## 2. domaća zadaća

# Linearizacija nelinearnih dinamičkih sustava

## 1. Uvod

Većina dinamičkih sustava je nelinearna, a za nelinearne sustave općenito nisu razvijene metode analize i sinteze. Da bi se mogle primijeniti metode analize i sinteze linearnih sustava i na nelinearne sustave, potrebno je model nelinearnog dinamičkog sustava prikazati približnim linearnim matematičkim modelom. Postupkom linearizacije dobiva se valjani linearni matematički model u okolini radne točke. Cilj je ove domaće zadaće spoznati korisnost postupka linearizacije nelinearnih dinamičkih sustava.

## Priprema za vježbu

### Zadatak 1

Zadana je nelinearna diferencijalna jednadžba drugog reda za grupe A i B:

$$\begin{aligned} \text{grupa A}: &\quad \ddot{y}(t) + 5\dot{y}(t) + y(t) + y^3(t) = \ln u(t); \\ \text{grupa B}: &\quad \ddot{y}(t) + 5\dot{y}(t) + 3y(t) + \tfrac{3}{2}y^2(t) = e^{u(t)}; \end{aligned} \tag{1}$$

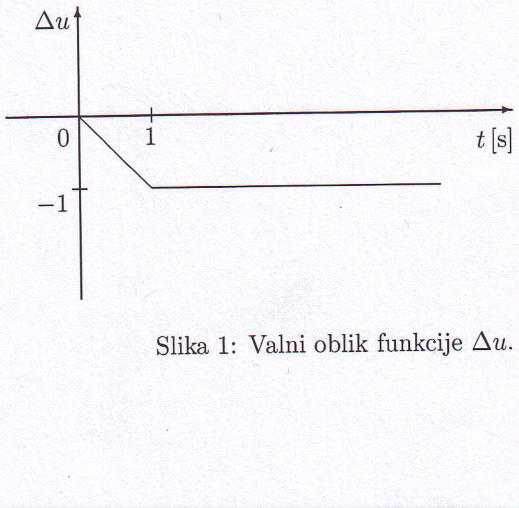
Potrebno je:

a) Linearizirati nelinearnu diferencijalnu jednadžbu u okolini radne točke određene s $y_0 = 1$;

b) Primijeniti Laplaceovu transformaciju na diferencijalnu jednadžbu dobivenu pod a) te odrediti prijenosnu funkciju $G(s) = \frac{Y(s)}{U(s)}$, pri čemu je $Y(s) = L(\Delta y(t))$ i $U(s) = L(\Delta u(t))$;

c) Odrediti odziv lineariziranog modela na pobudu $\Delta u(t)$ prikazanu slikom 1 te na temelju aproksimacije linearizacijom skicirati odziv nelinearnog modela na pobudu $u(t) = u_0 + \Delta u(t)$.

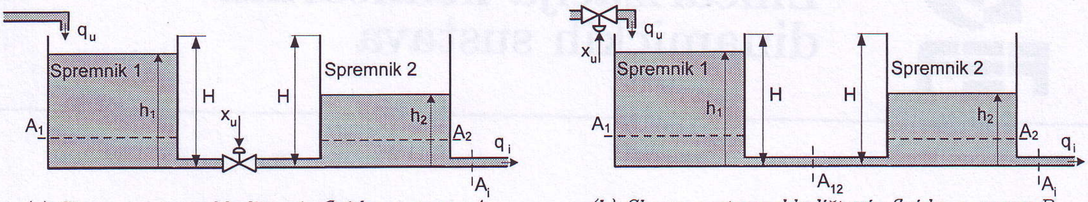
d) Odrediti stacionarnu vrijednost odziva lineariziranog modela te nagibe tog odziva u trenutcima $t = 0\,\text{s}$ i $t = 1\,\text{s}$.




Slika 1: Valni oblik funkcije $\Delta u$.

## ZADATAK 2

Za sustav skladištenja fluida modeliran nelinearnim modelom u domaćoj zadaći 1 (slika 2) potrebno je:




(a) Shema sustava skladištenja fluida za grupu A. (b) Shema sustava skladištenja fluida za grupu B.

