FAKULTET
ELEKTROTEHNIKE I
RAČUNARSTVA

MATEMATIKA 1

PREDAVANJA

DOMAĆE ZADAĆE

ZAGREB, 2006.

Sadržaj:

1. MATEMATIČKA LOGIKA

Samo je jedan Mali Ivica!

1.1. UVOD

TEMELJNE OZNAKE

= oznake skupova:

1. $\mathbb{N} = \{1,2,3,\dots\}$ - skup prirodnih brojeva
2. $\mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}$ - skup cijelih brojeva
3. $\mathbb{Q} = \{\frac{a}{b} \mid a\in\mathbb{Z},\ b\in\mathbb{N}\}$ - skup racionalnih brojeva (razlomaka)
4. $\mathbb{R}$ - skup realnih brojeva
   ↳ geometrijski je predočen brojevnim pravcem

= oznake zbroja i umnoška:

1. $\sum_{i=1}^{n} a_i := a_1 + a_2 + \dots + a_n$ ($:=$ ČITAJ: po definiciji jednako)
2. $\prod_{i=1}^{n} a_i := a_1 \cdot a_2 \cdot \dots \cdot a_n$

1.2. SUDOVI

= Temeljni pojam matematičke logike je sud. Primjerice:

a: „$5>3$" -> istinit sud
b: „$4>10$" -> lažan sud
c: „KOLIKO JE SATI" -> nije sud
d: „$x>2$" -> nije sud (predikat)

= Sud - smislena izreka kojoj možemo pridijeliti vrijednost istinitosti:

1.) istinit (eng. true) -> $\top$, T, 1
2.) lažan (eng. false) -> $\bot$, F, 0

- oznaka vrijednosti suda: $t(a)$ ili $v(a)$.

LOGIČKE OPERACIJE

= Definiranjem temeljnih operatora kao sudova stvara se u matematičkoj logici algebra sudova.

= Osnovni operatori:

1. $\neg$ - negacija } unarni operator
2. $\wedge$ - konjunkcija
3. $\vee$ - disjunkcija
4. $\Rightarrow$ - implikacija
5. $\Leftrightarrow$ - ekvivalencija

} binarni operatori (2.–5.)

= 1. Unarni operator -> operator koji za sebe veže samo jednu varijablu ($\neg X$)
2. Binarni operator -> operator koji za sebe veže dvije varijable ($X\wedge Y, X\vee Y, X\Rightarrow Y, X\Leftrightarrow Y$)

= **DEFINICIJA 1.**

↳ Negacija ($\neg$, čitaj: „non, nije")
Sud $\neg X$ je istinit ako je X lažan, a lažan ako je X istinit.

↳ DOKAZ: tablica istinitosti operatora negacije ($\neg$)

X	¬X
T	F
F	T

ili

X	¬X
1	0
0	1

= DEFINICIJA 2.

↳ **Konjunkcija** ($\wedge$, čitaj: „i, et")

Sud $x \wedge y$ je istinit ako su x i y istiniti, inače je lažan.

↳ DOKAZ: tablica istinitosti operatora konjunkcije ($\wedge$)

X	Y	X∧Y
T	T	T
T	F	F
F	T	F
F	F	F

ili

X	Y	X∧Y
1	1	1
1	0	0
0	1	0
0	0	0

= DEFINICIJA 3.

↳ **Disjunkcija** ($\vee$, čitaj: „ili, vel")

Sud $x \vee y$ je lažan ako su x i y lažni, inače je istinit.

↳ DOKAZ: tablica istinitosti operatora disjunkcije ($\vee$)

X	Y	X∨Y
T	T	T
T	F	T
F	T	T
F	F	F

ili

X	Y	X∨Y
1	1	1
1	0	1
0	1	1
0	0	0

= Uočimo da vrijedi:

1. $t(\neg x) = 1 - t(x)$
2. $t(x \wedge y) = t(x) \cdot t(y)$
3. $t(x \vee y) = t(x) + t(y) - t(x) \cdot t(y)$

= DEFINICIJA 4.

↳ **Implikacija** ($\Rightarrow$, čitaj: „x implicira y"
„x povlači y"
„iz x slijedi y"
„ako je x onda je y"
„x je dostatan uvjet za y"
„y je nuždan uvjet za x")

Sud x implicira y je lažan ako je X istinit, a Y lažan, inače je istinit

↳ DOKAZ: tablica istinitosti operatora implikacije ($\Rightarrow$)

X	Y	X⇒Y
1	1	1
1	0	0
0	1	1
0	0	1

↳ Ako imamo oznaku $x \Rightarrow y$, x se naziva pretpostavka (antecedenta), a y zaključak (konsekventa).

= PRIMJER 1.

$(x=2) \Rightarrow (x^2=4)$

$x=2$ je dostatan uvjet za $x^2=4$

$x^2=4$ je nuždan uvjet za $x=2$

= DEFINICIJA 5.

↳ **Ekvivalencija** ili obostrana implikacija ($\Leftrightarrow$, čitaj: „x ekvivalentno y"
„x je nuždan i dostatan uvjet za y")

Sud X ekvivalentno Y je istinit sud kada oba suda imaju iste istinitosti, tj. $t(x) = t(y)$.

↳ DOKAZ: tablica istinitosti operatora ekvivalencije ($\Leftrightarrow$)

X	Y	X⇔Y
1	1	1
1	0	0
0	1	0
0	0	1

Samo je jedan Mali Ivica!

= Može se pokazati da se svaki od operatora može izraziti samo s 2 operatora (negacijom (¬) i konjunkcijom (∧)).

= S pomoću logičkih operacija algebre sudova tvorimo složene izraze (formule) služeći se pri tome zagradama (,) i operacijama +, ·.

= PRIMJER 2.

Napiši tablicu istinitosti sljedeće formule: $(\neg a \vee b) \Rightarrow \neg(a \wedge c)$

a	b	c	¬a	¬a∨b	a∧c	¬(a∧c)	L(a,b,c)
1	1	1	0	1	1	0	0
1	1	0	0	1	0	1	1
1	0	1	0	0	1	0	1
1	0	0	0	0	0	1	1
0	1	1	1	1	0	1	1
0	1	0	1	1	0	1	1
0	0	1	1	1	0	1	1
0	0	0	1	1	0	1	1

= Funkcije u kojima su jedine vrijednosti varijabla (i ulaznih i izlaznih) 1 i 0 zovu se Booleove (logičke) funkcije.

= DEFINICIJA 6.

↳ Ako dvije formule, $A(x_1, \dots, x_n)$ i $B(x_1, \dots, x_n)$, imaju istu Booleovu funkciju kažemo da su istovrijedne i to zapisujemo: $A(x_1, \dots, x_n) \equiv B(x_1, \dots, x_n)$.

= PRIMJER 3.

Provjeri istovrijednost formule: $x \Rightarrow y \equiv \neg(x \wedge \neg y) \equiv \neg x \vee y$

x	y	x⇒y	¬y	x∧¬y	¬(x∧¬y)	¬x	¬x∨y
1	1	1	0	0	1	0	1
1	0	0	1	1	0	0	0
0	1	1	0	0	1	1	1
0	0	1	1	0	1	1	1

= PRIMJER 4.

Provjeri istovrijednost formule: $x \Leftrightarrow y \equiv (x \Rightarrow y) \wedge (y \Rightarrow x)$

x	y	x⇔y	x⇒y	y⇒x	L(x,y)
1	1	1	1	1	1
1	0	0	0	1	0
0	1	0	1	0	0
0	0	1	1	1	1

= Zamjenom formula ili njihovih dijelova s istovrijednim možemo ih transformirati u jednostavnije funkcije koje imaju istu Booleovu funkciju.

= STAVAK 1.

↳ Neke korisne formule i dokazi:

1.) $\neg(\neg x) \equiv x$ → dvostruka negacija

x	¬x	¬(¬x)
1	0	1

2.) $X \wedge X \equiv X$
$X \vee X \equiv X$

X	$X\wedge X$	$X\vee X$
1	1	1
0	0	0

3.) $X \wedge Y \equiv Y \wedge X$ → komutativnost
$X \vee Y \equiv Y \vee X$

X	Y	$X\wedge Y$	$Y\wedge X$	$X\vee Y$	$Y\vee X$
1	1	1	1	1	1
1	0	0	0	1	1
0	1	0	0	1	1
0	0	0	0	0	0

4.) $X \wedge (Y \wedge Z) \equiv (X \wedge Y) \wedge Z$
$X \vee (Y \vee Z) \equiv (X \vee Y) \vee Z$ → asocijativnost

X	Y	Z	$Y\wedge Z$	$X\wedge(Y\wedge Z)$	$X\wedge Y$	$(X\wedge Y)\wedge Z$	$Y\vee Z$	$X\vee(Y\vee Z)$	$X\vee Y$	$(X\vee Y)\vee Z$
1	1	1	1	1	1	1	1	1	1	1
1	1	0	0	0	1	0	1	1	1	1
1	0	1	0	0	0	0	1	1	1	1
1	0	0	0	0	0	0	0	1	1	1
0	1	1	1	0	0	0	1	1	1	1
0	1	0	0	0	0	0	1	1	1	1
0	0	1	0	0	0	0	1	1	0	1
0	0	0	0	0	0	0	0	0	0	0

5.) $X \wedge (Y \vee Z) \equiv (X \wedge Y) \vee (X \wedge Z)$
$X \vee (Y \wedge Z) \equiv (X \vee Y) \wedge (X \vee Z)$ - distributivnost

X	Y	Z	$Y\vee Z$	$X\wedge(Y\vee Z)$	$X\wedge Y$	$X\wedge Z$	$L(x,y,z)$	$Y\wedge Z$	$X\vee(Y\wedge Z)$	$X\vee Y$	$X\vee Z$	$L(x,y,z)$
1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1	0	1	1	1	1
1	0	1	1	1	0	1	1	0	1	1	1	1
1	0	0	0	0	0	0	0	0	1	1	1	1
0	1	1	1	0	0	0	0	1	1	1	1	1
0	1	0	1	0	0	0	0	0	0	1	0	0
0	0	1	1	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0

6.) $\neg(X \wedge Y) \equiv \neg X \vee \neg Y$
$\neg(X \vee Y) \equiv \neg X \wedge \neg Y$ - De Morganove formule

X	Y	$X\wedge Y$	$\neg(X\wedge Y)$	$\neg X$	$\neg Y$	$\neg X\vee\neg Y$	$X\vee Y$	$\neg(X\vee Y)$	$\neg X\wedge\neg Y$
1	1	1	0	0	0	0	1	0	0
1	0	0	1	0	1	1	1	0	0
0	1	0	1	1	0	1	1	0	0
0	0	0	1	1	1	1	0	1	1

7.) $X \Leftrightarrow Y \equiv (X \Rightarrow Y) \wedge (Y \Rightarrow X)$

→ Tablica u primjeru 4.

8.) $X \Rightarrow Y \equiv (\neg Y) \Rightarrow (\neg X)$ → obrat po kontrapoziciji

X	Y	$X\Rightarrow Y$	$\neg X$	$\neg Y$	$\neg Y \Rightarrow \neg X$
1	1	1	0	0	1
1	0	0	0	1	0
0	1	1	1	0	1
0	0	1	1	1	1

9.) $\neg(X \Rightarrow Y) \equiv X \wedge \neg Y$

X	Y	$X\Rightarrow Y$	$\neg(X\Rightarrow Y)$	$\neg Y$	$X\wedge\neg Y$
1	1	1	0	0	0
1	0	0	1	1	1
0	1	1	0	0	0
0	0	1	0	1	0

= DEFINICIJA 7.

↳ Ako je neka formula istinita za sve slogove kažemo da je identički istinita ili **tautologija** (pišemo $\equiv T$), u suprotnom je identički lažna ($\equiv F$). Identički istinite formule temelj su logičkog zaključivanja.

= STAVAK 2.

↳ Neke korisne i važnije tautologije i dokazi:

1.) $(X \wedge (X \Rightarrow Y)) \Rightarrow Y$ → modus ponens

X	Y	$X\Rightarrow Y$	$X\wedge(X\Rightarrow Y)$	$L(x,y)$
1	1	1	1	1
1	0	0	0	1
0	1	1	0	1
0	0	1	0	1

2.) $((X \Rightarrow Y) \wedge \neg Y) \Rightarrow \neg X$ → modus tollens

X	Y	$X\Rightarrow Y$	$\neg Y$	$(X\Rightarrow Y)\wedge\neg Y$	$\neg X$	$L(X,Y)$
1	1	1	0	0	0	1
1	0	0	1	0	0	1
0	1	1	0	0	1	1
0	0	1	1	1	1	1

3.) $((\neg X \Rightarrow Y) \wedge \neg Y) \Rightarrow X$ → dokaz iz protuslovlja

X	Y	$\neg X$	$\neg X\Rightarrow Y$	$\neg Y$	$(\neg X\Rightarrow Y)\wedge\neg Y$	$L(Y,Y)$
1	1	0	1	0	0	1
1	0	0	1	1	1	1
0	1	1	1	0	0	1
0	0	1	0	1	0	1

4.) $((X \Rightarrow Y) \land (Y \Rightarrow Z)) \Rightarrow (X \Rightarrow Z)$ → zakon silogizma

X	Y	Z	X⇒Y	Y⇒Z	1∧2	X⇒Z	L(x,y,z)
1	1	1	1	1	1	1	1
1	1	0	1	0	0	0	1
1	0	1	0	0	0	1	1
1	0	0	0	1	0	0	1
0	1	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	0	1	1	0	0	1	1
0	0	0	1	1	1	1	1

Samo je jedan Mali Ivica!

= Uz dogovor o jačem vezivanju konjunkcije ($\land$) i disjunkcije ($\lor$) od implikacije ($\Rightarrow$) formule se pišu bez zagrada.

ZADACI

① Dokaži da je $X \land (X \Rightarrow Y) \Rightarrow Y$ (modus ponens) identički istinita.
↳ VIDJETI: STAVAK 2.1.

② Dokaži da je $(X \Rightarrow Y) \land \neg Y \Rightarrow \neg X$ (modus tollens) identički istinita.
↳ VIDJETI: STAVAK 2.2.

③ Dokaži da je $(\neg X \Rightarrow Y) \land \neg Y \Rightarrow X$ (dokaz iz protuslovlja) identički istinita.
↳ VIDJETI: STAVAK 2.3.

④ Dokaži da je $(X \Rightarrow Y) \land (Y \Rightarrow Z) \Rightarrow (X \Rightarrow Z)$ (zakon silogizma) identički istinita.

1.3. PREDIKATI

= Skup svih promatranih objekata zvat ćemo univerzalnim skupom, i označavati ćemo ga sa $\mathcal{U}$ (ili M, ili Ω).

= DEFINICIJA 1.
↳ Sudna forma $P(x_1, x_2, \dots, x_n)$ koja za svaki slog elemenata $x_1, x_2, \dots, x_n \in \mathcal{U}$ postaje sud zove se n-mjesni **predikat** na $\mathcal{U}$.
↳ Skup svih elemenata univerzalnog skupa za koje je predikat $P(x)$ istinit zove se **karakterističan skup** predikata $P(x)$, i označava se sa $\mathcal{C}_P$.

= PRIMJER 1.
Neka je $\mathcal{U}$ zadan skup $\mathcal{U} = \{1, 3, 5\}$. Odredi karakterističan skup za $P(x) \equiv x < 3$.

$\mathcal{C}_P = \{1\}$.

= PRIMJER 2.
Neka je $\mathcal{U}$ zadan skup $\mathcal{U} = \{1, 3, 5\}$. Odredi karakterističan skup za $P(x_1, x_2) \equiv x_1 < x_2$.

$\mathcal{C}_P = \{(1,3), (1,5), (3,5)\}$.

= Iz predikata tvore se složeni predikati logičkim operacijama algebre sudova čime nastaje algebra predikata.

= PRIMJER 3.
Neka je $P(x) \equiv$ „$x \geq 2$", $Q(x) \equiv$ „$x \leq 5$", a $\mathcal{U} = \mathbb{R}$.
Definiramo novi predikat: $R(x) := P(x) \land Q(x) := (x \geq 2) \land (x \leq 5)$.
Za taj novi predikat $\mathcal{U} \in \mathbb{N}$ pa vrijedi $\mathcal{C}_R = \{2, 3, 4, 5\}$.

Samo je jedan Mali Ivica!

= DEFINICIJA 2.

↳ Kažemo da je predikat $P(x_1, x_2, \dots, x_n)$ identički istinit ako je istinit za sve slogove elemenata iz $\mathcal{U}$. To zapisujemo: $P(x_1, x_2, \dots, x_n) \equiv T$.

Kažemo da je predikat $P(x_1, x_2, \dots, x_n)$ identički lažan ako je lažan za sve slogove elemenata iz $\mathcal{U}$. To zapisujemo: $P(x_1, x_2, \dots, x_n) \equiv F$.

- DEFINICIJA 3.

↳ Neka je $P(x)$ jednomjesni predikat na $\mathcal{U}$.

1.) Sud $\forall x\, P(x)$ (čitaj: za svaki x je P od x) je istinit onda i samo onda ako je $P(x)$ identički istinit predikat, tj. $P(x) \equiv T$.

↳ Simbol $\forall$ (eng. all - svi) zove se univerzalni kvantifikator (kvant).

2.) Sud $\exists x\, P(x)$ (čitaj: postoji x takav da je P od x) je istinit onda i samo onda ako je $P(x)$ identički lažan predikat, tj. $P(x) \equiv F$.

↳ Simbol $\exists$ (eng. exist - postoji) zove se egzistencijalni kvantifikator (kvant)

↳ tablice istinitosti:

1) $\forall x\ P(x) \rightarrow$ istina

$x \in \mathcal{U}$	$P(x)$
x_1	T
x_2	T
⋮	⋮
⋮	⋮
⋮	⋮
x_n	T

2) $\exists x\ P(x) \rightarrow$ istina

$x \in \mathcal{U}$	$P(x)$
x_1	
x_2	
⋮	
⋮	T
⋮	
x_n	

= STAVAK 1.

↳ Iz definicije 3. direktnim zaključivanjem slijedi:

$$\left.\begin{array}{l} 1.)\ \neg \forall x\ P(x) \equiv \exists x\ \neg P(x) \\ 2.)\ \neg \exists x\ P(x) \equiv \forall x\ \neg P(x) \end{array}\right\} \text{De Morganove formule za kvantifikatore}$$

3.) $\forall x\ P(x) \Rightarrow \exists x\ P(x)$

= Očevidno je da ako u n-mjesnom predikatu $P(x_1, \dots, x_n)$ fiksiramo m objekata među $x_1, x_2, \dots, x_n$ dobivamo $(n-m)$-mjesni predikat.

Nulmjesni predikati su sudovi, pa i obratno, sudove možemo smatrati nulmjesnim predikatima.

= PRIMJER 4.

Neka je $\mathcal{U} = \{2, 3, 4\}$, $P(x,y)$: „$x \geq y$", $Q(x,y)$: „$x \mid y$". Nađi tablicu istinitosti za $P(x,y)$, $Q(x,y)$, $\forall x\ P(x,y)$, $\exists x\ P(x,y)$.

1. tablica istinitosti za $P(x,y)$:

x \ y	2	3	4
2	1	0	0
3	1	1	0
4	1	1	1

2. tablica istinitosti za $Q(x,y)$:

x \ y	2	3	4
2	1	0	1
3	0	1	0
4	0	0	1

3. tablica istinitosti za $\forall x P(x,y)$

↳ x - vezana varijabla pa:

$$\forall x\ P(x,y) \equiv P_1(y)$$

y	$P_1(y)$
2	1
3	0
4	0

4. tablica istinitosti za $\exists x\ P(x,y)$

↳ x - vezana varijabla pa:

$$\exists x\ P(x,y) \equiv P_2(y)$$

y	$P_2(y)$
2	1
3	1
4	1

= ZADATAK 1.

Napraviti tablicu istinitosti ako vrijede pravila iz prošlog zadatka za:
$R(x,y) \equiv P(x,y) \wedge \neg Q(x,y)$.

$P(x,y)$:

x \ y	2	3	4
2	1	0	0
3	1	1	0
4	1	1	1

$\wedge$

$\neg Q(x,y)$:

x \ y	2	3	4
2	0	1	0
3	1	0	1
4	1	1	0

$\equiv$

$R(x,y)$:

x \ y	2	3	4
2	0	0	0
3	1	0	0
4	1	1	0

= Ponovnom primjenom kvantifikatora na slobodnu varijablu nastaju od jednomjesnih predikata sudovi.

= PRIMJER 5.

Ako uzmemo podatke iz prošlog primjera dobivamo:

$\exists y \ \forall x \ P(x,y) \equiv T$
$\forall y \ \forall x \ P(x,y) \equiv F$
$\exists y \ \exists x \ P(x,y) \equiv T$
$\forall y \ \exists x \ P(x,y) \equiv T$

Samo je jedan Mali Ivica!

ZADACI (DZ 1. PRVI DIO)

5. a) $(\exists x \in \mathbb{N})(x^2 = 4) \equiv T$

- Rješenja jednadžbe $x^2=4$ su -2 i 2, pa postoji takav $x \in \mathbb{N}$ (2) za kojeg vrijedi dana jednakost, odnosno sud je istinit.

b) $(\exists x \in \mathbb{Q})(x^2 = 3) \equiv F$

- Rješenja jednadžbe $x^2=3$ su $-\sqrt{3}$ i $\sqrt{3}$, a to su iracionalni brojevi, pa ne postoji niti jedan $x \in \mathbb{Q}$ za kojeg bi vrijedila dana jednakost, odnosno sud je lažan.

c) $(\forall x, y \in \mathbb{R})(x^2 = y^2 \Rightarrow x = y) \equiv F$

- Rješenje jednadžbe $x^2=y^2$ je $|x|=|y|$, odnosno imamo 4 rješenja u skupu $\mathbb{R}$: $x=y$, $x=-y$, $-x=-y$, $-x=y$, što znači da dana implikacija ne vrijedi jer x i y očito ne moraju biti jednaki, odnosno sud je lažan.

d) $(\forall x, y \in \mathbb{R})(x^2 < y^2 \Longleftrightarrow |x| < |y|) \equiv T$

- Da bi dokazali da je ova ekvivalencija istinita moramo ju rastaviti na 2 implikacije.
  1. $x^2<y^2 \Rightarrow |x|<|y|$. Ako vrijedi da je $x^2<y^2$, onda logičkim zaključivanjem uviđamo da onda vrijedi da je $|x|<|y|$, $\forall x,y \in \mathbb{R}$.
  2. $|x|<|y| \Rightarrow x^2<y^2$. Ako vrijedi da je $|x|<|y|$, onda ponovno logičkim zaključivanjem uviđamo da onda vrijedi da je $x^2<y^2$ $\forall x,y \in \mathbb{R}$.

Kako vrijede oba slučaja, onda zadani sud mora biti istinit.

6. a.) $(\forall a \in \mathbb{Z})(\forall b \in \mathbb{Z})(\exists x \in \mathbb{Z})\ x^2+ax=b \equiv F$

- $x^2+ax-b=0$

$x_{1,2} = \dfrac{-a \pm \sqrt{a^2+4b}}{2}$

$(\forall a \in \mathbb{Z})(\forall b \in \mathbb{Z})\ a^2+4b>0$

- Kako a i b mogu biti negativni onda i izraz može biti negativan, tj. izraz ne vrijedi $\forall b \in \mathbb{Z}$.

b) $(\forall a \in \mathbb{Z})(\forall b \in \mathbb{Z})(\exists x \in \mathbb{R})\ x^2+ax=b \equiv F$

- $x^2+ax-b=0$

$x_{1,2} = \dfrac{-a \pm \sqrt{a^2+4b}}{2}$

$(\forall a \in \mathbb{Z})(\forall b \in \mathbb{Z})\ a^2+4b>0$

- Kako a i b mogu biti negativni onda i izraz može biti negativan, tj. izraz ne vrijedi $\forall b \in \mathbb{Z}$.

(7) a) $(\forall a\in\mathbb{Z})(\exists b\in\mathbb{Z})(\exists x\in\mathbb{R})\ x^2+ax=b \equiv T$

$-\ x^2+ax-b=0$ $\qquad(\forall a\in\mathbb{Z})(\exists b\in\mathbb{Z})\ a^2+4b>0$

$x_{1,2}=\frac{-a\pm\sqrt{a^2+4b}}{2}$ – Izraz vrijedi jer je a uvijek pozitivan, a za neki b će i izraz biti pozitivan.

b) $(\exists a\in\mathbb{Z})(\forall b\in\mathbb{Z})(\exists x\in\mathbb{R})\ x^2+ax-b \equiv F$

$-\ x^2+ax-b=0$ $\qquad(\exists a\in\mathbb{Z})(\forall b\in\mathbb{Z})\ a^2+4b>0$

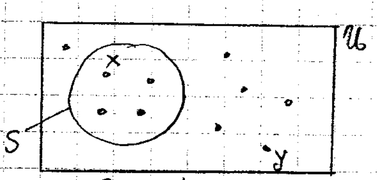
$x_{1,2}=\frac{-a\pm\sqrt{a^2+4b}}{2}$ – Kako je a uvijek pozitivan, a b može biti negativan, izraz ne vrijedi $\forall b\in\mathbb{Z}$.

$a^2+4b\geq 0$

1.4. SKUPOVI I PRESLIKAVANJA

= Neka je $\mathcal{U}$ univerzalni skup. Kažemo da je na $\mathcal{U}$ definiran skup S, ako se za svaki objekt x iz $\mathcal{U}$ znade pripada li skupu S, oznakom $x\in S$, ili mu ne pripada, oznakom $x\notin S$.

Simbolički to možemo predočiti Vennovim dijagramom skupa $\mathcal{U}$:



$x\in S,\ y\notin S$

= Svakom skupu S možemo pridružiti njegov predikat pripadanja $S(x)$, koji je istinit za one x koji pripadaju skupu S, a lažan za ostale. S je dakle karakteristični skup svoga predikata, tj. $S=\mathscr{C}_{S(x)}$. To često zapisujemo: $S=\{x\mid x\in S\}$.

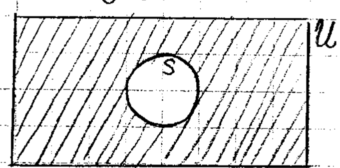
= DEFINICIJA 1.

↳ Neka su $S_1, S_2, S \in \mathcal{U}$. Definiramo operacije na skupovima.

1.) komplement (dopunak) skupa S $(\overline{S})$

↳ $\overline{S} := \{x \mid x\notin S\} = \{x \mid \neg(x\in S)\}$

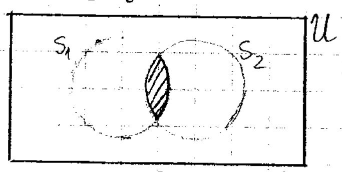
↳ Vennov dijagram:



2.) presjek skupova S_1 i S_2 $(S_1\cap S_2)$

↳ $S_1\cap S_2 := \{x \mid x\in S_1 \wedge x\in S_2\}$

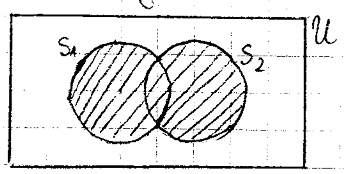
↳ Vennov dijagram:



3.) unija skupova S_1 i S_2 $(S_1\cup S_2)$

↳ $S_1\cup S_2 := \{x \mid x\in S_1 \vee x\in S_2\}$

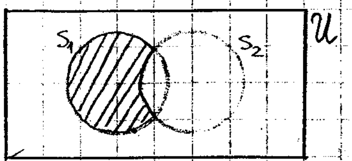
↳ Vennov dijagram:



4.) **razlika** između skupa S_1 i S_2 $(S_1 \setminus S_2)$

↳ $S_1 \setminus S_2 := \{x \mid x \in S_1 \wedge x \notin S_2\}$

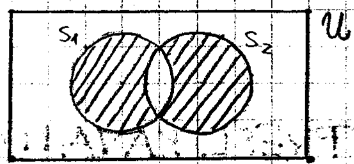
↳ Vennov dijagram:



5.) **simetrična razlika** između skupa S_1 i S_2 $(S_1 \Delta S_2)$

↳ $S_1 \Delta S_2 := (S_1 \setminus S_2) \cup (S_2 \setminus S_1)$

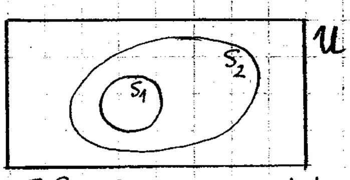
↳ Vennov dijagram:



6.) **relacije** među skupovima ($S_1 \subseteq S_2$ - podskup)

↳ $S_1 \subseteq S_2 \equiv \forall x \, (x \in S_1 \Rightarrow x \in S_2)$

↳ Vennov dijagram:



↳ $S_1 \subset S_2 \Rightarrow$ pravi podskup

= STAVAK 1.

↳ Među skupovima vrijedi:

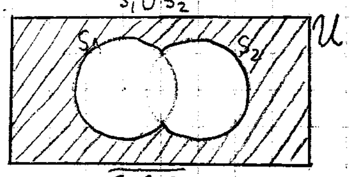
1.) $S_1 = S_2 \Leftrightarrow (S_1 \subseteq S_2) \wedge (S_2 \subseteq S_1)$

2.) $S_1 \subseteq S_2 \Leftrightarrow S_1 \cup S_2 = S_2 \Leftrightarrow S_1 \cap S_2 = S_1$

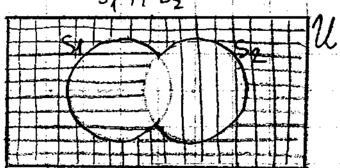
3.) $\left.\begin{array}{l} \overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2} \\ \overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2} \end{array}\right\}$ De Morganove formule za skupove

↳ DOKAZ:

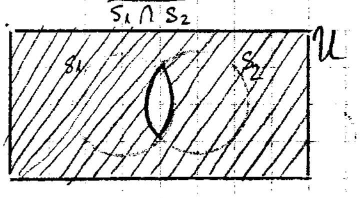
3.1. $\overline{S_1 \cup S_2}$



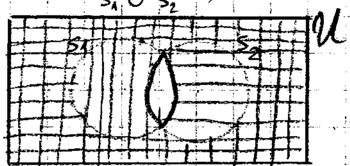
$\overline{S_1} \cap \overline{S_2}$



3.2. $\overline{S_1 \cap S_2}$



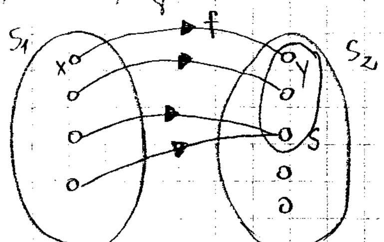
$\overline{S_1} \cup \overline{S_2}$



□ PRESLIKAVANJA

= DEFINICIJA 2.

↳ Neka su S_1 i S_2 skupovi na $\mathcal{U}$. Kažemo da je f - funkcija sa skupa S_1 u S_2, oznakom $f: S_1 \mapsto S_2$, ako je svakom $x \in S_1$ pridružen nekim propisom f jednoznačno određeni $y \in S_2$, oznakom: $f: x \mapsto f$, tj. $y = f(x)$.



- Vennov dijagram funkcije f:

S_1 = domena (područje definicije) - $\mathcal{D}_f$

S_2 = kodomena

x = original

y = f-slika originala x

S = slika skupa S_1 (područje vrijednosti)

$= f(S_1) = f(\mathcal{D}_f) := \{f(x) \mid x \in S_1\}$

= Realne funkcije realne varijable su takve funkcije kod kojih je $S_1 \subseteq \mathbb{R}$, a $S_2 = \mathbb{R}$.

= DEFINICIJA 3.

↳ Neka je $f: S_1 \mapsto S_2$, tj. $\mathcal{D}_f = S_1$ i $f(\mathcal{D}) \subseteq S_2$.

1.) Funkcija je **surjekcija** ako je $f(\mathcal{D}) = S_2$, tj. ako vrijedi:
$\forall y \in S_2 \;\; \exists x \in S_1 \;\; f(x) = y$.
Takvo preslikavanje je preslikavanja sa S_1 na S_2, a ne u S_2.

2.) Funkcija je **injekcija** ako različiti originali imaju različite slike, tj. ako vrijedi:
$\forall x_1, x_2 \in S_1 \;\; (x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2))$.
= Zbog poznate tautologije vrijedi:
$\forall x_1, x_2 \in S_1 \;\; (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$.

3.) Funkcija je **bijekcija** (ili 1-1 preslikavanje) ako je funkcija i injekcija i surjekcija zajedno.

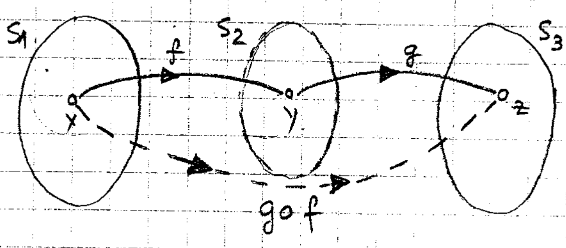
= DEFINICIJA 4.

↳ Neka je $f: S_1 \mapsto S_2$, $X \subseteq S_1$. Ondo je $f(X) := \{f(x) \mid x \in X\}$ f-slika podskupa X. Pa je onda $f(X) \subseteq S_2$. Ograničimo li se u djelovanju f na podskup X, dobivamo novo preslikavanje - **restrikciju** (suženje preslikavanja) f na X, oznakom $f|_X$. Vrijedi:
$f|_X : X \mapsto S_2$, $(f|_X)(x) = f(x) \;\; \forall x \in X$.

□ KOMPOZICIJA FUNKCIJA

= DEFINICIJA 5.

↳ Neka su S_1, S_2, S_3 skupovi, te $f: S_1 \mapsto S_2$, $g: S_2 \mapsto S_3$. **Kompozicija** tih funkcija (sastavak), oznakom $g \circ f : S_1 \mapsto S_3$, definiramo izrazom:
$\forall x \in S_1 \;\; (g \circ f)(x) := g[f(x)]$.



= Svojstva kompozicija funkcija:

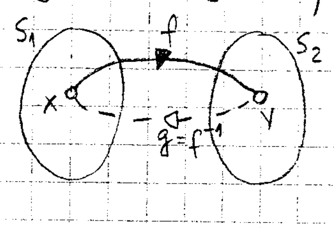
1.) Kompozicija bijekcija je opet bijekcija.

2.) Kompozicija funkcija je asocijativna:
$(h \circ g) \circ f = h \circ (g \circ f)$

3.) Bijekcija $f: S \mapsto S$ zove se permutacija skupa S, ili bijekcija na S.
Permutacija $id_S : S \to S$, $\forall x \in S \;\; id_S(x) = x$, koja svaki x iz S preslikava u njega samoga zove se identičko preslikavanje ili identitet na S.

= DEFINICIJA 6.

↳ Neka je $f: S_1 \mapsto S_2$. Preslikavanje $g: S_2 \mapsto S_1$ zove se **inverzno preslikavanje** od f (inverzna funkcija od f), ako je:
$\forall x \in S_1 : g[f(x)] = x$, tj. $g \circ f = id_{S_1}$
$\forall y \in S_2 : f[g(x)] = y$, tj. $f \circ g = id_{S_2}$



- STAVAK 2.
  - Za preslikavanje $f: S_1 \mapsto S_2$ opstoji inverzno preslikavanje $g: S_2 \mapsto S_1$ onda i samo onda, ako je f bijekcija. U tom slučaju je i g bijekcija. Inverzno preslikavanje je jedincato i označujemo ga f^{-1}.
- STAVAK 3.
  - Ako su $f: S_1 \mapsto S_2$ i $g: S_2 \mapsto S_3$ bijekcije, onda je: $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
  - DOKAZ:

    1. $(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g = (g \circ id_{S_2}) \circ g^{-1} = g \circ g^{-1} = id_{S_2}$.

    2. $(f^{-1} \circ g^{-1}) \circ (g \circ f) = id_{S_1}$.

    1. i 2. $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

ZADACI (DZ. 1. DRUGI DIO)

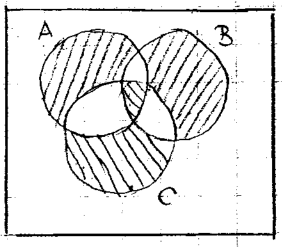
8. a) $A = \{1,2,3,4,5,6\}$
$B = \{3,4,5,6,7,8\}$
$C = \{2,4,5,8,9\}$

$(A \triangle B) \triangle C$
$\hookrightarrow D = (A \setminus B) \cup (B \setminus A) = \{1,2,7,8\}$
$RJ = (D \setminus C) \cup (C \setminus D) = \{1,4,5,7,9\}$

$A \triangle (B \triangle C)$
$\hookrightarrow D = (B \setminus C) \cup (C \setminus B) = \{2,3,6,7,9\}$
$RJ = (A \setminus D) \cup (D \setminus A) = \{1,4,5,7,9\}$

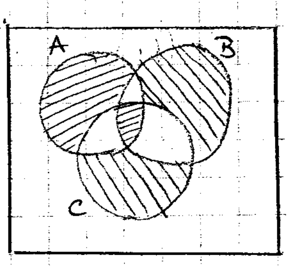
b) $(A \triangle B) \triangle C = A \triangle (B \triangle C)$

1.



=

2.

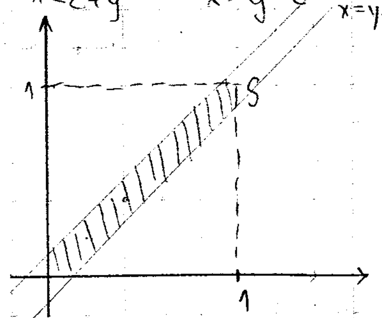


9. $\{x \in \mathbb{Z} \mid (\exists y \in \mathbb{Z})\ |x| + |y| = 5\}$ $\quad x = \{-5, -4, \dots, 4, 5\}$

$\to (-5,0), (-4,-1), (-4,1), (-3,-2), (-3,2), (-2,-3), (-2,3), (-1,-4), (-1,4), (0,-5), (0,5), (1,-4), (1,4), (2,-3), (2,3), (3,-2), (3,2), (4,-1), (4,1), (5,0)$.

10. $|x-y| < \varepsilon$, ε mali broj. Graf:

1° $x - y < \varepsilon$ $\quad$ 2° $x - y > -\varepsilon$
$x < \varepsilon + y$ $\quad$ $x > y - \varepsilon$



*) Za jako mali ε, to postaje pravac pa vrijedi:

$\Rightarrow$ Funkcija je simetrična, jer je $\forall x \forall y\ x \rho y$, odnosno ako je x u relaciji s y i y je u relaciji sa x.

$\Rightarrow$ Funkcija je refleksivna, jer je $\forall x\ x \rho x$, odnosno svaki x je u relaciji sa samim sobom.

$\Rightarrow$ Funkcija nije tranzitivna
$|x-y| < \varepsilon$ i $|y-z| < \varepsilon \not\Rightarrow |x-z| < \varepsilon$
što se dobiva namještanjem brojeva.

Samo je jedan Mali Ivica!

Samo je jedan Mali Ivica!

(11.) $f(x)=2x^2+4$
$g(x)=x+1$

$$f[g(x)]-g[f(x)] = 2(x+1)^2+4-(2x^2+4+1)=$$
$$=\cancel{2x^2}+4x+2+\cancel{4}-\cancel{2x^2}-\cancel{4}-1=$$
$$=4x+1$$

(12.) $f(x)=\frac{x}{x+1}$

$g(x)=1-\frac{1}{x}$

$$g[f(x+1)] = 1-\frac{1}{\frac{x+1}{x+2}} = 1-\frac{x+2}{x+1} = \frac{\cancel{x}+1-\cancel{x}-2}{x+1} = -\frac{1}{x+1}$$

(13.) $A=\{1,2,3\}$
$B=\{3,4,5,6\}$
$C=\{2,4,5\}$

a) $f: A \mapsto B$
Funkcija nije surjekcija, jer postoji bar 1 $y \in B$ koji nema sliku u A.
Funkcija je injekcija, jer različiti $x \in A$ imaju različite slike.

b) $f: B \mapsto C$
Funkcija nije injekcija, jer različiti $x \in A$ nemaju različite slike.
Funkcija je surjekcija, jer svaki $y \in C$ ima svoju sliku u B.

c) $f: A \mapsto C$.
Funkcija je bijekcija, jer je i injekcija i surjekcija istovremeno.

(14.) a) $f: \mathbb{N} \to \mathbb{Q}$, $f(x)=\frac{1}{x^2}$

= Funkcija nije bijekcija jer se 2 različita x preslikaju u isti y.

b) $f: \mathbb{R}_0^+ \to \mathbb{R}$, $f(x)=\sqrt{x}$

= Funkcija nije bijekcija jer se 2 različita x preslikaju u isti y.

c) $f: \mathbb{R} \to \langle -\infty, 1], f(x)=1-x^4$

= Funkcija nije bijekcija jer se 2 različita x preslikaju u isti y.

d) $f: \mathbb{R} \to \mathbb{R}, f(x)=x^5$

= Funkcija je bijekcija, jer se 2 različita x preslikaju u isti y.

(15.) DOKAZ:
- Funkcija f je injekcija, ako različiti originali imaju različite slike, odnosno ako vrijedi:
$$\forall x_1, x_2 \in S_1 \ (x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)).$$
Zbog poznate tautologije iz gornjeg izraza vrijedi:
$$\forall x_1, x_2 \in S_1 \ (f(x_1)=f(x_2) \Rightarrow x_1=x_2).$$
- Neka su zadane funkcije f i g injekcije, i neka je $h=g\circ f$. Tada je:
$h(x_1)=(g\circ f)(x_1)=g[f(x_1)]$;
$h(x_2)=(g\circ f)(x_2)=g[f(x_2)]$.
- Iz $h(x_1)=h(x_2)$ slijedi $g[f(x_1)]=g[f(x_2)]$. Kako je g injekcija, imat ćemo $f(x_1)=f(x_2)$. Kako je i f injekcija, imat ćemo $x_1=x_2$. Iz $h(x_1)=h(x_2)$ slijedi da je $x_1=x_2$, pa je onda i h, odnosno $g\circ f$ funkcija ponovno injekcija.

16. $f(x)=\frac{1}{x}-2$

$y=\frac{1}{x}-2$

$xy=1-2x$

$x(y+2)=1$

$x=\frac{1}{y+2}$ $\qquad f^{-1}(x)=\frac{1}{x+2}$

17. $f(x)=\frac{x+2}{3x-1}$

$y=\frac{x+2}{3x-1}$

$3xy-y=x+2$

$x(3y-1)=2+y$

$x=\frac{2+y}{3y-1}$ $\qquad f^{-1}(x)=\frac{2+x}{3x-1}$

18. $f(x)=\log_2\left(\frac{1}{3x-1}\right)$

$y=\log_2\left(\frac{1}{3x-1}\right)$

$\cancel{\log_2} 2^y=\cancel{\log_2}\left(\frac{1}{3x-1}\right)$

$3x\cdot 2^y-2^y=1$

$x=\frac{1+2^y}{3\cdot 2^y}$ $\qquad f^{-1}(x)=\frac{1+2^x}{3\cdot 2^x}$

19. $f(x)=3^{\frac{x+1}{x-1}}$

$y=3^{\frac{x+1}{x-1}}$

$\log_3 y=\frac{x+1}{x-1}\cdot\log_3 3$

$x\cdot\log_3 y-\log_3 y=x+1$

$x(\log_3 y-1)=1+\log_3 y$

$x=\frac{1+\log_3 y}{\log_3 y-1}$ $\qquad f^{-1}(x)=\frac{1+\log_3 x}{\log_3 x-1}$

20. $f(x)=\frac{2^x-2^{-x}}{2^x+2^{-x}}$ $\qquad (f\circ g)(x)=f[g(x)]=\frac{2^{\log_2 x^{\frac{1}{2}}}-2^{\log_2 x^{-\frac{1}{2}}}}{2^{\log_2 x^{\frac{1}{2}}}+2^{\log_2 x^{-\frac{1}{2}}}}=$

$g(x)=\log_4 x$

$=\frac{\sqrt{x}-\frac{1}{\sqrt{x}}}{\sqrt{x}+\frac{1}{\sqrt{x}}}=\frac{\frac{x-1}{\cancel{\sqrt{x}}}}{\frac{x+1}{\cancel{\sqrt{x}}}}=\frac{x-1}{x+1}$

$D(f\circ g)=?$

$1^\circ\ x>0$

$2^\circ\ 2^x+2^{-x}\neq 0$

$2^x+\frac{1}{2^x}\neq 0$

$\frac{2^{2x}+1}{2^x}\neq 0$

$\cancel{2^{2x}\neq -1}$

$3^\circ\ x+1\neq 0$

$x\neq -1$

$1^\circ\cup 2^\circ\cup 3^\circ = x>0$

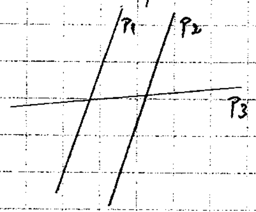
1.5. RELACIJE NA SKUPU

=DEFINICIJA 1.

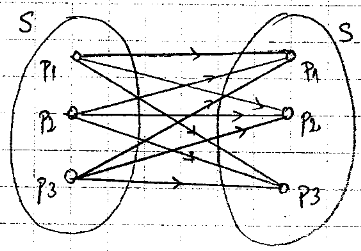
↳ Kažemo da je na skupu S definirana **relacija** ρ ako je $\rho \subseteq S \times S$. Za svaki $(x,y) \in S\times S$ je dakle $(x,y) \in \rho$ ("x i y su u relaciji ρ") ili $(x,y) \notin \rho$ ("x i y nisu u relaciji ρ"). Često pišemo: $x\rho y$.

=PRIMJER 1.

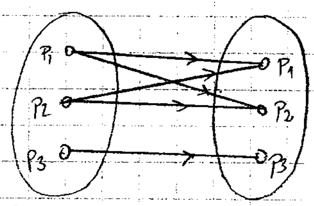
Zadan je skup $S_1 = \{p_1, p_2, p_3\}$ i relacija ρ_1 je relacija paralelnosti pravaca, tj. $x\rho_1 y :\Leftrightarrow x \parallel y$. Odredi ρ_1.



-> Iz zadane slike možemo načiniti Vennov dijagram skupa $S \times S$:



-> Sada možemo napraviti Vennov dijagram relacije ρ_1:



$\rho_1 = \{(p_1,p_1),(p_1,p_2),(p_2,p_1),(p_2,p_2),(p_3,p_3)\}$

=PRIMJER 2.

Zadan je skup $S_2 = \{1,2,4\}$ i relacija $x\rho_2 y :\Leftrightarrow x \mid y$. Odredi ρ_2.

$\rho_2 = \{(1,1),(1,2),(1,4),(2,2),(2,4),(4,4)\}$.

=DEFINICIJA 2.

1.) Relacija ρ je **refleksivna**, ako je $\forall x \; x\rho x$, tj. x je u relaciji sa samim sobom.
 ->PRIMJER 1. ρ_1 - refleksivna
 PRIMJER 2. ρ_2 - refleksivna

2.) Relacija ρ je **simetrična**, ako je $\forall x \, \forall y \; x\rho y \Rightarrow y\rho x$, tj. ako su x i y u relaciji onda slijedi da su i y i x u relaciji.
 ->PRIMJER 1. ρ_1 - simetrična
 PRIMJER 2. ρ_2 - nesimetrična

3.) Relacija ρ je **antisimetrična**, ako je $\forall x \, \forall y \; (x\rho y \wedge y\rho x \Rightarrow y = x)$, tj. za različite elemente x i y ne može istodobno biti i $x\rho y$ i $y\rho x$.
 ->PRIMJER 1. ρ_1 - nije antisimetrična
 PRIMJER 2. ρ_2 - antisimetrična

4.) Relacija ρ je **tranzitivna**, ako je $\forall x \, \forall y \, \forall z \; (x\rho y \wedge y\rho z \Rightarrow x\rho z)$, tj. ako vrijedi da je x u relaciji s y i y u relaciji sa z, onda je x u relaciji sa z.
 ->PRIMJER 1. ρ_1 - tranzitivna
 PRIMJER 2. ρ_2 - tranzitivna

=DEFINICIJA 3.

↳ Relacija ρ koja je istovremeno refleksivna, simetrična i tranzitivna zove se **relacija ekvivalencije**.
 ->PRIMJER 1. ρ_1 - relacija ekvivalencije

PRIMJER 2. ρ_2 – nije relacija ekvivalencije

→ Primjer 1. se može pisati i ovako zbog relacija:

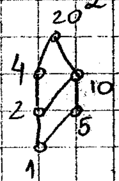
$$S_1 = \{p_1, p_2\} \cup \{p_3\},$$

pa se ti dijelovi nazivaju klase ekvivalencije od ρ_1, a skup klasa čini particiju od skupa S_1.

↳ Relacija koja je istovremeno refleksivna, antisimetrična i tranzitivna zove se **relacija poretka** (uređaja).

→ Prikaz relacije uređaja jasno je vidljiv u Primjeru 2.

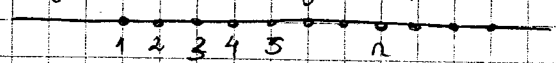
Uzmimo: $S_2 = \{1, 2, 4, 5, 10, 20\}$, pišemo:



2. REALNI I KOMPLEKSNI BROJEVI

1.1. PRIRODNI BROJEVI. MATEMATIČKA INDUKCIJA. BINOMNA FORMULA

= Prirodni brojevi
- ↳ oznaka $\mathbb{N}$
- ↳ definicija $\mathbb{N} = \{1, 2, 3, \ldots, n, \ldots\}$ ili $\mathbb{N} = \{n \mid n \in \mathbb{N}\}$
- ↳ predočavanje skupa $\mathbb{N}$ na brojevnom pravcu:

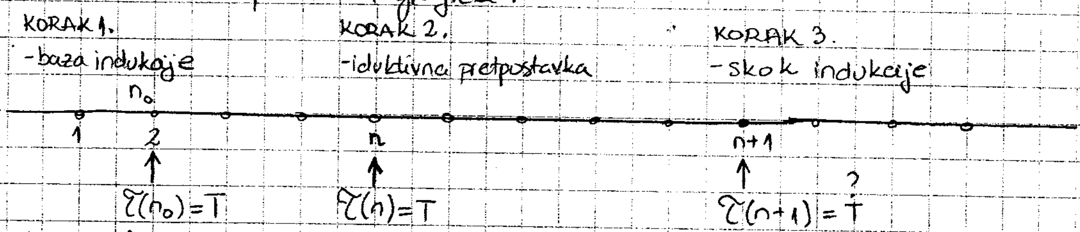


= STAVAK 1.

↳ Uz prirodne brojeve vezan je važan teorem: **načelo (princip) matematičke indukcije**:

Ako je neka tvrdnja $\tau(n)$ koja ovisi o prirodnom broju n istinita za neki prirodan broj n_0, i ako iz istinitosti te tvrdnje za $n \in \mathbb{N}$ $(n \geq n_0)$ slijedi da je istinita i za $n+1$, onda je ona istinita i za sve $n \in \mathbb{N}$ koji su veći od n_0.

↳ To možemo prikazati i grafički:



= PRIMJER 1.

Dokažite da je prirodni broj $N = n^2 + 11n$ uvijek paran, tj. $N = 2k, \forall n \in \mathbb{N}$.

1. KORAK: baza indukcije:
$n_0 = 1$
$N = 1^2 + 11 \cdot 1 = 12$ ✓ $12 = 2 \cdot 6$ (paran broj)

2. KORAK: iduktivna pretpostavka:
Pretpostavimo da je N paran za neki $n \in \mathbb{N}$. tj. $n^2 + 11n = 2k$.

3. KORAK: skok indukcije:
Treba pokazati da je tvrdnja istinita i za $n+1$:

$$(n+1)^2 + 11(n+1) =$$
$$n^2 + 2n + 1 + 11n + 1 =$$
$$\underbrace{(n^2 + 11n)}_{2k \text{ po pretp.}} + (2n + 12) =$$
$$2n + 12 =$$
$$\underbrace{2(n+6)}_{2k \text{ trivijalno}}$$

Zaključak: Zaista je uvijek $n^2 + 11n = 2k$ jer je skok bio uspješan.

□ BINOMNI KOEFICIJENTI

↳ Za zadani prirodan broj n i bilo koji $k \in \{0, 1, 2, \ldots, n\}$ definiramo:

$$\binom{n}{k} := \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!}, \quad n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n, \quad 0! = 1$$

$$\binom{n}{1} := 1$$

↳ Binomni koeficijenti čitaju se iz tzv. **Pascalovog trokuta**:

Samo je jedan Mali Ivica!

$n=0$	$(a+b)^0 = 1$	1
$n=1$	$(a+b)^1 = 1a + 1b$	1 1
$n=2$	$(a+b)^2 = 1a^2 + 2ab + 1b^2$	1 2 1
$n=3$	$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1 \cdot b^3$	1 3 3 1
$n=4$		1 4 6 4 1
$n=5$		1 5 10 10 5 1

⇒ Formula za tvorbu Pascalovog trokuta:

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$$

DOKAZ.:

$$\binom{n}{r} + \binom{n}{r-1} = \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!} =$$

$$= \frac{n!}{r(n-r)!} \cdot \frac{(n-r+1)}{(n-r+1)} + \frac{n!}{(r-1)!\,(n-r+1)!} \cdot \frac{r}{r} =$$

$$= \frac{n!\,(n-r+1+r)}{r!\,(n-r+1)!} =$$

$$= \frac{(n+1)!}{r!\,(n-r+1)!} =$$

$$= \binom{n+1}{r}$$

⇒ U Pascalovom trokutu vrijedi svojstvo simetrije:

$$\binom{n}{r} = \binom{n}{n-r}$$

= STAVAK 2.

↳ **Binomni teorem:**

Neka su $a, b \in \mathbb{R}$, $n \in \mathbb{N}$. Onda vrijedi binomna formula:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \cdot b^k$$

DOKAZ: – Pomoću matematičke indukcije:

1) baza indukcije:
$n=1 \ldots \quad (1+x)^1 = 1+x = 1 + \binom{1}{1} x \quad$ ✓

2) pretpostavka indukcije:
Pretpostavimo da za neki $n \in \mathbb{N}$ vrijedi: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

3) skok indukcije:
Treba dokazati da tvrdnja vrijedi i za $n+1$:

$$(1+x)^{n+1} = (1+x)^n (1+x) =$$

$$= \left[\sum_{k=0}^{n} \binom{n}{k} x^k\right](1+x) =$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k + \sum_{k=0}^{n} \binom{n}{k} x^{k+1} =$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k + \sum_{k=1}^{n+1} \binom{n}{k-1} x^k =$$

$$= \binom{n}{0} x^0 + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1}\right] x^k + \binom{n}{n} x^{n+1} =$$

$$= \binom{n+1}{0} x^0 + \sum_{k=1}^{n} \binom{n+1}{k} x^k + \binom{n+1}{n+1} x^{n+1} =$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k$$

= STAVAK 3.

↳ **Bernoullijeva nejednakost:**

Za svaki $x>0$ i za svaki $n\in\mathbb{N}$ vrijedi:

$$(1+x)^n \geq 1+nx$$

DOKAZ: - Pomoću matematičke indukcije:

1) baza indukcije:

$n=1 \ldots \quad (1+x)^1 \geq 1+x$

2) pretpostavka indukcije:

Pretpostavimo da za neki $n\in\mathbb{N}$ vrijedi: $(1+x)^n \geq 1+nx$.

3) skok indukcije:

Treba dokazati da tvrdnja vrijedi i za $n+1$:

$$(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x)$$
$$= 1+(n+1)x+nx^2 \geq 1+(n+1)x$$

ZADACI (DZ. 2. PRVI DIO)

(1) Dokaži: $1^3+2^3+3^3+\ldots+n^3 = \frac{n^2(n+1)^2}{4}$

(1) baza indukcije:

$n=1 \ldots \quad 1 = \frac{1^2(1+1)^2}{4} = \frac{1\cdot 2^2}{4} = \frac{4}{4} = 1$ ✓

(2) pretpostavka indukcije:

Pretpostavimo da za neki $n\in\mathbb{N}$ vrijedi: $1^3+2^3+\ldots+n^3 = \frac{n^2(n+1)^2}{4}$.

(3) skok indukcije:

Treba dokazati da tvrdnja vrijedi i za $n+1$ tj. moramo dokazati da vrijedi:

$$1^3+2^3+\ldots+n^3+(n+1)^3 = \frac{(n+1)^2\cdot(n+2)^2}{4}$$

$$\underbrace{1^3+2^3+\ldots+n^3}_{\text{PO PRETPOSTAVCI}}+(n+1)^3 =$$

$$\frac{n^2(n+1)^2}{4}+(n+1)^3 = \frac{n^2(n+1)^2+4(n+1)^3}{4} = \frac{(n+1)^2(n^2+4n+4)}{4} = \frac{(n+1)^2(n+2)^2}{4}$$

Iz toga slijedi da tvrdnja vrijedi za svaki $n\in\mathbb{N}$.

(2.) Dokaži: $\sum_{i=1}^{n}(-1)^{i-1}\cdot i^2 = (-1)^{n-1}\cdot\frac{n(n+1)}{2}$

$$\left.\begin{array}{l} i=1 \ldots (-1)^0\cdot 1^2 = 1 \\ i=2 \ldots (-1)^1\cdot 2^2 = -1\cdot 4 = -4 \\ i=3 \ldots (-1)^2\cdot 3^2 = 1\cdot 9 = 9 \\ i=4 \ldots (-1)^3\cdot 4^2 = -1\cdot 16 = -16 \end{array}\right\} \quad 1^2-2^2+3^2-4^2+\ldots+(-1)^{n-1}\cdot n^2$$

(1) baza indukcije:

$n=1 \ldots \quad 1 = (-1)^{1-1}\cdot\frac{1(1+1)}{2} = 1\cdot 1 = 1$ ✓

(2) pretpostavka indukcije:

Pretpostavimo da za neki $n\in\mathbb{N}$ vrijedi: $\sum_{i=1}^{n}(-1)^{i-1}\cdot i^2 = (-1)^{n-1}\cdot\frac{n(n+1)}{2}$

(3) skok indukcije:

Treba dokazati da tvrdnja vrijedi i za $n+1$, tj. moramo dokazati da vrijedi:

$$1^2-2^2+3^2-4^2+\ldots+(-1)^{n-1}\cdot n^2+(-1)^n\cdot(n+1)^2 = (-1)^n\cdot\frac{(n+1)(n+2)}{2}$$

$$\underbrace{1^2-2^2+3^2-4^2+\dots+(-1)^{n-1}\cdot n^2}_{\text{PO PRETPOSTAVCI}}+(-1)^n\cdot(n+1)^2=$$

$$(-1)^{n-1}\cdot\frac{n(n+1)}{2}+(-1)^n\,(n+1)^2=$$

$$=\frac{(-1)^{n-1}n(n+1)+2(-1)^n(n+1)^2}{2}=\frac{(-1)^n(n+1)(-1n+2n+2)}{2}=$$

$$=\frac{(-1)^n(n+1)(n+2)}{2}$$

Iz toga slijedi da tvrdnja vrijedi za svaki $n\in\mathbb{N}$.

(3.) Dokaži: $\prod_{i=1}^{n}\cos\left(\frac{x}{2^i}\right)=\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$

$$\left.\begin{matrix} i=1\dots\cos\frac{x}{2} \\ i=2\dots\cos\frac{x}{2^2} \\ i=3\dots\cos\frac{x}{2^3}\end{matrix}\right\}\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\cdot\cos\frac{x}{2^3}\cdot\dots\cdot\cos\left(\frac{x}{2^n}\right)$$

(1) baza indukcije:

$n=1\dots\quad \cos\frac{x}{2}=\frac{\sin x}{2^1\cdot\sin\left(\frac{x}{2^1}\right)}=\frac{\sin x}{2\sin\frac{x}{2}}=\frac{2\cancel{\sin\frac{x}{2}}\cos\frac{x}{2}}{2\cancel{\sin\frac{x}{2}}}=\cos\frac{x}{2}$ ✓

(2) pretpostavka indukcije:

Pretpostavimo da za neki $n\in\mathbb{N}$ vrijedi: $\prod_{i=1}^{n}\cos\left(\frac{x}{2^i}\right)=\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$.

(3) skok indukcije:

Treba dokazati da tvrdnja vrijedi i za $n+1$, tj. moramo dokazati da vrijedi:

$$\cos\frac{x}{2^1}\cdot\cos\frac{x}{2^{n+1}}\cdot\dots\cdot\cos\frac{x}{2^n}\cdot\cos\frac{x}{2^{n+1}}=\frac{\sin x}{2^{n+1}\cdot\sin\frac{x}{2^{n+1}}}$$

$$\underbrace{\cos\frac{x}{2^1}\cdot\cos\frac{x}{2^{n+1}}\cdot\dots\cdot\cos\frac{x}{2^n}}_{\text{PO PRETPOSTAVCI}}\cdot\cos\frac{x}{2^{n+1}}=$$

$$\frac{\sin x}{2^n\cdot\sin\frac{x}{2^n}}\cdot\cos\frac{x}{2^{n+1}}=\frac{\sin x\cdot\cancel{\cos\frac{x}{2^{n+1}}}}{2^n\cdot 2\sin\frac{x}{2\cdot2^n}\cdot\cancel{\cos\frac{x}{2\cdot 2^n}}}=$$

$$=\frac{\sin x}{2^{n+1}\cdot\sin\frac{x}{2^{n+1}}}$$

Iz toga slijedi da tvrdnja vrijedi za svaki $n\in\mathbb{N}$.

(4.) Dokaži: $\sum_{i=1}^{n}(2i+1)a_i=(n+1)^2a_n-\frac{1}{2}n(n+1)\,,\quad a_i=\sum_{j=1}^{i}\frac{1}{j}$

a) $a_i=\sum_{j=1}^{i}\frac{1}{j}\qquad a_1=1,\quad a_2=\frac{1}{2}+1=\frac{3}{2},\quad a_3=\frac{1}{3}+\frac{3}{2}=\frac{11}{6},\ \vdots$

$\Rightarrow a_n$ ćemo označiti: $1+\frac{1}{2}+\dots+\frac{1}{n}$, da ne pišemo u dokazu cijelu sumu

b) $\sum_{i=1}^{n}(2i+1)a_i\qquad \left.\begin{matrix}a_1=3\cdot1=3\\ a_2=5\cdot\frac{3}{2}=\frac{15}{2}\end{matrix}\right\}\quad 3+\frac{15}{2}+\dots+(2n+1)\cdot a_n$

(1) baza indukcije:

$n=1\dots\quad 3=(1+1)^2\cdot1-\frac{1}{2}\cdot1(1+1)=4-1=3$ ✓

(2) pretpostavka indukcije:

Pretpostavimo da za neki $n\in\mathbb{N}$ vrijedi:

$$3+\frac{15}{2}+\dots+(2n+1)\,a_n=(n+1)^2a_n-\frac{1}{2}n(n+1)$$

(3) skok indukcije:

Treba dokazati da tvrdnja vrijedi i za $n+1$, tj. moramo dokazati da vrijedi:

$$\underbrace{3+\frac{15}{2}+\dots+(2n+1)a_n}_{\text{PO PRETPOSTAVCI}}+(2(n+1)+1)a_{n+1}=(n+2)^2a_{n+1}-\frac{1}{2}(n+1)(n+2)$$

LIJEVA STRANA:

$$(n+1)^2a_n-\frac{1}{2}n(n+1)+(2n+3)\left(a_n+\frac{1}{n+1}\right)=$$

$$=(n^2+2n+1)a_n-\frac{1}{2}(n^2+n)+2na_n+3a_n+\frac{2n+3}{n+1}=$$

$$=a_n\cdot n^2+a_n\cdot 2n+a_n-\frac{n^2}{2}-\frac{n}{2}+2n\,a_n+3\,a_n+\frac{2n+3}{n+1}=$$

$$=a_n\cdot n^2+a_n\cdot 4n+a_n\cdot 4-\frac{n^2}{2}-\frac{n}{2}+\frac{2n+3}{n+1}$$

DESNA STRANA:

$$(n+2)^2\left(a_n+\frac{1}{n+1}\right)-\frac{1}{2}(n+1)(n+2)=$$

$$=(n^2+4n+4)\left(a_n+\frac{1}{n+1}\right)-\frac{1}{2}(n^2+3n+2)=$$

$$=n^2a_n+4na_n+4a_n+\frac{n^2+4n+4}{n+1}-\frac{n^2+3n+2}{2}=$$

$$=a_nn^2+a_n4n+a_n4+\frac{(n+2)^2}{n+1}-\frac{n^2}{2}-\frac{n}{2}-\frac{2n}{2}-1=$$

$$=a_nn^2+a_n4n+a_n4-\frac{n^2}{2}-\frac{n}{2}+\frac{2(n+2)^2-2n(n+1)-2(n+1)}{2(n+1)}=$$

$$=a_nn^2+a_n4n+a_n4-\frac{n^2}{2}-\frac{n}{2}+\frac{\cancel{2n^2}+3n+8-\cancel{2n^2}-2n-2n-2}{2(n+1)}=$$

$$=a_nn^2+a_n4n+a_n4-\frac{n^2}{2}-\frac{n}{2}+\frac{2(2n+3)}{2(n+1)}=$$

$$=a_n\cdot n^2+a_n\cdot 4n+a_n4-\frac{n^2}{2}-\frac{n}{2}+\frac{2n+3}{n+1}$$

Iz toga slijedi da tvrdnja vrijedi za svaki $n\in\mathbb{N}$.

(5.) Dokaži:

$$\sum_{i=1}^{n}\frac{1}{\sqrt{i}}\geq\sqrt{n}$$

$i=1\ \dots\ a_1=1$

$i=2\ \dots\ a_2=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$

$i=3\ \dots\ a_3=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$

$i=4\ \dots\ a_4=\frac{\sqrt{4}}{4}=\frac{1}{2}$

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}}$$

(1) baza indukcije:

$n=1\ \dots\ 1\geq\sqrt{1}\geq 1$ ✓

(2) pretpostavka indukcije:

Pretpostavimo da za neki $n\in\mathbb{N}$ vrijedi:

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}}\geq\sqrt{n}$$

(3) skok indukcije:

Treba dokazati da tvrdnja vrijedi i za $n+1$, tj. moramo dokazati da vrijedi:

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}\geq\sqrt{n+1}$$

$$\underbrace{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}}_{\text{PO PRETPOSTAVCI}}+\frac{1}{\sqrt{n+1}} \geq \sqrt{n+1}$$

$$\sqrt{n}+\frac{1}{\sqrt{n+1}} \geq \sqrt{n+1} \quad \Big/ \cdot \sqrt{n+1}$$

$$\sqrt{n^2+n} \cancel{+1} \geq n \cancel{+1}$$

$$\sqrt{n^2+n} \geq n \quad \text{jer je } \sqrt{n^2}=n \text{ pa je i } \sqrt{n^2+n} \geq n$$

Iz toga slijedi da tvrdnja vrijedi za svaki $n \in \mathbb{N}$.

6. Dokaži:

$$(2n)! > 3^n (n!)^2 \quad , n \geq 5$$

1. baza indukcije:

$n=5\ldots \quad (2\cdot 5)! > 3^5 \cdot (5!)^2 \quad 3628800 > 708588$ ✓

2. pretpostavka indukcije:

Pretpostavimo da za neki $n \in \mathbb{N}$ vrijedi: $(2n)! > 3^n (n!)^2$, $n \geq 5$.

3. skok indukcije:

Treba dokazati da tvrdnja vrijedi i za $n+1$, tj. treba dokazati da vrijedi:

$$(2(n+1))! > 3^{n+1}((n+1)!)^2$$

$$(2n+2)! > 3\cdot 3^n (n+1)!\,(n+1)!$$

$$\underbrace{(2n)!}_{\text{PO PRETPOSTAVCI}}(2n+1)(2n+2) > 3\cdot 3^n (n!)^2 \cdot (n+1)^2$$

$$\cancel{3^n (n!)^2}(2n+1)(2n+2) > \cancel{3^n (n!)^2} \cdot 3(n+1)^2$$

$$(2n+1)(2n+2) > 3\cdot(n+1)(n+1)$$

$$4n^2+6n+3 > 3n^2+6n+3 \quad \text{jer je } 4n^2 > 3n^2.$$

Iz toga slijedi da tvrdnja vrijedi za svaki $n \in \mathbb{N}$.

7. Dokaži:

$$5^{n-1}+2^n = 3k$$

1. baza indukcije:

$n=1\ldots \quad 5^{1-1}+2^1 = 5^0+2 = 3$ je djeljivo s 3 ✓

2. Pretpostavka indukcije:

Pretpostavimo da za neki $n \in \mathbb{N}$ vrijedi: $5^{n-1}+2^n=3k$

3. Skok indukcije:

Treba dokazati da tvrdnja vrijedi i za $n+1$, tj. treba dokazati da vrijedi: $5^{n+1-1}+2^{n+1}=3k$.

$$5^{n+1-1}+2^{n+1} = 5\cdot 5^{n-1}+2\cdot 2^n = 5(5^{n-1}+2^n)-5\cdot 2^n+2\cdot 2^n =$$

$$\underbrace{5(5^{n-1}+2^n)}_{3k \text{ PO PRETPOSTAVCI}} - \underbrace{3\cdot 2^n}_{3k \text{ TRIVIJALNO}}$$

Iz toga slijedi da tvrdnja vrijedi za svaki $n \in \mathbb{N}$.

8. Dokaži:

$$a_n = 2^{n-1}+1$$

$a_1 = 2$

$a_2 = 3$

$a_{n+2} = 3\cdot a_{n+1} - 2a_n$

① baza indukcije:

$n=1$ ...... $2 = 2^{1-1}+1 = 2^0+1 = 1+1 = 2$ ✓

$n=2$ ...... $3 = 2^{2-1}+1 = 2^1+1 = 2+1 = 3$ ✓

② pretpostavka indukcije:

Pretpostavimo da za neki $n \in \mathbb{N}$ vrijedi: $a_n = 2^{n-1}+1$, i da za neki $(n-1) \in \mathbb{N}$ vrijedi: $a_{n-1} = 2^{n-2}+1$.

③ skok indukcije:

Treba dokazati da tvrdnja vrijedi i za $n+1$, tj. treba dokazati da vrijedi: $a_{n+1} = 2^n+1$.

Ako vrijedi: $a_{n+2} = 3a_{n+1} - 2a_n$ onda vrijedi i: $a_{n+1} = 3a_n - 2a_{n-1}$, pa treba dokazati da je: $3a_n - 2a_{n-1} = 2^n+1$.

$$3a_n - 2a_{n-1} = 3\cdot(2^{n-1}+1) - 2\cdot(2^{n-2}+1) = 3\cdot 2^{n-1} + 3 - \cancel{2}\cdot\frac{2^{n-1}}{\cancel{2}} - 2 =$$

$$= 2^{n-1}(3-1)+1 = 2^{n-1}\cdot 2 + 1 = 2^n+1$$

Iz toga slijedi da tvrdnja vrijedi za svaki $n \in \mathbb{N}$.

2.2. CIJELI BROJEVI

= Cijeli brojevi

↳ oznaka $\mathbb{Z}$

↳ definicija $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$

= STAVAK 1.

↳ Teorem o dijeljenju:

Neka je $m \in \mathbb{Z}$, $n \in \mathbb{N}$. Onda postoje jednoznačno određeni cijeli brojevi q i r takvi da je:

$$m = nq + r, \quad 0 \le r < n.$$

Broj q zovemo nepotpunim kvocijentom pri dijeljenju m s n, a r ostatkom.

2.3. RACIONALNI BROJEVI

= Racionalni brojevi

↳ oznaka $\mathbb{Q}$

↳ definicija $\mathbb{Q} = \left\{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N}\right\}$

= Aritmetička sredina u skupu $\mathbb{Q}$:

$$\text{aritmetička sredina} = \frac{\frac{m_1}{n_1}+\frac{m_2}{n_2}}{2} = \frac{m_1n_2+m_2n_1}{2n_1n_2} = \frac{m_3}{n_3}$$

↳ Svojstvo **gustog skupa**:

Skup $\mathbb{Q}$ je gust skup, tj. između svaka 2 razlomka postoji još mnogo razlomaka.

Ali, sa druge strane čitav realni pravac ne može se „popuniti" racionalnim brojevima, jer je primjerice $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

= PRIMJER 1.

Dokaži da broj $\sqrt{2}$ nije racionalan broj, odnosno dokaži da $\sqrt{2} \notin \mathbb{Q}$, tj. $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

↳ Dokaz se izvodi koristeći modus tollens (dokaz protuslovlja) koji glasi:

$$((\neg X \Rightarrow Y) \wedge \neg Y) \Rightarrow X.$$

Ovdje se izvodi zaključivanje na istinitost suda „X", i to na sljedeći način: ako je istina da je „$\neg X \Rightarrow Y$" i ako je „Y" lažan, onda je sud „X" istinit.

↳ $X \ldots \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$

$\neg X \ldots \sqrt{2} \in \mathbb{Q}$

$\sqrt{2} = \dfrac{m}{n} \Big/^2$, $m, n \in \mathbb{N}$ i oba broja nisu parna.

$2 = \dfrac{m^2}{n^2} \Rightarrow m^2 = 2n^2$

↳ ako je m^2 paran, onda je i m paran, tj. $m = 2k$ pa:

$(2k)^2 = 2n^2$

$4k^2 = 2n^2$

$\Rightarrow n^2 = 2k^2$

↳ ako je n^2 paran, onda je i n paran, tj. $n = 2k$

Kako smo ustvrdili da ne smiju oba broja biti parna, onda je:

$X \ldots \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$

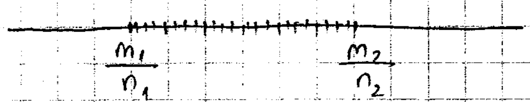
$\neg X \ldots \sqrt{2} \in \mathbb{Q}$

$Y \ldots$ tvrdnja: Nisu oba broja parna.

$\neg Y \ldots$ tvrdnja: Oba broja su parna.

Odnosno, naš dokaz nas je doveo u kontradikciju s uvjetom, što znači da $\sqrt{2}$ nije racionalan broj.

↳ Simbolički:



⇒ Usprkos tome što skup $\mathbb{Q}$ ima svojstvo gustoće, postoje brojevi koji se ne nalaze u njemu, odnosno na brojevnom pravcu postoje praznine u $\mathbb{R}$.

2.4. REALNI BROJEVI

= Realni brojevi

↳ oznaka $\mathbb{R}$

↳ geometrijski prikaz: na brojevnom pravcu

$(-\infty)$ —— 0 —— $r \in \mathbb{R}$ —— $(+\infty)$

= Svakom je realnom broju pridružena točka na brojevnom pravcu i obratno.

= Ne postoji najmanji i najveći realni broj.

= Skup $\mathbb{R}$ opisan je sa sljedećih 6 grupa aksioma:

= **AKSIOM 1.** = Aksiomi zbrajanja

↳ Na $\mathbb{R}$ je definirana binarna operacija zbrajanja (+). S obzirom na tu operaciju, $\mathbb{R}$ je komutativna grupa, tj. vrijedi:

Z1) $\forall a \ \forall b \ \forall c \quad (a+b)+c = a+(b+c)$ → asocijativnost zbrajanja

Z2) $\exists 0 \in \mathbb{R} \ \forall a \quad a+0 = 0+a = a$ → postojanje neutralnog elementa

Z3) $\forall a \ \exists (-a) \quad a+(-a) = (-a)+a = 0$ → postojanje suprotnog elementa

Z4) $\forall a \ \forall b \quad a+b = b+a$ → komutativnost

= **AKSIOM 2.** = Aksiomi množenja

↳ Na $\mathbb{R}$ je definirana binarna operacija množenja $(\cdot)$. S obzirom na tu operaciju $\mathbb{R} \setminus \{0\}$ je komutativna grupa, tj. vrijedi:

M1) $\forall a \ \forall b \ \forall c \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$ → asocijativnost množenja

M2) $\exists 1 \in \mathbb{R} \ \forall a \quad a \cdot 1 = 1 \cdot a = a$ → postojanje neutralnog elementa

M3) $\forall a \ \exists a^{-1} \quad a \cdot a^{-1} = a^{-1} \cdot a = 1$ → postojanje suprotnog elementa

M4) $\forall a\ \forall b \quad a\cdot b = b\cdot a$ → komutativnost

= AKSIOM 3. = Aksiomi (zakoni) distributivnosti

↳ Operacije zbrajanja i množenja povezane su međusobno na $\mathbb{R}$ zakonom distributivnosti.

D1) $\forall a\ \forall b\ \forall c \quad (a+b)\cdot c = ac+bc$ → desna distributivnost množenja prema zb.

D2) $\forall a\ \forall b\ \forall c \quad c\cdot(a+b) = ca+cb$ → lijeva distributivnost množenja prema zb.

= PRIMJEDBA 1.

↳ Skup u kojem vrijede aksiomi 1, 2, 3 zove se polje. Zato govorimo o polju realnih brojeva. Obzirom da isti aksiomi vrijede i u skupu racionalnih brojeva, onda je i taj skup polje.

= AKSIOM 4. = Aksiomi poretka (zakoni poretka)

↳ Na $\mathbb{R}$ je definirana relacija poretka $\leq$, tako da vrijedi:

P1) $\forall a \quad a \leq a$ → refleksivnost

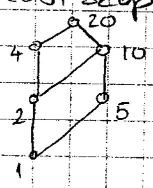
P2) $\forall a\ \forall b \quad (a\leq b)\wedge(b\leq a) \Rightarrow a=b$ → antisimetričnost

P3) $\forall a\ \forall b\ \forall c \quad (a\leq b)\wedge(b\leq c) \Rightarrow a\leq c$ → tranzitivnost

P4) $\forall a\ \forall b \quad a\leq b \vee b\leq a$ → potpuni poredak

= PRIMJEDBA 2.

↳ Skup u kojem vrijede aksiomi P1, P2, P3 zove se djelomično (parcijalno) uređen skup, tj. u njemu postoje neusporedivi elementi.



↳ Skup u kojemu vrijede aksiomi P1, P2, P3, P4 zove se potpuno uređen skup, tj. u njemu ne postoje neusporedivi elementi.

↳ Zbog toga se govori o uređenom polju realnih brojeva.

= AKSIOM 5. = aksiomi monotonije (zakoni monotonije)

↳ U $\mathbb{R}$ vrijede zakoni monotonije za zbrajanje i množenje tj.:

M1) $\forall a\ \forall b\ \forall c \quad a\leq b \Rightarrow a+c \leq b+c$ → monotonost zbrajanja

M2) $\forall a\ \forall b \quad (0\leq a)\wedge(0\leq b) \Rightarrow 0\leq a\cdot b$ → monotonost množenja

= PRIMJEDBA 3.

↳ Svi do sada nabrojeni aksiomi vrijede i u skupu $\mathbb{Q}$.

= AKSIOM 6. = Aksiomi potpunosti (neprekidnosti)

↳ Neka su S_d i $S_g \subset \mathbb{R}$ takvi da je:

1) $S_d \neq 0,\ S_g \neq 0$

2) $S_d \cup S_g = \mathbb{R}$

3) $\forall a\ \forall b\ (a\in S_d)\wedge(b\in S_g) \Rightarrow a\leq b$.

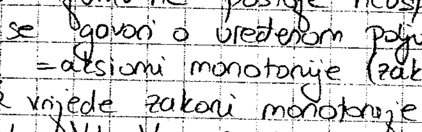
↳ Par podskupova S_d i S_g sa tim svojstvima zove se Dedekindov rez ili prerez, gdje je S_d donji skup reza, a S_g gornji skup reza.

↳ Tada postoji rezna točka sa sljedećim svojstvima:

A6.) Za svaki Dedekindov rez S_d, S_g postoji element $r\in\mathbb{R}$ tako da vrijedi: $\forall a\in S_d\ \forall b\in S_g \quad a\leq r\leq b$.

Broj r zove se rezna točka reza (S_d, S_g)

↳ Grafički:



= PRIMJER 1.

↳ Za Dedekindov rez $\langle -\infty, r] \cup [r, \infty\rangle$ gdje je $r^2 = 2$ pokazuje se da je rezna točka $r \notin \mathbb{Q}$, što znači da $\mathbb{Q}$ nije potpun (neprekidan).

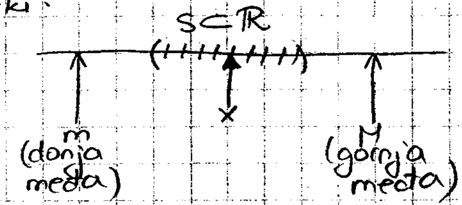
OMEĐENI SKUPOVI. SUPREMUM I INFIMUM SKUPA.

= DEFINICIJA 1.

↳ Za podskup $S \subseteq \mathbb{R}$ kažemo da je **omeđen odozgo** ako postoji $M \in \mathbb{R}$ takav da za sve $x \in S$ vrijedi $x \le M$. Takav M zovemo gornjom međom skupa S.

↳ Za podskup $S \subseteq \mathbb{R}$ kažemo da je **omeđen odozdo** ako postoji $m \in \mathbb{R}$ takav da za sve $x \in \mathbb{R}$ vrijedi $m \le x$. Takav m zovemo donjom međom skupa S.

↳ Grafički:



$\exists M \in \mathbb{R}$ takav da $\forall x \in S \Rightarrow x \le M$

$\exists m \in \mathbb{R}$ takav da $\forall x \in S \Rightarrow m \le x$

= DEFINICIJA 2.

↳ **Infimum** od S je najveća donja međa (ako postoji). Oznaka: $\inf S$. Ako je $\inf S$ sadržan u S onda ga zovemo minimumom skupa S i označavamo ga sa $\min S$.

↳ **Supremum** od S je najmanja gornja međa (ako postoji). Oznaka $\sup S$. Ako je $\sup S$ sadržan u S onda ga zovemo maksimumom skupa S i označavamo ga sa $\max S$.

= PRIMJER 1.

$S_1 = (3,5] \rightarrow \inf S_1 = 3$

$S_2 = [3,5) \rightarrow \inf S_2 = 3$, ali $3 = \min S_2$

= PRIMJER 3.

$S_1 = (3,5] \rightarrow \sup S_1 = 5$, ali $5 = \max S_1$

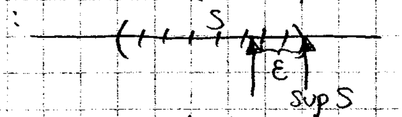
$S_2 = [3,5) \rightarrow \sup S_2 = 5$

= Svojstva supremuma (infimuma):

1) $\forall x \in S \Rightarrow x \le \sup S$

2) $\forall \varepsilon > 0 \; \exists x \in S \; (x > \sup S - \varepsilon)$

↳ grafički:



= AKSIOM 7. = Aksiom ekvivalentan aksiomu potpunosti (A6)

A6') Svaki odozgo omeđen podskup skupa $\mathbb{R}$ ima supremum u $\mathbb{R}$.

= PRIMJER 4.

$S_1 = \{x \in \mathbb{R} \mid x^2 < 2\} \rightarrow \sup S_1 = \sqrt{2}$

$S_2 = \{x \in \mathbb{Q} \mid x^2 < 2\} \rightarrow$ nema $\sup S_2$

= DEFINICIJA 3.

↳ Za $a, b \in \mathbb{R}$ definiramo razliku:

$$a - b := a + (-b)$$

↳ Za $a \in \mathbb{R}$ i $b \in \mathbb{R} \setminus \{0\}$ definiramo kvocijent:

$$\frac{a}{b} := a \cdot b^{-1}$$

= Svojstva: 1.) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

2.) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

= Vrste intervala u $\mathbb{R}$:

1.) otvoreni $\rightarrow (a,b) := \{x \in \mathbb{R} : a < x < b\}$, $a, b \in \mathbb{R} \cup \{\pm\infty\}$

2.) zatvoreni $\rightarrow [a,b] := \{x \in \mathbb{R} : a \le x \le b\}$, $a, b \in \mathbb{R}$

3.) poluotvoreni (poluzatvoreni) $\rightarrow (a,b], [a,b), (-\infty, b], [b, \infty)$, $a, b \in \mathbb{R}$

Samo je jedan Mali Ivica!

=STAVAK 1. = **Arhimedov teorem** ($\equiv A6, A6'$)
↳ Za svaki pozitivan realan broj x i y postoji takav n takav da je $nx > y$.

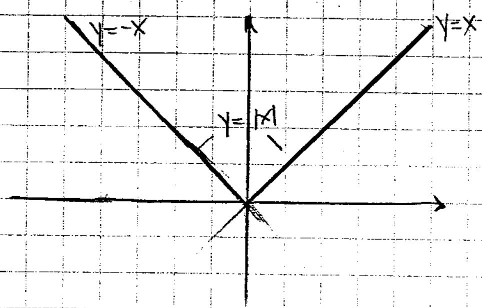
=STAVAK 2. = **Cantorov teorem** ($\equiv A6, A6'$)
↳ Neka je $[a_n, b_n]$ slijed omeđenih zatvorenih intervala u $\mathbb{R}$ takvih da je $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ za sve $n \in \mathbb{N}$. Onda je skup $\bigcap_{n=1}^{\infty} [a_n, b_n]$ neprazan tj. postoji $x \in \mathbb{R}$ koji se nalazi u svim intervalima.

□ APSOLUTNA VRIJEDNOST I UDALJENOST U $\mathbb{R}$

=DEFINICIJA 4.
↳ Funkcija koja realnom broju x pridružuje broj x ako je $x \geq 0$, a $-x$ ako je $x < 0$, zove se **apsolutna vrijednost** realnog broja, oznakom $|x|$.
Zapisujemo:

$$|x| := \begin{cases} x & \text{za } x \geq 0 \\ -x & \text{za } x < 0 \end{cases}$$

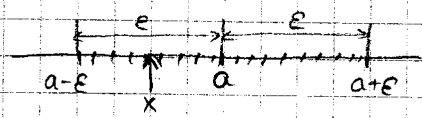
↳ Graf funkcije $y = |x|$ =



= Broj $|x|$ predstavlja udaljenost realnog broja x od 0 na realnom pravcu.
Broj $|x-a|$ predstavlja udaljenost točaka x i a. Vrijedi:

$$|x-a| < \varepsilon \iff -\varepsilon < x-a < \varepsilon \iff a-\varepsilon < x < \varepsilon + a$$

tj.:



= Za sve realne brojeve a i b vrijedi nejednakost trokuta:

$$|a+b| \leq |a| + |b|$$

DOKAZ. Rastavimo nejednakost na 4 slučaja:

1.) $a = +$ $b = +$
$|+a + (+b)| \leq |+a| + |+b|$
$a + b \leq a + b$ ✓

2.) $a = +$ $b = -$
$|+a + (-b)| \leq |+a| + |-b|$
$|a - b| \leq a + b$ ✓

3.) $a = -$ $b = +$
$|-a + (+b)| \leq |-a| + |+b|$
$|-a + b| \leq a + b$ ✓

4.) $a = -$ $b = -$
$|-a + (-b)| \leq |-a| + |-b|$
$|-a - b| \leq a + b$ ✓

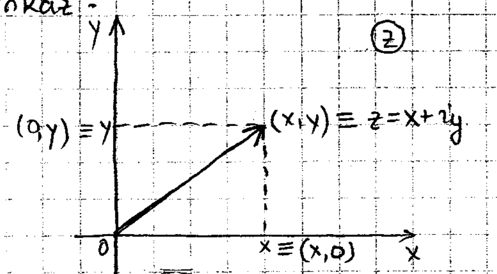
2.5. KOMPLEKSNI BROJEVI

= Kompleksni brojevi
- ↳ oznaka: $\mathbb{C}$
- ↳ definicija: $\mathbb{C} := \{a+bi : a,b \in \mathbb{R}\}$
- ↳ zapis: $z = x+yi$
  - $z \to$ kompleksni broj
  - $x \to$ realni dio broja
  - $y \to$ imaginarni dio broja
  - $i \to$ imaginarna jedinica

= Neka svojstva:
- ↳ 2 su broja jednaka $z_1 = z_2$ ($z_1 = a+bi$, $z_2 = c+di$) ako i samo ako su im realni dijelovi jednaki i imaginarni dijelovi jednaki, tj. ako vrijedi: $a=c \wedge b=d$ onda $a+bi = c+di$.

☐ GAUSSOVA RAVNINA

= Skup kompleksnih brojeva $\mathbb{C}$ možemo poistovjetiti s Gaussovom ravninom: $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ koju čine svi poredani dvojci realnih brojeva (a,b).
- ↳ Prikaz:



$x = Re[z] = (Re\, z)$
$y = Im[z] = (Im\, z)$
i = imaginarna jedinica
(z) = oznaka za kompleksnu (Gaussovu) ravninu

= DEFINICIJA $\boxed{1.}$
- ↳ Vrijedi:
  1) $(x_1,y_1) \oplus (x_2,y_2) := (x_1+x_2,\ y_1+y_2)$
  2) $(x_1,y_1) \odot (x_2,y_2) := (x_1\cdot x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$
- ↳ Posebice:
  $$(0,1)^2 = (0,1)(0,1) = (0\cdot 0 - 1\cdot 1,\ 0\cdot 1 + 1\cdot 0) = (-1,0)$$
  Sada uvodimo oznake:
  $(0,1) = i \Rightarrow i^2 = -1$
  $(x_1,y_1) = (x_1 + iy_1) \to$ zapis para
- ↳ Vrijedi:
  1) $z_1 + z_2 = (x+y_1 i) + (x_2+y_2 i) = (x_1+x_2) + (y_1+y_2)i$
  2) $z_1 \cdot z_2 = (x_1+y_1 i)(x_2+y_2 i) = (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i$

= Ostale operacije:

1) Konjugiranje kompleksnog broja $z = x+yi$

= DEFINICIJA $\boxed{2.}$
- ↳ Za zadani kompleksni broj $z = x+yi$ definiramo **konjugirano kompleksni broj** od z sa: $\bar{z} := x-yi$

= Svojstva:

1) $\bar{\bar{z}} = z$

DOKAZ: $\bar{\bar{z}} = \overline{\overline{x+yi}} = \overline{x-yi} = x+yi = z$ ✓

2) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

DOKAZ:
$$\overline{z_1+z_2} = \overline{x_1+y_1 i + x_2 + y_2 i} = \overline{(x_1+x_2)+(y_1+y_2)i} =$$
$$= (x_1+x_2) - (y_1+y_2)i = (x_1 - y_1 i) + (x_2 - y_2 i) =$$
$$= \bar{z}_1 + \bar{z}_2 \ ✓$$

3) $\overline{z_1\cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

DOKAZ: $\overline{z_1\cdot z_2} = \overline{(x_1+y_1 i)\cdot(x_2+y_2 i)} = \overline{(x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i} =$

$= (x_1x_2 - y_1y_2) - (x_1y_2 + x_2y_1)i = (x_1 - y_1 i)(x_2 - y_2 i) = \overline{z_1}\cdot\overline{z_2}$ ✓

4) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, \quad \overline{z_2} \neq 0$

DOKAZ: $\overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(z_1\cdot\frac{1}{z_2}\right)} = \overline{z_1}\cdot\overline{\left(\frac{1}{z_2}\right)} = \overline{z_1}\cdot\overline{\left(\frac{1}{x_2+y_2 i}\right)} =$

$= \overline{z_1}\cdot\overline{\left(\frac{x_2-y_2 i}{(x_2+y_2 i)(x_2-y_2 i)}\right)} = \overline{z_1}\cdot\overline{\left(\frac{x_2-y_2 i}{x_2^2+y_2^2}\right)} =$

$= \overline{z_1}\cdot\frac{x_2+y_2 i}{x_2^2+y_2^2} = \overline{z_1}\cdot\frac{x_2+y_2 i}{(x_2+y_2 i)(x_2-y_2 i)} = \overline{z_1}\cdot\frac{1}{x_2-y_2 i} =$

$= \overline{z_1}\cdot\frac{1}{\overline{z_2}} = \frac{\overline{z_1}}{\overline{z_2}}$ ✓

2) Apsolutna vrijednost kompleksnog broja $z = x+yi$

= DEFINICIJA 3.

↳ Za zadani kompleksni broj $z = x+yi$ definiramo **apsolutnu vrijednost kompleksnog broja** z kao: $|z| = \sqrt{x^2+y^2}$. To je udaljenost broja z od ishodišta.

= Svojstva:

1) $z\cdot\overline{z} = |z|^2$

DOKAZ: $z\cdot\overline{z} = (x+yi)(x-yi) = x^2+y^2 = |z|^2$ ✓

2) $|z_1\cdot z_2| = |z_1|\,|z_2|$

DOKAZ: $|z_1\cdot z_2|^2 = (z_1 z_2)\overline{(z_1 z_2)} = z_1 z_2\,\overline{z_1}\,\overline{z_2} = z_1\overline{z_1}\;z_2\overline{z_2} = |z_1|^2|z_2|^2$

Iz toga slijedi: $|z_1\cdot z_2| = |z_1|\,|z_2|$. ✓

3) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

DOKAZ: $\left|\frac{z_1}{z_2}\right|^2 = \frac{z_1}{z_2}\cdot\overline{\left(\frac{z_1}{z_2}\right)} = \frac{z_1}{z_2}\cdot\frac{\overline{z_1}}{\overline{z_2}} = \frac{|z_1|^2}{|z_2|^2} = \left(\frac{|z_1|}{|z_2|}\right)^2$

Iz toga slijedi: $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ ✓

4) $|z^n| = |z|^n$

DOKAZ: Pomoću matematičke indukcije:

1) baza indukcije:

$n=1 \ldots\quad z^1 = z^1 \Rightarrow z = z$ ✓

2) pretpostavka indukcije:

Pretpostavimo da za neki $n\in\mathbb{N}$ vrijedi: $|z^n| = |z|^n$.

3) Skok indukcije:

Treba dokazati da tvrdnja vrijedi i za $n+1$, tj. da vrijedi: $|z^{n+1}| = |z|^{n+1}$

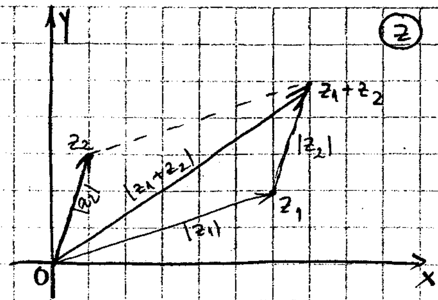
$|z^{n+1}| = |z^n\cdot z| = \underbrace{|z^n|}_{P.P.}\cdot|z| = |z|^n\cdot|z| = |z|^{n+1}$ ✓

Iz toga slijedi da tvrdnja vrijedi za svaki $n\in\mathbb{N}$.

5) Nejednakost trokuta:

$|z_1+z_2| \leq |z_1| + |z_2|$

DOKAZ: Pomoću grafa, Gaussove ravnine. Nacrtat ćemo trokut u Gaussovoj ravnini koji odgovara kompleksnim brojevima z_1, z_2 i z_1+z_2. Kako znamo da je u trokutu duljina svake stranice manja od zbroja duljina ostalih dviju, tvrdnja će biti dokazana:



= PRIMJER 1.

↳ Dokaži da je $\left|\frac{1}{z}\right| = \frac{1}{|z|}$

$$\left|\frac{1}{z}\right| = \left|\frac{1}{x+yi}\right| = \left|\frac{x-yi}{x^2+y^2}\right| = \sqrt{\left(\frac{x}{x^2+y^2}\right)^2+\left(\frac{-y}{x^2+y^2}\right)^2} = \sqrt{\frac{\cancel{x^2+y^2}}{(x^2+y^2)^{\cancel{2}}}} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{|z|} ✓$$

= PRIMJER 2.

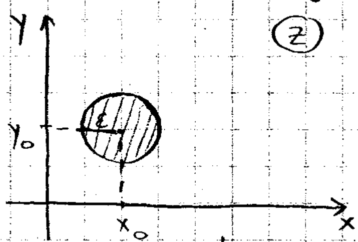
↳ Skicirati geometrijsko mjesto točaka $z \in \mathbb{C}$ za koje vrijedi $(z-z_0) \leq \varepsilon$, $z_0 \in \mathbb{C}$.

$$|z-z_0| = |x+yi-(x_0+y_0 i)| = |(x-x_0)+(y-y_0)i| = \sqrt{(x-x_0)^2+(y-y_0)^2}$$

$$\sqrt{(x-x_0)^2+(y-y_0)^2} \leq \varepsilon \quad /^2$$

$$(x-x_0)^2+(y-y_0)^2 \leq \varepsilon^2$$

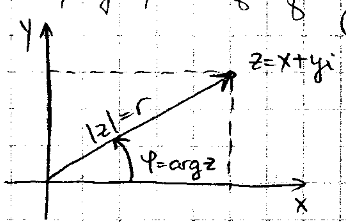
↳ zatvoreni krug $S(x_0, y_0)$, $r=\varepsilon$



↳ Dakle skup $\{z \mid |z-z_0| \leq \varepsilon\}$ predstavlja komplekshe brojeve unutar zatvorenog kruga sa središtem u točki z_0 sa radijusom ε.

□ TRIGONOMETRIJSKI PRIKAZ KOMPLEKSNOG BROJA

= Svakom kompleksnom broju $z=x+yi$ odgovara točka $T(x,y)$ u Gaussovoj ravni-ni, koji poistovjećujemo sa z.



= DEFINICIJA 4.

1) $|z|$ je udaljenost kompleksnog broja od ishodišta, tj.: $|z| := \sqrt{x^2+y^2}$.

2) $\arg z = \varphi$ je kut kojeg vektor $x+yi$ zatvara s pozitivnom realnom osi, tj. $\operatorname{tg}\varphi = \frac{y}{x}$.

= Sa slike vidimo da vrijedi:

$$x = r\cdot\cos\varphi$$
$$y = r\cdot\sin\varphi$$

Iz toga možemo dobiti trigonometrijski oblik broja z kao:

$$x+yi = r\cdot\cos\varphi + ir\sin\varphi = \underbrace{r(\cos\varphi + i\sin\varphi)}$$

Ako uvedemo novu oznaku:

$$\cos\varphi + i\sin\varphi = e^{i\varphi}$$

možemo pisati: $z = x+yi = r\cdot e^{i\varphi}$

Samo je jedan Mali Ivica!

- PRIMJER 3.

↳ Prikaži u trigonometrijskom obliku sljedeće brojeve:

a) $z=-1+i$

→ II kvadrant

→ $|z|=\sqrt{2}$

→ $\operatorname{tg}\varphi=\frac{1}{-1}=-1 \Rightarrow \varphi=\frac{3\pi}{4}, \cancel{\frac{7\pi}{4}}$

$z=\sqrt{2}\cdot\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

b) $z=1-i$

→ IV kvadrant

→ $|z|=\sqrt{2}$

→ $\operatorname{tg}\varphi=\frac{-1}{1}=-1 \Rightarrow \varphi=\cancel{\frac{3\pi}{4}}, \frac{7\pi}{4}$

$z=\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$

c) $z=-\sqrt{3}-i$

→ III kvadrant

→ $|z|=\sqrt{\sqrt{3}^2+1^2}=2$

→ $\operatorname{tg}\varphi=\frac{1}{\sqrt{3}} \Rightarrow \varphi=\frac{7\pi}{6}$

$z=2\cdot\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$

= Operacije s kompleksnim brojevima u trigonometrijskom obliku:

1) množenje 2 broja:

$$z_1\cdot z_2=r_1(\cos\varphi_1+i\sin\varphi_1)\cdot r_2(\cos\varphi_2+i\sin\varphi_2)=$$
$$=r_1r_2\left(\cos\varphi_1\cos\varphi_2-\sin\varphi_1\sin\varphi_2+i(\sin\varphi_1\cos\varphi_2+\cos\varphi_1\sin\varphi_2)\right)=$$
$$=r_1r_2\left(\cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2)\right)$$

↳ dakle:

$|z_1 z_2|=|z_1|\cdot|z_2|$

$\arg z_1z_2=\arg z_1+\arg z_2$

↳ poseban slučaj kad je $z_1=z_2=z$:

$z^2=|z|^n\left(\cos(n\varphi)+i\sin(n\varphi)\right)$

nazivamo prvom Moivre-ovom formulom za potenciranje

- PRIMJER 4.

↳ $(-\sqrt{3}-i)^{12}=\left(2\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)\right)^{12}=2^{12}\cdot(\cos14\pi+i\sin14\pi)=2^{12}(\cos0+i\sin0)=$
$=2^{12}$

□ KORIJEN KOMPLEKSNOG BROJA

= Neka je zadano $z=re^{i\varphi}$. Tražimo kompleksni broj w, takav da vrijedi: $w^n=z$. Traženi broj označujemo sa $w:=\sqrt[n]{z}$.

Dakle: $z=re^{i\varphi}$

$w=\rho\cdot e^{i\psi}$, $w^n=z$ → traži se

$w^n=z$

$\rho^n e^{in\psi}=re^{i\varphi} \equiv$ 1) $\rho^n=r \Rightarrow \rho=\sqrt[n]{|z|}$

2) $n\cdot\psi=\varphi+2k\pi \Rightarrow \psi=\frac{\varphi+2k\pi}{n}=\frac{\arg z+2k\pi}{n}$, $k=0,1,\dots,n-1$.

= Kompleksni broj $z\neq0$ ima točno n n-tih korijena koji leže u vrhovima pravilnog n-terokuta upisanog u kružnicu polumjera $\sqrt[n]{r}$.

= Iz toga izvodimo: **Moivreovu formulu** za radikaciju (korjenovanje):

$$\sqrt[n]{z}=\sqrt[n]{|z|}\cdot\left(\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right),\ k=0,1,2,\dots,n-1$$

ZADACI (DZ.2. DRUGI DIO)

9. Skicirati :

a) $|z-4+5i|=3$

$|x+yi-4+5i|=3$

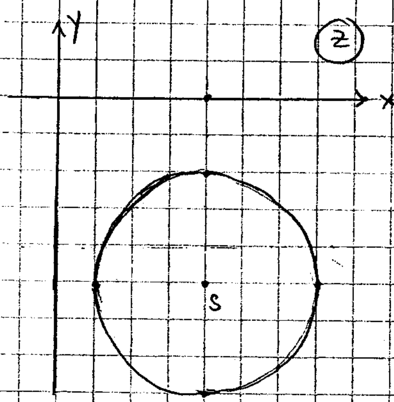
$|x-4+i(y+5)|=3$

$\sqrt{(x-4)^2+(y+5)^2}=3 \,/^2$

$(x-4)^2+(y+5)^2=9$

kružnica : $S(4,-5)$

$r=3$



b) $|z+i|+|z-i|=3$

$|x+yi+i|+|x+yi-i|=3$

$\sqrt{x^2+(y+1)^2}=3-\sqrt{x^2+(y-1)^2} \,/^2$

$\cancel{x^2}+(y+1)^2=9-6\sqrt{x^2+(y-1)^2}+\cancel{x^2}+(y-1)^2$

$(y+1)^2-(y-1)^2-9=-6\sqrt{x^2+(y-1)^2}$

$\cancel{y^2}+2y+\cancel{1}-\cancel{y^2}+2y-\cancel{1}-9=-6\sqrt{x^2+(y-1)^2}$

$4y-9=-6\sqrt{x^2+(y-1)^2} \,/^2$

$16y^2-72y+81=36x^2+36(y-1)^2$

$16y^2-\cancel{72y}+81-36x^2-36y^2+\cancel{72y}-36=0$

$-36x^2-20y^2+45=0$

$36x^2+20y^2=45$

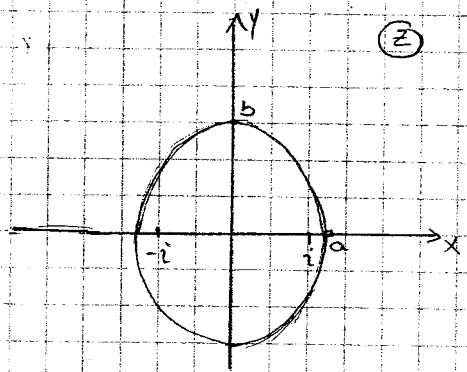
$$\frac{x^2}{\frac{45}{36}}+\frac{y^2}{\frac{45}{20}}=1$$

$$\frac{x^2}{\frac{5}{4}}+\frac{y^2}{\frac{9}{4}}=1$$

elipsa : $a^2=\frac{5}{4} \Rightarrow a=\frac{\sqrt{5}}{2}=1{,}12$

$b^2=\frac{9}{4} \Rightarrow b=\frac{3}{2}=1{,}5$

$e=\sqrt{\frac{5}{4}-\frac{9}{4}}=\sqrt{-\frac{4}{4}}=\pm i$



10. Skicirati :

$$\operatorname{Im}\left(\frac{1}{z+i}\right)=1$$

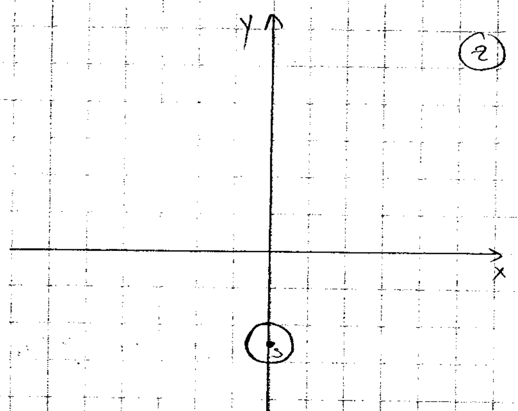
$$\frac{1}{z+i}=\frac{1}{x+(y+1)i}\cdot\frac{x-i(y+1)}{x-i(y+1)}=\frac{x-i(y+1)}{x^2+(y+1)^2}\Rightarrow \operatorname{Im}\left(\frac{1}{z+i}\right)=\frac{-y-1}{x^2+(y+1)^2}=1$$

$-y-1=x^2+y^2+2y+1$

$x^2+y^2+3y+2=0$

$x^2+\left(y+\frac{3}{2}\right)^2=\frac{1}{4}$

kružnica : $S\left(0,-\frac{3}{2}\right)$, $r=\frac{1}{2}$



11. $\arg z = \frac{2\pi}{3}$ $\rightarrow z = 2\sqrt{3}\cdot\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2\sqrt{3}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) =$

$|z| = 2\sqrt{3}$ $= -\sqrt{3} + 3i$

$z - 4i = ?$ TRIG.

$\rightarrow z - 4i = -\sqrt{3} + 3i - 4i = -\sqrt{3} - i$

Samo je jedan Mali Ivica!

$\rightarrow |z| = \sqrt{3+1} = 2$, III kvadrant

$tg\varphi = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \Rightarrow \varphi = \cancel{\frac{\pi}{6}}, \frac{7\pi}{6}$

$z - 4i = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

12.

a) $z = \dfrac{(\sqrt{3} - i)^7}{\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}}$

*) $\sqrt{3} - i = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$

$|z| = 2$, IV kvadrant

$tg\varphi = -\frac{\sqrt{3}}{3} \Rightarrow \varphi = -\frac{\pi}{6} + 2\pi = \frac{11\pi}{6}$

$$z = \frac{\left[2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)\right]^7}{\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}} = \frac{2^7\cdot\left(\cos\frac{77\pi}{6} + i\sin\frac{77\pi}{6}\right)}{\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}} =$$

$$= \frac{2^7\cdot\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}{\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}} = 2^7\left(\cos\left(\frac{5\pi}{6} - \frac{\pi}{12}\right) + i\sin\left(\frac{5\pi}{6} - \frac{\pi}{12}\right)\right) =$$

$$= 2^7\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = 2^7\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 2^6\left(-\sqrt{2} + i\sqrt{2}\right)$$

b) $z = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{17}\cdot\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$

*) $-\frac{1}{2} + i\frac{\sqrt{3}}{2} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$|z| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$, II kvadrant

$tg\varphi = \dfrac{\frac{\sqrt{3}}{\cancel{2}}}{-\frac{1}{\cancel{2}}} = -\sqrt{3} \Rightarrow \varphi = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$

$$z = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^{17}\cdot\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right) =$$

$$= \left(\cos\frac{34\pi}{3} + i\sin\frac{34\pi}{3}\right)\cdot\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right) =$$

$$= \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)\cdot\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right) =$$

$$= \cos\left(\frac{4\pi}{3}+\frac{5\pi}{12}\right)+i\sin\left(\frac{4\pi}{3}+\frac{5\pi}{12}\right)=\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}=\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$$

13.

