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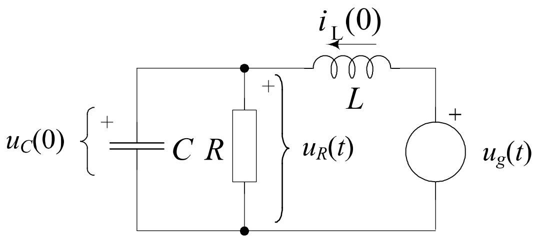
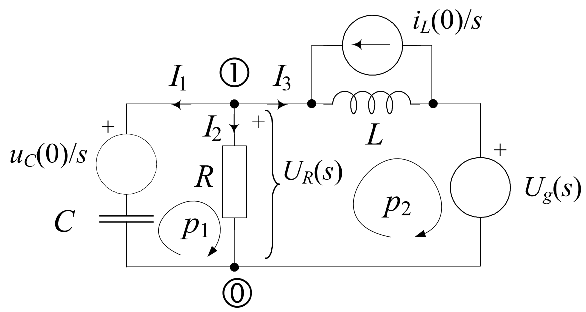
MEĐUISPIT IZ ELEKTRIČNIH KRUGOVA 2013-2014 – Rješenja

1. Za električni krug prikazan slikom primjenom Kirchhoffovih zakona izračunati valni oblik napona $u_R(t)$ kao odziv, ako je zadana pobuda $u_g(t)=\delta(t)$. Zadane su normalizirane vrijednosti elemenata $C=1$, $R=1$, $L=1/2$, te početni uvjeti $u_C(0)=8$ i $i_L(0)=2$.



Rješenje: Primjena Laplaceove transformacije



(1 bod)

Mreža ima N_b=3 grane i N_v=2 čvora

Jednadžbe KZN

$$\text{(p1)}\ -U_1(s)+U_2(s)=0$$
$$\text{(p2)}\ -U_2(s)+U_3(s)=0$$

Jednadžbe KZS

$$\text{(č1)}\ I_1(s)+I_2(s)+I_3(s)=0 \text{ (1 bod)}$$

Naponsko-strujne relacije grana

$$\text{(g1)}\ U_1(s)=\frac{1}{sC}I_1(s)+\frac{u_C(0)}{s}\Big/\cdot sC$$
$$\text{(g2)}\ U_2(s)=R\cdot I_2(s)$$
$$\text{(g3)}\ U_3(s)=sL\cdot\left[I_3(s)+\frac{i_L(0)}{s}\right]+U_g(s) \text{ (1 bod)}$$

$$\text{(g1)}\Rightarrow\ I_1(s)=sC\cdot U_1(s)-Cu_C(0)$$

$$\text{(g2)}\Rightarrow\ I_2(s)=U_2(s)\frac{1}{R}$$

$$\text{(g3)}\Rightarrow\ U_3(s)=sL\cdot I_3(s)+Li_L(0)+U_g(s)\ \Rightarrow\ I_3(s)=\frac{1}{sL}U_3(s)-\frac{i_L(0)}{s}-\frac{1}{sL}U_g(s)$$

$$\text{(č1)}\Rightarrow\ sC\cdot U_1(s)-Cu_C(0)+U_2(s)\frac{1}{R}+\frac{1}{sL}U_3(s)-\frac{i_L(0)}{s}-\frac{1}{sL}U_g(s)=0$$

$$\text{(p1), (p2)}\Rightarrow\ U_1(s)=U_2(s)=U_3(s),\ U_R(s)=U_2(s)$$

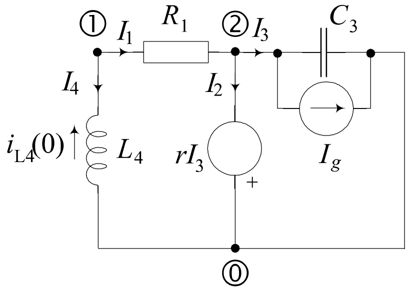
$$\Rightarrow \left(sC+\frac{1}{R}+\frac{1}{sL}\right)\cdot U_R(s)=Cu_C(0)+\frac{i_L(0)}{s}+\frac{1}{sL}U_g(s)$$

$$\Rightarrow U_R(s)=\frac{Cu_C(0)+\frac{i_L(0)}{s}+\frac{1}{sL}U_g(s)}{sC+\frac{1}{R}+\frac{1}{sL}}=\frac{8+\frac{2}{s}+\frac{2}{s}}{s+1+\frac{2}{s}}=\frac{8+\frac{4}{s}}{s+1+\frac{2}{s}}=\frac{8s+4}{s^2+s+2}$$ (1 bod)

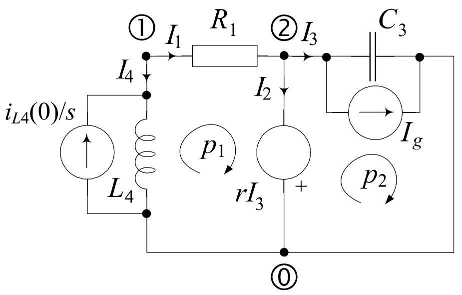
$$U_R(s)=8\cdot\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{7}{4}}=8\cdot\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}$$

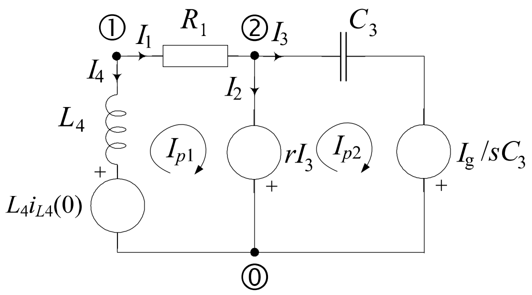
$$\Rightarrow \underline{u_R(t)=8\cdot e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{7}}{2}t\right)\cdot S(t)}$$ (1 bod)

2. Za električni krug na slici zadane su normalizirane vrijednosti elemenata R_1=1, C_3=1/2, L_4=1, r=2, $u_{C3}(0)$=0, $i_{L4}(0)$=1, te pobuda $i_g(t)$=$S(t)$. Koristeći metodu petlji te oznake grana i čvorova prema slici kao odziv izračunati napon grane 1 $u_1(t)$.



<u>Rješenje:</u> Primjena Laplaceove transformacije





(1 bod)

Vidljivo je:

$$I_1(s) = I_{p1}(s); \quad I_2(s) = I_{p1}(s) - I_{p2}(s); \quad I_3(s) = I_{p2}(s)$$

Jednadžbe petlji

$$1)\ I_{p1}(s)(R_1 + sL_4) = L_4 i_{L4}(0) + r \cdot I_{p2}(s)$$

$$2)\ I_{p2}(s)\frac{1}{sC_3} = -r \cdot I_{p2}(s) + \frac{1}{sC_3} \cdot I_g(s)$$ (1 bod)

$$2) \Rightarrow I_{p2}(s)\left(\frac{1}{sC_3} + r\right) = \frac{1}{sC_3} \cdot I_g(s) \Rightarrow I_{p2}(s) = \frac{\frac{1}{sC_3}}{\frac{1}{sC_3} + r} \cdot I_g(s) \Rightarrow I_{p2}(s) = \frac{1}{1 + rsC_3} \cdot I_g(s)$$

$$1) \Rightarrow I_1(s) = I_{p1}(s) = \frac{L_4 i_{L4}(0) + r \cdot I_{p2}(s)}{R_1 + sL_4} = \frac{L_4 i_{L4}(0) + r \cdot \frac{1}{1 + rsC_3} \cdot I_g(s)}{R_1 + sL_4}$$

Uz uvrštene vrijednosti elemenata:

$$I_1(s) = \frac{1 + 2 \cdot \frac{1}{1+s} \cdot \frac{1}{s}}{1+s} = \frac{1}{1+s} + \frac{2}{s} \cdot \frac{1}{(1+s)^2};$$

$$\Rightarrow U_1(s) = R_1 \cdot I_1(s) = \frac{1}{1+s} + \frac{2}{s} \cdot \frac{1}{(1+s)^2}$$ (1 bod)

Rastav na parcijalne razlomke:

$$\frac{2}{s} \cdot \frac{1}{(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

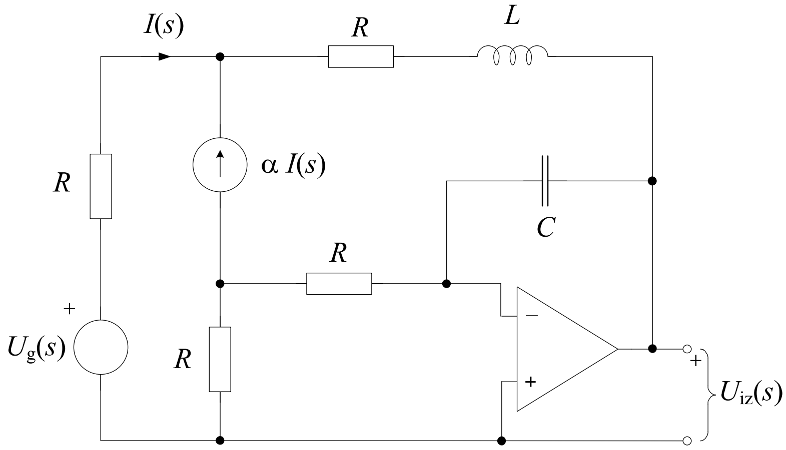
$$2 = A(s+1)^2 + Bs(s+1) + Cs$$

$$2 = As^2 + 2As + A + Bs^2 + Bs + Cs$$
$$2 = (A+B)s^2 + (2A+B+C)s + A$$

$$A + B = 0 \Rightarrow B = -A = -2$$
$$2A + B + C = 0 \Rightarrow C = -2A - B = -4 + 2 = -2$$
$$\underline{A = 2}$$

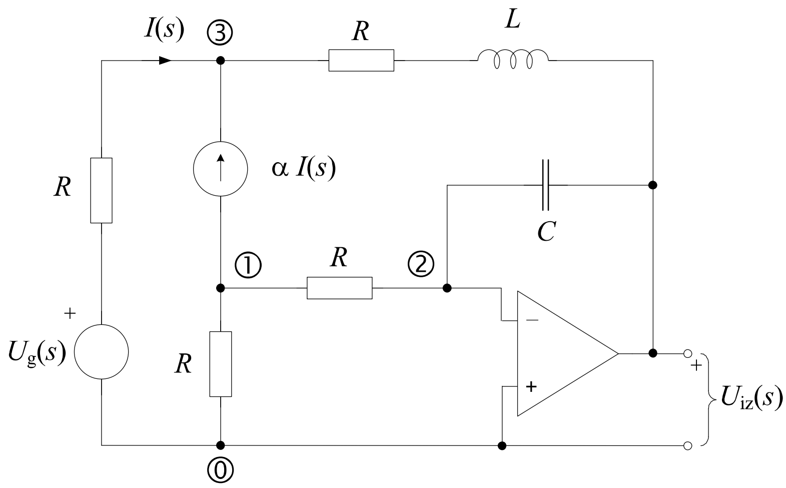
$$U_1(s) = \frac{1}{1+s} + \frac{2}{s} \cdot \frac{1}{(1+s)^2} = \frac{1}{1+s} + \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$
$$U_1(s) = \frac{1}{1+s} + \frac{2}{s} \cdot \frac{1}{(1+s)^2} = \frac{1}{1+s} + \frac{2}{s} - \frac{2}{s+1} - \frac{2}{(s+1)^2}$$
$$U_1(s) = \frac{2}{s} - \frac{1}{s+1} - \frac{2}{(s+1)^2} \quad \text{(1 bod)}$$
$$\Rightarrow \underline{u_1(t) = \left(2 - e^{-t} - 2 \cdot t \cdot e^{-t}\right) \cdot S(t)} \quad \text{(1 bod)}$$

4. Za električni krug prikazan slikom izračunati odziv u frekvencijskoj domeni $U_{\mathrm{iz}}(s)$ na pobudu $U_g(s)=E/s$. Zadane su normalizirane vrijednosti elemenata R=4, C=0.1 i L=1.25; te konstante E=6 i α=3. Operacijsko pojačalo je idealno. Početni uvjeti: $u_C(0)$=0, $i_L(0)$=0. Traženi odziv treba odrediti primjenom jednadžbi čvorišta.



Rješenje:

Primjenom Laplaceove transformacije dobivamo slijedeći električni krug u *frekvencijskoj* domeni. Postavimo jednadžbe čvorišta:



$$1)\ U_1\left(\frac{1}{R}+\frac{1}{R}\right)-U_2\frac{1}{R}=-\alpha I(s);\ \Rightarrow\ I(s)=\frac{U_g(s)-U_3(s)}{R};$$

$$2)\ -U_1\frac{1}{R}+U_2\left(sC+\frac{1}{R}\right)=U_{iz}(s)sC;$$

$$3)\ U_3\left(\frac{1}{R+sL}\right)=U_{iz}\frac{1}{R+sL}+\alpha\cdot I(s)+I(s);\ \textcolor{red}{(1\ \text{bod})}$$

$$U_2=0\ \Rightarrow$$

$$1)\ U_1\frac{2}{R}=-\alpha\frac{U_g(s)-U_3(s)}{R};$$

$$2)\ -U_1\frac{1}{R}=U_{iz}(s)sC;$$

$$3)\ U_3(s)\frac{1}{R+sL}=(1+\alpha)\cdot\frac{U_g(s)-U_3(s)}{R}+U_{iz}(s)\frac{1}{R+sL};\ \textcolor{red}{(1\ \text{bod})}$$

Nakon malo sređivanja:

$$1)\ 2U_1-\alpha U_3=-\alpha U_g(s)\ \Rightarrow\ U_1=\frac{\alpha}{2}U_3(s)-\frac{\alpha}{2}U_g(s);$$

$$2)\ U_{iz}(s)=-\frac{1}{sRC}U_1;$$

$$3)\ U_3(s)\left[\frac{1}{R+sL}+(1+\alpha)\cdot\frac{1}{R}\right]=U_g(s)\cdot\frac{1+\alpha}{R}+U_{iz}(s)\frac{1}{R+sL}$$

$$3)\Rightarrow\ U_3(s)\left[R+(1+\alpha)(R+sL)\right]=U_g(s)\cdot(1+\alpha)(R+sL)+U_{iz}(s)R;$$

$$\Rightarrow\ U_3(s)=U_g(s)\cdot\frac{(1+\alpha)(R+sL)}{R+(1+\alpha)(R+sL)}+U_{iz}(s)\frac{R}{R+(1+\alpha)(R+sL)};$$

$$2)\Rightarrow\ U_{iz}(s)=-\frac{1}{sRC}U_1=-\frac{1}{sRC}\frac{\alpha}{2}\left[U_3(s)-U_g(s)\right]$$

$$U_{iz}(s)=-\frac{1}{sRC}\frac{\alpha}{2}\left[\frac{(1+\alpha)(R+sL)}{R+(1+\alpha)(R+sL)}\cdot U_g(s)-U_g(s)+\frac{R}{R+(1+\alpha)(R+sL)}\cdot U_{iz}(s)\right]$$

$$-U_{iz}(s)\frac{2sRC}{\alpha}=\frac{-R}{R+(1+\alpha)(R+sL)}\cdot U_g(s)+\frac{R}{R+(1+\alpha)(R+sL)}\cdot U_{iz}(s)$$

$$U_{iz}(s)\left[\frac{2sRC}{\alpha}+\frac{R}{R+(1+\alpha)(R+sL)}\right]=\frac{R}{R+(1+\alpha)(R+sL)}\cdot U_g(s)$$

$$U_{iz}(s)=\frac{\dfrac{R}{R+(1+\alpha)(R+sL)}}{\dfrac{2sRC}{\alpha}+\dfrac{R}{R+(1+\alpha)(R+sL)}}\cdot U_g(s)=\frac{R\alpha}{2sRC\left[R+(1+\alpha)(R+sL)\right]+R\alpha}\cdot U_g(s)$$

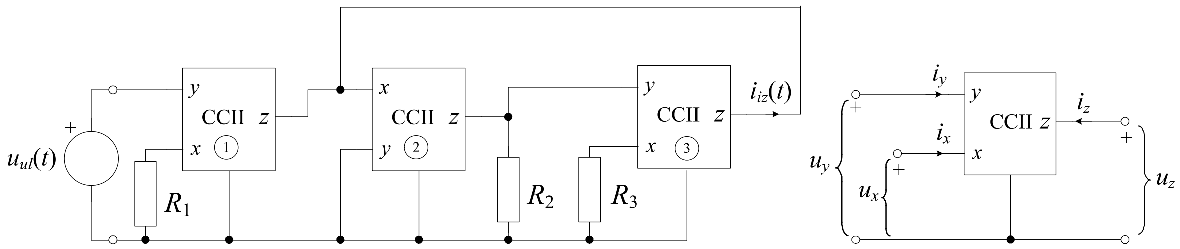
(2 boda)

Uz uvrštene vrijednosti elemenata:

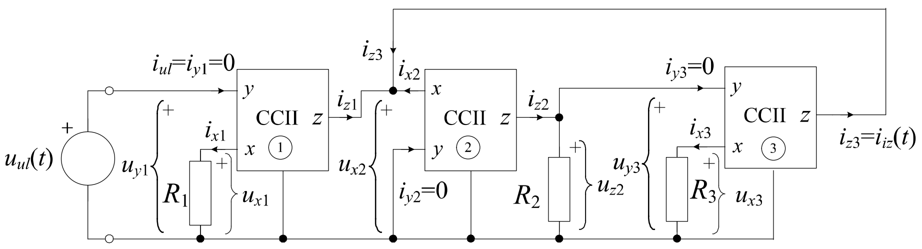
$$U_{iz}(s)=\frac{12}{2s0.4\left[4+(1+3)(4+s1.25)\right]+3\cdot 4}\cdot\frac{6}{s}=\frac{12}{0.8s\left[5s+20\right]+12}\cdot\frac{6}{s}=\frac{12}{4s^2+16s+12}\cdot\frac{6}{s}$$

$$\underline{U_{iz}(s)=\frac{3}{s^2+4s+3}\cdot\frac{6}{s}=\frac{18}{s\left(s^2+4s+3\right)}}\ \text{(1 bod)}$$

5. Za električni krug prikazan slikom izračunati valni oblik struje $i_{iz}(t)$ za $t>0$ kao odziv, ako je zadana pobuda $u_{ul}(t)=E\cdot S(t)$. Zadane su normalizirane vrijednosti elemenata $R_1=2$, $R_2=4$, $R_3=1$, te konstanta $E=10$. Za strujni prijenosnik druge generacije (CCII) vrijede slijedeće definicijske jednadžbe: $u_x=u_y$, $i_y=0$, $i_z=i_x$ uz referentna usmjerenja struja i napona prilaza prikazana na slici.



Rješenje:



Za prvi CCII vrijedi:

$$u_{x1}=u_{y1}=u_{ul},\ i_{y1}=0,\ i_{x1}=\frac{u_{x1}}{R_1}=\frac{u_{ul}}{R_1},\ i_{z1}=i_{x1}=\frac{u_{ul}}{R_1}\text{ (1 bod)}$$

Za drugi CCII vrijedi:

$$u_{x2}=u_{y2}=0,\ i_{x2}=-i_{z1}-i_{z3},\ i_{z2}=i_{x2}=-(i_{z1}+i_{z3})\text{ (1 bod)}$$

$$u_{z2}=R_2 i_{z2}=-R_2(i_{z1}+i_{z3})$$

Za treći CCII vrijedi:

$$u_{y3}=u_{z2},\ i_{y3}=0,\ u_{x3}=u_{y3}=R_3 i_{x3}\text{ (1 bod)}$$

$$i_{z3}=i_{x3}=\frac{u_{y3}}{R_3}=\frac{-(i_{z1}+i_{z3})R_2}{R_3}$$

$$i_{z3}=-i_{z1}\frac{R_2}{R_3}-i_{z3}\frac{R_2}{R_3}\ \Rightarrow\ i_{z3}+i_{z3}\frac{R_2}{R_3}=-i_{z1}\frac{R_2}{R_3}\ \Rightarrow\ i_{z3}\left(1+\frac{R_2}{R_3}\right)=-i_{z1}\frac{R_2}{R_3}$$

$$\Rightarrow i_{iz}=i_{z3}=-i_{z1}\cdot\frac{\frac{R_2}{R_3}}{1+\frac{R_2}{R_3}}=-\frac{u_{ul}}{R_1}\cdot\frac{\frac{R_2}{R_3}}{1+\frac{R_2}{R_3}}$$

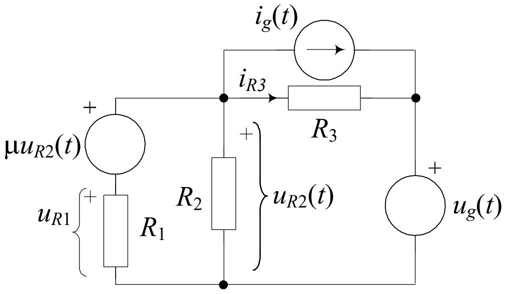
Uz uvrštene vrijednosti elemenata:

$$i_{iz}=-\frac{10}{2}\cdot\frac{\frac{4}{1}}{1+\frac{4}{1}}=-\frac{10}{2}\cdot\frac{4}{5}=-4\text{ (1 bod)}$$

$$\Rightarrow \underline{i_{iz}(t)=-4\cdot S(t)[A]}\text{ (1 bod)}$$

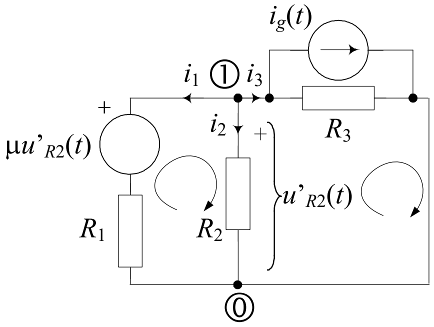
MEĐUISPIT IZ ELEKTRIČNIH KRUGOVA 2014-2015 – Rješenja

1. Za električni krug prikazan slikom primjenom metode superpozicije izračunati napon $u_{R2}(t)$. Zadane su normalizirane vrijednosti elemenata R_1=2, R_2=1, R_3=1/2 i μ=2 i pobude $u_g(t)$=$S(t)$ i $i_g(t)$=$S(t)$. Izračunati također napon $u_{R1}(t)$ na otporu R_1 i struju $i_{R3}(t)$ kroz R_3. U proračunu primijeniti Kirchhoffove zakone.



<u>Rješenje:</u> Primjena metode superpozicije.

a) Isključen naponski izvor u_g=0. Ovisni izvor (NONI) μ u_{R2} ostaje uključen.



Mreža ima N_b=3 grane i N_v=2 čvora

Jednadžbe KZN

$$\text{(p1)}\ -u_1(t)+u_2(t)=0$$

$$\text{(p2)}\ -u_2(t)+u_3(t)=0 \Rightarrow u_1(t)=u_2(t)=u_3(t),\ u'_{R2}(t)=u_2(t)$$

Naponsko-strujne relacije grana

$$\text{(g1)}\ u_1(t)=\mu u'_{R2}(t)+R_1 i_1(t) \Rightarrow i_1(t)=\frac{1}{R_1}\cdot\left[u_1(t)-\mu u_2(t)\right]$$

$$\text{(g2)}\ u'_{R2}(t)=u_2(t)=R\cdot i_2(t) \Rightarrow i_2(t)=u_2(t)\frac{1}{R_2}$$

$$\text{(g3)}\ u_3(s)=R_3\cdot\left[i_3(t)-i_g(t)\right] \Rightarrow i_3(t)=\frac{1}{R_3}u_3(t)+i_g(t)$$

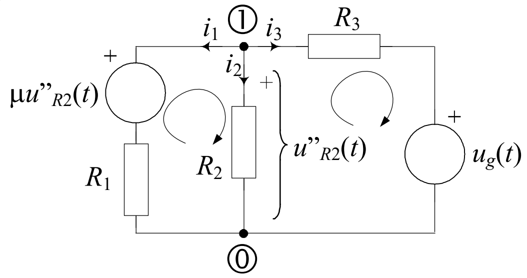
Jednadžbe KZS

$$\text{(č1)}\ i_1(t)+i_2(t)+i_3(t)=0 \Rightarrow u_1(t)\cdot\frac{1}{R_1}-\mu u_2(t)\frac{1}{R_1}+u_2(t)\frac{1}{R_2}+\frac{1}{R_3}u_3(t)+i_g(t)=0$$

$$\Rightarrow u'_{R2}(t)\cdot\left[\frac{1-\mu}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right]=-i_g(t)$$

$$\Rightarrow u'_{R2}(t)=\frac{-i_g(t)}{\frac{1-\mu}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}=\frac{-S(t)}{\frac{1-2}{2}+1+2}=\frac{-S(t)}{\frac{1}{2}+\frac{4}{2}}=\frac{-2}{5}S(t)\ \text{(1 bod)}$$

b) Isključen strujni izvor $i_g(t)$=0. Ovisni izvor (NONI) μ u_{R2} ostaje uključen.



Jednadžbe KZN (iste kao i u slučaju a)

$$\text{(p1)}\ -u_1(t)+u_2(t)=0$$

$$\text{(p2)}\ -u_2(t)+u_3(t)=0 \Rightarrow u_1(t)=u_2(t)=u_3(t),\ u''_{R2}(t)=u_2(t)$$

Naponsko-strujne relacije grana (g1 i g2 iste kao i u slučaju a)

$$\text{(g1)}\ u_1(t)=\mu u'_{R2}(t)+R_1 i_1(t) \Rightarrow i_1(t)=\frac{1}{R_1}\cdot\left[u_1(t)-\mu u_2(t)\right]$$

$$\text{(g2)}\ u''_{R2}(t)=u_2(t)=R\cdot i_2(t) \Rightarrow i_2(t)=u_2(t)\frac{1}{R_2}$$

$$\text{(g3)}\ u_3(s)=R_3\cdot i_3(t)+u_g(t) \Rightarrow i_3(t)=\frac{1}{R_3}\left[u_3(t)-u_g(t)\right]$$

Jednadžbe KZS

$$\text{(č1)}\ i_1(t)+i_2(t)+i_3(t)=0 \Rightarrow u_1(t)\cdot\frac{1}{R_1}-\mu u_2(t)\frac{1}{R_1}+u_2(t)\frac{1}{R_2}+\frac{1}{R_3}\left[u_3(t)-u_g(t)\right]=0$$

$$\Rightarrow u''_{R2}(t)\cdot\left[\frac{1-\mu}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right]=\frac{1}{R_3}u_g(t)$$

$$\Rightarrow u''_{R2}(t)=\frac{\frac{1}{R_3}u_g(t)}{\frac{1-\mu}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}=\frac{2S(t)}{\frac{1-2}{2}+1+2}=\frac{2S(t)}{\frac{1}{2}+\frac{4}{2}}=\frac{4}{5}S(t)\ \text{(1 bod)}$$

c) Superpozicija:

$$u_{R2}(t)=u'_{R2}(t)+u''_{R2}(t)=\frac{-i_g(t)+\frac{1}{R_3}u_g(t)}{\frac{1-\mu}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}=\frac{-S(t)+2S(t)}{\frac{1-2}{2}+1+2}=\frac{S(t)}{\frac{1}{2}+\frac{4}{2}}=\frac{2}{5}S(t)\ \text{(1 bod)}$$

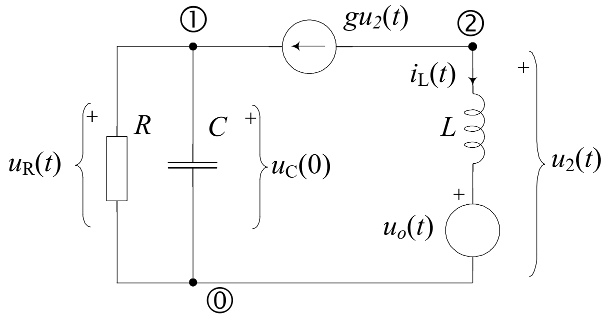
d) Napon u_{R1}

$$u_1=u_{R2}-\mu\cdot u_{R2}=(1-\mu)u_{R2}=-u_{R2}=-\frac{2}{5}S(t)\ \text{(1 bod)}$$

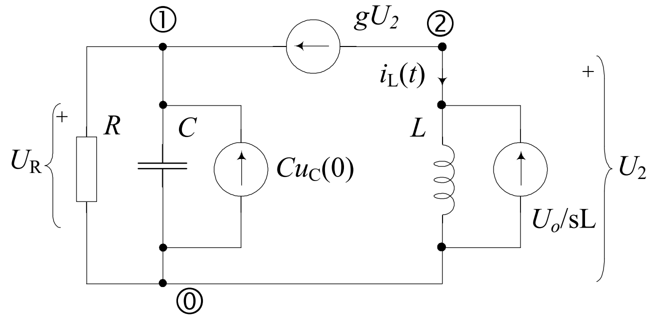
e) Struja i_{R3}

$$i_{R3}=\frac{u_{R2}-u_g}{R_3}=2\left(\frac{2}{5}-1\right)S(t)=-\frac{6}{5}S(t)\ \text{(1 bod)}$$

3. Za električni krug na slici zadane su normalizirane vrijednosti elemenata R=1, C=1, L=2, g=2, početni napon na kapacitetu $u_C(0)$=1, te pobuda $u_o(t)$=$S(t)$. Primjenom Laplaceove transformacije i koristeći metodu čvorišta izračunati napon $u_R(t)$ na otporu R i struju $i_L(t)$.



Rješenje: Primjena Laplaceove transformacije



(1 bod)

$$U_1(s)\left(\frac{1}{R}+sC\right)=Cu_C(0)+g\cdot U_2(s) \quad \Rightarrow \qquad U_1\left(\frac{1}{R}+sC\right)-g\cdot U_2=Cu_C(0)$$

$$U_2(s)\frac{1}{sL}=-g\cdot U_2(s)+\frac{1}{sL}\cdot U_o(s) \quad \Rightarrow \qquad U_2\left(\frac{1}{sL}+g\right)=\frac{U_o}{sL}\Rightarrow \; U_2=\frac{U_o}{1+sgL} \text{(1 bod)}$$

$$U_1\frac{1}{R}(1+sCR)=\frac{gU_o}{1+sgL}+Cu_C(0) \quad \Rightarrow \qquad U_1=\frac{R\left(gU_o+Cu_C(0)(1+sgL)\right)}{(1+sCR)(1+sgL)} \text{(1 bod)}$$

$$U_1=\frac{\frac{2}{s}+1+4s}{(1+s)(1+4s)}=\frac{s^2+\frac{s}{4}+\frac{1}{2}}{s(s+1)\left(s+\frac{1}{4}\right)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+\frac{1}{4}}$$

$$s^2+\frac{s}{4}+\frac{1}{2}=A(s+1)\left(s+\frac{1}{4}\right)+Bs\left(s+\frac{1}{4}\right)+Cs(s+1)$$

$$A+B+C=1$$

$$\frac{5A}{4}+\frac{B}{4}+C=\frac{1}{4}$$

$$\frac{A}{4}=\frac{1}{2}\Rightarrow \qquad A=2 \;\Rightarrow \qquad B+C=1-A=-1$$

$$\frac{5A}{4}+\frac{B}{4}+C=\frac{1}{4} \quad\Rightarrow\quad \frac{3}{4}C=\frac{1}{4}-\frac{5A}{4}-\frac{B+C}{4}=\frac{1}{4}-\frac{10}{4}-\frac{(-1)}{4} \quad\Rightarrow\quad C=-\frac{8}{3}$$

$$B = -1 - C = \frac{5}{3}$$

$$U_1 = \frac{2}{s} + \frac{5}{3} \cdot \frac{1}{s+1} - \frac{8}{3} \cdot \frac{1}{s + \frac{1}{4}} \quad \Rightarrow \quad u_R = u_1 = \left(2 + \frac{5}{3} \cdot e^{-t} - \frac{8}{3} \cdot e^{-t/4} \right) S(t)$$ (1 bod)

$$I_L = \frac{U_2 - U_o}{sL} = \frac{U_o}{sL}\left(\frac{1}{1+sgL} - 1\right) = \frac{-gU_o}{1+sgL} = \frac{-\frac{2}{s}}{1+4s} = -\frac{1}{2} \cdot \frac{1}{s(s+1/4)}$$

$$I_L = -\frac{1}{2} \cdot \frac{1}{s(s+1/4)} = \frac{A}{s} + \frac{B}{s+1/4}$$

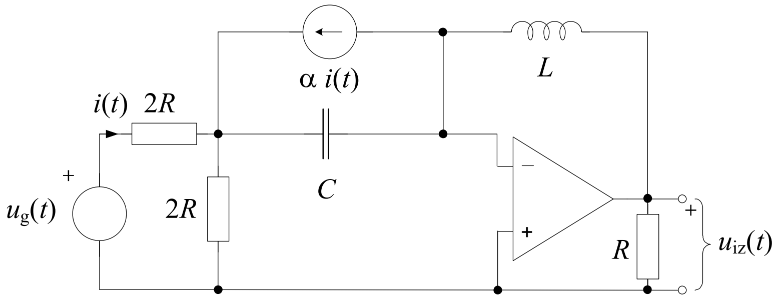
$$-\frac{1}{2} = A(s+1/4) + Bs$$

$$A + B = 0$$

$$\frac{A}{4} = -\frac{1}{2} \quad \Rightarrow \quad A = -2 \quad \Rightarrow \quad B = 2$$

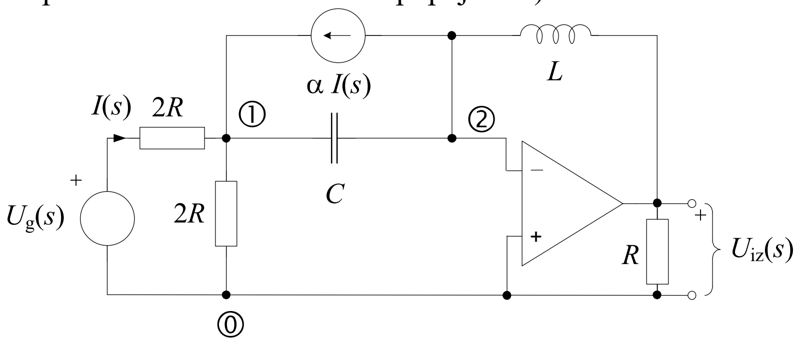
$$I_L = -\frac{2}{s} + \frac{2}{s+1/4} \quad \Rightarrow \quad i_L = -2\left(1 - e^{-t/4}\right) S(t)$$ (1 bod)

4. Za električni krug prikazan slikom izračunati odziv $u_{iz}(t)$ na pobudu $u_g(t)=S(t)$. Zadane su normalizirane vrijednosti elemenata $R=1$, $C=1$ i $L=2$; te konstanta ovisnog izvora $\alpha=1$. Operacijsko pojačalo je idealno. Početni uvjeti su jednaki nuli. Traženi odziv treba odrediti primjenom jednadžbi čvorišta.



Rješenje:

a) Primjenom Laplaceove transformacije dobivamo slijedeći električni krug u *frekvencijskoj* domeni. Postavimo jednadžbe čvorišta (otpor R na izlazu op. pojačala se zanemaruje, jer je paralelno spojen naponskom izvoru na izlazu op. pojačala):



$$1)\ U_1\left(\frac{1}{2R}+\frac{1}{2R}+sC\right)-U_2sC=\alpha I(s)+\frac{U_g(s)}{2R};\ \Rightarrow\ I(s)=\frac{U_g(s)-U_1(s)}{2R};$$

$$2)\ -U_1sC+U_2\left(sC+\frac{1}{sL}\right)=U_{iz}(s)\frac{1}{sL}-\alpha I(s);$$

Virtualni kratki spoj $\Rightarrow U_2=0 \Rightarrow$

$$1)\ U_1\left(\frac{2}{2R}+sC\right)=\alpha\frac{U_g(s)-U_1(s)}{2R}+\frac{U_g(s)}{2R};$$

$$2)\ -U_1sC=U_{iz}(s)\frac{1}{sL}-\alpha\frac{U_g(s)-U_1(s)}{2R};$$

Nakon malo sređivanja:

$$1)\ U_1\left(\frac{2+\alpha}{2R}+sC\right)=\frac{U_g(s)}{2R}(1+\alpha)$$

$$\Rightarrow U_1(s)=\frac{\frac{1}{2R}(1+\alpha)}{\frac{2+\alpha}{2R}+sC}U_g(s)=\frac{1+\alpha}{2+\alpha+s(2RC)}U_g(s);$$

$$2)\ U_1\left(\frac{\alpha}{2R}+sC\right)=-U_{iz}(s)\frac{1}{sL}+U_g(s)\frac{\alpha}{2R};$$

$$1),\,2)\ \Rightarrow\ \frac{1+\alpha}{2+\alpha+s(2RC)}\left(\frac{\alpha}{2R}+sC\right)U_g(s)=-U_{iz}(s)\frac{1}{sL}+U_g(s)\frac{\alpha}{2R};$$

$$U_{iz}(s)\frac{1}{sL}=-\frac{1+\alpha}{2+\alpha+s(2RC)}\left(\frac{\alpha}{2R}+sC\right)U_g(s)+\frac{\alpha}{2R}U_g(s)$$

$$U_{iz}(s)=-sL\left[\frac{1+\alpha}{2+\alpha+s(2RC)}\left(\frac{\alpha}{2R}+sC\right)-\frac{\alpha}{2R}\right]U_g(s)$$ (3 boda)

b) Uz uvrštene vrijednosti elemenata: R=1, C=1, L=2; $U_g(s)$=1/s i α=1

$$U_{iz}(s)=-2s\left[\frac{1+1}{2+1+2s}\left(\frac{1}{2}+s\right)-\frac{1}{2}\right]\cdot\frac{1}{s}=-2\left[\frac{2}{3+2s}\left(\frac{1+2s}{2}\right)-\frac{1}{2}\right]$$

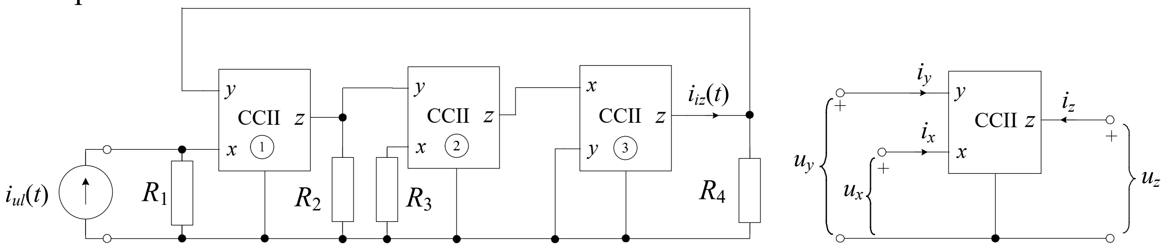
$$U_{iz}(s)=-2\left[\frac{1+2s}{3+2s}-\frac{1}{2}\right]$$

$$U_{iz}(s)=-2\left[\frac{3+2s-2}{3+2s}-\frac{1}{2}\right]=-2\left[1-\frac{2}{3+2s}-\frac{1}{2}\right]=-2\left[\frac{1}{2}-\frac{1}{s+\frac{3}{2}}\right]=-1+\frac{2}{s+\frac{3}{2}}$$ (1 bod)

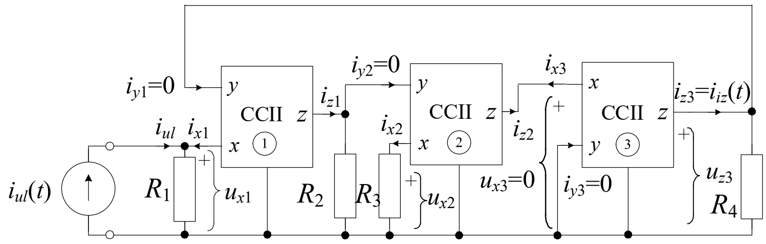
c) Inverzna Laplaceova transformacija izlaznog napona:

$$\underline{u_{iz}(t)=L^{-1}[U_{iz}(s)]=-\delta(t)+2e^{-3/2t}\cdot S(t)}$$ (1 bod)

5. Za električni krug prikazan slikom izračunati valni oblik struje $i_{iz}(t)$ za $t>0$ kao odziv, ako je zadana pobuda $i_{ul}(t)=E\cdot S(t)$[A]. Zadane su normalizirane vrijednosti elemenata $R_1=1$, $R_2=2$, $R_3=3$, $R_4=4$, te konstanta $E=5{,}5$. Za strujni prijenosnik druge generacije (CCII) vrijede slijedeće definicijske jednadžbe: $u_x=u_y$, $i_y=0$, $i_z=i_x$ uz referentna usmjerenja struja i napona prilaza prikazana na slici.



Rješenje:



a) Za prvi CCII vrijedi: (1 bod)

$$u_{x1}=u_{y1}=u_{z3},\ i_{y1}=0,\ i_{x1}+i_{ul}=\frac{u_{x1}}{R_1},\ i_{z1}=i_{x1}$$

b) Za drugi CCII vrijedi: (1 bod)

$$u_{x2}=u_{y2}=i_{z1}R_2,\ i_{x2}=\frac{u_{x2}}{R_3},\ i_{z2}=i_{x2}$$

$$i_{z2}=i_{x2}=\frac{u_{x2}}{R_3}=\frac{u_{y2}}{R_3}=i_{z1}\frac{R_2}{R_3}\Rightarrow\boxed{i_{z2}=i_{z1}\frac{R_2}{R_3}}$$

c) Za treći CCII vrijedi: (1 bod)

$$u_{x3}=u_{y3}=0,\ i_{z3}=i_{x3}=-i_{z2}\Rightarrow\boxed{i_{z3}=-i_{z2}}$$

$$u_{z3}=i_{z3}R_4$$

d) Nakon sređivanja do sada napisanih izraza: (1 bod)

$$\boxed{i_{iz}=i_{z3}},\ \boxed{i_{z3}=-i_{z1}\frac{R_2}{R_3}},\ i_{z1}=i_{x1}=\frac{u_{x1}}{R_1}-i_{ul}=\frac{u_{z3}}{R_1}-i_{ul}=i_{z3}\frac{R_4}{R_1}-i_{ul}$$

$$\Rightarrow i_{iz}=-\left(i_{iz}\frac{R_4}{R_1}-i_{ul}\right)\frac{R_2}{R_3}\Rightarrow i_{iz}\left(1+\frac{R_2R_4}{R_1R_3}\right)=i_{ul}\frac{R_2}{R_3}\Rightarrow\boxed{i_{iz}=\frac{\frac{R_2}{R_3}}{1+\frac{R_2R_4}{R_1R_3}}i_{ul}}$$

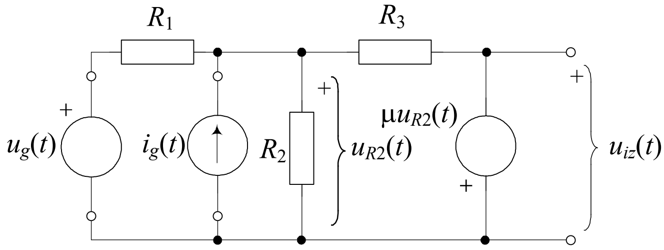
e) Uz uvrštene vrijednosti elemenata: (1 bod)

$$i_{iz}(t)=\frac{\frac{2}{3}}{1+\frac{2\cdot 4}{1\cdot 3}}i_{ul}(t)=\frac{2}{3+8}i_{ul}(t)=\frac{2}{11}i_{ul}(t)=\frac{2\cdot 5{,}5}{11}S(t)=1\cdot S(t)[A]$$

$$\Rightarrow \underline{i_{iz}(t)=1\cdot S(t)[A]}$$

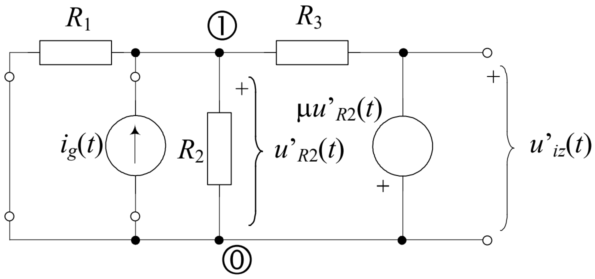
ZAVRŠNI ISPIT IZ ELEKTRIČNIH KRUGOVA 2014-2015 – Rješenja

1. Za električni krug prikazan slikom primjenom metode superpozicije izračunati valni oblik napona $u_{iz}(t)$ kao odziv, ako je zadana pobuda $u_g(t)=6S(t)$ i $i_g(t)=3\delta(t)$. Zadane su normalizirane vrijednosti elemenata $R_1=1$, $R_2=1$, $R_3=2$ i a) $\mu=1$; b) $\mu=\infty$.



Rješenje: Primjena metode superpozicije.

a) Isključen je naponski izvor: $u_g(t)=0$ (umjesto isključenog naponskog izvora je kratki spoj). Ovisni izvor (NONI) s parametrom μ (naponsko pojačanje) ostaje uključen.



Jednadžbe čvorišta (samo jedan čvor)

$$(1)\quad u'_1(t)\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)=i_g(t)+\frac{u'_{iz}(t)}{R_3}$$

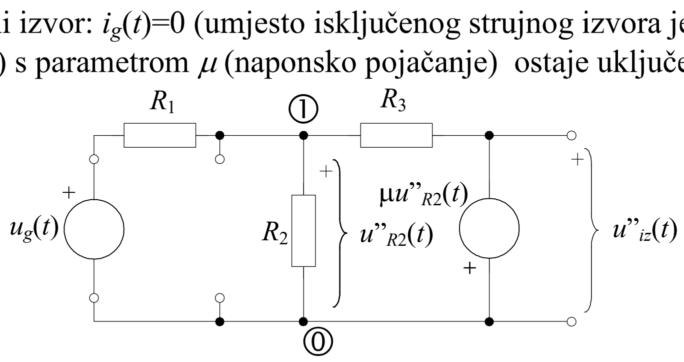
$$(2)\quad u'_{iz}(t)=-\mu u'_{R2}(t)=-\mu u'_1(t)\ \Rightarrow\ u'_1(t)=-\frac{u'_{iz}(t)}{\mu}$$

$$\Rightarrow\qquad -\frac{u'_{iz}(t)}{\mu}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)=i_g(t)+\frac{u'_{iz}(t)}{R_3}$$

$$\frac{u'_{iz}(t)}{\mu}\frac{1}{R_1}+\frac{u'_{iz}(t)}{\mu}\frac{1}{R_2}+\frac{u'_{iz}(t)}{R_3}\left(1+\frac{1}{\mu}\right)=-i_g(t)$$

$$u'_{iz}(t)=\frac{-i_g(t)}{\frac{1}{\mu R_1}+\frac{1}{\mu R_2}+\frac{1}{R_3}\left(1+\frac{1}{\mu}\right)}$$

b) Isključen je strujni izvor: $i_g(t)=0$ (umjesto isključenog strujnog izvora je prazni hod). Ovisni izvor (NONI) s parametrom μ (naponsko pojačanje) ostaje uključen.



Jednadžbe čvorišta (samo jedan čvor)

$$(1)\ u''_1(t)\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)=\frac{u_g(t)}{R_1}+\frac{u''_{iz}(t)}{R_3}$$

$$(2)\ u''_{iz}(t)=-\mu u''_{R2}(t)=-\mu u''_1(t)\ \Rightarrow\ u''_1(t)=-\frac{u''_{iz}(t)}{\mu}$$

$$\Rightarrow\qquad -\frac{u''_{iz}(t)}{\mu}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)=\frac{u_g(t)}{R_1}+\frac{u''_{iz}(t)}{R_3}$$

$$\frac{u''_{iz}(t)}{\mu}\frac{1}{R_1}+\frac{u''_{iz}(t)}{\mu}\frac{1}{R_2}+\frac{u''_{iz}(t)}{R_3}\left(1+\frac{1}{\mu}\right)=-\frac{u_g(t)}{R_1}$$

$$u''_{iz}(t)=\frac{-\dfrac{u_g(t)}{R_1}}{\dfrac{1}{\mu R_1}+\dfrac{1}{\mu R_2}+\dfrac{1}{R_3}\left(1+\dfrac{1}{\mu}\right)}$$

(3 boda)

c) Superpozicija:

$$u_{iz}(t)=u'_{iz}(t)+u''_{iz}(t)=\frac{-i_g(t)-\dfrac{u_g(t)}{R_1}}{\dfrac{1}{\mu R_1}+\dfrac{1}{\mu R_2}+\dfrac{1}{R_3}\left(1+\dfrac{1}{\mu}\right)}$$

(1 bod)

Uz uvrštene vrijednosti elemenata R_1=1, R_2=1, R_3=2 i μ=1 slijedi:

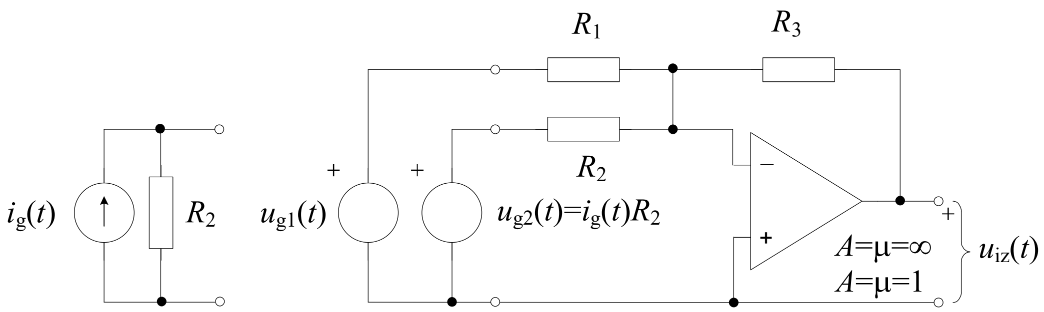
$$u_{iz}(t)=u'_{iz}(t)+u''_{iz}(t)=\frac{-i_g(t)-\dfrac{u_g(t)}{R_1}}{\dfrac{1}{R_1}+\dfrac{1}{R_2}+\dfrac{1}{R_3}\left(1+\dfrac{1}{1}\right)}=\frac{-i_g(t)-u_g(t)}{3}=-\delta(t)-2S(t)$$

Uz uvrštene vrijednosti elemenata R_1=1, R_2=1, R_3=2 i μ=∞ slijedi:

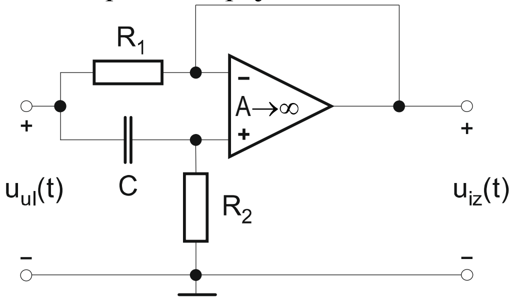
$$u_{iz}(t)=u'_{iz}(t)+u''_{iz}(t)=\frac{-i_g(t)-\dfrac{u_g(t)}{R_1}}{\dfrac{1}{R_3}}=-R_3 i_g(t)-\frac{R_3}{R_1}u_g(t)=-6\delta(t)-12S(t)$$

(1 bod)

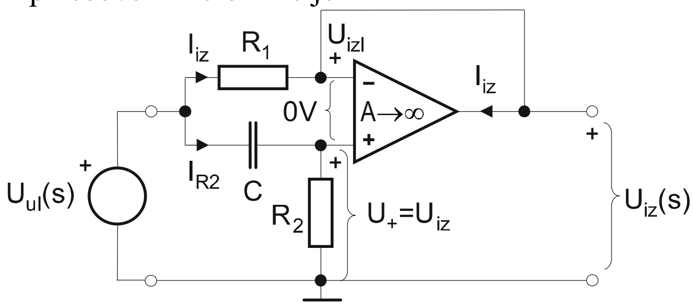
Električni krug u ovom zadatku također možemo prikazati pomoću operacijskog pojačala:



4. Odrediti odziv $u_{iz}(t)$ za mrežu prikazanu slikom ako je zadano: R_1=1kΩ, R_2=20kΩ, C=250nF te kao poticaj jedinična Step funkcija $u_{ul}(t)=S(t)$. Koji se ekvivalentni element može upotrijebiti umjesto kapaciteta C otpora R_2 i pojačala i koliko on iznosi ?



Rješenje: Primjena Laplaceove transformacije



$$U_+ \cdot \left(\frac{1}{R_2} + sC\right) - U_{ul} \cdot sC = 0 \;\Rightarrow\; U_+ = U_{ul} \cdot \frac{sC}{\frac{1}{R_2} + sC} = U_{ul} \cdot \frac{sR_2C}{1 + sR_2C} = U_{ul} \cdot \frac{R_2}{\frac{1}{sC} + R_2}$$

$$\frac{U_{iz} - U_{ul}}{R_1} = I_{iz}$$

$$(U_+ - U_{iz}) \cdot A = U_{iz} \;\Rightarrow\; U_+ = U_{iz}$$

$$U_{ul} - U_{iz} \cdot R_1 = U_{iz}$$

$$U_{ul} - I_{R2} \cdot \frac{1}{sC} = U_+$$

$$I_{R2} = \frac{U_{iz}}{\frac{1}{sC} + R_2}$$

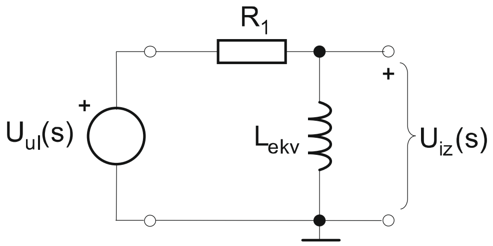
$$I_{R2} \cdot R_2 = U_+ = U_{iz}$$

$$U_{iz} = U_{ul} \cdot \frac{R_2}{\frac{1}{sC} + R_2} = U_{ul} \cdot \frac{1}{\frac{1}{sR_2C} + 1} = U_{ul} \cdot \frac{s}{s + \frac{1}{R_2C}}$$ (2 boda)

Što je isto kao i $U_{iz} = U_{ul} \cdot \frac{sL_{ekv}}{R_1 + sL_{ekv}} = U_{ul} \cdot \frac{s}{s + \frac{R_1}{L_{ekv}}}$ (vidi sliku)

Uspoređujući $\frac{R_1}{L_{ekv}} = \frac{1}{R_2C}$ slijedi $L_{ekv} = R_1R_2C = 1[\text{k}\Omega] \cdot 20[\text{k}\Omega] \cdot 250[\text{nF}] = 5[\text{H}]$

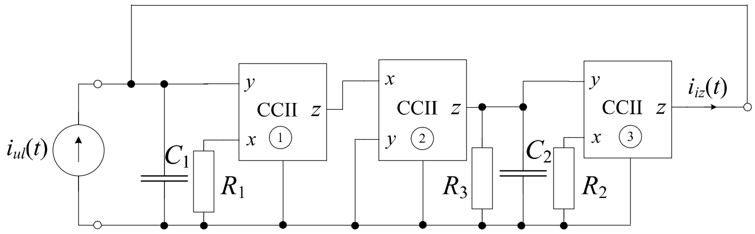
Dakle ekvivalentni element je induktivitet. (2 boda)

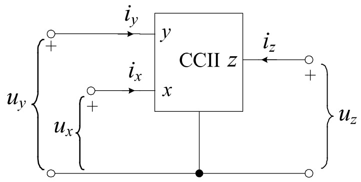


Odziv na poticaj $u_{ul}(t)=S(t)\Rightarrow U_{ul}(s)=\frac{1}{s}$ glasi: (1 bod)

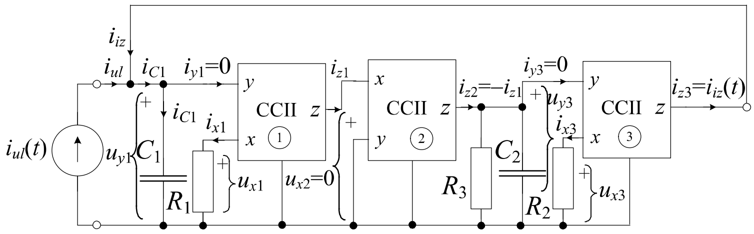
$$U_{iz}(s)=U_{ul}\cdot\frac{s}{s+\frac{R_1}{L_{ekv}}}=U_{ul}\cdot\frac{s}{s+\frac{10^3}{5}}=\frac{1}{s}\cdot\frac{s}{s+200}=\frac{1}{s+200}\Rightarrow \underline{u_{iz}(t)=e^{-200t}\cdot S(t)[\mathrm{V}]}$$

5. Za električni krug prikazan slikom izračunati valni oblik struje $i_{iz}(t)$ za $t>0$ kao odziv, ako je zadana pobuda $i_{ul}(t)=\delta(t)$[A]. Zadane su normalizirane vrijednosti elemenata $R_1=1$, $R_2=1$, $R_3=4$, $C_1=1$, $C_2=1/16$. Za strujni prijenosnik druge generacije (CCII) vrijede slijedeće definicijske jednadžbe: $u_x=u_y$, $i_y=0$, $i_z=i_x$ uz referentna usmjerenja struja i napona prilaza prikazana na slici.