Slika 2: Sustavi skladištenja fluida.

a) Linearizirati nelinearni matematički model u stacionarnoj radnoj točki određenoj otvorenošću ventila na polovici njegovog dozvoljenog radnog područja otvorenosti;

b) Odrediti matrice $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ i $\mathbf{D}$ iz zapisa lineariziranog sustava po varijablama stanja:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned} \tag{2}$$

pri čemu su vektori stanja, ulaza i izlaza definirani kao $\mathbf{x} = [\Delta h_1 \ \Delta h_2]^{\mathrm{T}}$, $\mathbf{u} = [\Delta x_u]$ i $\mathbf{y} = [\Delta h_1]$;

c) Odrediti prijenosnu funkciju $G(s) = \dfrac{H_1(s)}{X_u(s)}$, uz $H_1(s) = L(\Delta h_1(t))$ i $X_u(s) = L(\Delta x_u(t))$;

d) Odrediti odziv razine fluida u prvom spremniku lineariziranog modela na skokovitu promjenu otvorenosti ulaznog ventila $\Delta x_u = 5\%$ te ga skicirati;

e) Odrediti stacionarnu vrijednost odziva lineariziranog modela za pobudu pod d), kao i nagib odziva u trenutku $t = 0\,\mathrm{s}$. Odredite razliku u stacionarnom stanju između odziva dobivenog aproksimacijom sustava linearnim modelom u okolini radne točke i odziva stvarnog nelinearnog modela;

f) Nacrtati simulacijsku shemu lineariziranog modela sustava skladištenja fluida.

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· ZADATAK 1

$$\ddot{y}(t) + 5\dot{y}(t) + y(t) + y^3(t) = \ln(u(t))$$

(a) $y_0 = 1$

$$\ddot{y} = \dot{y} = 0 \rightarrow y_0 + y_0^3 = \ln(u_0) \rightarrow u_0 = e^2$$

$$y = y_0 + \Delta y, \quad \dot{y} = \Delta\dot{y}, \quad \ddot{y} = \Delta\ddot{y}$$

$$u = u_0 + \Delta u$$

$$\ddot{y}(t) = e^{u(t)} - y(t) - y^3(t) - 5\dot{y}(t) = f(u, y, \dot{y})$$

$$\ddot{y}(t) \approx \underset{=0}{f(u_0, y_0, 0)} + \frac{\partial f}{\partial u}\Big|_{S.T.}\Delta u + \frac{\partial f}{\partial y}\Big|_{S.T.}\Delta y + \frac{\partial f}{\partial \dot{y}}\Big|_{S.T.}\Delta\dot{y}$$

$$\Delta\ddot{y} = \frac{1}{u_0}\Delta u + (-1 - 3y_0^2)\Delta y - 5\Delta\dot{y}$$

$$\Delta\ddot{y} + 5\Delta\dot{y} + 4\Delta y = e^{-2}\Delta u$$

(b) $\mathcal{L}(\Delta y) = Y(s)$

$\mathcal{L}(\Delta u) = U(s)$

$$s^2 Y(s) + 5s\,Y(s) + 4Y(s) = e^{-2}U(s)$$

$$Y(s)(s^2 + 5s + 4) = e^{-2}U(s)$$

$$\frac{Y(s)}{U(s)} = G(s) = \frac{e^{-2}}{(s+1)(s+4)}$$

(c)

$$\Delta u(t) = \begin{cases} -t, & 0 < t < 1 \\ -1, & t > 1 \end{cases}$$

$$\Delta u(t) = -t(S(t) - S(t-1)) - 1\,S(t-1)$$

$$\Delta u(t) = -t\,S(t) + t\,S(t-1) - 1\,S(t-1)$$

$$\Delta u(t) = -t\,S(t) + (t-1)\,S(t-1)$$

$$\mathcal{L}(\Delta u(t)) = U(s) \rightarrow \Delta u(t) \circ\!\!-\!\!\bullet\, U(s) = -\frac{1}{s^2} + \frac{1}{s^2}e^{-s}$$

$$y(s) = U(s)\, G(s)$$

$$y(s) = \frac{1}{s^2}\left(e^{-s} - 1\right) \frac{e^{-2}}{(s+1)(s+4)}$$

$$G_1(s) = \frac{1}{s^2(s+1)(s+4)} = \frac{c_{11}}{s} + \frac{c_{12}}{s^2} + \frac{c_{21}}{s+1} + \frac{c_{31}}{s+4}$$

