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MATEMATIKA 0

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Algebarski izraz

- bilo koji izraz koji sadrži varijable, konstante, računske operacije...

$x+1$, $x-1$, x^2-1

primjer

(I) $\cancel{A}+B-\cancel{A}=B$

(III) $-x^2+x+5-x^2-2x-8=-x+2$

(II) $\frac{AB}{AC}=\frac{B}{C}$

(IV) $\frac{x\cdot\cancel{(x^2+x+1)}}{(x-3)\cdot\cancel{(x^2+x+1)}}=\frac{x}{x-3}$

- racionalizacija nazivnika

$$\frac{1}{\sqrt{3}-1}-\frac{\sqrt{3}}{2}=\frac{\sqrt{3}+1}{2}-\frac{\sqrt{3}}{2}=\frac{1}{2}$$

Zadatak 1.

$$3+\frac{2}{\sqrt{2}}-\frac{1}{1-\frac{1}{\sqrt{2}}}-\frac{1}{1-\frac{1}{1-\frac{1}{\sqrt{2}}}}=3+\frac{2\sqrt{2}}{2}-\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{1}{2}}-\frac{1}{1-\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{2}}}=$$

$$=3+\frac{2\sqrt{2}}{2}-2\cdot\left(1+\frac{\sqrt{2}}{2}\right)-\frac{1}{1-2^{\prime}\sqrt{2}}=3+\frac{2\sqrt{2}}{2}-2+\sqrt{2}+\frac{1}{1+\sqrt{2}}=$$

$$=3+\frac{2\sqrt{2}}{2}-2-\sqrt{2}+\frac{1-\sqrt{2}}{1-2}=3+\frac{2\sqrt{2}}{2}-2-\cancel{\sqrt{2}}-1+\cancel{\sqrt{2}}=\sqrt{2}$$

Zadatak 2. Pojednostavite izraz:

$$\frac{a-2b}{a^3+b^3}-\frac{a-b}{a^2b-ab^2+b^3}-\frac{1}{ab+b^2}=\frac{a-2b}{(a+b)(a^2-ab+b^2)}-\frac{a-b}{b(a^2-ab+b^2)}-\frac{1}{b(a+b)}$$

$$=\frac{ab-2b^2-a^2+\cancel{b^2}-a^2+ab-\cancel{b^2}}{b\cdot(a+b)(a^2-ab+b^2)}=\frac{-2a^2-2b^2+2ab}{b(a+b)(a^2-ab+b^2)}=$$

$$=-2\frac{\cancel{a^2-ab+b^2}}{b\cdot(a+b)\cancel{(a^2-ab+b^2)}}=\frac{-2}{ab+b^2}$$

Zadatak 3.

$$\left[\frac{\left(\frac{3}{2}\right)^{-3}+\left(\frac{2}{3}\right)^{-2}}{\left(\frac{3}{2}\right)^{-2}+\left(\frac{2}{3}\right)^{-3}}\right]^{-1}\cdot\left[\frac{\left(\frac{4}{3}\right)^{-4}+\left(\frac{3}{4}\right)^{-3}}{\left(\frac{4}{3}\right)^{-3}+\left(\frac{3}{4}\right)^{-4}}\right]^{-1}=\frac{\frac{4}{9}+\frac{27}{8}}{\frac{8}{27}+\frac{9}{4}}\cdot\frac{\frac{27}{64}+\frac{256}{81}}{\frac{81}{256}+\frac{64}{27}}=$$

$$=\frac{\frac{4\cdot 8+9\cdot 27}{9\cdot 8}}{\frac{4\cdot 8+9\cdot 27}{27\cdot 4}}\cdot\frac{\frac{27\cdot 81+256\cdot 64}{64\cdot 81}}{\frac{81\cdot 27+64\cdot 256}{256\cdot 27}}=\frac{3}{2}\cdot\frac{4}{3}=2$$

Zadatak 4.

$$\frac{\sqrt[3]{x^2}\cdot\sqrt[4]{x^3}}{\sqrt[12]{x^{11}}} = \frac{x^{\frac{2}{3}}\cdot x^{\frac{3}{4}}}{x^{\frac{11}{12}}} = \frac{x^{\frac{8+9}{12}}}{x^{\frac{11}{12}}} = x^{\frac{17}{12}-\frac{11}{12}} = x^{\frac{6}{12}} = \sqrt{x}$$

Zadatak 5.

$$\frac{x^2+2x-3}{x^2-x-2} - \frac{x^3+x^2-2x}{x^3-x^2-2x} + \frac{x^4-4x^3+3x^2}{x^4-x^3-2x^2} = \frac{x^2+2x-3}{x^2-x-2} - \frac{\cancel{x}(x^2+x-2)}{\cancel{x}(x^2-x-2)} + \frac{\cancel{x^2}(x^2-4x+3)}{\cancel{x^2}(x^2-x-2)} =$$

$$= \frac{x^2-3x+2}{x^2-x-2} = \frac{\cancel{(x-2)}(x-1)}{\cancel{(x-2)}(x+1)} = \frac{x-1}{x+1}$$

Zadatak 6.

$$\frac{1}{x^2}\sqrt{x^3\sqrt[3]{x\sqrt[4]{x^8}}} + \frac{11\sqrt{3}}{5\sqrt{3}-8} - \frac{24}{\sqrt{3}} = \frac{1}{x^2}\sqrt{x^3\sqrt[3]{x\cdot x^{\frac{8}{4}}}} + \frac{11\sqrt{3}\cdot(5\sqrt{3}+8)}{25\cdot 3-64} - 8\sqrt{3} =$$

$$= \frac{1}{x^2}\sqrt{x^3\sqrt[3]{x^3}} + \frac{55\sqrt{6}+88\sqrt{3}}{75-64} - 8\sqrt{3} =$$

$$= \frac{1}{x^2}\cdot x^2 + \frac{\cancel{11}(5\cdot 3+\cancel{8\sqrt{3}})}{\cancel{11}} - \cancel{8\sqrt{3}} = 1+15 = 16$$

Zadatak 7.

$$\left[\left(16^{\frac{3}{4}}\right)^{-\frac{2}{3}}\cdot\left(27^{\frac{1}{3}}\right)^{-2}\right]^{-\frac{1}{2}} = \left[8^{-\frac{2}{3}}\cdot 3^{-2}\right]^{-\frac{1}{2}} = \left(\frac{1}{\sqrt[3]{64}}\cdot\frac{1}{9}\right)^{-\frac{1}{2}} = \left(\frac{1}{4}\cdot\frac{1}{9}\right)^{-\frac{1}{2}} = \left(\frac{1}{36}\right)^{-\frac{1}{2}} = 6$$

Zadatak 8.

$$1-\frac{1}{1-\frac{1}{1-\frac{1}{x}}} = 1-\frac{1}{1-\frac{x}{x-1}} = 1-\frac{1}{\frac{-1}{x-1}} = 1+x-1 = x$$

POLINOMI I ALGEBARSKE JEDNADŽBE

- polinom $P(x) = a_n x^n + \dots + a_1 x + a_0$ (pr. $2x^3+4x-7$)
- primjer polinoma sa više varijabli: $P(x,y,z) = x^2 \cdot y - 7xyz^2 + y^4$
- dijeljenje polinoma: $\left(7:3 = \frac{7}{3} = 2 + \frac{1}{3}\right)$

$A(x), B(x)$ st(A) ≥ stupnja polinoma

$$\frac{A(x)}{B(x)} = Q(x) + \frac{R(x)}{B(x)}$$

$$A(x) = Q(x)B(x) + R(x)$$

pr. $(3x^5 - 5x^4 - 3x + 1) : (x^2 - x - 1) = (3x^3 - 2x^2 + x - 1) - \frac{3x}{x^2-x-1}$

$$\begin{array}{l}
3x^5 - 3x^4 - 3x^3 \\
\hline
-2x^4 + 3x^3 \\
-2x^4 + 2x^3 + 2x^2 \\
\hline
x^3 - 2x^2 - 3x \\
x^3 - x^2 - x \\
\hline
-x^2 - 2x + 1 \\
-x^2 + x + 1 \\
\hline
-3x
\end{array}$$

Zadatak 8.

$(x^{100} - x + 2) : (x^2 - 1)$ $\qquad R(x) = ?$

(I) $x^{100} - x + 2 = (x^2-1)Q(x) + ax + b$; if $x = 1$ $\qquad$ if $x = -1$

$1^{100} - 1 + 2 = (1^2-1)Q(x) + a + b$ $\qquad (-1)^{100} + 1 + 2 = -a + b$

$\qquad 4 = -a + b$

$$\left.\begin{array}{l} 2 = a + b \\ 4 = -a + b \end{array}\right\} +$$

$2b = 6 \;/ : 2$

$b = 3 \Rightarrow a = -1 \qquad \Rightarrow \qquad R(x) = -x + 3$

(II) $x^{100} - x + 2 = (x^2 - 1) \cdot S(x)$

$$\begin{array}{l}
x^{100} - x + 2 \\
-\ x^{100} \quad\; - 1 \\
\hline
\quad -x + 3
\end{array}$$

Kada je $A(x)$ djeljiv sa $B(x)$? Kada je ostatak 0

def. Nultočka polinoma je x_1 sa svojstvom $P(x_1) = 0$

primjer $P(x) = x^3 - 4x^2 + x + 2 \quad \Rightarrow \quad x_1 = 1$

$P(x)$ dijelimo sa $(x-1)$

$$\begin{array}{ccc} P(x) & = \underbrace{Q(x)(x-1)}_{0} & + R \\ 0 & & \overset{\shortparallel}{0} \end{array}$$

Zadatak 9.

$x^4 - 3x^3 - 5x^2 + 9x + 10 = 0$

$(x^3 - 4x^2 - x + 10)(x+1) = 0$

- pogađamo da su nultočke djelitelji slobodnog člana

$\pm 1, \pm 2, \pm 5, \pm 10 \Rightarrow -1$

$(x^4 - 3x^3 - 5x^2 + 9x + 10) : (x + (+1)) = x^3 - 4x^2 - x + 10$

$(x^3-4x^2-x+10):(x-2)=x^2-2x-5 \Rightarrow x_{1,2}=\frac{2\pm\sqrt{24}}{2}$

N.T ovog polinoma: $1\pm\sqrt{6}, -1, 2$

Zadatak 10.

$x^3-3x^2+x+1=0$

$(x^3-3x^2+x+1)(x-1)=x^2-2x-1 \quad \Rightarrow x_{1,2}=\frac{2\pm\sqrt{4+4}}{2}$

$\cancel{x^3}-x^2$

$-\cancel{2x^2}+x$

$-\cancel{2x^2}+2x$

$\cancel{-x+1}$

$\cancel{-x+1}$

0

$x_{1,2}=1\pm2\sqrt{2}$

N.T.: $1, 1\pm\sqrt{2}$

- zero product property $\rightarrow B\cdot C=0 \Rightarrow B=0 \; ; \; C=0$

oprez: $\frac{(x-1)(x-2)}{(x-3)(x-2)}=0$ 1 je nultočka, 2 nije!

Zadatak 11. Riješite jednadžbu

$\sqrt{x+4}-\sqrt{21-x}=-1$

$\sqrt{x+4}=-1+\sqrt{21-x} \quad /^2$

$x+4=1-2\sqrt{21-x}+21-x$

$2x-18=-2\sqrt{21-x} \quad /:(-2)$

$9-x=\sqrt{21-x} \quad /^2$

$81-18x+x^2=21-x$

$x^2-17x+60=0$

$(x-12)\cdot(x-5)=0 \quad \Rightarrow$ faktoriziramo; dva broja čiji je zbroj 17 i umnožak 60

$\cancel{x_1=12} \quad x_2=5$ ✓ $\rightarrow$ provjera!

Zadatak 12. Koliki je zbroj kvadrata jednadžbe $2x^4+x^2-2=0$

$t=x^2$

$2t^2+t-2=0$

$t_{1,2}=\frac{-1\pm\sqrt{1+16}}{4}$

$t_{1,2}=\frac{-1\pm\sqrt{17}}{4}$

$\left.\begin{array}{l} x_1^2=-1+\sqrt{17} \\ x_2^2=-1-\sqrt{17} \end{array}\right\} x_1^2+x_2^2=-1+\cancel{\sqrt{17}}-1-\cancel{\sqrt{17}}=-2$

Zadatak 13. $x^3+ax^2+bx+3=0$ N.T.: 1, 2 $a\cdot b=?$

① $1+a+b+3=0$
$a+b=-4$

② $8+4a+2b+3=0$
$4a+2b=-11$

$-2a-\cancel{2b}=8$
$4a+\cancel{2b}=-11$

$2a=-3$ $b=-\frac{5}{2}$
$a=-\frac{3}{2}$

Zadatak 14.

$x^2=1+x$ $x^{10}=a+bx$ $a+b=?$ $\Rightarrow a+b=89$

$(x^2)^2=(1+x)^2=1+2x+x^2=1+2x+1+x=2+3x$

$(x^4)=2+3x$

$x^8=4+12x+9x^2=4+12x+9+9x=13+21x$

$x^{10}=x^2\cdot(13+21x)=(x+1)(13+21x)=21x^2+34x+13=21+21x+34x+13=55x+34$ $a=34$ $b=55$

Zadatak 15. $(x^2+ax+3):(x-1)$ $a=?$ uz uvjet $R=0$ $=x+3$
x^2+x
$(a-1)x+3$
$(3x+3$

$a-1=3$

$\boxed{a=4}$

Zadatak 16. $f(x)=ax^2+bx+2$

$f(\frac{1}{2})=\frac{5}{4}$ $f(-1)=2$

$f(\sqrt{7}-1)=?$

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① $\frac{5}{4}=a\cdot\left(\frac{1}{2}\right)^2+b\cdot\frac{1}{2}+2$ $/\cdot4$

② $\cancel{2}=a-b+\cancel{2}$

$a=b$

$5=a+2b+4$

$a+2b=-3$

$a=b=-1$

$f(x)=-x^2-x+2$

$f(\sqrt{7}-1)=-(\sqrt{7}-1)^2-\sqrt{x}+1+2=-7+2\sqrt{7}\cancel{-1}-\sqrt{7}\cancel{+1}+2=-5+\sqrt{7}$

Zadatak 17.

Ako uvrstimo -2 dobijemo 0 (djeljivo)

$x^4-x^3+ax^2+x+6$ djeljivo sa $x+2$, ostatak pri dijeljenju sa $x-2$

$(x-2)Q(x)+r$ $r=-12$

Jednadžbe koje se svode na algebarske

→ iracionalne jednadžbe

pr1) $\sqrt[3]{x+1} = 2 \;/^3$

$x+1 = 8$

$x = 7$

pr) $\cancel{\sqrt{x} = -2}$

Nema rješenja

$(\sqrt{x})^2 = x$

$\sqrt{x^2} = |x|$

pr2) $x^2 = 4 \;/\sqrt{\ }$ $\Rightarrow |x| = 2$

$x_1 = 2 \quad x_2 = -2$

pr3) $x^2 \geq 4 \;/\sqrt{\ }$

$|x| \geq 2$

$x \geq 2 \qquad -x \geq 2$

$x \leq -2$

$x \in \langle -\infty, -2] \cup [2, +\infty\rangle$

- kod integrala često koristimo svođenje na potpuni kvadrat

Zadatak 18. Nađi najmanju vrijednost ovog izraza

$x^2 + y^2 - 6x + 4y + 24$

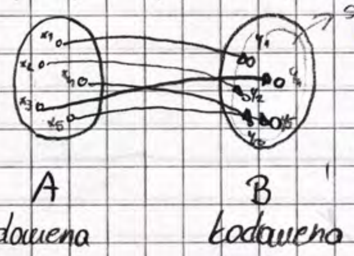
$= x^2 + 6x + y^2 + 4y + 24$

$= (x-3)^2 - 9 + (y+2)^2 - 4 + 24$

$= (x-3)^2 + (y+2)^2 + 11$ ⟶ Najmanja vrijednost je 11 u točki $T(3, -2)$

2. FUNKCIJE

2.1 Pojam funkcije i svojstva



slika funkcije

- $x_{1,2,3,\dots,n}$ → argumenti funkcije
- y_1, y_2, y_n → vrijednosti funkcije

- funkcija svakom x-u pridružuje neki y
- slika funkcije Im(f) → svi elementi skupa B u koji će se preslikati neki elementi iz A
- injekcija → ako svaka dva x (razl.) imaju razl. y ili ako različiti x-evi nemaju isti y
- surjekcija → ako svaki y ima neki x (tj. Im(f) = B)
- bijekcija → ako je i surjekcija i injekcija
  → na bijektivnim funkcijama jedino je moguće rješavati jednadžbe, jedine imaju inverz ($f^{-1}(x)$)

$f: \mathbb{R} \to \mathbb{R}$

$f: D(f) \to Im(f)$

$f^{-1}: Im(f) \to D(f)$

pr. 4) Odredi inverz funkcije $f(x)=\frac{x+2}{2x-1}$

$y=\frac{x+2}{2x-1} \quad / \cdot (2x-1)$

$f(x)^{-1}=\frac{x+2}{2y-1}$

$2xy-y=x+2$

$2xy-x=y+2$

$x(2y-1)=y+2 \quad / : (2y-1)$

$x=\frac{y+2}{2y-1}$

pr. 5.) Odredi domenu funkcije $\frac{\sqrt{x-2}}{x^2-9}$

① $x \geq 2$ ⑪ $x^2-9 \neq 0$ $D(f)=[2,+\infty\rangle \setminus \{3\}$

$x^2 \neq 9 \quad / \sqrt{\ }$

$x \neq \pm 3$

- kompozicija funkcije

pr. 6) $f(x)=x^2$ $(f \circ g)(x)=?$

$g(x)=x+1$

$(f \circ g)(x)=f(g(x))=(x+1)^2$

$(g \circ f)(x)=g(f(x))=x^2+1$

$(f \circ g \circ f)(x)=f(g(f(x)))=(x^2+1)^2$

Zadatak 19. $f(x+1)=\frac{x+2}{x-3}$, koliko je $f(x-1)$?