Rješenje:



a) Za prvi CCII vrijedi: (1 bod)

$$U_{x1}=U_{y1}=U_{C1}=I_{C1}\cdot\frac{1}{sC_1},\; I_{y1}=0$$

$$I_{z1}=I_{x1}=\frac{U_{x1}}{R_1}=I_{C1}\frac{1}{sR_1C_1}\Rightarrow \boxed{I_{z1}=I_{C1}\frac{1}{sR_1C_1}}$$

b) Za drugi CCII vrijedi: (1 bod)

$$I_{z2}=I_{x2}=-I_{z1}\Rightarrow \boxed{I_{z2}=-I_{z1}=-I_{C1}\frac{1}{sR_1C_1}}$$

$$U_{x2}=0,\; U_{y2}=0$$

c) Za treći CCII vrijedi: (1 bod)

$$U_{y3}=I_{z2}\cdot\frac{R_3\cdot 1/(sC_2)}{R_3+1/(sC_2)}=I_{z2}\cdot\frac{R_3}{sR_3C_2+1},$$

$$I_{iz}=I_{z3}=I_{x3}=\frac{U_{x3}}{R_2}=\frac{U_{y3}}{R_2}=I_{z2}\cdot\frac{R_3}{R_2}\cdot\frac{1}{sR_3C_2+1}\Rightarrow \boxed{I_{z3}=I_{z2}\cdot\frac{R_3}{R_2}\cdot\frac{1}{sR_3C_2+1}}$$

d) Nakon sređivanja do sada napisanih izraza: (1 bod)

$$I_{C1}=I_{ul}+I_{iz};\quad I_{iz}=I_{z3}=-(I_{ul}+I_{iz})\cdot\frac{1}{sR_1C_1}\cdot\frac{R_3}{R_2}\cdot\frac{1}{sR_3C_2+1}$$

$$I_{iz}\cdot\left[1+\frac{R_3}{sR_1C_1\cdot R_2\cdot(sR_3C_2+1)}\right]=-I_{ul}\cdot\frac{R_3}{sR_1C_1\cdot R_2\cdot(sR_3C_2+1)}$$

$$I_{iz}=-I_{ul}\cdot\frac{R_3}{s^2R_1R_2R_3C_1C_2+sR_1R_2C_1+R_3}=-I_{ul}\cdot\frac{1}{s^2R_1R_2C_1C_2+sR_1R_2C_1/R_3+1}$$

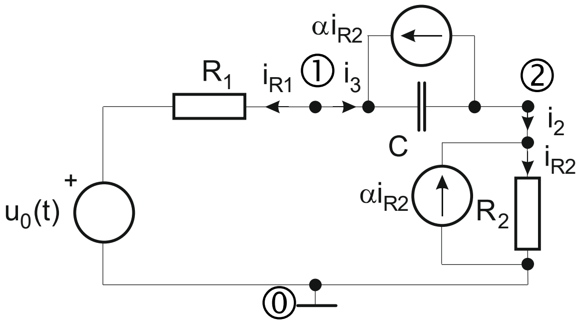
e) Uz uvrštene vrijednosti elemenata: (1 bod)

$$I_{iz}(s)=-I_{ul}\cdot\frac{16}{s^2+4s+16}=-1\cdot\frac{16}{(s+2)^2+12}=-\frac{16}{\sqrt{12}}\cdot\frac{\sqrt{12}}{(s+2)^2+(\sqrt{12})^2}$$

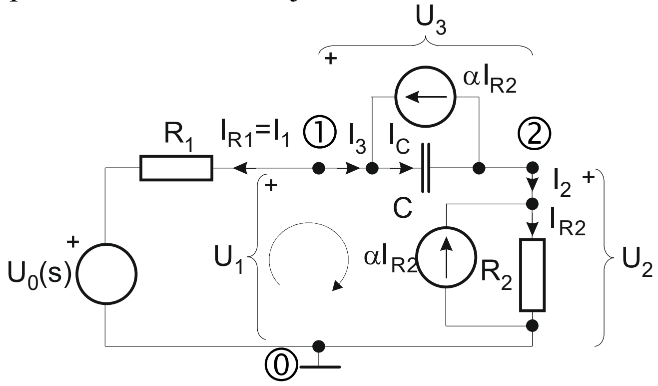
$$\Rightarrow i_{iz}(t)=-\frac{8\sqrt{3}}{3}\cdot e^{-2t}\cdot\sin(2\sqrt{3})\cdot S(t)[A]$$

MEĐUISPIT IZ ELEKTRIČNIH KRUGOVA 2016-2017 – Rješenja

1. Poštujući oznake čvorova i grana za električni krug na slici, napisati jednadžbe Kirchhoffovih zakona i izračunati odziv $i_{R1}(t)$ na poticaj $u_0(t)=S(t)$. Zadane su vrijednosti normaliziranih elemenata: $R_1=1$, $R_2=1$, $C=1$, te $\alpha=1/2$.



Rješenje: Primjena Laplaceove transformacije



a) Kirchhoffovi zakoni: (1 bod)

1) $I_1+I_3=0$; KZS

2) $-I_3+I_2=0$; KZS

$$\Rightarrow I_3=-I_1=I_2$$

3) $-U_1+U_2+U_3=0$; KZN

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b) Naponsko strujne relacije grana: (1 bod)

1) $U_1=I_1R_1+U_0$;

2) $U_2=I_2R_2+\alpha\cdot I_{R2}R_2$; $I_{R2}=\dfrac{U_2}{R_2}$;

3) $U_3=I_3\dfrac{1}{sC}+\alpha\cdot I_{R2}\dfrac{1}{sC}$;

$$U_2=I_2R_2+\alpha\cdot U_2 \Rightarrow U_2(1-\alpha)=I_2R_2 \Rightarrow U_2=\frac{R_2}{1-\alpha}\cdot I_2$$

$$U_3=I_3\frac{1}{sC}+\alpha\cdot\frac{1}{sR_2C}\cdot U_2 \Rightarrow U_3=I_3\frac{1}{sC}+\alpha\cdot\frac{1}{sR_2C}\cdot\frac{R_2}{1-\alpha}\cdot I_2$$

---

1) $U_1 = I_1 R_1 + U_0$;

2) $U_2 = \frac{R_2}{1-\alpha} \cdot I_2$;

3) $U_3 = \frac{\alpha}{1-\alpha} \cdot \frac{1}{sC} \cdot I_2 + \frac{1}{sC} \cdot I_3$

(1 bod)

Uvrstimo jedno u drugo:

$$-U_1 + U_2 + U_3 = 0,\ I_3 = -I_1 = I_2 \Rightarrow$$

$$-I_1 R_1 - U_0 + \frac{R_2}{1-\alpha} \cdot I_2 + \frac{\alpha}{1-\alpha} \cdot \frac{1}{sC} \cdot I_2 + \frac{1}{sC} \cdot I_3 = 0$$

$$-I_1 R_1 - U_0 - \frac{R_2}{1-\alpha} \cdot I_1 - \frac{\alpha}{1-\alpha} \cdot \frac{1}{sC} \cdot I_1 - \frac{1}{sC} \cdot I_1 = 0$$

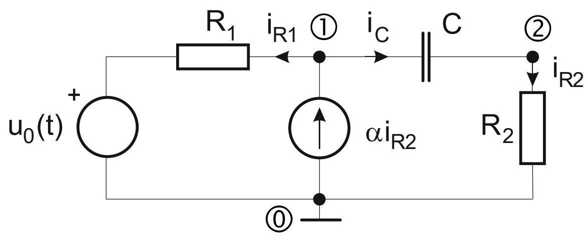
$$I_1 R_1 + \frac{R_2}{1-\alpha} \cdot I_1 + \left(\frac{\alpha}{1-\alpha} + 1\right) \frac{1}{sC} \cdot I_1 = -U_0$$

$$\left(R_1 + \frac{R_2}{1-\alpha} + \frac{1}{1-\alpha} \frac{1}{sC}\right) \cdot I_1 = -U_0$$

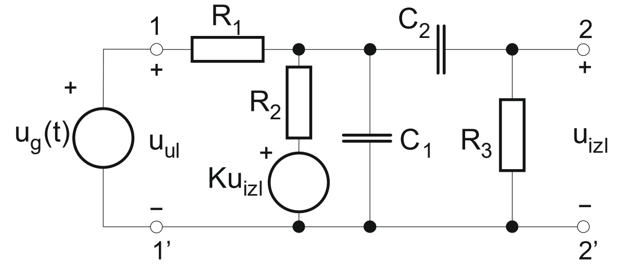
$$\Rightarrow I_1 = \frac{-U_0}{R_1 + \frac{1}{1-\alpha}\left(R_2 + \frac{1}{sC}\right)} = \frac{-\frac{1}{s}}{1 + \left(1 + \frac{1}{s}\right)2} = \frac{-\frac{1}{s}}{3 + \frac{2}{s}} = \frac{-1}{3s+2} = -\frac{1}{3} \cdot \frac{1}{s + \frac{2}{3}} \quad \text{(1 bod)}$$

$$\Rightarrow \underline{i_1(t) = -\frac{1}{3} \cdot e^{-\frac{2}{3}t} S(t)} \text{ (1 bod)}$$

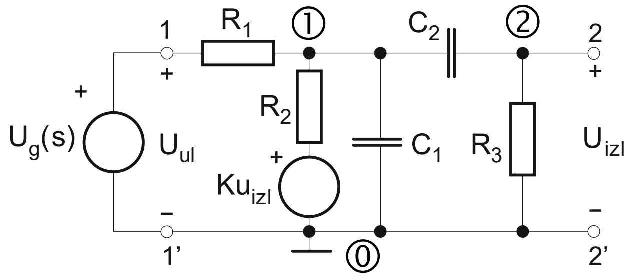
Napomena: Moguće je nacrtati ekvivalentni električni krug koji ima samo jedan strujno ovisni strujni izvor $\alpha\, i_{R2}$. (Izvršeno je posmicanje strujnog izvora.)



4. Odrediti funkciju napona na izlazu u frekvencijskoj $U_{izl}(s)$ i vremenskoj $u_{izl}(t)$ domeni za električni krug prikazan slikom. Koristiti metodu napona čvorišta. Zadano je: napon generatora na ulazu $u_g(t)=S(t)$, i normirane vrijednosti elemenata $R_1=R_2=2$, $R_3=1$, $C_1=C_2=1$ i $K=2$.



Rješenje: Metoda napona čvorišta i Laplaceova transformacija



1) $U_1\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_1+sC_2\right)-U_2sC_2=\frac{U_g(s)}{R_1}+K\frac{U_{iz}(s)}{R_2}$; (1 bod)

2) $-U_1sC_2+U_2\left(sC_2+\frac{1}{R_3}\right)=0$; (1 bod)

$$2)\Rightarrow U_1=U_2\left(1+\frac{1}{sC_2R_3}\right)\rightarrow 1)$$

$$U_2(s)\left[\left(1+\frac{1}{sC_2R_3}\right)\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_1+sC_2\right)-sC_2-\frac{K}{R_2}\right]=\frac{U_g(s)}{R_1}$$

$$U_2\left[\frac{1}{R_1}+\frac{1}{R_2}+sC_1+sC_2+\frac{1}{R_1R_3sC_2}+\frac{1}{R_2R_3sC_2}+\frac{sC_1}{sC_2R_3}+\frac{sC_2}{sC_2R_3}-sC_2-\frac{K}{R_2}\right]=\frac{U_g}{R_1}\Big/\cdot R_1R_2R_3sC_2$$

$$U_2\left[s^2C_1R_1R_2R_3C_2+sR_2R_3C_2+sR_1R_2(C_1+C_2)+s(1-K)R_1R_3C_2+R_1+R_2\right]=U_gR_2R_3sC_2$$

(1 bod račun)

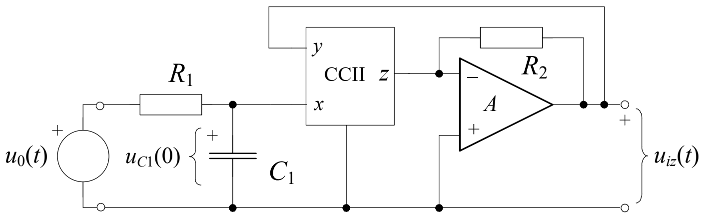
$$U_2(s)=U_g(s)\cdot\frac{R_2R_3sC_2}{s^2C_1R_1R_2R_3C_2+s\left[R_2R_3C_2+R_1R_2(C_1+C_2)+(1-K)R_1R_3C_2\right]+R_1+R_2}$$

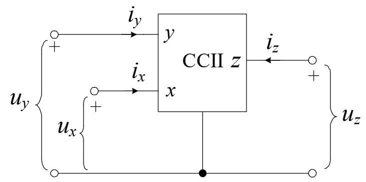
$$U_2(s)=\frac{2s}{4s^2+s[2+8-2]+4}\cdot U_g(s)=\frac{1}{4}\cdot\frac{2s}{s^2+2s+1}\cdot U_g(s)$$

$U_{iz}(s)=U_2(s)=\frac{1}{4}\cdot\frac{2s}{s^2+2s+1}\cdot\frac{1}{s}=\frac{1}{2}\cdot\frac{1}{(s+1)^2}$ (1 bod)

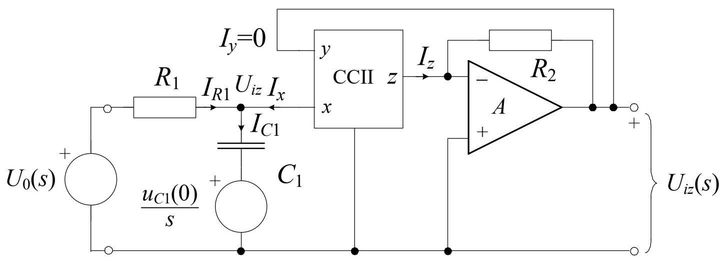
$u_{iz}(t)=\frac{1}{2}t\cdot e^{-t}\cdot S(t)$ (1 bod)

5. Za električni krug prikazan slikom izračunati valni oblik napona na izlazu $u_{iz}(t)$ za $t>0$ kao odziv, ako je zadana pobuda $u_0(t)=S(t)$ te početni napon na kapacitetu $u_{C1}(0)=1$. Zadane su normalizirane vrijednosti elemenata $R_1=1$, $R_2=1$, $C_1=1$. Operacijsko pojačalo je idealno, a za strujni prijenosnik druge generacije (CCII) vrijede slijedeće definicijske jednadžbe: $u_x=u_y$, $i_y=0$, $i_z=i_x$ uz referentna usmjerenja struja i napona prilaza prikazana na slici.





Rješenje: Laplaceova transformacija



Za CCII vrijedi:

$$I_{C1}=I_{R1}+I_x \;\Rightarrow\; I_x=I_{C1}-I_{R1}=I_z \text{ (1 bod)}$$

$$U_x=U_y=U_{iz}$$

$$I_{C1}=\left[U_{iz}-\frac{u_C(0)}{s}\right]\cdot sC_1,\; I_y=0$$

$$I_{R1}=\frac{U_0-U_{iz}}{R_1}$$

$$U_{iz}=-I_zR_2$$

$$U_{iz}=-(I_{C1}-I_{R1})R_2=-\left(U_{iz}sC_1-\frac{U_0-U_{iz}}{R_1}\right)R_2+C_1u_{C1}(0)R_2$$

$$U_{iz}=-U_{iz}sC_1R_2+\frac{U_0}{R_1}R_2-\frac{U_{iz}}{R_1}R_2+C_1u_{C1}(0)R_2$$

$$U_{iz}+U_{iz}sC_1R_2+U_{iz}\frac{R_2}{R_1}=U_0\frac{R_2}{R_1}+R_2C_1u_{C1}(0)$$

$$U_{iz}\left(1+sC_1R_2+\frac{R_2}{R_1}\right)=U_0\frac{R_2}{R_1}+R_2C_1u_{C1}(0)$$

$$U_{iz}(s)=\frac{\frac{R_2}{R_1}}{\left(1+sC_1R_2+\frac{R_2}{R_1}\right)}\cdot U_0+\frac{R_2C_1}{\left(1+sC_1R_2+\frac{R_2}{R_1}\right)}\cdot u_{C1}(0) \text{ (1 bod)}$$

Uz uvrštene vrijednosti elemenata

$$U_{iz}(s)=\frac{1}{(s+2)}\cdot\frac{1}{s}+\frac{1}{(s+2)} \text{ (1 bod)}$$

Rastav na parcijalne razlomke

$$\frac{1}{(s+2)}\cdot\frac{1}{s}=\frac{A}{s+2}+\frac{B}{s}=\frac{As+B(s+2)}{(s+2)\cdot s}=\frac{(A+B)s+2B}{(s+2)\cdot s}=\frac{1}{(s+2)\cdot s}$$

$$\begin{array}{l} A+B=0 \\ \underline{2B=1} \end{array} \Rightarrow \begin{array}{l} A=-B=-1/2 \\ \underline{2B=1 \Rightarrow B=1/2} \end{array}$$

$$U_{iz}(s)=\frac{1}{2}\left(\frac{-1}{s+2}+\frac{1}{s}\right)+\frac{1}{s+2}=\frac{1}{2}\left(\frac{1}{s+2}+\frac{1}{s}\right)$$ (1 bod)

$$\Rightarrow \underline{u_{iz}(t)=\frac{1}{2}\left(e^{-2t}+1\right)\cdot S(t)}$$ (1 bod)

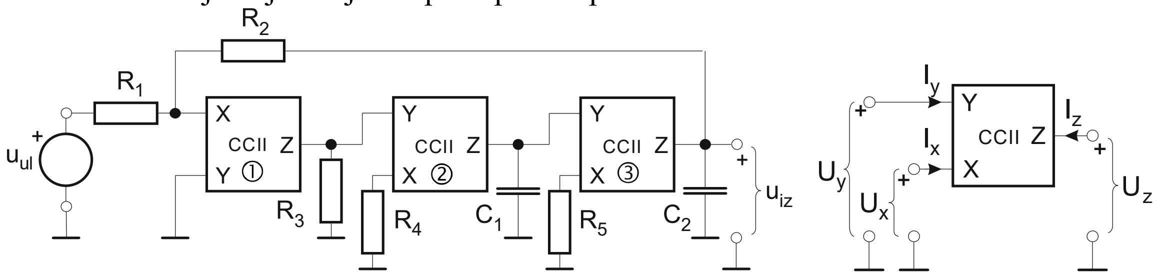
Ili imamo slijedeću mogućnost:

$$U_{iz}(s)=\frac{1}{(s+2)}\cdot\frac{1}{s}+\frac{1}{(s+2)}=\frac{1+s}{s(s+2)}=\frac{1+s}{s^2+2s}=\frac{1+s}{s^2+2s+1-1}=\frac{s+1}{(s+1)^2-1}$$

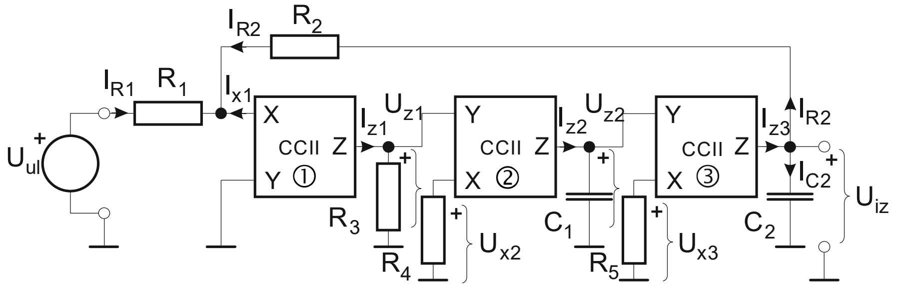
$$\underline{u_{iz}(t)=e^{-t}\cdot ch(t)\cdot S(t)=e^{-t}\cdot\frac{e^{-t}+e^{t}}{2}\cdot S(t)=\frac{1}{2}\left(e^{-2t}+1\right)\cdot S(t)}$$

ZAVRŠNI ISPIT IZ ELEKTRIČNIH KRUGOVA 2016-2017 – Rješenja

1. Za električni krug prikazan slikom izračunati valni oblik napona $u_{iz}(t)$ za $t>0$ kao odziv, ako je zadana pobuda $u_{ul}(t)=\delta(t)$[V], a početni naponi na kapacitetima su jednaki nula. Zadane su normalizirane vrijednosti elemenata $R_1=1$, $R_2=1$, $R_3=1$, $R_4=1$, $R_5=1$, $C_1=1$, $C_2=1$. Za strujni prijenosnik druge generacije (CCII) vrijede slijedeće definicijske jednadžbe: $u_x=u_y$, $i_y=0$, $i_z=i_x$ uz referentna usmjerenja struja i napona prilaza prikazana na slici.



Rješenje:



a) Za prvi CCII vrijedi: **(1 bod)**

$$U_{z1}=I_{z1}\cdot R_3,\; I_{y1}=0,\; U_{x1}=0,\; U_{y1}=0,$$

$$I_{z1}=I_{x1}=-\left(I_{R1}+I_{R2}\right)=-\left(\frac{U_{ul}}{R_1}+\frac{U_{iz}}{R_2}\right)\Rightarrow \boxed{U_{z1}=-U_{ul}\frac{R_3}{R_1}-U_{iz}\frac{R_3}{R_2}}$$

b) Za drugi CCII vrijedi: **(1 bod)**

$$U_{z2}=I_{z2}\cdot\frac{1}{sC_1},\; I_{y2}=0,\; U_{x2}=U_{y2}=U_{z1},\; I_{z2}=I_{x2}=\frac{U_{x2}}{R_4}\Rightarrow \boxed{U_{z2}=U_{z1}\frac{1}{sR_1C_1}}$$

c) Za treći CCII vrijedi: **(1 bod)**

$$I_{z3}=I_{C2}+I_{R2}=U_{iz}sC_2+\frac{U_{iz}}{R_2}=U_{iz}\left(sC_2+\frac{1}{R_2}\right)\Rightarrow U_{iz}=\frac{I_{z3}}{sC_2+1/R_2}$$

$$I_{z3}=I_{x3}=\frac{U_{x3}}{R_5}=\frac{U_{y3}}{R_5}=\frac{U_{z2}}{R_5}\Rightarrow \boxed{U_{iz}=\frac{1}{R_5}\cdot\frac{1}{sC_2+1/R_2}\cdot U_{z2}}$$

d) Nakon sređivanja do sada napisanih izraza: **(1 bod)**

$$U_{iz}=-\frac{1}{R_5}\cdot\frac{1}{sC_2+1/R_2}\cdot\frac{1}{sR_1C_1}\cdot\left(U_{ul}\frac{R_3}{R_1}+U_{iz}\frac{R_3}{R_2}\right)$$

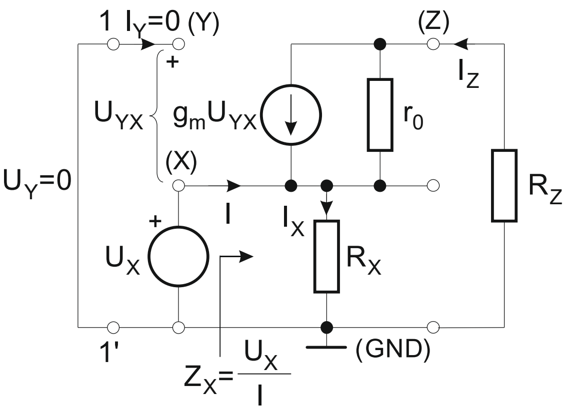
e) Uz uvrštene vrijednosti elemenata: **(1 bod)**

$$U_{iz}=-\frac{1}{s+1}\cdot\frac{1}{s}\cdot\left(U_{ul}+U_{iz}\right)\Rightarrow U_{iz}\left(1+\frac{1}{s+1}\cdot\frac{1}{s}\right)=-\frac{1}{s+1}\cdot\frac{1}{s}\cdot U_{ul}\Rightarrow$$

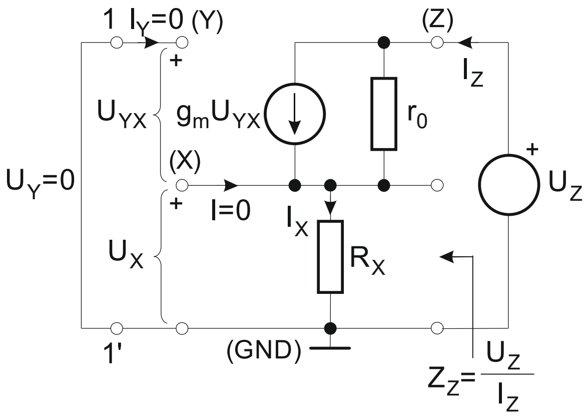
$$U_{iz}\frac{s^2+s+1}{s(s+1)}=-\frac{1}{s(s+1)}U_{ul}\Rightarrow U_{iz}(s)=-\frac{1}{s^2+s+1}U_{ul}(s);\; U_{ul}(s)=1\Rightarrow$$

$$U_{iz}(s)=-\frac{1}{s^2+s+1}=-\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}/2}{(s+1/2)^2+\left(\sqrt{3}/2\right)^2}\Rightarrow \underline{u_{iz}(t)=-\frac{2\sqrt{3}}{3}\cdot e^{-\frac{t}{2}}\cdot\sin\left(\frac{\sqrt{3}}{2}t\right)\cdot S(t)}$$

4. Za mrežu prikazanu slikom odrediti ulazni otpor na priključnicama X i GND $Z_X=U_X/I$ (slika a) i ulazni otpor na priključnicama Z i GND (slika b) $Z_Z=U_Z/I_Z$. Zadano je: $g_m=2\cdot10^{-4}$ [A/V], $r_0=2\cdot10^5[\Omega]$, $R_X=10^4[\Omega]$, $R_Z=10^2[\Omega]$.



Slika a)



Slika b)

Rješenje:

a) Ulazni otpor: $Z_X=\frac{U_X}{I}$ (slika a)

(1) $I_Z(r_0+R_Z)-g_mU_{YX}r_0+U_X=0$;

(2) $U_X=I_XR_X$, $U_{YX}=-U_X$, $I_X=I_Z+I$, $I_Y=0$ **(2 boda)**

(1) $\Rightarrow$ $I_Z=\frac{g_mU_{YX}r_0-U_X}{r_0+R_Z}=-U_X\frac{1+g_mr_0}{r_0+R_Z}$ $\Rightarrow$

$$U_X=I_XR_X=(I+I_Z)R_X=\left(I-U_X\frac{1+g_mr_0}{r_0+R_Z}\right)R_X$$

$$U_X\left(1+\frac{1+g_mr_0}{r_0+R_Z}R_X\right)=IR_X \Rightarrow Z_X=\frac{U_X}{I}=\frac{R_X}{1+\frac{1+g_mr_0}{r_0+R_Z}R_X}=\frac{1}{\frac{1}{R_X}+\frac{1+g_mr_0}{r_0+R_Z}}=3279{,}79[\Omega]$$

(1 bod)

b) Ulazni otpor: $Z_Z=\frac{U_Z}{I_Z}$ (slika b)

(1) $U_Z=(I_Z-g_mU_{YX})r_0+I_ZR_X$;

(2) $U_X=I_XR_X$, $U_{YX}=-U_X$, $I_X=I_Z$, $I_Y=0$, $I=0$

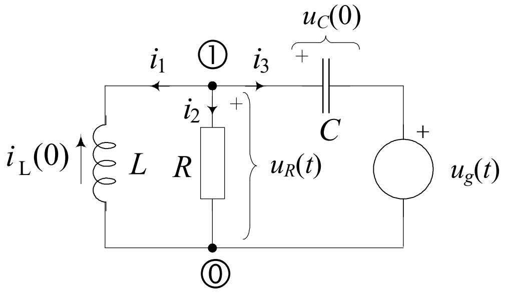
(1) $\Rightarrow$ $U_Z=I_Zr_0+g_mr_0R_XI_Z+I_ZR_X$

$$U_Z=I_Z[r_0+(1+g_mr_0)R_X] \Rightarrow R_Z=\frac{U_Z}{I_Z}=r_0+(1+g_mr_0)R_X=610[\text{k}\Omega]$$

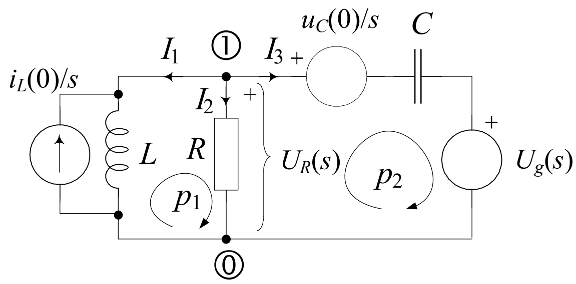
(2 boda)

MEĐUISPIT IZ ELEKTRIČNIH KRUGOVA 2017-2018 – Rješenja

1. Poštujući oznake čvorova i grana za električni krug na slici, primjenom Kirchhoffovih zakona izračunati valni oblik napona $u_R(t)$ kao odziv, ako je zadana pobuda $u_g(t)=\delta(t)$. Zadane su normalizirane vrijednosti elemenata C=1, R=1, L=1/2, te početni uvjeti $u_C(0)$=8 i $i_L(0)$=2.



<u>Rješenje:</u> Primjena Laplaceove transformacije



Mreža ima N_b=3 grane i N_v=2 čvora

Jednadžbe KZN

$$(\text{p1})\ \ -U_1(s)+U_2(s)=0$$
$$(\text{p2})\ \ -U_2(s)+U_3(s)=0$$

Jednadžbe KZS

$$(\text{č1})\ \ I_1(s)+I_2(s)+I_3(s)=0$$

——————————————————————— (1 bod)

Naponsko-strujne relacije grana

$$(\text{g1})\ \ U_1(s)=sL\cdot\left[I_1(s)+\frac{i_L(0)}{s}\right]\Big/:sL$$
$$(\text{g2})\ \ U_2(s)=R\cdot I_2(s)$$
$$(\text{g3})\ \ U_3(s)=\frac{1}{sC}I_3(s)+\frac{u_C(0)}{s}+U_g(s)\Big/\cdot sC$$

——————————————————————— (1 bod)

$$(\text{g1})\Rightarrow\ I_1(s)=\frac{U_1(s)}{sL}-\frac{i_L(0)}{s}$$
$$(\text{g2})\Rightarrow\ I_2(s)=U_2(s)\frac{1}{R}$$
$$(\text{g3})\Rightarrow\ I_3(s)=sCU_3(s)-Cu_C(0)-sCU_g(s)$$

$$(\text{č1})\Rightarrow\ \frac{U_1(s)}{sL}-\frac{i_L(0)}{s}+U_2(s)\frac{1}{R}+sCU_3(s)-Cu_C(0)-sCU_g(s)=0$$
$$(\text{p1}),\ (\text{p2})\Rightarrow\ U_1(s)=U_2(s)=U_3(s),\ U_R(s)=U_2(s)$$

$$\Rightarrow U_R(s)\left(\frac{1}{sL}+\frac{1}{R}+sC\right)=sCU_g(s)+\frac{i_L(0)}{s}+Cu_C(0)$$

$$\Rightarrow U_R(s)=\frac{sCU_g(s)+\frac{i_L(0)}{s}+Cu_C(0)}{\frac{1}{sL}+\frac{1}{R}+sC}=\frac{s+\frac{2}{s}+8}{s+1+\frac{2}{s}}=\frac{s^2+8s+2}{s^2+s+2}=1+\frac{7s}{s^2+s+2}$$ (2 boda)

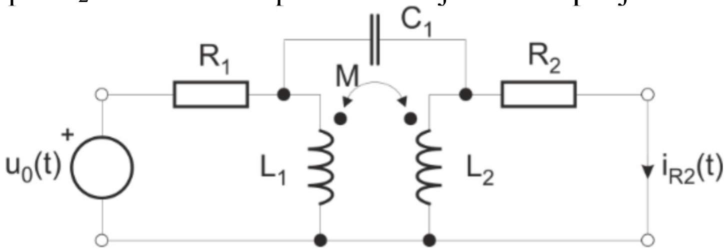
$$s^2+s+2=0\Rightarrow s_{1,2}=-\frac{1}{2}\pm\sqrt{\frac{1}{4}-2}=-\frac{1}{2}\pm j\frac{\sqrt{7}}{2}$$

$$U_R(s)=1+7\frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{7}{4}}=1+7\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{7}{4}}-\frac{7}{2}\frac{\frac{2}{\sqrt{7}}\frac{\sqrt{7}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{7}{4}}$$

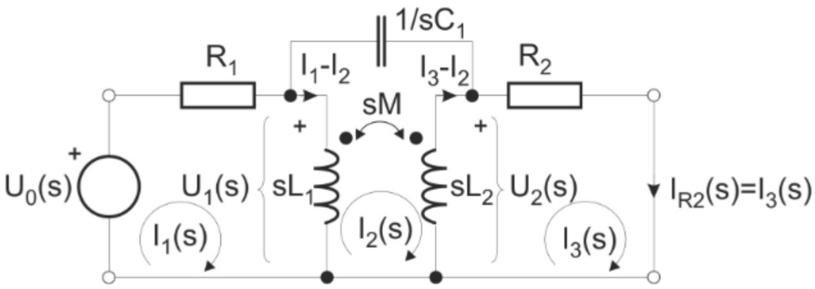
$$U_R(s)=1+7\cdot\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}-\sqrt{7}\cdot\frac{\frac{\sqrt{7}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}$$

$$\Rightarrow \underline{u_R(t)=\delta(t)+e^{-\frac{t}{2}}\cdot\left[7\cdot\cos\left(\frac{\sqrt{7}}{2}t\right)-\sqrt{7}\cdot\sin\left(\frac{\sqrt{7}}{2}t\right)\right]\cdot S(t)}$$ (1 bod)

2. Za električni krug na slici zadane su normalizirane vrijednosti elemenata $R_1=R_2=1$, $L_1=1$, $L_2=2$, $C_1=1$, $M=1$, početne struje kroz induktivitete i napon na kapacitetu su jednaki nuli, te pobuda $u_0(t)=\delta(t)$. Primjenom Laplaceove transformacije i koristeći metodu petlji izračunati struju $I_{R2}(s)$ kroz otpor R_2 kao odziv. Napisati sustav jednadžbi petlji u matričnom obliku.



Rješenje: Primjena Laplaceove transformacije



Metoda petlji:

1) $-U_0(s)+I_1(s)R_1+U_1(s)=0$;

2) $-U_1(s)+I_2(s)\dfrac{1}{sC_1}+U_2(s)=0$;

3) $-U_2(s)+I_3(s)R_2=0$; (1 bod)

---

1) $U_1=(I_1-I_2)sL_1+(I_2-I_3)sM$;

2) $U_2=(I_2-I_3)sL_2+(I_1-I_2)sM$; (1 bod)

Uvrštavanje jedno u drugo:

1) $-U_0(s)+I_1R_1+(I_1-I_2)sL_1+(I_2-I_3)sM=0$

2) $-\left[(I_1-I_2)sL_1+(I_2-I_3)sM\right]+I_2\dfrac{1}{sC_1}+\left[(I_2-I_3)sL_2+(I_1-I_2)sM\right]=0$

3) $-\left[(I_2-I_3)sL_2+(I_1-I_2)sM\right]+I_3R_2=0$

Nakon sređivanja:

1) $I_1(s)\left(R_1+sL_1\right)-I_2(s)(sL_1-sM)-I_3(s)sM=U_0(s)$

2) $-I_1(s)\left(sL_1-sM\right)+I_2(s)\left(\dfrac{1}{sC_1}+sL_1+sL_2-2sM\right)-I_3(s)(sL_2-sM)=0$

3) $-I_1(s)sM-I_2(s)\left(sL_2-sM\right)+I_3(s)\left(R_2+sL_2\right)=0$

$$\begin{bmatrix} R_1+sL_1 & -(sL_1-sM) & -sM \\ -(sL_1-sM) & \dfrac{1}{sC_1}+sL_1+sL_2-2sM & -(sL_2-sM) \\ -sM & -(sL_2-sM) & R_2+sL_2 \end{bmatrix}\begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \end{bmatrix}=\begin{bmatrix} U_0(s) \\ 0 \\ 0 \end{bmatrix}$$ (1 bod)

Uvrstimo elemente: R_1=R_2=1, L_1=1, L_2=2, C_1=1, M=1,

$$\begin{bmatrix} 1+s & 0 & -s \\ 0 & \dfrac{1}{s}+s & -s \\ -s & -s & 1+2s \end{bmatrix}\begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \end{bmatrix}=\begin{bmatrix} U_0(s) \\ 0 \\ 0 \end{bmatrix}$$

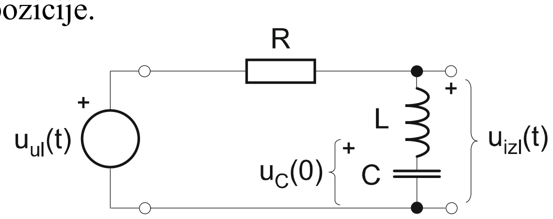
$$\Delta=\begin{vmatrix} 1+s & 0 & -s \\ 0 & \dfrac{1}{s}+s & -s \\ -s & -s & 1+2s \end{vmatrix}=(1+s)\begin{vmatrix} \dfrac{1}{s}+s & -s \\ -s & 1+2s \end{vmatrix}-s\begin{vmatrix} 0 & -s \\ \dfrac{1}{s}+s & -s \end{vmatrix}$$

$$\Delta=(1+s)\left[\left(\frac{1}{s}+s\right)(1+2s)-s^2\right]-s^2\left(\frac{1}{s}+s\right)=2s^2+2s+3+\frac{1}{s}$$

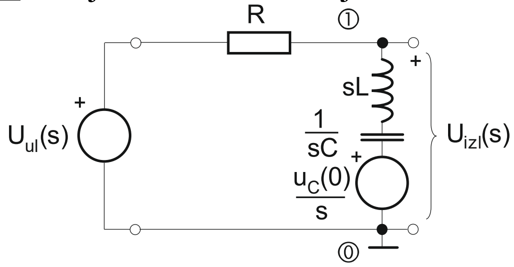
$$\Delta_3=\begin{vmatrix} 1+s & 0 & U_0 \\ 0 & \dfrac{1}{s}+s & 0 \\ -s & -s & 0 \end{vmatrix}=U_0\begin{vmatrix} 0 & \dfrac{1}{s}+s \\ -s & -s \end{vmatrix}=U_0\left(1+s^2\right)$$

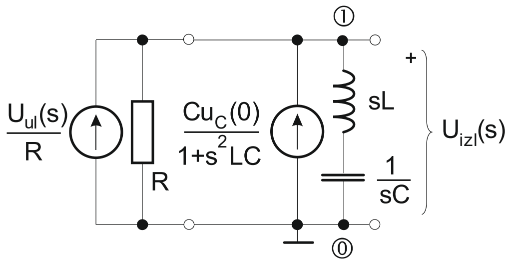
$$I_{R2}(s)=I_3(s)=\frac{\Delta_3}{\Delta}=\frac{U_0\left(1+s^2\right)}{2s^2+2s+3+\dfrac{1}{s}}=U_0\cdot\frac{s\left(s^2+1\right)}{2s^3+2s^2+3s+1}\,;\ U_0=1\,;$$ (2 boda)

3. Odrediti i skicirati valni oblik napona $u_{iz}(t)$ ako je zadano: $u_{ul}(t) = E \cdot S(t)$, E=5, R=1, L=1, C=1, te početni uvjeti $u_C(0)$=10V, $i_L(0)$=0A. Zadatak riješiti metodom napona čvorišta i primijeniti teorem superpozicije.



<u>Rješenje:</u> Primjena L-transformacije i metoda napona čvorišta.

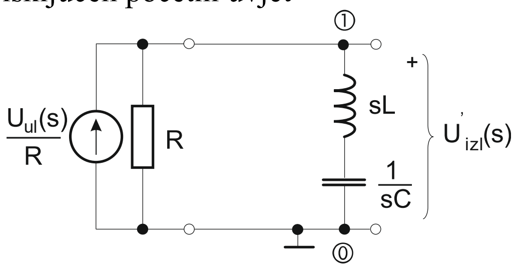




$E = 5[\mathrm{V}]$ ○—○ $U_{ul}(s) = \frac{E}{s} = \frac{5}{s}$; $I_{ul}(s) = \frac{U_{ul}(s)}{R} = \frac{E}{s \cdot R} = \frac{5}{s}$

Metoda superpozicije:

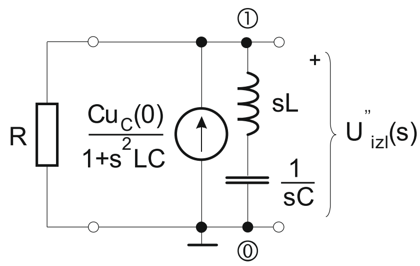
a) uključen neovisni izvor, isključen početni uvjet



$$U'_{izl}(s)\left(\frac{1}{R}+\frac{1}{sL+\frac{1}{sC}}\right)=\frac{U_{ul}(s)}{R} \Rightarrow U'_{izl}(s)=\frac{U_{ul}(s)}{R\left(\frac{1}{R}+\frac{sC}{s^2LC+1}\right)}=\frac{U_{ul}(s)}{1+\frac{sRC}{s^2LC+1}}$$

$$U'_{izl}(s)=U_{ul}(s)\frac{\left(s^2LC+1\right)}{s^2LC+sRC+1}=\frac{5}{s}\cdot\frac{\left(s^2+1\right)}{s^2+s+1}=5\cdot\frac{s}{s^2+s+1}+\frac{5}{s}\cdot\frac{1}{s^2+s+1}$$ (1 bod)

b) uključen početni uvjet, isključen neovisni izvor



$$U'_{izl}(s)\left(\frac{1}{R}+\frac{1}{sL+\frac{1}{sC}}\right)=\frac{Cu_C(0)}{1+s^2LC} \Rightarrow U''_{izl}(s)=\frac{Cu_C(0)}{\left(1+s^2LC\right)\left(\frac{1}{R}+\frac{sC}{s^2LC+1}\right)}$$

$$U''_{izl}(s)=\frac{RCu_C(0)}{s^2LC+sRC+1}=\frac{10}{s^2+s+1}$$ (1 bod)

c) zbroj dvaju doprinosa

$$U_{izl}(s) = U'_{izl}(s) + U''_{izl}(s) = 5 \cdot \frac{s}{s^2+s+1} + \frac{5}{s} \cdot \frac{1}{s^2+s+1} + \frac{10}{s^2+s+1} \quad \text{(1 bod)}$$

Rastav na parcijalne razlomke:

$$\frac{5}{s} \cdot \frac{1}{s^2+s+1} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1} = \frac{A(s^2+s+1)+s(Bs+C)}{s(s^2+s+1)} = \frac{(A+B)s^2+(A+C)s+A}{s(s^2+s+1)}$$

$A = 5;$

$A + B = 0;$

$A + C = 0.$

$A = 5; B = -A = -5; C = -A = -5$

$$\frac{5}{s} \cdot \frac{1}{s^2+s+1} = \frac{5}{s} + \frac{-5s-5}{s^2+s+1} = \frac{5}{s} - 5\frac{s}{s^2+s+1} - \frac{5}{s^2+s+1}$$

Konačno je

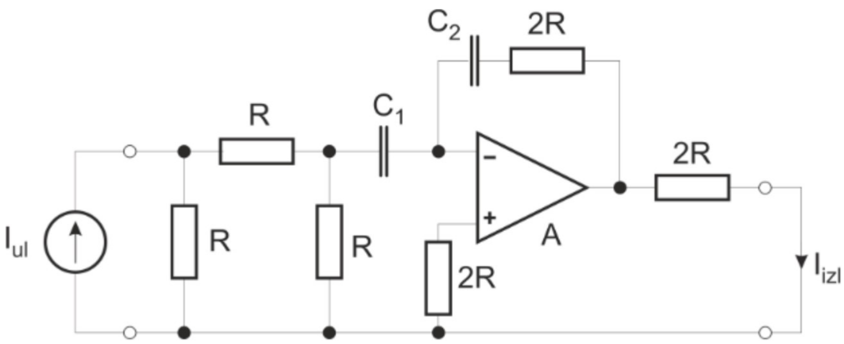
$$U_{izl}(s) = 5\frac{s}{s^2+s+1} + \frac{10}{s^2+s+1} + \frac{5}{s} - 5\frac{s}{s^2+s+1} - 5\frac{1}{s^2+s+1}$$

$$U_{izl}(s) = \frac{5}{s^2+s+1} + \frac{5}{s} \quad \text{(1 bod)}$$

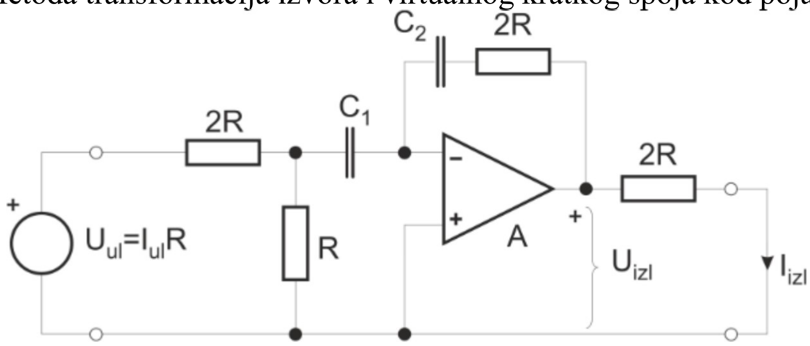
$$U_{izl}(s) = \frac{5}{s^2+s+\frac{1}{4}+\frac{3}{4}} + \frac{5}{s} = \frac{5}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} + \frac{5}{s} = 5\frac{2}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{5}{s}$$

$$u_{iz}(t) = \left[5 + \frac{10}{\sqrt{3}} e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t\right)\right] \cdot S(t) \quad \text{(1 bod)}$$

4. Odrediti funkciju struje na izlazu u frekvencijskoj $I_{iz}(s)$ i vremenskoj $i_{iz}(t)$ domeni za električni krug prikazan slikom, ako su zadane normirane vrijednosti elemenata R=1, C_1=1, C_2=1. Operacijsko pojačalo je idealno ($A\to\infty$). Valni oblik strujnog generatora na ulazu je $i_{ul}(t)=S(t)$. Primijeniti postupke transformacija izvora.

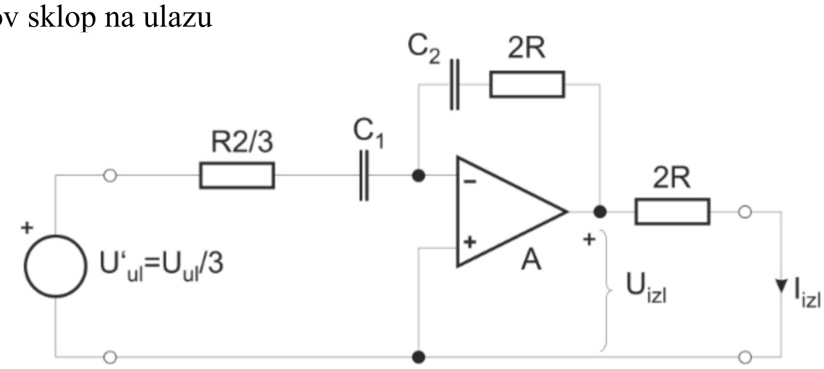


Rješenje: Metoda transformacija izvora i virtualnog kratkog spoja kod pojačala



(1 bod)

Theveninov sklop na ulazu



(1 bod)

Računamo napon U_{iz}, pa onda iz njega slijedi struja I_{iz}:

$$U_{iz}(s) = -\frac{2R+\frac{1}{sC_2}}{R\frac{2}{3}+\frac{1}{sC_1}}\cdot U'_{ul}(s)$$

$$U_{iz}(s) = -\frac{2+\frac{1}{s}}{\frac{2}{3}+\frac{1}{s}}\cdot\frac{U_{ul}(s)}{3} = -\frac{2+\frac{1}{s}}{2+\frac{3}{s}}\cdot U_{ul}(s) = -\frac{2s+1}{2s+3}\cdot U_{ul}(s) = -\frac{2s+1}{2s+3}\cdot I_{ul}(s)\cdot R;\quad R=1\Omega$$

$$I_{iz}(s) = \frac{U_{iz}(s)}{2R} = \frac{1}{2R}\cdot\left(-\frac{2s+1}{2s+3}\cdot I_{ul}(s)\cdot R\right);$$

$$I_{iz}(s) = -\frac{1}{2}\cdot\frac{2s+1}{2s+3}\cdot I_{ul}(s) = -\frac{1}{2}\cdot\frac{2s+1}{2s+3}\cdot\frac{1}{s}$$ (1 bod)

Rastav na parcijalne razlomke:

$$\frac{s+\frac{1}{2}}{s+\frac{3}{2}}\cdot\frac{1}{s}=\frac{A}{s+\frac{3}{2}}+\frac{B}{s}=\frac{As+B\left(s+\frac{3}{2}\right)}{s\left(s+\frac{3}{2}\right)}=\frac{(A+B)s+B\frac{3}{2}}{s\left(s+\frac{3}{2}\right)}$$

$$(A+B)=1$$

$$\underline{B\frac{3}{2}=\frac{1}{2}}\quad\Rightarrow\ \underline{B=\frac{1}{3};\ A=1-B=1-\frac{1}{3}=\frac{2}{3}}$$

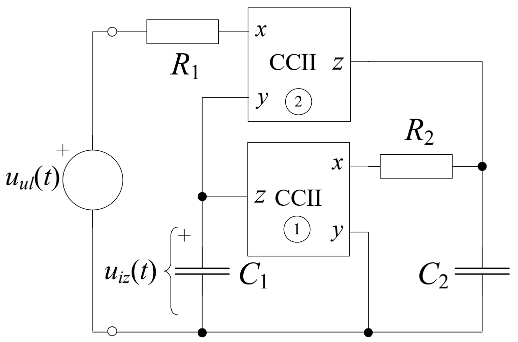
$$\frac{s+\frac{1}{2}}{s+\frac{3}{2}}\cdot\frac{1}{s}=\frac{A}{s+\frac{3}{2}}+\frac{B}{s}=\frac{\frac{2}{3}}{s+\frac{3}{2}}+\frac{\frac{1}{3}}{s}=\frac{1}{3}\left(\frac{2}{s+\frac{3}{2}}+\frac{1}{s}\right)$$ (1 bod)

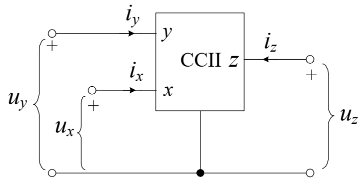
Tada je:

$$I_{iz}(s)=-\frac{1}{2}\cdot\frac{s+\frac{1}{2}}{s+\frac{3}{2}}\cdot\frac{1}{s}=-\frac{1}{2}\cdot\frac{1}{3}\left(\frac{2}{s+\frac{3}{2}}+\frac{1}{s}\right)=-\frac{1}{6}\cdot\left(\frac{2}{s+\frac{3}{2}}+\frac{1}{s}\right)$$

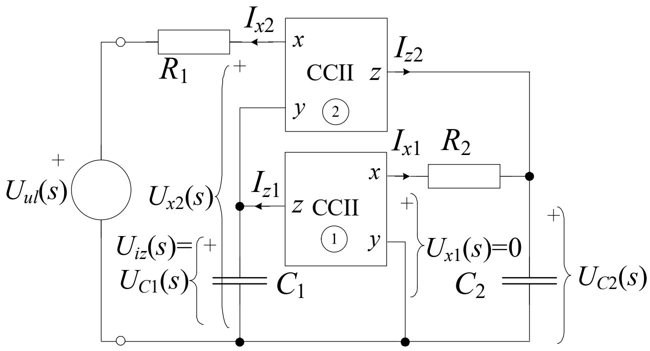
$$\Rightarrow\ \underline{i_{iz}(t)=-\frac{1}{6}\left(1+2e^{-3t/2}\right)\cdot S(t)}$$ (1 bod)

5. Za električni krug prikazan slikom izračunati valni oblik napona na izlazu $u_{iz}(t)$ za $t>0$ kao odziv, ako je zadana pobuda $u_{ul}(t)=S(t)$. Početni naponi na kapacitetima su jednaki nuli. Zadane su normalizirane vrijednosti elemenata $R_1=1$, $R_2=1$, $C_1=1$, $C_2=1$. Za strujni prijenosnik druge generacije (CCII) vrijede slijedeće definicijske jednadžbe: $u_x=u_y$, $i_y=0$, $i_z=i_x$ uz referentna usmjerenja struja i napona prilaza prikazana na slici.





Rješenje: Laplaceova transformacija



a) Za prvi CCII vrijedi: (1 bod)

$$U_{x1}=U_{y1}=0,\ I_{y1}=0$$

$$I_{z1}=I_{x1}=-\frac{U_{C2}}{R_2}$$

b) Za drugi CCII vrijedi: (1 bod)

$$I_{z2}=I_{x2}=\frac{U_{iz}-U_{ul}}{R_1},\ I_{y2}=0$$

$$U_{x2}=U_{y2}=U_{iz}$$

c) Također vrijedi:

$$U_{C2}=\left(I_{x1}+I_{z2}\right)\cdot\frac{1}{sC_2}$$

$$U_{iz}=U_{C1}=I_{z1}\cdot\frac{1}{sC_1}$$

d) Nakon sređivanja do sada napisanih izraza: (1 bod)

$$U_{C2}=\left(I_{x1}+I_{z2}\right)\cdot\frac{1}{sC_2}=\left(-\frac{U_{C2}}{R_2}+I_{z2}\right)\cdot\frac{1}{sC_2};\ U_{C2}\left(1+\frac{1}{sR_2C_2}\right)=I_{z2}\cdot\frac{1}{sC_2}$$

$$U_{C2}=I_{z2}\cdot\frac{R_2}{1+sR_2C_2}=I_{x2}\cdot\frac{R_2}{1+sR_2C_2}=\frac{U_{iz}-U_{ul}}{R_1}\cdot\frac{R_2}{1+sR_2C_2}$$

$$U_{iz} = U_{C1} = I_{z1} \cdot \frac{1}{sC_1} = I_{x1} \cdot \frac{1}{sC_1} = -\frac{U_{C2}}{R_2} \cdot \frac{1}{sC_1}$$

$$U_{iz} \cdot \left[1 + \frac{1}{sR_1C_1 \cdot (sR_2C_2 + 1)}\right] = U_{ul} \cdot \frac{1}{sR_1C_1 \cdot (sR_2C_2 + 1)}$$

$$U_{iz} = \frac{U_{ul}}{s^2 R_1R_2C_1C_2 + sR_1C_1 + 1}$$

e) Uz uvrštene vrijednosti elemenata:

$$U_{iz}(s) = \frac{1}{s} \cdot \frac{1}{s^2 + s + 1} \text{ (1 bod)}$$

Rastav na parcijalne razlomke

$$\frac{1}{s} \cdot \frac{1}{s^2 + s + 1} = \frac{A}{s} + \frac{Bs + C}{s^2 + s + 1} = \frac{A(s^2 + s + 1) + (Bs + C)s}{s(s^2 + s + 1)} = \frac{(A + B)s^2 (A + C)s + A}{s(s^2 + s + 1)}$$

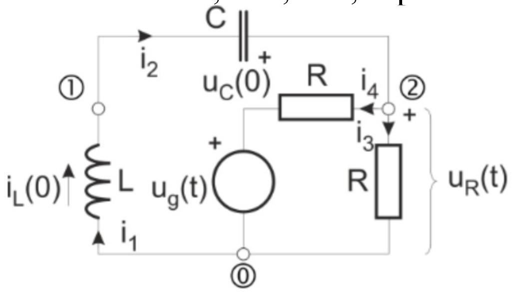
$$\begin{aligned} &A + B = 0 \\ &A + C = 0 \;\Rightarrow\; \underline{A = 1; \quad B = -A = -1; \quad C = -A = -1} \\ &\underline{A = 1} \end{aligned}$$

$$\frac{1}{s} \cdot \frac{1}{s^2 + s + 1} = \frac{1}{s} - \frac{s + 1}{s^2 + s + 1} = \frac{1}{s} - \frac{s + \frac{1}{2} + \frac{1}{2}}{s^2 + s + \frac{1}{4} + \frac{3}{4}} = \frac{1}{s} - \frac{\left(s + \frac{1}{2}\right)}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} \cdot \frac{\sqrt{3}}{\sqrt{3}}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

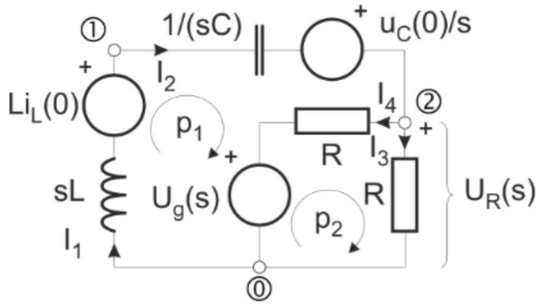
$$\Rightarrow \underline{u_{iz}(t) = \left\{1 - e^{-t/2}\left[\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]\right\} \cdot S(t)} \text{ (1 bod)}$$

MEĐUISPIT IZ ELEKTRIČNIH KRUGOVA 2018-2019 – Rješenja

1. Poštujući oznake čvorova i grana za električni krug na slici, primjenom Kirchhoffovih zakona izračunati valni oblik napona $u_R(t)$ kao odziv, ako je zadana pobuda $u_g(t)=\delta(t)$. Zadane su normalizirane vrijednosti elemenata $C=1$, $R=2$, $L=1$, te početni uvjeti $u_C(0)=1$ i $i_L(0)=1$.