$$c_{11} = \frac{d}{ds}\left(G_1(s)\cdot s^2\right)\Big|_{s=0} = -\frac{5}{16}$$

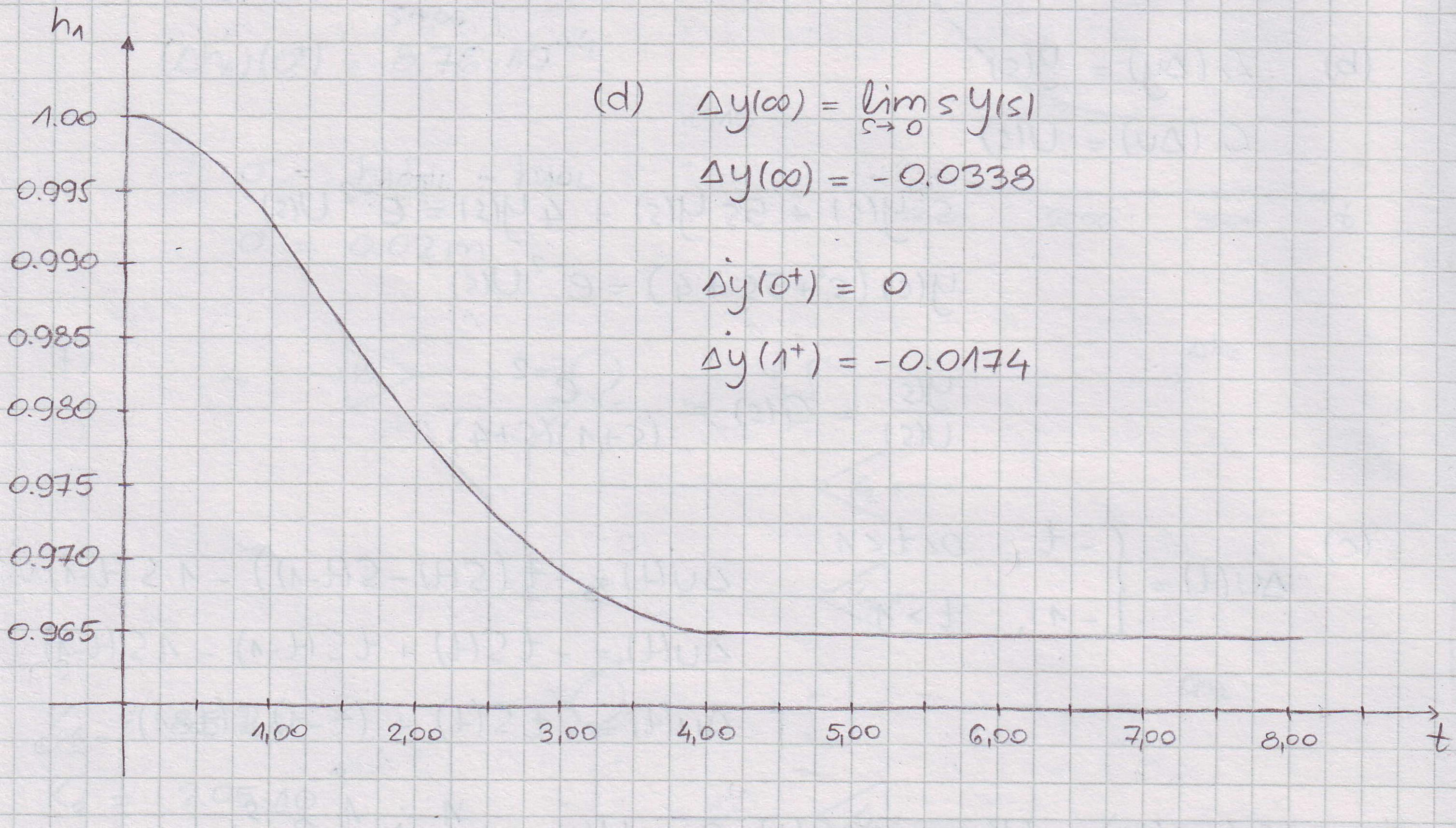
$$c_{12} = \left(G_1(s)\cdot s^2\right)\Big|_{s=0} = \frac{1}{4}$$

$$c_{21} = \left(G_1(s)\cdot (s+1)\right)\Big|_{s=-1} = \frac{1}{3}$$

$$c_{31} = \left(G_1(s)\cdot (s+4)\right)\Big|_{s=-4} = -\frac{1}{48}$$

$$y(s) \bullet\!\!-\!\!\circ \ \Delta y(t) = e^{-2}\left(\frac{5}{16} - \frac{1}{4}t - \frac{1}{3}e^{-t} + \frac{1}{48}e^{-4t}\right) S(t) -$$

$$- e^{-2}\left(\frac{5}{16} - \frac{1}{4}(t-1) - \frac{1}{3}e^{-(t-1)} + \frac{1}{48}e^{-4(t-1)}\right) S(t-1)$$




(d) $\Delta y(\infty) = \lim_{s\to 0} s\, y(s)$

$\Delta y(\infty) = -0.0338$

$\Delta \dot{y}(0^+) = 0$

$\Delta \dot{y}(1^+) = -0.0174$

• ZADATAK 2

$$\frac{dh_1}{dt} = \frac{q_u}{A_1 \rho} - \frac{A_v}{A_1} \sqrt{2g} \sqrt{h_1 - h_2}\, x_v$$

$$\frac{dh_2}{dt} = \frac{A_v}{A_2} \sqrt{2g} \sqrt{h_1 - h_2}\, x_v - \frac{A_i}{A_2} \sqrt{2g} \sqrt{h_2}$$