$z=x+1$ $f(z)=\frac{z+1}{z-4}$ $f(x-1)=\frac{x-1+1}{x-1-4}=\frac{x}{x-5}$

$x=z-1$

pr. 7) $f(x)=x^3+7$

a) prikaži kao komp dvije funkcije

$g(x)=x^3$ $f(x)=(h \circ g)(x)$

$h(x)=x+7$

b) tri funkcije

$f_1(x)=x$ $f(x)=(f_3 \circ f_2 \circ f_1)(x)$

$f_2(x)=x^3$

$f_3(x)=x+7$

c) pet funkcija

$f_1(x)=x^3$ $f(x)=(f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1)(x)$

$f_2(x)=x+1$

$f_3(x)=x+5$

$f_4(x)=x-2$

$f_5(x)=x+3$

Zadatak 20.

$$f(x)=\frac{1}{1-x} \; ; \; g(1-x)=x \; ; \; f\left(g\left(\frac{1}{x}\right)\right)=?$$

(I) $1-x=t$ $\qquad g(t)=1-t$

$x=1-t$

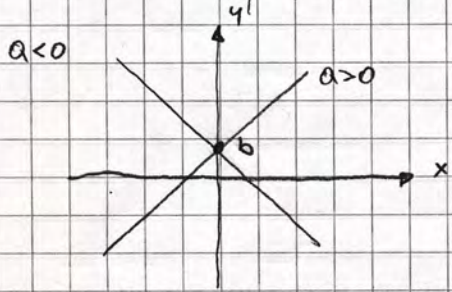
(II) $$(f\circ g)\left(\frac{1}{x}\right)=f\left(1-\frac{1}{x}\right)=\frac{1}{1-\left(1-\frac{1}{x}\right)}=x$$

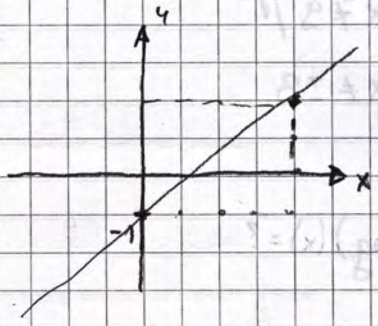
2.2. LINEARNA FUNKCIJA

$f(x)=ax+b$; graf linearne funkcije $\qquad f: R \to R$

a - koeficijent smjera $\qquad$ pr8) $y=\frac{3}{4}x-1$

b - odsječak na osi x





- sustav linearnih jednadžbi

pr.9) $\frac{1}{x}+\frac{1}{y}=5 \; /\cdot 3$

$\frac{2}{x}-\frac{3}{y}=13$

$\frac{3}{x}+\cancel{\frac{3}{y}}=15$

$\frac{2}{x}-\cancel{\frac{3}{y}}=13$

$\frac{5}{x}=28$

$x=\frac{5}{28}$

Zadatak 21. Za koju vrijednost a sustav nema rješenja

$x+2y=5 \; /\cdot(-3)$

$3x-ay=10$

$\cancel{-3x}-6y=-15$

$\cancel{3x}-ay=10$

$-y(6+a)=-5 \; /\cdot(-1)$

$y=\frac{5}{6+a}$

$a=-6$ sustav nema rješenja

- nejednadžbe

pr.10) $\frac{x+3}{x-1}\geq 2$

$\frac{x+3}{x-1}-2\geq 0$

$\frac{x+3-2x+2}{x-1}\geq 0$

$\frac{5-x}{x-1}\geq 0$

	$-\infty$ … 1	1 … 5	5 … ∞
$5-x$	+	+ 0	-
$x-1$	- 0	+	+
	-	+	-

Rj. $x\in\langle 1,5]$

Zadatak 22.

$$\frac{2x^2-1}{x^2} \leq 1 \quad / \cdot x^2 \text{ uz } x \neq 0$$

$$2x^2 - 1 \leq x^2$$

$$x^2 \leq 1$$

$$x \in [-1, 1] \setminus \{0\}$$

Zadatak 23. Za koje realne brojeve x $\sqrt{x^3+x^2-2x}$ je realan broj

$$x^3 + x^2 - 2x \geq 0$$

$$x(x^2+x-2) \geq 0$$

$$x(x-1)(x+2) \geq 0$$

N.T. $x_1 = 0$ $x_2 = 1$ $x_3 = -2$

$x \in [-2, 0] \cup [1, +\infty\rangle$

	$-\infty$ do -2	-2 do 0	0 do 1	1 do ∞
x	−	−	0 +	+
$x-1$	−	−	−	0 +
$x+2$	−	0 +	+	+
	−	+	−	+

Zadatak 24.

$$x+2 > \sqrt[3]{x+2} \quad /^3$$

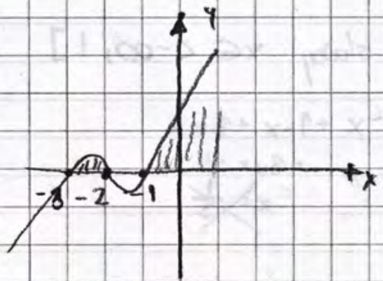
$$(x+2)^3 > x+2$$

$$(x+2)^3 - (x+2) > 0$$

$$(x+2)\left[(x+2)^2 - 1\right] > 0$$

$$(x+2)(x+1)(x+3) > 0$$

- kod padajućih funkcija ne smijemo raditi što hoćemo!



$x \in \langle -3, -2\rangle \cup \langle -1, +\infty\rangle$

Zadatak 25. $\frac{2}{x-1} \geq \frac{x+1}{x-2}$

$$\frac{2x-4-x^2+1}{(x-1)(x-2)} \geq 0$$

$$\frac{-x^2+2x-3}{(x-1)(x-2)} \geq 0$$

$$x_{1,2} = \cancel{\frac{-2 \pm \sqrt{4-12}}{2}}$$ → cijelo vrijeme je − nema nultočke

	$-\infty$ do 1	1 do 2	2 do ∞
$-x^2+2x-3$	−	−	−
$x-1$	−	0 +	+
$x-2$	−	−	0 +

$x \in \langle 1, 2\rangle$

Zadatak 26.

$$-1 \leq \frac{x+1}{x-1} < 2$$

$$\frac{x+1}{x-1} \geq -1 \qquad \frac{x+1}{x-1} < 2$$

$$\frac{x+1+x-1}{x-1} \geq 0 \qquad \frac{x+1-2x+2}{x-1} < 0$$

$$\frac{2x}{x-1} \geq 0 \qquad \frac{3-x}{x-1} < 0$$

	$-\infty$ do 0	0 do 1	1 do ∞
$2x$	−	0 +	+
$x-1$	−	−	0 +
	+		+

	$-\infty$ do 1	1 do 3	3 do $+\infty$
$3-x$	+	+	−
$x-1$	−	+	+
	−	+	−

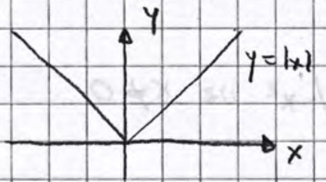
$x \in \langle -\infty, 1\rangle \cup \langle 3, +\infty\rangle$

APSOLUTNA VRIJEDNOST I JEDNADŽBE

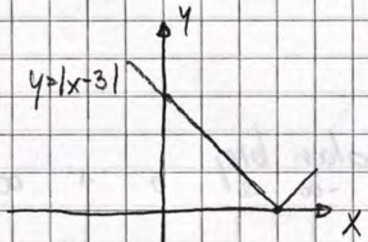
$|-\pi| = \pi$

$|\pi| = \pi$

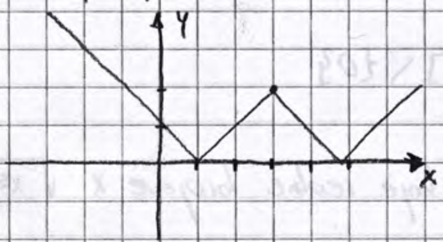
$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$$



pr.10) Nacrtaj $y = |x-3|$



pr.11) Nacrtaj $y = ||x-3|-2|$



pr.12) $|x+5| = 2$

$x+5 = 2$ $\qquad$ $x+5 = -2$

$x = -3$ $\qquad$ $x = -7$

Zadatak 27. Riješi jednadžbu: $|x-1| + |x-3| = x$

	$-\infty$ 1	3	$+\infty$
$x-1$	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$

1. slučaj $x \in \langle -\infty, 1]$

$-x+1-x+3 = x$

$-3x = -4$

$\cancel{-3x = \frac{4}{3}}$

2. slučaj $[1, 3]$

$\cancel{x}-1 \cancel{-x}+3 = x$

$x_1 = 2$ ✓

3. slučaj $x \in [3, +\infty\rangle$

$x-1+x-3 = x$

$x_2 = 4$ ✓

Zadatak 28. Izračunajte površinu lika kojeg omeđuje krivulja $y = 4 - |2x+1| - |2x-1|$ i pravac $y = 0$ (os x)

	$-\infty$ $-\frac{1}{2}$	$\frac{1}{2}$	∞
$2x+1$	$-$	$+$	$+$
$2x-1$	$-$	$-$	$+$

1. $x \in \langle -\infty, -\frac{1}{2}\rangle$

$y = 4+2x+1+2x-1$

$y = 4x+4$

2. $x \in [-\frac{1}{2}, \frac{1}{2}]$

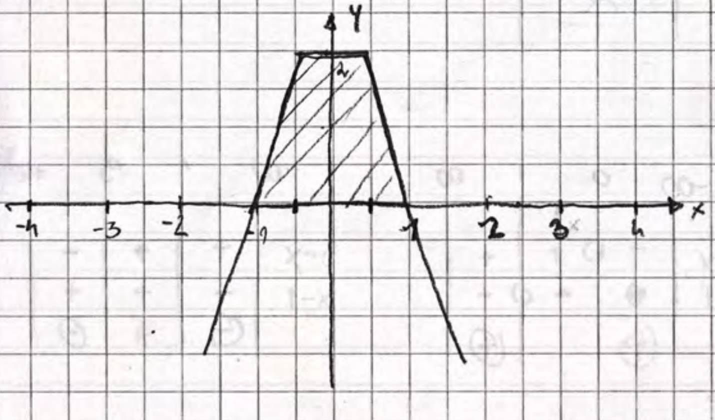
$y = 4-2x-1+2x-1$

$y = 2$

3. $x \in [\frac{1}{2}, \infty\rangle$

$y = 4-2x-1-2x+1$

$y = -4x+4$



$P = \frac{a+c}{2} \cdot v = \frac{3}{2} \cdot 2 = 3$

Zadatak 29.

$$||x+3| + |x+1| + |x-1| + |x-3| - 12| = 4$$

Ⓘ $|x+3| + |x+1| + |x-1| + |x-3| = 16$

Ⓘⓘ $|x+3| + |x+1| + |x-1| + |x-3| = 8$

	$-\infty$ a -3	b -1	c 1	d 3	e ∞
$x+3$	$-$ 0	$+$	$+$	$+$	$+$
$x+1$	$-$	$-$ 0	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$ 0	$+$	$+$
$x-3$	$-$	$-$	$-$	$-$ 0	$+$

a) $-x-3-x-1-x+1-x+3=16$ $\qquad -4x=8$

$-4x=16 \rightarrow \boxed{x=-4}$ W $\qquad \cancel{x=-2}$

b) $x+3-x-1-x+1-x+3=16$ $\qquad -2x=2$

$-2x=10 \rightarrow \cancel{x=-5}$ $\qquad \boxed{x=-1}$ W

c) $x+3+x+1-x+1-x+3=16$ $\qquad \cancel{x+x-x-x}+8=8$

$\cancel{8=16}$ nema rješenja $\qquad 8=8$ cijeli interval je rješenje $[-1,1]$

d) $x+3+x+1+x-1-x+3=16$ $\qquad 2x=2$

$2x=10 \rightarrow \cancel{x=5}$ $\qquad x=1$ W

e) $x+3+x+1+x-1+x-3=16$ $\qquad 4x=8 \ /:4$

$4x=16$ $\qquad \cancel{x=2}$

$\boxed{x=4}$ W

$$x \in [-1,1] \cup \{\pm 4\}$$

Zadatak 30.

$$\left|\frac{x^2+x-3}{x^2-x-5}\right| = 1$$

Ⓘ $\dfrac{x^2+x-3}{x^2-x-5} = 1$

Ⓘⓘ $\dfrac{x^2+x-3}{x^2-x-5} = -1$

$\cancel{x^2}+x-3=\cancel{x^2}-x-5$ $\qquad x^2+\cancel{x}-3=-x^2\cancel{+x}+5$

$2x=2$ $\qquad 2x^2=8$

$\boxed{x=-1}$ W $\qquad x=\pm 2$

Nejednadžbe s apsolutnom vrijednostima:

$|x| \leq 2$ $\quad\quad$ $|x| > 3$

$x \in [-2, 2]$ $\quad\quad$ $x \in \langle -\infty, -3 \rangle \cup \langle 3, +\infty \rangle$

Zadatak 31. Riješite nejednadžbu

$$1 < |2x-3| \leq 3$$

I $|2x-3| > 1$

a) $2x-3 > 1$

$2x > 4$

$x > 2$

b) $-2x+3 > 1$

$-2x > -2 \,/ \cdot (-2)$

$x < 1$

$\langle -\infty, 1 \rangle \cup \langle 2, \infty \rangle$

II $|2x-3| \leq 3$

a) $2x-3 \leq 3$

$2x \leq 6$

$x \leq 3$

b) $-2x+3 \leq 3$

$x \geq 0$

$[0, 3]$

$$x \in [0, 1 \rangle \cup \langle 2, 3]$$

Zadatak 32. $\left| \frac{1}{x-2} \right| < 1$

$$-1 < \frac{1}{x-2} < 1$$

I $\frac{1}{x-2} > -1$

$\frac{1+x-2}{x-2} > 0$

$\frac{x-1}{x+2} > 0$

...

II. način

-smijemo unožiti s $x-2$

$-1 < x-2 < 1$

$x-2 > -1$ $\quad\quad$ $x-2 < 1$

$x > 1$ $\quad\quad$ $x < 3$

$x \in \langle -\infty, 1 \rangle \cup \langle 3, \infty \rangle$

Zadatak 33.

$2x < |x+1|$

$x > -1$ $\quad\quad$ $x < -1$ $\quad\quad$ R. $x \in \langle -\infty, 1 \rangle$

I $2x < x+1$

$1x < 1$

$x \in \langle -1, 1 \rangle$

II $2x < -x-1$

$3x < -1$

$x < -\frac{1}{3}$

$x \in \langle -\infty, -1 \rangle$

Zadatak 34.

$$||x-2|-1| < \frac{1}{2}$$

Ⓘ $|x-2|-1 > -\frac{1}{2}$

$|x-2| > \frac{1}{2}$

a) $x-2 > \frac{1}{2}$ $\quad -x+2 > \frac{1}{2}$

$x > \frac{5}{2}$ $\quad x < \frac{3}{2}$

Ⓘ $|x-2|-1 < \frac{1}{2}$

$|x-2| < \frac{3}{2}$

a) $x-2 < \frac{1}{2}$

$x < \frac{7}{2}$

b) $-x+2 < \frac{3}{2}$

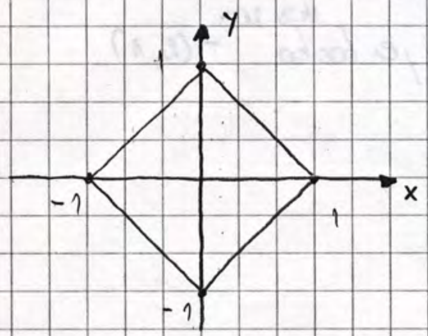
$-x < -\frac{1}{2} \quad / \cdot (-1)$

$x > \frac{1}{2}$

$\langle \frac{1}{2}, \frac{7}{2} \rangle$

$$x \in \langle \frac{1}{2}, \frac{3}{2} \rangle \cup \langle \frac{5}{2}, \frac{7}{2} \rangle$$

Zadatak 35. Riješi $|x|+|y| \leq 1$ grafički i računski



Ⓘ $+x, +y$

$x+y \leq 1$

$y \leq 1-x$

Ⓘ $+x, -y$

$x-y \leq 1$

$y \geq x-1$

Ⓘ $-x, +y$

$-x+y \leq 1$

$y \leq 1+x$

Ⓘ $-x, -y$

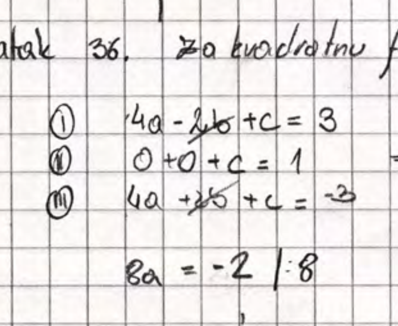
$-y-x \leq 1$

$y \geq -1-x$

2.4. KVADRATNA FUNKCIJA

$f(x) = ax^2+bx+c$

graf. parabola



$T(x_t, y_t)$ $\quad x_t = \frac{x_1+x_2}{2}$ $\quad x_1+x_2 = -\frac{b}{a}$

Viete

$D = b^2-4ac$ $\quad y_t = f(x_t)$ $\quad x_1 \cdot x_2 = \frac{c}{a}$

$ax^2+bx+c = 0$

$a(x-x_1)(x-x_2) = 0$

Zadatak 36. Za kvadratnu funkciju znamo: $f(-2)=3$; $f(0)=1$; $f(2)=-3$ $\quad f(1)=?$

Ⓘ $4a - \cancel{2b} + c = 3$

Ⓘ $0+0+c = 1$ $\quad \Rightarrow c = 1$

Ⓘ $4a + \cancel{2b} + c = -3$

$8a = -2 \quad / :8$

$a = -\frac{1}{4}$

$b = \frac{-4a-4}{2}$

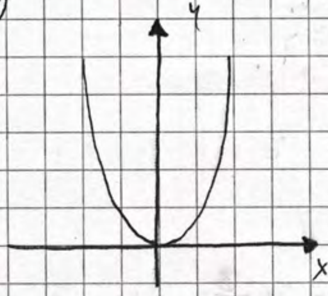
$b = \frac{1-4}{2}$

$b = -\frac{3}{2}$

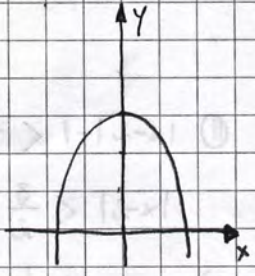
$f(x) = -\frac{1}{4}x^2 - \frac{3}{2}x + 1$

$f(1) = -\frac{3}{4}$

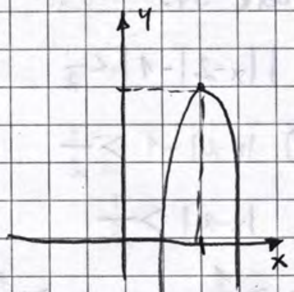
primjer 10.)