Rješenje: Primjena Laplaceove transformacije



Mreža ima N_b=4 grane i N_v=3 čvora

Jednadžbe KZN (N_b–N_v+1=2)

$$\text{(p1)}\ U_1(s)+U_2(s)+U_4(s)=0$$

$$\text{(p2)}\ U_3(s)-U_4(s)=0 \ \Rightarrow\ U_3(s)=U_4(s)$$

Jednadžbe KZS (N_v–1=2)

$$\text{(č1)}\ -I_1(s)+I_2(s)=0 \ \Rightarrow\ I_1(s)=I_2(s)$$

$$\text{(č2)}\ -I_2(s)+I_3(s)+I_4(s)=0 \ \Rightarrow\ I_2(s)=I_3(s)+I_4(s) \ \Rightarrow\ I_4(s)=I_2(s)-I_3(s)$$

——————————————————— (1 bod)

Naponsko-strujne relacije grana (N_b=2)

$$\text{(g1)}\ U_1(s)=sL\cdot I_1(s)-Li_L(0)$$

$$\text{(g2)}\ U_2(s)=\frac{1}{sC}\cdot I_2(s)-\frac{u_C(0)}{s}$$

$$\text{(g3)}\ U_3(s)=R\cdot I_3(s)$$

$$\text{(g4)}\ U_4(s)=R\cdot I_4(s)+U_g(s)$$

——————————————————— (1 bod)

$$\text{(p1)}\Rightarrow\ sL\cdot I_1(s)-Li_L(0)+\frac{1}{sC}\cdot I_2(s)-\frac{u_C(0)}{s}+R\cdot I_4(s)+U_g(s)=0$$

$$\text{(p2)}\Rightarrow\ R\cdot I_3(s)=R\cdot I_4(s)+U_g(s)$$

$$\text{(č1), (č2), (p1)}\Rightarrow\ sL\cdot I_2(s)-Li_L(0)+\frac{1}{sC}\cdot I_2(s)-\frac{u_C(0)}{s}+\underbrace{R\cdot\left[I_2(s)-I_3(s)\right]+U_g(s)}_{R\cdot I_3(s)}=0$$

$$\text{(č1), (č2), (p2)}\Rightarrow\ R\cdot I_3(s)=R\cdot\left[I_2(s)-I_3(s)\right]+U_g(s)$$

$$1)\ \left(sL+\frac{1}{sC}\cdot\right)I_2(s)+R\cdot I_3(s)=Li_L(0)+\frac{u_C(0)}{s}$$

$$2)\ -R\cdot I_2(s)+(R+R)\cdot I_3(s)=U_g(s)$$

(1 bod)

$$2)\Rightarrow\ I_2(s)=\frac{2R\cdot I_3(s)-U_g(s)}{R}=2\cdot I_3(s)-\frac{U_g(s)}{R}\ \rightarrow 1)$$

$$\left(sL+\frac{1}{sC}\cdot\right)\left(2\cdot I_3(s)-\frac{U_g(s)}{R}\right)+R\cdot I_3(s)=Li_L(0)+\frac{u_C(0)}{s}$$

$$\left(sL+\frac{1}{sC}\right)\cdot 2\cdot I_3(s)+R\cdot I_3(s)=\left(sL+\frac{1}{sC}\right)\cdot\frac{U_g(s)}{R}+Li_L(0)+\frac{u_C(0)}{s}$$

$$I_3(s)=\frac{\left(sL+\frac{1}{sC}\right)\cdot\frac{U_g(s)}{R}+Li_L(0)+\frac{u_C(0)}{s}}{\left(sL+\frac{1}{sC}\right)\cdot 2+R}\quad \text{(1 bod)}$$

$$U_R(s)=R\cdot I_3(s)$$

Uz uvrštene vrijednosti:

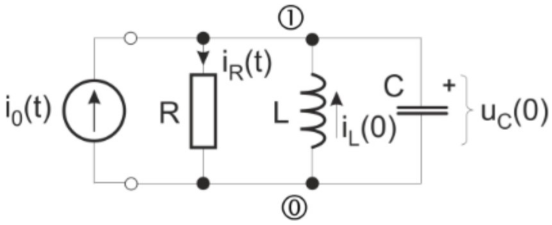
$$I_3(s)=\frac{\left(s+\frac{1}{s}\right)\cdot\frac{1}{2}+1+\frac{1}{s}}{\left(s+\frac{1}{s}\right)\cdot 2+2}=\frac{1}{2}\cdot\frac{\left(s+\frac{1}{s}\right)+2+\frac{2}{s}}{\left(s+1+\frac{1}{s}\right)\cdot 2}=\frac{1}{4}\cdot\frac{\left(s^2+2s+3\right)}{\left(s^2+s+1\right)}$$

$$U_R(s)=R\cdot I_3(s)=\frac{1}{2}\cdot\frac{\left(s^2+2s+3\right)}{\left(s^2+s+1\right)}=\frac{1}{2}\cdot\left(1+\frac{s+2}{s^2+s+1}\right)$$

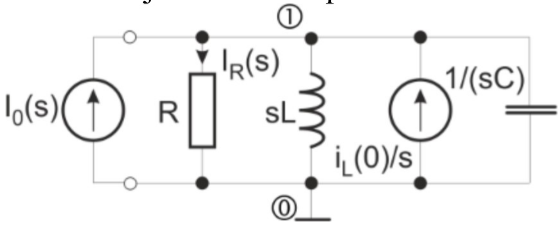
$$\frac{s+2}{s^2+s+1}=\frac{s+\frac{1}{2}+\frac{3}{2}}{s^2+s+\frac{1}{4}+\frac{3}{4}}=\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\sqrt{3}\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$u_R(t)=\frac{1}{2}\cdot\delta(t)+\frac{1}{2}e^{-\frac{t}{2}}\cdot\left[\cos\left(\frac{\sqrt{3}}{2}t\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]\cdot S(t)\quad \text{(1 bod)}$$

3. Odrediti i skicirati valni oblik struje $i_R(t)$ ako je zadano: $i_0(t)=S(t)$, R=1, L=1, C=1, te početni uvjeti $u_C(0)$=0V, $i_L(0)$=1A. Zadatak riješiti metodom napona čvorišta i primijeniti teorem superpozicije.



Rješenje: Primjena L-transformacije i metoda napona čvorišta

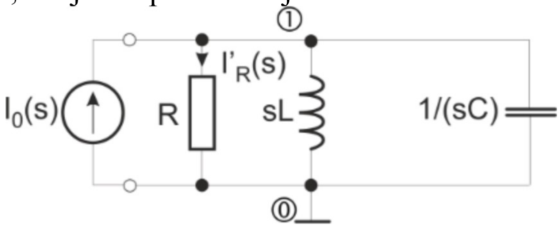


(1 bod)

$$i_0(t)=S(t)[\mathrm{A}] \circ\!\!-\!\!\circ\; I_0(s)=\frac{1}{s}$$

Metoda superpozicije:

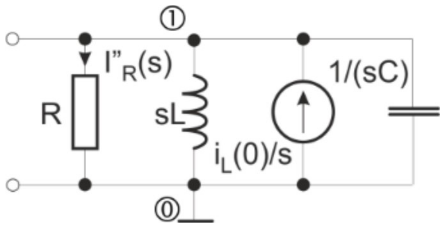
a) uključen neovisni izvor, isključen početni uvjet



$$\underline{U'_1(s)\left(\frac{1}{R}+\frac{1}{sL}+sC\right)=I_0(s)} \Rightarrow I'_R(s)=\frac{U'_1(s)}{R}=\frac{I_0(s)}{R\left(\frac{1}{R}+\frac{1}{sL}+sC\right)}$$

$$I'_R(s)=\frac{\frac{1}{s}}{1\left(1+\frac{1}{s}+s\right)}=\frac{1}{s^2+s+1}$$ (1 bod)

b) isključen neovisni izvor, uključen početni uvjet



$$\underline{U''_1(s)\left(\frac{1}{R}+\frac{1}{sL}+sC\right)=\frac{i_L(0)}{s}} \Rightarrow I''_R(s)=\frac{U''_1(s)}{R}=\frac{\frac{i_L(0)}{s}}{R\left(\frac{1}{R}+\frac{1}{sL}+sC\right)}$$

$$I''_R(s)=\frac{\frac{1}{s}}{1\left(1+\frac{1}{s}+s\right)}=\frac{1}{s^2+s+1}$$ (isti je kao pod a) (1 bod)

c) zbroj dvaju doprinosa

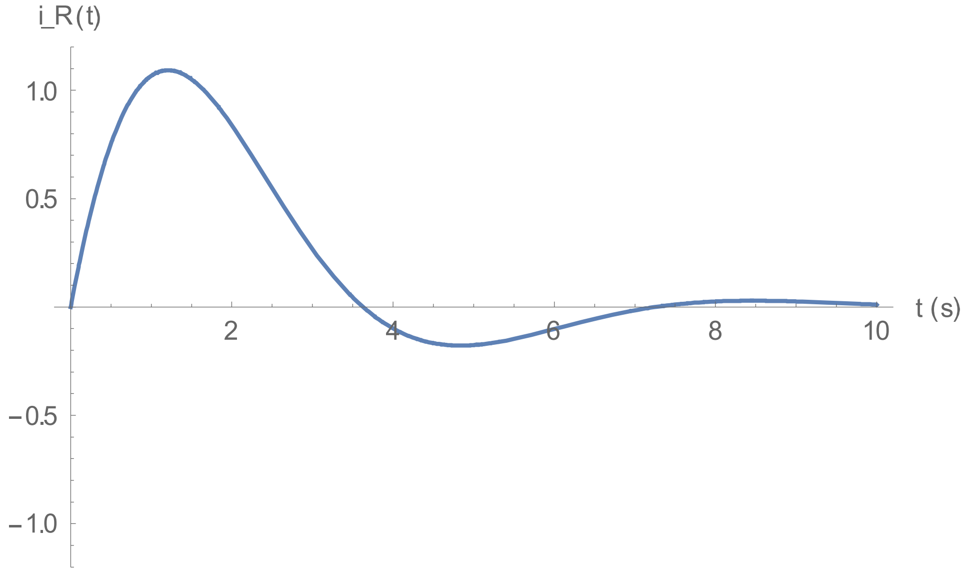
$$I_R(s)=I'_R(s)+I''_R(s)=\frac{1}{s^2+s+1}+\frac{1}{s^2+s+1}=\frac{2}{s^2+s+1}$$ (1 bod)

Tablična L-transformacija

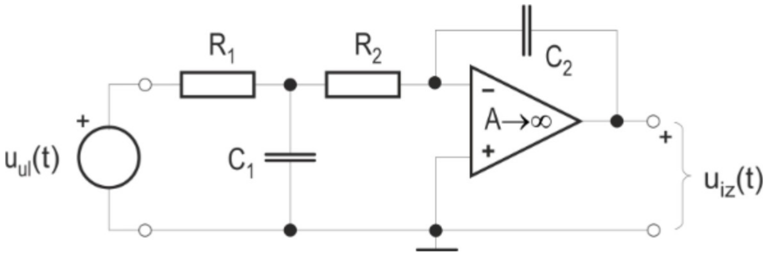
$$I_R(s)=\frac{2}{s^2+s+\frac{1}{4}+\frac{3}{4}}=\frac{2}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}=2\frac{2}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$i_R(t)=\left[\frac{4}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]\cdot S(t)$$ (1 bod)

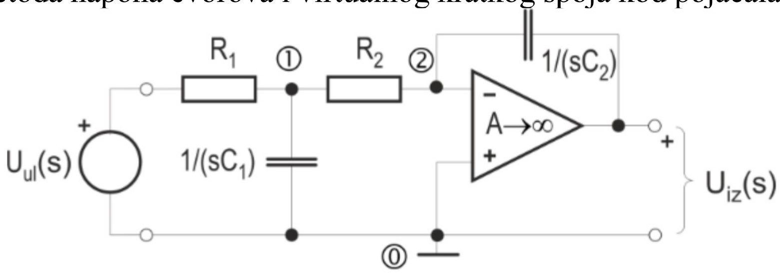
Skica:



4. Za krug prikazan slikom odrediti napon na izlazu operacijskog pojačala $u_{iz}(t)$, ako je pobuda $u_{ul}(t)=S(t)$. Zadane su normirane vrijednosti elemenata: $R_1=1$, $R_2=1$, $C_1=2$, $C_2=1$. Početni uvjeti su jednaki nula. Operacijsko pojačalo je idealno. Električni krug riješiti metodom prema izboru.



Rješenje: Metoda napona čvorova i virtualnog kratkog spoja kod pojačala



(1 bod)

Metoda čvorova:

1) $U_1(s)\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_1\right)-U_2(s)\frac{1}{R_2}=\frac{U_{ul}(s)}{R_1}$;

2) $-U_1(s)\frac{1}{R_2}+U_2(s)\left(\frac{1}{R_2}+sC_2\right)=U_{iz}(s)\cdot sC_2$;

Virtualni kratki spoj $U_2(s)=0$

1) $U_1(s)\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_1\right)=\frac{U_{ul}(s)}{R_1}$;

2) $-U_1(s)\frac{1}{R_2}=U_{iz}(s)\cdot sC_2$; (1 bod)

$$2)\Rightarrow U_1(s)=-U_{iz}(s)\cdot sR_2C_2 \rightarrow 1)\ -U_{iz}(s)\cdot sR_2C_2\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_1\right)=\frac{U_{ul}(s)}{R_1}$$

$$U_{iz}(s)=-\frac{U_{ul}(s)}{R_1\cdot sR_2C_2\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_1\right)}=-\frac{U_{ul}(s)}{sR_2C_2\left(1+\frac{R_1}{R_2}+sR_1C_1\right)}$$

Uz uvrštene vrijednosti elemenata

$$U_{iz}(s)=-\frac{\frac{1}{s}}{s(1+1+2s)}=-\frac{1}{s^2(2+2s)}=-\frac{1}{2}\cdot\frac{1}{s^2(s+1)}$$

(2 boda)

Rastav na parcijalne razlomke:

$$\frac{1}{s^2(s+1)}=\frac{A}{s+1}+\frac{B}{s^2}+\frac{C}{s}=\frac{As^2+B(s+1)+Cs(s+1)}{s^2(s+1)}$$

$$As^2+B(s+1)+Cs(s+1)=As^2+Bs+B+Cs^2+Cs=(A+C)s^2+(B+C)s+B=1$$

$$A+C=0$$
$$B+C=0 \quad \Rightarrow \quad \underline{B=1; \qquad C=-B=-1; \qquad A=-C=1;}$$
$$\underline{B=1}$$

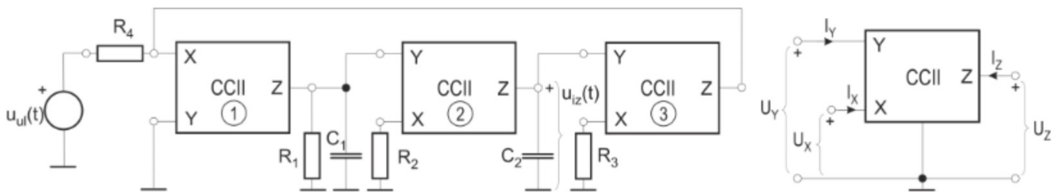
$$\frac{1}{s^2(s+1)}=\frac{1}{s+1}+\frac{1}{s^2}-\frac{1}{s}$$

Tada je:

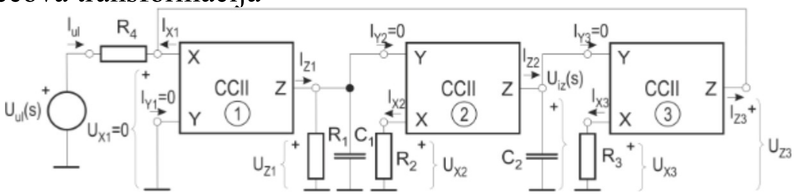
$$U_{iz}(s)=-\frac{1}{2}\cdot\frac{1}{s^2(s+1)}=-\frac{1}{2}\cdot\left(\frac{1}{s+1}+\frac{1}{s^2}-\frac{1}{s}\right)=-\frac{1}{2}\cdot\frac{1}{s+1}-\frac{1}{2}\cdot\frac{1}{s^2}+\frac{1}{2}\cdot\frac{1}{s}$$

$$\Rightarrow \underline{u_{iz}(t)=-\frac{1}{2}\left(e^{-t}+t-1\right)\cdot S(t)}$$ (1 bod)

5. Za električni krug prikazan slikom izračunati valni oblik napona na izlazu $u_{iz}(t)$ za $t>0$ kao odziv, ako je zadana pobuda $u_{ul}(t)=S(t)$. Početni naponi na kapacitetima su jednaki nuli. Zadane su normalizirane vrijednosti elemenata $R_1=1$, $R_2=1$, $R_3=1$, $R_4=1$, $C_1=1$, $C_2=1$. Za strujni prijenosnik druge generacije (CCII) vrijede slijedeće definicijske jednadžbe: $u_x=u_y$, $i_y=0$, $i_z=i_x$ uz referentna usmjerenja struja i napona prilaza prikazana na slici.



Rješenje: Laplaceova transformacija



a) Za prvi CCII vrijedi:

$$U_{x1}=U_{y1}=0,\ I_{y1}=0$$

$$I_{z1}=I_{x1}=-\left(I_{z3}+\frac{U_{ul}}{R_4}\right)$$

b) Za drugi CCII vrijedi:

$$I_{z2}=I_{x2}=\frac{U_{x2}}{R_1},\ I_{y2}=0$$

$$U_{x2}=U_{y2}=U_{z1}=I_{z1}\cdot\frac{R_1}{sR_1C_1+1}$$

c) Za treći CCII vrijedi:

$$U_{iz}=U_{C2}=U_{z2}=I_{z2}\cdot\frac{1}{sC_2}$$

$$I_{z3}=I_{x3}=\frac{U_{x3}}{R_3}=\frac{U_{y3}}{R_3}=\frac{U_{iz}}{R_3}\ \text{(1 bod)}$$

d) Nakon sređivanja do sada napisanih izraza:

$$U_{iz}=I_{x2}\cdot\frac{1}{sC_2}=\frac{U_{x2}}{R_1}\cdot\frac{1}{sC_2}=I_{z1}\cdot\frac{R_1}{sR_1C_1+1}\cdot\frac{1}{R_1}\cdot\frac{1}{sC_2}=-\left(I_{z3}+\frac{U_{ul}}{R_4}\right)\cdot\frac{1}{sR_1C_1+1}\cdot\frac{1}{sC_2}$$

$$U_{iz}=-\left(\frac{U_{iz}}{R_3}+\frac{U_{ul}}{R_4}\right)\cdot\frac{1}{sC_2\left(sR_1C_1+1\right)}$$

$$0=U_{iz}+\frac{U_{iz}}{R_3}\cdot\frac{1}{sC_2\left(sR_1C_1+1\right)}+\frac{U_{ul}}{R_4}\cdot\frac{1}{sC_2\left(sR_1C_1+1\right)}$$

$$U_{iz}\cdot\left(1+\frac{1}{sR_3C_2\left(sR_1C_1+1\right)}\right)=-U_{ul}\cdot\frac{1}{sR_4C_2\left(sR_1C_1+1\right)}$$

$$U_{iz}(s) = -U_{ul}(s)\cdot\frac{\dfrac{1}{sR_4C_2\left(sR_1C_1+1\right)}}{1+\dfrac{1}{sR_3C_2\left(sR_1C_1+1\right)}} = -U_{ul}(s)\cdot\frac{1}{sR_4C_2\left(sR_1C_1+1\right)+\dfrac{R_4}{R_3}}$$

$$U_{iz}(s) = -U_{ul}(s)\frac{1}{s^2R_1R_4C_1C_2+sR_4C_2+\dfrac{R_4}{R_3}} = -\frac{R_3}{R_4}\cdot\frac{1}{s^2R_1R_3C_1C_2+sR_3C_2+1}\cdot U_{ul}(s)$$ (2 boda)

e) Uz uvrštene vrijednosti elemenata:

$$U_{iz}(s) = -\frac{1}{s}\cdot\frac{1}{s^2+s+1}$$ (1 bod)

Rastav na parcijalne razlomke

$$\frac{1}{s}\cdot\frac{1}{s^2+s+1} = \frac{A}{s}+\frac{Bs+C}{s^2+s+1} = \frac{A\left(s^2+s+1\right)+\left(Bs+C\right)s}{s\left(s^2+s+1\right)} = \frac{\left(A+B\right)s^2\left(A+C\right)s+A}{s\left(s^2+s+1\right)}$$

$A+B=0$

$A+C=0 \Rightarrow \underline{A=1;\quad B=-A=-1;\quad C=-A=-1}$

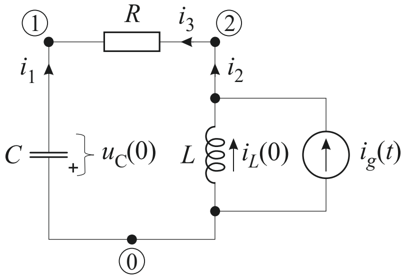
$\underline{A=1}$

$$\frac{1}{s}\cdot\frac{1}{s^2+s+1} = \frac{1}{s}-\frac{s+1}{s^2+s+1} = \frac{1}{s}-\frac{s+\frac{1}{2}+\frac{1}{2}}{s^2+s+\frac{1}{4}+\frac{3}{4}} = \frac{1}{s}-\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}-\frac{\frac{1}{2}\cdot\frac{\sqrt{3}}{\sqrt{3}}}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow \underline{u_{iz}(t) = \left\{-1+e^{-t/2}\left[\cos\left(\frac{\sqrt{3}}{2}t\right)+\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]\right\}\cdot S(t)}$$ (1 bod)

MEĐUISPIT IZ ELEKTRIČNIH KRUGOVA 2019-2020 – Rješenja

1. Poštujući oznake čvorova i grana za električni krug na slici, primjenom Kirchhoffovih zakona izračunati valni oblik struje kroz otpornik $i_3(t)$ kao odziv, ako je zadana pobuda $i_g(t)=\delta(t)$. Zadane su normalizirane vrijednosti elemenata R=1, C=1/2, L=1, te početni uvjeti $u_C(0)$=2 i $i_L(0)$=8.



Rješenje: Primjena Laplaceove transformacije

Mreža ima N_b=3 grane i N_v=3 čvora

Jednadžbe KZN

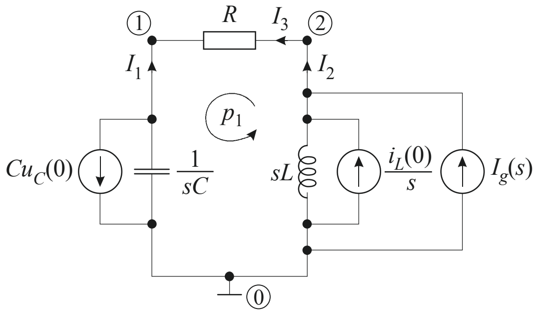
(p1) $-U_1(s)+U_2(s)+U_3(s)=0$

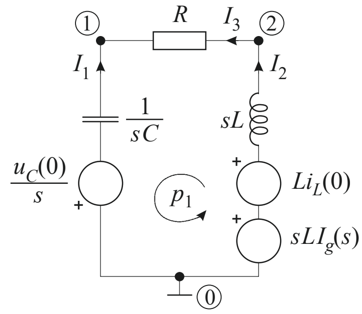
Jednadžbe KZS

(č1) $-I_1(s)-I_3(s)=0$

(č2) $-I_2(s)+I_3(s)=0$ **(1 bod)**

Naponsko-strujne relacije grana:





(1 bod)

(g1) $U_1(s)=I_1(s)\cdot\frac{1}{sC}+\frac{u_C(0)}{s}$

(g2) $U_2(s)=I_2(s)\cdot sL-Li_L(0)-sL\cdot I_g(s)$ **(1 bod)**

(g3) $U_3(s)=I_3(s)\cdot R$

Rješavanje:

$$(\text{p1})\Rightarrow -\left(I_1(s)\cdot\frac{1}{sC}+\frac{u_C(0)}{s}\right)+\left(I_2(s)\cdot sL-Li_L(0)-sL\cdot I_g(s)\right)+\left(I_3(s)\cdot R\right)=0$$

$(\text{č1})\Rightarrow I_1(s)=-I_3(s)$

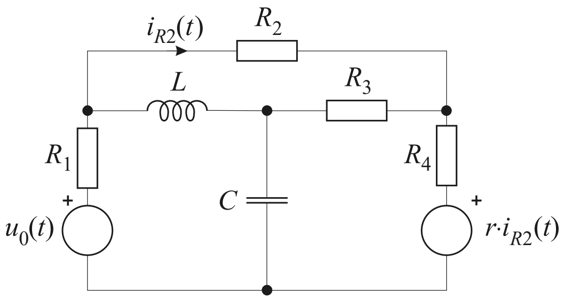
$(\text{č2})\Rightarrow I_2(s)=I_3(s)$

$$I_3(s)\cdot\left(\frac{1}{sC}+sL+R\right)=\left(\frac{u_C(0)}{s}+Li_L(0)+sL\cdot I_g(s)\right)$$

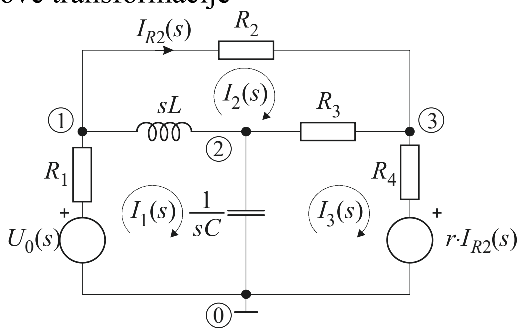
$$\Rightarrow I_R(s)=1+\frac{7s}{s^2+s+2} \quad \bullet\!\!-\!\!\circ \quad i_R(t)=\delta(t)+e^{-\frac{t}{2}}\cdot\left[7\cdot\cos\left(\frac{\sqrt{7}}{2}t\right)-\sqrt{7}\cdot\sin\left(\frac{\sqrt{7}}{2}t\right)\right]\cdot S(t)$$

(1 bod) **(1 bod)**

2. Za električni krug prikazan slikom napisati jednadžbe petlji. Konačni oblik jednadžbi prikazati u formi matrične jednadžbe. Izračunati i skicirati valni oblik struje $i_{R2}(t)$, ako je zadana pobuda $u_0(t)=\delta(t)$, konstanta r=1, a normirane vrijednosti elemenata su: R_1=1, R_2=1, R_3=1, R_4=1, L=1/2 i C=1. Početni uvjeti su jednaki nuli. Uputa: uvrstiti vrijednosti elemenata u jednadžbe petlji.



Rješenje: Primjena Laplaceove transformacije



Jednadžbe petlji

$$-U_0(s)+R_1I_1(s)+sL\left(I_1(s)-I_2(s)\right)+\frac{1}{sC}\left(I_1(s)-I_3(s)\right)=0$$

$$R_2I_2(s)+R_3\left(I_2(s)-I_3(s)\right)+sL\left(I_2(s)-I_1(s)\right)=0$$

$$\frac{1}{sC}\left(I_3(s)-I_1(s)\right)+R_3\left(I_3(s)-I_2(s)\right)+R_4I_3(s)+r\cdot I_2(s)=0;\ I_{R2}(s)=I_2(s)$$

$$\left(R_1+sL+\frac{1}{sC}\right)\cdot I_1-sL\cdot I_2-\frac{1}{sC}\cdot I_3=U_0$$

$$-sL\cdot I_1+\left(sL+R_2+R_3\right)\cdot I_2-R_3\cdot I_1=0$$

$$-\frac{1}{sC}\cdot I_1-\left(R_3-r\right)\cdot I_2+\left(\frac{1}{sC}+R_3+R_4\right)\cdot I_3=0 \quad \textbf{(1 bod)}$$

Jednadžbe petlji u matričnom obliku

$$\begin{bmatrix} R_1+sL+\frac{1}{sC} & -sL & -\frac{1}{sC} \\ -sL & sL+R_2+R_3 & -R_3 \\ -\frac{1}{sC} & -\left(R_3-r\right) & \frac{1}{sC}+R_3+R_4 \end{bmatrix}\cdot\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}=\begin{bmatrix} U_0 \\ 0 \\ 0 \end{bmatrix} \quad I_2(s)=\frac{\Delta_2}{\Delta} \quad \textbf{(1 bod)}$$

$$\Delta=\begin{vmatrix} R_1+sL+\frac{1}{sC} & -sL & -\frac{1}{sC} \\ -sL & sL+R_2+R_3 & -R_3 \\ -\frac{1}{sC} & -\left(R_3-r\right) & \frac{1}{sC}+R_3+R_4 \end{vmatrix}=\begin{vmatrix} 1+\frac{s}{2}+\frac{1}{s} & -\frac{s}{2} & -\frac{1}{s} \\ -\frac{s}{2} & \frac{s}{2}+2 & -1 \\ -\frac{1}{s} & 0 & \frac{1}{s}+2 \end{vmatrix}$$

$$= (-1)^{(1+2)} \cdot \left(-\frac{s}{2}\right) \begin{vmatrix} -\frac{s}{2} & -1 \\ -\frac{1}{s} & \frac{1}{s}+2 \end{vmatrix} + (-1)^{(2+2)} \cdot \left(\frac{s}{2}+2\right) \begin{vmatrix} 1+\frac{s}{2}+\frac{1}{s} & -\frac{1}{s} \\ -\frac{1}{s} & \frac{1}{s}+2 \end{vmatrix}$$

$$= \frac{s}{2}\left(-\frac{1}{2}-s-\frac{1}{s}\right)+\left(\frac{s}{2}+2\right)\left[\left(1+\frac{s}{2}+\frac{1}{s}\right)\left(\frac{1}{s}+2\right)-\frac{1}{s^2}\right]$$

$$= \frac{6}{s}+6+3s = 3\frac{2+2s+s^2}{s}$$

$$\Delta_2 = \begin{vmatrix} R_1+sL+\frac{1}{sC} & U_0 & -\frac{1}{sC} \\ -sL & 0 & -R_3 \\ -\frac{1}{sC} & 0 & \frac{1}{sC}+R_3+R_4 \end{vmatrix} = \begin{vmatrix} 1+\frac{s}{2}+\frac{1}{s} & 1 & -\frac{1}{s} \\ -\frac{s}{2} & 0 & -1 \\ -\frac{1}{s} & 0 & \frac{1}{s}+2 \end{vmatrix}$$

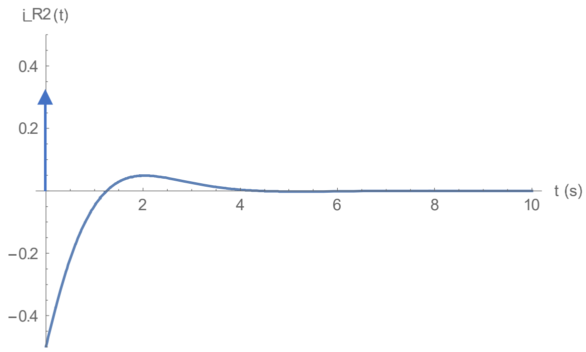
$$= (-1)^{(1+2)} \cdot (1) \begin{vmatrix} -\frac{s}{2} & -1 \\ -\frac{1}{s} & \frac{1}{s}+2 \end{vmatrix} = -\left(-\frac{1}{2}-s-\frac{1}{s}\right) = \frac{1}{2}+s+\frac{1}{s} = \frac{1+\frac{1}{2}s+s^2}{s}$$

$$\boxed{I_2(s) = \frac{\Delta_2}{\Delta} = \frac{1}{3}\cdot\frac{1+\frac{1}{2}s+s^2}{2+2s+s^2}} = \frac{1}{3}\cdot\frac{s^2+2s+2-\frac{3}{2}s-1}{s^2+2s+2}$$ **(1 bod)**

$$= \frac{1}{3}-\frac{1}{3}\cdot\frac{\frac{3}{2}s+1}{s^2+2s+2} = \frac{1}{3}-\frac{1}{2}\cdot\frac{s+\frac{2}{3}}{(s+1)^2+1} = \frac{1}{3}-\frac{1}{2}\cdot\frac{(s+1)-\frac{1}{3}}{(s+1)^2+1} = \frac{1}{3}-\frac{1}{2}\cdot\frac{(s+1)}{(s+1)^2+1}+\frac{1}{6}\cdot\frac{1}{(s+1)^2+1}$$

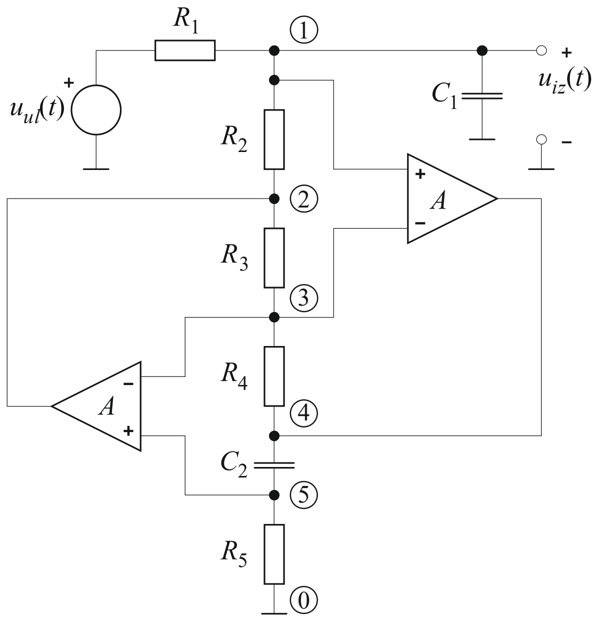
Odziv u vremenskoj domeni $i_2(t)$ slijedi primjenom inverzne Laplaceove-transformacije:

$$\boxed{i_{R2}(t) = \frac{1}{3}\delta(t)-\frac{1}{6}\cdot e^{-t}\left[3\cos(t)-\sin(t)\right]\cdot S(t)}$$ **(1 bod)**



(1 bod)

5. Za mrežu prikazanu slikom odrediti napon $u_{iz}(t)$, ako je pobuda $u_{ul}(t)=S(t)$. Zadatak riješiti metodom čvorišta. Zadane su normirane vrijednosti elemenata: $R_1=1$, $R_2=1$, $R_3=1$, $R_4=1$, $R_5=1$, $C_1=1$, $C_2=1$, $A\to\infty$.



Rješenje: Primjena Laplaceove transformacije

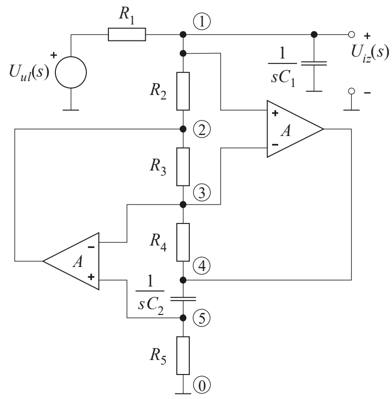
Metoda čvorova:

$$(\text{č1})\ U_1(s)\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_1\right)-U_2(s)\cdot\frac{1}{R_2}-U_{ul}(s)\cdot\frac{1}{R_1}=0$$

$$(\text{č3})\ U_3(s)\cdot\left(\frac{1}{R_3}+\frac{1}{R_4}\right)-U_2(s)\cdot\frac{1}{R_3}-U_4(s)\cdot\frac{1}{R_4}=0$$

$$(\text{č5})\ U_5(s)\cdot\left(\frac{1}{R_5}+sC_2\right)-U_4(s)\cdot sC_2=0$$

(2 bod)



$$(\text{č2})\ U_2(s)=A\cdot\left(U_5(s)-U_3(s)\right)$$

$$(\text{č4})\ U_4(s)=A\cdot\left(U_1(s)-U_3(s)\right)$$

(1 bod)

Nakon uvrštavanja vrijednosti:

$$(\text{č1})\ U_1\cdot(2+s)-U_2-U_{ul}=0$$

$$(\text{č3})\ U_3\cdot 2-U_2-U_4=0$$

$$(\text{č5})\ U_5\cdot(1+s)-U_4\cdot s=0$$

$$(\text{č2})\ U_5=U_3$$

$$(\text{č4})\ U_1=U_3$$

Nakon ovrštavanja jed. (č2) i (č4) u (č1), (č3) i (č5):

$$(\text{č1})\ U_1\cdot(2+s)-U_2-U_{ul}=0$$

$$(\text{č3})\ U_1\cdot 2-U_2-U_4=0$$

$$(\text{č5})\ U_1\cdot(1+s)-U_4\cdot s=0 \quad\Rightarrow\quad U_4=U_1\cdot\frac{(1+s)}{s}$$

$$(\text{č}5)\rightarrow(\text{č}3)\;\; U_1\cdot 2-U_2-U_1\cdot\frac{(1+s)}{s}=0\rightarrow U_2=U_1\cdot\frac{s-1}{s}$$

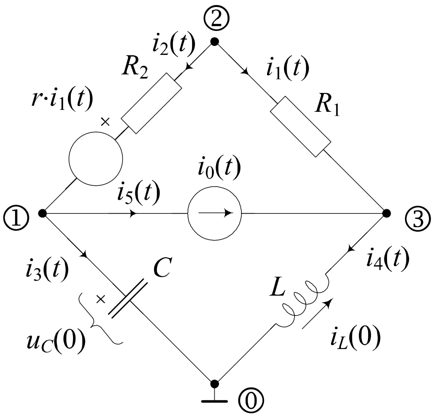
$$(\text{č}3)\rightarrow(\text{č}1)\;\; U_1\cdot(2+s)-U_1\cdot\frac{s-1}{s}-U_{ul}=0\rightarrow U_1=U_{ul}\cdot\frac{s}{s^2+s+1}$$

$$\Rightarrow \underline{U_{iz}(s)=U_1(s)=\frac{1}{s^2+s+1}} \qquad \bullet\!\!-\!\!\!-\!\!\circ \qquad \underline{u_{iz}(t)=\frac{2}{\sqrt{3}}\cdot e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)\cdot S(t)}$$

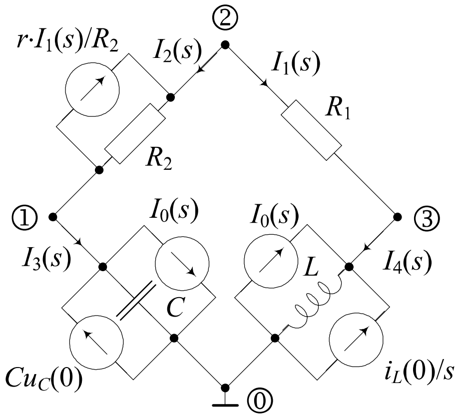
(1 bod) **(1 bod)**

GRAFOVI I MATRICE

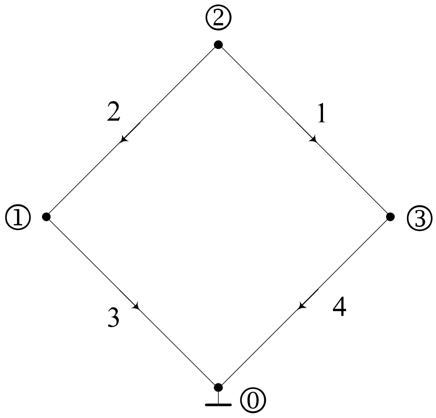
2. Zadan je električni krug prema slici. Poštujući oznake grana i čvorišta nacrtati pripadni orijentirani graf. Napisati reduciranu matricu incidencija **A**. Napisati strujno-naponske jednadžbe grana u matričnom obliku te ispisati matricu admitancija grana $\mathbf{Y}_b$ i vektor strujnih izvora grana $\mathbf{I}_{0b}$. Matrica $\mathbf{Y}_b$ mora biti regularna. Napisati sustav jednadžbi napona čvorova, odnosno odrediti matrice admitancija čvorova $\mathbf{Y}_v$ i vektor strujnih izvora u čvorovima $\mathbf{I}_{0v}$.



Rješenje: Uz primjenu Laplaceove transformacije i posmicanje strujnog izvora:



Električni krug



orijentirani graf

(1 bod)

Matrica incidencija (reducirana):

$$
\mathbf{A} = \begin{array}{c} \\ (1) \\ (2) \\ (3) \end{array}
\overbrace{\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \end{array}}^{grane}
$$

(1 bod)

$\uparrow$

cvorovi

Naponsko strujne relacije grana (struje izražene pomoću napona):

$$I_1 = U_1 \frac{1}{R_1}$$

$$I_2 = U_2 \frac{1}{R_2} - rI_1 \frac{1}{R_2} = -U_1 \frac{r}{R_1 R_2} + U_2 \frac{1}{R_2}$$

$$I_3 = U_3 sC + I_0 - Cu_C(0)$$

$$I_4 = U_4 \frac{1}{sL} - I_0 - \frac{i_L(0)}{s}$$

Naponsko-strujne relacije grana u matričnom obliku: $\mathbf{I}_b = \mathbf{Y}_b \cdot \mathbf{U}_b + \mathbf{I}_{0b}$

$$\underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}}_{\mathbf{I}_b} = \underbrace{\begin{bmatrix} \frac{1}{R_1} & 0 & 0 & 0 \\ -\frac{r}{R_1 R_2} & \frac{1}{R_2} & 0 & 0 \\ 0 & 0 & sC & 0 \\ 0 & 0 & 0 & \frac{1}{sL} \end{bmatrix}}_{\mathbf{Y}_b} \underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}}_{\mathbf{U}_b} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ I_0 - Cu_C(0) \\ -I_0 - \frac{i_L(0)}{s} \end{bmatrix}}_{\mathbf{I}_{0b}}$$ **(1 bod)**

Sustav jednadžbi napona čvorova u matričnom obliku $\mathbf{Y}_v \cdot \mathbf{U}_v = \mathbf{I}_{0v}$, gdje su (matrice $\mathbf{Y}_v$ i $\mathbf{I}_{0v}$):

$$\mathbf{Y}_v = \mathbf{A} \cdot \mathbf{Y}_b \cdot \mathbf{A}^T = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{R_1} & 0 & 0 & 0 \\ -\frac{r}{R_1 R_2} & \frac{1}{R_2} & 0 & 0 \\ 0 & 0 & sC & 0 \\ 0 & 0 & 0 & \frac{1}{sL} \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

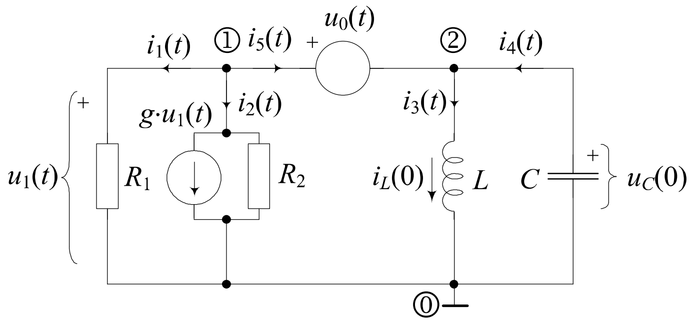
$$= \begin{bmatrix} \frac{r}{R_1 R_2} & -\frac{1}{R_2} & sC & 0 \\ \frac{1}{R_1} - \frac{r}{R_1 R_2} & \frac{1}{R_2} & 0 & 0 \\ -\frac{1}{R_1} & 0 & 0 & \frac{1}{sL} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{R_2} + sC & \frac{r}{R_1 R_2} - \frac{1}{R_2} & -\frac{r}{R_1 R_2} \\ -\frac{1}{R_2} & \frac{1}{R_1} + \frac{1}{R_2} - \frac{r}{R_1 R_2} & -\frac{1}{R_1} + \frac{r}{R_1 R_2} \\ 0 & -\frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{sL} \end{bmatrix}$$

(1 bod)

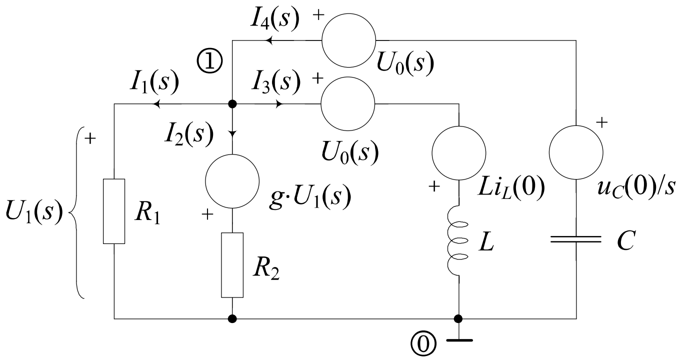
$$\mathbf{I}_{0v} = -\mathbf{A} \cdot \mathbf{I}_{0b} = -\begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ I_0 - Cu_C(0) \\ -I_0 - \frac{i_L(0)}{s} \end{bmatrix} = \begin{bmatrix} -I_0 + Cu_C(0) \\ 0 \\ I_0 + \frac{i_L(0)}{s} \end{bmatrix}$$

(1 bod)

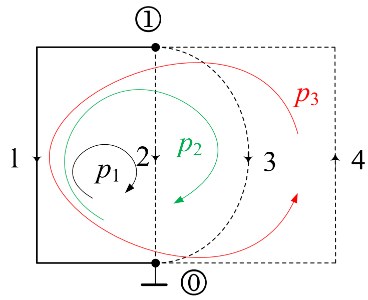
2. Zadan je električni krug prema slici. Poštujući oznake grana i čvorišta nacrtati pripadni orijentirani graf. Napisati spojnu matricu **S**. Napisati naponsko-strujne jednadžbe grana u matričnom obliku te ispisati matricu impedancija grana $\mathbf{Z}_b$ i vektor početnih uvjeta i nezavisnih izvora grana $\mathbf{U}_{0b}$. Matrica $\mathbf{Z}_b$ mora biti regularna. Napisati temeljni sustav jednadžbi petlji, odnosno odrediti matrice impedancija petlji $\mathbf{Z}_p$ i vektor početnih uvjeta i nezavisnih izvora petlji $\mathbf{U}_{0p}$.



<u>Rješenje:</u> Uz primjenu Laplaceove transformacije te posmicanje naponskog izvora (čvor (2) je nestao):



Električni krug



Orijentirani graf, stablo, spone i temeljne petlje

Spojna matrica (reducirana):

$$\mathbf{S} = \begin{array}{c} \\ (p_1) \\ (p_2) \\ (p_3) \end{array} \begin{array}{c} \overbrace{\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}}^{grane} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$\uparrow$

temeljne petlje

(2 boda)

Naponsko-strujne relacije grana (naponi izraženi pomoću struja):

$$U_1 = I_1 R_1$$

$$U_2 = I_2 R_2 - gU_1 R_2 = -gI_1 R_1 R_2 + I_2 R_2$$

$$U_3 = I_3 sL + U_0 - Li_L(0)$$

$$U_4 = I_4 \frac{1}{sC} - U_0 - \frac{u_C(0)}{s}$$

Naponsko-strujne relacije grana u matričnom obliku: $\mathbf{U}_b = \mathbf{Z}_b \cdot \mathbf{I}_b + \mathbf{U}_{0b}$

$$\underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}}_{\mathbf{U}_b} = \underbrace{\begin{bmatrix} R_1 & 0 & 0 & 0 \\ -gR_1R_2 & R_2 & 0 & 0 \\ 0 & 0 & sL & 0 \\ 0 & 0 & 0 & \dfrac{1}{sC} \end{bmatrix}}_{\mathbf{Z}_b} \cdot \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}}_{\mathbf{I}_b} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ U_0 - Li_L(0) \\ -U_0 - \dfrac{u_C(0)}{s} \end{bmatrix}}_{\mathbf{U}_{0b}}$$

Matrica $\mathbf{Z}_b$ je regularna.

(1 bod)

Temeljni sustav jednadžbi petlji u matričnom obliku: $\mathbf{Z}_p \cdot \mathbf{I}_p = \mathbf{U}_{0p}$, gdje su (matrice $\mathbf{Z}_p$ i $\mathbf{U}_{0p}$):

$$\mathbf{Z}_p = \mathbf{S} \cdot \mathbf{Z}_b \cdot \mathbf{S}^T = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_1 & 0 & 0 & 0 \\ -gR_1R_2 & R_2 & 0 & 0 \\ 0 & 0 & sL & 0 \\ 0 & 0 & 0 & \dfrac{1}{sC} \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

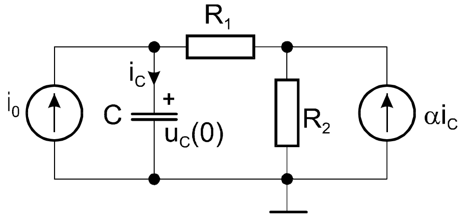
$$= \begin{bmatrix} -R_1 - gR_1R_2 & R_2 & 0 & 0 \\ -R_1 & 0 & sL & 0 \\ R_1 & 0 & 0 & \dfrac{1}{sC} \end{bmatrix} \cdot \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 + R_2 + gR_1R_2 & R_1 + gR_1R_2 & -R_1 - gR_1R_2 \\ R_1 & R_1 + sL & -R_1 \\ -R_1 & -R_1 & R_1 + \dfrac{1}{sC} \end{bmatrix}$$

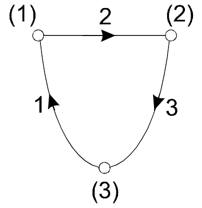
(1 bod)

$$\mathbf{U}_{0p} = -\mathbf{S} \cdot \mathbf{U}_{0b} = -\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ U_0 - Li_L(0) \\ -U_0 - \dfrac{u_C(0)}{s} \end{bmatrix} = \begin{bmatrix} 0 \\ -U_0 + Li_L(0) \\ U_0 + \dfrac{u_C(0)}{s} \end{bmatrix}$$

(1 bod)

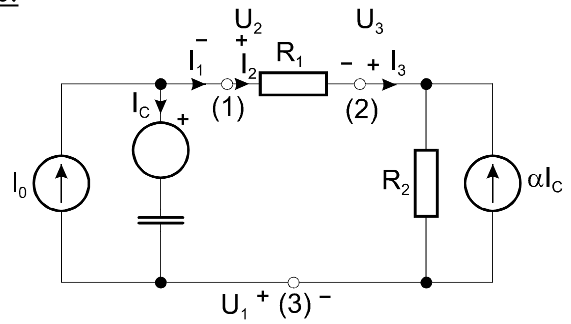
2. Zadana je mreža prema slici i njoj pripadni orijentirani graf. Odrediti sustav jednadžbi čvorova topološkom analizom.



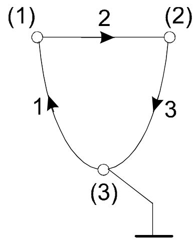


Rješenje:

Mreža:



Graf:



Matrica incidencije:

$$\mathbf{A} = \begin{array}{c} \\ (1) \\ (2) \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

Naponsko-strujne relacije grana:

$$U_1 = -I_C\frac{1}{sC} - \frac{u_C(0)}{s} = (I_1 - I_0)\frac{1}{sC} - \frac{u_C(0)}{s}, \text{ gdje je } I_0 = I_C + I_1 \Rightarrow I_C = I_0 - I_1$$

$$U_2 = R_1 I_2$$

$$U_3 = (\alpha I_C + I_3)R_2 = (\alpha I_0 - \alpha I_1 + I_3)R_2$$

$$U_1 = I_1\frac{1}{sC} - I_0\frac{1}{sC} - \frac{u_C(0)}{s}$$

$$U_2 = R_1 I_2$$

$$U_3 = -\alpha R_2 I_1 + R_2 I_3 + \alpha R_2 I_0$$

$$\mathbf{U}_b = \mathbf{I}_b \mathbf{Z}_b + \mathbf{U}_{0b}$$

$$\mathbf{Z}_b = \begin{bmatrix} \frac{1}{sC} & 0 & 0 \\ 0 & R_1 & 0 \\ -\alpha R_2 & 0 & R_2 \end{bmatrix} \qquad \mathbf{U}_{0b} = \begin{bmatrix} -I_0\frac{1}{sC} - \frac{u_C(0)}{s} \\ 0 \\ \alpha R_2 I_0 \end{bmatrix}$$

Strujno-naponske relacije grana:

$$U_1 = I_1\frac{1}{sC} - I_0\frac{1}{sC} - \frac{u_C(0)}{s} \Big/ \cdot sC$$

$$U_2 = R_1 I_2 \Big/ \cdot \frac{1}{R_1}$$

$$U_3 = -\alpha R_2 I_1 + R_2 I_3 + \alpha R_2 I_0 \Big/ \cdot \frac{1}{R_2}$$

$$sCU_1 = I_1 - I_0 - Cu_C(0)$$
$$\frac{1}{R_1}U_2 = I_2$$
$$\frac{1}{R_2}U_3 = -\alpha I_1 + I_3 + \alpha I_0$$

$$I_1 = sCU_1 + I_0 + Cu_C(0)$$
$$I_2 = \frac{1}{R_1}U_2$$
$$I_3 = \frac{1}{R_2}U_3 + \alpha I_1 - \alpha I_0 = \frac{1}{R_2}U_3 + \alpha(sCU_1 + I_0 + Cu_C(0)) - \alpha I_0 =$$
$$= \frac{1}{R_2}U_3 + \alpha sCU_1 + \alpha I_0 + \alpha Cu_C(0) - \alpha I_0 = \alpha sCU_1 + \frac{1}{R_2}U_3 + \alpha Cu_C(0)$$

$$\mathbf{I}_b = \mathbf{Y}_b\mathbf{U}_b + \mathbf{I}_{0b}$$

$$\mathbf{Y}_b = \mathbf{Z}_b^{-1} = \begin{bmatrix} sC & 0 & 0 \\ 0 & \frac{1}{R_1} & 0 \\ \alpha sC & 0 & \frac{1}{R_2} \end{bmatrix} \qquad \mathbf{I}_{0b} = \begin{bmatrix} I_0 + Cu_C(0) \\ 0 \\ \alpha Cu_C(0) \end{bmatrix}$$

Sustav jednadžbi čvorova glasi $\mathbf{Y}_v(s)\cdot\mathbf{U}_v(s) = \mathbf{I}_{0v}(s)$, gdje su:

$$\mathbf{Y}_v = \begin{bmatrix} sC + \frac{1}{R_1} & -\frac{1}{R_1} \\ -\alpha sC - \frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{R_2} \end{bmatrix} \qquad \mathbf{I}_{0v}(s) = \begin{bmatrix} I_0 + Cu_C(0) \\ -\alpha Cu_C(0) \end{bmatrix}$$

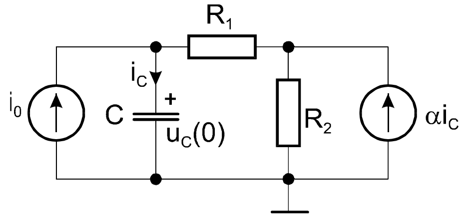
$$\mathbf{Y}_v = \mathbf{A}\cdot\mathbf{Y}_b(s)\cdot\mathbf{A}^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} sC & 0 & 0 \\ 0 & \frac{1}{R_1} & 0 \\ \alpha sC & 0 & \frac{1}{R_2} \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} -sC & \frac{1}{R_1} & 0 \\ \alpha sC & -\frac{1}{R_1} & \frac{1}{R_2} \end{bmatrix}}_{\mathbf{A}\cdot\mathbf{Y}_b(s)}\underbrace{\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}}_{\mathbf{A}^T} =$$

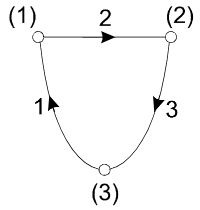
$$\mathbf{I}_{0v}(s) = \mathbf{A}\cdot\mathbf{Y}_b(s)\cdot\mathbf{U}_{0b} = \begin{bmatrix} -sC & \frac{1}{R_1} & 0 \\ \alpha sC & -\frac{1}{R_1} & \frac{1}{R_2} \end{bmatrix}\begin{bmatrix} -I_0\frac{1}{sC} - \frac{u_C(0)}{s} \\ 0 \\ \alpha R_2 I_0 \end{bmatrix} = \begin{bmatrix} I_0 + Cu_C(0) \\ -\alpha I_0 - \alpha Cu_C(0) + \alpha I_0 \end{bmatrix}$$

Rješenje $\mathbf{I}_{0v}(s)$ na drugi način:

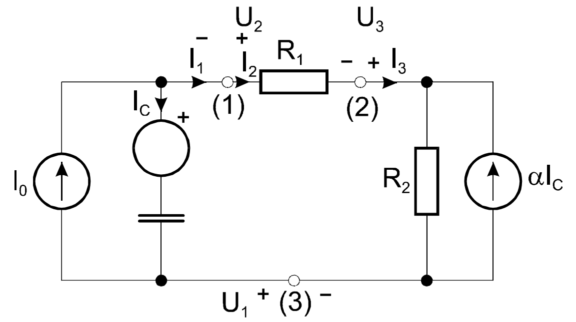
$$\mathbf{I}_{0v}(s) = -\mathbf{A}\cdot\mathbf{I}_{0b} = -\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} I_0 + Cu_C(0) \\ 0 \\ \alpha Cu_C(0) \end{bmatrix} = \begin{bmatrix} I_0 + Cu_C(0) \\ -\alpha Cu_C(0) \end{bmatrix}$$

2. Zadana je mreža prema slici i njoj pripadni orijentirani graf. Odrediti temeljni sustav jednadžbi petlji topološkom analizom.