(a) $x_{v0} = 0.7 \qquad h_{10} = 1.922\,m \qquad h_{20} = 0.4587\ m$

(1) $$\frac{dh_1}{dt} = \frac{q_u}{A_1 \rho} - \frac{A_v}{A_1} \sqrt{2g} \sqrt{h_1 - h_2}\, x_v = f(h_1, h_2, x_v)$$

$$h_1 = h_{10} + \Delta h_1 \ , \quad \dot{h}_1 = \Delta\dot{h}_1$$

$$h_2 = h_{20} + \Delta h_2 \ , \quad \dot{h}_2 = \Delta\dot{h}_2$$

$$x_v = x_{v0} + \Delta x_v$$

$$\dot{h}_1 \approx \underbrace{f(h_{10}, h_{20}, x_{v0})}_{=0} + \frac{\partial f}{\partial h_1}\bigg|_{S.T.} \Delta h_1 + \frac{\partial f}{\partial h_2}\bigg|_{S.T.} \Delta h_2 + \frac{\partial f}{\partial x_v}\bigg|_{S.T.} \Delta x_v$$

$$\Delta\dot{h}_1 = -2.05 \cdot 10^{-3} \Delta h_1 + 2.05 \cdot 10^{-3} \Delta h_2 - 8.57 \cdot 10^{-3} \Delta x_v$$

(2) $$\frac{dh_2}{dt} = \frac{A_v}{A_2} \sqrt{2g} \sqrt{h_1 - h_2}\, x_v - \frac{A_i}{A_2} \sqrt{2g} \sqrt{h_2} = f(h_1, h_2, x_v)$$

$$\dot{h}_2 \approx f(h_{10}, h_{20}, x_{v0}) + \frac{\partial f}{\partial h_1}\bigg|_{S.T.} \Delta h_1 + \frac{\partial f}{\partial h_2}\bigg|_{S.T.} \Delta h_2 + \frac{\partial f}{\partial x_v}\bigg|_{S.T.} \Delta x_v$$

$$\Delta\dot{h}_2 = 2.05 \cdot 10^{-3} \Delta h_1 - 8.57 \cdot 10^{-3} \Delta h_2 + 8.57 \cdot 10^{-3} \Delta x_v$$

(b) $x = [\Delta h_1\ \Delta h_2] \ , \quad u = [\Delta x_v] \ , \quad y = [\Delta h_1]$

$$\begin{bmatrix} \Delta\dot{h}_1 \\ \Delta\dot{h}_2 \end{bmatrix} = \begin{bmatrix} -2.05 \cdot 10^{-3} & 2.05 \cdot 10^{-3} \\ 2.05 \cdot 10^{-3} & -8.57 \cdot 10^{-3} \end{bmatrix} \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix} + \begin{bmatrix} -8.57 \cdot 10^{-3} \\ 8.57 \cdot 10^{-3} \end{bmatrix} [\Delta x_v]$$

$$[\Delta h_1] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix} + [\,0\,]\,[\Delta x_v]$$

(c) $a = 2.05 \cdot 10^{-3}$ $b = 8.57 \cdot 10^{-3}$

$$sH_1(s) = -aH_1(s) + aH_2(s) - bX_v(s)$$
$$sH_2(s) = aH_1(s) - bH_2(s) + bX_v(s)$$

$$H_2(s) = \frac{s}{a-b-s} H_1(s)$$

$$\frac{H_1(s)}{X_v(s)} = G(s) = -\frac{857s + 5.6}{100000s^2 + 1062s + 1.34}$$

(d) $\Delta x_v(t) = 0.05\, S(t) \rightarrow X_v(s) = 0.05 \frac{1}{s}$

$$H_1(s) = X_v(s)\, G(s)$$

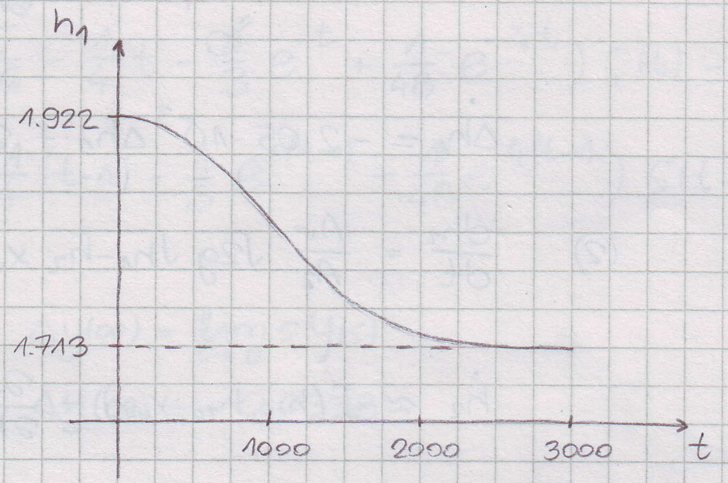
$$H_1(s) = \frac{-42.85s - 0.28}{100000s^3 + 1062s^2 + 1.34s}$$