$y = x^2$



$y = 3 - x^2$



$y = 2(x-1)^2 + 2 \Rightarrow$ y pomak (normalno + ↑ - ↓)

"širina" x-pomak (obrnut od predznaka - → + ←)

primjer 11.) $y = a(x - x_t)^2 + y_t$ $\qquad T\left(-\frac{1}{2}, \frac{13}{2}\right)$

$f(x) = 2x^2 + 2x + 7 =$

$= 2(x^2 + x) + 7 =$

$= 2\left(x + \frac{1}{2}\right)^2 - \frac{1}{2} + 7 = 2\left(x + \frac{1}{2}\right)^2 + \frac{13}{2}$

14.3.2014.

Zadatak 37. Točka $A(0,-1)$ na $y = ax^2 + bx + c$, a tjeme joj je točka $T(2,1)$

$-1 = 0 + 0 + c \Rightarrow \boxed{c = -1}$

$1 = 4a + 2b - 1$

① $2a + b = 1$ $\qquad x_T = -\frac{b}{2a}$

$-\frac{1}{2}b + b = 1$

$\frac{1}{2}b = 1 \,/\, 2$ $\qquad 4a = -b$

$b = 2 \quad a = -\frac{1}{2}$

Zadatak 38. Tjeme je $T(2,15)$, a udaljenost nultočaka je 10. $f(12) = ?$

$x_1 = 7 \quad x_2 = -3$ $\qquad f(x) = -\frac{3}{5}(x-7)(x+3)$

$f(x) = a(x - x_1)(x - x_2) = a(x-7)(x+3)$ $\qquad f(12) = -\frac{3}{5} \cdot 5 \cdot 15$

$15 = a \cdot (2-7) \cdot (2+3) =$ $\qquad f(12) = -45$

$15 = a \cdot (-25) \,/ : (-25)$

$a = -\frac{3}{5}$

Zad 39.) Za jednadžbu $x^2+4x+p=0$ vrijedi $x_1=x_2+1$. Koliki je p?

$x_1+x_2=-\frac{b}{a}$ $\quad x_2+x_2+1=-4$ $\quad \boxed{x_1=-\frac{3}{2}}$ $\quad x_1\cdot x_2=\frac{c}{a}$

$x_1+x_2=-4$ $\quad 2x_2=-5 \;/:2$ $\quad -\frac{5}{2}\cdot\frac{-3}{2}=p$

$\boxed{x_2=-\frac{5}{2}}$ $\quad \boxed{p=\frac{15}{4}}$

Zadatak 40.) $x^2-6x+13=0$; $(x_1-x_2)^2$

$x_1^2+x_2^2-2x_1x_2=(x_1-x_2)^2$ $\quad x_1+x_2=-\frac{b}{a}$ $\quad x_1\cdot x_2=\frac{c}{a}$

$\frac{36}{1}-4\cdot\frac{13}{a}=(x_1-x_2)^2$ $\quad x_1^2+2x_1x_2+x_2^2=\frac{b^2}{a^2}$

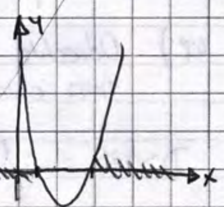
$36-52=(x_1-x_2)^2$

$-16=(x_1-x_2)^2$

Kvadratne nejednadžbe

primjer.) $2x^2-5x+2\geq 0$ $\quad x\in\langle-\infty,\frac{1}{2}]\cup[2,\infty\rangle$

$x_1=2$ $\quad x_2=\frac{1}{2}$



Zadatak 41. $x^2+2x-3>0$

$x^2+4x\leq 0$

$x_1=-3$ $\quad x_2=1$ $\quad x_3=0$ $\quad x_4=-4$

$\langle-\infty,-3\rangle\cup\langle 1,\infty\rangle$ $\quad [-4,0]$

$x\in[-4,-3\rangle$

Zadatak 42. Za koji parametar a je $[-1,3]$ domena funkcije $f(x)=\sqrt{-x^2+ax+3}$

$x_1+x_2=-\frac{b}{a}$

$2=-\frac{a}{-1}$

$\boxed{a=-2}$

Ova strana ne smije biti negativna

Zadatak 43. Riješi nejednadžbu $\sqrt{x+2}<4-x$ $\quad$ poč. uvjeti $\quad 4-x>0 \quad x<4$

$x+2>0 \quad x\geq -2$

$x+2<x^2-8x+16$ $\quad x_{1,2}=\frac{9\pm\sqrt{81-56}}{-1}$

$[-2,4\rangle$

$x^2-9x+14>0$ $\quad x_1=7 \quad x_2=2 \quad \langle-\infty,2]\cup[7,\infty\rangle$

$x\in\langle-2,2\rangle$

Zadatak 44. $\sqrt{x^2+2x-3}>x \;/^2$ $\quad x^2+2x-3\geq 0$ $\quad x\in\langle-\infty,-3]\cup\langle\frac{3}{2},+\infty\rangle$

$x_1=-1$ $\quad x_2=3$ $\quad x^2+2x-3>x^2$ $\quad$ x može biti $x<0$

$x\in\langle-\infty,-3]\cup[1,+\infty\rangle$ $\quad 2x\geq 3$

$x\geq\frac{3}{2}$

Zad. 45) Za koje vrijednosti parametra a rješenje sustava zadovoljava uvjet $y<x+2$;

① $-2x+y=a^2-1 \quad |\cdot(-2)$

$8x+2y=2a^2+7a+5$

$+\begin{cases} 4x-\cancel{2y}=-2a^2+2 \\ 3x+\cancel{2y}=\cancel{2a^2}+7a+5 \end{cases}$

$7x=7a+7 \;/:7$

$x=a+1 \Rightarrow$ ① $-2a-1+y=a^2-1$

$y=a^2+2a+1$

$a^2+2a+\cancel{1}<a+\cancel{1}+2$

$a^2+a-2<0$

$x_{1,2}=\frac{-1\pm\sqrt{1+8}}{2}$

$x_{1,2}=\frac{-1\pm3}{2} \quad x_1=1 \quad x_2=-2$

$a\in\langle-2,1\rangle$

Zad. 46) Odredite najveću vrijednost funk. $y=|x^2-2x-3|$, na intervalu $[-2,3]$

$x_{1,2}=\frac{2\pm\sqrt{4+12}}{2}=1\pm2 \quad x_1=3 \quad x_2=-1 \quad T_x=1 \quad T_y=-4$

$y(-2)=|4+4-3|=5$

Zad 47.) Odredite vrijednosti parametra a tako da $2x^2-(4a+1)x+3(2a-1)=0$ tako da ima dva realna različita rješenja

$D>0 \Rightarrow (4a+1)^2-4\cdot2\cdot3(2a-1)>0$ R: $\forall a\in\mathbb{R}\setminus\{\frac{5}{4}\}$

$16a^2+8a+1-48a-24>0$

$16a^2-40a-23>0$

$a_{1,2}=\frac{40\pm\sqrt{1600-1600}}{32}=\frac{40}{32}=\frac{5}{4}$

Zad 48.) Odredite m tako da funk. bude pozitivna za svaki broj x.

$f(x)>0 \quad a>0 \quad m>0 \quad (m-1)^2-4\cdot m\cdot(1-m)<0 \quad m\in\langle\frac{1}{5},1\rangle$

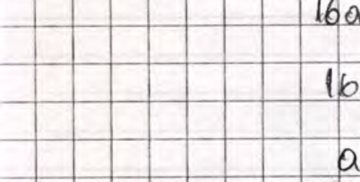
$D<0 \qquad m^2-2m+1-4m+4m^2<0$

$5m^2-6m+1<0$

$m_1=1 \quad m_2=\frac{1}{5}$

2.5 EKSPONENCIJALNE I LOGARITAMSKE FUNKCIJE

$f(x)=a^x$, $x\in\mathbb{R}$, $a>0$ $a\neq1$ bijekcija



$D(f)=\mathbb{R}$

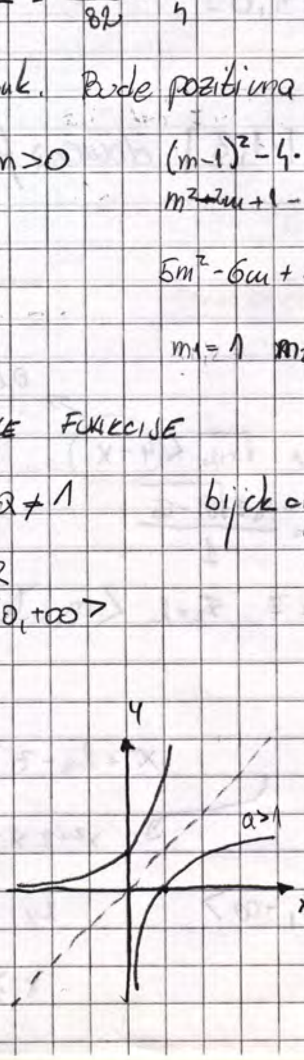
$Im(f)=\langle0,+\infty\rangle$

$f(x)=\log_a x$ $D(f)=\langle0,+\infty\rangle$

$Im(f)=\mathbb{R}$

$a\neq1; a>0$

$x>0$



$\log_a a=1$

$\log_a 1=0$

$\log x+\log y=\log xy$

$\log x-\log y=\log\frac{x}{y}$

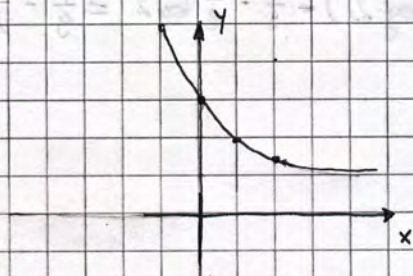
$\log_a x^n=n\log_a x$

$\log_{a^n}x=\frac{1}{n}\log_a x$

$\log_a x=\frac{\log_b x}{\log_b a}$

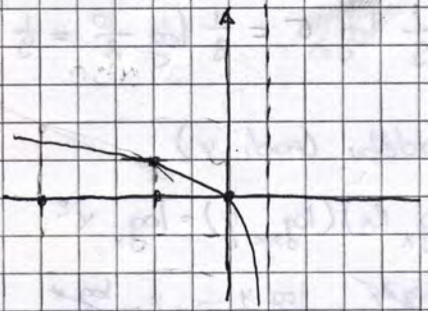
Zad. 49.) Skiciraj funkciju

a) $f(x) = 2\left(\frac{1}{2}\right)^x + 1$



x	y
-2	9
-1	5
0	3
1	2
2	$\frac{3}{2}$

b) $f(x) = \log_2 (1-x)$



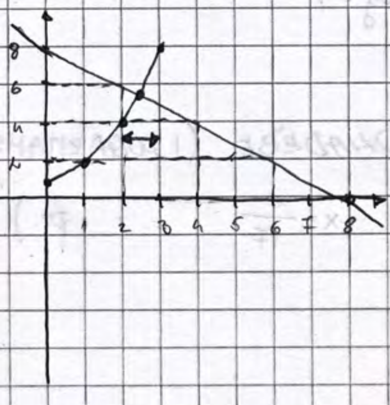
$x < 1$

x	y
0	0
-2	1
-8	2

Zad. 50) Odredite interval oblika $\langle n, n+1 \rangle, n \in \mathbb{Z}$ u kojem se nalaze rješenja jednadžbe

$2^x = 8 - x \quad / \log_2$

$x = \log_2 (8-x)$



2^x

x	y
0	1
1	2
2	4
3	8

$8-x$

x	y
4	4
8	0

Zad 51.) Izračunaj logaritam:

a) $\log_8 \sqrt[3]{16} = \log_{2^3} 2^{\frac{4}{3}} = \frac{\frac{4}{3}}{3} = \frac{4}{9} \log_2 2 = \frac{4}{9}$

b) $\left(\frac{1}{3}\right)^{-\log_{\sqrt{3}} 12 + 3\log_3 4} = \left(\frac{1}{3}\right)^{-2\log_3 12 + 3\log_3 4} = 3^{2\log_3 12 - 3\log_3 4} = 3^{\log_3 12^2 - \log_3 4^3} = 144 \cdot \frac{1}{8} = 18$

Zad 52) Odredite sve $a \in \mathbb{R}$ tako da je funkcija $f(x) = \log(x^2 - ax + 1)$ definirana $\forall x \in \mathbb{R}$

$x^2 - ax + 1 > 0$ $\qquad a^2 - 4 < 0$ $\qquad a \langle -2, 2 \rangle$

$x_{1,2} = \frac{a \pm \sqrt{a^2 - 4}}{2}$ $\qquad a^2 < 4$

Zad 53.) Odredite domenu funkcije $f(x) = \sqrt{\log(3 + x - x^2)}$

$\log(-x^2 + x + 3) \geq 0$ $\qquad x^2 - x + 3 > 0 \rightarrow$ ne treba rješavat

$\log(-x^2 + x + 3) \geq \log 1$ $\qquad x_{1,2} = \frac{1 \pm \sqrt{1+8}}{2}$ $\qquad x \in [-1, 2]$

$-x^2 + x + 3 \geq 1$ $\qquad x_{1,2} = \frac{1 \pm 3}{2}$

$-x^2 + x + 2 \geq 0$

$x_1 = -1, \; x_2 = 2$

Zad. 54) Ako je $4^x + 4^{-x} = 23$, koliko je $2^x + 2^{-x} = ?$

$2^x + 2^{-x} = y \quad /^2$ $\qquad 23 = y^2 - 2$

$4^x + 2 \cdot 2^x \cdot 2^{-x} + 4^{-x} = y^2$ $\qquad 25 = y^2$

$4^x + 4^{-x} = y^2 - 2$ $\qquad y = 5$ samo plus

Zad 55.) Ako je $\log 8 = a \Rightarrow \log 2 = \frac{a}{3}$

$\log \sqrt[9]{5} = ?$

$\log_{10} 5^{\frac{1}{9}} = \frac{1}{9}\log 5 = \frac{1}{9}\log\frac{10}{2} = \frac{1}{9}(\log 10 - \log 2) = \frac{1}{9} - \frac{1}{9}\log 2 = \frac{1}{9} - \frac{a}{27}$

Zad 56.) Riješi jednadžbu (naći y)

$(\log_3 x)(\log_x 2x)(\log_{2x} y) = \log_x x^2$

$\frac{\cancel{\log x}}{\log 3} \cdot \frac{\cancel{\log 2x}}{\cancel{\log x}} \cdot \frac{\log y}{\cancel{\log 2x}} = 2\frac{\cancel{\log x}}{\cancel{\log x}} \quad / \cdot \log 3$

$\log y = 2\log 3$

$y = 9$

EKSPONENCIJALNE JEDNADŽBE (I LOGARITAMSKE i nejednadžbe)

pr.) $25^{-3x+1} = \sqrt[3]{0{,}2^x} \qquad x = \frac{6}{17}$

$5^{-6x+2} = 5^{-\frac{x}{3}}$

$-6x + 2 = -\frac{x}{3} \quad / \cdot 3$

$-18x + 6 = -x$

$-17x = -6 \quad / : (-17)$

pr.) $\log_2 (1-x) = \frac{1}{2}$ poč. uvjet $1 - x > 0$, $x < 1$

$\log_2 (1-x) = \log_2 \sqrt{2}$

$1 - x = \sqrt{2}$

$x = 1 - \sqrt{2}$

$x \approx -0{,}414$

pr.) $2^x > \frac{1}{8}$ Baza > 1

$2^x > 2^{-3}$

$x > -3$

pr.) $\left(\frac{1}{3}\right)^x \leq 9$ Baza < 1

$\left(\frac{1}{3}\right)^x \leq \left(\frac{1}{3}\right)^{-2}$

$x \geq -2$

pr.) $\log_{\frac{1}{2}} x > 3$ Rj. $x \in \langle 0, \frac{1}{8} \rangle$

$-\log_2 x > \log_2 2^3$

$x^{-1} < 2^3 \quad / ^{-1}$

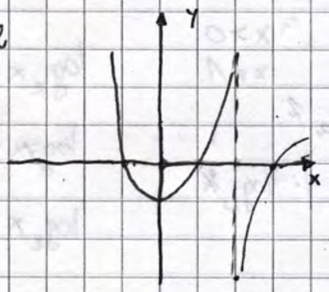
$x \leq \frac{1}{8}$

Zad 57.) $\log_2 |x-2| = x^2 - 1 \qquad x \neq 2$

2. slučaja a) $x-2>0$

$x>2$

$\log_2 (x-2) = x^2 - 1$

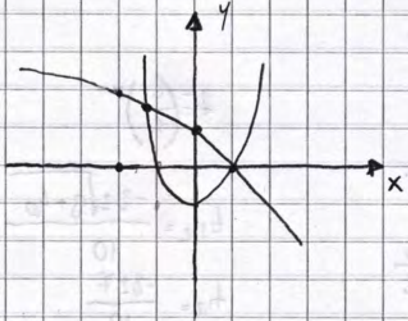


nema rješenja

b) $x-2<0$

$\log_2 (-x+2) = x^2 - 1$

$\boxed{x=1}$ $\quad x_2 \in \langle -2, -1 \rangle$



Zad 58.) $|x-2|^{x^2-9x+20} = 1$

1) $|x-2| = 1 \qquad x^2 - 9x + 20 = 0$

$x-2=1 \quad -x+2=1 \qquad x_{1,2} = \frac{9 \pm \sqrt{81-80}}{2}$

$x_1 = 3 \quad x_2 = 1 \qquad x_1 = 5 \quad x_2 = 4$

Zad 59.) $5\sqrt[x]{5} + 4\sqrt[x]{25} = \sqrt[x]{125}$

$5^{\frac{1}{x}+1} + 4 \cdot 5^{\frac{2}{x}} = 5^{\frac{3}{x}}$

$-5^{\frac{3}{x}} + 4 \cdot 5^{\frac{2}{x}} + 5^{\frac{1}{x}+1} = 0$

$5^{\frac{1}{x}} \left(-5^{\frac{2}{x}} + 4 \cdot 5^{\frac{1}{x}} + 5\right) = 0 \quad / : \left(5^{\frac{1}{x}}\right)$