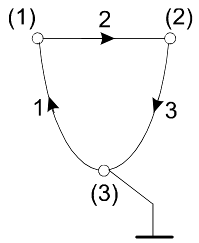




Rješenje:



Mreža:



Graf:

Spojna matrica:
$$\begin{array}{cc} & \begin{array}{ccc}1 & 2 & 3\end{array} \\ \mathbf{S} = (1) & \begin{bmatrix}1 & 1 & 1\end{bmatrix}\end{array}$$

Naponsko-strujne relacije grana:

$U_1 = -I_C \dfrac{1}{sC} - \dfrac{u_C(0)}{s} = (I_1 - I_0)\dfrac{1}{sC} - \dfrac{u_C(0)}{s}$, gdje je $I_0 = I_C + I_1 \Rightarrow I_C = I_0 - I_1$

$U_2 = R_1 I_2$

$U_3 = (\alpha I_C + I_3)R_2 = (\alpha I_0 - \alpha I_1 + I_3)R_2$

$U_1 = I_1 \dfrac{1}{sC} - I_0 \dfrac{1}{sC} - \dfrac{u_C(0)}{s}$

$U_2 = R_1 I_2$

$U_3 = -\alpha R_2 I_1 + R_2 I_3 + \alpha R_2 I_0$

Naponsko - strujne relacije grana u matričnom obliku: $\mathbf{U}_b = \mathbf{I}_b \mathbf{Z}_b + \mathbf{U}_{0b}$

$$\mathbf{Z}_b = \begin{bmatrix} \dfrac{1}{sC} & 0 & 0 \\ 0 & R_1 & 0 \\ -\alpha R_2 & 0 & R_2 \end{bmatrix} \qquad \mathbf{U}_{0b} = \begin{bmatrix} -I_0 \dfrac{1}{sC} - \dfrac{u_C(0)}{s} \\ 0 \\ \alpha R_2 I_0 \end{bmatrix}$$

Matrica $\mathbf{Z}_b$ je regularna.

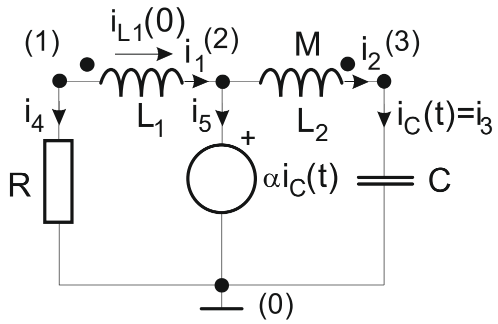
Sustav jednadžbi petlji glasi $\mathbf{Z}_p(s) \cdot \mathbf{I}_p(s) = \mathbf{U}_{0p}(s)$, gdje su:

$$\mathbf{Z}_p(s) = \mathbf{S} \cdot Z_b(s) \cdot \mathbf{S}^T = \begin{bmatrix}1 & 1 & 1\end{bmatrix} \cdot \begin{bmatrix} \dfrac{1}{sC} & 0 & 0 \\ 0 & R_1 & 0 \\ -\alpha R_2 & 0 & R_2 \end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{1}{sC} - \alpha R_2 & R_1 & R_2 \end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix} \dfrac{1}{sC} - \alpha R_2 + R_1 + R_2 \end{bmatrix}$$

$$\mathbf{U}_{0p}(s) = -\mathbf{S} \cdot \mathbf{U}_{0b}(s) = -\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -I_0 \dfrac{1}{sC} - \dfrac{u_C(0)}{s} \\ 0 \\ \alpha R_2 I_0 \end{bmatrix} = \begin{bmatrix} I_0 \dfrac{1}{sC} + \dfrac{u_C(0)}{s} - \alpha R_2 I_0 \end{bmatrix}$$

2. Zadan je električni krug prema slici. Poštujući oznake grana i čvorišta nacrtati pripadni orijentirani graf. Napisati spojnu matricu **S**. Napisati naponsko-strujne jednadžbe grana u matričnom obliku te ispisati matricu impedancija grana $\mathbf{Z}_b$ i vektor početnih uvjeta i nezavisnih izvora grana $\mathbf{U}_{0b}$. Matrica $\mathbf{Z}_b$ mora biti regularna. Napisati temeljni sustav jednadžbi petlji, odnosno odrediti matrice impedancija petlji $\mathbf{Z}_p$ i vektor početnih uvjeta i nezavisnih izvora petlji $\mathbf{U}_{0p}$.



Rješenje: Naponsko-strujne relacije grana, uz primjenu Laplaceove transformacije:

$$u_1(t) = L_1\frac{di_1(t)}{dt} - M\frac{di_2(t)}{dt}$$

$$u_2(t) = -M\frac{di_1(t)}{dt} + L_2\frac{di_2(t)}{dt} \qquad U_1(s) = sL_1I_1(s) - L_1i_1(0) - sMI_2(s) + Mi_2(0)$$

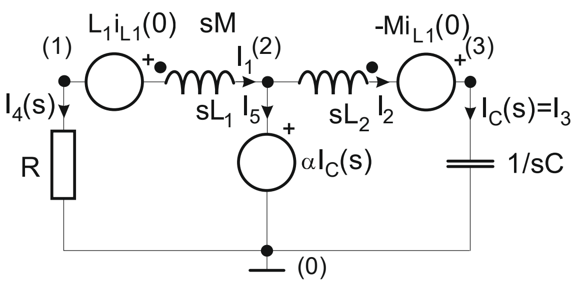
$$U_2(s) = -sMI_1(s) + Mi_1(0) + sL_2I_2(s) - L_2i_2(0)$$

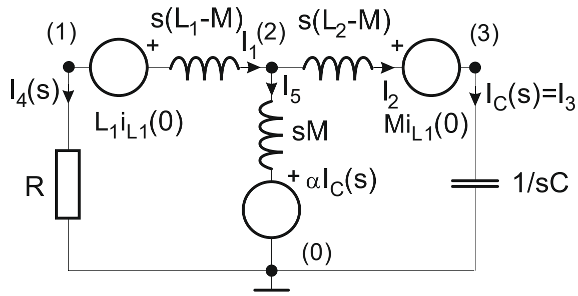
$$u_3(t) = \frac{1}{C}\int_0^t i_3(\tau)d\tau + u_3(0) \qquad U_3(s) = \frac{1}{sC}I_3(s) + \frac{u_3(0)}{s}$$

$$u_4(t) = R\cdot i_4(t) \qquad U_4(s) = R\cdot I_4(s)$$

$$u_5(t) = \alpha\cdot i_3(t) \qquad U_5(s) = \alpha\cdot I_3(s)$$

Od početnih uvjeta postoji jedino $i_{L1}(0)=i_1(0)$, a $i_{L2}(0)=0$ i $u_C(0)=u_3(0)=0$. Stoga električni krug u frekvencijskoj domeni i nakon transformacije mreže (da se izgubi M) izgleda ovako:





Konačno naponsko-strujne relacije grana glase:

$$U_1(s) = s(L_1 - M)I_1(s) - L_1i_{L1}(0)$$

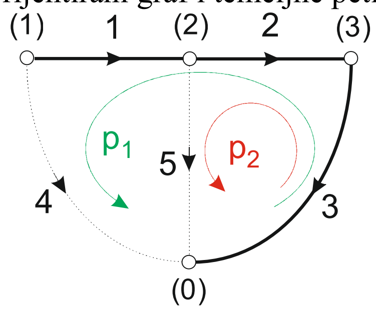
$$U_2(s) = s(L_2 - M)I_2(s) + Mi_{L1}(0)$$

$$U_3(s) = \frac{1}{sC}I_3(s)$$

$$U_4(s) = R\cdot I_4(s)$$

$$U_5(s) = \alpha\cdot I_3(s) + sMI_5(s)$$

Orijentirani graf i temeljne petlje



(2 boda)

$$
\text{Spojna matrica:}\qquad \mathbf{S}=\begin{array}{c} \\ (p_1) \\ (p_2) \end{array}\!\!\!\begin{array}{c} \overbrace{\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array}}^{grane} \\ \begin{bmatrix} -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & 0 & 1 \end{bmatrix} \end{array}
$$

$$
\underset{temeljne\ petlje}{\uparrow}
$$

Naponsko-strujne relacije grana u matričnom obliku: $\mathbf{U}_b=\mathbf{Z}_b\cdot\mathbf{I}_b+\mathbf{U}_{0b}$

$$
\underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix}}_{\mathbf{U}_b}=\underbrace{\begin{bmatrix} s(L_1-M) & 0 & 0 & 0 & 0 \\ 0 & s(L_2-M) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{sC} & 0 & 0 \\ 0 & 0 & 0 & R & 0 \\ 0 & 0 & \alpha & 0 & sM \end{bmatrix}}_{\mathbf{Z}_b}\cdot\underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix}}_{\mathbf{I}_b}+\underbrace{\begin{bmatrix} -L_1 i_{L1}(0) \\ M i_{L1}(0) \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{U}_{0b}}
$$

(1 bod)

Matrica $\mathbf{Z}_b$ je regularna. Temeljni sustav jednadžbi petlji u matričnom obliku: $\mathbf{Z}_p\cdot\mathbf{I}_p=\mathbf{U}_{0p}$, gdje su (matrice $\mathbf{Z}_p$ i $\mathbf{U}_{0p}$):

$$
\mathbf{Z}_p=\mathbf{S}\cdot\mathbf{Z}_b\cdot\mathbf{S}^T=\begin{bmatrix} -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} s(L_1-M) & 0 & 0 & 0 & 0 \\ 0 & s(L_2-M) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{sC} & 0 & 0 \\ 0 & 0 & 0 & R & 0 \\ 0 & 0 & \alpha & 0 & sM \end{bmatrix}\begin{bmatrix} -1 & -1 \\ -1 & -1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}=
$$

$$
=\begin{bmatrix} -s(L_1-M) & -s(L_2-M) & -\frac{1}{sC} & R & 0 \\ -s(L_1-M) & -s(L_2-M) & -\frac{1}{sC} & 0 & sM \end{bmatrix}\begin{bmatrix} -1 & -1 \\ -1 & -1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

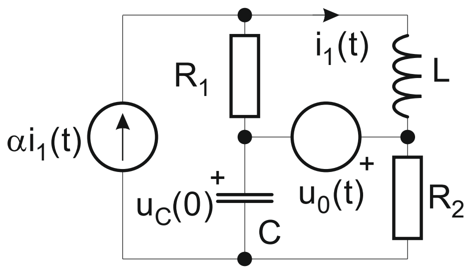
$$
=\begin{bmatrix} s(L_1+L_2-2M)+\frac{1}{sC}+R & s(L_1+L_2-2M) \\ s(L_1+L_2-2M) & s(L_1+L_2-M) \end{bmatrix}
$$

(1 bod)

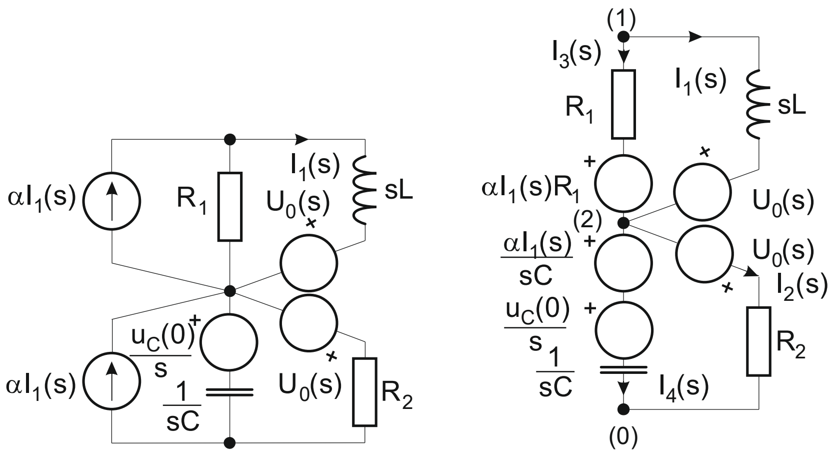
$$
\mathbf{U}_{0p}=-\mathbf{S}\cdot\mathbf{U}_{0b}=-\begin{bmatrix} -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -L_1 i_{L1}(0) \\ M i_{L1}(0) \\ 0 \\ 0 \\ 0 \end{bmatrix}=\begin{bmatrix} -L_1 i_{L1}(0)+M i_{L1}(0) \\ -L_1 i_{L1}(0)+M i_{L1}(0) \end{bmatrix}
$$

(1 bod)

2. Zadan je električni krug prema slici. Nacrtati pripadni orijentirani graf i napisati spojnu matricu **S**. Napisati naponsko-strujne jednadžbe grana u matričnom obliku te ispisati matricu impedancija grana $\mathbf{Z}_b$ i vektor početnih uvjeta i nezavisnih izvora grana $\mathbf{U}_{0b}$. Matrica $\mathbf{Z}_b$ mora biti regularna. Napisati temeljni sustav jednadžbi petlji, odnosno odrediti matrice impedancija petlji $\mathbf{Z}_p$ i vektor početnih uvjeta i nezavisnih izvora petlji $\mathbf{U}_{0p}$.



Rješenje: Posmicanje strujnog i naponskog izvora i primjena Laplaceove transformacije

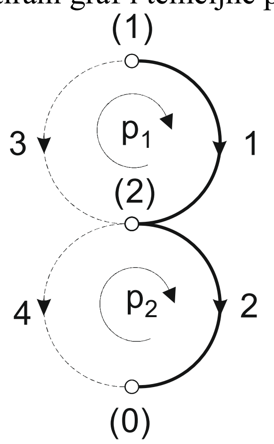


(1 bod)

Naponsko-strujne jednadžbe grana:

$$U_1 = I_1 \cdot sL + U_0$$
$$U_2 = I_2 \cdot R_2 - U_0$$
$$U_3 = I_1 \cdot \alpha R_1 + I_3 \cdot R_1$$
$$U_4 = I_1 \cdot \alpha \frac{1}{sC} + I_4 \cdot \frac{1}{sC} + \frac{u_C(0)}{s}$$

Orijentirani graf i temeljne petlje:



Spojna matrica:

$$\mathbf{S} = \begin{array}{c} \\ (p_1) \\ (p_2) \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \end{array}$$

(1 bod)

Naponsko-strujne relacije grana u matričnom obliku: $\mathbf{U}_b = \mathbf{Z}_b \cdot \mathbf{I}_b + \mathbf{U}_{0b}$

$$\underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}}_{\mathbf{U}_b} = \underbrace{\begin{bmatrix} sL & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ \alpha R_1 & 0 & R_1 & 0 \\ \dfrac{\alpha}{sC} & 0 & 0 & \dfrac{1}{sC} \end{bmatrix}}_{\mathbf{Z}_b} \cdot \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}}_{\mathbf{I}_b} + \underbrace{\begin{bmatrix} U_0 \\ -U_0 \\ 0 \\ \dfrac{u_C(0)}{s} \end{bmatrix}}_{\mathbf{U}_{0b}}$$ **(1 bod)**

Matrica $\mathbf{Z}_b$ je regularna. Temeljni sustav jednadžbi petlji u matričnom obliku: $\mathbf{Z}_p \cdot \mathbf{I}_p = \mathbf{U}_{0p}$, gdje su (matrice $\mathbf{Z}_p$ i $\mathbf{U}_{0p}$):

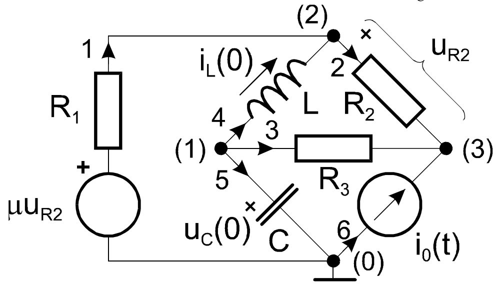
$$\mathbf{Z}_p = \mathbf{S} \cdot \mathbf{Z}_b \cdot \mathbf{S}^T = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} sL & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ \alpha R_1 & 0 & R_1 & 0 \\ \dfrac{\alpha}{sC} & 0 & 0 & \dfrac{1}{sC} \end{bmatrix} \cdot \mathbf{S}^T$$

$$= \begin{bmatrix} sL - \alpha R_1 & 0 & -R_1 & 0 \\ -\dfrac{\alpha}{sC} & R_2 & 0 & -\dfrac{1}{sC} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} sL + (1-\alpha)R_1 & 0 \\ -\dfrac{\alpha}{sC} & R_2 + \dfrac{1}{sC} \end{bmatrix}$$ **(1 bod)**

$$\mathbf{U}_{0p} = -\mathbf{S} \cdot \mathbf{U}_{0b} = -\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} U_0 \\ -U_0 \\ 0 \\ \dfrac{u_C(0)}{s} \end{bmatrix} = \begin{bmatrix} -U_0 \\ U_0 + \dfrac{u_C(0)}{s} \end{bmatrix}$$ **(1 bod)**

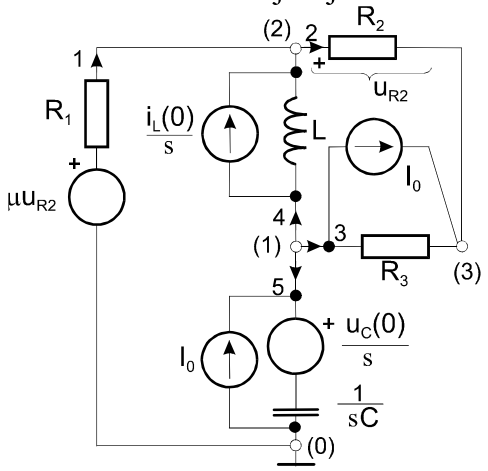
Rješenje: $\mathbf{Z}_p \cdot \mathbf{I}_p = \mathbf{U}_{0p} \;\Rightarrow\; \mathbf{I}_p = \begin{bmatrix} I_{p1} \\ I_{p2} \end{bmatrix}$

2. Za prikazanu mrežu topološkom analizom napisati sustav jednadžbi čvorova u matričnom obliku (matrice $\mathbf{Y}_v$ i $\mathbf{I}_{0v}$ pomoću matrica $\mathbf{Y}_b=\mathbf{Z}_b^{-1}$ i $\mathbf{I}_{0b}$). Nacrtati orijentirani graf (držati se oznaka čvorova i grana te pritom voditi računa da matrica $\mathbf{Z}_b$ mora biti regularna).

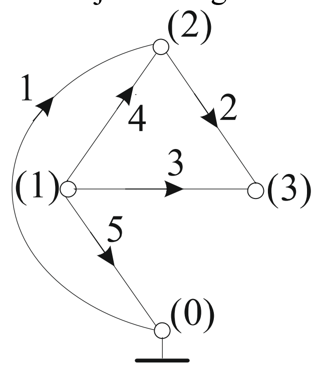


Rješenje: sustav jednadžbi čvorova glasi: $\mathbf{Y}_v \cdot \mathbf{U}_v = \mathbf{I}_v$ rješenje vektor napona čvorišta $\mathbf{U}_v$.

Mreža u frekvencijskoj domeni:



Orijentirani graf:



Matrica incidencija:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 \end{bmatrix}$$

Naponsko–strujne relacije grana glase:

$$U_1 = -\mu \cdot U_{R2} + I_1 \cdot R_1 = -\mu \cdot R_2 \cdot I_2 + I_1 \cdot R_1$$
$$U_2 = I_2 \cdot R_2$$
$$U_3 = I_3 \cdot R_3 - I_0 \cdot R_3$$
$$U_4 = I_4 \cdot sL - L \cdot i_L(0)$$
$$U_5 = I_5 \cdot \frac{1}{sC} + I_0 \cdot \frac{1}{sC} + \frac{u_C(0)}{s}$$

a u matričnom obliku glase:

$$\underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix}}_{\mathbf{U}_b} = \underbrace{\begin{bmatrix} R_1 & -\mu R_2 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 \\ 0 & 0 & R_3 & 0 & 0 \\ 0 & 0 & 0 & sL & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{sC} \end{bmatrix}}_{\mathbf{Z}_b} \cdot \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix}}_{\mathbf{I}_b} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ -I_0 \cdot R_3 \\ -L \cdot i_L(0) \\ I_0 \cdot \frac{1}{sC} + \frac{u_C(0)}{s} \end{bmatrix}}_{\mathbf{U}_{0b}}$$

Zavisni izvori (i međuinduktiviteti) se upisuju u matricu $\mathbf{Z}_b$, a nezavisni izvori i početni uvjeti u vektor $\mathbf{U}_{0b}$. Nađimo inverziju matrice $\mathbf{Z}_b$, tj. $\mathbf{Y}_b = \mathbf{Z}_b^{-1}$.

$$\begin{bmatrix} R_1 & -\mu R_2 \\ 0 & R_2 \end{bmatrix}^{-1} = \frac{1}{R_1 R_2}\begin{bmatrix} R_2 & 0 \\ \mu R_2 & R_1 \end{bmatrix}^{T} = \frac{1}{R_1 R_2}\begin{bmatrix} R_2 & \mu R_2 \\ 0 & R_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1} & \frac{\mu}{R_1} \\ 0 & \frac{1}{R_2} \end{bmatrix}$$

$$\mathbf{Y}_b = \begin{bmatrix} \frac{1}{R_1} & \frac{\mu}{R_1} & 0 & 0 & 0 \\ 0 & \frac{1}{R_2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{R_3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{sL} & 0 \\ 0 & 0 & 0 & 0 & sC \end{bmatrix}$$

$$\mathbf{Y}_v = \mathbf{A}\cdot\mathbf{Y}_b\cdot\mathbf{A}^T = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} \frac{1}{R_1} & \frac{\mu}{R_1} & 0 & 0 & 0 \\ 0 & \frac{1}{R_2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{R_3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{sL} & 0 \\ 0 & 0 & 0 & 0 & sC \end{bmatrix}\cdot\mathbf{A}^T =$$

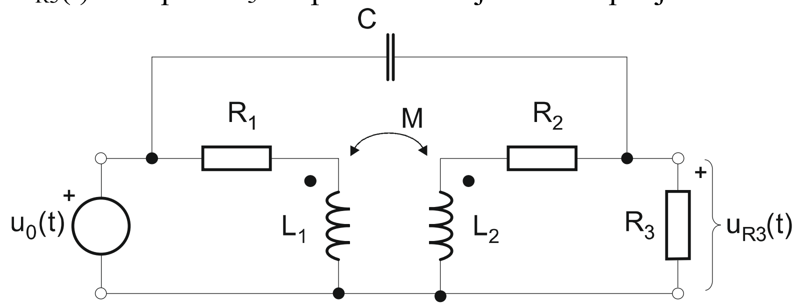
$$= \begin{bmatrix} 0 & 0 & \frac{1}{R_3} & \frac{1}{sL} & sC \\ -\frac{1}{R_1} & -\frac{\mu}{R_1}+\frac{1}{R_2} & 0 & -\frac{1}{sL} & 0 \\ 0 & -\frac{1}{R_2} & -\frac{1}{R_3} & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} \left(\frac{1}{R_3}+\frac{1}{sL}+sC\right) & -\frac{1}{sL} & -\frac{1}{R_3} \\ -\frac{1}{sL} & \left(\frac{1}{R_1}-\frac{\mu}{R_1}+\frac{1}{R_2}+\frac{1}{sL}\right) & \frac{\mu}{R_1}-\frac{1}{R_2} \\ -\frac{1}{R_3} & -\frac{1}{R_2} & \frac{1}{R_2}+\frac{1}{R_3} \end{bmatrix}$$

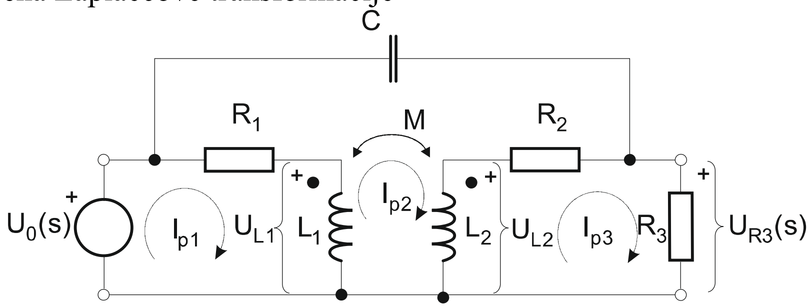
$$\mathbf{I}_v = \mathbf{A}\cdot\mathbf{Y}_b\cdot\mathbf{U}_{0b} = \begin{bmatrix} 0 & 0 & \frac{1}{R_3} & \frac{1}{sL} & sC \\ -\frac{1}{R_1} & -\frac{\mu}{R_1}+\frac{1}{R_2} & 0 & -\frac{1}{sL} & 0 \\ 0 & -\frac{1}{R_2} & -\frac{1}{R_3} & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 0 \\ -I_0\cdot R_3 \\ -L\cdot i_L(0) \\ I_0\cdot\frac{1}{sC}+\frac{u_C(0)}{s} \end{bmatrix} = \begin{bmatrix} -\frac{i_L(0)}{s}+C\cdot u_C(0) \\ \frac{i_L(0)}{s} \\ I_0 \end{bmatrix}$$

MEĐUISPIT IZ ELEKTRIČNIH KRUGOVA 2015-2016 – Rješenja

1. Za električni krug na slici zadane su normalizirane vrijednosti elemenata $R_1=R_2=R_3=1$, $C=1$, $L_1=L_2=1$, $M=1$, početne struje kroz induktivitete i početni napon na kapacitetu jednaki su nuli, te pobuda $u_0(t)=S(t)$. Primjenom Laplaceove transformacije i koristeći metodu petlji izračunati napon $u_{R3}(t)$ na otporu R_3. Napisati sustav jednadžbi petlji u matričnom obliku.



Rješenje: Primjena Laplaceove transformacije



a) Jednadžbe petlji: (1 bod)

1) $I_{p1}R_1 - I_{p2}R_1 = U_0 - U_{L1}$;

2) $-I_{p1}R_1 + I_{p2}\left(R_1 + R_2 + \dfrac{1}{sC}\right) - I_{p3}R_2 = U_{L1} - U_{L2}$;

3) $-I_{p2}R_2 + I_{p3}\left(R_2 + R_3\right) = U_{L2}$;

Jednadžbe vezanih induktiviteta:

4) $U_{L1} = (I_{p1} - I_{p2})sL_1 - (I_{p3} - I_{p2})sM$;

5) $U_{L2} = (I_{p1} - I_{p2})sM - (I_{p3} - I_{p2})sL_2$;

Nakon uvrštavanja 4) i 5) u 1), 2) i 3) te malo sređivanja: (1 bod)

1) $I_{p1}\left(R_1 + sL_1\right) - I_{p2}\left(R_1 + sL_1 - sM\right) - I_{p3}sM = U_0$;

2) $-I_{p1}\left(R_1 + sL_1 - sM\right) + I_{p2}\left(R_1 + R_2 + \dfrac{1}{sC} + sL_1 + sL_2 - 2sM\right) - I_{p3}\left(R_2 + sL_2 - sM\right) = 0$;

3) $-I_{p1}sM - I_{p2}\left(R_2 + sL_2 - sM\right) + I_{p3}\left(R_2 + R_3 + sL_2\right) = 0$;

U matričnom obliku:

$$\begin{bmatrix} R_1 + sL_1 & -\left(R_1 + sL_1 - sM\right) & -sM \\ -\left(R_1 + sL_1 - sM\right) & R_1 + R_2 + \dfrac{1}{sC} + sL_1 + sL_2 - 2sM & -\left(R_2 + sL_2 - sM\right) \\ -sM & -\left(R_2 + sL_2 - sM\right) & R_2 + R_3 + sL_2 \end{bmatrix} \begin{bmatrix} I_{p1} \\ I_{p2} \\ I_{p3} \end{bmatrix} = \begin{bmatrix} U_0 \\ 0 \\ 0 \end{bmatrix}$$

Na ovom mjestu se može napraviti provjera postupka ako je dobivena matrica impedancija petlji simetrična oko dijagonale. To vrijedi za recipročne mreže koje smiju sadržavati međuinduktivitete.

Uz uvrštene vrijednosti elemenata: (1 bod)

$$\begin{bmatrix} 1+s & -1 & -s \\ -1 & 2+\frac{1}{s} & -1 \\ -s & -1 & 2+s \end{bmatrix}\begin{bmatrix} I_{p1} \\ I_{p2} \\ I_{p3} \end{bmatrix}=\begin{bmatrix} U_0 \\ 0 \\ 0 \end{bmatrix}$$

Jadan od načina izračunavanja struje I_{p3}: $I_{p3}(s)=\frac{\Delta_3}{\Delta}$

Determinanta matrice npr. razvojem po prvom stupcu:

$$\Delta=\begin{vmatrix} 1+s & -1 & -s \\ -1 & 2+\frac{1}{s} & -1 \\ -s & -1 & 2+s \end{vmatrix}=(1+s)\cdot\begin{vmatrix} 2+\frac{1}{s} & -1 \\ -1 & 2+s \end{vmatrix}+1\cdot\begin{vmatrix} -1 & -s \\ -1 & 2+s \end{vmatrix}-s\cdot\begin{vmatrix} -1 & -s \\ 2+\frac{1}{s} & -1 \end{vmatrix}=$$

$$=(1+s)\cdot\left[\left(2+\frac{1}{s}\right)(2+s)-1\right]+1\cdot[-(2+s)-s]-s\cdot\left[1+s\left(2+\frac{1}{s}\right)\right]=$$

$$=(1+s)\left(2+\frac{1}{s}\right)(2+s)-(1+s)-(2+s)-s-s-s(2s+1)=$$

$$=(1+s)\left(5+2s+\frac{2}{s}\right)-3-5s-2s^2=$$

$$=5+2s+\frac{2}{s}+5s+2s^2+2-3-5s-2s^2=4+2s+\frac{2}{s}=2\left(2+s+\frac{1}{s}\right)$$

$$\Delta_3=\begin{vmatrix} 1+s & -1 & U_0 \\ -1 & 2+\frac{1}{s} & 0 \\ -s & -1 & 0 \end{vmatrix}=U_0\cdot\begin{vmatrix} -1 & -s \\ 2+\frac{1}{s} & -1 \end{vmatrix}=U_0\cdot\left[1+s\left(2+\frac{1}{s}\right)\right]=U_0\cdot 2(s+1)$$

uz R_3=1

$$U_{R3}(s)=I_{p3}(s)R_3=\frac{\Delta_3}{\Delta}=U_0(s)\cdot\frac{s+1}{2+s+\frac{1}{s}}=\frac{1}{s}\cdot\frac{s+1}{2+s+\frac{1}{s}}=\frac{s+1}{s^2+2s+1}$$ (1 bod)

Inverzna Laplaceova transformacija izlaznog napona: (1 bod)

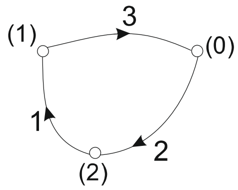
$$U_{R3}(s)=\frac{s+1}{s^2+2s+1}=\frac{s+1}{(s+1)^2}=\frac{1}{s+1}\quad\Rightarrow\quad\underline{u_{R3}(t)=e^{-t}\cdot S(t)}$$

2. Električni krug opisan je matricom incidencija **A**, matricom admitancija grana $\mathbf{Y}_b$, i vektorom strujnih izvora grana $\mathbf{I}_{0b}$. Nacrtati: a) pripadni orijentirani graf i b) električnu mrežu. c) Napisati sustav jednadžbi napona čvorova u matričnom obliku za dobivenu mrežu, d) odrediti matrice admitancija čvorova $\mathbf{Y}_v$ i e) vektor strujnih izvora u čvorovima $\mathbf{I}_{0v}$.

$$\mathbf{Y}_b = \begin{bmatrix} sC_1 & 0 & 0 \\ 0 & sC_2 & 0 \\ rGsC_1 & 0 & G \end{bmatrix}, \quad \mathbf{I}_{0b} = \begin{bmatrix} U_g sC_1 + C_1 u_{C1}(0) \\ 0 \\ rGU_g sC_1 + rGC_1 u_{C1}(0) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Rješenje:

a) Orijentirani graf slijedi iz matrice incidencija (reducirane) $\mathbf{A} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$



(1 bod)

b) Električna mreža: treba napisati strujno-naponske jednadžbe grana

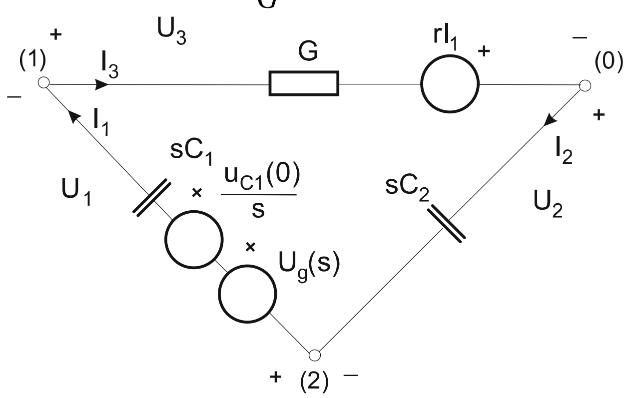
$$\mathbf{I}_b = \mathbf{Y}_b \cdot \mathbf{U}_b + \mathbf{I}_{0b}$$

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} sC_1 & 0 & 0 \\ 0 & sC_2 & 0 \\ rGsC_1 & 0 & G \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} + \begin{bmatrix} U_g sC_1 + C_1 u_{C1}(0) \\ 0 \\ rGU_g sC_1 + rGC_1 u_{C1}(0) \end{bmatrix}$$

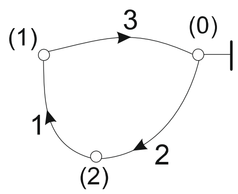
$$I_1 = U_1 \cdot sC_1 + U_g \cdot sC_1 + C_1 \cdot u_{C1}(0)$$
$$I_2 = U_2 \cdot sC_2$$
$$I_3 = U_1 rG \cdot sC_1 + U_3 G + rGU_g \cdot sC_1 + rGC_1 \cdot u_{C1}(0)$$

$$I_1 = \left[U_1 + U_g + \frac{u_{C1}(0)}{s} \right] \cdot sC_1$$
$$I_2 = U_2 \cdot sC_2$$
$$I_3 = U_3 G + rG \cdot sC_1 \left[U_1 + U_g + \frac{u_{C1}(0)}{s} \right] = (U_3 + rI_1) \cdot G$$

$G = \dfrac{1}{R}$ je vodljivost, odnosno otpor veličine $R = \dfrac{1}{G}$ je u mreži:



Orijentirani graf (čvor 0 je referentan):



c) Napisati sustav jednadžbi napona čvorova u matričnom obliku za dobivenu mrežu:
Sustav jednadžbi čvorova glasi: $\mathbf{Y}_v \cdot \mathbf{U}_v = \mathbf{I}_v$ rješenje vektor napona čvorišta $\mathbf{U}_v$.
b) + c) (2 boda)

d) matrica admitancija čvorova $\mathbf{Y}_v$:

$$\mathbf{Y}_v = \mathbf{A} \cdot \mathbf{Y}_b \cdot \mathbf{A}^T = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} sC_1 & 0 & 0 \\ 0 & sC_2 & 0 \\ rGsC_1 & 0 & G \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -sC_1 + rGsC_1 & 0 & G \\ sC_1 & -sC_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} sC_1 - rGsC_1 + G & -sC_1 + rGsC_1 \\ -sC_1 & sC_1 + sC_2 \end{bmatrix}$$

(1 bod)

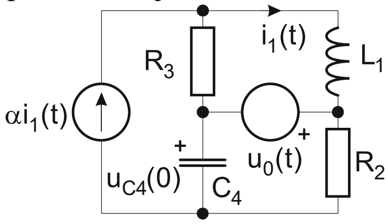
e) vektor strujnih izvora u čvorovima $\mathbf{I}_{0v}$:

$$\mathbf{I}_{0v} = -\mathbf{A} \cdot \mathbf{I}_{0b} = -\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} U_g sC_1 + C_1 u_{C1}(0) \\ 0 \\ rGU_g sC_1 + rGC_1 u_{C1}(0) \end{bmatrix} =$$

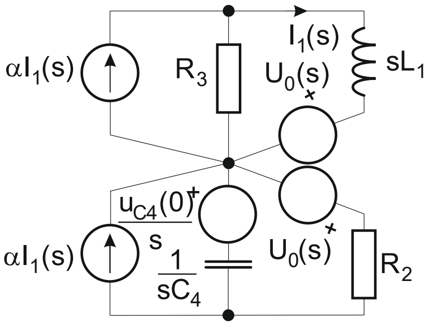
$$= \begin{bmatrix} U_g sC_1 + C_1 u_{C1}(0) - rGU_g sC_1 - rGC_1 u_{C1}(0) \\ -U_g sC_1 - C_1 u_{C1}(0) \end{bmatrix} = \begin{bmatrix} (1 - rG)(U_g sC_1 + C_1 u_{C1}(0)) \\ -U_g sC_1 - C_1 u_{C1}(0) \end{bmatrix}$$

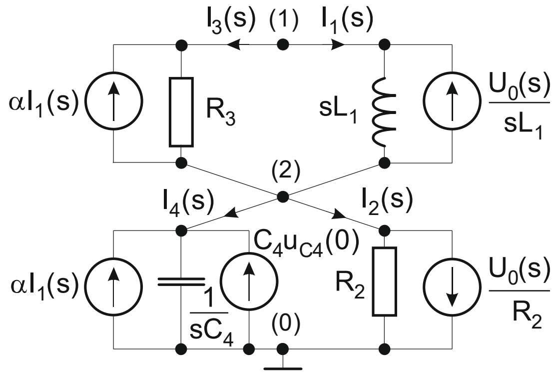
(1 bod)

2. Zadan je električni krug prema slici. Nacrtati pripadni orijentirani graf i napisati matricu incidencija **A**. Napisati strujno-naponske jednadžbe grana u matričnom obliku te ispisati matricu admitancija grana $\mathbf{Y}_b$ i vektor početnih uvjeta i nezavisnih strujnih izvora grana $\mathbf{I}_{0b}$. Matrica $\mathbf{Y}_b$ mora biti regularna. Napisati sustav jednadžbi čvorišta, odnosno odrediti matrice admitancija čvorova $\mathbf{Y}_v$ i vektor početnih uvjeta i nezavisnih izvora čvorova $\mathbf{I}_{0v}$.



Rješenje: Posmicanje strujnog i naponskog izvora i primjena Laplaceove transformacije. Pretvaranje svih izvora u strujne.





(1 bod)

Strujno-naponske jednadžbe grana (struje izražene pomoću napona):

$$I_1 = U_1\frac{1}{sL_1} - U_0\frac{1}{sL_1}$$

$$I_2 = U_2\frac{1}{R_2} + U_0\frac{1}{R_2}$$

$$I_3 = U_3\frac{1}{R_3} - \alpha\cdot I_1 = U_3\frac{1}{R_3} - \alpha\cdot\left(U_1\frac{1}{sL_1} - U_0\frac{1}{sL_1}\right) =$$

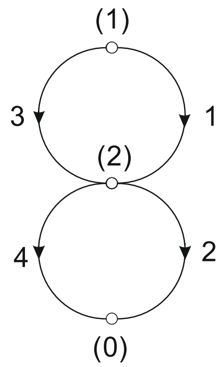
$$= -U_1\frac{\alpha}{sL_1} + U_3\frac{1}{R_3} + U_0\frac{\alpha}{sL_1}$$

$$I_4 = sC_4\cdot U_4 - I_1\cdot\alpha - C_4u_{C4}(0) =$$

$$= sC_4\cdot U_4 - \alpha\cdot\left(U_1\frac{1}{sL_1} - U_0\frac{1}{sL_1}\right) - C_4u_{C4}(0) =$$

$$= -\frac{\alpha}{sL_1}\cdot U_1 + sC_4\cdot U_4 + U_0\frac{\alpha}{sL_1} - C_4u_{C4}(0)$$

Orijentirani graf:



Matrica incidencija (nereducirana): $\mathbf{A}_a = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 \\ 0 & -1 & 0 & -1 \end{bmatrix}$ ili

Matrica incidencija (reducirana): $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}$ (1 bod)

Naponsko-strujne relacije grana u matričnom obliku: $\mathbf{I}_b = \mathbf{Y}_b \cdot \mathbf{U}_b + \mathbf{I}_{0b}$

$$\underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}}_{\mathbf{I}_b} = \underbrace{\begin{bmatrix} \frac{1}{sL_1} & 0 & 0 & 0 \\ 0 & \frac{1}{R_2} & 0 & 0 \\ -\frac{\alpha}{sL_1} & 0 & \frac{1}{R_3} & 0 \\ -\frac{\alpha}{sL_1} & 0 & 0 & sC_4 \end{bmatrix}}_{\mathbf{Y}_b} \underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}}_{\mathbf{U}_b} + \underbrace{\begin{bmatrix} -U_0 \frac{1}{sL_1} \\ U_0 \frac{1}{R_2} \\ U_0 \frac{\alpha}{sL_1} \\ U_0 \frac{\alpha}{sL_1} - C_4 u_{C4}(0) \end{bmatrix}}_{\mathbf{I}_{0b}}$$ (1 bod)

Matrica $\mathbf{Y}_b$ je regularna. Sustav jednadžbi napona čvorova u matričnom obliku $\mathbf{Y}_v \cdot \mathbf{U}_v = \mathbf{I}_{0v}$, gdje su (matrice $\mathbf{Y}_v$ i $\mathbf{I}_{0v}$):

$$\mathbf{Y}_v = \mathbf{A} \cdot \mathbf{Y}_b \cdot \mathbf{A}^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{sL_1} & 0 & 0 & 0 \\ 0 & \frac{1}{R_2} & 0 & 0 \\ -\frac{\alpha}{sL_1} & 0 & \frac{1}{R_3} & 0 \\ -\frac{\alpha}{sL_1} & 0 & 0 & sC_4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{sL_1} - \frac{\alpha}{sL_1} & 0 & \frac{1}{R_3} & 0 \\ -\frac{1}{sL_1} & \frac{1}{R_2} & -\frac{1}{R_3} & sC_4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1-\alpha}{sL_1} + \frac{1}{R_3} & -\frac{1-\alpha}{sL_1} - \frac{1}{R_3} \\ -\frac{1}{sL_1} - \frac{1}{R_3} & \frac{1}{sL_1} + \frac{1}{R_2} + \frac{1}{R_3} + sC_4 \end{bmatrix}$$

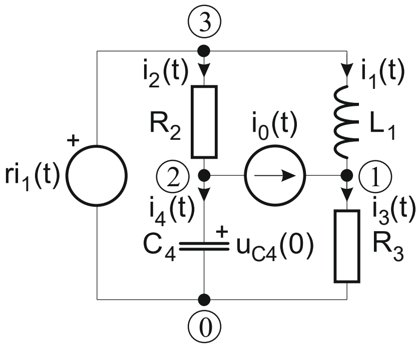
(1 bod)

$$\mathbf{I}_{0v} = -\mathbf{A} \cdot \mathbf{I}_{0b} = -\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -U_0 \frac{1}{sL_1} \\ U_0 \frac{1}{R_2} \\ U_0 \frac{\alpha}{sL_1} \\ U_0 \frac{\alpha}{sL_1} - C_4 u_{C4}(0) \end{bmatrix} = \begin{bmatrix} U_0 \frac{1-\alpha}{sL_1} \\ -U_0 \frac{1}{sL_1} - U_0 \frac{1}{R_2} + C_4 u_{C4}(0) \end{bmatrix}$$

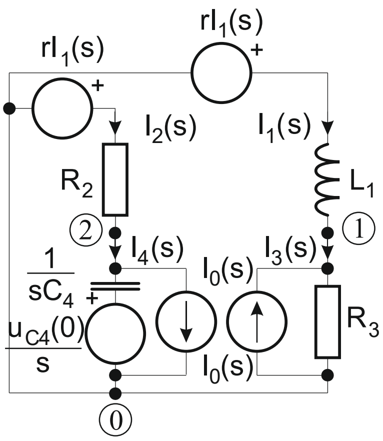
(1 bod)

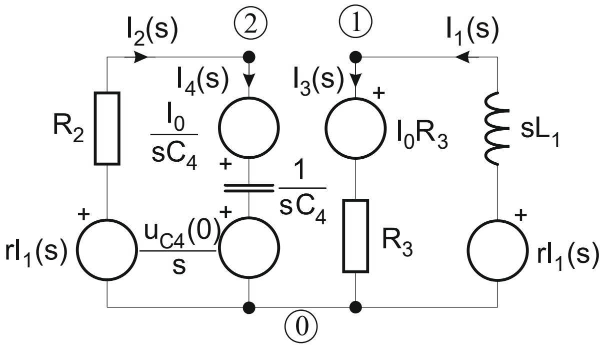
Rješenje: $\mathbf{Y}_v \cdot \mathbf{U}_v = \mathbf{I}_{0v} \quad \Rightarrow \quad \mathbf{U}_v = \begin{bmatrix} U_{v1} \\ U_{v2} \end{bmatrix}$

2. Zadan je električni krug prema slici. Poštujući oznake čvorišta i grana, nacrtati pripadni orijentirani graf i napisati spojnu matricu **S**. Napisati naponsko-strujne jednadžbe grana u matričnom obliku te ispisati matricu impedancija grana $\mathbf{Z}_b$ i vektor početnih uvjeta i nezavisnih izvora grana $\mathbf{U}_{0b}$. Matrica $\mathbf{Z}_b$ mora biti regularna. Napisati temeljni sustav jednadžbi petlji, odnosno odrediti matrice impedancija petlji $\mathbf{Z}_p$ i vektor početnih uvjeta i nezavisnih izvora petlji $\mathbf{U}_{0p}$.



Rješenje: Posmicanje strujnog i naponskog izvora (zato da bismo dobili regularnu matricu $\mathbf{Z}_b$) i zatim primjena Laplaceove transformacije. Čvor ③ je nestao i jedna petlja je nestala. Pretvaranje svih izvora u naponske.





(1 bod)

Naponsko-strujne jednadžbe grana:

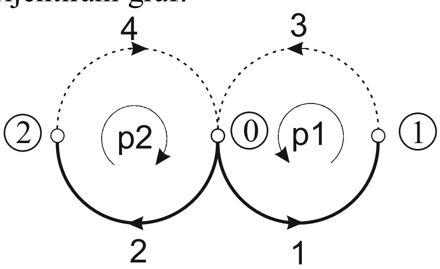
$$U_1 = sL_1 \cdot I_1 - r \cdot I_1$$

$$U_2 = -r \cdot I_1 + R_2 \cdot I_2$$

$$U_3 = R_3 \cdot I_3 + I_0 \cdot R_3$$

$$U_4 = \frac{1}{sC_4} \cdot I_4 - \frac{1}{sC_4} \cdot I_0 + \frac{u_{C4}(0)}{s}$$

Orijentirani graf:



Spojna matrica: $\mathbf{S} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ **(1 bod)**

Naponsko-strujne relacije grana u matričnom obliku: $\mathbf{U}_b = \mathbf{Z}_b \cdot \mathbf{I}_b + \mathbf{U}_{0b}$

$$\underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}}_{\mathbf{U}_b} = \underbrace{\begin{bmatrix} sL_1 - r & 0 & 0 & 0 \\ -r & R_2 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & \dfrac{1}{sC_4} \end{bmatrix}}_{\mathbf{Z}_b} \cdot \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}}_{\mathbf{I}_b} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ I_0 R_3 \\ -I_0 \dfrac{1}{sC_4} + \dfrac{u_{C4}(0)}{s} \end{bmatrix}}_{\mathbf{U}_{0b}} \quad \textbf{(1 bod)}$$

Matrica $\mathbf{Z}_b$ je regularna. Temeljni sustav jednadžbi petlji u matričnom obliku: $\mathbf{Z}_p \cdot \mathbf{I}_p = \mathbf{U}_{0p}$, gdje su

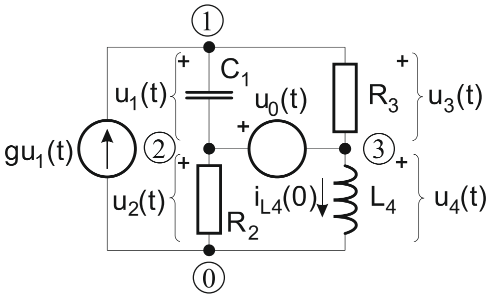
$$\mathbf{Z}_p = \mathbf{S} \cdot \mathbf{Z}_b \cdot \mathbf{S}^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} sL_1 - r & 0 & 0 & 0 \\ -r & R_2 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & \dfrac{1}{sC_4} \end{bmatrix} \cdot \mathbf{S}^T =$$

$$= \begin{bmatrix} sL_1 - r & 0 & R_3 & 0 \\ -r & R_2 & 0 & \dfrac{1}{sC_4} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} sL_1 - r + R_3 & 0 \\ -r & R_2 + \dfrac{1}{sC_4} \end{bmatrix} \quad \textbf{(1 bod)}$$

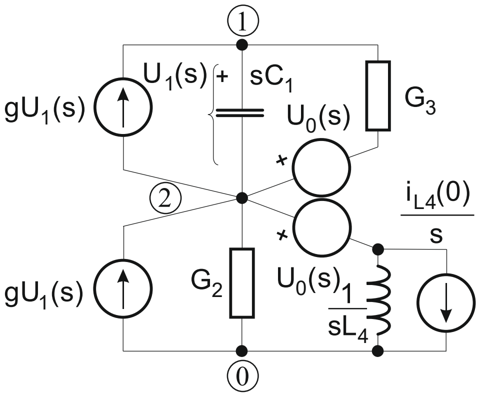
$$\mathbf{U}_{0p} = -\mathbf{S} \cdot \mathbf{U}_{0b} = -\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ I_0 R_3 \\ -I_0 \dfrac{1}{sC_4} + \dfrac{u_{C4}(0)}{s} \end{bmatrix} = \begin{bmatrix} -I_0 R_3 \\ I_0 \dfrac{1}{sC_4} - \dfrac{u_{C4}(0)}{s} \end{bmatrix} \quad \textbf{(1 bod)}$$

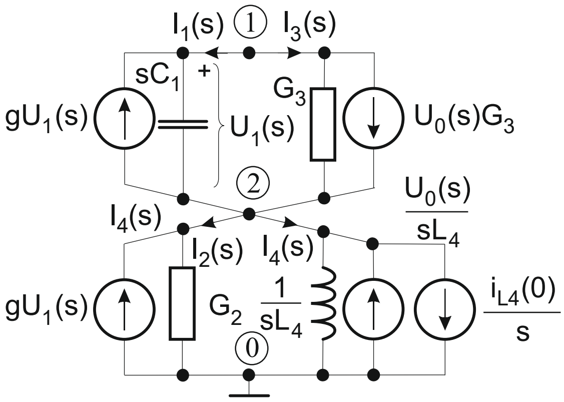
Rješenje: $\mathbf{Z}_p \cdot \mathbf{I}_p = \mathbf{U}_{0p} \;\Rightarrow\; \mathbf{I}_p = \begin{bmatrix} I_{p1} \\ I_{p2} \end{bmatrix}$

2. Zadan je električni krug prema slici. Poštujući oznake čvorišta i grana, nacrtati pripadni orijentirani graf i napisati matricu incidencija **A**. Napisati strujno-naponske jednadžbe grana u matričnom obliku te ispisati matricu admitancija grana $\mathbf{Y}_b$ i vektor početnih uvjeta i nezavisnih strujnih izvora grana $\mathbf{I}_{0b}$. Matrica $\mathbf{Y}_b$ mora biti regularna. Napisati sustav jednadžbi čvorišta, odnosno odrediti matrice admitancija čvorova $\mathbf{Y}_v$ i vektor početnih uvjeta i nezavisnih izvora čvorova $\mathbf{I}_{0v}$.



Rješenje: Posmicanje strujnog i naponskog izvora (zato da bismo dobili regularnu matricu $\mathbf{Y}_b$) i zatim primjena Laplaceove transformacije. Čvor ③ je nestao i jedna petlja je nestala. Pretvaranje svih izvora u strujne.





(1 bod)

Strujno-naponske jednadžbe grana (struje izražene pomoću napona):

$$I_1 = sC_1 \cdot U_1 - g \cdot U_1$$

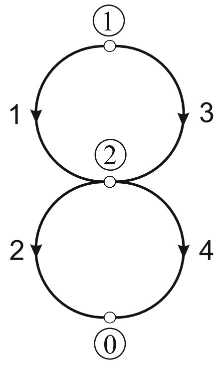
$$I_2 = -g \cdot U_1 + G_2 \cdot U_2$$

$$I_3 = G_3 \cdot U_3 + U_0 \cdot G_3$$

$$I_4 = \frac{1}{sL_4} \cdot U_4 - \frac{1}{sL_4} \cdot U_0 + \frac{i_{L4}(0)}{s}$$

$$G_i = \frac{1}{R_i}; i = 1,2,3.$$

Orijentirani graf:



Matrica incidencija (reducirana): $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}$ (1 bod)

Naponsko-strujne relacije grana u matričnom obliku: $\mathbf{I}_b = \mathbf{Y}_b \cdot \mathbf{U}_b + \mathbf{I}_{0b}$

$$\underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}}_{\mathbf{I}_b} = \underbrace{\begin{bmatrix} sC_1 - g_1 & 0 & 0 & 0 \\ -g_1 & G_2 & 0 & 0 \\ 0 & 0 & G_3 & 0 \\ 0 & 0 & 0 & \dfrac{1}{sL_4} \end{bmatrix}}_{\mathbf{Y}_b} \underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}}_{\mathbf{U}_b} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ G_3 U_0 \\ -\dfrac{1}{sL_4} U_0 + \dfrac{i_{L4}(0)}{s} \end{bmatrix}}_{\mathbf{I}_{0b}}$$ (1 bod)

Matrica $\mathbf{Y}_b$ je regularna. Sustav jednadžbi napona čvorova u matričnom obliku $\mathbf{Y}_v \cdot \mathbf{U}_v = \mathbf{I}_{0v}$, gdje su (matrice $\mathbf{Y}_v$ i $\mathbf{I}_{0v}$):

$$\mathbf{Y}_v = \mathbf{A} \cdot \mathbf{Y}_b \cdot \mathbf{A}^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} sC_1 - g_1 & 0 & 0 & 0 \\ -g_1 & G_2 & 0 & 0 \\ 0 & 0 & G_3 & 0 \\ 0 & 0 & 0 & \dfrac{1}{sL_4} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} sC_1 - g_1 & 0 & G_3 & 0 \\ -sC_1 & G_2 & -G_3 & \dfrac{1}{sL_4} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} sC_1 - g_1 + G_3 & -sC_1 + g_1 - G_3 \\ -sC_1 - G_3 & sC_1 + G_2 + G_3 + \dfrac{1}{sL_4} \end{bmatrix}$$

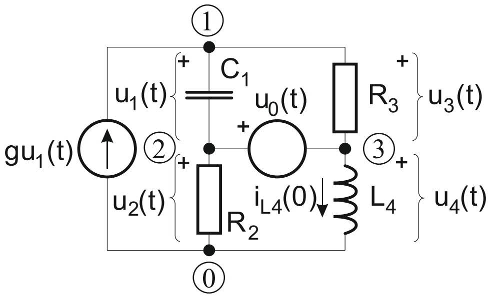
(1 bod)

$$\mathbf{I}_{0v} = -\mathbf{A} \cdot \mathbf{I}_{0b} = -\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ G_3 U_0 \\ -\dfrac{1}{sL_4} U_0 + \dfrac{i_{L4}(0)}{s} \end{bmatrix} = \begin{bmatrix} -G_3 U_0 \\ G_3 U_0 + \dfrac{1}{sL_4} U_0 - \dfrac{i_{L4}(0)}{s} \end{bmatrix}$$

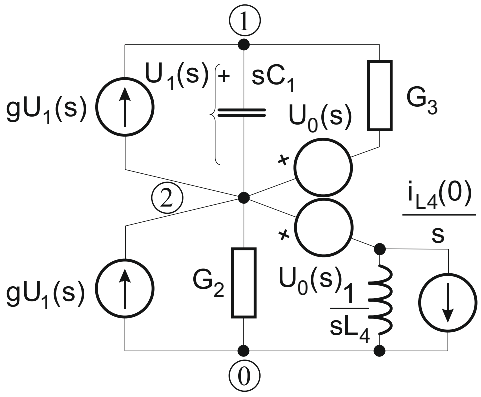
(1 bod)

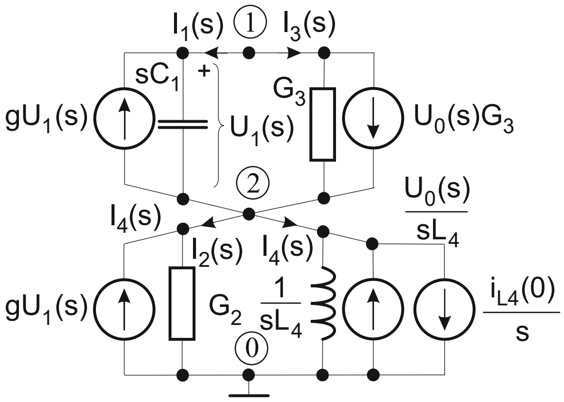
Rješenje: $\mathbf{Y}_v \cdot \mathbf{U}_v = \mathbf{I}_{0v} \;\Rightarrow\; \mathbf{U}_v = \begin{bmatrix} U_{v1} \\ U_{v2} \end{bmatrix}$

2. Zadan je električni krug prema slici. Poštujući oznake čvorišta i grana, nacrtati pripadni orijentirani graf i napisati matricu incidencija **A**. Napisati strujno-naponske jednadžbe grana u matričnom obliku te ispisati matricu admitancija grana $\mathbf{Y}_b$ i vektor početnih uvjeta i nezavisnih strujnih izvora grana $\mathbf{I}_{0b}$. Matrica $\mathbf{Y}_b$ mora biti regularna. Napisati sustav jednadžbi čvorišta, odnosno odrediti matrice admitancija čvorova $\mathbf{Y}_v$ i vektor početnih uvjeta i nezavisnih izvora čvorova $\mathbf{I}_{0v}$.