(e) $\Delta h_1(\infty) = \lim_{s \to 0} sH_1(s)$

$\Delta h_1(\infty) = -0.2089$ m

$(\dot{\Delta h_1})(0^+) = \lim_{s \to \infty} s^2 H_1(s)$

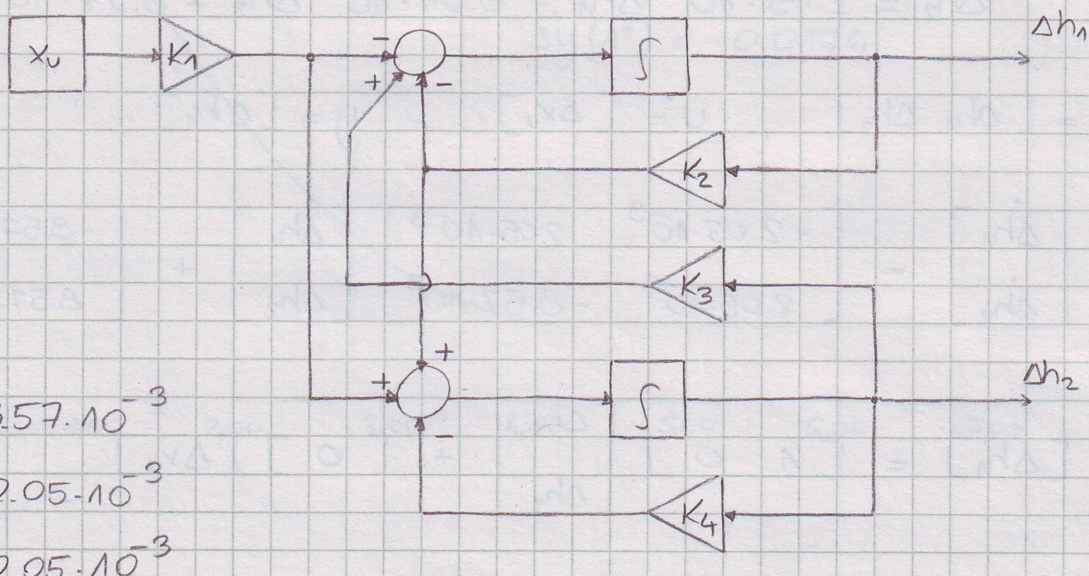
$(\dot{\Delta h_1})(0^+) = 8.76 \cdot 10^{-4}$




$\sigma = h_{10NL} - h_{10L}$

$\sigma = 0.02$ m

(f)




$K_1 = 8.57 \cdot 10^{-3}$

$K_2 = 2.05 \cdot 10^{-3}$

$K_3 = 2.05 \cdot 10^{-3}$

$K_4 = 8.57 \cdot 10^{-3}$

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• ZADATAK 1

$$\ddot{y}(t) + 5\dot{y}(t) + 3y(t) + \frac{3}{2}y^2(t) = e^{u(t)}$$

(a) $y_0 = 1$

$$\ddot{y} = \dot{y} = 0 \rightarrow 3y_0 + \frac{3}{2}y_0^2 = e^{u_0} \rightarrow u_0 = \ln\frac{9}{2}$$

$$y = y_0 + \Delta y \ , \ \dot{y} = \Delta\dot{y} \ , \ \ddot{y} = \Delta\ddot{y}$$
$$u = u_0 + \Delta u$$

$$\ddot{y}(t) = e^{u(t)} - 3y(t) - \frac{3}{2}y^2(t) - 5\dot{y}(t) = f(u, y, \dot{y})$$

$$\ddot{y}(t) \approx \underbrace{f(u_0, y_0, \dot{y})}_{=0} + \frac{\partial f}{\partial u}\Big|_{S.T.}\Delta u + \frac{\partial f}{\partial y}\Big|_{S.T.}\Delta y + \frac{\partial f}{\partial \dot{y}}\Big|_{S.T.}\Delta\dot{y}$$

$$\Delta\ddot{y} = e^{u_0}\Delta u + (-3 - 3y_0)\Delta y + (-5)\Delta\dot{y}$$

$$\Delta\ddot{y} + 5\Delta\dot{y} + 6\Delta y = \frac{9}{2}\Delta u$$

(b) $\mathcal{L}(\Delta y) = Y(s)$
$\mathcal{L}(\Delta u) = U(s)$

$$s^2Y(s) + 5sY(s) + 6Y(s) = \frac{9}{2}U(s)$$
$$Y(s)(s^2 + 5s + 6) = \frac{9}{2}U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{9}{2}\frac{1}{(s+2)(s+3)}$$