$t \cdot (-t^2 + 4t + 5) = 0$

$t = \frac{-4 \pm \sqrt{16+20}}{-2}$

$t_1 = 5 \quad \cancel{t_2 = -1}$

$5^{\frac{1}{x}} = t$

$5^{\frac{1}{x}} = 5$

$\frac{1}{x} = 1$

$\boxed{x=1}$

Zad 60.) $3^{x-1} + 2 \cdot 3^x + 2 \cdot 3^{x+1} - 9 \cdot 5^{x-1} = 0$

$3^x (3^{-1} + 2 + 2 \cdot 3) = 9 \cdot 5^{-1} \cdot 5^x \quad / : 3^x$

$(3^{-1} + 2 + 2 \cdot 3) = \left(\frac{5}{3}\right)^x \cdot \frac{9}{5}$

$\frac{25}{3} = \left(\frac{5}{3}\right)^x \cdot \frac{9}{5} \quad / \cdot \frac{5}{9}$

$\frac{125}{27} = \left(\frac{5}{3}\right)^x$

$\left(\frac{5}{3}\right)^3 = \left(\frac{5}{3}\right)^x$

$\boxed{x=3}$

Zad 61) $3\log_2 x - 2\log_x 2 = 1 \qquad x>0 \quad x\neq 1$

$3\log_2 x - 2\cdot\frac{\log_2 2}{\log_2 x} = \log_2 2$

$3\log_2 x - 2\cdot\frac{1}{\log_2 x} = 1 \quad /\cdot \log_2 x$

$3(\log_2 x)^2 - 2 = \log^2 x$

$3t^2 - t - 2 = 0 \qquad t_1 = 1 \quad t_2 = -\frac{2}{3}$

$t_{1,2} = \frac{1\pm\sqrt{1+24}}{6}$

$t_{1,2} = \frac{1\pm 5}{6}$

$\log_2 x = t \qquad x_1 = 2$

$\log_2 x = -\frac{2}{3}$

$\log_2 x = \log_2 2^{-\frac{2}{3}}$

$x = 2^{-\frac{2}{3}} = \frac{1}{\sqrt[3]{4}} = \frac{2\sqrt[3]{2}}{4} = \frac{\sqrt[3]{2}}{2}$

Zad 62.) $0{,}4^x - 2{,}5^{x+1} > 1{,}5$

$\left(\frac{2}{5}\right)^x - \left(\frac{5}{2}\right)^{x+1} > \frac{3}{2}$

$\left(\frac{5}{2}\right)^{-x} - \frac{5}{2}\left(\frac{5}{2}\right)^x > \frac{3}{2}$

$t^{-1} - \frac{5}{2}t > \frac{3}{2} \quad /\cdot 2t$

$2 - 5t^2 > 3t$

$-5t^2 - 3t + 2 > 0 \quad /\cdot(-1)$

$5t^2 + 3t - 2 < 0$

$t = \left(\frac{5}{2}\right)^x$

$t_{1,2} = \frac{-3\pm\sqrt{9+40}}{10}$

$t_{1,2} = \frac{-3\pm 7}{10}$

$t_1 = -1 \qquad t_2 = \frac{2}{5}$

$\left(\frac{5}{2}\right)^x > -1 \qquad \left(\frac{5}{2}\right)^x < \frac{2}{5}$

$x\in R \qquad x < -1$

$x \in \langle -\infty, -1\rangle$

Zad 63.) $\log_{\frac{1}{2}}\sqrt{x-4} + \log_{\frac{1}{2}}\sqrt{x-1} > \log_{\frac{1}{2}} 3$

$-\frac{1}{2}\log_2(x-4) - \frac{1}{2}\log(x-1) > -\log_2 3 \quad /\cdot(-1)$

$\frac{1}{2}\left(\log_2\left((x-4)\cdot(x-1)\right)\right) < \log_2 3 \quad /\cdot 2$

$\log_2(x^2+5x+4) < \log_2 9$

$x^2+5x+4 < 9$

$x^2+5x-5 < 0$

$x_{1,2} = \frac{-5\pm\sqrt{25+20}}{2}$

$x_1 = -\frac{5}{2} + \frac{3\sqrt{5}}{2}$

$x_2 = -\frac{5}{2} - \frac{3\sqrt{5}}{2}$

Rj. $x \in \left\langle 4, \frac{5+3\sqrt{5}}{2}\right\rangle$

Zad 64.) $6^{2x+3} < 2^{x+7}\cdot 3^{3x}$

$2^{2x+3}\cdot 3^{2x+3} < 2^{x+7}\cdot 3^{3x}$

$8\cdot 27\cdot 2^{2x}\cdot 3^{2x} < 2^7\cdot 2^x\cdot 3^{3x} \quad /:(2^x\cdot 3^{3x})\cdot(8\cdot 27)$

$2^x\cdot 3^{-x} < \frac{2^7}{2^3\cdot 3^3}$

$\left(\frac{2}{3}\right)^x < \frac{2^4}{3^3} \quad /\log_{\frac{2}{3}}$

$x > \log_{\frac{2}{3}}\frac{16}{27}$

$x \in \left\langle \log_{\frac{2}{3}}\frac{16}{27}, +\infty\right\rangle$

~~Zad~~ 65.) $\frac{\ln x + 1}{\ln x} < 0$ $\quad x \neq 1$, $x > 0$

I)

$\ln x + 1 > 0 \quad \ln x < 0$

$\quad x < 1$

$\ln x > -1$

$x > e^{-1} \quad \langle e^{-1}, 1 \rangle$

II) $\ln x + 1 < 0 \quad \ln x > 0$

$\ln x < -1 \quad \ln x > 1$

$\ln x < e^{-1}$

$\emptyset$

$x \in \langle e^{-1}, 1 \rangle$

II) $\left(\frac{1}{3}\right)^{x^2} > 3^{-|x|}$

$3^{-x^2} > 3^{-|x|}$

I) $x > 0$

$-x^2 > -x \quad / \cdot (-1)$

$x^2 - x < 0$

$x_1 = 0 \quad x_2 = 1$

$\langle 0, 1 \rangle$

II) $x < 0$

$-x^2 - x > 0 \quad / \cdot (-1)$

$x^2 + x < 0$

$\langle -1, 0 \rangle$

$x \in \langle -1, 1 \rangle \setminus \{0\}$

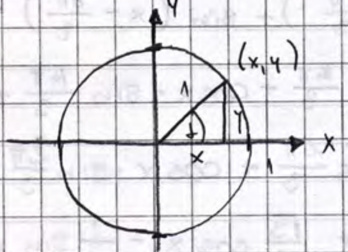
2.3. TRIGONOMETRIJA

$\pi = 180°$

	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
sin	0	1	2	3	4
cos	4	3	2	1	0

2

	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
sin x	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos x	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tg x	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞
ctg x	∞	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0



$\sin \alpha = \frac{y}{1} = y$

$\cos \alpha = \frac{x}{1} = x$

$tg\, \alpha = \frac{y}{x} = \frac{\sin \alpha}{\cos \alpha}$

$ctg\, \alpha = \frac{1}{tg\, \alpha} = \frac{\cos \alpha}{\sin \alpha}$

$\lim_{x \to \frac{\pi}{2}} tg\, x$ = NE POSTOJI !

$\sin^2 x + \cos^2 x = 1$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y$

$\sin^2 x = \frac{1 - \cos 2x}{2}$

$\cos^2 x = \frac{1 + \cos 2x}{2}$

Zad 66) $\sin x + \cos x = \frac{4}{3}$

a) $\sin x \cdot \cos x = ?$

$\sin x + \cos x = \frac{4}{3} \quad /^2$

$\sin^2 x + 2 \sin x \cdot \cos x + \cos^2 x = \frac{16}{9}$

$2 \sin x \cos x = \frac{7}{9}$

$\sin x \cdot \cos x = \frac{7}{18}$

b) $\sin^3 x + \cos^3 x = (\sin x + \cos x) \cdot (\sin^2 x - \sin x \cdot \cos x + \cos^2 x) =$

$= \frac{4}{3}\left(1 - \frac{7}{18}\right) = \frac{22}{27}$

Zad 67) $f(x) = \cos\left(\frac{\pi x}{3}\right)$

$g(x) = 2|x+1|$

$(g \circ f)(1990) = ?$

$(g \circ f)(1990) = g(f(1990)) = g\left(\cos \frac{1990\pi}{3}\right) = g\left(\cos \frac{4\pi}{3}\right) =$

$= g\left(-\frac{1}{2}\right) = 2\left|-\frac{1}{2} + 1\right| = 2 \cdot \frac{1}{2} = 1$

Zad 68) $\cos 2x = \frac{1}{2}$, $\sin 4x = ?$ $\quad x \in \langle \frac{\pi}{2}, \pi \rangle$

$\cos 2x = \cos \frac{\pi}{3}$

$2x = \frac{\pi}{3} + 2k\pi \quad /:2$

$x = \frac{\pi}{6} + k\pi$

$x = -\frac{\pi}{6} + k\pi$

u intervalu $\langle \frac{\pi}{2}, \pi \rangle$ rješenja je jedino $\frac{5\pi}{6}$

$\sin 4x = \sin \frac{20\pi}{6} =$

$= \sin \frac{10\pi}{3} = \sin\left(2\pi + \frac{4\pi}{3}\right) =$

$= \sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$

Zad 69) $\sin x + \sin\left(x + \frac{14\pi}{3}\right) + \sin\left(x - \frac{8\pi}{3}\right) =$

$= \sin x + \sin x \cdot \cos\frac{14\pi}{3} + \cos x \cdot \sin\frac{14\pi}{3} + \sin x \cos\frac{8\pi}{3} - \cos x \cdot \sin\frac{8\pi}{3} =$

$= \sin x + \sin x \cdot \cos\frac{2\pi}{3} + \cos x \cdot \sin\frac{2\pi}{3} + \sin x \cdot \cos\frac{2\pi}{3} - \cos x \cdot \sin\frac{2\pi}{3} =$

$= \sin x - \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x = 0$

Zad 70) Pojednostavni

$$\frac{4\sin t \cdot \cos t \cdot \cos 2t}{\cos^2(2t) - \sin(2t)} \cdot \operatorname{ctg} 4t = \frac{4\sin t \cdot \cos t \cdot \cos(2t)}{\cos 2t} \cdot \operatorname{ctg} 4t =$$

$$= \frac{\cancel{2\sin 2t \cdot \cos 2t}^{1}}{\cancel{\cos 4t}_{1}} \cdot \frac{\cancel{\cos 4t}^{1}}{\cancel{\sin 4a}_{1}} = 1$$

Zad 71) $\operatorname{tg}(-x) = 2{,}4 \qquad -\frac{15\pi}{2} < x < -7\pi \qquad \cos x = ?$

$-\operatorname{tg}(x) = 2{,}4$

$\boxed{\operatorname{tg}(x) = -2{,}4}$

$\operatorname{tg} t = \frac{\sin x}{\cos x} \quad /^2$

$\operatorname{tg}^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1 - \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1$

$\operatorname{tg}^2 x + 1 = \frac{1}{\cos^2 x}$

$(-2{,}4)^2 + 1 = \frac{1}{\cos^2 x}$

$\frac{144}{25} + 1 = \frac{1}{\cos^2 x}$

$\frac{169}{25} = \frac{1}{\cos^2}$

$\cos^2 x = \frac{25}{169} \quad /\sqrt{\ }$

$\boxed{\cos x = -\frac{5}{13}}$

Zad 72) provjeri parnost

$f(-x) = f(x) \to$ parna

$f(-x) = -f(x) \to$ neparna

a) $f(x) = \sin^2(3x) - \cos 3(2x)$

$f(-x) = \sin^2(-3x) - \cos^3(-2x) =$

$= \sin^2(3x) - \cos^3(2x) \rightarrow$ (parna)

b) $f(x) = \sin^3(-2x)$

$f(-x) = \sin^3(2x) = -\sin^3(-2x) = -f(x) \rightarrow$ NEPARNA

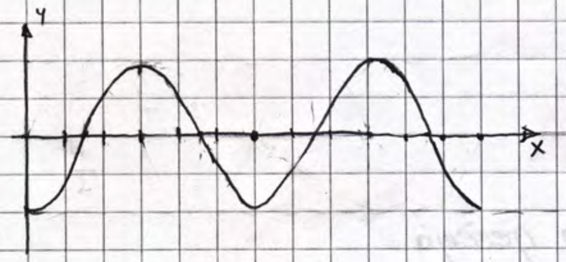
Grafovi trigonometrijskih funkcija

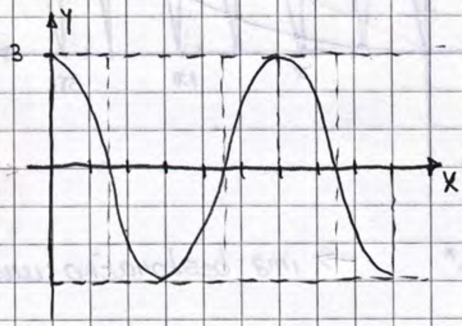
$y = A\sin(\omega x + \varphi)$ $\omega = 2\pi f = \frac{2\pi}{T}$ $\boxed{T = \frac{2\pi}{\omega}}$ pomak: $\boxed{\frac{\varphi}{\omega}}$

pr.) $f(x) = \sin\left(2x - \frac{\pi}{2}\right)$ $T = \frac{2\pi}{2} = \pi$

$f(x) = \sin\left(2\left(x - \frac{\pi}{4}\right)\right)$ x-pomak

$f(x) = -3\sin\left(2x - \frac{\pi}{2}\right)$

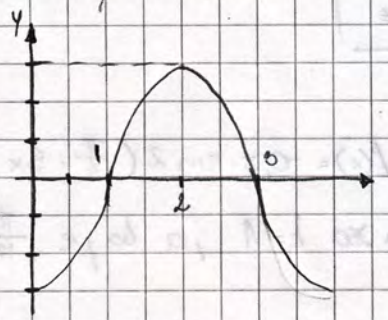




Zad 73) Koliki je period funkcije $f(x) = |4\cos(2x)|$

$T = \frac{2\pi}{2} \cdot \frac{1}{2} = \frac{\pi}{2}$ - jer apsolutna vrijednost okreće negativnu poluperiodu i frekv. je dvostruko veća.

Zad. 74) a) Zadana je sinusoida. Odredi jednadžbu



$y = A\sin(\omega x + \varphi)$

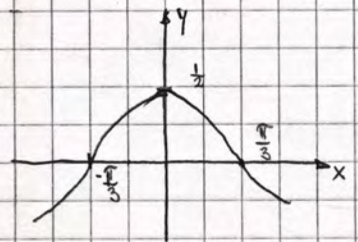
$A = 3$ $T = 4 \Rightarrow \omega = \frac{2\pi}{4} = \frac{\pi}{2}$

$1 = \frac{\varphi}{\omega} \Rightarrow \frac{\pi}{2} = \varphi$

VAŽNO!