Rješenje: Posmicanje strujnog i naponskog izvora (zato da bismo dobili regularnu matricu $\mathbf{Y}_b$) i zatim primjena Laplaceove transformacije. Čvor ③ je nestao i jedna petlja je nestala. Pretvaranje svih izvora u strujne.





(1 bod)

Strujno-naponske jednadžbe grana (struje izražene pomoću napona):

$$I_1 = sC_1 \cdot U_1 - g \cdot U_1$$

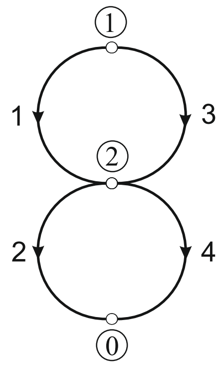
$$I_2 = -g \cdot U_1 + G_2 \cdot U_2$$

$$I_3 = G_3 \cdot U_3 + U_0 \cdot G_3$$

$$I_4 = \frac{1}{sL_4} \cdot U_4 - \frac{1}{sL_4} \cdot U_0 + \frac{i_{L4}(0)}{s}$$

$$G_i = \frac{1}{R_i}; i = 1,2,3.$$

Orijentirani graf:



Matrica incidencija (reducirana): $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}$ (1 bod)

Naponsko-strujne relacije grana u matričnom obliku: $\mathbf{I}_b = \mathbf{Y}_b \cdot \mathbf{U}_b + \mathbf{I}_{0b}$

$$\underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}}_{\mathbf{I}_b} = \underbrace{\begin{bmatrix} sC_1 - g_1 & 0 & 0 & 0 \\ -g_1 & G_2 & 0 & 0 \\ 0 & 0 & G_3 & 0 \\ 0 & 0 & 0 & \dfrac{1}{sL_4} \end{bmatrix}}_{\mathbf{Y}_b} \underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}}_{\mathbf{U}_b} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ G_3 U_0 \\ -\dfrac{1}{sL_4} U_0 + \dfrac{i_{L4}(0)}{s} \end{bmatrix}}_{\mathbf{I}_{0b}}$$ (1 bod)

Matrica $\mathbf{Y}_b$ je regularna. Sustav jednadžbi napona čvorova u matričnom obliku $\mathbf{Y}_v \cdot \mathbf{U}_v = \mathbf{I}_{0v}$, gdje su (matrice $\mathbf{Y}_v$ i $\mathbf{I}_{0v}$):

$$\mathbf{Y}_v = \mathbf{A} \cdot \mathbf{Y}_b \cdot \mathbf{A}^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} sC_1 - g_1 & 0 & 0 & 0 \\ -g_1 & G_2 & 0 & 0 \\ 0 & 0 & G_3 & 0 \\ 0 & 0 & 0 & \dfrac{1}{sL_4} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} sC_1 - g_1 & 0 & G_3 & 0 \\ -sC_1 & G_2 & -G_3 & \dfrac{1}{sL_4} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} sC_1 - g_1 + G_3 & -sC_1 + g_1 - G_3 \\ -sC_1 - G_3 & sC_1 + G_2 + G_3 + \dfrac{1}{sL_4} \end{bmatrix}$$

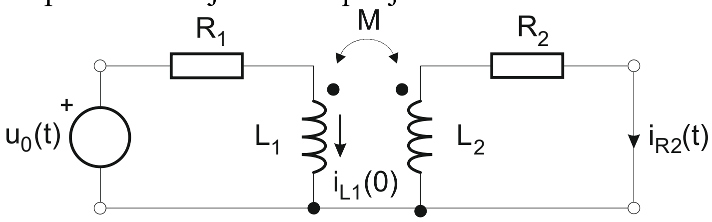
(1 bod)

$$\mathbf{I}_{0v} = -\mathbf{A} \cdot \mathbf{I}_{0b} = -\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ G_3 U_0 \\ -\dfrac{1}{sL_4} U_0 + \dfrac{i_{L4}(0)}{s} \end{bmatrix} = \begin{bmatrix} -G_3 U_0 \\ G_3 U_0 + \dfrac{1}{sL_4} U_0 - \dfrac{i_{L4}(0)}{s} \end{bmatrix}$$

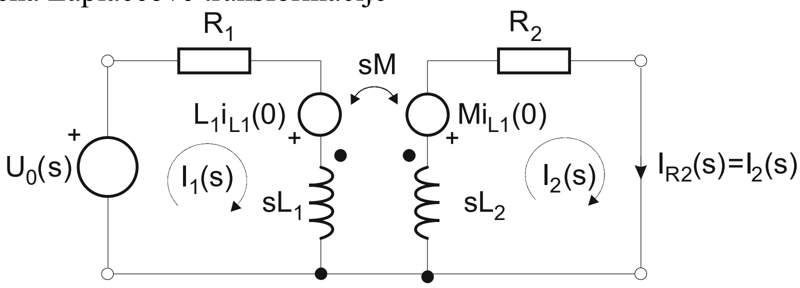
(1 bod)

Rješenje: $\mathbf{Y}_v \cdot \mathbf{U}_v = \mathbf{I}_{0v} \;\;\Rightarrow\;\; \mathbf{U}_v = \begin{bmatrix} U_{v1} \\ U_{v2} \end{bmatrix}$

2. Za električni krug na slici zadane su normalizirane vrijednosti elemenata $R_1=R_2=1$, $L_1=L_2=2$, $M=1$, početne struje kroz induktivitete $i_{L1}(0)=1$, $i_{L2}(0)=0$, te pobuda $u_0(t)=S(t)$. Primjenom Laplaceove transformacije i koristeći metodu petlji izračunati struju $i_{R2}(t)$ kroz otpor R_2 kao odziv. Napisati sustav jednadžbi petlji u matričnom obliku.



Rješenje: Primjena Laplaceove transformacije



Metoda petlji:

1) $I_1(R_1+sL_1)-I_2sM-L_1i_{L1}(0)-U_0(s)=0$;

2) $-I_1sM+I_2(R_2+sL_2)+Mi_{L1}(0)=0$;

---

1) $I_1(R_1+sL_1)-I_2sM=L_1i_{L1}(0)+U_0(s)$; (1 bod)

2) $-I_1sM+I_2(R_2+sL_2)=-Mi_{L1}(0)$; (1 bod)

$$\begin{bmatrix} R_1+sL_1 & -sM \\ -sM & R_2+sL_2 \end{bmatrix}\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix}=\begin{bmatrix} L_1i_{L1}(0)+U_0(s) \\ -Mi_{L1}(0) \end{bmatrix}$$

$$\Delta=\begin{vmatrix} R_1+sL_1 & -sM \\ -sM & R_2+sL_2 \end{vmatrix}=(R_1+sL_1)(R_2+sL_2)-(sM)^2$$

$$\Delta_2=\begin{vmatrix} R_1+sL_1 & L_1i_{L1}(0)+U_0(s) \\ -sM & -Mi_{L1}(0) \end{vmatrix}=-Mi_{L1}(0)(R_1+sL_1)+sM\left(L_1i_{L1}(0)+U_0(s)\right)$$

$$I_2(s)=\frac{\Delta_2}{\Delta}=\frac{-Mi_{L1}(0)(R_1+sL_1)+sM\left(L_1i_{L1}(0)+U_0(s)\right)}{(R_1+sL_1)(R_2+sL_2)-(sM)^2}$$ (1 bod)

a) Uz uvrštene vrijednosti elemenata i uz pobudu $u_0(t)=S(t)$:

$$I_2(s)=\frac{-(1+2s)+s(2+1/s)}{(1+2s)^2-s^2}=0$$ (2 boda)

b) Uz uvrštene vrijednosti elemenata i uz pobudu $u_0(t)=\delta(t)$:

$$I_2(s)=\frac{-(1+2s)+s(2+1)}{(1+2s)^2-s^2}=\frac{s-1}{3s^2+4s+1}=\frac{1}{3}\cdot\frac{s-1}{s^2+\frac{4}{3}s+\frac{1}{3}}=\frac{1}{3}\cdot\frac{s-1}{\left(s+\frac{2}{3}\right)^2-\frac{1}{9}}$$ (1 bod)

$$s^2+\frac{4}{3}s+\frac{1}{3}=0\Rightarrow s_{1,2}=-\frac{2}{3}\pm\sqrt{\left(\frac{2}{3}\right)^2-\frac{1}{3}}=-\frac{2}{3}\pm\sqrt{\frac{1}{9}}=-\frac{2}{3}\pm\frac{1}{3}\Rightarrow s_1=-\frac{1}{3};s_2=-1.$$

Rastav na parcijalne razlomke:

$$\frac{s-1}{s^2+\frac{4}{3}s+\frac{1}{3}}=\frac{A}{s+\frac{1}{3}}+\frac{B}{s+1}\Rightarrow\frac{s-1}{s^2+\frac{4}{3}s+\frac{1}{3}}=\frac{A(s+1)+B\left(s+\frac{1}{3}\right)}{\left(s+\frac{1}{3}\right)(s+1)}=\frac{(A+B)s+\left(A+B\frac{1}{3}\right)}{s^2+\frac{4}{3}s+\frac{1}{3}}$$

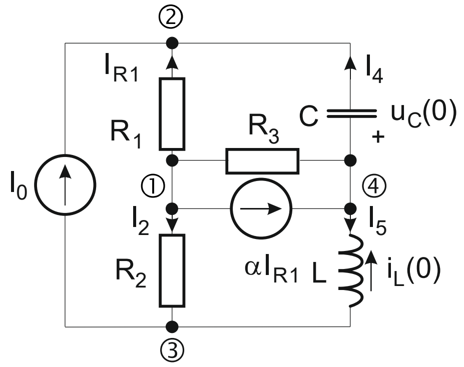
$$\begin{aligned}&A+B=1 && A=1-B=-2\\ &\underline{A+B\frac{1}{3}=-1} &\Rightarrow\quad& \underline{B\frac{2}{3}=2\Rightarrow B=3}\end{aligned}$$

$$I_2(s)=\frac{1}{3}\left(\frac{A}{s+\frac{1}{3}}+\frac{B}{s+1}\right)=\frac{1}{3}\left(\frac{-2}{s+\frac{1}{3}}+\frac{3}{s+1}\right)=-\frac{2}{3}\frac{1}{s+\frac{1}{3}}+\frac{1}{s+1}$$

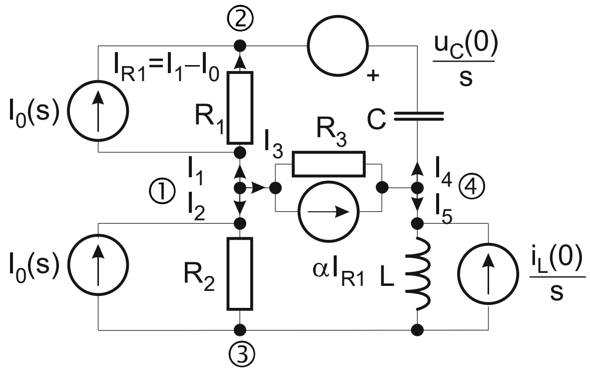
$$\Rightarrow\underline{i_2(t)=\left(-\frac{2}{3}e^{-\frac{1}{3}t}+e^{-t}\right)S(t)}$$ (1 bod)

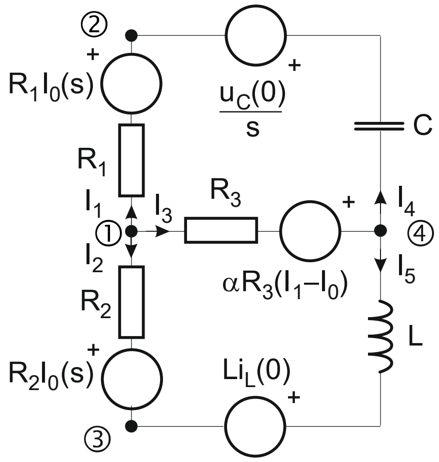
Napomena: Ovaj dio zadatka je bodovan i ako su studenti uvrstili bilo koju (npr. pogrešnu $u_0(t)=\delta(t)$) pobudu i/ili ako su zaboravili početni uvjet $Mi_{L1}(0)$, dobili su bodove na pretvaranje iz Laplaceove u vremensku domenu. Dobili su bodove na ispravan postupak.

2. Zadan je električni krug prema slici. Poštujući oznake čvorišta i grana nacrtati pripadni orijentirani graf i napisati spojnu matricu **S**. Napisati naponsko-strujne jednadžbe grana u matričnom obliku te ispisati matricu impedancija grana $\mathbf{Z}_b$ i vektor početnih uvjeta i nezavisnih izvora grana $\mathbf{U}_{0b}$. Matrica $\mathbf{Z}_b$ mora biti regularna. Napisati temeljni sustav jednadžbi petlji, odnosno odrediti matrice impedancija petlji $\mathbf{Z}_p$ i vektor početnih uvjeta i nezavisnih izvora petlji $\mathbf{U}_{0p}$.



Rješenje: Posmicanje strujnog izvora i primjena Laplaceove transformacije **(1 bod)**
Da bismo na ispravan način postavili naponsko strujne jednadžbe grana, moramo paziti da nakon transformacije (posmicanja strujnog izvora) mreža ostane ista i da naponsko strujni odnosi unutar mreže ostanu nepromijenjeni. Odnosno, moramo paziti kako ćemo označiti pojedine struje grana.





(1 bod)

Naponsko-strujne jednadžbe grana:

$$U_1 = I_1 \cdot R_1 - I_0 \cdot R_1$$

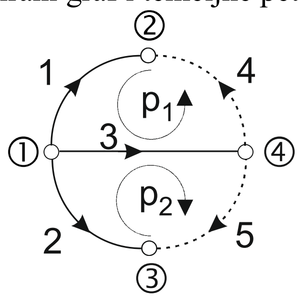
$$U_2 = I_2 \cdot R_2 + I_0 \cdot R_2$$

$$U_3 = -I_1 \cdot \alpha R_3 + I_3 \cdot R_3 + I_0 \cdot \alpha R_3$$

$$U_4 = I_4 \cdot \frac{1}{sC} + \frac{u_C(0)}{s}$$

$$U_5 = I_5 \cdot sL + Li_L(0)$$

Orijentirani graf i temeljne petlje:



Spojna matrica:

$$\mathbf{S} = \begin{array}{c} \\ (p_1) \\ (p_2) \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Naponsko-strujne relacije grana u matričnom obliku: $\mathbf{U}_b = \mathbf{Z}_b \cdot \mathbf{I}_b + \mathbf{U}_{0b}$

$$\underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix}}_{\mathbf{U}_b} = \underbrace{\begin{bmatrix} R_1 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 \\ -\alpha R_3 & 0 & R_3 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{sC} & 0 \\ 0 & 0 & 0 & 0 & sL \end{bmatrix}}_{\mathbf{Z}_b} \cdot \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix}}_{\mathbf{I}_b} + \underbrace{\begin{bmatrix} -I_0 R_1 \\ I_0 R_2 \\ \alpha R_3 I_0 \\ \frac{u_C(0)}{s} \\ Li_L(0) \end{bmatrix}}_{\mathbf{U}_{0b}}$$ **(1 bod)**

Matrica $\mathbf{Z}_b$ je regularna, jer nema niti jedan stupac niti redak jednak nuli. Temeljni sustav jednadžbi petlji u matričnom obliku: $\mathbf{Z}_p \cdot \mathbf{I}_p = \mathbf{U}_{0p}$, gdje su (matrice $\mathbf{Z}_p$ i $\mathbf{U}_{0p}$):

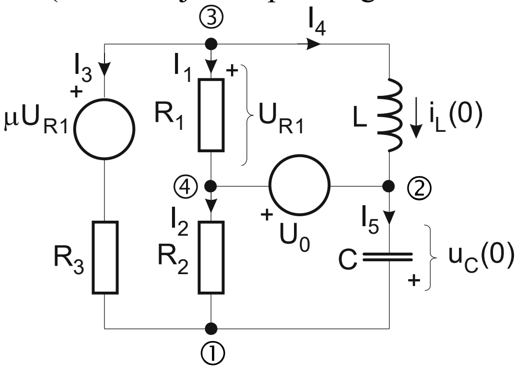
$$\mathbf{Z}_p = \mathbf{S} \cdot \mathbf{Z}_b \cdot \mathbf{S}^T = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 \\ -\alpha R_3 & 0 & R_3 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{sC} & 0 \\ 0 & 0 & 0 & 0 & sL \end{bmatrix} \cdot \mathbf{S}^T =$$

$$= \begin{bmatrix} -R_1 - \alpha R_3 & 0 & R_3 & \frac{1}{sC} & 0 \\ -\alpha R_3 & -R_2 & R_3 & 0 & sL \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 + \alpha R_3 + R_3 + \frac{1}{sC} & R_3 \\ \alpha R_3 + R_3 & R_2 + R_3 + sL \end{bmatrix}$$ **(1 bod)**

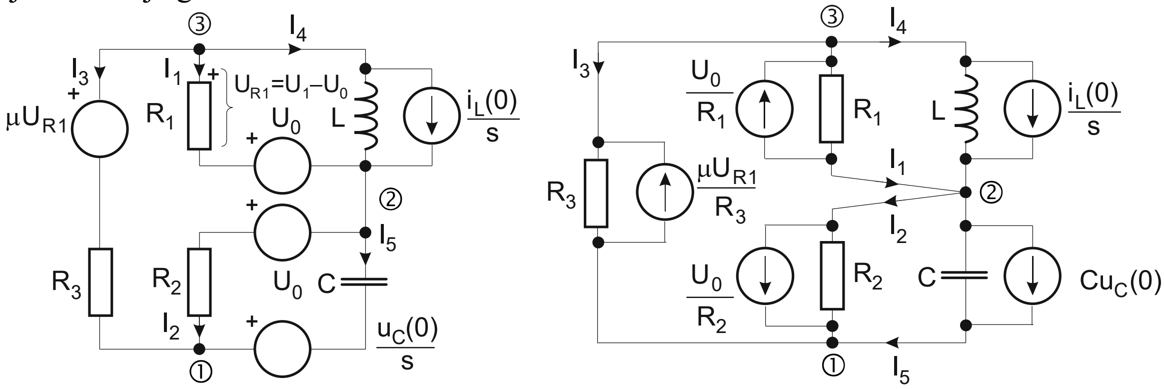
$$\mathbf{U}_{0p} = -\mathbf{S} \cdot \mathbf{U}_{0b} = -\begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -I_0 R_1 \\ I_0 R_2 \\ \alpha R_3 I_0 \\ \frac{u_C(0)}{s} \\ Li_L(0) \end{bmatrix} = -\begin{bmatrix} I_0 R_1 + \alpha R_3 I_0 + \frac{u_C(0)}{s} \\ -I_0 R_2 + \alpha R_3 I_0 + Li_L(0) \end{bmatrix}$$ **(1 bod)**

Rješenje: $\mathbf{Z}_p \cdot \mathbf{I}_p = \mathbf{U}_{0p} \;\Rightarrow\; \mathbf{I}_p = \begin{bmatrix} I_{p1} \\ I_{p2} \end{bmatrix}$

2. Zadan je električni krug prema slici. Poštujući oznake čvorišta i grana nacrtati pripadni orijentirani graf i napisati matricu incidencija **A**. Napisati strujno-naponske jednadžbe grana u matričnom obliku te ispisati matricu admitancija grana $\mathbf{Y}_b$ i vektor početnih uvjeta i nezavisnih strujnih izvora grana $\mathbf{I}_{0b}$. Matrica $\mathbf{Y}_b$ mora biti regularna. Napisati sustav jednadžbi čvorišta, odnosno odrediti matrice admitancija čvorova $\mathbf{Y}_v$ i vektor početnih uvjeta i nezavisnih izvora čvorova $\mathbf{I}_{0v}$. (Posmicanjem naponskog izvora U_0 nestaje čvor 4).



Rješenje: Posmicanje naponskog izvora i primjena Laplaceove transformacije **(1 bod)**
Da bismo na ispravan način postavili naponsko strujne jednadžbe grana, moramo paziti da nakon transformacije (posmicanja naponskog izvora) mreža ostane ista i da naponsko strujni odnosi unutar mreže ostanu nepromijenjeni. Odnosno, moramo paziti kako ćemo označiti pojedine struje grana.



(1 bod)

Strujno-naponske jednadžbe grana (struje izražene pomoću napona):

$$I_1 = U_1 \cdot 1/R_1 - U_0 \cdot 1/R_1$$

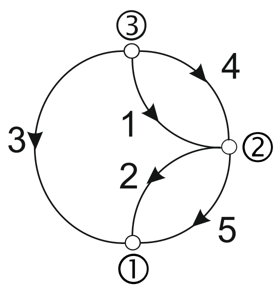
$$I_2 = U_2 \cdot 1/R_2 + U_0 \cdot 1/R_2$$

$$I_3 = -U_1 \cdot \mu/R_3 + U_3 \cdot 1/R_3 + U_0 \cdot \mu/R_3$$

$$I_4 = U_4 \cdot \frac{1}{sL} + \frac{i_L(0)}{s}$$

$$I_5 = U_5 \cdot sC + Cu_C(0)$$

Orijentirani graf:



Matrica incidencija (reducirana): $\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 & -1 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$ (1 bod)

Strujno-naponske relacije grana u matričnom obliku: $\mathbf{I}_b = \mathbf{Y}_b \cdot \mathbf{U}_b + \mathbf{I}_{0b}$

$$\underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix}}_{\mathbf{I}_b} = \underbrace{\begin{bmatrix} \frac{1}{R_1} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R_2} & 0 & 0 & 0 \\ -\frac{\mu}{R_3} & 0 & \frac{1}{R_3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{sL} & 0 \\ 0 & 0 & 0 & 0 & sC \end{bmatrix}}_{\mathbf{Y}_b} \cdot \underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix}}_{\mathbf{U}_b} + \underbrace{\begin{bmatrix} -U_0 \frac{1}{R_1} \\ U_0 \frac{1}{R_2} \\ U_0 \frac{\mu}{R_3} \\ \frac{i_L(0)}{s} \\ Cu_C(0) \end{bmatrix}}_{\mathbf{I}_{0b}}$$ **(1 bod)**

Matrica $\mathbf{Y}_b$ je regularna. Sustav jednadžbi napona čvorova u matričnom obliku $\mathbf{Y}_v \cdot \mathbf{U}_v = \mathbf{I}_{0v}$, gdje su (matrice $\mathbf{Y}_v$ i $\mathbf{I}_{0v}$):

$$\mathbf{Y}_v = \mathbf{A} \cdot \mathbf{Y}_b \cdot \mathbf{A}^T = \begin{bmatrix} -1 & 1 & 0 & -1 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{R_1} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R_2} & 0 & 0 & 0 \\ -\frac{\mu}{R_3} & 0 & \frac{1}{R_3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{sL} & 0 \\ 0 & 0 & 0 & 0 & sC \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{1}{R_1} & \frac{1}{R_2} & 0 & -\frac{1}{sL} & sC \\ \frac{1}{R_1} - \frac{\mu}{R_3} & 0 & \frac{1}{R_3} & \frac{1}{sL} & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{sL} + sC & -\frac{1}{R_1} - \frac{1}{sL} \\ -\frac{1}{R_1} + \frac{\mu}{R_3} - \frac{1}{sL} & \frac{1}{R_1} + \frac{1-\mu}{R_3} + \frac{1}{sL} \end{bmatrix}$$

(1 bod)

$$\mathbf{I}_{0v} = -\mathbf{A} \cdot \mathbf{I}_{0b} = -\begin{bmatrix} -1 & 1 & 0 & -1 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -U_0 \frac{1}{R_1} \\ U_0 \frac{1}{R_2} \\ U_0 \frac{\mu}{R_3} \\ \frac{i_L(0)}{s} \\ Cu_C(0) \end{bmatrix} = -\begin{bmatrix} U_0 \frac{1}{R_1} + U_0 \frac{1}{R_2} - \frac{i_L(0)}{s} + Cu_C(0) \\ -U_0 \frac{1}{R_1} + U_0 \frac{\mu}{R_3} + \frac{i_L(0)}{s} \end{bmatrix}$$

(1 bod)

Rješenje: $\mathbf{Y}_v \cdot \mathbf{U}_v = \mathbf{I}_{0v} \;\Rightarrow\; \mathbf{U}_v = \begin{bmatrix} U_{v1} \\ U_{v2} \end{bmatrix}$

2. Zadana je reducirana matrica incidencija grafa u obliku:

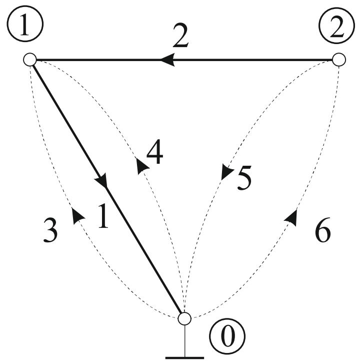
$$\mathbf{A}=\begin{bmatrix}\mathbf{A}_{\mathbf{ST}} & \mathbf{A}_{\mathbf{SP}}\end{bmatrix} \qquad \mathbf{A}=\begin{bmatrix}1 & -1 & -1 & -1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & -1\end{bmatrix}$$

pri čemu je $\mathbf{A}_{\mathbf{ST}}$ incidentna stablu, a $\mathbf{A}_{\mathbf{SP}}$ sponama. Nacrtati graf. Odrediti temeljnu spojnu matricu **S** i temeljnu rastavnu matricu **Q** ovog grafa. Kojeg ranga je graf?

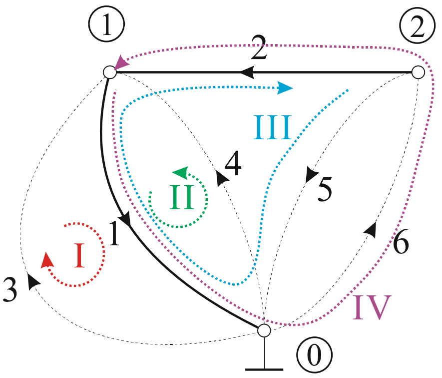
$N_{č}=2+1=3$

$N_{ST}=2, N_{SP}=4$

Graf

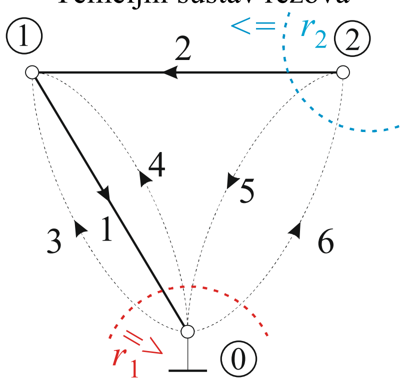


Temeljni sustav petlji



(1 bod)

Temeljni sustav rezova



(1 bod)

Temeljna spojna matrica **S**

$$\mathbf{S}=\left[\begin{array}{cc|cccc}1 & 0 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 1 & 0 & 0\\ -1 & -1 & 0 & 0 & 1 & 0\\ 1 & 1 & 0 & 0 & 0 & 1\end{array}\right]$$

(1 bod)

Temeljna rastavna matrica **Q**

$$\mathbf{Q}=\left[\begin{array}{cc|cccc}1 & 0 & -1 & -1 & 1 & -1\\ 0 & 1 & 0 & 0 & 1 & -1\end{array}\right]$$

(1 bod)

Rang grafa $R=N_{ST}=2$

Nulitet grafa $0=N_{SP}=4$

(1 bod)

2. Zadana je reducirana matrica incidencija grafa u obliku:

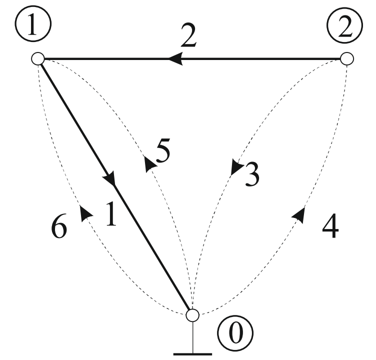
$$\mathbf{A} = \left[\mathbf{A}_{\mathbf{ST}} \quad \mathbf{A}_{\mathbf{SP}}\right] \qquad \mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 & 0 & 0 \end{bmatrix}$$

pri čemu je $\mathbf{A}_{\mathbf{ST}}$ incidentna stablu, a $\mathbf{A}_{\mathbf{SP}}$ sponama. Nacrtati graf. Odrediti temeljnu spojnu matricu **S** i temeljnu rastavnu matricu **Q** ovog grafa. Kojeg ranga je graf?

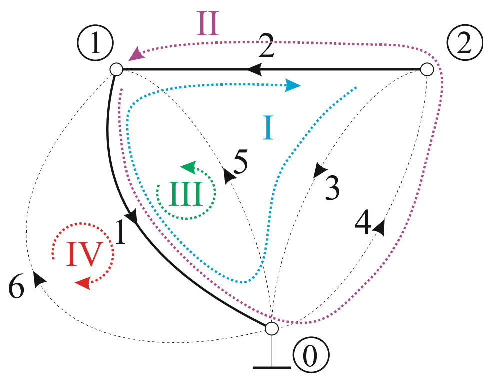
$N_{\check{c}} = 2 + 1 = 3$

$N_{ST} = 2, N_{SP} = 4$

Graf

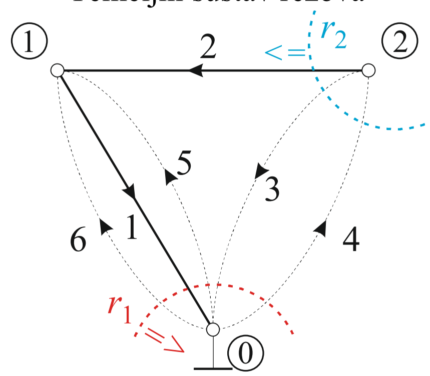


Temeljni sustav petlji



(1 bod)

Temeljni sustav rezova



(1 bod)

Temeljna spojna matrica **S**

$$\mathbf{S} = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(1 bod)

Temeljna rastavna matrica **Q**

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 1 & 1 & -1 & 0 & 0 \end{bmatrix}$$

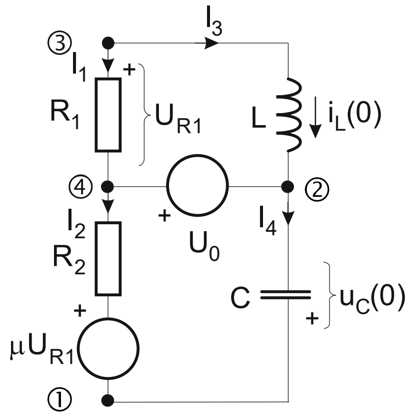
(1 bod)

Rang grafa $R = N_{ST} = 2$

Nulitet grafa $0 = N_{SP} = 4$

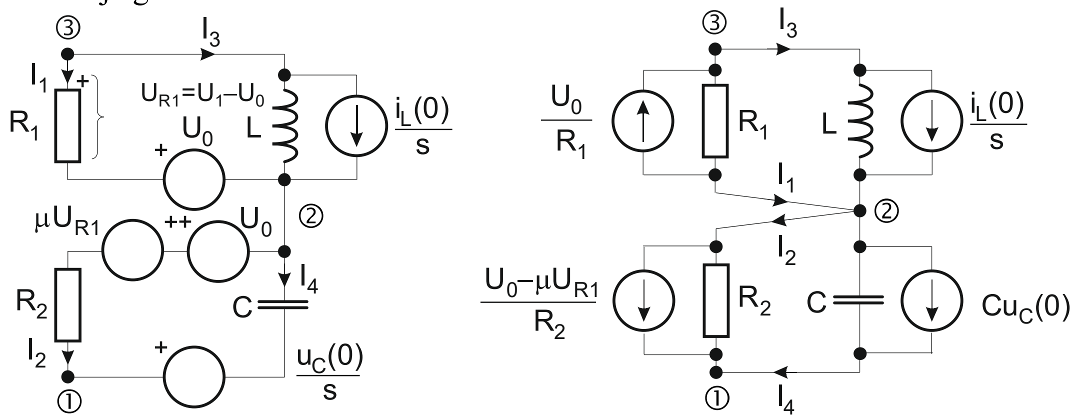
(1 bod)

2. Zadan je električni krug prema slici. Poštujući oznake čvorišta i grana nacrtati pripadni orijentirani graf i napisati matricu incidencija **A**. Napisati strujno-naponske jednadžbe grana u matričnom obliku te ispisati matricu admitancija grana $\mathbf{Y}_b$ i vektor početnih uvjeta i nezavisnih strujnih izvora grana $\mathbf{I}_{0b}$. Matrica $\mathbf{Y}_b$ mora biti regularna. Napisati sustav jednadžbi čvorišta, odnosno odrediti matrice admitancija čvorova $\mathbf{Y}_v$ i vektor početnih uvjeta i nezavisnih izvora čvorova $\mathbf{I}_{0v}$. (Posmicanjem naponskog izvora U_0 nestaje čvor 4, čvor 1 je referentan).



Rješenje: Posmicanje naponskog izvora i primjena Laplaceove transformacije

Da bismo na ispravan način postavili naponsko strujne jednadžbe grana, moramo paziti da nakon transformacije (posmicanja naponskog izvora) mreža ostane ista i da naponsko strujni odnosi unutar mreže ostanu nepromijenjeni. Odnosno, moramo paziti kako ćemo označiti pojedine struje grana.



(1 bod)

Strujno-naponske jednadžbe grana (struje izražene pomoću napona):

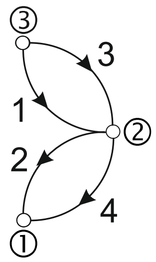
$$I_1 = U_1 \cdot 1/R_1 - U_0 \cdot 1/R_1$$

$$I_2 = -U_1 \cdot \mu / R_2 + U_2 \cdot 1/R_2 + U_0 \cdot (1+\mu)/R_2$$

$$I_3 = U_3 \cdot \frac{1}{sL} + \frac{i_L(0)}{s}$$

$$I_4 = U_4 \cdot sC + Cu_C(0)$$

Orijentirani graf:



Matrica incidencija (reducirana):
$$\mathbf{A} = \begin{array}{c} (2) \\ (3) \end{array} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$ (1 bod)

Strujno-naponske relacije grana u matričnom obliku: $\mathbf{I}_b = \mathbf{Y}_b \cdot \mathbf{U}_b + \mathbf{I}_{0b}$

$$\underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}}_{\mathbf{I}_b} = \underbrace{\begin{bmatrix} \frac{1}{R_1} & 0 & 0 & 0 \\ -\frac{\mu}{R_2} & \frac{1}{R_2} & 0 & 0 \\ 0 & 0 & \frac{1}{sL} & 0 \\ 0 & 0 & 0 & sC \end{bmatrix}}_{\mathbf{Y}_b} \cdot \underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}}_{\mathbf{U}_b} + \underbrace{\begin{bmatrix} -U_0 \frac{1}{R_1} \\ U_0 \frac{1+\mu}{R_2} \\ \frac{i_L(0)}{s} \\ Cu_C(0) \end{bmatrix}}_{\mathbf{I}_{0b}}$$ (1 bod)

Matrica $\mathbf{Y}_b$ je regularna. Sustav jednadžbi napona čvorova u matričnom obliku $\mathbf{Y}_v \cdot \mathbf{U}_v = \mathbf{I}_{0v}$, gdje su (matrice $\mathbf{Y}_v$ i $\mathbf{I}_{0v}$):

$$\mathbf{Y}_v = \mathbf{A} \cdot \mathbf{Y}_b \cdot \mathbf{A}^T = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{R_1} & 0 & 0 & 0 \\ -\frac{\mu}{R_2} & \frac{1}{R_2} & 0 & 0 \\ 0 & 0 & \frac{1}{sL} & 0 \\ 0 & 0 & 0 & sC \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{1}{R_1} - \frac{\mu}{R_2} & \frac{1}{R_2} & -\frac{1}{sL} & sC \\ \frac{1}{R_1} & 0 & \frac{1}{sL} & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1} + \frac{\mu}{R_2} + \frac{1}{R_2} + \frac{1}{sL} + sC & -\frac{1}{R_1} - \frac{\mu}{R_2} - \frac{1}{sL} \\ -\frac{1}{R_1} - \frac{1}{sL} & \frac{1}{R_1} + \frac{1}{sL} \end{bmatrix}$$

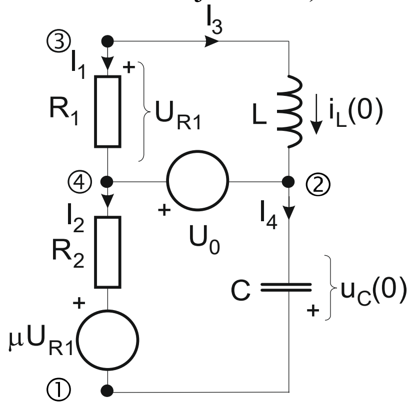
(1 bod)

$$\mathbf{I}_{0v} = -\mathbf{A} \cdot \mathbf{I}_{0b} = -\begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -U_0 \frac{1}{R_1} \\ U_0 \frac{1+\mu}{R_2} \\ \frac{i_L(0)}{s} \\ Cu_C(0) \end{bmatrix} = -\begin{bmatrix} U_0 \frac{1}{R_1} + U_0 \frac{1+\mu}{R_2} - \frac{i_L(0)}{s} + Cu_C(0) \\ -U_0 \frac{1}{R_1} + \frac{i_L(0)}{s} \end{bmatrix}$$

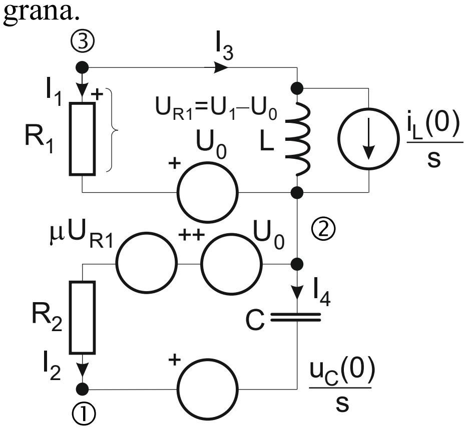
(1 bod)

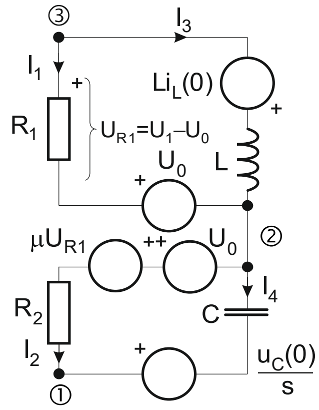
Rješenje: $\mathbf{Y}_v \cdot \mathbf{U}_v = \mathbf{I}_{0v} \Rightarrow \mathbf{U}_v = \begin{bmatrix} U_{v1} \\ U_{v2} \end{bmatrix}$

2. Zadan je električni krug prema slici. Poštujući oznake čvorišta i grana nacrtati pripadni orijentirani graf i napisati spojnu matricu **S**. Napisati naponsko-strujne jednadžbe grana u matričnom obliku te ispisati matricu impedancija grana $\mathbf{Z}_b$ i vektor početnih uvjeta i nezavisnih izvora grana $\mathbf{U}_{0b}$. Matrica $\mathbf{Z}_b$ mora biti regularna. Napisati temeljni sustav jednadžbi petlji, odnosno odrediti matrice impedancija petlji $\mathbf{Z}_p$ i vektor početnih uvjeta i nezavisnih izvora petlji $\mathbf{U}_{0p}$. (Posmicanjem naponskog izvora U_0 nestaje čvor 4).



Rješenje: Posmicanje naponskog izvora i primjena Laplaceove transformacije
Da bismo na ispravan način postavili naponsko strujne jednadžbe grana, moramo paziti da nakon transformacije (posmicanja naponskog izvora) mreža ostane ista i da naponsko strujni odnosi unutar mreže ostanu nepromijenjeni. Odnosno, moramo paziti kako ćemo označiti pojedine struje grana.



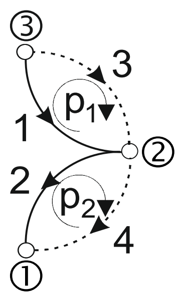


(1 bod)

Naponsko-strujne jednadžbe grana (naponi izražen pomoću struja):

$$U_1 = R_1 \cdot I_1 + U_0$$
$$U_2 = R_2 \cdot I_2 + \mu \cdot U_{R1} - U_0 = R_2 \cdot I_2 + \mu \cdot (U_1 - U_0) - U_0$$
$$= R_2 I_2 + \mu U_1 - \mu U_0 - U_0 = R_2 I_2 + \mu (R_1 I_1 + U_0) - \mu U_0 - U_0$$
$$= \mu R_1 \cdot I_1 + R_2 \cdot I_2 - U_0$$
$$U_3 = sL \cdot I_3 - L i_L(0)$$
$$U_4 = \frac{1}{sC} \cdot I_4 - \frac{u_C(0)}{s}$$

Orijentirani graf:



Spojna matrica:

$$\mathbf{S} = \begin{array}{c} \\ (p_1) \\ (p_2) \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

(1 bod)

Strujno-naponske relacije grana u matričnom obliku: $\mathbf{U}_b = \mathbf{Z}_b \cdot \mathbf{I}_b + \mathbf{U}_{0b}$

$$\underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}}_{\mathbf{U}_b} = \underbrace{\begin{bmatrix} R_1 & 0 & 0 & 0 \\ \mu R_1 & R_2 & 0 & 0 \\ 0 & 0 & sL & 0 \\ 0 & 0 & 0 & \dfrac{1}{sC} \end{bmatrix}}_{\mathbf{Z}_b} \cdot \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}}_{\mathbf{I}_b} + \underbrace{\begin{bmatrix} U_0 \\ -U_0 \\ -Li_L(0) \\ -\dfrac{u_C(0)}{s} \end{bmatrix}}_{\mathbf{U}_{0b}} \quad \text{(1 bod)}$$

Matrica $\mathbf{Z}_b$ je regularna, jer nema niti jedan stupac niti redak jednak nuli. Temeljni sustav jednadžbi petlji u matričnom obliku: $\mathbf{Z}_p \cdot \mathbf{I}_p = \mathbf{U}_{0p}$, gdje su (matrice $\mathbf{Z}_p$ i $\mathbf{U}_{0p}$):

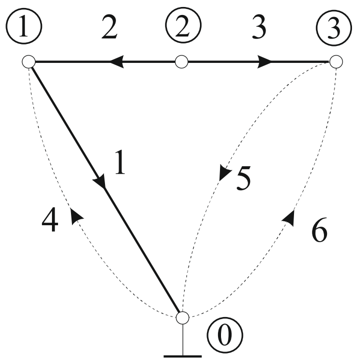
$$\mathbf{Z}_p = \mathbf{S} \cdot \mathbf{Z}_b \cdot \mathbf{S}^T = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R_1 & 0 & 0 & 0 \\ \mu R_1 & R_2 & 0 & 0 \\ 0 & 0 & sL & 0 \\ 0 & 0 & 0 & \dfrac{1}{sC} \end{bmatrix} \cdot \mathbf{S}^T =$$

$$= \begin{bmatrix} -R_1 & 0 & sL & 0 \\ -\mu R_1 & -R_2 & 0 & \dfrac{1}{sC} \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 + sL & 0 \\ \mu R_1 & R_2 + \dfrac{1}{sC} \end{bmatrix} \quad \textbf{(1 bod)}$$

$$\mathbf{U}_{0p} = -\mathbf{S} \cdot \mathbf{U}_{0b} = -\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} U_0 \\ -U_0 \\ -Li_L(0) \\ -\dfrac{u_C(0)}{s} \end{bmatrix} = -\begin{bmatrix} -U_0 - Li_L(0) \\ U_0 - \dfrac{u_C(0)}{s} \end{bmatrix} = \begin{bmatrix} U_0 + Li_L(0) \\ -U_0 + \dfrac{u_C(0)}{s} \end{bmatrix} \quad \textbf{(1 bod)}$$

Rješenje: $\mathbf{Z}_p \cdot \mathbf{I}_p = \mathbf{U}_{0p} \;\Rightarrow\; \mathbf{I}_p = \begin{bmatrix} I_{p1} \\ I_{p2} \end{bmatrix}$

2. Zadan je orijentirani graf sa stablom. Napisati reduciranu matricu incidencija grafa u obliku: **A**=[**A**ST **A**SP], pri čemu je **A**ST incidentna stablu, a **A**SP sponama. Odrediti temeljnu spojnu matricu **S** i temeljnu rastavnu matricu **Q** ovog grafa: a) grafički, b) računski (množenjem matrica). Kojeg ranga je graf?



Rješenje:

$N_{č} = 3+1=4$

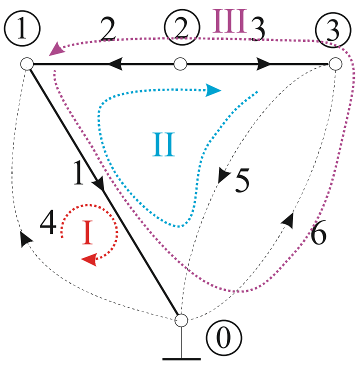
$N_{ST}=3, N_{SP}=3$

a) grafički:

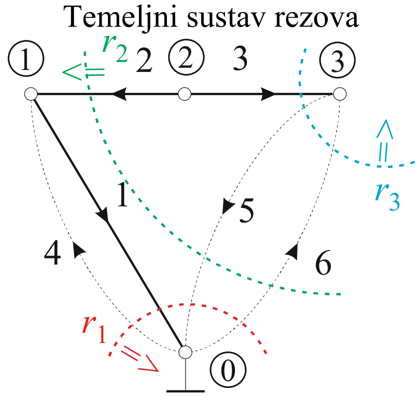
Matrica incidencija

$$\mathbf{A}=\left[\begin{array}{ccc|ccc}1&-1&0&-1&0&0\\0&1&1&0&0&0\\0&0&-1&0&1&-1\end{array}\right]=\left[\mathbf{A}_{\mathbf{ST}}\mid\mathbf{A}_{\mathbf{SP}}\right]\ \textbf{(1 bod)}$$

Temeljni sustav petlji



Temeljni sustav rezova



Temeljna spojna matrica **S**

$$\mathbf{S}=\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\-1&-1&1&0&1&0\\1&1&-1&0&0&1\end{array}\right]$$

(1 bod)

Temeljna rastavna matrica **Q**

$$\mathbf{Q}=\left[\begin{array}{ccc|ccc}1&0&0&-1&1&-1\\0&1&0&0&1&-1\\0&0&1&0&-1&1\end{array}\right]$$

(1 bod)

Rang grafa $R=N_{ST}=3$

Nulitet grafa $N=N_{SP}=3$

b) računski (množenjem matrica):

Znamo da vrijedi: $\mathbf{A}=\left[\mathbf{A}_{\mathbf{ST}}\quad\mathbf{A}_{\mathbf{SP}}\right], \mathbf{S}=\left[\mathbf{S}_{\mathbf{ST}}\quad\mathbf{E}\right];\ \mathbf{Q}=\left[\mathbf{E}\quad\mathbf{Q}_{\mathbf{SP}}\right]$

-Matrica **S**:

Iz $\mathbf{S}\cdot\mathbf{A}^t=\begin{bmatrix}\mathbf{S}_{ST} & \mathbf{E}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{A}_{ST}^t\\ \mathbf{A}_{SP}^t\end{bmatrix}=\mathbf{S}_{ST}\cdot\mathbf{A}_{ST}^t+\mathbf{A}_{SP}^t=\mathbf{0}\Rightarrow\boxed{\mathbf{S}_{ST}=-\mathbf{A}_{SP}^t\cdot\left[\mathbf{A}_{ST}^t\right]^{-1}}$;

$$\mathbf{S}_{ST}=-\mathbf{A}_{SP}^t\cdot\mathbf{A}_{ST}^t=-\begin{bmatrix}-1&0&0\\0&0&1\\0&0&-1\end{bmatrix}\cdot\begin{bmatrix}1&0&0\\-1&1&0\\0&1&-1\end{bmatrix}^{-1}=\begin{bmatrix}1&0&0\\0&0&-1\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0\\1&1&0\\1&1&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&-1&1\\1&1&-1\end{bmatrix}$$

$$\mathbf{S}=\begin{bmatrix}\mathbf{S}_{ST} & \mathbf{E}\end{bmatrix}=\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\-1&-1&1&0&1&0\\1&1&-1&0&0&1\end{array}\right]\ \textbf{(1 bod)}$$

Pritom je izvršeno invertiranje matrice:

$$\begin{bmatrix}1&0&0\\-1&1&0\\0&1&-1\end{bmatrix}^{-1}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&-1\end{bmatrix}$$

- prvi redak pomnožen s 1 i dodan drugom retku

$$\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\-1&1&0&0&1&0\\0&1&-1&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&1&1&0\\0&1&-1&0&0&1\end{array}\right]$$

- drugi redak pomnožen s -1 i dodan drugom retku

$$\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&1&1&0\\0&1&-1&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&1&1&0\\0&0&-1&-1&-1&1\end{array}\right]$$

- treći redak pomnožen s -1

$$\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&1&1&0\\0&0&-1&-1&-1&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&1&1&0\\0&0&1&1&1&-1\end{array}\right]$$

- Matrica **Q**:

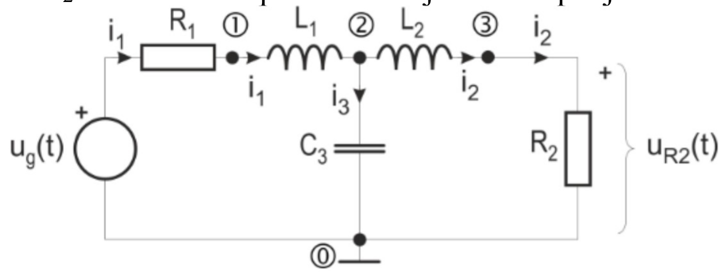
Iz $\mathbf{Q}\cdot\mathbf{S}^t=\begin{bmatrix}\mathbf{E} & \mathbf{Q}_{SP}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{S}_{ST}^t\\ \mathbf{E}\end{bmatrix}=\mathbf{S}_{ST}^t+\mathbf{Q}_{SP}=\mathbf{0}\Rightarrow\boxed{\mathbf{Q}_{SP}=-\mathbf{S}_{ST}^t}$

$$\mathbf{Q}_{SP}=-\mathbf{S}_{ST}^t=-\begin{bmatrix}1&0&0\\-1&-1&1\\1&1&-1\end{bmatrix}^t=-\begin{bmatrix}1&-1&1\\0&-1&1\\0&1&-1\end{bmatrix}=\begin{bmatrix}-1&1&-1\\0&1&-1\\0&-1&1\end{bmatrix}$$

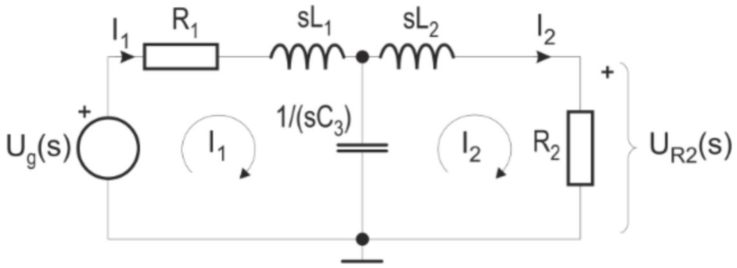
$$\mathbf{Q}=\begin{bmatrix}\mathbf{E} & \mathbf{Q}_{SP}\end{bmatrix}=\left[\begin{array}{ccc|ccc}1&0&0&-1&1&-1\\0&1&0&0&1&-1\\0&0&1&0&-1&1\end{array}\right]\ \textbf{(1 bod)}$$

<u>Zaključak:</u> uz poznato stablo grafa iz matrice **A** moguće je izračunati matrice **S** i **Q**.
Napomena: Formule koje su uokvirene su glavne formule koje trebamo koristiti u ovom b) računskom dijelu zadatka. One su izvedene na predavanjima, a također su izvedene ovdje.

2. Za električni krug na slici zadane su normalizirane vrijednosti elemenata R_1=R_2=1, L_1=1, L_2=1, C_3=1, početne struje kroz induktivitete i napon na kapacitetu su jednaki nuli, te pobuda $u_g(t)$=δ(t). Primjenom Laplaceove transformacije i koristeći metodu petlji izračunati napon $u_{R2}(t)$ na otporu otpor R_2 kao odziv. Napisati sustav jednadžbi petlji u matričnom obliku.



Rješenje: Primjena Laplaceove transformacije



Metoda petlji:

$$1)\ I_1(s)\left(R_1+sL_1+\frac{1}{sC_3}\right)-I_2(s)\frac{1}{sC_3}=U_g(s);$$

$$2)\ -I_1(s)\frac{1}{sC_3}+I_2(s)\left(R_2+sL_2+\frac{1}{sC_3}\right)=0;$$

(1 bod)

$$\begin{bmatrix} R_1+sL_1+\frac{1}{sC_3} & -\frac{1}{sC_3} \\ -\frac{1}{sC_3} & R_2+sL_2+\frac{1}{sC_3} \end{bmatrix}\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix}=\begin{bmatrix} U_g(s) \\ 0 \end{bmatrix}$$

Uvrstimo elemente: R_1=R_2=1, L_1=1, L_2=1, C_1=1,

$$\begin{bmatrix} 1+s+\frac{1}{s} & -\frac{1}{s} \\ -\frac{1}{s} & 1+s+\frac{1}{s} \end{bmatrix}\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix}=\begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix} \quad \text{(1 bod)}$$

$$\Delta=\begin{vmatrix} 1+s+\frac{1}{s} & -\frac{1}{s} \\ -\frac{1}{s} & 1+s+\frac{1}{s} \end{vmatrix}=\left(1+s+\frac{1}{s}\right)^2-\left(\frac{1}{s}\right)^2$$

$$\Delta=\left(1+s+\frac{1}{s}\right)^2-\left(\frac{1}{s}\right)^2=(1+s)\left(1+s+\frac{2}{s}\right)=s^3+5s^2+3s+3+\frac{1}{s}$$

$$\Delta_2 = \begin{vmatrix} 1+s+\frac{1}{s} & U_g \\ -\frac{1}{s} & 0 \end{vmatrix} = U_g \frac{1}{s}$$

$$U_{R2}(s) = R_2 I_{R2}(s) = R_2 \frac{\Delta_2}{\Delta} = \frac{U_g \frac{1}{s}}{(1+s)\left(1+s+\frac{2}{s}\right)} = \frac{U_g(s)}{(1+s)(s^2+s+2)} = \frac{1}{(1+s)(s^2+s+2)};$$

(2 boda)

Rastav na parcijalne razlomke:

$$\frac{1}{(1+s)(s^2+s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+2} = \frac{A(s^2+s+2)+(Bs+C)(s+1)}{(1+s)(s^2+s+2)}$$

$$(A+B)s^2 + (A+B+C)s + (2A+C) = 1$$

$$\begin{aligned} &(A+B) = 0 \\ &(A+B+C) = 0 \;\Rightarrow\; \underline{A = \frac{1}{2}; B = -\frac{1}{2}; C = 0} \\ &\underline{(2A+C) = 1} \end{aligned}$$

$$U_{R2}(s) = \frac{1}{(1+s)(s^2+s+2)} = \frac{1}{2} \cdot \frac{1}{s+1} - \frac{1}{2} \cdot \frac{s}{s^2+s+2}$$

gdje je

$$\frac{1}{2} \cdot \frac{s}{s^2+s+2} = \frac{1}{2} \cdot \frac{s}{s^2+s+\frac{1}{4}+\frac{7}{4}} = \frac{1}{2} \cdot \frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} =$$

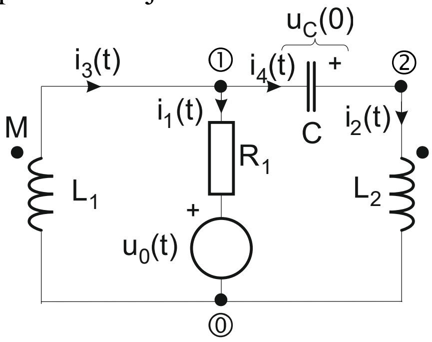
$$= \frac{1}{2} \cdot \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} - \frac{1}{4} \frac{2}{\sqrt{7}} \frac{\frac{\sqrt{7}}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2}$$

pa je

$$U_{R2}(s) = \frac{1}{2} \frac{1}{s+1} - \frac{1}{2} \cdot \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} + \frac{1}{2\sqrt{7}} \frac{\frac{\sqrt{7}}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2}$$

$$\underline{u_{R2}(t) = \frac{1}{2}\left[e^{-t} - e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{7}}{2} t\right) + \frac{1}{\sqrt{7}} e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{7}}{2} t\right)\right] \cdot S(t)}$$ (1 bod)

2. Zadan je električni krug prema slici. Poštujući oznake čvorišta i grana nacrtati pripadni orijentirani graf i napisati spojnu matricu **S** ako su 1 i 2 grane stabla. Napisati naponsko-strujne jednadžbe grana u matričnom obliku te ispisati matricu impedancija grana $\mathbf{Z}_b$ i vektor početnih uvjeta i nezavisnih izvora grana $\mathbf{U}_{0b}$. Matrica $\mathbf{Z}_b$ mora biti regularna. Pomoću matrica **S**, $\mathbf{Z}_b$ i $\mathbf{U}_{0b}$ napisati temeljni sustav jednadžbi petlji, odnosno odrediti matrice impedancija petlji $\mathbf{Z}_p$ i vektor početnih uvjeta i nezavisnih izvora petlji $\mathbf{U}_{0p}$.