(c)
$$\Delta u(t) = \begin{cases} -t, & 0 < t < 1 \\ -1, & t > 1 \end{cases}$$

$$\Delta u(t) = -t\,(S(t) - S(t-1)) - S(t-1)$$
$$\Delta u(t) = -t\,S(t) + t\,S(t-1) - S(t-1)$$
$$\Delta u(t) = -t\,S(t) + (t-1)\,S(t-1)$$

$$\mathcal{L}(\Delta u(t)) = U(s) \rightarrow \Delta u(t) \circ\!\!-\!\!\bullet -\frac{1}{s^2} + \frac{1}{s^2}e^{-s}$$

$$Y(s) = U(s)\,G(s)$$

$$Y(s) = \frac{1}{s^2}\left(e^{-s}-1\right)\frac{9}{2}\,\frac{1}{(s+2)(s+3)}$$

$$Y(s) = \frac{9}{2}\,\frac{1}{s^2(s+2)(s+3)}\left(e^{-s}-1\right)$$

$$G_1(s) = \frac{9}{2}\,\frac{1}{s^2(s+2)(s+3)} = \frac{c_{11}}{s} + \frac{c_{12}}{s^2} + \frac{c_{21}}{s+2} + \frac{c_{31}}{s+3}$$

$$c_{11} = \frac{d}{ds}\left(G_1(s)\cdot s^2\right)\Big|_{s=0} = -\frac{5}{8}$$

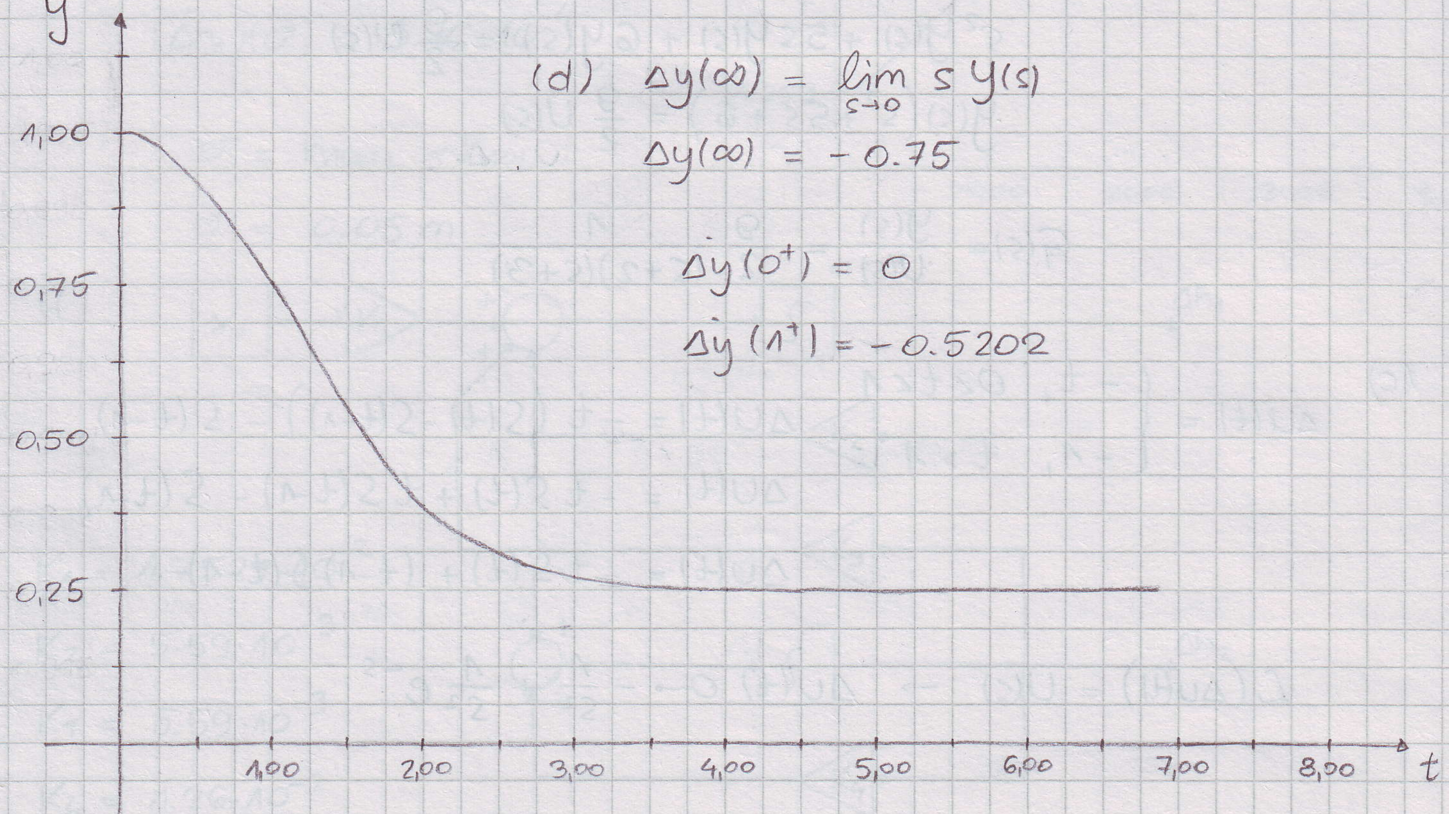
$$c_{12} = \left(G_1(s)\cdot s^2\right)\Big|_{s=0} = \frac{3}{4}$$