$y = 3 \cdot \sin\left(\frac{\pi}{2}x - \frac{\pi}{2}\right)$

b)



$y = A\sin(\omega x + \varphi)$

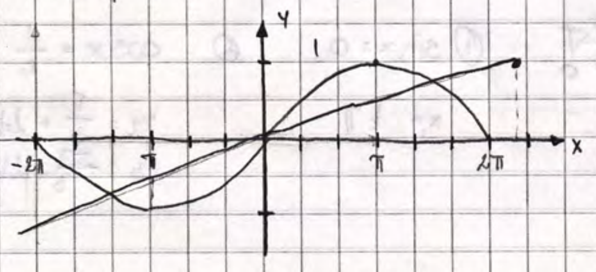
$T = \frac{4\pi}{3}$ $A = \frac{1}{2}$ $\omega = \frac{2\pi}{\frac{4\pi}{3}} = \frac{3}{2}$

$\frac{-\pi}{3} = \frac{-\varphi}{\frac{3}{2}} \Rightarrow \varphi = \frac{\pi}{2}$

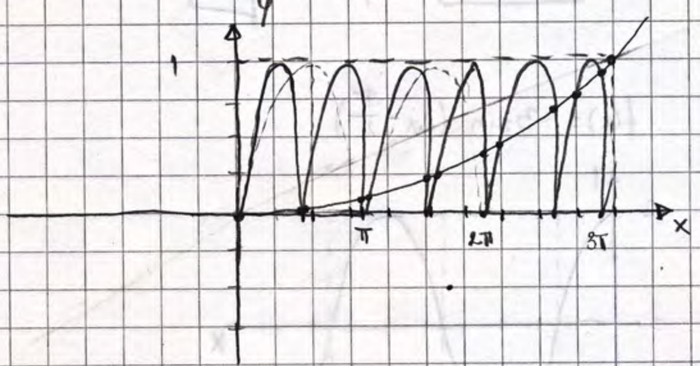
$y = \frac{1}{2}\sin\left(\frac{3}{2}x + \frac{\pi}{2}\right)$

Zad 75) Koliko rješenja ima jednadžba?

a) $\sin\frac{x}{2} = \frac{1}{4}x$



b) $|\sin 2x| = \frac{1}{100}x^2$



- 11 rješenja za $x \geq 0$

c) $\sin x = 2^x$ → ima beskonačno mnogo rješenja

TRIGONOMETRIJSKE JEDNADŽBE

Zad F6) $\sin\frac{x}{2} = -\frac{\sqrt{3}}{2}$

$\frac{x}{2} = \frac{4\pi}{3} + 2k\pi \ /\cdot 2$

$\boxed{x = \frac{8\pi}{3} + 4k\pi \quad k \in \mathbb{Z}}$

$\frac{x}{2} = \frac{5\pi}{3} + 2k\pi$

$\boxed{x = \frac{10\pi}{3} + 2k\pi \quad k \in \mathbb{Z}}$

Zad F7) Odredite najmanju pozitivnu nultočku funkcije $f(x) = -0{,}5 \cdot \sin 2\left(\frac{\pi}{4} + 3x\right)$

$-\frac{1}{2}\sin 2\cdot\left(\frac{\pi}{4} + 3x\right) = 0 \ /:(-0{,}5)$

Najmanja za $k = 1$, a to je $\frac{\pi}{12}$

$\sin 2\cdot\left(\frac{\pi}{4} + 3x\right) = 0$

$2\left(\frac{\pi}{4} + 3x\right) = k\pi \ /:2$

$\frac{\pi}{4} + 3x = \frac{k\pi}{2}$

$x = \frac{k\pi}{6} - \frac{\pi}{12} \quad k \in \mathbb{Z}$

Zad F8) $\sqrt{3}\,\mathrm{tg}\left(x - \frac{\pi}{6}\right) = 1 \ /:\sqrt{3}$

$\mathrm{tg}\left(x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

$x - \frac{\pi}{6} = \frac{\pi}{6} + k\pi$

$\boxed{x = \frac{\pi}{3} + k\pi \quad k \in \mathbb{Z}}$

Zad F9) $\sin x = \sin 2x$

To NE

$\cancel{\sin x} = 2\cancel{\sin x}\cos x$

$\sin x - 2\sin x \cdot \cos x = 0$

$\sin x \cdot (1 - 2\cos x) = 0$

I $\sin x = 0$

$x_1 = k\pi$

II $\cos x = \frac{1}{2}$

$x_2 = \frac{\pi}{3} + 2k\pi$

$x_3 = \frac{5\pi}{3} + 2k\pi$

Zad 80) $\sqrt{3}\sin 2x = 2\cos^2 x$ u $[0,2\pi]$ Zbroj rješenja?

$2\sqrt{3}\sin x\cdot\cos x - 2\cos^2 x) = 0 \,/ :2$

$\cos x\cdot(\sqrt{3}\sin x - \cos x) = 0$

① $\cos x = 0$

$x_1 = \frac{\pi}{2} + k\pi$

u $[0,2\pi] \to \frac{\pi}{2}, \frac{3\pi}{2}$

② $\sqrt{3}\sin x - \cos x = 0 \,/ :\cos x$

$\sqrt{3}\,tg\,x - 1 = 0$

$tg\,x = \frac{\sqrt{3}}{3}$ $\quad \frac{\pi}{6}, \frac{7\pi}{6}$

$x = \frac{\pi}{6} + k\pi$

$x \in \left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}\right\}$ $\quad x_1 + x_2 + x_3 + x_4 = \frac{\pi + 3\pi + 7\pi + 9\pi}{6} = \frac{10\pi}{3}$

Zad 81) $2\cos^2 x - \sin x = 1$

$2\cdot(1 - \sin^2 x) - \sin x = 1$

$2 - 2\sin^2 x - \sin x - 1 = 0 \,/ \cdot(-1)$

$2\sin^2 x + \sin x - 1 = 0$

$(\sin x)_{1,2} = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4}$

$\sin x = \frac{1}{2}$ $\quad \sin x = -1$

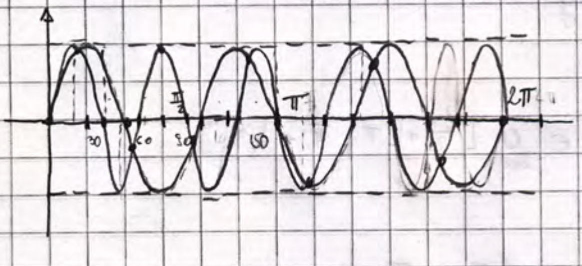
$x_1 = \frac{\pi}{6} + 2k\pi$ $\quad x_3 = \frac{3\pi}{2} + 2k\pi \quad k \in \mathbb{Z}$

$x_2 = \frac{5\pi}{6} + 2k\pi$

Zad 82) Koliko rješenja ima jednadžba $\sin 5x = \sin 3x$ u intervalu $[0, 2\pi]$?

$y_1 = \sin 5x$, $T = \frac{2\pi}{5}$ $\quad y_2 = \sin 3x$, $T = \frac{2\pi}{3}$

$180 : 5 =$
$36 \cdot 2 = 72$



11 rješenja

$\sin 5x - \sin 3x = 0$

$2\cos\frac{8x}{2}\cdot\sin\frac{2x}{2} = 0$

$2\cos 4x\cdot\sin x = 0$

$\cos 4x = 0$

$4x = \frac{\pi}{2} + k\pi$

$x = \frac{\pi}{8} + \frac{k\pi}{4}$

$\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$, $\frac{13\pi}{8}, \frac{15\pi}{8}$

$\sin x = 0$

$x = k\pi$

$(0, \pi, 2\pi$

Zad 83) $\sin^2 x - 2\sin x\cos x - 3\cos^2 x = 0 \,/ :\cos^2 x$; $\cos x \neq 0$ $\quad x \neq \frac{\pi}{2} + k\pi$ $\quad 1 = 0 \Rightarrow$ smijemo dijeliti

$tg^2 x - 2tg\,x - 3 = 0$

$(tg\,x)_{1,2} = \frac{2 \pm \sqrt{4+12}}{2} = 1 \pm 2$

$tg\,x = 3$ $\quad x = arctg\,3 + k\pi$

$tg\,x = -1$ $\quad x = \frac{3\pi}{4} + k\pi$

Zad 84) Za koje vrijednosti a jedn. $\sin^4 x + \cos^4 x = a$

$(\sin^2 x + \cos^2 x)^2 - 2\cos^2 x \cdot \sin^2 x = a$

$1 - 2\cos^2 x \cdot \sin^2 x = a$

$1 - 2 \cdot \left(\frac{1}{2}\sin 2x\right)^2 = a$

$\sin^2 2x = 2 - 2a$

$0 \leq 2 - 2a \leq 1$

$a \geq \frac{1}{2}$ $\quad 2a \leq 2$

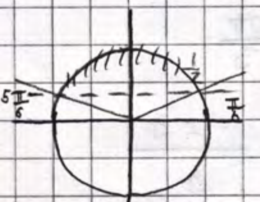
$a \leq 1$

$a \in [\frac{1}{2}, 1]$

TRIGONOMETRIJSKE NEJEDNADŽBE

primjer.) $\sin x > \frac{1}{2}$ $\quad x \in \langle \frac{\pi}{6}, \frac{5\pi}{6} \rangle$

$x \in \bigcup_{k \in \mathbb{Z}} \langle \frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi \rangle$



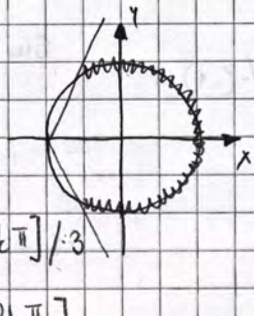
Zadatak 85) $2\cos(3x) + \sqrt{3} \geq 0$

$\cos(3x) \geq -\frac{\sqrt{3}}{2}$

$3x \in [-\frac{5\pi}{6}, \frac{5\pi}{6}]$

$3x \in \bigcup_{k \in \mathbb{Z}} [-\frac{5\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi] / :3$

$x \in \bigcup_{k \in \mathbb{Z}} [-\frac{5\pi}{18} + \frac{2k\pi}{3}, \frac{5\pi}{18} + \frac{2k\pi}{3}]$



Zadatak 86) $\sqrt{3}\, tg(x - \frac{\pi}{2}) - 1 \geq 0$ na $[0, 2\pi]$

$tg(x - \frac{\pi}{2}) \geq \frac{1}{\sqrt{3}}$ / arctg

$x - \frac{\pi}{2} \geq \frac{\pi}{6} + k\pi$ $\rightarrow x \in \bigcup_{k \in \mathbb{Z}} [\frac{2\pi}{3} + k\pi, \pi + k\pi]$

$x \in [\frac{2\pi}{3}, \pi\rangle \cup [\frac{5\pi}{3}, 2\pi\rangle$

Zadatak 87) $ctg(2x) \geq 1$ $\quad x \in \bigcup_{k \in \mathbb{Z}} [\frac{k\pi}{2}, \frac{\pi}{8} + k\pi\rangle$

$2x \leq \frac{\pi}{4}$

Zadatak 88.) $\sin x + \sqrt{3}\cos x > 0$ $/ : \cos x$ na $\langle 0, 2\pi \rangle$

I $\cos x > 0$

$x \in \langle 0, \frac{\pi}{2} \rangle \cup \langle \frac{3\pi}{2}, 2\pi \rangle$

$tg\, x + \sqrt{3} > 0$

$tg\, x > -\sqrt{3}$

$x > -\frac{\pi}{3}$

$x \in \langle -\frac{\pi}{3} + k\pi, \frac{\pi}{2} + k\pi \rangle$

$x \in \langle -\frac{\pi}{3}, \frac{\pi}{2} \rangle \cup \langle \frac{\pi}{3}, \frac{3\pi}{2} \rangle \cup \langle \frac{5\pi}{3}, \frac{5\pi}{2} \rangle$

II $\cos x < 0$

$x \in \langle \frac{\pi}{2}, \frac{3\pi}{2} \rangle$

$tg\, x < -\sqrt{3}$

$x < -\frac{\pi}{3} + k\pi$

$x \in \langle -\frac{\pi}{2} + k\pi, -\frac{\pi}{3} + k\pi \rangle$

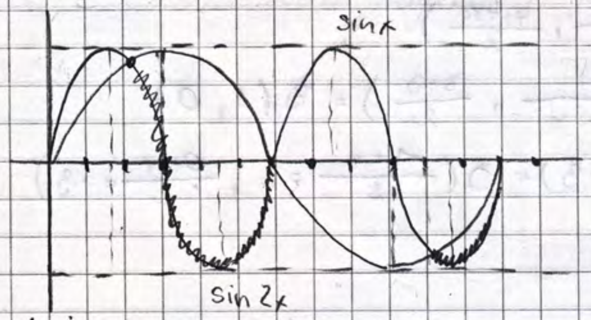
$x \in \langle -\frac{\pi}{2}, -\frac{\pi}{3} \rangle \cup \langle \frac{\pi}{2}, \frac{2\pi}{3} \rangle \cup \langle \frac{3\pi}{2}, \frac{5\pi}{3} \rangle$

I $x \in \langle 0, \frac{\pi}{2} \rangle \cup \langle \frac{5\pi}{3}, 2\pi \rangle$

II $x \in \langle \frac{\pi}{2}, \frac{2\pi}{3} \rangle$

Rj. $x \in \langle 0, \frac{2\pi}{3} \rangle \cup \langle \frac{5\pi}{3}, 2\pi \rangle$

Zad 89.) $\sin 2x \leq \sin x$ na $[0, 2\pi]$



$2\sin x \cos x - \sin x = 0$

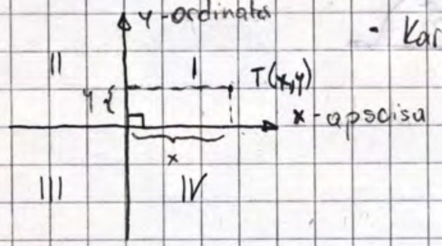
$\sin x \, (2\cos - 1) = 0$

$\cos x = \frac{1}{2}$ $\qquad$ $\sin x = 0$

$x \in \langle \frac{\pi}{3}, \pi \rangle \cup \langle \frac{5\pi}{3}, 2\pi \rangle$

3. KOORDINATNI SUSTAV U RAVNINI

3.1. Točke i pravci

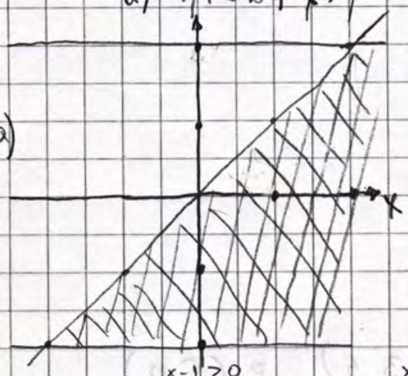


- Kartezijev (pravokutni) sustav

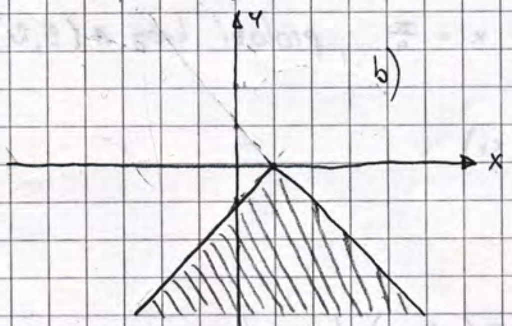
Zad 90) Skiciraj sljedeće skupove

a) $|y| \leq 2$, $x \geq y$ $\qquad$ $(y \leq x) \rightarrow y$ je manji od pravca

a)



b)



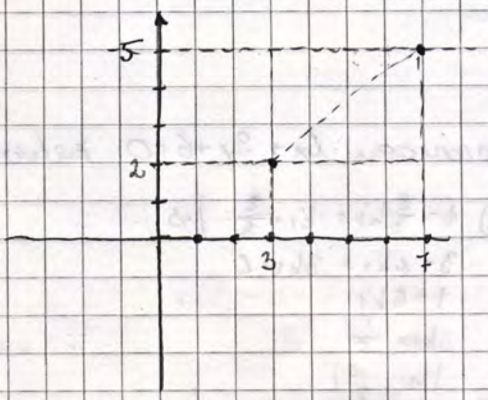
$x - 1 > 0$ $\qquad$ $x - 1 < 0$

b) $y \leq -|x - 1|$ $\qquad$ $y \leq x - 1$

$y \leq -x + 1$

Zad 91) Točka $A(x, 5)$ nalazi se u I. kvadrantu i jednako je udaljena od osi apscisa i $B(3, 2)$

$A(7, 5)$



- udaljenost dvije točke:

$$\boxed{d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

$5 = \sqrt{(3 - x)^2 + (5 - 2)^2}$

$25 = (3 - x)^2 + 9$

$16 = (3 - x)^2 \quad / \sqrt{\ }$

$-4 = 3 - x$

$\boxed{x = 7}$

zad.92) C je simetrična točki A s obzirom na B, a D je simetrična točki B s obzirom na C. Ako je $A(-1,3)$, $C(3,-3)$. Koliko je D

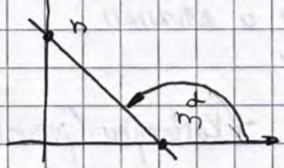
polovište dužine: $P(A,B) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$$P(A,B) = B\left(\frac{3-1}{2}, \frac{3-3}{2}\right) = B(1,0)$$

$$P(B,D) = C(3,-3) = D\left(\frac{1+x}{2} = 3, \frac{0+y}{2} = -3\right)$$

$$D(5,-6)$$

pravac: eksplicitna jedžb.: $y = kx + l$
implicitna jedžb.: $Ax + By + C = 0$
segmentni $\frac{x}{m} + \frac{y}{n} = 1$



$\boxed{k = tg\,\alpha}$ - pozitivni dio x osi

kut između dva pravca: $tg\,\varphi = \left|\frac{k_2 - k_1}{1 + k_1 k_2}\right|$

ako su pravci paralelni $p_1 \parallel p_2$: $k_1 = k_2$

ako su pravci okomiti $p_1 \perp p_2$: $k_1 = -\frac{1}{k_2}$

zad 93) kut s osi $x = \frac{\pi}{4}$, prolazi kroz $A(3,2)$

$k = tg\,\frac{\pi}{4} = 1$

$y - y_1 = k\cdot(x - x_1)$
$y - 2 = 1(x-3)$
$y = x - 1$

zad 94) odredi točku $F(-1,y)$ koja leži na pravcu AB, $A(-3,4)$, $B(5,0)$

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ $\qquad y = -\frac{1}{2}x + \frac{5}{2}$

$y - 4 = \frac{0-4}{5+3}(x+3)$ $\qquad F(-1,3)$

$y - 4 = -\frac{1}{2}x - \frac{3}{2}$

zad 95) Točkom $T(2,-1)$ položite pravac koji s pravcem $2x + 3y + 6 = 0$ zatvara kut od 45°

$3y = -2x - 6 \;/ :3$

$y = -\frac{2}{3}x - 2$

$$1 = \left|\frac{k_1 + \frac{2}{3}}{1 - \frac{2}{3}k_1}\right|$$

I) $1 - \frac{2}{3}k_1 = k_1 + \frac{2}{3} \;/ \cdot 3$
$3 - 2k_1 = 3k_1 + 2$
$1 = 5k_1$
$k_1 = \frac{1}{5}$

II) $-1 + \frac{2}{3}k_1 = k_1 + \frac{2}{3}$

II) $k_1 = -5$

p_1: $y + 1 = -5x + 10$
$y = -5x + 9$

p_2: $y + 1 = \frac{1}{5}x - \frac{2}{5}$
$y = \frac{1}{5}x - \frac{7}{5}$

Zad 96) Pravac $x - 2y + 3 = 0$ treba zaokrenuti za kut α oko sjecišta s $y = 3x - 1$ tako bi se poklopio s tim pravcem.

