix} x_1^2 + x_2^2 - 1 \\ x_1^2 - x_2^2 + 0.5 \end{bmatrix}$$

$$DF(x) = \begin{bmatrix} 2x_1 & 2x_2 \\ 2x_1 & -2x_2 \end{bmatrix}$$

1.

$$DF(x_0) \cdot \delta_0 = -F(x_0)$$

$$DF(1,3)\,\delta_0 = -F(1,3)$$

$$\begin{bmatrix} 2 & 6 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} \delta_1^0 \\ \delta_2^0 \end{bmatrix} = \begin{bmatrix} -9 \\ 7.5 \end{bmatrix} \qquad \delta^0 = \begin{bmatrix} -0.375 \\ -1.375 \end{bmatrix}$$

$$x^1 = x^0 + \delta^0 = \begin{bmatrix} 0,625 \\ 1,625 \end{bmatrix}$$

2. korak

$$DF(x^1)s^1 = -F(x^1)$$

$$s^1 = \begin{bmatrix} -0.1125 \\ -0.5917 \end{bmatrix} \qquad x^2 = \begin{bmatrix} 0.5125 \\ 1.0433 \end{bmatrix}$$

$$x^5 = \begin{bmatrix} 0.5 \\ 0.8560254 \end{bmatrix}$$

P1.

$$e^{x_1} = 1$$
$$e^{x_2} = 1 \qquad x^* = [0\ 0]^T$$

$$x^0 = \begin{bmatrix} -10 \\ -10 \end{bmatrix} \qquad F(x) = \begin{bmatrix} e^{x_1}-1 \\ e^{x_2}-1 \end{bmatrix}$$

$$DF(x) = \begin{bmatrix} e^{x_1} & 0 \\ 0 & e^{x_2} \end{bmatrix}$$

$$DF(x^0)s^0 = -F(x^0)$$

$$s^0 = \begin{bmatrix} e^{10}-1 \\ e^{10}-1 \end{bmatrix} \qquad x^1 = \begin{bmatrix} e^{10}-11 \\ e^{10}-11 \end{bmatrix}$$ obič. jako daleko od rješenja.

globalno konvergentna Newtonova metoda u više dim.

treba pokazati da u okolini x^n u smjeru traženja s^n norma fj. $F(x)$ strogo pada

$$\|F(x^n+\mu s^n)\| < \|F(x^n)\| \quad \text{za dovoljno mali } \mu<0$$

treba pokazati da $\left.\frac{d}{d\lambda}\|F(x^n+\lambda s^n)\|^2\right|_{\lambda=0} < 0$

$$\left.\frac{d}{d\lambda}\|F(x^n+\lambda s^n)\|^2\right|_{\lambda=0} = \frac{d}{d\lambda}\left(F(x^n+\lambda s^n), F(x^n+\lambda s^n)\right) = 2\left(DF(x^n)s^n, F(x^n)\right) =$$

$$= -2\left(F(x^n), F(x^n)\right) = -2\|F(x^n)\|^2 < 0$$

$\|x\|^2 = (x,x)$

Kvazi - Newtonova metoda

- parcijalne derivacije u Jakobijanu aproksimiramo centralnim diferencijama (konvergencija sporija)

Napomena:

$$F : \mathbb{R}^d \to \mathbb{R}^d$$

Smjer u kojem vrijednost $\|F(x)\|^2$ najbrže pada je dan s $\hat{s}^n = -\nabla \|F($

$$= -2\left[DF(x^n)\right] F(x^n)$$

Iterativna metoda s korakom $\hat{s}^n$ je metoda najbržeg silaska

$$\boxed{x^{n+1} = x^n + \hat{s}^n}$$

Rok 2. 2009.

$$x_{n+1} = f(x_n)$$
$$x_0 = 0$$

$$x_1 = f(x_0) = \frac{7}{15} \quad , \quad \max |f'(x)| \leq L = \frac{3}{10}$$

apriorna ocjena

$$|x_n - \overline{x}| \leq \frac{L^n}{1-L} |x_1 - x_0|$$

1) koliko je iteracija potrebno napraviti do udaljenosti od x_n do $\overline{x}$ bude $< = 10^{-3}$

2) pokazati da metoda jedn. iteracija konv. linearno

(1) $|x_n - \overline{x}| \leq 10^{-3}$ · potrebna apriorna ocjena

$$|x_n - \overline{x}| \leq \boxed{\frac{L^n}{1-L} |x_1 - x_0| \leq 10^{-3}}$$

$$\frac{\left(\frac{3}{10}\right)^n}{1 - \frac{3}{10}} \left| \frac{7}{15} - 0 \right| \leq 10^{-3}$$

obavezno znati
dokaz za lin. i kvadratne konv.