Rješenje:

Naponsko-strujne jednadžbe grana:

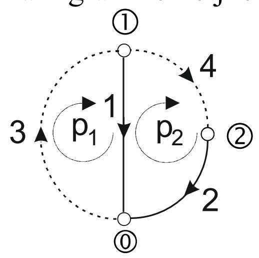
$$U_1 = I_1 \cdot R_1 + U_0$$

$$U_2 = I_2 \cdot sL_2 - I_3 \cdot sM$$

$$U_3 = -I_2 \cdot sM + I_3 \cdot sL_1$$

$$U_4 = I_4 \cdot \frac{1}{sC} - \frac{u_C(0)}{s}$$

Orijentirani graf i temeljne petlje:



(1 bod)

Spojna matrica:

$$\mathbf{S} = \begin{array}{c} \\ (p_1) \\ (p_2) \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right] \end{array}$$ **(1 bod)**

Naponsko-strujne relacije grana u matričnom obliku: $\mathbf{U}_b = \mathbf{Z}_b \cdot \mathbf{I}_b + \mathbf{U}_{0b}$

$$\underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}}_{\mathbf{U}_b} = \underbrace{\begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & sL_2 & -sM & 0 \\ 0 & -sM & sL_1 & 0 \\ 0 & 0 & 0 & \frac{1}{sC} \end{bmatrix}}_{\mathbf{Z}_b} \cdot \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}}_{\mathbf{I}_b} + \underbrace{\begin{bmatrix} U_0 \\ 0 \\ 0 \\ -\frac{u_C(0)}{s} \end{bmatrix}}_{\mathbf{U}_{0b}}$$ **(1 bod)**

Matrica $\mathbf{Z}_b$ je regularna, jer nema niti jedan stupac niti redak jednak nuli. Temeljni sustav jednadžbi petlji u matričnom obliku: $\mathbf{Z}_p \cdot \mathbf{I}_p = \mathbf{U}_{0p}$, gdje su (matrice $\mathbf{Z}_p$ i $\mathbf{U}_{0p}$):

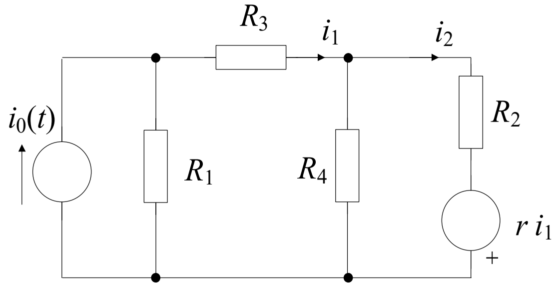
$$\mathbf{Z}_p = \mathbf{S} \cdot \mathbf{Z}_b \cdot \mathbf{S}^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & sL_2 & -sM & 0 \\ 0 & -sM & sL_1 & 0 \\ 0 & 0 & 0 & \frac{1}{sC} \end{bmatrix} \cdot \mathbf{S}^T =$$

$$=\begin{bmatrix} R_1 & -sM & sL_1 & 0 \\ -R_1 & sL_2 & -sM & \dfrac{1}{sC} \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 + sL_1 & -R_1 - sM \\ -R_1 - sM & R_1 + sL_2 + \dfrac{1}{sC} \end{bmatrix}$$ **(1 bod)**

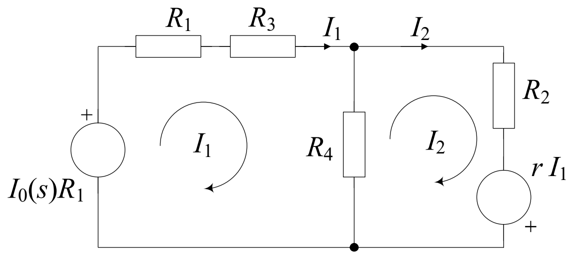
$$\mathbf{U}_{0p} = -\mathbf{S} \cdot \mathbf{U}_{0b} = -\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} U_0 \\ 0 \\ 0 \\ -\dfrac{u_C(0)}{s} \end{bmatrix} = -\begin{bmatrix} U_0 \\ -U_0 - \dfrac{u_C(0)}{s} \end{bmatrix} = \begin{bmatrix} -U_0 \\ U_0 + \dfrac{u_C(0)}{s} \end{bmatrix}$$ **(1 bod)**

Rješenje: $\mathbf{Z}_p \cdot \mathbf{I}_p = \mathbf{U}_{0p} \;\Rightarrow\; \mathbf{I}_p = \begin{bmatrix} I_{p1} \\ I_{p2} \end{bmatrix}$

2. Za krug prikazan slikom napisati jednadžbe petlji. Konačni oblik jednadžbi prikazati u formi matrične jednadžbe. Izračunati struju $i_2(t)$, ako je zadana pobuda $i_0(t)=S(t)$, konstanta $r=1/2$, a normirane vrijednosti elemenata su: $R_1=2$, $R_2=1$, $R_3=1/2$ i $R_4=1$.



Rješenje: Transformacija strujnog izvora u naponski i metoda jednadžbi petlji



(1 bod)

Jednadžbe petlji

$$-I_0(s)R_1 + I_1(s)(R_1+R_3) + (I_1(s)-I_2(s))R_4 = 0$$

$$-(I_1(s)-I_2(s))R_4 + R_2 I_2(s) - rI_1(s) = 0$$

(1 bod)

Sustav jednadžbi glasi:

$$I_0R_1 = (R_1+R_3+R_4)I_1 - R_4I_2$$

$$0 = -I_1(r+R_4) + I_2(R_2+R_4)$$

$$\begin{bmatrix} R_1+R_3+R_4 & -R_4 \\ -(r+R_4) & R_2+R_4 \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} I_0R_1 \\ 0 \end{bmatrix}$$

(1 bod)

$$\Delta = \begin{vmatrix} R_1+R_3+R_4 & -R_4 \\ -(r+R_4) & R_2+R_4 \end{vmatrix} = (R_1+R_3+R_4)(R_2+R_4) - R_4(r+R_4)$$

$$\Delta = (R_1+R_3+R_4)R_2 + (R_1+R_3+R_4)R_4 - R_4(r+R_4)$$

$$= (R_1+R_3+R_4)R_2 + (R_1+R_3-r)R_4$$

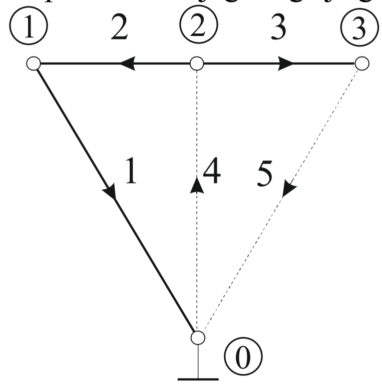
$$\Delta_2 = \begin{vmatrix} R_1+R_3+R_4 & I_0R_1 \\ -(r+R_4) & 0 \end{vmatrix} = I_0R_1(r+R_4)$$

$$I_2(s) = \frac{\Delta_2}{\Delta} = \frac{I_0(s)R_1(r+R_4)}{(R_1+R_3+R_4)R_2 + (R_1+R_3-r)R_4}$$ **(1 bod)**

Uvrstimo: $r=1/2$, $R_1=2$, $R_2=1$, $R_3=1/2$ i $R_4=1$.

$$I_2(s) = \frac{I_0(s)2\left(\frac{1}{2}+1\right)}{\left(2+\frac{1}{2}+1\right)+\left(2+\frac{1}{2}-\frac{1}{2}\right)} = \frac{3}{5+\frac{1}{2}}I_0(s) = \frac{6}{11}I_0(s) \quad \Rightarrow \quad \underline{i_2(t) = \frac{6}{11}S(t)} \text{ [A]}$$ **(1 bod)**

2. Zadan je orijentirani graf sa stablom. Napisati reduciranu matricu incidencija grafa u obliku: **A**=[**A**$\mathbf{A}_{ST}$ $\mathbf{A}_{SP}$], pri čemu je $\mathbf{A}_{ST}$ incidentna stablu, a $\mathbf{A}_{SP}$ sponama. Odrediti grafički temeljnu spojnu matricu **S**=[$\mathbf{S}_{ST}$ $\mathbf{S}_{SP}$] i temeljnu rastavnu matricu **Q**=[$\mathbf{Q}_{ST}$ $\mathbf{Q}_{SP}$] ovog grafa, te njihove dijelove koji su incidentni stablu i sponama. Kojeg ranga je graf?



Rješenje:

$N_{č} = 3 + 1 = 4$

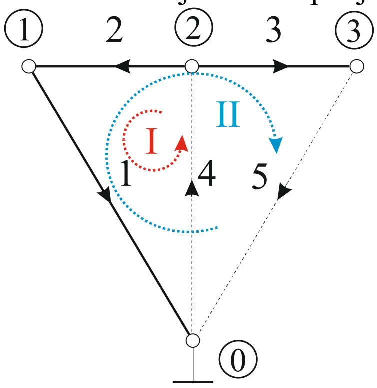
$N_{ST} = 3, N_{SP} = 2$

a) grafički:

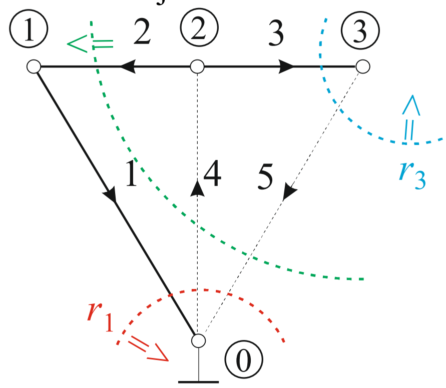
Matrica incidencija

$$\mathbf{A} = \left[\begin{array}{ccc|cc} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{array}\right] = \left[\mathbf{A}_{\mathbf{ST}} \mid \mathbf{A}_{\mathbf{SP}}\right] \quad \textbf{(1 bod)}$$

Temeljni sustav petlji



Temeljni sustav rezova



(1 bod)

Temeljna spojna matrica **S**

$$\mathbf{S} = \left[\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 1 \end{array}\right] = \left[\mathbf{S}_{\mathbf{ST}} \mid \mathbf{S}_{\mathbf{SP}}\right]$$

(1 bod)

Temeljna rastavna matrica **Q**

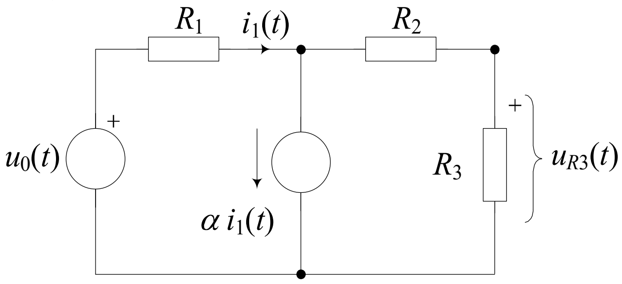
$$\mathbf{Q} = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right] = \left[\mathbf{Q}_{\mathbf{ST}} \mid \mathbf{Q}_{\mathbf{SP}}\right]$$

(1 bod)

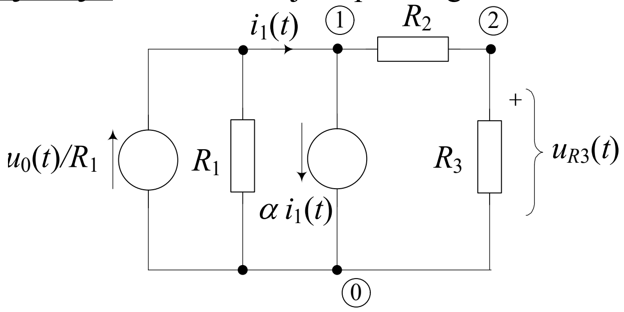
Rang grafa $R = N_{ST} = 3$

Nulitet grafa $N = N_{SP} = 2$ **(1 bod)**

3. Za mrežu prikazanu slikom napisati jednadžbe čvorišta i prikazati ih u formi matrične jednadžbe. Izračunati a) struju $i_1(t)$, i b) napon $u_{R3}(t)$ ako je pobuda $u_0(t)=S(t)$, $\alpha=1/2$ i normirane vrijednosti elemenata $R_1=2$, $R_2=1$ i $R_3=1/2$.



Rješenje: Transformacija naponskog izvora u strujni i metoda jednadžbi čvorišta



Jednadžbe za čvorišta 1 i 2

$$\frac{U_0(s)}{R_1}-\alpha I_1(s)=\frac{U_1(s)}{R_1}+\left(U_1(s)-U_2(s)\right)\frac{1}{R_2}$$

$$0=-\left(U_1(s)-U_2(s)\right)\frac{1}{R_2}+U_2(s)\frac{1}{R_3}$$

(1 bod)

(1 bod)

$$I_1(s)=\left(U_0(s)-U_1(s)\right)\frac{1}{R_1}\ \rightarrow (1)\Rightarrow \frac{U_0(s)}{R_1}-\alpha\left(U_0(s)-U_1(s)\right)\frac{1}{R_1}=\frac{U_1(s)}{R_1}+\left(U_1(s)-U_2(s)\right)\frac{1}{R_2}$$

$$U_0(s)\frac{1}{R_1}-\alpha U_0(s)\frac{1}{R_1}+\alpha U_1(s)\frac{1}{R_1}=U_1(s)\frac{1}{R_1}+U_1(s)\frac{1}{R_2}-U_2(s)\frac{1}{R_2}$$

Sustav jednadžbi glasi:

$$(1-\alpha)\frac{U_0(s)}{R_1}=\left(\frac{(1-\alpha)}{R_1}+\frac{1}{R_2}\right)U_1(s)-\frac{1}{R_2}U_2(s)$$

$$0=-\frac{1}{R_2}U_1(s)+\left(\frac{1}{R_2}+\frac{1}{R_3}\right)U_2(s)$$

$$\begin{bmatrix}\frac{1-\alpha}{R_1}+\frac{1}{R_2} & -\frac{1}{R_2}\\ -\frac{1}{R_2} & \frac{1}{R_2}+\frac{1}{R_3}\end{bmatrix}\cdot\begin{bmatrix}U_1\\U_2\end{bmatrix}=\begin{bmatrix}\frac{U_0(1-\alpha)}{R_1}\\0\end{bmatrix}\quad\Rightarrow\quad U_1=\frac{\Delta_1}{\Delta};\ U_2=\frac{\Delta_2}{\Delta}$$

$$\Delta=\begin{vmatrix}\frac{1-\alpha}{R_1}+\frac{1}{R_2} & -\frac{1}{R_2}\\ -\frac{1}{R_2} & \frac{1}{R_2}+\frac{1}{R_3}\end{vmatrix}=\left(\frac{1-\alpha}{R_1}+\frac{1}{R_2}\right)\left(\frac{1}{R_2}+\frac{1}{R_3}\right)-\frac{1}{R_2^2}=\frac{1}{R_2R_3}+\frac{1-\alpha}{R_1}\left(\frac{1}{R_2}+\frac{1}{R_3}\right)$$

$$\Delta_1=\begin{vmatrix}\frac{U_0(1-\alpha)}{R_1} & -\frac{1}{R_2}\\ 0 & \frac{1}{R_2}+\frac{1}{R_3}\end{vmatrix}=\frac{U_0(1-\alpha)}{R_1}\left(\frac{1}{R_2}+\frac{1}{R_3}\right);\Delta_2=\begin{vmatrix}\frac{1-\alpha}{R_1}+\frac{1}{R_2} & \frac{U_0(1-\alpha)}{R_1}\\ -\frac{1}{R_2} & 0\end{vmatrix}=\frac{U_0(1-\alpha)}{R_1R_2}$$

(1 bod)

a) struja $i_1(t)$

$$I_1(s)=\frac{U_0-U_1}{R_1}=\frac{U_0-\Delta_1/\Delta}{R_1}=\frac{U_0}{R_1}-\frac{1}{R_1}\frac{U_0\frac{1-\alpha}{R_1}\left(\frac{1}{R_2}+\frac{1}{R_3}\right)}{\frac{1}{R_2R_3}+\frac{1-\alpha}{R_1}\left(\frac{1}{R_2}+\frac{1}{R_3}\right)}$$

Uvrstimo: α=1/2, R_1=2, R_2=1 i R_3=1/2

$$I_1(s)=\frac{U_0}{2}-\frac{1}{2}\cdot\frac{U_0\frac{3}{4}}{2+\frac{1}{4}(1+2)}=U_0\left(\frac{1}{2}-\frac{3}{22}\right)=\frac{4}{11}U_0 \quad\Rightarrow\quad \underline{i_1(t)=\frac{4}{11}S(t)}\ \text{[A]}$$ **(1 bod)**

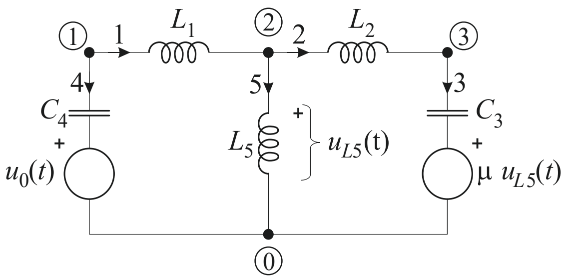
b) napon $u_{R3}(t)$

$$U_2=\frac{\Delta_2}{\Delta}=\frac{\frac{U_0(1-\alpha)}{R_1R_2}}{\frac{1}{R_2R_3}+\frac{1-\alpha}{R_1}\left(\frac{1}{R_2}+\frac{1}{R_3}\right)}$$

Uvrstimo: α=1/2, R_1=2, R_2=1 i R_3=1/2

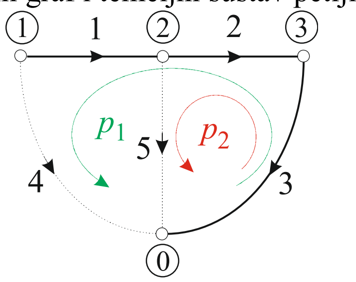
$$U_2=\frac{U_0\frac{1}{4}}{2+\frac{1}{4}(1+2)}=\frac{1}{11}U_0 \quad\Rightarrow\quad \underline{u_2(t)=\frac{1}{11}S(t)}\ \text{[V]}$$ **(1 bod)**

4. Zadan je električni krug prema slici. Poštujući oznake grana i čvorišta nacrtati pripadni orijentirani graf. Napisati naponsko-strujne jednadžbe grana u matričnom obliku te ispisati matricu impedancija grana $\mathbf{Z}_b$ i vektor izvora grana $\mathbf{U}_{0b}$. Napisati temeljni sustav jednadžbi petlji, odnosno odrediti matrice impedancija petlji $\mathbf{Z}_p$ i vektor izvora petlji $\mathbf{U}_{0p}$. (Uputa: grane 1, 2 i 3 su grane stabla.)



Rješenje:

Orijentirani graf i temeljni sustav petlji



(1 bod)

Spojna matrica:

$$\mathbf{S} = \begin{bmatrix} -1 & -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{bmatrix}$$ **(1 bod)**

Naponsko–strujne relacije grana:

$$U_1 = I_1 \cdot sL_1$$

$$U_2 = I_2 \cdot sL_2$$

$$U_3 = I_3 \cdot \frac{1}{sC_3} + \mu \cdot U_{L5}$$

$$U_4 = I_4 \cdot \frac{1}{sC_4} + U_0$$

$$U_5 = I_5 \cdot sL_5$$

$$U_{L5} = U_5 \qquad \Rightarrow U_3 = I_3 \cdot \frac{1}{sC} + I_5 \cdot \mu sL$$

$$\underset{\mathbf{U}_b}{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix}} = \underbrace{\begin{bmatrix} sL_1 & 0 & 0 & 0 & 0 \\ 0 & sL_2 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{sC_3} & 0 & \mu sL_5 \\ 0 & 0 & 0 & \frac{1}{sC_4} & 0 \\ 0 & 0 & 0 & 0 & sL_5 \end{bmatrix}}_{\mathbf{Z}_b} \underset{\mathbf{I}_b}{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix}} + \underset{\mathbf{U}_{0b}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ U_0 \\ 0 \end{bmatrix}}$$

(1 bod)

Temeljni sustav jednadžbi petlji: $\mathbf{Z}_p \cdot \mathbf{I}_p = \mathbf{U}_{0p}$

$$\mathbf{Z}_p = \mathbf{S}\mathbf{Z}_b\mathbf{S}^T = \begin{bmatrix} -1 & -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} sL_1 & 0 & 0 & 0 & 0 \\ 0 & sL_2 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{sC_3} & 0 & \mu sL_5 \\ 0 & 0 & 0 & \dfrac{1}{sC_4} & 0 \\ 0 & 0 & 0 & 0 & sL_5 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & -1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -sL_1 & -sL_2 & -\dfrac{1}{sC_3} & \dfrac{1}{sC_4} & -\mu sL_5 \\ 0 & -sL_2 & -\dfrac{1}{sC_3} & 0 & (1-\mu)sL_5 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & -1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

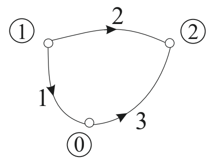
$$= \begin{bmatrix} sL_1 + sL_2 + \dfrac{1}{sC_3} + \dfrac{1}{sC_4} & sL_2 + \dfrac{1}{sC_3} - \mu sL_5 \\ sL_2 + \dfrac{1}{sC_3} & sL_2 + \dfrac{1}{sC_3} + (1-\mu)sL_5 \end{bmatrix}$$

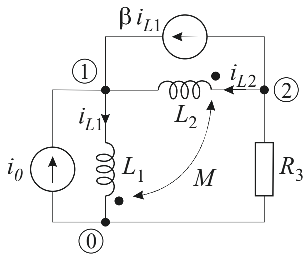
(1 bod)

$$\mathbf{U}_{0p} = -\mathbf{S}\mathbf{U}_{0b} = -\begin{bmatrix} -1 & -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ U_0 \\ 0 \end{bmatrix} = \begin{bmatrix} -U_0 \\ 0 \end{bmatrix}$$

(1 bod)

2. Za mrežu na slici napisati matricu incidencija **A** te strujno-naponske jednadžbe grana u matričnom obliku poštujući oznake čvorova i grana. Istaknuti matricu admitancija grana $\mathbf{Y}_b$ i vektor početnih uvjeta i nezavisnih strujnih izvora grana $\mathbf{I}_{0b}$.





Rješenje:
Reducirana matrica incidencija: **(1 bod)**

$$\mathbf{A}=\begin{bmatrix}1 & 1 & 0\\ 0 & -1 & -1\end{bmatrix}$$

Strujno-Naponske jednadžbe grana: **(4 boda)**

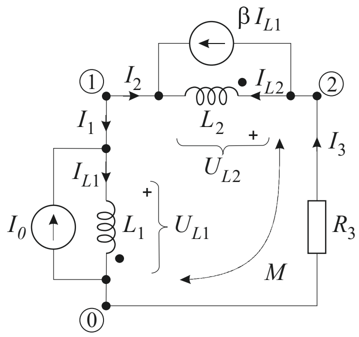
$$U_{L1}=\quad I_{L1}\cdot sL_1-I_{L2}\cdot sM$$

$$U_{L2}=-I_{L1}\cdot sM+I_{L2}\cdot sL_2$$

$$\begin{bmatrix}U_{L1}\\ U_{L2}\end{bmatrix}=\begin{bmatrix}sL_1 & -sM\\ -sM & sL_2\end{bmatrix}\cdot\begin{bmatrix}I_{L1}\\ I_{L2}\end{bmatrix}$$

$$\rightarrow\begin{bmatrix}I_{L1}\\ I_{L2}\end{bmatrix}=\frac{1}{\Delta M}\cdot\begin{bmatrix}sL_2 & sM\\ sM & sL_1\end{bmatrix}\cdot\begin{bmatrix}U_{L1}\\ U_{L2}\end{bmatrix}$$

ako je $\Delta M=s^2\cdot\left(L_1L_2-M^2\right)$



KZS:

$$I_1=I_{L1}-I_0$$

$$I_2=-I_{L2}-\beta\cdot I_{L1}$$

$$U_{L1}=U_1;\qquad U_{L2}=-U_2$$

$$I_1=\frac{sL_2}{\Delta M}\cdot U_1-\frac{sM}{\Delta M}\cdot U_2-I_0$$

$$I_2=-\left(\frac{sM}{\Delta M}+\frac{\beta sL_2}{\Delta M}\right)\cdot U_1+\left(\frac{sL_1}{\Delta M}+\frac{\beta sM}{\Delta M}\right)\cdot U_2$$

$$I_3=\frac{1}{R_3}\cdot U_3$$

Matrični oblik:

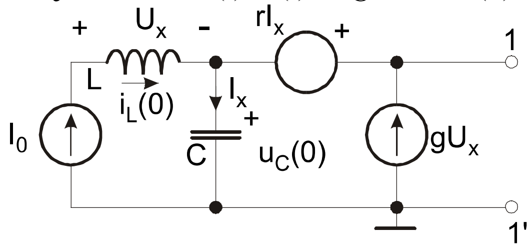
$$\mathbf{I}_b=\mathbf{Y}_b\cdot\mathbf{U}_b+\mathbf{I}_{0b}$$

$$\underbrace{\begin{bmatrix}I_1\\ I_2\\ I_3\end{bmatrix}}_{\mathbf{I}_b}=\underbrace{\begin{bmatrix}\frac{sL_2}{\Delta M} & -\frac{sM}{\Delta M} & 0\\ -\frac{sM+\beta sL_2}{\Delta M} & \frac{sL_1+\beta sM}{\Delta M} & 0\\ 0 & 0 & \frac{1}{R_3}\end{bmatrix}}_{\mathbf{Y}_b}\cdot\underbrace{\begin{bmatrix}U_1\\ U_2\\ U_3\end{bmatrix}}_{\mathbf{U}_b}+\underbrace{\begin{bmatrix}-I_0\\ 0\\ 0\end{bmatrix}}_{\mathbf{I}_{0b}}$$

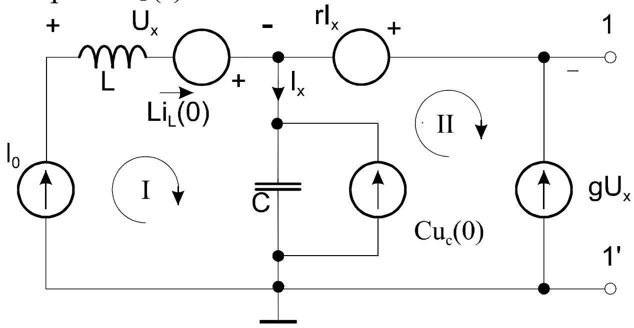
THEVENIN

DEKANSKI ISPIT IZ ELEKTRIČNIH KRUGOVA Jesen 2014 – Rješenja

1. Za mrežu prikazanu slikom odrediti $U_T(s)$ i $Z_T(s)$ nadomjesne sheme po Teveninu s obzirom na stezaljke 1–1' ako je zadano: $i_0(t)=S(t)$, $r=g=1/2$, $u_C(0)=i_L(0)=1$, $L=C=1$.

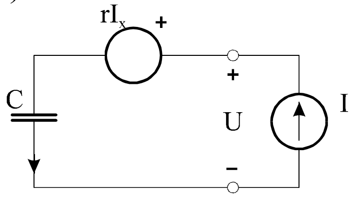


<u>Rješenje:</u> a) Teveninov napon: $U_T(s)$



$$I_I = I_0$$
$$I_{II} = -gU_x$$
$$U_x = I_0 sL - Li_L(0)$$
$$I_x = I_0 + gU_x$$
$$U_1 = \left[I_x + Cu_c(0)\right]\frac{1}{sC}$$
$$\underline{U_T = U_1 + rI_x}$$
$$I_x = I_0 + gI_0 sL - gLi_L(0)$$
$$I_x = I_0(1+gsL) - gLi_L(0)$$
$$U_1 = \frac{I_x}{sC} + \frac{u_C(0)}{s}$$
$$U_T = \frac{I_x}{sC} + \frac{u_C(0)}{s} + rI_x = I_x\left(r + \frac{1}{sC}\right) + \frac{u_C(0)}{s}$$
$$U_T = \left[I_0 + I_0 gsL - gLi_L(0)\right]\left(r + \frac{1}{sC}\right) + \frac{u_C(0)}{s}$$
$$U_T = I_0 r + I_0 rgsL - grLi_L(0) + \frac{I_0}{sC} + I_0 g\frac{L}{C} - gLi_L(0)\frac{1}{sC} + \frac{u_C(0)}{s}$$
$$= \frac{1}{2s} + \frac{1}{4} - \frac{1}{4} + \frac{1}{s^2} + \frac{1}{2s} - \frac{1}{2s} + \frac{1}{s} = \frac{1}{s^2} + \frac{3}{2s} = \frac{3s+2}{2s^2}$$

b) Theveninova impedancija: $Z_T(s)$

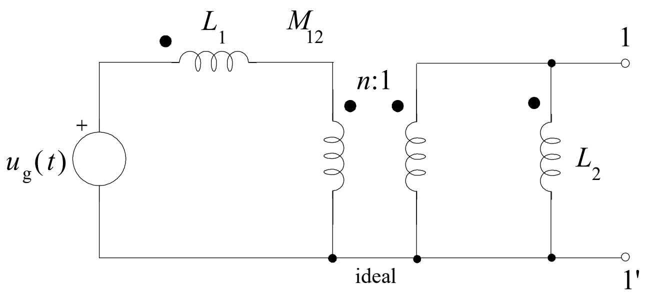


$$I = I_x$$

$$U = I\frac{1}{sC} + rI$$

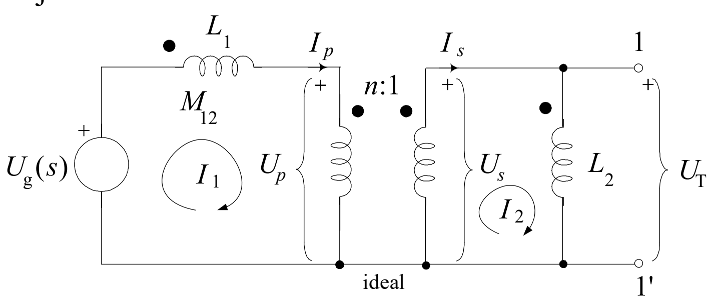
$$Z_T = \frac{U}{I} = \frac{1}{sC} + r = \frac{1}{s} + \frac{1}{2} = \frac{2+s}{2s}$$

2. Za električni krug na slici zadane su normalizirane vrijednosti elemenata L_1=2, L_2=1, M_{12}=1 te n=2, $u_g(t)$=$S(t)$. Odrediti nadomjesne parametre mreže po Theveninu s obzirom na polove 1–1'. Koristiti metodu struja petlji. U zadatku je potrebno: a) Nacrtati sklop za izračunavanje Theveninovog napona i postaviti jednadžbe petlji; b) Odrediti Theveninov napon $U_T(s)$; c) Nacrtati sklop za izračunavanje Theveninove impedancije i postaviti jednadžbe petlji; d) Odrediti Theveninovu impedanciju $Z_T(s)$. e) Da li je električni krug recipročan? Zašto?



Rješenje:

a) Jednadžbe petlji:



Iz sheme je vidljivo da vrijedi: $I_p = I_1, \;\; I_s = I_2, \;\; U_s = U_T$

$$1)\;\; I_1 sL_1 + I_2 sM_{12} + U_p(s) = U_g(s)$$

$$2)\;\; I_1 sM_{12} + I_2 sL_2 = U_s(s)$$

$$3)\;\; U_s = \frac{1}{n}U_p \;\Rightarrow\; U_p = nU_s$$

$$4)\;\; I_s = nI_p \;\Rightarrow\; I_2 = nI_1$$

$$1)\;\; I_1 sL_1 + I_2 sM_{12} + nU_s(s) = U_g(s)$$

$$2)\;\; I_1 sM_{12} + I_2 sL_2 = U_s(s) \qquad \text{(1 bod)}$$

---

b) Theveninov napon $U_T(s)$=$U_s(s)$:

$$1), 2) \Rightarrow\; I_1 sL_1 + I_2 sM_{12} + n(I_1 sM_{12} + I_2 sL_2) = U_g(s)$$

$$I_1 sL_1 + nI_1 sM_{12} + n(I_1 sM_{12} + nI_1 sL_2) = U_g(s)$$

$$I_1(sL_1 + 2nsM_{12} + n^2 sL_2) = U_g(s)$$

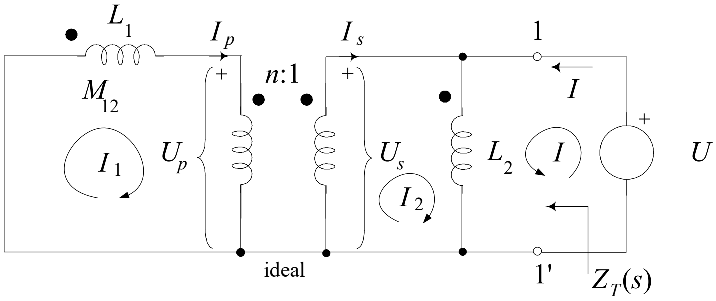
Uvrstimo vrijednosti: L_1=2, L_2=1, M_{12}=1, n=2, $u_g(t)$=$S(t)$.

$$I_1(s) = \frac{U_g(s)}{s(L_1 + 2nM_{12} + n^2 L_2)} = \frac{1}{s}\cdot\frac{1}{s(2+4+4)} = \frac{1}{s}\cdot\frac{1}{s(2+4+4)} = \frac{1}{10s^2}$$

$$I_2(s) = n\cdot I_1(s) = 2\cdot\frac{1}{10s^2} = \frac{1}{5s^2}$$

$$U_T(s)=U_s(s)=sM_{12}I_1(s)+sL_2I_2(s)=s\frac{1}{10s^2}+s\frac{2}{10s^2}=\frac{3}{10s} \quad \text{(1 bod)}$$

c) Izračunavanje Theveninove impedancije pomoću jednadžbi petlji



Iz sheme je vidljivo da vrijedi: $I_p=I_1,\ \ I_s=I_2,\ U(s)=U_s(s),\ Z_T(s)=\dfrac{U(s)}{I(s)}$

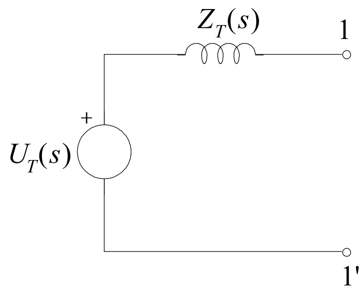
1) $I_1sL_1+(I_2+I)sM_{12}+U_p(s)=0$
2) $I_1sM_{12}+(I_2+I)sL_2=U_s(s)$
3) $U_s(s)=U(s)$
4) $U_s=\dfrac{1}{n}U_p \Rightarrow U_p=nU_s$
5) $I_s=nI_p \Rightarrow I_2=nI_1$

---

1) $I_1sL_1+(nI_1+I)sM_{12}+nU_s(s)=0$
2) $I_1sM_{12}+(nI_1+I)sL_2=U_s(s)$

---

1) $I_1(sL_1+nsM_{12})+IsM_{12}+nU(s)=0$
2) $I_1(sM_{12}+nsL_2)+IsL_2=U(s)$ (1 bod)



d) Theveninova impedancija $Z_T(s)$=$U(s)/I(s)$:

$$2)\Rightarrow I_1=\frac{U(s)-IsL_2}{sM_{12}+nsL_2} \quad \rightarrow 1)\ \frac{U(s)-IsL_2}{sM_{12}+nsL_2}(sL_1+nsM_{12})+IsM_{12}+nU(s)=0$$

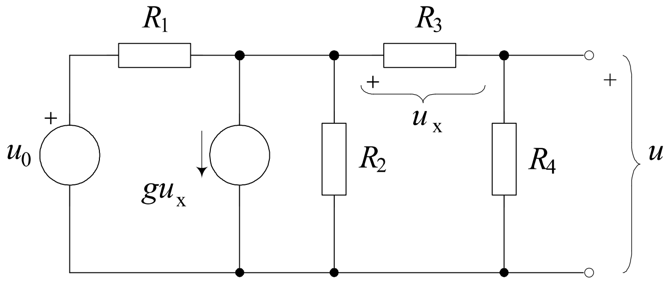
$$U(s)\frac{L_1+nM_{12}}{M_{12}+nL_2}+nU(s)=I(s)sL_2\frac{L_1+nM_{12}}{M_{12}+nL_2}-I(s)sM_{12}$$

$$Z_T(s)=\frac{U(s)}{I(s)}=\frac{sL_2\dfrac{L_1+nM_{12}}{M_{12}+nL_2}-sM_{12}}{\dfrac{L_1+nM_{12}}{M_{12}+nL_2}+n}=\frac{s\dfrac{2+2}{1+2}-s}{\dfrac{2+2}{1+2}+2}=\frac{s\dfrac{4}{3}-s}{\dfrac{4}{3}+\dfrac{6}{3}}=\frac{4s-3s}{10}=\frac{s}{10}=sL_T$$

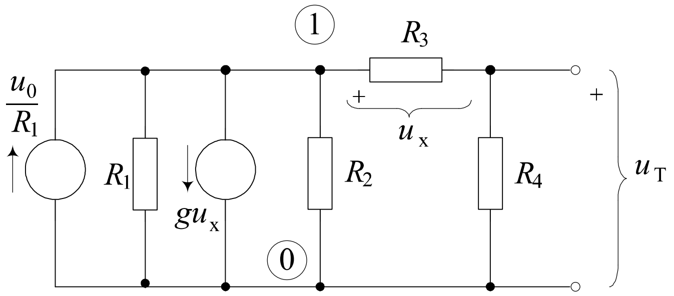
(1 bod)

e) Da li je električni krug recipročan? Zašto?
DA! Jer ima idealni transformator i vezane induktivitete (nema ovisne izvore ili girator).
(1 bod)

2. Za električni krug na slici odrediti ekvivalentni dvopol po Theveninu. Zadane su vrijednosti elemenata: $R_1 = 1/6\ \mathrm{k\Omega}$, $R_2 = 1/4\ \mathrm{k\Omega}$, $R_3 = 1/3\ \mathrm{k\Omega}$, $R_4 = 1\ \mathrm{k\Omega}$, $g = 2\ \mathrm{mS}$, te $u_0 = 2{,}5\ V$. Odrediti: a) Theveninov napon u_T i b) Theveninov otpor R_T, c) napon u_x, d) napon na otporu R_2 i e) struju kroz R_2.



Rješenje: a) Theveninov napon



$$\frac{u_0}{R_1} - g u_x = u_1\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) - u_T\frac{1}{R_3}$$

$$\frac{u_T}{R_4} = \frac{u_x}{R_3} \;\Rightarrow\; u_x = \frac{u_T R_3}{R_4}$$

$$u_1 = u_x + u_T = u_T\left(1 + \frac{R_3}{R_4}\right)$$

$$\frac{u_0}{R_1} - g\frac{u_T R_3}{R_4} = u_T\left(1 + \frac{R_3}{R_4}\right)\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) - u_T\frac{1}{R_3}$$

$$\frac{u_0}{R_1} = u_T\left(g\frac{R_3}{R_4} + \left(1 + \frac{R_3}{R_4}\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{1}{R_4}\right)$$

$$u_T = \frac{\frac{1}{R_1}u_0}{g\frac{R_3}{R_4} + \left(1 + \frac{R_3}{R_4}\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{1}{R_4}} = \frac{6\cdot 2{,}5}{\frac{2}{3} + \left(1 + \frac{1}{3}\right)(6+4) + 1} = \frac{15}{\frac{2}{3} + \frac{4}{3}10 + 1} = 1\ \mathrm{V}$$

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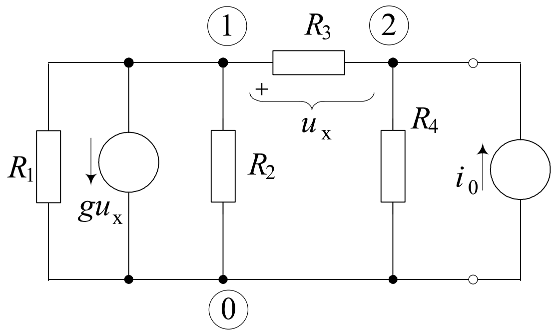
b) Theveninov otpor



$$-gu_x = u_1\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)-u_2\frac{1}{R_3} \qquad u_x = u_1 - u_2$$

$$i_0 = -u_1\frac{1}{R_3}+u_2\left(\frac{1}{R_3}+\frac{1}{R_4}\right)$$

$$-g(u_1-u_2) = u_1\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)-u_2\frac{1}{R_3} \Rightarrow u_1 = u_2\frac{g+\frac{1}{R_3}}{g+\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}$$

$$i_0 = -u_2\frac{g+\frac{1}{R_3}}{g+\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}\frac{1}{R_3}+u_2\left(\frac{1}{R_3}+\frac{1}{R_4}\right) = u_2\frac{\frac{1}{R_3}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\frac{1}{R_4}\left(g+\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)}{g+\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}$$

$$R_T = \frac{u_2}{i_0} = \frac{g+\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}{\frac{1}{R_3}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\frac{1}{R_4}\left(g+\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)} = \frac{2+6+4+3}{3(6+4)+1(2+6+4+3)} = \frac{1}{3}\ \mathrm{k\Omega}$$

Bodovi: a) + b) = (3 boda)

c) napon u_x

$$u_x = \frac{u_T R_3}{R_4} = \frac{1}{3}\ V$$

d) napon na R_2

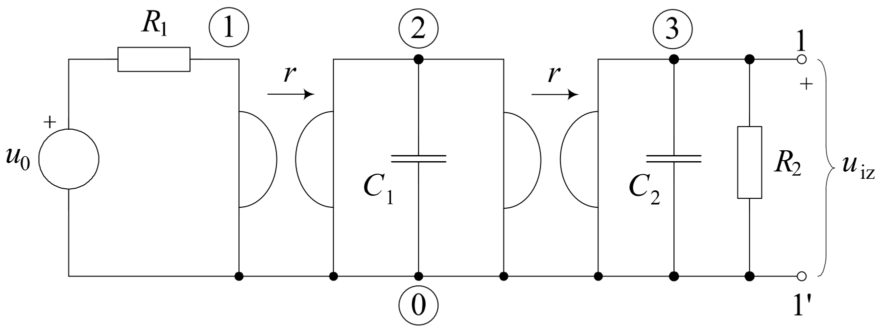
$$u_{R2} = u_1 = u_x + u_T = \frac{4}{3}\ V$$

e) struja kroz R_2

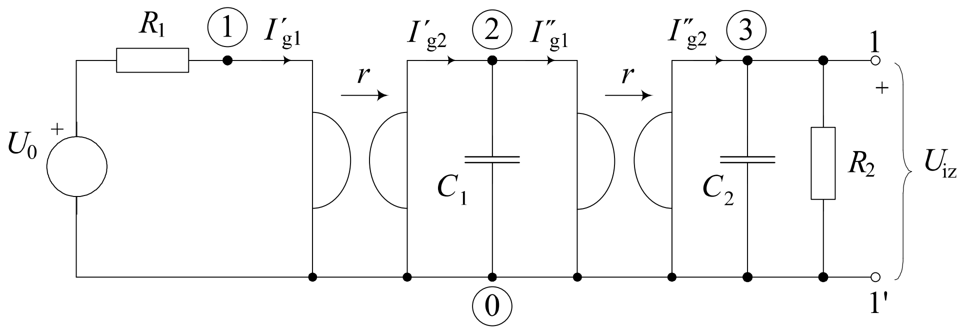
$$i_{R2} = \frac{u_{R2}}{R_2} = \frac{4}{3}\cdot 4 = \frac{16}{3}\ mA$$

Bodovi: c) + d) + e) = (2 boda)

3. Za električni krug na slici zadane su normalizirane vrijednosti elemenata $C_1 = C_2 = \sqrt{2}$, $R_1 = R_2 = 1$, te $r = 1$. Odrediti: nadomjesne parametre mreže po Theveninu obzirom na polove 1–1': a) Theveninov napon U_T i b) Theveninovu impedanciju Z_T. Također izračunati: c) napon u_1; d) napon u_2. Koristiti metodu napona čvorišta (čvorište 0 je referentno).



Rješenje: a) Jednadžbe napona za čvorišta 1, 2 i 3 (1 bod)



a) Theveninov napon

$$\frac{U_0}{R_1} - I'_{g1} = \frac{U_1}{R_1} \qquad I'_{g1} = -\frac{U_2}{r}, \quad I'_{g2} = -\frac{U_1}{r}$$

$$I'_{g2} - I''_{g1} = U_2 s C_1 \qquad I''_{g1} = -\frac{U_3}{r}, \quad I''_{g2} = -\frac{U_2}{r}$$

$$I''_{g2} = U_3\left(sC_2 + \frac{1}{R_2}\right)$$

$$\frac{U_0}{R_1} = \frac{U_1}{R_1} - \frac{U_2}{r}$$

$$0 = \frac{U_1}{r} + U_2 s C_1 - \frac{U_3}{r} \quad \Rightarrow \quad U_1 = -rU_2 s C_1 + U_3$$

$$0 = \frac{U_2}{r} + U_3\left(sC_2 + \frac{1}{R_2}\right) \quad \Rightarrow \quad U_2 = -rU_3\left(sC_2 + \frac{1}{R_2}\right)$$

$$U_1 = -r\left(-rU_3\left(sC_2 + \frac{1}{R_2}\right)\right)sC_1 + U_3 = \left(r^2\left(sC_2 + \frac{1}{R_2}\right)sC_1 + 1\right)U_3$$

$$\frac{U_0}{R_1}=\frac{U_1}{R_1}-\frac{U_2}{r}=\frac{U_3}{R_1}\cdot\left(r^2\left(sC_2+\frac{1}{R_2}\right)sC_1+1\right)+U_3\left(sC_2+\frac{1}{R_2}\right)$$

$$U_3=\frac{U_0}{\left(r^2\left(sC_2+\frac{1}{R_2}\right)sC_1+1\right)+R_1\left(sC_2+\frac{1}{R_2}\right)}$$

$$U_T=U_3=\frac{U_0R_2}{s^2r^2R_2C_1C_2+s\left(r^2C_1+R_1R_2C_2\right)+R_1+R_2}$$

$$U_T=\frac{1}{2}\cdot\frac{U_0}{s^2+\sqrt{2}s+1}$$ (1 bod)

b) Theveninova impedancija

$$Z_{iz1}=\frac{r^2}{R_1}$$

$$Z_{iz2}=\frac{r^2\left(Z_{iz1}+\frac{1}{sC_2}\right)}{Z_{iz1}\frac{1}{sC_2}}=\frac{r^2\left(Z_{iz1}sC_2+1\right)}{Z_{iz1}}=r^2sC_2+R_1$$

$$Z_T=\frac{Z_{iz2}R_2}{Z_{iz2}\left(sC_3R_2+1\right)+R_2}=\frac{\left(r^2sC_1+R_1\right)R_2}{s^2r^2C_1C_2R_2+s\left(R_1R_2C_2+r^2C_1\right)+R_1+R_2}=\frac{\sqrt{2}s+1}{2\left(s^2+\sqrt{2}s+2\right)}$$ (1 bod)

c) Napon U_1

$$U_1=\left(r^2\left(sC_2+\frac{1}{R_2}\right)sC_1+1\right)U_3=\frac{1}{R_2}\left(r^2\left(sR_2C_2+1\right)sC_1+R_2\right)U_3$$

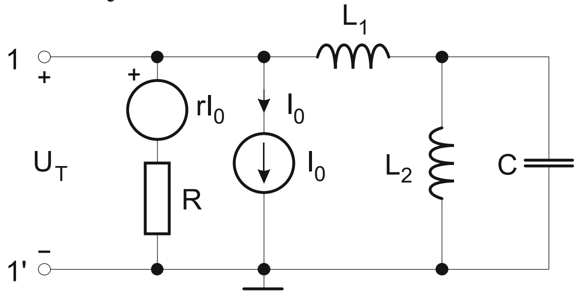
$$U_1=\frac{U_0\left(r^2\left(sC_2R_2+1\right)sC_1+R_2\right)}{s^2r^2R_2C_1C_2+s\left(r^2C_1+R_1R_2C_2\right)+R_1+R_2}=\frac{U_0\left(2s^2+\sqrt{2}s+1\right)}{2s^2+2\sqrt{2}s+2}$$ (1 bod)

d) Napon U_2

$$U_2=\frac{-r\left(sC_2R_2+1\right)U_0}{s^2r^2C_1C_2R_1+s\left(r^2C_1+C_2R_1R_2\right)+R_1+R_2}=\frac{-\left(s\sqrt{2}+1\right)U_0}{2s^2+2\sqrt{2}s+2}$$ (1 bod)

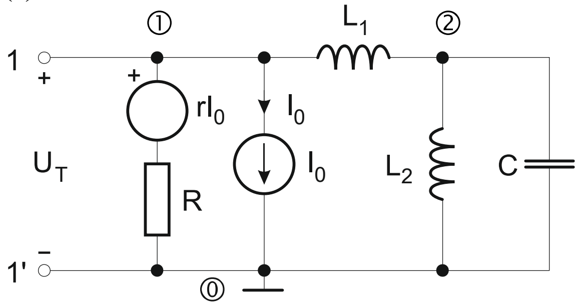
PISMENI ISPIT IZ ELEKTRIČNIH KRUGOVA 2016-2017 – Rješenja

1. Za mrežu prikazanu slikom odrediti nadomjesne parametre po Theveninu $U_T(s)$ i $Z_T(s)$ s obzirom na polove 1−1'. Zadano je: r=1, R=1/2, L_1=1, L_2=2, C=1, te strujni izvor $I_0(s)$=1/s.



Rješenje:

a) Theveninov napon $U_T(s)$:



Na priključnicama (polovima) 1−1' računamo (ili mjerimo) napon praznog hoda. To je Theveninov napon.

Naponi čvorišta:

$$(1)\ U_1\left(\frac{1}{sL_1}+\frac{1}{R}\right)-U_2\frac{1}{sL_1}=\frac{r}{R}I_0-I_0;$$

$$(2)\ -U_1\frac{1}{sL_1}+U_2\left(\frac{1}{sL_1}+\frac{1}{sL_2}+sC\right)=0$$

$$(2)\Rightarrow\ U_2\left(1+\frac{sL_1}{sL_2}+s^2L_1C\right)=U_1\ \Rightarrow\ U_2=\frac{U_1}{(1+L_1/L_2)+s^2L_1C}\ \rightarrow(1)\Rightarrow$$

$$U_1\left(\frac{1}{sL_1}+\frac{1}{R}\right)-\frac{U_1}{(1+L_1/L_2)+s^2L_1C}\cdot\frac{1}{sL_1}=\left(\frac{r}{R}-1\right)I_0$$

$$U_T(s)=U_1(s)=\frac{\left(\frac{r}{R}-1\right)}{\left(\frac{1}{sL_1}+\frac{1}{R}\right)-\frac{1}{(1+L_1/L_2)+s^2L_1C}\cdot\frac{1}{sL_1}}\cdot I_0(s)$$

Uz uvrštene vrijednosti elemenata i nakon malo sređivanja slijedi

$$U_T(s)=\frac{1}{\left(\frac{1}{s}+2\right)-\frac{1}{3/2+s^2}\cdot\frac{1}{s}}\cdot\frac{1}{s}=\frac{1}{(1+2s)-\frac{1}{3/2+s^2}}=\frac{\frac{3}{2}+s^2}{(1+2s)\left(\frac{3}{2}+s^2\right)-1}$$

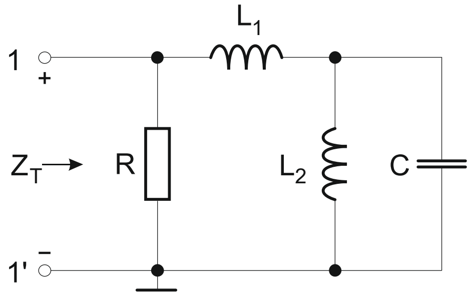
$$U_T(s)=\frac{s^2+\frac{3}{2}}{2s^3+s^2+3s+\frac{1}{2}}$$ **(3 boda)**

b) Theveninova impedancija $Z_T(s)$:

Kod izračunavanja Theveninove impedancije ovisni izvori su uključeni, a neovisni izvori i početni uvjeti moraju biti isključeni. Izvana se dodaje pomoćni neovisni izvor.

U ovom zadatku $I_0(s)=0 \Rightarrow r\cdot I_0(s)=0$ pa je isključen i ovisni izvor $r\cdot I_0(s)$ i neovisni izvor $I_0(s)$.

Impedanciju možemo odrediti na jednostavan način.



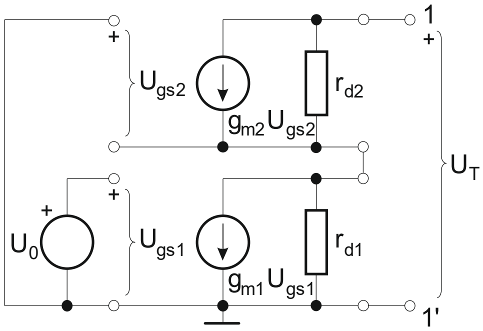
$$Z_T(s)=\frac{1}{\dfrac{1}{R}+\dfrac{1}{sL_1+\dfrac{1}{\dfrac{1}{sL_2}+sC}}}=\frac{1}{2+\dfrac{1}{s+\dfrac{1}{\dfrac{1}{2s}+s}}}=\frac{1}{2+\dfrac{1}{s+\dfrac{2s}{1+2s^2}}}$$

$$Z_T(s)=\frac{1}{2+\dfrac{1+2s^2}{s(1+2s^2)+2s}}=\frac{1}{2+\dfrac{1+2s^2}{3s+2s^3}}=\frac{3s+2s^3}{2(3s+2s^3)+1+2s^2}=\frac{s\left(s^2+\dfrac{3}{2}\right)}{2s^3+s^2+3s+\dfrac{1}{2}}$$

(2 boda)

PISMENI ISPIT IZ ELEKTRIČNIH KRUGOVA 2017-2018 – Rješenja

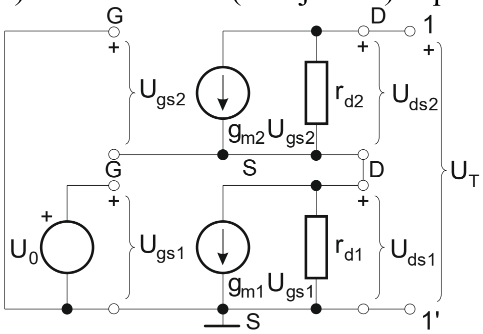
1. Za mrežu prikazanu slikom odrediti nadomjesne parametre po Theveninu $u_T(t)$ i R_T s obzirom na polove 1–1'. Zadano je: $g_{m1}=g_{m2}=0{,}2$[mS], $r_{d1}=r_{d2}=200$[kΩ], te naponski izvor $u_0(t)=\cos(t)$[mV].



<u>Rješenje:</u> (Cascode Amplifier with Open Output)

a) Theveninov napon $U_T(s)$:

Na priključnicama (polovima) 1–1' računamo (ili mjerimo) napon praznog hoda.



$$U_{ds1}=-g_{m1}U_{in}\cdot r_{ds1}$$
$$U_{gs2}=-U_{ds1}$$
$$U_{ds2}=-g_{m2}U_{gs2}\cdot r_{ds2}=g_{m2}U_{ds1}\cdot r_{ds2}$$
$$U_{ds2}=-g_{m1}g_{m2}r_{ds1}r_{ds2}\cdot U_{in}$$

$$U_T=U_{ds1}+U_{ds2}=\left(-g_{m1}r_{ds1}-g_{m1}g_{m2}r_{ds1}r_{ds2}\right)\cdot U_{in}$$
$$U_T=-g_{m1}r_{ds1}\left(1+g_{m2}r_{ds2}\right)\cdot U_{in}$$

Uvrstimo zadane vrijednosti

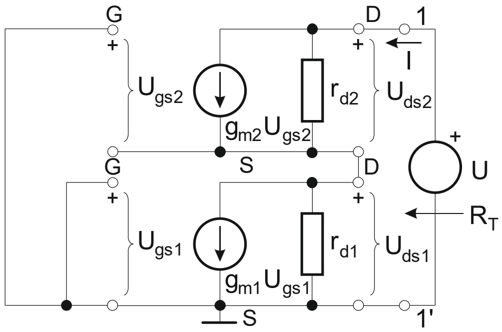
$$U_T=-0,2\cdot10^{-3}\cdot200\cdot10^{3}\left(1+0,2\cdot10^{-3}\cdot200\cdot10^{3}\right)\cdot U_{in}=-40\left(1+40\right)\cdot U_{in}=-1640\cdot U_{in}$$

$U_T=-1640\cdot1\cdot10^{-3}=-1,64\left[\mathrm{V}\right]\Rightarrow u_T(t)=-1,64\cos(t)$[V] To je Theveninov napon.