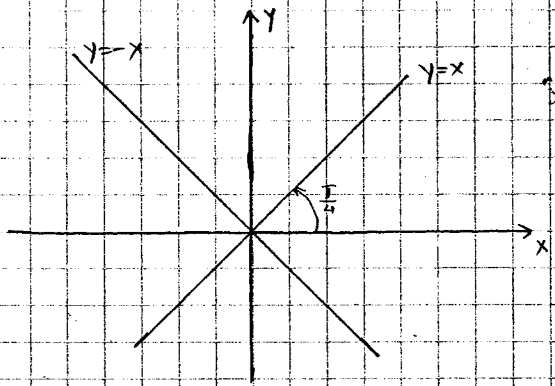
a) $\arg(z^4)=\pi$

$4\arg z=\pi+2k\pi$

$\arg z=\frac{\pi}{4}+\frac{k\pi}{2}$

↳ pravac $-y=x$

$-y=-x$



b) $\arg(z^6)=\pi$

$6\arg z=\pi+2k\pi$

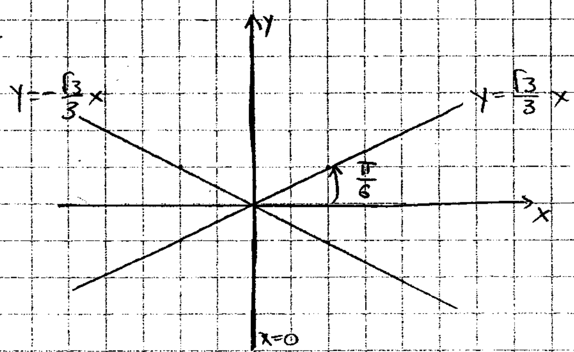
$\arg z=\frac{\pi}{6}+\frac{k\pi}{3}$

↳ pravac $-x=0$

$\frac{y}{x}=\frac{\sqrt{3}}{3}$

$-\ y=\frac{\sqrt{3}}{3}x$

$-\ y=-\frac{\sqrt{3}}{3}y$



14.

a) $z=\sqrt[3]{i}=\sqrt[3]{\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}}$

$*)\ i=?$

$|z|=1$

$\operatorname{tg}\varphi=\frac{1}{0}\Rightarrow\varphi=\frac{\pi}{2}$

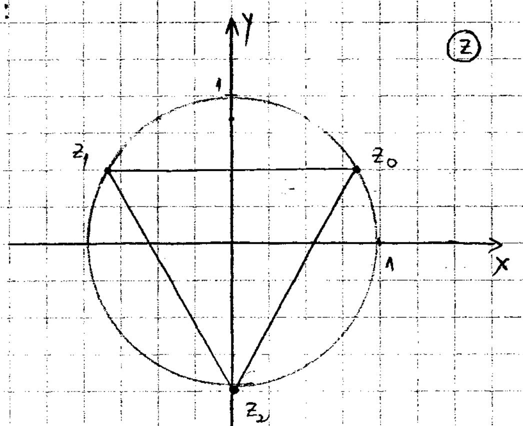
opći: $\sqrt[3]{1}\left(\cos\frac{\frac{\pi}{2}+2k\pi}{3}+i\sin\frac{\frac{\pi}{2}+2k\pi}{3}\right),\ k=0,1,2$

$k=0\ \ldots\ z_0=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}+i\frac{1}{2}$

$k=1\ \ldots\ z_1=\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}+i\frac{1}{2}$

$k=2\ \ldots\ z_2=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}=-i$

skica:



b) $z = \sqrt[7]{1} = \sqrt[7]{\cos\pi + i\sin\pi}$

*) $1 = ?$

$|z| = 1$

$tg\varphi = \frac{0}{1} \Rightarrow \varphi = \pi$

opći: $\sqrt[7]{1}\left(\cos\frac{\pi + 2k\pi}{7} + i\sin\frac{\pi+2k\pi}{7}\right)$, $k = 0,1,2,3,4,5,6$

$k=0 \ldots\ z_0 = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$

$k=1 \ldots\ z_1 = \cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}$

$k=2 \ldots\ z_2 = \cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}$

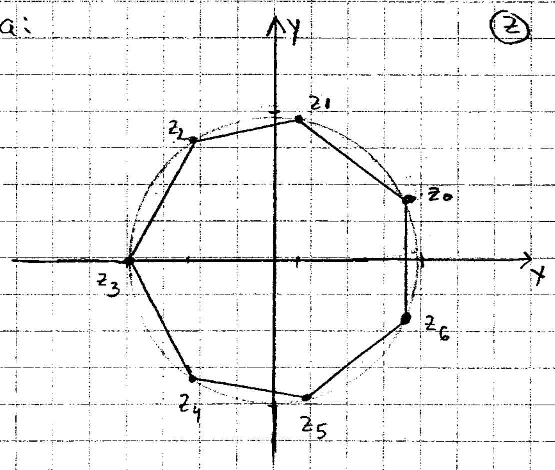
$k=3 \ldots\ z_3 = \cos\pi + i\sin\pi$

$k=4 \ldots\ z_4 = \cos\frac{9\pi}{7} + i\sin\frac{9\pi}{7}$

$k=5 \ldots\ z_5 = \cos\frac{11\pi}{7} + i\sin\frac{9\pi}{7}$

$k=6 \ldots\ z_6 = \cos\frac{13\pi}{7} + i\sin\frac{9\pi}{7}$

skica:



c) $z = \sqrt[8]{-1-\sqrt{3}i} = \sqrt[8]{2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)}$

*) $-1-\sqrt{3}i = ?$

$|z| = 2$, III kvadrant

$tg\varphi = \sqrt{3} \Rightarrow \varphi = \cancel{\frac{\pi}{3}}, \frac{4\pi}{3}$

opći: $\sqrt[8]{2}\cdot\left(\cos\frac{\frac{4\pi}{3}+2k\pi}{8} + i\sin\frac{\frac{4\pi}{3}+2k\pi}{8}\right)$, $k=0,1,2,3,4,5,6,7$

$k=0 \ldots\ z_0 = \sqrt[2]{8}\cdot\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$k=1 \ldots\ z_1 = \sqrt[2]{8}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$

$k=2 \ldots\ z_2 = \sqrt[2]{8}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$k=3 \ldots\ z_3 = \sqrt[2]{8}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$

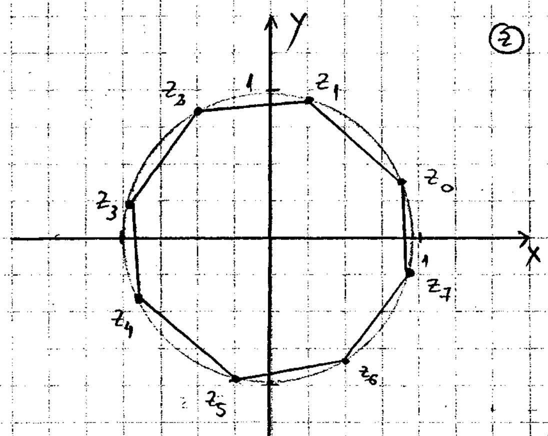
$k=4 \ldots\ z_4 = \sqrt[2]{8}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

$k=5 \ \dots \ z_5 = \sqrt[2]{8}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$

$k=6 \ \dots \ z_6 = \sqrt[2]{8}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

$k=7 \ \dots \ z_7 = \sqrt[2]{8}\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right)$

skica:



(15.) $z^4 = \left(\left(\sqrt{3}+i\right)\left(1-i\right)^7\right)^8$

*) $\sqrt{3}+i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$|z| = 2$

$tg\varphi = \frac{\sqrt{3}}{3} \Rightarrow \varphi = \frac{\pi}{6}$

*) $1-i = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

$|z| = \sqrt{2}$, $\overline{\underline{IV}}$ kv.

$tg\varphi = -1 \Rightarrow \varphi = -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$

$z^4 = \left(2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\cdot\left(\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)\right)^7\right)^8 =$

$= \left(2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\cdot\left(8\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)\right)^8 =$

$= \left(16\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)\right)^8 =$

$= 16^9\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$\Rightarrow z = \sqrt[4]{16^9}\cdot\left(\cos\frac{\frac{4\pi}{3}+2k\pi}{4} + i\sin\frac{\frac{4\pi}{3}+2k\pi}{4}\right)$, $k = 0,1,2,3$

$k=0 \ \dots \ z_0 = 512\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$k=1 \ \dots \ z_1 = 512\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$k=2 \ \dots \ z_2 = 512\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$k=3 \ \dots \ z_3 = 512\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$

(16.) $z^6 + z^4 + z^2 + 1 = 0$

$z^4(z^2+1) + (z^2+1) = 0$

$(z^4+1)(z^2+1) = 0$

*) $z^4 + 1 = 0$ $\qquad -1 = ?$

$z^4 = -1$ $\qquad |z| = 1$

$z = \sqrt[4]{-1}$ $\qquad tg\varphi = \frac{0}{1} \Rightarrow \varphi = \pi$

$z = \sqrt[4]{\cos\pi + i\sin\pi}$

opći : $z = \cos\frac{\pi + 2k\pi}{4} + i\sin\frac{\pi + 2k\pi}{4}$, $k = 0, 1, 2, 3$

$k=0 \ldots\ z_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$k=1 \ldots\ z_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$k=2 \ldots\ z_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$k=3 \ldots\ z_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

*) $z^2 + 1 = 0$ $\qquad z = \sqrt{\cos\pi + i\sin\pi}$

$z^2 = -1$

$z = \sqrt{-1}$

opći : $z = \cos\frac{\pi + 2k\pi}{2} + i\sin\frac{\pi + 2k\pi}{2}$, $k = 0, 1$

$k=0 \ldots\ z_4 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$k=1 \ldots\ z_5 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$

17. $z^8 + 2x^4 + 4 = 0$

$z^4 = t$

$t^2 + 2t + 4 = 0$

$t_{1,2} = \frac{-2 \pm \sqrt{4-16}}{2} = \frac{-2 \pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}i$

$t_1 = -1 + \sqrt{3}i$

$z^4 = -1 + \sqrt{3}i$

$z = \sqrt[4]{-1+\sqrt{3}i}$

$\hookrightarrow |z| = 2$, II kv.

$tg\,\varphi = \sqrt{3} \to \varphi = \cancel{\frac{\pi}{3}}, \frac{2\pi}{3}$

$z = \sqrt[4]{2}\left(\cos\frac{\frac{2\pi}{3} + 2k\pi}{4} + i\sin\frac{\frac{2\pi}{3} + 2k\pi}{4}\right)$

$k=0 \ldots \sqrt[4]{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$k=1 \ldots \sqrt[4]{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$k=2 \ldots \sqrt[4]{2}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

$k=3 \ldots \sqrt[4]{2}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

$t_2 = -1 - \sqrt{3}i$

$z^4 = -1 - \sqrt{3}i$

$z = \sqrt[4]{-1-\sqrt{3}i}$

$\hookrightarrow |z| = 2$, III kv.

$tg\,\varphi = \sqrt{3} \to \varphi = \frac{4\pi}{3}$

$z = \sqrt[4]{2}\left(\cos\frac{\frac{4\pi}{3} + 2k\pi}{4} + i\sin\frac{\frac{4\pi}{3} + 2k\pi}{4}\right)$

$k=0 \ldots \sqrt[4]{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$k=1 \ldots \sqrt[4]{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$k=2 \ldots \sqrt[4]{2}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$k=3 \ldots \sqrt[4]{2}\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$

18. $z^{10} + \bar{z} = 0 \quad / \cdot z$

$z^{11} + |z|^2 = 0$

$z^{11} = -|z|^2 \qquad \to z_0 = 0$

$r^{\cancel{11}}(\cos 11\varphi + i\sin 11\varphi) = -\cancel{r^2}$

$r^9(\cos 11\varphi + i\sin 11\varphi) = -1$

$r^9(\cos 11\varphi + i\sin 11\varphi) = \cos\pi + i\sin\pi$

$\hookrightarrow r^9 = 1 \qquad 11\varphi = \pi + 2k\pi$

$r = 1 \qquad \varphi = \frac{\pi}{11} + \frac{2k\pi}{11}$, $k = 0 - 10$

$z_1 = \cos\frac{\pi}{11} + i\sin\frac{\pi}{11}$

$z_2 = \cos\frac{3\pi}{11} + i\sin\frac{3\pi}{11}$

$z_3 = \cos\frac{5\pi}{11} + i\sin\frac{5\pi}{11}$

$z_4 = \cos\frac{7\pi}{11} + i\sin\frac{7\pi}{11}$

$z_5 = \cos\frac{9\pi}{11} + i\sin\frac{9\pi}{11}$

$z_6 = \cos\pi + i\sin\pi$

$z_7 = \cos\frac{13\pi}{11} + i\sin\frac{13\pi}{11}$

$z_8 = \cos\frac{15\pi}{11} + i\sin\frac{15\pi}{11}$

$z_9 = \cos\frac{17\pi}{11} + i\sin\frac{17\pi}{11}$

$z_{10} = \cos\frac{19\pi}{11} + i\sin\frac{19\pi}{11}$

$z_{11} = \cos\frac{21\pi}{11} + i\sin\frac{21\pi}{11}$

19. $\arg z = \frac{2\pi}{3}$, II kvadrant !!

$|z+1| = \sqrt{7}$

I) $\frac{y}{x} = -\sqrt{3} \Rightarrow y = -\sqrt{3}x$

II) $|z+1| = \sqrt{7}$

$|x+yi+1| = \sqrt{7}$

$\sqrt{(x+1)^2+y^2} = \sqrt{7} \quad |^2$

$(x+1)^2+y^2 = 7$

I i II

$(x+1)^2 + 3x^2 = 7$

$x^2+2x+1+3x^2-7=0$

$4x^2+2x-6=0$

$2x^2+x-3=0$

$x_{1,2} = \frac{-1 \pm \sqrt{1+24}}{4} = \frac{-1 \mp 5}{4}$

$x_1 = 1 \rightarrow y_1 = -\sqrt{3} \qquad \cancel{z_1 = 1 - i\sqrt{3}}$

$x_2 = -\frac{3}{2} \rightarrow y_2 = \frac{3\sqrt{3}}{2} \qquad z_2 = -\frac{3}{2} + i\frac{3\sqrt{3}}{2}$

20. $|z^4+i| = |z^4+1| = 1$

I) $|z+i| = |z+1|$

$|R+Ii+i| = |R+Ii+1|$

$\sqrt{R^2+(I+1)^2} = \sqrt{(R+1)^2+I^2} \quad |^2$

$\cancel{R^2}+\cancel{I^2}+2I+\cancel{1} = \cancel{R^2}+2R+\cancel{1}+\cancel{I^2}$

$\cancel{2}I = \cancel{2}R$

$I = R$

II $|z+i| = 1$

$|R+Ii+i| = 1$

$\sqrt{R^2+(I+1)^2} = 1 \quad |^2$

$R^2+I^2+2I+\cancel{1} = \cancel{1}$

$R^2+I^2+2I=0$

- iz I $\rightarrow I=R$

$R^2+R^2+2R=0$

$2R^2+2R=0$

$2R(R+1)=0$

$R=0 \rightarrow I=0$

$R=-1 \rightarrow I=-1$

1) $z^4=0 \qquad \rightarrow z_0=0$

2) $z^4=-1-i$

$z = \sqrt[4]{-1-i}$

$\hookrightarrow |z| = \sqrt{2}$, 3kv.

$tg\varphi = 1 \Rightarrow \varphi = \cancel{\frac{\pi}{4}}, \frac{3\pi}{4}$

$z = \sqrt[8]{2}\left(\cos\frac{\frac{3\pi}{4}+2k\pi}{4} + i\sin\frac{\frac{3\pi}{4}+2k\pi}{4}\right)$, $k=0,1,2,3$

$k=0 \ldots z_1 = \sqrt[8]{2}\left(\cos\frac{3\pi}{16} + i\sin\frac{3\pi}{16}\right)$

$k=1 \ldots z_2 = \sqrt[8]{2}\left(\cos\frac{11\pi}{16} + i\sin\frac{11\pi}{16}\right)$

$k=2 \ldots z_3 = \sqrt[8]{2}\left(\cos\frac{19\pi}{16} + i\sin\frac{19\pi}{16}\right)$

$k=3 \ldots z_4 = \sqrt[8]{2}\left(\cos\frac{27\pi}{16} + i\sin\frac{27\pi}{16}\right)$

3. MATRICE

Samo je jedan Mali Ivica!

3.1. DEFINICIJA I PRIMJERI MATRICA

= DEFINICIJA 1.

↳ **Matrica** je pravokutna tablica realnih brojeva sa m redaka i n stupaca. Posebice, ako je $m=n$, matrica se zove kvadratna matrica reda n.

= PRIMJER 1.

↳ Matrice primjeri:

$A=\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 3 \end{bmatrix}$ → matrica tipa (formata) 2×3, (2×3), tzv. pravokutna matrica

$B=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ → matrica tipa 2×1, (2×1), tzv. vektor stupac

= Zapis matrice:

$$A=\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & c_{ij} & \vdots \\ c_{m1} & \cdots & \cdots & c_{mn} \end{bmatrix} \qquad A=\begin{bmatrix} & & a_{1j} & & \\ & & \vdots & & \\ a_{i1} & a_{i2} \ldots & a_{ij} & \ldots & a_{in} \\ & & \vdots & & \\ & & a_{nj} & & \end{bmatrix}$$

→ i-ti redak od A

↳ j-ti stupac od A

= Zapis elementa matrice:

$A=(a_{ij})$

- A → matrica
- i → indeks retka
- j → indeks stupca
- a_{ij} → opći element matrice

$m\Big\{\boxed{\;A\;}$ (širina n) → za isticanje formata matrice A, $m\times n$

= DEFINICIJA 2.

↳ Jednakost matrica:

Dvije matrice, A i B su jednake, oznakom $A=B$, ako su istog tipa i ako je $(A)_{ij}=(B)_{ij}$ $\forall i\ \forall j$.

□ PRIMJERI MATRICA

1) Nul-matrica

↳ Matrica čiji su svi elementi nule naziva se nul-matrica, oznaka **0**.

↳ Zapis:

$$\mathbf{0}=\begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}$$

2) Dijagonalna matrica

↳ Matrica koja je isključivo kvadratna i kojoj su svi elementi van dijagonale jednaki nuli $(a_{ij}=0,\ i\neq j)$

↳ Zapis:

$$A=\begin{bmatrix} c_{11} & & 0 \\ & c_{22} & \\ 0 & & \ddots \;\; c_{nn} \end{bmatrix}$$

↳ Primjer:

$$A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

3) Jedinična matrica

↳ Jedinična matrica je dijagonalna matrica čiji su dijagonalni elementi jednaki jedan, oznaka **I, E.**

↳ Zapis:

$$I=(\delta_{ij}),\quad \delta_{ij}=\begin{cases}1 & za\ i=j\\ 0 & za\ i\neq j\end{cases}\quad (\delta_{ij})\text{- Kroneckerov simbol}$$

↳ Primjer:

$$A=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

4) Trokutasta matrica

→ Kvadratna matrica je gornja trokutasta ako su svi elementi ispod glavne dijagonale jednaki nula.

Zapis:

$$A=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\0&a_{22}&&\vdots\\0&0&\ddots&\vdots\\0&0&0&a_{nn}\end{bmatrix}\rightarrow \text{gornji trokut}$$

→ Kvadratna matrica je donja trokutasta ako su svi elementi iznad glavne dijagonale jednaki nula.

Zapis:

$$A=\begin{bmatrix}a_{11}&0&0&0\\a_{21}&a_{22}&0&0\\\vdots&&\ddots&0\\a_{n1}&\cdots&\cdots&a_{nn}\end{bmatrix}\rightarrow \text{donji trokut}$$

5) Transponirana matrica

↳ Matrica B je transponirana matrica matrice A ako vrijedi da je: $(B)_{ij}=(A)_{ji},\ \forall i\ \forall j$ ili $b_{ij}=a_{ji}$.

↳ Postupak: Elemente prvog retka matrice A zapišemo na mjesto prvog stupca matrice B, i tako redom.

↳ Zapis:

$$A=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix}\qquad A^T=\begin{bmatrix}a_{11}&a_{21}&\cdots&a_{m1}\\a_{12}&a_{22}&\cdots&a_{m2}\\\vdots&\vdots&\ddots&\vdots\\a_{1n}&a_{2n}&\cdots&a_{mn}\end{bmatrix}$$

6) Vektor matrica

↳ Matrica koja ima samo 1 redak/stupac, pa govorimo o vektor-retku, vektor-stupcu.

↳ Zapis:

$$A=\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix},\qquad A=\begin{bmatrix}a_1&a_2&\cdots&a_n\end{bmatrix}$$

↳ Transponiranje:

$$A=\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}\qquad A^T=\begin{bmatrix}a_1&a_2&\cdots&a_n\end{bmatrix}$$

7) Simetrična matrica

↳ Kvadratna matrica koja se zrcaljenjem s obzirom na dijagonalu ne mijenja, tj. $A^T=A$.

↳ Primjer:

$$A=\begin{bmatrix}1&0&2\\0&1&0\\2&0&1\end{bmatrix}$$

= PRIMJER 2.

↳ Transponiraj:

$$A = \begin{bmatrix} 2 & 7 \\ 8 & 3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 8 \\ 7 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{(2\times 3)} \qquad B^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{(3\times 2)}$$

3.2. OPERACIJE S MATRICAMA

= Skup svih matrica istog tipa $m \times n$ označavamo sa $\mathcal{M}_{mn}$.

= DEFINICIJA 1.

↳ Na skupu $\mathcal{M}_{mn}$ definiraju se dvije operacije:

1) zbrajanje matrica

→ zbroj dviju matrica A i B je matrica s elementima:

$$(A+B)_{ij} := (A)_{ij} + (B)_{ij}.$$

Da bi zbroj matrica bio definiran, obje matrice moraju biti istog tipa. Rezultat zbrajanja je opet matrica, istog tipa $m \times n$.

2) množenje matrice skalarom

→ Neka je $\lambda \in \mathbb{R}$ bilo koji skalar, te $A \in \mathcal{M}_{mn}$. Umnožak matrice A skalarom λ je matrica $\lambda \cdot A$ tipa $m \times n$ tako da je:

$$(\lambda A)_{ij} = \lambda \cdot (A)_{ij} \quad \forall_i \ \forall_j$$

Dakle, matrica se množi skalarom tako da se svaki njezin element pomnoži tim skalarom.

= PRIMJER 1.

a) zbrajanje matrica:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 4 & 6 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \text{nije definiran}$$

b)

$$5 \cdot \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 15 \\ 0 & 20 & 35 \end{bmatrix}$$

$$(-1) \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} = -A$$

= PRIMJEDBA 1.

↳ Matrice tipa $\mathcal{M}_{mn}$ čine komutativnu grupu s obzirom na operaciju zbrajanja matrica.

Postoji:
- neutralni element je nul-matrica
- suprotni element je $-A$ matrica

Vrijedi: $A + (-A) = 0$

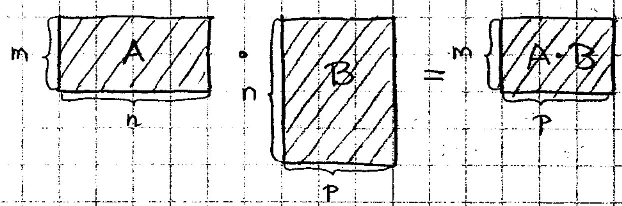
3.3. ALGEBRA MATRICA

= DEFINICIJA 1.

↳ Množenje matrica:

→ Da bi postojao umnožak dviju matrica, one moraju biti ulančane, tj. broj stupaca prve matrice mora biti jednak broju redaka druge matrice.

Grafički:



Znači matrica A se množi s B po tipovima ovako:

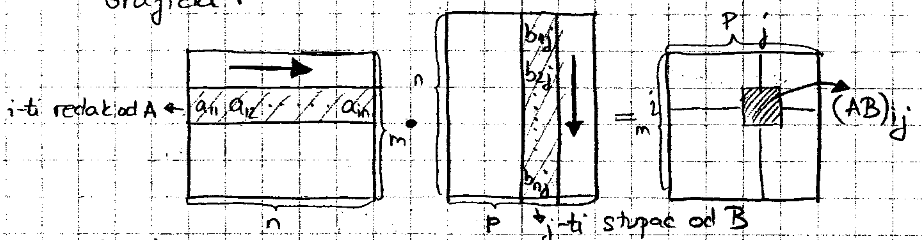
$$(m,n)\cdot(n,p)=(m,p)$$

→ Neka je $A=(a_{ij})$ matrica tipa $m\times n$, i matrica $B=(b_{ij})$ matrica tipa $n\times p$. Opći član umnoška $A\cdot B$ dan je formulom:

$$(A\cdot B)_{ij} := a_{i1}\cdot b_{1j}+a_{i2}\cdot b_{2j}+\dots+a_{in}\cdot b_{nj}$$

$$(A\cdot B)_{ij} := \sum_{k=1}^{n} a_{ik}\cdot b_{kj}$$

Grafički:



→ $(AB)_{ij}$ je skalarni produkt dva vektora i to i-tog retka od A sa j-tim stupcem od B.

= PRIMJER 1.

a)

$$\underset{(2,2)}{\begin{bmatrix}1&2\\0&1\end{bmatrix}}\cdot\underset{(2,2)}{\begin{bmatrix}0&0\\1&0\end{bmatrix}}=\underset{(2,2)}{\begin{bmatrix}1\cdot0+2\cdot1&1\cdot0+2\cdot0\\0\cdot0+1\cdot1&0\cdot0+1\cdot0\end{bmatrix}}=\begin{bmatrix}2&0\\1&0\end{bmatrix}$$

b)

$$\underset{(2,2)}{\begin{bmatrix}0&0\\1&0\end{bmatrix}}\cdot\underset{(2,2)}{\begin{bmatrix}1&2\\0&1\end{bmatrix}}=\underset{(2,2)}{\begin{bmatrix}0\cdot1+0\cdot0&0\cdot2+0\cdot1\\1\cdot1+0\cdot0&1\cdot2+0\cdot1\end{bmatrix}}=\begin{bmatrix}0&0\\1&2\end{bmatrix}$$

= PRIMJER 2.

$$\begin{bmatrix}-1&1\\3&-3\end{bmatrix}\cdot\begin{bmatrix}2&3\\2&3\end{bmatrix}=\begin{bmatrix}-2+2&-3+3\\6-6&9-9\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}=0$$

= PRIMJER 3.

$$\underset{(2,3)}{\begin{bmatrix}2&3&-1\\0&1&2\end{bmatrix}}\cdot\underset{(3,2)}{\begin{bmatrix}3&-1\\2&0\\-1&1\end{bmatrix}}=\underset{(2,2)}{\begin{bmatrix}6+6+1&-2+0-1\\0+2-2&0+0+2\end{bmatrix}}=\begin{bmatrix}13&-3\\0&2\end{bmatrix}$$

= Neuobičajena svojstva matrica:

1) ako je umnožak $A\cdot B$ definiran, onda $B\cdot A$ ne mora biti definiran.
2) ako postoje matrice AB i BA vrijedi $AB\neq BA$.
3) ako je $A\cdot B=0$, tada ne mora biti $A=0$ niti $B=0$.

Samo je jedan Mali Ivica!

SVOJSTVA MATRIČNOG MNOŽENJA

1) asocijativnost matričnog množenja

= STAVAK 1.

↳ Vrijedi: $(AB)C = A(BC)$, ako su umnošci definirani s obje strane znaka jednakosti.

DOKAZ:

Neka je $A=(a_{ij})$ tipa $m \times n$
$B=(b_{jk})$ tipa $n \times p$
$C=(c_{kl})$ tipa $p \times r$

I provjera tipa:

LIJEVA STRANA: $[(m,n)(n,p)]\cdot(p,r) = (m,p)(p,r) = (m,r)$

DESNA STRANA: $(m,n)\cdot[(n,p)(p,r)] = (m,n)(n,r) = (m,r)$ ✓

II provjera općih elemenata

LIJEVA STRANA:

$$[(AB)\cdot C]_{il} = \sum_{k=1}^{p} (AB)_{ik}\cdot c_{kl} = \sum_{k=1}^{p}\left(\sum_{j=1}^{n} a_{ij} b_{jk}\right)\cdot c_{kl} =$$

$$= \sum_{k=1}^{p}\sum_{j=1}^{n} a_{ij} b_{jk} c_{kl}$$

DESNA STRANA:

$$[A(BC)]_{il} = \sum_{k=1}^{n} a_{ik}\cdot (BC)_{kl} = \sum_{k=1}^{n}\sum_{s=1}^{p} b_{ks} c_{sl} = \sum_{k=1}^{n}\sum_{s=1}^{p} a_{ik} b_{ks} c_{sl} =$$

$$\left(\text{zamjena: } k \to j,\ s \to k\right) =$$

$$= \sum_{k=1}^{p}\sum_{j=1}^{n} a_{ij} b_{jk} c_{kl}$$

2) distributivnost množenja prema zbrajanju

↳ Vrijedi:

a) $(A+B)\cdot C = A\cdot C + B\cdot C$ → desna distributivnost

b) $C\cdot(A+B) = C\cdot A + C\cdot B$ → lijeva distributivnost

DOKAZ:

$$[(A+B)\cdot C]_{ik} = \sum_{j=1}^{n} (A+B)_{ij}\cdot c_{jk} = \sum_{j=1}^{n} (a_{ij}+b_{ij})c_{jk} =$$

$$= \sum_{j=1}^{n} a_{ij} c_{jk} + \sum_{j=1}^{n} b_{ij} c_{jk} = (AC)_{ik} + (BC)_{ik} =$$

$$= [AC + BC]_{ik}$$

3.) umnožak s jediničnom matricom

= STAVAK 2.

↳ Ako je I jedinična matrica reda n, te A kvadratna matrica reda n, tada je:

$$A\cdot I = I\cdot A = A$$

Odnosno, jedinična matrica je neutralni element.

DOKAZ:

$$(A\cdot I)_{ij} = \sum_{k=1}^{n} a_{ik}(\delta_{kj}) = 0+0+\ldots+a_{ij}\cdot 1 + 0 + 0 = a_{ij} = (A)_{ij}$$

$$(I A)_{ij} = \sum_{k=1}^{n}(\delta_{kj})a_{ik} = 0+0+\ldots+1\cdot a_{ij}+0+0 = a_{ij} = (A)_{ij}$$ ✓

4) **transponiranje umnoška matrica**
↳ Vrijedi:
$$(A\cdot B)^T = B^T\cdot A^T$$

DOKAZ:

I provjera tipa: A tipa (m,n)
B tipa (n,p)

LIJEVA STRANA: $AB = (m,n)(n,p) = (m,p)$
$(AB)^T = (p,m)$

DESNA STRANA: $B^TA^T = (p,n)(n,m) = (p,m)$

II provjera općeg elementa:
$$\left[(AB)^T\right]_{ik} = (AB)_{ki} = \sum_{j=1}^{n} a_{kj}\, b_{ji} = \sum_{j=1}^{n} (B^T)_{ij}(A^T)_{jk} = (B^TA^T)_{ik}$$

5) **potenciranje matrica**
↳ Kvadrat matrice definiran je samo za kvadratne matrice, i to indukt. tivno sve potencije matrice A definirane su samo za kvadratne matrice, na način:
$$A^2 := A\cdot A$$
$$A^s := A^{s-1}\cdot A = \underbrace{A\cdot A\cdots A}_{s\text{-faktora}}$$

6) **matrični polinom**
= DEFINICIJA 2.
↳ Neka je $f(x) = a_p x^p + \dots + a_1 x + a_0$, bilo koji polinom stupnja p, i neka je $A^0 = I$. Tada definiramo:
$$f(A) := a_p A^p + \dots + a_1 A + a_0 I$$

= PRIMJER 4.
$$f(x) = x^2 - 4x + 7,\quad A = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix},\quad f(A) = ?$$

$$f(A) = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\cdot\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} - 4\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} + 7\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4-3 & 2+2 \\ -6-6 & -3+4 \end{bmatrix} - \begin{bmatrix} 8 & 4 \\ -12 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} =$$
$$= \begin{bmatrix} 1 & 4 \\ -12 & 1 \end{bmatrix} + \begin{bmatrix} -8 & -4 \\ 12 & -8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = 0$$

= Vrijedi:
$$A^n = \begin{bmatrix} a^n & n\cdot a^{n-1} \\ 0 & a^n \end{bmatrix},\quad A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$$

Dokaz: Matematičkom indukcijom:

① baza indukcije:
$n=1$ ...
$$\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^1 & 1\cdot a^0 \\ 0 & a^1 \end{bmatrix} = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \checkmark$$

② pretpostavka indukcije:
Pretpostavimo da za neki $n \in \mathbb{N}$ vrijedi: $A^n = \begin{bmatrix} a^n & n\cdot a^{n-1} \\ 0 & a^n \end{bmatrix}$.

③ skok indukcije
Treba dokazati da tvrdnja vrijedi i za $n+1$:
$$A^{n+1} = A^n\cdot A = \begin{bmatrix} a^n & na^{n-1} \\ 0 & a^n \end{bmatrix}\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^n a & a^n + na^{n-1}\cdot a \\ 0 & a^n\cdot a \end{bmatrix} =$$
$$= \begin{bmatrix} a^{n+1} & (n+1)\,a^n \\ 0 & a^{n+1} \end{bmatrix} \checkmark$$

Samo je jedan Mali Ivica!

ZADACI (DZ. 3. PRVI DIO)

1. $X = ?$ $\quad AX = A, \quad A = \begin{bmatrix} 1 & 2 \end{bmatrix}$

$$X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow \quad A \cdot X = A$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} + 2a_{21} & a_{12} + 2a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$a_{11} + 2a_{21} = 1 \quad \Rightarrow a_{11} = 1 - 2a_{21}$$
$$a_{12} + 2a_{22} = 2 \quad \Rightarrow a_{12} = 2 - 2a_{22}$$

$$X = \begin{bmatrix} 1-2a_{21} & 2-2a_{22} \\ a_{21} & a_{22} \end{bmatrix}, \quad a_{21}, a_{22} \in \mathbb{R}$$

2. $X = ?$

$$X \cdot \begin{bmatrix} 3 & -2 & 1 \\ -3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -4 & -1 \\ 3 & 0 & 1 \\ 6 & -8 & 2 \end{bmatrix}$$

Tip: $(?,?) \cdot (2,3) = (3,3)$
$(?,?) = (3,2)$

$$X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 & 1 \\ -3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -4 & -1 \\ 3 & 0 & 1 \\ 6 & -8 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3a_{11} - 3a_{12} & -2a_{11} & a_{11} - a_{12} \\ 3a_{21} - 3a_{22} & -2a_{21} & a_{21} - a_{22} \\ 3a_{31} - 3a_{32} & -2a_{31} & a_{31} - a_{32} \end{bmatrix} = \begin{bmatrix} -3 & -4 & -1 \\ 3 & 0 & 1 \\ 6 & -8 & 2 \end{bmatrix}$$

1. $3a_{11} - 3a_{12} = -3$
   $-2a_{11} = -4 \quad \Rightarrow a_{11} = 2$
   $a_{11} - a_{12} = -1 \quad \Rightarrow a_{12} = 2 + 1 = 3$

2. $3a_{21} - 3a_{22} = 3$
   $-2a_{21} = 0 \quad \Rightarrow a_{21} = 0$
   $a_{21} - a_{22} = 1 \quad \Rightarrow a_{22} = 0 - 1 = -1$

3. $3a_{31} - 3a_{32} = 6$
   $-2a_{31} = -8 \quad \Rightarrow a_{31} = 4$
   $a_{31} - a_{32} = 2 \quad a_{32} = 4 - 2 = 2$

$$X = \begin{bmatrix} 2 & 3 \\ 0 & -1 \\ 4 & 2 \end{bmatrix}$$

3. $X = ?$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot X = \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Tip: $(3,3) \cdot (?,?) = (3,2)$
$(?,?) = (3,2)$

$$X=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\\a_{31}&a_{32}\end{bmatrix}\rightarrow\begin{bmatrix}0&1&0\\1&0&1\\0&0&0\end{bmatrix}\cdot\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\\a_{31}&a_{32}\end{bmatrix}=\begin{bmatrix}3&5\\0&1\\0&0\end{bmatrix}$$

$$\begin{bmatrix}a_{21}&a_{22}\\a_{11}+a_{31}&a_{12}+a_{32}\\0&0\end{bmatrix}=\begin{bmatrix}3&5\\0&1\\0&0\end{bmatrix}$$

$a_{21}=3$

$a_{22}=5$

$a_{11}+a_{31}=0 \Rightarrow a_{31}=-a_{11}$

$a_{12}+a_{32}=1 \quad a_{32}=1-a_{12}$

$$X=\begin{bmatrix}a_{11}&a_{12}\\3&5\\-a_{11}&1-a_{12}\end{bmatrix},\ a_{11},a_{12}\in\mathbb{R}$$

4. $X=?$

$$\begin{bmatrix}1&2\\2&4\end{bmatrix}\cdot X=\begin{bmatrix}1&2\\3&4\end{bmatrix}$$

tip: $(2,2)\cdot(?,?)=(2,2)$

$(?,?)=(2,2)$

$$X=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\rightarrow\begin{bmatrix}1&2\\2&4\end{bmatrix}\cdot\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}=\begin{bmatrix}1&2\\3&4\end{bmatrix}$$

$$\begin{bmatrix}a_{11}+2a_{21}&a_{12}+2a_{22}\\2a_{11}+4a_{21}&2a_{12}+4a_{22}\end{bmatrix}=\begin{bmatrix}1&2\\3&4\end{bmatrix}$$

$\underline{1}$ $a_{11}+2\cdot a_{21}=1$

$2a_{11}+4a_{21}=3$

$2(1-2a_{21})+4a_{21}=3$

$2-\cancel{4a_{21}}+\cancel{4a_{21}}=3$

$2=3$

$\underline{2}$ $a_{12}+2a_{22}=2$

$2a_{12}+4a_{22}=4$

$2(2-2a_{22})+4a_{22}=4$

$\cancel{4}-\cancel{4a_{22}}+\cancel{4a_{22}}=\cancel{4}$

$0=0$

Matrica X ne postoji.

5. $P(x)=x^2-6x+5 \qquad A=?$ (dijagonalna) $\rightarrow p(A)=0$

$$A=\begin{bmatrix}a_{11}&0\\0&a_{22}\end{bmatrix}\rightarrow P(A)=0$$

$$\begin{bmatrix}a_{11}&0\\0&a_{22}\end{bmatrix}\cdot\begin{bmatrix}a_{11}&0\\0&a_{22}\end{bmatrix}-6\begin{bmatrix}a_{11}&0\\0&a_{22}\end{bmatrix}+5\cdot\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\begin{bmatrix}a_{11}^2&0\\0&a_{22}^2\end{bmatrix}-\begin{bmatrix}6a_{11}&0\\0&6a_{22}\end{bmatrix}+\begin{bmatrix}5&0\\0&5\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\begin{bmatrix}a_{11}^2-6a_{11}+5&0\\0&a_{22}^2-6a_{22}+5\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$\underline{1}$ $a_{11}^2-6a_{11}+5=0$

$$a_{11(1,2)}=\frac{6\pm\sqrt{36-20}}{2}=\frac{6\pm4}{2}$$

$a_{11(1)}=1$

$a_{11(2)}=5$

$\underline{2}$ $a_{22}^2-6a_{22}+5=0$

$a_{22(1)}=1$

$a_{22(2)}=5$

$A_1=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $A_2=\begin{bmatrix}1&0\\0&5\end{bmatrix}$, $A_3=\begin{bmatrix}5&0\\0&1\end{bmatrix}$, $A_4=\begin{bmatrix}5&0\\0&5\end{bmatrix}$

Samo je jedan Mali Ivica!

6. $X=?$

$A\cdot X=X\cdot A$, $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$

$X=\begin{bmatrix}a&b\\c&d\end{bmatrix} \rightarrow \begin{bmatrix}1&2\\3&4\end{bmatrix}\cdot\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}\cdot\begin{bmatrix}1&2\\3&4\end{bmatrix}$

$\begin{bmatrix}a+2c&b+2d\\3a+4c&3b+4d\end{bmatrix}=\begin{bmatrix}a+3b&2a+4b\\c+3d&2c+4d\end{bmatrix}$

$a+2c=a+3b \Rightarrow 3b=2c \rightarrow b=\frac{2c}{3}=\frac{2\cdot 3b}{3}=2b$
$b+2d=2a+4b \Rightarrow 2a+3b=2d$
$3a+4c=c+3d \Rightarrow a+c=d \rightarrow d=a+3b$
$3b+4d=2c+4d \Rightarrow 3b=2c \rightarrow c=\frac{3b}{2}=\frac{3\cdot 2b}{2}=3b$

$X=\begin{bmatrix}a&2b\\3b&a+3b\end{bmatrix}$, $a,b\in\mathbb{R}$

7. $X=?$

$A\cdot X=X\cdot A$, $A=\begin{bmatrix}3&1&0\\0&3&1\\0&0&3\end{bmatrix}$

$X=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix} \rightarrow \begin{bmatrix}3&1&0\\0&3&1\\0&0&3\end{bmatrix}\cdot\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}\cdot\begin{bmatrix}3&1&0\\0&3&1\\0&0&3\end{bmatrix}$

$\begin{bmatrix}3a+d&3b+e&3c+f\\3d+g&3e+h&3f+i\\3g&3h&3i\end{bmatrix}=\begin{bmatrix}3a&a+3b&b+3c\\3d&d+3e&e+3f\\3g&g+3h&h+3i\end{bmatrix}$

$3a+d=3a \rightarrow d=0$
$3d+g=3d \rightarrow g=0$
$3g=3g \rightarrow /$
$3b+e=a+3b \rightarrow e=a$
$3e+h=d+3e \rightarrow h=d=0$
$3h=g+3h \rightarrow g=0$
$3c+f=b+3c \rightarrow f=b$
$3f+i=e+3f \rightarrow i=e=a$
$3i=h+3i \rightarrow h=0$

$X=\begin{bmatrix}a&b&c\\0&a&b\\0&0&a\end{bmatrix}$, $a,b,c\in\mathbb{R}$

8. $B=?$

$(A+B)^2=A^2+2AB+B^2$, $A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$

$\parallel$

$(A+B)^2=A^2+AB+BA+B^2$

↳ treba naći matricu B za koju vrijedi: $AB=BA$

$B=\begin{bmatrix}a&b\\c&d\end{bmatrix} \rightarrow AB=BA$

$$\begin{bmatrix}1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}$$

$$\begin{bmatrix}a+c&b+d\\c&d\end{bmatrix}=\begin{bmatrix}a&a+b\\c&c+d\end{bmatrix}$$

$\cancel{a}+c=\cancel{a} \quad \to c=0$
$\cancel{b}+d=a+\cancel{b} \quad \to a=d$
$c=c$
$\cancel{d}=c+\cancel{d} \quad \to c=0$

$$B=\begin{bmatrix}a&b\\0&a\end{bmatrix},\ a,b\in\mathbb{R}$$

9. Dokaži, opovrgni. Ako za matrice A i B vrijedi $A^2=B^2$ onda je $A=B$ ili $A=-B$.

↳ Tvrdnja ne vrijedi. Dovoljno je naći jedan protuprimjer. Uzmimo na primjer matrice:

$$A=\begin{bmatrix}0&1\\0&0\end{bmatrix} \text{ i } B=\begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

Vidimo da: $A\neq B$, $A\neq -B$.
Izračunajmo A^2 i B^2.

$$\left.\begin{aligned}A^2&=\begin{bmatrix}0&1\\0&0\end{bmatrix}\cdot\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}\\ B^2&=\begin{bmatrix}0&0\\0&0\end{bmatrix}\end{aligned}\right\} A^2=B^2.$$

Time je tvrdnja opovrgnuta.

10. Dokaži. Matrice A i B komutiraju ako i samo ako je $(A-B)(A+B)=A^2-B^2$.

$(A-B)(A+B)=A^2-B^2$
$A^2+AB-BA-B^2=A^2-B^2$
↳ Te dvije strane su jednake ako je:
$AB-BA=0$
odnosno:
$AB=BA$,
tj. matrice komutiraju.

11. Matrica A → involutarna ako je $A^2=I$. Neka su A i B sa svojstvom $A=B-I$. Dokaži da je A involutarna ako i samo ako je $B^2=2B$.
↳ A je involutarna $\Leftrightarrow B^2=2B$. Treba dokazati obje strane:
I) A je involutarna $\Rightarrow B^2=2B$:
↳ Pretpostavimo da je $A^2=I$. Iz $A=B-I$ vrijedi: $B=A+I$. Dokazujemo $B^2=2B$:

$$B^2=(A+I)^2=\underbrace{A^2}_{P.P.}+2A+I=I+2A+I=2I+2A=2(A+I)=2B\ \checkmark$$

II $B^2=2B \Rightarrow A$ je involutarna:
↳ Pretpostavimo da je $B^2=2B$. Vrijedi: $A=B-I$. Dokazujemo $A^2=I$:

$$A^2=(B-I)^2=\underbrace{B^2}_{P.P.}-2B+I=\cancel{2B}-\cancel{2B}+I=I\ \checkmark$$

(12) Dokaz: Ako su A i B simetrične, onda je i AB+BA simetrična matrica.
↳ Ako su matrice simetrične vrijedi:
$A = A^T$
$B = B^T$
Treba dokazati jednakost:
$(AB+BA)^T = AB+BA.$

Samo je jedan Mali Ivica!

$$(AB+BA)^T = (AB)^T + (BA)^T = \underbrace{B^T A^T + A^T B^T}_{\text{iz } A=A^T,\ B=B^T} = BA + AB = AB+BA \checkmark$$

(13.) $\begin{bmatrix} 2 & \lambda \\ 0 & 1 \end{bmatrix}^n, \ n \in \mathbb{N} = ?$

$n=1 \ldots \begin{bmatrix} 2 & \lambda \\ 0 & 1 \end{bmatrix}$

$n=2 \ldots \begin{bmatrix} 2 & \lambda \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & \lambda \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3\lambda \\ 0 & 1 \end{bmatrix}$

$n=3 \ldots \begin{bmatrix} 4 & 3\lambda \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & \lambda \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 7\lambda \\ 0 & 1 \end{bmatrix}$

opći član: $\begin{bmatrix} 2^n & (2^n-1)\lambda \\ 0 & 1 \end{bmatrix}$

Kako smo samo predvidjeli rješenje, potrebno ga je dokazati matematičkom indukcijom.

(1) baza indukcije:
$n=1 \ldots \begin{bmatrix} 2 & (2-1)\lambda \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & \lambda \\ 0 & 1 \end{bmatrix} \checkmark$

(2) pretpostavka indukcije:
Pretpostavimo da za neki $n \in \mathbb{N}$ vrijedi: $A^n = \begin{bmatrix} 2^n & (2^n-1)\lambda \\ 0 & 1 \end{bmatrix}$.

(3) skok indukcije:
Treba dokazati da tvrdnja vrijedi i za $n+1$.

$$A^{n+1} = A^n \cdot A = \begin{bmatrix} 2^n & (2^n-1)\lambda \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & \lambda \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 2^{n+1} & 2^n \cdot \lambda + (2^n-1)\lambda \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 2^{n+1} & \lambda(2^{n+1}-1) \\ 0 & 1 \end{bmatrix}$$

Zaključak: Time smo dokazali da je:

$$\begin{bmatrix} 2 & \lambda \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 2^n & (2^n-1)\lambda \\ 0 & 1 \end{bmatrix}$$

3.4. DEFINICIJA DETERMINANTE

= Svakoj je kvadratnoj matrici pridružen skalar, njezina determinanta, oznakom $\det A$ ili $|A|$.

= Determinante se definiraju induktivno:

1) determinanta matrice prvog reda – $n=1$:

$$A=[a_{11}] \to \det A = |a_{11}| = a_{11}$$

2) determinanta matrice drugog reda – $n=2$:

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix} \to \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix} := a_{11}\cdot a_{22} - a_{21}\cdot a_{12}$$

3) determinanta matrice trećeg reda – $n=3$:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix} \to \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix} := a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{vmatrix}$$

↳ označimo:

$$M_{11} := \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{vmatrix}, \quad M_{12} := \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{vmatrix}, \quad M_{13} := \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{vmatrix}$$

M_{11} – minora elementa a_{11}
M_{12} – minora elementa a_{12}
M_{13} – minora elementa a_{13}

= Općenito: **Minora** (M_{ij}) elementa a_{ij} je determinanta matrice koja se dobije izbacivanjem i-tog retka i j-tog stupca matrice A.

↳ označimo:

$$A_{11} := +M_{11}, \quad A_{12} := -M_{12}, \quad A_{13} := +M_{13}$$

A_{11} – algebarski komplement elementa a_{11}
A_{12} – algebarski komplement elementa a_{12}
A_{13} – algebarski komplement elementa a_{13}

= Općenito: **Algebarski komplement** (A_{ij}) elementa a_{ij}, tj. ko-faktor od a_{ij}, je:

$$A_{ij} = (-1)^{i+j}\cdot M_{ij}$$

= Uz ove oznake, definicija determinante trećeg reda je:

$$\det A = a_{11}\cdot M_{11} + a_{12}\cdot(-M_{12}) + a_{13}\cdot M_{13} =$$
$$= a_{11}\cdot A_{11} + a_{12} A_{12} + a_{13}\cdot A_{13} =$$
$$= \sum_{j=1}^{3} (-1)^{1+j}\cdot a_{1j}\cdot M_{1j}$$

↳ **Sarrusovo pravilo** za računanje determinanata trećeg reda:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix}\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32}\end{matrix} = a_{11}\cdot a_{22}\cdot a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$$

→ postupak: prva dva stupca determinante se prepišu iza trećeg i potom se izmnože po tri broja koja se nalaze na istovrsnim dijagonalama. Padajuće dijagonale nose pozitivan, a rastuće negativan predznak.

4) determinanta matrice četvrtog reda – $n=4$:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix} \to \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44}\end{vmatrix} = a_{12}\cdot(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44}\end{vmatrix} +$$

$$+ a_{22} \cdot (-1)^{2+2} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + a_{32} \cdot (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + a_{42} \cdot (-1)^{4+2} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \end{vmatrix}$$

= PRIMJER 1.

↳ det A = ?

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 7 \\ 4 & 2 & 2 & 5 \\ -1 & 3 & 0 & -1 \end{bmatrix} \longrightarrow \det A = 0 \cdot A_{13} + 0 \cdot A_{23} + 2 \cdot (-1)^{3+3} \cdot \begin{vmatrix} 0 & 1 & 2 \\ 0 & 1 & 7 \\ -1 & 3 & -1 \end{vmatrix} + 0 \cdot A_{43} =$$

$$= 2 \cdot \left\{ 0 \cdot A_{11} + 0 \cdot A_{12} + (-1) \cdot (-1)^{3+1} \cdot \begin{vmatrix} 1 & 2 \\ 1 & 7 \end{vmatrix} \right\} =$$

$$= 2 \cdot (-5) = -10$$

□ DETERMINANTA MATRICE n-TOGA REDA – LAPLACEOV RAZVOJ

= Determinanta matrice se inače definira razvojem po nekom retku, odnosno stupcu, pa se takav razvoj naziva Laplaceov razvoj determinante.

= Neka je M_{ij} minora, a A_{ij} algebarski komplement elementa a_{ij}. Tada se determinanta matrice reda n definira kao:

$$\det A = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \cdot M_{ij}$$

↳ razvoj po elementima i-tog retka

$$\det A = \sum_{i=1}^{n} a_{ij} A_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

↳ razvoj po elementima j-tog stupca.

= Shema predznaka:

$n=3$ ....
$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$n=4$ ....
$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

3.5. SVOJSTVA DETERMINANTI

= Svojstva determinanata biti će iskazana za retke determinante, ali isto vrijedi i za stupce.

= SVOJSTVO 1.

↳ Ako matrica A ima redak sastavljen od samih nula, onda je det A = 0

= SVOJSTVO 2.

↳ Determinanta trokutaste matrice jednaka je umnošku elemenata na dijagonali.

= SVOJSTVO 3.

↳ Ako matrica A ima dva jednaka retka, onda je det A = 0.

DOKAZ: Matematičkom indukcijom.

1) $n=1$ ....

$$\det A \begin{vmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{vmatrix} = a_{11} \cdot a_{12} - a_{11} a_{12} = 0 ✓$$

(2) Pretpostavimo da su u matrici reda n dva reda jednaka, npr. l-ti i s-ti redak. Tada je $\det A = 0$, tj.

$$\det A_n = \begin{vmatrix} \vdots & & & \vdots \\ a_{l1} & \cdots & \cdots & a_{ln} \\ \vdots & & & \vdots \\ a_{s1} & \cdots & \cdots & a_{sn} \\ \vdots & & & \vdots \end{vmatrix} = 0$$

(3) Treba dokazati da je $\det A_{n+1} = 0$.

$$\det A_{n+1} = \begin{vmatrix} \vdots & & \vdots & \vdots \\ a_{i1} & \cdots & a_{in} & a_{in+1} \\ \vdots & & \vdots & \vdots \\ a_{j1} & \cdots & a_{jn} & a_{jn+1} \end{vmatrix}, \quad i \neq 1, j \neq 1$$

↳ razvojem po elementima 1. retka, koristeći induktivnu pretpostavku:

$$\det A_{n+1} = \sum_{k=1}^{n+1} (-1)^{1+k} a_{1k} M_{1k} = 0$$

= SVOJSTVO 4.

↳ Transponiranjem matrice vrijednost determinante se ne mijenja, tj. $\det A = \det A^T$

= SVOJSTVO 5.

↳ Determinanta se množi skalarom tako da se jedan redak (bilo koji) pomnoži tim skalarom.

DOKAZ:

Neka je A početna matrica, a A' matrica u kojoj je jedan redak pomnožen sa λ. Tada vrijedi:

$$\det A' = \sum_{j=1}^{n} (\lambda a_{1j}) A_{1j} = \lambda \cdot \sum_{j=1}^{n} a_{1j} A_{1j} = \det A$$

↳ Ovo svojstvo koristi se za izlučivanje faktora iz matrice.

= PRIMJER 1.

$$\begin{vmatrix} 4 & 5 & 6 \\ 20 & 40 & 80 \\ 16 & 4 & 2 \end{vmatrix} = 4 \begin{vmatrix} 1 & 5 & 6 \\ 5 & 40 & 80 \\ 4 & 4 & 2 \end{vmatrix} = 4 \cdot 5 \begin{vmatrix} 1 & 5 & 6 \\ 1 & 8 & 6 \\ 4 & 4 & 2 \end{vmatrix} = 4 \cdot 5 \cdot 2 \begin{vmatrix} 1 & 5 & 3 \\ 1 & 8 & 3 \\ 4 & 4 & 1 \end{vmatrix}$$

= SVOJSTVO 6.

↳ Rastave li se svi elementi nekog retka matrice na zbroj dvaju elemenata, onda je determinanta jednaka zbroju dviju odgovarajućih determinanata.

= SVOJSTVO 7.

↳ Ako zamijenimo dva retka matrice, determinanta mijenja predznak.

= SVOJSTVO 8.

↳ Ako nekom retku matrice dodamo neki drugi redak pomnožen skalarom, vrijednost determinante neće se promijeniti.

= PRIMJER 2.

$$D = \begin{vmatrix} 200 & 5 & 99 & -3 \\ 202 & 6 & 100 & -3 \\ 204 & -2 & 101 & -3 \\ 206 & 7 & 102 & -3 \end{vmatrix} = \left(1.s. - 3.s. \times 2 \right) = \begin{vmatrix} 2 & 5 & 99 & -3 \\ 2 & 6 & 100 & -3 \\ 2 & -2 & 101 & -3 \\ 2 & 7 & 102 & -3 \end{vmatrix} = \left(\begin{matrix} \text{2 ista} \\ \text{stupca} \end{matrix} \right) = 0$$

= PRIMJER 3.

$$D = \begin{vmatrix} \sqrt{2} & \sqrt{3} & \sqrt{5} & \sqrt{3} \\ \sqrt{6} & \sqrt{21} & \sqrt{10} & -2\sqrt{3} \\ \sqrt{10} & 2\sqrt{15} & 5 & \sqrt{6} \\ 2 & 2\sqrt{6} & \sqrt{10} & \sqrt{15} \end{vmatrix} = \sqrt{2}\sqrt{3}\sqrt{5}\sqrt{3} \begin{vmatrix} 1 & 1 & 1 & 1 \\ \sqrt{3} & \sqrt{7} & \sqrt{2} & -2 \\ \sqrt{5} & 2\sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{2} & 2\sqrt{2} & \sqrt{2} & \sqrt{5} \end{vmatrix} = \left(\begin{matrix} 2.s - 1.s \\ 3.s - 1.s \\ 4.s - 1.s \end{matrix} \right) =$$

$$= 3\sqrt{10}\begin{vmatrix} 1 & 0 & 0 & 0 \\ \sqrt{3} & \sqrt{7}-\sqrt{3} & \sqrt{2}-\sqrt{3} & -2 \\ \sqrt{5} & \sqrt{5} & 0 & \sqrt{2}-\sqrt{5} \\ \sqrt{2} & \sqrt{2} & 0 & \sqrt{5}-\sqrt{2} \end{vmatrix} = 3\sqrt{10}\begin{vmatrix} \sqrt{7}-\sqrt{3} & \sqrt{2}-\sqrt{3} & -2-\sqrt{3} \\ \sqrt{5} & 0 & \sqrt{2}-\sqrt{5} \\ \sqrt{2} & 0 & \sqrt{5}-\sqrt{2} \end{vmatrix} =$$

$$= 3\sqrt{10}\cdot(-1)(\sqrt{2}-\sqrt{3})\cdot\begin{vmatrix} \sqrt{5} & \sqrt{2}-\sqrt{5} \\ \sqrt{2} & \sqrt{5}-\sqrt{2} \end{vmatrix} = 3\sqrt{10}(\sqrt{3}-\sqrt{2})(5-2) = 9\sqrt{10}(\sqrt{3}-\sqrt{2})$$

= SVOJSTVO 9.

↳ Binet-Cauchyjev teorem:

Determinanta umnoška dviju kvadratnih matrica jednaka je umnošku determinanata faktora, tj.

$$\det(AB) = \det A \cdot \det B$$

ZADACI (DZ 3. DRUGI DIO)

14. a)
$$\begin{vmatrix} 2 & 0 & 5 \\ 1 & 0 & 3 \\ 8 & 3 & -2 \end{vmatrix} = -3\begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} = -3(6-5) = -3$$

b)
$$\begin{vmatrix} 2 & 3 & 0 & -4 \\ 1 & 0 & 0 & 1 \\ 9 & 8 & 2 & -5 \\ 6 & -2 & 0 & 7 \end{vmatrix} = 2\left|\begin{array}{ccc|cc} 2 & 3 & -4 & 2 & 3 \\ 1 & 0 & 1 & 1 & 0 \\ 6 & -2 & 7 & 6 & -2 \end{array}\right| = 2(0+18+8+0+4-21) = 18$$

15. a)
$$\begin{vmatrix} 2 & 4 & 6 & 8 & 10 \\ -2 & 0 & 6 & 8 & 10 \\ -2 & -4 & 0 & 8 & 10 \\ -2 & -4 & -6 & 0 & 10 \\ -2 & -4 & -6 & -8 & 0 \end{vmatrix} = \begin{pmatrix} 2.r. + 1.r. \\ 3.r. + 1.r. \\ 4.r. + 1.r. \\ 5.r. + 1.r. \end{pmatrix} = \begin{vmatrix} 2 & 4 & 6 & 8 & 10 \\ 0 & 4 & 12 & 16 & 20 \\ 0 & 0 & 6 & 16 & 20 \\ 0 & 0 & 0 & 8 & 20 \\ 0 & 0 & 0 & 0 & 10 \end{vmatrix} = \begin{pmatrix} \text{trokutasta} \end{pmatrix} =$$

$$= 2\cdot4\cdot6\cdot8\cdot10 = 3840$$

b)
$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & -5 \\ 1 & 1 & 1 & 1 & -5 & 1 \\ 1 & 1 & 1 & -5 & 1 & 1 \\ 1 & 1 & -5 & 1 & 1 & 1 \\ 1 & -5 & 1 & 1 & 1 & 1 \\ -5 & 1 & 1 & 1 & 1 & 1 \end{vmatrix} = \begin{pmatrix} 1+2+3+4+ \\ 5+6\,r \to 1 \end{pmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & -5 & 1 \\ 1 & 1 & 1 & -5 & 1 & 1 \\ 1 & 1 & -5 & 1 & 1 & 1 \\ 1 & -5 & 1 & 1 & 1 & 1 \\ -5 & 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

16.
$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & x & x & x \\ 1 & x & 0 & x & x \\ 1 & x & x & 0 & x \\ 1 & x & x & x & 0 \end{vmatrix} = \begin{pmatrix} 2.s.-3.s. \\ 4.s.-5.s. \end{pmatrix} = \begin{vmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & -x & x & 0 & x \\ 1 & x & 0 & 0 & x \\ 1 & 0 & x & -x & x \\ 1 & 0 & x & x & 0 \end{vmatrix} = \begin{pmatrix} 2.r.+3.r. \end{pmatrix} = \begin{vmatrix} 0 & 0 & 1 & 0 & 1 \\ 2 & 0 & x & 0 & 2x \\ 1 & x & 0 & 0 & x \\ 1 & 0 & x & -x & x \\ 1 & 0 & x & x & 0 \end{vmatrix} =$$

$$= -x\begin{vmatrix} 0 & 1 & 0 & 1 \\ 2 & x & 0 & 2x \\ 1 & x & -x & x \\ 1 & x & x & 0 \end{vmatrix} = \begin{pmatrix} 3.r.-4.r. \\ 2.r.-3.r.\cdot(2) \end{pmatrix} = -x\begin{vmatrix} 0 & 1 & 0 & 1 \\ 0 & -x & 2x & 0 \\ 0 & 0 & -2x & x \\ 1 & x & x & 0 \end{vmatrix} = +x\begin{vmatrix} 1 & 0 & 1 \\ -x & 2x & 0 \\ 0 & -2x & x \end{vmatrix} = \begin{pmatrix} 1.s.-3.r. \end{pmatrix} =$$

$$= x\begin{vmatrix} 0 & 0 & 1 \\ -x & 2x & 0 \\ -x & -2x & x \end{vmatrix} = x\begin{vmatrix} -x & 2x \\ -x & -2x \end{vmatrix} = x(2x^2+2x^2) = 4x^3$$

Samo je jedan Mali Ivica!

17.

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ -x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & x & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & x & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & x & 0 & 0 \\ 0 & 0 & 0 & 0 & -x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & -x & x \end{vmatrix} = a_1 \begin{vmatrix} x & 0 & 0 & 0 & 0 & 0 \\ -x & x & 0 & 0 & 0 & 0 \\ 0 & -x & x & 0 & 0 & 0 \\ 0 & 0 & -x & x & 0 & 0 \\ 0 & 0 & 0 & -x & x & 0 \\ 0 & 0 & 0 & 0 & -x & x \end{vmatrix} - a_2 \begin{vmatrix} -x & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 0 \\ 0 & -x & x & 0 & 0 & 0 \\ 0 & 0 & -x & x & 0 & 0 \\ 0 & 0 & 0 & -x & x & 0 \\ 0 & 0 & 0 & 0 & -x & x \end{vmatrix} +$$

$$+ a_3 \begin{vmatrix} -x & x & 0 & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & -x & x & 0 & 0 \\ 0 & 0 & 0 & -x & x & 0 \\ 0 & 0 & 0 & 0 & -x & x \end{vmatrix} - a_4 \begin{vmatrix} -x & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & x & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & -x & x & 0 \\ 0 & 0 & 0 & 0 & -x & x \end{vmatrix} + a_5 \begin{vmatrix} -x & x & 0 & 0 & 0 & 0 \\ 0 & -x & x & 0 & 0 & 0 \\ 0 & 0 & -x & x & 0 & 0 \\ 0 & 0 & 0 & -x & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & -x & x \end{vmatrix} - a_6 \begin{vmatrix} -x & x & 0 & 0 & 0 & 0 \\ 0 & -x & x & 0 & 0 & 0 \\ 0 & 0 & -x & x & 0 & 0 \\ 0 & 0 & 0 & -x & x & 0 \\ 0 & 0 & 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 & 0 & x \end{vmatrix} +$$

$$+ a_7 \begin{vmatrix} -x & x & 0 & 0 & 0 & 0 \\ 0 & -x & x & 0 & 0 & 0 \\ 0 & 0 & -x & x & 0 & 0 \\ 0 & 0 & 0 & -x & x & 0 \\ 0 & 0 & 0 & 0 & -x & x \\ 0 & 0 & 0 & 0 & 0 & -x \end{vmatrix} = a_1 \cdot x^6 - a_2 \cdot (-x^6) + a_3 \cdot x^6 - a_4 \cdot (-x^6) + a_5 \cdot x^6 - a_6 \cdot (-x^6) +$$

$$+ a_7 \cdot x^6 = x^6 (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7)$$

18.

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2-x & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3-x & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4-x & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 5-x & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 6-x & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 7-x \end{vmatrix} = 0 \qquad \begin{pmatrix} 2r - 1r \\ 3r - 1r \\ 4r - 1r \\ 5r - 1r \\ 6r - 1r \\ 7r - 1r \end{pmatrix}$$

$$\downarrow$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1-x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2-x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3-x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4-x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5-x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6-x \end{vmatrix} = 0$$

$$1 \cdot \begin{vmatrix} 1-x & 0 & 0 & 0 & 0 & 0 \\ 0 & 2-x & 0 & 0 & 0 & 0 \\ 0 & 0 & 3-x & 0 & 0 & 0 \\ 0 & 0 & 0 & 4-x & 0 & 0 \\ 0 & 0 & 0 & 0 & 5-x & 0 \\ 0 & 0 & 0 & 0 & 0 & 6-x \end{vmatrix} = 0$$

$$(1-x)(2-x)(3-x)(4-x)(5-x)(6-x) = 0$$

$$\left.\begin{aligned} &\hookrightarrow 1-x=0 \rightarrow x=1 \\ &2-x=0 \rightarrow x=2 \\ &3-x=0 \rightarrow x=3 \\ &4-x=0 \rightarrow x=4 \\ &5-x=0 \rightarrow x=5 \\ &6-x=0 \rightarrow x=6 \end{aligned}\right\} x \in \{1,2,3,4,5,6\}$$

(19.) $\det\begin{vmatrix}3&5&5&5&5\\5&3&5&5&5\\5&5&3&5&5\\5&5&5&3&5\\5&5&5&5&3\end{vmatrix} \approx \left(1r+2r+3r+4r+5r\right) = \begin{vmatrix}23&23&23&23&23\\5&3&5&5&5\\5&5&3&5&5\\5&5&5&3&5\\5&5&5&5&3\end{vmatrix} = 23\begin{vmatrix}1&1&1&1&1\\5&3&5&5&5\\5&5&3&5&5\\5&5&5&3&5\\5&5&5&5&3\end{vmatrix} \approx$

$\begin{pmatrix}2s-1s\\3s-1s\\4s-1s\\5s-1s\end{pmatrix} \Rightarrow 23\begin{vmatrix}1&0&0&0&0\\5&2&0&0&0\\5&0&2&0&0\\5&0&0&2&0\\5&0&0&0&2\end{vmatrix} = 23\begin{vmatrix}2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&2\end{vmatrix} = 23\cdot 2^4 = 368$

(20.) Dokazi. Ako za A vrijedi $A^2=I$, $\det(A-1)\neq 0$ onda vrijedi $\det(A+I)=0$.

$A^2-I=(A-I)(A+I)$

$\Rightarrow$ Po Binet Chauchyjevom teoremu:

$\underbrace{\det(A^2-I)}_{0}=\det(A-I)\cdot\det(A+I)$

$0=\underbrace{\det(A-I)}_{\neq 0}\cdot\underbrace{\det(A+I)}_{=0}$

↳ Kako je $\det(A^2-I)$ nula, onda da bi tvrdnja bila jednaka mora jedan od dva člana ($\det(A-I)$ ili $\det(A+I)$) s desne strane biti jednak nuli. Kako je zadano da vrijedi $\det(A-1)\neq 0$, onda je očito da je $\det(A+I)=0$, čime je tvrdnja dokazana.

Samo je jedan Mali Ivica!

4. RANG I INVERZ MATRICE

4.1. INVERZNA MATRICA

- DEFINICIJA 1.
  - Neka je $A \in \mathcal{M}_n$ zadana matrica. Matrica A' za koju vrijedi:
  $$A' \cdot A = A \cdot A' = I \qquad (*)$$
  naziva se **inverzna matrica** od A, a označava se A^{-1}.
  - Za matricu A kažemo da je **regularna** ukoliko postoji njezina inverzna matrica A^{-1}. Ako ona ne postoji, kažemo da je A **singularna** matrica.

= STAVAK 1.
  - Neka je $A \in \mathcal{M}_n$. Ako A^{-1} postoji onda je ona jednoznačno određena.

DOKAZ.

Pretpostavimo da postoje 2 matrice A' i A'' za koje vrijedi uvjet (*). Onda je:

$$A' \cdot A = I \;/\; A'' \qquad\qquad A \cdot A' = I \;/\cdot A''$$
$$(A'A)A'' = IA'' \qquad\qquad (A \cdot A')A'' = I \cdot A''$$
$$A'\underbrace{(AA'')}_{PP} = IA'' \qquad\qquad A \cdot \underbrace{(A'A'')}_{PP} = I \cdot A''$$
$$A' \cdot I = I \cdot A'' \qquad\qquad A \cdot I = I \cdot A''$$
$$A' = A'' \qquad\qquad A = A''$$

= PRIMJER 1.
  - Odredi je li matrica A regularna ili singularna, ako je $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Pretpostavimo da A^{-1} postoji. Tada je $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a, b, c, d \in \mathbb{R}$.

Iz uvjeta (*) $A \cdot A^{-1} = I$ slijedi:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ⋇$$

~ oznaka za kontradikciju, tj. A^{-1} ne postoji jer $0 \neq 1$, pa je matrica A singularna.

= STAVAK 2.
  - Ako su A i B regularne matrice istog reda ($A, B \in \mathcal{M}_n$), tada je i $A \cdot B$ regularna, i vrijedi:
  $$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$$

DOKAZ.: Direktnom provjerom uvjeta (*) za obje strane uvjeta:

LIJEVA STRANA:
$$(AB)(AB)^{-1} = (AB) \cdot (B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = (AI) \cdot A^{-1} = A \cdot A^{-1} = I \checkmark$$

DESNA STRANA:
$$(AB)^{-1}(AB) = (B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = (B^{-1}I)B = B^{-1} \cdot B = I \checkmark$$

4.2. ELEMENTARNE TRANSFORMACIJE I REDUCIRANI OBLIK MATRICE

= DEFINICIJA 1.

↳ **Elementarne transformacije** nad retcima matrice su:

1) zamjena dvaju redaka,
2) množenje nekog retka skalarom različitog od nule,
3) dodavanje nekog retka nekom drugom retku.

↳ Ove transformacije koriste se kod:

1) računanja determinanti,
2) rješavanja linearnih sustava,
3) određivanja ranga matrice,
4) nalaženja inverzne matrice.

↳ Ove transformacije služe za dovođenje matrice u tzv. reducirani oblik.

= DEFINICIJA 2.

↳ **Reducirani oblik matrice**:

I grafički:

$$\begin{bmatrix} 1 & x & y & 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

II riječima:

Mora vrijediti: a) Prvi stožerni element (nenul element) svakog retka iznosi jedan. Svi preostali elementi u stupcu tog stožernog elementa jednaki su nula.

b) Svaki naredni stožer (gledajući po retcima počevši od prvog) nalazi se desno od prethodnog stožera, tj. ako su:

$$a_{i_1 j_1} = 1 \;,\; a_{i_2 j_2} = 1,$$

tada:

$$i_2 > i_1 \Rightarrow j_2 > j_1.$$

c) Ako ima redaka sa samim nulama one se nalaze ispod redaka u kojima se nalazi stožerni element.

↳ Primjeri reduciranih oblika matrice:

$$A_R = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} , \quad B_R = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} , \quad I_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

= Algoritam svođenja matrice na reducirani oblik:

= 1. korak: Izaberimo u 1. stupcu neki element različit od nule. Primjenjujući prvu elementarnu transformaciju, možemo ga dovesti na poziciju stožernog elementa a_{11} (ako je $a_{11} \neq 0$, prvi je korak nepotreban). Ako su svi elementi 1. stupca nula prelazimo na sljedeći i ponavljamo postupak.

= 2. korak: Podijelimo elemente 1. retka s a_{11} (druga elementarna transformacija). Time stožerni element postaje jednak 1.

= 3. korak: Pomoću stožernog elementa poništavamo sve preostale elemente u njegovom stupcu, primjenjujući treću elementarnu transformaciju.

=PRIMJER 2.

↳ Reduciraj:

$$A=\begin{bmatrix}2&7&5&3\\1&0&0&2\\-1&-1&1&2\\3&1&2&0\end{bmatrix}\sim\begin{pmatrix}1\to2\end{pmatrix}\sim\begin{bmatrix}1&0&0&2\\2&7&5&3\\-1&-1&1&2\\3&1&2&0\end{bmatrix}\sim\begin{pmatrix}2r-1r\times2\\3r-1r\times(-1)\\4r-1r\times3\end{pmatrix}\sim\begin{bmatrix}1&0&0&2\\0&7&5&-1\\0&-1&1&4\\0&1&2&-6\end{bmatrix}\sim$$

$$\begin{pmatrix}2\to4\end{pmatrix}\sim\begin{bmatrix}1&0&0&2\\0&1&2&-6\\0&-1&1&4\\0&7&5&-1\end{bmatrix}\sim\begin{pmatrix}3r-2r\times(-1)\\4r-2r\times7\end{pmatrix}\sim\begin{bmatrix}1&0&0&2\\0&1&2&-6\\0&0&3&-2\\0&0&-9&41\end{bmatrix}\sim\begin{pmatrix}3r/3\end{pmatrix}\sim$$

$$\begin{bmatrix}1&0&0&2\\0&1&2&-6\\0&0&1&-\frac{2}{3}\\0&0&-9&41\end{bmatrix}\sim\begin{pmatrix}2r-3r\times2\\4r-3r\times(-9)\end{pmatrix}\sim\begin{bmatrix}1&0&0&2\\0&1&0&-\frac{14}{3}\\0&0&1&-\frac{2}{3}\\0&0&0&35\end{bmatrix}\sim\begin{pmatrix}4r/35\end{pmatrix}\sim\begin{bmatrix}1&0&0&2\\0&1&0&-\frac{14}{3}\\0&0&1&-\frac{2}{3}\\0&0&0&1\end{bmatrix}\sim$$

$$\begin{pmatrix}1r-4r\times2\\2r-4r\times(-\frac{14}{3})\\3r-4r\times(-\frac{2}{3})\end{pmatrix}\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

=PRIMJER 3.

↳ Reduciraj:

$$B=\begin{bmatrix}0&1&2&3\\0&0&0&2\\0&-1&0&0\end{bmatrix}\sim\begin{pmatrix}3r-1r\times(-1)\end{pmatrix}\sim\begin{bmatrix}0&1&2&3\\0&0&0&2\\0&0&2&3\end{bmatrix}\sim\begin{pmatrix}2\to3\end{pmatrix}\sim\begin{bmatrix}0&1&2&3\\0&0&2&3\\0&0&0&2\end{bmatrix}\sim\begin{pmatrix}2r/2\end{pmatrix}\sim$$

$$\begin{bmatrix}0&1&2&3\\0&0&1&\frac{3}{2}\\0&0&0&2\end{bmatrix}\sim\begin{pmatrix}1r-2r\times2\end{pmatrix}\sim\begin{bmatrix}0&1&0&0\\0&0&1&\frac{3}{2}\\0&0&0&2\end{bmatrix}\sim\begin{pmatrix}3r/2\end{pmatrix}\sim\begin{bmatrix}0&1&0&0\\0&0&1&\frac{3}{2}\\0&0&0&1\end{bmatrix}\sim\begin{pmatrix}2r-3r\times\frac{3}{2}\end{pmatrix}\sim$$

$$\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

4.3. ELEMENTARNE I EKVIVALENTNE MATRICE. RANG I INVERZ MATRICE.

= Elementarne transformacije mogu se opisati pomoću množenja s matricama triju tipova, koje nazivamo elementarnim matricama.

= DEFINICIJA 1.

↳ Pretpostavimo da je matrica B dobivena iz matrice A nizom elementarnih transformacija. Tada za matrice A i B kažemo da su **ekvivalentne**, i to zapisujemo **B ~ A**.

↳ Pokazuje se da je reducirana forma matrice jednoznačno određena (bez dokaza). Izbor elementarnih transformacija nije jednoznačan, ali konačan rezultat jest.

= DEFINICIJA 2.

↳ Rang matrice je broj ne-nul redaka u reduciranom obliku matrice A_R. Rang se iskazuje simbolom rang (A).

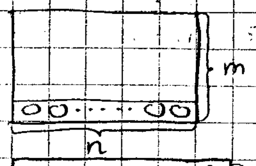
↳ Moguće su još dvije definicije ranga. Pomoću:

1) linearno nezavisnih redaka matrice
2) determinanata

Samo je jedan Mali Ivica!

= Iz definicije 2. slijedi:

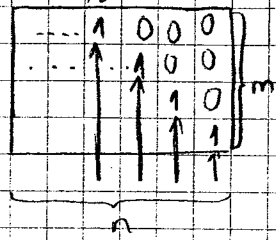
a)



rang (A) ≤ m

↳ maksimalan broj ne-nul redaka je m

b)



rang (A) ≤ n

↳ najviše je n mjesta za raspored stožernih elemenata po stupcima

= STAVAK 1.

↳ Da bi izveli algoritam za računanje inverzne matrice moramo izvesti 2 leme. Zbog jednoznačnosti i definicije reducirane forme je:

= LEMA 1.

↳ Kvadratna matrica A reda n ima rang jednak n onda i samo onda ako je A_R jedinična matrica, tj. $A_R = I$.

= LEMA 2.

↳ Ako je kvadratna matrica A regularna i matrica B ekvivalentna s njom, tada je i matrica B regularna.

↳ Zbog toga imamo: algoritam za računanje inverzne matrice

→ Neka je A kvadratna regularna matrica reda n. Po lemi je $A_R = I$. To znači da se jedinična matrica može dobiti iz matrice A nizom elementarnih transformacija. Možemo napisati:

$$I = (E_r E_{r-1} \ldots E_1) A$$

gdje su $E_r E_{r-1} \ldots E_1$ elementarne matrice.

→ Sada se vidi da je A^{-1} jednaka umnošku elementarnih matrica koje su sudjelovale u transformaciji, tj.

$$A^{-1} = E_r E_{r-1} \ldots E_2 E_1$$

→ Dakle, inverzna matrica A^{-1} jednaka je produktu elementarnih matrica upotrebljenih da se matrica A svede na reduciranu formu: $A_R = I$. Međutim, pri računu A^{-1} nije potrebno ispisivati eksplicitno matrice E_i i računati njihove produkte. Naime, te se matrice dobivaju istim transformacijama koje su načinjene na matrici A.

= Elementarne matrice:

↳ 1. elementarna matrica: - i-ti redak zamijenimo s j-tim retkom.

$$E_{ij} = \begin{array}{c} \begin{bmatrix} \ddots & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \ddots \end{bmatrix} \end{array} \begin{array}{l} \\ \leftarrow i \\ \leftarrow j \\ \\ \end{array}$$

↳ svi elementi su kao i jedinični, osim $e_{ii} = e_{jj} = 0$, $e_{ij} = e_{ji} = 1$.

↳ 2. elementarna matrica: - i-ti redak pomnožimo skalarom λ različitim od nule

$$E_i(\lambda) = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \lambda & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

↳ svi ostali elementi jednaki su onima u jediničnoj matrici.

↳ 3. elementarna matrica: - i-ti redak pomnožen skalarom λ dodajemo j-tom retku.

$$E_{ij}(\lambda)=\begin{bmatrix}\ddots & & \\ & 1 & \\ \lambda & & 1 \\ & & & \ddots\end{bmatrix}\begin{matrix}\\ \leftarrow i\\ \leftarrow j\\ \\ \end{matrix} \quad \text{ili} \quad E_{ji}(\lambda)=\begin{bmatrix}\ddots & & \\ & 1 & \lambda \\ & & 1 \\ & & & \ddots\end{bmatrix}\begin{matrix}\\ \leftarrow j\\ \leftarrow i\\ \\ \end{matrix}$$

↳ matrica je istovjetna jediničnoj, osim što vrijedi: $e_{ji}=\lambda$

= PRIMJER 1.

Zamjena 1. i 3. retka matrice $A=\begin{bmatrix}2&1\\3&-1\\4&2\end{bmatrix}$ dobije se množenjem s elementarnom matricom oblika:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\sim\begin{pmatrix}1\mapsto 3\\ \\ \end{pmatrix}\sim\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}.$$

Tj.:

$$A\sim\underbrace{\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}}_{\text{elementarna matrica tipa } E_I}\cdot\begin{bmatrix}2&1\\3&-1\\4&2\end{bmatrix}=\begin{bmatrix}4&2\\3&-1\\2&1\end{bmatrix}$$

= PRIMJER 2.

Produkt 2. i 3. retka matrice $A=\begin{bmatrix}2&1\\3&-1\\4&2\end{bmatrix}$ dobije se množenjem s elementarnom matricom oblika:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\sim\begin{pmatrix}2.r\times 3\end{pmatrix}\sim\begin{bmatrix}1&0&0\\0&3&0\\0&0&1\end{bmatrix}$$

Tj.:

$$A\sim\underbrace{\begin{bmatrix}1&0&0\\0&3&0\\0&0&1\end{bmatrix}}_{\text{elementarna matrica tipa } E_{II}}\cdot\begin{bmatrix}2&1\\3&-1\\4&2\end{bmatrix}=\begin{bmatrix}2&1\\9&-3\\4&2\end{bmatrix}$$

= PRIMJER 3.

Dodavanje 1 retka pomnoženog s -2 trećem retku matrice $A=\begin{bmatrix}2&1\\3&-1\\4&2\end{bmatrix}$ dobije se množenjem s elementarnom matricom oblika:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\sim\begin{pmatrix}3.r+1.r\times(-2)\\ \\ \end{pmatrix}\sim\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}.$$

Tj.:

$$A\sim\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}}_{\text{elementarna matrica tipa } E_{III}}\cdot\begin{bmatrix}2&1\\3&-1\\4&2\end{bmatrix}=\begin{bmatrix}2&1\\3&-1\\0&0\end{bmatrix}$$

= Stoga možemo koristiti sljedeći algoritam:

↳ Korak 1.: Napišemo matricu tipa $n\times 2n$ u kojoj je lijevo ispisana matrica A, a desno matrica I:

$$n\left\{\overbrace{\left[\begin{array}{ccc|ccc}a_{11}&\cdots&a_{1n}&1&0\cdots&0\\ \vdots&A&\vdots&\vdots&I&\vdots\\ a_{n1}&\cdots&a_{nn}&0&\cdots&1\end{array}\right]}^{2n}\right.=[A\,|\,I] \rightarrow \text{tzv. proširena matrica}$$

↳ Korak 2: Vršimo elementarne transformacije na proširenoj matrici, dok matricu A ne dovedemo u reduciranu formu, tj.:

$$[A \vdots I] \sim [E_1 A \vdots E_1] \sim [E_2 E_1 A \vdots E_2 E_1] \sim \dots \sim [A_R \vdots E_r \cdots E_2 E_1] \sim [A_R \vdots B]$$

↳ Korak 3: Ako je $A_R = I$ tada je matrica A regularna, i na desnoj proširene matrice nalazi se A^{-1}.
Ako je pak $A_R \neq I$ matrica A nije regularna i ne postoji njena inverzna matrica, a iz broja ne-nul redaka od A_R može se zaključiti koliki je rang od A.

Dakle:
Početna matrica: $[A \vdots I]$. Nakon prve elementarne transformacije kad je A regularna i svedena na reduciranu formu:
$$[A \vdots I] \sim [A_R \vdots B] = [I \vdots B] = [E_r \cdots E_1 A \vdots E_r \cdots E_1].$$
Kako je $B = E_r \cdots E_1$, a matrica $I = E_r \cdots E_1 A \equiv B \cdot A$.
Zbog toga slijedi: $B = A^{-1}$.

=PRIMJER 4.
Odredi inverz matrice:
$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

$$[A \vdots I] = \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \sim \begin{pmatrix} 1 \mapsto 2 \\ \\ \\ \end{pmatrix} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \sim \begin{pmatrix} 2r - 1r \times (2) \\ \\ \\ \end{pmatrix} \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & -5 & 1 & -2 & 0 \\ 0 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \sim \begin{pmatrix} 2 \mapsto 3 \\ \\ \\ \end{pmatrix} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 & 0 & 1 \\ 0 & 0 & -5 & 1 & -2 & 0 \end{array}\right] \sim \begin{pmatrix} 2r/2 \\ \\ \\ \end{pmatrix} \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -5 & 1 & -2 & 0 \end{array}\right] \sim \begin{pmatrix} 2r/(-5) \\ \\ \\ \end{pmatrix} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{5} & \frac{2}{5} & 0 \end{array}\right] \sim \begin{pmatrix} 1r - 3r \times (3) \\ 2r - 3r \times (2) \\ \\ \end{pmatrix} \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{5} & -\frac{1}{5} & 0 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{4}{5} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{5} & \frac{2}{5} & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} & 0 \\ \frac{2}{5} & -\frac{4}{5} & \frac{1}{2} \\ -\frac{1}{5} & \frac{2}{5} & 0 \end{bmatrix} = \frac{1}{10} \cdot \begin{bmatrix} 6 & -2 & 0 \\ 4 & -8 & 5 \\ -2 & 4 & 0 \end{bmatrix}$$

4.4. DETERMINANTE I INVERZNA MATRICA

= Iz relacije $A \cdot A^{-1} = A^{-1} \cdot A = I$ koja vrijedi za regularne matrice, primjenom Binet-Cauchyjevog teorema slijedi:
$$\det(A^{-1} \cdot A) = \det A^{-1} \cdot \det A = \det I = 1.$$
Odatle zaključujemo da je za regularnu matricu $\det A \neq 0$ i $\det A^{-1} \neq 0$.

= STAVAK 1.
↳ Ako je $\det A \neq 0$ onda je matrica A regularna.
DOKAZ: Izvodi se konstruktivno, tj. bit će izvedena eksplicitna formula za računanje A^{-1}!

→ Po definiciji determinante je: (Laplaceov razvoj po elementima k-tog stupca):

$$\det A = \sum_{j=1}^{n} a_{jk} A_{jk}$$

- Ako u ovoj formuli A_{jk} (algebarski komplement od a_{jk}) zamijenimo algebarskim komplementom nekog drugog stupca, onda je:

$$\sum_{j=1}^{n} a_{jk} A_{ji} = 0 \quad , \text{ za } i \neq k .$$

Obrazloženje: Ova je suma Laplaceov razvoj determinante po i-tom stupcu koja ima dva jednaka stupca, tj. i-ti i k-ti. Stoga slijedi:

$$\sum_{j=1}^{n} A_{ji} a_{jk} = \det A \cdot \delta_{ik} . \quad (*)$$

Označimo li matricu kofaktora (A_{ij}) sa $\tilde{A}$ možemo element A_{ji} shvatiti kao transpoziciju elementarne matrice $(\tilde{A})^T$, tj.:

$$\sum_{j=1}^{n} A_{ij}^{T} a_{jk} = \det A \cdot \delta_{ik} .$$

Kako $A_{ij}^{T} a_{jk}$ možemo shvatiti kao (i,k)-ti element matrice koja se dobiva množenjem $\tilde{A}^T$ i matrice A, dobivamo zapis formule $(*)$ kao:

$$\tilde{A}^T \cdot A = \det A \cdot I ,$$ gdje je $\tilde{A}^T$ tzv. adjungirana matrica od A.

Odnosno:

$$\frac{\tilde{A}^T}{\det A} \cdot A = I$$

Kako je $A \cdot A^{-1} = I$ vrijedi:

$$A^{-1} = \frac{\tilde{A}^T}{\det A}$$

=STAVAK 2.

↳ Cramerovo pravilo za inverz matrice:

→ Neka je A kvadratna matrica reda n. Ona je regularna ako i samo ako vrijedi $\det A \neq 0$.

Elementi inverzne matrice su:

$$(A^{-1})_{ij} = a'_{ij} = \frac{A_{ji}}{\det A} .$$

Umjesto ove formule postoji eksplicitni zapis inverzne matrice:

$$A^{-1} = \frac{1}{\det A} \cdot \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T$$

Samo je jedan Mali Ivica!

=PRIMJER 1.

Odredi inverz matrice Cramerovim pravilom:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 3 \\ 0 & 2 & 4 \end{bmatrix} .$$

1.) determinanta mora biti različita od 0 da bi matrica imala inverz.

$$\det A = \begin{vmatrix} 2 & 0 & 1 \\ 1 & 0 & 3 \\ 0 & 2 & 4 \end{vmatrix} \begin{matrix} 2 & 0 \\ 1 & 0 \\ 0 & 2 \end{matrix} = 0+0+2-0-12-0 = -10 \quad \text{pa } \det A \neq 0 .$$

Samo je jedan Mali Ivica!

2) Cramerovo pravilo:

$$A^{-1} = \frac{\tilde{A}^T}{\det A}$$

$A_{11} = +\begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = -6$ $\quad A_{21} = -\begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} = 2$ $\quad A_{31} = +\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} = 0$

$A_{12} = -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -4$ $\quad A_{22} = +\begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} = 8$ $\quad A_{32} = -\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = -5$

$A_{13} = +\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2$ $\quad A_{23} = -\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = -4$ $\quad A_{33} = +\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} = 0$

$$A^{-1} = \frac{\begin{bmatrix} -6 & -4 & 2 \\ 2 & 8 & -4 \\ 0 & -5 & 0 \end{bmatrix}^T}{-10} = \frac{\begin{bmatrix} -6 & 2 & 0 \\ -4 & 8 & -5 \\ 2 & -4 & 0 \end{bmatrix}}{-10} = \frac{1}{10}\begin{bmatrix} 6 & -2 & 0 \\ 4 & -8 & 5 \\ -2 & 4 & 0 \end{bmatrix}$$

⇒ Dakle: možemo koristiti sljedeći algoritam:

↳ Korak 1: Izračunati determinantu od matrice A. Ako je ona različita od nule krećemo dalje sa korakom 2, a ako nije, matrica nema inverza.

↳ Korak 2: Odrediti algebarski komplement svakog matričnog elementa i zapisati ih u odgovarajuće mjesto u matrici.

↳ Korak 3: Transponirati dobivenu matricu i podijeliti je s determinantom.

ZADACI (DZ4. PRVI DIO)

1. a) $A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \sim \left(1r/2\right) \sim \begin{bmatrix} 1 & \frac{3}{2} \\ 6 & 9 \end{bmatrix} \sim \left(2r - 1r \times 6\right) \sim \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 0 \end{bmatrix}$

rang A = 1

b) $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \sim \begin{pmatrix} 2r - 1r \times (2) \\ 3r - 1r \times (3) \end{pmatrix} \sim \begin{bmatrix} 1 & 4 \\ 0 & -3 \\ 0 & -6 \end{bmatrix} \sim \left(2r/(-3)\right) \sim \begin{bmatrix} 1 & 4 \\ 0 & 1 \\ 0 & -6 \end{bmatrix} \sim \begin{pmatrix} 1r - 2r \times (4) \\ 3r - 2r \times (-6) \end{pmatrix} \sim$

$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

rang A = 2

c) $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ -1 & 3 & 6 \end{bmatrix} \sim \left(2 \leftrightarrow 1\right) \sim \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 3 \\ -1 & 3 & 6 \end{bmatrix} \sim \begin{pmatrix} 2r - 1r \times (-2) \\ 3r - 1r \times (-1) \end{pmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 9 \\ 0 & 5 & 9 \end{bmatrix} \sim$

$\left(2r/5\right) \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{9}{5} \\ 0 & 5 & 9 \end{bmatrix} \sim \begin{pmatrix} 1r - 2r \times (2) \\ 3r - 2r \times (5) \end{pmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{3}{5} \\ 0 & 1 & \frac{9}{5} \\ 0 & 0 & 0 \end{bmatrix}$

rang A = 2

2. a)

$$A=\begin{bmatrix}1&-1&0\\2&-2&0\\1&3&-3\\1&4&-4\end{bmatrix}\sim\begin{pmatrix}2r-1r\times(2)\\3r-1r\times(1)\\4r-1r\times(1)\end{pmatrix}\sim\begin{bmatrix}1&-1&0\\0&0&0\\0&4&-3\\0&5&-4\end{bmatrix}\sim\begin{pmatrix}3r/4\end{pmatrix}\sim\begin{bmatrix}1&-1&0\\0&0&0\\0&1&-\frac{3}{4}\\0&5&-4\end{bmatrix}\sim$$

$$\begin{pmatrix}1r-3r\times(-1)\\4r-3r\times(5)\end{pmatrix}\sim\begin{bmatrix}1&0&-\frac{3}{4}\\0&0&0\\0&1&-\frac{3}{4}\\0&0&-\frac{1}{4}\end{bmatrix}\sim\begin{pmatrix}4r/\left(-\frac{1}{4}\right)\end{pmatrix}\sim\begin{bmatrix}1&0&-\frac{3}{4}\\0&0&0\\0&1&-\frac{3}{4}\\0&0&1\end{bmatrix}$$

$$\begin{pmatrix}1r-4r\times\left(-\frac{3}{4}\right)\\3r-4r\times\left(-\frac{3}{4}\right)\end{pmatrix}\sim\begin{bmatrix}1&0&0\\0&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

rang $A=3$

b)

$$A=\begin{bmatrix}1&2&5&3\\0&1&0&0\\2&4&1&2\end{bmatrix}\sim\begin{pmatrix}3r-1r\times(2)\end{pmatrix}\sim\begin{bmatrix}1&2&5&3\\0&1&0&0\\0&0&-9&-4\end{bmatrix}\sim\begin{pmatrix}1r-2r\times(2)\end{pmatrix}\sim$$

$$\begin{bmatrix}1&0&5&3\\0&1&0&0\\0&0&-9&-4\end{bmatrix}\sim\begin{pmatrix}3r/(-9)\end{pmatrix}\sim\begin{bmatrix}1&0&5&3\\0&1&0&0\\0&0&1&\frac{4}{9}\end{bmatrix}\sim\begin{pmatrix}1r-3r\times(5)\end{pmatrix}\sim$$

$$\begin{bmatrix}1&0&0&\frac{7}{9}\\0&1&0&0\\0&0&1&\frac{4}{9}\end{bmatrix}$$

rang $A=3$

3.

$$A=\begin{bmatrix}3&1&-1&2\\1&0&0&0\\0&2&-2&9\end{bmatrix}\sim\begin{pmatrix}1\mapsto2\end{pmatrix}\sim\begin{bmatrix}1&0&0&0\\3&1&-1&2\\0&2&-2&9\end{bmatrix}\dots\quad E_1=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\sim$$

$$\sim\begin{pmatrix}2r-1r\times(3)\end{pmatrix}\sim\begin{bmatrix}1&0&0&0\\0&1&-1&2\\0&2&-2&9\end{bmatrix}\dots\quad E_2=\begin{bmatrix}0&1&0\\1&-3&0\\0&0&1\end{bmatrix}\sim$$

$$\sim\begin{pmatrix}3r-2r\times(2)\end{pmatrix}\sim\begin{bmatrix}1&0&0&0\\0&1&-1&2\\0&0&0&5\end{bmatrix}\dots\quad E_3=\begin{bmatrix}0&1&0\\1&-3&0\\-2&6&1\end{bmatrix}\sim$$

$$\sim\begin{pmatrix}3r/5\end{pmatrix}\sim\begin{bmatrix}1&0&0&0\\0&1&-1&2\\0&0&0&1\end{bmatrix}\dots\quad E_4=\begin{bmatrix}0&1&0\\1&-3&0\\-\frac{2}{5}&\frac{6}{5}&\frac{1}{5}\end{bmatrix}\sim$$

$$\sim\begin{pmatrix}2r-3r\times(2)\end{pmatrix}\sim\begin{bmatrix}1&0&0&0\\0&1&-1&0\\0&0&0&1\end{bmatrix}\dots\quad E_5=\begin{bmatrix}0&1&0\\1&-\frac{27}{5}&-\frac{2}{5}\\-\frac{2}{5}&\frac{6}{5}&\frac{1}{5}\end{bmatrix}=\frac{1}{5}\begin{bmatrix}0&5&0\\5&-27&-2\\-2&6&1\end{bmatrix}$$

rang $A=3$

4.

$$A=\begin{bmatrix}1&0&-1&0&3\\1&1&0&3&2\\4&1&-3&3&9\\-1&1&2&3&5\\-1&0&1&0&3\end{bmatrix}\sim\begin{pmatrix}2r-1r\times(1)\\3r-1r\times(4)\\4r-1r\times(-1)\\5r-1r\times(-1)\end{pmatrix}\sim\begin{bmatrix}1&0&-1&0&3\\0&1&1&3&-1\\0&1&1&3&-3\\0&1&1&3&8\\0&0&0&0&6\end{bmatrix}\sim\begin{pmatrix}3r-2r\times(1)\\4r-2r\times(1)\end{pmatrix}\sim$$

$$\begin{bmatrix}1&0&-1&0&3\\0&1&1&3&-1\\0&0&0&0&-2\\0&0&0&0&9\\0&0&0&0&6\end{bmatrix}\sim\begin{pmatrix}3r|(-2)\end{pmatrix}\sim\begin{bmatrix}1&0&-1&0&3\\0&1&1&3&-1\\0&0&0&0&1\\0&0&0&0&9\\0&0&0&0&6\end{bmatrix}\sim\begin{pmatrix}1r-2r\times(3)\\2r-2r\times(-1)\\4r-2r\times(9)\\5r-2r\times(6)\end{pmatrix}\sim$$

$$\begin{bmatrix}1&0&-1&0&0\\0&1&1&3&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

$\text{rang}(A)=3$

5.

$$A=\begin{bmatrix}2\lambda-1&\lambda&1\\\lambda&\lambda&1\\1&1&\lambda\end{bmatrix}\sim\begin{pmatrix}1\mapsto 3\end{pmatrix}\sim\begin{bmatrix}1&1&\lambda\\\lambda&\lambda&1\\2\lambda-1&\lambda&1\end{bmatrix}\sim\begin{pmatrix}2r-1r\times(\lambda)\\3r-1r\times(2\lambda-1)\end{pmatrix}\sim$$

$$\begin{bmatrix}1&1&\lambda\\0&0&1-\lambda^2\\0&-\lambda-1&-2\lambda^2-\lambda+1\end{bmatrix}\sim\begin{pmatrix}2\mapsto 3\end{pmatrix}\sim\begin{bmatrix}1&1&\lambda\\0&-\lambda-1&-2\lambda^2-\lambda+1\\0&0&1-\lambda^2\end{bmatrix}\sim\begin{pmatrix}2r|(-\lambda-1)\end{pmatrix}\sim$$

$$\begin{bmatrix}1&1&\lambda\\0&1&1-2\lambda\\0&0&1-\lambda^2\end{bmatrix}\sim\begin{pmatrix}1r-2r\times(1)\end{pmatrix}\sim\begin{bmatrix}1&0&1-\lambda\\0&1&1-2\lambda\\0&0&1-\lambda^2\end{bmatrix}\sim\begin{pmatrix}3r|(1-\lambda^2)\end{pmatrix}\sim$$

$$\begin{bmatrix}1&0&1-\lambda\\0&1&1-2\lambda\\0&0&1\end{bmatrix}\sim\begin{pmatrix}1r-3r\times(1-\lambda)\\2r-3r\times(1-2\lambda)\end{pmatrix}\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

1° $\text{rang}(A)=3$ za $\lambda\neq-1,\ \lambda\neq1$

2° $\lambda=-1$ *) uvrštavaš ispred dijeljenja matrice sa $(-\lambda-1)$:

$$\begin{bmatrix}1&1&-1\\0&0&0\\0&0&0\end{bmatrix}\sim\text{rang}(A)=1 \text{ za } \lambda=-1$$

3° $\lambda=1$

$$\begin{bmatrix}1&0&0\\0&1&-1\\0&0&0\end{bmatrix}\sim\text{rang}(A)=2 \text{ za } \lambda=1$$

6.

$$A=\begin{bmatrix}1&3&2&3\\-1&\lambda&\lambda&\lambda\\2&6&\lambda&6\end{bmatrix}\sim\begin{pmatrix}2r-1r\times(-1)\\3r-1r\times(2)\end{pmatrix}\sim\begin{bmatrix}1&3&2&3\\0&3+\lambda&2+\lambda&3+\lambda\\0&0&\lambda-4&0\end{bmatrix}\sim\begin{pmatrix}2r/(3+\lambda)\end{pmatrix}\sim$$

$$\begin{bmatrix}1&3&2&3\\0&1&\frac{2+\lambda}{3+\lambda}&1\\0&0&\lambda-4&0\end{bmatrix}\sim\begin{pmatrix}1r-2r\times(3)\end{pmatrix}\sim\begin{bmatrix}1&0&-\frac{\lambda}{3+\lambda}&0\\0&1&\frac{2+\lambda}{3+\lambda}&1\\0&0&\lambda-4&0\end{bmatrix}\sim\begin{pmatrix}3r/(\lambda-4)\end{pmatrix}\sim$$

$$\begin{bmatrix}1&0&-\frac{\lambda}{3+\lambda}&0\\0&1&\frac{2+\lambda}{3+\lambda}&1\\0&0&1&0\end{bmatrix}\sim\begin{pmatrix}1r-3r\times\left(-\frac{\lambda}{3+\lambda}\right)\\2r-3r\times\left(\frac{2+\lambda}{3+\lambda}\right)\end{pmatrix}\sim\begin{bmatrix}1&0&0&0\\0&1&0&1\\0&0&1&0\end{bmatrix}$$

1° $\text{rang}(A)=3$ za $\lambda\neq4,\ \lambda\neq-3$

2° $\lambda=-3$

$$\begin{bmatrix}1&3&2&3\\0&0&-1&0\\0&0&-7&0\end{bmatrix}\sim\begin{pmatrix}2r/(-1)\end{pmatrix}\sim\begin{bmatrix}1&3&2&3\\0&0&1&0\\0&0&-7&0\end{bmatrix}\sim\begin{pmatrix}1r-2r\times(2)\\3r-2r\times(-7)\end{pmatrix}\sim\begin{bmatrix}1&3&0&3\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

$\text{rang}(A)=2$ za $\lambda=-3$

3° $\lambda=4$

$$\begin{bmatrix}1&0&-\frac{4}{7}&0\\0&1&\frac{6}{7}&1\\0&0&0&0\end{bmatrix}$$

$\text{rang}(A)=2$ za $\lambda=4$

7. $A,B\in\mathcal{M}_n$, $\text{rang}\,A<n$, da li je $\text{rang}(AB)=n$?

- Da bi rang A bio strogo manji od n, matrica A ne smije biti regularna, odnosno mora vrijediti da je $\det A=0$.
- Za matricu AB vrijedi:

$$\det AB=\det A\cdot\det B=0\cdot\det B=0,$$

tj. matrica AB nije regularna, pa vrijedi: $\text{rang}(AB)<n$.