$$\left(\frac{3}{10}\right)^n \leq \frac{3}{2} \cdot 10^{-4}$$

$$n \geq \frac{\log 3 - \log 2 - 4}{\log 3 - \log 10} = 7{,}3132\ldots$$

$$\boxed{n = 8}$$

(2) linearna konvergencija

$$\exists\, n_0 \in \mathbb{N} \text{ td } |x_{n+1} - \overline{x}| \leq L |x_n - \overline{x}| \quad \forall n \geq n_0$$

$$|x_{n+1} - \overline{x}| = |f(x_n) - f(\overline{x})|$$

po koristi teorem srednje vrijednosti

$$f(x) - f(y) = f'(\xi)(x-y) \quad \boxed{x < \xi < y}$$

$$f(\overline{x}) = \overline{x} \quad = |f'(\xi) \cdot (x_n - \overline{x})|$$

$$= |f'(\xi)| \cdot |x_n - \overline{x}| \leq L |x_n - \overline{x}|$$

dokaz ovo

PZI 2009.

$f(x) = e^x - a = 0 \quad a > 0$

(i) napišite newtonovu metodu za traženje nultočke za traženje f uz $x_0 = a$. Dobiveni označite s (x_n)

(ii) pokažite da je NM lokalno kvadratično konvergentna

(iii) koristeći činjenicu da za niz x_n nekog iz (i) vrijedi $f(x_n) > 0$ tre... pokažite da je niz x_n monotono padajući i ograničen odozdo s...

(IV) nađite limes niza

Rješenje

(i) $f(x) = e^x - a$

$f'(x) = e^x \qquad x_{n+1} = x_n - \frac{e^{x_n} - a}{e^{x_n}} \qquad x_0 = a$

(ii) metoda jednostavnih iteracija

$$x_{n+1} = g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$g(x) = x - \frac{e^x - a}{e^x}$$

TM $g'(x^*) = 0$, $g''(x^*) \neq 0$ to moramo dokazati

$$g'(x) = 1 - \frac{f'(x)^2 - f(x) f''(x)}{(f'(x))^2} = 1 - 1 + \frac{f(x) f''(x)}{(f'(x))^2}$$

$$g'(x) = \frac{f(\bar{x}) f''(\bar{x})}{f'(x)^2} = 0$$

$$g''(x) = \frac{(g'(x) f''(x) + f(x) f'''(x)] f'(x)^2 - f(x) f''(x)^2 \cdot 2f'(x)}{g'(x)'}$$

$f(\bar{x}) = 0$

$$f''(\bar{x}) = \frac{\cancel{f'(\bar{x})^3} \cdot f''(\bar{x})}{f'(\bar{x})^{\cancel{4}}} = \frac{f''(\bar{x})}{f'(\bar{x})} = \frac{a}{a} = 1 \neq 0$$

$f(x) = e^x - a = 0 \rightarrow e^x = a \quad x = \ln a$

$\boxed{\text{kvadratična konv.}}$

$f'(x) = e^x \qquad f'(\bar{x}) = a$

$f''(x) = e^x \qquad f''(\bar{x}) = a$

iii

$$\begin{cases} x_{n+1} = x_n - \dfrac{e^{x_n} - a}{e^{x_n}} \\ x_0 = a \end{cases}$$

$f(x_n) > 0, \forall n$

T1: niz monotono padajući

$x_{n+1} < x_n$

$$x_{n+1} = x_n - \frac{e^{x_n} - a}{e^{x_n}} = x_n - 1 + \frac{a}{e^{x_n}} < x_n$$

$$1 - \frac{a}{e^{x_n}} > 0$$

$$\frac{a}{e^{x_n}} < 1$$

$$e^{x_n} > a$$

$$x_n > \ln a$$

$$\boxed{\begin{array}{l} f(x_n) > 0 \\ e^{x_n} - a > 0 \end{array}}$$

će vrijedi

weeeee

T2: ograničen $\quad x_n > \ln a$ za svaki n

(iv) limes

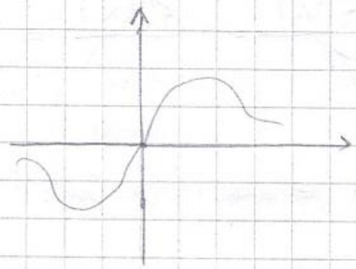
$$x_{n+1} = x_n - 1 + \frac{a}{e^{x_n}}$$

$$\lim_n x_{n+1} = \lim x_n - 1 + \frac{a}{\lim e^{x_n}}$$

$$x^* = x^* - 1 + \frac{a}{x^*} \rightarrow \boxed{x^* = \ln a}$$

$\boxed{\text{ZI 2009.}}$

$$f(x)=\begin{cases}\frac{1}{x}e^{-\frac{1}{x^2}} & x\neq 0\\ 0 & x=0\end{cases}$$