(3 boda)

b) Theveninova impedancija $Z_T(s)$:

Kod izračunavanja Theveninove impedancije ovisni izvori su uključeni, a neovisni izvori i početni uvjeti moraju biti isključeni. Izvana se dodaje pomoćni neovisni izvor.



$$U_{ds1} = -I \cdot r_{ds1}$$

$$U_{gs2} = -U_{ds1}$$

$$U_{ds2} = r_{ds2} \cdot (I - g_{m2} U_{gs2}) = r_{ds2} \cdot (I + g_{m2} U_{ds1})$$

$$U = U_{ds1} + U_{ds2} = U_{ds1} + r_{ds2} \cdot (I + g_{m2} U_{ds1})$$

$$U = r_{ds1} \cdot I + r_{ds2} \cdot (I + g_{m2} \cdot r_{ds1} \cdot I)$$

$$R_T = \frac{U}{I} = r_{ds1} + r_{ds2} \cdot (1 + g_{m2} \cdot r_{ds1}) = r_{ds1} + r_{ds2} + g_{m2} \cdot r_{ds1} \cdot r_{ds2}$$

Uvrstimo zadane vrijednosti

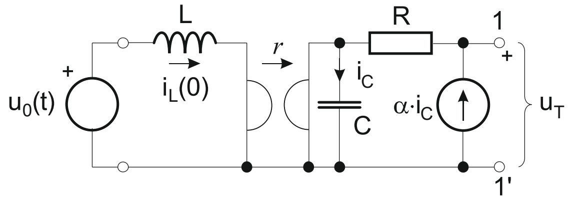
$$R_T = r_{ds1} + r_{ds2} + g_{m2} \cdot r_{ds1} \cdot r_{ds2} = 200 \cdot 10^3 + 200 \cdot 10^3 + 0{,}2 \cdot 10^{-3} \cdot 200 \cdot 10^3 \cdot 200 \cdot 10^3$$

$R_T = 400 \cdot 10^3 + 8 \cdot 10^6 = 8{,}4 [\mathrm{M}\Omega]$ To je Theveninov otpor.

(2 boda)

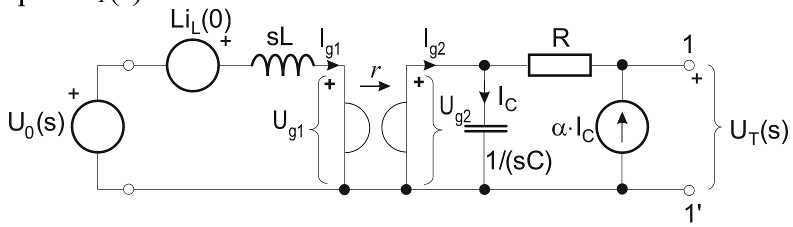
ZAVRŠNI ISPIT IZ ELEKTRIČNIH KRUGOVA 2018-2019 – Rješenja

1. Za prikazanu mrežu odrediti nadomjesne parametre po Theveninu $U_T(s)$ i $Z_T(s)$ na stezaljkama 1−1'. Zadane su normalizirane vrijednosti elemenata $L=C=R=1$, $\alpha=1/2$, $r=2$, $i_L(0)=1$, $u_0(t)=S(t)$.



Rješenje:

a) Theveninov napon $U_T(s)$:



Na priključnicama (polovima) 1−1' računamo (ili mjerimo) napon praznog hoda. To je Theveninov napon.

(1) $I_{g1}(s)sL+U_{g1}(s)=U_0(s)+Li_L(0)$;

(2) $U_{g2}(s)=I_C(s)\dfrac{1}{sC}$

(3) $I_C(s)=I_{g2}(s)+\alpha I_C(s)$

(4) $U_T(s)=U_{g2}(s)+\alpha I_C(s)R$

(5) $U_{g1}(s)=-rI_{g2}(s)$

(6) $U_{g2}(s)=-rI_{g1}(s)$

$$(3)\Rightarrow\ I_{g2}(s)=(1-\alpha)I_C(s)\ \Rightarrow\ I_C(s)=\frac{I_{g2}(s)}{1-\alpha}\ \to(2)\Rightarrow\ U_{g2}(s)=I_{g2}(s)\frac{1}{1-\alpha}\cdot\frac{1}{sC}$$

$$(1)\Rightarrow\ U_{g1}(s)=U_0(s)+Li_L(0)-I_{g1}(s)sL$$

$$I_C(s)=-\frac{1}{r}\frac{U_{g1}(s)}{1-\alpha}=-\frac{1}{r}\frac{1}{1-\alpha}\left[U_0(s)+Li_L(0)-I_{g1}(s)sL\right]=-\frac{1}{r}\frac{1}{1-\alpha}\left[U_0(s)+Li_L(0)+\frac{U_{g2}(s)}{r}sL\right]$$

$$I_C(s)=-\frac{1}{r}\frac{1}{1-\alpha}\left[U_0(s)+Li_L(0)+\frac{1}{r}I_C(s)\frac{sL}{sC}\right]$$

$$I_C(s)+\frac{1}{r}\frac{1}{1-\alpha}\frac{1}{r}I_C(s)\frac{sL}{sC}=-\frac{1}{r}\frac{1}{1-\alpha}\left[U_0(s)+Li_L(0)\right]$$

$$I_C(s)=\frac{-\frac{1}{r}\frac{1}{1-\alpha}\left[U_0(s)+Li_L(0)\right]}{1+\frac{1}{r^2}\frac{1}{1-\alpha}\frac{L}{C}}=\frac{-rC\left[U_0(s)+Li_L(0)\right]}{r^2C\left(1-\alpha\right)+L}$$

$$U_T(s)=U_{g2}(s)+\alpha I_C(s)R=I_C(s)\frac{1}{sC}+\alpha I_C(s)R=I_C(s)\left(\frac{1}{sC}+\alpha R\right)$$

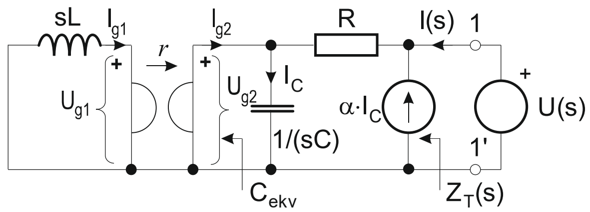
$$U_T(s)=\frac{-rC\left[U_0(s)+Li_L(0)\right]}{r^2C\left(1-\alpha\right)+L}\left(\frac{1}{sC}+\alpha R\right)$$

Uz uvrštene vrijednosti elemenata i nakon malo sređivanja slijedi

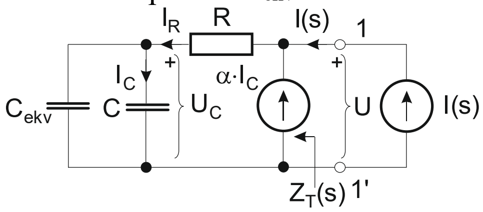
$$U_T(s)=\frac{-2\cdot\left(\frac{1}{s}+1\right)}{4\cdot\frac{1}{2}+1}\left(\frac{1}{s}+\frac{1}{2}\right)=-\frac{2}{3}\left(\frac{1}{s}+1\right)\left(\frac{1}{s}+\frac{1}{2}\right)=-\frac{(s+1)(s+2)}{3s^2}=-\frac{s^2+3s+2}{3s^2}$$ **(3 boda)**

b) Theveninova impedancija $Z_T(s)$:

Kod izračunavanja Theveninove impedancije ovisni izvori su uključeni, a neovisni izvori i početni uvjeti moraju biti isključeni. Izvana se dodaje pomoćni neovisni izvor.



U ovom zadatku isključen početni uvjet $i_L(0)$ i neovisni izvor $U_0(s)$. Na ulazu u girator je vidljiv nadomjesni element, odnosno kapacitet $C_{ekv}=L/r^2=1/4$.



(1) $U_C(s)=I_C(s)\frac{1}{sC}$

(2) $U_C(s)=I_R(s)\frac{1}{sC+sC_{ekv}}$

(3) $\underline{I_R(s)=I(s)+\alpha I_C(s)}$

(1), (2) $\Rightarrow$ $I_C(s)=I_R(s)\frac{sC}{sC+sC_{ekv}}$, (3) $I_C(s)=\left[I(s)+\alpha I_C(s)\right]\frac{sC}{sC+sC_{ekv}}$

$$I_C(s)\left(1-\alpha\frac{sC}{sC+sC_{ekv}}\right)=I(s)\frac{sC}{sC+sC_{ekv}} \Rightarrow I_C(s)=I(s)\frac{sC}{(1-\alpha)sC+sC_{ekv}}$$

$$U_C(s)+I_R(s)R=U(s) \Rightarrow I_C(s)\frac{1}{sC}+\left[I(s)+\alpha I_C(s)\right]R=U(s) \Rightarrow$$

$$I_C(s)\left(\frac{1}{sC}+\alpha R\right)+I(s)R=U(s) \Rightarrow I(s)\frac{sC}{(1-\alpha)sC+sC_{ekv}}\left(\frac{1}{sC}+\alpha R\right)+I(s)R=U(s)$$

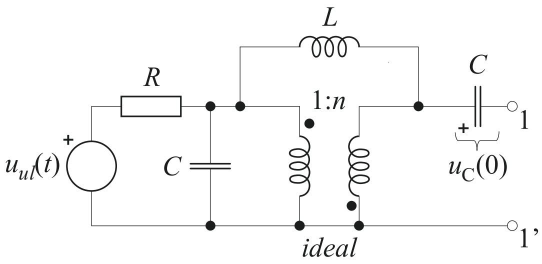
$$Z_T(s)=\frac{U(s)}{I(s)}=R+\frac{sC}{(1-\alpha)sC+sC_{ekv}}\left(\frac{1}{sC}+\alpha R\right)$$

Uz uvrštene vrijednosti elemenata i nakon malo sređivanja slijedi

$$Z_T(s)=\frac{U(s)}{I(s)}=1+\frac{s}{\frac{1}{2}s+s\frac{1}{4}}\left(\frac{1}{s}+\frac{1}{2}\right)=1+\frac{4}{3}\left(\frac{1}{s}+\frac{1}{2}\right)=1+\frac{2}{3}+\frac{4}{3}\frac{1}{s}=\frac{5}{3}+\frac{4}{3}\cdot\frac{1}{s}$$

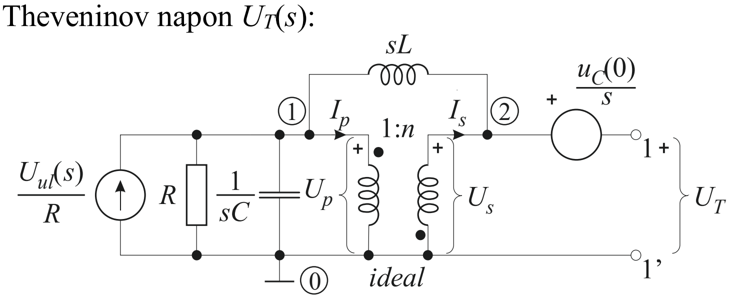
(2 boda)

3. Za mrežu prikazanu slikom odrediti nadomjesne parametre mreže po Theveninu obzirom na polove 1–1', korištenjem metode napona čvorišta. Zadane su normalizirane vrijednosti elemenata: L=1, C=3, R=1/3, n=2 $u_C(0)$=1, te izvor $u_{ul}(t)$=(1/2)·S(t).



Rješenje:

Theveninov napon $U_T(s)$:



Jednadžbe idealnog transformatora:

$$U_p = -\frac{1}{n}\cdot U_s \quad \textbf{(1 bod)}$$

$$I_p = -n\cdot I_s$$

$$\text{(č1)}\ U_1\cdot\left(\frac{1}{R}+sC+\frac{1}{sL}\right)-U_2\cdot\frac{1}{sL}=\frac{U_{ul}}{R}-I_p$$

$$\text{(č2)}\ U_2\cdot\frac{1}{sL}-U_1\cdot\frac{1}{sL}=I_s \quad \textbf{(1 bod)}$$

$$U_1 = U_p;\quad U_2 = U_s$$

$$U_1 = -\frac{1}{n}\cdot U_s$$

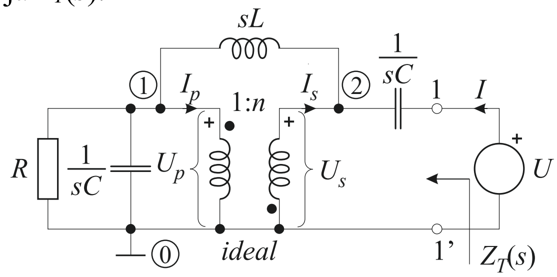
$$\text{(č1)}\ -\frac{1}{n}\cdot U_s\cdot\left(\frac{1}{R}+sC+\frac{1}{sL}\right)-U_s\cdot\frac{1}{sL}=\frac{U_{ul}}{R}+n\cdot I_s$$

$$\text{(č2)}\ U_s\cdot\frac{1}{sL}+\frac{1}{n}\cdot U_s\cdot\frac{1}{sL}=I_s$$

$$\Rightarrow\ U_s=\frac{-sL\cdot U_{ul}}{s^2\frac{RLC}{n}+s\frac{L}{n}+\frac{R\cdot\left(n^2+2n+1\right)}{n}}$$

$$U_T = U_s-\frac{u_C(0)}{s}\qquad\Rightarrow\qquad \underline{U_T=-\left(\frac{1}{s^2+s+3}+\frac{1}{s}\right)}\ \textbf{(1 bod)}$$

Theveninova impedancija $Z_T(s)$:



$$(\check{c}1)\ U_1 \cdot \left(\frac{1}{R}+sC+\frac{1}{sL}\right)-U_2 \cdot \frac{1}{sL}=-I_p$$

(1 bod)

$$(\check{c}2)\ U_2 \cdot \left(\frac{1}{sL}+sC\right)-U_1 \cdot \frac{1}{sL}-U \cdot sC=I_s$$

$$U_1=-\frac{1}{n} \cdot U_s\,; \quad U_2=U_s$$

$$(\check{c}1)\ -\frac{1}{n} \cdot U_s \cdot \left(\frac{1}{R}+sC+\frac{1}{sL}\right)-U_s \cdot \frac{1}{sL}=n \cdot I_s$$

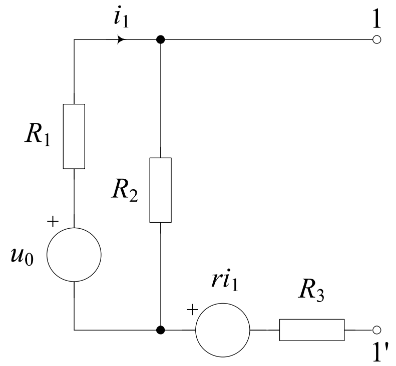
$$(\check{c}2)\ U_s \cdot \left(\frac{1}{sL}+sC\right)+\frac{1}{n} \cdot U_s \cdot \frac{1}{sL}-U \cdot sC=I_s$$

$$\Rightarrow U_s=U \cdot \frac{s^2 RLC \cdot n^2}{s^2 RLC \cdot \left(1+n^2\right)+sL+R \cdot \left(n^2+2n+1\right)}$$

$$U=U_s+I \cdot \frac{1}{sC} \qquad \Rightarrow \qquad \underline{Z_T=\frac{U}{I}=\frac{1}{s} \cdot \left(\frac{1}{3}+\frac{4s}{5s^2+s+3}\right)}$$ **(1 bod)**

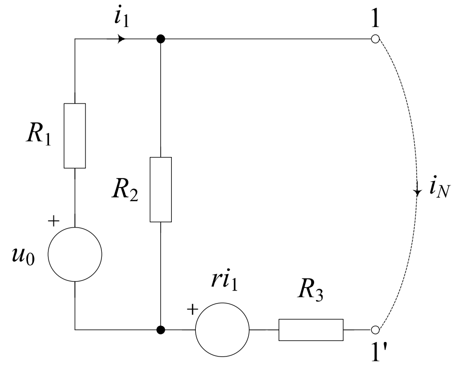
NORTON

3. Za električni krug na slici obzirom na priključnice 1–1' odrediti:
   a) izraz za Nortonovu struju i_N;
   b) izraz za Nortonovu admitanciju G_N;
   c) iznos konstante r ako je G_N=1S;
   d) struju i_1 kad se na priključnice 1–1' spoji otpor R=2Ω;
   e) iznos konstante r za koji je G_N=1/2.
   Zadano je: u_0=2V, R_1=3Ω, R_2=2Ω, R_3=1Ω.



Rješenje:

a) Nortonova struja i_N:



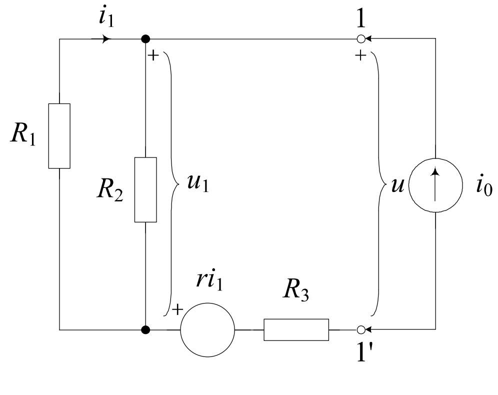
$$u_0 = i_1 \cdot (R_1 + R_2) - i_N \cdot R_2$$

$$0 = -i_1 \cdot R_2 - r \cdot i_1 + i_N \cdot (R_2 + R_3) \;\Rightarrow\; i_1 = i_N \cdot \frac{R_2 + R_3}{r + R_2}$$

$$u_0 = i_N \cdot \frac{(R_1 + R_2)(R_2 + R_3)}{r + R_2} - i_N \cdot R_2$$

$$\boxed{i_N = u_0 \cdot \frac{r + R_2}{R_1 R_2 + (R_1 + R_2) R_3 - r R_2}}$$ (1 bod)

b) Nortonova admitancija G_N:



$$i_1 = -\frac{u_1}{R_1} = -i_0 \frac{R_1 R_2}{R_1 + R_2} \cdot \frac{1}{R_1} = -i_0 \frac{R_2}{R_1 + R_2}$$

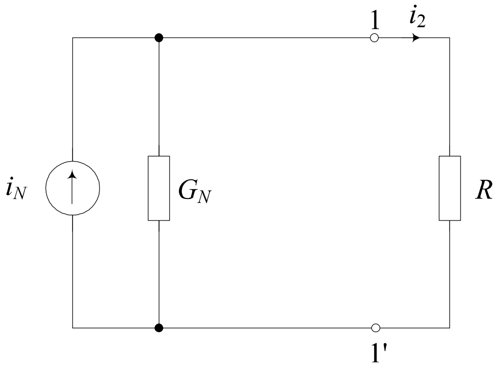
$$u = u_1 + ri_1 + i_0 R_3 = i_0 \frac{R_1 R_2}{R_1 + R_2} - ri_0 \frac{R_2}{R_1 + R_2} + i_0 R_3$$

$$\frac{u}{i_0} = \frac{R_1 R_2 + R_1 R_3 + R_2 R_3 - rR_2}{R_1 + R_2} \quad \Rightarrow \boxed{G_N = \frac{R_1 + R_2}{R_1 R_2 + (R_1 + R_2) R_3 - rR_2}}$$ (1 bod)

c) iznos konstante r ako je G_N=1S.

$$G_N = \frac{5}{11 - 2r} = 1 \quad \Rightarrow \boxed{r = 3}$$ (1 bod)

d) struju i_1 kad se na priključnice 1–1' spoji otpor R=2Ω



$$i_N = u_0 \frac{r + R_2}{R_1 R_2 + (R_1 + R_2) R_3 - rR_2} \quad \Rightarrow \quad i_N = 2 \cdot \frac{3 + 2}{3 \cdot 2 + (3 + 2) \cdot 1 - 3 \cdot 2} = 2 \cdot \frac{5}{5} = 2$$

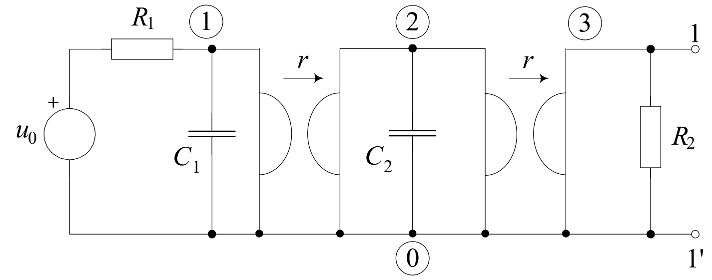
$$i_2 \cdot R = \frac{i_N - i_2}{G_N} \Rightarrow i_2 = i_N \frac{1}{R \cdot G_N + 1} = \frac{i_N}{3} \quad \Rightarrow \quad i_2 = \frac{2}{3}$$

$$i_1 = i_2 \cdot \frac{R_2 + R_3 + R}{r + R_2} = \frac{2}{3} \cdot \frac{5}{5} = \frac{2}{3}$$ (1 bod)

e) iznos konstante r za koji je G_N=1/2.

$$G_N = \frac{5}{11 - 2r} = \frac{1}{2} \Rightarrow r = \frac{1}{2}$$ (1 bod)

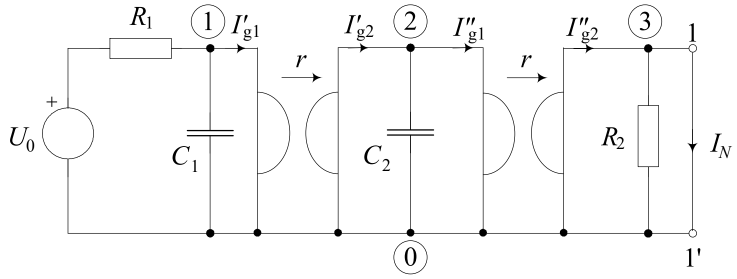
3. Za električni krug na slici zadane su normalizirane vrijednosti elemenata $C_1 = C_2 = \sqrt{2}$, $R_1 = R_2 = 1$, te $r = 1$. Odrediti: nadomjesne parametre mreže po Nortonu obzirom na polove 1–1': a) Nortonovu struju $I_N(s)$ i b) Nortonovu admitanciju $Y_N(s)$.



Rješenje:

a) Nortonova struja:

Jednadžbe napona za čvorišta i Laplaceova transfomacija.



$$\frac{U_0 - U_1}{R_1} = U_1 sC_1 + I'_{g1} \qquad I'_{g1} = -\frac{U_2}{r}, \quad I'_{g2} = -\frac{U_1}{r}$$

$$I'_{g2} - I''_{g1} = U_2 sC_2 \qquad I''_{g1} = -\frac{U_3}{r}, \quad I''_{g2} = -\frac{U_2}{r}$$

$$I_N = I''_{g2}$$

$$\frac{U_0}{R_1} = U_1\left(\frac{1}{R_1} + sC_1\right) - \frac{U_2}{r}$$

$$0 = \frac{U_1}{r} + U_2 sC_2 - \frac{U_3}{r}; \; U_3 = 0 \qquad \Rightarrow \qquad U_1 = -rU_2 sC_2$$

$$I_N = I''_{g2} = -\frac{U_2}{r}$$

$$\frac{U_0}{R_1} = -U_2 srC_2\left(sC_1 + \frac{1}{R_1}\right) - \frac{U_2}{r} \Big/ \cdot rR_1$$

$$rU_0 = -U_2 sr^2 R_1 C_2\left(sC_1 + \frac{1}{R_1}\right) - U_2 R_1$$

$$rU_0 = -U_2\left[sr^2 C_2 (sR_1 C_1 + 1) + R_1\right]$$

$$U_2 = -\frac{rU_0}{sr^2 C_2 (sR_1 C_1 + 1) + R_1}$$

$$I_N(s) = -\frac{U_2}{r} = \frac{U_0}{sr^2C_2(sR_1C_1+1)+R_1} = \frac{U_0}{s^2r^2R_1C_1C_2+sr^2C_2+R_1}$$

$$I_N(s) = \frac{U_0}{2s^2+s\sqrt{2}+1} = \frac{1}{2}\cdot\frac{U_0}{s^2+s\sqrt{2}/2+1/2}$$ (3 boda)

b) Nortonova admitancija $Y_N(s)$:

$$Z_{iz1} = \frac{r^2}{R_1 \| \frac{1}{sC_1}} = \frac{r^2}{R_1\cdot\frac{1}{sC_1}}\left(R_1+\frac{1}{sC_1}\right) = r^2\left(sC_1+\frac{1}{R_1}\right)$$

$$Z_{iz2} = \frac{r^2}{Z_{iz1} \| \frac{1}{sC_2}} = \frac{r^2}{Z_{iz1}\cdot\frac{1}{sC_2}}\left(Z_{iz1}+\frac{1}{sC_2}\right) = r^2\left(sC_2+\frac{1}{Z_{iz1}}\right) = r^2\left[\frac{1}{r^2(1/R_1+sC_1)}+sC_2\right]$$

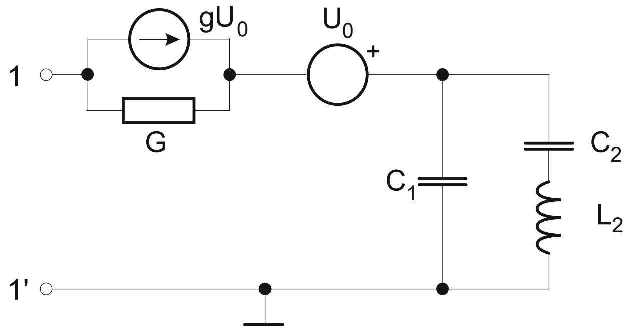
$$Y_N(s) = \frac{1}{R_2}+\frac{1}{Z_{iz2}} = \frac{1}{R_2}+\frac{1}{\frac{1}{1/R_1+sC_1}+sr^2C_2} = \frac{1}{R_2}+\frac{1/R_1+sC_1}{1+sr^2C_2(1/R_1+sC_1)}$$

$$Y_N(s) = \frac{1+sr^2C_2(1/R_1+sC_1)+R_2(1/R_1+sC_1)}{R_2\left[1+sr^2C_2(1/R_1+sC_1)\right]} = \frac{1+s\sqrt{2}(1+s\sqrt{2})+(1+s\sqrt{2})}{1+s\sqrt{2}(1+s\sqrt{2})}$$

$$Y_N(s) = \frac{2s^2+s2\sqrt{2}+2}{2s^2+s\sqrt{2}+1} = \frac{s^2+s\sqrt{2}+1}{s^2+s\sqrt{2}/2+1/2}$$ (2 boda)

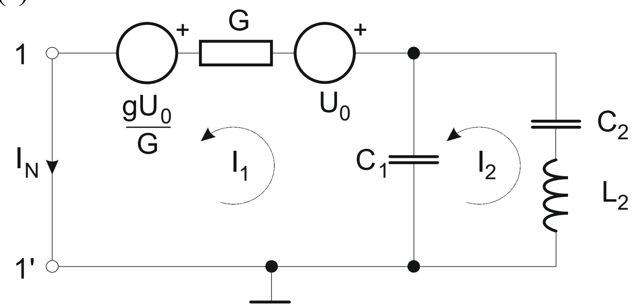
PISMENI ISPIT IZ ELEKTRIČNIH KRUGOVA Jesen 2017 – Rješenja

1. Za mrežu prikazanu slikom odrediti nadomjesne parametre po Nortonu $I_N(s)$ i $Y_N(s)$ s obzirom na polove 1–1'. Zadano je: g=1, G=2, C_1=1, C_2=1/2, L_2=1, te naponski izvor $U_0(s)$=1/s.



Rješenje:

a) Nortonova struja $I_N(s)$:



Na priključnicama (polovima) 1−1' računamo (ili mjerimo) struju kratkog spoja.
To je Nortonova struja.
Struje petlji:

$$(1)\ I_1\left(\frac{1}{sC_1}+\frac{1}{G}\right)-I_2\frac{1}{sC_1}=-\frac{g}{G}U_0-U_0;$$

$$(2)\ -I_1\frac{1}{sC_1}+I_2\left(\frac{1}{sC_1}+\frac{1}{sC_2}+sL_2\right)=0$$ **(1 bod)**

$$(2)\Rightarrow\ I_2\left(1+\frac{sC_1}{sC_2}+s^2C_1L_2\right)=I_1\ \Rightarrow\ I_2=\frac{I_1}{(1+C_1/C_2)+s^2C_1L_2}\ \rightarrow(1)\Rightarrow$$

$$I_1\left(\frac{1}{sC_1}+\frac{1}{G}\right)-\frac{I_1}{(1+C_1/C_2)+s^2C_1L_2}\cdot\frac{1}{sC_1}=\left(-\frac{g}{G}-1\right)U_0$$

$$I_N(s)=I_1(s)=\frac{-\left(\frac{g}{G}+1\right)}{\left(\frac{1}{sC_1}+\frac{1}{G}\right)-\frac{1}{(1+C_1/C_2)+s^2C_1L_2}\cdot\frac{1}{sC_1}}\cdot U_0(s)$$

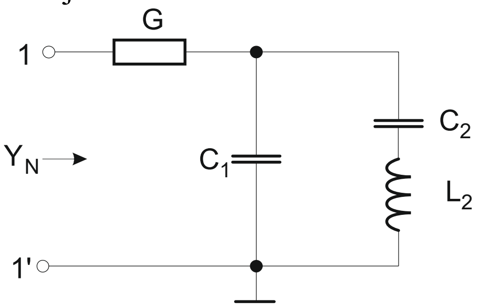
Uz uvrštene vrijednosti elemenata i nakon malo sređivanja slijedi

$$I_N(s)=\frac{-\frac{3}{2}}{\left(\frac{1}{s}+\frac{1}{2}\right)-\frac{1}{3+s^2}\cdot\frac{1}{s}}\cdot\frac{1}{s}=\frac{-\frac{3}{2}}{\left(1+\frac{s}{2}\right)-\frac{1}{3+s^2}}=\frac{-3}{(2+s)-\frac{2}{3+s^2}}=\frac{-3(3+s^2)}{(2+s)(3+s^2)-2}$$

$$I_N(s)=\frac{-3(s^2+3)}{2s^2+6+s^3+3s-2}=\frac{-3s^2-9}{s^3+2s^2+3s+4}$$

(2 boda)

b) Nortonova admitancija $Y_N(s)$:
Kod izračunavanja Nortonove admitancije ovisni izvori su uključeni, a neovisni izvori i početni uvjeti moraju biti isključeni. Izvana se dodaje pomoćni neovisni izvor.
U ovom zadatku $U_0(s)=0 \Rightarrow g\cdot U_0(s)=0$ pa je isključen i ovisni izvor $g\cdot U_0(s)$ i neovisni izvor $U_0(s)$.
Admitanciju možemo odrediti na jednostavan način.

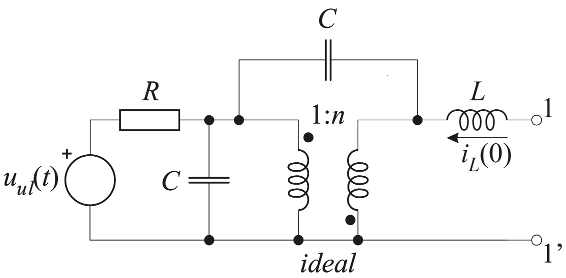


$$Y_N(s)=\frac{1}{\frac{1}{G}+\frac{1}{sC_1+\frac{1}{\frac{1}{sC_2}+sL_2}}}=\frac{1}{\frac{1}{2}+\frac{1}{s+\frac{1}{\frac{2}{s}+s}}}=\frac{1}{\frac{1}{2}+\frac{1}{s+\frac{s}{2+s^2}}}$$

$$Y_N(s)=\frac{1}{\frac{1}{2}+\frac{2+s^2}{s^3+2s+s}}=\frac{1}{\frac{1}{2}+\frac{s^2+2}{s^3+3s}}=\frac{2(s^3+3s)}{s^3+3s+2s^2+4}$$

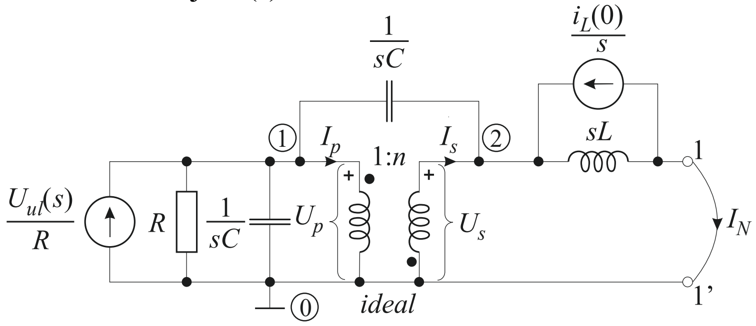
(2 boda)

2. Za mrežu prikazanu slikom odrediti nadomjesne parametre mreže po Northonu obzirom na polove 1–1'. Zadane su normalizirane vrijednosti elemenata: C=2/5, L=1, R=1/4, n=2 $i_L(0)$=1/2, te izvor $u_{ul}(t)$=−1/4·$S(t)$.



Rješenje:

Northonova struja $I_N(s)$:



Jednadžbe idealnog transformatora:

$$U_p = -\frac{1}{n}\cdot U_s$$ **(1 bod)**

$$I_p = -n\cdot I_s$$

$$(\text{č}1)\ U_1\cdot\left(\frac{1}{R}+2\cdot sC\right)-U_2\cdot sC=\frac{U_{ul}}{R}-I_p$$

$$(\text{č}2)\ U_2\cdot\left(sC+\frac{1}{sL}\right)-U_1\cdot sC=I_s+\frac{i_L(0)}{s}$$

(1 bod)

$$(\text{č}3)\ \frac{U_2}{sL}-\frac{i_L(0)}{s}-I_N=0$$

$$U_1=U_p;\quad U_2=U_s$$

$$U_1=-\frac{1}{n}\cdot U_s$$

$$(\text{č}1)\ -\frac{1}{n}\cdot U_s\cdot\left(\frac{1}{R}+2\cdot sC\right)-U_s\cdot sC=\frac{U_{ul}}{R}+n\cdot I_s$$

$$(\text{č}2)\ U_s\cdot\left(sC+\frac{1}{sL}\right)+\frac{1}{n}\cdot U_s\cdot sC=I_s+\frac{i_L(0)}{s}\rightarrow I_s=U_s\cdot\left(sC+\frac{1}{sL}+\frac{sC}{n}\right)-\frac{i_L(0)}{s}$$

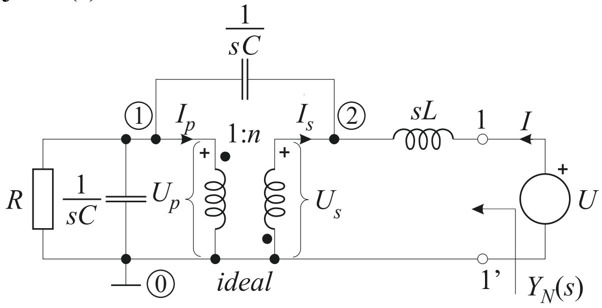
$$\Rightarrow\ -\frac{1}{n}\cdot U_s\cdot\left(\frac{1}{R}+2\cdot sC\right)-U_s\cdot sC=\frac{U_{ul}}{R}+n\cdot\left(U_s\cdot\left(sC+\frac{1}{sL}+\frac{sC}{n}\right)-\frac{i_L(0)}{s}\right)$$

$$U_s\cdot\left[-\frac{1}{n}\cdot\left(\frac{1}{R}+2\cdot sC\right)-sC-n\cdot\left(sC+\frac{1}{sL}+\frac{sC}{n}\right)\right]=\frac{U_{ul}}{R}-\frac{n\cdot i_L(0)}{s}$$

$$U_s=\frac{\frac{U_{ul}}{R}-n\cdot\frac{i_L(0)}{s}}{-\frac{1}{n}\cdot\left(\frac{1}{R}+2\cdot sC\right)-sC-n\cdot\left(sC+\frac{1}{sL}+\frac{sC}{n}\right)}=\ldots=\frac{1}{s^2+s+1}$$

(č3) $I_N=\frac{U_s}{sL}+\frac{i_L(0)}{s}=\frac{1}{s\cdot\left(s^2+s+1\right)}-\frac{1}{2s}$ **(1 bod)**

Northonova admitancija $Y_N(s)$:



(č1) $U_1\cdot\left(\frac{1}{R}+2\cdot sC\right)-U_2\cdot sC=-I_p$

(č2) $U_2\cdot\left(sC+\frac{1}{sL}\right)-U_1\cdot sC-U\cdot\frac{1}{sL}=I_s$ **(1 bod)**

$U_1=-\frac{1}{n}\cdot U_s\,;\quad U_2=U_s$

(č1) $-\frac{1}{n}\cdot U_s\cdot\left(\frac{1}{R}+2\cdot sC\right)-U_s\cdot sC=n\cdot I_s\rightarrow I_s=U_s\cdot\left(-\frac{1}{n^2}\cdot\left(\frac{1}{R}+2\cdot sC\right)-\frac{1}{n}\cdot sC\right)$

(č2) $U_s\cdot\left(sC+\frac{1}{sL}\right)+\frac{1}{n}\cdot U_s\cdot sC-U\cdot\frac{1}{sL}=I_s\rightarrow I_s=U_s\cdot\left(sC+\frac{1}{sL}+\frac{sC}{n}\right)-U\cdot\frac{1}{sL}$

$$U_s\cdot\left(sC+\frac{1}{sL}+\frac{sC}{n}+\frac{1}{n^2}\cdot\left(\frac{1}{R}+2\cdot sC\right)+\frac{1}{n}\cdot sC\right)=U\cdot\frac{1}{sL}$$

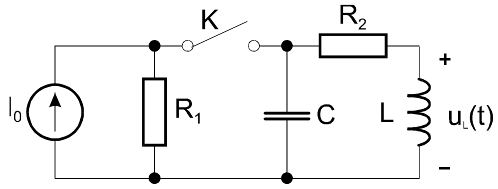
$$\Rightarrow I=\frac{U-U_s}{sL}=\frac{U}{sL}-\frac{U}{(sL)^2}\cdot\frac{1}{sC+\frac{1}{sL}+\frac{sC}{n}+\frac{1}{n^2}\cdot\left(\frac{1}{R}+2\cdot sC\right)+\frac{1}{n}\cdot sC}$$

$Y_N=\frac{I}{U}=\ldots=\frac{s+1}{s^2+s+1}$ **(1 bod)**

PRIJELAZNE POJAVE

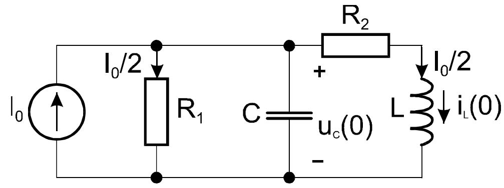
PISMENI ISPIT IZ ELEKTRIČNIH KRUGOVA Jesen 2014 – Rješenja

1. Odrediti odziv $u_L(t)$ u mreži nakon što se otvori sklopka. Skicirati dobiveni odziv. Zadano je $i_0(t)=5, R_1=R_2=L=C=1$.



Rješenje:

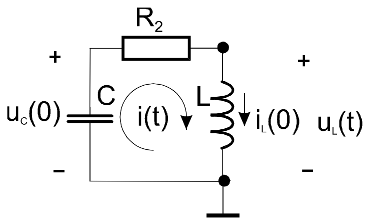
$a) t<0$



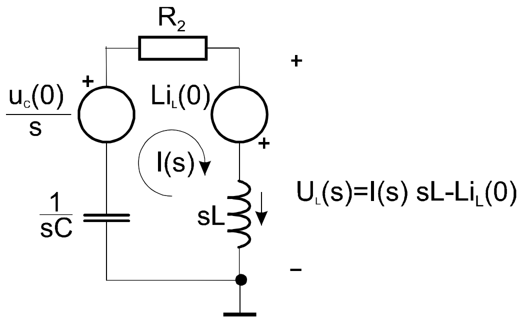
Početni uvjeti: $i_L(0)=\frac{5}{2}=2{,}5\ [\mathrm{A}]$, $\quad u_C(0)=\frac{I_0}{2}R_1=\frac{5}{2}\cdot 1=2{,}5\ [\mathrm{V}]$

$b) t\geq 0$

Za $t\geq 0$ mreža izgleda ovako:



U vremenskoj domeni s početnim uvjetima



U frekvencijskoj domeni (primjenom Laplaceove transformacije)

$$I(s)\left(\frac{1}{sC}+\frac{1}{R_2}+sL\right)=\frac{u_C(0)}{s}+Li_L(0)$$

$$\Rightarrow\quad I(s)=\frac{\frac{u_C(0)}{s}+Li_L(0)}{\frac{1}{sC}+\frac{1}{R_2}+sL}=\frac{\frac{2{,}5}{s}+2{,}5}{\frac{1}{s}+1+s}=2{,}5\frac{s+1}{s^2+s+1}$$

$s^2+s+1=0$, $\Delta=b^2-4ac=1-4=-3$,

$$s_{1,2}=\frac{-b\pm\sqrt{\Delta}}{2a}=-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}$$

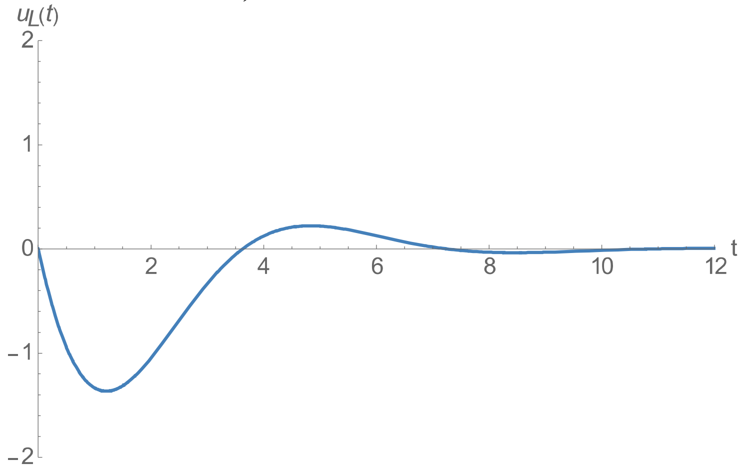
$$U_L(s)=I(s)\cdot sL-Li_L(0)=2{,}5\frac{(1+s)s}{s^2+s+1}-2{,}5=2{,}5\left(\frac{s^2+s+1-1}{s^2+s+1}-1\right)=$$

$$U_L(s)=2{,}5\left(1-\frac{1}{s^2+s+1}-1\right)=2{,}5\left(-\frac{1}{s^2+s+\frac{1}{4}+\frac{3}{4}}\right)=-\frac{5}{2}\cdot\frac{\frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

Rješenje:

$$u_L(t) = -\frac{5}{\sqrt{3}} e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) S(t)$$

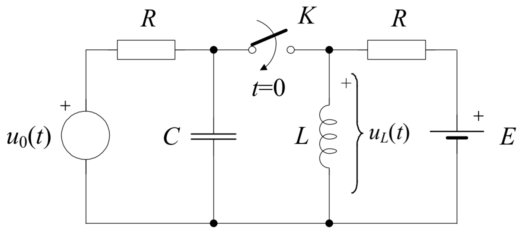
Skica odziva (Wolfram Mathematica):



PISMENI ISPIT IZ ELEKTRIČNIH KRUGOVA 2014-2015 – Rješenja

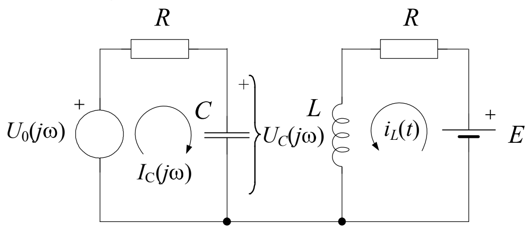
1. Za mrežu prikazanu slikom odrediti valni oblik napona na induktivitetu $u_L(t)$ za $-\infty<t<\infty$, ako se u trenutku $t=0$ zatvori sklopka K. Zadane su normalizirane vrijednosti elemenata: $L=2$, $C=1/2$, $R=1$, napon baterije $E=2$ za $-\infty<t<\infty$, te pobuda naponskog izvora:

$$u_0(t)=\begin{cases}10\sqrt{2}\sin(t+\pi/4), & \text{za } t<0;\\ 0 & \text{, za } t\geq 0.\end{cases}$$



Rješenje:

a) za $t<0$ izračunavamo početne uvjete:



$$i_L(t)=\frac{E}{R}=\frac{2}{1}=2 \quad\Rightarrow\quad i_L(0)=2[\mathrm{A}]$$

$$U_C(j\omega)=U_0(j\omega)\cdot\frac{\dfrac{1}{j\omega C}}{R+\dfrac{1}{j\omega C}}=U_0(j\omega)\cdot\frac{1}{1+j\omega RC}$$

$$U_0(j\omega)=10\sqrt{2}\cdot e^{j\frac{\pi}{4}}=10\sqrt{2}\cdot\left(\cos\frac{\pi}{4}+j\sin\frac{\pi}{4}\right)=10\sqrt{2}\cdot\left(\frac{\sqrt{2}}{2}+j\frac{\sqrt{2}}{2}\right)=10\cdot(1+j)$$

$$U_C(j\omega)=10\cdot(1+j)\cdot\frac{1}{1+j\cdot 1\cdot 1\cdot\frac{1}{2}}=20\cdot\frac{1+j}{2+j}\cdot\frac{2-j}{2-j}=20\cdot\frac{2+2j-j+1}{4+1}=4\cdot(3+j)$$

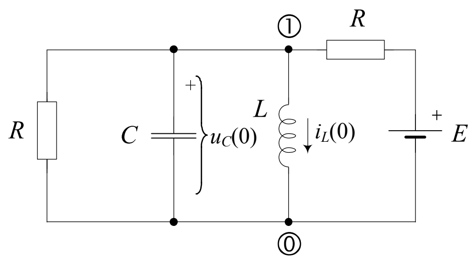
$$|U_C(j\omega)|=4\cdot\sqrt{9+1}=4\cdot\sqrt{10}=12{,}649$$

$$\varphi_C(\omega)=\arctan\left(\frac{\mathrm{Im}[U_C(j\omega)]}{\mathrm{Re}[U_C(j\omega)]}\right)=\arctan\left(\frac{1}{3}\right)=18{,}436°$$

$$u_C(t)=12{,}649\cdot\sin(t+18{,}436°)$$

$$u_C(0)=12{,}649\cdot\sin(18{,}436°)=4[\mathrm{V}]$$ **(2 boda)**

b) za $t\geq 0$ primijenimo Laplaceovu transformaciju na slijedeći električni krug koji ima zadane početne uvjete:



$$(1)\quad \underline{U_1(s)\left(sC+\frac{1}{sL}+\frac{1}{R}+\frac{1}{R}\right)=\frac{E/R}{s}+C\cdot u_C(0)-\frac{i_L(0)}{s};}$$

$$U_1(s)=U_L(s)$$

Uz zadane normalizirane vrijednosti elemenata L=2, C=1/2, R=1, E=2 slijedi:

$$U_1(s)=\frac{\frac{E/R}{s}+C\cdot u_C(0)-\frac{i_L(0)}{s}}{sC+\frac{1}{sL}+\frac{1}{R}+\frac{1}{R}}=\frac{\frac{2}{s}+\frac{1}{2}\cdot 4-\frac{2}{s}}{\frac{s}{2}+\frac{1}{2s}+1+1}=\frac{2}{\frac{s}{2}+\frac{1}{2s}+2}=\frac{4s}{s^2+4s+1}$$ **(2 boda)**

$$s^2+4s+1=0\Rightarrow s_{p1,2}=\frac{-4\pm\sqrt{16-4}}{2}=-2\pm\sqrt{3}$$

$$\Rightarrow s_{p1}=-2+\sqrt{3}=-0{,}2679;\quad s_{p2}=-2-\sqrt{3}=-3{,}73205$$

$$U_L(s)=\frac{4s}{s^2+4s+1}=\frac{4s}{(s-2-\sqrt{3})(s-2+\sqrt{3})}=\frac{A}{s-2-\sqrt{3}}+\frac{B}{s-2+\sqrt{3}}$$

$$U_L(s)=\frac{4s}{s^2+4s+1}=\frac{A(s-2+\sqrt{3})+B(s-2-\sqrt{3})}{(s-2-\sqrt{3})(s-2+\sqrt{3})}=\frac{As+2A-A\sqrt{3}+Bs+2B+B\sqrt{3}}{s^2+4s+1}$$

$$A+B=4$$
$$\underline{2(A+B)+(B-A)\sqrt{3}=0}$$

$$A+B=4$$
$$\underline{A-B=\frac{8}{\sqrt{3}}}$$

$$2A=4+\frac{8}{\sqrt{3}}\Rightarrow A=2+\frac{4}{\sqrt{3}}=4{,}3094$$

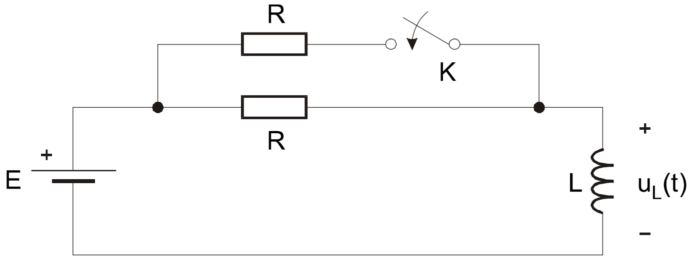
$$\underline{B=4-A=2-\frac{4}{\sqrt{3}}=-0{,}3094}$$

$$U_L(s)=\frac{4s}{s^2+4s+1}=\frac{2+4/\sqrt{3}}{s-2-\sqrt{3}}+\frac{2-4/\sqrt{3}}{s-2+\sqrt{3}}=\frac{4{,}3094}{s+0{,}2679}-\frac{0{,}3094}{s+3{,}73205}$$

$$u_L(t)=\left(4{,}3094\cdot e^{-0{,}2679\cdot t}-0{,}3094\cdot e^{-3{,}73205\cdot t}\right)\cdot S(t)$$ **(1 bod)**

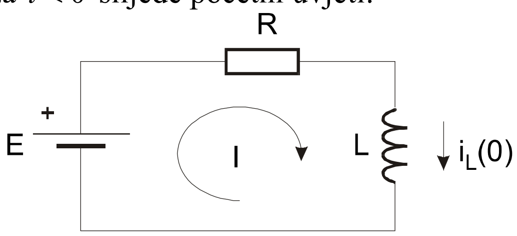
PISMENI ISPIT IZ ELEKTRIČNIH KRUGOVA Jesen 2015 – Rješenja

1. U trenutku t=0 zatvara se sklopka K u prikazanoj mreži. Odrediti i skicirati valni oblik napona $u_L(t)$ ako je zadano: $E = 10[\mathrm{V}]$; $L = 1$; $R = 2$.



Rješenje:

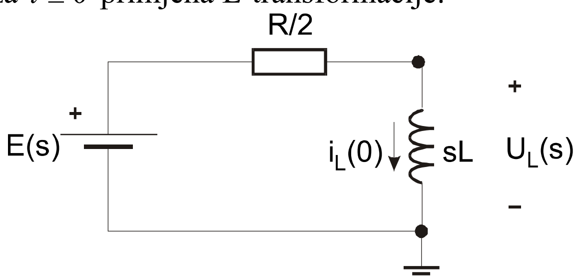
a) za $t < 0$ slijede početni uvjeti:



$$i(t) = \frac{E}{R} = \frac{10}{2} = 5[\mathrm{A}]$$

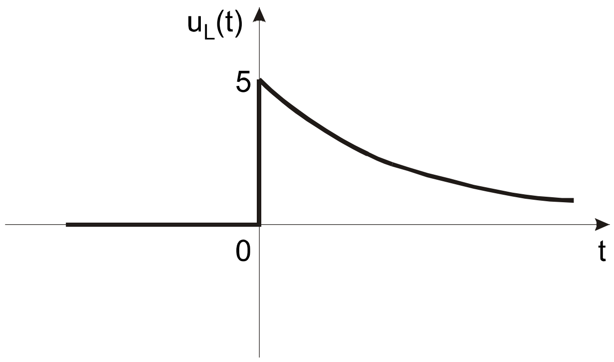
$$i_L(0) = 5[\mathrm{A}]$$

b) za $t \geq 0$ primjena L-transformacije:



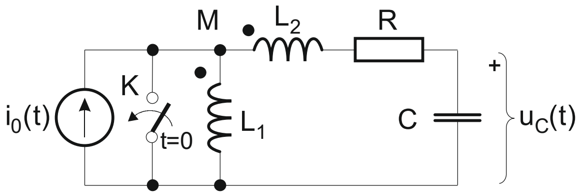
$$e(t) = E = 10[\mathrm{V}] \circ\!\!-\!\!\circ \; E(s) = \frac{10}{s}; \; U_L(s)\left(\frac{2}{R} + \frac{1}{sL}\right) = \frac{E(s)}{R/2} - \frac{i_L(0)}{s}$$

$$U_L(s) = \frac{\frac{10}{s} - \frac{5}{s}}{1 + \frac{1}{s}} = \frac{10-5}{1+s} = \frac{5}{1+s}; \; U_L(s) = \frac{5}{1+s} \Rightarrow u_L(t) = 5 \cdot e^{-t} \cdot S(t)$$



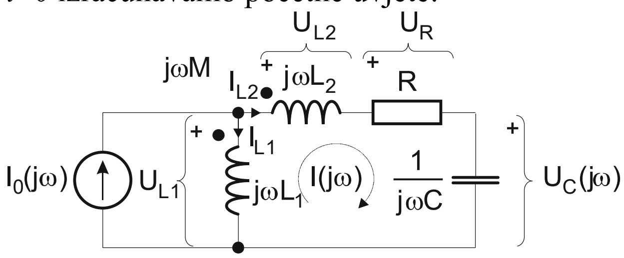
PISMENI ISPIT IZ ELEKTRIČNIH KRUGOVA 2015-2016 – Rješenja

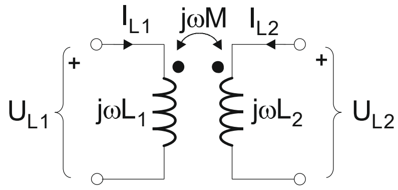
1. Za mrežu prikazanu slikom odrediti valni oblik napona na kapacitetu $u_C(t)$ za $-\infty<t<\infty$, ako se u trenutku $t=0$ zatvori sklopka K. Zadane su normalizirane vrijednosti elemenata: $L_1=2$, $L_2=2$, $M=1$, $C=1$, $R=1$, te pobuda strujnog izvora $i_0(t)=2\sin t$ za $-\infty<t<\infty$ (stacionarni sinusni signal).