8.

$$[A|I]=\left[\begin{array}{ccc|ccc}2&1&3&1&0&0\\2&4&0&0&1&0\\0&3&1&0&0&1\end{array}\right]\sim\begin{pmatrix}1\leftrightarrow2\end{pmatrix}\sim\left[\begin{array}{ccc|ccc}2&4&0&0&1&0\\2&1&3&1&0&0\\0&3&1&0&0&1\end{array}\right]\sim\begin{pmatrix}1r/2\end{pmatrix}\sim\left[\begin{array}{ccc|ccc}1&2&0&0&\frac12&0\\2&1&3&1&0&0\\0&3&1&0&0&1\end{array}\right]\sim$$

$$\begin{pmatrix}2r-1r\times(2)\end{pmatrix}\sim\left[\begin{array}{ccc|ccc}1&2&0&0&\frac12&0\\0&-3&3&1&-1&0\\0&3&1&0&0&1\end{array}\right]\sim\begin{pmatrix}2r/(-3)\end{pmatrix}\sim\left[\begin{array}{ccc|ccc}1&2&0&0&\frac12&0\\0&1&-1&-\frac13&\frac13&0\\0&3&1&0&0&1\end{array}\right]\sim$$

$$\begin{pmatrix}1r-2r\times(2)\\3r-2r\times(3)\end{pmatrix}\sim\left[\begin{array}{ccc|ccc}1&0&2&\frac23&-\frac16&0\\0&1&-1&-\frac13&\frac13&0\\0&0&4&1&-1&1\end{array}\right]\sim\begin{pmatrix}3r/4\end{pmatrix}\sim\left[\begin{array}{ccc|ccc}1&0&2&\frac23&-\frac16&0\\0&1&-1&-\frac13&\frac13&0\\0&0&1&\frac14&-\frac14&\frac14\end{array}\right]\sim$$

$$\begin{pmatrix}1r-3r\times(2)\\2r-3r\times(-1)\end{pmatrix}\sim\left[\begin{array}{ccc|ccc}1&0&0&\frac16&\frac13&-\frac12\\0&1&0&-\frac1{12}&\frac1{12}&\frac14\\0&0&1&\frac14&-\frac14&\frac14\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}1/6&1/3&-1/2\\-1/12&1/12&1/4\\1/4&-1/4&1/4\end{bmatrix}$$

9.

$$[A\,|\,I]=\left[\begin{array}{cccc|cccc}1&1&0&0&1&0&0&0\\0&1&1&0&0&1&0&0\\0&0&1&1&0&0&1&0\\0&0&0&1&0&0&0&1\end{array}\right]\sim\begin{pmatrix}1r-2r\times(1)\end{pmatrix}\sim\left[\begin{array}{cccc|cccc}1&0&-1&0&1&-1&0&0\\0&1&1&0&0&1&0&0\\0&0&1&1&0&0&1&0\\0&0&0&1&0&0&0&1\end{array}\right]\sim$$

$$\begin{pmatrix}1r-3r\times(-1)\\2r-3r\times(1)\end{pmatrix}\sim\left[\begin{array}{cccc|cccc}1&0&0&1&1&-1&1&0\\0&1&0&-1&0&1&-1&0\\0&0&1&1&0&0&1&0\\0&0&0&1&0&0&0&1\end{array}\right]\sim\begin{pmatrix}1r-4r\times(1)\\2r-4r\times(-1)\\3r-4r\times(1)\end{pmatrix}\sim$$

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&-1&1&-1\\0&1&0&0&0&1&-1&1\\0&0&1&0&0&0&1&-1\\0&0&0&1&0&0&0&1\end{array}\right]\qquad A^{-1}=\begin{bmatrix}1&-1&1&-1\\0&1&-1&1\\0&0&1&-1\\0&0&0&1\end{bmatrix}$$

10.

$$[A\,|\,I]\sim\left[\begin{array}{cccc|cccc}1&2&0&1&1&0&0&0\\-1&3&-1&2&0&1&0&0\\0&2&0&1&0&0&1&0\\1&-1&0&-1&0&0&0&1\end{array}\right]\sim\begin{pmatrix}2r-1r\times(-1)\\4r-1r\times(1)\end{pmatrix}\sim\left[\begin{array}{cccc|cccc}1&2&0&1&1&0&0&0\\0&5&-1&3&1&1&0&0\\0&2&0&1&0&0&1&0\\0&-3&0&-2&-1&0&0&1\end{array}\right]\sim\begin{pmatrix}2r\,|\,5\end{pmatrix}\sim$$

$$\left[\begin{array}{cccc|cccc}1&2&0&1&1&0&0&0\\0&1&-\frac{1}{5}&\frac{3}{5}&\frac{1}{5}&\frac{1}{5}&0&0\\0&2&0&1&0&0&1&0\\0&-3&0&-2&-1&0&0&1\end{array}\right]\sim\begin{pmatrix}1r-2r\times(2)\\3r-2r\times(2)\\4r-2r\times(-3)\end{pmatrix}\sim\left[\begin{array}{cccc|cccc}1&0&\frac{2}{5}&-\frac{1}{5}&\frac{3}{5}&-\frac{2}{5}&0&0\\0&1&-\frac{1}{5}&\frac{3}{5}&\frac{1}{5}&\frac{1}{5}&0&0\\0&0&\frac{2}{5}&-\frac{1}{5}&-\frac{2}{5}&-\frac{2}{5}&1&0\\0&0&-\frac{3}{5}&-\frac{1}{5}&-\frac{2}{5}&\frac{3}{5}&0&1\end{array}\right]\sim$$

$$\begin{pmatrix}3r\,|\,\frac{2}{5}\end{pmatrix}\sim\left[\begin{array}{cccc|cccc}1&0&\frac{2}{5}&-\frac{1}{5}&\frac{3}{5}&-\frac{2}{5}&0&0\\0&1&-\frac{1}{5}&\frac{3}{5}&\frac{1}{5}&\frac{1}{5}&0&0\\0&0&1&-\frac{1}{2}&-1&-1&\frac{5}{2}&0\\0&0&+\frac{3}{5}&-\frac{1}{5}&-\frac{2}{5}&\frac{3}{5}&0&1\end{array}\right]\sim\begin{pmatrix}1r-3r\times(\frac{2}{5})\\2r-3r\times(-\frac{1}{5})\\4r-3r\times(-\frac{3}{5})\end{pmatrix}\sim$$

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&-1&0\\0&1&0&\frac{1}{2}&0&0&\frac{1}{2}&0\\0&0&1&-\frac{1}{2}&-1&-1&\frac{5}{2}&0\\0&0&0&-\frac{1}{2}&-1&0&\frac{3}{2}&1\end{array}\right]\sim\begin{pmatrix}4r\,/\,(-\frac{1}{2})\end{pmatrix}\sim\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&-1&0\\0&1&0&\frac{1}{2}&0&0&\frac{1}{2}&0\\0&0&1&-\frac{1}{2}&-1&-1&\frac{5}{2}&0\\0&0&0&1&2&0&-3&-2\end{array}\right]\sim$$

$$\begin{pmatrix}2r-4r\times(\frac{1}{2})\\3r-4r\times(-\frac{1}{2})\end{pmatrix}\sim\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&-1&0\\0&1&0&0&-1&0&2&1\\0&0&1&0&0&-1&1&-1\\0&0&0&1&2&0&-3&-2\end{array}\right]\qquad A^{-1}=\begin{bmatrix}1&0&-1&0\\-1&0&2&1\\0&-1&1&-1\\2&0&-3&-2\end{bmatrix}$$

11.

$$A=\begin{bmatrix}a&1&-a\\2&0&3\\1&2a&-a^2\end{bmatrix}$$, $a=?$ da postoji A^{-1} , $A^{-1}=?$

- Da postoji A^{-1} : $\det A\neq 0$

$$\left|\begin{array}{ccc}a&1&-a\\2&0&3\\1&2a&-a^2\end{array}\right|\begin{array}{cc}a&1\\2&0\\1&2a\end{array}=(0+3-4a^2)-(0+6a^2-2a^2)=3-4a^2-4a^2=3-8a^2\neq0$$

$$a^2\neq\frac{3}{8}$$

$$a\neq\pm\frac{\sqrt{3}}{2\sqrt{2}}$$

- Po Crameru slijedi:

$$A^{-1}=\frac{\tilde{A}^T}{\det A}$$

$$\tilde{A}^T=\begin{bmatrix}\begin{vmatrix}0&3\\2a&-a^2\end{vmatrix}&-\begin{vmatrix}2&3\\1&-a^2\end{vmatrix}&\begin{vmatrix}2&0\\1&2a\end{vmatrix}\\-\begin{vmatrix}1&-a\\2a&-a^2\end{vmatrix}&\begin{vmatrix}a&-a\\1&-a^2\end{vmatrix}&-\begin{vmatrix}a&1\\1&2a\end{vmatrix}\\\begin{vmatrix}1&-a\\0&3\end{vmatrix}&-\begin{vmatrix}a&-a\\2&3\end{vmatrix}&\begin{vmatrix}a&1\\2&0\end{vmatrix}\end{bmatrix}^T=\begin{bmatrix}-6a&2a^2+3&4a\\-a^2&-a^3+a&-2a^2+2\\3&-5a&-2\end{bmatrix}^T=\begin{bmatrix}-6a&-a^2&3\\2a^2+3&-a^3+a&-5a\\4a&-2a^2+2&-2\end{bmatrix}$$

$$A^{-1}=\frac{1}{3-8a^2}\cdot\begin{bmatrix}-6a&-a^2&3\\2a^2+3&-a^3+a&-5a\\4a&-2a^2+2&-2\end{bmatrix}$$

12. $A=\begin{bmatrix}1&-1&1\\0&1&-1\\0&0&1\end{bmatrix},\ B=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix},\ (A^{-1}BA)^n=?$

$$A^{-1}\Rightarrow[A\,\vdots\,I]=\left[\begin{array}{ccc|ccc}1&-1&1&1&0&0\\0&1&-1&0&1&0\\0&0&1&0&0&1\end{array}\right]\sim\Big(1r-2r\times(-1)\Big)\sim\left[\begin{array}{ccc|ccc}1&0&0&1&1&0\\0&1&-1&0&1&0\\0&0&1&0&0&1\end{array}\right]\sim\Big(2r-3r\times(-1)\Big)\sim$$

$$\left[\begin{array}{ccc|ccc}1&0&0&1&1&0\\0&1&0&0&1&1\\0&0&1&0&0&1\end{array}\right]\qquad A^{-1}=\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$$

$$(A^{-1}\cdot B\cdot A)^1=\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}\cdot\begin{bmatrix}1&-1&1\\0&1&-1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&2&0\\0&2&3\\0&0&3\end{bmatrix}\cdot\begin{bmatrix}1&-1&1\\0&1&-1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&1&-1\\0&2&1\\0&0&3\end{bmatrix}$$

$$(A^{-1}\cdot B\cdot A)^2=\begin{bmatrix}1&1&-1\\0&2&1\\0&0&3\end{bmatrix}\cdot\begin{bmatrix}1&1&-1\\0&2&1\\0&0&3\end{bmatrix}=\begin{bmatrix}1&3&-3\\0&4&5\\0&0&9\end{bmatrix}$$

$$(A^{-1}\cdot B\cdot A)^3=\begin{bmatrix}1&3&-3\\0&4&5\\0&0&9\end{bmatrix}\begin{bmatrix}1&1&-1\\0&2&1\\0&0&3\end{bmatrix}=\begin{bmatrix}1&7&-7\\0&8&19\\0&0&27\end{bmatrix}$$

$$(A^{-1}\cdot B\cdot A)^n=\begin{bmatrix}1&2^n-1&-2^n+1\\0&2^n&3^n-2^n\\0&0&3^n\end{bmatrix}$$

= Kako smo rješenje naslutili, potrebno ga je dokazati matematičkom indukcijom.

(I) baza indukcije:

$n=1$.

$$\begin{bmatrix}1&1&-1\\0&2&1\\0&0&3\end{bmatrix}=\begin{bmatrix}1&1&-1\\0&2&1\\0&0&3\end{bmatrix}\checkmark$$

(II) pretpostavka indukcije:

Pretpostavimo da za neki $n\in N$ vrijedi $(A^{-1}BA)^n=\begin{bmatrix}1&2^n-1&-2^n+1\\0&2^n&3^n-2^n\\0&0&3^n\end{bmatrix}$.

III) skok indukcije:

Treba dokazati da tvrdnja vrijedi i za n+1:

$$C^{n+1}=C^n\cdot C=\begin{bmatrix}1 & 2^n-1 & -2^n+1\\0 & 2^n & 3^n-2^n\\0 & 0 & 3^n\end{bmatrix}\cdot\begin{bmatrix}1 & 1 & -1\\0 & 2 & 1\\0 & 0 & 3\end{bmatrix}=$$

$$\begin{bmatrix}1 & 1+2(2^n-1) & -1+2^n-1+3(-2^n+1)\\0 & 2^n\cdot 2 & 2^n+3(3^n-2^n)\\0 & 0 & 3\cdot 3^n\end{bmatrix}=$$

$$\begin{bmatrix}1 & 1+2\cdot 2^n-2 & -1-1+3+2^n-3\cdot 2^n\\0 & 2\cdot 2^n & 2^n-3\cdot 2^n+3\cdot 3^n\\0 & 0 & 3\cdot 3^n\end{bmatrix}=$$

$$\begin{bmatrix}1 & 2^{n+1}-1 & -2^{n+1}+1\\0 & 2^{n+1} & -2^{n+1}+3^{n+1}\\0 & 0 & 3^{n+1}\end{bmatrix}$$

Zaključak: Tvrdnja vrijedi za svaki $n\in\mathbb{N}$.

13. $A,B,X\in\mathcal{M}_n$ su regularne i vrijedi $A^{-1}\cdot X^2=B^{-1}\cdot A\cdot X$.

= $X=?$
$\det X=?$ } izrazi

= $\det A=6$
$\det B=9$
$\det X=?$

$$A^{-1}X^2=B^{-1}A\,X \quad /\cdot X^{-1}\ (L)$$
$$A^{-1}X^2X^{-1}=B^{-1}A \quad /\cdot A\ (D)$$
$$X^2\cdot X^{-1}=A\cdot B^{-1}A$$
$$X=A\cdot B^{-1}A$$

$$\det X=\det A\cdot\frac{1}{\det B}\cdot\det A=\frac{\det^2 A}{\det B}$$

$$=\det X=\frac{6^2}{9}=4$$

14. $A=\begin{bmatrix}1 & -2\\3 & 1\end{bmatrix}$, $B=\begin{bmatrix}0 & 3\\-1 & 2\end{bmatrix}$, regularne=? Riješi: $(BX^{-1}A)^{-1}=AB$

= $\det A\neq 0$ $\qquad$ $\det B\neq 0$
$\det A=1+6=7\neq 0$ $\qquad$ $\det B=0+3\neq 0$

$$=(BX^{-1}A)^{-1}=AB$$
$$A^{-1}XB^{-1}=AB \quad /\cdot A\ (L)$$
$$XB^{-1}=A^2\cdot B \quad /\cdot B\ (D)$$
$$X=A^2\cdot B^2$$

$$X=\begin{bmatrix}1 & -2\\3 & 1\end{bmatrix}\begin{bmatrix}1 & -2\\3 & 1\end{bmatrix}\begin{bmatrix}0 & 3\\-1 & 2\end{bmatrix}\begin{bmatrix}0 & 3\\-1 & 2\end{bmatrix}=\begin{bmatrix}-5 & -4\\6 & -5\end{bmatrix}\begin{bmatrix}-3 & 6\\-2 & 1\end{bmatrix}=\begin{bmatrix}23 & -34\\-8 & 31\end{bmatrix}$$

15.

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 3 \\ -2 & 0 & 2 \end{bmatrix} X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad X = ?$$

$$AX = B \;/\cdot A^{-1} \,(L)$$
$$X = A^{-1} B$$

$$A^{-1} \Rightarrow [A \,\vdots\, I] = \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ -2 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \begin{pmatrix} 3r - 1r \times (-2) \end{pmatrix} \sim \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & -2 & -2 & 2 & 0 & 1 \end{array}\right] \sim$$

$$\begin{pmatrix} 1r - 2r \times (-1) \\ 3r - 2r \times (-2) \end{pmatrix} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 4 & 2 & 2 & 1 \end{array}\right] \sim \begin{pmatrix} 3r / 4 \end{pmatrix} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{array}\right] \sim$$

$$\begin{pmatrix} 1r - 3r \times (1) \\ 2r - 3r \times (3) \end{pmatrix} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{3}{2} & -\frac{1}{2} & -\frac{3}{4} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{array}\right] \qquad A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{3}{2} & -\frac{1}{2} & -\frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{3}{2} & -\frac{1}{2} & -\frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{3}{4} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

16.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix} X = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad X = ?$$

$$AX = B \;/\cdot A^{-1} \,(L)$$
$$X = A^{-1} B$$

$$A^{-1} \Rightarrow [A \,\vdots\, I] = \left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \sim \begin{pmatrix} 2 \leftrightarrow 1 \end{pmatrix} \sim \left[\begin{array}{cccc|cccc} -1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\begin{pmatrix} 1r / (-1) \end{pmatrix} \sim \left[\begin{array}{cccc|cccc} 1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \sim \begin{pmatrix} 3r - 1r \times (-1) \\ 4r - 1r \times (-1) \end{pmatrix} \sim$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & -1 & 0 & 1 \end{array}\right] \sim \begin{pmatrix} 3r - 2r \times (-1) \\ 4r - 2r \times (-1) \end{pmatrix} \sim$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \sim \begin{pmatrix} 3 \leftrightarrow 4 \end{pmatrix} \sim \left[\begin{array}{cccc|cccc} 1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \end{array}\right] \sim$$

$$\begin{pmatrix} 3r / (-1) \end{pmatrix} \sim \left[\begin{array}{cccc|cccc} 1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \end{array}\right] \sim \begin{pmatrix} 1r - 3r \times (-1) \\ 2r - 3r \times (1) \end{pmatrix} \sim$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \end{array}\right] \sim \begin{pmatrix} 1r - 4r \times (-1) \\ 2r - 4r \times (1) \end{pmatrix} \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \end{array}\right] \quad A^{-1} = \begin{bmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

17. a) Dokaži da za regularne matrice $A, B, C \in M_n$ je: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

$$(ABC)X = I$$
$$ABCX = I \quad / \cdot A^{-1} \ (L)$$
$$BCX = A^{-1} \quad / \cdot B^{-1} \ (L)$$
$$CX = B^{-1}A^{-1} \quad / \cdot C^{-1} \ (L)$$
$$X = C^{-1}B^{-1}A^{-1} \ \checkmark$$

b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$

$$BX^{-1}A = B^{-1}A^{-1} \quad / \cdot B^{-1} \ (L)$$
$$X^{-1}A = B^{-1} \cdot B^{-1} \cdot A^{-1} \quad / \cdot A^{-1} \ (D)$$
$$X^{-1} = B^{-1} \cdot B^{-1} \cdot A^{-1} \cdot A^{-1} \quad /^{-1}$$
$$X = A^{1} \cdot A^{1} \cdot B^{1} \cdot B^{1}$$
$$X = A^2 \cdot B^2$$

$$X = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 7 \\ 1 & 1 & 1 \\ 3 & 5 & 3 \end{bmatrix}$$

18. $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix}$

$$(X^{-1}A)^{-1} = I + BX$$
$$A^{-1}X = I + BX \quad / \cdot X^{-1} \ (D)$$
$$A^{-1} = X^{-1} + B$$
$$X^{-1} = A^{-1} - B$$

$$A^{-1}: \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 7 & 0 & 1 \end{array}\right] \sim \begin{pmatrix} 2r - 1r \times 3 \end{pmatrix} \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{array}\right] \sim \begin{pmatrix} 1r - 2r \times 2 \end{pmatrix} \sim \left[\begin{array}{cc|cc} 1 & 0 & 7 & -2 \\ 0 & 1 & -3 & 1 \end{array}\right]$$

$$X^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$$

$$X = \left[\begin{array}{cc|cc} 5 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{array}\right] \sim \begin{pmatrix} 1r / 5 \end{pmatrix} \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{5} & 0 \\ 0 & -2 & 0 & 1 \end{array}\right] \sim \begin{pmatrix} 2r / (-2) \end{pmatrix} \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{5} & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \end{array}\right]$$

$$X = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

(19.) $A=\begin{bmatrix}1&2&0\\0&0&1\\1&0&0\end{bmatrix}$, $B=\begin{bmatrix}1&-2&1\\1&0&2\\0&1&1\end{bmatrix}$

→ A, B - regularne ?

→ $AX^{-1}B = (B^T)^{-1}A$

= $\det A \neq 0$ $\qquad \det B \neq 0$

$\det A = \begin{vmatrix}1&2&0\\0&0&1\\1&0&0\end{vmatrix}\begin{matrix}1&2\\0&0\\1&0\end{matrix} = 2$ ✓ $\qquad \det B = \begin{vmatrix}1&-2&1\\1&0&2\\0&1&1\end{vmatrix}\begin{matrix}1&-2\\1&0\\0&1\end{matrix} = 1-2+2=1$ ✓

$= \quad AX^{-1}\cdot B = (B^T)^{-1}A \quad |\cdot A^{-1}\,(L)$

$X^{-1}\cdot B = A^{-1}\cdot(B^T)^{-1}A \quad |\cdot B^{-1}\,(D)$

$X^{-1} = A^{-1}\cdot(B^T)^{-1}\cdot A\cdot B^{-1} \quad |^{-1}$

$X = B\cdot A^{-1}\cdot B^T\cdot A$

$[A\,|\,I] = \left[\begin{array}{ccc|ccc}1&2&0&1&0&0\\0&0&1&0&1&0\\1&0&0&0&0&1\end{array}\right] \sim \left(3r - 1r\times(1)\right) \sim \left[\begin{array}{ccc|ccc}1&2&0&1&0&0\\0&0&1&0&1&0\\0&-2&0&-1&0&1\end{array}\right] \sim \left(2 \mapsto 3\right) \sim$

$\left[\begin{array}{ccc|ccc}1&2&0&1&0&0\\0&-2&0&-1&0&1\\0&0&1&0&1&0\end{array}\right] \sim \left(2r\,|\,(-2)\right) \sim \left[\begin{array}{ccc|ccc}1&2&0&1&0&0\\0&1&0&\frac{1}{2}&0&-\frac{1}{2}\\0&0&1&0&1&0\end{array}\right] \sim \left(1r - 2r\times(2)\right) \sim$

$\left[\begin{array}{ccc|ccc}1&0&0&0&0&1\\0&1&0&\frac{1}{2}&0&-\frac{1}{2}\\0&0&1&0&1&0\end{array}\right] \qquad A^{-1} = \begin{bmatrix}0&0&1\\\frac{1}{2}&0&-\frac{1}{2}\\0&1&0\end{bmatrix}$

$X = \begin{bmatrix}1&-2&1\\1&0&2\\0&1&1\end{bmatrix}\cdot\begin{bmatrix}0&0&1\\\frac{1}{2}&0&-\frac{1}{2}\\0&1&0\end{bmatrix}\cdot\begin{bmatrix}1&1&0\\-2&0&1\\1&2&1\end{bmatrix}\cdot\begin{bmatrix}1&2&0\\0&0&1\\1&0&0\end{bmatrix} = \begin{bmatrix}-1&1&2\\0&2&1\\\frac{1}{2}&1&-\frac{1}{2}\end{bmatrix}\cdot\begin{bmatrix}1&2&1\\-1&-4&0\\2&2&2\end{bmatrix} =$

$= \begin{bmatrix}2&-2&3\\0&-6&2\\\frac{3}{2}&-4&-\frac{1}{2}\end{bmatrix}$

(20.) $\begin{cases}AX+Y=I & |\cdot B\,(D)\\ XB+A^{-1}Y=0 & |\cdot A\,(L)\end{cases}$ $\qquad A=\begin{bmatrix}3&-2\\-4&3\end{bmatrix}$, $B=\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$, $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $0=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

$\begin{cases}AXB+YB=B\\ AXB+Y=0\end{cases}$ $\qquad X, Y = ?$

$YB-B=-Y$

$YB+Y=B$

$Y(B+I)=B$

$Y=B\cdot(B+I)^{-1}$

$Y=B \quad \Rightarrow \quad Y=\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$

$AX=I-Y \quad |\cdot A^{-1}\,(L)$

$X=A^{-1}(I-Y)$

$A^{-1} = \left[\begin{array}{cc|cc}3&-2&1&0\\-4&3&0&1\end{array}\right] \sim \left(1r\,|\,3\right) \sim \left[\begin{array}{cc|cc}1&-\frac{2}{3}&\frac{1}{3}&0\\-4&3&0&1\end{array}\right] \sim \left(2r-1r\times(-4)\right) \sim \left[\begin{array}{cc|cc}1&-\frac{2}{3}&\frac{1}{3}&0\\0&\frac{1}{3}&\frac{4}{3}&1\end{array}\right] \sim$

$\left(2r\,/\,\frac{1}{3}\right) \sim \left[\begin{array}{cc|cc}1&-\frac{2}{3}&\frac{1}{3}&0\\0&1&4&3\end{array}\right] \sim \left(1r-2r\times(-\frac{2}{3})\right) \sim \left[\begin{array}{cc|cc}1&0&3&2\\0&1&4&3\end{array}\right]$

$$X = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

4.5. LINEARNA NEZAVISNOST VEKTORA I RANG MATRICE

= Sa V^n označavat ćemo prostor svih vektor-stupaca duljine n, odnosno:

$$V^n \to \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

= **DEFINICIJA 1.**

↳ Neka su $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_k$ bilo koji vektori iz prostora V^n. Linearna kombinacija vektora $\vec{a}_1, \dots, \vec{a}_k$ je vektor oblika:

$$x = \lambda_1 \cdot \vec{a}_1 + \lambda_2 \cdot \vec{a}_2 + \dots + \lambda_k \cdot \vec{a}_k,$$

gdje su $\lambda_1, \lambda_2, \dots, \lambda_k$ skalari iz $\mathbb{R}$.

Skup svih ovakvih linearnih kombinacija nazivamo prostor razapet vektorima $\vec{a}_1, \dots, \vec{a}_k$ i označavamo s:

$$L(\vec{a}_1, \dots \vec{a}_k) = \{\vec{x} \mid \vec{x} = \lambda_1 \vec{a}_1 + \dots + \lambda_k \vec{a}_k,\ \lambda_i \in \mathbb{R}\}.$$

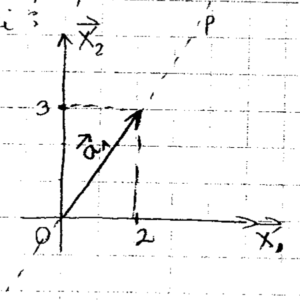
= **PRIMJER 1.**

Neka je $\vec{a}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in V^2$. Naći $L_1(\vec{a}_1) = \lambda$, i interpretirati vektor geometrijski.

$$= L(\vec{a}_1) = \{\vec{x} = \lambda \cdot \vec{a}_1,\ \lambda \in \mathbb{R}\} = \left\{\begin{bmatrix} 2\lambda \\ 3\lambda \end{bmatrix}, \lambda \in \mathbb{R}\right\}$$

= vektor $\vec{a}_1$ pišemo i kao: $2\lambda \vec{i} + 3\lambda \vec{j} = \lambda(2\vec{i} + 3\vec{j})$

Grafički:



→ $L(\vec{a}_1)$ je jednodimenzionalni podprostor u V^2; predstavlja pravac p kroz ishodište određen vektorom $2\vec{i} + 3\vec{j}$ tzv. vektor nosioc.

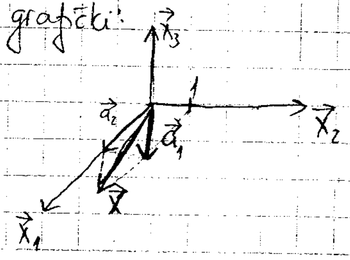
= **PRIMJER 2.**

U skupu V^3 naći prostor razapet vektorima: $\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ i $\vec{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$= L(\vec{a}_1, \vec{a}_2) = \{\vec{x} \mid \vec{x} = \lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2\} = \left\{\vec{x} \mid \vec{x} = \begin{bmatrix} \lambda_1 \\ \lambda_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \lambda_2 \\ 0 \\ 0 \end{bmatrix}\right\} = \left\{\vec{x} \mid \vec{x} = \begin{bmatrix} \lambda_1 + \lambda_2 \\ \lambda_1 \\ 0 \end{bmatrix},\ \lambda_1, \lambda_2 \in \mathbb{R}\right\}$$

↳ $\vec{x} = (\lambda_1 + \lambda_2)\vec{i} + \lambda_1 \vec{j}$.

= grafički:



- Ovo je podprostor vektora u XOY ravnini, jer se svaki vektor iz XOY može napisati pomoću linearne kombinacije od $\vec{a}_1$ i $\vec{a}_2$.

Dokaz:

1) geometrijski - sa slike:

$$\vec{x} = \lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2$$

2) analitički:

Uzmimo bilo koji vektor $\vec{x} \in xoy$; recimo $\vec{x} = \begin{bmatrix} \vec{x}_1 \\ \vec{x}_2 \\ 0 \end{bmatrix}$.

$\hookrightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\hookrightarrow \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_2 + \lambda_1 \\ \lambda_1 \\ 0 \end{bmatrix}$

$\hookrightarrow$

$x_1 = \lambda_1 + \lambda_2 \quad \Rightarrow \lambda_2 = x_1 - x_2$

$x_2 = \lambda_1 \quad \Rightarrow \lambda_1 = x_2$

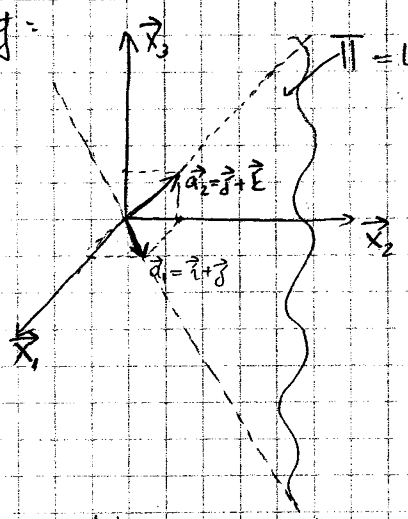
$0 = 0$

Dakle za svaki vektor iz ravnine xoy možemo pronaći odgovarajuće koeficijente λ_1 i λ_2 tako da je $\vec{x} = \lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2$.

= PRIMJER 3.

a) Neka su zadani $\vec{a}_1 = [1,1,0]^T$, $\vec{a}_2 = [0,1,1]^T$. Naći $L(\vec{a}_1, \vec{a}_2)$ i skiciraj ga.

-> graf:



$\Pi = L(\vec{a}_1, \vec{a}_2)$: skup svih radij-vektora koji leže u ravnini Π razapetoj s vektorima $\vec{a}_1$ i $\vec{a}_2$.

b) 1 PITANJE: Ako uzmemo treći vektor $\vec{a}_3 = [2,-1,-3]^T$, da li vrijedi:

$L_1(\vec{a}_1, \vec{a}_2, \vec{a}_3) = L_2(\vec{a}_1, \vec{a}_2)$?

$\hookrightarrow$ Vrijedi $L_1(\vec{a}_1, \vec{a}_2, \vec{a}_3) = L_2(\vec{a}_1, \vec{a}_2)$ jer vektor $\vec{a}_3$ leži u ravnini razapetoj vektorima $\vec{a}_1$ i $\vec{a}_2$!

Naime:

$\vec{a}_3 = 2\vec{a}_1 - 3\vec{a}_2$, tj. $\vec{a}_3$ je linearna kombinacija tih dvaju vektora.

2 PITANJE: Ako uzmemo treći vektor $\vec{a}_3 = [0,1,0]^T$, da li vrijedi:

$L_1(\vec{a}_1, \vec{a}_2, \vec{a}_3) = L_2(\vec{a}_1, \vec{a}_2)$?

$\hookrightarrow$ Ne vrijedi $L_1(\vec{a}_1, \vec{a}_2, \vec{a}_3) = L_2(\vec{a}_1, \vec{a}_2)$, jer vektor $\vec{a}_3$ ne leži u ravnini razapetoj vektorima $\vec{a}_1$, $\vec{a}_2$, tj.

$L_1(\vec{a}_1, \vec{a}_2, \vec{a}_3) = V^3$

Naime, svaki vektor $\vec{x} = [\vec{x}_1, \vec{x}_2, \vec{x}_3]^T$ možemo pokazati kao linearnu kombinaciju vektora $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, tj.:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \lambda_3 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_1 + \lambda_2 + \lambda_3 \\ \lambda_2 \end{bmatrix} \Leftrightarrow$$

$x_1 = \lambda_1 \qquad\qquad\quad \lambda_1 = x_1$

$x_2 = \lambda_1 + \lambda_2 + \lambda_3 \quad \longrightarrow \quad \lambda_2 = x_2 - x_1 - x_3$

$x_3 = \lambda_2 \qquad\qquad\quad \lambda_3 = x_3$

= DEFINICIJA 2.

↳ Kažemo da su vektori $\vec{a}_1, \dots, \vec{a}_k$ linearno nezavisni ako iz jednakosti:

$$\lambda_1 \vec{a}_1 + \dots + \lambda_k \vec{a}_k = \vec{0}$$

Slijedi da svi skalari $\lambda_1, \dots, \lambda_k$ moraju biti jednaki 0, tj. : $\lambda_1 = \lambda_2 = \dots = \lambda_k = \emptyset$.
Drugim riječima, vektori su linearno nezavisni ako njihova linearna kombinacija iščezava samo na trivijalan način.

= DEFINICIJA 3.

↳ Kažemo da su vektori $\vec{a}_1, \dots, \vec{a}_k$ linearno zavisni ako postoje skalari $\lambda_1, \dots, \lambda_k$ kod kojih barem jedan nije jednak nuli, tako da vrijedi:

$$\lambda_1 \vec{a}_1 + \dots + \lambda_k \vec{a}_k = \vec{0}$$

Drugim riječima, vektori su linearno zavisni ako njihova linearna kombinacija iščezava na ne-trivijalan način.

= PRIMJEDBA 1.

↳ U slučaju linearno zavisnih vektora, jedan od vektora može se izraziti kao linearna kombinacija ostalih. Naime, ako je primjerice $\lambda_i \neq 0$, onda:

$$\lambda_1 \vec{a}_1 + \dots + \lambda_i \vec{a}_i + \lambda_k \vec{a}_k = \vec{0} \quad / : \lambda_i$$

$$\vec{a}_i = \frac{\lambda_1}{\lambda_i} \vec{a}_1 - \dots - \frac{\lambda_k}{\lambda_i} \vec{a}_k .$$

= PRIMJER 4.

Ispitaj da li su vektori zavisni ili nezavisni ako vrijedi:

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{a}_1, \vec{a}_2 \in V^3.$$

$$\lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2 = \vec{0}$$

$$\lambda_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \lambda_2 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_1 + \lambda_2 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{matrix} \lambda_1 = 0 \\ \lambda_1 + \lambda_2 = 0 \\ \lambda_2 = 0 \end{matrix} \iff \lambda_1 = \lambda_2 = 0$$

↳ trivijalno rješenje, vektori su linearno nezavisni

= PRIMJER 5.

Ispitaj da li su vektori zavisni ili nezavisni ako vrijedi:

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}, \quad \vec{a}_1, \vec{a}_2, \vec{a}_3 \in V^3.$$

$$\lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2 + \lambda_3 \vec{a}_3 = \vec{0}$$

$$\lambda_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \lambda_3 \cdot \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 + 2\lambda_3 \\ \lambda_1 + \lambda_2 - \lambda_3 \\ \lambda_2 - 3\lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{array}{l} \lambda_1 + 2\lambda_3 = 0 \\ \lambda_1 + \lambda_2 - \lambda_3 = 0 \\ \lambda_2 - 3\lambda_3 = 0 \end{array} \iff \begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = -3 \\ \lambda_3 = -1 \end{array}$$

$\hookrightarrow$ $\lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2 + \lambda_3 \vec{a}_3 = 0$

$2\vec{a}_1 - 3\vec{a}_2 - \vec{a}_3 = 0$

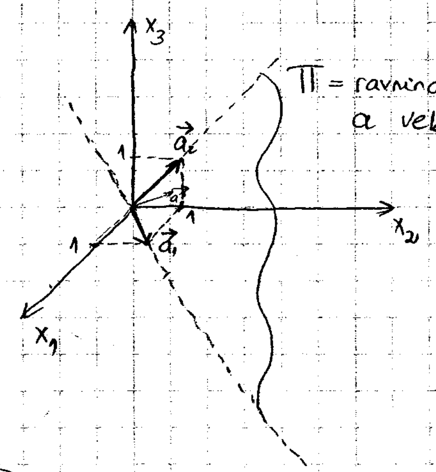
$\vec{a}_3 = 2\vec{a}_1 - 3\vec{a}_2$

$\hookrightarrow$ vektori su zavisni jer se jedan vektor može izraziti kao kombinacija preostalih vektora.

= **PRIMJER 6.**

Odredi dimenziju prostora $L(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ i nacrtaj graf, ako vrijedi:
$\vec{a}_1 = [1, 1, 0]^T$, $\vec{a}_2 = [0, 1, 1]^T$, $\vec{a}_3 = [2, -1, -3]^T$.

Dimenzija prostora $L(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ je 2, jer postoje vektori $\vec{a}_1, \vec{a}_2, \vec{a}_3$ koji su linearno zavisni, dok su npr. $\vec{a}_1$ i $\vec{a}_2$ linearno nezavisni, grafički:



Π = ravnina razapeta vektorima $\vec{a}_1, \vec{a}_2$, a vektor $\vec{a}_3 \in \Pi$

= Za zadane vektore $\vec{a}_1, \dots, \vec{a}_k$ uvijek je jednoznačno određen broj linearno nezavisnih vektora iz skupa $\{\vec{a}_1, \dots, \vec{a}_k\}$. Taj broj nazivamo dimenzijom prostora kojeg označavamo $L(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n)$.

= **PRIMJER 7.**

Odredi da li su vektori zavisni ili nezavisni ako vrijedi:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \quad \dots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

$$\lambda_1 \vec{e}_1 + \lambda_2 \vec{e}_2 + \dots + \lambda_n \vec{e}_n = \vec{0}$$

$$\lambda_1 \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \lambda_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} + \dots + \lambda_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \iff \begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \vdots \\ \lambda_n = 0 \end{array}$$ $\rightarrow$ vektori su linearno nezavisni

= To znači da se svaki vektor $\vec{x} = [x_1, \dots, x_n]^T$ može prikazati u obliku linearne kombinacije zadanih vektora $e_1, \dots, e_n$ tj.:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \dots + x_n \vec{e}_n$$

= Broj linearno nezavisnih redaka neke matrice A dobije se tako da se ta matrica svede na reduciranu formu A_R. Onda je broj linearno nezavisnih redaka od matrice A jednak rangu matrice A_R, tj. broju ne-nul redaka od A_R.

= STAVAK 1.

↳ Broj linearno nezavisnih redaka bilo koje matrice jednak je broju njenih linearno nezavisnih stupaca.

5. LINEARNI SUSTAVI

5.1. GAUSSOVA METODA ELIMINACIJE

= PRIMJER 1.

Napiši vektorski zapis sustava:
$$\begin{aligned} x+3y-z&=-4\\ -x+2y+3z&=5\\ 2x+y+z&=6 \end{aligned}$$

↳ Zadanom sustavu pripišemo matricu:

$$\begin{bmatrix} 1 & 3 & -1\\ -1 & 2 & 3\\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} -4\\ 5\\ 6 \end{bmatrix}$$

↳ Iz sustava vrijedi: $x=2$, $y=-1$, $z=3$ $\Rightarrow$ $\vec{x}=\begin{bmatrix} 2\\ -1\\ 3 \end{bmatrix}$

= Općenito, sustav od m jednadžbi s n nepoznanica zapisuje se:

$$\begin{aligned} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n&=b_1\\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n&=b_2\\ &\vdots\\ a_{m1}x_1+a_{m2}x_2+\dots+a_{mn}x_n&=b_m \end{aligned}$$

Njegov zapis dan je u obliku matrične jednadžbe $A\vec{x}=\vec{b}$ i to da vrijedi:

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},\quad \vec{x}=\begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n \end{bmatrix},\quad \vec{b}=\begin{bmatrix} b_1\\ b_2\\ \vdots\\ b_n \end{bmatrix}.$$

A = matrica koeficijenata sustava
$\vec{x}$ = vektor nepoznanica
$\vec{b}$ = desna strana sustava, vektor slobodnih koeficijenata

Rješenje sustava je svaki n-terac $(x_1, x_2, \dots, x_n)$ koji zadovoljava sve jednadžbe sustava. Mogu nastupiti sljedeći slučajevi za broj rješenja:

1.) nula rješenja $\Rightarrow$ kontradiktoran (protuslovan) sustav

= PRIMJER 2.

Sustav:
$$\begin{aligned} 2x+3y&=4\\ 4x+6y&=7 \end{aligned}$$
je protuslovan.

$$\begin{aligned} 2x+3y&=4 \mid \cdot 2\\ 4x+6y&=7\\ \hline 4x+6y&=8\\ 4x+6y&=7\\ \hline 7&=8 \ ?? \end{aligned}$$

2.) jedno rješenje $\Rightarrow$ jednoznačno (jedinstveno) rješenje

= PRIMJER 3.

Sustav:
$$\begin{aligned} 2x+3y&=4\\ -x+y&=3 \end{aligned}$$
ima jedinstveno rješenje.

$2x+3y=4$
$-x+y=3 \to x=y-3$
$2(y-3)+3y=4 \quad x=-1$
$5y=10$
$y=2$

3.) beskonačno mnogo rješenja

=PRIMJER 4.
Sustav: $\begin{array}{l} 2x+3y=4 \\ 4x+6y=8 \end{array}$ ima beskonačno mnogo rješenja.

$2x+3y=4 \to 2x=4-3y$
$4x+6y=8 \qquad x=2-\frac{3}{2}y \to x=2-\frac{3}{2}\lambda,\ \lambda\in\mathbb{R}$
$4\left(2-\frac{3}{2}y\right)+6y=8$

$8-\cancel{6y}+\cancel{6y}=8$
$0=0 \quad (\infty)$
$y=\lambda,\ \lambda\in\mathbb{R}$

= Gaussova metoda eliminacije
↳ Metoda se sastoji u tome da se sustav prevede elementarnim transformacijama u ekvivalentni sustav kojem se lako nalaze rješenja.
Navedene transformacije dostaje provoditi nad proširenom matricom koeficijenata sustava: $[A \vdots \vec{B}]$.
↳ Algoritam:
- 1. korak - Sustav $A\vec{x}=\vec{B}$ napišemo u matričnom obliku $[A|\vec{B}]$.
- 2. korak - Matricu A svedemo na reducirani oblik A_R čime se dobiva ekvivalentan sustav: $[A_R \vdots \vec{B}']$.
Ako postoji nulredak od A_R takav da je odgovarajući član u stupcu $\vec{B}$ različit od nule, sustav tada nema rješenja jer je $\text{rang}(A)<\text{rang}(A,\vec{B})$.
- 3. korak - Ispisujemo sustav jednadžbi koji odgovara sustavu $[A_R|\vec{b}']$. Nepoznanice pridružene stupcima sa stožernim elementima ostavljamo na lijevoj strani jednakosti (to su tzv. "vezane" nepoznanice), a ostale elemente prebacujemo na desnu stranu jednakosti (to su tzv. "slobodne" nepoznanice).
- 4. korak - Prebačene nepoznanice parametriziramo, jer mogu poprimiti sve moguće vrijednosti iz $\mathbb{R}$, te zapišemo rješenje sustava u vektorskom obliku.

= Postoje dvije vrste sustava:
1.) homogeni sustav
↳ oblik:
$$\begin{array}{l} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=0 \\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n=0 \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+\dots+a_{mn}x_n=0 \end{array}$$
↳ uvijek ima rješenje $(\vec{0})$
2.) nehomogeni sustav
↳ ostali sustavi
↳ ne mora uvijek imati rješenje

= PRIMJER 5.

Riješi sustav:
$$\begin{aligned} x_1+2x_2+x_3+x_4&=0\\ 2x_1+x_2+x_3+2x_4&=0\\ x_1+2x_2+2x_3+x_4&=0\\ x_1+x_2+x_3+x_4&=0 \end{aligned}$$

$$\hookrightarrow [A|\vec{b}]=\left[\begin{array}{cccc|c}1&2&1&1&0\\2&1&1&2&0\\1&2&2&1&0\\1&1&1&1&0\end{array}\right]\sim\begin{pmatrix}2r-1r\times(2)\\3r-1r\times(1)\\4r-1r\times(1)\end{pmatrix}\sim\left[\begin{array}{cccc|c}1&2&1&1&0\\0&-3&-1&0&0\\0&0&1&0&0\\0&-1&0&0&0\end{array}\right]\sim\begin{pmatrix}2r\,|\,(-3)\end{pmatrix}\sim$$

$$\left[\begin{array}{cccc|c}1&2&1&1&0\\0&1&\frac{1}{3}&0&0\\0&0&1&0&0\\0&-1&0&0&0\end{array}\right]\sim\begin{pmatrix}1r-2r\times(2)\\4r-2r\times(-1)\end{pmatrix}\sim\left[\begin{array}{cccc|c}1&0&\frac{1}{3}&1&0\\0&1&\frac{1}{3}&0&0\\0&0&1&0&0\\0&0&0&0&0\end{array}\right]\sim\begin{pmatrix}1r-3r\times(\frac{1}{3})\\2r-3r\times(\frac{1}{3})\end{pmatrix}\sim$$

$$\left[\begin{array}{cccc|c}1&0&0&1&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\end{array}\right]$$

$\hookrightarrow$
$$\begin{aligned} x_1+x_4&=0\\ x_2&=0\\ x_3&=0 \end{aligned}$$

$x_1, x_2, x_3 \to$ "vezane" nepoznanice

$x_4 \to$ "slobodna" nepoznanica $\quad x_4=\lambda,\ \lambda\in\mathbb{R}$

$$\Updownarrow$$

$$\begin{aligned} x_1&=-x_4\\ x_2&=0\\ x_3&=0 \end{aligned}\quad\Longleftrightarrow\quad\begin{aligned} x_1&=-\lambda\\ x_2&=0\\ x_3&=0 \end{aligned}\ ,\ \lambda\in\mathbb{R}$$

$$\hookrightarrow\quad \vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-\lambda\\0\\0\\\lambda\end{bmatrix}=\lambda\begin{bmatrix}-1\\0\\0\\1\end{bmatrix},\ \lambda\in\mathbb{R}$$

= PRIMJER 6.

Riješi sustav:
$$\begin{aligned} x_1+x_2+x_3&=3\\ -2x_1+x_2+x_3&=0\\ 3x_2+3x_3&=5 \end{aligned}$$

$$\hookrightarrow [A|\vec{b}]=\left[\begin{array}{ccc|c}1&1&1&3\\-2&1&1&0\\0&3&3&5\end{array}\right]\sim\begin{pmatrix}2r-1r\times(-2)\end{pmatrix}\sim\left[\begin{array}{ccc|c}1&1&1&3\\0&3&3&6\\0&3&3&5\end{array}\right]\sim\begin{pmatrix}2r\,|\,3\end{pmatrix}\sim$$

$$\left[\begin{array}{ccc|c}1&1&1&3\\0&1&1&2\\0&3&3&5\end{array}\right]\sim\begin{pmatrix}1r-2r\times(1)\\3r-2r\times(3)\end{pmatrix}\sim\left[\begin{array}{ccc|c}1&0&0&-1\\0&1&1&2\\0&0&0&-\frac{1}{3}\end{array}\right]$$

$\hookrightarrow$
$$\begin{aligned} x_1&=-1\\ x_2+x_3&=2\\ 0&=-\tfrac{1}{3} \end{aligned}$$

$\hookrightarrow$ protuslovan sustav jer $0\neq-\frac{1}{3}$ pa sustav nema rješenja

$\hookrightarrow$ $\underset{2}{\text{rang}(A)} < \underset{3}{\text{rang}(A|\vec{b})}$

$$2 < 3$$

Samo je jedan Mali Ivica!

=PRIMJER 7.

Riješi sustav: $$\begin{aligned} x_1+2x_2+x_3+2x_4+3x_5&=0\\ 2x_1+4x_2+3x_3+3x_4+7x_5&=0\\ 2x_1+4x_2+x_3+5x_4+5x_5&=0 \end{aligned}$$

$$[A\,|\,\vec{b}\,]=\left[\begin{array}{ccccc|c} 1&2&1&2&3&0\\ 2&4&3&3&7&0\\ 2&4&1&5&5&0\end{array}\right]\sim\begin{pmatrix}2r-1r\times(2)\\3r-1r\times(2)\end{pmatrix}\sim\left[\begin{array}{ccccc|c} 1&2&1&2&3&0\\ 0&0&1&-1&1&0\\ 0&0&-1&1&-1&0\end{array}\right]\sim\begin{pmatrix}1r-2r\times(1)\\3r-2r\times(-1)\end{pmatrix}\sim$$

$$\left[\begin{array}{ccccc|c} 1&2&0&3&2&0\\ 0&0&1&-1&1&0\\ 0&0&0&0&0&0\end{array}\right]$$

$$\begin{aligned} x_1+2x_2+3x_4+2x_5&=0\\ x_3-x_4+x_5&=0\end{aligned}$$

x_1, x_3 – "vezane" nepoznanice

x_2, x_4, x_5 – "slobodne" nepoznanice $x_2=\alpha,\ \alpha\in\mathbb{R}$; $x_4=\beta,\ \beta\in\mathbb{R}$; $x_5=\gamma,\ \gamma\in\mathbb{R}$

$$\Updownarrow$$

$$\begin{aligned} x_1&=-2x_2-3x_4-2x_5\\ x_3&=x_4-x_5\end{aligned}\iff\begin{aligned} x_1&=-2\alpha-3\beta-2\gamma\\ x_3&=\beta-\gamma\end{aligned},\ \alpha,\beta,\gamma\in\mathbb{R}$$

$$\vec{X}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}-2\alpha-3\beta-2\gamma\\ \alpha\\ \beta-\gamma\\ \beta\\ \gamma\end{bmatrix}=\alpha\cdot\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+\beta\cdot\begin{bmatrix}-3\\0\\1\\1\\0\end{bmatrix}+\gamma\cdot\begin{bmatrix}-2\\0\\-1\\0\\1\end{bmatrix},\ \alpha,\beta,\gamma\in\mathbb{R}$$

dobili smo troparametarsko rješenje u kojem vektori: $[-2\ 1\ 0\ 0\ 0]^T$, $[-3\ 0\ 1\ 1\ 0]^T$, $[-2\ 0\ -1\ 0\ 1]^T$ razapinju trodimenzionalni prostor.

=PRIMJER 8.

Riješi sustav u ovisnosti o parametru λ: $$\begin{aligned} \lambda x+y+z&=1\\ x+\lambda y+z&=\lambda\\ x+y+\lambda z&=\lambda\end{aligned}$$

$$[A\,|\,\vec{b}\,]=\left[\begin{array}{ccc|c} \lambda&1&1&1\\ 1&\lambda&1&\lambda\\ 1&1&\lambda&\lambda\end{array}\right]\sim\begin{pmatrix}1\mapsto 3\end{pmatrix}\sim\left[\begin{array}{ccc|c} 1&1&\lambda&\lambda\\ 1&\lambda&1&\lambda\\ \lambda&1&1&1\end{array}\right]\sim\begin{pmatrix}2r-1r\times(1)\\3r-1r\times(\lambda)\end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1&1&\lambda&\lambda\\ 0&\lambda-1&1-\lambda&0\\ 0&1-\lambda&1-\lambda^2&1-\lambda^2\end{array}\right]\sim\begin{pmatrix}2r\,|(\lambda-1)\end{pmatrix}$$

1° $\lambda-1=0$

$\lambda=1$

$$[A\,|\,\vec{b}\,]=\left[\begin{array}{ccc|c} 1&1&1&1\\ 0&0&0&0\\ 0&0&0&0\end{array}\right]$$

$$x+y+z=1$$
$$\Updownarrow$$
$$x=1-y-z$$
$$\Updownarrow$$
$$x=1-\alpha-\beta$$

x – "vezana" nepoznanica

y, z – "slobodne" nepoznanice

$y=\alpha,\ z=\beta,\ \alpha,\beta\in\mathbb{R}$

$$\vec{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-\alpha-\beta+1\\ \alpha\\ \beta\end{bmatrix}=\alpha\cdot\begin{bmatrix}-1\\1\\0\end{bmatrix}+\beta\cdot\begin{bmatrix}-1\\0\\1\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \alpha,\beta\in\mathbb{R}$$ //

2° $\lambda-1\neq 0$

$\lambda\neq 1$

$$[A\,\vdots\,\vec{b}]=\left[\begin{array}{ccc|c}1&1&\lambda&\lambda\\0&\lambda-1&1-\lambda&0\\0&1-\lambda&1-\lambda^2&1-\lambda^2\end{array}\right]\sim\begin{pmatrix}2r\,/\,(\lambda-1)\end{pmatrix}\sim\left[\begin{array}{ccc|c}1&1&\lambda&\lambda\\0&1&-1&0\\0&1-\lambda&1-\lambda^2&1-\lambda^2\end{array}\right]\sim$$

$$\begin{pmatrix}3r\,/\,(1-\lambda)\end{pmatrix}\sim\left[\begin{array}{ccc|c}1&1&\lambda&\lambda\\0&1&-1&0\\0&1&1+\lambda&1+\lambda\end{array}\right]\sim\begin{pmatrix}1r-2r\times(1)\\3r-2r\times(1)\end{pmatrix}\sim$$

$$\left[\begin{array}{ccc|c}1&0&\lambda+1&\lambda\\0&1&-1&0\\0&0&2+\lambda&1+\lambda\end{array}\right]\sim\begin{pmatrix}3r\,/\,(\lambda+2)\end{pmatrix}$$

2a° $\lambda+2=0$

$\lambda=-2$

$$[A\,\vdots\,\vec{b}]=\left[\begin{array}{ccc|c}1&0&-1&-2\\0&1&-1&0\\0&0&0&-1\end{array}\right]$$

$$\begin{aligned}x-z&=-2\\y-z&=0\\0&=-1\end{aligned}$$

↳ protuslovan sustav

↳ nema rješenja //

2b° $\lambda+2\neq 0$

$\lambda\neq -2$

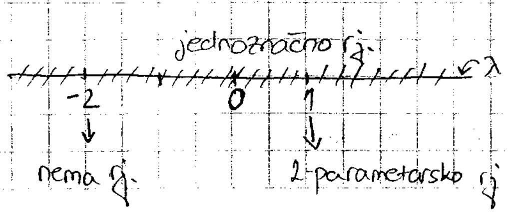
$$[A\,\vdots\,\vec{b}]=\left[\begin{array}{ccc|c}1&0&\lambda+1&\lambda\\0&1&-1&0\\0&0&2+\lambda&1+\lambda\end{array}\right]\sim\begin{pmatrix}3r\,/\,(2+\lambda)\end{pmatrix}\sim\left[\begin{array}{ccc|c}1&0&\lambda+1&\lambda\\0&1&-1&0\\0&0&1&\frac{1+\lambda}{2+\lambda}\end{array}\right]\sim$$

$$\begin{pmatrix}1r-3r\times(\lambda+1)\\2r-3r\times(-1)\end{pmatrix}\sim\left[\begin{array}{ccc|c}1&0&0&-\frac{1}{\lambda+2}\\0&1&0&\frac{\lambda+1}{\lambda+2}\\0&0&1&\frac{\lambda+1}{\lambda+2}\end{array}\right]$$

$$x=-\frac{1}{\lambda+2}$$
$$y=\frac{\lambda+1}{\lambda+2}$$
$$z=\frac{\lambda+1}{\lambda+2}$$

$$\vec{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-\frac{1}{\lambda+2}\\ \frac{\lambda+1}{\lambda+2}\\ \frac{\lambda+1}{\lambda+2}\end{bmatrix}$$

↳ Shematski:



Samo je jedan Mali Ivica!

5.2. CRAMEROVO PRAVILO

= STAVAK 1.

↳ Svaka komponenta x_i rješenja $\vec{x}$ sustava $A\vec{x}=\vec{b}$ s regularnom matricom A može se napisati u obliku razlomka u kojem je u nazivniku determinanta od matrice A, a u brojniku determinanta matrice u kojoj je i-ti stupac zamijenjen s vektorom b, tj.:

$$x_i = \frac{D_i}{D}$$

↳ Korisno samo za matrice reda 2 i 3.

= PRIMJER 1.

Riješi sustav Cramerovim pravilom:

$$\begin{aligned} x+2y+3z&=5\\ 2x-y-z&=1\\ x+3y+4z&=6 \end{aligned}$$

$$↳ D = \left|\begin{matrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ 1 & 3 & 4 \end{matrix}\right|\begin{matrix} 1 & 2 \\ 2 & -1 \\ 1 & 3 \end{matrix} = -4-2+18+3+3-16=2$$

$$↳ D_x = \left|\begin{matrix} 5 & 2 & 3 \\ 1 & -1 & -1 \\ 6 & 3 & 4 \end{matrix}\right|\begin{matrix} 5 & 2 \\ 1 & -1 \\ 6 & 3 \end{matrix} = -20-12+9+18+15-8=2$$

$$D_y = \left|\begin{matrix} 1 & 5 & 3 \\ 2 & 1 & -1 \\ 1 & 6 & 4 \end{matrix}\right|\begin{matrix} 1 & 5 \\ 2 & 1 \\ 1 & 6 \end{matrix} = 4-5+36-3+6-40=-2$$

$$D_z = \left|\begin{matrix} 1 & 2 & 5 \\ 2 & -1 & 1 \\ 1 & 3 & 6 \end{matrix}\right|\begin{matrix} 1 & 2 \\ 2 & -1 \\ 1 & 3 \end{matrix} = -6+2+30+5-3-24=4$$

↳

$$x = \frac{D_x}{D} = \frac{2}{2} = 1$$

$$y = \frac{D_y}{D} = \frac{-2}{2} = -1$$

$$z = \frac{D_z}{D} = \frac{4}{2} = 2$$

5.3. KARAKTERISTIČNI POLINOM I SVOJSTVENA VRIJEDNOST

= DEFINICIJA 1.

↳ Vektor $\vec{v} \neq 0$ zovemo svojstvenim (vlastitim) vektorom kvadratne matrice A ako postoji skalar λ takav da vrijedi:

$$A\cdot\vec{v} = \lambda\cdot\vec{v}$$

Skalar λ zovemo svojstvena (vlastita) vrijednost matrice A, koja odgovara svojstvenom vektoru $\vec{v}$.

= STAVAK 1.

↳ Ako su $\vec{x}, \vec{y}$ dva nekolinearna vektora koja odgovaraju istoj svojstvenoj vrijednosti λ, onda je i vektor $\alpha\cdot\vec{x}+\beta\cdot\vec{y}$ svojstveni vektor za vrijednost λ.

DOKAZ:

$$A(\alpha\cdot\vec{x}+\beta\cdot\vec{y})=\alpha\cdot A\vec{x}+\beta\cdot A\vec{y}=\alpha\lambda\vec{x}+\beta\lambda\vec{y}=\lambda(\alpha\vec{x}+\beta\vec{y})$$

↳ To znači da je svaki vektor iz podprostora $L(\vec{x},\vec{y})$ razapetog vektorima $\vec{x}$ i $\vec{y}$ svojstven. Taj podprostor nazivamo svojstveni podprostor koji pripada svojstvenoj vrijednosti λ.

=PRIMJER 1.

Za matricu $A=I$ vrijedi: Svaki je vektor svojstveni, a zajednička svojstvena vrijednost je $\lambda=1$, jer očevidno vrijedi: $I\cdot\vec{v}=1\cdot\vec{v}$

=PRIMJER 2.

Nađi svojstvenu vrijednost i svojstvene vektore matrice $A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$.

↳ $A\cdot\vec{v}=\lambda\cdot\vec{v}$

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}=\lambda\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$\begin{bmatrix}x_2\\x_1\end{bmatrix}=\begin{bmatrix}\lambda x_1\\\lambda x_2\end{bmatrix}\iff \begin{aligned}x_2&=\lambda x_1\\x_1&=\lambda x_2\end{aligned}$$

$x_1=\lambda(\lambda x_1)=\lambda^2 x_1$

$x_1(\lambda^2-1)=0$

1° $\lambda^2-1=0$ $\quad$ 2° $x_1=0$

$\lambda_1=1$ $\quad \lambda_2=-1$ $\quad$ $x_2=0\to$ otpada, $\vec{v}\neq\vec{0}$

1° $\lambda_1=1\ldots$ $\begin{aligned}x_2&=x_1\\x_1&=x_2\end{aligned}\iff\vec{v}=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\x_1\end{bmatrix}=x_1\begin{bmatrix}1\\1\end{bmatrix},\quad x_1\in\mathbb{R}$

↳ Za $\lambda_1=1$ dobijemo svojstveni vektor $\vec{v}=\begin{bmatrix}1\\1\end{bmatrix}$. Taj vektor je svojstveni vektor koji odgovara svojstvenoj vrijednosti $\lambda=1$.

2° $\lambda_2=-1\ldots$ $\begin{aligned}x_2&=-x_1\\x_1&=-x_2\end{aligned}\iff\vec{v}=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\-x_1\end{bmatrix}=x_1\begin{bmatrix}1\\-1\end{bmatrix},\quad x_1\in\mathbb{R}$

↳ Za $\lambda_2=-1$ dobijemo svojstveni vektor $\vec{v}=\begin{bmatrix}1\\-1\end{bmatrix}$. Taj vektor je svojstveni vektor koji odgovara svojstvenoj vrijednosti $\lambda=-1$.

□ NALAŽENJE SVOJSTVENIH VEKTORA

= Svodi se na rješavanje homogenog sustava.

$$A\vec{v}=\lambda\vec{v}\iff(A-\lambda I)\vec{v}=\vec{0},\quad\vec{v}\neq 0$$

Slijedi: $\vec{v}$ je svojstveni vektor onda i samo onda ako homogeni sustav $(A-\lambda I)\vec{v}=\vec{0}$ ima netrivijalno rješenje.

Da bi to bilo ispunjeno, matrica $A-\lambda I$ nesmije biti regularna, pa zato njena determinanta mora biti različita od nule, tj.

$$\det(\lambda I-A)=\begin{vmatrix}\lambda-a_{11}&-a_{12}&\cdots&-a_{1n}\\-a_{21}&\lambda-a_{22}&\cdots&-a_{2n}\\\vdots&&&\vdots\\-a_{n1}&-a_{n2}&\cdots&\lambda-a_{nn}\end{vmatrix}=0$$

=DEFINICIJA 2.

↳ Karakteristični polinom matrice A, oznakom $K(\lambda)$ je:

$$K(\lambda):=\det(\lambda I-A)=\lambda^n-\sigma_1\lambda^{n-1}-\ldots-\sigma_{n-1}\lambda-\sigma_n$$

Jednadžba:

$$K(\lambda)=\det(\lambda I-A)=0$$

zove se karakteristična jednadžba matrice A, a njena rješenja su svojstvene vrijednosti matrice A.

= Algoritam za traženje svojstvenih vektora:

- 1. korak – Izračunamo karakteristični polinom:
$$K(\lambda) = \det(\lambda I - A).$$
- 2. korak – Odredimo svojstvene vrijednosti (sva rješenja karakteristične jednadžbe $K(\lambda) = 0$)
- 3. korak – za svaku svojstvenu vrijednost λ odredimo pripadni svojstveni vektor. To su linearno nezavisna i različita od nule rješenja homogenog sustava:
$$(\lambda I - A)\vec{v} = \vec{0}$$

= PRIMJER 3.

Odredi svojstvene vrijednosti i svojstvene vektore matrice $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 4 & 4 & -1 \end{bmatrix}$.

↳ karakteristični polinom:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-4 & -1 & 1 \\ -2 & \lambda-5 & 2 \\ -4 & -4 & \lambda+1 \end{vmatrix} \begin{matrix} \lambda-4 & -1 \\ -2 & \lambda-5 \\ -4 & -4 \end{matrix} = (\lambda-4)(\lambda-5)(\lambda+1) +$$

$$+8+8+4(\lambda-5)+8(\lambda-4)-2(\lambda+1) =$$
$$= (\lambda^2-5\lambda-4\lambda+20)(\lambda+1)+16+4\lambda-20+8\lambda-32-2\lambda-2 =$$
$$= \lambda^3+\lambda^2-9\lambda^2-9\lambda+20\lambda+20+16+4\lambda-20+8\lambda-32-2\lambda+2 =$$
$$= \lambda^3 - 8\lambda^2 + 21\lambda - 18$$

↳ karakteristična jednadžba:

$$K(\lambda) = \lambda^3 - 8\lambda^2 + 21\lambda - 18 = 0$$

- rješenja jednadžbe mogu biti samo cjelobrojni djelitelji broja 18, pa dobivamo da je $\lambda_1 = 2$ i $\lambda_2 = 3$. pa pišemo gornju jednadžbu kao:
$$K(\lambda) = (\lambda-2)(\lambda-3)^2 = 0$$
$$\lambda_1 = 2 \qquad \lambda_2 = 3$$

1° $\lambda_1 = 2$ - svojstvena vrijednost (jednostruka)

Mora biti $(\lambda_1 I - A)\vec{v} = 0$

$$\begin{bmatrix} -2 & -1 & 1 \\ -2 & -3 & 2 \\ -4 & -4 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ -2 & -3 & 2 & 0 \\ -4 & -4 & 3 & 0 \end{array}\right] \sim \begin{pmatrix} 1r \mid (-2) \\ \\ \\ \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ -2 & -3 & 2 & 0 \\ -4 & -4 & 3 & 0 \end{array}\right] \sim \begin{pmatrix} 2r - 1r \times (-2) \\ 3r - 1r \times (-4) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right]$$

$$\begin{pmatrix} 2r \mid (-2) \\ \\ \\ \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & -2 & 1 & 0 \end{array}\right] \sim \begin{pmatrix} 1r - 2r \times \left(\frac{1}{2}\right) \\ 3r - 2r \times (-2) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \quad x_1 = \frac{1}{4}x_3 = \frac{1}{4}t$$
$$x_2 = \frac{1}{2}x_3 = \frac{1}{2}t$$
$$x_3 = t$$

$$\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t \\ \frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}, \text{ za } t = 4 \ldots \quad X_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

2° $\lambda_{2,3}=3$ - svojstvena vrijednost (dvostruka)
Mora biti : $(\lambda_{2,3}I - A)\vec{v} = \vec{0}$

$$\begin{bmatrix} -1 & -1 & 1 \\ -2 & -2 & 2 \\ -4 & -4 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ -2 & -2 & 2 & 0 \\ -4 & -4 & 4 & 0 \end{array}\right] \sim \begin{pmatrix} 1r \,/\, (-1) \\ \\ \\ \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -2 & -2 & 2 & 0 \\ -4 & -4 & 4 & 0 \end{array}\right] \sim \begin{pmatrix} 2r - 1r \times (-2) \\ 3r - 1r \times (-4) \\ \\ \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$\Rightarrow$ $x_1 + x_2 - x_3 = 0$ $\qquad x_1 = s - t$
$x_2 = t$
$x_3 = s$

$$\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s-t \\ t \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow$ postoje 2 svojstvena vektora :

$v_1 = [1\ 0\ 1]^T$ i $v_2 = [-1\ 1\ 0]^T$.

ZADACI (DZ 5. PRVI DIO)

1.
$$\begin{cases} 2x_1 + x_2 + 4x_3 + x_4 = 0 \\ 3x_1 + 2x_2 - x_3 - 6x_4 = 0 \\ 7x_1 + 4x_2 + 6x_3 - 5x_4 = 0 \\ x_1 \qquad + 8x_3 + 7x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 2 & 1 & 4 & 1 & 0 \\ 3 & 2 & -1 & -6 & 0 \\ 7 & 4 & 6 & -5 & 0 \\ 1 & 0 & 8 & 7 & 0 \end{array}\right] \sim \begin{pmatrix} 1 \to 4 \\ \\ \\ \\ \end{pmatrix} \sim \left[\begin{array}{cccc|c} 1 & 0 & 8 & 7 & 0 \\ 3 & 2 & -1 & -6 & 0 \\ 7 & 4 & 6 & -5 & 0 \\ 2 & 1 & 4 & 1 & 0 \end{array}\right] \sim \begin{pmatrix} 2r - 1r \times (3) \\ 3r - 1r \times (7) \\ 4r - 1r \times (2) \\ \\ \end{pmatrix} \sim \left[\begin{array}{cccc|c} 1 & 0 & 8 & 7 & 0 \\ 0 & 2 & -25 & -27 & 0 \\ 0 & 4 & -50 & -54 & 0 \\ 0 & 1 & -12 & -13 & 0 \end{array}\right] \sim$$

$$\begin{pmatrix} 2 \to 4 \\ \\ \\ \\ \end{pmatrix} \sim \left[\begin{array}{cccc|c} 1 & 0 & 8 & 7 & 0 \\ 0 & 1 & -12 & -13 & 0 \\ 0 & 4 & -50 & -54 & 0 \\ 0 & 2 & -25 & -27 & 0 \end{array}\right] \sim \begin{pmatrix} 3r - 2r \times (4) \\ 4r - 2r \times (2) \\ \\ \\ \end{pmatrix} \sim \left[\begin{array}{cccc|c} 1 & 0 & 8 & 7 & 0 \\ 0 & 1 & -12 & -13 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right] \sim \begin{pmatrix} 3r \,/\, (-2) \\ \\ \\ \\ \end{pmatrix} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 8 & 7 & 0 \\ 0 & 1 & -12 & -13 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right] \sim \begin{pmatrix} 1r - 3r \times (8) \\ 2r - 3r \times (-12) \\ 4r - 3r \times (-1) \\ \\ \end{pmatrix} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$\Rightarrow$ $x_1 - x_4 = 0 \qquad x_1 = x_4 = \alpha \qquad x_4 = \alpha,\ \alpha \in \mathbb{R}$
$x_2 - x_4 = 0 \qquad x_2 = x_4 = \alpha$
$x_3 + x_4 = 0 \qquad x_3 = -x_4 = -\alpha$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \\ -\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

(2) Gramerom:
$$\begin{cases} x + y + 3z = -7 \\ -x - y + 3z = 1 \\ x - y + z = 1 \end{cases}$$

$$D = \det\begin{bmatrix} 1 & 1 & 3 \\ -1 & -1 & 3 \\ 1 & -1 & 1 \end{bmatrix} = \begin{vmatrix} 1 & 1 & 3 \\ -1 & -1 & 3 \\ 1 & -1 & 1 \end{vmatrix}\begin{matrix} 1 & 1 \\ -1 & -1 \\ 1 & -1 \end{matrix} = -1+3+3-(-3-3-1) = 12$$

$$D_x = \begin{vmatrix} -7 & 1 & 3 \\ 1 & -1 & 3 \\ 1 & -1 & 1 \end{vmatrix}\begin{matrix} -7 & 1 \\ 1 & -1 \\ 1 & -1 \end{matrix} = 7+3\cancel{-3}-(-3+21+1) = -12$$

$$D_y = \begin{vmatrix} 1 & -7 & 3 \\ -1 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix}\begin{matrix} 1 & -7 \\ -1 & 1 \\ 1 & 1 \end{matrix} = 1+21-3-(3+3+7) = -36$$

$$D_z = \begin{vmatrix} 1 & 1 & -7 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \end{vmatrix}\begin{matrix} 1 & 1 \\ -1 & -1 \\ 1 & -1 \end{matrix} = -1+1-7-(7-1-1) = -12$$

$$x = \frac{D_x}{D} = -1 \qquad y = \frac{D_y}{D} = -3 \qquad z = \frac{D_z}{D} = -1$$

(3.)
$$\begin{cases} 2x + y + 2z = 3 \\ x + 2y - 2z = 3 \\ x + 2z = 1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 1 & 2 & -2 & 3 \\ 1 & 0 & 2 & 1 \end{array}\right] \sim \left(1 \to 3\right) \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 1 & 2 & -2 & 3 \\ 2 & 1 & 2 & 3 \end{array}\right] \sim \begin{pmatrix} 2r-1r\times(1) \\ 3r-1r\times(2) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 2 & -4 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right] \sim \left(2 \to 3\right) \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 2 & -4 & 2 \end{array}\right] \sim \left(3r-2r\times(2)\right) \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_1 + 2x_3 = 1$ $\qquad x_1 = 1-2\alpha$

$x_2 - 2x_3 = 1$ $\qquad x_2 = 1+2\alpha$

$x_3 = \alpha$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1-2\alpha \\ 1+2\alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \ \alpha \in \mathbb{R}$$

(4.)
$$\begin{cases} -2x_1 + x_2 + 3x_4 = -5 \\ 3x_1 + 2x_3 - 2x_4 = 1 \\ 3x_1 + 2x_3 + 2x_4 = -1 \\ x_1 + x_2 + 2x_3 + x_4 = -4 \end{cases}$$

$$\left[\begin{array}{cccc|c} -2 & 1 & 0 & 3 & -5 \\ 3 & 0 & 2 & -2 & 1 \\ 3 & 0 & 2 & 2 & -1 \\ 1 & 1 & 2 & 1 & -4 \end{array}\right] \sim \left(1 \to 4\right) \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & -4 \\ 3 & 0 & 2 & -2 & 1 \\ 3 & 0 & 2 & 2 & -1 \\ -2 & 1 & 0 & 3 & -5 \end{array}\right] \sim \left(\begin{array}{l} 2r - 1r \times (3) \\ 3r - 1r \times (3) \\ 4r - 1r \times (-2) \end{array}\right) \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & -4 \\ 0 & -3 & -4 & -5 & 13 \\ 0 & -3 & -4 & -1 & 11 \\ 0 & 3 & 4 & 5 & -13 \end{array}\right] \sim$$

$$\left(2r / (-3)\right) \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & -4 \\ 0 & 1 & \frac{4}{3} & \frac{5}{3} & -\frac{13}{3} \\ 0 & -3 & -4 & -1 & 11 \\ 0 & 3 & 4 & 5 & -13 \end{array}\right] \sim \left(\begin{array}{l} 1r - 2r \times (1) \\ 3r - 2r \times (-3) \\ 4r - 2r \times (3) \end{array}\right) \sim \left[\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 1 & \frac{4}{3} & \frac{5}{3} & -\frac{13}{3} \\ 0 & 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left(3r / (4)\right) \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 1 & \frac{4}{3} & \frac{5}{3} & -\frac{13}{3} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left(\begin{array}{l} 1r - 3r \times (-\frac{2}{3}) \\ 2r - 3r \times (\frac{5}{3}) \end{array}\right) \sim \left[\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 1 & \frac{4}{3} & 0 & -\frac{7}{2} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{l} x_1 + \frac{2}{3} x_3 = 0 \\ x_2 + \frac{4}{3} x_3 = -\frac{7}{2} \\ x_4 = -\frac{1}{2} \\ x_3 = \alpha \end{array} \qquad \begin{array}{l} x_1 = -\frac{2}{3}\alpha \\ x_2 = -\frac{4}{3}\alpha - \frac{7}{2} \end{array}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}\alpha \\ -\frac{4}{3}\alpha - \frac{7}{2} \\ \alpha \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{7}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} + \alpha \begin{bmatrix} -\frac{2}{3} \\ -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix}, \quad \alpha \in \mathbb{R}$$

5.

$$\begin{cases} 2x_1 + x_2 + x_3 + 3x_4 + x_5 = 1 \\ x_1 + x_2 - x_3 + x_4 - x_5 = 0 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 2 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 0 \end{array}\right] \sim (1 \to 2) \sim \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & -1 & 0 \\ 2 & 1 & 1 & 3 & 1 & 1 \end{array}\right] \sim \left(2r - 1r \times (2)\right) \sim \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & -1 & 0 \\ 0 & -1 & 3 & 1 & 3 & 1 \end{array}\right] \sim$$

$$\left(2r / (-1)\right) \sim \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & -3 & -1 & -3 & -1 \end{array}\right] \sim \left(1r - 2r \times (1)\right) \sim \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 2 & 2 & 1 \\ 0 & 1 & -3 & -1 & -3 & -1 \end{array}\right]$$

$$\Rightarrow \begin{array}{l} x_1 + 2x_3 + 2x_4 + 2x_5 = 1 \\ x_2 - 3x_3 - x_4 - 3x_5 = -1 \end{array} \qquad \begin{array}{l} x_3 = \alpha \\ x_4 = \beta \\ x_5 = \gamma \end{array} \qquad \begin{array}{l} x_1 = -2\alpha - 2\beta - 2\gamma + 1 \\ x_2 = 3\alpha + \beta + 3\gamma - 1 \end{array}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2\alpha - 2\beta - 2\gamma + 1 \\ 3\alpha + \beta + 3\gamma - 1 \\ \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R}$$

6.

$$\begin{cases} x_1 + 3x_2 + 5x_3 - x_4 = 1 \\ x_1 + 3x_2 + 2x_3 - 2x_4 + x_5 = -1 \\ x_1 - 2x_2 + x_3 - x_4 - x_5 = 3 \\ x_1 - 4x_2 + x_3 + x_4 - x_5 = 3 \\ x_1 + 2x_2 + x_3 - x_4 + x_5 = -1 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & 3 & 5 & -1 & 0 & 1 \\ 1 & 3 & 2 & -2 & 1 & -1 \\ 1 & -2 & 1 & -1 & -1 & 3 \\ 1 & -4 & 1 & 1 & -1 & 3 \\ 1 & 2 & 1 & -1 & 1 & -1 \end{array}\right] \sim \left(\begin{array}{l} 2r - 1r \times (1) \\ 3r - 1r \times (1) \\ 4r - 1r \times (1) \\ 5r - 1r \times (1) \end{array}\right) \sim \left[\begin{array}{ccccc|c} 1 & 3 & 5 & -1 & 0 & 1 \\ 0 & 0 & -3 & -1 & 1 & -2 \\ 0 & -5 & -4 & 0 & -1 & 2 \\ 0 & -7 & -4 & 2 & -1 & 2 \\ 0 & -1 & -4 & 0 & 1 & -2 \end{array}\right] \sim (2 \to 5) \sim \left[\begin{array}{ccccc|c} 1 & 3 & 5 & -1 & 0 & 1 \\ 0 & -1 & -4 & 0 & 1 & -2 \\ 0 & -5 & -4 & 0 & -1 & 2 \\ 0 & -7 & -4 & 2 & -1 & 2 \\ 0 & 0 & -3 & -1 & 1 & -2 \end{array}\right] \sim$$

$$\begin{pmatrix} 2r/-1 \end{pmatrix} \sim \left[\begin{array}{ccccc|c} 1 & 3 & 5 & -1 & 0 & 1 \\ 0 & 1 & 4 & 0 & -1 & 2 \\ 0 & -5 & -4 & 0 & -1 & 2 \\ 0 & -7 & -4 & 2 & -1 & 2 \\ 0 & 0 & -3 & -1 & 1 & -2 \end{array}\right] \sim \begin{pmatrix} 1r - 2r \times (3) \\ 3r - 2r \times (-5) \\ 4r - 2r \times (-7) \end{pmatrix} \sim \left[\begin{array}{ccccc|c} 1 & 0 & -7 & -1 & 3 & -5 \\ 0 & 1 & 4 & 0 & -1 & 2 \\ 0 & 0 & 16 & 0 & -6 & 12 \\ 0 & 0 & 24 & 2 & -8 & 16 \\ 0 & 0 & -3 & -1 & 1 & -2 \end{array}\right] \sim \begin{pmatrix} 3r/16 \end{pmatrix} \sim$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -7 & -1 & 3 & -5 \\ 0 & 1 & 4 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & -\frac{3}{8} & \frac{3}{4} \\ 0 & 0 & 24 & 2 & -8 & 16 \\ 0 & 0 & -3 & -1 & 1 & -2 \end{array}\right] \sim \begin{pmatrix} 1r - 3r \times (-7) \\ 2r - 3r \times (4) \\ 4r - 3r \times (24) \\ 5r - 3r \times (-3) \end{pmatrix} \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & \frac{3}{8} & \frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 & 0 & -\frac{3}{8} & \frac{3}{4} \\ 0 & 0 & 0 & 2 & 1 & -2 \\ 0 & 0 & 0 & -1 & -\frac{1}{8} & \frac{1}{4} \end{array}\right] \sim \begin{pmatrix} 4r/2 \end{pmatrix} \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & \frac{3}{8} & \frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 & 0 & -\frac{3}{8} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -1 \\ 0 & 0 & 0 & -1 & -\frac{1}{8} & \frac{1}{4} \end{array}\right] \sim$$

$$\begin{pmatrix} 1r - 4r \times (-1) \\ 5r - 4r \times (-1) \end{pmatrix} \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & \frac{7}{8} & -\frac{3}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 & 0 & -\frac{3}{8} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 0 & \frac{3}{8} & -\frac{3}{4} \end{array}\right] \sim \begin{pmatrix} 5r \,|\, \frac{3}{8} \end{pmatrix} \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & \frac{7}{8} & -\frac{3}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 & 0 & -\frac{3}{8} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right] \sim \begin{pmatrix} 1r - 5r \times (\frac{7}{8}) \\ 2r - 5r \times (\frac{1}{2}) \\ 3r - 5r \times (-\frac{3}{8}) \\ 4r - 5r \times (\frac{1}{2}) \end{pmatrix} \sim$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}$$

7. A $(0,3)$ B $(1,4)$ C $(2,9)$ $\quad y = ax^2 + bx + c$

$$\begin{aligned} &\ldots\ 3 = c \\ &\ldots\ 4 = a + b + c \\ &\ldots\ 9 = 4a + 2b + c \end{aligned}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 3 \\ 1 & 1 & 1 & 4 \\ 4 & 2 & 1 & 9 \end{array}\right] \sim \begin{pmatrix} 1 \leftrightarrow 2 \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 4 & 2 & 1 & 9 \end{array}\right] \sim \begin{pmatrix} 3r - 1r \times (4) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & -2 & -3 & -7 \end{array}\right] \sim \begin{pmatrix} 2 \leftrightarrow 3 \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -3 & -7 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim$$

$$\begin{pmatrix} 2r/(-2) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \begin{pmatrix} 1r - 2r \times (1) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \begin{pmatrix} 1r - 3r \times (-\frac{1}{2}) \\ 2r - 3r \times (\frac{3}{2}) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \Longrightarrow y = 2x^2 - x + 3$$

8. $P(0) = 1$, $P(1) = 4$, $P(2) = 15$, $P(-1) = 0$ $\quad y = ax^3 + bx^2 + cx + d$

$$\begin{aligned} &\ldots\ 1 = d \\ &\ldots\ 4 = a + b + c + d \\ &\ldots\ 15 = 8a + 4b + 2c + d \\ &\ldots\ 0 = -1a + b - c + d \end{aligned}$$

$$\left[\begin{array}{cccc|c} 0&0&0&1&1\\ 1&1&1&1&4\\ 8&4&2&1&15\\ -1&1&-1&1&0 \end{array}\right] \sim \left(1\to 2\right) \sim \left[\begin{array}{cccc|c} 1&1&1&1&4\\ 0&0&0&1&1\\ 8&4&2&1&15\\ -1&1&-1&1&0 \end{array}\right] \sim \left(\begin{array}{c} 3r-1r\times(8)\\ 4r-1r\times(-1) \end{array}\right) \sim \left[\begin{array}{cccc|c} 1&1&1&1&4\\ 0&0&0&1&1\\ 0&-4&-6&-7&-17\\ 0&2&0&2&4 \end{array}\right] \sim \left(2\to 4\right) \sim$$

$$\left[\begin{array}{cccc|c} 1&1&1&1&4\\ 0&2&0&2&4\\ 0&-4&-6&-7&-17\\ 0&0&0&1&1 \end{array}\right] \sim \left(2r/2\right) \sim \left[\begin{array}{cccc|c} 1&1&1&1&4\\ 0&1&0&1&2\\ 0&-4&-6&-7&-17\\ 0&0&0&1&1 \end{array}\right] \sim \left(\begin{array}{c} 1r-2r\times(1)\\ 3r-2r\times(-4) \end{array}\right) \sim \left[\begin{array}{cccc|c} 1&0&1&0&2\\ 0&1&0&1&2\\ 0&0&-6&-3&-9\\ 0&0&0&1&1 \end{array}\right] \sim \left(3r/(-6)\right) \sim$$

$$\left[\begin{array}{cccc|c} 1&0&1&0&2\\ 0&1&0&1&2\\ 0&0&1&\frac{1}{2}&\frac{3}{2}\\ 0&0&0&1&1 \end{array}\right] \sim \left(1r-3r\times(1)\right) \sim \left[\begin{array}{cccc|c} 1&0&0&-\frac{1}{2}&\frac{1}{2}\\ 0&1&0&1&2\\ 0&0&1&\frac{1}{2}&\frac{3}{2}\\ 0&0&0&1&1 \end{array}\right] \sim \left(\begin{array}{c} 1r-4r\times(-\frac{1}{2})\\ 2r-4r\times(1)\\ 3r-4r\times(\frac{1}{2}) \end{array}\right) \sim \left[\begin{array}{cccc|c} 1&0&0&0&1\\ 0&1&0&0&1\\ 0&0&1&0&1\\ 0&0&0&1&1 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} a\\ b\\ c\\ d \end{bmatrix} = \begin{bmatrix} 1\\ 1\\ 1\\ 1 \end{bmatrix} \Rightarrow y = x^3 + x^2 + x + 1 \,/\!/$$

9. $P(0)=1$, $P(1)=3$, $P(2)=21$, $P(-2)=21$

$y = ax^4 + bx^3 + cx^2 + dx + e \;\ldots\; 1 = e$

$\ldots\; 3 = a + b + c + d + e$

$\ldots\; 21 = 16a + 8b + 4c + 2d + e$

$\ldots\; 21 = 16a - 8b + 4c - 2d + e$

$$\left[\begin{array}{ccccc|c} 0&0&0&0&1&1\\ 1&1&1&1&1&3\\ 16&8&4&2&1&21\\ 16&-8&4&-2&1&21 \end{array}\right] \sim \left(1\to 2\right) \sim \left[\begin{array}{ccccc|c} 1&1&1&1&1&3\\ 0&0&0&0&1&1\\ 16&8&4&2&1&21\\ 16&-8&4&-2&1&21 \end{array}\right] \sim \left(\begin{array}{c} 3r-1r\times(16)\\ 4r-1r\times(16) \end{array}\right) \sim \left[\begin{array}{ccccc|c} 1&1&1&1&1&3\\ 0&0&0&0&1&1\\ 0&-8&-12&-14&-15&-27\\ 0&-24&-12&-18&-15&-27 \end{array}\right] \sim \left(2\to 3\right) \sim$$

$$\left[\begin{array}{ccccc|c} 1&1&1&1&1&3\\ 0&-8&-12&-14&-15&-27\\ 0&0&0&0&1&1\\ 0&-24&-12&-18&-15&-27 \end{array}\right] \sim \left(2r/(-8)\right) \sim \left[\begin{array}{ccccc|c} 1&1&1&1&1&3\\ 0&1&\frac{3}{2}&\frac{7}{4}&\frac{15}{8}&\frac{27}{8}\\ 0&0&0&0&1&1\\ 0&-24&-12&-18&-15&-27 \end{array}\right] \sim \left(\begin{array}{c} 1r-2r\times(1)\\ 4r-2r\times(-24) \end{array}\right) \sim \left[\begin{array}{ccccc|c} 1&0&-\frac{1}{2}&-\frac{3}{4}&-\frac{7}{8}&-\frac{3}{8}\\ 0&1&\frac{3}{2}&\frac{7}{4}&\frac{15}{8}&\frac{27}{8}\\ 0&0&0&0&1&1\\ 0&0&24&24&30&54 \end{array}\right] \sim \left(\begin{array}{c} 3\to 4\\ 4r/24 \end{array}\right) \sim$$

$$\left[\begin{array}{ccccc|c} 1&0&-\frac{1}{2}&-\frac{3}{4}&-\frac{7}{8}&-\frac{3}{8}\\ 0&1&\frac{3}{2}&\frac{7}{4}&\frac{15}{8}&\frac{27}{8}\\ 0&0&1&1&\frac{5}{4}&\frac{9}{4}\\ 0&0&0&0&1&1 \end{array}\right] \sim \left(\begin{array}{c} 1r-3r\times(-\frac{1}{2})\\ 2r-3r\times(\frac{3}{2}) \end{array}\right) \sim \left[\begin{array}{ccccc|c} 1&0&0&-\frac{1}{4}&-\frac{1}{4}&\frac{3}{4}\\ 0&1&0&\frac{1}{4}&0&0\\ 0&0&1&1&\frac{5}{4}&\frac{9}{4}\\ 0&0&0&0&1&1 \end{array}\right] \sim \left(\begin{array}{c} 1r-4r\times(-\frac{1}{4})\\ 3r-4r\times(\frac{5}{4}) \end{array}\right) \sim \left[\begin{array}{ccccc|c} 1&0&0&-\frac{1}{4}&0&1\\ 0&1&0&\frac{1}{4}&0&0\\ 0&0&1&1&0&1\\ 0&0&0&0&1&1 \end{array}\right]$$

$x_1 = \frac{1}{4}x_4 + 1$

$x_2 = -\frac{1}{4}x_4$

$x_3 = -x_4 + 1$

$x_4 = \alpha$

$x_5 = 1$

$$\vec{x} = \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\alpha + 1\\ -\frac{1}{4}\alpha\\ -\alpha + 1\\ \alpha\\ 1 \end{bmatrix} = \begin{bmatrix} 1\\ 0\\ 1\\ 0\\ 1 \end{bmatrix} + \alpha \cdot \begin{bmatrix} \frac{1}{4}\\ -\frac{1}{4}\\ -1\\ 1\\ 0 \end{bmatrix}, \alpha \in \mathbb{R} \Rightarrow y = x^4 + x^2 + 1 \,/\!/$$

10. $a \in \mathbb{R} = ?$ $(x, y, z) = (1, -1, 2)$

$$\begin{cases} a^2 y + 2z = 0\\ ax + y + 3z = 3 \end{cases}, \; a = ?$$

I) $\to -a^2 + 4 = 0$ $\quad a + 1 + 6 = 3$ $\quad \Rightarrow a = -2$, dakle postoji //

$+a^2 = -4$ $\quad a = -2$

$a^2 = 4$

$a = \pm 2$

$\downarrow$

$4y + 2z = 0$

$-2x + y + 3z = 3$

$$\left[\begin{array}{ccc|c} 0&4&2&0\\ -2&1&3&3 \end{array}\right] \sim \left(1\to 2\right) \sim \left[\begin{array}{ccc|c} -2&1&3&3\\ 0&4&2&0 \end{array}\right] \sim \left(1r/(-2)\right) \sim \left[\begin{array}{ccc|c} 1&-\frac{1}{2}&-\frac{3}{2}&-\frac{3}{2}\\ 0&4&2&0 \end{array}\right] \sim \left(2r/4\right) \sim \left[\begin{array}{ccc|c} 1&-\frac{1}{2}&-\frac{3}{2}&-\frac{3}{2}\\ 0&1&\frac{1}{2}&0 \end{array}\right] \sim$$

$$\left(1r-2r\times(-\tfrac{1}{2})\right) \sim \left[\begin{array}{ccc|c} 1&0&-\frac{5}{4}&-\frac{3}{2}\\ 0&1&\frac{1}{2}&0 \end{array}\right]$$

$x_1 = \frac{5}{4}x_3 - \frac{3}{2}$

$x_2 = -\frac{1}{2}x_3$

$x_3 = \alpha$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{4}\alpha - \frac{3}{2} \\ -\frac{1}{2}\alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} \frac{5}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}$$

Samo je jedan Mali Ivica!

11.
$$\begin{cases} x_1 - 2x_2 + 2x_3 + x_4 = 3 \\ -x_1 + 3x_2 - x_3 + x_4 = -2 \\ -4x_1 + 9x_2 - 7x_3 - 2x_4 = \lambda \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & 1 & 3 \\ -1 & 3 & -1 & 1 & -2 \\ -4 & 9 & -7 & -2 & \lambda \end{array}\right] \sim \begin{pmatrix} 2r - 1r \times (-1) \\ 3r - 1r \times (-4) \end{pmatrix} \sim \left[\begin{array}{cccc|c} 1 & -2 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 & \lambda+12 \end{array}\right] \sim \begin{pmatrix} 1r - 2r \times (-2) \\ 3r - 2r \times (1) \end{pmatrix} \sim \left[\begin{array}{cccc|c} 1 & 0 & 4 & 5 & 5 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & \lambda+11 \end{array}\right]$$

1° za $\lambda \neq 11$ → sustav nema rješenja

2° za $\lambda = 11$:

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 5 & 5 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{ll} x_1 = -4x_3 - 5x_4 + 5 & = -4\alpha - 5\beta + 5 \\ x_2 = -x_3 - 2x_4 + 1 & = -\alpha - 2\beta + 1 \\ x_3 = \alpha & \\ x_4 = \beta & \end{array}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4\alpha - 5\beta + 5 \\ -\alpha - 2\beta + 1 \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -4 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -5 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}$$

12.
$$\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ 3x_1 - 2x_2 + 4x_3 = 0 \\ -x_1 + 2x_2 + \lambda x_3 = \lambda \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 3 & -2 & 4 & 0 \\ -1 & 2 & \lambda & \lambda \end{array}\right] \sim \begin{pmatrix} 2r - 1r \times (3) \\ 3r - 1r \times (-1) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 1 & \lambda+2 & \lambda+1 \end{array}\right] \sim \begin{pmatrix} 1r - 2r \times (-1) \\ 3r - 2r \times (1) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & \lambda+4 & \lambda+4 \end{array}\right]$$

1° za $\lambda = -4$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = -2 \\ x_2 = 2x_3 - 3 = 2\alpha - 3 \\ x_3 = \alpha \end{array} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2\alpha - 3 \\ \alpha \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}$$

13.
$$\begin{cases} x_1 - x_2 - x_3 = -1 \\ 2x_1 - x_2 - \lambda x_3 = -1 \\ 2\lambda x_1 - \lambda x_2 - 4x_3 = -2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 2 & -1 & -\lambda & -1 \\ 2\lambda & -\lambda & -4 & -2 \end{array}\right] \sim \begin{pmatrix} 2r - 1r \times (2) \\ 3r - 1r \times (2\lambda) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 0 & 1 & -\lambda+2 & 1 \\ 0 & \lambda & -4+2\lambda & -2+2\lambda \end{array}\right] \sim \begin{pmatrix} 1r - 2r \times (-1) \\ 3r - 2r \times (\lambda) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda & 1 \\ 0 & 0 & \lambda^2-4 & \lambda-2 \end{array}\right]$$

1° a) $\lambda = -2$: → sustav nema rješenja

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

b) za $\lambda = 2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_1 = x_3 = \alpha \\ x_2 = 1 \\ x_3 = \alpha \end{array} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 1 \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \alpha \in \mathbb{R}$$

2° a) za $\lambda \neq -2$, $\lambda \neq 2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda & 1 \\ 0 & 0 & \lambda^2-4 & \lambda-2 \end{array}\right] \sim \left(3r / (\lambda^2-4)\right) \sim \left[\begin{array}{ccc|c} 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda & 1 \\ 0 & 0 & 1 & \frac{1}{\lambda+2} \end{array}\right] \sim \begin{pmatrix} 1r - 3r \times (1-\lambda) \\ 2r - 3r \times (2-\lambda) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{\lambda-1}{\lambda+2} \\ 0 & 1 & 0 & \frac{2\lambda}{\lambda+2} \\ 0 & 0 & 1 & \frac{1}{\lambda+2} \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{\lambda-1}{\lambda+2} \\ \frac{2\lambda}{\lambda+2} \\ \frac{1}{\lambda+2} \end{bmatrix}$$

14.

$$\begin{cases} (2\lambda-1)\, x_1 + \lambda x_2 + x_3 = 2-\lambda \\ \lambda x_1 + \lambda x_2 + x_3 = 1 \\ x_1 + x_2 + \lambda x_3 = 1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2\lambda-1 & \lambda & 1 & 2-\lambda \\ \lambda & \lambda & 1 & 1 \\ 1 & 1 & \lambda & 1 \end{array}\right] \sim \left(1 \to 3\right) \sim \left[\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ \lambda & \lambda & 1 & 1 \\ 2\lambda-1 & \lambda & 1 & 2-\lambda \end{array}\right] \sim \begin{pmatrix} 2r - 1r \times \lambda \\ 3r - 1r \times (2\lambda-1) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & 0 & 1-\lambda^2 & 1-\lambda \\ 0 & 1-\lambda & -2\lambda^2+\lambda+1 & 3-3\lambda \end{array}\right] \sim$$

$$\left(2 \to 3\right) \left[\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & 1-\lambda & -2\lambda^2+\lambda+1 & 3-3\lambda \\ 0 & 0 & 1-\lambda^2 & 1-\lambda \end{array}\right]$$

1° za $\lambda = 1$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\alpha-\beta+1 \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \alpha, \beta \in \mathbb{R}$$

2° za $\lambda \neq 1$:

$$\left[\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & 1-\lambda & -2\lambda^2+\lambda+1 & 3-3\lambda \\ 0 & 0 & 1-\lambda^2 & 1-\lambda \end{array}\right] \sim \left(2r / (1-\lambda)\right) \sim \left[\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & 1 & 2\lambda+1 & 3 \\ 0 & 0 & 1-\lambda^2 & 1-\lambda \end{array}\right] \sim \left(1r - 2r \times (1)\right) \sim \left[\begin{array}{ccc|c} 1 & 0 & -\lambda-1 & -2 \\ 0 & 1 & 2\lambda+1 & 3 \\ 0 & 0 & 1-\lambda^2 & 1-\lambda \end{array}\right] \sim$$

$$\left(3r / (1-\lambda)\right) \sim \left[\begin{array}{ccc|c} 1 & 0 & -\lambda-1 & -2 \\ 0 & 1 & 2\lambda+1 & 3 \\ 0 & 0 & 1+\lambda & 1 \end{array}\right]$$

1°° za $\lambda = -1$: $\to$ sustav nema rješenja

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

2°° za $\lambda \neq -1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\lambda-1 & -2 \\ 0 & 1 & 2\lambda+1 & 3 \\ 0 & 0 & 1+\lambda & 1 \end{array}\right] \sim \left(3r / (1+\lambda)\right) \sim \left[\begin{array}{ccc|c} 1 & 0 & -\lambda-1 & -2 \\ 0 & 1 & 2\lambda+1 & 3 \\ 0 & 0 & 1 & \frac{1}{1+\lambda} \end{array}\right] \sim \begin{pmatrix} 1r - 3r \times (-\lambda-1) \\ 2r - 3r \times (2\lambda+1) \end{pmatrix} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{\lambda+2}{\lambda+1} \\ 0 & 0 & 1 & \frac{1}{\lambda+1} \end{array}\right] \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{\lambda+2}{\lambda+1} \\ \frac{1}{\lambda+1} \end{bmatrix}$$

15.
$$\begin{cases} \lambda x_1 + x_2 + x_3 = 1 \\ x_1 + \lambda x_2 + x_3 = \lambda \\ x_1 + x_2 + \lambda x_3 = \lambda^2 \end{cases}$$

$$\left[\begin{array}{ccc|c} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & \lambda \\ 1 & 1 & \lambda & \lambda^2 \end{array}\right] \sim \begin{pmatrix} 1 \to 2 \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & \lambda & 1 & \lambda \\ \lambda & 1 & 1 & 1 \\ 1 & 1 & \lambda & \lambda^2 \end{array}\right] \sim \begin{pmatrix} 2r - 1r \times (\lambda) \\ 3r - 1r \times (1) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & \lambda & 1 & \lambda \\ 0 & 1-\lambda^2 & 0 & 1-\lambda^2 \\ 0 & 1-\lambda & \lambda-1 & \lambda^2-\lambda \end{array}\right]$$

1° za $\lambda = 1$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\alpha-\beta+1 \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \alpha, \beta \in \mathbb{R}$$

2° za $\lambda = -1$:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 2 \end{array}\right] \sim \begin{pmatrix} 2 \to 3 \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{pmatrix} 2r/2 \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{pmatrix} 1r - 2r \times (-1) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1+\alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ \alpha \in \mathbb{R}$$

3° za $\lambda \neq 1, \lambda \neq -1$:

$$\left[\begin{array}{ccc|c} 1 & \lambda & 1 & \lambda \\ 0 & 1-\lambda^2 & 0 & 1-\lambda^2 \\ 0 & 1-\lambda & \lambda+1 & \lambda^2-\lambda \end{array}\right] \sim \begin{pmatrix} 2r / (1-\lambda^2) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & \lambda & 1 & \lambda \\ 0 & 1 & 0 & 1 \\ 0 & 1-\lambda^2 & \lambda-1 & \lambda^2-\lambda \end{array}\right] \sim \begin{pmatrix} 1r - 2r \times \lambda \\ 3r - 2r \times (1-\lambda^2) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \lambda-1 & -\lambda-1 \end{array}\right]$$

$$\begin{pmatrix} 3r / (\lambda-1) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{-\lambda-1}{\lambda-1} \end{array}\right] \sim \begin{pmatrix} 1r - 3r \times (1) \end{pmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{\lambda+1}{\lambda-1} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{-\lambda-1}{\lambda-1} \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{\lambda+1}{\lambda-1} \\ 1 \\ \frac{-\lambda-1}{\lambda-1} \end{bmatrix}$$

16.
$$\begin{cases} \lambda x_1 + x_2 + x_3 + x_4 = \lambda \\ x_1 + \lambda x_2 + x_3 + x_4 = \lambda \\ x_1 + x_2 + \lambda x_3 + x_4 = 1 \\ x_1 + x_2 + x_3 + \lambda x_4 = 1 \end{cases}$$

$$\left[\begin{array}{cccc|c} \lambda & 1 & 1 & 1 & \lambda \\ 1 & \lambda & 1 & 1 & \lambda \\ 1 & 1 & \lambda & 1 & 1 \\ 1 & 1 & 1 & \lambda & 1 \end{array}\right] \sim \begin{pmatrix} 1 \to 4 \end{pmatrix} \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & \lambda & 1 \\ 1 & \lambda & 1 & 1 & \lambda \\ 1 & 1 & \lambda & 1 & 1 \\ \lambda & 1 & 1 & 1 & \lambda \end{array}\right] \sim \begin{pmatrix} 2r - 1r \times (1) \\ 3r - 1r \times (1) \\ 4r - 1r \times (\lambda) \end{pmatrix} \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & \lambda & 1 \\ 0 & \lambda-1 & 0 & 1-\lambda & \lambda+1 \\ 0 & 0 & \lambda+1 & 1-\lambda & 0 \\ 0 & 1-\lambda & 1-\lambda & 1-\lambda^2 & 0 \end{array}\right]$$

1° za $\lambda = 1$:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\alpha-\beta-\gamma+1 \\ \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ \alpha, \beta, \gamma \in \mathbb{R}$$

2° za $\lambda \neq 1$:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & \lambda & 1 \\ 0 & \lambda-1 & 0 & 1-\lambda & \lambda-1 \\ 0 & 0 & \lambda-1 & 1-\lambda & 0 \\ 0 & 1-\lambda & 1-\lambda & 1-\lambda^2 & 0 \end{array}\right] \sim \begin{pmatrix} 2r \,|\, (\lambda-1) \\ 4r \,|\, (1-\lambda) \end{pmatrix} \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & \lambda & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & \lambda-1 & 1-\lambda & 0 \\ 0 & 1 & 1 & 1+\lambda & 0 \end{array}\right] \sim \begin{pmatrix} 1r - 2r \times (1) \\ 4r - 2r \times (1) \end{pmatrix} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & \lambda+1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & \lambda+1 & 1-\lambda & 0 \\ 0 & 0 & 1 & 2+\lambda & -1 \end{array}\right] \sim \begin{pmatrix} 3r \,|\, (\lambda-1) \end{pmatrix} \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & \lambda+1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2+\lambda & -1 \end{array}\right] \sim \begin{pmatrix} 1r - 3r \times (1) \\ 4r - 3r \times (1) \end{pmatrix} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & \lambda+2 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & \lambda+3 & -1 \end{array}\right]$$

1°° za $\lambda = -3$: $\rightarrow$ sustav nema rješenja

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

2°° za $\lambda \neq -3$:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & \lambda+2 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & \lambda+3 & -1 \end{array}\right] \sim \begin{pmatrix} 4r \,|\, (\lambda+3) \end{pmatrix} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & \lambda+2 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{\lambda+3} \end{array}\right] \sim \begin{pmatrix} 1r - 4r \times (\lambda+2) \\ 2r - 4r \times (-1) \\ 3r - 4r \times (-1) \end{pmatrix} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{\lambda+2}{\lambda+3} \\ 0 & 1 & 0 & 0 & \frac{\lambda+2}{\lambda+3} \\ 0 & 0 & 1 & 0 & -\frac{1}{\lambda+3} \\ 0 & 0 & 0 & 1 & -\frac{1}{\lambda+3} \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{\lambda+2}{\lambda+3} \\ \frac{\lambda+2}{\lambda+3} \\ -\frac{1}{\lambda+3} \\ -\frac{1}{\lambda+3} \end{bmatrix}, \ \lambda \in \mathbb{R}$$

(17.) Svojstvena vrijednost / vektor = ? , $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

I) polinom:

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\right) = \det\begin{bmatrix} \lambda-1 & -2 \\ -2 & \lambda-4 \end{bmatrix} = (\lambda-1)(\lambda-4)-4 =$$

$$= \lambda^2 - 4\lambda - \lambda + 4 - 4 = \lambda^2 - 5\lambda = \lambda(\lambda-5) \rightarrow \lambda_1 = 0$$

$$\rightarrow \lambda_2 = 5 //$$

II) 1° $\lambda_1 = 0$ ... $\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} -1 & -2 & 0 \\ -2 & -4 & 0 \end{array}\right] \sim \begin{pmatrix} 1r \,|\, (-1) \end{pmatrix} \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ -2 & -4 & 0 \end{array}\right] \sim \begin{pmatrix} 2r - 1r \times (-2) \end{pmatrix} \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2\alpha \\ \alpha \end{bmatrix} = \alpha \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \ \alpha \in \mathbb{R} \ //$$

2° $\lambda_2 = 5$ ... $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ -2 & 1 & 0 \end{array}\right] \sim \begin{pmatrix} 1r \,|\, 4 \end{pmatrix} \sim \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ -2 & 1 & 0 \end{array}\right] \sim \begin{pmatrix} 2r - 1r \times (-2) \end{pmatrix} \sim \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2\,\alpha \\ \alpha \end{bmatrix} = \alpha \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}$$

Samo je jedan Mali Ivica!

(18.) Svojstvena vrijednost/vektor? ... $A = \begin{bmatrix} -3 & 0 \\ 1 & 3 \end{bmatrix}$

I) polinom:

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 1 & 3 \end{bmatrix}\right) = \det \begin{bmatrix} \lambda+3 & 0 \\ -1 & \lambda-3 \end{bmatrix} = (\lambda+3)(\lambda-3) \quad \begin{matrix} \rightarrow \lambda_1 = -3 \\ \rightarrow \lambda_2 = 3 \end{matrix}$$

II) 1° $\lambda_1 = -3$ ... $\begin{bmatrix} 0 & 0 \\ -1 & -6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & -6 & 0 \end{array}\right] \sim (1 \leftrightarrow 2) \sim \left[\begin{array}{cc|c} -1 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim (1r / (-1)) \sim \left[\begin{array}{cc|c} 1 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -6\alpha \\ \alpha \end{bmatrix} = \alpha \cdot \begin{bmatrix} -6 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}$$

2° $\lambda_2 = 3$ .... $\begin{bmatrix} 6 & 0 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 6 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right] \sim (1r / 6) \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right] \sim (2r - 1r \times (-1)) \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}$$

(19.) Svojstvena vrijednost/vektor? ... $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

I) polinom:

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}\right) = \det \left[\begin{array}{ccc|cc} \lambda-1 & 0 & 0 & \lambda-1 & 0 \\ 0 & \lambda & 0 & 0 & \lambda \\ 0 & -2 & \lambda & 0 & -2 \end{array}\right] = (\lambda-1)\lambda^2 \quad \begin{matrix} \rightarrow \lambda_1 = 1 \\ \rightarrow \lambda_2 = 0 \end{matrix}$$

II) 1° $\lambda_1 = 1$ ... $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right] \sim (3r - 2r \times (1)) \sim \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}$$

2° $\lambda_2 = 0$ .... $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c}-1&0&0&0\\0&0&0&0\\0&-2&0&0\end{array}\right]\sim\begin{pmatrix}1r\,|\,(-1)\\ \\ \end{pmatrix}\sim\left[\begin{array}{ccc|c}1&0&0&0\\0&0&0&0\\0&-2&0&0\end{array}\right]\sim\begin{pmatrix}2r\to 1r\\2r\,|\,(-2)\end{pmatrix}\sim\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&0&0\end{array}\right]$$

Samo je jedan Mali Ivica!

$$\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\\alpha\end{bmatrix}=\alpha\begin{bmatrix}0\\0\\1\end{bmatrix},\ \alpha\in\mathbb{R}$$

(20.) Svojstvena vrijednost / vektor? .... $A=\begin{bmatrix}1&-1&1\\1&0&-1\\2&-1&0\end{bmatrix}$

I) polinom:

$$\det(\lambda I-A)=\det\left(\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}-\begin{bmatrix}1&-1&1\\1&0&-1\\2&-1&0\end{bmatrix}\right)=\det\begin{bmatrix}\lambda-1&-1&-1\\-1&\lambda&1\\-2&1&\lambda\end{bmatrix}\begin{matrix}\lambda-1&-1\\-1&\lambda\\-2&1\end{matrix}=$$

$$=(\lambda-1)\lambda^2+2+1-2\lambda-\lambda+1-\lambda=\lambda^3-\lambda^2-4\lambda+4\ \to\lambda_1=1$$

$$\to\lambda_2=2$$

$$\to\lambda_3=-2$$

II) $1^\circ\ \lambda_1=1\ \dots\ \begin{bmatrix}1&-1&1\\1&0&-1\\2&-1&0\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\left[\begin{array}{ccc|c}1&-1&1&0\\1&0&-1&0\\2&-1&0&0\end{array}\right]\sim\begin{pmatrix}2r-1r\times(1)\\3r-1r\times(2)\end{pmatrix}\sim\left[\begin{array}{ccc|c}1&-1&1&0\\0&1&-2&0\\0&1&-2&0\end{array}\right]\sim\begin{pmatrix}1r-2r\times(-1)\\3r-2r\times(1)\end{pmatrix}\sim\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&-2&0\\0&0&0&0\end{array}\right]$$

$$\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}\alpha\\2\alpha\\\alpha\end{bmatrix}=\alpha\begin{bmatrix}1\\2\\1\end{bmatrix},\ \alpha\in\mathbb{R}$$

$2^\circ\ \lambda_2=2\ \dots\ \begin{bmatrix}1&-1&-1\\-1&2&1\\-2&1&2\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\left[\begin{array}{ccc|c}1&-1&-1&0\\-1&2&1&0\\-2&1&2&0\end{array}\right]\sim\begin{pmatrix}2r\times1r\times(-1)\\3r-1r\times(-2)\end{pmatrix}\sim\left[\begin{array}{ccc|c}1&-1&-1&0\\0&1&0&0\\0&-1&0&0\end{array}\right]\sim\begin{pmatrix}1r-2r\times(-1)\\3r-2r\times(-1)\end{pmatrix}\sim\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&0&0\\0&0&0&0\end{array}\right]$$

$$\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}\alpha\\-\alpha\\\alpha\end{bmatrix}=\alpha\begin{bmatrix}1\\-1\\1\end{bmatrix},\ \alpha\in\mathbb{R}$$

$3^\circ\ \lambda_3=-2\ \dots\ \begin{bmatrix}-3&-1&-1\\-1&-2&1\\-2&1&-2\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\left[\begin{array}{ccc|c}-3&-1&-1&0\\-1&-2&1&0\\-2&1&-2&0\end{array}\right]\sim\begin{pmatrix}2\,(\times(-1))\to1\\ \\ \end{pmatrix}\sim\left[\begin{array}{ccc|c}1&2&-1&0\\-3&-1&-1&0\\-2&1&-2&0\end{array}\right]\sim\begin{pmatrix}2r-1r\times(-3)\\3r-1r\times(-2)\end{pmatrix}\sim\left[\begin{array}{ccc|c}1&2&-1&0\\0&5&-4&0\\0&5&-4&0\end{array}\right]\sim\begin{pmatrix}2r/5\\ \\ \end{pmatrix}\sim$$

$$\left[\begin{array}{ccc|c}1&2&-1&0\\0&1&-\frac{4}{5}&0\\0&5&-4&0\end{array}\right]\sim\begin{pmatrix}1r-2r\times(2)\\3r-2r\times(5)\end{pmatrix}\sim\left[\begin{array}{ccc|c}1&0&\frac{3}{5}&0\\0&1&-\frac{4}{5}&0\\0&0&0&0\end{array}\right]$$

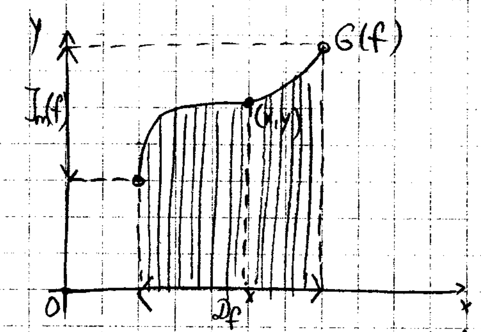
$$\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-\frac{3}{5}\alpha\\\frac{4}{5}\alpha\\\alpha\end{bmatrix}=\alpha\begin{bmatrix}-3/5\\4/5\\1\end{bmatrix},\ \alpha\in\mathbb{R}$$

Samo je jedan Mali Ivica!