$$-2y = -x + 3 \,/ : (-2) \qquad \operatorname{tg} \alpha_{12} = \left| \frac{k_2 - k_1}{1 + k_1 \cdot k_2} \right|$$

$$y = \frac{x}{2} + \frac{3}{2}$$

$$k_1 = \frac{1}{2} \quad k_2 = 3 \qquad \operatorname{tg} \alpha_{12} = \left| \frac{3 - \frac{1}{2}}{1 + \frac{3}{2}} \right|$$

$$\operatorname{tg} \alpha_{1,2} = \left| \frac{\cancel{\frac{5}{2}}\, 1}{\cancel{\frac{5}{2}}\, 1} \right|$$

$$\operatorname{tg} \alpha_{1,2} = 1 \quad \Rightarrow \alpha_{1,2} = \frac{\pi}{4}$$

Zad 97) površina trokuta kojeg $3x + ay = 12$ zatvara s koordinatnim osima je 6. Koliko je a?

$$\frac{x}{4} + \frac{ay}{12} = 1 \qquad a > 0 \qquad a < 0$$

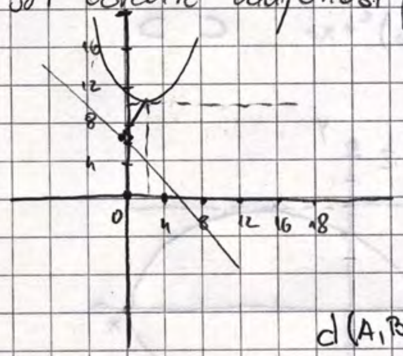
$$P = \frac{m \cdot n}{2} \qquad 3a = 12 \,/ : 3 \qquad -3a = 12 \,/ : (-3)$$

$$6 = \frac{4 \cdot \frac{12}{a}}{2} \,/ \cdot 2 \qquad a = 4 \qquad a = -4$$

$$12 = 4 \cdot \frac{12}{a} \,/ : 4$$

$$3 = \frac{12}{a}$$

Zad 98) Odredite udaljenost pravca $y = \frac{-4x}{3} + 6$ od tjemena parabole $y = x^2 - 4x + 14$



$$k = \frac{3}{4} \quad T(2, 10)$$

$$y - 10 = \frac{3}{4}x - \frac{3}{2}$$

$$y = \frac{3}{4}x + \frac{17}{2}$$

$$-\frac{4x}{3} + 6 = \frac{3}{4}x + \frac{17}{2} \,/ \cdot 12$$

$$-16x + 72 = 9x + 102$$

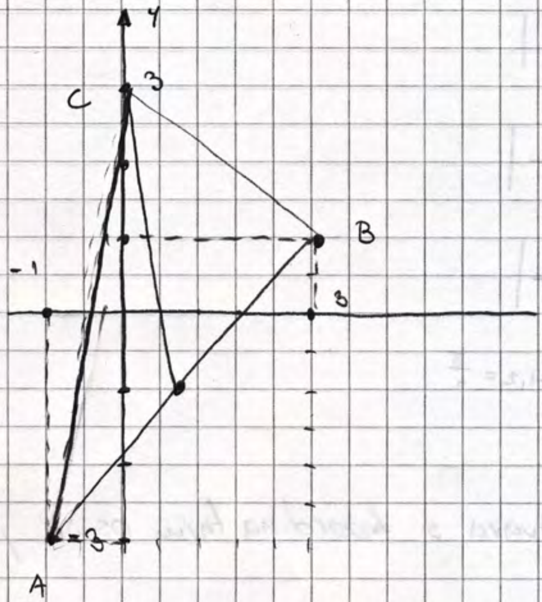
$$-25x = 30$$

$$x = -\frac{6}{5} \quad y = \frac{38}{5}$$

$$d(A, B) = \sqrt{\left(2 + \frac{6}{5}\right)^2 + \left(10 - \frac{38}{5}\right)^2} =$$

$$= \sqrt{\frac{256}{25} + \frac{144}{25}} = \sqrt{\frac{400}{25}} = \frac{20}{5} = 4$$

zad 99) $A(-1,-3)$, $B(3,1)$, $C(0,3)$. odredi težišnicu iz vrha C



$P_{AB}\left(\frac{-1+3}{2}, \frac{-3+1}{2}\right)$

$P_{AB}(1,-1)$

$y-3=\frac{-1-3}{1-0}(x-0)$

$y-3=-4x$

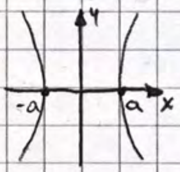
$\boxed{y+4x-3=0}$

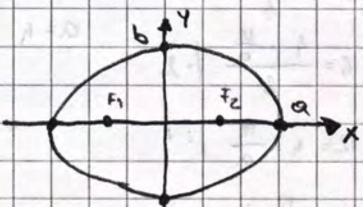
3.2. Krivulje drugog reda

kružnica: $(x-x_0)^2+(y-y_0)^2=r^2$

elipsa: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

hiperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$





- a - govori odkuda kreće
- b - govori koliko je "široka"

parabola: $y=ax^2+bx+c=a(x-x_1)\cdot(x-x_2)=a(x+x_0)^2+y_0$ ∪ ∩

$x=ay^2+by+c=a(y-y_1)\cdot(y-y_2)=a(y-y_0)^2+x_0$ ⊂ ⊃

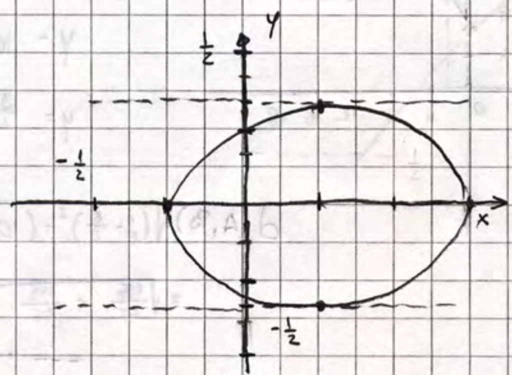
pr.) a) Skiciraj $x^2+2y^2=x$

$x^2-x+2y^2=0$

$(x-\frac{1}{2})^2+2y^2=\frac{1}{4}$ $|\cdot 4$

$4(x-\frac{1}{2})^2+8y^2=1$

$\frac{(x-\frac{1}{2})^2}{\frac{1}{4}}+\frac{y^2}{\frac{1}{8}}=1$

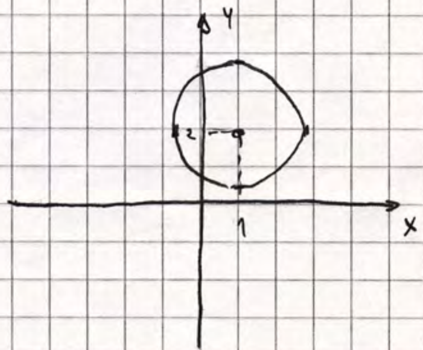


b) $2x^2+2y^2-4y-8y=-3$ $|:2$

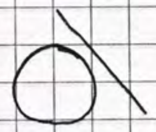
$x^2+y^2-2y-4y=-\frac{3}{2}$

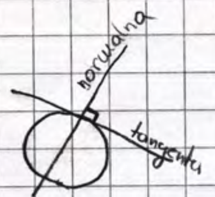
$(x-1)^2+(y-2)^2-1-4=-\frac{3}{2}$

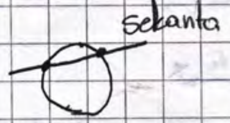
$(x-1)^2+(y-2)^2=\frac{7}{2}$ $r=\sqrt{\frac{7}{2}}$



Odnosi pravca i kružnice







ne sijeku se

zad 99.) Odredi jedžb. tangenata koje su paralelne s pravcom $3x-4y+8=0$

$k: x^2+y^2=9$

$-4y=-3x-8 \;/\cdot(-4)$ $\qquad k=\frac{3}{4}$

$y=\frac{3}{4}x+2$

$k_{norm}=-\frac{4}{3} \qquad y=-\frac{4}{3}x$

$x^2+\frac{16}{9}x^2=9 \;/\cdot 9 \qquad T_1\left(\frac{9}{5}, -\frac{12}{5}\right)$

$25x^2=81 \qquad T_2\left(\frac{-9}{5}, \frac{12}{5}\right)$

$x_{1,2}=\frac{\pm 9}{5}$

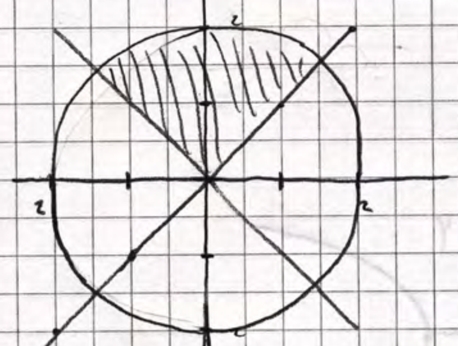
$y=kx+l \qquad l_1=y-\frac{3}{4}x$

$y=\frac{3}{4}x+l \qquad t_1: y-\frac{12}{5}=\frac{3}{4}\left(x+\frac{9}{5}\right)$

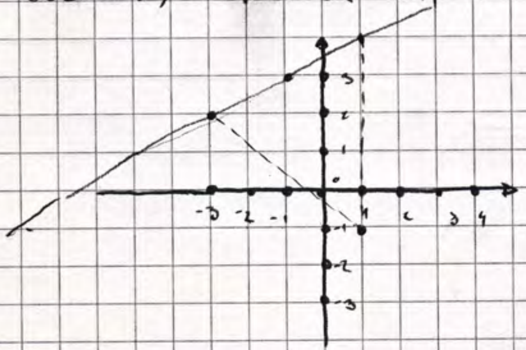
$t_2: y+\frac{12}{5}=\frac{3}{4}\left(x-\frac{9}{5}\right)$

zad 100.) $x^2+y^2\le 4$ i $|x|-y\le 0$ izračunaj površinu lika.

$P=\frac{r^2\pi}{4}=\pi$



zad 101.) Kružnica $S(1,-1)$ odsjeca na pravcu $x-2y+7=0$ tetivu duljine 8. Odredite



$A(-1, 3)$

$B(-3, 2)$

$(x-1)^2+(y+1)^2=r^2$

$x-2y+7=0$

$x=2y-7$

$(2y-8)^2+(y+1)^2=r^2$

$4y^2-32y+64+y^2+2y+1=r^2 \;/:5$

$y^2-6y+13-\frac{r^2}{5}=0$

$x_{1,2}=2\cdot\frac{6\pm\sqrt{36-4\left(13-\frac{r^2}{5}\right)}}{2}-7 \qquad y_{1,2}=\frac{6\pm\sqrt{36-4\left(13-\frac{r^2}{5}\right)}}{2}$

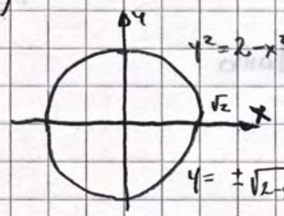
$x_{1,2}=-1\pm\sqrt{\frac{4r^2}{5}-16}$

$y_{1,2}=3\pm\frac{1}{2}\sqrt{\frac{4r^2}{5}-16}$

$d(D_1,D_2)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=8 \;/^2$

$r^2=36 \qquad r=6$

pr.)

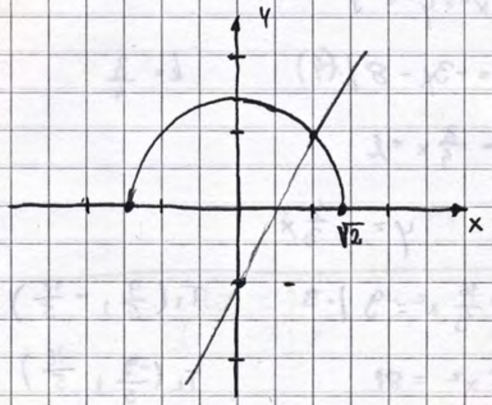


$y = \pm\sqrt{2-x^2}$ → kružnica nije funkcija

Zad 102) Koliko ima rješenja ima?

$$2x - 1 - \sqrt{2-x^2} = 0$$

$$2x - 1 = \sqrt{2-x^2}$$

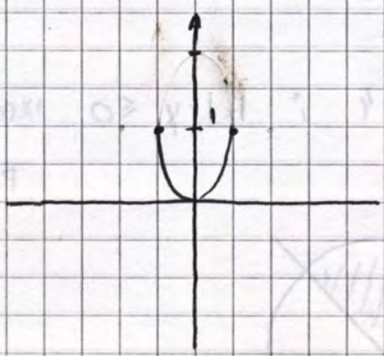


Zad 103.) Skiciraj $y = 1 - \sqrt{1-2x^2}$

$$y-1 = -\sqrt{1-2x^2} \quad /^2$$

$$(y-1)^2 = 1-2x^2$$

$$2x^2 + (y-1)^2 = 1$$

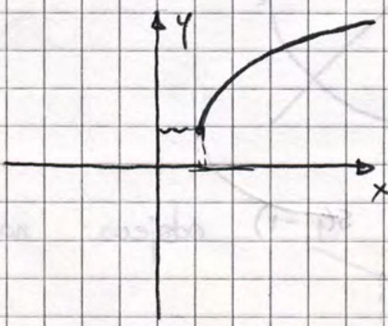


Zad 104.) Skicirajte $y = 1 + 2\sqrt{x-1}$

$$y - 1 = 2\sqrt{x-1} \quad /:2$$

$$(y-1)^2 = 4x-4 \quad /:4$$

$$\frac{(y-1)^2}{4} - x = 1$$



ARITMETIČKI I GEOMETRIJSKI NIZ

$S \quad a: \mathbb{N} \to S$

$a_1, a_2, a_3, \dots, a_n, \dots$

$S = \mathbb{R}$ niz realnih brojeva

$(a_n) \to$ niz
$\{a_n\} \to$ skup vrijednosti niza

- rastući niz
$\forall n \quad a_n \leq a_{n+1}$
- padajući niz
$\forall n \quad a_n \geq a_{n+1}$

} monotoni nizovi

- ograničeni niz: niz je ograničen ako postoje realni brojevi m i M tako da vrijedi $m \leq a_n \leq M; \forall n$
(m – donja međa, M – gornja međa)

primjer) a) $a_n = n^2 \quad 1, 4, 9, 16 \dots$

monotono rastući i nije ograničen

b) $a_n = \frac{(-1)^n}{n} \quad -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4} \dots$

nije monoton, ograničen je $m = -1 \quad M = \frac{1}{2}$

c) $a_n = \frac{n}{n+1} \to \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \dots$

monotono rastući niz, omeđen sa $m = \frac{1}{2}, M = 1$

ARITMETIČKI NIZ

def. Niz je aritmetički ako je razlika susjednih članova stalna i iznosi d
d – razlika (diferencija) aritm. niza

$a_n - a_{n-1} = d \quad n \geq 2$

$a_1, \underbrace{a_1 + d}_{a_2}, \underbrace{a_1 + 2d}_{a_3}, \dots, \underbrace{a_1 + (n-1)d}_{n}$

Zbroj prvih n članova niza $S_n = \frac{n}{2}(a_1 + a_n)$

pr.) a) $1 + 2 + 3 + \dots + n = \frac{(n+1) \cdot n}{2}$

b) $\underbrace{1 + 4 + 7 + \dots + (3n+1)}_{(n+1) \text{ pribrojnika}} = \frac{n+1}{2}(1 + 3n + 1) = \frac{(n+1)(3n+2)}{2}$

105.
zad) Godine starosti petero braće čine aritmetički niz. Zbroj godina najstarije dvojice jednak je zbroju godina ostale trojice. Dok svi osim najmlađeg imaju ukupno 78 godina. Koliko godina ima najmlađ. brat

$a_1, a_2, a_3, a_4, a_5 \qquad a_1 + 3d + a_1 + 4d = a_1 + a_1 + d + a_1 + 2d$

$a_4 + a_5 = a_1 + a_2 + a_3 \qquad a_1 - 4d = 0 \Rightarrow a_1 = 4d$

$a_2 + a_3 + a_4 + a_5 = 78$

$4a_1 + 10d = 78$

$d = 3 \Rightarrow a_1 = 4 \cdot 3 = 12$

zad 106) Koliko članova aritm. niza trebamo zbrojiti da njihov zbroj bude 0

niz: 21, 18, 15, ... $a_1 = 21$ $d = -3$ rj: $n = 15$

$S_n = \frac{n}{2}(a_1 + a_n)$

$0 = \frac{n}{2}(a_1 + (n-1)d + a_1) / \cdot 2$

$0 = n \cdot a_1 + n(n-1) \cdot (-3) + a_1 \cdot n$

$-42n = (n^2 - n) \cdot (-3) / : (-3)$

$-n^2 + 12n = 0$

zad 107) Koliko brojeva treba umetnuti između -22 i 18 tako da bi se dobio aritm. niz čiji je zbroj -42

Rj: Treba umetnuti 19 brojeva

$a_1 = -22$ $a_n = 18$ $S = -42$

$S = \frac{n}{2}(a_1 + a_n) / \cdot 2 \frac{1}{a_1 + a_n}$

$\frac{2S}{a_1 + a_n} = n$

$n = \frac{-84}{-22 + 18} = \frac{-84}{-4} = 21$

zad 108) Odredite x tako da $\sqrt{x-1}, \sqrt{5x-1}, \sqrt{12x+1}$ budu 3 uzastopna člana aritm. niza

Rj: $x_1 = 2$ $x_2 = 10$

$\sqrt{5x-1} - \sqrt{x-1} = \sqrt{12x+1} - \sqrt{5x-1}$

$2\sqrt{5x-1} = \sqrt{12x+1} + \sqrt{x-1} /^2$

$20x - 4 = (12x+1) + 2\sqrt{(12x+1)\cdot(x-1)} + (x-1)$

$7x - 4 = 2\sqrt{12x^2 - 11x - 1} /^2$

$49x^2 - 56x + 16 = 4 \cdot (12x^2 - 11x - 1)$

$49x^2 - 56x + 16 = 48x^2 - 44x - 4$

$x^2 - 12x + 20 = 0$

zad. 109) Odredi zbroj svih troznamenkastih brojeva koji su djeljivi s dva ili s 3 (ne oba !)