(i) NM za traženje nultočke fj. $f(x)$ prikaži kao metodu jednostavnih iteracija s pripadnom iteracijskom fjom $\varphi(x)$

(ii) odredi sve $x_0 \in \mathbb{R}$ za koje NM konvergira

Uputa: zapiši $\varphi(x) = x\cdot\psi(x)$

Rješ.

iz grafa $\rightarrow$ $x^* = 0$ jedinstvena nultočka

(i)

$$f(x)=\frac{1}{x}e^{-\frac{1}{x^2}} \qquad f'(x) = -\frac{1}{x^2}\cdot e^{-\frac{1}{x^2}} + \frac{1}{x}\cdot\left(-\frac{2}{x^3}\,e^{-\frac{1}{x^2}}\right)$$

$$= e^{-\frac{1}{x^2}}\left[\frac{2-x^2}{x^4}\right]$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$\varphi(x_n)=x_n-\frac{f(x_n)}{f'(x_n)} = x_n-\frac{\frac{1}{\cancel{x_n}}\cdot\cancel{e^{-\frac{1}{x_n^2}}}}{\cancel{e^{-\frac{1}{x_n^2}}}\left[\frac{2-x_n^2}{x_n^3}\right]} = x_n-\frac{x_n^3}{2-x_n^2} = x_n\left[1-\frac{x_n^2}{2-x_n^2}\right]$$

$$\boxed{\varphi(x_n) = 2x_n\cdot\frac{1-x_n^2}{2-x_n^2}}$$

$x_{n+1}=\varphi(x_n)$

(ii) koliko blizu nule moramo biti da bi išao u nulu

$$|x_{n+1}|<|x_n| \quad \forall n$$

$$x_{n+1}=\varphi(x_n) = x_n\cdot\psi(x_n)$$

$$|x_{n+1}| = |x_n|\cdot|\psi(x_n)| < |x_n|$$

$$|\psi(x_n)|<1$$

graf funkcije :

$$\varphi(x) = 2 \cdot \frac{1-x_n^2}{2-x_n^2} = 1 - \frac{x^2}{2-x^2} > 1$$

onaj uvjet $|\varphi(x)| < 1$ mora biti zadovoljen

$$|x| > \sqrt{2}$$

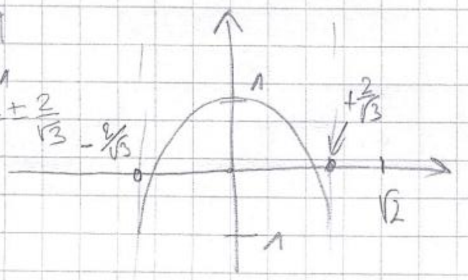
$$x_0 \in \langle -\sqrt{2}, \sqrt{2} \rangle$$

Nacrtaj graf fje φ

$\varphi(0) = 1$

$\varphi(x) = 1$

$\hookrightarrow x = \pm \frac{2}{\sqrt{3}}$



$$\varphi'(\,) = -\frac{4x}{(2-x^2)^2} \begin{cases} >0, & x<0 \\ =0, & x=0 \\ <0, & x>0 \end{cases}$$

$$x \in \left\langle -\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}} \right\rangle$$

$\boxed{\text{Z} \mid 2010.}$

3 zad.

Rješiti pr 2 iteracije NM

$$x_1 + 2x_2 = 2$$
$$x_1^2 + 4x_2^2 = 4$$

$$x^0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Moži rješenje

$$x_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$F(x) = \begin{bmatrix} x_1 + 2x_2 - 2 \\ x_1^2 + 4x^2 - 4 \end{bmatrix}$$

$$DF(x^n) s^n = -F(x^n)$$

$$x^{n+1} = x^n + s^n$$

$$DF = \begin{bmatrix} 1 & 2 \\ 2x_1 & 8x_2 \end{bmatrix}$$

1. korak

$$x^0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$DF(x^0) s^0 = -F(x^0)$$

$$s^0 = \begin{bmatrix} -1.83 \\ -0.58 \end{bmatrix}$$

$$x^1 = \begin{bmatrix} -0.83 \\ 1.42 \end{bmatrix}$$

2. korak

$$DF(x^1) s^1 = -F(x^1)$$

$$x^2 = \begin{bmatrix} -0.19 \\ 1.10 \end{bmatrix}$$

konvergira u $[0\ 1]^T$

- velike uvj. kvadrata
- lin, kvad.)   bilo kojeg oblika

ponovi izvod

$$\sum_{k=1}^{m} \| f(x_k) - y_k \|^2$$

deriviramo po nepoznanicama
teorija

QR faktorizacija

teorija slabo gradivo