6. FUNKCIJE

6.1. SVOJSTVA REALNE FUNKCIJE REALNE VARIJABLE

= Ako su skupovi $S_1, S_2 \subseteq \mathbb{R}$ kažemo da je $f: S_1 \mapsto S_2$ realna funkcija realne varijable.
- ↳ domena (područje definicije funkcije): $\mathcal{D}_f = S_1$.
- ↳ slika funkcije: $\mathcal{J}m(f) = f(\mathcal{D}_{(f)}) = f(S_1)$
- ↳ graf funkcije: $G(f) := \{(x,y) \mid y = f(x)\}$
  - crtanjem grafa $G(f)$ dobivamo krivulju u ravnini $\mathbb{R}^2$, Primjerice:



↳ znamo da za inverznu funkciju f^{-1} (ukoliko postoji, a postoji ako i samo ako je f bijekcija) vrijedi:

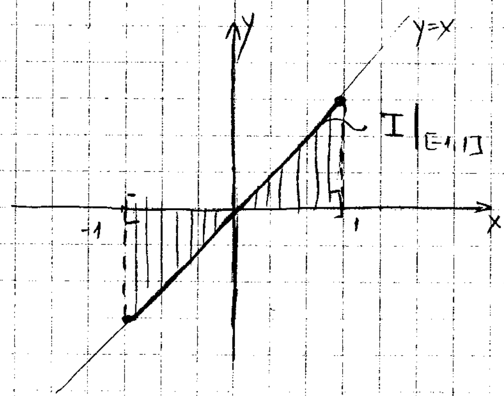
(1.) $\mathcal{D}(f^{-1}) = \mathcal{J}m(f)$
$\mathcal{D}(f) = \mathcal{J}m(f^{-1})$

(2.) $f^{-1}(f(x)) = x \quad , \forall x \in \mathcal{D}(f)$
$f(f^{-1}(x)) = x \quad , \forall x \in \mathcal{D}(f^{-1})$

= PRIMJER 1.

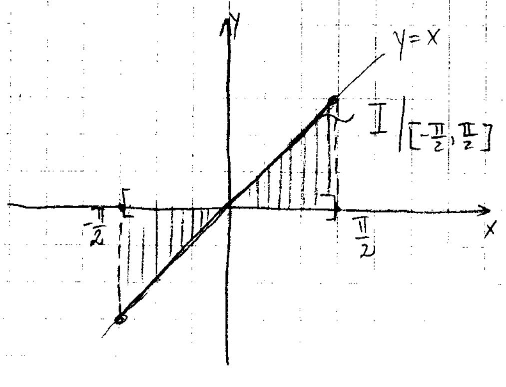
a) ↳ Nacrtaj funkciju i odredi domenu: $\sin(\arcsin x) = x$.

$x \in [-1, 1]$



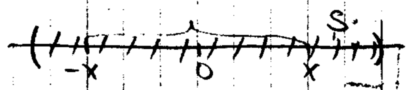
b) ↳ Nacrtaj funkciju i odredi domenu: $\arcsin(\sin x) = x$.

$x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$



= Za funkcije definirane na simetričnom intervalu oko ishodišta imamo katkad neka posebna svojstva.

- Simetrični interval oko nule je $S \subseteq \mathbb{R}$ takav da je : $x \in S \Rightarrow -x \in S$, grafički :



*) ako je $3 \in S$ onda $-3 \in S$
ako je $5 \in S$ onda $-5 \in S$

= Pojam simetrije : - postoje 2 vrste simetrije :

1) osna simetrija

↳ simetrija s obzirom na os simetrije

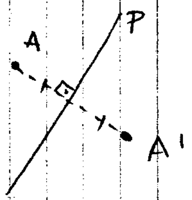
↳ Točke A i A' su simetrične s obzirom na os simetrije p ako vrijedi :

i) A i A' su sa suprotnih strana od p

ii) A i A' su okomiti na p

iii) udaljenost točke A od pravca p i točke A' od pravca p je jednaka (tj. $d(A,p) = d(A',p)$)

↳ Skica :



2) centralna simetrija

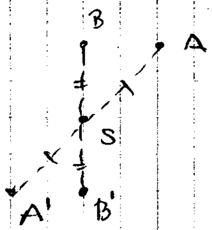
↳ simetrija s obzirom na točku (centar)

↳ Točke A i A' su simetrične s obzirom na točku simetrije ako vrijedi :

i) centar simetrije nalazi se na dužini $\overline{AA'}$ (tj. $S \in \overline{AA'}$)

ii) udaljenost točaka A i A' od centra je jednaka (tj. $d(A,S) = d(A',S)$)

↳ Skica :



□ PARNOST I NEPARNOST

- DEFINICIJA 1.

↳ Funkcija $f(x)$ je parna (+â ka) ako je domena od f simetrična oko nule i ako vrijedi :

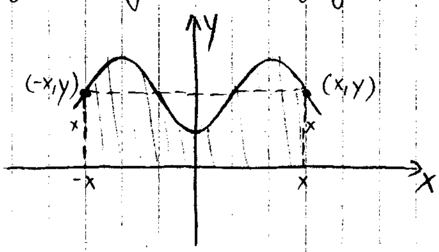
$$f(-x) = f(x), \ \forall x \in \mathcal{D}(f).$$

↳ Funkcija $f(x)$ je neparna (Liha) ako je domena od f simetrična oko nule i ako vrijedi :

$$f(-x) = -f(x), \ \forall x \in \mathcal{D}(f).$$

= Geometrijska interpretacija parnosti i neparnosti :

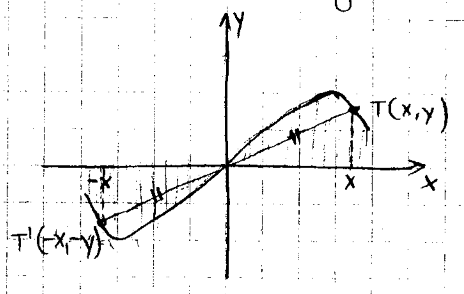
1) Ako je funkcija parna, graf te funkcije je simetričan s obzirom na os y.



$\rightarrow f(x) = y \Rightarrow f(-x) = y$, tj. :

$(x,y) \in G(f) \Rightarrow (-x,y) \in G(f)$

2) Ako je funkcija neparna, graf te funkcije je simetričan s obzirom na ishodište kao centar simetrije.

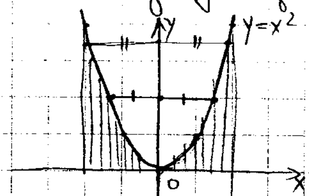


$\Rightarrow f(x)=y \Rightarrow f(-x)=-y$ tj.

$(x,y)\in G(f) \Rightarrow (-x,-y)\in G(f)$

=PRIMJER 2.

↳ Odredi da li je funkcija x^2 parna ili neparna i dokaži općenito za x^{2n}.



$f(x)=x^2$

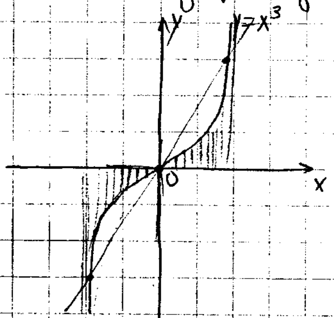
$f(-x)=(-x)^2=x^2=f(x)$ - Parna funkcija

Općenito vrijedi i za $f(x)=x^{2n}$. Dokaz:

$f(-x)=(-x)^{2n}=(-1)^{2n}\cdot x^{2n}=x^{2n}=f(x)$ - Parna funkcija

=PRIMJER 3.

↳ Odredi da li je funkcija x^3 parna ili neparna i dokaži općenito za x^{2n+1}.



$f(x)=x^3$

$f(-x)=(-x)^3=-x^3=-f(x)$ - Neparna funkcija

Općenito vrijedi i za $f(x)=x^{2n+1}$. Dokaz:

$f(-x)=(-x)^{2n+1}=(-1)^{2n+1}\cdot x^{2n+1}=-x^{2n+1}=-f(x)$

Neparna funkcija.

=PRIMJER 4.

↳ Odredi da li je funkcija parna ili neparna.

$y=\frac{x+\cos x}{x^2+1}$ $\quad f(-x)=\frac{-x+\cos(-x)}{(-x)^2+1}=\frac{-x+\cos x}{x^2+1}\neq\frac{x+\cos x}{x^2+1}\neq f(x)$

$\neq-\frac{x+\cos x}{x^2+1}\neq -f(x)$

Ni parna, ni neparna funkcija.

= STAVAK 1.

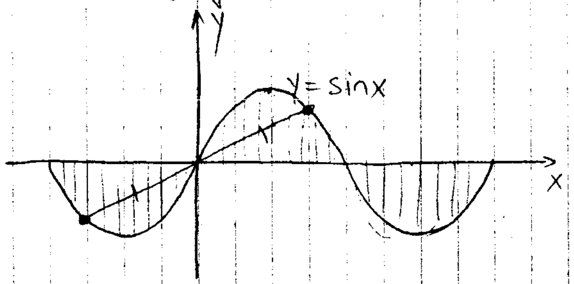
↳ Neka je $y=f(x)$ neparna funkcija. Ako je $0\in \mathcal{D}(f)$ tada je $f(0)=0$, odnosno graf $G(f)$ mora prolaziti ishodištem $(0,0)$.

Dokaz:

$$\left.\begin{array}{l} f(0)=y \\ f(-0)=f(0)=-y \end{array}\right\} \Rightarrow \begin{array}{l} y=-y \\ 2y=0 \\ y=0 \quad \Rightarrow f(0)=0 \quad \text{Q.E.D.}\end{array}$$

= PRIMJER 5.

↳ Odredi da li je funkcija parna ili neparna:



$\sin(-x) = -\sin(x)$

↳ Neparna funkcija

= STAVAK 2.

i) ↳ Neka je $y=f(x)$ parna funkcija, i neka je $y=g(x)$ bilo kakva realna funkcija realne varijable. Tada je kompozicija tih dviju funkcija $(g\circ f)(x)$ parna funkcija.

ii) ↳ Neka je $y=f(x)$ neparna funkcija. Ako je:

1) $y=g(x)$ parna funkcija i kompozicija $(g\circ f)(x)$ je parna funkcija.
2) $y=g(x)$ neparna funkcija, kompozicija $(g\circ f)(x)$ je neparna funkcija.

DOKAZ:

i) $(g\circ f)(x)$

$(g\circ f)(-x) = g[f(-x)] = g[f(x)] = (g\circ f)(x)$ → Parna funkcija.

ii) 1) Iz pretpostavki: $f(-x) = -f(x)$

$g(-x) = g(x)$

slijedi: $(g\circ f)(-x) = g[f(-x)] = g[f(x)] = g[-f(x)]$

$$\Downarrow$$

$$(g\circ f)(-x) = (g\circ f)(x)$$

2) Iz pretpostavki: $f(-x) = -f(x)$

$g(-x) = -g(x)$

slijedi: $(g\circ f)(-x) = g[f(-x)] = g[-f(x)] = -g[f(x)]$

$$\Downarrow$$

$$(g\circ f)(-x) = -(g\circ f)(x)$$

= PRIMJER 6.

↳ Dokažite da je produkt parne i neparne funkcije neparna funkcija.

$y_1 = f(x)$ - Neparna
$y_2 = g(x)$ - Parna

$y_3 = y_1 \cdot y_2 = f(x)\cdot g(x)$

$y_3(-x) = f(-x)\cdot g(-x) = [-f(x)]\cdot g(x) = -f(x)g(x) = -y_3(x)$

Q.E.D.

□ PERIODIČNOST

= DEFINICIJA 2.

↳ Funkcija $y=f(x)$ je periodička ako postoji realan broj P takav da vrijedi:

$$f(x+P) = f(x), \quad \forall x \in \mathbb{R}$$

Najmanji takav broj P, ako postoji, zovemo temeljnim periodom i obično označujemo sa T.

= PRIMJER 7.

a) $\left.\begin{array}{l} y=\sin x \\ y=\cos x \end{array}\right\} T = 2\pi$

b) $\left.\begin{array}{l} y=\operatorname{tg} x \\ y=\operatorname{ctg} x \end{array}\right\} T = \pi$

c) $y = A\cos(nx) + B\sin(nx) \qquad T = \frac{2\pi}{n}$

↳ harmonik u Fourierovom redu

= Odnos kompozicije i periodičnosti funkcije dana je sljedećim teoremom.

= STAVAK 3.

↳ Neka je $y=f(x)$ periodička funkcija s temeljnim periodom T, te neka je $y=g(x)$ injektivna realna funkcija realne varijable. Onda je kompozicija $(g\circ f)(x) = g[f(x)]$ također periodička funkcija s istim temeljnim periodom T.

=PRIMJER 8.

a) ↳ $f(x)=e^{tgx}$

Funkcija je periodička sa temeljnim periodom $T=\pi$.

b) ↳ $f(x)=sh(\sin 3x)$

Funkcija je periodička sa temeljnim periodom $T=\frac{2\pi}{3}$.

=PRIMJEDBA 1.

↳ Uvjet injektivnosti za funkciju $y=g(x)$ u stavku 3 je potreban jer se u protivnom temeljni period može promijeniti.

=PRIMJER 9.

↳ $f(x)=\sin^2 x$

Funkcija $y=x^2$ nije injektivna. Ova funkcija ima temeljni period $T=\pi$, tj. period nutarnje funkcije koji iznosi 2π je smanjen.

☐ MONOTONOST

=DEFINICIJA 3.

↳ Funkcija $y=f(x)$ je monotono rastuća na intervalu $[a,b]$ ako vrijedi:

$$x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2) \quad , \forall x_1, x_2 \in [a,b].$$

↳ Funkcija $y=f(x)$ je monotono padajuća na intervalu $[a,b]$ ako vrijedi:

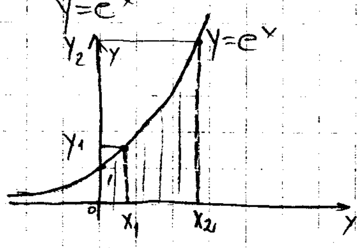
$$x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2) \quad , \forall x_1, x_2 \in [a,b].$$

↳ Funkcija $y=f(x)$ je monotona na intervalu $[a,b]$ ako je ili monotono rastuća ili monotono padajuća na tom intervalu.

↳ Pri tome, ako vrijede stroge nejednakosti kažemo da je funkcija $y=f(x)$ strogo monotona.

=PRIMJER 10.

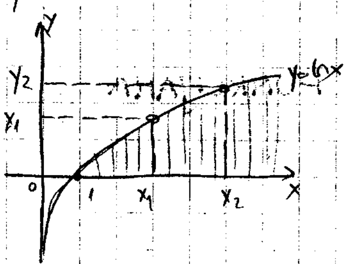
a) $y=e^x$



$x_1 < x_2 \Rightarrow y_1 < y_2$

↳ Funkcija je striktno monotono rastuća na intervalu $\mathbb{R}$.

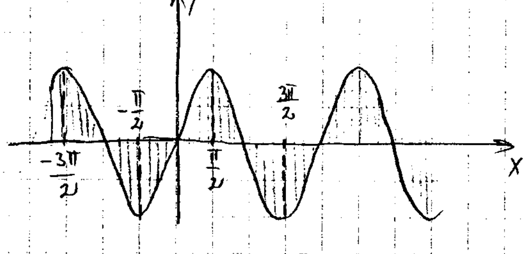
b) $y=\ln x$



$x_1 < x_2 \Rightarrow y_1 < y_2$

↳ Funkcija je striktno monotono rastuća na intervalu $\langle 0, \infty \rangle$

c) $y=\sin x$



↳ Funkcija nije monotona jer je monotona samo po dijelovima tj.:

$[-\frac{\pi}{2}+2k\pi, \frac{\pi}{2}+2k\pi]$ - monotono rastuća

$[\frac{\pi}{2}+2k\pi, \frac{3\pi}{2}+2k\pi]$ - monotono padajuća

= STAVAK 4.

↳ Odnos kompozicije i monotonosti funkcija:

i) Neka je $f(x)$ rastuća funkcija na intervalu $[a,b]$. Ako je $g(x)$ rastuća funkcija na $\mathbb{R}$ tada je i kompozicija $(g\circ f)(x)$ rastuća funkcija na intervalu $[a,b]$.

ii) Neka je $f(x)$ rastuća funkcija na intervalu $[a,b]$. Ako je $g(x)$ padajuća funkcija na $\mathbb{R}$ tada je i kompozicija $(g\circ f)(x)$ padajuća funkcija na intervalu $[a,b]$.

iii) Neka je $f(x)$ padajuća funkcija na intervalu $[a,b]$. Ako je $g(x)$ rastuća funkcija na $\mathbb{R}$ tada je i kompozicija $(g\circ f)(x)$ padajuća funkcija na intervalu $[a,b]$.

iiii) Neka je $f(x)$ padajuća funkcija na intervalu $[a,b]$. Ako je $g(x)$ padajuća funkcija na $\mathbb{R}$ tada je i kompozicija $(g\circ f)(x)$ rastuća funkcija na intervalu $[a,b]$.

· PRIMJER 11.

a) Funkcija $h(x)=e^{2x}$ sastavljena je od 2 funkcije: $f(x)=2x$, $g(x)=e^{x}$.
- Funkcija $f(x)=2x$ je rastuća funkcija, a funkcija $g(x)=e^{x}$ je također rastuća funkcija. Kompozicija tih dviju funkcija je: $(g\circ f)(x)=g[f(x)]=e^{2x}$. Funkcija e^{2x} je također rastuća funkcija.

b) Funkcija $h(x)=e^{-2x}$ sastavljena je od 2 funkcije: $f(x)=2x$, $g(x)=e^{-x}$.
- Funkcija $f(x)=2x$ je rastuća funkcija, a funkcija $g(x)=e^{-x}$ je padajuća funkcija. Kompozicija tih dviju funkcija je: $(g\circ f)(x)=g[f(x)]=e^{-2x}$. Funkcija e^{-2x} je padajuća funkcija.

c) Funkcija $h(x)=e^{-2x}$ sastavljena je od 2 funkcije: $f(x)=-2x$, $g(x)=e^{x}$.
- Funkcija $f(x)=-2x$ je padajuća funkcija, a funkcija $g(x)=e^{x}$ je rastuća funkcija. Kompozicija tih dviju funkcija je: $(g\circ f)(x)=g[f(x)]=e^{-2x}$. Funkcija e^{-2x} je padajuća funkcija.

d) Funkcija $h(x)=e^{2x}$ sastavljena je od 2 funkcije: $f(x)=-2x$, $g(x)=e^{-x}$.
- Funkcija $f(x)=-2x$ je padajuća funkcija, a funkcija $g(x)=e^{-x}$ je padajuća funkcija. Kompozicija tih dviju funkcija je: $(g\circ f)(x)=g[f(x)]=e^{2x}$. Funkcija e^{2x} je rastuća funkcija.

= Pomagalo za monotonost funkcija:

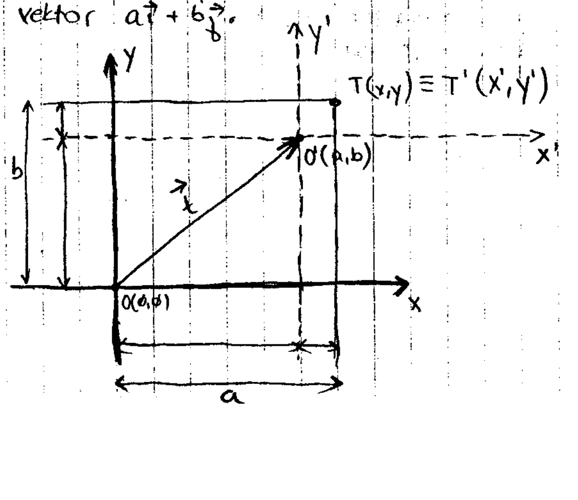
$+ \; + \; = \; +$
$+ \; - \; = \; -$
$- \; + \; = \; -$
$- \; - \; = \; +$

6.2. TRANSFORMACIJE NAD FUNKCIJAMA

- Pod transformacijama nad realnim funkcijama realne varijable podrazumijevamo horizontalnu i vertikalnu translaciju, promjenu amplitude i skaliranje.

□ TRANSLACIJA

- Ako je točka $O(0,0)$ ishodište u sustavu xOy, onda je točka $O'(a,b)$ ishodište u sustavu $x'O'y'$. Da bi došli iz točke O u točku O' moramo izvršiti translaciju za vektor $a\vec{i}+b\vec{j}$.

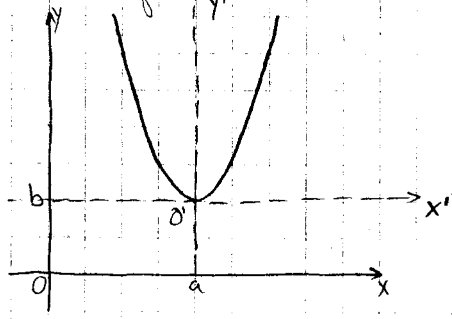


- Sa slike čitamo: $\left.\begin{array}{l} x=x'+a \\ y=y'+b \end{array}\right\}$ $\begin{array}{l} x'=x-a \\ y'=y-a \end{array}$

↳ jednadžbe translacije

= PRIMJER 1.

↳ Zadana je parabola u sustavu XOY, odredi njenu jednadžbu.



$y' = \alpha \cdot x'^2$

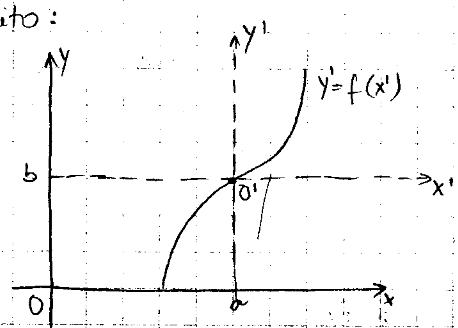
$y - b = \alpha (x-a)^2$

$\Downarrow$

$y = \alpha (x-a)^2 + b$

↳ jednadžba parabole u početnom sustavu XOY.

= Općenito:



= Ako je $y' = f(x')$ jednadžba u sustavu X'O'Y', uz $O'(a,b)$, onda je:

$$y - b = f(x-a).$$

Posebni slučajevi:

1) horizontalni pomak za a

↳ Za funkciju $y=f(x)$ njena horizontalna translacija za broj $a \in \mathbb{R}$ je funkciju $y=g(x)$ definirana na sljedeći način:

$$g(x) = f(x-a) \ ; \ x \in \mathcal{D}(f)$$

↳ Ako je $c>0$ tada graf nastaje translacijom u desno, a ako je $c<0$ translacijom u lijevo, za a.

2) vertikalni pomak za b

↳ Za funkciju $y=f(x)$ njena vertikalna translacija za broj $b \in \mathbb{R}$ je funkcija $y=g(x)$ definirana na sljedeći način:

$$g(x) = f(x) + b \ , x \in \mathcal{D}(f)$$

↳ Ako je $b>0$ tada graf nastaje translacijom prema gore, a ako je $b<0$ translacijom prema dolje, za b.

□ SKALIRANJE

= Rastezanje funkcije $f(x)$ u smjeru osi Ox.

1) skaliranje u smjeru osi y:

↳ Promjenom amplitude funkcije $y=f(x)$ za faktor $A>0$ dobivamo funkciju $y=g(x)$ definiranu:

$$g(x) = A \cdot f(x) \ ; \ x \in \mathcal{D}(f)$$

2) skaliranje u smjeru osi x:

↳ Skaliranjem funkcije $y=f(x)$ za broj $\lambda > 0$ dobivamo novu funkciju $y=g(x)$ koja je definirana na sljedeći način:

$$g(x) = f(\lambda x) \ , \ \forall x \in \mathcal{D}(f)$$

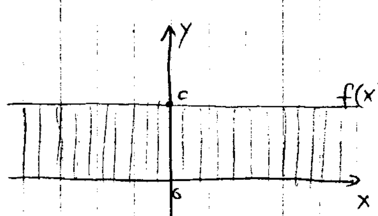
↳ Ako je $\lambda > 1$ - stezanje

$\lambda < 1$ - rastezanje

6.3. POPIS ELEMENTARNIH FUNKCIJA I NJIHOVIH KARAKTERISTIČNIH OBILJEŽJA

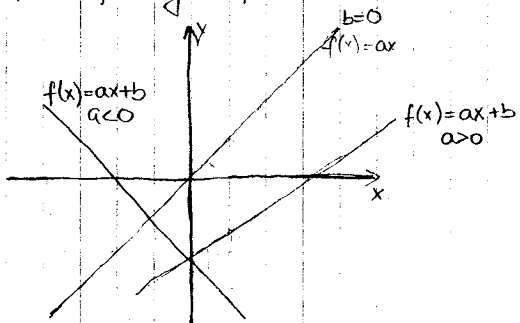
POLINOMI

1) Konstanta $f(x)=c$



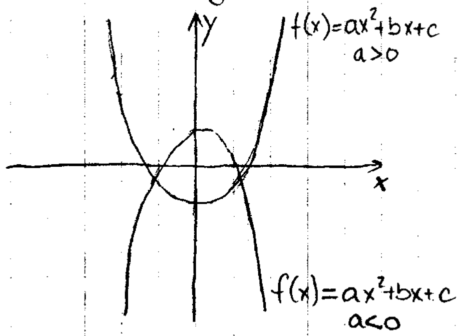
- $\mathcal{D}(f)=\mathbb{R}$
- $\mathcal{I}m(f)=\{c\}$
- nultočka postoji ako je $c=0$, i iznosi $\mathbb{R}$
- parna funkcija

2) Afina funkcija $f(x)=ax+b$



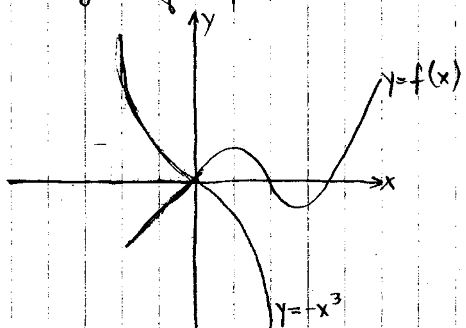
- $\mathcal{D}(f)=\mathbb{R}$
- $\mathcal{I}m(f)=\mathbb{R}$, uz uvjet da je $a\neq 0$
- nultočka: $x=-\frac{b}{a}$, $a\neq 0$

3) Kvadratna funkcija $f(x)=ax^2+bx+c$, $a\neq 0$



- $\mathcal{D}(f)=\mathbb{R}$
- $\mathcal{I}m(f)=\left[\frac{4ac-b^2}{4a},\infty\right\rangle$, $a>0$
- nultočke: $x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, $b^2-4ac\geq 0$
- parna funkcija $f(-x)=(-x)^2=x^2=f(x)$

4) Kubna funkcija $f(x)=ax^3+bx^2+cx+d$, $a\neq 0$



- $\mathcal{D}(f)=\mathbb{R}$
- $\mathcal{I}m(f)=\mathbb{R}$
- nultočka $x=0$

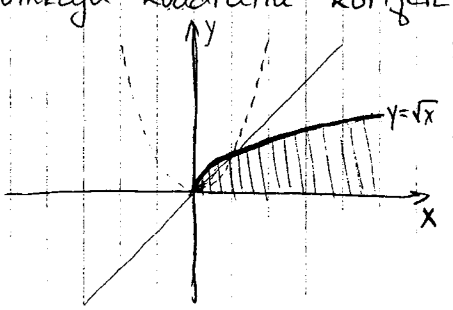
5) Polinom n-tog stupnja $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$, $a_n\neq 0$

- $\mathcal{D}(f)=\mathbb{R}$
- $\mathcal{I}m(f)=\mathbb{R}$

IRACIONALNE FUNKCIJE

inverzne funkcije polinoma

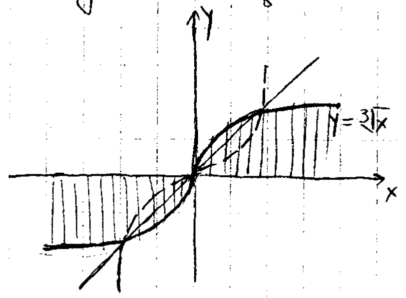
1) Funkcija kvadratni korijen $f(x)=\sqrt{x}$



- simetrična funkcija funkcije $y=x^2$ s obzirom na pravac $y=x$ nakon uvođenja restrikcije $f(x)=x^2\big|_{[0,\infty\rangle}$.
- $\mathcal{D}(f)=[0,\infty\rangle$
- $\mathcal{I}m(f)=[0,\infty\rangle$
- nultočka $x=0$

↳ vrijedi: $\sqrt{x^2} = x \quad , \forall x \in [0, \infty\rangle$
$(\sqrt{x})^2 = x \quad , \forall x \in [0, \infty\rangle$
↳ primjedba: $\sqrt{x^2} = |x|$

2) Funkcija treći korijen $f(x) = \sqrt[3]{x}$

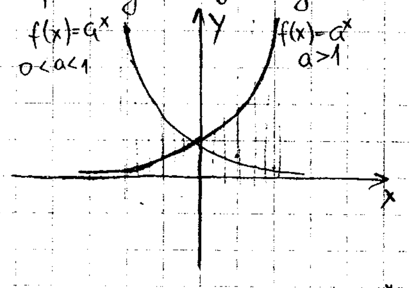


- inverzna funkcija funkcije $y=x^3$ s obzirom na pravac $y=x$ jer funkcija $y=x^3$ je injektivna
- $\mathcal{D}(f) = \mathbb{R}$
- $\mathrm{Im}(f) = \mathbb{R}$
- nultočka $x=0$

↳ vrijedi: $\sqrt[3]{x^3} = x \quad , \forall x \in \mathbb{R}$
$(\sqrt[3]{x})^3 = x \quad , \forall x \in \mathbb{R}$

□ EKSPONENCIJALNA I LOGARITAMSKA FUNKCIJA

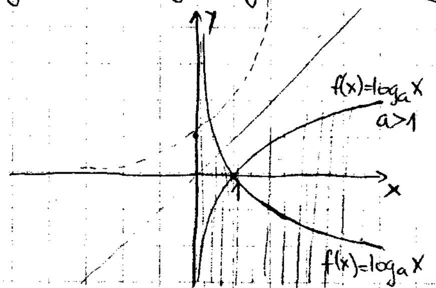
1) Eksponencijalna funkcija $f(x) = a^x \quad , a>0, a\neq 1$



- $\mathcal{D}(f) = \mathbb{R}$
- $\mathrm{Im}(f) = \langle 0, \infty\rangle$
- nultočaka nema

↳ vrijedi: $a^{x_1+x_2} = a^{x_1}\cdot a^{x_2} \quad , \forall x_1, x_2 \in \mathbb{R}$

2) Logaritamska funkcija $f(x) = \log_a x \quad , a>0, a\neq 1$



- inverzna funkcija eksponencijalne funkcije $y=a^x$ s obzirom na pravac $y=x$ jer funkcija $y=a^x$ je injektivna
- $\mathcal{D}(f) = \langle 0, \infty\rangle$
- $\mathrm{Im}(f) = \mathbb{R}$
- nultočka $x=1$

↳ vrijedi: $\log_a(x_1 x_2) = \log_a x_1 + \log_a x_2 \quad , \forall x_1, x_2 \in \langle 0, \infty\rangle$

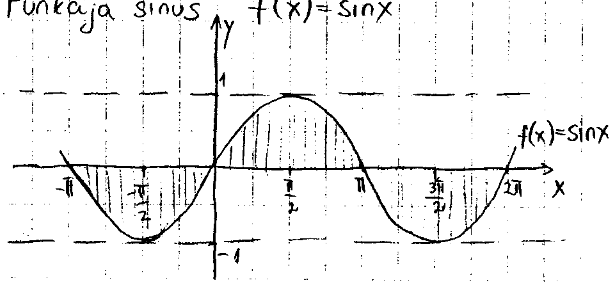
$\log_a\left(\frac{1}{x}\right) = -\log_a x \quad , \forall x \in \langle 0, \infty\rangle$

$\log_a(a^x) = x \quad , \forall x \in \mathbb{R}$

$a^{\log_a x} = x \quad , \forall x \in \langle 0, \infty\rangle$

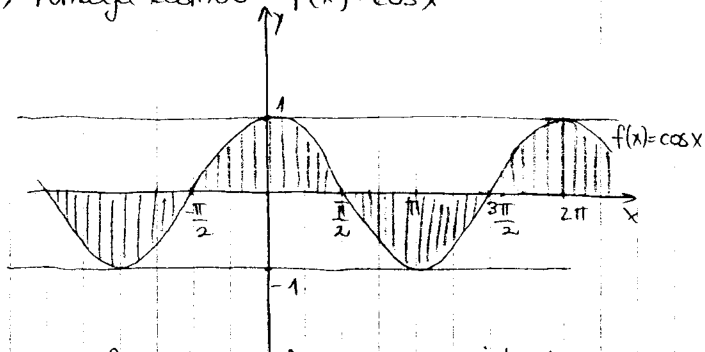
□ TRIGONOMETRIJSKE FUNKCIJE

1) Funkcija sinus $f(x) = \sin x$



- $\mathcal{D}(f) = \mathbb{R}$
- $\mathrm{Im}(f) = [-1, 1]$
- nultočke $x = k\pi, k\in\mathbb{Z}$
- periodička funkcija: $T=2\pi$
- neparna funkcija: $\sin(-x) = -\sin x$

2) Funkcija kosinus $f(x)=\cos x$

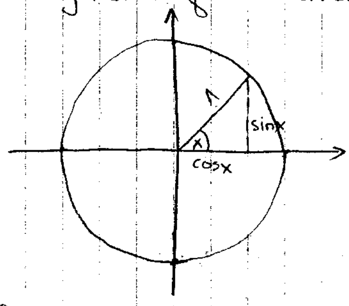


- $D(f)=\mathbb{R}$
- $J_m(f)=[-1,1]$
- nultočke $x=\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$
- periodička funkcija $T=2\pi$
- parna funkcija: $\cos(-x)=\cos x$

- Osnovne formule za funkcije sinus i kosinus:

1) $\sin^2 x+\cos^2 x=1$

Dokaz: Iz trigonometrijske kružnice primjenjujući Pitagorin poučak:



2) adicione formule:

A) $\sin(x\pm y)=\sin x\cos y\pm\cos x\sin y$ (1 i 2)

B) $\cos(x\pm y)=\cos x\cos y\mp\sin x\sin y$ (3 i 4)

3) formule za polovični i dvostruki argument:

A) $\sin^2 x=\frac{1-\cos 2x}{2}$ $\quad\cos^2 x=\frac{1+\cos 2x}{2}$ (7 i 8)

B) $\sin 2x=2\sin x\cos x$ $\quad\cos 2x=\cos^2 x-\sin^2 x$ (5 i 6)

4) formule zbroja i razlike funkcije:

A) $\sin x\pm\sin y=2\sin\frac{x\pm y}{2}\cdot\cos\frac{x\mp y}{2}$ (12 i 13)

B) $\cos x+\cos y=2\cos\frac{x+y}{2}\cdot\cos\frac{x-y}{2}$ (14)

C) $\cos x-\cos y=-2\sin\frac{x+y}{2}\cdot\sin\frac{x-y}{2}$ (15)

5) formule produkta funkcija:

A) $\sin x\sin y=\frac{1}{2}\left(\cos(x-y)-\cos(x+y)\right)$ (9)

B) $\cos x\cos y=\frac{1}{2}\left(\cos(x-y)+\cos(x+y)\right)$ (10)

C) $\sin x\cos y=\frac{1}{2}\left(\sin(x-y)+\sin(x+y)\right)$ (11)

Dokazi:

(1) Koristeći redukciju i adicijske teoreme za kosinus funkciju: (ili vidi dokaz 3)

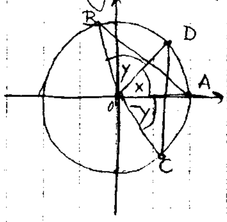
$$\sin(x+y)=\cos\left(\frac{\pi}{2}-(x+y)\right)=\cos\left(\left(\frac{\pi}{2}-x\right)-y\right)=$$
$$=\cos\left(\frac{\pi}{2}-x\right)\cos y+\sin\left(\frac{\pi}{2}-x\right)\sin y=$$
$$=\sin x\cos y+\cos x\sin y$$

(2) Koristeći adicijski teorem za sinus funkciju i parnost-neparnost:

$$\sin(x-y)=\sin(x+(-y))=\sin x\cos(-y)+\cos x\sin(-y)=$$

$= \sin x \cos y - \cos x \sin y$

(3) Pomoću trigonometrijske kružnice:



- Luk $\widehat{AB}$ ima duljinu $x+y$, kao i luk $\widehat{CD}$. Kako jednakim lukovima i pripadne tetive imaju jednaku duljinu, vrijedi:

$$|AB| = |CD|$$

- Koordinate tih točaka su:

$A\,(1, 0)$
$B\,(\cos(x+y), \sin(x+y))$
$C\,(\cos(-y), \sin(-y)) = (\cos y, -\sin y)$
$D\,(\cos x, \sin x)$

- Vrijedi: $|AB|^2 = (\cos(x+y) - 1)^2 + (\sin(x+y))^2 =$

$= \cos^2(x+y) - 2\cos(x+y) + 1 + \sin^2(x+y) =$

$= 2 - 2\cos(x+y)$

$|CD|^2 = [\cos x - \cos y]^2 + [\sin x + \sin y]^2 =$

$= \cos^2 x - 2\cos x \cos y + \cos^2 y + \sin^2 x + 2\sin x \sin y + \sin^2 y =$

$= 2 - 2\cos x \cos y + 2\sin x \sin y$

- Kako je $|AB|^2 = |CD|^2$ vrijedi:

$$\cancel{2} - 2\cos(x+y) = \cancel{2} - 2\cos x \cos y + 2\sin x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

(4) Koristeći adicijske formule za kosinus i parnost-neparnost:

$$\cos(x-y) = \cos(x+(-y)) = \cos x \cos(-y) - \sin x \sin(-y) =$$
$$= \cos x \cos y + \sin x \sin y$$

(5) Koristeći adicijski teorem za sinus:

$$\sin 2x = \sin(x+x) = \sin x \cos x + \cos x \sin x = 2\sin x \cos x$$

(6) Koristeći adicijski teorem za kosinus

$$\cos 2x = \cos(x+x) = \cos x \cdot \cos x - \sin x \sin x = \cos^2 x - \sin^2 x$$

(7) Koristeći formulu za kosinus dvostrukog kuta:

$$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1$$
$$\Downarrow$$
$$\cos^2 x = \frac{\cos 2x + 1}{2}$$

(8) Koristeći formulu za kosinus dvostrukog kuta:

$$\cos 2x = \cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x$$
$$\Downarrow$$
$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

(9) Koristeći adicijske teoreme:

$$\frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y) = \cancel{\frac{1}{2}\cos x \cos y} + \frac{1}{2}\sin x \sin y - \cancel{\frac{1}{2}\cos x \cos y} + \frac{1}{2}\sin x \sin y =$$
$$= \sin x \cdot \sin y$$

(10) Koristeći adicijske teoreme:

$$\frac{1}{2}\cos(x-y)+\frac{1}{2}\cos(x+y)=\frac{1}{2}\cos x\cos y+\cancel{\frac{1}{2}\sin x\sin y}+\frac{1}{2}\cos x\cos y-\cancel{\frac{1}{2}\sin x\sin y}=$$
$$=\cos x\cos y$$

(11) Koristeći adicijske teoreme:

$$\frac{1}{2}\sin(x-y)+\frac{1}{2}\sin(x+y)=\frac{1}{2}\sin x\cos y-\cancel{\frac{1}{2}\cos x\sin y}+\frac{1}{2}\sin x\cos y+\cancel{\frac{1}{2}\sin y\cos x}=$$
$$=\sin x\cos y$$

(12 i 13) Koristeći formule transformacije umnoška u zbroj i supstitucijom: $x+y=\alpha$, $x-y=\beta$, tj. $x=\frac{\alpha+\beta}{2}$ i $y=\frac{\alpha-\beta}{2}$:

I $\sin(x+y)=\sin x\cos y+\cos x\sin y$
SUPS: $\sin\alpha=\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}$

II $\sin(x-y)=\sin x\cos y-\cos x\sin y$
SUPS: $\sin\beta=\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

I+II $\rightarrow \sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

I−II $\rightarrow \sin\alpha-\sin\beta=2\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}$

(14 i 15) Koristeći formule transformacije umnoška u zbroj i supstitucijom gore navedenom:

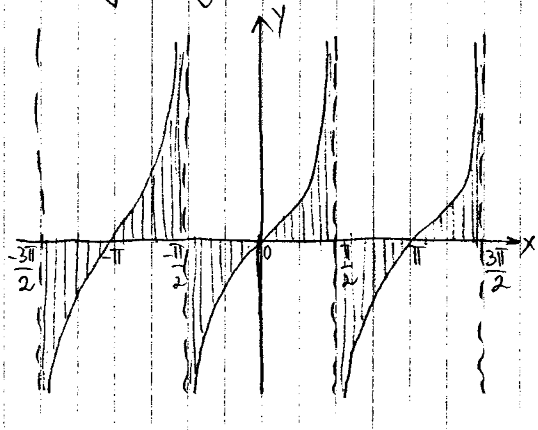
I $\cos(x+y)=\cos x\cos y-\sin x\sin y$
SUPS: $\cos\alpha=\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}-\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

II $\cos(x-y)=\cos x\cos y+\sin x\sin y$
SUPS: $\cos\beta=\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}+\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}$

I+II $\rightarrow \cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}$

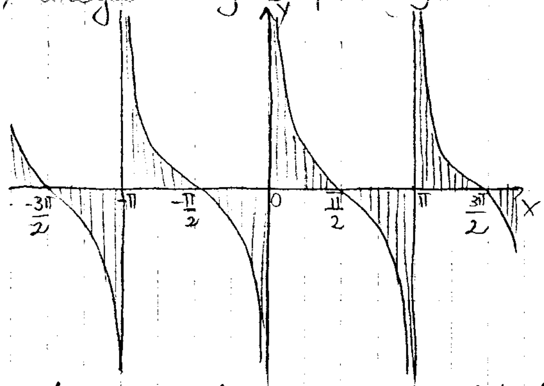
I−II $\rightarrow \cos\alpha-\cos\beta=-2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

3) Funkcija tangens $f(x)=tg\,x$



- definirana formulom: $tg\,x:=\frac{\sin x}{\cos x}$, $x\neq\frac{\pi}{2}+k\pi$, $k\in\mathbb{Z}$
- $\mathcal{D}(f)=\bigcup\left(-\frac{\pi}{2}+k\pi,\frac{\pi}{2}+k\pi\right)$ $k\in\mathbb{Z}$
- $Im(f)=\mathbb{R}$
- nultočke $x=k\pi$, $k\in\mathbb{Z}$
- periodična funkcija $T=\pi$
- neparna funkcija: $tg(-x)=-tg\,x$

3) Funkcija kotangens $f(x)=ctg\,x$



- definirana formulom: $ctg\,x := \frac{1}{tg\,x} := \frac{\cos x}{\sin x}$, $x \neq k\pi,\ k \in \mathbb{Z}$
- $\mathcal{D}(f) = \bigcup (k\pi, \pi + k\pi),\ k \in \mathbb{Z}$
- $Im(f) = \mathbb{R}$
- nultočke $x = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$
- periodička funkcija, $T = \pi$
- neparna funkcija: $ctg(-x) = -ctg\,x$

= Osnovne formule za funkcije tangens i kotangens:

1) $tg\,x \cdot ctg\,x = 1$

Dokaz: $\frac{\cancel{\sin x}}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{\cancel{\sin x}} = 1$

2) adicione formule:

$$tg(x \pm y) = \frac{tg\,x \pm tg\,y}{1 \mp tg\,x\,tg\,y} \qquad (1\ i\ 2)$$

$$ctg(x \pm y) = \frac{ctg\,x\,ctg\,y \mp 1}{ctg\,x \pm ctg\,y} \qquad (3\ i\ 4)$$

3) formule polovičnog i dvostrukog argumenta:

$$tg^2 x = \frac{1-\cos 2x}{1+\cos 2x} \qquad ctg^2 x = \frac{1+\cos 2x}{1-\cos 2x} \qquad (7\ i\ 8)$$

$$tg\,2x = \frac{2\,tg\,x}{1-tg^2 x} \qquad ctg\,2x = \frac{ctg^2 x - 1}{2\,ctg\,x} \qquad (5\ i\ 6)$$

4) formule zbroja i razlike argumenata: (adicione formule)

$$tg(x \pm y) = \frac{tg\,x \pm tg\,y}{1 \mp tg\,x\,tg\,y}$$

$$ctg(x \pm y) = \frac{ctg\,x\,ctg\,y \mp 1}{ctg\,x \pm ctg\,y}$$

5) formule zbroja i razlike funkcije:

$$tg\,x \pm tg\,y = \frac{\sin(x \pm y)}{\cos x \cos y} \qquad (9\ i\ 10)$$

$$ctg\,x \pm ctg\,y = \frac{\pm \sin(x \pm y)}{\sin x \sin y} \qquad (11\ i\ 12)$$

$$tg\,x + ctg\,y = \frac{\cos(x-y)}{\cos x \sin y} \qquad (13)$$

$$ctg\,x - tg\,y = \frac{\cos(x+y)}{\sin x \cos y} \qquad (14)$$

Samo je jedan Mali Ivica!

DOKAZI:

(1) Korištenjem adicijskih teorema za sinus i kosinus:

$$tg(x+y)=\frac{\sin(x+y)}{\cos(x+y)}=\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}\begin{matrix}:\cos x\cos y\\:\cos x\cos y\end{matrix}\Bigg\}\text{jer: } \begin{matrix}\cos x\neq 0\\ \cos y\neq 0\end{matrix}$$

$$=\frac{\frac{\sin x}{\cos x}+\frac{\sin y}{\cos y}}{1-\frac{\sin x}{\cos x}\cdot\frac{\sin y}{\cos y}}=\frac{tg x+tg y}{1-tg x\, tg y}$$

(2) Korištenjem neparnosti funkcije tangens:

$$tg(x-y)=tg(x+(-y))=\frac{tg x+tg(-y)}{1-tg x\, tg(-y)}=\frac{tg x-tg y}{1+tg x\, tg y}$$

(3) Korištenjem adicijskog teorema za tangens:

$$ctg(x+y)=\frac{\cos(x+y)}{\sin(x+y)}=\frac{\cos x\cos y-\sin x\sin y}{\sin x\cos y+\cos x\sin y}\begin{matrix}:\sin x\sin y\\:\sin x\sin y\end{matrix}\Bigg\}\text{jer: } \begin{matrix}\sin x\neq 0\\ \sin y\neq 0\end{matrix}$$

$$=\frac{\frac{\cos x}{\sin x}\cdot\frac{\cos y}{\sin y}-1}{\frac{\cos x}{\sin x}+\frac{\cos y}{\sin y}}=\frac{ctg x\cdot ctg y-1}{ctg x+ctg y}$$

(4) Korištenjem neparnosti funkcije kotangens:

$$ctg(x-y)=ctg(x+(-y))=\frac{ctg x\cdot ctg(-y)-1}{ctg x+ctg(-y)}=\frac{-ctg x\, ctg y-1}{ctg x-ctg y}=$$

$$=\frac{ctg x\, ctg y+1}{ctg y-ctg x}$$

(5 i 6) Korištenjem adicijskih teorema:

$$tg 2x=tg(x+x)=\frac{tg x+tg x}{1-tg x\, tg x}=\frac{2tg x}{1-tg^2x}$$

$$ctg 2x=ctg(x+x)=\frac{ctg x\, ctg x-1}{ctg x+ctg x}=\frac{ctg^2x-1}{2ctg x}$$

(7 i 8) Korištenjem formula polovičnog argumenta za sinus i kosinus:

$$tg^2x=\frac{\sin^2x}{\cos^2x}=\frac{\frac{1-\cos 2x}{2}}{\frac{1+\cos 2x}{2}}=\frac{1-\cos 2x}{1+\cos 2x}$$

$$ctg^2x=\frac{1+\cos 2x}{1-\cos 2x}$$

(9 i 10) Korištenjem adicionih formula za sinus:

$$tg x\pm tg y=\frac{\sin x}{\cos x}\pm\frac{\sin y}{\cos y}=\frac{\sin x\cos y\pm\sin y\cos x}{\cos x\cos y}=\frac{\sin(x\pm y)}{\cos x\cos y}$$

(11 i 12) Korištenjem adicionih formula za sinus:

$$ctg x\pm ctg y=\frac{\cos x}{\sin x}\pm\frac{\cos y}{\sin y}=\frac{\cos x\sin y\pm\cos y\sin x}{\sin x\sin y}=\frac{\pm\sin(x\pm y)}{\sin x\sin y}$$

(13) Korišteujem adicionih formula za kosinus:

$$tgx+ctgy=\frac{\sin x}{\cos x}+\frac{\cos y}{\sin y}=\frac{\sin x\sin y+\cos x\cos y}{\cos x\sin y}=\frac{\cos(x-y)}{\cos x\sin y}$$

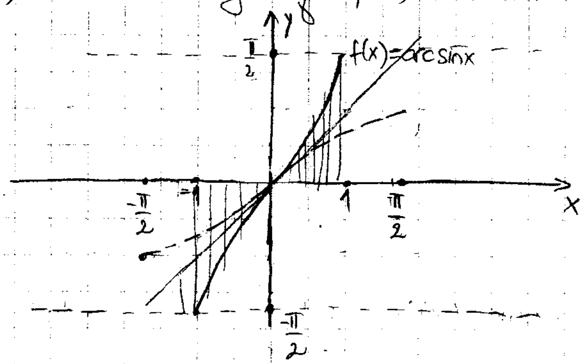
(14) Korišteujem adicionih formula za kosinus:

$$ctgx-tgy=\frac{\cos x}{\sin x}-\frac{\sin y}{\cos y}=\frac{\cos x\cos y-\sin x\sin y}{\sin x\cos y}=\frac{\cos(x+y)}{\sin x\cos y}$$

CIKLOMETRIJSKE FUNKCIJE

↳ inverzne funkcije trigonometrijskih funkcija

1) Arkus-sinus funkcija $f(x)=\arcsin x$

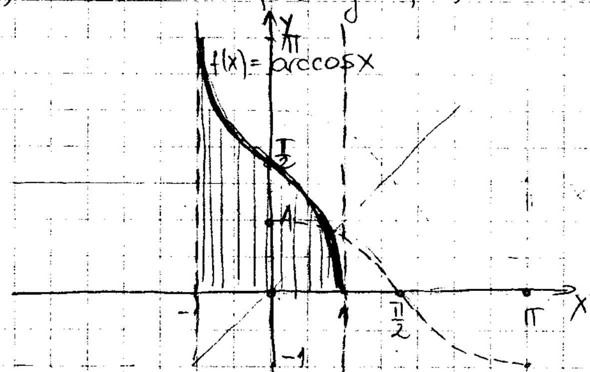


- inverzna funkcija funkciji $y=\sin x$ nakon uvođenja restrikcije $f(x)=\sin x|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$.
- $\mathcal{D}(f)=[-1,1]$
- $\mathcal{I}m(f)=[-\frac{\pi}{2},\frac{\pi}{2}]$
- nultočka $x=0$
- neparna funkcija $\arcsin(-x)=-\arcsin x$

↳ vrijedi: $\arcsin(\sin x)=x \quad , \forall x\in[-\frac{\pi}{2},\frac{\pi}{2}]$

$\sin(\arcsin x)=x \quad , \forall x\in[-1,1]$

2) Arkus-kosinus funkcija $f(x)=\arccos x$



- inverzna funkcija funkciji $y=\cos x$ nakon uvođenja restrikcije $f(x)=\cos x|_{[0,\pi]}$.
- $\mathcal{D}(f)=[-1,1]$
- $\mathcal{I}m(f)=[0,\pi]$
- nultočka $x=1$
- ni parna ni neparna funkcija

↳ vrijedi: $\arccos(\cos x)=x \quad , \forall x\in[0,\pi]$

$\cos(\arccos x)=x \quad , \forall x\in[-1,1]$

= Osnovne formule za funkcije arccos i arcsin:

1) $\arcsin x+\arccos x=\frac{\pi}{2}, \ \forall x\in[-1,1]$

2) $\arccos(-x)=\pi-\arccos x, \ \forall x\in[-1,1]$

DOKAZ:

1) $\left.\begin{array}{l}\arcsin x=y_1 \to x=\sin y_1\\ \arccos x=y_2 \to x=\cos y_2\end{array}\right\}$ $\sin \underline{y_1}=\cos y_2=\sin\left(\underline{\frac{\pi}{2}-y_2}\right)$

$\Downarrow$

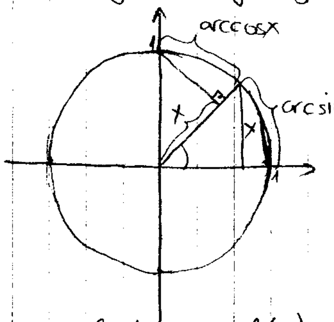
$y_1=\frac{\pi}{2}-y_2 \Rightarrow y_1+y_2=\frac{\pi}{2}$ // Q.E.D.

2) $\left.\begin{array}{l}\arccos(-x)=y_1 \to -x=\cos y_1\\ \arccos x=y_2 \quad\ \ x=\cos y_2\end{array}\right\}$ $-\cos \underline{y_1}=\cos y_2=\cos(\underline{\pi-y_2})$

$\Downarrow$

$y_1=\pi-y_2 \Rightarrow y_1+y_2=\pi$ // Q.E.D.

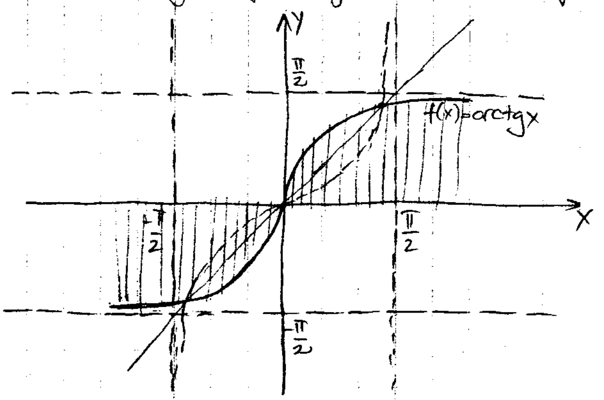
ili na trigonometrijskoj kružnici:



$\Rightarrow \arcsin x + \arccos x = \frac{\pi}{2}$

Samo je jedan Mali Ivica!

3) Arkus-tangens funkcija $f(x) = \operatorname{arctg} x$

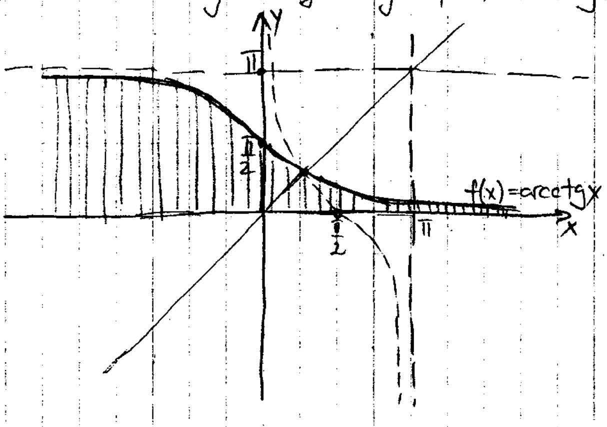


- inverzna funkcija funkcije $y = \operatorname{tg} x$ nakon uvođenja restrikcije $f(x) = \operatorname{tg} x |_{\langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle}$
- $D(f) = \mathbb{R}$
- $Im(f) = \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$
- nultočka $x = 0$
- neparna funkcija: $\operatorname{arctg}(-x) = -\operatorname{arctg} x$

↳ vrijedi: $\operatorname{arctg}(\operatorname{tg} x) = x, \ \forall x \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$

$\operatorname{tg}(\operatorname{arctg} x) = x, \ \forall x \in \mathbb{R}$

4) Arkus-kotangens funkcija $f(x) = \operatorname{arcctg} x$



- inverzna funkcija funkcije $y = \operatorname{ctg} x$ nakon uvođenja restrikcije $f(x) = \operatorname{ctg} x |_{\langle 0, \pi \rangle}$.
- $D(f) = \mathbb{R}$
- $Im(f) = \langle 0, \pi \rangle$
- nultočka nema
- ni parna ni neparna funkcija

↳ vrijedi: $\operatorname{arcctg}(\operatorname{ctg} x) = x, \ \forall x \in \langle 0, \pi \rangle$

$\operatorname{ctg}(\operatorname{arcctg} x) = x, \ \forall x \in \mathbb{R}$

- Osnovne formule za funkcije arctg i arcctg:

1) $\operatorname{arctg} x + \operatorname{arcctg} x = \frac{\pi}{2}, \forall x \in \mathbb{R}$

2) $\operatorname{arctg}(-x) = \pi - \operatorname{arcctg}(x), \ \forall x \in \mathbb{R}$

DOKAZ:

1) $\left. \begin{array}{l} y_1 = \operatorname{arctg} x \rightarrow x = \operatorname{tg} y_1 \\ y_2 = \operatorname{arcctg} x \rightarrow x = \operatorname{ctg} y_2 \end{array} \right\} \operatorname{tg} \underline{y_1} = \operatorname{ctg} y_2 = \operatorname{tg} \underline{(\frac{\pi}{2} - y_2)}$

$\Downarrow$

$y_1 = \frac{\pi}{2} - y_2 \Rightarrow y_1 + y_2 = \frac{\pi}{2}$ Q.E.D.

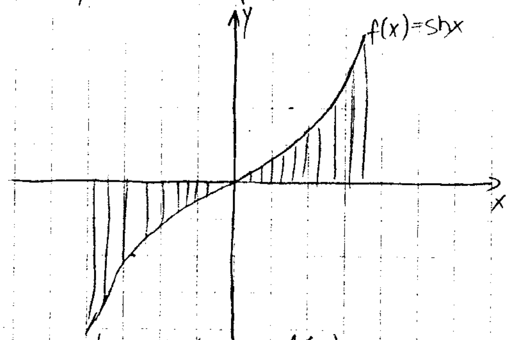
2) $\left. \begin{array}{l} y_2 = \operatorname{arctg}(-x) \rightarrow -x = \operatorname{tg} y_2 \\ y_1 = \operatorname{arcctg} x \rightarrow x = \operatorname{ctg} y_1 \end{array} \right\} \operatorname{ctg} \underline{y_1} = -\operatorname{tg} y_2 = \operatorname{tg} \underline{(\pi - y_2)}$

$\Downarrow$

$y_1 = \pi - y_2 \Rightarrow y_1 + y_2 = \pi$ Q.E.D.

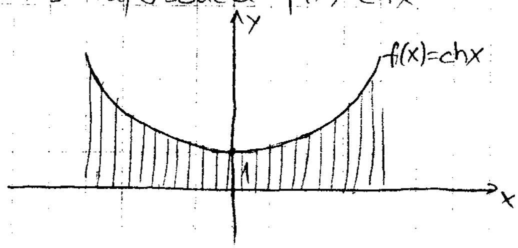
□ HIPERBOLIČKE FUNKCIJE

1) Sinus hiperbolički $f(x)=\operatorname{sh}x$



- definiran formulom: $\operatorname{sh}x := \frac{e^x-e^{-x}}{2}$
- $\mathcal{D}(f)=\mathbb{R}$
- $\mathcal{I}m(f)=\mathbb{R}$
- nultočka $x=0$
- neparna funkcija: $\operatorname{sh}(-x)=-\operatorname{sh}x$

2) Kosinus hiperbolički $f(x)=\operatorname{ch}x$



- definiran formulom: $\operatorname{ch}x := \frac{e^x+e^{-x}}{2}$
- parna funkcija $f(-1)=\operatorname{ch}(-x)=\operatorname{ch}(x)\overset{?}{=}f(x)$
- $\mathcal{D}(f)=\mathbb{R}$
- $\mathcal{I}m(f)=[1,+\infty\rangle$
- nultočka nema
- graf: „lančanica"

= Osnovne formule za funkcije sh i ch:

1) $\operatorname{ch}^2x-\operatorname{sh}^2x=1$ (1)

2) adicione formule:

$$\operatorname{sh}(x\pm y)=\operatorname{sh}x\operatorname{ch}y\pm\operatorname{ch}x\operatorname{sh}y \qquad (2;3)$$
$$\operatorname{ch}(x\pm y)=\operatorname{ch}x\operatorname{ch}y\pm\operatorname{sh}x\operatorname{sh}y \qquad (4;5)$$

3) formule polovičnog i dvostrukog argumenta:

$$\operatorname{sh}^2x=\frac{1}{2}(\operatorname{ch}2x-1) \qquad \operatorname{ch}^2x=\frac{1}{2}(\operatorname{ch}2x+1) \qquad (6;7)$$
$$\operatorname{sh}2x=2\operatorname{sh}x\operatorname{ch}x \qquad \operatorname{ch}2x=\operatorname{sh}^2x+\operatorname{ch}^2x \qquad (8;9)$$

4) formule zbroja i razlike funkcija

$$\operatorname{sh}x\pm\operatorname{sh}y=2\operatorname{sh}\frac{x\pm y}{2}\operatorname{ch}\frac{x\mp y}{2} \qquad (10;11)$$
$$\operatorname{ch}x+\operatorname{ch}y=2\operatorname{ch}\frac{x+y}{2}\operatorname{ch}\frac{x-y}{2} \qquad (12)$$
$$\operatorname{ch}x-\operatorname{ch}y=2\operatorname{sh}\frac{x+y}{2}\operatorname{sh}\frac{x-y}{2} \qquad (13)$$

DOKAZI:

(1) Po definiciji:

$$\operatorname{ch}^2x-\operatorname{sh}^2x=\left(\frac{e^x+e^{-x}}{2}\right)^2-\left(\frac{e^x-e^{-x}}{2}\right)^2=\frac{1}{4}(e^{2x}+2e^x\cdot e^{-x}+e^{-2x}-e^{2x}+2e^xe^{-x}-e^{-2x})=1$$

(2;3) Po definiciji:

$$LS: \operatorname{sh}(x+y)=\frac{e^{x+y}-e^{-x-y}}{2}$$

$$DS: \operatorname{sh}x\operatorname{ch}y+\operatorname{ch}x\operatorname{sh}y=\frac{e^x-e^{-x}}{2}\cdot\frac{e^y+e^{-y}}{2}+\frac{e^x+e^{-x}}{2}\cdot\frac{e^y-e^{-y}}{2}=$$
$$=\frac{e^xe^y+\cancel{e^xe^{-y}}-\cancel{e^{-x}e^y}-e^{-x}e^{-y}+e^xe^y-\cancel{e^xe^{-y}}+\cancel{e^{-x}e^y}-e^{-x}e^{-y}}{4}=$$
$$=\frac{2(e^xe^y-e^{-x}e^{-y})}{4}=\frac{e^{x+y}-e^{-x-y}}{2}$$

$LS=DS$ w

Analogno formula 3.

(4 i 5) Pogledati izvod formule (2).

(6 i 7) Pomoću definicije:

$$sh^2x = shx\cdot shx = \frac{e^x-e^{-x}}{2}\cdot\frac{e^x-e^{-x}}{2} = \frac{e^{2x}+e^{-2x}-e^{x-x}-e^{x-x}}{2} = \frac{e^{2x}+e^{-2x}-1-1}{2} =$$

$$= \frac{ch2x-1}{2}$$

Analogno formula 7.

(8 i 9) Pomoću definicije:

$$2shx\,chx = 2\cdot\frac{e^x-e^{-x}}{\cancel{2}}\cdot\frac{e^x+e^{-x}}{2} = \frac{e^{2x}+\cancel{1}-\cancel{1}-e^{-2x}}{2} = sh2x$$

$$sh^2x+ch^2x = \frac{e^x-e^{-x}}{2}\cdot\frac{e^x-e^{-x}}{2}+\frac{e^x+e^{-x}}{2}\cdot\frac{e^x+e^{-x}}{2} = \frac{e^{2x}\cancel{-2e^{x-x}}+e^{-2x}+e^{2x}\cancel{+2e^{x-x}}+e^{-2x}}{4} =$$

$$= \frac{2(e^{2x}+e^{-2x})}{4} = \frac{e^{2x}+e^{-2x}}{2} = ch2x$$

(10 i 11) Pomoću supstitucije i adicionih formula:

$$x = \frac{\alpha+\beta}{2},\quad y = \frac{\alpha-\beta}{2}$$

I) $sh(x+y) = shx\,chy + chx\,shy$

SUPS: $sh\alpha = sh\frac{\alpha+\beta}{2}\cdot ch\frac{\alpha-\beta}{2} + ch\frac{\alpha+\beta}{2}\cdot sh\frac{\alpha-\beta}{2}$

II) $sh(x-y) = shx\,chy - chx\,shy$

SUPS: $sh\beta = sh\frac{\alpha+\beta}{2}\cdot ch\frac{\alpha-\beta}{2} - ch\frac{\alpha+\beta}{2}\cdot sh\frac{\alpha-\beta}{2}$

I+II) $sh\alpha+sh\beta = 2sh\frac{\alpha+\beta}{2}\cdot ch\frac{\alpha-\beta}{2}$

I−II) $sh\alpha-sh\beta = 2ch\frac{\alpha+\beta}{2}\cdot sh\frac{\alpha-\beta}{2}$

(12 i 13) Pomoću supstitucije i adicionih formula:

$$x = \frac{\alpha+\beta}{2},\quad y = \frac{\alpha-\beta}{2}$$

I $ch(x+y) = chx\,chy + shx\,shy$

SUPS: $ch\alpha = ch\frac{\alpha+\beta}{2}\cdot ch\frac{\alpha-\beta}{2} + sh\frac{\alpha+\beta}{2}\,sh\frac{\alpha-\beta}{2}$

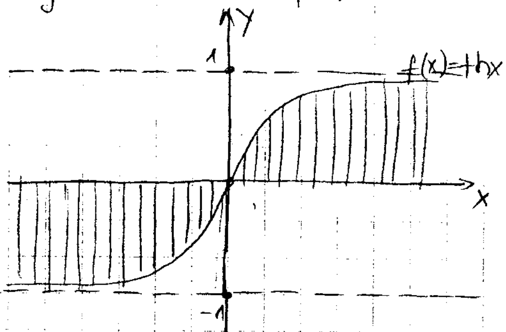
II $ch(x-y) = chx\,chy - shx\,shy$

SUPS: $ch\beta = ch\frac{\alpha+\beta}{2}\,ch\frac{\alpha-\beta}{2} - sh\frac{\alpha+\beta}{2}\,sh\frac{\alpha-\beta}{2}$

I+II) $ch\alpha+ch\beta = 2ch\frac{\alpha+\beta}{2}\,ch\frac{\alpha-\beta}{2}$

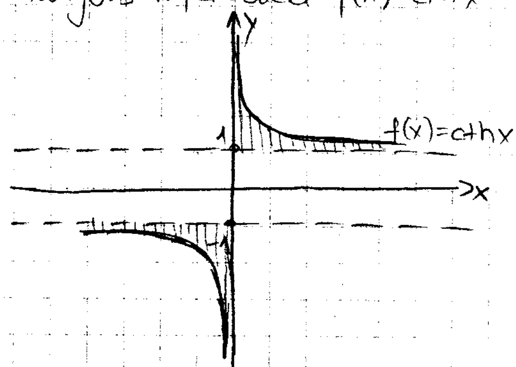
I−II) $ch\alpha-ch\beta = 2sh\frac{\alpha+\beta}{2}\,sh\frac{\alpha-\beta}{2}$

3) Tangens hiperbolički $f(x)=\text{th}x$



- definiran formulom: $\text{th}x := \frac{\text{sh}x}{\text{ch}x} := \frac{e^x-e^{-x}}{e^x+e^{-x}}$
- $D(f)=\mathbb{R}$
- $Im(f)=\langle -1,1\rangle$
- nultočka $x=0$
- neparna funkcija: $\text{th}(-x)=-\text{th}x$

4) Kotangens hiperbolički $f(x)=\text{cth}x$



- definiran formulom: $\text{cth}x := \frac{\text{ch}x}{\text{sh}x} := \frac{e^x+e^{-x}}{e^x-e^{-x}}$
- $D(f)=\mathbb{R}\setminus\{0\}$
- $Im(f)=\langle -\infty,-1\rangle\cup\langle 1,\infty\rangle$
- nultočka nema
- neparna funkcija: $\text{cth}(-x)=-\text{cth}x$

= Osnovne formule za funkcije $\text{th}x$ i $\text{cth}x$:

1) $\text{th}x\cdot\text{cth}x=1$

2) adicione formule:

$$\text{th}(x\pm y)=\frac{\text{th}x\pm\text{th}y}{1\mp\text{th}x\,\text{th}y}$$

$$\text{cth}(x\pm y)=\frac{1\pm\text{cth}x\,\text{cth}y}{\text{cth}x\pm\text{cth}y}$$

3) formule polovičnog i dvostrukog kuta:

$$\text{th}\frac{x}{2}=\frac{\text{sh}x}{1+\text{ch}x} \qquad \text{cth}\frac{x}{2}=\frac{1+\text{ch}x}{\text{sh}x}$$

$$\text{th}2x=\frac{2\text{th}x}{1+\text{th}^2x} \qquad \text{cth}2x=\frac{\text{cth}^2x+1}{2\text{cth}x}$$

4) formule zbroja i razlike:

$$\text{th}x\pm\text{th}y=\frac{\text{sh}(x\pm y)}{\text{ch}x\,\text{ch}y}$$

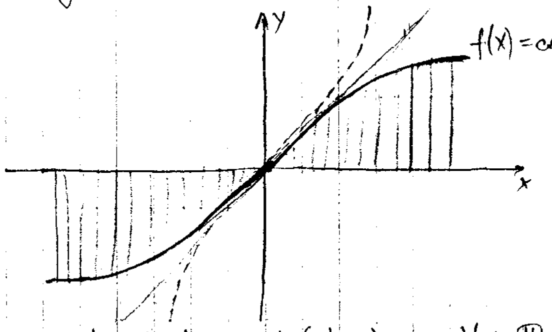
DOKAZI:

Analogni za tangens i kotangens funkcije na stranici: 112.

AREA-FUNKCIJE

↳ inverzne funkcije hiperboličkih funkcija

1) Funkcija area-sinus $f(x)=\operatorname{arsh} x$



$f(x)=\operatorname{arsh}x$ – inverzna funkcija funkcije $y=\operatorname{sh}x$
- $\mathcal{D}(f)=\mathbb{R}$
- $\operatorname{Im}(f)=\mathbb{R}$
- nultočka $x=0$
- neparna funkcija: $\operatorname{arsh}(-x)=-\operatorname{arsh}x$

↳ vrijedi: $\operatorname{arsh}(\operatorname{sh}x)=x,\ \forall x\in\mathbb{R}$
$\operatorname{sh}(\operatorname{arsh}x)=x,\ \forall x\in\mathbb{R}$

↳ može se prikazati kao: $\operatorname{arsh}x=\ln\left(x+\sqrt{x^2+1}\right)$

DOKAZ:

$$x=\operatorname{sh}y=\frac{e^y-e^{-y}}{2}\ \Big/\cdot 2e^y$$

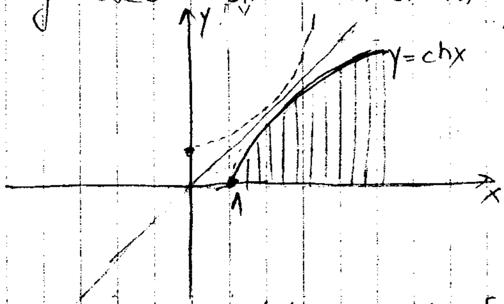
$$2xe^y=e^{2y}-1$$

$$(e^y)^2+(e^y)2x-1=0$$

$$e^y_{1,2}=\frac{2x\pm\sqrt{4x^2-4}}{2}=x\pm\sqrt{x^2+1}$$

Dobivamo 2struki predznak, ali kako je $e^y>0,\ \forall y\in\mathbb{R}$, onda uzimamo rješenje: $e^y=x+\sqrt{x^2+1}$ Q.E.D.

2) Funkcija area-kosinus $f(x)=\operatorname{arch}x$



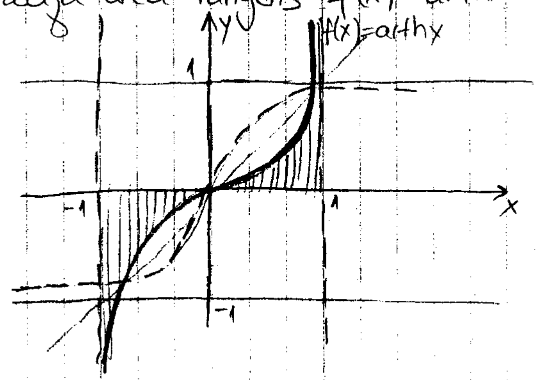
- inverzna funkcija funkcije $y=\operatorname{ch}x\big|_{[0,\infty\rangle}$
- $\mathcal{D}(f)=[1,\infty\rangle$
- $\operatorname{Im}(f)=[0,+\infty\rangle$
- nultočka $x=1$

↳ vrijedi: $\operatorname{arch}(\operatorname{ch}x)=x,\ \forall x\in[0,\infty\rangle$
$\operatorname{ch}(\operatorname{arch}x)=x,\ \forall x\in[1,\infty\rangle$

↳ može se prikazati kao: $\operatorname{arch}x=\ln\left(x+\sqrt{x^2-1}\right)$

DOKAZ: analogan prethodnom

3) Funkcija area-tangens $f(x)=\operatorname{arth}x$

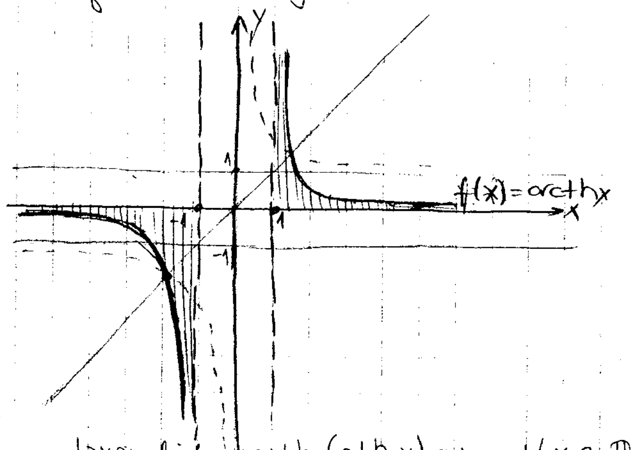


- inverzna funkcija funkcije $y=\operatorname{th}x$
- $\mathcal{D}(f)=\langle-1,1\rangle$
- $\operatorname{Im}(f)=\mathbb{R}$
- nultočke $x=0$
- neparna funkcija: $\operatorname{arth}(-x)=-\operatorname{arth}x$

↳ vrijedi: $\operatorname{arth}(\operatorname{th}x)=x,\ \forall x\in\mathbb{R}$
$\operatorname{th}(\operatorname{arth}x)=x,\ \forall x\in\langle-1,1\rangle$

↳ može se prikazati kao: $\operatorname{arth}x=\frac{1}{2}\ln\frac{1+x}{1-x},\ x\in\langle-1,1\rangle$

4) Funkcija area-kotangens $f(x)=\text{arcth}\,x$



- inverzna funkcija funkcije $y=\text{cth}\,x$
- $\mathcal{D}(f)=\langle-\infty,-1\rangle\cup\langle1,\infty\rangle$
- $\text{Im}(f)=\mathbb{R}\setminus\{0\}$
- nema nultočke
- neparna funkcija: $\text{arcth}(-x)=-\text{arcth}\,x$

↳ slijedi: $\text{arcth}(\text{cth}\,x)=x \quad , \forall x\in\mathbb{R}\setminus\{0\}$

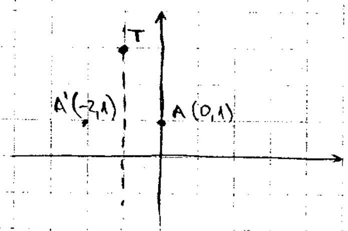
$\text{cth}(\text{arcth}\,x)=x \quad , \forall x\in\langle-\infty,-1\rangle\cup\langle1,\infty\rangle$

↳ može se prikazati kao:

$$\text{arcth}\,x=\frac{1}{2}\ln\frac{x+1}{x-1} \quad , \forall x\in\langle-\infty,-1\rangle\cup\langle1,\infty\rangle$$

ZADACI (DZ 6. PRVI DIO)

1. $T(-1,3)$
$A(0,1)$
os simetrije y



$y=ax^2+bx+c=?$
$y_{1,2}=?$

I) $1=c$
II) $1=a\cdot4+b(-2)+c$
III) $T(x,y)=T(-1,3)$

a) $x=-\frac{b}{2a} \rightarrow -1=-\frac{b}{2a}$
$-2a=-b$
$2a=b$

b) $y=\frac{4ac-b^2}{4a} \rightarrow 3=\frac{4ac-b^2}{4a}$
$12a=4ac-b^2$
$12a=4a-b^2$

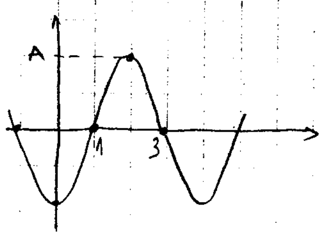
a) i b) $12a=4a-(2a)^2$
$12a=4a-4a^2$
$4a^2+8a=0$
$4a(a+2)=0$
$a=0 \quad a=-2$

$b=-2\cdot2=-4$

$y=-2x^2-4b+1$

$Y_{1,2}=\frac{4\pm\sqrt{16+8}}{-4}=\frac{4\pm2\sqrt{6}}{-4}=\frac{2\pm\sqrt{6}}{-2} \qquad Y_1=\frac{2+\sqrt{6}}{-2} \qquad Y_2=\frac{2-\sqrt{6}}{-2}$

2. $N_1(1,0)$
$N_2(3,0)$
$M(2,2)$
$A\sin(\omega x+\alpha)$



1) $A=2$
2) $P=\frac{2\pi}{\omega} \qquad 4=\frac{2\pi}{\omega} \Rightarrow 4\omega=2\pi$
$\omega=\frac{\pi}{2}$
3) $2=2\sin\left(\frac{\pi}{2}\cdot2+\alpha\right) \Rightarrow \sin\alpha=-1 \Rightarrow \alpha=\frac{3\pi}{2}+2k\pi$

$$y=2\sin\left(\frac{\pi}{2}x+\frac{3\pi}{2}\right)$$

Samo je jedan Mali Ivica!

3. $y = 2x^2 - 1$

a) ZA 1 U DESNO $\rightarrow y = 2(x-1)^2 - 1 = 2x^2 - 4x + 2 - 1 = 2x^2 - 4x + 1$

b) ZA 2 GORE $\rightarrow (y-2) = 2x^2 - 1$

$y = 2x^2 + 1$

c) ZRCALNO-Y-OS $\rightarrow y = 2(-x)^2 - 1 = 2x^2 - 1$

4. $y = x^2 + x$

a) ZA 3 U LIJEVO $\rightarrow y = (x+3)^2 + (x+3) = x^2 + 6x + 9 + x + 3 = x^2 + 7x + 12$

b) ZA 1 DOLJE $\rightarrow (y+1) = x^2 + x$

$y = x^2 + x - 1$

c) ZRCALIMO-X-OS $\rightarrow -y = x^2 + x$

$y = -x^2 - x$

5. $y = \frac{1}{x}$

a) ZA 2 LIJEVO $\rightarrow y = \frac{1}{x+2}$

b) ZA 1 GORE $\rightarrow y - 1 = \frac{1}{x+2}$

$$y = \frac{x+2+1}{x+2} = \frac{x+3}{x+2}$$

c) ZRCALIMO-X-OS $\rightarrow -y = \frac{x+3}{x+2}$

$$y = -\frac{x+3}{x+2}$$

6. a) $f(x) = \sqrt{x^3 - 3x^2 - 10x + 24}$, $D(f) = ?$

$x^3 - 3x^2 - 10x + 24 \geq 0$

↳ Rješenja su cjelobrojni djelitelji broja 24 $(\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24)$
Dobiju se tako da se nađe jedno i da se ostala nađu dijeljenjem polinoma:

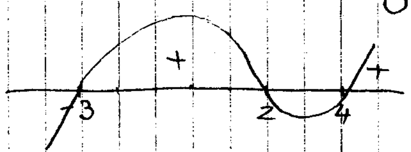
$x_1 = 2$

$$(x^3 - 3x^2 - 10x + 24) : (x-2) = x^2 - x - 12$$
$$\begin{array}{l} \underline{-\cancel{x^3} + 2x^2} \\ \quad -\cancel{x^2} - 10x + 24 \\ \quad \underline{\pm \cancel{x^2} \pm 2x} \\ \qquad -12x + 24 \\ \qquad \underline{\pm \cancel{12x} \pm \cancel{24}} \\ \qquad\qquad 0 \end{array}$$

$$x_{2,3} = \frac{1 \pm \sqrt{1+48}}{2} = \frac{1 \pm 7}{2}$$

$x_2 = 4$

$x_3 = -3$



$x \in [-3, 2] \cup [4, \infty\rangle$ //

b) $f(x) = \frac{4x - 1}{2 - \sqrt{x^2 - 3x}}$, $D(f) = ?$

1° $2 - \sqrt{x^2 - 3x} \neq 0$

$\sqrt{x^2 - 3x} \neq 2 \;/^2$

$x^2 - 3x \neq 4$

$x^2 - 3x - 4 \neq 0$

$x_{1,2} \neq \frac{3 \pm \sqrt{9+16}}{2} \neq \frac{3 \pm 5}{2}$

$x_1 \neq 4$

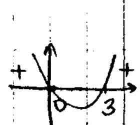
$x_2 \neq -1$

2° $x^2 - 3x \geq 0$

$x(x-3) \geq 0$

$x_1 = 0$

$x_2 = 3$



$x \in \langle -\infty, 0] \cup [3, \infty\rangle \setminus \{-1, 4\}$ //

(7.) a) $f(x)=\sqrt{3-|x-1|}$, $\mathcal{D}(f)=?$

$3-|x-1|\geq 0$

$-|x-1|\geq -3 \quad /(-1)$

$|x-1|\leq 3$

$1^\circ \; x-1\leq 3 \qquad 2^\circ \; x-1\geq -3$

$x\leq 4 \qquad x\geq -2$

$x\in[-2,4]$ //

b) $f(x)=\sqrt{|x^2-3x-1|-3}$, $\mathcal{D}(f)=?$

$|x^2-3x-1|-3\geq 0$

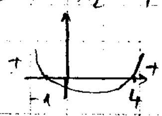
$|x^2-3x-1|\geq 3$

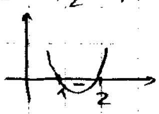
$1^\circ \; x^2-3x-1\geq 3 \qquad 2^\circ \; x^2-3x-1\leq -3$

$x^2-3x-4\geq 0 \qquad x^2-3x+2\leq 0$

$x_{1,2}=\frac{3\pm\sqrt{9+16}}{2}=\frac{3\pm 5}{2} \qquad x_{1,2}=\frac{3\pm\sqrt{9-8}}{2}=\frac{3\pm 1}{2}$

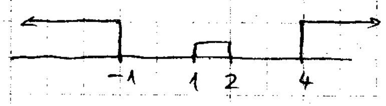
$x_1=4 \quad x_2=-1 \qquad x_1=2 \quad x_2=1$





$x\in\langle-\infty,-1]\cup[4,\infty\rangle \qquad x\in[1,2]$

1° ili 2°



$x\in\langle-\infty,-1]\cup[1,2]\cup[4,\infty\rangle$ //

(8.) a) $f(x)=\frac{1}{\ln(x^2-3)}$, $\mathcal{D}(f)=?$

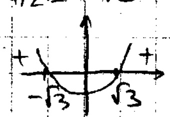
$1^\circ \; \ln(x^2-3)\neq 0 \qquad 2^\circ \; x^2-3>0$

$\cancel{\ln}(x^2-3)\neq \cancel{\ln} e^0 \qquad x_{1,2}=\pm\sqrt{3}$

$x^2-3\neq 1$

$x^2\neq 4$

$x_1\neq 2 \quad x_2\neq -2$



$x\in\langle-\infty,-\sqrt{3}\rangle\cup\langle\sqrt{3},\infty\rangle$

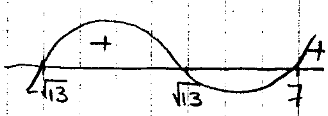
1° i $2^\circ \qquad x\in\langle-\infty,-\sqrt{3}\rangle\cup\langle\sqrt{3},\infty\rangle\setminus\{-2,2\}$ //

b) $f(x)=\sqrt{\ln\left(\frac{x^2-13}{x-7}\right)}$, $\mathcal{D}(f)=?$

$1^\circ \; x-7\neq 0 \qquad 2^\circ \; \frac{x^2-13}{x-7}>0$

$x\neq 7 \qquad x_{1,2}=\pm\sqrt{13}$

$x_3=7$



$x\in\langle-\sqrt{13},\sqrt{13}\rangle\cup\langle 7,\infty\rangle$

$3^\circ \ \ln\left(\frac{x^2-13}{x-7}\right) \geq 0$

$\cancel{\ln}\left(\frac{x^2-13}{x-7}\right) \geq \cancel{\ln} e^0$

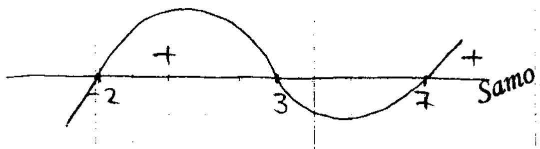
$\frac{x^2-13}{x-7} - 1 \geq 0$

$\frac{x^2-13-x+7}{x-7} \geq 0$

$\frac{x^2-x-6}{x-7} \geq 0$

$x_{1,2} = \frac{1 \pm \sqrt{1+24}}{2} = \frac{1\pm 5}{2} \begin{cases} x_1 = 3 \\ x_2 = -2 \end{cases}$

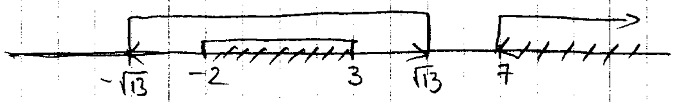
$x_3 = 7$



Samo je jedan Mali Ivica!

$x \in [-2, 3] \cup [7, \infty\rangle$

1° i 2° i 3°



$x \in [-2, 3] \cup \langle 7, \infty\rangle$ //

9. a) $f(x) = \arcsin\left(\frac{1}{x-2}\right)$, $\mathcal{D}(f) = ?$

1° $x - 2 \neq 0$

$x \neq 2$

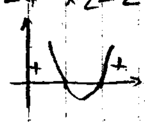
2° $-1 \leq \frac{1}{x-2} \leq 1$

a) $\frac{1}{x-2} \geq -1$

$\frac{1+x-2}{x-2} \geq 0$

$\frac{x-1}{x-2} \geq 0$

$x_1 = 1 \quad x_2 = 2$



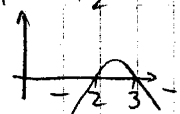
$x \in \langle -\infty, 1] \cup [2, \infty\rangle$

b) $\frac{1}{x-2} \leq 1$

$\frac{1-x+2}{x-2} \leq 0$

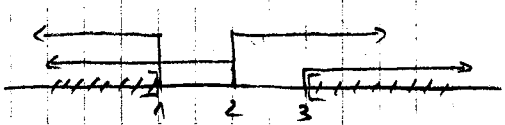
$\frac{-x+3}{x-2} \leq 0$

$x_1 = 3 \quad x_2 = 2$



$x \in \langle -\infty, 2] \cup [3, \infty\rangle$

1° i $2^\circ a$ i $2^\circ b$



$x \in \langle -\infty, 1] \cup [3, \infty\rangle$ //

b) $f(x) = \operatorname{arcth}\left(\frac{1}{x^2-x-1}\right)$, $\mathcal{D}(f) = ?$

1° $x^2 - x - 1 \neq 0$

$x_{1,2} \neq \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

$x_1 \neq \frac{1+\sqrt{5}}{2}$

$x_2 \neq \frac{1-\sqrt{5}}{2}$

2° $1 < \dfrac{1}{x^2-x-1} < -1$

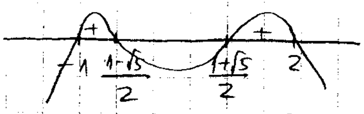
a) $\dfrac{1}{x^2-x-1} > 1$

$\dfrac{1-x^2+x+1}{x^2-x-1} > 0$

$\dfrac{-x^2+x+2}{x^2-x-1} > 0$

$x_{1,2} = \dfrac{-1 \pm \sqrt{1+8}}{-2} = \dfrac{-1\pm 3}{-2}$; $x_1 = -1$, $x_2 = 2$

$x_{3,4} = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1\pm\sqrt{5}}{2}$; $x_3 = \dfrac{1+\sqrt5}{2}$, $x_4 = \dfrac{1-\sqrt5}{2}$



$x \in \langle -1, \dfrac{1-\sqrt5}{2} \rangle \cup \langle \dfrac{1+\sqrt5}{2}, 2 \rangle$

b) $\dfrac{1}{x^2-x-1} < -1$

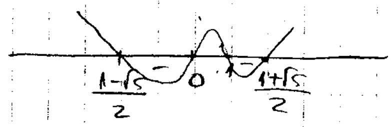
$\dfrac{\cancel{1}+x^2-x-\cancel{1}}{x^2-x-1} < 0$

$\dfrac{x(x-1)}{x^2-x-1} < 0$

$x_1 = 0$

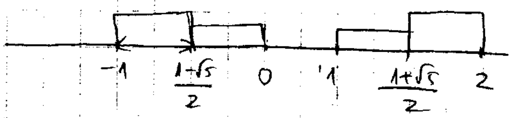
$x_2 = 1$

$x_{3,4} = \dfrac{1 \pm \sqrt5}{2}$; $x_3 = \dfrac{1+\sqrt5}{2}$, $x_4 = \dfrac{1-\sqrt5}{2}$



$x \in \langle \dfrac{1-\sqrt5}{2}, 0 \rangle \cup \langle 1, \dfrac{1+\sqrt5}{2} \rangle$

1° i (2°a ili 2°b)



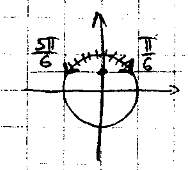
$x \in \langle -1, 0 \rangle \cup \langle 1, 2 \rangle$ //

(10) a) $f(x) = \ln(2\sin x - 1)$

$2\sin x - 1 > 0$

$2\sin x > 1$

$\sin x > \dfrac{1}{2}$



$x \in \langle \dfrac{\pi}{6} + 2k\pi, \dfrac{5\pi}{6} + 2k\pi \rangle$

b) $f(x) = \sqrt{\arccos x - \dfrac{\pi}{3}}$

1° $\arccos x - \dfrac{\pi}{3} > 0$

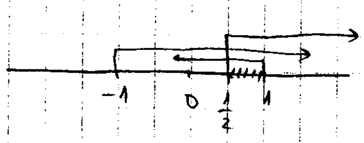
$\arccos x > \dfrac{\pi}{3}$

$x > \dfrac{1}{2}$

2° $-1 \le x \le 1$

a) $x \le 1$ b) $x \ge -1$

1° i 2°a i 2°b



$x \in \langle \dfrac{1}{2}, 1]$ //

11. $f(x)=\dfrac{2^x+2^{-x}}{2^x-2^{-x}}$, $D(f)=?$, $f^{-1}=?$, $D(f^{-1})=?$, f-injekcija

1)

$x \to y$, $y \to x$ $\longrightarrow$ $x=\dfrac{2^y+2^{-y}}{2^y-2^{-y}} \quad / \cdot (2^y-2^{-y})$

$x\cdot 2^y - x\cdot 2^{-y} = 2^y + 2^{-y}$

$x\,2^y - 2^y = 2^{-y} + x\cdot 2^{-y}$

$2^y(x-1) = 2^{-y}(1+x) \quad / : 2^{-y} \; : (x-1)$

$\dfrac{2^y}{2^{-y}} = \dfrac{1+x}{x-1}$

$2^{2y} = \dfrac{1+x}{x-1}$

$4^y = \dfrac{1+x}{x-1} \quad / \cdot \log_4$

$y\log_4 4 = \log_4 \dfrac{1+x}{x-1}$

$y = \log_4 \dfrac{1+x}{x-1} = \dfrac{1}{2}\log_2 \dfrac{1+x}{x-1}$ //

2) $D(f)=?$

$2^x-2^{-x}\neq 0$

$2^x \neq 2^{-x} \quad / \log$

$\log 2^x \neq \log 2^{-x}$

$x\cancel{\log 2} \neq -x\cancel{\log 2}$

$x+x\neq 0$

$2x \neq 0$

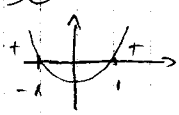
$x\neq 0$

$x\in \mathbb{R}\setminus\{0\}$ //

3) 1° $\dfrac{1+x}{x-1}>0$

$x_1=1$

$x_2=-1$



$x\in\langle-\infty,-1\rangle\cup\langle 1,\infty\rangle$

2° $x-1\neq 0$

$x\neq 1$

1° i 2° $\quad x\in\langle-\infty,-1\rangle\cup\langle 1,\infty\rangle$ //

4) f-injekcija:

$f(x_1)=f(x_2) \Rightarrow x_1=x_2$

12. $f(x)=\dfrac{2e^x-1}{e^x+2}$, $D(f)=?$, $f^{-1}=?$, $D(f^{-1})=?$, f-injekcija

1)

$x \to y$, $y \to x$ $\longrightarrow$ $x=\dfrac{2\cdot e^y-1}{e^y+2} \quad / \cdot (e^y+2)$

$x\cdot e^y + 2x = 2\cdot e^y - 1$

$x\cdot e^y - 2e^y = -1-2x$

$e^y(x-2) = -(1+2x)$

$e^y = \dfrac{1+2x}{2-x} \quad / \ln$

$y\cdot \ln e = \ln \dfrac{1+2x}{2-x}$

$y = \ln \dfrac{1+2x}{2-x}$ //

2) $D(f)=?$

$e^x+2\neq 0$

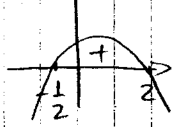
$e^x \neq -2$

$x\in\mathbb{R}$ //

3) 1° $\dfrac{1+2x}{2-x}>0$

$x_1=-\dfrac{1}{2}$

$x_2=2$



$x\in\langle-\tfrac{1}{2},2\rangle$

2° $2-x\neq 0$

$x\neq 2$

1° i 2° $\quad x\in\langle-\tfrac{1}{2},2\rangle$ //

4) f-injekcija

$f(x_1)=f(x_2)$

$\dfrac{2e^{x_1}-1}{e^{x_1}+2}=\dfrac{2e^{x_2}-1}{e^{x_2}+2}$

$\cancel{2e^{x_1+x_2}}+4e^{x_1}-e^{x_2}-\cancel{2} = \cancel{2e^{x_1+x_2}}-e^{x_1}+4e^{x_2}-\cancel{2}$

$5e^{x_1}=5e^{x_2}$

$\Downarrow$

$x_1=x_2$

Samo je jedan Mali Ivica!

13. $f(x)=\arcsin\left(\frac{1}{\sqrt{x-1}}\right)$, $f^{-1}=?$, $\mathcal{D}(f)=?$, $\mathcal{D}(f^{-1})=?$

1)

$x \to y$, $y \to x$ $\longrightarrow$ $x=\arcsin\left(\frac{1}{\sqrt{y-1}}\right)$

$\sin x = \frac{1}{\sqrt{y-1}}$

$\sin^2 x\,(y-1)=1$

$y\sin^2 x - \sin^2 x = 1$

$y\sin^2 x = 1+\sin^2 x$

$y=\frac{1+\sin^2 x}{\sin^2 x}$

2) $\mathcal{D}(f)=?$

1° $-1 \le \frac{1}{\sqrt{x-1}} \le 1$

a) $\frac{1}{\sqrt{x-1}} \ge -1 \;/\cdot\sqrt{x-1}$, $\sqrt{x-1}>0$, $x>1$

$1 \ge -\sqrt{x-1} \;/^2$

$1 \ge x-1$

$x \le 2$

$x\in\langle 1,2]$

b) $\frac{1}{\sqrt{x-1}} \le 1 \;/\sqrt{x-1}$, $\sqrt{x-1}>0$, $x>1$

$1 \le \sqrt{x-1} \;/^2$

$1 \le x-1$

$x \ge 2$

$x\in[2,\infty\rangle$

2° $\sqrt{x-1}\neq 0$

$x-1\neq 0$

$x\neq 1$

3° $x-1>0$

$x>1$

$\mathcal{D}_f \in \langle 1,\infty\rangle$

3) $\mathcal{D}(f^{-1})=?$

$\sin^2 x \neq 0$

$2\sin x\cos x \neq 0$

$\sin x \neq 0$ $\quad$ $\cos x \neq 0$

$x_1 \neq 2k\pi$ $\quad$ $x_1 \neq \frac{\pi}{2}+2k\pi$

$x_2 \neq \pi+2k\pi$ $\quad$ $x_2 \neq -\frac{\pi}{2}+2k\pi$

$\mathcal{D}(f^{-1}) \in \mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi\right\}, k\in\mathbb{Z}$

14. Dokaži:

a) $\sin x\cos y = \frac{1}{2}\left[\sin(x-y)+\sin(x+y)\right]$

$DS=\frac{1}{2}\left(\sin(x-y)+\sin(x+y)\right)=\frac{1}{2}\left(\sin x\cos y \cancel{-\cos x\sin y} + \sin x\cos y \cancel{+\cos x\sin y}\right)=$

$=\frac{1}{\cancel{2}}\,\cancel{2}\sin x\cos y = \sin x\cos y = LS$ Q.E.D.

b) $\sin x+\sin y = 2\sin\frac{x+y}{2}\cdot\cos\frac{x-y}{2}$

$x=\frac{\alpha+\beta}{2}$, $y=\frac{\alpha-\beta}{2}$

(I) $\sin(x+y)=\sin x\cos y+\cos x\sin y$

SUPS: $\sin\alpha = \sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}$

(II) $\sin(x-y)=\sin x\cos y-\cos x\sin y$

SUPS: $\sin\beta = \sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha+\beta}{2}$

(I)+(II) $=\sin\alpha+\sin\beta = 2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}$

15. Dokaži: $\operatorname{arctg}x+\operatorname{arcctg}x=\frac{\pi}{2}$, $\forall x\in\mathbb{R}$

$y_1=\operatorname{arctg}x \to x=\operatorname{tg}y_1$

$y_2=\operatorname{arcctg}x \to x=\operatorname{ctg}y_2$

$\Big\}\ \operatorname{tg}\underline{\underline{y_1}}=\operatorname{ctg}y_2=\operatorname{tg}\left(\underline{\underline{\frac{\pi}{2}-y_2}}\right)$

$\Downarrow$

$y_1=\frac{\pi}{2}-y_2$

$y_1+y_2=\frac{\pi}{2}$ Q.E.D.

Samo je jedan Mali Ivica!

16. $\cos(2\arccos x) = 2x \quad , x \in [-1,1]$

$\cos^2(\arccos x) - \sin^2(\arccos x) = 2x$

$\cos^2(\arccos x) - 1 + \cos^2(\arccos x) = 2x$

$2\cos^2(\arccos x) - 1 = 2x$

$\Downarrow$

$\cos(\arccos x) = x \quad , x \in [-1,1]$

$\Downarrow$

$2x^2 - 2x - 1 = 0$

$x_{1,2} = \frac{2 \pm \sqrt{4+8}}{4} = \frac{1 \pm \sqrt{3}}{2}$

$\cancel{x_1 = \frac{1+\sqrt{3}}{2}} \qquad x_2 = \frac{1-\sqrt{3}}{2}$ //

jer $x \in [-1,1]$

17. a) $y = \arccos(\cos x) \to$ graf

b) $\arccos(\cos x) = \frac{x}{3} \quad , x = ?$

a) $\to \cos x$ ima $P = 2\pi \Rightarrow \arccos(\cos x)$ ima $P = 2\pi$

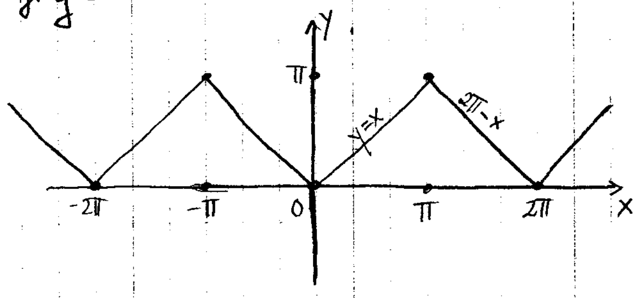
1) $[0,\pi] \to \arccos(\cos x) = y$

$x = y$

2) $[\pi, 2\pi] \to \arccos(\cos(2\pi - x)) = y$

$2\pi - x = y$

$\to$ graf:



b) $\arccos(\cos x) = \frac{x}{3} \quad / \cos \quad \Rightarrow \frac{x}{3} \in [0,\pi] \Rightarrow x \in [0, 3\pi]$

$\cos x = \cos\frac{x}{3}$

$\cos x - \cos\frac{x}{3} = 0$

$-2\sin\frac{2x}{3}\sin\frac{x}{3} = 0$

I) $\sin\frac{2x}{3} = 0$

$\frac{2x}{3} = k\pi$

$x = \frac{3k\pi}{2}$

$k=0 \ldots \ x_1 = 0$

$k=1 \ldots \ x_2 = \frac{3\pi}{2}$

$k=2 \ldots \ x_3 = 3\pi$

II) $\sin\frac{x}{3} = 0$

$\frac{x}{3} = k\pi$

$x = 3k\pi$

$x_1 = 0$

$x_2 = 3\pi$

$\cancel{x_3 = 6\pi}$

jer $x \in [0, 3\pi]$

$x \in \left\{0, \frac{3\pi}{2}, 3\pi\right\}$ //

(18.) $\Rightarrow \mathrm{sh}(x \pm y) = \mathrm{sh}x\,\mathrm{ch}y \pm \mathrm{sh}y\,\mathrm{ch}x$, dokaži!
↳ Vidi str. 115. dokazi (2 i 3)

(19.) $4\,\mathrm{sh}(\ln x) = x$, $x = ?$ $\Rightarrow x > 0$

$$\cancel{4} \cdot \frac{e^{\ln x} - e^{-\ln x}}{\cancel{2}} = x$$

$$2(x - x^{-1}) = x$$

$$2x - \frac{2}{x} = x \;/\cdot x$$

$$2x^2 - 2 = x^2$$

$$x^2 = 2$$

$$x = \pm\sqrt{2} \qquad x = +\sqrt{2}$$

(20.) $\mathrm{ch}(\ln x) = x$, $x = ?$ $\Rightarrow x > 0$

$$\frac{e^{\ln x} + e^{-\ln x}}{2} = x \;/\cdot 2$$

$$x + x^{-1} = 2x$$

$$x + \frac{1}{x} = 2x \;/\cdot x$$

$$x^2 + 1 = 2x^2$$

$$-x^2 = -1$$

$$x^2 = 1$$

$$x = \pm 1 \qquad x = 1$$

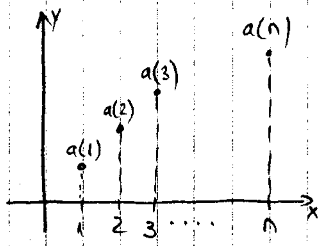
7. NIZOVI

Samo je jedan Mali Ivica!

7.1. POJAM NIZA. PODNIZ. OMEĐEN NIZ.

= DEFINICIJA 1.

↳ Funkcija $a: \mathbb{N} \to S$ zove se niz (slijed) u skupu S. Ako je $S = \mathbb{R}$ onda je to niz (slijed) realnih brojeva.



= Oznake: $a(n) = a_n$ → opći član niza

(a_n) → oznaka niza

$\{a_n\}$ → skup vrijednosti niza (a_n)

= DEFINICIJA 2.

↳ Kažemo da je niz realnih brojeva (b_n) podniz niza (a_n), ako postoji strogo rastuća funkcija $f: \mathbb{N} \to \mathbb{N}$ takva da je: $b_n = a_{f(n)}$.

= PRIMJER 1.

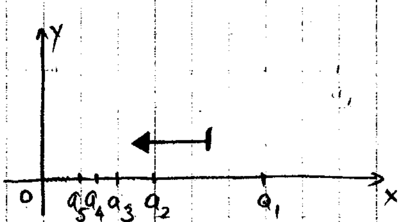
↳ Navedimo nekoliko nizova:

a) $a_n = \frac{1}{n}$

↳ $(a_n) = 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}$

↳ $\{a_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$

↳ grafički:



b) $a_n = \frac{1}{2n}$

↳ $(a_n) = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots, \frac{1}{2n}$

↳ $\{a_n\} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots, \frac{1}{2n}, \dots\}$

↳ Niz $(a_n) = \frac{1}{2n}$ je očito podniz niza iz primjera a jer je funkcija $f(n) = 2n$ monotono rastuća.

c) $a_n = (-1)^{n+1} \frac{n}{n+1}$

↳ $(a_n) = \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \dots$

↳ $\{a_n\} = \{\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \dots\}$

d) $a_n = \sqrt{n}$

↳ $(a_n) = 1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \dots, \sqrt{n}$

↳ $\{a_n\} = \{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \dots, \sqrt{n}, \dots\}$

= DEFINICIJA 3.

↳ Kažemo da je niz (a_n) omeđen odozgo (odozdo) ako je skup $\{a_n\}$ omeđen odozgo (odozdo). Niz je omeđen, ako je omeđen i odozgo i odozdo.