① $a_1 = 100$ $d = 2$ $a_n = 998$

$a_n = 998 = 100 + (n-1) \cdot 2 \Rightarrow n = 450$

② $b_1 = 102$ $d = 3$ $a_n = 999 \Rightarrow n = 300$

③ $c_1 = 102$ $d = 6$ $a_n = 996 \Rightarrow n = 150$

Suma: $S_a = \frac{450}{2}(100 + 998) = 247\,050$

$S_b = \frac{300}{2}(102 + 999) = 165\,150$

$S_c = \frac{150}{2}(102 + 996) = 82\,350$

$S = 329\,850$

GEOMETRIJSKI NIZ

def. Niz je geometrijski ako je omjer susjednih članova stalan i iznosi q

$\frac{a_n}{a_{n-1}} = q$ $n \geq 2$ → kvocijent ili količnik geom. niza

$a_1, \underset{a_2}{a_2}, a_1 \cdot q^2, \ldots, a_1 \cdot q^n, \ldots$

$a_n = a_1 q^{n-1}$

$S_n = a_1 \frac{1-q^n}{1-q}$

pr.) a) $3 + 3^2 + \ldots + 3^n = \frac{3(3^n - 1)}{2}$

$a_1 = 3 \quad q = 3$

b) $1 + 2 + \ldots + 2^n = 1 \frac{1-2^{n+1}}{1-2} = 2^{n+1} - 1$

$a_1 = 1 \qquad q = 2$

zad. 10) Umnožak prva tri člana rastućeg geom. niza iznosi 1728, a njihov zbroj 63. Koliki je treći član tog niza?

$a_1 \cdot a_2 \cdot a_3 = 1728$

$a_1^3 \cdot q^3 = 1728 \quad / \sqrt[3]{\ }$

$a_1 \cdot q = 12 \Rightarrow a_2 = 12$

$q = \frac{12}{a_1}$

$a_1 + a_2 + a_3 = 63$

$\cancel{3a_1 + 3d = 63 / 3}$

$\cancel{a_1 + d = 21} \quad \cancel{\Rightarrow \boxed{a_2 = 21}}$

$a_1 \cdot (1 + q_1 + q^2) = 63$

$\frac{12}{q} + 12 + 12q = 63 \quad / \cdot q$

$12 + 12q + 12q^2 = 63q$

$12q^2 - 51q + 12$

$q_{1,2} = \ldots$

$\cancel{q_1 = \frac{1}{4}} \quad q_2 = 4$

Niz mora biti rastući

$a_1 = 3$
$a_2 = 12$
$a_3 = 48$

zad 11) U geometrijskom nizu od 20 članova zbroj prvih 10 članova 10 puta je manji od zbroja preostalih 10. Koliki je q?

$n = 20 \quad \rightarrow$ koristimo $n = 10$

$10 \cdot S_{10} = S_{11 \ldots 20}$

$10 \cdot a_1 \cdot \frac{\cancel{1-q^{10}}}{\cancel{1-q}} = a_{11} \cdot \frac{\cancel{1-q^{10}}}{\cancel{1-q}}$

$10 \cdot \cancel{a_1} = \cancel{a_1} \cdot q^{10} \cdot \cancel{\frac{1-q^{10}}{1-q}}$

$q^{10} = 10 \quad / \sqrt[10]{\ }$

$q = \sqrt[10]{10}$

Zad 112) 4 broja čine aritmetički niz i imaju zbroj 2,5. Izostavimo li 2 član, preostali 3 čine geometrijski niz. Koji su to brojevi?

$a_1+a_2+a_3+a_4=2{,}5$

$4a_1+6d=\frac{5}{2}$

$8a_1+12d=5$

$8a_1=5-12d \quad /:8$

$a_1=\frac{5-12d}{8}$

$a_1=-4d$

$-4d_1=\frac{5-12d}{8} \quad /\cdot 8$

$-32d=5-12d$

$-20d=5 \quad /:20$

$d=\frac{-1}{4}$

$a_1=1$

$a_1, a_3, a_4 \Rightarrow$ geom. niz

$g_1\cdot g=a_3 \qquad a_3\cdot g=4$

$a_1\cdot g=a_1+2d \qquad (a_1+2d)\cdot g=a_1+3d$

$g_1=\frac{a_1+2d}{a_1} \qquad g=\frac{a_1+3d}{a_1+2d}$

$\frac{a_1+2d}{a_1}=\frac{a_1+3d}{a_1+2d} \quad /\cdot(a_1)\cdot(a_1+2d)$

$(a_1+2d)^2=a_1^2+3da_1$

$\cancel{a_1^2}+4a_1d+4d^2=\cancel{a_1^2}+3a_1d$

$4a_1d+4d^2=0 \quad /d$

$d(a_1+4d)=0$

$1,\ 0{,}75,\ 0{,}5,\ 0{,}25$

GEOMETRIJSKI RED

$x_1+x_2+x_3\ldots+x_n+\ldots$

$\sum_{n=1}^{\infty} x_n$

Geometrijski red dobije se zbrajanjem članova geometrijskog niza

$$\sum_{n=0}^{\infty} a_1 g^n = a_1+a_1\cdot g+\ldots+a_1\cdot g^{n-1}+\ldots=\frac{a_1}{1-g} \quad \text{ako je } |g|<1$$

pr.) $\frac{1}{3}-\frac{2}{9}+\frac{4}{27}-\frac{8}{81}+\ldots=\sum_{n=0}^{\infty}\frac{1}{3}\left(-\frac{2}{3}\right)^n=\frac{\frac{1}{3}}{1+\frac{2}{3}}=\frac{1}{5}$

Zad. 113) Izračunaj:

$$\sqrt[3]{2}\cdot\sqrt[6]{2}\cdot\sqrt[12]{2}\cdot\ldots=2^{\frac{1}{3}}\cdot 2^{\frac{1}{6}}\cdot 2^{\frac{1}{12}}\cdot\ldots=2^{\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\ldots}=2^{\sum_{n=1}^{\infty}\frac{1}{3}\cdot\left(\frac{1}{2}\right)^n}=2^{\frac{\frac{1}{3}}{1-\frac{1}{2}}}=2^{\frac{2}{3}}=\sqrt[3]{4}$$

Zad 114) Riješi jedž.

$2^{x-1}+2^{x-2}+2^{x-4}+\ldots=6{,}5+3{,}25+1{,}65+\ldots \qquad x=4$

$a_1=6{,}5 \qquad g=\frac{1}{2}$

$2^x\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{16}\right)=6{,}5\,\frac{1}{1-\frac{1}{2}}$

$2^x\,\frac{13}{16}=13$

Zad 118) $z^2 + i = \bar{z}$

$(x+iy)^2 + 1 = x - iy$

$x^2 + 2xyi - y^2 + 1 = x - iy$

$x^2 - y^2 + 1 = x$ $\qquad$ $2xy = -y$

$y=0$

① $x^2 + 1 = x$ $\qquad$ $2xy + y = 0$

$y(2x+1) = 0$

$x^2 - x + 1 = 0$

$y = 0 \qquad x = -\frac{1}{2}$

$\cancel{x_1 = \frac{1}{2} + i\sqrt{3}} \quad \cancel{x_2 = \frac{1}{2} - i\sqrt{3}}$

zato jer su $x, y \in \mathbb{R}$

⑪ $\frac{1}{4} - y^2 + 1 = -\frac{1}{2} \,/\cdot(-1)$

$y^2 = -\frac{1}{2} - \frac{1}{4} - 1$

$y^2 = \frac{7}{4} \,/\sqrt{\ }$

$y_{1,2} = \pm\frac{\sqrt{7}}{2}$

$z_1 = -\frac{1}{2} + i\frac{\sqrt{7}}{2}$

$z = -\frac{1}{2} - i\frac{\sqrt{7}}{2}$

Zad 119) Riješi sustav

$z^2 + \bar{z}^2 = 6$

$z \cdot \bar{z} = 5$

$(x+yi)^2 + (x-yi)^2 = 6$

$(x+iy)\cdot(x-iy) = 5$

$x^2 + \cancel{2xyi} - y^2 + x^2 - \cancel{2xyi} - y^2 = 6$

$x^2 + y^2 = 5 \,/\cdot 2$

$2x^2 - \cancel{2y^2} = 6$

$2x^2 + \cancel{2y^2} = 10$

$4x^2 = 16 \,/:4$

$x = \pm 2$

$y^2 = 5 - 4$

$y = \pm 1$

$z_1 = 2 + i$

$z_2 = -2 + i$

$z_3 = 2 - i$

$z_4 = -2 - i$

Zad 120) Kompleksni broj iz prvog kvadranta ima $Re(z) = 4\,Im(z)$. Koliko puta je $Re\, z^2 > Im\, z^2$

$x = 4y$

$Re(z^2) = k \cdot Im(z^2)$

$Re(x^2 + 2xyi - y^2) = k \cdot Im(x^2 + 2xyi - y^2)$

$x^2 - y = k \cdot 2xy$

$15y^2 = k \cdot 8y^2 \,/: 8y^2$

$k = \frac{15y^2}{8y^2}$

$\boxed{k = \frac{15}{8}}$

potencije od i: $\quad i^0 = 1 = i^{4k}$

$i^1 = i = i^{4k+1}$

$i^2 = -1 = i^{4k+2}$

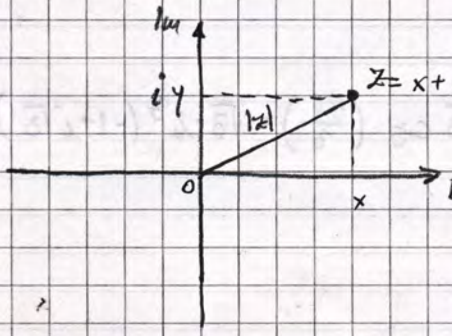
$i^3 = -i = i^{4k+3}$

pr.) $i^{314} = i^{4\cdot 78 + 2} = i^2 = -1$

Zad 121) $\sum_{k=0}^{99} i^k = ?$; Izračunaj prvih 100 potencija broja i

$$i^0+i^1+i^2+i^3+i^4+\ldots+i^{99} = \underbrace{1+i-1-i}_{0}+1+i+\ldots-i=0$$

PRIKAZ COMPLEX. BROJA U KOMPLEKSNOJ RAVNINI



$|z| \to$ modul; udaljenost od ishodišta

$|z| = \sqrt{x^2+y^2}$ $\qquad |z^n| = |z|^n$

$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Zad 122) $z = \frac{(1+i\sqrt{3})\cdot(\sqrt{3}+i)\cdot(\sqrt{3}+2i)}{\sqrt{13}+i\sqrt{3}}$; $|z| = ?$

$$|z| = \frac{|1+i\sqrt{3}|\cdot|\sqrt{3}+i|\cdot|\sqrt{3}+2i|}{|\sqrt{13}+i\sqrt{3}|}$$

$$|z| = \frac{\cancel{2}^{\,1}\cdot 2\cdot 3}{\cancel{4}_{\,1}} = 3$$

Zad 123) $\frac{z}{1-2i} - \frac{2\bar{z}}{1+2i} = 5i$ $\qquad |z| = ?$

$$\frac{(x+iy)\cdot(1+2i) - 2(x-iy)(1-2i)}{5} = 5i \;/\cdot 5$$

$$x+iy+2xi-2y-2x+4xi+2iy+4y = 25i$$

$-x+2y=0 \Rightarrow x=2y$

$2x+y+4x+2y=25$

$4y+y+8y+2y=25$

$15y=25 \;/:15$

$y=\frac{25}{15}=\frac{5}{3}$

$x=\frac{10}{3}$

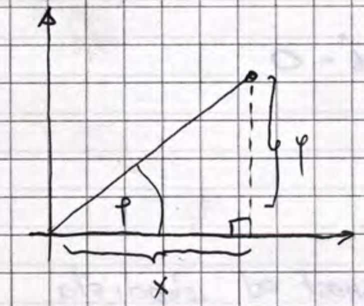
$z=\frac{5}{3}+\frac{10}{3}i$

$|z|=\sqrt{\frac{25}{9}+\frac{100}{9}}=\frac{5\sqrt{5}}{3}$

Zad 124) Izračunaj udaljenost točaka $z=1-2i$, $w=2+3i$

$$d(z,w)=|z-w|=|1-2i-2-3i|=|-1-5i|=\sqrt{1+25}=\sqrt{26}$$

TRIGONOMETRIJSKI PRIKAZ KOMPLEKSNOG BROJA



$x = r \cdot \cos \varphi$

$y = r \cdot \sin \varphi$

$z = r(\cos \varphi + \sin \varphi)$

$z = r \cdot \operatorname{cis} \varphi$

$z_1 \cdot z_2 = r_1 \cdot r_2 \operatorname{cis}(\varphi_1 + \varphi_2)$

$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot \operatorname{cis}(\varphi_1 - \varphi_2)$

$z^n = r^n \cdot \operatorname{cis}(n\varphi)$

Zad 125) $z = \frac{(2+2i)^{10}}{1 - \frac{i}{\sqrt{3}}} = \frac{(2\sqrt{2} \cdot \operatorname{cis}\frac{\pi}{4})^{10}}{\frac{2}{\sqrt{3}} \cdot \operatorname{cis}(-\frac{\pi}{6})} = \frac{2^{15} \operatorname{cis}\frac{5\pi}{2}}{\frac{2}{\sqrt{3}} \operatorname{cis}\frac{-\pi}{6}} = 2^{14}\sqrt{3} \operatorname{cis}\left(\frac{8\pi}{3}\right) = \sqrt{3} \cdot 2^{13}(-1 + i\sqrt{3})$

$z_1 = 2 + 2i = 2\sqrt{2} \operatorname{cis}\frac{\pi}{4}$

$z_2 = 1 - \frac{i}{\sqrt{3}} = \frac{2}{\sqrt{3}} \operatorname{cis}\left(-\frac{\pi}{6}\right)$

Množenje kompl. broja sa π mu dodaje π

Zad 126) $z = -2\left(\cos\frac{\pi}{6} - i \sin\frac{5\pi}{6}\right)^4 = -2 \operatorname{cis}\frac{\pi}{6} = -2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)^4 = -2 \operatorname{cis}\frac{4\pi}{6} =$

$= 2 \operatorname{cis}\frac{5\pi}{3}$

Zad 127) $(1+i)^6 - (1-i)^6 = (\sqrt{2} \operatorname{cis}\frac{\pi}{4})^6 - (\sqrt{2} \operatorname{cis}(-\frac{\pi}{4}))^6 = 8 \operatorname{cis}\frac{3\pi}{2} - 8 \operatorname{cis}\left(-\frac{3\pi}{2}\right) =$

$= 8 \cdot (0 - i) - 8(0 + i) = -16i$

KORJENOVANJE KOMPLEKSNIH BROJEVA

$\underbrace{\sqrt[n]{z} = \sqrt[n]{r} \operatorname{cis}\frac{\varphi + 2k\pi}{n}}_{n \text{ rješenja}}, \quad k = 0, 1, 2, \dots, n-1$

Zad 128) $z^6 = (1+i)^2$

$z^6 = \left(\sqrt{2} \operatorname{cis}\frac{\pi}{4}\right)^2$

$z^6 = 2 \operatorname{cis}\frac{\pi}{2} \quad / \sqrt[6]{}$

$z = \sqrt[6]{2} \operatorname{cis}\frac{\pi + 4k\pi}{12}$

$z_1 = \sqrt[6]{2} \operatorname{cis}\frac{\pi}{12}$

$z_2 = \sqrt[6]{2} \operatorname{cis}\frac{5\pi}{12}$

$z_3 = \sqrt[6]{2} \operatorname{cis}\frac{3\pi}{4}$

$z_4 = \sqrt[6]{2} \operatorname{cis}\frac{13\pi}{12}$

$z_5 = \sqrt[6]{2} \operatorname{cis}\frac{17\pi}{12}$

$z_6 = \sqrt[6]{2} \operatorname{cis}\frac{7\pi}{3}$

Zad 129) Riješi jednadžbu i nađi rješenja koja imaju realni dio veći od 0 ($\operatorname{Re} z > 0$)

$\left(z + \frac{3}{4}i\right)^3 + i = 0$

$\left(z + \frac{3}{4}i\right)^3 = -i \quad / \sqrt[3]{}$

$z + \frac{3}{4}i = \sqrt[3]{-i}$

$z = \sqrt[3]{-i} - \frac{3}{4}i$

$\sqrt[3]{-i} = \sqrt[3]{\operatorname{cis}\frac{3\pi}{2}} = \operatorname{cis}\frac{3\pi + 4k\pi}{6}; \quad k = 0, 1, 2$

$z_1 = \operatorname{cis}\frac{\pi}{2} - \frac{3}{4}i = i - \frac{3}{4}i = \frac{1}{4}i$

$z_2 = \operatorname{cis}\frac{7\pi}{6} - \frac{3}{4}i = \left(-\frac{\sqrt{3}}{2} - i\frac{1}{2} - \frac{3}{4}i\right) = -\frac{\sqrt{3}}{2} - \frac{5}{4}i$

$z_3 = \operatorname{cis}\frac{11\pi}{6} - \frac{3}{4}i = \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i - \frac{3}{4}i\right) = \frac{\sqrt{3}}{2} - \frac{5}{4}i$

$\boxed{z_3 = \text{rješenje.}}$

Zad 130) $z^8+z^6+2z^4+z^2+1=0$

$z^8+z^6+z^4+z^4+z^2+1=0$

$z^4(z^4+z^2+1)+(z^4+z^2+1)=0$

$(z^4+z^2+1)\cdot(z^4+1)=0$

$z^2=t$

$z^4+1=0$

$z^4=-1$

$z^4=\text{cis}\,\pi$

$z=\text{cis}\,\frac{\pi+2k\pi}{4}$

$z_{1,2,3,4}=\pm\frac{\sqrt{2}}{2}\pm\frac{\sqrt{2}}{2}$

$z^4+z^2+1=0$

$t^2+t+1=0$

$t_{1,2}=\frac{-1\pm\sqrt{1-4}}{2}$

$t_1=-\frac{1}{2}+\frac{\sqrt{3}}{2}=\text{cis}\,\frac{2\pi}{3}$

$t_2=-\frac{1}{2}-\frac{\sqrt{3}}{2}=\text{cis}\,\frac{5\pi}{3}$

$z^2=\text{cis}\,\frac{2\pi}{3}\ /\sqrt{\ }$ $\qquad z^2=\text{cis}\,\frac{5\pi}{3}$