$$c_{21} = \left(G_1(s)\,(s+2)\right)\Big|_{s=-2} = \frac{9}{8}$$

$$c_{31} = \left(G_1(s)\,(s+3)\right)\Big|_{s=-3} = -\frac{1}{2}$$

$$Y(s) \multimap \Delta y(t) = \left(\frac{5}{8} - \frac{3}{4}t - \frac{9}{8}e^{-2t} + \frac{1}{2}e^{-3t}\right)S(t) -$$

$$-\left(\frac{5}{8} - \frac{3}{4}(t-1) - \frac{9}{8}e^{-2(t-1)} + \frac{1}{2}e^{-3(t-1)}\right)S(t-1)$$




(d) $\Delta y(\infty) = \lim_{s\to 0} s\,Y(s)$

$\Delta y(\infty) = -0.75$

$\Delta \dot{y}(0^+) = 0$

$\Delta \dot{y}(1^+) = -0.5202$

• ZADATAK 2

$$\frac{dh_1}{dt} = \frac{A_v}{A_1\sqrt{\rho}}\sqrt{2\Delta p}\,x_v - \frac{A_{12}}{A_1}\sqrt{2g}\sqrt{h_1-h_2}$$

$$\frac{dh_2}{dt} = \frac{A_{12}}{A_2}\sqrt{2g}\sqrt{h_1-h_2} - \frac{A_i}{A_2}\sqrt{2g}\sqrt{h_2}$$

(a) $x_{v0} = 0.4 \qquad h_{10} = 1.253\,m \qquad h_{20} = 0.626\,m$

(1) $$\frac{dh_1}{dt} = \frac{A_v}{A_1\sqrt{\rho}}\sqrt{2\Delta p}\,x_v - \frac{A_{12}}{A_1}\sqrt{2g}\sqrt{h_1-h_2} = f(h_1, h_2, x_v)$$

$$h_1 = h_{10} + \Delta h_1 \quad , \quad \dot{h}_1 = \Delta\dot{h}_1$$
$$h_2 = h_{20} + \Delta h_2 \quad , \quad \dot{h}_2 = \Delta\dot{h}_2$$
$$x_v = x_{v0} + \Delta x_v$$

$$\dot{h}_1 \approx \underbrace{f(h_{10}, h_{20}, x_{v0})}_{=0} + \left.\frac{\partial f}{\partial h_1}\right|_{S.T.}\Delta h_1 + \left.\frac{\partial f}{\partial h_2}\right|_{S.T.}\Delta h_2 + \left.\frac{\partial f}{\partial x_v}\right|_{S.T.}\Delta x_v$$

$$\Delta\dot{h}_1 = -5.59\cdot10^{-3}\,\Delta h_1 + 5.59\cdot10^{-3}\,\Delta h_2 + 1.7527\cdot10^{-2}\,\Delta x_v$$

(2) $$\frac{dh_2}{dt} = \frac{A_{12}}{A_2}\sqrt{2g}\sqrt{h_1-h_2} - \frac{A_i}{A_2}\sqrt{2g}\sqrt{h_2} = f(h_1, h_2)$$

$$\dot{h}_2 \approx \underbrace{f(h_{10}, h_{20})}_{=0} + \left.\frac{\partial f}{\partial h_1}\right|_{S.T.}\Delta h_1 + \left.\frac{\partial f}{\partial h_2}\right|_{S.T.}\Delta h_2$$

$$\Delta\dot{h}_2 = 5.59\cdot10^{-3}\,\Delta h_1 - 1.26\cdot10^{-2}\,\Delta h_2$$

(b) $x = [\Delta h_1 \;\; \Delta h_2] \quad , \quad u = [\Delta x_v] \quad , \quad y = [\Delta h_1]$

$$\begin{bmatrix}\Delta\dot{h}_1 \\ \Delta\dot{h}_2\end{bmatrix} = \begin{bmatrix}-5.59\cdot10^{-3} & 5.59\cdot10^{-3} \\ 5.59\cdot10^{-3} & -1.26\cdot10^{-2}\end{bmatrix}\begin{bmatrix}\Delta h_1 \\ \Delta h_2\end{bmatrix} + \begin{bmatrix}1.75\cdot10^{-2} \\ 0\end{bmatrix}[\Delta x_v]$$

$$[\Delta h_1] = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\Delta h_1 \\ \Delta h_2\end{bmatrix} + [0]\,[\Delta x_v]$$