↳ Primjeri: a, b, c su omeđeni nizovi, a d nije

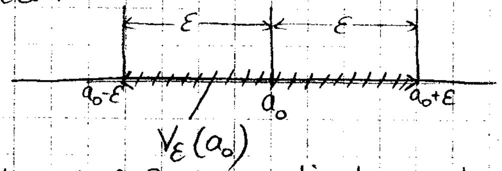
7.2. POJAM GOMILIŠTA NIZA

= DEFINICIJA 1.

↳ Neka je $a_0 \in \mathbb{R}$, i $\varepsilon > 0$. ε-okoliš broja a_0 je interval oblika:

$$V_\varepsilon(a_0) = \{a \in \mathbb{R} \mid |a - a_0| < \varepsilon\} \equiv (a_0 - \varepsilon, a_0 + \varepsilon)$$

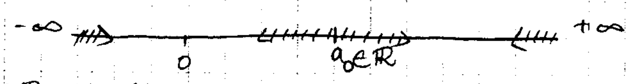
ili grafički:



↳ Okolišem broja a_0 nazivamo svaki skup za kojeg postoji ε-okoliš koji je u njemu sadržan.

= Radi prikladnijeg daljnjeg iskazivanja definicija i teorema dodajemo skupu $\mathbb{R}$ još dva objekta, tj. „točke" $+\infty$ i $-\infty$.

↳ grafički:



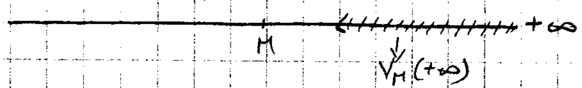
↳ za $a \in \mathbb{R}$ vrijedi:

$$-\infty < a < +\infty .$$

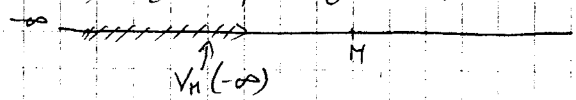
↳ Skup $\mathbb{R} \cup \{-\infty, +\infty\} = \overline{\mathbb{R}}$; zove se prošireni brojevni pravac.

= DEFINICIJA 2.

↳ Skup $V_M(+\infty) = \{a \in \mathbb{R} \mid a > M\} = \langle M, +\infty\rangle$ naziva se M-okoliš točke $+\infty$.



↳ Skup $V_M(-\infty) = \{a \in \mathbb{R} \mid a < M\} = \langle -\infty, M\rangle$ naziva se M-okoliš točke $-\infty$.



↳ Okoliš točke $+\infty$ $(-\infty)$ je svaki skup za kojeg postoji M-okoliš koji je u njemu sadržan.

= DEFINICIJA 3.

↳ Realan broj $a \in \mathbb{R}$ je gomilište niza (a_n) ako se unutar svakog okoliša nalazi beskonačno mnogo članova niza (a_n).

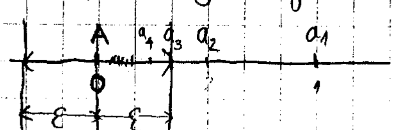
↳ Gornja definicija može se proširiti u smislu da $a \in \mathbb{R}$ možemo zamijeniti beskonačno dalekim točkama $+\infty$ ili $-\infty$.

= PRIMJER 1.

↳ Navedimo gomilišta iz prethodnog primjera:

a) $a_n = \frac{1}{n}$

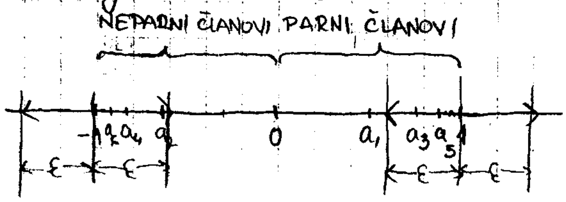
$(a_n) = 1, \frac{1}{2}, \frac{1}{3}, \ldots$



↳ niz ima gomilište: $A = 0$

b) $a_n = (-1)^{n+1} \frac{n}{n+1}$

$(a_n) = \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{5}{4}, \ldots$

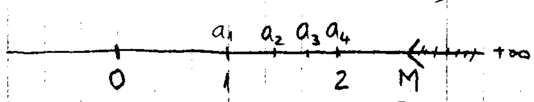


↳ niz ima 2 gomilišta : $A_1 = -1$ i $A_2 = 1$

c) $a_n = \sqrt{n}$

Samo je jedan Mali Ivica!

$(a_n) = 1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \dots$



↳ niz ima gomilište : $A = +\infty$, jer za $n > M^2$ su svi $a_n = \sqrt{n}$ unutar okoliša $\langle M, +\infty \rangle$

= STAVAK 1.

↳ Svaki omeđen niz realnih brojeva ima barem jedno gomilište, i to gomilište je realan broj.

Dokaz :

- Neka se svi članovi nalaze unutar intervala $[m, M]$. Ako podijelimo taj interval na dva podintervala: $\left[m, \frac{m+M}{2}\right]$, $\left[\frac{m+M}{2}, M\right]$. U barem jednom od ta dva podintervala mora biti beskonačno mnogo članova niza. Ako odaberemo jedan od podintervala u kojem ima beskonačno mnogo članova niza i podijelimo ga na dva manja podintervala onda opet u jednom od njih ima beskonačno mnogo članova. Nastavljajući taj postupak dolazimo do niza zatvorenih intervala $[a_i, b_i]$ takvih da je $[a_{i+1}, b_{i+1}] \subset [a_i, b_i]$, čiji presjek neka je broj a . U okolišu tog broja a nalazi se beskonačno mnogo članova niza, što znači da je upravo taj broj a gomilište niza a_n.

= Ako dozvolimo da gomilište može biti i $+\infty$ ili $-\infty$ gornji stavak možemo proširiti na:

↳ Svaki niz (a_n) ima gomilište u $\overline{\mathbb{R}}$. Posebice ima :

1) najmanje gomilište → limes inferior ($\liminf a_n$)
2) najveće gomilište → limes superior ($\limsup a_n$)

= PRIMJER 2.

↳ Navedimo lim sup i lim inf iz prethodnog primjera :

a) $a_n = (-1)^{n+1} \frac{n}{n+1}$

$(a_n) = \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \dots$ → $\liminf a_n = -1$
→ $\limsup a_n = +1$

b) $a_n = \frac{1}{n}$

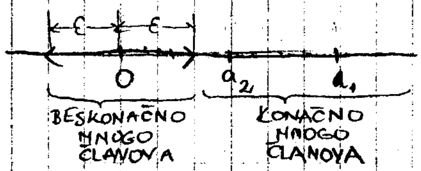
$(a_n) = 1, \frac{1}{2}, \frac{1}{3}, \dots$ → $\limsup a_n = \liminf a_n = 0$

7.3. KONVERGENCIJA NIZA REALNIH BROJEVA

= PRIMJER 1.

↳ $a_n = \frac{1}{n}$ ... $(a_n) = 1, \frac{1}{2}, \frac{1}{3}, \dots$

↳ Unutar okoliša $\langle -\varepsilon, \varepsilon \rangle$ postoji beskonačno mnogo članova, a izvan konačno mnogo članova.



= DEFINICIJA 1.

↳ Niz realnih brojeva (a_n) konvergira (teži) k realnom broju L ako svaki svaki okoliš od $\mathbb{R}$ sadrži beskonačno mnogo članova niza (a_n), a izvan bilo kojeg okoliša ima samo konačno mnogo članova niza.

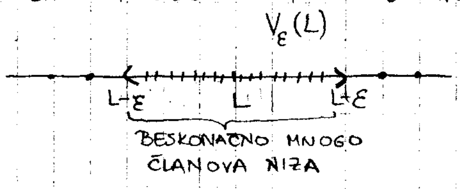
Za L kažemo da je limes niza (a_n) i to zapisujemo : $L = \lim_{n \to \infty} a_n$ ili $L = \lim a_n$ ili $a_n \to L$.

= PRIMJEDBA 1.

↳ U gornjoj definiciji dostatno je postaviti samo drugi zahtjev (da izvan okoliša ima samo konačno mnogo članova), jer je prvi zahtjev (da unutar ima beskonačno mnogo članova) njegova posljedica.

= Ako niz ima limes onda je konvergentan, inače je divergentan.

= Ako je $\varepsilon > 0$ onda možemo napisati ε-iziranu definiciju limesa:



↳ Među konačno mnogo članova izvan okoliša $V_\varepsilon(L)$ neki ima najveći indeks n_1, pa za $n_0 = n_1 + 1$, i za sve $n \geq n_0$ vrijedi da je svaki daljnji član je unutar okoliša $V_\varepsilon(L)$.

Dakle vrijedi:

$$\lim_{n\to\infty} a_n = L \Leftrightarrow (\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N})[n \geq n_0 \Rightarrow |a_n - L| < \varepsilon].$$

$$\Leftrightarrow [n \geq n_0 \Rightarrow a_n \in V_\varepsilon(L)].$$

= **PRIMJER 2.**

↳ Koristeći ε-iziranu definiciju limesa dokažite da je: $\lim_{n\to\infty} \frac{1}{n} = 0$.

- Arhimedov stavak glasi:

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})\left(n_0 > \frac{1}{\varepsilon}\right)$$

- Uvrštavanjem $a_n = \frac{1}{n}$ i $L = 0$ dobivamo:

$$(\forall \varepsilon > 0)(\exists n_0)(\forall n)\left(n \geq n_0 \Rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon\right),$$

što znači da je: $\lim_{n\to\infty} \frac{1}{n} = 0$ jer za $n \geq n_0$ imamo:

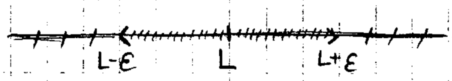
$$\frac{1}{n} \leq \frac{1}{n_0} < \varepsilon \quad \text{Q.E.D.}$$

= **STAVAK 1.**

↳ Konvergentan niz je omeđen.

Dokaz:

- Među konačno mnogo članova s desna i s lijeva od intervala $\langle L-\varepsilon, L+\varepsilon \rangle$ postoji najveći, odnosno najmanji, broj, pa su to međe skupa $\{a_n\}$.
- Grafički:

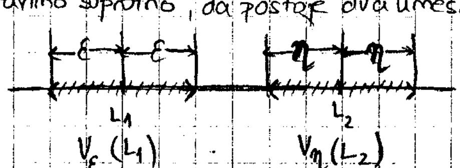


= **STAVAK 2.**

↳ Konvergentan niz ima samo jedan limes.

Dokaz:

- Pretpostavimo suprotno, da postoje dva limesa. Grafički:



- Izvan $V_\varepsilon(L_1)$ ima samo konačno mnogo članova, pa u okolišu $V_\eta(L_2)$ ne može biti beskonačno mnogo članova, pa L_2 nije limes, što smo trebali i dokazati.

= **STAVAK 3.**

↳ Ako postoji $n_0 \in \mathbb{N}$ takav da je $a_n \leq b_n$ za svaki $n \geq n_0$, onda je i $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

= **STAVAK 4.**

↳ tzv. „Sendvič-teorem":

Ako postoji $n_0 \in \mathbb{N}$ takav da je $a_n \leq b_n \leq c_n$ za svaki $n \geq n_0$, te ako je:
$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, onda je i $\lim_{n\to\infty} b_n = L$.

Dokaz:

- Direktna posljedica stavka 3, tako da se on upotrijebi 2 puta.

= STAVAK 5.

↳ Vrijedi:

$$\lim_{n\to\infty} a_n = 0 \iff \lim_{n\to\infty} |a_n| = 0$$

Dokaz:

- Neposredna posljedica očevidne ekvivalencije: $|a_n - 0| < \varepsilon \iff ||a_n| - 0| < \varepsilon$.

7.4. PRAVILA ZA RAČUNANJE S LIMESIMA

= STAVAK 1.

↳ Ako su a_n i b_n konvergentni nizovi, onda vrijedi:

a) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

b) $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

c) $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, za $b_n \neq 0 \Rightarrow \lim_{n\to\infty} b_n \neq 0$

Dokaz:

a) Uzmimo da je $\varepsilon > 0$. Tada vrijedi:

$$\exists n_1 : \forall n \geq n_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}.$$

$$\exists n_2 : \forall n \geq n_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2}.$$

Neka je $n_3 = \max\{n_1, n_2\}$ onda za $n \geq n_3$ vrijedi:

$$|(a_n + b_n) - (a+b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

Q.E.D.

b) Konvergentni niz je omeđen, pa za $\exists M_1$ vrijedi da je $|a_n| < M_1$.
Neka je $M = \max\{M, |b|\}$.
Za $\varepsilon > 0$ vrijedi:

$$\exists n_1 \ldots n \geq n_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2M}.$$

$$\exists n_2 \ldots n \geq n_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2M}$$

Za $n_3 = \max\{n_1, n_2\}$, vrijedi za $n \geq n_3$:

$$|a_n b_n - ab| = |(a_n b_n - a_n b) + (a_n b - ab)| \leq |a_n||b_n - b| + |b||a_n - a| <$$

$$< \cancel{M} \cdot \frac{\varepsilon}{2\cancel{M}} + \cancel{M} \cdot \frac{\varepsilon}{2\cancel{M}} = \varepsilon.$$

Q.E.D.

7.5. NIZOVI S BESKONAČNIM LIMESIMA

= DEFINICIJA 1.

↳ Niz realnih brojeva (a_n) divergira k $+\infty$, i to zapisujemo: $\lim_{n\to\infty} a_n = +\infty$ ($\lim a_n = +\infty$, $a_n \to +\infty$), ako za svaki $M > 0$ postoji $n_0 \in \mathbb{N}$ takav da za $n \geq n_0$ vrijedi $a_n > M$.

↳ Niz realnih brojeva (a_n) divergira k $-\infty$, i to zapisujemo: $\lim_{n\to\infty} a_n = -\infty$ ($\lim a_n = -\infty$, $a_n \to -\infty$), ako za svaki $m < 0$ postoji $n_0 \in \mathbb{N}$ takav da za $n \geq n_0$ vrijedi $a_n < m$.

= PRIMJER 1.

↳ $\lim_{n\to\infty}\sqrt{n}=+\infty$, jer:

$$(\forall M>0)(\exists n_0\in\mathbb{N})(n_0>M^2)\text{, a onda vrijedi i:}$$

$$(\forall n, n\geq n_0)\Rightarrow(n>M^2)\Rightarrow(\sqrt{n}>M)$$

Q.E.D.

= STAVAK 1.

↳ Neka su nizovi (a_n) i (b_n) takvi da je $\lim_{n\to\infty}a_n=a>0$, $b_n>0$ i $\lim_{n\to\infty}b_n=0$. Tada je: $\lim_{n\to\infty}\frac{a_n}{b_n}=+\infty$. (simbolički: $\frac{a}{0}=+\infty$).

= STAVAK 2.

↳ Neka su nizovi (a_n), (b_n) i (c_n) takvi da je $\lim_{n\to\infty}a_n=+\infty$, $\lim_{n\to\infty}b_n=+\infty$ i $\lim_{n\to\infty}c_n=c$. Tada vrijedi:

SIMBOLIČKI:

a) $\lim_{n\to\infty}(a_n+b_n)=+\infty \longrightarrow (+\infty)+(+\infty)=+\infty$

b) $\lim_{n\to\infty}(a_n+c_n)=+\infty \longrightarrow (+\infty)+(c)=+\infty$

c) $\lim_{n\to\infty}(a_n\cdot b_n)=+\infty \longrightarrow (+\infty)\cdot(+\infty)=+\infty$

d) $\lim_{n\to\infty}(a_n\cdot c_n)=+\infty$, za $c>0$ $\longrightarrow +\infty\cdot c=+\infty$, $c>0$
$\qquad\qquad\qquad\quad =-\infty$, za $c<0$ $\longrightarrow +\infty\cdot c=-\infty$, $c<0$

e) $\lim_{n\to\infty}\frac{c_n}{a_n}=0 \longrightarrow \frac{c}{\infty}=0$

= PRIMJER 2.

↳ Izračunaj limese:

a) $\lim_{n\to\infty}\frac{3n+1}{2n^2+4n}\left(\frac{+\infty}{+\infty}\right)\begin{matrix}:n^2\\:n^2\end{matrix}=\lim_{n\to\infty}\frac{\frac{3}{n}+\frac{1}{n^2}}{2+\frac{4}{n}}=\frac{0}{2}=0$

b) $\lim_{n\to\infty}\frac{3n^2+1}{2n^2+4n}\left(\frac{+\infty}{+\infty}\right)\begin{matrix}:n^2\\:n^2\end{matrix}=\lim_{n\to\infty}\frac{3+\frac{1}{n^2}}{2+\frac{4}{n}}=\frac{3}{2}$

c) $\lim_{n\to\infty}\frac{3^n+2^n}{3^{n+1}+2^{n+1}}=\lim_{n\to\infty}\frac{3^n+2^n}{3^n\cdot 3+2^n\cdot 2}\left(\frac{+\infty}{+\infty}\right)\begin{matrix}:3^n\\:3^n\end{matrix}=\lim_{n\to\infty}\frac{1+\left(\frac{2}{3}\right)^n}{3+2\cdot\left(\frac{2}{3}\right)^n}=\frac{1}{3}$

7.6. MONOTONI NIZOVI

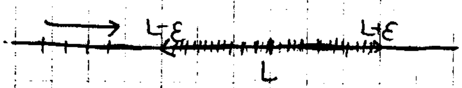
= Niz $a:\mathbb{N}\to\mathbb{R}$ je monoton ako je pripadna funkcija a monotona.

= STAVAK 1.

↳ Ako je niz realnih brojeva monoton i omeđen, onda je niz konvergentan.

Dokaz:

- Neka je a_n monotono rastući niz, i omeđen odozgo. Neka je $\sup\{a_n\}=L$.
- Grafički:



- Za svaki $\varepsilon>0$, $L-\varepsilon$ nije supremum od $\{a_n\}$, pa postoji $n_0\in\mathbb{N}$ takav da je $a_{n_0}>L-\varepsilon$. Kako je niz (a_n) rastući tada i za sve $n\geq n_0$ vrijedi:

$a_n \in \langle L-\varepsilon, L+\varepsilon \rangle$, tj. ε-okoliš od L sadrži beskonačno mnogo članova niza, dok je izvan tog okoliša konačno mnogo članova niza $(n_0 - 1)$, tj. prema osnovnoj definiciji : $L = \lim_{n\to\infty} a_n$. Q.E.D.

- Analogno se izvodi i druga monotonost.

- **PRIMJER 1.**

↳ Ispitati konvergenciju geometrijskog niza $a_n = q^n$, $q \in \langle 0, 1 \rangle$.

- $(a_n) = (q^n) = 1, q, q^2, \ldots, q^n$.

a) Niz je padajući, dokaz : $a_{n+1} = q^{n+1} = q^n \cdot q = a_n \cdot q < a_n$

$$\Downarrow$$
$$a_{n+1} < a_n \quad \text{Q.E.D.}$$

b) Niz je ograđen odozdo brojem 0, dokaz je očit (grafički)

- Kako smo dokazali da je monoton i omeđen, onda je niz konvergentan i ima limes : $\lim_{n\to\infty} q^n = L$.

Iz rekurzivne formule za opći član niza : $a_{n+1} = a_n \cdot q$ i primjenom limes postupka na lijevu i desnu stranu formule dobije se :

$$\underbrace{\lim_{n\to\infty} (a_{n+1})}_{L} = \lim_{n\to\infty} (a_n \cdot q) = q \cdot \underbrace{\lim_{n\to\infty} a_n}_{L} = q \cdot L$$

$$\Downarrow$$

$$L = L \cdot q$$
$$L - L \cdot q = 0$$
$$L(1-q) = 0$$

1° $L = 0$ $\qquad$ 2° $1 - q = 0$ ↳ jer $q \in \langle 0, 1 \rangle$

- Preostali slučajevi lako se ispitaju iz :

I) $q \in \langle -1, 0 \rangle \rightarrow \lim_{n\to\infty} q^n = \lim_{n\to\infty} |q^n| = \lim_{n\to\infty} |q|^n = 0$ jer je $|q| \in \langle 0, 1 \rangle$

II) $q = 0 \rightarrow \lim_{n\to 0} 0^n = 0$

III) $q = 1 \rightarrow \lim_{n\to 0} q^n = \lim_{n\to\infty} 1 = 1$

IV) $q = -1 \rightarrow \lim_{n\to\infty} q^n = 1, -1, 1, -1, \ldots$ → niz je divergentan

V) $q \in \langle 1, \infty \rangle \rightarrow \lim_{n\to\infty} q^n = \lim_{n\to\infty} \left(\frac{1}{q'}\right)^n = \frac{1}{\lim_{n\to\infty} (q')^n} = \frac{1}{0} = +\infty$ → niz divergira k $+\infty$

- Tablica za geometrijski niz $a_n = q^n$:

$$q = \begin{cases} \langle -\infty, -1] & \ldots \text{ divergira} \\ \langle -1, 1 \rangle & \ldots \text{ konvergira}, L = 0 \\ 1 & \ldots \text{ konvergira}, L = 1 \\ \langle 1, \infty \rangle & \ldots \text{ divergira} \end{cases}$$

- **PRIMJER 2.**

↳ Ispitati konvergenciju niza : $a_n = \dfrac{n^2 \cdot \cos n}{n^3 + 1}$.

- Uzmimo $b_n = |a_n| = \left| \dfrac{n^2 \cdot \cos n}{n^3 + 1} \right| = \dfrac{n^2 \cdot |\cos n|}{n^3 + 1}$.

- Iz sendvič teorema :

$$0 \le \lim_{n\to\infty} \frac{n^2 |\cos n|}{n^3 + 1} \le \lim_{n\to\infty} \frac{n^2}{n^3 + 1} \begin{matrix} :n^3 \\ :n^3 \end{matrix} = \frac{0}{1} = 0$$

Samo je jedan Mali Ivica!

- Pa: $\lim_{n\to\infty}\left|\frac{n^2\cos n}{n^3+1}\right|=0$, iz čega slijedi: $\lim_{n\to\infty}\frac{n^2\cos n}{n^3+1}=0$.

= PRIMJER 3.

↳ Ispitati konvergenciju niza s općim članom: $a_n=\sqrt{2+\sqrt{2+\sqrt{2+\dots}}}$.

- $(a_n)=\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \dots$ Iz toga uočimo rekurzivnu formulu za:

$a_{n+1}=\sqrt{2+a_n}$.

- a) Niz je omeđen odozgo što se dokaže matematičkom indukcijom:
  - I) baza indukcije: $a_1=\sqrt{2}<2$ ✓.
  - II) pretpostavka: $a_n<2$.
  - III) skok: $a_{n+1}=\sqrt{2+a_n}<\sqrt{2+2}=2<\sqrt{2}$ ✓ Q.E.D.

  Dakle $a_n<2$.

  b) Niz je monotono rastući:
  - I) baza indukcije: $a_1=\sqrt{2}$, $a_2=\sqrt{2+\sqrt{2}}$ $\Rightarrow a_2>a_1$
  - II) pretpostavka: $a_n>a_{n-1}$
  - III) skok: $2+a_n>2+a_{n-1}\ /\sqrt{\ }$

    $\sqrt{2+a_n}>\sqrt{2+a_{n-1}}$

    $a_{n+1}>a_n$ ✓ Q.E.D.

- Niz je monoton i omeđen, pa je konvergentan, tj. ima limes: $L=\lim_{n\to\infty}a_n$.

Iz rekurzivne formule: $a_{n+1}=\sqrt{2+a_n}$ primjenom limesa slijedi:

$$\lim_{n\to\infty}a_{n+1}=\lim_{n\to\infty}\sqrt{2+a_n}=\sqrt{\lim_{n\to\infty}(2+a_n)}=\sqrt{2+\lim_{n\to\infty}a_n}\Rightarrow$$

$$L=\sqrt{2+L}\ \Big|^2$$

$$L^2=2+L$$

$$L^2-L-2=0$$

$$L_{1,2}=\frac{1\pm\sqrt{1+8}}{2}=\frac{1\pm3}{2}\ \begin{cases}L_1=2\\ \cancel{L_2=-1}\end{cases}\Rightarrow\lim_{n\to\infty}a_n=2$$

= Ovdje smo primjenili činjenicu da je $\lim_{n\to\infty}\sqrt{2+a_n}=\sqrt{2+\lim_{n\to\infty}a_n}$, što vrijedi i općenitije:

$\lim_{n\to\infty}f(a_n)=f(\lim_{n\to\infty}a_n)$ za neprekinute funkcije f, a takve su sve elementarne funkcije.

7.7. NEKI VAŽNI LIMESI

⇒ 1) $\lim_{n\to\infty}q^n=\begin{cases}0 & \dots\ q\in\langle-1,1\rangle\\ 1 & \dots\ q=1\\ \infty & \dots\ q>1\end{cases}$

Dokaz: Pogledati stranicu 134.

⇒ 2) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$

Dokaz: Niz je rastući i omeđen je odozdo $a_1<3$.
Neodređen oblik: 1^∞.

Samo je jedan Mali Ivica!

⇒ 3) $\lim_{n\to\infty} \frac{a^n}{n!} = 0$, $\forall a \in \mathbb{R}$

Dokaz: Niz je padajući i omeđen je odozdo s nulom.
Neodređen oblik: $\frac{\infty}{\infty}$.

- Uočiti rekurzivnu formulu: $a_{n+1} = a_n \cdot \left(\frac{a}{n+1}\right)$.

= Faktorijele brže teže u beskonačno od bilo koje eksponencijalne funkcije.

⇒ 4) $\lim_{n\to\infty} \frac{n^p}{a^n} = 0$, $\forall p \in \mathbb{Q}^+$, $\forall a > 1$

Dokaz: Niz je padajući i omeđen je odozdo s nulom.
Neodređen oblik: $\frac{\infty}{\infty}$.

- Uočiti rekurzivnu formulu: $a_{n+1} = \frac{\left(1+\frac{1}{n}\right)^p}{a} \cdot a_n$

= Eksponencijalna funkcija brže teži u beskonačno od bilo koje potencije.

⇒ 5) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Dokaz: Koristeći sendvič-teorem.
Neodređeni oblik: ∞^0.

⇒ 6) $\lim_{n\to\infty} \sqrt[n]{a} = 1$, $\forall a > 0$

Dokaz: Koristeći sendvič-teorem.

⇒ 7) $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$

=PRIMJER 1.

$$\lim_{n\to\infty} \frac{3n+1}{n+\sqrt{n^2+1}} \left(\frac{\infty}{\infty}\right) \begin{matrix}:n\\:n\end{matrix} = \lim_{n\to\infty} \frac{3+\frac{1}{n}}{1+\sqrt{\frac{n^2}{n^2}+\frac{1}{n^2}}} = \frac{3+0}{1+1} = \frac{3}{2}$$

=PRIMJER 2.

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n} i}{n^2} = \lim_{n\to\infty} \frac{1+2+\ldots+n}{n^2} = \lim_{n\to\infty} \frac{\frac{n(n+1)}{2}}{n^2} = \lim_{n\to\infty} \frac{n^2+n}{2n^2} \left(\frac{\infty}{\infty}\right) \begin{matrix}:n^2\\:n^2\end{matrix} =$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2} = \frac{1}{2}$$

=PRIMJER 3.

$$\lim_{n\to\infty} \frac{1}{n\left(\sqrt{n^2+1}-n\right)} \left(\frac{1}{\infty(\infty-\infty)}\right) \cdot \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \lim_{n\to\infty} \frac{\sqrt{n^2+1}+n}{n(n^2+1-n^2)} = \lim_{n\to\infty} \frac{\sqrt{n^2+1}+n}{n} \begin{matrix}:n\\:n\end{matrix} =$$

$$= \lim_{n\to\infty} \frac{\sqrt{\frac{n^2}{n^2}+\frac{1}{n^2}}+1}{\frac{n}{n}} = 2$$

=PRIMJER 4.

$$\lim_{n\to\infty} \left(\sqrt[3]{n^3+2n}-2n\right) \left(\infty-\infty\right) = \lim_{n\to\infty} \frac{n^3+2n-(2n)^3}{\left(\sqrt[3]{n^3+2n}\right)^2+\sqrt[3]{n^3+2n}\cdot 2n+(2n)^2} =$$

$$= \lim_{n\to\infty} \frac{n^3+2n-8n^3}{\left(\sqrt[3]{n^3+2n}\right)^2+2n\sqrt[3]{n^3+2n}+4n^2} \begin{matrix}:n^3\\:n^3\end{matrix} = \frac{-7}{0} = -\infty$$

= PRIMJER 5.

$$\lim_{n\to\infty}\left(\frac{3n+3}{3n+2}\right)^n \quad (1^\infty) = \lim_{n\to\infty}\left(\frac{3n+2+1}{3n+2}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{3n+2}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{3n+2}\right)^{\frac{n(3n+2)}{(3n+2)}} =$$

$$= e^{\lim\limits_{n\to\infty}\frac{n}{3n+2}:n} = e^{\frac{1}{3}} = \sqrt[3]{e}$$

= PRIMJER 6.

$$\lim_{n\to\infty}\left(\frac{2n+3}{2n+1}\right)^{3n-2} \quad (1^\infty) = \lim_{n\to\infty}\left(\frac{2n+1+2}{2n+1}\right)^{3n-2} = \lim_{n\to\infty}\left(1+\frac{2}{2n+1}\right)^{3n-2} =$$

$$= \lim_{n\to\infty}\left(1+\frac{2}{2n+1}\right)^{\frac{(3n-2)(2n+1)}{(2n+1)}} = e^{2\cdot\lim\limits_{n\to\infty}\frac{3n-2}{2n+1}\ :n} = e^{2\cdot\frac{3}{2}} = e^3$$

= PRIMJER 7.

$$\lim_{n\to\infty}\left(\frac{2n+3}{3n+2}\right)^{3n+4} \sim \left(\frac{2}{3}\right)^\infty = 0$$

= PRIMJER 8.

$$\lim_{n\to\infty}\left(\frac{3n+2}{2n+3}\right)^{n^2+4} \sim \left(\frac{3}{2}\right)^\infty = \infty$$

= PRIMJER 9.

$$\lim_{n\to\infty}\frac{10^n+8^n+3}{n^{100}+n^{98}+2} = \lim_{n\to\infty}\frac{10^n}{n^{100}} = \infty$$

7.8. EKVIVALENTNE NEIZMJERNO VELIKE VELIČINE

= DEFINICIJA 1.

↳ Kažemo da su a_n i b_n beskonačno velike veličine istog reda ako je:
$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \pm\infty$, $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = c$, $c \neq 0, c \neq \pm\infty$.

U posebnom slučaju a_n i b_n su ekvivalentne beskonačno velike veličine ako vrijedi: $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \pm\infty$, $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = 1$, što zapisujemo $a_n \sim b_n$.

= STAVAK 1.

↳ Ako su a_n i b_n beskonačno velike veličine i (c_n) bilo koja beskonačno velika veličina, tada je: $\lim\limits_{n\to\infty}\frac{a_n}{c_n} = \lim\limits_{n\to\infty}\frac{b_n}{c_n} = \frac{\infty}{\infty} \Rightarrow a_n \sim b_n$, tj. u kvocjentu $\frac{\infty}{\infty}$ neizmjerno velika veličina može se zamijeniti sa sebi ekvivalentnom.

= PRIMJER 1.

$$\lim_{n\to\infty}\frac{5n^3+3n^2+n+1}{4n^3+n^2+2n+3} = \lim_{n\to\infty}\frac{5n^3}{4n^3} = \frac{5}{4}$$

jer je: $(5n^3+3n^2+n+1) \sim (5n^3)$,
$(4n^3+n^2+2n+3) \sim (4n^3)$ kad $n\to\infty$.

= PRIMJER 2.

$$\lim_{n\to\infty}\frac{\sqrt[3]{n^2+n+1}+\sqrt[3]{n+1}}{\sqrt[4]{n^3+n+3}+\sqrt{n}} = \lim_{n\to\infty}\frac{\sqrt[3]{n^2}}{\sqrt[4]{n^3}} = \lim_{n\to\infty}\frac{1}{\sqrt[12]{n}} = 0$$

jer je: $\left(\sqrt[3]{n^2+n+1}+\sqrt[3]{n+1}\right) \sim \left(\sqrt[3]{n^2}\right)$,
$\left(\sqrt[4]{n^3+n+3}+\sqrt{n}\right) \sim \left(\sqrt[4]{n^3}\right)$ kad $n\to\infty$.

Samo je jedan Mali Ivica!

ZADACI (DZ. 7. PRVI DIO)

(1.) I) $a_n = (-1)^{n+1} \frac{n}{2n+5}$ → gomilišta, limes = ?

II) $b_n = (-1)^{n+1} \frac{n}{2n^2+5}$

I) $\lim\limits_{n\to\infty} \frac{n}{2n+5} \begin{matrix} :n \\ :n \end{matrix} = \frac{1}{2}$, ako je $n=2k \ldots \frac{1}{2}$
, ako je $n=2k+1 \ldots -\frac{1}{2}$

↳ niz nema limes jer ima dva gomilišta $A_1 = \frac{1}{2}$ i $A_2 = -\frac{1}{2}$.

II $\lim\limits_{n\to\infty} \frac{n}{2n^2+5} \begin{matrix} :n^2 \\ :n^2 \end{matrix} = \frac{0}{2} = 0$, ako je $n=2k \ldots 0$
, ako je $n=2k+1 \ldots 0$

↳ niz ima limes jer ima jedno gomilište $A_1 = 0$, $\lim\limits_{n\to\infty} \frac{n}{n^2+5} = 0$.

(2.) a) $\lim\limits_{n\to\infty} \frac{2n^2+1}{3n^2+3n+4} \begin{matrix} :n^2 \\ :n^2 \end{matrix} = \lim\limits_{n\to\infty} \frac{2+\frac{1}{n^2}}{3+\frac{3}{n}+\frac{4}{n^2}} = \frac{2}{3}$

b) $\lim\limits_{n\to\infty} \frac{2n+1}{n^2+4} \begin{matrix} :n^2 \\ :n^2 \end{matrix} = \lim\limits_{n\to\infty} \frac{\frac{2}{n}+\frac{1}{n^2}}{1+\frac{4}{n^2}} = \frac{0}{1} = 0.$

c) $\lim\limits_{n\to\infty} \left(n - \frac{n^2+2n}{n+5}\right) = \lim\limits_{n\to\infty} \left(\frac{\cancel{n^2}+5n-\cancel{n^2}-2n}{n+5}\right) = \lim\limits_{n\to\infty} \frac{3n}{n+5} \begin{matrix} :n \\ :n \end{matrix} = \frac{3}{1} = 3$

(3.) a) $\lim\limits_{n\to\infty} \frac{4n^3+n}{n^4+1} \begin{matrix} :n^4 \\ :n^4 \end{matrix} = \lim\limits_{n\to\infty} \frac{\frac{4}{n}+\frac{1}{n^3}}{1+\frac{1}{n^4}} = \frac{0}{1} = 0$

b) $\lim\limits_{n\to\infty} \frac{n^2+1}{2n^2+3n+4} \begin{matrix} :n^2 \\ :n^2 \end{matrix} = \lim\limits_{n\to\infty} \frac{1+\frac{1}{n^2}}{2+\frac{3}{n}+\frac{4}{n^2}} = \frac{1}{2}$

c) $\lim\limits_{n\to\infty} \left(n^2 - \frac{n^3+n^2}{n+4}\right) = \lim\limits_{n\to\infty} \left(\frac{\cancel{n^3}+4n^2-\cancel{n^3}-n^2}{n+4}\right) = \lim\limits_{n\to\infty} \frac{3n^2}{n+4} \begin{matrix} :n^2 \\ :n^2 \end{matrix} = \lim\limits_{n\to\infty} \frac{3}{\frac{1}{n}+\frac{4}{n^2}} =$

$= \frac{3}{0} = +\infty$

(4.) a) $\lim\limits_{n\to\infty} \frac{3n^2+4}{2n^3+3} \begin{matrix} :n^3 \\ :n^3 \end{matrix} = \lim\limits_{n\to\infty} \frac{\frac{3}{n}+\frac{4}{n^3}}{2+\frac{3}{n^3}} = \frac{0}{2} = 0$

b) $\lim\limits_{n\to\infty} \left(n - \frac{n^2+3n}{n+4}\right) = \lim\limits_{n\to\infty} \left(\frac{\cancel{n^2}+4n-\cancel{n^2}-3n}{n+4}\right) = \lim\limits_{n\to\infty} \frac{n}{n+4} \begin{matrix} :n \\ :n \end{matrix} = \frac{1}{1} = 1$

c) $\lim\limits_{n\to\infty} \frac{\sqrt[3]{n}+1}{\sqrt[4]{n}+4} \begin{matrix} :\sqrt[3]{n} \\ :\sqrt[3]{n} \end{matrix} = \lim\limits_{n\to\infty} \frac{1+\frac{1}{\sqrt[3]{n}}}{\frac{1}{\sqrt[12]{n}}+\frac{4}{\sqrt[3]{n}}} = \frac{1}{0} = +\infty$

(5.) a) $\lim\limits_{n\to\infty} \frac{5n^3+3n^2+3n+5}{3n^3+5n^2+5n+3} \begin{matrix} :n^3 \\ :n^3 \end{matrix} = \frac{5}{3}$

b) $\lim\limits_{n\to\infty} \frac{2n+1}{\sqrt{n^3+1}} \begin{matrix} :\sqrt{n^3} \\ :\sqrt{n^3} \end{matrix} = \lim\limits_{n\to\infty} \frac{\frac{2}{\sqrt{n}}+\frac{1}{\sqrt{n^3}}}{\sqrt{1+\frac{1}{n^3}}} = \frac{0}{1} = 0$

c) $\lim_{n\to\infty}\left(n-\frac{(n-1)^5}{(n+1)^4}\right)=\lim_{n\to\infty}\left(\frac{n(n+1)^4-(n-1)^5}{(n+1)^4}\right)=\lim_{n\to\infty}\frac{\cancel{n^5}+4n^4+6n^3+4n^2+n-\cancel{n^5}+5n^4-10n^3+10n^2-5n+1}{n^4+4n^3+6n^2+4n+1}$

$=\lim_{n\to\infty}\frac{9n^4}{n^4}\begin{array}{l}:n^4\\:n^4\end{array}=\frac{9}{1}=9$

(6.) $\lim_{n\to\infty}\left(n-\frac{(n-a)^3}{(n+1)^2}\right)=\lim_{n\to\infty}\left(\frac{n(n+1)^2-(n-a)^3}{(n+1)^2}\right)=\lim_{n\to\infty}\left(\frac{n(n^2+2n+1)-(n^3-3n^2a+3na^2-a^3)}{n^2+2n+1}\right)=$

$=\lim_{n\to\infty}\frac{\cancel{n^3}+2n^2+n-\cancel{n^3}+3n^2a-3na^2+a^3}{n^2+2n+1}=\lim_{n\to\infty}\frac{2n^2+3n^2a+n-3na^2+a^3}{n^2+2n+1}\begin{array}{l}:n^2\\:n^2\end{array}=$

$=\lim_{n\to\infty}\frac{2+3a+\frac{1}{n}-\frac{3a^2}{n}+\frac{a^3}{n^2}}{1+\frac{2}{n}+\frac{1}{n^2}}=\frac{2+3a}{1}=2+3a$

(7.) a) Dokazati: $\lim_{n\to\infty}\frac{\sin n}{n}=0$.

$\hookrightarrow (\forall\varepsilon>0)\,(\exists n_0\in\mathbb{N})\,(\forall n\geq n_0)\left(\left|\frac{\sin n}{n}-0\right|<\varepsilon\right)$

$\left|\frac{\sin n}{n}\right|\leq\frac{1}{n}\quad,\quad n_0\geq\frac{1}{\varepsilon}$

$\left|\frac{\sin n}{n}\right|\leq\frac{1}{n}\leq\frac{1}{n_0}<\varepsilon$ w Q.E.D.

b) $\lim_{n\to\infty}\frac{n\cdot\sin n}{n^2+n+1}\begin{array}{l}:n^2\\:n^2\end{array}=\lim_{n\to\infty}\frac{\frac{\sin n}{n}}{1}=\frac{0}{1}=0$

(8.) a) $\lim_{n\to\infty}\frac{3^n+6^n}{2^n+7^n}\begin{array}{l}:7^n\\:7^n\end{array}=\lim_{n\to\infty}\frac{\left(\frac{3}{7}\right)^n+\left(\frac{6}{7}\right)^n}{\left(\frac{2}{7}\right)^n+1}=\frac{0}{1}=0$

b) $\lim_{n\to\infty}\frac{2^{n+1}-1}{2^{n-1}+1}=\lim_{n\to\infty}\frac{2^n\cdot2-1}{2^n\cdot2^{-1}+1}\begin{array}{l}:2^n\\:2^n\end{array}=\lim_{n\to\infty}\frac{2-\frac{1}{2^n}}{2^{-1}+\frac{1}{2^n}}=\frac{2}{\frac{1}{2}}=4$

(9.) a) $\lim_{n\to\infty}\left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^n}\right)$

$\hookrightarrow a_1=1\ ,\ a_2=\frac{1}{3}\ ,\ a_3=\frac{1}{9}\ \Rightarrow$ g-niz $q=\frac{1}{3}$

$S_q=a_1\frac{q^n-1}{q-1}=\frac{\frac{1}{3}^n-1}{\frac{1}{3}-1}=\frac{3^{-n}-1}{-\frac{2}{3}}=\frac{-3(3^{-n}-1)}{2}=\frac{3(1-3^{-n})}{2}$

$\hookrightarrow$ za g-niz gleda se (n+1) član:

$\lim_{n\to\infty}\frac{3-3\cdot3^{-n-1}}{2}=\frac{3}{2}-\lim_{n\to\infty}\frac{\cancel{3}\cdot3^{-n}\cdot\cancel{3^{-1}}}{2}=\frac{3}{2}-\lim_{n\to\infty}\frac{\frac{1}{3^n}}{2}=\frac{3}{2}-\lim_{n\to\infty}\frac{1}{2\cdot3^n}=$

$=\frac{3}{2}-\lim_{n\to\infty}\left(\frac{1}{2}\cdot\left(\frac{1}{3}\right)^n\right)=\frac{3}{2}-0=\frac{3}{2}$

b) $\lim_{n\to\infty}\frac{1+2^n}{1+2+2^2+\ldots+2^n}$

$\hookrightarrow a_1=1,\ a_2=2\ ,\ a_3=4\ ,\ a_4=8\ \Rightarrow$ g-niz $q=2$

$S_q=a_1\cdot\frac{q^n-1}{q-1}=1\cdot\frac{2^n-1}{2-1}=2^n-1$

↳ za g-niz gleda se (n+1) član:

$$\lim_{n\to\infty} \frac{1+2^n}{2^{n+1}-1} = \lim_{n\to\infty} \frac{1+2^n}{2^n \cdot 2 - 1} \begin{array}{l} :2^n \\ :2^n \end{array} = \lim_{n\to\infty} \frac{\left(\frac{1}{2}\right)^n + 1}{2 - \left(\frac{1}{2}\right)^n} = \frac{1}{2}$$

10. a) $\lim_{n\to\infty} \left(1 - \frac{2}{5} + \frac{4}{25} + \ldots + \left(-\frac{2}{5}\right)^n\right)$

↳ $a_1 = 1$, $a_2 = -\frac{2}{5}$, $a_3 = \frac{4}{25}$ $\Rightarrow$ g-niz $q = -\frac{2}{5}$

$$S_q = a_1 \cdot \frac{q^n - 1}{q - 1} = \frac{\left(-\frac{2}{5}\right)^n - 1}{-\frac{2}{5} - 1} = \frac{\frac{(-2)^n - 5^n}{5^n}}{-\frac{7}{5}} = \frac{5\left((-2)^n - 5^n\right)}{-7 \cdot 5^n} = \frac{5^n \cdot 5 - (-2)^n \cdot 5}{-7 \cdot 5^n}$$

↳ za g-niz gleda se (n+1) član:

$$\lim_{n\to\infty} \frac{5^{n+1} \cdot 5 - (-2)^{n+1} \cdot 5}{7 \cdot 5^{n+1}} = \lim_{n\to\infty} \frac{5^n \cdot 5 \cdot \cancel{5} - (-2)^n \cdot (-2) \cdot \cancel{5}}{7 \cdot 5^n \cdot \cancel{5}} = \lim_{n\to\infty} \frac{5^n \cdot 5 + 2 \cdot (-2)^n}{7 \cdot 5^n} \begin{array}{l} :5^n \\ :5^n \end{array} =$$

$$= \lim_{n\to\infty} \frac{5 + 2 \cdot \left(\frac{-2}{5}\right)^n}{7} = \frac{5}{7}$$

b) $\lim_{n\to\infty} \frac{1 + 10^n}{1 + 10 + 10^2 + \ldots + 10^n}$

↳ $a_1 = 1$, $a_2 = 10$, $a_3 = 100$ $\Rightarrow$ g-niz $q = 10$

$$S_q = a_1 \cdot \frac{q^n - 1}{q - 1} = \frac{10^n - 1}{10 - 1} = \frac{10^n - 1}{9}$$

↳ za g-niz gleda se (n+1) član:

$$\lim_{n\to\infty} \frac{1 + 10^n}{\frac{10^{n+1} - 1}{9}} = \lim_{n\to\infty} \frac{9 + 9 \cdot 10^n}{10^n \cdot 10 - 1} \begin{array}{l} :10^n \\ :10^n \end{array} = \lim_{n\to\infty} \frac{\left(\frac{9}{10}\right)^n + 9}{10 - \left(\frac{1}{10}\right)^n} = \frac{9}{10}$$

11. a) $\lim_{n\to\infty} \frac{1 + 2 + 3 + \ldots + 2n}{n^2 + 1}$

↳ $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ $\Rightarrow$ a-niz $d = 1$

$$S_n = \frac{2n}{2}(a_1 + a_n) = \frac{2n}{2}(1 + 2n) = n(1 + 2n)$$

$$\lim_{n\to\infty} \frac{n(1 + 2n)}{(n^2 + 1)} = \lim_{n\to\infty} \frac{n + 2n^2}{n^2 + 1} \begin{array}{l} :n^2 \\ :n^2 \end{array} = \lim_{n\to\infty} \frac{\frac{1}{n} + 2}{1 + \frac{1}{n^2}} = 2$$

b) $\lim_{n\to\infty} \frac{1 + 3 + 5 + \ldots + (2n - 1)}{n^2 + 1}$

↳ $a_1 = 1$, $a_2 = 3$, $a_3 = 5$ $\Rightarrow$ a-niz $d = 2$

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(1 + 2n - 1) = \frac{n}{2} \cdot 2n = \frac{2n^2}{2} = n^2$$

$$\lim_{n\to\infty} \frac{n^2}{n^2 + 1} \begin{array}{l} :n^2 \\ :n^2 \end{array} = 1$$

12. a) $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n+1} = \lim_{n\to\infty} \left(\frac{n + 1 - 1}{n + 1}\right)^{n+1} = \lim_{n\to\infty} \left(1 + \frac{-1}{n+1}\right)^{n+1} = e^{-1} = \frac{1}{e}$

b) $\lim_{n\to\infty} \left(\frac{2n+1}{2n+5}\right)^n = \lim_{n\to\infty} \left(\frac{2n + 1 + 4 - 4}{2n + 5}\right)^n = \lim_{n\to\infty} \left(1 + \frac{-4}{2n+5}\right)^{\frac{n(2n+5)}{2n+5}} = e^{-4 \cdot \lim_{n\to\infty} \frac{n}{2n+5}} =$

$= e^{-4 \cdot \frac{1}{2}} = e^{-2} = \frac{1}{e^2}$

(13.) a) $\lim\limits_{n\to\infty}\left(\frac{2n}{2n+1}\right)^{2n+1}=\lim\limits_{n\to\infty}\left(\frac{2n+1-1}{2n+1}\right)^{2n+1}=\lim\limits_{n\to\infty}\left(1+\frac{-1}{2n+1}\right)^{2n+1}=e^{-1}=\frac{1}{e}$

b) $\lim\limits_{n\to\infty}\left(\frac{n^2}{n^2+1}\right)^{n^2}=\lim\limits_{n\to\infty}\left(\frac{n^2+1-1}{n^2+1}\right)^{n^2}=\lim\limits_{n\to\infty}\left(1+\frac{-1}{n^2+1}\right)^{n^2}=\lim\limits_{n\to\infty}\left(1+\frac{-1}{n^2+1}\right)^{\frac{n^2(n^2+1)}{n^2+1}}=$

$=e^{-1\lim\limits_{n\to\infty}\frac{n^2}{n^2+1}}=e^{-1}=\frac{1}{e}$

(14.) a) $\lim\limits_{n\to\infty}\left(\frac{3n-1}{3n+2}\right)^{n}=\lim\limits_{n\to\infty}\left(\frac{3n-1+2-2}{3n+2}\right)^{n}=\lim\limits_{n\to\infty}\left(1+\frac{-3}{3n+2}\right)^{n}=\lim\limits_{n\to\infty}\left(1+\frac{-3}{3n+2}\right)^{\frac{n(3n+2)}{3n+2}}=$

$=e^{-3\lim\limits_{n\to\infty}\frac{n}{3n+2}}=e^{-3\cdot\frac{1}{3}}=e^{-1}=\frac{1}{e}$

b) $\lim\limits_{n\to\infty}\left(\frac{n+2}{n+3}\right)^{n+4}=\lim\limits_{n\to\infty}\left(\frac{n+2+1-1}{n+3}\right)^{n+4}=\lim\limits_{n\to\infty}\left(1+\frac{-1}{n+3}\right)^{n+4}=\lim\limits_{n\to\infty}\left(1+\frac{-1}{n+3}\right)^{\frac{(n+4)(n+3)}{(n+3)}}$

$=e^{-1\cdot\lim\limits_{n\to\infty}\frac{n+4}{n+3}}=e^{-1}=\frac{1}{e}$

(15.) a) $\lim\limits_{n\to\infty}\frac{n+1}{n+\sqrt{n^2+n+1}}\begin{matrix}:n\\:n\end{matrix}=\lim\limits_{n\to\infty}\frac{1+\frac{1}{n}}{1+\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}}=\frac{1}{2}$

b) $\lim\limits_{n\to\infty}\frac{n+1}{\sqrt{n^4+n^3+n^2+n+1}-n^2}\cdot\frac{\sqrt{n^4+n^3+n^2+n+1}+n^2}{\sqrt{n^4+n^3+n^2+n+1}+n^2}=\lim\limits_{n\to\infty}\frac{(n+1)\left(n^2+\sqrt{n^4+n^3+n^2+n+1}\right)}{\cancel{n^4}+n^3+n^2+n+1-\cancel{n^4}}=$

$=\lim\limits_{n\to\infty}\frac{n^3+n^2+n\sqrt{n^4+n^3+n^2+n+1}+\sqrt{n^4+n^3+n^2+n+1}}{n^3+n^2+n+1}=\lim\limits_{n\to\infty}\frac{n^3+n^2+\sqrt{n^6+n^5+n^4+n^3+n^2}+\sqrt{n^4+n^3+n^2+n+1}}{n^3+n^2+n+1}$

$=\lim\limits_{n\to\infty}\frac{1+\frac{1}{n}+\sqrt{1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}+\frac{1}{n^4}}+\sqrt{\frac{1}{n^2}+\frac{1}{n^3}+\frac{1}{n^4}+\frac{1}{n^5}+\frac{1}{n^6}}}{1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}}=\frac{2}{1}=2$

(16.) $\lim\limits_{n\to\infty}\left(\sqrt{n+\sqrt{n+\sqrt{n}}}-\sqrt{n}\right)\cdot\frac{\sqrt{n+\sqrt{n+\sqrt{n}}}+\sqrt{n}}{\sqrt{n+\sqrt{n+\sqrt{n}}}+\sqrt{n}}=\lim\limits_{n\to\infty}\frac{\cancel{n}+\sqrt{n+\sqrt{n}}-\cancel{n}}{\sqrt{n+\sqrt{n+\sqrt{n}}}+\sqrt{n}}\begin{matrix}:\sqrt{n}\\:\sqrt{n}\end{matrix}=$

$=\lim\limits_{n\to\infty}\frac{\sqrt{\frac{n+\sqrt{n}}{n}}}{\sqrt{\frac{n+\sqrt{n+\sqrt{n}}}{n}}+1}=\lim\limits_{n\to\infty}\frac{\sqrt{1+\frac{1}{\sqrt{n}}}}{\sqrt{1+\sqrt{\frac{1}{n}+\sqrt{\frac{1}{n^3}}}}+1}=\frac{1}{1+1}=\frac{1}{2}$

(17.) $\lim\limits_{n\to\infty}\left(\sqrt[4]{n^4+n^3+n^2+n+1}-n\right)=\lim\limits_{n\to\infty}\frac{\cancel{n^4}+n^3+n^2+n+1-\cancel{n^4}}{\left(\sqrt[4]{n^4+n^3+n^2+n+1}+n\right)\left(\sqrt{n^4+n^3+n^2+n+1}+n^2\right)}=$

$\lim\limits_{n\to\infty}\frac{n^3+n^2+n+1}{\left(\sqrt[4]{n^4+n^3+n^2+n+1}\right)^3+\sqrt[4]{n^{12}\cdot n^{11}\cdot n^{10}\cdot n^9+n^8}+\sqrt{n^6\cdot n^5+n^4+n^3+n^2}+n^3}$

$\lim\limits_{n\to\infty}\frac{n^3}{\left(\sqrt[4]{n^4}\right)^3+\sqrt[4]{n^{12}}+\sqrt{n^6}+n^3}\begin{matrix}:n^3\\:n^3\end{matrix}=\lim\limits_{n\to\infty}\frac{1}{1+1+1+1}=\frac{1}{4}$

(18) $a_n=\sqrt{3\cdot\sqrt{3\cdot\ldots\cdot\sqrt{3}}}$. Dokaži konvergenciju i nađi limes.

↳ $a_1=\sqrt{3}$, $a_2=\sqrt{3\sqrt{3}}$ } rekurzivna formula $=$ $a_{n+1}=\sqrt{3\cdot a_n}$

↳ Da bi niz bio konvergentan mora biti monoton i omeđen.

a) Pretpostavimo da je niz rastući i dokažimo to matematičkom indukcijom.

I) baza: $a_2>a_1$

$\sqrt{3\cdot\sqrt{3}}>\sqrt{3}$ ✓

II) pretpostavka: $a_n>a_{n-1}$

III) skok: $a_{n+1}>a_n$

$\sqrt{3\cdot a_n}>\sqrt{3\cdot a_{n-1}} \;/^2$

$3\cdot a_n>3\cdot a_{n-1} \;/:3$

$a_n>a_{n-1} \Rightarrow a_{n+1}>a_n$ ✓ Q.E.D.

b) Pretpostavimo da je niz omeđen odozgo sa 3 i dokažimo to indukcijom.

I) baza: $a_1<3$

$\sqrt{3}<3$ ✓

II) pretpostavka: $a_n<3$

III) skok: $a_{n+1}<3$

$\sqrt{3\cdot a_n}<3 \;/^2$

$3\cdot a_n<9 \;/:3$

$a_n<3$ ✓ Q.E.D.

Niz a_n je konvergentan i ima limes $L=\lim\limits_{n\to\infty}a_n$.

$a_{n+1}=\sqrt{3\cdot a_n} \;/\lim\limits_{n\to\infty}$

$$\underbrace{\lim_{n\to\infty}a_{n+1}}_{L}=\lim_{n\to\infty}\sqrt{3\cdot a_n}=\sqrt{\lim_{n\to\infty}(3a_n)}=\sqrt{3\cdot\underbrace{\lim_{n\to\infty}a_n}_{L}}$$

$$\Downarrow$$

$L=\sqrt{3\cdot L} \;/^2$

$L^2=3L$

$L^2-3L=0$

$L(L-3)=0$

$\cancel{L_1=0}$ $L_2=3$ ✓ $\lim\limits_{n\to\infty}a_n=3$

(19) $a_n=\sqrt{6+\sqrt{6+\ldots+\sqrt{6}}}$. Dokaži konvergenciju i nađi limes.

↳ $a_1=\sqrt{6}$, $a_2=\sqrt{6+\sqrt{6}}$ } rekurzivna formula $=$ $a_{n+1}=\sqrt{6+a_n}$

↳ Da bi niz bio konvergentan mora biti monoton i omeđen.

a) Pretpostavimo da je niz rastući i dokažimo to indukcijom!

I) baza: $a_2>a_1$

$\sqrt{6+\sqrt{6}}>\sqrt{6}$ ✓

II) pretpostavka: $a_n>a_{n-1}$

III) skok: $a_{n+1}>a_n$

$\sqrt{6+a_n}>\sqrt{6+a_{n-1}} \;/^2$

$\cancel{6}+a_n>\cancel{6}+a_{n-1}$

$a_n>a_{n-1} \Rightarrow a_{n+1}>a_n$ ✓ Q.E.D.

b) Pretpostavimo da je niz omeđen odozgo sa 3 i dokažimo to indukcijom:

I) baza: $a_1<3$

$\sqrt{6}<3$ ✓

II) pretpostavka: $a_n<3$

III) skok: $a_{n+1}<3$

$\sqrt{6+a_n}<3 \;/^2$

$6+a_n<9$

$a_n<3$ ✓ Q.E.D.

Niz a_n je konvergentan i ima limes $L=\lim_{n\to\infty} a_n$.

$$a_{n+1}=\sqrt{6+a_n} \quad \Big/ \lim_{n\to\infty}$$

$$\underbrace{\lim_{n\to\infty} a_{n+1}}_{L} = \lim_{n\to\infty}\sqrt{6+a_n} = \sqrt{\lim_{n\to\infty}(6+a_n)} = \sqrt{6+\underbrace{\lim_{n\to\infty} a_n}_{L}}$$

$$\Downarrow$$

$$L=\sqrt{6+L} \quad /^2$$
$$L^2=6+L$$
$$L^2-L-6=0$$
$$L_{1,2}=\frac{1\pm\sqrt{1+24}}{2}=\frac{1\pm 5}{2} \quad \begin{cases} L_1=3 \\ \cancel{L_2=-2}\end{cases} \quad \lim_{n\to\infty} a_n=3$$

20. $a_1=1$ Dokaži konvergentnost i nađi limes.

$$a_{n+1}=3-\frac{1}{a_n+3}$$

↳ Da bi niz bio konvergentan mora biti monoton i omeđen.

a) Pretpostavimo da je niz rastući i dokažimo to indukcijom.

I) baza: $a_2>a_1$

$$\frac{11}{4}>1 \checkmark$$

II) pretpostavka: $a_{n-1}<a_n$

III) skok: $a_n<a_{n+1}$

$$\cancel{3}-\frac{1}{a_{n-1}+3}<\cancel{3}-\frac{1}{a_n+3} \quad /\cdot(-1)$$

$$a_n+\cancel{3}>a_{n-1}+\cancel{3}$$

$$a_n>a_{n-1} \Rightarrow a_{n+1}>a_n \checkmark \text{ Q.E.D.}$$

b) Pretpostavimo da je niz omeđen odozgo i da je gornja međa 3, što je očito i trivijalno i ne treba se dokazivati.

Niz je konvergentan pa ima limes $\lim_{n\to\infty} a_n=L$.

$$a_{n+1}=3-\frac{1}{a_n+3}$$

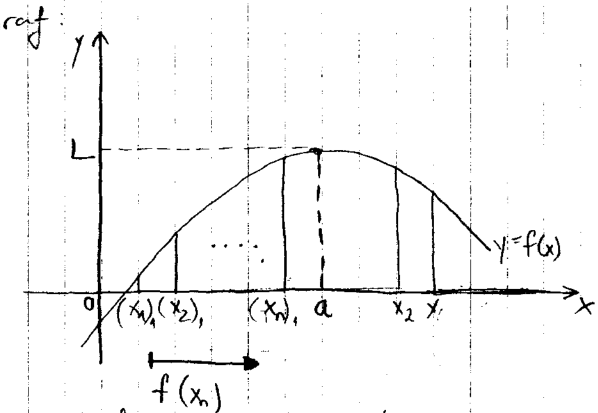
$$L=3-\frac{1}{L+3}=\frac{3L+9-1}{L+3}=\frac{3L+8}{L+3}$$

$$L^2+\cancel{3L}-\cancel{3L}-8=0$$
$$L^2-8=0$$
$$L^2=8$$
$$L_1=2\sqrt{2} \quad \cancel{L_2=-2\sqrt{2}} \qquad \lim_{n\to\infty} a_n=2\sqrt{2}$$

8. LIMES FUNKCIJE. NEPREKINUTE FUNKCIJE.

8.1. LIMES FUNKCIJE

= Promotrimo graf:

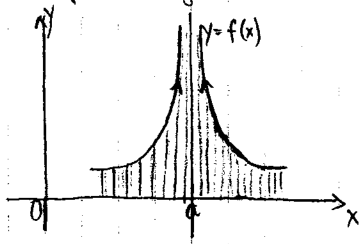


↳ funkcija ima konačni limes u konačnosti

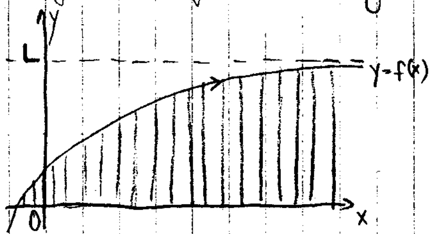
= DEFINICIJA 1.

↳ Broj L je limes funkcije $y=f(x)$ u točki $x=a$ onda i samo onda ako za svaki niz (x_n), $\{x_n\} \subseteq D(f) \setminus \{a\}$, s $\lim\limits_{n\to\infty} x_n = a$.
Sada vrijedi: $L = \lim\limits_{n\to\infty} f(x_n)$, $\forall x_n$, što označujemo $\lim\limits_{x\to a} f(x) = L$.

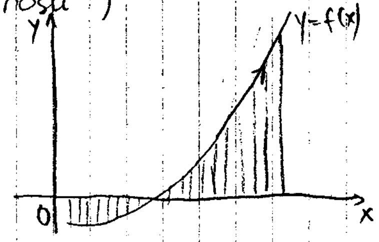
↳ Ova se definicija proširuje i na slučaj $L = \pm\infty$ („beskonačni limes u konačnosti")



↳ Ova se definicija proširuje i na slučaj $a = \pm\infty$, $L \in \mathbb{R}$ („konačni limes u beskonačnosti")



↳ Ova se definicija proširuje i na slučaj $a = \pm\infty$, $L = \pm\infty$ („beskonačni limes u beskonačnosti")



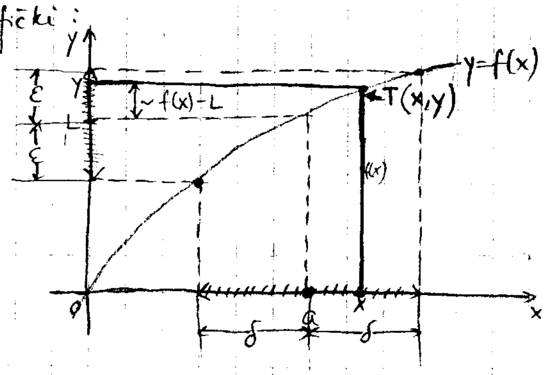
= Za praktično računanje ponekad je umjesto definicije 1 bolje koristiti njenu epsilonističku ekvivalenciju.

= STAVAK 1.

↳ Realan broj L je limes funkcije $y=f(x)$ u točki $x=a$, oznakom $L = \lim\limits_{x\to a} f(x)$ ako vrijedi:

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in D(f) \setminus \{a\})\left[|x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon\right]$$

↳ grafički :



- x - zadovoljava $|x-a|<\delta$
- $\lim\limits_{x\to a} f(x) \neq L$

= STAVAK 2.

↳ Ako funkcija ima limes onda je on jednoznačno određen.

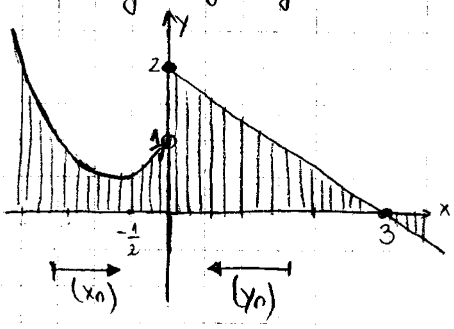
= PRIMJER 1.

↳ Funkcija $f(x)$ dana je izrazom:

$$f(x)=\begin{cases} x^2+x+1 & \dots\ x<0 \\ -x+3 & \dots\ x\geq 0 \end{cases}$$

. Ispitati limes u točki $x=0$.

- Nacrtajmo funkciju :



- Funkcija nema limesa u točki $x=0$, jer za rastući niz $\{x_n\}$, $x_n \to 0$ je: $\lim f(x_n)=1$, a za padajući niz $\{y_n\}$, $y_n \to 0$ je: $\lim f(y_n)=2$.
- Obzirom da je $L_1 \neq L_2$, funkcija nema limesa u točki $x=0$.

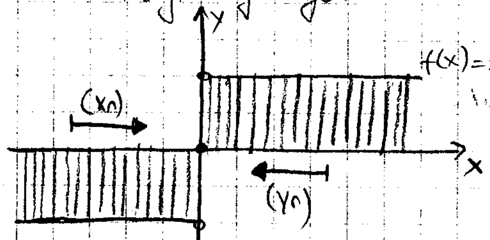
= PRIMJER 2.

↳ Funkcija $f(x)$ dana je izrazom:

$$f(x)=\begin{cases} -1 & \dots\ x<0 \\ 0 & \dots\ x=0 \\ 1 & \dots\ x>0 \end{cases}$$

. Ispitati limes u točki $x=0$.

- Nacrtajmo funkciju :



$f(x)=\operatorname{sgn} x$

- Funkcija nema limesa u točki $x=0$, jer za rastući niz $\{x_n\}$, $x_n \to 0$ je: $\lim f(x_n)=-1$, a za padajući niz $\{y_n\}$, $y_n \to 0$ je: $\lim f(y_n)=1$.
- Obzirom da je $L_1 \neq L_2$, funkcija nema limesa u točki $x=0$.

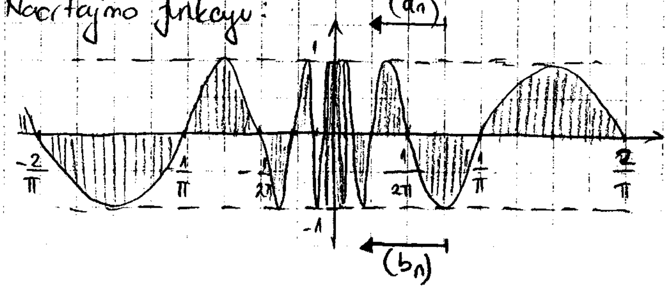
= PRIMJER 3

↳ Funkcija $f(x)$ dana je izrazom:

$$f(x)=\begin{cases} \sin\frac{1}{x} & \dots\ x\neq 0 \\ 0 & \dots\ x=0 \end{cases}$$

- Ispitati limes u točki $x=0$.

- Nacrtajmo funkciju :



$A=1$

nultočke: $\frac{1}{x}=k\pi$

$x=\frac{1}{k\pi}$

- $a_n = \frac{1}{\frac{\pi}{2} + 2n\pi} \longrightarrow f(a_n) = \sin\left(\frac{\pi}{2} + 2n\pi\right) = 1 \longrightarrow f(a_n) = 1, 1, 1, \dots, 1 \longrightarrow 1$

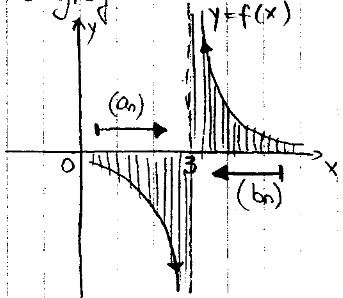
$b_n = \frac{1}{\frac{3\pi}{2} + 2n\pi} \longrightarrow f(b_n) = \sin\left(\frac{3\pi}{2} + 2n\pi\right) = -1 \longrightarrow f(b_n) = -1, -1, \dots, -1 \longrightarrow -1$

- Kako postoje 2 gomilišta (-1) i (1) funkcija nema limes u točki $x=0$.

= PRIMJER 4.

↳ Ispitaj da li funkcija $f(x) = \frac{1}{x-3}$, ima limes u točki $x=3$.

- Nacrtajmo graf:

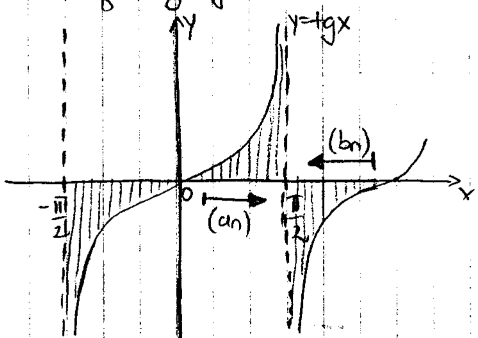


- $\left.\begin{array}{l} f(a_n) \to -\infty \\ f(b_n) \to +\infty \end{array}\right\}$ - Kako postoje 2 gomilišta $(-\infty)$ i $(+\infty)$ funkcija nema limes u točki $x=3$.

= PRIMJER 5.

↳ Ispitaj da li funkcija $f(x) = \operatorname{tg} x$, ima limes u točki $x = \frac{\pi}{2}$.

- Nacrtajmo graf:



- $\left.\begin{array}{l} f(a_n) \to +\infty \\ f(b_n) \to -\infty \end{array}\right\}$ - Kako postoje 2 gomilišta $(-\infty)$ i $(+\infty)$ funkcija nema limes u točki $x = \frac{\pi}{2}$.

8.2. RAČUNANJE LIMESA

= STAVAK 1.

↳ Neka su f i g realne funkcije s istim domenama $(\mathcal{D}(f) = \mathcal{D}(g))$ i neka je $\lim_{x\to x_0} f(x) = a$, $\lim_{x\to x_0} g(x) = b$, $a, b \in \mathbb{R}$. Onda vrijedi:

1) $\lim_{x\to x_0} [f(x) \pm g(x)] = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x) = a \pm b$

2) $\lim_{x\to x_0} [f(x) \cdot g(x)] = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x) = a \cdot b$

3) $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{a}{b}$, za $\lim_{x\to x_0} g(x) = b \neq 0$

4) $\lim_{x\to x_0} \left[f(x)^{g(x)}\right] = \left[\lim_{x\to x_0} f(x)\right]^{\lim_{x\to x_0} g(x)} = a^b$

Dokaz:

Sva ova pravila su s obzirom na definiciju direktna posljedica za limese nizova.

Uz odgovarajuće "računanje" s $(+\infty)$ i $(-\infty)$ tvrdnje stavka vrijede i u slučajima $a, b \in \mathbb{R}$ kad izrazi na desnim stranama nisu neodređeni.

= Neodređeni oblici:

1) $\frac{\infty}{\infty}$ ⇒ brojnik i nazivnik dijelimo s najvećom potencijom

2) $\frac{0}{0}$ ⇒ brojnik i nazivnik skratimo

3) $\infty - \infty$ ⇒ primjena algebarskih operacija kojima se funkcija transformira u odgovarajuću racionalnu funkciju

4) 1^{∞} ⇒ potrebno upotrijebiti transformaciju: $f(a)^{g(a)} = e^{\ln(f(a)^{g(a)})} = e^{g(a)\cdot\ln f(a)}$

5) 0^0

6) ∞^0

= Osnovni limesi:

1) $\lim\limits_{x\to\infty} \frac{1}{x^n} = 0$

2) $\lim\limits_{x\to 0} x^n = 0$

3) $\lim\limits_{x\to\pm\infty} \left(1+\frac{1}{x}\right)^x = e$

4) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

5) $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x} = 1$

6) $\lim\limits_{x\to 0} \frac{e^x-1}{x} = 1$

7) $\lim\limits_{x\to 0^-} \operatorname{cth} x = -\infty$

8) $\lim\limits_{x\to 0^+} \operatorname{cth} x = +\infty$

9) $\lim\limits_{x\to 0^-} e^{\frac{1}{x}} = 0$

10) $\lim\limits_{x\to 0^+} e^{\frac{1}{x}} = +\infty$

11) $\lim\limits_{x\to 1^-} \operatorname{arcth} x = -\infty$

12) $\lim\limits_{x\to 1^+} \operatorname{arcth} x = +\infty$

13) $\lim\limits_{x\to 0} \frac{\operatorname{tg} x}{x} = 1$

14) $\lim\limits_{x\to 0} \frac{\arcsin x}{x} = 1$

15) $\lim\limits_{x\to\infty} \frac{\operatorname{arctg} x}{x} = 1$

= PRIMJER 1.

a) $\lim\limits_{x\to 0} \frac{3\sin x}{\sin 2x} \left(\frac{0}{0}\right) \begin{matrix} :2x \\ :2x \end{matrix} = \lim\limits_{x\to 0} \frac{\frac{3\sin x}{2x}}{\frac{\sin 2x}{2x}} = \frac{3}{2}$

ili

$$\lim_{x\to 0} \frac{3\sin x}{\sin 2x} = \lim_{x\to 0} \frac{3\sin x}{2\sin x\cos x} = \frac{3}{2}\lim_{x\to 0}\frac{\sin x}{\sin x}\cdot\lim_{x\to 0}\frac{1}{\cos x} = \frac{3}{2\cos 0} = \frac{3}{2}$$

b) $\lim\limits_{x\to 0}\left(\frac{2x}{2x+3}\right)^{4x} = \lim\limits_{x\to 0}\left(\frac{2x+3-3}{2x+3}\right)^{4x} = \lim\limits_{x\to 0}\left(1+\frac{-3}{2x+3}\right)^{4x} = \lim\limits_{x\to 0}\left(1+\frac{-3}{2x+3}\right)^{\frac{4x(2x+3)}{2x+3}} =$

$= e^{-3\cdot\lim\limits_{x\to 0}\frac{4x}{2x+3}} = e^{-6}$

c) $\lim\limits_{x\to 0}\frac{\sin 2x}{x^3}\left(\frac{0}{0}\right) \begin{matrix} :2x \\ :2x \end{matrix} = \lim\limits_{x\to 0}\frac{\frac{\sin 2x}{2x}}{\frac{x^3}{2x}} = \lim\limits_{x\to 0}\frac{2}{x^2} = \frac{2}{0} = +\infty$

= STAVAK 2.

↳ Računanje limesa supstitucijom:

Neka postoji limes funkcije $y=f(x)$ u točki $x=x_0$, u oznaci $L=\lim\limits_{x\to x_0} f(x)$. Neka je $f(D(f)) \subseteq D(g)$ te neka funkcija $y=g(x)$ ima limes u L, tako da je $\lim\limits_{y\to L} g(y) = g(L)$. Onda vrijedi:

$$\lim_{x\to x_0}(g\circ f)(x) = \lim_{y\to L} g(y).$$

=PRIMJER 2.

a) $\lim\limits_{x\to 1} \frac{1-\sqrt{x}}{1-\sqrt[3]{x}} \left(\frac{0}{0}\right) = \left| \begin{array}{l} \text{SUPST: } x=t^6 \\ x\to 1 \Leftrightarrow t\to 1 \end{array} \right| = \lim\limits_{t\to 1} \frac{1-t^3}{1-t^2} = \lim\limits_{t\to 1} \frac{\cancel{(1-t)}(1+t+t^2)}{\cancel{(1-t)}(1+t)} = \frac{3}{2}$

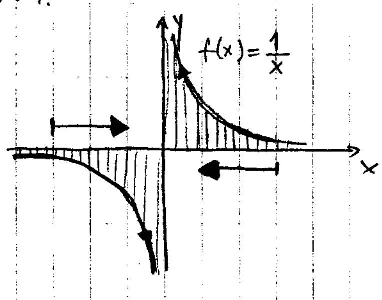
b) $\lim\limits_{x\to 0} \frac{e^x-1}{x} \left(\frac{0}{0}\right) = \left| \begin{array}{l} \text{SUPST: } e^x-1=t \\ e^x=t+1 \\ x=\ln(t+1) \\ x\to 0 \Leftrightarrow t\to e^0-1\to 0 \end{array} \right| = \lim\limits_{t\to 0} \frac{t}{\ln(t+1)} = \lim\limits_{t\to 0} \frac{1}{\frac{\ln(1+t)}{t}} =$

$= \lim\limits_{t\to 0} \frac{1}{\ln(1+t)^{\frac{1}{t}}} = \frac{1}{\ln\left(\lim\limits_{t\to 0}(1+t)^{\frac{1}{t}}\right)} = \left| \begin{array}{l} \text{SUPST: } \frac{1}{t}=z \\ t=\frac{1}{z} \\ t\to 0 \Leftrightarrow z\to \pm\infty \end{array} \right| = \frac{1}{\ln\left(\lim\limits_{z\to\infty}\left(1+\frac{1}{z}\right)^z\right)} =$

$= \frac{1}{\ln e} = 1$

8.3. JEDNOSTRANI LIMESI

=PRIMJER 1.



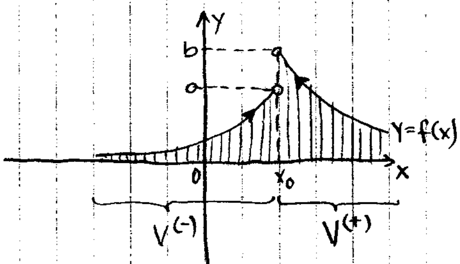
- Ova se funkcija ponaša različito za pozitivne x i za negativne x u okolišu $x_0=0$. To zapisujemo:

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

=DEFINICIJA 1.

Neka je f realna funkcija i $x_0 \in \mathbb{R}$. Neka je k tomu: $V^{(-)} = (-\infty, x_0) \cap \mathcal{D}(f)$
$V^{(+)} = (x_0, +\infty) \cap \mathcal{D}(f)$



- Kažemo da je a lijevi limes funkcije $f(x)$ kad $x\to x_0$ i to zapisujemo ovako:

$$\lim_{x\to x_0^-} f(x) = a,$$

ako je: $\lim\limits_{x\to x_0} \left(f|_{V^{(-)}}\right) = a.$

- Kažemo da je b desni limes funkcije $f(x)$ kad $x\to x_0$, i to zapisujemo ovako:

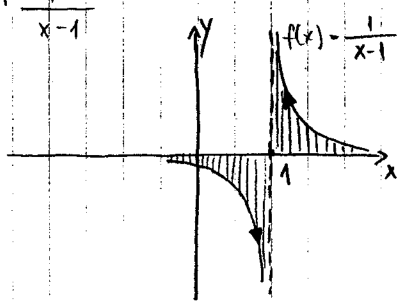
$$\lim_{x\to x_0^+} f(x) = b,$$

ako je: $\lim\limits_{x\to x_0} \left(f|_{V^{(+)}}\right) = b.$

=PRIMJER 2.

Izračunajte lijevi i desni limes funkcije u točki:

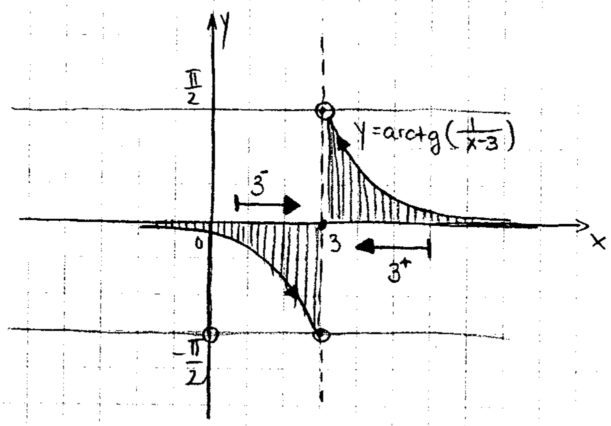
a) $\lim\limits_{x\to 1\pm} \frac{1}{x-1}$



$$\lim_{x\to 1^-} \frac{1}{x-1} = \frac{1}{(1-0)-1} = \frac{1}{-0} = -\infty$$

$$\lim_{x\to 1^+} \frac{1}{x-1} = \frac{1}{(1+0)-1} = \frac{1}{+0} = +\infty$$

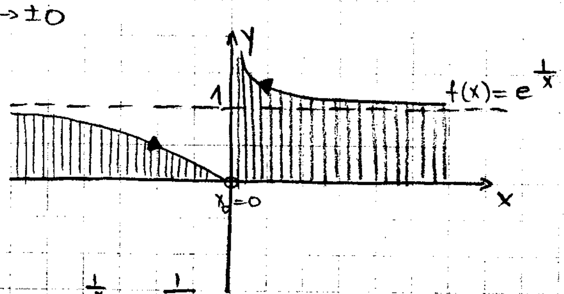
b) $\lim\limits_{x \to 3^{\pm}} \operatorname{arctg}\left(\frac{1}{x-3}\right)$



$$\lim_{x \to 3^-} \operatorname{arctg}\left(\frac{1}{x-3}\right) = \operatorname{arctg}\frac{1}{3-0-3} = \operatorname{arctg}\frac{1}{-0} = \operatorname{arctg}(-\infty) = -\frac{\pi}{2}$$

$$\lim_{x \to 3^+} \operatorname{arctg}\left(\frac{1}{x-3}\right) = \operatorname{arctg}\frac{1}{3+0-3} = \operatorname{arctg}\frac{1}{+0} = \operatorname{arctg}(+\infty) = \frac{\pi}{2}$$

c) $\lim\limits_{x \to \pm 0} e^{\frac{1}{x}}$



$$\lim_{x \to 0^-} e^{\frac{1}{x}} = e^{\frac{1}{-0}} = e^{-\infty} = 0$$

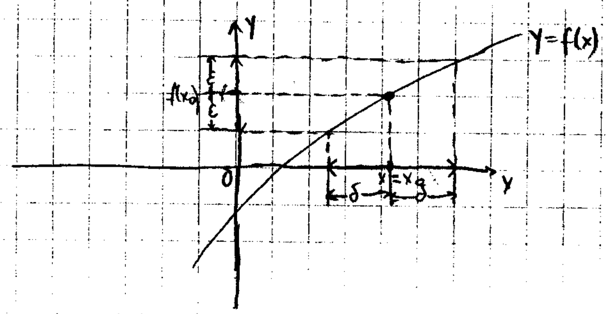
$$\lim_{x \to 0^+} e^{\frac{1}{x}} = e^{\frac{1}{+0}} = e^{+\infty} = +\infty$$

=STAVAK 1.

↳Ako postoje lijevi i desni limes funkcije $f(x)$ kod $x \to x_0$, postoji $\exists \lim\limits_{x \to x_0^+} f(x)$, $\exists \lim\limits_{x \to x_0^-} f(x)$, i ako su međusobno jednaki, onda postoji i limes funkcije u točki $x=x_0$ koji je jednak tim limesima:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$$

8.4. NEPREKINUTE FUNKCIJE I LIMESI



= DEFINICIJA 1.

↳ Funkcija $y=f(x)$ neprekinuta je u točki $x_0=x$, $x_0 \in \mathbb{R}$, ako vrijedi:

$$(\forall \varepsilon>0)(\exists \delta>0)(\forall x \in \mathcal{D}(f))\left[|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon\right]$$

↳ Ako funkcija nije neprekinuta u točki $x=x_0$ kažemo da u točki $x=x_0$ ima prekid.

↳ Funkcija $f(x)$ je neprekinuta na skupu S ako je neprekinuta u svakoj točki skupa S.

= STAVAK 1.

↳ Neka postoji limes funkcije $y=f(x)$ u točki $x=x_0$, te neka je $x_0 \in \mathcal{D}(f)$, i $f(x_0)=\lim\limits_{x\to x_0} f(x)$. Onda je $f(x)$ neprekinuta funkcija u točki $x=x_0$.

= STAVAK 2.

↳ Ako su $f(x)$ i $g(x)$ neprekinute funkcije u x_0, onda su u x_0 neprekinute i funkcije:
a) $f(x) \pm g(x)$
b) $f(x)\cdot g(x)$
c) $\dfrac{f(x)}{g(x)}$, za $g(x_0)\neq 0$.

= PRIMJER 1.

a) Sve elementarne funkcije su neprekinute funkcije u svakoj točki svoje domene.

b) Ako je funkcija f neprekinuta funkcija i $x_0 \in \mathcal{D}(f)$, onda je:
$$\lim_{x\to x_0} f(x) = f\left(\lim_{x\to x_0} x\right)$$

c) $f(x)=\begin{cases} x\cdot\sin\frac{1}{x} & \ldots\ x\neq 0 \\ 0 & \ldots\ x=0\end{cases}$. Dokazati da je f neprekidna na $\mathbb{R}$.

I) $x\neq 0$ ... funkcija je neprekinuta jer su neprekinute i funkcije: $\frac{1}{x}$, $\sin\frac{1}{x}$, $x\cdot\sin\frac{1}{x}$

II) $x=0$ .... $\lim\limits_{x\to 0}\left(x\cdot\sin\frac{1}{x}\right)=?$

↳ Sendvič teoremom:

$$0 \leq \left|x\cdot\sin\frac{1}{x}\right| = |x|\cdot\underbrace{\left|\sin\frac{1}{x}\right|}_{\leq 1} \leq |x|$$

Za $x\to 0$:

$$0 \leq \lim_{x\to 0}\left|x\cdot\sin\frac{1}{x}\right| \leq \lim_{x\to 0}|x| = 0$$

$$\Downarrow$$

$$\lim_{x\to 0}\left(x\cdot\sin\frac{1}{x}\right)=0$$

- funkcija f je neprekinuta u točki $x=0$ jer je $f(0)=0=\lim\limits_{x\to 0} f(x)$.

d) $f(x)=\dfrac{x+e^{\frac{1}{x}}+1}{x-e^{\frac{1}{x}}-1}$. Koliko treba biti $f(0)$ da bi funkcija bila neprekinuta u točki $x=0$.

↳ $f(0)=\lim\limits_{x\to 0^\pm} f(x)=?$

I) lijevi limes: $\lim\limits_{x\to 0^-}\dfrac{x+e^{\frac{1}{x}}+1}{x-e^{\frac{1}{x}}-1}=\dfrac{0+e^{-\infty}+1}{0-e^{-\infty}-1}=-1$

II desni limes: $\lim\limits_{x\to 0^+}\dfrac{x+e^{\frac{1}{x}}+1}{x-e^{\frac{1}{x}}-1}=\lim\limits_{x\to 0^+}\dfrac{x\cdot e^{-\frac{1}{x}}+1+e^{-\frac{1}{x}}}{x\cdot e^{-\frac{1}{x}}-1-e^{-\frac{1}{x}}}=$
$=\dfrac{0\cdot e^{-\infty}+1+e^{-\infty}}{0\cdot e^{-\infty}-1-e^{-\infty}}=\dfrac{0+1+0}{0-1-0}=-1$

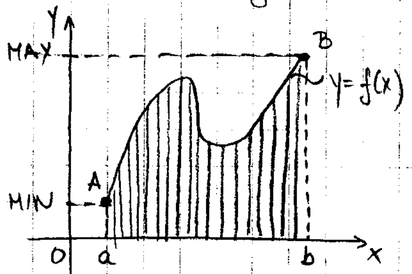
↳ Treba definirati da je $f(0)=-1$.

8.5. SVOJSTVA NEPREKINUTIH FUNKCIJA

= STAVAK 1.

↳ Weierstrass-ov teorem:

Funkcija f koja je neprekinuta na zatvorenom intervalu $I=[a,b]$, na tom je intervalu i omeđena i prima na njemu minimalnu vrijednost $\min_I f$ i maksimalnu vrijednost $\max_I f$.

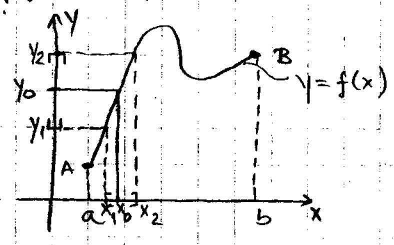


= STAVAK 2.

↳ Stavak o međuvrijednosti:

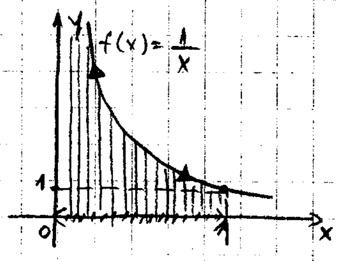
Ako je za x_1 i $x_2 \in I=[a,b]$, tako da je $f(x_1)=y_1 < f(x_2)=y_2$. Onda za svaku vrijednost y_0, $y_1<y_0<y_2$, postoji $x_0 \in [x_1, x_2]$, tako da je $f(x_0)=y_0$.

- grafički:



= PRIMJEDBA 1.

↳ Ovi stavci općenito ne vrijede na područjima koja nisu zatvoreni intervali, npr. $y=\frac{1}{x}$, na $\langle 0,1\rangle$ jest neprekinuta funkcija, ali ne poprima niti minimum, niti maksimum.



ZADACI

1\. a) $\lim\limits_{x\to-\infty}\frac{2x^2+1}{3x^2+3x+4} = \lim\limits_{x\to\infty}\frac{2x^2+1}{3x^2+3x+4}\frac{:x^2}{:x^2} = \lim\limits_{x\to\infty}\frac{2+\frac{1}{x^2}}{3+\frac{3}{x}+\frac{4}{x^2}} = \frac{2}{3}$

b) $\lim\limits_{x\to+\infty}\frac{2x^2+1}{3x^2+3x+4}\frac{:x^2}{:x^2} = \lim\limits_{x\to+\infty}\frac{2+\frac{1}{x^2}}{3+\frac{3}{x}+\frac{4}{x^2}} = \frac{2}{3}$

c) $\lim\limits_{x\to-\infty}\frac{2x^3+1}{3x^2+3x+4} = \lim\limits_{x\to\infty}\frac{-2x^3+1}{3x^2-3x+4}\frac{:x^3}{:x^3} = \lim\limits_{x\to\infty}\frac{-2+\frac{1}{x^3}}{\frac{3}{x}-\frac{3}{x^2}+\frac{4}{x^3}} = \frac{-2}{0} = -\infty$

d) $\lim\limits_{x\to+\infty}\frac{2x^3+1}{3x^2+3x+4}\frac{:x^3}{:x^3} = \lim\limits_{x\to+\infty}\frac{2+\frac{1}{x^3}}{\frac{3}{x}+\frac{3}{x^2}+\frac{4}{x^3}} = \frac{2+0}{0+0+0} = \frac{2}{0} = +\infty$

2. a) $\lim\limits_{x\to-\infty} \dfrac{2x+1}{\sqrt{4x^2+3}} = \lim\limits_{x\to\infty} \dfrac{-2x+1}{\sqrt{4x^2+3}} \begin{matrix}:x\\:x\end{matrix} = \lim\limits_{x\to\infty} \dfrac{-2+\frac{1}{x}}{\sqrt{4+\frac{3}{x^2}}} = \dfrac{-2}{2} = -1$

b) $\lim\limits_{x\to+\infty} \dfrac{2x+1}{\sqrt{4x^2+3}} \begin{matrix}:x\\:x\end{matrix} = \lim\limits_{x\to+\infty} \dfrac{2+\frac{1}{x}}{\sqrt{4+\frac{3}{x^2}}} = \dfrac{2}{2} = 1$

3. a) $\lim\limits_{x\to-\infty} \dfrac{2x+1}{\sqrt[3]{x^3+x^2+x+1}} = \lim\limits_{x\to+\infty} \dfrac{-2x+1}{\sqrt[3]{-x^3+x^2-x+1}} \begin{matrix}:x\\:x\end{matrix} = \lim\limits_{x\to+\infty} \dfrac{-2+\frac{1}{x}}{\sqrt[3]{-1+\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}}} = \dfrac{-2}{-1} = 2$

b) $\lim\limits_{x\to+\infty} \dfrac{2x+1}{\sqrt[3]{x^3+x^2+x+1}} \begin{matrix}:x\\:x\end{matrix} = \lim\limits_{x\to+\infty} \dfrac{2+\frac{1}{x}}{\sqrt[3]{1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}}} = \dfrac{2}{1} = 2$

4. a) $\lim\limits_{x\to-\infty} \dfrac{2x+1}{x+\sqrt{4x^2+3}} = \lim\limits_{x\to\infty} \dfrac{-2x+1}{-x+\sqrt{4x^2+3}} \begin{matrix}:x\\:x\end{matrix} = \lim\limits_{x\to\infty} \dfrac{-2+\frac{1}{x}}{-1+\sqrt{4+\frac{3}{x^2}}} = \dfrac{-2}{-1+2} = -2$

b) $\lim\limits_{x\to\infty} \dfrac{2x+1}{x+\sqrt{4x^2+3}} \begin{matrix}:x\\:x\end{matrix} = \lim\limits_{x\to\infty} \dfrac{2+\frac{1}{x}}{1+\sqrt{4+\frac{3}{x}}} = \dfrac{2}{1+2} = \dfrac{2}{3}$

5. a) $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^4+x^3+1}-x^2}{x+1} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x^4-x^3+1}-x^2}{-x+1} \cdot \dfrac{\sqrt{x^4-x^3+1}+x^2}{\sqrt{x^4-x^3+1}+x^2} = \lim\limits_{x\to\infty} \dfrac{\cancel{x^4}-x^3+1-\cancel{x^4}}{(-x+1)(x^2+\sqrt{x^4-x^3+1})} =$

$= \lim\limits_{x\to\infty} \dfrac{-x^3+1}{-x^3-\sqrt{x^6-x^5-x^2}+x^2+\sqrt{x^4-x^3+1}} \begin{matrix}:x^3\\:x^3\end{matrix} = \lim\limits_{x\to\infty} \dfrac{-1+\frac{1}{x^3}}{-1-\sqrt{1-\frac{1}{x}-\frac{1}{x^4}}+\frac{1}{x}+\sqrt{\frac{1}{x^2}-\frac{1}{x^3}+\frac{1}{x^6}}} =$

$= \dfrac{-1}{-2} = \dfrac{1}{2}$

b) $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^4+x^3+1}-x^2}{x+1} \cdot \dfrac{\sqrt{x^4+x^3+1}+x^2}{\sqrt{x^4+x^3+1}+x^2} = \lim\limits_{x\to\infty} \dfrac{x^4+x^3+1-\cancel{x^4}}{(x+1)(x^2+\sqrt{x^4+x^3+1})} =$

$= \lim\limits_{x\to\infty} \dfrac{x^3+1}{x^3+\sqrt{x^4+x^3+1}+x^2+\sqrt{x^6+x^5+x^2}} \begin{matrix}:x^3\\:x^3\end{matrix} = \lim\limits_{x\to\infty} \dfrac{1+\frac{1}{x^3}}{1+\sqrt{\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^6}}+\frac{1}{x}+\sqrt{1+\frac{1}{x}+\frac{1}{x^4}}} =$

$= \dfrac{1}{2}$

6. $\lim\limits_{x\to\infty} \left(\sqrt[3]{x^3+3x^2+3x+3}-x\right) = \lim\limits_{x\to\infty} \dfrac{\cancel{x^3}+3x^2+3x+3-\cancel{x^3}}{\sqrt[3]{x^3+3x^2+3x+3}^{\,2}+x\sqrt[3]{x^3+3x^2+3x+3}+x^2} =$

$= \lim\limits_{x\to\infty} \dfrac{3x^2+3x+3}{\sqrt[3]{x^3+3x^2+3x+3}^{\,2}+\sqrt[3]{x^6+3x^5+3x^4+3x^3}+x^2} \begin{matrix}:x^2\\:x^2\end{matrix} =$

$= \lim\limits_{x\to\infty} \dfrac{3+\frac{3}{x}+\frac{3}{x^2}}{\sqrt[3]{1+\frac{3}{x}+\frac{3}{x^2}+\frac{3}{x^3}}^{\,2}+\sqrt{1+\frac{3}{x}+\frac{3}{x^2}+\frac{3}{x^3}}+1} = \dfrac{3}{3} = 1$

7. $\lim\limits_{x\to-\infty} \left(\sqrt[4]{x^4+x^3+x^2+x+1}+x\right) = \lim\limits_{x\to\infty} \left(\sqrt[4]{x^4-x^3+x^2-x+1}-x\right) =$

$= \lim\limits_{x\to\infty} \dfrac{\cancel{x^4}-x^3+x^2-x+1-\cancel{x^4}}{\left(\sqrt[4]{x^4-x^3+x^2-x+1}+x\right)\left(\sqrt{x^4-x^3+x^2-x+1}+x^2\right)}$

$= \lim\limits_{x\to\infty} \dfrac{-x^3+x^2-x+1}{\sqrt[4]{x^4-x^3+x^2-x+1}^{\,3}+\sqrt[4]{x^{12}-x^{11}+x^{10}-x^9+x^8}+\sqrt{x^6-x^5+x^4-x^3+x^2}+x^3} \begin{matrix}:x^3\\:x^3\end{matrix} =$

$$= \lim_{x\to\infty} \frac{-1+\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}}{\sqrt{1-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}+\frac{1}{x^4}}+\sqrt{1-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}+\frac{1}{x^4}}+\sqrt{1-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}+\frac{1}{x^4}}+1} =$$

$$= \frac{-1}{1+1+1+1} = -\frac{1}{4}$$

(8.) a) $\lim_{x\to -2^-} \frac{x+1}{x+2} = \frac{-2-0+1}{-2-0+2} = \frac{-1}{-0} = +\infty$

b) $\lim_{x\to -2^+} \frac{x+1}{x+2} = \frac{-2+0+1}{-2+0+2} = \frac{-1}{+0} = -\infty$

c) $\lim_{x\to -2^-} \frac{x+1}{(x+2)^2} = \lim_{x\to -2^-} \frac{x+1}{x^2+4x+4} = \frac{-2+1}{+0} = \frac{-1}{+0} = -\infty$

d) $\lim_{x\to -2^+} \frac{x+1}{(x+2)^2} = \lim_{x\to -2^+} \frac{x+1}{x^2+4x+4} = \frac{-2+1}{+0} = \frac{-1}{+0} = -\infty$

(9.) a) $\lim_{x\to 1^-} \frac{1}{x^2-3x+2} = \frac{1}{(1-0)^2-3(1-0)+2} = \frac{1}{1-\cancel{0}+\cancel{0}-3+0+2} = \frac{1}{+0} = +\infty$

b) $\lim_{x\to 1^+} \frac{1}{x^2-3x+2} = \frac{1}{(1+0)^2-3(1+0)+2} = \frac{1}{1+0+\cancel{0}-3-0+2} = \frac{1}{+0} = +\infty$

(10.) a) $\lim_{x\to 2^-} \frac{x^2-2x}{x^2-4} = \lim_{x\to 2^-} \frac{x\cancel{(x-2)}}{\cancel{(x-2)}(x+2)} = \lim_{x\to 2^-} \frac{x}{x+2} = \frac{2-0}{2-0+2} = \frac{2}{4} = \frac{1}{2}$

b) $\lim_{x\to 2^+} \frac{x^2-2x}{x^2-4} = \lim_{x\to 2^+} \frac{x\cancel{(x-2)}}{\cancel{(x-2)}(x+2)} = \lim_{x\to 2^+} \frac{x}{x+2} = \frac{2+0}{2+0+2} = \frac{2}{4} = \frac{1}{2}$

(11.) a) $\lim_{x\to 1^-} \frac{x^2-3x+2}{x^2-2x+1} = \lim_{x\to 1^-} \frac{\cancel{(x-1)}(x-2)}{(x-1)^{\cancel{2}}} = \lim_{x\to 1^-} \frac{x-2}{x-1} = \frac{1-0-2}{1-0-1} = \frac{-1}{-0} = +\infty$

b) $\lim_{x\to 1^+} \frac{x^2-3x+2}{x^2-2x+1} = \lim_{x\to 1^+} \frac{\cancel{(x-1)}(x-2)}{(x-1)^{\cancel{2}}} = \lim_{x\to 1^+} \frac{x-2}{x-1} = \frac{1+0-2}{1+0-1} = \frac{-1}{+0} = -\infty$

(12.) a) $\lim_{x\to -2^-} x\cdot e^{\frac{x+1}{x+2}} = \lim_{x\to -2^-} x \cdot \lim_{x\to -2^-} e^{\frac{x+1}{x+2}} = -2\cdot e^{\frac{-2-0+1}{-2-0+2}} = -2\cdot e^{\frac{-1}{-0}} = -2\cdot e^{\infty} = -2\cdot\infty = +\infty$

b) $\lim_{x\to -2^+} x\cdot e^{\frac{x+1}{x+2}} = \lim_{x\to -2^+} x \cdot \lim_{x\to -2^+} e^{\frac{x+1}{x+2}} = -2\cdot e^{\frac{-2+0+1}{-2+0+2}} = -2\cdot e^{\frac{-1}{+0}} = -2\cdot e^{-\infty} = -2\cdot \frac{1}{e^{\infty}} = 0$

(13.) a) $\lim_{x\to 2^-} \operatorname{arctg}\left(\frac{x+1}{x-2}\right) = \operatorname{arctg}\frac{2-0+1}{2-0-2} = \operatorname{arctg}\frac{3}{-0} = \operatorname{arctg}(-\infty) = -\frac{\pi}{2}$

b) $\lim_{x\to 2^+} \operatorname{arctg}\left(\frac{x+1}{x-2}\right) = \operatorname{arctg}\frac{2+0+1}{2+0-2} = \operatorname{arctg}\frac{3}{+0} = \operatorname{arctg}(+\infty) = \frac{\pi}{2}$

(14.) a) $\lim_{x\to 0^-} \operatorname{cth}(x^2-x) = \lim_{x\to 0^-} \operatorname{cth}(x(x-1)) = \operatorname{cth}((0-0)(0-0-1)) = \operatorname{cth}(-0\cdot(-1)) = \operatorname{cth}(+0) = +\infty$

b) $\lim_{x\to 0^+} \operatorname{cth}(x^2-x) = \lim_{x\to 0^+} \operatorname{cth}(x(x-1)) = \operatorname{cth}((0+0)(0+0-1)) = \operatorname{cth}(+0(-1)) = \operatorname{cth}(-0) = -\infty$

15. a) $\lim\limits_{x\to\infty}\left(\frac{x}{x+3}\right)^x=\lim\limits_{x\to\infty}\left(\frac{x+3-3}{x+3}\right)^x=\lim\limits_{x\to\infty}\left(1+\frac{-3}{x+3}\right)^x=\lim\limits_{x\to\infty}\left(1+\frac{-3}{x+3}\right)^{\frac{x(x+3)}{x+3}}=$

$=e^{-3\cdot\lim\limits_{x\to\infty}\frac{x}{x+3}}=e^{-3\cdot 1}=e^{-3}=\frac{1}{e^3}$

b) $\lim\limits_{x\to\infty}\left(\frac{2x+1}{2x+5}\right)^{2x+3}=\lim\limits_{x\to\infty}\left(\frac{2x+5-4}{2x+5}\right)^{2x+3}=\lim\limits_{x\to\infty}\left(1+\frac{-4}{2x+5}\right)^{2x+3}=$

$=\lim\limits_{x\to\infty}\left(1+\frac{-4}{2x+5}\right)^{\frac{(2x+3)(2x+5)}{2x+5}}=e^{-4\lim\limits_{x\to\infty}\frac{2x+3}{2x+5}}=e^{-4\cdot 1}=\frac{1}{e^4}$

16. a) $\lim\limits_{x\to 0}\frac{x}{\ln(2x+1)}=\lim\limits_{x\to 0}\frac{1}{\frac{\ln(2x+1)\cdot 2}{x\cdot 2}}=\frac{1}{\lim\limits_{x\to 0}\frac{2\cdot\ln(2x+1)}{2x}}=\frac{1}{2\cdot\lim\limits_{x\to 0}\frac{\ln(2x+1)}{2x}}=\frac{1}{2}.$

b) $\lim\limits_{x\to 0}\frac{x}{e^{3x}-1}=\lim\limits_{x\to 0}\frac{1}{\frac{e^{3x}-1\cdot 3}{x\cdot 3}}=\frac{1}{\lim\limits_{x\to 0}\frac{3(e^{3x}-1)}{3x}}=\frac{1}{3\cdot\lim\limits_{x\to 0}\frac{e^{3x}-1}{3x}}=\frac{1}{3}$

17. a) $\lim\limits_{x\to 0}\frac{\arcsin(2x)\ :2x\ :3x}{\arcsin(3x)\ :2x\ :3x}=\lim\limits_{x\to 0}\frac{\frac{\arcsin 2x}{2x}\cdot\frac{1}{3x}}{\frac{\arcsin 3x}{3x}\cdot\frac{1}{2x}}=\lim\limits_{x\to 0}\frac{2x}{3x}=\frac{2}{3}$

b) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=\lim\limits_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\lim\limits_{x\to 0}\frac{2\sin\frac{x}{2}\cdot\sin\frac{x}{2}}{\frac{x}{2}\cdot\frac{x}{2}\cdot 4}=\frac{1}{2}$

18. a) $\lim\limits_{x\to\infty}\frac{x^2+2x}{x+1}\operatorname{arctg}\left(\frac{1}{x}\right)=\lim\limits_{x\to\infty}\frac{x(x+2)}{x+1}\cdot\operatorname{arctg}\left(\frac{1}{x}\right)=\lim\limits_{x\to\infty}\left(\frac{x+2}{x+1}\cdot\frac{\operatorname{arctg}\frac{1}{x}}{\frac{1}{x}}\right)=$

$=\lim\limits_{x\to\infty}\frac{x+2}{x+1}\cdot\lim\limits_{x\to\infty}\frac{\operatorname{arctg}\frac{1}{x}}{\frac{1}{x}}=1\cdot 1=1$

b) $\lim\limits_{x\to\infty}\frac{x\sqrt{x}}{x+2}\ln\left(\frac{x}{x+1}\right)=\lim\limits_{x\to\infty}\left(\frac{x\sqrt{x}}{x+2}\cdot\left(-\ln\frac{x+1}{x}\right)\right)=\lim\limits_{x\to\infty}\left(\frac{-x\sqrt{x}}{x+2}\cdot\ln\left(1+\frac{1}{x}\right)\right)=$

$=\lim\limits_{x\to\infty}\left(\frac{-\sqrt{x}}{x+2}\cdot\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\right)=\lim\limits_{x\to\infty}\frac{-\sqrt{x}\ :x}{x+2\ :x}\cdot 1=\lim\limits_{x\to\infty}\frac{-\frac{1}{\sqrt{x}}}{1+\frac{2}{\sqrt{x}}}=\frac{0}{1}\cdot 1=0$

19. $a=?$ – da funkcija bude neprekinuta

$$\begin{cases} f(x)=a\cdot\operatorname{th}\left(\frac{x}{x-1}\right), & \text{ako } x>1\\ f(x)=x^2+x, & \text{ako } x\le 1\end{cases}$$

→ uvjet: $\lim\limits_{x\to 1^+}f(x)=\lim\limits_{x\to 1^-}f(x)=f(1)$

$$\begin{cases} \text{I} \ \lim\limits_{x\to 1^+}f(x)=\lim\limits_{x\to 1^+}a\operatorname{th}\left(\frac{x}{x-1}\right)=a\cdot\lim\limits_{x\to 1^+}\operatorname{th}\left(\frac{x}{x-1}\right)=a\cdot\operatorname{th}\left(\frac{1^+}{0^+}\right)=a\cdot\operatorname{th}(+\infty)=a\cdot 1=a\\ \text{II} \ f(1)=1+1=2\end{cases}$$

I i II $\Rightarrow a=2$

20. $a,b=?$ – da funkcija bude neprekinuta

$f(x)=\frac{3e^{\frac{2}{x}}+2\cdot e^{\frac{1}{x}}+a}{b\cdot e^{\frac{2}{x}}+e^{\frac{1}{x}}+2}$ za $x=0$ i $f(0)=1$

→ uvjet: $\lim\limits_{x\to 0^+}f(x)=\lim\limits_{x\to 0^-}f(x)=f(0)=1$

$$\lim_{x\to 0^+} f(x) = \left(\frac{\infty}{\infty}\right) = \lim_{x\to 0^+} \frac{3+2\cdot e^{-\frac{1}{x}}+a\cdot e^{-\frac{2}{x}}}{5+e^{-\frac{1}{x}}+2\cdot e^{-\frac{2}{x}}} = \frac{3+2\cdot 0+a\cdot 0}{5+0+2\cdot 0} = \frac{3}{b}$$

$$\lim_{x\to 0^-} f(x) = \frac{a}{2}$$

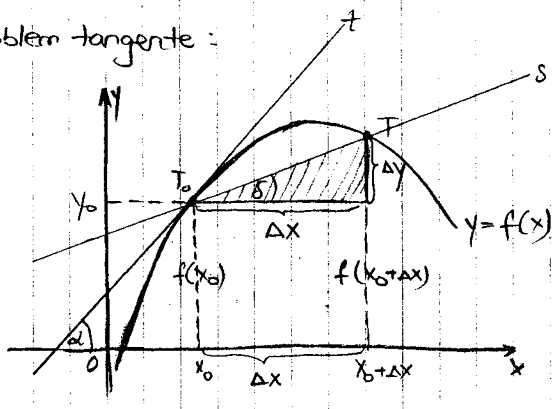
$\frac{3}{b} = f(0) = 1 \quad \Rightarrow b = 3$

$\frac{a}{2} = f(0) = 1 \quad \Rightarrow a = 2$

9. DIFERENCIJALNI RAČUN I

9.1. TANGENTA NA KRIVULJU

= Problem tangente :



t - tangenta
s - sekanta
Δx - prirast argumenta
Δy - prirast funkcije $(\Delta y = f(x_0+\Delta x) - f(x_0))$
$\overline{T_0T}$ - tetiva

↳ sa slike: $\operatorname{tg}\delta = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ $\Rightarrow k_s$, tj. koeficijent smjera sekante

$\operatorname{tg}\alpha = \lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x} = \lim\limits_{x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \to k_t$, tj. koeficijent smjera tangente

9.2. DEFINICIJA DERIVACIJE

= DEFINICIJA 1.