$z=\text{cis}\,\frac{2\pi+6k\pi}{6}$

$z_1=\text{cis}\,\frac{\pi}{4}$
$z_2=\text{cis}\,\frac{3\pi}{4}$
$z_3=\text{cis}\,\frac{5\pi}{4}$
$z_4=\text{cis}\,\frac{7\pi}{4}$

$z_5=\text{cis}\,\frac{\pi}{3}$
$z_6=\text{cis}\,\frac{4\pi}{3}$
$z_7=\text{cis}\,\frac{2\pi}{3}$
$z_8=\text{cis}\,\frac{5\pi}{3}$

Zad 131) Vrijedi da je $z^3=1$, a $\text{Im}\,z>0$, $z^{20}=?$

$z^3=\text{cis}\,0\ /\sqrt[3]{\ } \Rightarrow z=\text{cis}\,\frac{2k\pi}{3}$ $\qquad z^{20}=\left(\text{cis}\,\frac{4\pi}{3}\right)^{20}=\text{cis}\,\frac{40\pi}{3}=\text{cis}\,\frac{4\pi}{3}$

$z_1=\text{cis}\,0=1$
$z_2=\text{cis}\,\frac{2\pi}{3}=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$
$z_3=\text{cis}\,\frac{4\pi}{3}=\frac{1}{2}-i\frac{\sqrt{3}}{2}$

7. DERIVACIJE

- definicija: $f'(x)=\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$

Zad. 132 Koristeći def. deriv. izračunaj derivaciju funk. $f(x)=x^3+2x+3$

$f'(x)=\lim\limits_{\Delta x\to 0}\frac{(x+\Delta x)^3+2(x+\Delta x)+\cancel{3}-x^3-2x-\cancel{3}}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{\cancel{x^3}+3x^2\Delta x+3x\Delta x^2+\Delta x^3+\cancel{2x}+2\Delta x-\cancel{x^3}-\cancel{2x}}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{\Delta x(3x^2+3x\Delta x+\Delta x^2+2)}{\Delta x}$

$=\lim\limits_{\Delta x\to 0}3x^2+3\Delta x+\Delta x+2=3x^2+2$

osnovna pravila deriviranja:

$c'=0 \qquad (a^x)'=a^x\ln a \qquad (\log_a x)'=\frac{1}{x\ln a} \qquad (f\cdot g)'=f'\cdot g+f\cdot g'$

$(x^n)'=n\cdot x^{n-1} \qquad (\log x)'=\frac{1}{x\ln 10}$

Zad 133) $f(x)=2x^7+\sqrt[3]{x^5}+\frac{1}{\sqrt[3]{x}}+\underbrace{\ln\sqrt{3}}_{\text{KONSTANTA}}$

$f'(x)=14x^6+\frac{5}{3}\sqrt[3]{x^2}-\frac{1}{3x\sqrt[3]{x}}$

zad 134.) $f(x)=x^2 \cdot e^x \cdot \ln x$

$$f'(x)=2x\cdot e^x\cdot \ln x + x^2\cdot e^x \cdot \ln x + \frac{1}{x}\cdot x^2 \cdot e^x$$

zad 135.) $f(x)=\dfrac{\sqrt{x}+\sin x}{x\ln x}$

$$f'(x)=\frac{\left(\frac{1}{2\sqrt{x}}+\cos x\right)\cdot x\ln x-(\sqrt{x}+\sin x)\left(\ln x + x\cdot\frac{1}{x}\right)}{x^2\ln^2 x}$$

pr. 8) Łańcuchowe derivowanie:

$$f(g(h(x)))' = f'\cdot g'\cdot h'$$

zad 136) $f(x)=\cos^3(\pi x)$

$$f(x)=3\cos^2(\pi x)\cdot(-\sin(\pi x))\cdot(0\cdot x+\pi)=-3\pi\cdot\cos^2(\pi x)\cdot\sin(\pi x)$$

zad 137) $f(x)=\arcsin^2(e^{-x})$

$$f'(x)=2\arcsin(e^{-x})\cdot\frac{1}{\sqrt{1-e^{-2x}}}\cdot e^{-x}\cdot(-1)=-2\cdot\frac{e^{-x}\cdot\arcsin(e^{-x})}{\sqrt{1-e^{-2x}}}$$

zad 138) $f(x)=\cos\frac{x}{2}$

$f''''(x)=?$

$$f'(x)=-\sin\frac{x}{2}\cdot\frac{1}{2}$$

$$f''(x)=-\frac{1}{2}\cos\frac{x}{2}\cdot\frac{1}{2}=-\frac{1}{4}\cos\frac{x}{2}$$

$$f'''(x)=\frac{1}{4}\sin\frac{x}{2}\cdot\frac{1}{2}=\frac{1}{8}\sin\frac{x}{2}$$

$$f^{IV}(x)=\frac{1}{16}\cos\frac{x}{2}$$

zad) $f(x)=(2x+5)^{-3}$

$$f'(x)=-3(2x+5)^{-4}\cdot 2=-\frac{6}{(2x+5)^4}$$

zad 139) $f(x)=\sin\sqrt{1+e^{2x}}$, $f'(0)=?$

$$f'(x)=\cos\sqrt{1+e^{2x}}\cdot\frac{1}{2\sqrt{1+e^{2x}}}\cdot e^{2x}\cdot 2$$

$$f'(0)=\cos\sqrt{1+1}\cdot\frac{1}{2\sqrt{2}}\cdot 1\cdot 2=\frac{\cos\sqrt{2}}{\sqrt{2}}$$

zad 140) $f(x)=\sqrt{g(x)}$ $f'(1)=4$, $g'(1)=1$, $f(1)=?$

$$f'(x)=\frac{1}{2\sqrt{g(x)}}\cdot g'(x)$$

$$\boxed{f(1)=\frac{1}{8}}$$

$$f'(1)=\frac{1}{2\sqrt{g(1)}}\cdot g'(1)$$

$$4=\frac{1}{2\sqrt{g(1)}}\cdot 1 \quad \Big|\cdot\frac{\sqrt{g(1)}}{4}$$

$$\sqrt{g(1)}=\frac{1}{8}\ \Big|^2 \qquad g(1)=\frac{1}{64}$$

Zad 141) $f(x)=\frac{1}{\sqrt[3]{5^{x^2}}}=5^{-\frac{x^2}{3}}$

$$f'(x)=5^{-\frac{x^2}{3}}\cdot \ln 5\cdot\left(-\frac{2x}{3}\right)$$

$$\boxed{a^b=e^{b\ln a}}$$

Zad 142) $f(x)=\ln\frac{1+x}{1-x}$ $\quad f^{(5)}(0)=?$

$f(x)=\ln(1+x)+\ln(1-x)$

$f'(x)=\frac{1}{1+x}+\frac{1}{1-x}$ $\quad f'''(x)=\frac{2}{(1+x)^3}+\frac{2}{(1+x)^3}$ $\quad f^{(v)}(x)=\frac{24}{(1+x)^5}+\frac{24}{(1-x)^5}$

$f''(x)=-\frac{1}{(1+x)^2}+\frac{1}{(1-x)^2}$ $\quad f^{IV}(x)=\frac{-6}{(1+x)^4}+\frac{6}{(1-x)^4}$ $\quad \boxed{f^{V}(0)=48}$

LOGARITAMSKO DERIVIRANJE

Zad 143) $f(x)=x^{\sin x}\ /\ln$

$\ln f(x)=\ln x^{\sin x}$

$\ln f(x)=\sin x\cdot\ln x\ /'$

$\frac{1}{f(x)}\cdot f'(x)=\cos x\cdot\ln x+\sin x\cdot\frac{1}{x}\ /\cdot f(x)$

$f'(x)=\left(\cos x\cdot\ln x+\sin x\cdot\frac{1}{x}\right)\cdot f(x)$

pr.) $f(x)=(2x+1)^5\cdot(3x+6)^{10}\cdot(x+2)^{12}\ /\ln$

$\ln f(x)=5\ln(2x+1)+10\ln(3x+6)+12\ln(x+2)\ /'$

$\frac{1}{f(x)}\cdot f'(x)=\frac{5}{2x+1}+\frac{10}{3x+6}+\frac{12}{x+2}\ /\cdot f(x)$

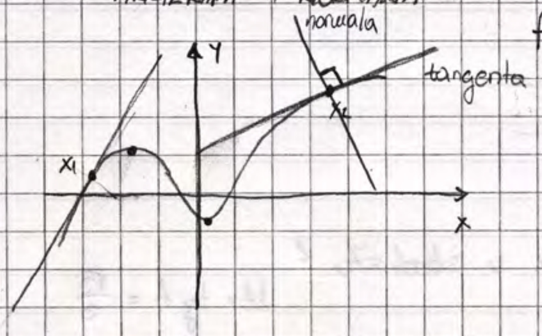
$f'(x)=f(x)\left(\frac{5}{2x+1}+\frac{10}{3x+6}+\frac{12}{x+2}\right)$

Zad 144) $f(x)=(\operatorname{arctg}x)^{\ln x}=e^{\ln x\cdot\ln(\operatorname{arctg}x)}$

$f'(x)=e^{\ln x\cdot\ln(\operatorname{arctg}x)}\cdot\left[\frac{1}{x}\ln(\operatorname{arctg}(x))+\ln x\cdot\frac{1}{\operatorname{arctg}x}\cdot\frac{1}{1+x^2}\right]=$

$=(\operatorname{arctg}x)^{\ln x}\cdot\left[\frac{\ln\operatorname{arctg}(x)}{x}+\frac{\ln x}{\operatorname{arctg}(x)(1+x^2)}\right]$

TANGENTA I NORMALA



$f'(x_1)>f'(x_2)$

$k=f'(x)$

$t\ldots\ y-y_0=k_t(x-x_0)$ $\quad k_t=y'(x_0)$

$n\ldots\ y-y_0=k_n(x-x_0)$ $\quad k_n=-\frac{1}{y'(x_0)}$

Zad. 145) Odredi jedžb. tang. i normale na funk. $y=\sqrt[3]{x^2-1}$ u točki

a) $x_0=0$

$y_0=-1$

norm: $x=0$

$y'=\frac{1}{3}\cdot(x^2-1)^{-\frac{2}{3}}\cdot 2x$

$k=y'(0)=0$

$y+1=0\cdot(x-0)$

tang: $y=-1$

b) $x_0=1$

$y_0=0$

$y'=\frac{1}{3}\cdot(1^2-1)^{-\frac{2}{3}}\cdot 2x=\frac{1}{3}\cdot\cancel{0^{\frac{2}{3}}}\cdot 2$

$-\lim_{x=1}y'=1$

tang: $x=1$

norm: $y=0$

Zad 146) Odredi jednadžbu tangente na funk. koja je okomita na pravac p

$f(x) = x^2 - x + 4$

$p: x + 3y + 1 = 0 \Rightarrow k_p = -\frac{1}{3} \quad k_t = 3$

$t: y - 6 = 3(x-2) \qquad x_0 = 2$

$\boxed{y = 3x} \qquad y_0 = 6$

U kojim je točkama na $y = 2e^{-x} + 1$ tang. paralelna s pravcem $2x + y = 2$

$y' = -2e^{-x} \qquad k_p = -2 \qquad k_t = \frac{1}{2}$

$e^{-x} = 1 \Rightarrow x = 0 \; ; y = 3 \qquad T(0, 3)$

Za koju vrijednost parametra b je $y = 7x + b$ će biti tangenta na krivulji $y = x^3 - 5x + 4$

$k_t = 7 = 3x^2 - 5 \qquad x_1 = 2 \quad y_1 = -9$

$x_2 = -2 \quad y_2 = -5$

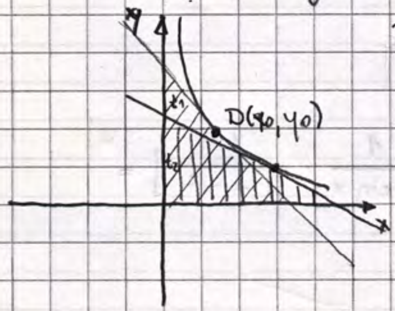
$3x^2 = 12 \,/ : 3 \qquad -9 = 14 + b \qquad -5 = -14 + 9$

$x^2 = 4 \,/ \sqrt{\ }$

$b_1 = -23 \qquad b_2 = 9$

$x = \pm 2$

Zad 147) Dokažite da sve tangente na krivulju $y = \frac{1}{x}$ u bilo kojoj točki u prvom kvadrantu zatvaraju s koordinatnim osima trokut jednake površine.



$P(t_1) = P(t_2) \qquad P = \frac{1}{2} \cdot m \cdot n$

$y' = -\frac{1}{x^2}$

$t: y - y_0 = k_t (x - x_0)$

$y - \frac{1}{x_0} = -\frac{1}{x_0^2} \cdot (x - x_0)$

$y - \frac{1}{x_0} = -\frac{x}{x_0^2} + \frac{1}{x_0}$

$\frac{x}{x_0^2} + y = \frac{2}{x_0} \,/ : \frac{2}{x_0}$

$\frac{x}{2x_0} + \frac{y}{\frac{2}{x_0}} = 1$

$m = 2x_0$

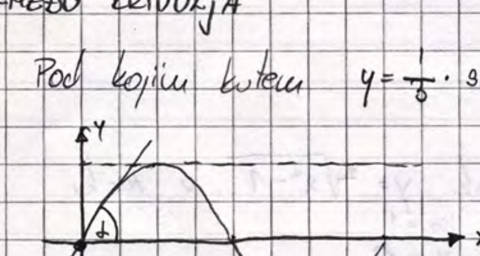
$n = \frac{2}{x_0}$

$P = \frac{1}{\cancel{2}} \cancel{2x_0} \cdot \frac{2}{\cancel{x_0}}$

$\boxed{P = 2}$

KUT IZMEĐU KRIVULJA

Zad. 148) Pod kojim kutem $y = \frac{1}{3} \cdot \sin(\sqrt{3}x)$ siječe os x u ishodištu?



$k_t = \operatorname{tg} \alpha$

$y' = \frac{1}{3} \cos(\sqrt{3}x) \cdot \sqrt{3}$

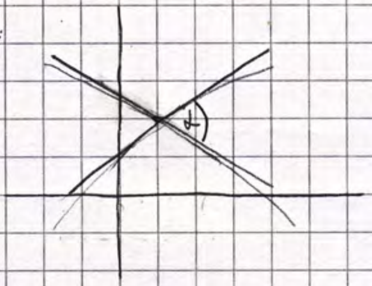
$y'(0) = \frac{1}{3} \cos 0 \cdot \sqrt{3} = \frac{\sqrt{3}}{3}$

$k_t = \operatorname{tg} \alpha = \frac{\sqrt{3}}{3}$

$\alpha = \frac{\pi}{6}$

Zad 149) Pod kojim kutom se sijeku $y_1 = 2-x$, $y_2 = \sqrt{x}$ i u kojoj točki? 47.

kut:



$tg\,\alpha = \left| \frac{k_1 - k_2}{1 + k_1 \cdot k_2} \right|$

p.u. $x > 0$

$2 - x > 0 \Rightarrow x < 2$

$2 - x = \sqrt{x} \;/^2$

$4 - 4x + x^2 = x$

$x^2 - 5x + 4 = 0$

$x_1 = 1$ $\quad \cancel{x_2 = 4}$

$k_1 = -1$

$k_2 = y_2'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$

$tg\,\alpha = \left| \frac{-1 - \frac{1}{2}}{1 + \frac{1}{2}} \right| = \left| \frac{-\frac{3}{2}}{\frac{1}{2}} \right| = 3$

$\alpha = arc\,tg\,3$

Zad 150) Za koju vrijednost parametra $a < 0$ se $y_1 = ax^2$ i $y_2 = a(x-2)^2$ sijeku pod pravim kutem

① $y_1' = 2ax$ $\quad y_2' = 2a(x-2)$

② $\cancel{a}x^2 = \cancel{a}(x-2)^2$

$x^2 = x^2 - 4x + 4$

$4x = 4 \;/:4$

$x = 1$

③ $k_1 = 2a$

$k_2 = -2a$

$k_1 = -\frac{1}{k_2}$

$2a = -\frac{1}{2a}$

$a^2 = \frac{1}{4} \;/\sqrt{\ }$

$\boxed{a = \frac{1}{2}}$

Zad 151) Odredite kut između tang. povučenih iz točke $(0, -3)$ povučene na parabolu $y = \frac{1}{4}x^2 - \frac{1}{2}x - \frac{3}{4}$

$y' = \frac{1}{2}x - \frac{1}{2}$

$y - y_0 = y'(x_0)(x - x_0)$

$-3 - y_0 = y'(x_0)(0 - x_0)$

$-3 - \left(\frac{1}{4}x_0^2 - \frac{1}{2}x_0 - \frac{3}{4}\right) = \left(\frac{1}{2}x_0 - \frac{1}{2}\right) \cdot (-x_0)$

$-3 - \frac{1}{4}x_0^2 - \frac{1}{2}x_0 - \frac{3}{4} = -\frac{1}{2}x_0^2 - \frac{1}{2}x_0 \;/\cdot 4$

$-12 - x_0^2 - \cancel{2x_0} - 3 = -2x_0^2 - \cancel{2x_0}$

$x_0^2 = 9 \;/\sqrt{\ }$

$x_{01} = 3$ $\quad y_{01} = 0$

$x_{02} = -3$ $\quad y_{02} = 3$

t_1: $y = x - 3$

t_2: $y = -2x - 3$

$tg\,\alpha = \left| \frac{-2 - 1}{1 + 2} \right| = 3$

$\alpha = arctg\,3$