(c) $a = 5.59 \cdot 10^{-3}$ $\quad b = 1.26 \cdot 10^{-2}$ $\quad c = 1.7527 \cdot 10^{-2}$

$$sH_1(s) = -aH_1(s) + aH_2(s) + cX_v(s)$$

$$sH_2(s) = aH_1(s) - bH_2(s)$$

$$H_2(s) = \frac{a}{s+b} H_1(s)$$

$$\frac{H_1(s)}{X_v(s)} = G(s) = \frac{1.7527 \cdot 10^{-2} s + 2.208 \cdot 10^{-4}}{s^2 + 1.819 \cdot 10^{-2} s + 3.918 \cdot 10^{-5}}$$

$$G(s) = \frac{447\,s + 5.63}{25519\,s^2 + 464\,s + 1}$$

(d) $\Delta x_v(t) = 0.05\,S(t) \rightarrow X_v(s) = 0.05 \frac{1}{s}$

$$H_1(s) = X_v(s)\,G(s)$$

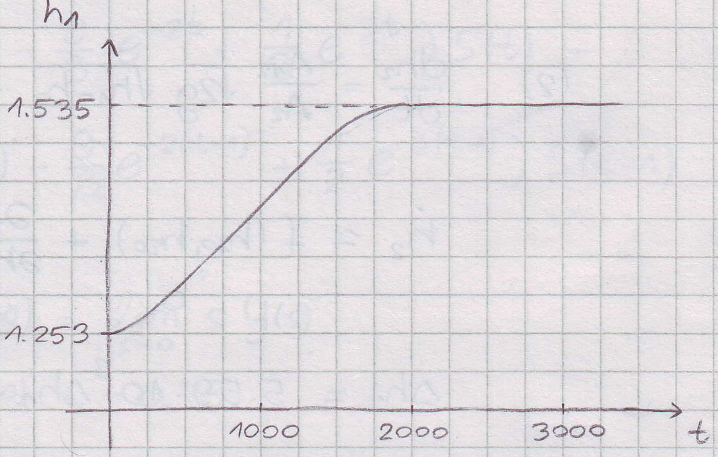
$$H_1(s) = \frac{22.35s + 0.2815}{25519s^3 + 464s^2 + s}$$

(e) $\Delta h_1(\infty) = \lim_{s\to 0} sH_1(s)$

$\Delta h_1(\infty) = 0.2818\ m$

$(\dot{\Delta h_1})(0^+) = \lim_{s\to\infty} s^2 H(s)$

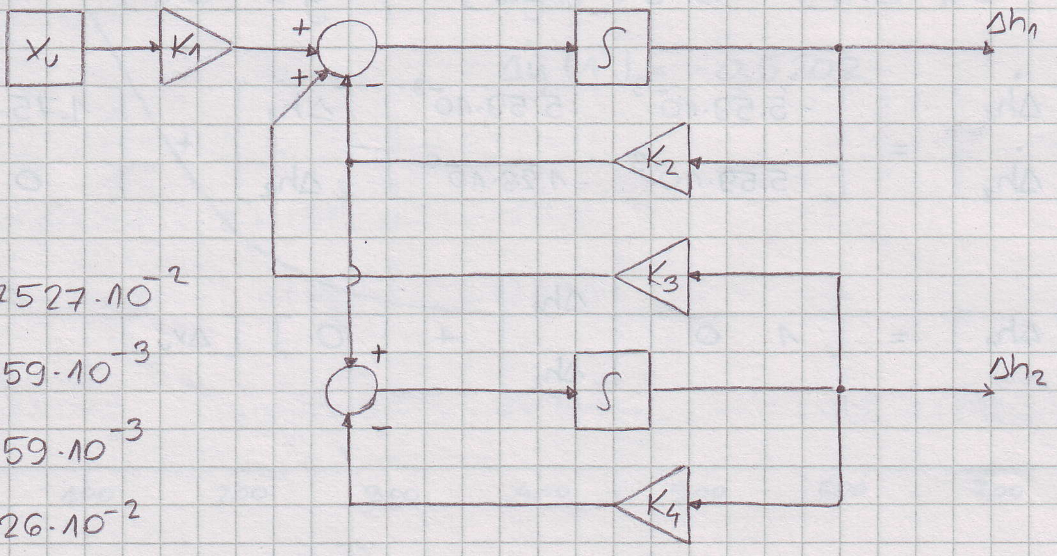
$(\dot{\Delta h_1})(0^+) = 8.76 \cdot 10^{-4}$




$\sigma = h_{10NL} - h_{10L}$

$\sigma = 0.05\,m$

(f)




$K_1 = 1.7527 \cdot 10^{-2}$

$K_2 = 5.59 \cdot 10^{-3}$

$K_3 = 5.59 \cdot 10^{-3}$

$K_4 = 1.26 \cdot 10^{-2}$