↳ Neka je zadana realna funkcija f na otvorenom intervalu $I \subseteq \mathbb{R}$ i neka je $x_0 \in I$. Ako postoji broj $f'(x_0)$:

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}, \quad \ldots \quad (*)$$

onda taj broj zovemo derivacijom funkcije f u točki x_0, i označujemo s $f'(x_0)$.

↳ Za funkciju kažemo da je derivabilna ili diferencijabilna u točki x_0, ako postoji limes dan formulom (*) u toj točki. Nadalje, funkcija je derivabilna ili diferencijabilna na intervalu I ako u svakoj točki tog intervala ima derivaciju.

↳ Oznake: - $f'(x_0) = \lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x} = \dfrac{dy(x_0)}{dx}$ ... derivacija

- $\Delta y = f(x_0+\Delta x) - f(x_0)$ ... prirast funkcije f u točki x_0
- Δx ... prirast argumenta u točki x_0.

↳ Derivacija je granična vrijednost kvocijenta prirasta funkcije i prirasta argumenta, kad prirast argumenta teži k nuli, tj.:

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

= Jednadžba tangente i normale na graf funkcije $f(x)$ u točki $T_0(x_0, y_0)$ je:

$$t \ldots \quad y - y_0 = f'(x_0)(x - x_0)$$

$$n \ldots \quad y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$$

= PRIMJER 1.

↳ Koristeći definiciju derivacije izračunati derivacije sljedećih funkcija:

a) $f(x) = c$

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{c-c}{\Delta x} = 0$$

↳ $(c)' = 0$

Samo je jedan Mali Ivica!

b) $f(x)=x$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\cancel{x}+\Delta x-\cancel{x}}{\Delta x}=\lim_{\Delta x\to 0}1=1$$

$\hookrightarrow (x)'=1$

c) $f(x)=\sqrt{x}$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}\cdot\frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}}=$$

$$=\lim_{\Delta x\to 0}\frac{\cancel{x+\Delta x-x}}{\cancel{\Delta x}(\sqrt{x+\Delta x}+\sqrt{x})}=\lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}=\frac{1}{\sqrt{x+0}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

$\hookrightarrow (\sqrt{x})'=\frac{1}{2\sqrt{x}}$

d) $f(x)=\sin x$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin x}{\Delta x}=\lim_{\Delta x\to 0}\frac{2\sin\frac{x+\Delta x-x}{2}\cdot\cos\frac{x+\Delta x+x}{2}}{\Delta x}=$$

$$=\lim_{\Delta x\to 0}\frac{2\sin\frac{\Delta x}{2}\cdot\cos\left(x+\frac{\Delta x}{2}\right)}{\Delta x}=\lim_{\Delta x\to 0}\left(\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\cdot\cos\left(x+\frac{\Delta x}{2}\right)\right)=$$

$$=1\cdot\lim_{\Delta x\to 0}\cos\left(x+\frac{\Delta x}{2}\right)=\cos\left(x+\frac{0}{2}\right)=\cos x$$

$\hookrightarrow (\sin x)'=\cos x$

e) $f(x)=|x|$, u točki $x_0=0$

$$f'(0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x)-f(0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(\Delta x)-0}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(\Delta x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{|\Delta x|}{\Delta x}=$$

$$=\begin{cases}\lim\limits_{\Delta x\to 0^+}\frac{\Delta x}{\Delta x}=1\\[2mm] \lim\limits_{\Delta x\to 0^-}\frac{-\Delta x}{\Delta x}=-1\end{cases}$$

$\hookrightarrow$ Dakle, obzirom da su lijevi i desni limes različiti, nema limesa, tj. nema derivacije u točki x_0. Geometrijski to znači da ne postoji tangenta na promatrani graf u promatranoj točki.

9.3. DERIVACIJA I NEPREKINUTOST

= STAVAK 1.

$\hookrightarrow$ Ako je funkcija f diferencijabilna u točki x, onda je f neprekinuta u toj točki x. Obrat stavka ne vrijedi.

Dokaz:

Prema pretpostavci u točki x postoji limes $f'(x)=\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$. Vrijedi:

$$\lim_{\Delta x\to 0}[f(x+\Delta x)-f(x)]\cdot\frac{\Delta x}{\Delta x}=\lim_{\Delta x\to 0}\Delta x\cdot\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=0\cdot f'(x)=0.$$

- To jest:

$$\lim_{\Delta x\to 0}(f(x+\Delta x)-f(x))=\lim_{\Delta x\to 0}f(x+\Delta x)-\lim_{\Delta x\to 0}f(x)=\lim_{\Delta x\to 0}f(x+\Delta x)-f(x)=0$$

- Iz toga slijedi:

$$\lim_{\Delta x\to 0}(x+\Delta x)=f(x)\Rightarrow f\text{ je neprekinuta u } x.\quad \text{Q.E.D.}$$

↳ Geometrijsko značenje:
↳ - To znači da "glatka" funkcija (tj. funkcija na čiji graf možemo povući tangentu) je ujedno i neprekinuta, što možemo i očekivati.
- Obrat ne vrijedi, tj. postoje neprekinute funkcije koje nisu "glatke", npr. $f(x)=|x|$.

9.4. OSNOVNA PRAVILA DERIVIRANJA

= STAVAK 1.
↳ Ako su f i g funkcije diferencijabilne u točki x, onda vrijedi:

a) $[f(x)\pm g(x)]' = f'(x) \pm g'(x)$

b) $[f(x)\cdot g(x)]' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

c) $\left[\dfrac{1}{f(x)}\right]' = \dfrac{-f'(x)}{f^2(x)}$, $f(x)\neq 0$

d) $\left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$, $g(x)\neq 0$

DOKAZ:
- Pomoću definicije derivacije:

a) $$[f(x)+g(x)]' = \lim_{\Delta x\to 0}\frac{[f(x+\Delta x)+g(x+\Delta x)]-[f(x)+g(x)]}{\Delta x} =$$
$$= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} + \lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x} =$$
$$= f'(x)+g'(x)$$
- Analogno i za minus.

b) $$[f(x)\cdot g(x)]' = \lim_{\Delta x\to 0}\frac{[f(x+\Delta x)\,g(x+\Delta x)]-[f(x)\cdot g(x)]}{\Delta x} =$$
$$= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x)g(x+\Delta x)+f(x)\cdot g(x+\Delta x)-f(x)\cdot g(x)}{\Delta x} =$$
$$= \lim_{\Delta x\to 0}\left[g(x+\Delta x)\cdot\frac{f(x+\Delta x)-f(x)}{\Delta x} + f(x)\,\frac{g(x+\Delta x)-g(x)}{\Delta x}\right] =$$
$$= f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

*) $\lim\limits_{\Delta x\to 0} g(x+\Delta x) = g(x)$ jer je po pretpostavci $g(x)$ diferencijabilna, i neprekinuta u točki x

c) $$\left[\frac{1}{f(x)}\right]' = \lim_{\Delta x\to 0}\frac{\frac{1}{f(x+\Delta x)}-\frac{1}{f(x)}}{\Delta x} = \lim_{\Delta x\to 0}\frac{\frac{f(x)-f(x+\Delta x)}{f(x)\cdot f(x+\Delta x)}}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x)-f(x+\Delta x)}{\Delta x f(x)\cdot f(x+\Delta x)} =$$
$$= -\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\cdot\frac{1}{\lim\limits_{\Delta x\to 0} f(x)}\cdot\frac{1}{\lim\limits_{\Delta x\to 0} f(x+\Delta x)} =$$
$$= -f'(x)\cdot\frac{1}{f(x)}\cdot\frac{1}{f(x)} = \frac{-f'(x)}{f^2(x)}$$

d) $$\left[\frac{f(x)}{g(x)}\right]' = \left(f(x)\cdot\frac{1}{g(x)}\right)' = f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\left(\frac{1}{g(x)}\right)' =$$
$$= \frac{f'(x)}{g(x)} + f(x)\cdot\left(-\frac{g'(x)}{g^2(x)}\right) = \frac{f'(x)}{g(x)} - \frac{f(x)\,g'(x)}{g^2(x)} =$$
$$= \frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{g^2(x)}$$

= PRIMJEDBA 1.

↳ Vrijedi: a) $(f(x)+C)' = f'(x)+0 = f'(x)$

b) $c' \cdot f(x) + c \cdot f'(x) = c \cdot f'(x)$

Aditivna konstanta prilikom deriviranja nestaje, a multiplikativna konstanta se kod deriviranja prepisuje.

• PRIMJER 1.

$$y = 2\sqrt{x}\,\sin x + 3\pi$$

$$y' = 2(\sqrt{x}\sin x)' + (3\pi)' = 2\cdot\left((\sqrt{x})'\cdot \sin x + \sqrt{x}\cdot(\sin x)'\right) + 0 = 2\cdot\frac{1}{2\sqrt{x}}\sin x + \sqrt{x}\cos x =$$

$$= \frac{\sin x}{\sqrt{x}} + \sqrt{x}\cos x$$

= PRIMJER 2.

$$y = 2x + \frac{3}{\sin x} + \frac{1}{e}$$

$$y' = 2 + 3\cdot\frac{-\cos x}{\sin^2 x} + 0 = 2 - \frac{3\cos x}{\sin^2 x}$$

9.5. DERIVACIJA SLOŽENE FUNKCIJE I INVERZNE FUNKCIJE

= STAVAK 1.

↳ Neka je složena funkcija $g\circ f$ definirana u točki x. Neka je funkcija f diferencijabilna u točki x i g diferencijabilna u točki $f(x)$. Tada funkcija $g\circ f$ ima derivaciju u x i vrijedi:

$$(g\circ f)'(x) = g'[f(x)]\cdot f'(x).$$

To se naziva tzv. "lančano pravilo".

Dokaz:

- Označimo: $y = (g\circ f)(x) = g[f(x)] = g(u) \quad \ldots \quad u = f(x)$

Iz izraza:

$$g'(u) = \lim_{\Delta u\to 0}\frac{\Delta g}{\Delta u} \Rightarrow \frac{\Delta g}{\Delta u} = g'(u) + \omega(\Delta u) \,/\cdot \Delta u$$

! $-g'(u)$ je broj !
! $-\lim_{\Delta u\to 0}\omega(\Delta u) = 0$

$$\Delta g = g'(u)\cdot\Delta u + \omega(\Delta u)\cdot\Delta u \,/ : \Delta x$$

$$\frac{\Delta g}{\Delta x} = \frac{g'(u)\cdot\Delta u}{\Delta x} + \frac{\omega(\Delta u)\cdot\Delta u}{\Delta x} \,/\cdot \lim_{\Delta x\to 0}$$

$$\lim_{\Delta x\to 0}\frac{\Delta g}{\Delta x} = \lim_{\Delta x\to 0} g'(u)\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} + \lim_{\Delta x\to 0}\omega(\Delta u)\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}$$

Kako promjena Δu nastaje radi promjene x-a zbog pretpostavke da postoji derivacija od $f(x)=0$ slijedi da je funkcija f neprekinuta u x pa ako Δx teži k nuli onda i Δu teži k nuli pa vrijedi:

$$\lim_{\Delta x\to 0}\omega(\Delta u) = \lim_{\Delta u\to 0}\omega(\Delta u) = 0$$

Zbog toga pišemo:

$$\lim_{\Delta x\to 0}\frac{\Delta g}{\Delta x} = g'(u)\cdot f'(x) + 0\cdot f'(x)$$

$$\frac{\Delta g(f(x))}{\Delta x} = \frac{\Delta(g\circ f)(x)}{\Delta x} = \frac{\Delta y}{\Delta x}$$

Samo je jedan Mali Ivica!

Dakle:

$$(g\circ f)'(x) = g'[f(x)]\cdot f'(x)$$

↳ Ovaj stavak pišemo i u obliku:

- $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$, gdje je $\begin{matrix} y=g(u) \\ u=f(x)\end{matrix}$
- $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dv}\cdot\dfrac{dv}{dx}$, gdje je $\begin{matrix} y=f(u) \\ u=g(v) \\ v=h(x)\end{matrix}$, te $y=(f\circ g\circ h)(x)$

PRIMJER 1.

↳ Primjenjujući derivaciju kompozicije funkcija ili definiciju naći sljedeće derivacije:

a) $y=\cos x$, $y'=?$

I) $y=\cos x = \sin\left(x+\frac{\pi}{2}\right) \Rightarrow \begin{matrix} y=\sin u \\ u=x+\frac{\pi}{2}\end{matrix}$

II) $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = \cos u\,(1+0) = \cos\left(x+\frac{\pi}{2}\right) = -\sin x$

↳ $(\cos x)' = -\sin x$

b) $y=e^x$, $y'=?$

$$y' = \lim_{\Delta x\to 0}\frac{e^{x+\Delta x}-e^x}{\Delta x} = \lim_{\Delta x\to 0}\frac{e^x(e^{\Delta x}-1)}{\Delta x} = e^x\cdot\lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x} = e^x\cdot 1 = e^x$$

↳ $(e^x)' = e^x$

c) $y=e^{-x}$, $y'=?$

I) $y=e^{-x} \Rightarrow \begin{matrix} y=e^u \\ u=-x\end{matrix}$

II) $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = e^u\cdot(-1) = -e^u = -e^{-x}$

↳ $(e^{-x})' = -e^{-x}$

d) $y=\mathrm{sh}$, $y'=?$

$$y' = \frac{1}{2}(e^x-e^{-x})' = \frac{1}{2}(e^x-(-e^{-x})) = \frac{1}{2}(e^x+e^{-x}) = \mathrm{ch}\,x$$

↳ $(\mathrm{sh}\,x)' = \mathrm{ch}\,x$

e) $y=\mathrm{ch}\,x$, $y'=?$

$$y' = \frac{1}{2}(e^x+e^{-x})' = \frac{1}{2}\left(e^x+(-e^{-x})\right) = \frac{1}{2}(e^x-e^{-x}) = \mathrm{sh}\,x$$

↳ $(\mathrm{ch}\,x)' = \mathrm{sh}\,x$

f) $y=\mathrm{tg}\,x$, $y'=?$

$$y' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\sin' x\cdot\cos x - \sin x\cos' x}{\cos^2 x} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = 1+\mathrm{tg}^2 x = \frac{1}{\cos^2 x}$$

↳ $(\mathrm{tg}\,x)' = \dfrac{1}{\cos^2 x}$

g) $y=a^x$, $y'=?$

I) $y=a^x=e^{\ln(a^x)}=e^{x\ln a} \Rightarrow y=e^u$, $u=x\ln a$

II) $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=e^u\cdot\ln a\cdot 1=e^{x\ln a}\cdot\ln a=e^{\ln a^x}\cdot\ln a=a^x\cdot\ln a$

↳ $(a^x)'=a^x\cdot\ln a$

h) $y=x^\alpha$, $\alpha\in\mathbb{R}$, $y'=?$

I) $y=x^\alpha:=e^{\ln(x^\alpha)}=e^{\alpha\ln(x)} \Rightarrow y=e^u$, $u=\alpha\cdot\ln x$

II) $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=e^u\cdot\alpha\cdot\frac{1}{x}=e^{\alpha\cdot\ln x}\cdot\alpha\cdot\frac{1}{x}=\frac{x^\alpha\cdot\alpha}{x}$

↳ $(x^\alpha)'=\alpha\cdot x^{\alpha-1}$

= PRIMJER 2.

↳ Deriviraj:

a) $(x^4)'=4x^3$

b) $\left(\frac{1}{x^3}\right)'=(x^{-3})'=-3x^{-4}=\frac{-3}{x^4}$

c) $(\sqrt{x})'=x^{\frac{1}{2}}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$

d) $\frac{1}{\sqrt[3]{x}}=\frac{1}{x^{\frac{1}{3}}}=x^{-\frac{1}{3}}=-\frac{1}{3}x^{-\frac{4}{3}}=\frac{-1}{3\sqrt[3]{x^4}}$

= PRIMJER 3.

↳ Deriviraj: $f(x)=\operatorname{tg}\ln(x^2+1)=\frac{1}{\cos^2\ln(x^2+1)}\cdot\frac{1}{x^2+1}\cdot 2x$

Dokaz: $\left.\begin{array}{l} y=\operatorname{tg}u \\ u=\ln v \\ v=x^2+1 \end{array}\right\}$ $y'=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}=\frac{1}{\cos^2 u}\cdot\frac{1}{v}\cdot 2x=\frac{1}{\cos^2\ln(x^2+1)}\cdot\frac{2x}{x^2+1}$

= STAVAK 2.

↳ Derivacija inverzne funkcije:

- Neka je f diferencijabilna u točki $x\in I$, te $f'(x)\neq 0$. Onda je i inverzna funkcija f^{-1} (ako je definirana) diferencijabilna u točki $y=f(x)$ i vrijedi:

$$\left(f^{-1}(y)\right)'=\frac{1}{f'(x)}$$

Dokaz:

- $(f^{-1}\circ f)(x)=f^{-1}(f(x))=x$

$f^{-1}(y)=x$, $y=f(x)$

- Korištenjem derivacije dobivamo:

$$\frac{df^{-1}(y)}{dy}\cdot\frac{dy}{dx}=\frac{dx}{dx}=1$$

a uz oznaku $x=f^{-1}(y)$ slijedi:

$$\frac{df^{-1}(y)}{dy}=\frac{1}{\frac{dy}{dx}}=\frac{1}{\frac{df(x)}{dx}}$$

- što češće zapisujemo:

$$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}} \quad \text{ili} \quad \frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$$

Q.E.D

=PRIMJER 4.

↳ Nađi derivaciju inverzne funkcije od sljedećih funkcija:

a) $y=\ln x$

$\Downarrow$

$x=e^y \qquad \frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{e^y}=\frac{1}{x}$

$(\ln x)'=\frac{1}{x}\ ,\ x>0$

b) $y=\log_a x$

$\Downarrow$

$x=a^y \ /\ln$

$\ln x=y\cdot\ln a$

$y=\frac{\ln x}{\ln a} \qquad \frac{dy}{dx}=\left(\frac{\ln x}{\ln a}\right)'=\frac{1}{\ln a}\circ(\ln x)'=\frac{1}{x\cdot\ln a}$

$(\log_a x)'=\frac{1}{x\cdot\ln a}\ ,\ x>0$

c) $y=\arcsin x$

$\Downarrow$

$x=\sin y \,\Big|_{[-\frac{\pi}{2},\frac{\pi}{2}]} \qquad \frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{\cos y}=\frac{1}{\pm\sqrt{1-\sin^2 y}}=\frac{+1}{\sqrt{1-x^2}}$

*) rješenje sa negativnim predznakom odbacujemo jer je kosinus pozitivan u 1 i 4 kvadrantu

$(\arcsin x)'=\frac{1}{\sqrt{1-x^2}}\ ,\ |x|<1$

d) $y=\operatorname{arsh}x$

$\Downarrow$

$x=\operatorname{sh}y \qquad$ 1. način: $\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{\operatorname{ch}y}=\frac{1}{\pm\sqrt{1+\operatorname{sh}^2 y}}=\frac{+1}{\sqrt{1+x^2}}$

*) negativno rješenje odbacujemo jer je $\operatorname{ch}y\geq 1$

$(\operatorname{arsh}x)'=\frac{1}{\sqrt{1+x^2}}$

2. način: $\operatorname{arsh}x=\ln(x+\sqrt{1+x^2})$

$(\operatorname{arsh}x)'=(\ln(x+\sqrt{1+x^2}))'=\frac{1}{x+\sqrt{1+x^2}}\cdot\left(1+\frac{1}{\cancel{2}\sqrt{1+x^2}}\cdot\cancel{2}x\right)=$

$=\frac{1}{x+\sqrt{1+x^2}}\cdot\frac{\cancel{x+\sqrt{1+x^2}}}{\sqrt{1+x^2}}=\frac{1}{\sqrt{1+x^2}}$

$(\operatorname{arsh}x)'=\frac{1}{\sqrt{1+x^2}}$

e) $y=\operatorname{arth}x$

$\Downarrow$

$x=\operatorname{th}y \qquad$ 1. način: $\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{\frac{1}{\operatorname{ch}^2 y}}=\frac{1}{\frac{\operatorname{ch}^2 y-\operatorname{sh}^2 y}{\operatorname{ch}^2 y}}=\frac{1}{1-\operatorname{th}^2 y}=\frac{1}{1-x^2}$

$$(\operatorname{arth} x)' = \frac{1}{1-x^2}, \quad |x|<1$$

2. način: $\operatorname{arth} x = \frac{1}{2} \ln \frac{1+x}{1-x}$

$$(\operatorname{arth} x)' = \left(\frac{1}{2} \ln \frac{1+x}{1-x}\right)' = \frac{1}{2} \cdot \frac{1-x}{1+x} \cdot \left(\frac{1+x}{1-x}\right)' = \frac{1}{2} \cdot \frac{\cancel{1-x}}{1+x} \cdot \frac{(1-x)-(1+x)\cdot(-1)}{(1-x)^{\cancel{2}}}$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{1-x+1+x}}{(1+x)(1-x)} = \frac{1}{1-x^2}$$

$$(\operatorname{arth} x)' = \frac{1}{1-x^2}, \quad |x|<1$$

□ DERIVACIJA LOGARITAMSKE I EKSPONENCIJALNE FUNKCIJE

= 1. način: – Primjenjuje se prvenstveno kod oblika $y = f(x)^{g(x)}$.
– Osnovna ideja je da se računa derivacija od $\ln y$, a ne od y, pa vrijedi:

$$y = f(x)^{g(x)} \quad / \ln$$
$$\ln y = g(x) \cdot \ln f(x) \quad /'$$
$$\frac{1}{y} \cdot y' = g'(x) \cdot \ln f(x) + g(x) \cdot \frac{1}{f(x)} \cdot (f(x))' \quad / \cdot y$$
$$y' = f(x)^{g(x)} \cdot \left[g'(x) \cdot \ln f(x) + \frac{g(x) \cdot f'(x)}{f(x)} \right]$$

= 2. način: – Ova mogućnost je bolja.
– Osnovna ideja je da se kod zapisa funkcije koju dobivamo koristi identitet: $A = e^{\ln A}$ pa slijedi:

$$f(x)^{g(x)} := e^{(\ln f(x))^{g(x)}} = e^{g(x) \cdot \ln f(x)} \quad /'$$
$$y' = e^{g(x) \cdot \ln f(x)} \cdot (g(x) \cdot \ln f(x))' =$$
$$= e^{g(x) \ln f(x)} \cdot (g'(x) \ln f(x) + g(x) \cdot \ln' f(x))$$

= PRIMJER 5

↳ Deriviraj funkcije:

a) 1) $y = x^{\sin x} \quad / \ln$

$$\ln y = \sin x \ln x \quad /'$$
$$\frac{1}{y} \cdot y' = \cos x \cdot \ln x + \sin x \cdot \frac{1}{x} \quad / \cdot y$$
$$y' = y \cdot \left[\cos x \ln x + \frac{\sin x}{x}\right]$$
$$y' = x^{\sin x} \cdot \left[\cos x \ln x + \frac{\sin x}{x}\right]$$

2) $y = x^{\sin x} \Rightarrow y = e^{\ln x^{\sin x}} = e^{\sin x \ln x}$

$$y' = (e^{\sin x \ln x})' = e^{\sin x \ln x} \cdot (\sin x \ln x)' = e^{\sin x \ln x} \cdot \left(\cos x \ln x + \sin x \frac{1}{x}\right) =$$
$$= x^{\sin x} \cdot \left[\cos x \ln x + \frac{\sin x}{x}\right]$$

b) $y = (1+x)^x + \sin x$

$$y' = (e^{x \ln(1+x)} + \sin x)' = e^{x \ln(1+x)} \cdot (x \cdot \ln(1+x))' + \cos x =$$

$$= e^{x\ln(1+x)} \cdot \left[\ln(1+x) + \frac{x}{1+x}\right] + \cos x =$$
$$= (1+x)^x \left(\ln(1+x) + \frac{x}{1+x}\right) + \cos x$$

◻ DERIVACIJE VIŠEG REDA

=DEFINICIJA 1.

↳ Derivacija funkcije je općenito opet funkcija, i kao takva može biti ponovno diferencijabilna, što označavamo $(f'(x))' = f''(x)$ i nazivamo drugom derivacijom funkcije f.

↳ Općenito je n-ta derivacija od funkcije f, $n \geq 2$, derivacija $(n-1)$-te derivacije, ukoliko ta postoji i diferencijabilna je, tj.:

$$f^{(n)} = (f^{(n-1)})', \quad n \geq 2.$$

=PRIMJER 6.

↳ Odredi n-tu derivaciju sljedećih funkcija:

a) $(a^x)^{(n)} = ?$

$y = a^x \quad \rightarrow \quad y' = a^x \cdot \ln a$

$y'' = a^x \cdot \ln a \cdot \ln a = a^x \cdot \ln^2 a$

$y''' = a^x \cdot \ln^2 a \cdot \ln a = a^x \cdot \ln^3 a$

$\vdots$

$y^{(n)} = a^x (\ln a)^n$

Dokaz: - matematičkom indukcijom:

I) baza: $n_0 = 1$

$y' = a^x \cdot \ln a$ ✓

II) pretpostavka:

Pretpostavimo da za neki $n \in \mathbb{N}$ vrijedi: $y^{(n)} = a^x \cdot (\ln a)^n$

III) skok: Treba dokazati da vrijedi i za $n+1$:

$$y^{(n+1)} = (y^{(n)})' = (a^x (\ln a)^n)' =$$
$$= a^x \cdot \ln a \cdot (\ln a)^n =$$
$$= a^x \cdot (\ln a)^{n+1}$$

Q.E.D.

b) $(\sin x)^{(n)} = ?$

$y = \sin x \quad \rightarrow \quad y' = \cos x$

$y'' = -\sin x$

$y''' = -\cos x$

$y^{IV} = \sin x$

$$y^{(n)} = \begin{cases} \cos x, & \text{za } n = 2k+1 \\ -\sin x, & \text{za } n = 2k+2 \\ -\cos x, & \text{za } n = 2k+3 \\ \sin x, & \text{za } n = 2k+4 \end{cases}$$

9.6. TABLICA DERIVACIJA ELEMENTARNIH FUNKCIJA

funkcija: $f(x)=$	derivacija: $f'(x)=$	napomena:
a	0	$(a\cdot x)' = a$, $(a+x)' = 1$
x^a	ax^{a-1}	$\frac{1}{x^n} = x^{-n}$, $\sqrt[m]{x^n} = x^{\frac{n}{m}}$
e^x	e^x	%
$\ln x$	$\frac{1}{x}$	%
a^x	$a^x \cdot \ln a$	$\ln a$ = konst.
$\log_a x$	$\frac{1}{x\ln a}$	$\ln a$ = konst.

funkcija: $f(x)=$	derivacija: $f'(x)=$	napomena:
$\sin x$	$\cos x$	%
$\cos x$	$-\sin x$	%
$\operatorname{tg} x$	$\frac{1}{\cos^2 x}$	%
$\operatorname{ctg} x$	$-\frac{1}{\sin^2 x}$	%
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$f'(x) = -(\arccos x)'$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	$f'(x) = -(\arcsin x)'$

funkcija: $f(x)=$	derivacija: $f'(x)=$	napomena:
$\operatorname{arctg} x$	$\frac{1}{1+x^2}$	$f'(x)=-(\operatorname{arcctg} x)'$
$\operatorname{arcctg} x$	$-\frac{1}{1+x^2}$	$f'(x)=-(\operatorname{arctg} x)'$
$\operatorname{sh} x$	$\operatorname{ch} x$	%
$\operatorname{ch} x$	$\operatorname{sh} x$	%
$\operatorname{th} x$	$\frac{1}{\operatorname{ch}^2 x}$	%
$\operatorname{cth} x$	$-\frac{1}{\operatorname{sh}^2 x}$	%

funkcija: $f(x)=$	derivacija: $f'(x)=$	napomena:
$\operatorname{arsh} x$	$\frac{1}{\sqrt{1+x^2}}$	%
$\operatorname{arch} x$	$\frac{1}{\sqrt{x^2-1}}$	%
$\operatorname{arth} x$	$\frac{1}{1-x^2}$	$f'(x)=(\operatorname{arcth}(x))'$
$\operatorname{arcth} x$	$\frac{1}{1-x^2}$	$f'(x)=(\operatorname{arth} x)'$
$\lvert x\rvert$	$\frac{x}{\lvert x\rvert}$	$\frac{x}{\lvert x\rvert}=\frac{\lvert x\rvert}{x}$

ZADACI (Dz. 9. PRVI DIO)

1. $y=\operatorname{tg}^3\frac{\pi x}{3}$, $y'=?$

$$y'=\cancel{3}\operatorname{tg}^2\frac{\pi x}{3}\cdot\frac{1}{\cos^2\frac{\pi x}{3}}\cdot\frac{\pi}{\cancel{3}}=\pi\cdot\frac{\operatorname{tg}^2\frac{\pi x}{3}}{\cos^2\frac{\pi x}{3}}$$

2. $f(x)=2^x\cdot\operatorname{sh}x+2\operatorname{ch}x+\sqrt[3]{\operatorname{sh}x}$, $f''(0)=?$

$$f'(x)=2^x\ln 2\cdot\operatorname{sh}x+2^x\cdot\operatorname{ch}x+2\operatorname{sh}x+\frac{1}{3}\operatorname{sh}x^{-\frac{2}{3}}\cdot\operatorname{ch}x$$

$$f''(x)=\ln 2\left(2^x\ln 2\cdot\operatorname{sh}x+2^x\operatorname{ch}x\right)+2^x\ln 2\cdot\operatorname{ch}x+2^x\operatorname{sh}x+2\operatorname{ch}x-\frac{2}{9}\operatorname{sh}x^{-\frac{5}{3}}\cdot\operatorname{ch}^2x+\frac{1}{3}\operatorname{sh}x^{-\frac{2}{3}}\operatorname{sh}x$$

$$=2^x\operatorname{sh}x\left(\ln^2 2+1\right)+2^x\operatorname{ch}x\cdot 2\ln 2+2\operatorname{ch}x-\frac{2\operatorname{ch}^2x}{9\sqrt[3]{\operatorname{sh}x^5}}+\frac{\sqrt[3]{\operatorname{sh}x}}{3}$$

$$f''(0)=\cancel{2^0\operatorname{sh}0\left(\ln^2 2+1\right)}+2^0\operatorname{ch}0\cdot 2\ln 2+2\operatorname{ch}0-\frac{2\operatorname{ch}^2 0}{9\cdot\sqrt[3]{\operatorname{sh}0^5}}+\frac{\sqrt[3]{\operatorname{sh}0}}{3}=$$

$$=2\ln 2+2-\frac{2}{0}+\frac{0}{3}=-\infty$$

3. $y=x\cdot e^{-x}$, dokaži da je $x\cdot y'=(1-x)y$

$$y'=e^{-x}-x\cdot e^{-x}$$

$$x\cdot y'=(1-x)\cdot y$$
$$xe^{-x}(1-x)=(1-x)\cdot x\cdot e^{-x}\quad\text{Q.E.D.}$$

4. $f(x)=x^2\cdot 2^x+(x+1)^x$, $f'(x)=?$

$$f'(x)=2x\cdot 2^x+x^2\cdot 2^x\cdot\ln 2+e^{x\ln(x+1)}\cdot\left(\ln(x+1)+\frac{x}{x+1}\right)=$$

$$=2^x x\left(2+x\ln 2\right)+(x+1)^x\left(\ln(x+1)+\frac{x}{x+1}\right)$$

(5.) $f(x)=\ln^3(\sin 5x)$, $f'(x)=?$

$$f'(x)=3\ln^2(\sin 5x)\cdot\frac{1}{\sin 5x}\cdot\cos 5x\cdot 5 = 15\ln^2(\sin 5x)\cdot ctg 5x$$

(6.) $y=x\cdot|x|$, $y'=?$

$$y'=|x|+x\cdot\frac{|x|}{x}=|x|+|x|=2|x|$$

(7.) dokaži da za $x\neq 0 \Rightarrow (\ln|x|)'=\frac{1}{x}$

$$(\ln|x|)'=\frac{1}{\cancel{|x|}}\cdot\frac{\cancel{|x|}}{x}=\frac{1}{x}$$

(8.) $f^{(n)}(x)=?$ $f(x)=\sin x$

$f'(x)=\cos x$
$f''(x)=-\sin x$
$f'''(x)=-\cos x$
$f^{IV}(x)=\sin x$
$\vdots$

$$f^{(n)}(x)=\begin{cases}\cos x & za\ n=2k+1\\ -\sin x & za\ n=2k+2\\ -\cos x & za\ n=2k+3\\ \sin x & za\ n=2k+4\end{cases}$$

(9.) $f^{(n)}(x)=?$ $f(x)=\cos x$

$f'(x)=-\sin x$
$f''(x)=-\cos x$
$f'''(x)=\sin x$
$f^{IV}(x)=\cos x$
$\vdots$

$$f^{(n)}(x)=\begin{cases}-\sin x & za\ n=2k+1\\ -\cos x & za\ n=2k+2\\ \sin x & za\ n=2k+3\\ \cos x & za\ n=2k+4\end{cases}$$

(10.) $f^{(n)}(x)=?$ $f(x)=a^x$

$f'(x)=a^x\cdot\ln a$
$f''(x)=a^x\cdot\ln^2 a$
$f'''(x)=a^x\cdot\ln^3 a$
$\vdots$
$f^{(n)}(x)=a^x\cdot\ln^n a$

Dokaz indukcijom:
1) baza: $n=1$ ... $f'(x)=a^x\cdot\ln^1 a$ ✓
2) pretpostavka: $\exists n\in N$ vrijedi: $f^{(n)}(x)=a^x(\ln a)^n$
3) skok: dokazati: $f^{(n+1)}(x)=a^x(\ln a)^{n+1}$

$$(f^{(n)}(x))'=(a^x(\ln a)^n)'=a^x\cdot\ln a(\ln a)^n=$$
$$=a^x(\ln a)^{n+1}\ ✓\quad Q.E.D.$$

(11.) $y=x$ .... t
$y=\lambda e^x$, $\lambda=?$
$T(x_0,y_0)=?$

I) $T(x_0,y_0)$...
$x_0=\lambda e^{x_0}$
$\lambda=\frac{x_0}{e^{x_0}}$

II) $\begin{cases}k_T=1\\ k_T=y'(x_0)=\lambda e^{x_0}\end{cases}$
$1=\lambda e^{x_0}$
$\lambda=\frac{1}{e^{x_0}}$

I i II) $\frac{x_0}{e^{x_0}}=\frac{1}{e^{x_0}}$
$x_0=1$

$$\lambda=\frac{1}{e^{x_0}}=\frac{1}{e^1}=\frac{1}{e}$$

(12.) $y=x$ .... n
$y=\lambda x(1-x)$, $\lambda=?$ $x\neq 0$

I) $T(x_0,y_0)$....
$\cancel{x_0}=\lambda\cancel{x_0}(1-x_0)$
$\lambda=\frac{1}{1-x_0}$

II) $k_N=1$
$\begin{cases}k_N=-\frac{1}{k_T}\Rightarrow k_T=-1\\ k_T=y'(x_0)=\lambda[(1-x_0)+x_0(-1)]=\lambda(1-x_0-x_0)=\\ =\lambda(1-2x_0)\end{cases}$
$-1=\lambda(1-2x_0)$
$$\lambda=\frac{1}{2x_0-1}$$

I i II) $\frac{1}{1-x_0} = \frac{1}{2x_0-1}$ $\qquad \lambda = \frac{1}{1-x_0} = \frac{1}{1-\frac{2}{3}} = \frac{1}{\frac{1}{3}} = 3$

$1-x_0 = 2x_0 - 1$

$-3x_0 = -2$

$x_0 = \frac{2}{3}$

13. $y = x \ldots n$

$y = \lambda e^x$, $\lambda = ?$, $\lambda < 0$

I) $T(x_0, y_0) \ldots$

$x_0 = \lambda e^{x_0}$

$\lambda = \frac{x_0}{e^{x_0}}$

II) $k_N = 1$

$k_N = -\frac{1}{k_T} \Rightarrow k_T = -1$

$k_T = y'(x_0) = \lambda e^{x_0}$

$-1 = \lambda e^{x_0}$

$\lambda = -\frac{1}{e^{x_0}}$

I i II) $\frac{x_0}{e^{x_0}} = -\frac{1}{e^{x_0}}$ $\qquad \lambda = -\frac{1}{e^{-1}} = -e$

$x_0 = -1$

14. $y = x \ldots t$

$y = x^2 + c$, $c = ?$

I) $T(x_0, y_0)$

$x_0 = x_0^2 + c$

$c = x_0 - x_0^2$

II) $k_T = 1$

$k_T = y'(x_0) = 2x_0$

$2x_0 = 1$

$x_0 = \frac{1}{2}$

I i II) $c = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

15. a) $A = ?$

b) $\omega = ?$

$y = A \sin(\omega x) \rightarrow t \ldots \; T(0,0)$, $k_T = 1$

$\rightarrow T = 4\pi$

b) $\omega = \frac{2\pi}{T} = \frac{2\pi}{4\pi} = \frac{1}{2}$

a) $t \ldots \; y - y_0 = k_T (x - x_0)$

$y = x$

$y'(0) = 1$

$A \cos(\omega x) \cdot \omega = 1$

$A \cos\left(\frac{1}{2} \cdot 0\right) \cdot \frac{1}{2} = 1$

$A \cdot 1 \cdot 1 = 2$

$A = 2$

Samo je jedan Mali Ivica!

16. $t \ldots = ?$

$y = \sqrt[3]{x-1}\ ,\ T(1,0)$

$y' = \left((x-1)^{\frac{1}{3}}\right)' = \frac{1}{3}(x-1)^{-\frac{2}{3}} = \frac{1}{3\cdot\sqrt[3]{(x-1)^2}}$

$k_T = y'(1) = \frac{1}{3\cdot\sqrt[3]{1-1}^2} = \frac{1}{3\cdot 0} = \frac{1}{0} = \infty$ ⇒ mora biti okomica na X os

$t \ldots\ T(1,0) \Rightarrow x = 1$ //

17. $t_1, t_2 = ?$ $T(-2,-3)$, na $y = \frac{1}{4}x^2 + \frac{1}{2}x - \frac{3}{4}$

→ $y' = \frac{1}{2}x + \frac{1}{2} \Rightarrow k_T = y'(x_0) = \frac{1}{2}x_0 + \frac{1}{2}$

$t_1, t_2 \ldots\ y+3 = \left(\frac{1}{2}x_0 + \frac{1}{2}\right)(x+2)$

→ $T_1(x_0, y_0)$, $T_2(x_0, y_0) = ?$

(1) $T_{1,2}$ → leži na tangenti: $y_0 + 3 = \frac{1}{2}(x_0+1)(x_0+2)$

(2) $T_{1,2}$ → leži na paraboli: $y_0 = \frac{1}{4}x_0^2 + \frac{1}{2}x_0 - \frac{3}{4}$

$3 = \frac{1}{2}x_0^2 + \frac{3}{2}x_0 + 1 - \frac{1}{4}x_0^2 - \frac{1}{2}x_0 + \frac{3}{4}\ /\cdot 4$

$12 = 2x_0^2 + 6x_0 + 4 - x_0^2 - 2x_0 + 3$

$-x_0 - 4x_0 + 5 = 0$

$x_0 + 4x_0 - 5 = 0$

$x_{0\,1,2} = \frac{-4 \pm \sqrt{16+20}}{2} = \frac{-4 \pm 6}{2}$

$x_{01} = 1 \qquad x_{02} = -5$

$y_{01} = 0 \qquad y_{02} = 3$

$t_1 \ldots\ y+3 = \left(\frac{1}{2}\cdot 1 + \frac{1}{2}\right)(x+2)$

$y+3 = x+2$

$y = x-1$ //

$t_2 \ldots\ y+3 = \left(\frac{1}{2}\cdot(-5) + \frac{1}{2}\right)(x+2)$

$y+3 = -2(x+2)$

$y+3 = -2x-4$

$y = -2x-7$ //

18. $a = ?$ $f_1(x) = \ln x$ dira $f_2(x) = ax^2$

↳ dirati će graf funkcije ukoliko oba grafa imaju zajedničku tangentu:

$\left.\begin{aligned} y_1' &= \frac{1}{x} \\ y_2' &= 2ax \end{aligned}\right\}$

$y_1'(x_0) = y_2'(x_0)$

$\frac{1}{x_0} = 2ax_0$

$\frac{1-2ax_0^2}{x_0} = 0$

$1 - 2ax_0^2 = 0$

$2ax_0^2 = 1$

$a = \frac{1}{2x_0^2}$ //

(19.) $f'(0)=?\quad f(x)=\sqrt{1-\sqrt{1-x^2}}$

$$f'(x)=\frac{1}{2\sqrt{1-\sqrt{1-x^2}}}\cdot\left(\cancel{-}\frac{1}{\cancel{2}\sqrt{1-x^2}}\right)\cdot\left(\cancel{-2}x\right)=$$

$$=\frac{x}{2\sqrt{1-\sqrt{1-x^2}}\ \sqrt{1-x^2}}$$

$$f'(0)=\frac{0}{2\sqrt{1-\sqrt{1-0}}\ \sqrt{1-0}}=\frac{0}{2\cdot 0\cdot 1}=\frac{0}{0}\ \Rightarrow \text{ne postoji } f'(0)$$

(20) $f'(0)=?\qquad f(x)=x\cdot\sin\frac{1}{x}\quad,\ x\neq 0,\ f(0)=0$

$$f'(x)=\sin\frac{1}{x}+x\cdot\cos\frac{1}{x}\cdot(-x^{-2})=\sin\frac{1}{x}-\frac{\cos\frac{1}{x}}{x}$$

$$f'(0)=\sin\frac{1}{0}-\frac{\cos\frac{1}{0}}{0}\quad\Rightarrow \text{ne postoji}$$

$g'(0)=?\quad g(x)=x^2\sin\frac{1}{x},\ \ x\neq 0,\ g(0)=0$

$$g'(x)=2x\sin\frac{1}{x}+\cancel{x^2}\cos\frac{1}{x}\cdot(-\cancel{x^{-2}})=2x\sin\frac{1}{x}-\cos\frac{1}{x}$$

$$g'(0)=\cancel{2\cdot 0\cdot\sin\frac{1}{0}}-\cos\frac{1}{0}\quad\Rightarrow \text{ne postoji}$$

10. DIFERENCIJALNI RAČUN II

10.1. DERIVACIJA IMPLICITNO I PARAMETARSKI ZADANE FUNKCIJE

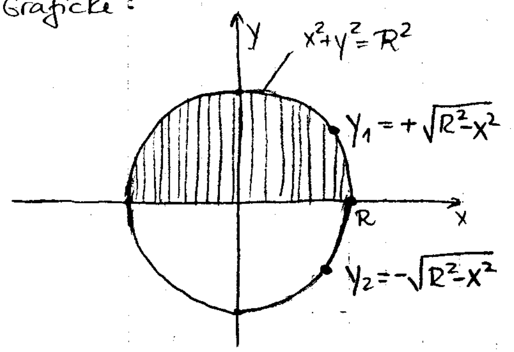
DERIVACIJA IMPLICITNO ZADANE FUNKCIJE

= Ako je skup točaka $(x,y) \in \mathbb{R}^2$ zadan jednadžbom $F(x,y)=0$, tj. $\{(x,y) \mid (x,y)=0\} \subseteq \mathbb{R}^2$, onda taj graf ne mora biti graf funkcije.

- PRIMJER 1.

↳ Ako je $F(x,y) \equiv x^2+y^2-R^2=\emptyset$, onda je to jednadžba kružnice s rješenjima: $y_1=\sqrt{R^2-x^2}$
$y_2=-\sqrt{R^2-x^2}$

↳ Grafički:



↳ Pitanje je: Može li se naći eksplicitni izraz za y' bez da se zna eksplicitno izraziti y_1 i y_2.

Odgovor je: - Može, i to tako da shvatimo y kao funkciju od x i zapisujemo:

$$F(x,y) \equiv x^2+y^2(x)-R^2=0 \,\Big/\frac{d}{dx}$$

$$2x+2y\cdot y'=0$$

$$y'=-\frac{x}{y}=-\frac{x}{y(x)}$$

1) za $y_1=f_1(x)=\sqrt{R^2-x^2}$ slijedi $y_1'(x)=\frac{-x}{\sqrt{R^2-x^2}}$

2) za $y_2=f_2(x)=-\sqrt{R^2-x}$ slijedi $y_2'(x)=\frac{x}{\sqrt{R^2-x^2}}$

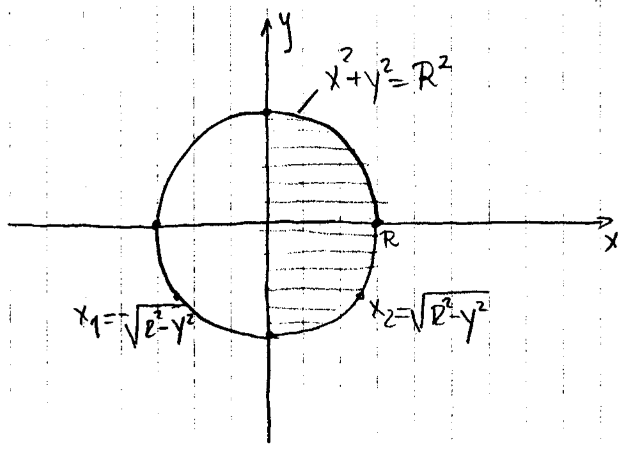
- Ako pak u izrazu $F(x,y)=0$ shvatimo $x=x(y)$ dobit ćemo na analogan način derivaciju inverzne funkcije:

$$F(x,y)\equiv x^2(y)+y^2-R^2=0 \,\Big/\frac{dx}{dy}$$

$$2x\cdot x'+2y=0$$

$$x'=-\frac{y}{x}$$

↳ grafički:



=PRIMJER 2.

↳Naći jednadžbu tangente u točki T (3,4) kružnice zadane jednadžbom $x^2+y^2=25$.

$$x^2+y^2=25 \Big/ \frac{dy}{dx}$$
$$2x+2y\cdot y'=0$$
$$y'=-\frac{x}{y}$$

$$k_T=y'(3,4)=-\frac{3}{4}$$
$$T(3,4)$$

$$y-y_1=k_T(x-x_1)$$
$$y-4=-\frac{3}{4}(x-3)$$
$$y=-\frac{3}{4}x+\frac{25}{4}$$

=PRIMJER 3.

↳Naći $y''(x)$ ako je $\ln y+x^4-3y-5=0$.

1.) $$\ln y+x^4-3y-5=0 \Big/ \frac{dy}{dx}$$
$$\frac{1}{y}\cdot y'+4x^3-3y'=0$$
$$\frac{y'}{y}-3y'+4x^3=0$$
$$y'\left(\frac{1}{y}-3\right)=-4x^3$$
$$y'=4\cdot\frac{x^3\cdot y}{3y-1}$$

2) $$y''=4\cdot\frac{(x^3y)'(3y-1)-(x^3y)(3y-1)'}{(3y-1)^2}=$$
$$=4\,\frac{(3x^2y+x^3y')(3y-1)-x^3y\,3y'}{(3y-1)^2}=$$
$$=4\cdot\frac{\left(3x^2y+x^3\cdot\frac{4x^3y}{3y-1}\right)(3y-1)-\left(x^3y\cdot3\cdot\frac{4x^3y}{3y-1}\right)}{(3y-1)^2}$$

□ DERIVACIJA PARAMETARSKI ZADANE FUNKCIJE

=Neka je $$\mathcal{C}=\begin{cases}x=\varphi(t)\\ y=\psi(t)\end{cases},\ t\in\langle\alpha,\beta\rangle.$$

Pretpostavimo da su funkcije $\varphi(t)$ i $\psi(t)$ diferencijabilne na $\langle\alpha,\beta\rangle$, što označujemo : $\dot{x}=\frac{dx}{dt}$ i $\dot{y}=\frac{dy}{dt}$. Pretpostavimo da funkcija $x=\varphi(t)$ ima inverznu funkciju $t=\varphi^{-1}(x)$, tj. $y=\psi[\varphi^{-1}(x)]$. Ako taj izraz shvatimo kao kompletnu funkciju, tada imamo :

$$y'(x)=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{dy}{dt}\cdot\frac{1}{\frac{dx}{dt}}=\dot{y}\cdot\frac{1}{\dot{x}};$$

iz čega slijedi :

$$y'=\frac{dy}{dx}=\frac{\dot{y}}{\dot{x}}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

=PRIMJER 4.

↳Naći jednadžbu normale u točki u kojoj je parametar $t=\frac{\pi}{4}$, ako je funkcija zadana parametarski $\begin{cases}x=R\cos t\\ y=R\sin t\end{cases}$, $t\in[0,2\pi]$.

1.) Naći točku za $t=\frac{\pi}{4}$:

$$x(T)=R\cdot\cos\frac{\pi}{4}=\frac{R}{\sqrt{2}}\ ;\quad y(T)=R\sin\frac{\pi}{4}=\frac{R}{\sqrt{2}}\quad\Rightarrow T\left(\frac{R}{\sqrt{2}},\frac{R}{\sqrt{2}}\right)$$

2.)

$$y' = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{R\cos t}{-R\sin t} = -\operatorname{ctg} t$$

$$k_T = y'(T) = y'\left(t=\frac{\pi}{4}\right) = -\operatorname{ctg}\frac{\pi}{4} = -1$$

$$k_N = -\frac{1}{k_T} = 1$$

$n \ldots \quad y - y_1 = k_N(x - x_1)$

$$y - \cancel{\frac{R}{\sqrt{2}}} = 1 \cdot \left(x - \cancel{\frac{R}{\sqrt{2}}}\right)$$

$$y = x$$

= Više derivacije parametarski zadane funkcije :

↳ Ako je poznata $(n-1)$ derivacija $y^{(n-1)}(x)$, onda je :

$$y^{(n)}(x) = \frac{dy^{(n-1)}(x)}{dx} \cdot \frac{dt}{dt} = \frac{dt}{dx} \cdot \frac{dy^{(n-1)}(x)}{dt} =$$

$$= \frac{1}{\frac{dx}{dt}} \cdot \frac{dy^{(n-1)}(x)}{dt} = \frac{1}{\dot{x}} \cdot \frac{d}{dt}\left(y^{(n-1)}\right) = y^{(n)}(x)$$

↳ Posebice :

$$y''(x) = \frac{1}{\dot{x}} \cdot (\dot{y'}) = \frac{1}{\dot{x}} \cdot \frac{(\dot{\frac{\dot{y}}{\dot{x}}})}{} = \frac{1}{\dot{x}} \cdot \frac{\ddot{y}\cdot\dot{x} - \dot{y}\cdot\ddot{x}}{(\dot{x})^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x})^3}$$

= **PRIMJER 5.**

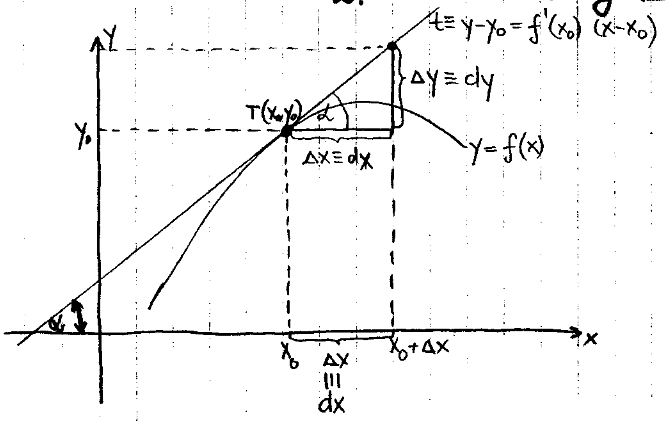
↳ Nađi $y'''(x)$ ako je $\begin{cases} x = t - \ln t \\ y = t + \ln t \end{cases}$, $t > 0$.

$$y'(x) = \frac{\dot{y}}{\dot{x}} = \frac{1 + \frac{1}{t}}{1 - \frac{1}{t}} = \frac{t+1}{t-1}$$

$$y''(x) = \frac{1}{\dot{x}} \cdot \dot{\left(y'(x)\right)} = \frac{1}{1 - \frac{1}{t}} \cdot \frac{1(t-1) - (t+1)\cdot 1}{(t-1)^2} = \frac{t}{t-1} \cdot \frac{-2}{(t-1)^2} = \frac{-2t}{(t-1)^3}$$

$$y'''(x) = \frac{1}{\dot{x}} \cdot \dot{\left(y''(x)\right)} = \frac{1}{1 - \frac{1}{t}} \cdot (-2) \cdot \frac{(t-1)^{\cancel{3}\,1} - t \cdot 3\,\cancel{(t-1)^2}}{(t-1)^{\cancel{6}\,4}} = \frac{4t^2 + 2t}{(t-1)^4}$$

10.2. DIFERENCIJAL FUNKCIJE



$$\operatorname{tg}\alpha = \frac{dy}{dx}$$

= Neka je $y = f(x)$ funkcija koja je diferencijabilna u točki $x = x_0$. Onda je :

$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. Odavdje slijedi :

$\Delta y = f'(x_0)\cdot\Delta x + \omega(\Delta x)\cdot\Delta x$, pri čemu je drugi član zanemariv u odnosu na prvi sumand jer je $\lim_{\Delta x \to 0} \omega(\Delta x) = 0$.

Sada Δy zovemo prirast funkcije, a tim izrazom je definiran diferencijal funkcije u točki $x = x_0$.

=DEFINICIJA 1.

↳ Neka je $y=f(x)$ diferencijabilna u točki x. Prvi diferencijal od f u x je izraz:

$$df = f'(x)\cdot dx$$

gdje smo umjesto Δx pisali dx, jer za $\Delta x \ll$ (jako mali) možemo identificirati $\Delta x \equiv dx$.

↳ Geometrijska interpretacija:

- Prvi diferencijal jednak je promjeni na tangenti povučenoj u točki x, ako se nezavisna varijabla promijeni za dx.

= Izračunavanje približne vrijednosti funkcije, aproksimiranjem prirasta funkcije njenim diferencijalom vrši se formulom:

$$f(x+\Delta x) \approx f(x) + f'(x)\cdot \Delta x \ , \text{ za } \Delta x \ll ,$$

jer za $\Delta x \ll$ vrijedi: $\Delta f \approx df$, te $\Delta f \approx f'(x)\cdot \Delta x$.

=PRIMJER 1.

↳ Izračunati približnu vrijednost $\sqrt{38}$.

$f(x)=\sqrt{x}$ (funkcija koju tražimo)

$x = 36$ (1 najbliža vrijednost iz koje znamo izvoditi korijen)

$\Delta x = 2$ (razlika zadane i najbliže vrijednosti)

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$$f(x+\Delta x) \approx f(x) + f'(x)\cdot\Delta x$$
$$f(38) \approx f(36) + f'(36)\cdot 2 \approx 6 + \frac{1}{12}\cdot 2 \approx \frac{37}{6} \approx 6{,}1\dot{6}$$

=PRIMJER 2.

↳ Izračunati približnu vrijednost $ctg\, 44^\circ$.

$f(x) = ctg x$

$x = 45^\circ$

$\Delta x = -1^\circ$

$f'(x) = -\dfrac{1}{\sin^2 x}$

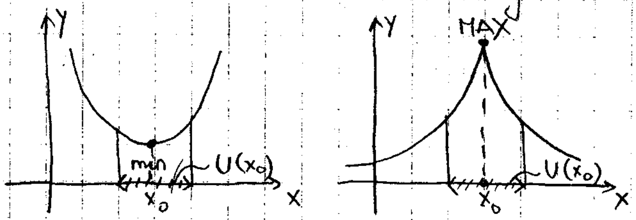
$$f(x+\Delta x) \approx f(x) + f'(x)\cdot \Delta x$$
$$f(44^\circ) \approx f(45^\circ) + f'(45^\circ)\cdot(-1^\circ) \approx$$
$$\approx ctg 45^\circ + \frac{1}{\sin^2 45^\circ}\cdot\frac{\pi}{180^\circ} \approx 1 + \frac{\pi}{90^\circ} \approx 1.0349$$

10.3. OSNOVNI TEOREMI DIFERENCIJALNOG RAČUNA

=DEFINICIJA 1.

↳ Kažemo da funkcija f ima u točki lokalni minimum (maksimum) ako postoji okoliš $U = U(x_0)$ točke x_0 takav da je $f(x_0)$ minimum funkcije, oznakom $\underset{U}{\min} f$, nad tim okolišem (maksimum funkcije, oznakom $\underset{U}{MAX} f$).

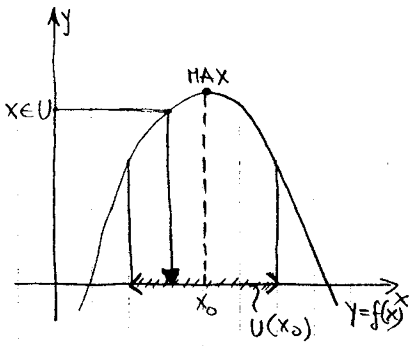
↳ Lokalni minimum i maksimum zajedno se zovu lokalni ekstremi.



= STAVAK 1. Fermatov teorem

↳ Ako funkcija f diferencijabilna na otvorenom intervalu I ima u točki $x_0 \in I$ lokalni ekstrem, onda vrijedi da je: $f'(x_0) = 0$.

↳ DOKAZ:



1) Neka f ima lokalni maksimum u točki x_0. Vrijedi: $(\exists U(x_0))(\forall x)(f(x) \le f(x_0))$, tj.:
$$f(x) - f(x_0) \le 0.$$
Prema pretpostavci je funkcija diferencijabilna u x_0, tj. postoji limes:
$$f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}.$$

a) Ako je $x > x_0$ tj. $x - x_0 > 0$, onda:
$$\frac{f(x)-f(x_0)}{x-x_0} = \frac{\ominus}{\oplus} = \ominus \Rightarrow \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \le 0$$

b) Ako je $x < x_0$, tj. $x - x_0 < 0$, onda:
$$\frac{f(x)-f(x_0)}{x-x_0} = \frac{\ominus}{\ominus} = \oplus \Rightarrow \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \ge 0$$

Obzirom da mora biti:
$$\lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0},$$
to je moguće jedino u slučaju kad je $f'(x_0) = 0$.

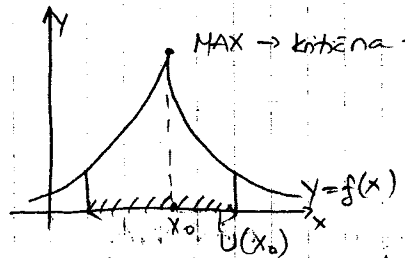
2) Za funkciju koja ima lokalni minimum, dokaz je analogan.

Q.E.D.

↳ Geometrijska interpretacija:
- U svakoj točki lokalnog ekstrema tangenta na graf diferencijabilne funkcije je horizontalna.

= PRIMJEDBA 1.

↳ Ako f nebi bila diferencijabilna na I moguće je da se ekstrem postiže u tzv. kritičnoj točki u kojoj nije $f'(x_0) = 0$, već derivacija u x_0 ne postoji. Takve, kritične točke, zovu se još i stacionarne točke, a dobivaju se tako da se $f'(x_0)$ izjednači s nulom.

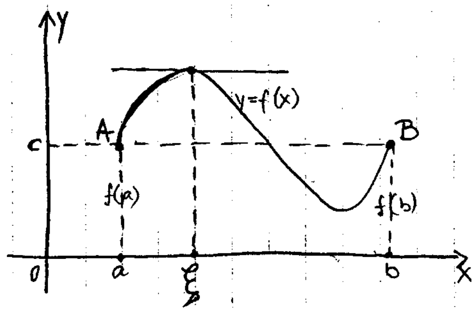


↳ Dakle, Fermatov teorem kaže da ako je funkcija f diferencijabilna na I, ima ekstreme, onda su ti ekstremi stacionarne točke.

- **STAVAK 2.** Rolleov teorem.

↳ Neka je funkcija f neprekinuta na intervalu $\bar{I} = [a, b]$ i diferencijabilna na otvorenom intervalu $I = \langle a, b \rangle$, te neka je $f(a) = f(b)$. Onda postoji točka ξ (ksi) $\in I$ takva da je $f'(\xi) = 0$.

↳ DOKAZ:

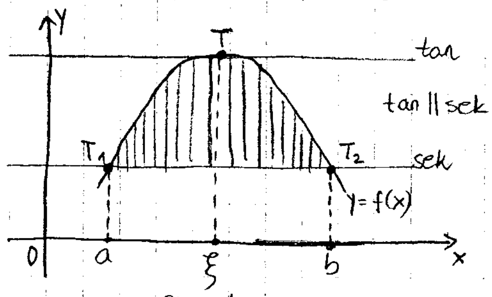


1) - Za $f(\bar{I}) = \{f(a)\} \Rightarrow f(x) = \text{konst. } c \Rightarrow f'(x) = 0$ na I, pa stavak vrijedi.

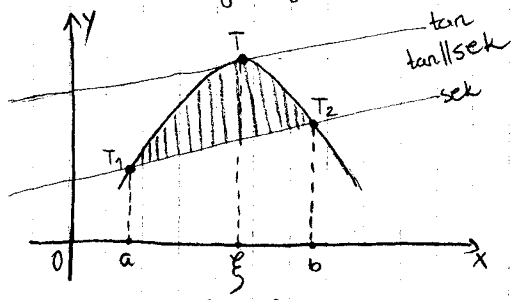
2) - Za netrivijalne sustave vrijedi:
↳ Prema Weierstrassovom teoremu za neprekinute funkcije nad zatvorenim intervalom, funkcija poprima $\min_I f$, $\max_I f$, od kojih je bar jedan ekstrem različit od $f(a) = f(b)$, te ga funkcija poprima u točki $\xi \in I$. Prema stavku 1. $\Rightarrow f'(\xi) = 0$

Q.E.D.

↳ Geometrijska interpretacija:
- Stavak 2. kaže da za svaku horizontalnu sekantu $\overline{T_1T_2}$, grafa diferencijabilne funkcije, postoji među točka T grafa, u kojoj je tangenta usporedna s tom sekantom. (SKICA 1.)
- To vrijedi i općenitije, za sekantu $\overline{T_1T_2}$ koja nije horizontalna. (SKICA 2.)



SKICA 1.



SKICA 2.

= **STAVAK 3.** Lagrangeov teorem o srednjoj vrijednosti.
↳ Neka je funkcija f neprekinuta na $\overline{I}=[a,b]$ i diferencijabilna na $I=\langle a,b\rangle$. Onda postoji točka $\xi\in I$ takva da je:

$$\frac{f(b)-f(a)}{b-a}=f'(\xi).$$

↳ DOKAZ:
- Definirajmo pomoćnu funkciju $F(x):=f(x)-\frac{f(b)-f(a)}{b-a}(x-a)$. Za $F(x)$ ispunjeni su uvjeti Rolleovog teorema. Posebice je: $F(a)=F(b)=f(a)$. Stoga postoji $\xi\in I$ takav da je $F'(\xi)=0$, tj.:

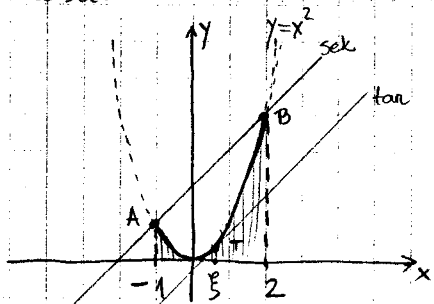
$$F(x)=\frac{f(b)-f(a)}{b-a}(x-a)$$

$$F'(\xi)=f'(\xi)-\frac{f(b)-f(a)}{b-a}=0$$

Q.E.D.

↳ Geometrijska interpretacija:
- Vidi Skicu 2.

= **PRIMJER 1.**
↳ Pronađi ξ za funkciju $y=x^2$ nad $I=[-1,2]$ primjenom stavka srednje vrijednosti.



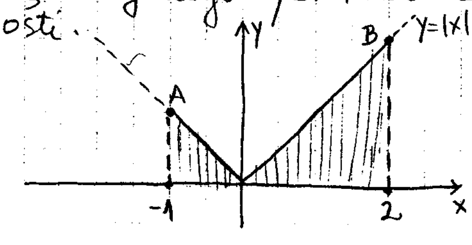
$$\frac{f(b)-f(a)}{b-a}=\frac{2^2-(-1)^2}{2-(-1)}=\frac{4-1}{2+1}=1$$

$$f'(\xi)=2\xi=1$$

$$\xi=\frac{1}{2}$$

$$T\left(\frac{1}{2},\frac{1}{4}\right)$$

= **PRIMJER 2.**
↳ Pronađi ξ za funkciju $y=|x|$ nad $I=[-1,2]$ primjenom stavka srednje vrijednosti.



- Obzirom da nisu ispunjeni uvjeti stavka 3 (tj. f. nije diferencijabilna) stavak nije moguće primjeniti.

= Stavak srednje vrijednosti ima sljedeći važan korolar:

KOROLAR 1. Obrat formule $c'=0$

↳ Ako je funkcija f neprekinuta na $\overline{I}=[a,b]$ i diferencijabilna na $I=\langle a,b\rangle$, te ako je $f'(x)=0$ na I, onda je funkcija konstanta na I.

↳ DOKAZ:

- Za koje god $x_1, x_2 \in \overline{I}$, $x_1 \neq x_2$, vrijede uvjeti stavka srednje vrijednosti, te postoji $\xi \in \langle x_1, x_2\rangle$ takav da je:

$$f'(\xi)=\frac{f(x_2)-f(x_1)}{x_2-x_1}=0 \Rightarrow f(x_2)=f(x_1)$$

Budući da to vrijedi za bilo koje $x_1, x_2 \in \overline{I}$, očevidno je f konstanta na $\overline{I}$.

Q.E.D.

= Općenitije vrijedi:

KOROLAR 2.

↳ Neka je funkcija f diferencijabilna na $I=\langle a,b\rangle$.

a) Ako je $f'(x)>0$ na I onda je f monotono (strogo) rastuća na I.

b) Ako je $f'(x)<0$ na I onda je f monotono (strogo) padajuća na I.

↳ DOKAZ:

-(a) Uzmimo bilo koje x_1 i $x_2 \in I$, $x_1<x_2$. Prema stavku srednje vrijednosti postoji neki $\xi \in \langle x_1, x_2\rangle$ takav da je:

$$f(x_2)-f(x_1)=f'(\xi)\,(x_2-x_1)=\oplus\cdot\oplus=\oplus>0$$

Dakle $f(x_1)<f(x_2)$, tj. funkcija je monotono rastuća.

(b) Analogan dokaz.

Q.E.D.

= **STAVAK 4.** Cauchyjev teorem srednje vrijednosti.

↳ Neka su f i g funkcije neprekinute na $\overline{I}=[a,b]$ i diferencijabilne na $I=\langle a,b\rangle$, te neka je $g'(x)\neq 0$, $\forall x\in I$. Onda je $g(b)\neq g(a)$, i postoji $\xi\in I$ takav da je:

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}.$$

= **PRIMJEDBA 2.**

↳ Za $g(x)=x$ dobiva se iz Cauchyjevog teorema, Lagrangeov teorem srednje vrijednosti, tj.:

$$\frac{f(b)-f(a)}{b-a}=f'(\xi).$$

10.4. TAYLOROVA FORMULA

= Ako izvodimo obradu neke funkcije čija je jednadžba komplicirana, dolazimo na ideju o aproksimaciji promatrane funkcije, na način da je u okolišu neke točke aproksimiramo funkcijom s kojom lakše računamo. Taylorova formula daje prikaz zadane funkcije kao zbroj nekog polinoma i ostatka.

= **DEFINICIJA 1.**

↳ Neka je funkcija f definirana na $I=\langle a,b\rangle$ koja je u točki $x_0\in I$ n-puta diferencijabilna. Polinom n-tog stupnja:

$$T_n(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\dots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$=\sum_{r=0}^{n}\frac{f^{(r)}(x_0)}{r!}(x-x_0)^r$$

zove se Taylorov polinom n-tog stupnja od f u točki x_0.

Samo je jedan Mali Ivica!

↳ Ako je $x_0=0$ Taylorov polinom nazivamo Maclaurinovim polinomom:

$$T_n(x)=\sum_{r=0}^{n}\frac{f^{(r)}(0)}{r!}(x^r)$$

= PRIMJER 1.

↳ Naći Maclaurinove funkcije, funkcije $f(x)=e^x$, za različite $n\leq 4$. Skicirati funkciju i grafove polinoma T_1, T_2, T_3, T_4.

$f(x)=e^x$

$f^n(x)=0\ ,\ \forall n$

$f^n(0)=1\ ;\ \forall n$

$$T_0(x)=f(0)=e^0=1$$

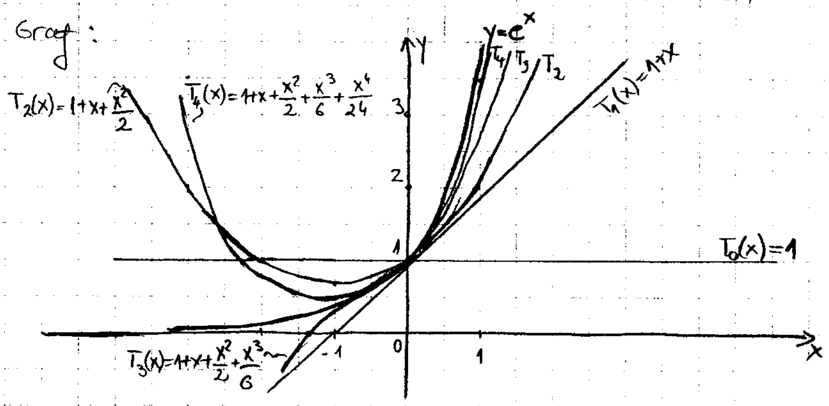
$$T_1(x)=f(0)+\frac{f'(0)}{1!}\cdot x=1+\frac{1}{1!}x=1+x$$

$$T_2(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f'(0)}{2!}x^2=1+x+\frac{1}{2}x^2$$

$$T_3(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}$$

$$T_4(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}$$

Graf:



– Vidi se: porast stupnja polinoma postiže da se graf polinoma sve jače približava grafu funkcije e^x u okolini točke $x_0=0$.

= STAVAK 1.

↳ Neka je funkcija f definirana na I, i neka je na I $(n+1)$-puta diferencijabilna, te neka je $x_0\in I$. Onda se f može na I prikazati s pomoću Taylorove formule:

$$f(x)=T_n(x)+R_n(x),$$

dakle:

$$\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+R_n(x),$$

pri čemu je $R_n(x)$ tzv. n-ti ostatak, kojeg zapisujemo u Lagrangeovom obliku kao:

$$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}\cdot(x-x_0)^{n+1},$$

gdje je ξ neka točka iz intervala $\langle x_0,x\rangle$, tj. $\xi=x_0+\vartheta(x-x_0)$, $\vartheta\in\langle 0,1\rangle$.

= PRIMJER 2.

↳ Primijeniti Maclaurinovu formulu, te procjenom ostatka od $R_4(x)$ približno izračunati $\sqrt{e}$ i e^2.

– Očito koristimo funkciju $f(x)=e^x$, pa vrijedi:

$$e^x=\sum_{k=0}^{4}\frac{x^k}{k!}+R_4(x)\ ,\ R_4(x)=\frac{x^5}{5!}\cdot e^{\xi}\ ,\ \xi\in\langle 0,x\rangle$$

– $\sqrt{e}=e^{\frac{1}{2}}$

$$R_4\left(\tfrac{1}{2}\right)=\frac{e^{\xi}}{5!}\left(\frac{1}{2}\right)^5\ ,\ \xi\in\langle 0,\tfrac{1}{2}\rangle \qquad R_4\left(\tfrac{1}{2}\right)\leq\frac{2}{5!}\left(\frac{1}{2}\right)^5$$

$$e^{\xi}\in\langle 1,\sqrt{e}\rangle\subseteq\langle 1,2\rangle \qquad \approx 0{,}0005$$

Ostatak:

$$\sqrt{e} = 1 + \frac{1}{2!}\left(\frac{1}{2}\right)^2 + \frac{1}{3!}\left(\frac{1}{2}\right)^3 + \frac{1}{4!}\left(\frac{1}{2}\right)^4 \approx 1{,}6484$$

- e^2

$$R_4(2) = \frac{e^{\xi}}{5!} \cdot 2^5 \qquad \xi \in \langle 0,2 \rangle \qquad R_4(2) \le \frac{9}{5!} \cdot 2^5 \approx 2.4$$

$$e^{\xi} \in \langle 1, e^2 \rangle$$

Ostatak:

$$e^2 = 1 + \frac{1}{2!} \cdot 2^2 + \frac{1}{3!} \cdot 2^3 + \frac{1}{4!} \cdot 2^4 = 5$$

10.5. L'HOSPITALOVO PRAVILO

= L'Hospitalovo pravilo služi za računanje limesa kvocijenta dviju funkcija, tj. $\lim \frac{f(x)}{g(x)}$, kada on ima neodređeni oblik $\frac{0}{0}$ ili $\frac{\infty}{\infty}$.

= **STAVAK 1.** L'Hospitalovo pravilo $\frac{0}{0}$

↳ Neka su funkcije f i g diferencijabilne na $\langle a, x_0 \rangle \cup \langle x_0, b \rangle = S$, te neka je $g'(x) \neq 0$ na S. Ako vrijedi:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0,$$

i ako postoji:

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} \in \overline{\mathbb{R}},$$

onda je:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

= **STAVAK 2.** L'Hospitalovo pravilo $\frac{\infty}{\infty}$

↳ Neka su funkcije f i g diferencijabilne na $\langle a, x_0 \rangle \cup \langle x_0, b \rangle = S$, te neka je $g'(x) \neq 0$ na S. Ako vrijedi:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \infty,$$

i postoji:

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} \in \overline{\mathbb{R}},$$

onda je:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

= **PRIMJER 1.**

↳ Izračunajte sljedeće limese:

a) $$\lim_{x \to 0} \frac{\operatorname{arth} x}{\operatorname{sh}(2x)} = \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{\frac{1}{1-x^2}}{2\cos 2x} = \frac{\frac{1}{1-0}}{2 \cdot 1} = \frac{1}{2}$$

b) $$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{1 - \cos x}{3x^2} = \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{\sin x}{6x} = \left(\frac{0}{0}\right) =$$
$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}$$

c) $$\lim_{x \to 0^+} \frac{\ln x}{\operatorname{ctg} x} = \left(\frac{-\infty}{+\infty}\right) = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{\sin^2 x}} = -\lim_{x \to 0^+} \frac{\sin x}{x} \cdot \lim_{x \to 0^+} \sin x = -1 \cdot 0 = 0$$

d) $\lim\limits_{x\to+\infty} \frac{shx}{chx} = \left(\frac{\infty}{\infty}\right) = \lim\limits_{x\to+\infty} \frac{chx}{shx} = \left(\frac{\infty}{\infty}\right) = \lim\limits_{x\to+\infty} \frac{shx}{chx} \dots$

↳ očito je da ovu funkciju nije moguće rastaviti (riješiti) na ovaj način pa treba paziti kada L'Hospitalovo pravilo pomaže, a kada ne
↳ u ovom slučaju:

$$\lim_{x\to+\infty} \frac{shx}{chx} = \lim_{x\to+\infty} thx = 1$$

e) $\lim\limits_{x\to 0} \frac{x^2\cos\frac{1}{x}}{\sin x} = \lim\limits_{x\to 0} \frac{x}{\sin x} \cdot \lim\limits_{x\to 0} x\cdot\cos\frac{1}{x} = 1\cdot 0 = 0$

- Drugi neodređeni oblici ($0\cdot\infty$; $\infty-\infty$; $0^0, 1^\infty, \infty^0$) svodimo već na spomenute oblike po sljedećim pravilima:

(1) oblici tipa $0\cdot\infty$:

$$\lim_{x\to x_0} [f(x)\cdot g(x)] = (\infty\cdot 0) = \lim_{x\to x_0} \frac{f(x)}{\frac{1}{g(x)}} = \left(\frac{\infty}{\infty}\right)$$

ili

$$= \lim_{x\to x_0} \frac{g(x)}{\frac{1}{f(x)}} = \left(\frac{0}{0}\right)$$

(2) oblici tipa $\infty - \infty$:

$$\lim_{x\to x_0} [f(x)-g(x)] = (\infty-\infty) = \lim_{x\to x_0} f(x)\left(1-\frac{g(x)}{f(x)}\right) = \begin{cases} \infty\cdot 0 & \lim\limits_{x\to x_0}\frac{g(x)}{f(x)} = 1 \\ \pm\infty & \lim\limits_{x\to x_0}\frac{g(x)}{f(x)} \neq 1 \end{cases}$$

(3) oblici tipa: $\infty^0, 0^0, 1^\infty$:

$$\lim_{x\to x_0} \left[f(x)^{g(x)}\right] = (\infty^0 \text{ ili } 0^0 \text{ ili } 1^\infty) = \lim_{x\to x_0} e^{g(x)\ln f(x)} = e^{\lim\limits_{x\to x_0} g(x)\ln f(x)}$$

= PRIMJER 2.
↳ Izračunajte sljedeće limese:

a) $\lim\limits_{x\to 0^+} x\ln x = (0^+\cdot(-\infty)) = \lim\limits_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \left(\frac{-\infty}{+\infty}\right) = \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0^+}(-x) = 0$

b) $\lim\limits_{x\to\infty} (x\,arth\,x - x) = (\infty-\infty) = \lim\limits_{x\to\infty} \frac{arth\,x - 1}{\frac{1}{x}} = \left(\frac{0}{0}\right) = \lim\limits_{x\to\infty} \frac{\frac{1}{1-x^2}}{-\frac{1}{x^2}} =$

$$= \lim_{x\to\infty} \frac{-x^2}{1-x^2} = 1$$

Samo je jedan Mali Ivica!

c) $\lim\limits_{x\to\infty}(\operatorname{ch}x-\operatorname{sh}x)=(\infty-\infty)=\lim\limits_{x\to\infty}\operatorname{ch}x\,(1-\operatorname{th}x)=(\infty\cdot 0)=$

$$=\lim_{x\to\infty}\frac{1-\operatorname{th}x}{\frac{1}{\operatorname{ch}x}}=\left(\frac{0}{0}\right)=\lim_{x\to\infty}\frac{-\frac{1}{\operatorname{ch}^2x}}{\frac{-\operatorname{sh}x}{\operatorname{ch}^2x}}=$$

$$=\lim_{x\to\infty}\frac{1}{\operatorname{sh}x}=0$$

d) $\lim\limits_{x\to 0}x^{\sin x}=(0^0)=\lim\limits_{x\to 0}e^{\sin x\ln x}=e^{\lim\limits_{x\to 0}\sin x\ln x}=e^0=1$

$$\hookrightarrow\lim_{x\to 0}\sin x\ln x=(0(-\infty))=\lim_{x\to 0}\frac{\ln x}{\frac{1}{\sin x}}=\left(\frac{\infty}{\infty}\right)=0$$

e) $\lim\limits_{x\to\infty}x\cdot\ln\left(1+\frac{1}{x}\right)=(\infty\cdot 0)=\lim\limits_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}=\left(\frac{0}{0}\right)=$

$$=\lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot(-1\cdot x^{-2})}{-1\cdot x^{-2}}=\lim_{x\to\infty}\frac{-\frac{1}{x^{\cancel{2}}}\cdot\frac{\cancel{x}}{x+1}}{-\frac{1}{x^2}}=$$

$$=\lim_{x\to\infty}\frac{\frac{\cancel{-}1}{\cancel{x}(x+1)}}{\cancel{-}\frac{1}{x^{\cancel{2}}}}=\lim_{x\to\infty}\frac{x}{x+1}\begin{matrix}:x\\:x\end{matrix}=\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}=1$$

Samo je jedan Mali Ivica!

ZADACI

1. $y' = ?$ $\quad x^3 + y^3 = 3axy \; /'$

$$3x^2 + 3y^2 \cdot y' = 3a(y + x \cdot y')$$
$$3x^2 + 3y^2 y' = 3ay + 3axy'$$
$$y'(3y^2 - 3ax) = 3ay - 3x^2$$
$$y' = \frac{ay - x^2}{y^2 - ax}$$

2. $y' = ?$ $\quad x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \; /'$

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}} \cdot y' = 0$$
$$y' = \frac{-\cancel{\frac{2}{3}}x^{-\frac{1}{3}}}{\cancel{\frac{2}{3}}y^{-\frac{1}{3}}} = \frac{-\frac{1}{\sqrt[3]{x}}}{\frac{1}{\sqrt[3]{y}}} = -\sqrt[3]{\frac{y}{x}}$$

3. $t \ldots = ?$ $\quad y^4 = 4x^4 + 6xy$

$A(1,2)$

$$y = f(x) \; /'$$
$$4y^3 \cdot y' = 16x^3 + 6y + 6xy'$$
$$y'(4y^3 - 6x) = 16x^3 + 6y$$
$$y' = \frac{8x^3 + 3y}{2y^3 - 3x}$$

$t \ldots$ $\quad y - y_1 = k_T(x - x_1)$

$$y - 2 = -11(x - 1)$$
$$y = -11x + 11 + 2$$
$$y = -11x + 13$$

$$k_T = y'(1,2) = \frac{8+3}{2-3} = \frac{11}{-1} = -11$$

4. $y = 3x^4 + 4x^3 - 12x^2 + 20$, točke$=?$, $t \parallel$ os x

$$y' = 12x^3 + 12x^2 - 24x$$
$$y' = 0 \Rightarrow 12x^3 + 12x^2 - 24x = 0$$
$$x(x^2 + x - 2) = 0$$
$$x_1 = 0 \qquad x_{2,3} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$$
$$x_2 = 1 \qquad x_3 = -2$$

$x_1 = 0 \ldots \quad y_1 = 20 \quad \Rightarrow T_1(0, 20)$

$x_2 = 1 \ldots \quad y_2 = 3 + 4 - 12 + 20 = 15 \quad \Rightarrow T_2(1, 15)$

$x_3 = -2 \quad y_3 = 3 \cdot (-2)^4 + 4 \cdot (-2)^3 - 12(-2)^2 + 20 = -12 \quad \Rightarrow T_3(-2, -12)$

5. $x=\frac{1+t}{t^3}$, $y=\frac{3}{2t^2}+\frac{1}{2t}$, $t=?$ $A(2,2)$.

$$y'(x)=\frac{\dot y}{\dot x}=\frac{\left(\frac{3}{2t^2}+\frac{1}{2t}\right)'}{\left(\frac{1+t}{t^3}\right)'}=\frac{\frac{3}{2}\cdot\frac{-2t}{t^4}+\frac{1}{2}\cdot\frac{-1}{t^2}}{\frac{t^3-3t^2(1+t)}{t^6}}=\frac{-\frac{3}{t^3}-\frac{1}{2t^2}}{\frac{-2t-3}{t^4}}=\frac{(6+t)t}{2(2t+3)}$$

$t=?\ldots$ $2=\frac{1+t}{t^3}\ /\cdot t^3$

$2t^3-t-1=0$

$t_1=1$ ... $(2t^3-t-1):(t-1)=2t^2+2t+1=0$

$-2t^3+2t^2$

$2t^2-t-1$

$-2t^2+2t$

$t-1$

$-t+1$

0

$\cancel{t_{2,3}=\frac{-2\pm\sqrt{4-9}}{4}}$

$k_T=y'(1)=\frac{(6+1)1}{2(2\cdot1+3)}=\frac{7}{10}$, $A(2,2)$

$t\ldots$ $y-y_0=k_T(x-x_0)$

$y-2=\frac{7}{10}(x-2)$

$y=\frac{7}{10}x+\frac{3}{5}$

6. $y''=?$ $T\left(\frac{3\sqrt2}{2},\frac{3\sqrt2}{2}\right)$, $x=3\cos t$, $y=3\sin t$, $0\le t<2\pi$

$t=?\ldots$ $x=3\cos t$

$3\sqrt2=6\cos t$

$\cos t=\frac{\sqrt2}{2}\Rightarrow t=\frac{\pi}{4}$, $\in[0,2\pi\rangle$

$$y'=\frac{\dot y}{\dot x}=\frac{3\cos t}{-3\sin t}=-\operatorname{ctg} t$$

$$y''=\frac{\dot{(y')}}{\dot x}=\frac{1}{\sin^2 t}\cdot\frac{1}{-3\sin t}=\frac{-1}{3\sin^3 t}$$

$$y''\left(\frac{\pi}{4}\right)=\frac{-1}{3\sin^3\frac{\pi}{4}}=\frac{-1}{3\cdot\left(\frac{\sqrt2}{2}\right)^3}=-\frac{8}{3\cdot2\sqrt2}=\frac{-4}{3\sqrt2}=\frac{-4\sqrt2}{6}=\frac{-2\sqrt2}{3}$$

7. $t\ldots=?$ $x=a\cos^3 t$, $y=a\sin^3 t$, $t=\frac{\pi}{4}$

$$y'(x)=\frac{\dot y}{\dot x}=\frac{\cancel{a}\cdot\cancel{3}\sin^2 t\cos t}{-\cancel{a}\cancel{3}\cos^2 t\sin t}=-\operatorname{tg} t$$

$k_T=y'(x)=y'\left(\frac{\pi}{4}\right)=-\operatorname{tg}\frac{\pi}{4}=-1$

$t\ldots$ $y-y_0=k_T(x-x_0)$

$y-\frac{a\sqrt2}{4}=-1\left(x-\frac{a\sqrt2}{4}\right)$

$y=-x+\frac{a\sqrt2}{2}$

$y=x=a\cos^3\frac{\pi}{4}=\frac{a}{2\sqrt2}=\frac{a\sqrt2}{4}$ $T\left(\frac{a\sqrt2}{4},\frac{a\sqrt2}{4}\right)$

(8.) $y', y'', y''' = ?$ $\quad x = t^2+1$, $\quad y = 3t+e^t$

$$y' = \frac{\dot{y}}{\dot{x}} = \frac{3+e^t}{2t}$$

$$y'' = \frac{1}{\dot{x}} \cdot (\dot{y'}) = \frac{1}{\cancel{2}t} \cdot \frac{e^t \cdot \cancel{2}t + (3+e^t)\cancel{2}}{4t^2} = \frac{te^t + e^t + 3}{4t^3}$$

$$y''' = \frac{1}{2t} \cdot \frac{(e^t + te^t + e^t)4t^3 + 12t^2(te^t + e^t + 3)}{16t^6} =$$

$$= \frac{[(te^t + 2e^t)t + 3(te^t + e^t + 3)] \cdot \cancel{4t^2}}{\cancel{32t^7}} = \frac{te^t(t+2) + 3(te^t + e^t + 3)}{8t^5} \ //$$

(9.) $c = ?$ Lagrangeov teorem : $f(x) = \sin x$, $I = [0, \frac{\pi}{2}]$.

$a = 0$

$b = \frac{\pi}{2}$

$$\frac{f(b)-f(a)}{b-a} = \frac{\sin\frac{\pi}{2} - \sin 0}{\frac{\pi}{2}} = \frac{2}{\pi}$$

$f'(x) = \cos x$ $\qquad f'(c) = \cos c$

$$\cos c = \frac{2}{\pi} \Rightarrow c = \arccos\frac{2}{\pi} \ //$$

(10.) $c = ?$ Lagrangeov teorem : $f(x) = \arcsin x$, $I = [-1, 1]$

$a = -1$

$b = 1$

$$\frac{f(b)-f(a)}{b-a} = \frac{\arcsin 1 - \arcsin(-1)}{1+1} = \frac{\frac{\pi}{2} + \frac{\pi}{2}}{2} = \frac{\pi}{2}$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} \qquad f'(c) = \frac{1}{\sqrt{1-c^2}}$$

$$\frac{1}{\sqrt{1-c^2}} = \frac{\pi}{2} \Rightarrow 2 = \pi\sqrt{1-c^2} \ /^2$$

$$4 = \pi^2(1-c^2)$$

$$4 = \pi^2 - \pi^2 c^2$$

$$\pi^2 c^2 = \pi^2 - 4$$

$$c^2 = \frac{\pi^2-4}{\pi^2} \Rightarrow c = \frac{\sqrt{\pi^2-4}}{\pi} \ //$$

(11.) $f(x) = \operatorname{ch} x$, Taylor - oko 0.

$c = 0$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} \cdot (x-c)^k$$

1) $f(0) = \operatorname{ch} 0 = 1$

2) $f'(0) = \operatorname{sh} x = \operatorname{sh} 0 = 0$

3) $f''(0) = \operatorname{ch} x = \operatorname{ch} 0 = 1$

$$T_n = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots + \frac{x^{2n}}{(2n)!} = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} \ //$$

Samo je jedan Mali Ivica!

12. $f(x)=\operatorname{sh}x$, Taylor-oko 0.
$c=0$

$$T_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k$$

1) $f(0)=\operatorname{sh}0=0$
2) $f'(0)=\operatorname{ch}x=\operatorname{ch}0=1$
3) $f''(0)=\operatorname{sh}x=\operatorname{sh}0=0$

$$T_n(x)=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots+\frac{x^{2n-1}}{(2n-1)!}=\sum_{k=0}^{n}\frac{x^{2k-1}}{(2k-1)!}$$

13. $f(x)=x^3+2x^2-x+1$, Taylor, polinom u $(x-1)$
$c=1$

$$T_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k$$

1) $f(1)=1+2-1+1=3$
2) $f'(1)=3x^2+4x-1=3+4-1=6$
3) $f''(1)=6x+4=6+4=10$
4) $f'''(1)=6$

$$T_n(x)=3+6(x-1)+5(x-1)^2+(x-1)^3$$

14. $$\lim_{x\to0}\left[\frac{\sin(3x)}{3x}\right]^{\frac1x}=(0^\infty)=\lim_{x\to0}e^{\ln\left[\frac{\sin(3x)}{3x}\right]^{\frac1x}}=\lim_{x\to0}e^{\frac1x\ln\frac{\sin(3x)}{3x}}=$$

$$=\lim_{x\to0}e^{\lim\limits_{x\to0}\frac1x\ln\frac{\sin(3x)}{3x}}=\lim_{x\to0}e^{A}=\lim_{x\to0}e^{-\infty}=0$$

$$A=\lim_{x\to0}\frac{\ln\frac{\sin(3x)}{3x}}{x}=\left(\frac00\right)=\lim_{x\to0}\frac{\frac{\cancel{3x}}{\sin(3x)}\cdot\frac{\cancel{3}x\cos(3x)-\sin(3x)\cdot\cancel{3}}{\cancel{9}x^2}}{1}=\lim_{x\to0}\frac{x\cos(3x)-\sin(3x)}{x\sin(3x)}=$$

$$=\left(\frac00\right)=\lim_{x\to0}\frac{\cos3x-\cancel{3x\sin(3x)}-3\cos(3x)}{\cancel{\sin(3x)}+3x\cos(3x)}=\frac{1-3}{0}=-\infty$$

15. $$\lim_{x\to0}\sqrt{x}\ln x=\lim_{x\to0}\frac{\ln x}{\frac{1}{\sqrt x}}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to0}\frac{\frac{1}{\cancel{x}}}{-\frac{1}{2\cancel{x}\sqrt x}}=\lim_{x\to0}(-2)\sqrt x=0$$

16. $$\lim_{x\to\frac\pi2}(\cos x)^{\frac\pi2-x}=\left|\begin{array}{l}\text{SUPST:}\\ \frac\pi2-x=y\Rightarrow y\to0\\ x=\frac\pi2-y\end{array}\right|=\lim_{y\to0}\left(\cos\left(\frac\pi2-y\right)\right)^y=\lim_{y\to0}(\sin y)^y=\lim_{y\to0}e^{\ln(\sin y)^y}=$$

$$=\lim_{y\to0}e^{\lim\limits_{y\to0}y\cdot\ln(\sin y)}=\lim_{y\to0}e^{A}=e^{+\infty}=+\infty$$

$$A=\lim_{y\to0}\frac{\ln(\sin y)}{\frac1y}=\left(\frac\infty\infty\right)=\lim_{y\to0}\frac{\frac{1}{\sin y}\cdot\cos y}{-\frac{1}{y^2}}=-\lim_{y\to0}\frac{y^2\cos y}{\sin y}=\left(\frac00\right)=$$

$$=\lim_{y\to0}\frac{2y\cos y-y^2\cdot\sin y}{\cos y}=\frac01=+\infty$$

17. $$\lim_{x\to0}\left(\frac{\operatorname{tg}x}{x}\right)^{\frac1x}=\left(\left(\frac00\right)^\infty\right)=\lim_{x\to0}e^{\ln\left(\frac{\operatorname{tg}x}{x}\right)^{\frac1x}}=\lim_{x\to0}e^{\lim\limits_{x\to0}\frac1x\ln\frac{\operatorname{tg}x}{x}}=\lim_{x\to0}e^{A}=$$

$$=e^0=1$$

$$A=\lim_{x\to 0}\frac{1}{x}\ln\frac{tgx}{x}=\lim_{x\to 0}\frac{\ln\frac{tgx}{x}}{x}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\frac{\cancel{x}}{tgx}\cdot\frac{\frac{1}{\cos^2x}\cdot x-tgx}{x^2}}{1}=$$

$$=\lim_{x\to 0}\frac{\frac{x-\sin x\cos x}{\cos^2x}}{x\,tgx}=\lim_{x\to 0}\frac{x-\sin x\cos x}{x\sin x\cos x}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{x-\frac{1}{2}\sin(2x)}{x\cdot\frac{1}{2}\sin(2x)}=$$

$$=\lim_{x\to 0}\frac{\cancel{2}(2x-\sin(2x))}{\cancel{2}x\cdot\sin(2x)}=\lim_{x\to 0}\frac{2-2\cos 2x}{\sin(2x)+2x\cos(2x)}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{4\sin(2x)}{2\cos(2x)+2\cos(2x)-4x\sin(2x)}$$

$$=\frac{0}{2+2}=0$$

(18.) $$\lim_{x\to-\infty}\frac{\ln(ch(x+3))}{x}=\lim_{x\to+\infty}\frac{\ln(ch(3-x))}{-x}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{\frac{1}{ch(3-x)}sh(3-x)\cdot(-1)}{-1}=$$

$$=\lim_{x\to\infty}th(3-x)=th(-\infty)=-1$$

(19.) $$\lim_{x\to 0}\frac{\ln\frac{\sin x}{x}}{x^2}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\frac{\cancel{x}}{\sin x}\cdot\frac{x\cos x-\sin x}{x^2}}{2\cancel{x}}=\lim_{x\to 0}\frac{x\cos x-\sin x}{2x^2\sin x}=\left(\frac{0}{0}\right)=$$

$$=\lim_{x\to 0}\frac{\cancel{\cos x}-x\sin x-\cancel{\cos x}}{2(2x\sin x+2x^2\cos x)}=\lim_{x\to 0}\frac{-x\sin x}{4(x\sin x+x^2\cos x)}=\left(\frac{0}{0}\right)=$$

$$=\lim_{x\to 0}\frac{-\sin x-x\cos x}{4(\sin x+x\cos x+2x\cos x-x^2\sin x)}=\left(\frac{0}{0}\right)=$$

$$=\lim_{x\to 0}\frac{-\cos x-\cos x+x\sin x}{4(\cos x+\cos x-x\sin x+2\cos x-2x\sin x-2x\sin x-x^2\cos x)}=$$

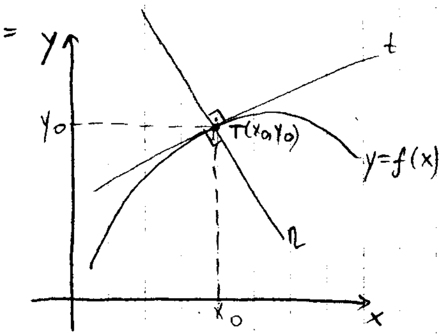
$$=\lim_{x\to 0}\frac{-2\cos x+x\sin x}{4(4\cos x-5x\sin x-x^2\cos x)}=\frac{-2}{4(4-0)}=\frac{-2}{16}=-\frac{1}{8}$$

(20.) $$\lim_{x\to-\infty}\left(xe^{-\frac{1}{x^2}}-x\right)=\lim_{x\to\infty}\left(x-xe^{-\frac{1}{x^2}}\right)=\lim_{x\to\infty}\left[x\left(1-e^{-\frac{1}{x^2}}\right)\right]=\lim_{x\to\infty}\frac{1-e^{-\frac{1}{x^2}}}{\frac{1}{x}}=$$

$$=\left(\frac{0}{0}\right)=\lim_{x\to\infty}\frac{-e^{-\frac{1}{x^2}}\cdot\left(+\frac{2}{x^3}\right)}{-\frac{1}{\cancel{x^2}}}=\lim_{x\to\infty}\frac{e^{-\frac{1}{x^2}}\cdot 2}{\cancel{x}}=\frac{e^0\cdot 2}{\infty}=\frac{2}{\infty}=0$$

11. PRIMJENE DIFERENCIJALNOG RAČUNA

11.1. TANGENTA I NORMALA NA GRAF FUNKCIJE



→ Tangenta na graf funkcije f u točki (x_0, y_0) ima jednadžbu:

$$t \ldots \quad y-y_0 = f'(x_0)(x-x_0)$$

→ Normala na graf funkcije f u točki (x_0, y_0) ima jednadžbu:

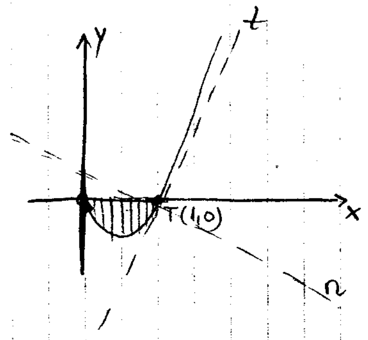
$$n \ldots \quad y-y_0 = -\frac{1}{f'(x_0)}(x-x_0)$$

=PRIMJER 1.

↳ Nađi jednadžbu tangente i normale na graf funkcije $y = x\ln x$ u točki $x_0 = 1$. Nacrtaj skicu.

$$y' = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

$$\left.\begin{array}{l} k_T = y'(1) = \ln 1 + 1 = 1 \\ T(1,0) \end{array}\right\} \begin{array}{l} t \ldots \; y-0 = 1(x-1) \\ \qquad y = x-1 \\ n \ldots \; y-0 = -1\,(x-1) \\ \qquad y = 1-x \end{array}$$



=PRIMJER 2.

↳ Nađi jednadžbu tangente i normale na graf funkcije $x^4 + y^4 - y\ln x - 2 = 0$ u točki grafa $T(1,1)$.

$$x^4 + y^4 - y\ln x - 2 = 0 \;\Big/ \frac{d}{dx}$$

$$4x^3 + 4y^3 y' - y'\ln x - y\frac{1}{x} = 0$$

$$y'(4y^3 - \ln x) = \frac{y}{x} - 4x^3$$

$$y' = \frac{\frac{y}{x} - 4x^3}{4y^3 - \ln x}$$

$$\left.\begin{array}{l} k_T = y'(1,1) = \dfrac{1-4}{4-0} = -\dfrac{3}{4} \\ T(1,1) \end{array}\right\} \begin{array}{l} t \ldots \; y-1 = -\frac{3}{4}(x-1) \\ n \ldots \; y-1 = \frac{4}{3}(x-1) \end{array}$$

=PRIMJER 3.

↳ Nađi jednadžbu tangente i normale na graf funkcije zadane parametarski formulom: $\begin{cases} x = t + e^t \\ y = t - e^t \end{cases}$ u točki kojoj odgovara parametar $t = 0$.

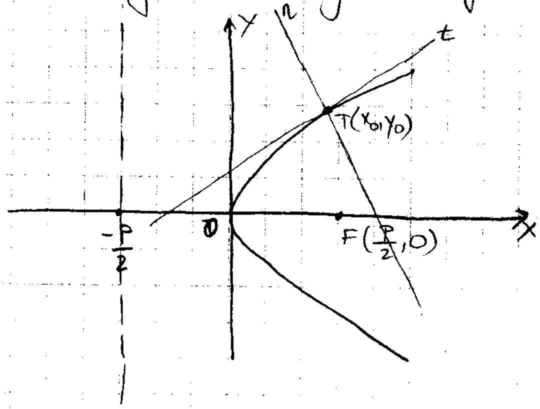
$$T(x_0, y_0) \ldots \left.\begin{array}{l} x_0 = 0 + e^0 = 1 \\ y_0 = 0 - e^0 = -1 \end{array}\right\} T(1,-1)$$

$$y'(x) = \frac{\dot{y}}{\dot{x}} = \frac{1-e^t}{1+e^t}$$

$$\left.\begin{array}{l} k_T = y'(T) = \frac{1-1}{1+1} = 0 \\ T(1,-1) \end{array}\right\} \begin{array}{l} t \ldots y+1 = 0(x-1) \\ \quad y=-1 \\ n \ldots y+1 = -\frac{0}{0}(x-1) \\ \quad \Downarrow \\ \quad x = 1 \end{array}$$

=PRIMJER 4.

↳Nađi jednadžbu tangente na funkciju: $y^2 = 2px$ u točki $T(x_0, y_0)$.



$\boxed{p>0}$

$$y^2 = 2px \Big/ \frac{d}{dx}$$

$$2yy' = 2p$$

$$y' = \frac{2p}{2y} = \frac{p}{y}$$

$$k_T = y'(x_0, y_0) = \frac{p}{y_0}$$

$$T(x_0, y_0)$$

$$t \ldots \; y - y_0 = \frac{p}{y_0}(x - x_0)$$

$$y - y_0 = \frac{px}{y_0} - \frac{px_0}{y_0}$$

⇒ kako je $T(x_0, y_0)$ na funkciji slijedi:

$$2px_0 = y_0^2$$

$$x_0 = \frac{y_0^2}{2p}$$

$$y - y_0 = \frac{px}{y_0} - \frac{\frac{y_0^2}{2}}{y_0}$$

$$y - y_0 = \frac{px}{y_0} - \frac{y_0}{2} \Rightarrow y = \frac{p}{y_0}x + \frac{y_0}{2} \Big/ \cdot y_0$$

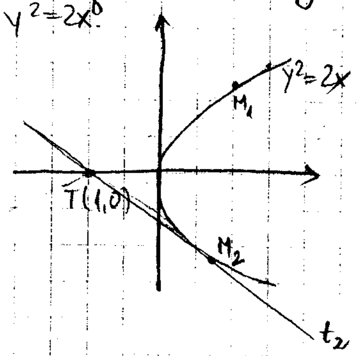
$$yy_0 = px + \frac{y_0^2}{2}$$

$$yy_0 = px + \frac{2px_0}{2}$$

$$yy_0 = p(x + x_0)$$

=PRIMJER 5.

↳Nađi jednadžbu tangenata povučenih iz točke $T(-1,0)$ na graf parabole $y^2 = 2x$.



-Označimo točku $M(x_0, y_0)$ na paraboli u kojoj tangenta tangira parabolu.

$$y^2 = 2x \Big/ \frac{d}{dx}$$

$$2yy' = 2$$

$$y' = \frac{1}{y}$$

$$\left.\begin{array}{l} y'(M) = \frac{1}{y_0} \\ T(x_0, y_0) \end{array}\right\} \; t \ldots \; y - y_0 = \frac{1}{y_0}(x - x_0)$$

- Za točku $T(-1,0)$:

I) $0-y_0=\frac{1}{y_0}(-1-x_0) \;/\cdot y_0$ II) $y_0^2=2x_0$

$y_0^2=1+x_0$

- I) i II) $\Rightarrow$

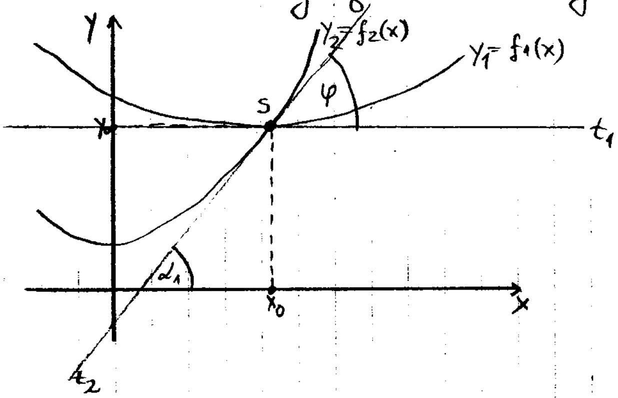
$$1+x_0=2x_0$$
$$1-x_0=0$$
$$-x_0=-1$$
$$x_0=1 \qquad y_0^2=2$$
$$y_0=\pm\sqrt{2}$$

$M_1(1,\sqrt{2}) \qquad M_2(1,-\sqrt{2})$

- $M_1(1,\sqrt{2})$
  - ↳ $t_1 \ldots \; y-\sqrt{2}=\frac{1}{\sqrt{2}}(x-1)$ //
- $M_2(1,-\sqrt{2})$
  - ↳ $t_2 \ldots \; y+\sqrt{2}=-\frac{1}{\sqrt{2}}(x-1)$ //

□ KUT IZMEĐU KRIVULJA

= Kut između krivulja je kut između njihovih tangenata u sjecištu.



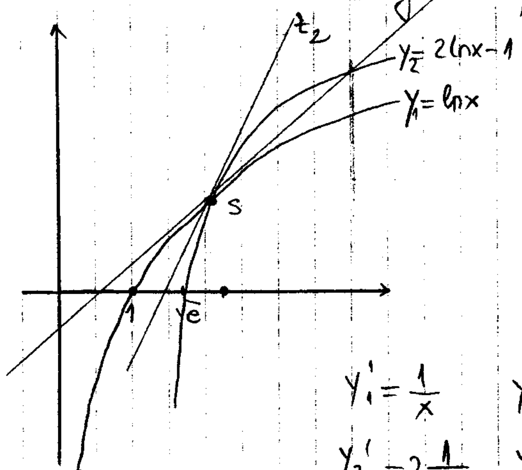
$$\varphi=\alpha_2-\alpha_1$$

$$tg\,\varphi=tg(\alpha_2-\alpha_1)=\frac{tg\alpha_2-tg\alpha_1}{1+tg\alpha_2\cdot tg\alpha_1}=$$

$$=\frac{k_2-k_1}{1+k_2k_1}=\frac{f_2'(S)-f_1'(S)}{1+f_1'(S)\cdot f_2'(S)}$$

= **PRIMJER 6.**

↳ Naći kut među krivuljama $y=\ln x$ i $y=2\ln x-1$ i priloži sliku.



$$\begin{cases} y=\ln x \\ y=2\ln x-1 \end{cases}$$
$$\ln x=2\ln x-1$$
$$-\ln x=-1$$
$$\ln x=1$$
$$\cancel{\ln} x=\cancel{\ln} e^1$$
$$x=e \qquad y=\ln e=1$$
$$S(e,1)$$

$$y_1'=\frac{1}{x} \qquad y_1'(e,1)=\frac{1}{e}$$

$$y_2'=2\frac{1}{x} \qquad y_2'(e,1)=\frac{2}{e}$$

$$tg\,\varphi=\frac{\frac{2}{e}-\frac{1}{e}}{1+\frac{2}{e}\cdot\frac{1}{e}}=\frac{\frac{1}{e}}{\frac{e^2+2}{e^2}}=\frac{e^2}{e(e^2+2)}=\frac{e}{e^2+2} \Rightarrow \varphi=arctg\frac{e}{e^2+2}$$ //

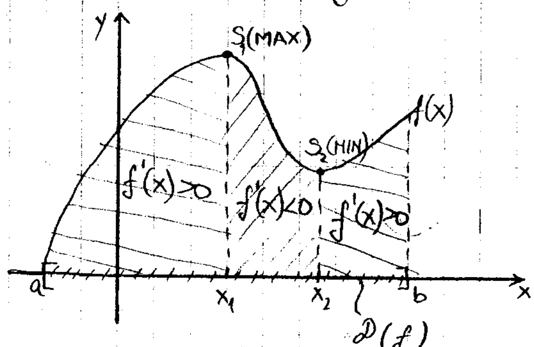
11.2. PAD I RAST FUNKCIJE. EKSTREMI

= STAVAK 1.

↳ Funkcija f raste na intervalu $\langle a, b\rangle$ onda i samo onda ako je $f'(x) \geq 0$ u svakoj točki x tog intervala.

↳ Funkcija f pada na intervalu $\langle a, b\rangle$ onda i samo onda ako je $f'(x) \leq 0$ u svakoj točki x tog intervala.

= Krajnje točke intervala monotonosti su ili rubovi domene, ili su stacionarne točke, u kojima je $f'(x)=0$, primjerice:



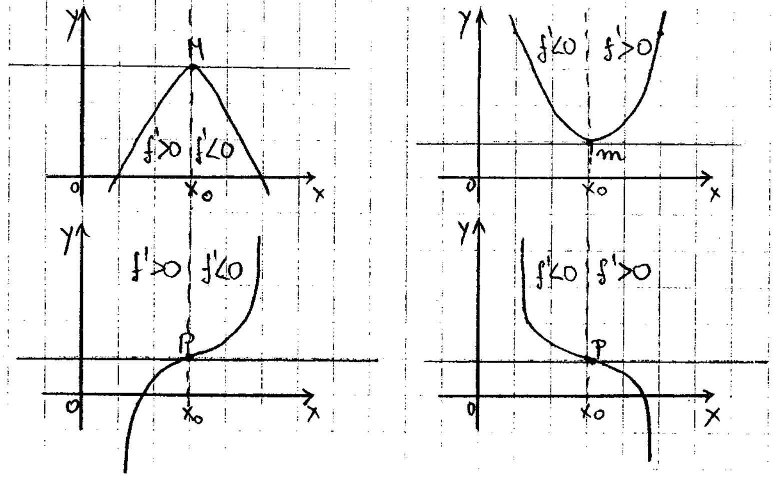
$[a, x_1]$ - interval rasta
$[x_1, x_2]$ - interval pada
$[x_2, b]$ - interval rasta

= Nalaženje stacionarnih točaka i intervala monotonosti:

1. izračunamo derivaciju $f'(x)$
2. riješimo jednadžbu $f'(x)=0$ i odredimo stacionarne točke funkcije f
3. stacionarnim točkama podijelimo područje definicije na intervale monotonosti
4. provjerom predznaka prve derivacije određujemo jesu li oni intervali rasta ili pada zadane funkcije, tj.:

	lijevo od x_0	desno od x_0	karakter točke
derivacija $f'(x)$	$f'(x)>0$	$f'(x)<0$	maximum
derivacija $f'(x)$	$f'(x)<0$	$f'(x)>0$	minimum
derivacija $f'(x)$	$f'(x)>0$	$f'(x)>0$	horizontalna infleksija
derivacija $f'(x)$	$f'(x)<0$	$f'(x)<0$	horizontalna infleksija

ili slikovito:



Samo je jedan Mali Ivica!

=PRIMJER 1.

↳ Naći intervale monotonosti i ekstreme funkcije $x \cdot \ln x$, priloži skicu.

1.) $f'(x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1$

$f'(x) = 0$
$\ln x = 1$
$\ln x = -1$
$\ln x = \ln e^{-1}$
$x = \frac{1}{e}$, $D_f \in \langle 0, \infty \rangle$

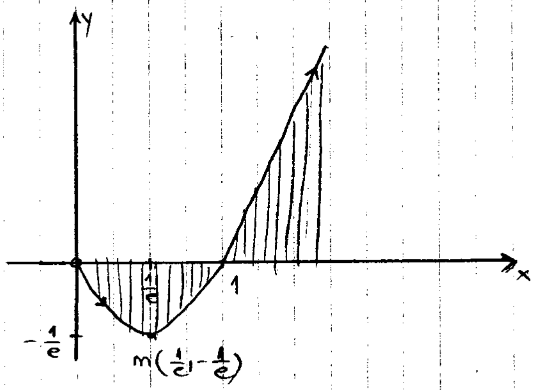
	$0 \quad \frac{1}{e} \quad \infty$
$f'(x)$	$- \; 0 \; +$
$f(x)$	$\searrow -\frac{1}{e} \nearrow$
	min

2.) graf:

$\lim\limits_{x\to 0^+} y = \lim\limits_{x\to 0^+} (x \ln x) = (0 \cdot (-\infty)) = \lim\limits_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} =$

$= \left(\frac{-\infty}{+\infty}\right) = \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -0$

$\lim\limits_{x\to \infty} y = \lim\limits_{x\to \infty} (x \cdot \ln x) = \infty \cdot \infty = +\infty$



=PRIMJER 2.

↳ Naći intervale monotonosti i ekstreme funkcije $y = \frac{x}{x^2+1}$, priloži skicu.

1.) $f'(x) = \frac{x^2+1-x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$

$f'(x) = 0$
$-x^2+1 = 0$
$-x^2 = -1$
$x^2 = 1$
$x = \pm 1$

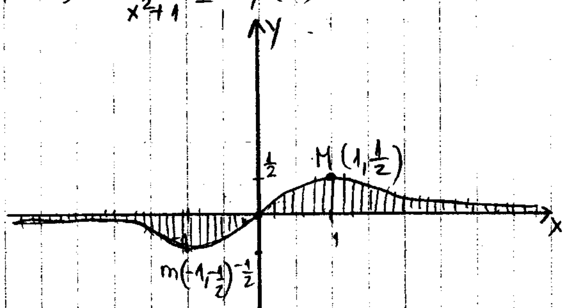
	$-\infty \quad -1 \quad 1 \quad \infty$
$f'(x)$	$- \; 0 \; + \; 0 \; -$
$f(x)$	$\searrow -\frac{1}{2} \nearrow \frac{1}{2} \searrow$
	min max

2.) graf:

$\lim\limits_{x\to -\infty} y = \frac{-\infty}{+\infty} = \lim\limits_{x\to -\infty} \frac{1}{2x} = -0$

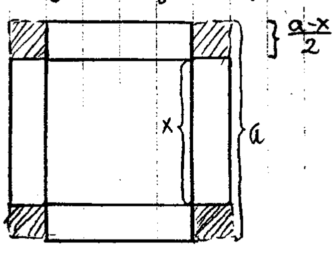
$\lim\limits_{x\to +\infty} y = \frac{\infty}{\infty} = \lim\limits_{x\to \infty} \frac{1}{2x} = +0$

$y(-x) = \frac{-x}{x^2+1} = -y(x)$ ⇒ Neparna



-PRIMJER 3.

↳ Od kartona kvadratnog oblika sa stranicom dužine a treba načiniti otvorenu kutiju što je moguće većeg volumena. Koliki je taj volumen?



$V = V(x) = x^2 \cdot \frac{a-x}{2}$

$V(x) = \frac{a}{2} x^2 - \frac{1}{3} x^3$ → funkcija čiji maksimum tražimo, $x \in [0, a]$

$$V'(x) = \frac{a}{2}\cdot 2x - \frac{3}{2}x^2 = ax - \frac{3}{2}x^2 = x\left(a - \frac{3}{2}x\right)$$

$V'(x) = 0$

$x\left(a - \frac{3}{2}x\right) = 0$

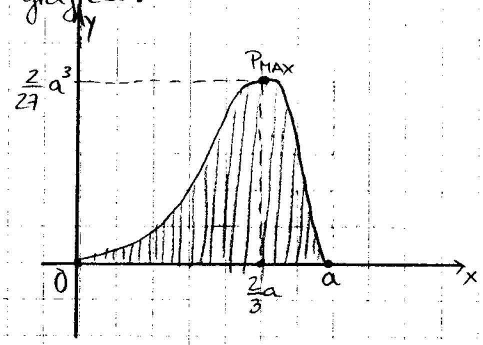
$x_1 = 0$ → trivijalni minimum

$a - \frac{3}{2}x = 0 \;/\cdot 2$

$2a - 3x = 0$

$x_2 = \frac{2a}{3}$ → maksimum $\in [0, a]$

grafički:



$$V_{MAX} = V\left(\frac{2}{3}a\right) = \left(\frac{2}{3}a\right)^2 \cdot \frac{a - \frac{2}{3}a}{2} = \frac{2}{27}a^3$$

→ Provjera: $x_2 = x_{max}$

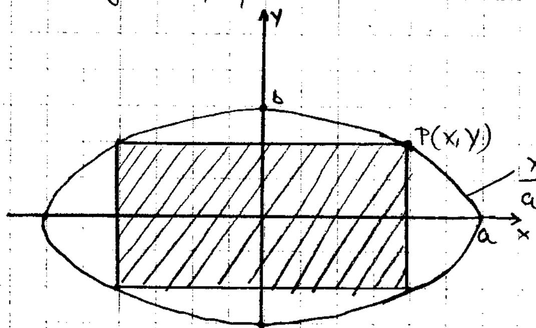
$P'(x)$ za $x \in \langle 0, \frac{2}{3}a\rangle \Rightarrow \operatorname{sgn}\left(\frac{a}{3}\left(a - \frac{3}{2}\cdot\frac{a}{3}\right)\right) = + \longrightarrow P(x) \nearrow$

$P'(x)$ za $x \in \langle \frac{2}{3}a, a\rangle \Rightarrow \operatorname{sgn}\left(\frac{5}{6}a\left(a - \frac{3}{2}\cdot\frac{5}{6}a\right)\right) = - \longrightarrow P(x) \searrow$

$\Longrightarrow x_2 \to MAX$

=PRIMJER 4.

↳ Odredimo pravokutnik najveće površine koji je upisan u elipsu s poluosima duljina a, b, tako da su mu stranice paralelne s osima elipse.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$P = P(x) = x\cdot y\cdot 4 = 4xb\sqrt{1 - \frac{x^2}{a^2}} =$$

$$= \frac{4b}{a}\,x\sqrt{a^2 - x^2}$$ → funkcija čiji maksimum tražimo, $x \in [0, a]$

$$P'(x) = \frac{4b}{a}\left(\sqrt{a^2 - x^2} + x\cdot\frac{1}{2\sqrt{a^2 - x^2}}(-2x)\right)$$

$P'(x) = 0$

$\sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}} = 0 \;/\cdot\sqrt{a^2 - x^2}$, za $a \neq x$

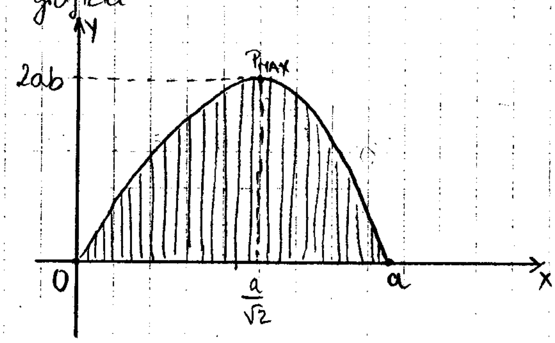
$a^2 - x^2 - x^2 = 0$

$a^2 - 2x^2 = 0$

$x^2 = \frac{a^2}{2}$

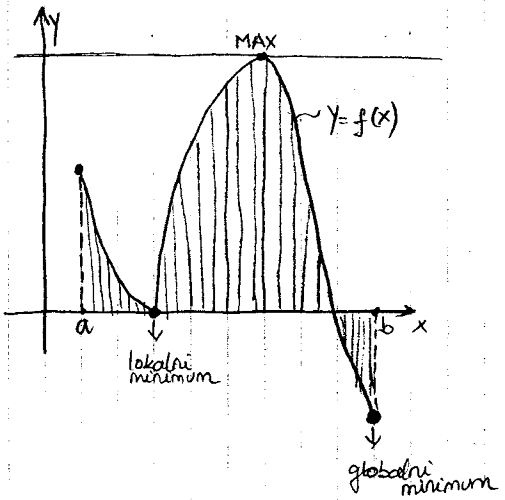
$\cancel{x_1 = -\frac{a}{\sqrt{2}}}$ $\qquad x_2 = \frac{a}{\sqrt{2}}$ → maksimum $\in [0, a]$

grafički:



$$\ldots P = P_{MAX}\left(\frac{a}{\sqrt{2}}\right) = \frac{4b}{\cancel{a}} \cdot \frac{\cancel{a}}{\sqrt{2}} \cdot \frac{a}{\sqrt{2}} = 2ab$$

□ GLOBALNI EKSTREM. EKSTREM FUNKCIJE NA ZATVORENOM INTERVALU.



- Ako tražimo maksimum ili minimum funkcije na cijelom intervalu $[a,b]$ onda govorimo o globalnim ekstremima. Takav se ekstrem mora postizati u stacionarnoj točki unutar intervala, već možda i na njegovu rubu, ili u točki u kojoj derivacija ne postoji (tzv. točka loma).

PRIMJER 5.

↳ Naći najveću i najmanju vrijednost funkcije $y = x^2 - x + 1$ na intervalu $[0,1]$.

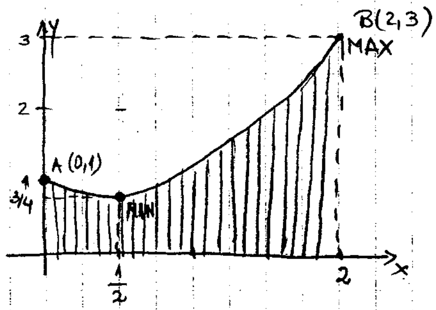
$\rightarrow y' = 2x - 1$

$y' = 0 \Rightarrow 2x - 1 = 0$

$2x = 1$

$x = \frac{1}{2} \quad y = \frac{3}{4} \quad T\left(\frac{1}{2}, \frac{3}{4}\right)$

→ graf:



→ Dakle iz grafa se vidi da je najmanja vrijednost funkcije u točki $T\left(\frac{1}{2}, \frac{3}{4}\right)$ a najveća vrijednost funkcije je u točki $(2,3)$, koja nije stacionarna točka.

PRIMJER 6.

↳ Naći najveću i najmanju vrijednost funkcije $f(x) = 3 \cdot \sqrt[3]{x^2} - 2x$ na intervalu $[-1,2]$.

$\rightarrow y' = 3 \cdot \frac{2}{3} \cdot x^{-\frac{1}{3}} - 2 = \frac{2}{\sqrt[3]{x}} - 2$

$y' = 0 \Rightarrow \frac{2}{\sqrt[3]{x}} = 2 \,/\cdot \sqrt[3]{x}, \quad x \neq 0$

$\cancel{2} = \cancel{2}\sqrt[3]{x}$

$\sqrt[3]{x} = 1$

$x = 1 \quad \rightarrow S(1,1)$

→ točke u kojima funkcija nije definirana: $T(0,0)$

$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-}\left(\frac{2}{\sqrt[3]{x}} - 2\right) = \frac{2}{-0} = -\infty$$

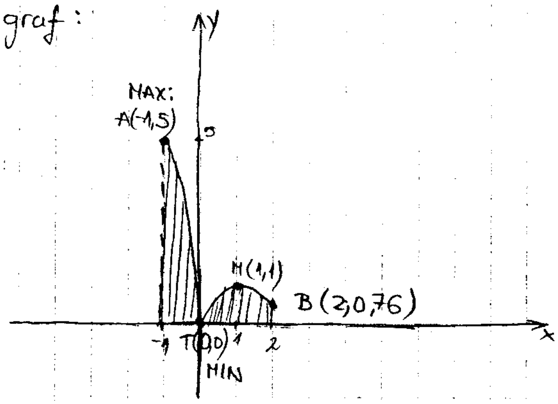
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}\left(\frac{2}{\sqrt[3]{x}} - 2\right) = \frac{2}{+0} = +\infty$$

→ rubne vrijednosti:

$x_1 = -1 \quad f(x_1) = f(-1) = 5$

$x_2 = 2 \quad f(x_2) = f(2) = 0{,}76$

→graf:



= Najveća vrijednost → A(−1,5); najmanja vrijednost → T(0,0), što nisu stacionarne točke.

□ DRUGA DERIVACIJA I EKSTREMI. DERIVACIJE VIŠEG REDA I EKSTREMI.

= Ispitivanje karaktera ekstrema pomoću druge derivacije:

1. Izračunamo derivacije $f'(x)$ i $f''(x)$
2. riješimo jednadžbu $f'(x)=0$, a njezina rješenja su stacionarne točke
3. a) ako je $f''(x_0)>0$, onda je x_0 minimum;
   b) ako je $f''(x_0)<0$, onda je x_0 maksimum;
   c) ako je $f''(x_0)=0$, onda karakter stacionarne točke istražujemo pomoću predznaka prve derivacije

= STAVAK 2.

↳ Ako vrijedi:

$$f'(c)=f''(c)=\ldots=f^{(n-1)}(c)=0,$$

a vrijedi:

$$f^{(n)}(c)\neq 0,$$

onda c nije točka ekstrema ukoliko je n neparan, a jest točka ekstrema ukoliko je n paran broj. Karakter ekstrema ovisi o predznaku derivacije, pa je:

c − točka minimuma za $f^{(n)}(c)>0$,
c − točka maksimuma za $f^{(n)}(c)<0$.

= PRIMJER 7.

a) ↳ Naći ekstreme funkcije $f(x)=x^4-2x^2$.

→ $f'(x)=4x^3-4x$

$f'(x)=0$
$4x^3-4x=0$
$4x(x^2-1)=0$
$x_1=0 \quad x_{2,3}=\pm 1$

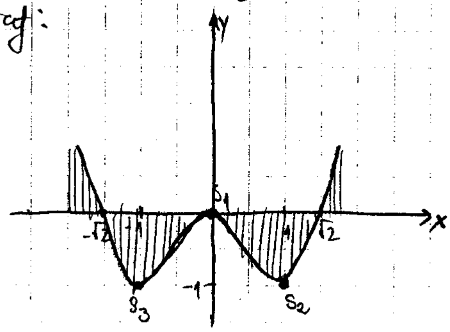
stacionarne točke: $S_1(0,0)$, $S_2(1,-1)$, $S_3(-1,-1)$

→ $f''(x)=12x^2-4$

$f''(S_1)=-4<0$ → maksimum
$f''(S_2)=8>0$ → minimum
$f''(S_3)=-8<0$ → maksimum

nultočke: $x^4-2x^2=0$
$x^2(x^2-2)=0$
$x_1=0 \quad x_{2,3}=\pm\sqrt{2}$

→graf:



= S_1 − nultočka parnog reda
S_2 i S_3 − nultočka neparnog reda

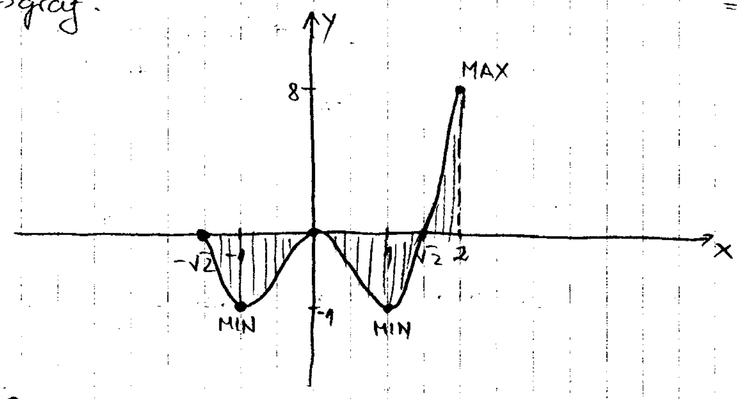
b) ↳ Naći ekstreme prethodne funkcije na intervalu $[-\sqrt{2}, 2]$.

→ stacionarne točke: $S_1(0,0)$ – max.
$S_2(1,-1)$ – min.
$S_3(-1,-1)$ – max.

→ nultočke: $N_1(-\sqrt{2},0)$
$N_2(\sqrt{2},0)$

→ na rubovima funkcije:
$f(-\sqrt{2}) = 0$ $A(-\sqrt{2},0)$
$f(2) = 8$ $B(2,8)$

→ graf:



= Maksimum: $(2,8)$, a minimumi: $(1,-1)$ i $(-1,-1)$.

- PRIMJER 8.
↳ Odredite maksimum i minimum funkcije $f(x)=x^4-4x^3$ nad intervalom $[-1,5]$.

→ $f'(x)=4x^3-12x^2$
$f'(x)=0$
$4x^3-12x^2=0$
$4x^2(x-3)=0$
$x_1=0 \quad x_2=3 \Rightarrow T_1(0,0)$
$T_2(3,-27)$

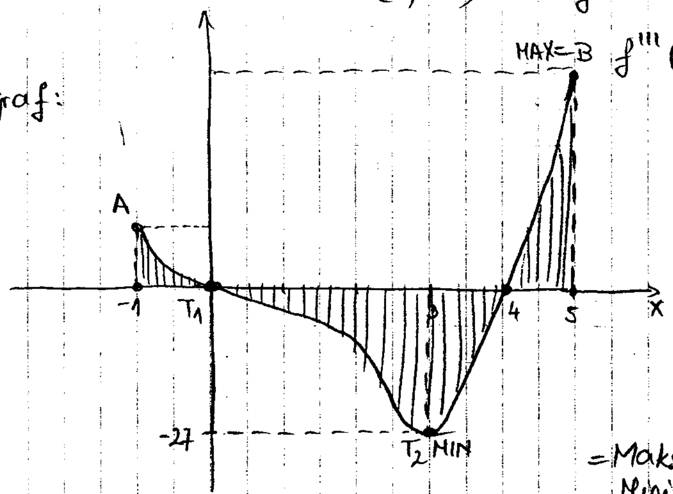
→ $f''(x)=12x^2-24x$

$f''(T_1)=0 \quad =0 \quad ?$
$f''(T_2)=36 \quad >0$ minimum

→ $f'''(x)=24x-24$

$f'''(T_1)=-24 \neq 0$ → nije točka ekstrema nego je točka infleksije

→ graf:



→ nultočke: $x^4-4x^3=0$
$x^3(x-4)=0$
$x_1=0 \quad x_2=4$

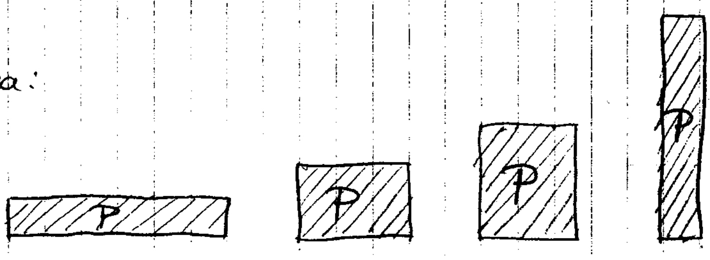
→ rubne točke:
$f(5)=125$ $B(5,125)$
$f(-1)=5$ $A(-1,5)$

= Maksimum: $B(5,125)$
Minimum: $T_2(3,-27)$

11.3. PRIMJENE

= PRIMJER 1.
↳ Dokažite da među svim pravokutnicima s površinom P kvadrat ima najmanji opseg.

→ skica:



P = konst.

→ opseg pravokutnika:

$$O = 2x + 2y\,,\quad P = xy \Rightarrow y = \frac{P}{x}$$

$$O = 2x + 2\frac{P}{x} = 2\left(x + \frac{P}{x}\right)$$ → funkcija čiji minimum tražimo $x \in \langle 0, \infty \rangle$

$$O' = 2 - \frac{2P}{x^2}$$

$$O' = 0$$

$$2 - \frac{2P}{x^2} = 0$$

$$\frac{\cancel{2}P}{x^2} = \cancel{2}$$

$$x^2 = P$$

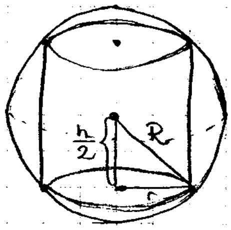
$$\cancel{x_1 = -\sqrt{P}} \qquad x_2 = \sqrt{P} \qquad y_2 = \frac{P}{x} = \frac{P}{\sqrt{P}} = \sqrt{P}$$ //

$x_2 = y_2 \Rightarrow$ pravokutnik je kvadrat //

= PRIMJER 2.

↳ U zadanu kuglu polumjera R upišimo valjak najvećeg volumena.

→ Skica:



→ volumen valjka:

(I) $V = B \cdot h = r^2\pi \cdot h$

(II) $h^2 + (2r)^2 = (2R)^2$

$$\left(\frac{h}{2}\right)^2 + r^2 = R^2$$

$$\frac{h^2}{4} + r^2 = R^2 \Rightarrow h = 2\sqrt{R^2 - r^2}$$

$V(r) = r^2\pi \cdot 2\sqrt{R^2 - r^2}$ → funkcija čiji maksimum tražimo

$$V'(r) = 2\pi\left(2r\sqrt{R^2 - r^2} + \frac{1}{2\sqrt{R^2 - r^2}}(-2r)\cdot r^2\right) = 2\pi\,\frac{2r(R^2 - r^2) - r^3}{\sqrt{R^2 - r^2}} =$$

$$= \frac{2\pi r\left(2(R^2 - r^2) - r^2\right)}{\sqrt{R^2 - r^2}}$$

$$V'(r) = 0$$

$$\cancel{2\pi r}\left(2(R^2 - r^2) - r^2\right) = 0$$

$$2R^2 - 2r^2 - r^2 = 0$$

$$2R^2 - 3r^2 = 0$$

$$r^2 = \frac{2R^2}{3}$$

$$\cancel{r_1 = -\frac{\sqrt{2}}{\sqrt{3}}R} \qquad r_2 = \frac{\sqrt{2}}{\sqrt{3}}R = \frac{\sqrt{6}}{3}R$$

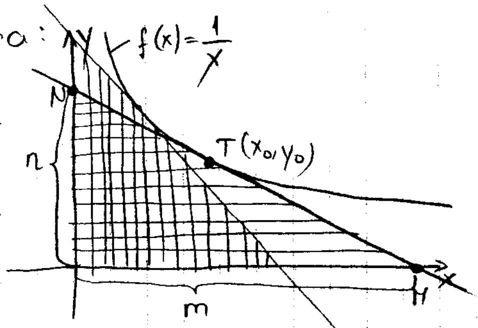
$$\to V_{MAX} = V\left(\frac{\sqrt{6}}{3}R\right) = 2\pi \cdot \frac{2}{3}R^2\sqrt{\frac{R^2}{3}} =$$

$$= \frac{4}{3}\pi R^2\,\frac{R\sqrt{3}}{3} = \frac{4\sqrt{3}\pi}{9}R^3$$ //

PRIMJER 3.

Dokažite da sve tangente na graf funkcije $f(x)=\frac{1}{x}$ u bilo kojoj točki u prvom kvadrantu zatvaraju s koordinatnim osima trokut konstantne površine. Izračunajte tu površinu.

→ skica:



→ jednadžba tangente:

$$t \ldots \quad y-y_0 = y'_0 (x-x_0)$$

$$y'_0 = f'(x) = -\frac{1}{x_0^2}$$

$$t \ldots \quad y-y_0 = -\frac{1}{x_0^2}(x-x_0)$$

(I) $\left.\begin{matrix} x=0 \\ \Downarrow \\ y=n \end{matrix}\right\}$ $\quad y_0 + \frac{x_0}{x_0^2} = \frac{1}{x_0} + \frac{1}{x_0} = \frac{2}{x_0} \rightarrow N\left(0, \frac{2}{x_0}\right)$

(II) $\left.\begin{matrix} y=0 \\ \Downarrow \\ x=m \end{matrix}\right\}$ $\quad x_0^2 y_0 + x_0 = x_0^{\cancel{2}} \frac{1}{\cancel{x_0}} + x_0 = 2x_0 \rightarrow M(2x_0, 0)$

→ maksimalna površina:

$$P_\triangle = \frac{m \cdot n}{2} = \frac{\cancel{2x_0} \cdot \frac{2}{\cancel{x_0}}}{\cancel{2}} = 2$$

ZADACI

1. $t, n = ?$

$y = \arcsin x \qquad T\left(\frac{1}{2}, y\right) \qquad y' = \frac{1}{\sqrt{1-x^2}}$

$T \ldots \quad y = \arcsin\frac{1}{2} = \frac{\pi}{6}$

$T\left(\frac{1}{2}, \frac{\pi}{6}\right)$

$$k_t = y'\left(\frac{1}{2}\right) = \frac{1}{\sqrt{1-\frac{1}{4}}} = \frac{1}{\sqrt{\frac{3}{4}}} = \frac{2\sqrt{3}}{3}$$

$t \ldots \quad y - y_0 = k_t(x-x_0)$

$y - \frac{\pi}{6} = \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$

$y = \frac{2\sqrt{3}}{3}x - \frac{\sqrt{3}}{3} + \frac{\pi}{6}$

$n \ldots \quad y - y_0 = -\frac{1}{k_t}(x-x_0)$

$y - \frac{\pi}{6} = -\frac{\sqrt{3}}{2}\left(x - \frac{1}{2}\right)$

$y = -\frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{4} + \frac{\pi}{6}$

2. $t, n = ?$

$xy^2 + e^{x+y} = 2 \qquad T(1, -1)$

$xy^2 + e^{x+y} = 2 \quad \Big| \frac{d}{dx}$

$y^2 + 2yy'x + e^{x+y}(x+y)' = 0$

$y^2 + 2yy'x + e^{x+y}(y'+1) = 0$

$y'(2yx + e^{x+y}) = -e^{x+y} - y^2$

$$y' = \frac{-e^{x+y} - y^2}{2yx + e^{x+y}}$$

$$k_T = y'(T) = y'(1,-1) = \frac{-1-1}{-2+1} = \frac{-2}{-1} = 2$$

$t \ldots \quad y - y_0 = k_T(x-x_0)$

$y + 1 = 2(x-1)$

$y = 2x - 3$

$n \ldots \quad y - y_0 = -\frac{1}{k_T}(x-x_0)$

$y - 1 = -\frac{1}{2}(x-1)$

$y = -\frac{1}{2}x - \frac{1}{2}$

3. $y^x + \sin(x-1) = \sqrt{y}$

$T(1,1)$

$t\ldots = ?$ $\quad n\ldots = ?$

$t\ldots \equiv y - y_0 = y'(T)\,(x - x_0)$

$\hookrightarrow y'(T) = ?$

$y^x + \sin(x-1) = \sqrt{y}$

$e^{x\ln y} + \sin(x-1) = \sqrt{y} \;\Big|\; \frac{d}{dy}$

$y'\left(\ln y + \frac{1}{y}\cdot y'\cdot x\right) + \cos(x-1) = \frac{1}{2\sqrt{y}}\cdot y'$

$y^x \ln y + \frac{x}{y}\, y^x y' + \cos(x-1) - \frac{1}{2\sqrt{y}}\, y' = 0$

$y' = \dfrac{-y^x \ln y - \cos(x-1)}{x\cdot y^{x-1} - \frac{1}{2\sqrt{y}}}$

$k_T = y'(T) = y'(1,1) = = \dfrac{-1\cdot 0 - 1}{1 - \frac{1}{2}} = \dfrac{-1}{\frac{1}{2}} = -2$

$t\ldots \quad y - y_0 = k_T(x - x_0)$

$y - 1 = -2(x-1)$

$y = -2x + 3$ //

$n\ldots \quad y - y_0 = \frac{-1}{k_T}(x - x_0)$

$y - 1 = \frac{1}{2}(x-1)$

$y = \frac{1}{2}x + \frac{1}{2}$ //

4. $\begin{cases} x = \sin^2 t - \cos t \\ y = \ln\left(\operatorname{tg}\frac{t}{2}\right) \end{cases}$, $t = \frac{\pi}{2}$, $t, n = ?$

$T(x_0, y_0)\ldots \quad x = \sin^2\frac{\pi}{2} - \cos\frac{\pi}{2} = 1 - 0 = 1$ $\qquad T(1,0)$ //

$y = \ln\left(\operatorname{tg}\frac{\pi}{4}\right) = \ln 1 = 0$

$y' = \dfrac{\dot y}{\dot x} = \dfrac{\frac{1}{\operatorname{tg}\frac{t}{2}}\cdot\frac{1}{\cos^2\frac{t}{2}}\cdot\frac{1}{2}}{2\sin t\cos t + \sin t} = \dfrac{\frac{1}{\operatorname{tg}\frac{t}{2}\,\cos^2\frac{t}{2}\cdot 2}}{2\sin t\cos t + \sin t} = \dfrac{1}{(\sin 2t + \sin t)\left(2\operatorname{tg}\frac{t}{2}\cdot\cos^2\frac{t}{2}\right)}$

$k_T = y'\left(\frac{\pi}{2}\right) = \dfrac{1}{(0+1)\left(2\cdot 1\cdot\frac{1}{2}\right)} = 1$

$t\ldots \quad y - y_0 = k_T(x - x_0)$

$y - 0 = 1\,(x+1)$

$y = x - 1$ //

$n\ldots \quad y - y_0 = -\frac{1}{k_T}(x - x_0)$

$y - 0 = -1\,(x-1)$

$y = -x + 1$ //

5. a) $y = x^2 - 1$, najbližu $T(3,-1)$

(1) $y + 1 = k_N(x-3)$ $\quad$ (2) $k_N = -\frac{1}{k_T} = -\frac{1}{2x}$

$k_N = \frac{y+1}{x-3}$

(1) i (2)... $\frac{y+1}{x+3} = -\frac{1}{2x}$

$(y+1)\,2x = -x + 3$

$(x^2-1)\,2x = -x + 3$

$2x^3 - 2x + x - 3 = 0$

$2x^3 - x - 3 = 0$

$x_1 = 1$ (dijeljenjem nema drugih nultočaka)

$x_1 = 1 \quad y_1 = 0 \qquad T(1,0)$ //

b) $y_1 = x^2 + x - 2$, najbližu pravcu $y_2 = 5x - 10$

$y_1' = 2x + 1 \qquad y_2' = 5$

$2x + 1 = 5$

$2x = 4$

$x = 2 \qquad y = 4 \qquad T(2,4)$ //

6. $y_1 = e^{2x}$ $\qquad \varphi = ?$
$y_2 = e^{-3x}$

$T(x_0, y_0) \ldots \quad \cancel{e}^{2x} = \cancel{e}^{-3x}$

$2x = -3x$

$5x = 0$

$x = 0 \quad , \quad y = 1 \qquad T(0,1)$

$$\left.\begin{aligned} k_1 &= y_1' = e^{2x} \cdot 2 = 2 \\ k_2 &= y_2' = e^{-3x} \cdot (-3) = -3 \end{aligned}\right\} \quad tg\varphi = \left|\frac{k_2 - k_1}{1 + k_1 k_2}\right| = \left|\frac{-3-2}{1-6}\right| = 1 \Rightarrow \varphi = 45^\circ$$

7. $\left.\begin{aligned} y_1 &= 4x^2 + 2x - 8 \\ y_2 &= x^3 - x + 10 \end{aligned}\right\}$ tangiraju u točki $T(3,34)$, da li i u $T(-2,4)$

$y_1' = 8x + 2$
$y_2' = 3x^2 - 1$

$k_1 = k_2$

$8x + 2 = 3x^2 - 1$

$3x^2 - 8x - 3 = 0$

$x_{1,2} = \frac{8 \pm \sqrt{64+36}}{6} = \frac{8 \pm 10}{6} \qquad x_1 = 3, \quad x_2 = -\frac{1}{3}$

$\Rightarrow$ derivacije su im jednake za $x = 3$ i $x = -\frac{1}{3}$

↳ $T_1\ (3,34) \rightarrow$ tangiraju se
$T_2\ (-2,4) \rightarrow$ ne tangiraju se

8. $f(x) = -2x^3 + 3x^2 + 12x - 1$
↳ 1. derivacija

$f'(x) = -6x^2 + 6x + 12$

$f'(x) = 0$

$-6x^2 + 6x + 12 = 0$

$-x^2 + x + 2 = 0$

$x^2 - x - 2 = 0$

$x_{1,2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} \qquad x_1 = 2, \quad x_2 = -1$

	$-\infty$	-1		2	∞
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	↘		↗		↘
		MIN		MAX	

MIN $\rightarrow S_1(-1, -8)$
MAX $\rightarrow S_2(2, 19)$

9. $f(x) = x^3 - 3x + 2$
↳ 1. derivacija

$f'(x) = 3x^2 - 3$

$f'(x) = 0$

$3x^2 - 3 = 0$

$x^2 - 1 = 0$

$x^2 = 1 \qquad x_1 = 1, \quad x_2 = -1$

	$-\infty$	-1		1	∞
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗		↘		↗
		MAX		MIN	

MIN $\rightarrow S_1(-1, 4)$
MAX $\rightarrow S_2(1, 0)$

10. $f(x) = \frac{2x^2}{1-x^2}$, $D_f \in \mathbb{R} \setminus \{-1, 1\}$
↳ 1. derivacija

$$f'(x) = \frac{4x(1-x^2) - 2x^2(-2x)}{(1-x^2)^2} = \frac{4x - \cancel{4x^3} + \cancel{4x^3}}{(1-x^2)^2} = \frac{4x}{(1-x^2)^2}$$

$f'(x) = 0$

$4x = 0$

$x = 0$

	$-\infty$	-1	0	1	∞
$f'(x)$	$-$	$-$	$+$	$+$	
$f(x)$	↘	↘	↗	↗	
			MIN		

MIN $\rightarrow S_1(0, 0)$

11. $f(x)=x\ln(x^2)$, $D_f \in \mathbb{R}\setminus\{0\}$

$f'(x)=\ln(x^2)+x\cdot\frac{1}{\cancel{x^2}}\cdot 2\cancel{x}=\ln(x^2)+2$

$f'(x)=0$

$\ln(x^2)+2=0$

$\ln(x^2)=-2$

$\cancel{\ln}(x^2)=\cancel{\ln}e^{-2}$

$x^2=e^{-2}$ $\quad x_1=1/e$ // $\quad x_2=-1/e$ //

	$-\infty$ … $-\frac{1}{e}$	$-\frac{1}{e}$ … 0	0 … $\frac{1}{e}$	$\frac{1}{e}$ … ∞
$f'(x)$	+	−	−	+
$f(x)$	↗	↘	↘	↗
	MAX		MIN	

//

MAX $S_1\left(-\frac{1}{e},\frac{2}{e}\right)$

MIN $S_2\left(\frac{1}{e},-\frac{2}{e}\right)$ //

12. $f(x)=x^3\cdot e^{-x^2}$

$f'(x)=3x^2e^{-x^2}+x^3\cdot e^{-x^2}(-2x)=e^{-x^2}(3x^2-2x^4)$

$f'(x)=0$

$e^{-x^2}(3x^2-2x^4)=0$

$x^2(3-2x^2)=0$

$x_1=0$ // $\quad 3-2x^2=0$

$2x^2=3$

$x_2=\sqrt{\frac{3}{2}}$ // $\quad x_3=-\sqrt{\frac{3}{2}}$ //

	$-\infty$ … $-\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$ … 0	0 … $\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$ … ∞
$f'(x)$	−	+	+	−
$f(x)$	↘	↗	↗	↘
	MIN		MAX	

//

MIN $S_1\left(-\sqrt{\frac{3}{2}},-\sqrt{\frac{3}{2}}^3e^{-\frac{3}{2}}\right)$

MAX $S_2\left(\sqrt{\frac{3}{2}},\sqrt{\frac{3}{2}}^3e^{-\frac{3}{2}}\right)$

13. Dokaži: $\sin x<x<\operatorname{tg}x$, $x\in\langle 0,\frac{\pi}{2}\rangle$

I) $f(x)=\operatorname{tg}x-x$ $\quad f'(x)=\frac{1}{\cos^2x}-1<0\Rightarrow$ funkcija je padajuća $\Rightarrow f(x)<f(0),\ \forall x\in\langle 0,\frac{\pi}{2}\rangle$

↳ Iz toga slijedi da je $x<\operatorname{tg}x$ za svaki $x\in\langle 0,\frac{\pi}{2}\rangle$.

II) $g(x)=x-\sin x$ $\quad g'(x)=1-\cos x>0\Rightarrow$ funkcija je rastuća $\Rightarrow f(x)>f(0),\ \forall x\in\langle 0,\frac{\pi}{2}\rangle$

↳ Iz toga slijedi da je $x>\sin x$ za svaki $x\in\langle 0,\frac{\pi}{2}\rangle$.

Q.E.D.

14. Dokaži: $x-\frac{x^2}{2}<\ln(1+x)<x,\ x>0$

I) $f(x)=x-\ln(1+x)$ $\quad f'(x)=1-\frac{1}{1+x}>0\Rightarrow$ funkcija je rastuća $\Rightarrow f(x)>f(0),\ \forall x>0$

↳ Iz toga slijedi da je $x>\ln(1+x)$ za $x>0$.

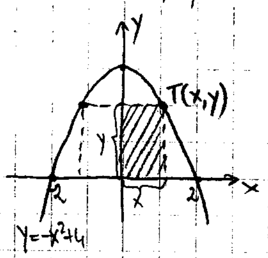
II) $g(x)=\ln(1+x)-x+\frac{x^2}{2}$ $\quad g'(x)=\frac{1}{1+x}-1+\frac{1}{2}\cdot 2x=\frac{1-1-x+x+x^2}{1+x}=\frac{x^2}{1+x}>0$

$\Rightarrow$ funkcija f je rastuća $\Rightarrow f(x)>f(0),\ \forall x>0$

↳ Iz toga slijedi da je $\ln(1+x)>x-\frac{x^2}{2}$ za svaki $x>0$.

Q.E.D.

15. $P_{max}=?$, $y=-x^2+4$, segment $[-2,2]$



$P=2xy=2x(-x^2+4)=-2x^3+8x$ $\Rightarrow$ funkcija čiji maksimum tražimo

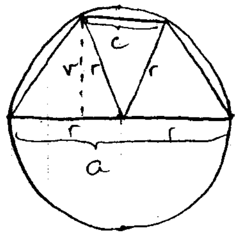
$P'(x)=-6x^2+8$

$P'(x)=0\Rightarrow -6x^2+8=0$

$x^2=\frac{4}{3}$ $\quad \cancel{x_1=\frac{-2\sqrt{3}}{3}}$ $\quad x_2=\frac{2\sqrt{3}}{3}$

$P_{MAX}=P\left(\frac{2\sqrt{3}}{3}\right)=-2\cdot\frac{8\cdot\cancel{3}\sqrt{3}}{\cancel{27}}+8\cdot\frac{2\sqrt{3}}{3}=\frac{-16\sqrt{3}}{9}+\frac{16\sqrt{3}}{3}=\frac{-16\sqrt{3}+48\sqrt{3}}{9}=\frac{32\sqrt{3}}{9}$ //

16. $P_{trap\,max}=?$



$$P=\frac{a+c}{2}\cdot v \quad , \quad a=2r$$
$$v^2+\left(\frac{c}{2}\right)^2=r^2 \Rightarrow c^2=4(r^2-v^2)$$

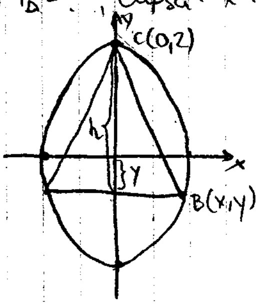
$$P=\frac{\cancel{2}r+\cancel{2}\sqrt{r^2-v^2}}{\cancel{2}}\cdot v = v\left(r+\sqrt{r^2-v^2}\right) \rightarrow$$ funkcija čiji maksimum tražimo

$$P'(v)= r+\sqrt{r^2-v^2}+v\cdot\left(\frac{1}{\cancel{2}\sqrt{r^2-v^2}}\cdot(-\cancel{2}v)\right)=r+\sqrt{r^2-v^2}-\frac{v^2}{\sqrt{r^2-v^2}}$$

$$P'(x)=0$$
$$r+\sqrt{r^2-v^2}-\frac{v^2}{\sqrt{r^2-v^2}}=0 \quad /\cdot\sqrt{r^2-v^2}$$
$$r\sqrt{r^2-v^2}+r^2-v^2-v^2=0$$
$$r\sqrt{r^2-v^2}=2v^2-r^2 \quad /^2$$
$$r^2(r^2-v^2)=4v^4-4v^2r^2+r^4$$
$$\cancel{r^4}-r^2v^2=4v^4-4v^2r^2+\cancel{r^4}$$
$$3r^2\cancel{v^2}=4v^{\cancel{4}}$$
$$3r^2=4v^2 \Rightarrow v=\frac{\sqrt{3}}{2}r$$

$$P_{max}=P\left(\frac{\sqrt{3}}{2}r\right)=\frac{\sqrt{3}}{2}r\left(r+\sqrt{r^2-\frac{3}{4}r^2}\right)=\frac{3}{2}r\cdot\frac{\sqrt{3}}{2}r=\frac{3\sqrt{3}}{4}r^2$$

17. $P_\Delta=?$, elipsa: $x^2+\frac{y^2}{4}=1$, vrh $C(0,2)$, $\overline{AB}=2x$



$$a=AB=2x$$
$$h=2+y=2+2\sqrt{1-x^2}$$

$$P=\frac{a\cdot h}{2}=\frac{2x\cdot\cancel{2}(1+\sqrt{1-x^2})}{\cancel{2}}=2x\left(1+\sqrt{1-x^2}\right)$$

$P(x)=2x(1+\sqrt{1-x^2}) \rightarrow$ funkcija čiji maksimum tražimo

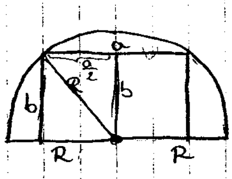
$$P'(x)=2\left(1+\sqrt{1-x^2}\right)+\left(\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)\right)\cdot 2x=2+2\sqrt{1-x^2}-\frac{4x^2}{2\sqrt{1-x^2}}=2+2\sqrt{1-x^2}-\frac{2x^2}{\sqrt{1-x^2}}$$

$$P'(x)=0$$
$$2\sqrt{1-x^2}+2(1-x^2)-2x^2=0$$
$$2\sqrt{1-x^2}=4x^2-2 \quad /^2$$
$$1-x^2=(2x^2-1)^2$$
$$1-x^2=4x^4-4x^2+1$$
$$4x^4-3x^2=0$$
$$x^2(4x^2-3)=0$$

$\cancel{x_1=0}$ $\quad \cancel{x_2=-\frac{\sqrt{3}}{2}}$ $\quad x_3=\frac{\sqrt{3}}{2}$

$$P_{max}=\cancel{2}\frac{\sqrt{3}}{\cancel{2}}\left(1+\sqrt{1-\frac{3}{4}}\right)=\sqrt{3}\left(1+\frac{1}{2}\right)=\frac{3\sqrt{3}}{2}$$

18. $P_{\square\,max}=?$



$$b^2+\frac{a^2}{4}=R^2$$
$$a^2=4(R^2-b^2)$$
$$a=2\sqrt{R^2-b^2}$$

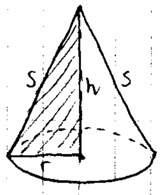
$P=a\cdot b=2\sqrt{R^2-b^2}\cdot b \rightarrow$ funkcija čiji maksimum tražimo

$$P'(b)=2\left(\sqrt{R^2-b^2}+\frac{1}{2\sqrt{R^2-b^2}}\cdot b(-2b)\right)=$$
$$=2\left(\sqrt{R^2-b^2}-\frac{b^2}{\sqrt{R^2-b^2}}\right)$$

$$P'(b)=0 \Rightarrow \sqrt{R^2-b^2}=\frac{b^2}{\sqrt{R^2-b^2}} \quad /\cdot\sqrt{R^2-b^2}$$
$$R^2-b^2=b^2$$
$$2b^2=R^2$$
$$b=\frac{R\sqrt{2}}{2}$$

$$P_{max}=2\cdot\sqrt{R^2-\frac{R^2}{2}}\cdot\frac{R}{\sqrt{2}}=2\,\frac{R^2}{\sqrt{2}\cdot\sqrt{2}}=R^2$$

(19.) $V_{max} = ?$



$$V = \frac{B \cdot v}{3} = \frac{r^2 \pi \cdot v}{3} = \frac{(s^2 - v^2)\pi v}{3} = \frac{\pi}{3}(s^2 - v^2)v \Rightarrow \text{funkcija čiji maksimum tražimo}$$

$\hookrightarrow s^2 = v^2 + r^2 \Rightarrow r^2 = s^2 - v^2$

$$V'(v) = \frac{\pi}{3}(-2v \cdot v + s^2 - v^2) = \frac{\pi}{3}(s^2 - 3v^2)$$

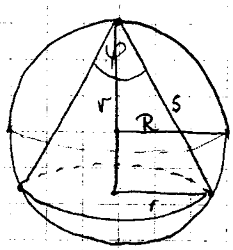
$$V'(v) = 0 \Rightarrow s^2 - 3v^2 = 0$$

$$s^2 = 3v^2$$

$$v^2 = \frac{s^2}{3} \Rightarrow \cancel{v_1 = -\frac{s}{\sqrt{3}}} \quad v_2 = \frac{s}{\sqrt{3}}$$

$$V_{max} = \frac{\pi}{3}\left(s^2 - \frac{s^2}{3}\right) \cdot \frac{s}{\sqrt{3}} = \frac{\pi}{3} \cdot \frac{2s^2}{3} \cdot \frac{s}{\sqrt{3}} = \frac{2\pi s^3}{9\sqrt{3}} = \frac{2\sqrt{3}}{27}\pi s^3 //$$

(20.) $V_{max} = ?$



$$V = \frac{B \cdot v}{3} = \frac{r^2 \pi v}{3}$$

$\hookrightarrow s^2 = v^2 + r^2$ $\qquad$ $\hookrightarrow s = 2R\cos\varphi = 2R\frac{v}{s} \ / \cdot s$

$$s^2 = 2Rv$$

$$v^2 + r^2 = 2Rv$$

$$r^2 = 2Rv - v^2$$

$$V = (2Rv - v^2)v\,\frac{\pi}{3} = v^2(2R - v)\frac{\pi}{3} \Rightarrow \text{funkcija čiji maksimum tražimo}$$

$$V'(v) = (2v(2R - v) + (-1) \cdot v^2)\frac{\pi}{3} = (4vR - 2v^2 - v^2)\frac{\pi}{3} = (4vR - 3v^2)\frac{\pi}{3}$$

$$V'(v) = 0 \Rightarrow 4vR = 3v^2$$

$$4vR - 3v^2 = 0$$

$$v(4R - 3v) = 0 \qquad \cancel{v_1 = 0}$$

$$v_2 = \frac{4R}{3}$$

$$V_{max} = \frac{16}{9}R^2\left(2R - \frac{4R}{3}\right)\frac{\pi}{3} = \frac{16R^2}{9}\left(\frac{2R}{3}\right)\frac{\pi}{3} = \frac{32}{81}R^3\pi //$$

12. ISPITIVANJE TIJEKA FUNKCIJE

12.1. ASIMPTOTE

= DEFINICIJA 1.

↳ Neka se točka T neprekinuto giba po grafu Γ_f funkcije f, tako da barem jedna od njezinih koordinata teži u $+\infty$ ili $-\infty$. Ako pri tom njezina udaljenost od pravca p teži k nuli, onda se taj pravac naziva asimptota funkcije.

= Treba razlikovati dva bitno različita slučaja:

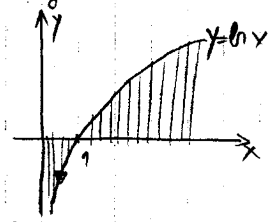
I) vertikalne asimptote grafa funkcije

↳ Ako za funkciju f vrijedi: $\lim_{x\to x_0^-} f(x)=\pm\infty$ ili $\lim_{x\to x_0^+} f(x)=\pm\infty$, onda je pravac $x=x_0$ njezina vertikalna asimptota.

= PRIMJER 1.

↳ Naći vertikalne asimptote funkcije $y=\ln x$.

→ $\mathcal{D}_f \in \langle 0,\infty\rangle$

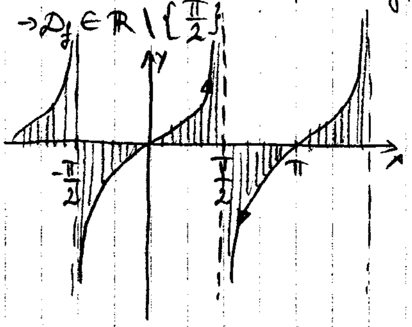


$x=0$ → desna vertikalna asimptota

= PRIMJER 2.

↳ Naći vertikalne asimptote funkcije: $y=tg\,x$, na intervalu $[0,\pi]$

→ $\mathcal{D}_f \in \mathbb{R}\setminus\{\frac{\pi}{2}\}$



$x=\frac{\pi}{2}$ → obostrana vertikalna asimptota

II) kose asimptote grafa funkcije

↳ Desna kosa asimptota je pravac $p\equiv y=kx+l$ za kojeg vrijedi:

$$\lim_{x\to+\infty}\left(f(x)-(kx+l)\right)=0.$$

↳ Ako ovakav limes postoji kod $x\to-\infty$, govorimo o lijevoj kosoj asimptoti.

↳ Izračunavanje nepoznatih koeficijenata k i l:

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x}\quad ; \quad l=\lim_{x\to\pm\infty}\left(f(x)-kx\right)$$

DOKAZ: → Iz osnovnog uvjeta kojim je asimptota definirana slijedi:

$$\lim_{x\to\pm\infty}\left[f(x)-(kx+l)\right]=0\ /:x$$

$$\lim_{x\to\pm\infty}\frac{f(x)}{x}-\underbrace{\lim_{x\to\pm\infty}k}_{=k}+\underbrace{\lim_{x\to\pm\infty}\frac{l}{x}}_{=0}=0$$

$$\Downarrow$$

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x}$$

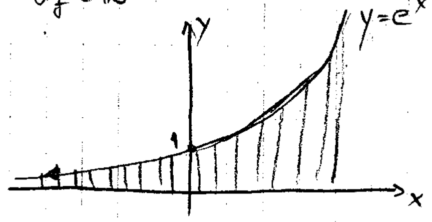
→ Sada se l izračuna iz osnovnog uvjeta za asimptotu:

$$\lim_{x\to\pm\infty} [f(x) - (kx+l)] = 0$$

Q.E.D.

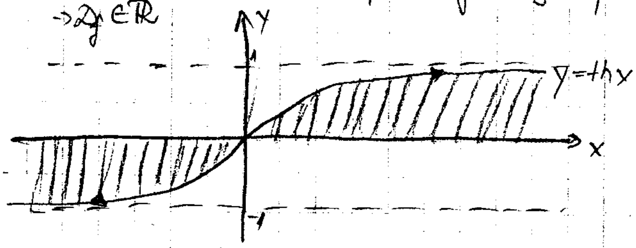
= Specijalni slučaj: Horizontalne asimptote grafa funkcije
↳ Kose asimptote sa koeficijentom smjera $k=0$.

= PRIMJER 3.
↳ Naći horizontalne asimptote funkcije $y=e^x$.
→ $\mathcal{D}_f \in \mathbb{R}$



$y=0$ → lijeva horizontalna asimptota

= PRIMJER 4.
↳ Naći horizontalne asimptote funkcije $y=\text{th}x$.
→ $\mathcal{D}_f \in \mathbb{R}$



$y=1$ → desna horizontalna asimptota
$y=-1$ → lijeva horizontalna asimptota

= PRIMJER 5.
↳ Naći asimptote funkcije: $y=\dfrac{x^3}{x^2+x+1}$.

I) vertikalne asimptote:

$$x^2+x+1=0$$

$$x_{1,2}=\frac{-1\pm\sqrt{1-4}}{2}=\cancel{\frac{-1\pm\sqrt{-3}}{2}}$$

↳ NEMA B//

II) kose asimptote:

$$y=kx+l$$

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{x^3}{x^3+x^2+x}=\frac{1}{1}=1$$

$$l=\lim_{x\to\pm\infty}(f(x)-kx)=\lim_{x\to\pm\infty}\left(\frac{x^3}{x^2+x+1}-x\right)=\lim_{x\to\pm\infty}\frac{\cancel{x^3}-\cancel{x^3}-x^2-x}{x^2+x+1}=-1$$

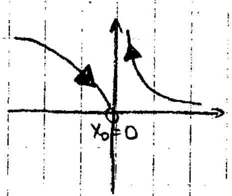
↳ $y=x-1$ //

= PRIMJER 6.
↳ Naći asimptote funkcije: $y=e^{\frac{1}{x}}-x$.

I) vertikalne asimptote:

$x\neq 0$ → $\lim\limits_{x\to 0^-} y=\lim\limits_{x\to 0^-}\left(e^{\frac{1}{x}}-x\right)=e^{-\infty}+0=0$

→ $\lim\limits_{x\to 0^+} y=\lim\limits_{x\to 0^+}\left(e^{\frac{1}{x}}-x\right)=e^{+\infty}-0=+\infty$



↳ $x=0$ → desna vertikalna asimptota

II) kose asimptote:

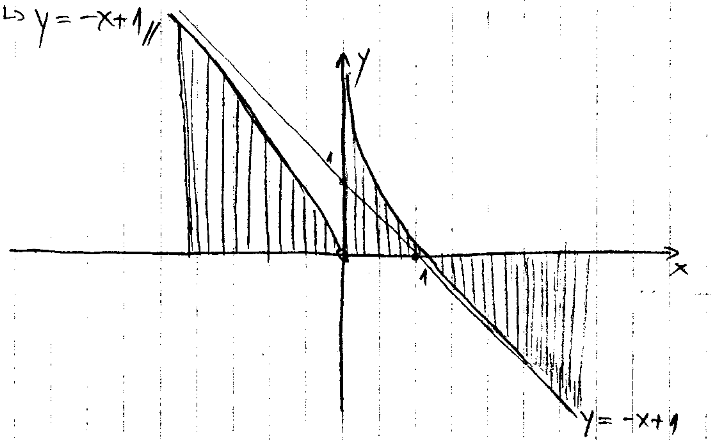
$y = kx + l$

$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \left(\frac{e^{\frac{1}{x}}}{x} - 1\right) = 0 - 1 = -1$$

$$l = \lim_{x\to\pm\infty} (f(x) - kx) = \lim_{x\to\pm\infty} (e^{\frac{1}{x}} \cancel{-x} \cancel{+x}) = \lim_{x\to\pm\infty} e^{\frac{1}{x}} = 1$$

↳ $y = -x + 1$

graf:



PRIMJER 7.

↳ Naći asimptote funkcije: $y = x - \text{arctg}\,x$

I) vertikalne asimptote:

$D_f \in \mathbb{R}$

↳ NEMA!

II) kose asimptote:

$y = kx + l$

$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \left(1 - \frac{\text{arctg}\,x}{x}\right) = 1$$

$$l_1 = \lim_{x\to+\infty} (f(x) - kx) = \lim_{x\to+\infty} -\text{arctg}\,x = -\frac{\pi}{2}$$

$$l_2 = \lim_{x\to-\infty} (f(x) - kx) = \lim_{x\to-\infty} -\text{arctg}\,x = \frac{\pi}{2}$$

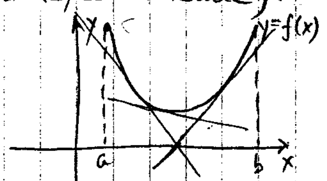
↳ $y_1 = x - \frac{\pi}{2}$ (desna)

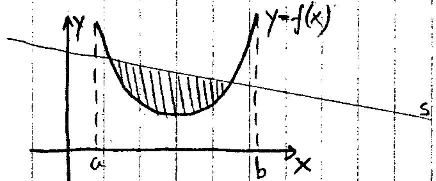
$y_2 = x + \frac{\pi}{2}$ (lijeva)

12.2. KONVEKSNOST I KONKAVNOST

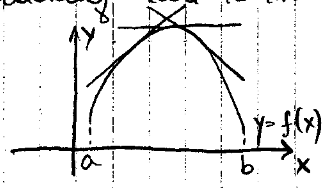
DEFINICIJA 1.

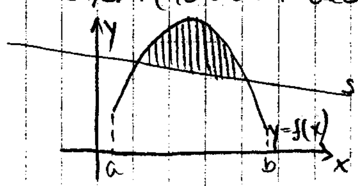
↳ Funkcija $y = f(x)$ je konveksna na intervalu $\langle a, b\rangle$ ako se njen graf nalazi iznad tangente u po volji odabranoj točki iz intervala $\langle a, b\rangle$ (Također i ako se njen graf nalazi ispod sekante).





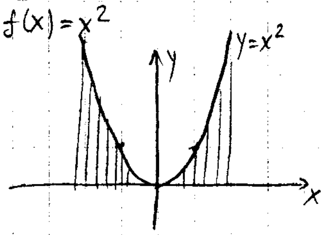
↳ Funkcija $y = f(x)$ je konkavna na intervalu $\langle a, b\rangle$ ako se njen graf nalazi ispod tangente u po volji odabranoj točki iz intervala $\langle a, b\rangle$. (Također i ako se njen graf nalazi ispod sekante).





=PRIMJER 1.

a) $f(x)=x^2$

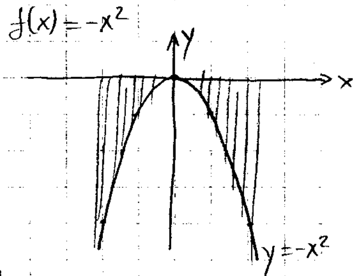


⇒ konveksna funkcija

$y' = 2x$

$y'' = 2 = \oplus > 0$ (∪)

b) $f(x)=-x^2$

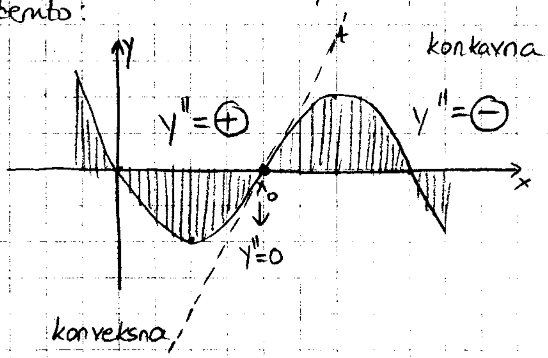


⇒ konkavna funkcija

$y' = -2x$

$y'' = -2 = \ominus < 0$ (∩)

=Općenito:



→ Ako je druga derivacija funkcije $f''(x)>0$ onda je funkcija konveksna.

→ Ako je druga derivacija funkcije $f''(x)<0$ onda je funkcija konkavna.

□ TOČKA PREGIBA

=DEFINICIJA 2.

↳ Točka $T(x_0, y_0)$ je točka infleksije (pregiba) funkcije $f(x)$, ako vrijedi:

1.) $f''(x_0)=0$ ili $f''(x_0)$ = nije definirano.

2.) $f''(x_0)$ = mijenja predznak pri prolazu kroz točku x_0.

= Određivanje intervala konveksnosti i točaka pregiba:

1. izračunamo $f''(x)$
2. riješimo jednadžbu $f''(x)=0$, a njezina rješenja su eventualne točke pregiba
3. na onim intervalima na kojima je $f''(x)>0$ funkcija je konveksna, a na ostalim intervalima ($f''(x)<0$) je konkavna.

↳ Na granici konveksnosti i konkavnosti nalazi se točka infleksije.

=PRIMJER 2.

a) ↳ Ispitati konveksnost i konkavnost funkcije $f(x)=x^2-3x+2$

$f'(x)=2x+3$

$f''(x)=2=\oplus>0 \Rightarrow$ konveksna na $\mathbb{R}$

b) ↳ Ispitati konveksnost i konkavnost funkcije $f(x)=\ln x$.

$f'(x)=\dfrac{1}{x}$

$f''(x)=-\dfrac{1}{x^2}=\ominus<0 \Rightarrow$ konkavna na $\mathbb{R}$

=PRIMJER 3.

↳ Nađi točku infleksije i područja konveksnosti i konkavnosti te jednadžbu infleksione tangente, ako je $f(x)=x^3-x$.

$\rightarrow f'(x) = 3x^2 - 1$
$f''(x) = 6x$

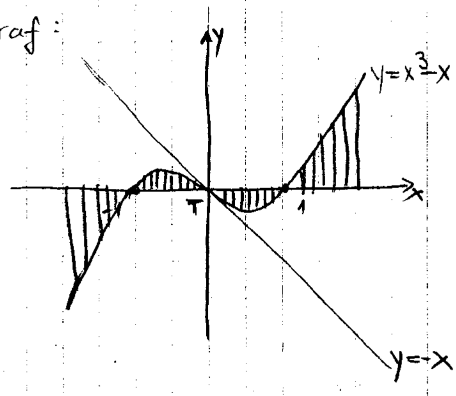
$f''(x) = 0$
$6x = 0$
$x = 0 \;,\; y = 0 \quad T(0,0)$

	$-\infty$ 0 ∞
$f''(x)$	$-$ 0 $+$
$f(x)$	$\cap$ 0 $\cup$

točka infleksije $T(0,0)$

$\langle -\infty, 0 \rangle \rightarrow f(-1) = \ominus \rightarrow$ konkavna
$\langle 0, \infty \rangle \rightarrow f(1) = \oplus \rightarrow$ konveksna

$\rightarrow$ graf:



tangenta... $T(0,0)$ $\quad k = y'(0) = -1$

t.... $y - y_0 = k(x - x_0)$
$y - 0 = -1(x - 0)$
$y = -x$

= PRIMJER 4.

$\hookrightarrow$ Nađi točku infleksije (ako postoji); odredi intervale konveksnosti i konkavnosti i nacrtaj graf funkcije $f = x|x|$.

$\rightarrow f'(x) = 1 \cdot |x| + \cancel{x} \cdot \frac{|x|}{\cancel{x}} = 2|x|$

$f''(x) = 2\frac{|x|}{x} \;,\; x \neq 0$

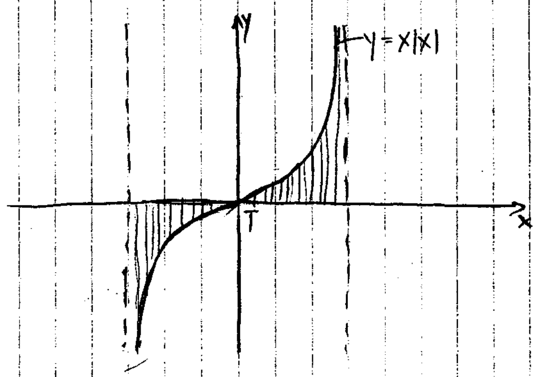
$f''(x) = 0$
$2\frac{|x|}{x} = 0$
$2|x| = 0$
$x = 0 \;,\; y = 0 \quad T(0,0)$

	$-\infty$ 0 ∞
$f''(x)$	$-$ 0 $+$
$f(x)$	$\cap$ 0 $\cup$

točka infleksije $T(0,0)$

$\langle -\infty, 0 \rangle \rightarrow f(-1) = \ominus \rightarrow$ konkavna
$\langle 0, \infty \rangle \rightarrow f(1) = \oplus \rightarrow$ konveksna

$\rightarrow$ graf:



12.3. TIJEK FUNKCIJE

= Postupak crtanja grafa funkcije:

1.) Istraživanje funkcije f:

- $\rightarrow$ odredi se domena
- $\rightarrow$ odredi se limes funkcije u rubnim točkama domene
- $\rightarrow$ ispitaju se svojstva parnosti, neparnosti i periodičnosti
- $\rightarrow$ odrede se nultočke, i sjecišta s osi y
- $\rightarrow$ odrede se asimptote

2) istraživanje funkcije f':
→ nađu se ekstremi i područja rasta i pada
3) istraživanje funkcije f'':
→ nađu se točke infleksije i područja konveksnosti i konkavnosti
4) po potrebi se izračuna vrijednost funkcije u nekoliko po volji odabranih točaka

= PRIMJER 1.
↳ Nacrtajte graf (kvalitativni) funkcije: $y = e^{\frac{1}{x}} - x$.

1) $\mathcal{D}_f \in \mathbb{R} \setminus \{0\}$
– asimptote: $y = -x+1$ (obostrana kosa), $x = 0$ (desna okomita) } VIDI: PRIMJER 6. STR. 205

2) $f'(x) = -\dfrac{e^{\frac{1}{x}}}{x^2} - 1$

$f'(x) = 0 \Rightarrow \cancel{e^{\frac{1}{x}} = -x^2}$
nema stacionarnih točaka

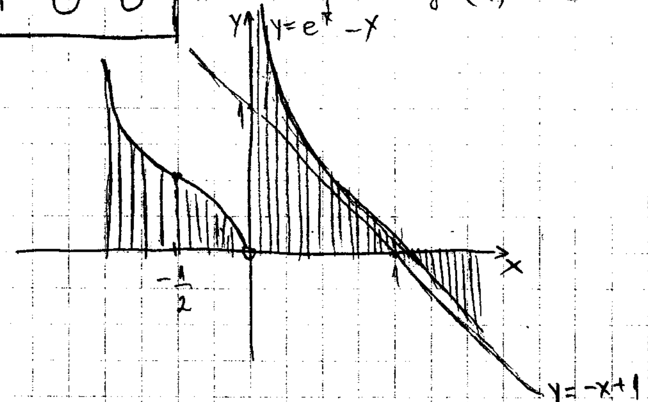
3) $f''(x) = \dfrac{e^{\frac{1}{x}}}{x^2} \cdot \dfrac{1}{x^2} - e^{\frac{1}{x}} \dfrac{(-2)}{x^3} = \dfrac{e^{\frac{1}{x}}}{x^4}(1+2x)$

$f''(x) = 0 \Rightarrow 1 + 2x = 0$
$x = -\frac{1}{2}$, $y = \frac{1}{e^2} + \frac{1}{2}$

	$-\infty$ … $-\frac{1}{2}$	$-\frac{1}{2}$ … 0	0 … ∞
$f'(x)$	−	+	+
$f(x)$	∩	∪	∪

$\langle -\infty, -\frac{1}{2} \rangle$ $f''(-1) = \ominus$ → konkavna
$\langle -\frac{1}{2}, 0 \rangle$ $f''(-\frac{1}{3}) = \oplus$ → konveksna
$\langle 0, \infty \rangle$ $f''(1) = \oplus$ → konveksna

→ graf:



= PRIMJER 2.
↳ Nacrtaj graf funkcije: $y = \dfrac{\ln^2 x}{x}$

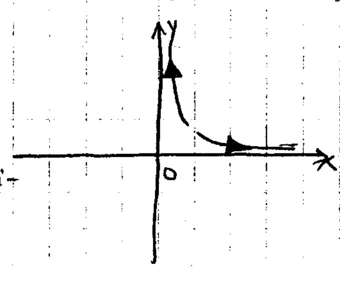
1) – domena $\mathcal{D}_f \in \langle 0, \infty \rangle$
– na rubu:

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{\ln^2 x}{x} = \dfrac{(-\infty)^2}{+0} = +\infty$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{\ln^2 x}{x} = \left(\dfrac{\infty}{\infty}\right) \overset{L'H}{=} \lim\limits_{x\to 0^-} \dfrac{2\ln x \cdot \frac{1}{x}}{1} = 2 \cdot \lim\limits_{x\to 0^-} \dfrac{\ln x}{x} = \left(\dfrac{\infty}{\infty}\right) =$

$\overset{L'H}{=} \lim\limits_{x\to 0^-} \dfrac{\frac{1}{x}}{1} = +0$

– asimptote: 1) vertikalna → $x = 0$ (desna)
2) horizontalna → $y = 0$ (desna)
↳ vidi se iz desnog grafa, kada skiciramo gornje limese



-nultočke: $\frac{\ln^2 x}{x}=0$

$\ln^2 x=0$

$\ln x=0$

$\cancel{\ln} x=\cancel{\ln} e^0$

$x=1$, $y=0$ $N(1,0)$

2) - $y'=\frac{2\ln x\cdot\frac{1}{x}\cdot x-\ln^2 x\cdot 1}{x^2}=\frac{\ln x\ (2-\ln x)}{x^2}$

$y'=0$

$\Downarrow$

$\ln x\ (2-\ln x)=0$

1.) $\ln x=0$ | 2.) $2-\ln x=0$

$\cancel{\ln} x=\cancel{\ln} e^0$ | $\ln x=2$

$x=1$ | $\cancel{\ln} x=\cancel{\ln} e^2$

$y=0$ | $x=e^2$

$S_1(1,0)$ | $y=\frac{4}{e^2}$

$S_2(e^2,\frac{4}{e^2})$

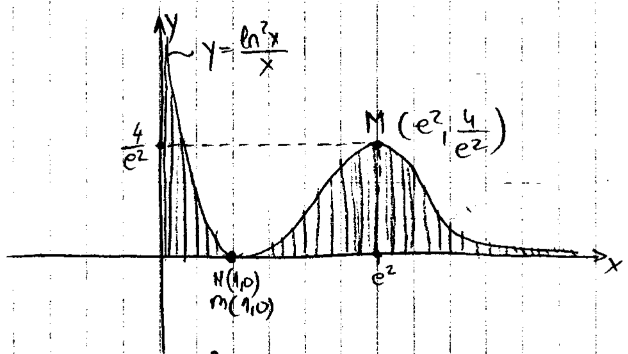
-intervali pomoću predznaka 2. derivacije:

$$y''=\frac{(2\frac{1}{x}-2\ln x\cdot\frac{1}{x})\cdot x^2-(2\ln x-\ln^2 x)2x}{x^4}=\frac{2(1-3\ln x+\ln^2 x)}{x^3}$$

$y''(S_1)=y''(1)=2>0 \Rightarrow$ minimum $m(1,0)$

$y''(S_2)=y''(e^2)=-\frac{2}{e^6}<0 \Rightarrow$ maksimum $M(e^2,\frac{4}{e^2})$

-graf:



= **PRIMJER 3.**

↳ Nacrtajte graf funkcije: $f(x)=e^{-\frac{x^2}{2}}$ (TZV. Gaussova krivulja)

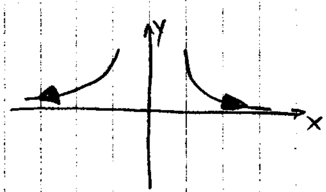
1) -domena $\mathcal{D}_f \in \mathbb{R}$

- na rubu: $\lim_{x\to\mp\infty} e^{-\frac{x^2}{2}}=0$

- asimptote: -> vertikalne: NEMA

-> horizontalne: $y=0$ (obostrana)

↳ iz desnog grafa o ponašanju limesa gornjih



-nultočke: NEMA

-parnost: $f(-x)=e^{-\frac{(-x)^2}{2}}=e^{-\frac{x^2}{2}}=f(x)\to$ Parna.

2)

$f'(x)=-x\cdot e^{-\frac{x^2}{2}}$

$f'(x)=0 \Rightarrow -x\cdot e^{-\frac{x^2}{2}}=0$

$x=0$, $y=1$ $S(0,1)$

	$-\infty$ 0 ∞
$f'(x)$	− 0 +
$f(x)$	↗ 1 ↘

MAX (0,1)

$\langle-\infty,0\rangle$ $f'(-1)=\ominus$

$\langle 0\ \infty\rangle$ $f'(1)=\oplus$

3) $f''(x)=(x^2-1)\ e^{-\frac{x^2}{2}}$

$f''(x)=0 \Rightarrow (x^2-1)e^{-\frac{x^2}{2}}=0$

$x^2-1=0$

$x_1=1$ | $x_2=-1$

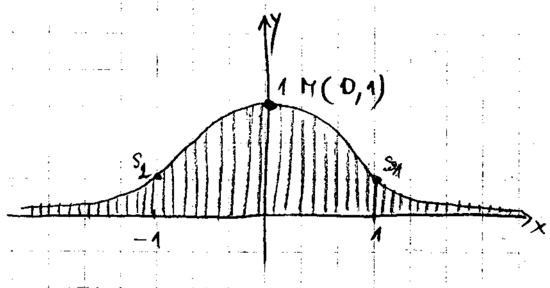
$y_1=e^{-\frac{1}{4}}$ | $y_2=e^{-\frac{1}{4}}$

$S_1(1,e^{-\frac{1}{4}})$, $S_2(-1,e^{-\frac{1}{4}})$

	$-\infty$ -1 1 $+\infty$
$f'(x)$	$+ \; 0 \; - \; 0 \; +$
$f(x)$	$\cup \; e^{-\frac{1}{4}} \; \cap \; e^{-\frac{1}{4}} \; \cup$

$\langle -\infty, -1 \rangle$ $f''(-2) = \oplus \to$ konveksna
$\langle -1, 1 \rangle$ $f''(0) = \ominus \to$ konkavna
$\langle 1, \infty \rangle$ $f''(2) = \oplus \to$ konveksna

-graf:



☐ GRAFOVI RACIONALNIH FUNKCIJA

= Racionalna funkcija $f(x) = \frac{P_n(x)}{Q_m(x)}$ može biti:

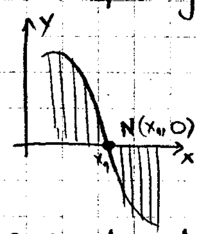
1) prava racionalna : za $n < m$
2) neprava racionalna : za $n \geq m$
↳ u ovom slučaju diobom brojnika i nazivnika dobije se:

$$\frac{P_n(x)}{Q_m(x)} = \underbrace{R_{n-m}(x)}_{\text{rezultat}} + \underbrace{\frac{\sigma(x)}{Q_m(x)}}_{\text{ostatak-prava racionalna funkcija}}$$

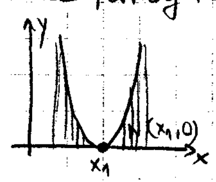
= Nultočke racionalne funkcije:

$P_n(x) = 0 \Rightarrow x = x_1$
$\quad x = x_2$

1) nultočke neparnog reda :



2.) nultočke parnog reda :

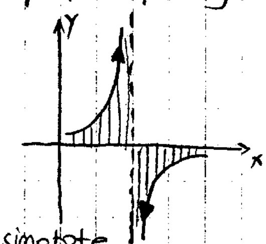


= Asimptote racionalne funkcije:

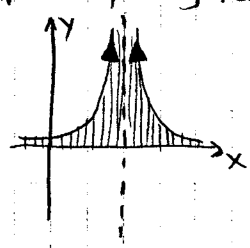
1) vertikalne asimptote
↳ nalaze se u nultočkama nazivnika:

$Q_m(x) = 0 \Rightarrow x = x_1$
$\quad x = x_2$

(a) asimptota neparnog reda:



(b) asimptota parnog reda:



2) horizontalne asimptote
↳ postoje onda i samo onda ako je stupanj brojnika manji ili jednak stupnju nazivnika, pa vrijedi:

$$\lim_{x \to \pm\infty} f(x) = 0,$$

pa je $y = 0$ asimptota.

↳ako je stupanj brojnika jednak stupnju nazivnika racionalna funkcija imat će horizontalnu asimptotu $y=\frac{a_n}{b_n}$ gdje su a_n i b_n vodeći koeficijenti polinoma u brojniku i nazivniku

3) kose asimptote

↳ako je stupanj brojnika za jedan veći od stupnja nazivnika

= PRIMJER 4.

↳Nacrtajmo graf funkcije: $f(x)=\frac{3x-x^2}{x-4}$.

1.) - domena: $\mathcal{D}_f \in \mathbb{R} \setminus \{4\}$

– na rubovima:

I) $\frac{-x^2+3x}{x-4} = (-\cancel{x^2}+3x):(x-4) = -x-1$

$$\begin{array}{l} \underline{+\cancel{x^2} \pm 4x} \\ \quad -\cancel{x} \\ \quad \underline{+\cancel{x} \pm 4} \\ \qquad -4 \end{array}$$

$P = -x-1+\frac{-4}{x-4}$

↳ $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty}\left(\frac{-x^2+3x}{x-4}\right)\begin{array}{l}:x^2\\:x^2\end{array} = \lim\limits_{x\to-\infty}\frac{-1+\frac{3}{x}}{\frac{1}{x}-\frac{4}{x^2}} = +\infty$

↳ $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}\left(\frac{-x^2+3x}{x-4}\right)\begin{array}{l}:x^2\\:x^2\end{array} = \lim\limits_{x\to\infty}\frac{-1+\frac{3}{x}}{\frac{1}{x}-\frac{4}{x^2}} = -\infty$

↳ $\lim\limits_{x\to4^+} f(x) = \lim\limits_{x\to4^+}\frac{-x^2+3x}{x-4} = \frac{-(4+0)^2+3(4+0)}{\cancel{4}+0-\cancel{4}} = \frac{-16-0-0+12+0}{+0} = \frac{-4}{+0} = -\infty$

↳ $\lim\limits_{x\to4^-} f(x) = \lim\limits_{x\to4^-}\frac{-x^2+3x}{x-4} = \frac{-(4-0)^2+3(4-0)}{\cancel{4}-0-\cancel{4}} = \frac{-16+0+0+12-0}{-0} = \frac{-4}{-0} = +\infty$

– asimptote: → vertikalne: $x=4$

→ kose: $k=\lim\limits_{x\to\pm\infty}\frac{f(x)}{x} = \lim\limits_{x\to\pm\infty}\frac{3x-x^2}{x^2-4x} = -1$

$l=\lim\limits_{x\to\pm\infty}[f(x)-kx] = \lim\limits_{x\to\pm\infty}\left(\frac{3x-x^2}{x-4}+x\right) = \lim\limits_{x\to\pm\infty}\frac{-x}{x-4} = -1$

$y=-x-1$

– parnost: $f(-x) = \frac{3(-x)-(-x)^2}{(-x)-4} = \frac{-3x-x^2}{-x-4} = \frac{x^2+3x}{x+4}$ ⇒ ni parna ni neparna

– nultočke: $\frac{3x-x^2}{x-4}=0$

$3x-x^2=0$

$x(3-x)=0$

$x_1=0 \quad x_2=3$

$N(0,0)$

$N(3,0)$

2.) –

$f'(x) = \frac{(3-2x)(x-4)-(3x-x^2)}{(x-4)^2} = \frac{(x-2)(x-6)}{(x-4)^2}$

$f'(x)=0 \Rightarrow (x-2)(x-6)=0$

$x_1=2 \quad x_2=6$

$y_1=-1 \quad y_2=-9$

$S_1(2,-1)$

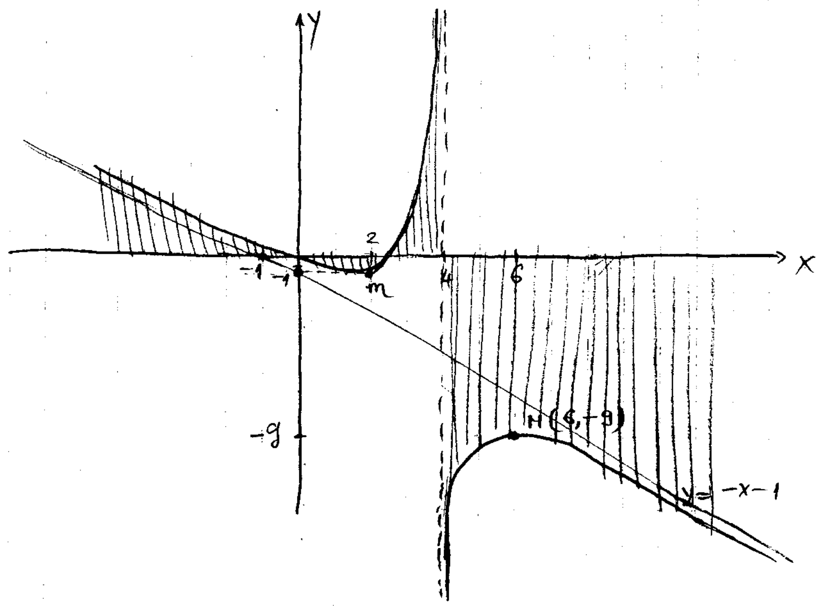
$S_2(6,-9)$

	$-\infty$	2		6	$+\infty$
$f'(x)$	−	0	+	0	−
$f(x)$	↘	−1	↗	−9	↘
		MIN		MAX	

$m(2,-1) \quad M(6,-9)$

Samo je jedan Mali Ivica!

-graf:



ZADACI

① asimptote: $f(x)=\frac{-x^3-x^2+x+5}{(x+1)^2}$

→ vertikalne: $D_f \in \mathbb{R} \setminus \{-1\}$

$x=1$ (provjera: $\lim_{x\to -1\pm} \frac{-x^3-x^2+x+5}{(x+1)^2}=\infty$)

→ kose:

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\left(\frac{-x^3-x^2+x+5}{x(x+1)^2}\right)=\lim_{x\to\pm\infty}\left(\frac{-x^3-x^2+x+5}{x^3+2x^2+5}\right)=-1$$

$$l=\lim_{x\to\pm\infty}\left(f(x)-xk\right)=\lim_{x\to\pm\infty}\left(\frac{-x^3-x^2+x+5}{x^2+2x+1}+x\right)=\lim_{x\to\pm\infty}\frac{\cancel{-x^3}-x^2+x+5+\cancel{x^3}+2x^2+x}{x^2+2x+1}=-1$$

$y=kx+l=-x-1$

② asimptote: $f(x)=e^{\frac{1}{x-2}}$

→ vertikalne: $D_f \in \mathbb{R} \setminus \{2\}$

$x=2$ (provjera: $\lim_{x\to 2^+} e^{\frac{1}{x-2}}=+\infty$, $\lim_{x\to 2^-} e^{\frac{1}{x-2}}=0$)

→ kose:

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{e^{\frac{1}{x-2}}}{x}=0$$

NEMA

→ horizontalne:

$$l=\lim_{x\to\pm\infty}f(x)=\lim_{x\to\pm\infty}e^{\frac{1}{x-2}}=1 \qquad y=1$$

③ asimptote: $f(x)=\frac{x}{\ln^2 x}$

→ vertikalne: $D_f \in \mathbb{R} \setminus \{1\}$

$x=1$ (provjera: $\lim_{x\to 1^-}\frac{x}{\ln^2 x}=+\infty$, $\lim_{x\to 1^+}\frac{x}{\ln^2 x}=+\infty$)

→ kose:

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{1}{\ln^2 x}=0$$

NEMA!//

→ horizontalne:

$$l=\lim_{x\to\pm\infty}f(x)=\lim_{x\to\pm\infty}\frac{1}{2\ln x\cdot\frac{1}{x}}=\lim_{x\to\pm\infty}\frac{x}{2\ln x}=\lim_{x\to\pm\infty}\frac{1}{2\cdot\frac{1}{x}}=+\infty$$

NEMA!//

(4.) asimptote:

$$f(x)=\operatorname{arctg}\frac{x^2}{x+1}$$

→ vertikalne: $D_f\in\mathbb{R}\setminus\{1\}$

$$\lim_{x\to 1^-}f(x)=\operatorname{arctg}(-\infty)=-\frac{\pi}{2}\qquad x=-\frac{\pi}{2}\,//$$

$$\lim_{x\to 1^+}f(x)=\operatorname{arctg}(+\infty)=\frac{\pi}{2}\qquad x=\frac{\pi}{2}\,//$$

→ kose:

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{\operatorname{arctg}\left(\frac{x^2}{x+1}\right)}{x}=\frac{\frac{\pi}{2}}{\infty}=0$$

NEMA!//

→ horizontalne:

$$l=\lim_{x\to\pm\infty}f(x)=\lim_{x\to\pm\infty}\operatorname{arctg}\left(\frac{x^2}{x+1}\right)=\operatorname{arctg}\left(\lim_{x\to\pm\infty}\frac{x^2}{x+1}\right)=\operatorname{arctg}(\pm\infty)=\pm\frac{\pi}{2}$$

$$y_1=\frac{\pi}{2}\,//\qquad y_2=-\frac{\pi}{2}\,//$$

(5.) asimptote: $f(x)=2x-1-\sqrt{x^2-x+1}$

→ vertikalne: $D_f\in\mathbb{R}$

NEMA!

→ kose:

$$k_1=\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\left(2-\cancel{\frac{1}{x}}^{0}-\sqrt{1-\cancel{\frac{1}{x}}^{0}+\cancel{\frac{1}{x^2}}^{0}}\right)=1$$

$$k_2=\lim_{x\to-\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\left(2-\cancel{\frac{1}{x}}^{0}+\sqrt{1-\cancel{\frac{1}{x}}^{0}+\cancel{\frac{1}{x^2}}^{0}}\right)^{0}=3$$

$$l_1=\lim_{x\to\infty}(f(x)-kx)=\lim_{x\to\infty}\left(2x-1-\sqrt{x^2-x+1}-x\right)=\lim_{x\to+\infty}\frac{(x-1)^2-(x^2-x+1)}{x-1+\sqrt{x^2-x+1}}=$$

$$=\lim_{x\to\infty}\frac{\cancel{x^2}-2x+\cancel{1}-\cancel{x^2}+x-\cancel{1}}{x+1+\sqrt{x^2-x+1}}=\lim_{x\to\infty}\frac{-x}{x-1+\sqrt{x^2-x+1}}=-\frac{1}{2}$$

$$l_2=\lim_{x\to\infty}\left(2x-1-\sqrt{x^2-x+1}-3x\right)=\lim_{x\to\infty}\frac{(x-1)^2-(x^2+x+1)}{x-1+\sqrt{x^2-x+1}}=\lim_{x\to\infty}\frac{\cancel{x^2}-2x+\cancel{1}-\cancel{x^2}-x-\cancel{1}}{x-1+\sqrt{x^2-x+1}}=$$

$$=-\frac{3}{2}$$

$$y_1=x-\frac{1}{2}\,//\qquad y_2=3x-\frac{3}{2}\,//$$

(6.) $f(x)=x^3+x^2-5x+10$

↳ 2. derivacija

$$f'(x)=3x^2+2x-5$$
$$f''(x)=6x+2=0$$
$$f''(x)=0\Rightarrow 6x+2=0$$
$$2(3x+1)=0$$
$$x=-\frac{1}{3}\,//$$

	$-\infty$ … $-\frac{1}{3}$	$-\frac{1}{3}$ … ∞
$f''(x)$	−	+
$f(x)$	∩	∪

INF.

(7.) $f(x)=\frac{x^2}{2}+\ln x$
↳ 2. derivacija

$f'(x)=\frac{\cancel{2}x}{\cancel{2}}+\frac{1}{x}=x+\frac{1}{x}$

$f''(x)=1-\frac{1}{x^2}$

$f''(x)=0 \Rightarrow \frac{x^2-1}{x^2}=0$

$x^2-1=0 \quad x_1=1 \quad \cancel{x_2=-1} \quad + D_f \in \langle 0,\infty\rangle$

	0 ... 1	1 ... ∞
$f''(x)$	$-$	$+$
$f(x)$	$\cap$	$\cup$

INF

(8.) $f(x)=\frac{1}{x+3}$
↳ 2. derivacija

$f'(x)=\frac{-x}{(x+3)^2}$

$f''(x)=\frac{-1(x+3)^{\cancel{2}}+x\cdot 2\cancel{(x+3)}\cdot 1}{(x+3)^{\cancel{4}}}=\frac{-(x+3)+2x}{(x+3)^3}=\frac{x-3}{(x+3)^3}$

$f''(x)=0 \Rightarrow x-3=0$
$x=3$, $D_f \in \mathbb{R}\setminus\{-3\}$

	$-\infty$ ... 3	3 ... $+\infty$
$f''(x)$	$-$	$+$
$f(x)$	$\cap$	$\cup$

INF

(9.) $f(x)=\sqrt[3]{x+2}$
↳ 2. derivacija

$f'(x)=\frac{1}{3}(x+2)^{-\frac{2}{3}}$

$f''(x)=-\frac{2}{9}(x+2)^{-\frac{5}{3}}$

$f''(x)=0 \Rightarrow -\frac{2}{9}(x+2)^{-\frac{5}{3}}=0$
$x=-2$

	$-\infty$ ... -2	-2 ... $+\infty$
$f''(x)$	$+$	$-$
$f(x)$	$\cup$	$\cap$

INF

(10.) $f(x)=(1+x^2)e^x$
↳ 2. derivacija

$f'(x)=2xe^x+(1+x^2)e^x$
$f''(x)=2e^x+2xe^x+2xe^x+(1+x^2)e^x=e^x(3+4x+x^2)$
$f''(x)=0 \Rightarrow e^x(3+4x+x^2)=0$
$x^2+4x+3=0 \quad x_{1,2}=\frac{-4\pm\sqrt{16-12}}{2}=\frac{-4\pm 2}{2} \quad x_1=+3 \quad x_2=-1$

	$-\infty$ ... -3	-3 ... -1	-1 ... ∞
$f''(x)$	$+$	$-$	$+$
$f(x)$	$\cup$	$\cap$	$\cup$

INF INF

11. $f(x)=x^3-3x$

↳ tok

1) - domena: $D_f \in \mathbb{R}$

- na rubu:

$$\lim_{x\to-\infty}(x^3-3x)=-\infty$$

$$\lim_{x\to+\infty}(x^3-3x)=+\infty$$

- asimptote: → vertikalne: NEMA!

→ kose:

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}(x^2-3)=+\infty$$

NEMA!

→ horizontalne: NEMA!

- parnost:

$$f(-x)=(-x)^3-3(-x)=-x^3+3x=-(x^3-3x)=-f(x) \Rightarrow \text{Neparna}$$

- nultočke:

$x^3-3x=0$

$x(x^2-3)=0$

$x_1=0 \quad x_2=\sqrt{3} \quad x_3=-\sqrt{3}$

2) -

$f'(x)=3x^2-3$

$f'(x)=0 \Rightarrow 3(x^2-1)=0$

$x_1=1 \quad x_2=-1$

$y_1=-2 \quad y_2=2$

	$-\infty$ … -1	-1 … 1	1 … ∞
$f'(x)$	+	−	+
$f(x)$	↗	↘	↗
		MAX	MIN

MIN → $S_1(1,-2)$

MAX → $S_2(-1,2)$

3) -

$f''(x)=6x$

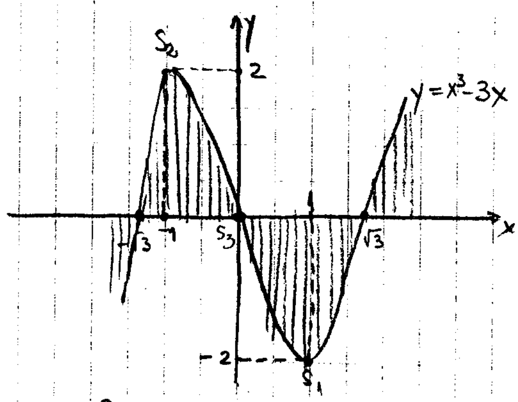
$f''(x)=0 \Rightarrow 6x=0$

$x=0$

	$-\infty$ … 0	0 … ∞
$f''(x)$	−	+
$f(x)$	$\cap$	$\cup$
	INF	

INF → $S_3(0,0)$

- graf:



12. $f(x)=\dfrac{x^2-2x+2}{x-1}$, tok = ?

1) - domena: $D_f \in \mathbb{R}\setminus\{1\}$

- na rubu:

$$\lim_{x\to1^+}f(x)=+\infty$$

$$\lim_{x\to1^-}f(x)=-\infty$$

- asimptote: → vertikalne $x=1$

→ kose:

$$k=\lim_{x\to\pm\infty}\frac{x^2-2x+2}{x^2-x}=1$$

$$l=\lim_{x\to\pm\infty}\frac{x^2-2x-x^2+x}{x-1}=-1$$

$y=x-1$

- parnost: $f(-x)=\dfrac{x^2+2x+2}{-x-1} \Rightarrow$ ni parna, ni neparna

- nultočke: $x^2-2x+2=0$

$$x_{1,2}=\frac{2\pm\sqrt{4-8}}{2} \Rightarrow \text{nema}$$

2)
$$f'(x)=\frac{(2x-2)(x-1)-(x^2-2x+2)}{(x-1)^2}=\frac{2x^2-\cancel{2x}-2x+\cancel{2}-x^2+\cancel{2x}-\cancel{2}}{(x-1)^2}=\frac{x^2-2x}{(x-1)^2}$$

$f'(x)=0 \Rightarrow x^2-2x=0$
$x(x-2)=0$
$x_1=0 \quad x_2=2$

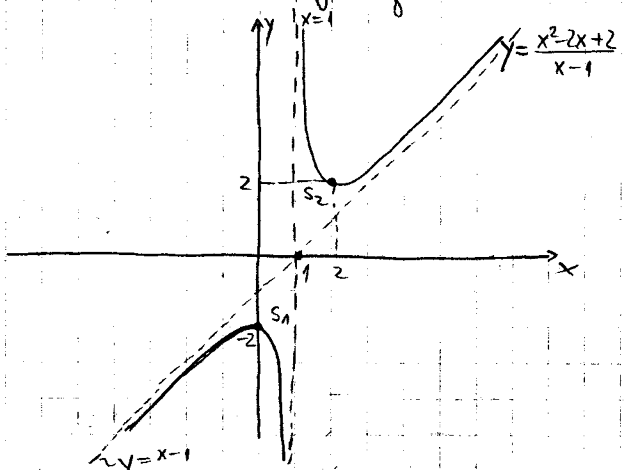
	$-\infty$ … 0	0 … 1	1 … 2	2 … ∞
$f'(x)$	+	−	−	+
$f(x)$	↗	↘	↘	↗
		MAX		MIN

MAX $S_1(0,-2)$
MIN $S_2(2,2)$

3)
$$f''(x)=\frac{(2x-2)(x-1)^2-(x^2-2x)2(x-1)}{(x-1)^4}=\frac{2\cancel{(x-1)}\left[(x-1)^2-x^2+2x\right]}{(x-1)^4}=\frac{2(\cancel{x^2}-\cancel{2x}+2-\cancel{x^2}+\cancel{2x})}{(x-1)^3}=\frac{4}{(x-1)^3}$$

$f''(x)=0 \Rightarrow 4=0 \Rightarrow$ nema točke infleksije

→ graf:



(13.) $f(x)=\frac{8}{x^2-4}$, tok=?

1) – domena: $D_f \in \mathbb{R} \setminus \{-2,2\}$
– na rubu:
$$\lim_{x\to 2^+}\frac{8}{x^2-4}=+\infty$$
$$\lim_{x\to 2^-}\frac{8}{x^2-4}=-\infty$$
$$\lim_{x\to -2^-}\frac{8}{x^2-4}=-\infty$$
$$\lim_{x\to -2^+}\frac{8}{x^2-4}=+\infty$$

– asimptote: → vertikalne:
$x=2,\ x=-2$
→ kose:
$k_1=\lim_{x\to\pm\infty}\frac{8}{x^3-4x}=0$ NEMA!
→ horizontalne:
$l_1=\lim_{x\to\pm\infty}\frac{8}{x^2-4}=0 \quad y=0$

– parnost: $f(-x)=\frac{8}{(-x)^2-4}=\frac{8}{x^2-4}=f(x) \Rightarrow$ parna

– nultočke: $x=0$

2) $f'(x)=\frac{-8\cdot 2x}{x^2-4}$ $\quad f'(x)=0$
$-16x=0$
$x=0$

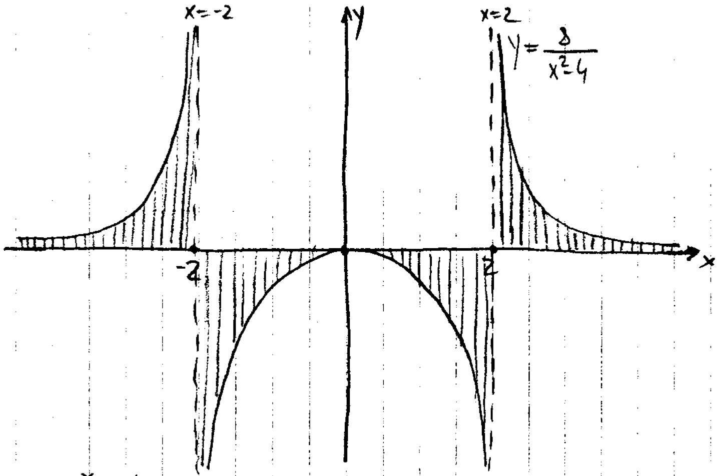
	$-\infty$ … −2	−2 … 0	0 … 2	2 … ∞
$f'(x)$	+	+	−	−
$f(x)$	↗	↗	↘	↘
			MAX	

MAX → $S_1(0,-2)$

3)
$$f''(x)=\frac{-16(x^2-4)^{\cancel{2}}+16x\cdot 2\cancel{(x^2-4)}2x}{(x^2-4)^{\cancel{4}}}=\frac{16(3x^2+4)}{(x^2-4)^3}=0$$

$f''(x)\neq 0 \Rightarrow$ ~~$3x^2=-4$~~

-graf:



(14.) $f(x)=xe^{-x}$, tok=?

1) -domena: $D_f \in \mathbb{R}$

-na rubu:

$\lim\limits_{x\to-\infty} f(x)=-\infty$

$\lim\limits_{x\to+\infty} f(x)=0$

-parnost: $f(-x)=-xe^{x} \Rightarrow$ ni parna ni neparna

-nultočke: $xe^{-x}=0$

$x=0 \quad y=0 \quad N(0,0)$

-asimptote: → vertikalne: NEMA

→ kose:

$k_1=\lim\limits_{x\to\infty}\frac{f(x)}{x}=\lim\limits_{x\to\infty}e^{-x}=0$

$k_2=\lim\limits_{x\to-\infty}\frac{f(x)}{x}=\lim\limits_{x\to-\infty}e^{-x}=+\infty$ NEMA

$l_1=\lim\limits_{x\to+\infty}\frac{x}{e^x}=\lim\limits_{x\to+\infty}\frac{1}{e^x}=0$

$l_2=\lim\limits_{x\to-\infty}xe^{-x}=-\infty$

2) $f'(x)=e^{-x}-xe^{-x}=e^{-x}(1-x)$

$f'(x)=0 \Rightarrow e^{-x}(1-x)=0$

$x=1$

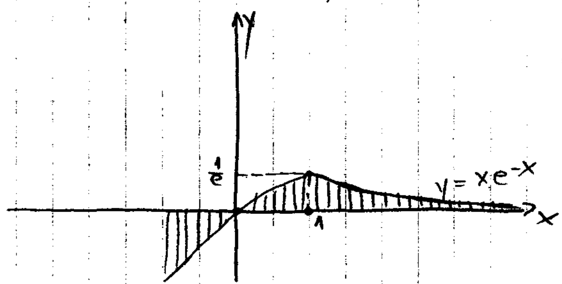
	$-\infty$ … 1	1 … ∞
$f'(x)$	+	−
$f(x)$	↗	↘

MAX

MAX $\Rightarrow S(1,\frac{1}{e})$

3.) $f''(x)=-e^{-x}(1-x)+e^{-x}=e^{-x}\neq 0,\ \forall x\in\mathbb{R}$

-graf:



(15.) $f(x)=\frac{\ln x}{x}$, tok=?

1) -domena: $D_f \in \langle 0,\infty\rangle$

-na rubu:

$\lim\limits_{x\to0^+} f(x)=+\infty$

$\lim\limits_{x\to\infty} f(x)=0$

-asimptote: → vertikalne: $x=0$

→ kose:

$k=\lim\limits_{x\to+\infty}\frac{\ln x}{x^2}=0$ NEMA

→ horizontalne:

$l=\lim\limits_{x\to\infty}\frac{\ln x}{x}=0 \quad y=0$

-nultočke: $\ln x=0$

$\ln x=\ln e^0 \Rightarrow x=1 \quad N(1,0)$

-parnost: ni parna ni neparna

2) $f'(x)=\frac{1}{x^2}(1-\ln x)$

$f'(x)=0 \Rightarrow 1-\ln x=0$

$\ln x=1$

$\cancel{\ln}x=\cancel{\ln}e \quad x=e$

	0 … e	e … ∞
$f'(x)$	+	−
$f(x)$	↗	↘

MAX

MAX $\Rightarrow S(e,\frac{1}{e})$

3.) $f''(x) = \frac{-2}{x^3}(1-\ln x) + \frac{1}{x^2}\left(\frac{-1}{x}\right) = \frac{1}{x^3}(2\ln x - 3)$

$f''(x) = 0 \Rightarrow 2\ln x = 3$

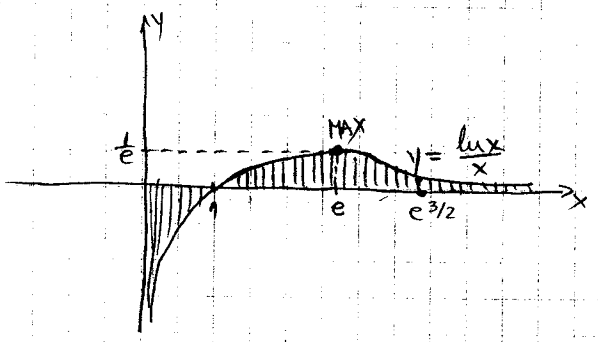
$\ln x = \frac{3}{2}$

$x = e^{\frac{3}{2}}$

	0	$e^{3/2}$	∞
$f''(x)$	$-$	$+$	
$f(x)$	$\frown$	$\smile$	
		INF	

$\text{INF} \Rightarrow S_1\left(e^{\frac{3}{2}}, \frac{3}{2e^{3/2}}\right)$

-graf:



(16.) $f(x) = \sqrt{x}\ln x$, 1. derivacija

1) - domena: $D_f \in \langle 0, \infty\rangle$

- na rubu:

$\lim_{x\to 0^+} \sqrt{x}\ln x = 0$

$\lim_{x\to\infty} \sqrt{x}\ln x = +\infty$

- asimptote: → vertikalne: $x = 0$

→ kose:

$k = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = -\infty$ NEMA

- parnost: ni parna ni neparna

- nultočke: $\sqrt{x}\ln x = 0$

$\ln x = 0 \Rightarrow x = 1$ $\quad N(1,0)$

2) $f'(x) = \frac{1}{2}\frac{\sqrt{x}}{x}\ln x + \frac{\sqrt{x}}{x} = \frac{\sqrt{x}}{x}\left(\frac{1}{2}\ln x + 1\right)$

$f'(x) = 0 \Rightarrow \frac{1}{2}\ln x + 1 = 0$

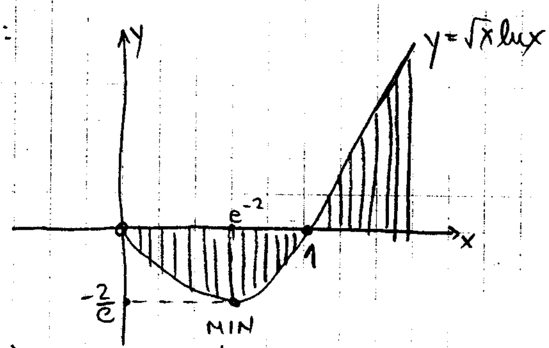
$\ln x = -2$

$x = e^{-2}$

	0	e^{-2}	∞
$f'(x)$	$-$	$+$	
$f(x)$	↘	↗	
		MIN	

$\text{MIN} \Rightarrow S\left(e^{-2}, \frac{-2}{e}\right)$

-graf:



(17.) $f(x) = x\,\mathrm{arctg}\,x$, 1. derivacija

1) - domena: $D_f \in \mathbb{R}$

- na rubu:

$\lim_{x\to-\infty} f(x) = +\infty$

$\lim_{x\to+\infty} f(x) = +\infty$

- asimptote: → vertikalne: NEMA

→ kosa:

$k_1 = \lim_{x\to+\infty} \mathrm{arctg} = \frac{\pi}{2}$

$k_2 = \lim_{x\to-\infty} \mathrm{arctg} = -\frac{\pi}{2}$

$l_1 = \lim_{x\to+\infty}\left(x\,\mathrm{arctg}\,x - \frac{\pi}{2}x\right) = \lim_{x\to+\infty} x\left(\mathrm{arctg}\,x - \frac{\pi}{2}\right) =$

$= (\infty \cdot 0) = \lim_{x\to\infty} \frac{\operatorname{arctg} x - \frac{\pi}{2}}{\frac{1}{x}} = \left(\frac{0}{0}\right) = \lim_{x\to\infty} \frac{-x^2}{1+x^2} = -1$ $y = \frac{\pi}{2}x - 1$

$l_2 = \lim (x \operatorname{arctg} x + \frac{\pi}{2}x) = -1$ $y = -\frac{\pi}{2}x - 1$

-nultočke: $x=0$

2) $f'(x) = \operatorname{arctg} x + \frac{x}{1+x^2}$

$f'(x) = 0$

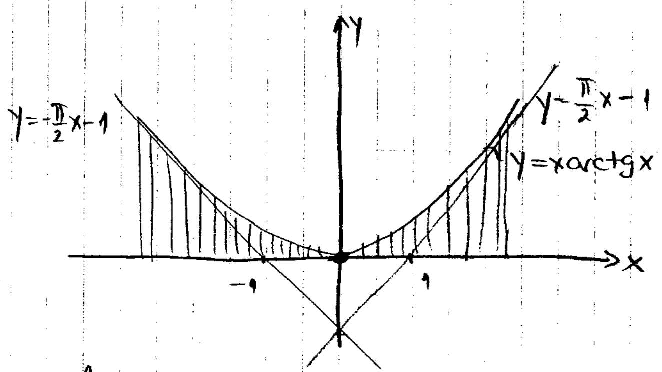
$\operatorname{arctg} x = -x \Rightarrow x = 0$

	$-\infty$ … 0	0 … ∞
$f'(x)$	−	+
$f(x)$	↘	↗

MIN

MIN $\Rightarrow S(0,0)$

-graf:



(18.) $f(x) = \operatorname{th} \frac{1}{1-x}$, 1. derivacija

1) -domena: $\mathbb{R} \setminus \{1\}$

- na rubu:

$\lim_{x\to 1^-} f(x) = 1$

$\lim_{x\to 1^+} f(x) = -1$

$\lim_{x\to\infty} = 0$

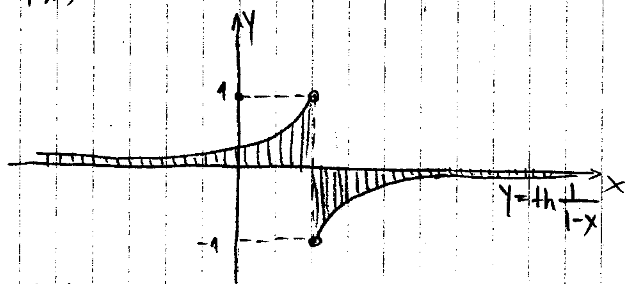
$\lim_{x\to-\infty} = 0$

-asimptote: → vertikalne: NEMA
→ kose: NEMA
→ horizontalne: $y=0$

-nultočke: nema

2) $f'(x) = \frac{1}{\operatorname{ch}^2\left(\frac{1}{1-x}\right)} \cdot \frac{1}{(1-x)^2} \neq 0, \forall x \in \mathcal{D}_f \Rightarrow$ nema extrema

-graf:



(19.) $f(x) = \operatorname{cth}(x^2-1)$, 1. derivacija

1) -domena: $\mathcal{D}_f \in \mathbb{R} \setminus \{-1, 1\}$

- na rubu:

$\lim_{x\to -1^-} f(x) = +\infty$ $\lim_{x\to\pm\infty} f(x) = 1$

$\lim_{x\to -1^+} f(x) = -\infty$

$\lim_{x\to 1^-} f(x) = -\infty$

$\lim_{x\to 1^+} f(x) = +\infty$

-asimptote: → vertikalne: $x = 1$
→ kose: NEMA
→ horizontalne: $y = 1$

-nultočke: NEMA

-parnost: $f(-x) = \operatorname{cth}(x^2-1) \Rightarrow$ parna

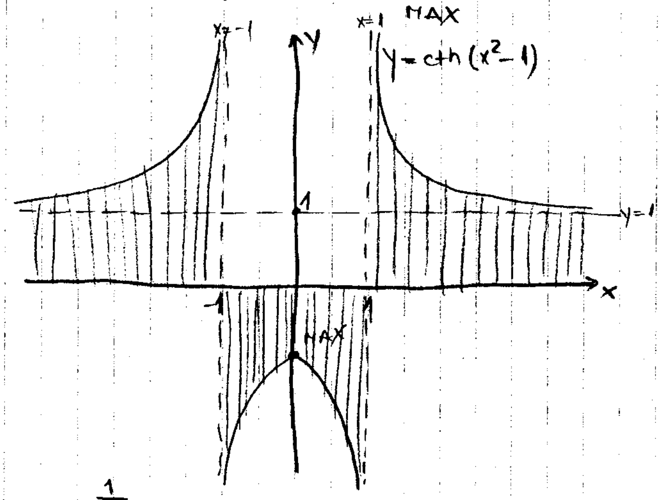
2) $f'(x) = \dfrac{-2x}{sh^2(x^2-1)}$

$f'(x)=0 \Rightarrow x=0$

	$-\infty$ … -1	-1 … 0	0 … 1	1 … ∞
$f'(x)$	+	+	−	−
$f(x)$	↗	↗	↘	↘
		MAX		

MAX $\Rightarrow S\left(0, \dfrac{1+e^2}{1-e^2}\right)$

-graf:



(20.) $f(x) = (x+1)e^{\frac{1}{2x}}$, 1. derivacija

1) - domena: $D_f \in \mathbb{R} \setminus \{0\}$

- na rubu:

$\lim\limits_{x\to 0^-} f(x) = 0$

$\lim\limits_{x\to 0^+} f(x) = +\infty$

$\lim\limits_{x\to +\infty} f(x) = +\infty$

$\lim\limits_{x\to -\infty} f(x) = -\infty$

- asimptote: → vertikalne: $x=0$

→ kose:

$k_1 = \lim\limits_{x\to\pm\infty} \dfrac{f(x)}{x} = 1 + \lim\limits_{x\to\pm\infty} \dfrac{e^{\frac{1}{2x}}}{x} = 1$

$l = \lim\limits_{x\to\pm\infty} \left(x e^{\frac{1}{2x}} + e^{\frac{1}{2x}} - x\right) = \dfrac{3}{2}$

$y = x + \dfrac{3}{2}$

- nultočke: $x+1=0$

$x=-1 \quad N(-1,0)$

2) $f'(x) = e^{\frac{1}{2x}} + (x+1)e^{\frac{1}{2x}} \cdot \left(-\frac{1}{2}\right)\frac{1}{x^2} = \dfrac{e^{\frac{1}{2x}}}{2x^2}(2x^2-x-1)$

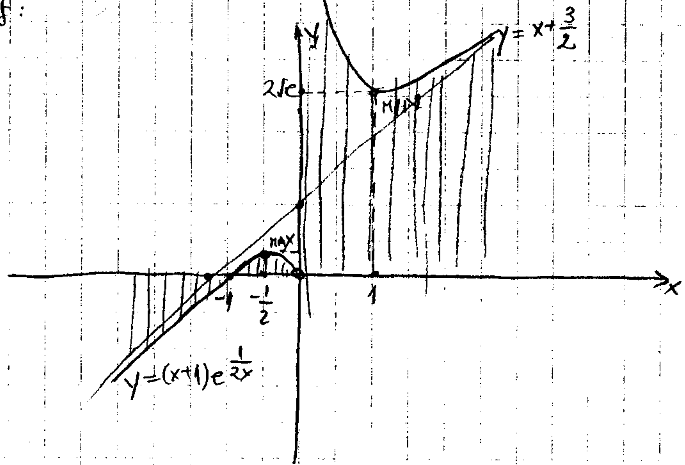
$f'(x)=0 \Rightarrow 2x^2-x-1=0 \quad x_{1,2} = \dfrac{1\pm 3}{4} \quad x_1 = 1 \quad x_2 = -\dfrac{1}{2}$

	$-\infty$ … $-\frac{1}{2}$	$-\frac{1}{2}$ … 0	0 … 1	1 … ∞
$f'(x)$	+	−	−	+
$f(x)$	↗	↘	↘	↗
		MAX		MIN

MAX: $S_1\left(-\frac{1}{2}, \frac{1}{2e}\right)$

MIN: $S_2\left(1, 2\sqrt{e}\right)$

→ graf:



13. INTEGRALNI RAČUN

13.1. PRIMITIVNA FUNKCIJA I NEODREĐENI INTEGRAL

= Pitanje: Neka je zadana funkcija f sa $f:\langle a,b\rangle \to \mathbb{R}$. Postoji li funkcija ($F:\langle a,b\rangle \to \mathbb{R}$) za koju vrijedi $F'(x)=f(x)$, $\forall x \in \langle a,b\rangle$?

- DEFINICIJA 1.

↳ Kažemo da je $F(x)$ primitivna funkcija od $f(x)$ na intervalu $I=\langle a,b\rangle$ ako je $F'(x)=f(x)$, $\forall x \in I$.

↳ To zapisujemo:

$$\int \underbrace{f(x)}_{\substack{\text{integrant} \\ \text{(podintegralna} \\ \text{funkcija)}}} dx = \underbrace{F(x)}_{\substack{\text{primitivna} \\ \text{funkcija}}}$$

= PRIMJER 1.

a) ↳ Ako je $f(x)=e^x$, onda je $F(x)=e^x+7$ ili $F(x)=e^x+\frac{1}{e}$

b) ↳ Ako je $f(x)=x^2$, onda je $F(x)=\frac{x^3}{3}+2$ ili $F(x)=\frac{x^3}{3}+\ln 2$

c) ↳ Ako je $f(x)=\sqrt{1-x^2}$, onda je $F(x)=-\frac{x}{\sqrt{1-x^2}}+1$

= STAVAK 1.

↳ Ako su $F_1(x)$ i $F_2(x)$ primitivne funkcije za isti integrant $f(x)$ na $I=\langle a,b\rangle$, onda je:

$$F_1(x)-F_2(x)=C \quad , \quad C\in\mathbb{R}.$$

DOKAZ:

- Označimo li $\Phi(x)=F_1(x)-F_2(x)$, treba dokazati da je $\Phi(x)=\text{konst.}$

Vrijedi:

$$\Phi'(x)=[F_1(x)-F_2(x)]'=F_1'(x)-F_2'(x)=f(x)-f(x)=0$$

$$\Downarrow$$

$$\Phi'(x)=0$$

$$\Downarrow$$

$$\Phi(x)=\text{konst.}=C \qquad \text{Q.E.D.}$$

= Dakle vrijedi: $F_1(x)=F_2(x)+C$, $\forall x\in I$

Ako je dakle $F(x)$ neka primitivna funkcija na I, onda je skup svih primitivnih funkcija na I:

$$\{F(x)+C \mid_{C\in\mathbb{R}}\}=\int f(x)\,dx$$

= DEFINICIJA 2.

↳ Skup svih primitivnih funkcija dane funkcije f na intervalu $I=\langle a,b\rangle$ nazivamo neodređenim integralom funkcije f na intervalu I i označavamo:

$$\int f(x)\,dx=\left\{F(x)+C \,\Big|_{C\in\mathbb{R}}\right\}=F(x)+C,$$

što kraće zapisujemo:

$$\int f(x)\,dx=F(x)+C \iff F'(x)=f(x)$$

Ova ekvivalencija vrijedi za $I=\langle a,b\rangle$ na kojem je $F(x)$ derivabilna.

= STAVAK 2.

↳ Temeljna svojstva neodređenih integrala:

1.) $\frac{d}{dx}\int f(x)\,dx=f(x)$ DOKAZ: $\frac{d}{dx}[F(x)+C]=F'(x)=f(x)$

2) $d\left(\int f(x)dx\right) = f(x)dx$

3) $\int d\, f(x) = f(x) + C$

ili

$\int f'(x)\,dx = f(x) + C$

4.) $\int c\cdot f(x)\,dx = c\cdot\int f(x)\,dx$, $c\in\mathbb{R}$

5.) $\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

DOKAZ:

$$\frac{d}{dx}\left[\int f(x)dx + \int g(x)dx\right] = \frac{d}{dx}\left[F(x)+C_1+G(x)+C_2\right] = F'(x)+G'(x) =$$
$$= f(x)+g(x) \;/\!/ \text{ QED.}$$

↳ Svojstva 4. i 5. iskazuju da je neodređeni integral tzv. linearni operator jer ima svojstva koje ima i linearna funkcija $L(x)=kx$:

a) aditivnost:
$$L(x+y) = k(x+y) = kx+ky = L(x)+L(y)$$

b) homogenost:
$$L(ax) = k(ax) = kax = a(kx) = a\cdot L(x)$$

= Na temelju stavka 2 i obrata tablice deriviranja, lako dobijemo tablicu neodređenih integrala:

integral= $\int f(x)dx$:	rezultat:
$\int 0\,dx$	C
$\int 1\,dx$	$x+C$
$\int \frac{dx}{x}$	$\ln\lvert x\rvert + C$
$\int x^{\alpha}dx$	$\frac{x^{\alpha+1}}{\alpha+1}+C$
$\int e^{x}dx$	$e^{x}+C$
$\int a^{x}dx$	$\frac{a^{x}}{\ln a}+C$
$\int \sin x\,dx$	$-\cos x + C$
$\int \cos x\,dx$	$\sin x + C$
$\int \frac{dx}{\sin^2 x}$	$-\operatorname{ctg} x + C$

integral= $\int f(x)dx$:	rezultat:
$\int \frac{dx}{\cos^2 x}$	$\operatorname{tg} x + C$
$\int \frac{dx}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right)+C$
$\int \frac{dx}{x^2+a^2}$	$\frac{1}{a}\operatorname{arctg}\left(\frac{x}{a}\right)+C$
$\int \operatorname{sh} x\,dx$	$\operatorname{ch} x + C$
$\int \operatorname{ch} x\,dx$	$\operatorname{sh} x + C$
$\int \frac{dx}{\operatorname{sh}^2 x}$	$-\operatorname{cth} x + C$
$\int \frac{dx}{\operatorname{ch}^2 x}$	$\operatorname{th} x + C$
$\int \frac{dx}{\sqrt{x^2+a}}$	$\ln\left\lvert x+\sqrt{x^2+a}\right\rvert + C$
$\int \frac{dx}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert + C$

DOKAZI FORMULA:

1) $\int \frac{dx}{x} = \ln\lvert x\rvert + C$

↳ $\frac{d}{dx}\left[\ln\lvert x\rvert + C\right] = \left(\ln\lvert x\rvert\right)' = \frac{1}{\lvert x\rvert}\cdot\lvert x\rvert' = \frac{1}{\cancel{\lvert x\rvert}}\cdot\frac{\cancel{\lvert x\rvert}}{x} = \frac{1}{x}$ // QED

2) $\int \frac{dx}{\sqrt{x^2+a}} = \ln\left\lvert x+\sqrt{x^2+a}\right\rvert + C$

↳ budući da je: $\left(\ln\lvert f(x)\rvert\right)' = \frac{1}{\cancel{\lvert f(x)\rvert}}\cdot\frac{\cancel{\lvert f(x)\rvert}}{f(x)}\cdot f'(x) = \frac{f'(x)}{f(x)}$

vrijedi: $$(\ln|x+\sqrt{x^2+a}|+c)' = \frac{1}{x+\sqrt{x^2+a}}\left(1+\frac{x}{\sqrt{x^2+a}}\right) = \frac{1}{\cancel{x+\sqrt{x^2+a}}}\cdot\frac{\cancel{x+\sqrt{x^2+a}}}{\sqrt{x^2+a}} = \frac{1}{\sqrt{x^2+a}} \quad // \text{ QED}$$

=PRIMJER 2.
↳ Izračunati sljedeće integrale:

a) $\int(ax^2+bx+c)\,dx = \int ax^2dx + \int bx\,dx + \int c\,dx = a\frac{x^3}{3} + b\frac{x^2}{2} + cx + C$

b) $\int\left(x\sqrt{x}+1+\frac{3}{x}-\frac{4}{x^2+1}\right)dx = \int\left(x^{\frac{3}{2}}+1+\frac{3}{x}-\frac{4}{x^2+1}\right)dx = \frac{2}{5}x^{5/2}+x+3\ln|x|-4\,\mathrm{arctg}\,x+C$

c) $\int\frac{dx}{\sqrt{x^2+5}} = \ln|x+\sqrt{x^2+5}|+C = \ln(x+\sqrt{x^2+5})+C$

d) $\int\frac{dx}{\sqrt{x^2-5}} = \ln|x+\sqrt{x^2-5}|+C$

e) $\int\frac{dx}{99-x^2} = -\int\frac{dx}{x^2-99} = -\frac{1}{2\sqrt{99}}\ln\left|\frac{x-\sqrt{99}}{x+\sqrt{99}}\right|+C$

=Ako neodređeni integral nije izraz preko elementarnih funkcija, kažemo da se radi o neelementarnom integralu, primjerice:
$$\int e^{-x^2}dx,\ \int\frac{\sin x}{x}dx,\ \int\frac{dx}{\sqrt{1-x^4}},\ \int\sin(x^2)\,dx\ \ldots$$

=STAVAK 3. Metoda supstitucije
↳ Neka je $f(x)$ integrabilna na $\langle a,b\rangle$, a $\varphi(u)$ neprekinuto diferencijabilna funkcija na $\langle\alpha,\beta\rangle$. Tada je: $\varphi:[\alpha,\beta]\to[a,b]$ pri čemu je φ injektivna funkcija. Uz supstituciju $x=\varphi(u)$ vrijedi:
$$\int f(x)\,dx = \left|\begin{array}{l}\text{SUPST:}\\ x=\varphi(u)\\ dx=\varphi'(u)du\end{array}\right| = \int f(\varphi(u))\cdot\varphi(u)\,du = \int g(u)\,du = G(u)+C =$$
$$= G[\varphi'(x)]+C.$$

=PRIMJER 3.
↳ Izračunati metodom supstitucije integrale:

a) $\int\sin(3x+2)\,dx = \left|\begin{array}{l}\text{SUPST:}\\ 3x+2=u\\ x=\frac{1}{3}(u-2)\,/'\\ dx=\frac{1}{3}du\end{array}\right| = \frac{1}{3}\int\sin u\,du = \frac{1}{3}(-\cos u)+C = -\frac{1}{3}\cos(3x+2)+C$

b) $\int(\sqrt{x}+1)^{30} = \left|\begin{array}{l}\text{SUPST:}\\ \sqrt{x}+1=u\\ \sqrt{x}=u-1\,/^2\\ x=(u-1)^2\,/'\\ dx=2(u-1)du\end{array}\right| = \int u^{30}\cdot 2(u-1)du = 2\cdot\int(u^{31}-u^{30})\,du = 2\frac{u^{32}}{32}-2\frac{u^{31}}{31}+C =$
$= \frac{u^{32}}{16}-\frac{2u^{31}}{31}+C$

c) $\int\frac{dx}{3x-1} = \left|\begin{array}{l}\text{SUPST:}\\ 3x-1=u\\ x=\frac{u+1}{3}\,/'\\ dx=\frac{1}{3}du\end{array}\right| = \int\frac{\frac{1}{3}du}{u} = \frac{1}{3}\ln|u|+C = \frac{1}{3}\ln|3x-1|+C$

d) $\int e^{x^2+x+1}\cancel{(2x+1)}dx = \left|\begin{array}{l}\text{SUPST:}\\ x^2+x+1=u\,/'\\ (2x+1)dx=du\\ dx=\frac{du}{\cancel{2x+1}}\end{array}\right| = \int e^u du = e^u+C = e^{x^2+x+1}+C$

=PRIMJEDBA 1.
↳ Formulu supstitucije iz stavka 3. primjenjujemo i u obratnom smjeru onda kada je integrand umnožak složene funkcije $f[\varphi(x)]$ i diferencijala $\varphi'(x)dx$.
U tom slučaju je:

$$\int f(\varphi(x))\,\varphi'(x)\,dx = \left| \begin{array}{l} \text{SUPST:} \\ u=\varphi(x) \;/' \\ du=\varphi'(x)dx \end{array} \right| = \int f(u)\,du = F(u)+C = F(\varphi(x))+C$$

↳ Ovaj se postupak često zove i svođenje pod znak diferencijala.

=PRIMJER 4.

↳ Riješi sljedeće integrale:

a) $$\int tg\,x\,dx = \int \frac{\sin x}{\cos x}dx = \left| \begin{array}{l} \text{SUPST:} \\ \cos x = u \;/' \\ \sin x\,dx = du \\ dx = \frac{du}{\sin x} \end{array} \right| = \int \frac{du}{u} = \ln|u| + C = \ln|\cos x| + C$$

b) $$\int \frac{\ln^5 x}{\cancel{x}}dx = \left| \begin{array}{l} \text{SUPST:} \\ \ln x = u \;/' \\ \frac{1}{x}dx = du \\ dx = \cancel{x}du \end{array} \right| = \int u^5 du = \frac{u^6}{6} + C = \frac{\ln^6 x}{6} + C$$

c) $$\int \cancel{x} e^{x^2} dx = \left| \begin{array}{l} \text{SUPST:} \\ e^{x^2} = u \;/' \\ e^{x^2}\cdot 2x\,dx = du \\ dx = \frac{du}{2\cancel{x}\cdot e^{x^2}} \end{array} \right| = \int \frac{du\cdot \cancel{y}}{2\cdot\cancel{y}} = \int \frac{du}{2} = \frac{1}{2}u + C = \frac{1}{2}e^{x^2} + C$$

d) $$\int \frac{\cancel{e^x}dx}{e^{2x}+3} = \left| \begin{array}{l} \text{SUPST:} \\ e^x = u \;/' \\ e^x dx = du \\ dx = \frac{du}{\cancel{e^x}} \end{array} \right| = \int \frac{du}{u^2+\sqrt{3}^2} = \frac{1}{\sqrt{3}}\,arctg\frac{u}{\sqrt{3}} + C = \frac{1}{\sqrt{3}}\,arctg\frac{e^x}{\sqrt{3}} + C$$

e) $$\int \frac{\sin^3 x\,dx}{\cos^5 x} = \int \frac{\sin^3 x}{\cos^3 x}\cdot\frac{dx}{\cos^2 x} = \int tg^3 x\cdot\frac{dx}{\cancel{\cos^2 x}} = \left| \begin{array}{l} \text{SUPST:} \\ tg\,x = u \;/' \\ \frac{1}{\cos^2 x}dx = du \\ dx = \cancel{\cos^2 x}\,du \end{array} \right| = \int u^3 du = \frac{u^4}{4} + C = \frac{tg^4 x}{4} + C$$

f) $$\int \frac{3x\,dx}{x^4+2} = \frac{3}{2}\int \frac{2x\,dx}{x^4+2} = \frac{3}{2}\int \frac{d(x^2)}{(x^2)^2+\sqrt{2}^2} = \frac{3}{2}\cdot\frac{1}{\sqrt{2}}\,arctg\left(\frac{x^2}{\sqrt{2}}\right) + C$$

=STAVAK 4. Parcijalna ili djelomična integracija

↳ Neka su f i g diferencijabilne funkcije na otvorenom intervalu I, onda vrijedi:

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

Označimo li: $u=f(x)$ i $v=g(x)$ imamo:

$du=f'(x)dx$, $dv=g'(x)dx$,

pa gornju formulu možemo zapisati kao:

$$\int u\,dv = u\cdot v - \int v\,du \;,$$

pri čemu je $v=\int dv$.

DOKAZ:

$$\frac{d}{dx}\left[f(x)g(x) - \int f'(x)\,g(x)\,dx\right] = \cancel{f'(x)g(x)} + f(x)g'(x) - \cancel{f'(x)g(x)} = f(x)g'(x) \;// \quad \text{QED}$$

=PRIMJER 5.

↳ Izračunaj metodom parcijalne integracije sljedeće integrale:

a) $$\int x e^x dx = \left| \begin{array}{ll} u = x & du = 1\,dx \\ dv = e^x dx & v = \int e^x dx = e^x \end{array} \right| = xe^x - \int e^x dx = xe^x - e^x + C = e^x(x-1)+C$$

↳ Da smo uzeli obrnuto: $u=e^x$, a $dv=x$, parcijalna integracija ne bi imala svrhu jer bi se integral (podintegralna funkcija) zakomplicirao, tj. bio bi kompliciraniji od zadanog.

*) TRIKOVI : → ako je u integralu jedna od funkcija lnx, arc...x, ar...x, onda je to uvijek u, a dv je ostatak
→ ako je u integralu x sa nekom od gore ne spomenutih funkcija, onda je on uvijek u, a dv je ostatak

b) $\int \ln x\,dx = \left| \begin{matrix} u=\ln x & du=\frac{1}{x}dx \\ dv=dx & v=\int dx = x \end{matrix} \right| = x\ln x - \int x\cdot\frac{1}{x}dx = x\ln x - x + C$

c) $\int \ln^2 x\,dx = \left| \begin{matrix} u=\ln^2 x & du=2\ln x\frac{1}{x}dx \\ dv=dx & v=\int dx = x \end{matrix} \right| = x\ln^2 x - \int 2x\ln x\frac{1}{x}dx =$

$= x\ln^2 x - \int 2x\ln x\frac{1}{x}dx = x\ln^2 x - \int 2\ln x\,dx = x\ln^2 x - 2\int \ln x\,dx =$

$= \left| \begin{matrix} u=\ln x & du=\frac{1}{x}dx \\ dv=dx & v=\int dx = x \end{matrix} \right| = x\ln^2 x - 2\left(x\ln x - \int x\frac{1}{x}dx\right) =$

$= x\ln^2 x - 2x\ln x + 2x + C = x(\ln^2 x - 2\ln x + 2) + C$

d) $\int \arcsin x\,dx = \left| \begin{matrix} u=\arcsin x & du=\frac{1}{\sqrt{1-x^2}}dx \\ dv=dx & v=\int dx = x \end{matrix} \right| = x\arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}} = \left| \begin{matrix} \text{SUPST:} \\ 1-x^2=t \\ -2x\,dx=dt \\ dx=\frac{-dt}{2x} \end{matrix} \right| =$

$= x\arcsin x - \int \frac{-dt}{2\sqrt{t}} = x\arcsin x + \frac{1}{2}\cdot 2\sqrt{t} + C =$

$= x\arcsin x + \sqrt{1-x^2} + C$

e) $\int \text{arsh}\,x\,dx = \left| \begin{matrix} u=\text{arsh}\,x & du=\frac{1}{\sqrt{x^2+1}}dx \\ dv=dx & v=\int dx = x \end{matrix} \right| = x\,\text{arsh}\,x - \int \frac{x}{\sqrt{x^2+1}}dx = \left| \begin{matrix} \text{SUPST:} \\ x^2+1=t \\ 2x\,dx=dt \\ dx=\frac{dt}{2x} \end{matrix} \right| =$

$= x\,\text{arsh}\,x - \int \frac{dt}{2\sqrt{t}} = x\,\text{arsh}\,x - \frac{1}{2}\cdot 2\sqrt{t} + C = x\,\text{arsh}\,x - \sqrt{x^2+1} + C$

f) $\int x^2\sin x\,dx = \left| \begin{matrix} u=x^2 & du=2x\,dx \\ dv=\sin x\,dx & v=\int \sin x\,dx = -\cos x \end{matrix} \right| = -x^2\cos x - \int -2x\cos x\,dx =$

$= \left| \begin{matrix} u=2x & du=2\,dx \\ dv=\cos x\,dx & v=\int \cos x\,dx = \sin x \end{matrix} \right| = -x^2\cos x + 2x\sin x - \int 2\sin x\,dx =$

$= -x^2\cos x + 2x\sin x + 2\cos x + C$

=PRIMJER 6.
↳ Izračunaj integral:

$I = \int e^{\alpha x}\cos(\beta x)dx = \left| \begin{matrix} u=e^{\alpha x} & du=\alpha e^{\alpha x}dx \\ dv=\cos(\beta x)dx & v=\int \cos(\beta x)dx = \frac{1}{\beta}\sin(\beta x) \end{matrix} \right| =$

$= e^{\alpha x}\frac{\sin(\beta x)}{\beta} - \int \frac{\alpha e^{\alpha x}}{\beta}\sin(\beta x)\,dx = e^{\alpha x}\frac{\sin(\beta x)}{\beta} - \frac{\alpha}{\beta}\int e^{\alpha x}\sin(\beta x)\,dx =$

Samo je jedan Mali Ivica!

$$= \left| \begin{matrix} u = e^{\alpha x} & du = \alpha e^{\alpha x}\,dx \\ dv = \sin(\beta x)dx & v = \int \sin(\beta x)dx = -\dfrac{\cos(\beta x)}{\beta} \end{matrix} \right| =$$

$$= e^{\alpha x}\,\frac{\sin(\beta x)}{\beta} + \frac{\alpha e^{\alpha x}}{\beta^2}\cdot\cos(\beta x) - \frac{\alpha^2}{\beta^2}\underbrace{\int e^{\alpha x}\cos(\beta x)\,dx}_{I}$$

$$\Downarrow$$

$$I = \frac{e^{\alpha x}\beta\sin(\beta x) + e^{\alpha x}\alpha\cos(\beta x)}{\beta^2} - \frac{\alpha^2}{\beta^2} I$$

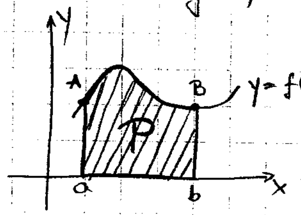
$$I\beta^2 = e^{\alpha x}\left(\beta\sin(\beta x) + \alpha\cos(\beta x)\right) - \alpha^2 I$$

$$I(\beta^2+\alpha^2) = e^{\alpha x}\left(\beta\sin(\beta x) + \alpha\cos(\beta x)\right)$$

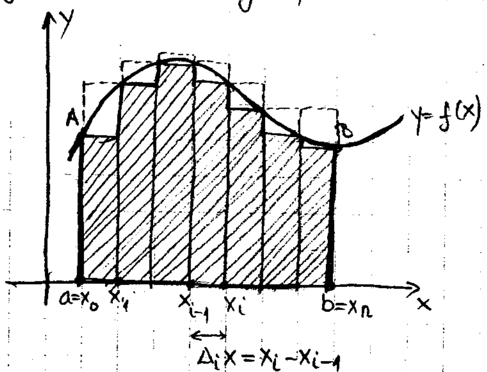
$$I = \frac{e^{\alpha x}\left[\beta\sin(\beta x) + \alpha\cos(\beta x)\right]}{\alpha^2+\beta^2} + C$$

13.2. ODREĐENI INTEGRAL

= Problem nalaženja ploštine lika koji omeđuju krivulje (lik = pseudotrapez)



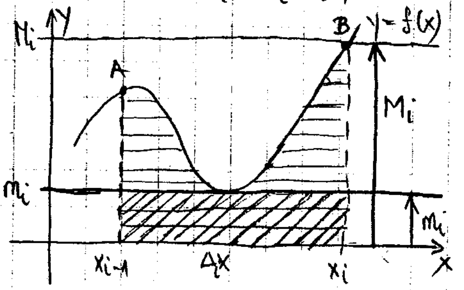
= Algoritam nalaženja ploštine P :



→ Ideja: Podijelimo interval $[a,b]$ na više dijelova, u svakome nađemo najmanju i najveću vrijednost funkcije i nacrtamo odgovarajuće pravokutnike.

↳ Zbroj pravokutnika s minimalnim visinama bit će manji od tražene površine, a zbroj pravokutnika s maksimalnim visinama veći od nje.

– Što manje intervale uzimamo zbroj će biti bliži stvarnoj površini.



= Neka je f omeđena funkcija na intervalu $[a,b]$, te Δ neka razdioba intervala I. Sada označimo sa $m_i = \min\limits_{I_i} f$, $M_i = \max\limits_{I_i} f$, minimum i maksimum funkcije f na I_i, za $i = 1, \dots, n$.

↳ Vrijedi : $s_\Delta = \sum\limits_{i=1}^{n} m_i \cdot \Delta_i x$ → donja integralna suma

$S_\Delta = \sum\limits_{i=1}^{n} M_i \cdot \Delta_i x$ → gornja integralna suma

$\sigma_\Delta = \sum\limits_{i=1}^{n} f(\xi_i)\cdot\Delta_i x$ → integralna suma

↳ Sada očevidno vrijedi:

$$m(b-a) \leq s_\Delta \leq \sigma_\Delta \leq S_\Delta \leq M(b-a)$$

=DEFINICIJA 1.

↳ Neka je funkcija f omeđena na intervalu $[a,b]=I$. Realni broj:

(1) $I_* = I_*(f,[a,b]) = \sup\{s_\Delta \mid \Delta\text{-razdioba na } I\}$
- zove se donji Riemannov integral

(2) $I^* = I^*(f,[a,b]) = \inf\{S_\Delta \mid \Delta\text{-razdioba na } I\}$
- zove se gornji Riemannov integral

↳ Očevidno je:

$$s_\Delta \leq I_* \leq I^* \leq S_\Delta,$$

za svaku razdiobu Δ.

↳ Kažemo da je f integrabilna na $I=[a,b]$ (u Riemannovom smislu), ako je $I_* = I^* = I$. U tom slučaju pišemo:

$$I = \int_a^b f(x)\,dx.$$

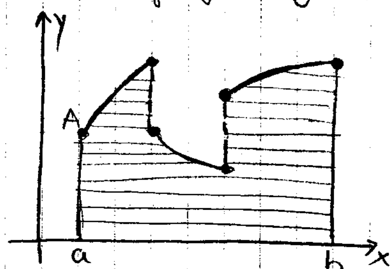
=PRIMJEDBA 1.

↳ Ova oznaka: $\int_a^b f(x)\,dx$ dolazi od izraza: $\sigma_\Delta = \sum_{i=1}^{n} f(\xi_i)\Delta_i x$, zamjenjujući grčka slova: Σ, ξ, Δ, sa latinskim: $\int, x, d$.

= STAVAK 1.

a) Ako je funkcija f neprekinuta funkcija na $I=[a,b]$, onda je funkcija f integrabilna na $[a,b]$.

b) Ako je funkcija f omeđena na $I=[a,b]$ i na njemu ima samo konačan broj točaka prekida, onda je funkcija f integrabilna na $[a,b]$, primjerice:



=DEFINICIJA 2.

↳ Vrijedi: a) $\int_a^a f(x)\,dx := 0$

b) $\int_a^b f(x) := -\int_b^a f(x)\,dx$

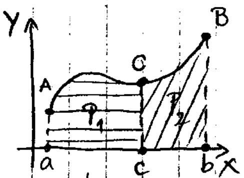
=STAVAK 2. Pravila integriranja

1) Ako je f integrabilna na $[a,b]$ onda je integrabilna na svakom intervalu: $[x_1, x_2] \subseteq [a,b]$.

2) Ako je $a<c<b$ i ako je f integrabilna na $[a,c]$ i $[a,b]$ onda je:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

↳ grafički:



3) Vrijedi aditivnost: $\int_a^b [f_1(x)+f_2(x)]\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx$

4) Vrijedi homogenost: $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx, \ \forall c \in \mathbb{R}$

13.3 STAVAK O SREDNJOJ VRIJEDNOSTI INTEGRALNOG RAČUNA. LEIBNIZ-NEWTONOVA FORMULA.

= STAVAK 1. Stavak o srednjoj vrijednosti integralnog računa

↳ Neka je funkcija f neprekinuta na intervalu $I=[a,b]$. Onda postoji $\xi\in\langle a,b\rangle$ takav da je:

$$\int_a^b f(x)\,dx=(b-a)\,f(\xi)$$

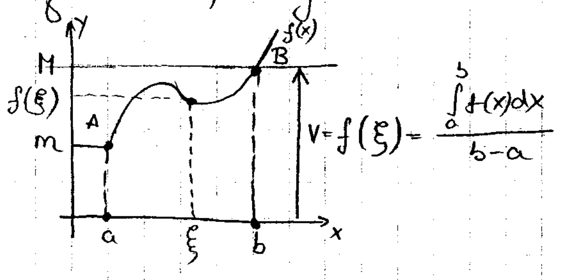
DOKAZ:

↳ $m(b-a)\le s_\Delta\le\int_a^b f(x)\,dx\le S_\Delta\le M(b-a)\ \Big/ :(b-a)$

$$m\le\frac{\int_a^b f(x)\,dx}{b-a}\le M$$

↳ Znamo da funkcija neprekinuta na zatvorenom intervalu prima na tom intervalu svaku vrijednost među m i M. Dakle, za neki $\xi\in\langle a,b\rangle$ vrijednost: $f(\xi)=\dfrac{\int_a^b f(x)\,dx}{b-a}$. Q.E.D.

↳ Geometrijska interpretacija:



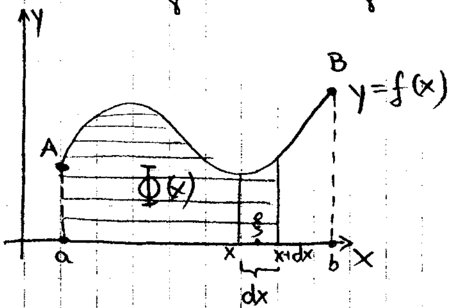
= STAVAK 2.

↳ Neka je funkcija f neprekinuta na $I=[a,b]$, te $x\in I$. Onda je:

$$\Phi(x)=\int_a^x f(x)\,dx$$

diferencijabilna funkcija na I i vrijedi: $\Phi'(x)=f(x)$.

↳ Skica:



DOKAZ:

→ $\Phi(x+\Delta x)-\Phi(x)=\int_a^{x+\Delta x} f(x)\,dx-\int_a^x f(x)\,dx=\cancel{\int_a^x f(x)\,dx}+\int_x^{x+\Delta x} f(x)\,dx-\cancel{\int_a^x f(x)\,dx}=$

$$=\int_a^{x+\Delta x} f(x)\,dx=\Delta x\,f(\xi)\quad,\quad \xi\in\langle x,x+\Delta x\rangle.$$

→ Sada je: $\Phi'(x)=\lim\limits_{\Delta x\to0}\dfrac{\Phi(x+\Delta x)-\Phi(x)}{\Delta x}=\lim\limits_{\Delta x\to0}\dfrac{\cancel{\Delta x}\cdot f(\xi)}{\cancel{\Delta x}}=\lim\limits_{\xi\to x} f(\xi)=f(x)$

Q.E.D.

=STAVAK 3.

↳ Opisuje vezu između određenog i neodređenog integrala:

$$\int_a^x f(x)dx = \int f(x)dx + C,$$

tj. neodređeni integral jednak je određenom kome je gornja granica upravo nezavisna varijabla.

=STAVAK 4. Leibniz-Newtonova formula

↳ Neka je funkcija f neprekinuta na $[a,b]$, te F bilo koja primitivna funkcija od $f(x)$ na $[a,b]$. Onda je:

$$\int_a^b f(x)\,dx = F(b) - F(a) \equiv F(x)\Big|_a^b \quad , \text{ za } F'(x) = f(x)$$

DOKAZ: → Po stavku 3 slijedi:

$$\int_a^x f(x)\,dx = \Phi(x) = F(x) + C,$$

jer su F i Φ primitivne funkcije iste funkcije f, pa se razlikuju za konstantu C.

→ Iz izraza:

$$0 = \int_a^a f(x)\,dx = \Phi(a) = F(a) + C,$$

slijedi da je $C = -F(a)$

→ Dakle je:

$$\Phi'(x) = F(x) - F(a), \text{ tj.}$$

$$\Phi(b) = \int_a^b f(x)\,dx = F(b) - F(a).$$

Q.E.D.

=PRIMJER 1.

↳ Izračunaj određeni integral:

$$\int_a^b x^2 dx = \frac{x^3}{3}\Big|_a^b = \frac{b^3}{3} - \frac{a^3}{3} = \frac{b^3 - a^3}{3}$$

13.4. PARCIJALNA INTEGRACIJA I SUPSTITUCIJA U ODREĐENOM INTEGRALU

=STAVAK 1. Parcijalna integracija

↳ Neka su f i g neprekinuto diferencijabilne funkcije na $[a,b]$. Onda je:

$$\int_a^b f(x)\cdot g'(x)\,dx = \big[f(x)\,g(x)\big]\Big|_a^b - \int_a^b f'(x)\,g(x)\,dx$$

- STAVAK 2. Supstitucija

↳ Neka je f neprekinuta na $I = [a,b]$, diferencijabilna, neprekinuta i injektivna na intervalu $I^* = [\alpha, \beta]$, te neka je $\varphi(I^*) \leq I$. Onda je, uz zamjenu: $x = \varphi(u)$:

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \left| \begin{array}{l} \text{SUPST:} \\ x = \varphi(u) \\ dx = \varphi'(u)\,du \\ G: \varphi(\beta) \Rightarrow \cancel{\varphi}(u_2) = \cancel{\varphi}(\beta) \Rightarrow u_2 = \beta \\ D: \varphi(\alpha) \Rightarrow \cancel{\varphi}(u_1) = \cancel{\varphi}(\alpha) \Rightarrow u_1 = \alpha \end{array} \right| = \int_\alpha^\beta f[\varphi(u)]\,\varphi'(u)\,du$$

-PRIMJER 1.

↳ Riješi integrale:

a)
$$\int_0^1 x\,e^x\,dx = xe^x\Big|_0^1 - \int_0^1 e^x \cdot 1\,dx = 1\cdot e^1 - 0\cdot e^0 - \int_0^1 e^x dx = e - e^x\Big|_0^1 = e - e^1 + e^0 = 1$$

b) $\int_0^1 \sqrt{1-x^2}\, dx = \left| \begin{array}{l} \text{SUPST:} \\ x=\cos t \\ dx=-\sin t\, dt \\ x_1=0 \to 0=\cos t \Rightarrow t_1=\frac{\pi}{2} \\ x_2=1 \to 1=\cos t \Rightarrow t_2=0 \end{array} \right| = \int_{\frac{\pi}{2}}^{0} \sqrt{1-\cos^2 t}\,(-\sin t)\, dt =$

$= -\int_{\frac{\pi}{2}}^{0} \sqrt{\sin^2 t}\, \sin t\, dt = \int_0^{\pi/2} \sin^2 t\, dt = \int_0^{\pi/2} \frac{1-\cos(2t)}{2} dt = \frac{1}{2}\left(t - \frac{\sin(2t)}{2}\right)\Big|_0^{\pi/2} =$

$= \frac{1}{2}\left(\frac{\pi}{2}-0\right) - \frac{1}{2}(0-0) = \frac{\pi}{4}$

=PRIMJER 2.

↳ Nađi rekurzivnu formulu za:

$I_n = \int_0^{\pi/2} \sin^n x\, dx \quad , n \geq 2.$

$- \ I_0 = \int_0^{\pi/2} \sin^0 x\, dx = \frac{\pi}{2}$

$I_1 = \int_0^{\pi/2} \sin x\, dx = -\cos x \Big|_0^{\pi/2} = -0+1=1$

$I_2 = \int_0^{\pi/2} \sin^{n-1} x\, \sin x\, dx = \sin^{n-1} x\,(-\cos x)\Big|_0^{\pi/2} - \int (-\cos x)(n-1)\sin^{n-2} x\, \cos x\, dx =$

$= (n-1) \int_0^{\pi/2} \sin^{n-2} x \cos^2 x\, dx = (n-1) \int_0^{\pi/2} \sin^{n-2} x \cdot (1-\sin^2 x)\, dx =$

$= (n-1) \int_0^{\pi/2} \sin^{n-2} x\, dx - (n-1) \int_0^{\pi/2} \sin^n x\, dx =$

$= (n-1) I_{n-2} - (n-1) I_n \ , \quad I_n = (1+n-1) = (n-1)\, I_{n-2} \,/ : n$

$\Downarrow$

$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n} I_{n-2}, \quad n \geq 2$

$I_0 = \frac{\pi}{2}, \ I_1 = 1.$

=PRIMJER 3.

$I = \int_0^{\pi/2} \sin^6 x\, dx = \frac{5}{6} I_4 = \frac{5}{6} \cdot \frac{3}{4} \cdot I_0 = \frac{5\pi}{32}$

=PRIMJEDBA 1.

↳ Ista formula vrijedi i za:

$\int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n} I_{n-2}\ , \ n \geq 2$

$I_0 = \frac{\pi}{2}, \ I_1 = 1.$

=PRIMJER 4.

$I = \int_0^{\pi/2} \cos^7 x\, dx = \frac{6}{7} I_5 = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot I_1 = \frac{16}{35}$

ZADACI

1.

a) $$\int \frac{2x+1}{x^4}\,dx = \int \frac{2x}{x^4}\,dx + \int \frac{1}{x^4}\,dx = \int 2x^{-3}\,dx + \int x^{-4}\,dx = 2\,\frac{x^{-2}}{-2} + \frac{x^{-3}}{-3} + C = -\frac{1}{x^2} - \frac{1}{3x^3} + C$$

b) $$\int \left(1-\frac{1}{x^2}\right)\sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx - \int \frac{x^{\frac{1}{2}}}{x^2}\,dx = \int x^{\frac{1}{2}}\,dx - \int x^{-\frac{3}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}} + 2x^{-\frac{1}{2}} + C = \frac{2}{3}\sqrt{x^3} + \frac{2}{\sqrt{x}} + C$$

2.

a) $$\int_0^1 x(2x+1)^5\,dx = \left|\begin{array}{l} \text{SUPST:} \\ 2x+1=t \Rightarrow x=\frac{t-1}{2} \\ 2dx=dt \\ dx=\frac{dt}{2} \\ G:\ t=3 \\ D:\ t=1 \end{array}\right| = \int_1^3 \frac{t-1}{2}\,t^5\,\frac{dt}{2} = \frac{1}{4}\int_1^3 (t^6-t^5)\,dt = \frac{1}{4}\left(\frac{t^7}{7}-\frac{t^6}{6}\right)\Bigg|_1^3 = \frac{1}{4}\left[\left(\frac{3^7}{7}-\frac{3^6}{6}\right)-\left(\frac{1}{7}-\frac{1}{6}\right)\right] = \frac{1}{4}\left[\frac{3^7-1}{7}+\frac{1-3^6}{6}\right]$$

b) $$\int_0^1 x\sqrt{3x+1}\,dx = \left|\begin{array}{l} \text{SUPST:} \\ 3x+1=t \Rightarrow x=\frac{t-1}{3} \\ 3dx=dt \\ dx=dt \\ G:\ t=4^3 \\ D:\ t=1 \end{array}\right| = \int_1^4 \frac{t-1}{3}\sqrt{t}\,\frac{dt}{2} = \frac{1}{6}\left(\int_1^4 t^{\frac{3}{2}}\,dt - \int_1^4 t^{\frac{1}{2}}\,dt\right) =$$

$$= \frac{1}{6}\left(\frac{2}{5}t^{\frac{5}{2}} - \frac{2}{3}t^{\frac{3}{2}}\right)\Bigg|_1^4 = \frac{1}{6}\left(\frac{2\sqrt{t^5}}{5} - \frac{2\sqrt{t^3}}{3}\right)\Bigg|_1^4 = \frac{1}{6}\left[\left(\frac{2\sqrt{4}^5}{5} - \frac{2\sqrt{4}^3}{3}\right) - \left(\frac{2}{5}-\frac{2}{3}\right)\right] = \frac{1}{6}\left(\frac{2\sqrt{4}^5-2}{5} + \frac{2-2\sqrt{4}^3}{3}\right)$$

3.

a) $$\int_0^{\sqrt{3}} x\sqrt{4-x^2}\,dx = \left|\begin{array}{l} \text{SUPST:} \\ 4-x^2=t \\ -2x\,dx=dt \\ dx=\frac{-dt}{2} \\ G:\ t=1 \\ D:\ t=4 \end{array}\right| = \int_4^1 \frac{-dt}{2}\sqrt{t} = -\frac{1}{2}\int_4^1 t^{\frac{1}{2}} = -\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\,t^{3/2}\Bigg|_4^1 = -\frac{1}{3}\sqrt{t}^3\Bigg|_4^1 = -\frac{1}{3}\left(1-\sqrt{4^3}\right) = \frac{7}{3}$$

b) $$\int_0^{\sqrt{3}} \frac{x\,dx}{\sqrt{x^2+1}} = \left|\begin{array}{l} \text{SUPST:} \\ x^2+1=t \\ 2x\,dx=dt \\ dx=\frac{dt}{2x} \\ G:\ t=4 \\ D:\ t=1 \end{array}\right| = \int_1^4 \frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_1^4 t^{-\frac{1}{2}}\,dt = \frac{1}{\cancel{2}}\cdot\cancel{2}\,t^{\frac{1}{2}}\Bigg|_1^4 = \sqrt{t}\Bigg|_1^4 = 2-1 = 1$$

4.

a) $$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx = \left|\begin{array}{l} \text{SUPST:} \\ e^x+1=t \Rightarrow e^x=t-1 \\ e^x\,dx=dt \\ dx=\frac{dt}{e^x} \end{array}\right| = \int \frac{t-1}{\sqrt{t}}\,dt = \int t^{\frac{1}{2}}\,dt - \int t^{-\frac{1}{2}}\,dt = \frac{2}{3}t^{\frac{3}{2}} - 2t^{\frac{1}{2}} = \frac{2\sqrt{t}^3}{3} - 2\sqrt{t} = \frac{2\sqrt{(e^x+1)^3}}{3} - 2\sqrt{e^x+1} + C$$

b) $$\int \frac{\ln^3 x}{x}\,dx = \left|\begin{array}{l} \text{SUPST:} \\ \ln x=t \\ \frac{1}{x}\,dx=dt \\ dx=x\,dt \end{array}\right| = \int t^3\,dt = \frac{t^4}{4} + C = \frac{\ln^4 x}{4} + C$$

5.

a) $$\int \mathrm{ctg}^2 x\,dx = \int \frac{\cos^2 x}{\sin^2 x}\,dx = \int \frac{1-\sin^2 x}{\sin^2 x}\,dx = \int \frac{1}{\sin^2 x}\,dx - \int dx = -\mathrm{ctg}\,x - x + C$$

b) $$\int \mathrm{th}(2x)\,dx = \int \frac{\mathrm{sh}\,2x}{\mathrm{ch}\,2x}\,dx = \left|\begin{array}{l} \text{SUPST:} \\ \mathrm{ch}\,2x=t \\ 2\,\mathrm{sh}\,2x\,dx=dt \\ dx=\frac{dt}{2\,\mathrm{sh}\,2x} \end{array}\right| = \int \frac{dt}{2t} = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln|\mathrm{ch}(2x)| + C$$

(6) $$\int_1^e (x^2-2x+3)\ln x\,dx = \left|\begin{array}{ll} u=\ln x & du=\frac{1}{x}\,dx \\ dv=(x^2-2x+3)\,dx & v=\int(x^2-2x+3)\,dx=\frac{x^3}{3}-x^2+3x \end{array}\right| =$$

$$=\left[\ln x\left(\frac{x^3}{3}-x^2+3x\right)\right]\Bigg|_1^e-\int_1^e\frac{1}{x}\left(\frac{x^3}{3}-x^2+3x\right)dx=\ln e\left(\frac{e^3}{3}-e^2+3e\right)-\cancel{\ln 1\left(\frac{1^3}{3}-1+3\right)}-$$

$$-\int_1^e\frac{x^2}{3}dx+\int_1^e x\,dx-\int_1^e 3\,dx=\frac{e^3}{3}-e^2+3e-\frac{1}{3}\frac{x^3}{3}\Bigg|_1^e+\frac{x^2}{2}\Bigg|_1^e-3x\Bigg|_1^e=$$

$$=\frac{e^3}{3}-e^2+3e-\frac{e^3}{9}+\frac{1}{9}+\frac{e^2}{2}-\frac{1}{2}-3e+3=\frac{2e^3}{9}-\frac{e^2}{2}-\frac{47}{18}$$

(7) $$\int_0^{\sqrt3}x^5\sqrt{x^2+1}\,dx=\left|\begin{array}{l}\text{SUPST:}\\ x^2+1=t \Rightarrow x=\sqrt{t-1}\\ 2x\,dx=dt\\ dx=\frac{dt}{2x}\\ D: t=1\\ G: t=4\end{array}\right|=\int_1^4\sqrt{t-1}^{\,4}\cdot\sqrt{t}\,\frac{dt}{2}=\frac{1}{2}\int_1^4(t-1)^2\sqrt{t}\,dt=$$

$$=\frac{1}{2}\int_1^4(t^2-2t+1)\sqrt{t}\,dt=\frac{1}{2}\left[\frac{2}{7}t^{\frac{7}{2}}-2\cdot\frac{2}{5}t^{\frac{5}{2}}+\frac{2}{3}t^{\frac{3}{2}}\right]\Bigg|_1^4=$$

$$=\frac{1}{2}\left(\frac{2}{7}\sqrt{t}^{\,7}-\frac{4}{5}\sqrt{t}^{\,5}+\frac{2}{3}\sqrt{t}^{\,3}\right)\Bigg|_1^4=$$

$$=\frac{2^7-1}{7}+\frac{2-2^6}{5}+\frac{2^3-1}{3}$$

(8) $$\int_0^{16}\frac{dx}{(1+\sqrt[4]{x})^2}=\left|\begin{array}{l}\text{SUPST:}\\ t=\sqrt[4]{x}\Rightarrow x=t^4\\ dx=4t^3dt\\ G: t=2\\ D: t=0\end{array}\right|=\int_0^2\frac{4t^3}{(1+t)^2}dt=\left|\begin{array}{l}\text{SUPST:}\\ 1+t=u\\ dt=du\\ G: t=3\\ D: t=1\end{array}\right|=\int_1^3\frac{4(u-1)^3}{u^2}du=\int_1^3\frac{4(u^3-3u^2+3u-1)}{u^2}du=$$

$$=4\int_1^3\left(u-3+\frac{3}{u}-\frac{1}{u^2}\right)du=4\left(\frac{u^2}{2}-3u+3\ln|u|+\frac{1}{u}\right)\Bigg|_1^3=4\left(\frac{9}{2}-\frac{1}{2}-9+3+3\ln3+\right.$$

$$\left.+\frac{1}{3}-1\right)=-\frac{32}{3}+12\ln3$$

(9) $$\int_0^1\frac{x^2dx}{x^6+1}=\int_0^1\frac{x^2dx}{(x^3)^2+1}=\left|\begin{array}{l}\text{SUPST:}\\ x^3=t\\ 3x^2dx=dt\\ dx=\frac{dt}{3x^2}\\ G: t=1\\ D: t=0\end{array}\right|=\int_0^1\frac{dt}{3(t^2+1)}=\frac{1}{3}\,\mathrm{arctg}\,t\Bigg|_0^1=\frac{1}{3}\cdot\frac{\pi}{4}=\frac{\pi}{12}$$

(10) $$\int_0^1\ln(2x+1)\,dx=\left|\begin{array}{l}\text{SUPST:}\\ 2x+1=t\\ 2dx=dt\\ dx=\frac{dt}{2}\\ G: 3=t\\ D: 1=t\end{array}\right|=\int_1^3\frac{\ln t\,dt}{2}=\frac{1}{2}\int_1^3\ln t\,dt=\left|\begin{array}{ll}u=\ln t & du=\frac{1}{t}dt\\ dv=dt & v=t\end{array}\right|=$$

$$=\frac{1}{2}\left[t\ln t-\int\cancel{t}\,\frac{1}{\cancel{t}}\,dt\right]\Bigg|_1^3=\frac{1}{2}\left[t\ln t-t\right]\Bigg|_1^3=$$

$$=\frac{1}{2}(3\ln3-3-0+1)=\frac{3\ln3-2}{2}=\frac{3}{2}\ln3-1$$

(11) $$\int\frac{\cos^3x\,dx}{\sqrt{\sin x}}=\int\frac{\cos x}{\sqrt{\sin x}}\cdot\cos^2x\,dx=\int\frac{\cancel{\cos x}\,(1-\sin^2x)}{\sqrt{\sin x}}dx=\left|\begin{array}{l}\text{SUPST:}\\ t=\sin x\\ dt=\cos x\,dx\\ dx=\frac{dt}{\cancel{\cos x}}\end{array}\right|=\int\frac{1-t^2}{\sqrt{t}}dt=$$

$$=\int t^{-\frac{1}{2}}dt-\int t^{\frac{3}{2}}dt=2t^{\frac{1}{2}}-\frac{2}{5}t^{\frac{5}{2}}+C=2\sqrt{\sin x}-\frac{2}{5}\sqrt{\sin^5x}+C$$

12. $$\int x^2\sin(3x)\,dx = \left|\begin{array}{ll} u=x^2 & du=2x\,dx \\ dv=\sin(3x)\,dx & v=\int\sin(3x)\,dx=\frac{-\cos(3x)}{3}\end{array}\right| = -\frac{x^2\cos(3x)}{3}+$$

$$+\int\frac{2x\cos(3x)}{3} = \left|\begin{array}{ll} u=x & du=dx \\ dv=\cos(3x)\,dx & v=\int\cos(3x)\,dx=\frac{\sin(3x)}{3}\end{array}\right| = \frac{-x^2\cos(3x)}{3}+\frac{2}{3}\left[\frac{x\sin(3x)}{3}-\right.$$

$$\left.-\int\frac{\sin(3x)}{3}\,dx\right] = \frac{-x^2\cos(3x)}{3}+\frac{2x\sin(3x)}{9}+\frac{2\cos 3x}{27}+C$$

13. $$\int x\sin^2x\,dx = \left|\begin{array}{ll} u=\sin^2x & du=2\sin x\cdot\cos x=\sin(2x)\,dx \\ dv=x\,dx & v=\int x\,dx=\frac{x^2}{2}\end{array}\right| = \frac{x^2\sin^2x}{2}-\int\frac{x^2\sin(2x)}{2}\,dx=$$

$$=\left|\begin{array}{ll} u=x^2 & du=2x\,dx \\ dv=\sin(2x)\,dx & v=\int\sin 2x\,dx=\frac{-\cos(2x)}{2}\end{array}\right| = \frac{x^2\sin^2x}{2}-\frac{1}{2}\left[\frac{-\cos(2x)\,x^2}{2}-\int x\cos(2x)\,dx\right]=$$

$$=\left|\begin{array}{ll} u=x & du=dx \\ dv=\cos(2x)\,dx & v=\int\cos(2x)\,dx=\frac{\sin(2x)}{2}\end{array}\right| = \frac{x^2\sin^2x}{2}+\frac{1}{2}\left[\frac{\cos(2x)\,x^2}{2}-\left(\frac{x\sin(2x)}{2}-\right.\right.$$

$$\left.\left.-\int\frac{\sin(2x)}{2}\,dx\right)\right] = \frac{x^2\sin^2x}{2}+\frac{x^2\cos(2x)}{4}-\frac{x\sin(2x)}{4}-\frac{\cos(2x)}{8}+C$$

14. $$\int\frac{x\,dx}{\sin^2x} = \left|\begin{array}{ll} u=x & du=dx \\ dv=\frac{1}{\sin^2x}\,dx & v=\int\frac{1}{\sin^2x}\,dx=-\mathrm{ctg}\,x\end{array}\right| = -x\,\mathrm{ctg}\,x+\int\mathrm{ctg}\,x\,dx=$$

$$=-x\,\mathrm{ctg}\,x+\int\frac{\cancel{\cos x}}{\sin x}\,dx = \left|\begin{array}{l} \text{SUPST} \\ \sin x=t \\ \cos x\,dx=dt \\ dx=\frac{dt}{\cancel{\cos x}}\end{array}\right| = -x\,\mathrm{ctg}\,x+\int\frac{dt}{t} = -x\,\mathrm{ctg}\,x+$$

$$+\ln|\sin x|+C$$

15. $$\int x\,\mathrm{tg}^2x\,dx = \int x\frac{\sin^2x}{\cos^2x}\,dx = \int\frac{x(1-\cos^2x)}{\cos^2x}\,dx = \int\frac{x}{\cos^2x}\,dx-\int x\,dx=$$

$$=\left|\begin{array}{ll} (1): & \\ u=x & du=dx \\ dv=\frac{1}{\cos^2x}\,dx & v=\int\frac{1}{\cos^2x}\,dx=\mathrm{tg}\,x\end{array}\right| = x\,\mathrm{tg}\,x-\int\mathrm{tg}\,x\,dx-\frac{x^2}{2} = x\,\mathrm{tg}\,x-\frac{x^2}{2}-\int\frac{\cancel{\sin x}}{\cos x}\,dx=$$

$$=\left|\begin{array}{l} \text{SUPST:} \\ \cos x=t \\ -\sin x\,dx=dt \\ dx=\frac{-dt}{\cancel{\sin x}}\end{array}\right| = x\,\mathrm{tg}\,x-\frac{x^2}{2}+\int\frac{dt}{t} = x\,\mathrm{tg}\,x-\frac{x^2}{2}+\ln|\cos x|+C$$

16. $$\int_0^1 e^{-\sqrt{x}}\,dx = \left|\begin{array}{l} \text{SUPST:} \\ -\sqrt{x}=t\ /^2 \\ x=t^2 \\ dx=2t\,dt \\ G: t=-1 \\ D: t=0\end{array}\right| = 2\int_0^{-1}te^t\,dt = \left|\begin{array}{ll} u=t & du=dt \\ dv=e^t\,dt & v=\int e^t\,dt=e^t\end{array}\right| = 2\left(te^t\Big|_0^{-1}-\int_0^{-1}e^t\,dt\right)=$$

Samo je jedan Mali Ivica!

$$=2\left(te^t\Big|_0^{-1}-e^t\Big|_0^{-1}\right)=2\left[\left(-1e^{-1}-0\right)-\left(e^{-1}-e^0\right)\right]=\left[-\frac{1}{e}-\frac{1}{e}+1\right]\cdot 2=-\frac{4}{e}+2$$

17.

$$\int \ln(\sin x)\cos x\,dx=\left|\begin{matrix} u=\ln(\sin x) & du=\frac{1}{\sin x}\cdot\cos x\,dx \\ dv=\cos x\,dx & v=\int\cos x\,dx=\sin x\end{matrix}\right|=$$

$$=\ln(\sin x)\cdot\sin x-\int\frac{\cancel{\sin x}\cos x}{\cancel{\sin x}}\,dx=\sin x\ln(\sin x)-\sin x+C$$

18.

$$\int \sin x\,\mathrm{ch}x\,dx=I$$

$$I=\int\sin x\,\mathrm{ch}x\,dx=\left|\begin{matrix} u=\sin x & du=\cos x\,dx \\ dv=\mathrm{ch}x\,dx & v=\int\mathrm{ch}x\,dx=\mathrm{sh}x\end{matrix}\right|=\sin x\,\mathrm{sh}x-\int\cos x\,\mathrm{sh}x\,dx=$$

$$=\left|\begin{matrix} u=\cos x & du=-\sin x\,dx \\ dv=\mathrm{sh}x\,dx & v=\int\mathrm{sh}x\,dx=\mathrm{ch}x\end{matrix}\right|=\sin x\,\mathrm{sh}x-\cos x\,\mathrm{ch}x-\underbrace{\int\sin x\,\mathrm{ch}x\,dx}_{I}$$

$$\Downarrow$$

$$I=\sin x\,\mathrm{sh}x-\cos x\,\mathrm{ch}x-I$$
$$2I=\sin x\,\mathrm{sh}x-\cos x\,\mathrm{ch}x$$

$$I=\frac{\sin x\,\mathrm{sh}x-\cos x\,\mathrm{ch}x}{2}+C$$

19.

$$I=\int\sin(\ln x)dx=\left|\begin{matrix} u=\sin(\ln x) & du=\cos(\ln x)\cdot\frac{1}{x}\,dx \\ dv=dx & v=\int dx=x\end{matrix}\right|=x\sin(\ln x)-\int\frac{\cancel{x}\cos(\ln x)}{\cancel{x}}\,dx=$$

$$=x\sin(\ln x)-\int\cos(\ln x)dx=\left|\begin{matrix} u=\cos(\ln x) & du=-\sin(\ln x)\cdot\frac{1}{x}\,dx \\ dv=dx & v=\int dx=x\end{matrix}\right|=$$

$$=x\sin(\ln x)-x\cos(\ln x)-\int\frac{x\sin(\ln x)}{x}\,dx=x\sin(\ln x)-x\cos(\ln x)-\underbrace{\int\sin(\ln x)dx}_{I}$$

$$\Downarrow$$

$$I=x\sin(\ln x)-x\cos(\ln x)-I$$
$$2I=x\sin(\ln x)-x\cos(\ln x)$$

$$I=\frac{x}{2}\left(\sin(\ln x)-\cos(\ln x)\right)+C$$

20. $I=\int_{\frac{1}{e}}^{e} |\ln x| dx = \begin{vmatrix} u=|\ln x| & du=\frac{|\ln x|}{\ln x}\cdot\frac{1}{x}dx \\ dv=dx & v=\int dx = x \end{vmatrix} = x|\ln x| \Big|_{\frac{1}{e}}^{e} - \int_{\frac{1}{e}}^{e} \frac{|\ln x|}{\ln x} dx$

1. slučaj: $x>1 \Rightarrow$

$$I = x\ln x \Big|_{\frac{1}{e}}^{e} - \int_{\frac{1}{e}}^{e} \frac{\ln x}{\ln x}dx = x\ln x \Big|_{\frac{1}{e}}^{e} - x\Big|_{\frac{1}{e}}^{e} =$$

$$= \cancel{e\ln e} - \frac{1}{e}\ln e^{-1} - \cancel{e} + \frac{1}{e} = \frac{1}{e}+\frac{1}{e} = \frac{2}{e}$$

2. slučaj: $0<x<1 \Rightarrow$

$$I = -x\ln x \Big|_{\frac{1}{e}}^{e} - \int_{\frac{1}{e}}^{e} \frac{-\ln x}{\ln x}dx = -x\ln x\Big|_{\frac{1}{e}}^{e} + x\Big|_{\frac{1}{e}}^{e} =$$

$$= -\cancel{e\ln e} + \frac{1}{e}\ln e^{-1} + \cancel{e} - \frac{1}{e} = -\frac{1}{e} - \frac{1}{e} = -\frac{2}{e}$$

Samo je jedan Mali Ivica!

14. INTEGRIRANJE NAJVAŽNIJIH TIPOVA INTEGRALA

14.1. INTEGRALI RACIONALNIH FUNKCIJA

= Racionalna funkcija $f(x)$ je funkcija oblika: $f(x) = \frac{P_n(x)}{Q_m(x)}$.

Ako je: 1) $n<m$, onda je $f(x)$ prava racionalna funkcija;
2) $n \geq m$, onda je $f(x)$ neprava racionalna funkcija.

Neprava racionalna funkcija se diobom brojnika s nazivnikom svodi na cijeli dio i pravu razlomljenu cijelu funkciju:

$$\frac{P_n(x)}{Q_m(x)} = R_{n-m}(x) + \frac{\vartheta(x)}{Q_m(x)}$$

= **PRIMJER 1.**

↳ Podijeli: $\frac{x^4-2x^3+x^2-1}{x^2+3}$

$$(x^4-2x^3+x^2-1):(x^2+3) = x^2-2x-2 \quad \Rightarrow \quad \frac{x^4-2x^3+x^2-1}{x^2+3} = (x^2-2x-2) + \frac{6x+5}{x^2+3}$$

$$\begin{array}{l}
\underline{-\cancel{x^4} \quad \pm 3x^2} \\
-2\cancel{x^3}-2x^2-1 \\
\underline{+\cancel{2x^3}+6x} \\
-2\cancel{x^2}+6x-1 \\
\underline{+\cancel{2x^2} \quad +6} \\
6x+5
\end{array}$$

= S obzirom da je integriranje cijelog dijela racionalne funkcije elementarno, problem integriranja racionalne funkcije svodi se na integral pravog racionalnog razlomka.

= **STAVAK 1.**

↳ Svaki realni polinom $Q_m(x)$ ima rastav na jednoznačno određene linearne i kvadratne realne faktore u obliku:

$$Q_m(x) = a_m \prod_{i=1}^{r} (x-x_i)^{\alpha_i} \cdot \prod_{j=1}^{s} (x^2+p_j x+q_j)^{\beta_j},$$

uz:

$$\sum_{i=1}^{r} \alpha_i + 2\sum_{j=1}^{s} \beta_j = m, \quad p_j^2 - 4q_j < 0 \text{ za } j = 1,2,\dots,s.$$

= **PRIMJER 2.**

↳ Rastavi na faktore:

a) $x^6-1 = (x^3-1)(x^3+1) = (x-1)(x^2+x+1)(x+1)(x^2-x+1) =$
$= (x-1)(x+1)(x^2+x+1)(x^2-x+1)$

b) $x^9+x^6-x^3-1 = x^6(x^3+1)-(x^3+1) = (x^3+1)(x^6-1) = (x^3+1)(x^3+1)(x^3-1) = (x^3+1)^2(x^3-1) =$
$= [(x+1)(x^2-x+1)]^2 \cdot (x-1)(x^2+x+1) =$
$= (x+1)^2(x-1)^2(x^2-x+1)^2(x^2+x+1)$

= **STAVAK 2.**

↳ Neka je: $Q_m(x) = \prod_{i=1}^{r} (x-x_i)^{\alpha_i} \cdot \prod_{j=1}^{s} (x^2+p_j x+q_j)^{\beta_j}$. Onda za pravu racionalnu funkciju $f(x) = \frac{P_n(x)}{Q_m(x)}$ imamo ovaj jednoznačno određen rastav na parcijalne razlomke:

$$(*)\quad \frac{P_n(x)}{Q_m(x)}=\frac{P_n(x)}{\prod_{i=1}^{r}(x-x_i)^{\alpha_i}\cdot\prod_{j=1}^{s}(x^2+p_jx+q_j)^{\beta_j}}=$$

$$=\sum_{i=1}^{r}\left[\frac{A_1^{(i)}}{x-x_i}+\cdots+\frac{A_{\alpha_i}^{(i)}}{(x-x_i)^{\alpha_i}}\right]+\sum_{j=1}^{s}\left[\frac{B_1^{(j)}x+C_1^{(j)}}{x^2+p_jx+q_j}+\cdots+\frac{B_{\beta_j}^{(j)}x+C_{\beta_j}^{(j)}}{(x^2+p_jx+q_j)^{\beta_j}}\right].$$

↳U formuli (*) koeficijenti $A_1^{(1)},\dots,C_{\beta_j}^{(s)}$ jednoznačno su određeni, a dobivamo ih tako da obje strane gornjeg identiteta svedemo na cijeli oblik i zatim primijenimo metode neodređenih koeficijenata, tj. izjednačujemo koeficijente s istim stupnjem varijable x ili tako da u jednadžbu uvrštavamo za x prikladno odabrane brojeve.

= PRIMJER 3.

↳Rastavi na parcijalne razlomke:

a) $$\frac{4x^2+1}{(x-1)^3(x^2+x+1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{Dx+E}{(x^2+x+1)}+\frac{Fx+G}{(x^2+x+1)^2}$$

b) $$\frac{x^9+11}{x^3(x+1)(x-1)^2(x^2+x+4)^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{(x+1)}+\frac{E}{(x-1)}+\frac{F}{(x-1)^2}+\frac{Gx+H}{x^2+x+4}+\frac{Ix+J}{(x^2+x+4)^2}$$

= PRIMJER 4.

↳Rastavi na parcijalne razlomke i izračunaj koeficijente i integral.

a) $f(x)=\frac{1}{x^2-a^2}$

$$\frac{1}{x^2-a^2}=\frac{1}{(x-a)(x+a)}=\frac{A}{(x-a)}+\frac{B}{(x+a)}\quad /\cdot(x^2-a^2)$$

$$1=A(x+a)+B(x-a)$$

→ metoda neodređenih koeficijenata:

(I) 1. način:

$$1=A(x+a)+B(x-a)=x(A+B)+Aa-Ba$$

(1) $A+B=0$

(2) $Aa-Ba=1$

$a(A-B)=0$

$$A=\frac{1}{2a},\quad B=-\frac{1}{2a}$$

(II) 2. način: $x=a\to 1=A\cdot 2a+0\Rightarrow A=\frac{1}{2a}$

$x=-a\to 1=B(-2a)\Rightarrow B=-\frac{1}{2a}$

Dakle: $$\frac{1}{x^2-a^2}=\frac{\frac{1}{2a}}{x-a}-\frac{\frac{1}{2a}}{x+a}$$

$$\int\frac{1}{x^2-a^2}dx=\int\frac{1}{2a(x-a)}dx-\int\frac{1}{2a(x+a)}dx=\frac{1}{2a}\int\frac{1}{x-a}dx-\frac{1}{2a}\int\frac{1}{x+a}dx=$$

$$=\frac{1}{2a}\left[\ln|x-a|-\ln|x+a|\right]+C$$

b) $f(x)=\frac{1}{x^3-2x^2+x}$

$$\frac{1}{x^3-2x^2+x}=\frac{1}{x(x^2-2x+1)}=\frac{1}{x(x-1)^2}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}\quad /\cdot(x^3-2x^2+x)$$

$$1=A(x-1)^2+Bx(x-1)+Cx$$

→ metoda neodređenih koeficijenata:

(I) 1. način:

$$1=x^2(A+B)+x(-2A-B+C)+A$$

(1) $A+B=0$
(2) $-2A-B+C=0$
(3) $A=1$

$A=1,\ B=-1,\ C=1$

(II) 2. način: $x=0 \rightarrow 1=A(-1)^2+0+0 \Rightarrow A=1$
$x=1 \rightarrow 1=0+0+C\cdot 1 \Rightarrow C=1$
$x=2 \rightarrow 1=A+2B+2C \Rightarrow B=-1$

Dakle: $\dfrac{1}{x^3-2x^2+x}=\dfrac{1}{x}-\dfrac{1}{x-1}+\dfrac{1}{(x-1)^2}$

$$\int\frac{1}{x^3-2x^2+x}dx=\int\frac{1}{x}dx-\int\frac{1}{x-1}dx+\int\frac{1}{(x-1)^2}dx=\ln|x|-\ln|x-1|-\frac{1}{x-1}+C$$

= S obzirom na stavak 2. i činjenicu da je integral zbroja jednak zbroju integrala, integracija racionalne funkcije svodi se na integraciju parcijalnih razlomaka.
↳ Osnovni primjeri:

(1) $\displaystyle\int\frac{dx}{x-x_0}=\left|\begin{matrix}\text{SUPST:}\\ x-x_0=t\\ dx=dt\end{matrix}\right|=\int\frac{dt}{t}=\ln|t|+C=\ln|x-x_0|+C$

(2) $\displaystyle\int\frac{dx}{(x-x_0)^2}=\left|\begin{matrix}\text{SUPST:}\\ x-x_0=t\\ dx=dt\end{matrix}\right|=\int\frac{dt}{t^n}=\frac{t^{-n+1}}{-n+1}+C=-\frac{1}{(n-1)(x-x_0)^{n-1}}+C$

(3) $\displaystyle\int\frac{dx}{x^2+px+q}=\left|p^2-4q<0\right|=\int\frac{dx}{(x+\frac{p}{2})^2-\frac{p^2}{4}+q}=\left|\begin{matrix}\text{SUPST:}\\ x+\frac{p}{2}=t\\ dx=dt\end{matrix}\right|=$

$\displaystyle=\int\frac{dt}{t^2+a^2}=\frac{1}{a}\operatorname{arctg}\left(\frac{t}{a}\right)+C=\frac{1}{a}\operatorname{arctg}\left(\frac{x+\frac{p}{2}}{a}\right)+C$

(3)' $\displaystyle\int\frac{Ax+B}{x^2+px+q}dx=\int\frac{Ax+B}{(x+\frac{p}{2})^2-\frac{p^2}{4}+q}dx=\left|\begin{matrix}\text{SUPST:}\\ x+\frac{p}{2}=t\\ dx=dt\\ \text{OZNAKA: } q-\frac{p^2}{4}=a^2\end{matrix}\right|=\int\frac{A(t-\frac{p}{2})+B}{t^2+a^2}dt=$

$\displaystyle=\frac{A}{2}\int\frac{2t\,dt}{t^2+a^2}+\left(B-\frac{Ap}{2}\right)\int\frac{dt}{t^2+a^2}=\frac{A}{2}\ln|t^2+a^2|+\left(B-\frac{Ap}{2}\right)\cdot\frac{1}{a}\operatorname{arctg}\left(\frac{t}{a}\right)+C=$

$\displaystyle=\frac{A}{2}\ln(x^2+px+q)+\left(B-\frac{Ap}{2}\right)\frac{1}{a}\operatorname{arctg}\left(\frac{x+\frac{p}{2}}{a}\right)+C$

(4) $\displaystyle\int\frac{Ax+B}{(x^2+px+q)^n}dx=\left|\begin{matrix}n\geq 2;\\ p^2-4q<0\end{matrix}\right|=\dots=\frac{A}{2}\int\frac{2t\,dt}{(t^2+a^2)^n}+\left(B-\frac{Ap}{2}\right)I_n=$

$\displaystyle=-\frac{A}{2(n-1)(x^2+px+q)^{n-1}}+\left(B-\frac{Ap}{2}\right)I_n\,,$

gdje je: $I_n=\displaystyle\int\frac{dx}{(x^2+a^2)^n}$, $n\geq 2$

Za integral I_n izvodimo rekurzivnu formulu:

$$I_n=\int\frac{dx}{(x^2+a^2)^n}=\frac{1}{a^2}\int\frac{(x^2+a^2)-x^2}{(x^2+a^2)^n}dx=\frac{1}{a^2}\int\frac{dx}{(x^2+a^2)^{n-1}}-\frac{1}{2a^2}\int \underset{\substack{\downarrow\\ u}}{x}\cdot\underset{\substack{\downarrow\\ dv}}{\frac{2x\,dx}{(x^2+a^2)^n}}=$$

$$= \frac{1}{a^2} I_{n-1} - \frac{1}{2a^2}\left[x\,\frac{(x^2+a^2)^{-n+1}}{-n+1} - \int \frac{(x^2+a^2)^{-n+1}}{-n+1}\,dx\right] =$$

$$= \frac{1}{a^2} I_{n-1} + \frac{1}{2a^2(n-1)} \cdot \frac{x}{(x^2+a^2)^{n-1}} - \frac{1}{2a^2(n-1)} \cdot \int \frac{dx}{(x^2+a^2)^{n-1}}$$

Dakle:

$$I_n = \int \frac{dx}{(x^2+a^2)^n} = \frac{2n-3}{2a^2(n-1)} \cdot I_n + \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}}, \quad \text{za } n \geq 2, \text{ da vrijedi:}$$

$$I_1 = \frac{1}{a}\,\mathrm{arctg}\left(\frac{x}{a}\right)$$

Posebice:

$$n=2 \Rightarrow I_2 = \int \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^3}\,\mathrm{arctg}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)} + C$$

= STAVAK 3.

↳ Sve su racionalne funkcije elementarno integrabilne, što smo dokazali razmatranjem 5 osnovnih primjera.

= Algoritam integriranja racionalne funkcije:

1) uspoređujemo stupanj brojnika i nazivnika te u slučaju $n \geq m$ dijelimo brojnik s nazivnikom čime se integral svodi na integral prave racionalne funkcije $R_n(x)$
2) provodimo faktorizaciju nazivnika $Q_m(x)$ u obliku $(*)$
3) prikazujemo $R(x)$ u obliku parcijalnih razlomaka, te integriramo svaki od tih razlomaka posebno

= PRIMJER 5.

↳ Izračunati integral: $\displaystyle\int \frac{2x^3+4x^2+x+2}{(x-1)^2(x^2+x+1)}\,dx.$

$$\frac{2x^3+4x^2+x+2}{(x-1)^2(x^2+x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+x+1} \quad \Big/ \cdot (x-1)^2(x^2+x+1)$$

$$2x^3+4x^2+x+2 = A(x-1)(x^2+x+1) + B(x^2+x+1) + (Cx+D)(x-1)^2$$

$$2x^3+4x^2+x+2 = Ax^3 + \cancel{Ax^2} + \cancel{Ax} - \cancel{Ax^2} - \cancel{Ax} - A + Bx^2 + Bx + B + (Cx+D)(x^2-2x+1)$$

$$2x^3+4x^2+x+2 = Ax^3 - A + Bx^2 + Bx + B + Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D$$

$$2x^3+4x^2+x+2 = x^3(A+C) + x^2(B-2C+D) + x(B+C-2D) + (-A+B+D)$$

(1) $A+C=2 \Rightarrow C = 2-A = 0$
(2) $B-2C+D=4$
(3) $B+C-2D=1$
(4) $-A+B+D=2 \Rightarrow A = B+D-2 = 3+1-2 = 2$

(1) → (2) $B-4+2A+D=4$
(1) → (3) $B+2-A-2D=1$

(4) → (2) $B+D+2B+2D-4-4=4$
(4) → (3) $\cancel{B}+2D-\cancel{B}-D+2+2=1$

$3B+3D=12$
$-3D=-3$

$B+D=4 \Rightarrow B=3$
$D=1$

$$\int \frac{2x^3+4x^2+x+2}{(x-1)^2(x^2+x+1)}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{3\,dx}{(x-1)^2} + \int \frac{1}{x^2+x+1}\,dx = 2\ln|x-1| - \frac{3}{x-1} +$$

$$+ \frac{2}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

14.2. INTEGRIRANJE NEKIH IRACIONALNIH FUNKCIJA

□ INTEGRALI OBLIKA: $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx$

= Integral se rješava tako da se pretpostavi njegovo rješenje u obliku:

$$(*)\quad \int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx = Q_{n-1}(x)\sqrt{ax^2+bx+c} + \lambda \int \frac{dx}{\sqrt{ax^2+bx+c}},$$

gdje je $Q_{n-1}(x)$ polinom stupnja $(n-1)$ s neodređenim koeficijentima, a λ neodređena konstanta.
Svi nepoznati koeficijenti i konstanta λ određuju se metodom neodređenih koeficijenata nakon deriviranja identiteta $(*)$.

= PRIMJER 1.
↳ Izračunaj: $\int \sqrt{x^2+A}\,dx = I$

$$I = \int \sqrt{x^2+A} = \int \frac{x^2+A}{\sqrt{x^2+A}}\,dx$$

↳ pretpostavimo rješenje:

$$\int \frac{x^2+A}{\sqrt{x^2+A}}\,dx = (A_1x+B_1)\sqrt{x^2+A} + \lambda \int \frac{dx}{\sqrt{x^2+A}} \quad \Big/ \cdot \frac{d}{dx}$$

$$\frac{x^2+A}{\sqrt{x^2+A}} = A_1\sqrt{x^2+A} + (A_1x+B_1)\cdot\frac{1}{2\sqrt{x^2+A}}\cdot 2x + \lambda\cdot\frac{1}{\sqrt{x^2+A}} \quad \Big/ \cdot \sqrt{x^2+A}$$

$$x^2+A = A_1(x^2+A) + (A_1x+B_1)x + \lambda$$

$$x^2+A = 2A_1x^2 + B_1x + A_1A + \lambda$$

$$\Downarrow$$

(1) $1 = 2A_1$
(2) $0 = B_1$
(3) $A = A_1A + \lambda$

(1) $\Rightarrow A_1 = \frac{1}{2}$

(2) $\Rightarrow B_1 = 0$

(3) $\Rightarrow \lambda = \frac{1}{2}A$

$$I = \left(\frac{1}{2}x+0\right)\sqrt{x^2+A} + \frac{1}{2}A\int\frac{dx}{\sqrt{x^2+A}} = \frac{1}{2}x\sqrt{x^2+A} + \frac{A}{2}\ln\left|x+\sqrt{x^2+A}\right| + C$$

= PRIMJER 2.
↳ Izračunaj: $\int \sqrt{k^2-x^2}\,dx$

= Analogno se dokazuje kao u primjeru 1. da vrijedi:

$$\int \sqrt{k^2-x^2}\,dx = \frac{1}{2}x\sqrt{k^2-x^2} + \frac{k^2}{2}\arcsin\left(\frac{x}{k}\right) + C$$

= PRIMJER 3.
↳ Izračunaj: $I = \int \sqrt{2-x-x^2}\,dx$

$$\int\sqrt{2-x-x^2}\,dx = \int\sqrt{(-x^2+x)+2}\,dx = \int\sqrt{-\left(x+\frac{1}{2}\right)^2+\frac{1}{4}+2}\,dx = \int\sqrt{\left(\frac{3}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}\,dx =$$

$$= \left| \begin{array}{l} \text{SUPST:} \\ x+\frac{1}{2}=t \\ dx=dt \end{array} \right| = \int \sqrt{\left(\frac{3}{2}\right)^2 - t^2}\, dt = \frac{1}{2} t \sqrt{\left(\frac{3}{2}\right)^2 - t^2} + \frac{\frac{9}{4}}{2} \arcsin\left(\frac{t}{\frac{3}{2}}\right) + C =$$

$$= \frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{2-x-x^2} + \frac{9}{8} \arcsin\left(\frac{2x+1}{3}\right) + C$$

= PRIMJER 4.

Izračunati integral: $\int x^2 \sqrt{x^2+4}\, dx$.

$$I = \int x^2 \sqrt{x^2+4}\, dx = \int \frac{x^4+4x^2}{\sqrt{x^2+4}} dx$$

= Pretpostavimo rješenje:

$$\int \frac{x^4+4x^2}{\sqrt{x^2+4}} dx = (Ax^3+Bx^2+Cx+D)\sqrt{x^2+4} + \lambda \int \frac{dx}{\sqrt{x^2+4}} \quad \Big/ \frac{d}{dx}$$

$$\frac{x^4+4x^2}{\sqrt{x^2+4}} = (3Ax^2+2Bx+C)\sqrt{x^2+4} + (Ax^3+Bx^2+Cx+D)\frac{x}{\sqrt{x^2+4}} + \lambda \frac{1}{\sqrt{x^2+4}} \quad \Big/ \cdot \sqrt{x^2+4}$$

$$x^4+4x^2 = (3Ax^2+2Bx+C)(x^2+4) + (Ax^4+Bx^3+Cx^2+Dx) + \lambda$$

$$x^4+4x^2 = x^4(4A) + x^3(3B) + x^2(12A+2C) + x(8B+D) + (4C+\lambda)$$

$$\Downarrow$$

(1) $4A=1 \Rightarrow A=\frac{1}{4}$

(2) $3B=0 \Rightarrow B=0$

(3) $12A+2C=4 \Rightarrow C=\frac{1}{2}$

(4) $8B+D=0 \Rightarrow D=0$

(5) $4C+\lambda=0 \Rightarrow \lambda=-2$

$$\Downarrow$$

$$\int x^2\sqrt{x^2+4}\, dx = \left(\frac{1}{4}x^3+\frac{1}{2}x\right)\sqrt{x^2+4} - 2\int \frac{dx}{\sqrt{x^2+4}} = \frac{1}{4}(x^3+2x)\sqrt{x^2+4} - 2\ln\left(x+\sqrt{x^2+4}\right) + C$$

□ INTEGRALI OBLIKA: $\int R\left[x, \sqrt[q_1]{\left(\frac{ax+b}{cx+d}\right)^{p_1}}, \sqrt[q_2]{\left(\frac{ax+b}{cx+d}\right)^{p_2}}, \ldots\right] dx$

= Integrali gore navedenog oblika imaju racionalnu funkciju R i cijele brojeve $p_1, p_2, q_1, q_2, \ldots$ te realne brojeve a, b, c, d.
Ovi integrali svode se na integrale racionalne funkcije supstitucijom: $\frac{ax+b}{cx+d} = t^n$, gdje je n najmanji zajednički višekratnik brojeva $q_1, q_2, \ldots$, primjerice ako je zadan integral: $\int R(x, \sqrt{x}, \sqrt[3]{x}, \sqrt[4]{x})$ uvest ćemo supstituciju $x=t^{12}$.

= PRIMJER 5.

Izračunati integral: $\int \frac{dx}{\sqrt{2x-1} - \sqrt[4]{2x-1}}$.

$$I = \int \frac{dx}{\sqrt{2x-1} - \sqrt[4]{2x-1}} = \left| \begin{array}{l} \text{SUPST:} \\ 2x-1=t^4 \\ x=\frac{t^4+1}{2} \\ dx=2t^3dt \end{array} \right| = \int \frac{2t^3 dt}{t^2-t} = 2\int \frac{t^2 dt}{t-1} = 2\int \left(t+1+\frac{1}{t-1}\right) dt =$$

$$= (t+1)^2 + 2\ln|t-1| + C = \left(1+\sqrt[4]{2x-1}\right)^2 + 2\ln\left|\sqrt[4]{2x-1}-1\right| + C$$

14.3. INTEGRALI NEKIH TRIGONOMETRIJSKIH I HIPERBOLIČKIH FUNKCIJA

□ INTEGRIRANJE NEKIH TRIGONOMETRIJSKIH FUNKCIJA

= U općem slučaju radi se o integralu tipa:

$$\int R(\sin x, \cos x)\,dx$$

gdje je R racionalna funkcija.
Takav integral rješava se univerzalnom supstitucijom:

$$\operatorname{tg}\left(\frac{x}{2}\right)=t \Rightarrow x = 2\operatorname{arctg} t \qquad \sin x = \frac{2t}{1+t^2},$$

$$dx = \frac{2dt}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2}$$

DOKAZ:

$$\sin x = 2\sin\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right) = 2\,\frac{t}{\sqrt{1+t^2}}\cdot\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$$

$$\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2 = \frac{1-t^2}{1+t^2}$$

Uvrštavanjem upotrebljene supstitucije u gornji integral dobivamo:

$$I = \int R\left[\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right]\frac{2}{1+t^2}\,dt = \int R_1(t)\,dt$$

= U posebnom slučaju, ako vrijedi identitet:

$$R(-\sin x, -\cos x) = R(\sin x, \cos x)$$

prikladnija je supstitucija:

$$\operatorname{tg} x = t \rightarrow x = \operatorname{arctg} t \qquad \sin x = \frac{t}{\sqrt{1+t^2}}$$

$$dx = \frac{dt}{1+t^2} \qquad \cos x = \frac{1}{\sqrt{1+t^2}}$$

Uvrštavanjem upotrebljene supstitucije dobivamo:

$$I = \int R\left[\frac{1}{\sqrt{1+t^2}}, \frac{t}{\sqrt{1+t^2}}\right]\frac{dt}{1+t^2}$$

= PRIMJEDBA 1.

↳ Osim ove dvije navedene supstitucije postoje još neke u posebnim slučajevima, primjerice:

1) ako je $R(-\sin x, \cos x) = -R(\sin x, \cos x)$ prikladnija je supstitucija: $\cos x = t$

2) ako je $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ prikladnija je supstitucija $\sin x = t$

↳ U nekim slučajevima korisnije je koristiti formule:

$$\sin^2 x = \frac{1-\cos(2x)}{2}, \quad \cos^2 x = \frac{1+\cos(2x)}{2}, \quad \sin x \cos x = \frac{\sin(2x)}{2}$$

= PRIMJER 1.

↳ Izračunaj: $\int \frac{dx}{1+\sin x+\cos x}$

UNIV. SUPST.:

$\operatorname{tg}\frac{x}{2} = t \Rightarrow x = 2\operatorname{arctg} t$, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = 2\frac{dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$

$$\int \frac{dx}{1+\sin x+\cos x} = \int \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} = \int \frac{\frac{2dt}{\cancel{1+t^2}}}{\frac{1+\cancel{t^2}+2t+1-\cancel{t^2}}{\cancel{1+t^2}}}$$

$$= \int \frac{2dt}{2t+2} = \int \frac{dt}{t+1} = \ln|t+1| + C = \ln\left|tg\left(\frac{x}{2}\right)+1\right| + C$$

=PRIMJER 2.

↳ Izračunaj integral: $\int \frac{dx}{\sin^2 x - \sin x \cos x}$

$$\Rightarrow R(-\sin x, -\cos x) \Rightarrow \frac{1}{\sin^2 x - \sin x\cos x} = \frac{1}{(-\sin x)^2 - (-\sin x)(-\cos x)} = \frac{1}{\sin^2 x - \sin x \cos x}$$

$\Downarrow$

$R(\sin x, \cos x)$

$$\int \frac{dx}{\sin^2 x - \sin x\cos x} = \left| \begin{array}{ll} \text{SUPST:} & \\ tg\left(\frac{x}{2}\right) = t \Rightarrow x = 2arctg\, t & \cos x = \frac{1-t^2}{1+t^2} \\ dx = 2\frac{dt}{1+t^2} & \sin x = \frac{2t}{1+t^2} \end{array} \right| = \int \frac{\frac{dt}{1+t^2}}{\left(\frac{t}{\sqrt{1+t^2}}\right)^2 - \frac{t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}}} =$$

$$= \int \frac{\frac{dt}{1+t^2}}{\frac{t^2-t}{1+t^2}} = \int \frac{dt}{t(t-1)} = \int \frac{t-(t-1)}{t(t-1)}dt = \int \frac{dt}{t-1} - \int \frac{dt}{t} =$$

$$= \ln|t-1| - \ln|t| + C = \ln\left|\frac{t-1}{t}\right| + C = \ln\left|\frac{tg x - 1}{tg x}\right| + C$$

=PRIMJER 3.

↳ Izračunaj integral: $\int \frac{\sin^3 x}{\cos^4 x}dx$.

$$\int \frac{\sin^3 x}{\cos^4 x}dx = \int \frac{\sin^2 x \sin x}{\cos^4 x}dx = \int \frac{(1-\cos^2 x)dx}{\cos^4 x} = \left| \begin{array}{l} \text{SUPST:} \\ \cos x = t \\ dx(-\sin x) = dt \\ dx = \frac{-dt}{\sin x} \end{array} \right| = \int \frac{(1-t^2)dt}{t^4} =$$

$$= -\int \frac{dt}{t^4} + \int \frac{t^2}{t^4}dt = \frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C$$

=PRIMJER 4.

↳ Izračunaj integral: $\int \sin^2 x \cos^2 x \, dx$

$$\int \sin^2 x \cos^2 x\, dx = \int \frac{1}{4}\sin^2(2x)dx = \frac{1}{4}\int \frac{1-\cos(4x)}{2}dx = \frac{1}{8}\left(\int dx - \int \cos(4x)dx\right) =$$

$$= \frac{x}{8} - \frac{\sin(4x)}{32} + C$$

☐ INTEGRIRANJE NEKIH HIPERBOLIČKIH FUNKCIJA

= Ako se radi o integralima tipa: $\int R(shx, chx)dx$, integriranje je potpuno analogno integriranju trigonometrijskih funkcija uvođenjem univerzalne supstitucije:

$$th\left(\frac{x}{2}\right) = t \Rightarrow x = 2arth\, t \qquad shx = \frac{2t}{1-t^2}$$

$$dx = \frac{2dt}{1-t^2}$$

$$chx = \frac{1+t^2}{1-t^2}$$

DOKAZ:

$$shx = 2\,sh\left(\frac{x}{2}\right)ch\left(\frac{x}{2}\right) = 2ch^2\left(\frac{x}{2}\right)\frac{sh\left(\frac{x}{2}\right)}{ch\left(\frac{x}{2}\right)} = \frac{2\,th\left(\frac{x}{2}\right)}{1-th^2\left(\frac{x}{2}\right)} = \frac{2t}{1-t^2}$$

$$chx = ch^2\left(\frac{x}{2}\right) + sh^2\left(\frac{x}{2}\right) = ch^2\left(\frac{x}{2}\right)\left(1 + th^2\left(\frac{x}{2}\right)\right) = \frac{1+th^2\left(\frac{x}{2}\right)}{1-th^2\left(\frac{x}{2}\right)} = \frac{1+t^2}{1-t^2}$$

= Potpuno analogno se primjenjuju i druge supstitucije kao i kod trigonometrijskih funkcija.
= Dodatna mogućnost je još i supstitucija:

$$e^x = t \Rightarrow x = \ln t \qquad \operatorname{sh} x = \frac{t - \frac{1}{t}}{2}$$
$$dx = \frac{dt}{t} \qquad \operatorname{ch} x = \frac{t + \frac{1}{t}}{2}$$

= PRIMJER 5.
↳ Izračunaj : $\int \frac{dx}{1+2\operatorname{sh}x+3\operatorname{ch}x}$

a)

$$\int \frac{dx}{1+2\operatorname{sh}x+3\operatorname{ch}x} = \left| \begin{array}{l} \text{UNIV. SUPST:} \\ \operatorname{th}\left(\frac{x}{2}\right) = t \Rightarrow x = 2\operatorname{arth} t \qquad \operatorname{sh}x = \frac{2t}{1-t^2} \\ \qquad dx = \frac{2dt}{1-t^2} \qquad \operatorname{ch}x = \frac{1+t^2}{1-t^2} \end{array} \right| = \int \frac{2\,dt}{(1-t^2)\left(1 + \frac{4t}{1-t^2} + \frac{3+3t^2}{1-t^2}\right)} =$$

$$= \int \frac{2\,dt}{1-t^2+4t+3+3t^2} = \int \frac{2dt}{2t^2+4t+4} = \int \frac{dt}{t^2+2t+2} = \int \frac{dt}{(t+1)^2+1} = \left| \begin{array}{l} \text{SUPST:} \\ t+1 = z \\ dt = dz \end{array} \right| =$$

$$= \int \frac{dz}{z^2+1} = \operatorname{arctg} z + C = \operatorname{arctg}\left(\operatorname{th}\left(\frac{x}{2}\right)+1\right) + C //$$

b)

$$\int \frac{dx}{1+2\operatorname{sh}x+3\operatorname{ch}x} = \int \frac{dx}{1+2\,\frac{e^x-e^{-x}}{2} + 3\,\frac{e^x+e^{-x}}{2}} = \int \frac{dx}{\frac{2+2e^x-2e^{-x}+3e^x+3e^{-x}}{2}} =$$

$$= \int \frac{2\,dx}{2+5e^x+e^{-x}} = \left| \begin{array}{l} \text{SUPST:} \\ e^x = t \\ dx\,e^x = dt \\ dx = \frac{dt}{e^x} \end{array} \right| = \int \frac{2\,\frac{dt}{t}}{2+5t+t^{-1}} = \int \frac{2dt}{2t+5t^2+1} = \frac{2}{5}\int \frac{dt}{(t+\frac{1}{5})^2+\frac{4}{25}} =$$

$$= \frac{2}{5}\cdot\frac{5}{2}\,\operatorname{arctg}\left(\frac{t+\frac{1}{5}}{\frac{1}{5}}\right) + C = \operatorname{arctg}\left(\frac{5t+1}{2}\right) + C = \operatorname{arctg}\left(\frac{5e^x+1}{2}\right) + C //$$

□ INTEGRIRANJE NEKIH IRACIONALNIH FUNKCIJA POMOĆU TRIGONOMETRIJSKIH ILI HIPERBOLIČKIH SUPSTITUCIJA

1) integrali tipa: $\int R(x, \sqrt{k^2-x^2})\,dx, \; k>0$

rješavaju se supstitucijom: $x = k\sin t$.

2) integrali tipa: $\int R(x, \sqrt{x^2+k^2})\,dx, \; k>0$

rješavaju se supstitucijom: $x = k\operatorname{sh} t$.

3) integrali tipa: $\int R(x, \sqrt{x^2-k^2})\,dx, \; k>0$

rješavaju se supstitucijom: $x = k\operatorname{ch} t$.

= Općenito: integrali tipa: $\int R(x, \sqrt{ax^2+bx+c})\,dx$, se svode na jedan od prethodnih oblika transformacijom:

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

= PRIMJER 6.

↳ Izračunaj integral: $\int x^2\sqrt{x^2+4}\,dx$

$$I = \int x^2\sqrt{x^2+4}\,dx = \left|\begin{array}{l}\text{SUPST:}\\ x=2\,\mathrm{sh}t\\ dx = 2\,\mathrm{ch}t\,dt\end{array}\right| = \int 4\,\mathrm{sh}^2t\cdot 2\,\mathrm{ch}t\cdot 2\,\mathrm{ch}t\,dt = 16\int \mathrm{sh}^2t\,\mathrm{ch}^2t\,dt =$$

$$= 4\int \mathrm{sh}^2(2t)\,dt = 4\int \frac{\mathrm{ch}(4t)-1}{2}\,dt = \frac{1}{2}\,\mathrm{sh}(4t) - 2t + C = \mathrm{sh}(2t)\,\mathrm{ch}(2t) - 2t + C =$$

$$= 2\,\mathrm{sh}t\,\mathrm{ch}t\,(2\,\mathrm{sh}^2t+1) - 2t + C = \frac{1}{4}(x^3+2x)\sqrt{x^2+4} - 2\ln\left(x+\sqrt{x^2+4}\right) + C$$

ZADACI

(1.)

$$\int \frac{x\,dx}{x^2+x+1} = \int \frac{x\,dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = \left|\begin{array}{l}\text{SUPST:}\\ x+\frac{1}{2}=t\\ dx=dt\\ x=t-\frac{1}{2}\end{array}\right| = \int \frac{t-\frac{1}{2}}{t^2+\frac{3}{4}}\,dt = \int \frac{t\,dt}{t^2+\frac{3}{4}} - \frac{1}{2}\int \frac{dt}{t^2+\frac{3}{4}} =$$

$$\left|\begin{array}{l}\text{SUPST: (1)}\\ t^2+\frac{3}{4}=u\\ 2t\,dt=du\\ dt=\frac{du}{2t}\end{array}\right| = \int \frac{\frac{du}{2}}{u} - \frac{1}{2}\int \frac{dt}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2}\ln\left(t^2+\frac{3}{4}\right) - \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C =$$

$$= \frac{1}{2}\ln|x^2+x+1| - \frac{\sqrt{3}}{3}\,\mathrm{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

(2.) $\int \frac{x^3\,dx}{x^2+2x+4}$

$$(x^3):(x^2+2x+4) = x-2$$
$$\underline{-x^3 \pm 2x^2 \pm 4x}$$
$$-2x^2-4x$$
$$\underline{+2x^2+4x+8}$$
$$8$$

$$\int \frac{x^3\,dx}{x^2+2x+4} = \int \left(x-2+\frac{8}{x^2+2x+4}\right)dx = \int x\,dx - 2\int dx +$$

$$+8\int \frac{dx}{x^2+2x+4} = \frac{x^2}{2} - 2x + 8\int \frac{dx}{(x+1)^2+3} = \left|\begin{array}{l}\text{SUPST:}\\ x+1=t\\ dx=dt\end{array}\right| =$$

$$= \frac{x^2}{2} - 2x + 8\int \frac{dt}{t^2+(\sqrt{3})^2} = \frac{x^2}{2} - 2x + 8\cdot\frac{\sqrt{3}}{3}\,\mathrm{arctg}\left(\frac{x+1}{\sqrt{3}}\right) + C$$

(3.) $\int \frac{dx}{x^5-x^2}$

$$\frac{1}{x^5-x^2} = \frac{1}{x^2(x^3-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+x+1} \quad /\cdot x^2(x^3-1)$$

$$↳\ 1 = Ax(x^3-1) + B(x^3-1) + Cx^2(x^2+x+1) + (Dx+E)x^2(x-1)$$
$$1 = Ax^4 - Ax + Bx^3 - B + Cx^4 + Cx^3 + Cx^2 + Dx^4 - Dx^3 + Ex^3 - Ex^2$$
$$1 = x^4(A+C+D) + x^3(B+C-D+E) + x^2(C-E) + x(A) - B$$

(1) $A+C+D=0 \Rightarrow C=-D \Rightarrow D=-\frac{1}{3}$

(2) $B+C-D+E=0 \Rightarrow E=-\frac{1}{3}$

(3) $C-E=0 \Rightarrow C=E=\frac{1}{3}$

(4) $A=0 \Rightarrow A=0$

(5) $-B=1 \Rightarrow B=-1$

$$\int \frac{dx}{x^5-x^2} = \int -\frac{1}{x^2}\,dx - \frac{1}{3}\int \frac{dx}{x-1} + \int \frac{\frac{1}{3}x-\frac{1}{3}}{x^2+(x+1)}\,dx = \frac{1}{x} - \frac{1}{3}\ln|x-1| + \frac{1}{3}\int \frac{x\,dx}{(x+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2} -$$

$$- \int \frac{dx}{(x+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$= \left| \begin{array}{l} \text{SUPST:} \\ x+\frac{1}{2}=t \\ dx=dt \\ x=t-\frac{1}{2} \end{array} \right| = \frac{1}{x} - \frac{1}{3}\ln|x-1| + \frac{1}{3}\left(\int \frac{(t-\frac{1}{2})dt}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \int \frac{dt}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2} \right) =$$

$$= \frac{1}{x} - \frac{1}{3}\ln|x-1| + \frac{1}{3}\left(\int \frac{t\,dt}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{3}{2}\int \frac{dt}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2} \right) = \left| \begin{array}{l} \text{SUPST:} \\ t^2=u \\ 2t\,dt=du \\ dt=\frac{du}{2t} \end{array} \right| =$$

$$= \frac{1}{x} - \frac{1}{3}\ln|x-1| + \frac{1}{3}\left(\frac{1}{2}\int \frac{du}{u+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{3}{2}\int \frac{dt}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2} \right) =$$

$$= \frac{1}{x} - \frac{1}{3}\ln|x-1| + \frac{1}{6}\ln|x^2+x+1| - \frac{\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

4.

6. $$\int_0^1 \frac{dx}{(x^2+1)^2} = \int_0^1 \frac{x^2+1-x^2}{(x^2+1)^2}dx = \int_0^1 \frac{x^2+1}{(x^2+1)^2}dx - \int_0^1 \frac{x^2dx}{(x^2+1)^2} = \int_0^1 \frac{dx}{x^2+1} - \frac{1}{2}\int_0^1 x\cdot\frac{2x}{(x^2+1)^2}dx =$$

$$= \left| \begin{array}{ll} u = x & du = dx \\ dv = \frac{2x}{(x^2+1)^2}dx & v = \int \frac{2x}{(x^2+1)^2} = \left| \begin{array}{l} \text{SUPST.} \\ x^2+1=t \\ 2x\,dx = dt \\ dx = \frac{dt}{2x} \end{array} \right| = \int \frac{dt}{t^2} = -\frac{1}{x^2+1} \end{array} \right| =$$

$$= \int_0^1 \frac{dx}{(x^2+1)} - \frac{1}{2}\left[\frac{x}{x^2+1}\Big|_0^1 + \int_0^1 \frac{1}{x^2+1}dx \right] = \frac{3}{2}\int_0^1 \frac{dx}{x^2+1} - \frac{x}{x^2+1}\cdot\frac{1}{2}\Big|_0^1 =$$

$$= \left[\frac{3}{2}\operatorname{arctg} x - \frac{1}{2}\left(\frac{x}{x^2+1}\right)\right]\Big|_0^1 = \frac{3}{2}\cdot\frac{\pi}{2} - \frac{1}{4} - 0 + 0 = \frac{3\pi}{4} - \frac{1}{4}$$

7. $$\int_0^1 \frac{dx}{(x+1)(x^2+x+1)^2}$$

Samo je jedan Mali Ivica!