Rješenje:

a) za $t<0$ izračunavamo početne uvjete:





$$I_{L1}(j\omega)=I_0(j\omega)-I(j\omega)$$
$$I_{L2}(j\omega)=I(j\omega)$$
$$-U_{L1}(j\omega)+U_{L2}(j\omega)+U_R(j\omega)+U_C(j\omega)=0$$

$$U_C(j\omega)=I(j\omega)\cdot 1/(j\omega C); U_R(j\omega)=I(j\omega)\cdot R;$$
$$U_{L1}(j\omega)=j\omega L_1\cdot I_{L1}(j\omega)+j\omega M\cdot I_{L2}(j\omega)$$
$$U_{L2}(j\omega)=j\omega M\cdot I_{L1}(j\omega)+j\omega L_2\cdot I_{L2}(j\omega)$$

Uvrstimo izraze:

$$-j\omega L_1\cdot\left[I_0(j\omega)-I(j\omega)\right]-j\omega M\cdot I(j\omega)+j\omega M\cdot\left[I_0(j\omega)-I(j\omega)\right]+$$
$$+j\omega L_2\cdot I(j\omega)+R\cdot I(j\omega)+I(j\omega)\cdot 1/(j\omega C)=0$$

$$I_0(j\omega)\left[j\omega L_1-j\omega M\right]=I(j\omega)\left[j\omega L_1-j\omega M-j\omega M+j\omega L_2+R+1/(j\omega C)\right];$$
$$I_0(j\omega)=2\angle 0^\circ;$$

$$I(j\omega)=\frac{j\omega(L_1-M)}{j\omega(L_1+L_2-2M)+R+1/(j\omega C)}I_0(j\omega)=\frac{j(2-1)}{j(2+2-2)+1-j}I_0(j\omega)=\frac{j}{1+j}I_0(j\omega)=$$
$$=\frac{j}{1+j}\cdot\frac{1-j}{1-j}\cdot I_0(j\omega)=\frac{1}{2}(1+j)\cdot I_0(j\omega)=\frac{1}{2}(1+j)\cdot 2=1+j=\sqrt{2}\cdot e^{j\frac{\pi}{4}}$$

$$I_{L1}(j\omega)=I_0(j\omega)-I(j\omega)=I_0(j\omega)\left(1-\frac{1+j}{2}\right)=I_0(j\omega)\frac{1-j}{2}=2\frac{1-j}{2}=1-j=\sqrt{2}\cdot e^{-j\frac{\pi}{4}}$$

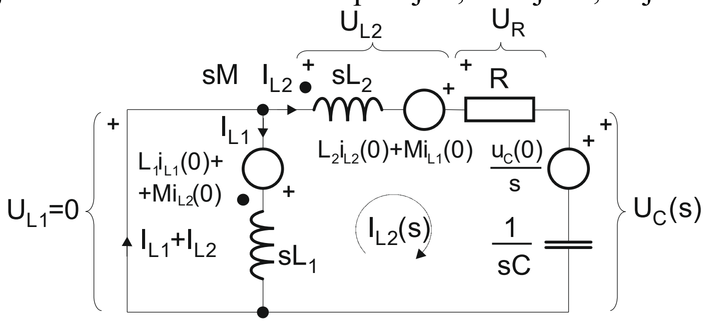
$$I_{L2}(j\omega)=I(j\omega)=1+j=\sqrt{2}\cdot e^{j\frac{\pi}{4}}$$

$$U_C(j\omega)=I(j\omega)\frac{1}{j\omega C}=-j\cdot I(j\omega)=-j\cdot(1+j)=1-j=e^{-j\frac{\pi}{2}}\cdot\sqrt{2}e^{j\frac{\pi}{4}}=\sqrt{2}e^{-j\frac{\pi}{4}}=\sqrt{2}\angle-45^\circ$$

$$i_{L1}(t)=\sqrt{2}\cdot\sin(t-\pi/4) \qquad i_{L1}(0)=\sqrt{2}\cdot\sin(-\pi/4)=\sqrt{2}\cdot\left(-\sqrt{2}/2\right)=-1[\mathrm{A}]$$
$$i_{L2}(t)=\sqrt{2}\cdot\sin(t+\pi/4) \quad\Rightarrow\quad i_{L2}(0)=\sqrt{2}\cdot\sin(\pi/4)=1[\mathrm{A}]$$
$$u_C(t)=\sqrt{2}\cdot\sin(t-\pi/4) \qquad u_C(0)=\sqrt{2}\cdot\sin(-\pi/4)=-1[\mathrm{V}]$$

(2 boda)

b) za $t \geq 0$ primijenimo Laplaceovu transformaciju na slijedeći električni krug koji ima zadane početne uvjete: $i_{L1}(0) = -1$, $i_{L2}(0) = 1$; $u_C(0) = -1$ (vezane induktivitete s početnim uvjetima vidjeti na predavanjima br. 09 Grafovi i mreže primjeri, Primjer 2, slajdovi 32 i 33):



$$sL_1 \cdot I_{L1}(s) + sM \cdot I_{L2}(s) - L_1 i_{L1}(0) - M i_{L2}(0) = 0$$

$$sM \cdot I_{L1}(s) + sL_2 \cdot I_{L2}(s) - L_2 i_{L2}(0) - M i_{L1}(0) + I_{L2}(s) \cdot \left(R + \frac{1}{sC} \right) + \frac{u_C(0)}{s} = 0$$

$$U_C(s) = I_{L2}(s) \cdot \frac{1}{sC} + \frac{u_C(0)}{s}$$

$$sL_1 \cdot I_{L1}(s) + sM \cdot I_{L2}(s) = L_1 i_{L1}(0) + M i_{L2}(0)$$

$$sM \cdot I_{L1}(s) + I_{L2}(s) \cdot \left(sL_2 + R + \frac{1}{sC} \right) = L_2 i_{L2}(0) + M i_{L1}(0) - \frac{u_C(0)}{s}$$

Uz zadane normalizirane vrijednosti elemenata $L_1=2$, $L_2=2$, $M=1$, $C=1$, $R=1$, slijedi:

$$2s \cdot I_{L1}(s) + s \cdot I_{L2}(s) = -2 + 1$$

$$s \cdot I_{L1}(s) + \left(2s + 1 + \frac{1}{s} \right) \cdot I_{L2}(s) = 2 - 1 + \frac{1}{s} \Big/ \cdot 2$$

$$2s \cdot I_{L1}(s) + s \cdot I_{L2}(s) = -2 + 1$$

$$-2s \cdot I_{L1}(s) - 2\left(2s + 1 + \frac{1}{s} \right) \cdot I_{L2}(s) = -2 - \frac{2}{s} \Big/ +$$

$$s \cdot I_{L2}(s) - \left(4s + 2 + \frac{2}{s} \right) \cdot I_{L2}(s) = -1 - 2 - \frac{2}{s}$$

$$\left(s - 4s - 2 - \frac{2}{s} \right) \cdot I_{L2}(s) = -3 - \frac{2}{s} \quad \Rightarrow \quad I_{L2}(s) = \frac{3 + \frac{2}{s}}{3s + 2 + \frac{2}{s}} = \frac{3s+2}{3s^2 + 2s + 2} = \frac{s + \frac{2}{3}}{s^2 + \frac{2}{3}s + \frac{2}{3}}$$

$$U_C(s) = \frac{s + \frac{2}{3}}{s^2 + \frac{2}{3}s + \frac{2}{3}} \cdot \frac{1}{s} - \frac{1}{s} = \frac{s + \frac{2}{3} - s^2 - \frac{2}{3}s - \frac{2}{3}}{s\left(s^2 + \frac{2}{3}s + \frac{2}{3} \right)} = \frac{-s^2 + \frac{1}{3}s}{s\left(s^2 + \frac{2}{3}s + \frac{2}{3} \right)} = -\frac{s - \frac{1}{3}}{s^2 + \frac{2}{3}s + \frac{2}{3}}$$ (2 boda)

$$s^2 + \frac{2}{3}s + \frac{2}{3} = 0 \Rightarrow s_{p1,2} = -\frac{1}{3} \pm \sqrt{\left(\frac{1}{3}\right)^2 - \frac{2}{3}} = -\frac{1}{3} \pm \sqrt{\frac{1}{9} - \frac{6}{9}} = -\frac{1}{3} \pm j\frac{\sqrt{5}}{3}$$

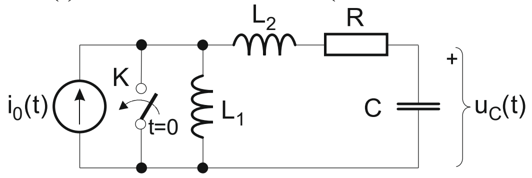
$$U_C(s) = -\frac{s - \frac{1}{3}}{s^2 + \frac{2}{3}s + \frac{2}{3}} = -\frac{s + \frac{1}{3} - \frac{2}{3}}{\left(s + \frac{1}{3} \right)^2 + \left(\frac{\sqrt{5}}{3} \right)^2} = -\frac{s + \frac{1}{3}}{\left(s + \frac{1}{3} \right)^2 + \left(\frac{\sqrt{5}}{3} \right)^2} + \frac{\frac{2}{3} \cdot \frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{3}}{\left(s + \frac{1}{3} \right)^2 + \left(\frac{\sqrt{5}}{3} \right)^2}$$

$$u_C(t) = e^{-\frac{1}{3}t} \left(-\cos\frac{\sqrt{5}}{3} + \frac{2}{\sqrt{5}} \cdot \sin\frac{\sqrt{5}}{3} \right) S(t) = e^{-0{,}3333 \cdot t} \left(-\cos 0{,}74536 + 0{,}89443 \cdot \sin 0{,}74536 \right) \cdot S(t)$$

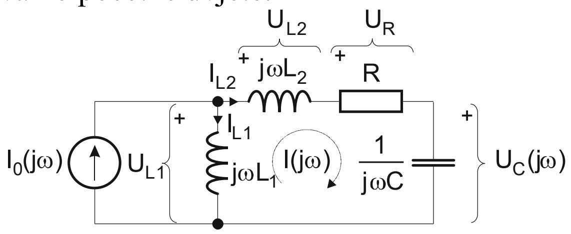
(1 bod)

PISMENI ISPIT IZ ELEKTRIČNIH KRUGOVA 2015-2016 – Rješenja

1. Za mrežu prikazanu slikom odrediti valni oblik napona na kapacitetu $u_C(t)$ za $-\infty<t<\infty$, ako se u trenutku $t=0$ zatvori sklopka K. Zadane su normalizirane vrijednosti elemenata: $L_1=2$, $L_2=2$, $C=1$, $R=1$, te pobuda strujnog izvora $i_0(t)=2\sin t$ za $-\infty<t<\infty$ (stacionarni sinusni signal).



Rješenje: a) za $t<0$ izračunavamo početne uvjete:



$$I_{L1}(j\omega)=I_0(j\omega)-I(j\omega) \qquad U_C(j\omega)=I(j\omega)\cdot 1/(j\omega C); U_R(j\omega)=I(j\omega)\cdot R;$$

$$I_{L2}(j\omega)=I(j\omega) \qquad U_{L1}(j\omega)=j\omega L_1\cdot I_{L1}(j\omega)$$

$$-U_{L1}(j\omega)+U_{L2}(j\omega)+U_R(j\omega)+U_C(j\omega)=0 \qquad U_{L2}(j\omega)=+j\omega L_2\cdot I_{L2}(j\omega)$$

Uvrstimo izraze:

$$-j\omega L_1\cdot[I_0(j\omega)-I(j\omega)]+j\omega L_2\cdot I(j\omega)+R\cdot I(j\omega)+I(j\omega)\cdot 1/(j\omega C)=0$$

$$I_0(j\omega)j\omega L_1=I(j\omega)[j\omega L_1+j\omega L_2+R+1/(j\omega C)];\ I_0(j\omega)=2\angle 0°;$$

$$I(j\omega)=\frac{j\omega L_1}{j\omega(L_1+L_2)+R+1/(j\omega C)}I_0(j\omega)=\frac{j2}{j(2+2)+1-j}I_0(j\omega)=\frac{2j}{1+3j}I_0(j\omega)=$$

$$=\frac{2j}{1+3j}\cdot\frac{1-3j}{1-3j}\cdot I_0(j\omega)=\frac{1}{5}(3+j)\cdot I_0(j\omega)=\frac{1}{5}(3+j)\cdot 2=\frac{2}{5}(3+j)=2\sqrt{\frac{2}{5}}\cdot e^{j\arctan(1/3)}$$

$$I_{L1}(j\omega)=I_0(j\omega)-I(j\omega)=I_0(j\omega)\left(1-\frac{3+j}{5}\right)=I_0(j\omega)\frac{2-j}{5}$$

$$I_{L1}(j\omega)=2\frac{2-j}{5}=\frac{2}{5}(2-j)=\frac{2\sqrt{5}}{5}e^{-j\arctan(1/2)}$$

$$I_{L2}(j\omega)=I(j\omega)=\frac{2}{5}(3+j)=2\sqrt{\frac{2}{5}}\cdot e^{j\arctan(1/3)}$$

$$U_C(j\omega)=I(j\omega)\frac{1}{j\omega C}=-j\cdot I(j\omega)=-j\cdot\frac{2}{5}(3+j)=e^{-j\frac{\pi}{2}}\cdot 2\sqrt{\frac{2}{5}}\cdot e^{j\arctan(1/3)}=$$

$$=2\sqrt{\frac{2}{5}}\cdot e^{j(\arctan(1/3)-\pi/2)}=2\sqrt{\frac{2}{5}}\angle(\arctan(1/3)-\pi/2)$$

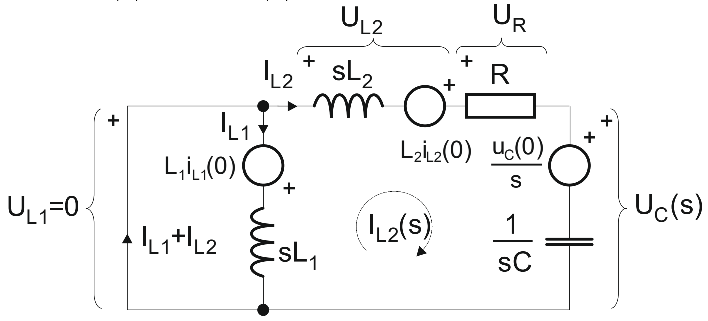
$$i_{L1}(t)=\frac{2}{\sqrt{5}}\cdot\sin(t-\arctan(1/2)) \qquad i_{L1}(0)=\frac{2}{\sqrt{5}}\cdot\sin(-\arctan(1/2))=-0{,}4[\mathrm{A}]$$

$$i_{L2}(t)=2\sqrt{\frac{2}{5}}\cdot\sin(t+\arctan(1/3)) \qquad \Rightarrow\ i_{L2}(0)=2\sqrt{\frac{2}{5}}\cdot\sin(\arctan(1/3))=0{,}4[\mathrm{A}]$$

$$u_C(t)=2\sqrt{\frac{2}{5}}\cdot\sin(t+\arctan(1/3)-\pi/2) \qquad u_C(0)=2\sqrt{\frac{2}{5}}\cdot\sin(\arctan(1/3)-\pi/2)=-1{,}2[\mathrm{V}]$$

(2 boda)

b) za $t \geq 0$ primijenimo Laplaceovu transformaciju na slijedeći električni krug koji ima zadane početne uvjete: $i_{L1}(0)=-0{,}4$; $i_{L2}(0)=0{,}4$; $u_C(0)=-1{,}2$:



$$sL_1 \cdot I_{L1}(s) - L_1 i_{L1}(0) = 0 \;\Rightarrow\; sL_1 \cdot I_{L1}(s) = L_1 i_{L1}(0) \Rightarrow I_{L1}(s) = i_{L1}(0)/s \;\; \text{(ne treba)}$$

$$sL_2 \cdot I_{L2}(s) - L_2 i_{L2}(0) + I_{L2}(s) \cdot \left(R + \frac{1}{sC}\right) + \frac{u_C(0)}{s} = 0$$

$$\boxed{I_{L2}(s) \cdot \left(sL_2 + R + 1/(sC)\right) = L_2 i_{L2}(0) - u_C(0)/s} \Rightarrow I_{L2}(s)$$

$$\boxed{U_C(s) = I_{L2}(s) \cdot \frac{1}{sC} + \frac{u_C(0)}{s}}$$

Uz zadane normalizirane vrijednosti elemenata L_1=2, L_2=2, C=1, R=1, slijedi:

$$\left(2s + 1 + \frac{1}{s}\right) \cdot I_{L2}(s) = 2 \cdot 0{,}4 + \frac{1{,}2}{s} \Big/ \cdot s \qquad \left(2s^2 + s + 1\right) \cdot I_{L2}(s) = 0{,}8 \cdot s + 1{,}2$$

$$U_C(s) = I_{L2}(s) \cdot \frac{1}{sC} + \frac{u_C(0)}{s} \;\Rightarrow \qquad I_{L2}(s) = \frac{0{,}8 \cdot s + 1{,}2}{2s^2 + s + 1}$$

$$U_C(s) = \frac{0{,}8 \cdot s + 1{,}2}{2s^2 + s + 1} \cdot \frac{1}{s} - \frac{1{,}2}{s} = \frac{0{,}8 \cdot s + 1{,}2 - 1{,}2\left(2s^2 + s + 1\right)}{s\left(2s^2 + s + 1\right)} = \frac{0{,}8 \cdot s + 1{,}2 - 2{,}4s^2 - 1{,}2s - 1{,}2}{s\left(2s^2 + s + 1\right)}$$

$$U_C(s) = -\frac{2{,}4s^2 + 0{,}4s}{s\left(2s^2 + s + 1\right)} = -\frac{2{,}4s + 0{,}4}{2s^2 + s + 1} = -\frac{1{,}2s + 0{,}2}{s^2 + (1/2)s + 1/2}$$

(2 boda)

c) odziv u vremenskoj domeni:

$$s^2 + \frac{1}{2}s + \frac{1}{2} = 0 \Rightarrow s_{p1,2} = -\frac{1}{4} \pm \sqrt{\left(\frac{1}{4}\right)^2 - \frac{1}{2}} = -\frac{1}{4} \pm \sqrt{\frac{1}{16} - \frac{8}{16}} = -\frac{1}{4} \pm j\frac{\sqrt{7}}{4}$$

$$U_C(s) = -\frac{1{,}2\left(s + \frac{1}{4}\right) - \frac{1{,}2}{4} + \frac{0{,}8}{4}}{s^2 + (1/2)s + 1/2} = -1{,}2\frac{\left(s + \frac{1}{4}\right)}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2} + 0{,}1 \cdot \frac{4}{\sqrt{7}} \frac{\left(\frac{\sqrt{7}}{4}\right)}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2}$$

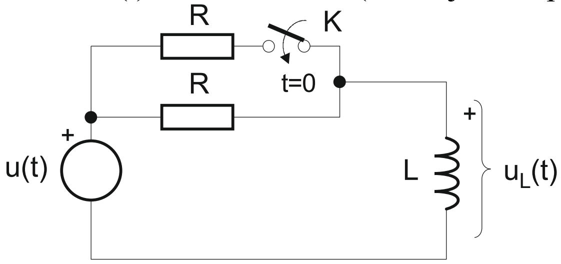
$$u_C(t) = e^{-\frac{1}{4}t}\left[-\frac{6}{5}\cos\left(\frac{\sqrt{7}}{4}t\right) + \frac{0{,}4}{\sqrt{7}} \cdot \sin\left(\frac{\sqrt{7}}{4}t\right)\right] \cdot S(t)$$

$$u_C(t) = e^{-0{,}25 \cdot t}\left[-1{,}2 \cdot \cos(0{,}6614t) + 0{,}1512 \cdot \sin(0{,}6614t)\right] \cdot S(t)$$

(1 bod)

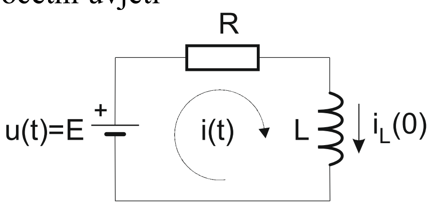
DEKANSKI ISPIT IZ ELEKTRIČNIH KRUGOVA Jesen 2016–Rješenja

1. U trenutku t=0 zatvara se sklopka K u mreži prikazanoj slikom. Odrediti i skicirati valni oblik napona $u_L(t)$ ako je zadano $u(t)$=10 za $-\infty<t<\infty$ (istosmjerni napon baterije), R=2, L=1.



Rješenje:

a) t<0 početni uvjeti

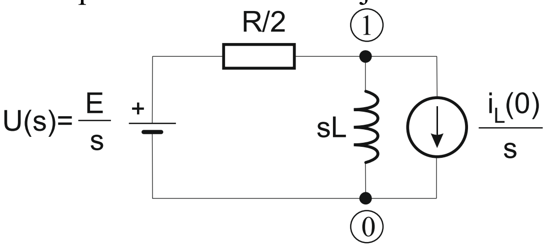


$$i(t) = \frac{E}{R} = \frac{10}{2} = 5[\mathrm{A}]$$

$$\underline{i_L(0) = 5[\mathrm{A}]}$$

(2 boda)

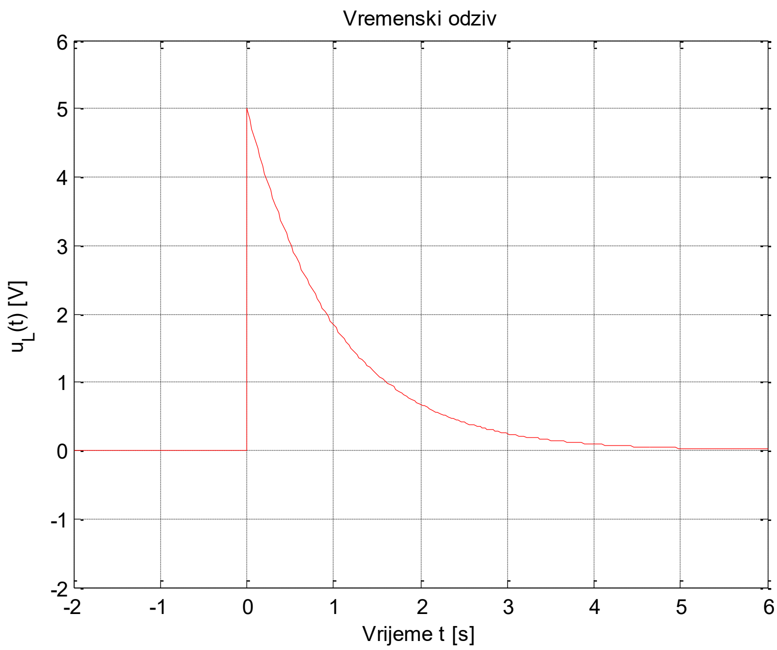
b) t≥0 Laplaceova transformacija



$$U_L(s)\left(\frac{2}{R}+\frac{1}{sL}\right) = \frac{E/(R/2)}{s} - \frac{i_L(0)}{s}$$

$$U_L(s) = \frac{\frac{2}{R}\cdot\frac{E}{s} - \frac{i_L(0)}{s}}{\frac{2}{R}+\frac{1}{sL}} = \frac{\frac{10}{s}-\frac{5}{s}}{1+\frac{1}{s}} = \frac{5}{1+s}$$

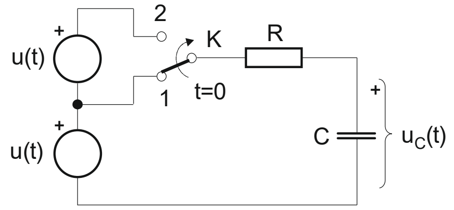
$\Rightarrow \underline{u_L(t) = 5\cdot e^{-t}\cdot S(t)}$ Matlab



(3 boda)

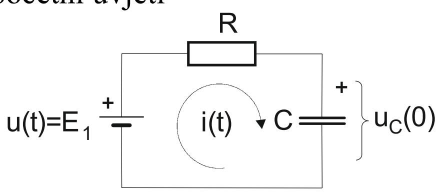
DEKANSKI ISPIT IZ ELEKTRIČNIH KRUGOVA Jesen 2017 – Rješenja

1. U trenutku t=0 preklapa se sklopka K iz položaja 1 u položaj 2 u mreži prikazanoj slikom. Odrediti i skicirati valni oblik napona $u_C(t)$ ako je zadano $u(t)$=5; $-\infty<t<\infty$ (istosmjerni napon baterije) te R=1, C=1.



Rješenje:

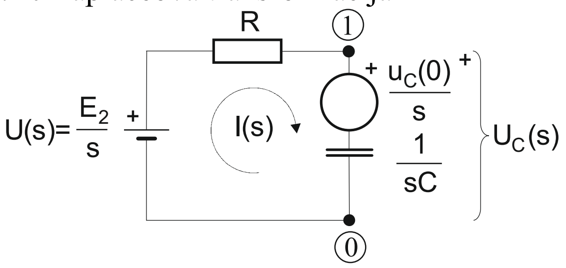
a) t<0 početni uvjeti



$$i(t)=0[\mathrm{A}]$$
$$\underline{u_C(0)=E_1=5[\mathrm{V}]}$$

(2 boda)

b) $t\geq 0$ Laplaceova transformacija



$$I(s)\left(R+\frac{1}{sC}\right)=\frac{E_2}{s}-\frac{u_C(0)}{s}$$

$$I(s)=\frac{\frac{E_2}{s}-\frac{u_C(0)}{s}}{R+\frac{1}{sC}}=\frac{\frac{10}{s}-\frac{5}{s}}{1+\frac{1}{s}}=\frac{5}{s}\cdot\frac{1}{1+\frac{1}{s}}$$

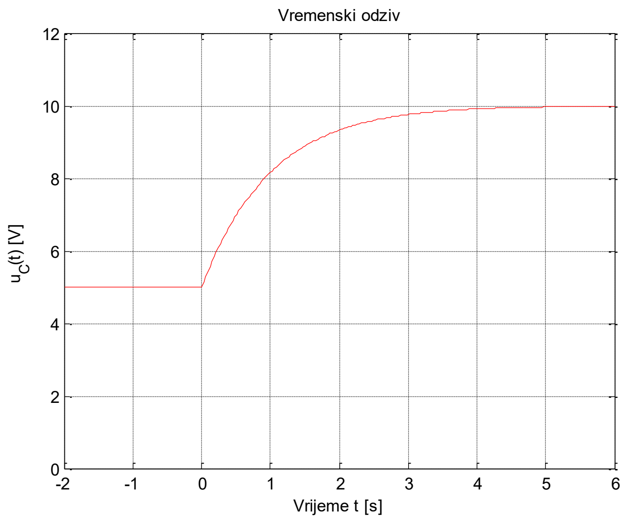
$$U_C(s)=I(s)\cdot\frac{1}{sC}+\frac{u_C(0)}{s}=\frac{5}{s}\cdot\frac{1}{1+\frac{1}{s}}\cdot\frac{1}{s}+\frac{5}{s}$$

$$\Rightarrow U_C(s)=\frac{5}{s}\cdot\frac{1}{s+1}+\frac{5}{s}=\frac{A}{s}+\frac{B}{s+1}+\frac{5}{s}=\frac{A(s+1)+Bs}{s(s+1)}+\frac{5}{s}=\frac{(A+B)s+A}{s(s+1)}+\frac{5}{s}$$

$$A=1;A+B=0\Rightarrow B=-A=-1$$

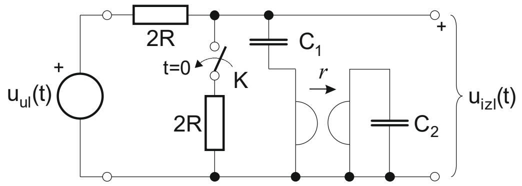
$$U_C(s)=5\left(\frac{1}{s}-\frac{1}{s+1}\right)+\frac{5}{s}=\frac{10}{s}-\frac{5}{s+1}\quad\Rightarrow \underline{u_C(t)=10S(t)-5e^{-t}S(t)}$$ **(3 boda)**

Matlab



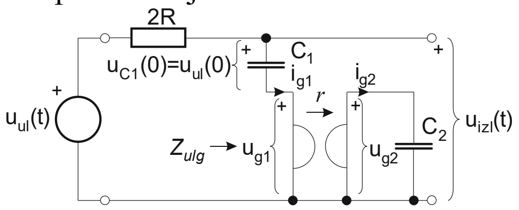
ZAVRŠNI ISPIT IZ ELEKTRIČNIH KRUGOVA 2017-2018 – Rješenja

1. U trenutku t=0 zatvara se sklopka K u prikazanoj mreži. Odrediti i skicirati valni oblik napona $u_{iz}(t)$ ako je zadano: $u_{ul}(t)=E=10[\mathrm{V}];-\infty<t<\infty$, R=1, r=1, C_1=1, C_2=1.



Rješenje:

a) za $t<0$ slijede početni uvjeti:



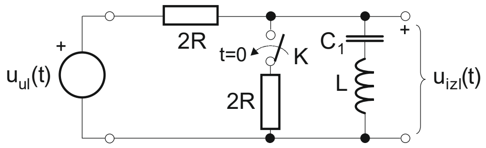
$$u_{g1}(t)=-r\cdot i_{g2}(t)$$
$$u_{g2}(t)=-r\cdot i_{g1}(t)$$

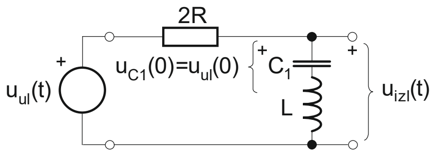
Za istosmjerni generator vrijedi

$$i_{g2}(t)=0\Rightarrow u_{g1}(t)=0; i_{g1}(t)=0\Rightarrow u_{g2}(t)=u_{C2}(t)=0\ \Rightarrow u_{C2}(0)=0;\ u_{C1}(0)=10[\mathrm{V}]$$

Zadatak se može pojednostavniti ako se mreža nacrta s ekvivalentnim induktivitetom L

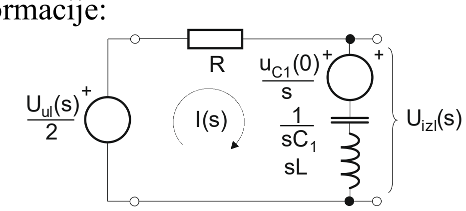
$$Z_{ulg}(s)=\frac{U_{g1}(s)}{I_{g1}(s)}=\frac{-r\cdot I_{g2}(s)}{\frac{1}{-r}\cdot U_{g2}(s)}=\frac{r^2}{\frac{U_{g2}(s)}{I_{g2}(s)}}=\frac{r^2}{Z_2(s)}=r^2sC_2=sL\,;\ L=r^2C_2;\ Z_{ulg}(s)=sL$$





$u_{C1}(0)=10[\mathrm{V}],\ i_L(0)=0[\mathrm{A}]$ (2 boda)

b) za $t\geq 0$ primjena L-transformacije:



$$e(t)=E=10[\mathrm{V}]\ \circ\!\!-\!\!\circ\ E(s)=\frac{10}{s};I(s)=\frac{\frac{E(s)}{2}-\frac{u_C(0)}{s}}{R+sL+\frac{1}{sC}}=\frac{\frac{10}{2s}-\frac{10}{s}}{1+s+\frac{1}{s}}=\frac{-\frac{5}{s}}{1+s+\frac{1}{s}}=\frac{-5}{s^2+s+1}$$ (1 bod)

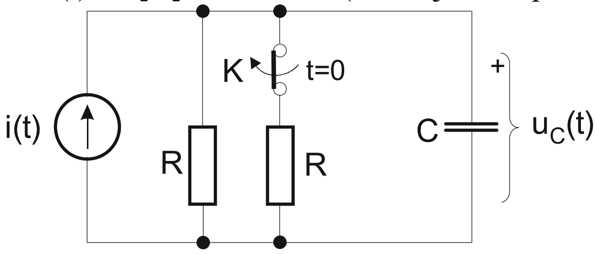
Odakle slijedi $U_{iz}(s)=\dfrac{U_{ul}(s)}{2}-I(s)R=\dfrac{5}{s}+\dfrac{5}{s^2+s+1}$ (1 bod)

$$U_{izl}(s)=\frac{5}{s^2+s+\frac{1}{4}+\frac{3}{4}}+\frac{5}{s}=\frac{5}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{5}{s}=5\frac{2}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{5}{s}$$

$$u_{iz}(t)=\left[5+\frac{10}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]\cdot S(t)$$ (1 bod)

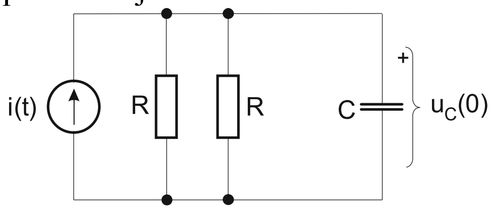
PISMENI ISPIT IZ ELEKTRIČNIH KRUGOVA 2017-2018 – Rješenja

1. U trenutku t=0 otvara se sklopka K u mreži prikazanoj slikom. Odrediti i skicirati valni oblik napona $u_C(t)$ ako je zadano $i(t)$=10[A] za $-\infty<t<\infty$ (istosmjerni napon baterije), R=2, C=1.



Rješenje:

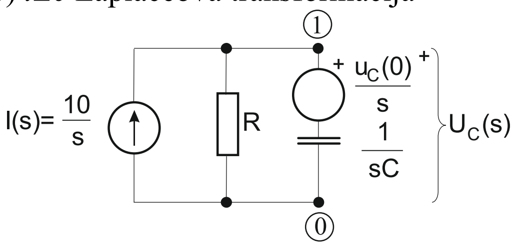
a) t<0 početni uvjeti



$$u(t)=i(t)\frac{R}{2}=10\frac{2}{2}=10[\mathrm{V}]$$

$$\underline{u_C(0)=10[\mathrm{V}]}$$ **(2 boda)**

b) t≥0 Laplaceova transformacija



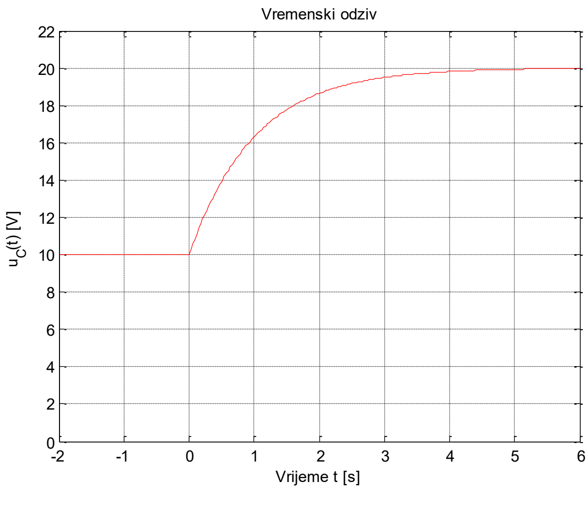
$$U_C(s)\left(\frac{1}{R}+sC\right)=I(s)+Cu_C(0)$$

$$U_C(s)=\frac{I(s)+Cu_C(0)}{\frac{1}{R}+sC}=\frac{\frac{10}{s}+10}{\frac{1}{2}+s}=10\left(\frac{1}{s}\cdot\frac{1}{\frac{1}{2}+s}+\frac{1}{\frac{1}{2}+s}\right)$$

$$\frac{1}{s\left(\frac{1}{2}+s\right)}=\frac{A}{s}+\frac{B}{\frac{1}{2}+s}=\frac{A\left(\frac{1}{2}+s\right)+Bs}{s\left(\frac{1}{2}+s\right)}=\frac{A\frac{1}{2}+s(A+B)}{s\left(\frac{1}{2}+s\right)};\quad \left.\begin{array}{l}A\frac{1}{2}=1\\ A+B=0\end{array}\right\}\Rightarrow \begin{array}{l}A=2\\ B=-A=-2\end{array}$$

$$U_C(s)=10\left(\frac{2}{s}-\frac{2}{\frac{1}{2}+s}+\frac{1}{\frac{1}{2}+s}\right)=10\left(\frac{2}{s}-\frac{1}{\frac{1}{2}+s}\right)\Rightarrow \underline{u_C(t)=10\cdot\left(2-e^{-\frac{1}{2}t}\right)\cdot S(t)}$$

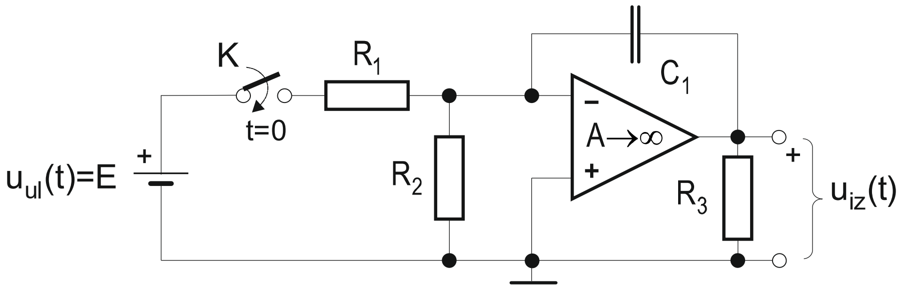
Matlab



(3 boda)

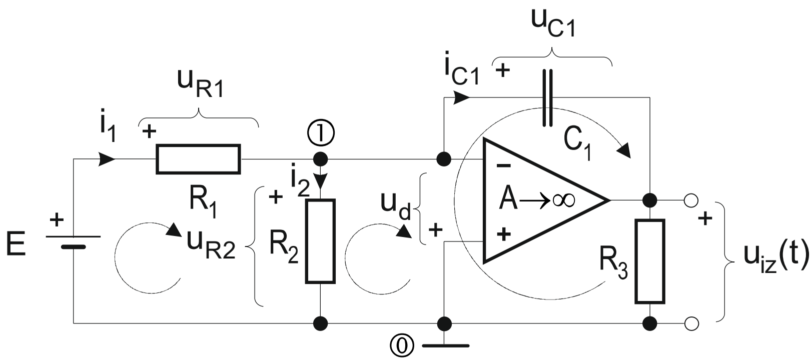
PISMENI ISPIT IZ ELEKTRIČNIH KRUGOVA 2018-2019 – Rješenja

1. U trenutku t=0 zatvara se sklopka K u mreži prikazanoj slikom. Odrediti i skicirati valni oblik napona $u_{iz}(t)$ ako je zadano $u(t)=E=1\text{V}$ za $-\infty<t<\infty$ (istosmjerni napon baterije), $R_1=1\text{k}\Omega$, $R_2=2\text{k}\Omega$, $R_3=3\text{k}\Omega$, $C_1=1\mu\text{F}$. Početni napon na kapacitetu je $u_{C1}(0)=0\text{V}$. Operacijsko pojačalo je idealno. Napon napajanja operacijskog pojačala je $E_{sup}=\pm 12\text{V}$. Zadatak riješiti u vremenskoj domeni (ne koristiti Laplaceovu transformaciju) pomoću Kirchhoffovih zakona.



Rješenje:

a) Jednadžbe mreže



Kirchhoffovi zakoni

(1) $E-u_{R1}+u_d=0 \Rightarrow E=u_{R1}$

(2) $u_{R2}+u_d=0 \Rightarrow u_{R2}=0$

(3) $u_{iz}+u_{C1}+u_d=0 \Rightarrow u_{iz}=-u_{C1}$

(4) $-i_1+i_2+i_{C1}=0 \Rightarrow i_1=i_{C1}$

Naponsko-strujne jednadžbe grana

(5) $u_{R1}=i_1\cdot R_1$

(6) $u_{R2}=i_2\cdot R_2$

(7) $u_{C1}=\frac{1}{C_1}\int\limits_{-\infty}^{t} i_{C1}(\tau)d\tau=u_{C1}(0^-)+\frac{1}{C_1}\int\limits_{0^-}^{t} i_{C1}(\tau)d\tau=0+\frac{1}{C_1}\int\limits_{0^-}^{t} i_{C1}(\tau)d\tau$; jer je $u_{C1}(0^-)=0$.

(8) $u_{iz}=u_d\cdot A;\ A\to\infty \Rightarrow u_d=\frac{u_{iz}}{A}=0$ jer je $|u_{iz}|\le E_{sup}<\infty$, E_{sup} je napon napajanja OP

(1 bod)

Uvrštenje jednadžbi: (koristimo prividni kratki spoj na ulazu operacijskog pojačala $u_d(t)=0$)

$$E-u_{R1}=E-i_1R_1=0 \Rightarrow i_1=\frac{E}{R_1}$$

$$u_{R2}=0 \Rightarrow i_{R2}=0$$

$$u_{iz}+u_{C1}=0 \Rightarrow u_{iz}=-u_{C1}$$

$\underline{i_1 - i_{C1} = 0} \Rightarrow \underline{i_{C1} = i_1 = \frac{E}{R_1}}$ **(1 bod)**

Otpor R_2 na sebi ima napon $u_{R2} = -u_d = 0$ pa ga možemo zanemariti. Također otpor R_3 je spojen na naponski izvor na izlazu (operacijskog pojačala) pa i njega možemo zanemariti.

Napon na izlazu OP glasi:

$$u_{iz}(t) = -u_{C1}(t) = -\left[\frac{1}{C_1}\int_{0^-}^{t} i_{C1}(\tau)d\tau + u_{C1}(0^-)\right] = -\frac{1}{C_1}\int_{0^-}^{t}\frac{E}{R_1}d\tau = -\frac{E}{R_1C_1}(t-0) = -\frac{E}{R_1C_1}\cdot t;\quad t \geq 0$$

(1 bod)

Ovaj valni oblik traje do trenutka t_1 kada je $u_{iz}(t_1) = -E_{\sup} = -12[\mathrm{V}]$ napon izlaza dosegao napon napajanja (baterije) operacijskog pojačala, gdje je

$$-E_{\sup} = -\frac{E}{R_1C_1}\cdot t_1 \Rightarrow \underline{t_1 = \frac{E_{\sup}}{E} = R_1C_1}$$

Stoga je napon na izlazu OP:

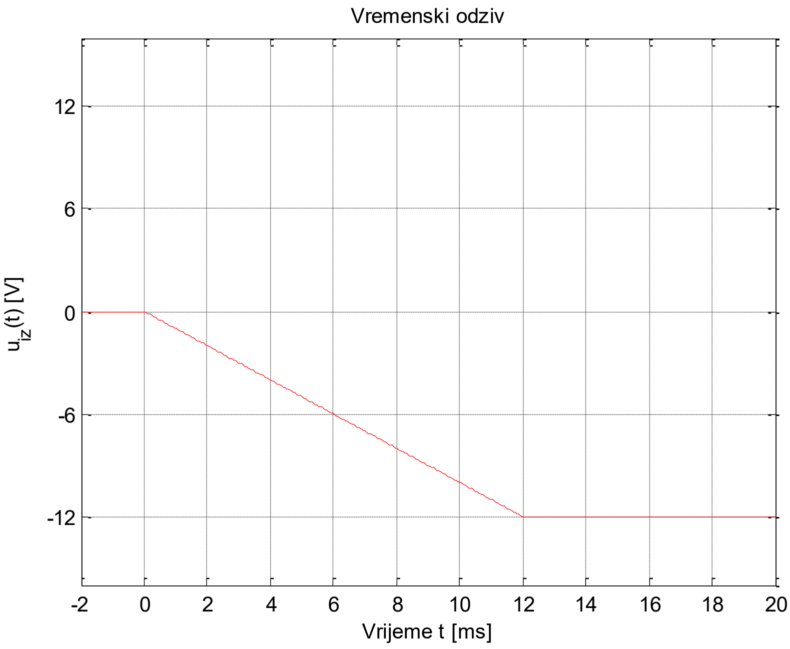
$$u_{iz}(t) = \begin{cases} 0\,; & 0 \leq t \\ -\frac{E}{R_1C_1}\cdot t\,; & 0 \leq t \leq t_1 \\ -E_{\sup}\,; & t_1 \leq t \end{cases}$$

(1 bod)

Uvrstivši vrijednosti elemenata:

$$t_1 = \frac{E_{\sup}}{E}\cdot R_1C_1 = \frac{12[\mathrm{V}]}{1[\mathrm{V}]}\cdot 1[\mathrm{k\Omega}]\cdot 1[\mu F] = 12[\mathrm{ms}]$$

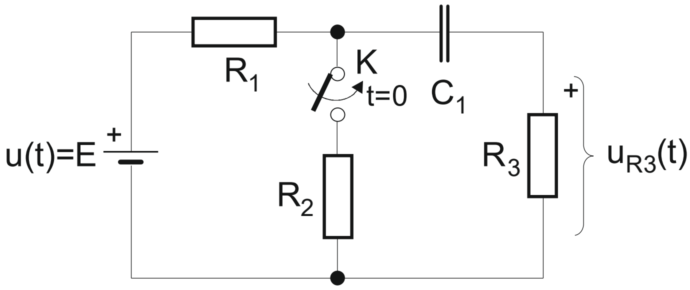
Matlab



(1 bod)

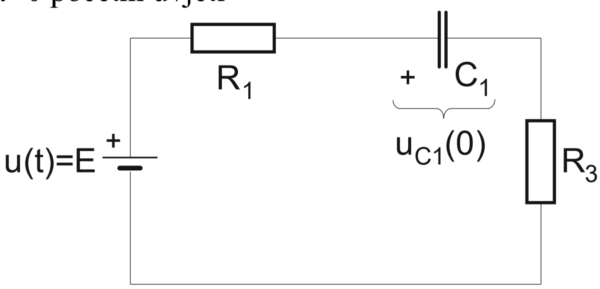
DEKANSKI ISPIT IZ ELEKTRIČNIH KRUGOVA 2018-2019 – Rješenja

1. U trenutku t=0 zatvara se sklopka K u mreži prikazanoj slikom. Odrediti i skicirati valni oblik napona $u_{R3}(t)$ ako je zadano $u(t)=E=10$[V] za $-\infty<t<\infty$ (istosmjerni napon baterije), $R_1=2$, $R_2=1$, $R_3=1/2$, $C_1=1$.



Rješenje:

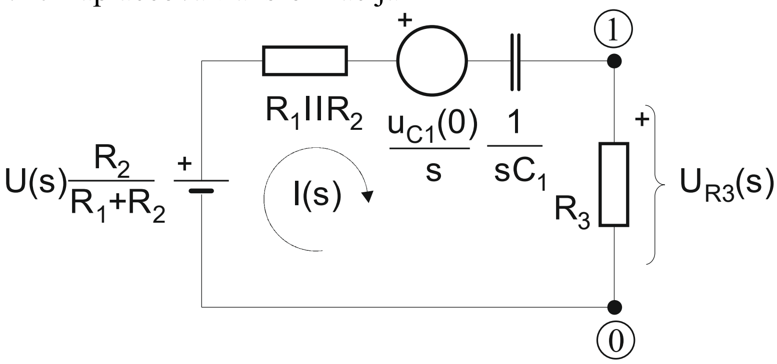
a) t<0 početni uvjeti



$$\underline{u_{C1}(0)=u(0)=E=10[\text{V}]}$$

(1 bod)

b) $t\geq 0$ Laplaceova transformacija



$$u(t)=S(t)\Rightarrow U(s)=\frac{E}{s},$$

$$I(s)=\frac{U(s)\dfrac{R_2}{R_1+R_2}-\dfrac{u_{C1}(0)}{s}}{\dfrac{R_1R_2}{R_1+R_2}+R_3+\dfrac{1}{sC_1}},$$

$$U_{R3}(s)=R_3I(s)$$

$$U_{R3}(s)=R_3\cdot I(s)=R_3\cdot\frac{\dfrac{E}{s}\cdot\dfrac{R_2}{R_1+R_2}-\dfrac{u_{C1}(0)}{s}}{\dfrac{R_1R_2}{R_1+R_2}+R_3+\dfrac{1}{sC_1}}=R_3\cdot\frac{\dfrac{E}{s}\cdot\dfrac{R_2}{R_1+R_2}-\dfrac{E}{s}}{R+\dfrac{1}{sC_1}};$$ **(1 bod)**

gdje smo uveli oznaku $R=\dfrac{R_1R_2}{R_1+R_2}+R_3$

$$U_{R3}(s)=R_3\cdot\frac{\dfrac{E}{s}\cdot\left(\dfrac{R_2}{R_1+R_2}-1\right)}{R+\dfrac{1}{sC_1}}=R_3\cdot\frac{-\dfrac{E}{s}\cdot\dfrac{R_1}{R_1+R_2}}{R+\dfrac{1}{sC_1}}=\frac{-\dfrac{E}{s}\cdot\dfrac{R_3}{R}\cdot\dfrac{R_1}{R_1+R_2}}{1+\dfrac{1}{sRC_1}}=\frac{-E\cdot\dfrac{R_3}{R}\cdot\dfrac{R_1}{R_1+R_2}}{s+\dfrac{1}{RC_1}}$$

$$U_{R3}(s)=\frac{-E}{\frac{R(R_1+R_2)}{R_3R_1}}\cdot\frac{1}{s+\frac{1}{RC_1}}=\frac{-E}{\frac{\left(\frac{R_1R_2}{R_1+R_2}+R_3\right)(R_1+R_2)}{R_3R_1}}\cdot\frac{1}{s+\frac{1}{RC_1}}$$

$$U_{R3}(s)=\frac{-E}{\frac{R_1R_2+R_3(R_1+R_2)}{R_3R_1}}\cdot\frac{1}{s+\frac{1}{RC_1}}=\frac{-E}{\frac{R_2}{R_3}+\frac{R_2}{R_1}+1}\cdot\frac{1}{s+\frac{1}{RC_1}}$$ **(1 bod)**

U vremenskoj domeni:

$$u_{R3}(t)=-\frac{E}{\frac{R_2}{R_3}+\frac{R_2}{R_1}+1}\cdot e^{-\frac{t}{RC_1}}\cdot S(t)$$

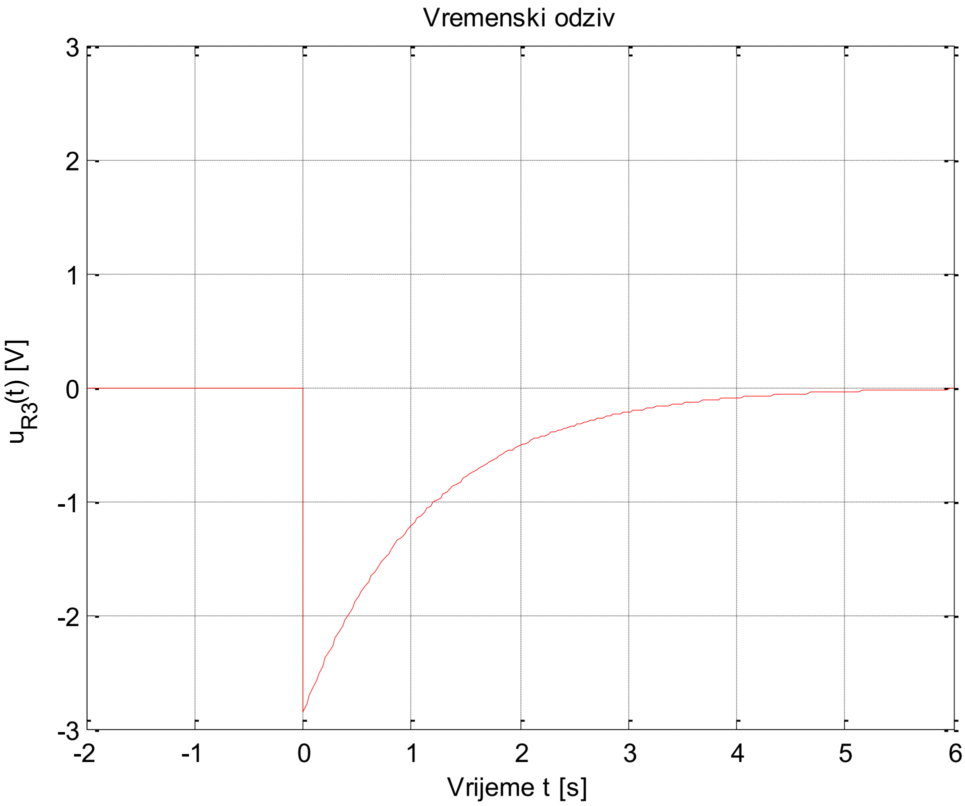
Uz uvrštene vrijednosti elemenata: R_1=2, R_2=1, R_3=1/2, C_1=1, te E=10 [V] slijedi

$$R=\frac{R_1R_2}{R_1+R_2}+R_3=\frac{2}{2+1}+\frac{1}{2}=\frac{2}{3}+\frac{1}{2}=\frac{4}{6}+\frac{3}{6}=\frac{7}{6}=1{,}166$$

$$u_{R3}(t)=-\frac{10}{\frac{1}{1/2}+\frac{1}{2}+1}\cdot e^{-\frac{t}{RC_1}}\cdot S(t)=\underline{-\frac{20}{7}\cdot e^{-\frac{6}{7}t}\cdot S(t)=-2{,}85714\cdot e^{-0{,}85714\cdot t}\cdot S(t)}$$

(1 bod)

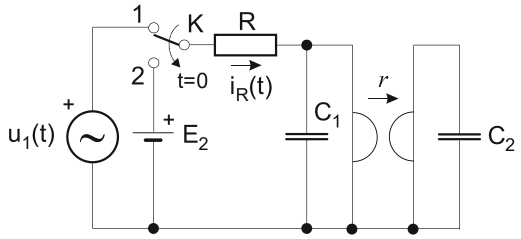
Matlab



(1 bod)

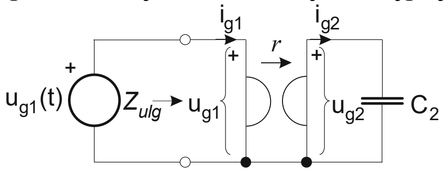
ZAVRŠNI ISPIT IZ ELEKTRIČNIH KRUGOVA 2019-2020 – Rješenja

1. U trenutku t=0 preklapa se sklopka K iz položaja 1 u položaj 2 u mreži prikazanoj slikom. Odrediti i skicirati valni oblik struje $i_R(t)$ kroz otpor R ako je zadano: $u_1(t)=\cos(t)$; $-\infty<t<\infty$ (sinusoidalno stacionarno stanje), $E_2=1$; $-\infty<t<\infty$ (istosmjerni napon baterije) te $R=1$, $r=1$, $C_1=1$, $C_2=1$.



Rješenje:

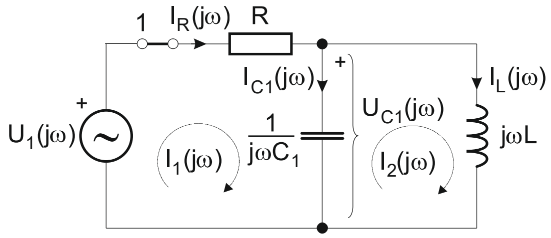
a) za $t<0$ računamo početne uvjete. Izračunajmo najprije ulaznu impedanciju u girator:



$$u_{g1}(t)=-r\cdot i_{g2}(t)$$
$$u_{g2}(t)=-r\cdot i_{g1}(t)$$

$$Z_{ulg}(s)=\frac{U_{g1}(s)}{I_{g1}(s)}=\frac{-r\cdot I_{g2}(s)}{\frac{1}{-r}\cdot U_{g2}(s)}=\frac{r^2}{\frac{U_{g2}(s)}{I_{g2}(s)}}=\frac{r^2}{Z_2(s)}=r^2sC_2=sL\text{ ; } L=r^2C_2\text{; } Z_{ulg}(s)=sL$$

Mrežu možemo prikazati kao:



Za sinusni generator vrijedi: $u_1(t)=\cos(t)\left[\mathrm{V}\right];-\infty<t<\infty\;\Rightarrow\;U_1(j\omega)=1\angle 0°$ fazor napona

Struje petlji glase:

$$I_1(j\omega)\left(R+\frac{1}{j\omega C_1}\right)-I_2(j\omega)\frac{1}{j\omega C_1}=U_1(j\omega)\qquad I_1(j\omega)\left(1+\frac{1}{j}\right)-I_2(j\omega)\frac{1}{j}=1$$

$$-I_1(j\omega)\frac{1}{j\omega C_1}+I_2(j\omega)\left(j\omega L+\frac{1}{j\omega C_1}\right)=0\qquad -I_1(j\omega)\frac{1}{j}+I_2(j\omega)\left(j+\frac{1}{j}\right)=0$$

$\Rightarrow\; I_1(j\omega)=0; I_2(j\omega)=-j=1\angle-90°$ ili

$I_{C1}(j\omega)=I_1(j\omega)-I_2(j\omega)=j=1\angle 90°$, $I_L(j\omega)=I_2(j\omega)=-j=1\angle-90°$

$I_R(j\omega)=I_1(j\omega)=I_{C1}(j\omega)+I_L(j\omega)=j-j=0$

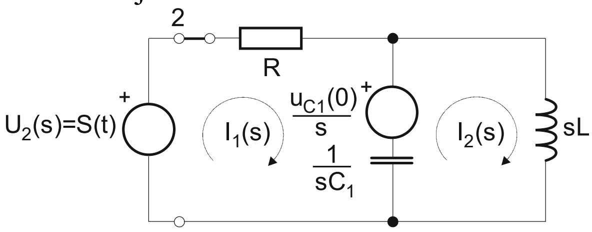
Rješenje koje daje struju I_R kroz R jednaku nuli je očekivano jer C_1-L čini paralelni titrajni krug s rezonantnom frekvencijom ω=1 (što je ujedno i frekvencija pobude $u_1(t)$) i s impedancijom jednakoj neizmjerno.

$\Rightarrow\; U_{C1}(j\omega)=I_{C1}(j\omega)\frac{1}{j\omega C_1}=j\frac{1}{j}=1\angle 0°\;\Rightarrow\; U_{C1}(j\omega)=U_1(j\omega)$ jer je $I_R(j\omega)=0$.

u vremenskoj domeni $\Rightarrow$ $\underline{i_L(t) = i_2(t) = \cos(t - 90°)}$; $\underline{u_{C1}(t) = u_1(t) = \cos(t)}$

u trenutku t=0 vrijedi (početni uvjeti): $\underline{i_L(0) = \cos(-90) = 0[\mathrm{A}]}$, $\underline{u_{C1}(0) = \cos(0) = 1[\mathrm{V}]}$ (2 boda)

b) za $t \geq 0$ primjena L-transformacije:



Struje petlji glase:

$$I_1(s)\left(R + \frac{1}{sC_1}\right) - I_2(s)\frac{1}{sC_1} = U_2(s) - \frac{u_{C1}(0)}{s} \qquad I_1(s)\left(1 + \frac{1}{s}\right) - I_2(s)\frac{1}{s} = \frac{1}{s} - \frac{1}{s} = 0$$

$$\underline{-I_1(s)\frac{1}{sC_1} + I_2(s)\left(sL + \frac{1}{sC_1}\right) = \frac{u_{C1}(0)}{s}} \qquad \underline{-I_1(s)\frac{1}{s} + I_2(s)\left(s + \frac{1}{s}\right) = \frac{1}{s}}$$

$$\begin{aligned} &I_1(s)(s+1) - I_2(s) = 0 \\ &\underline{-I_1(s) + I_2(s)(s^2+1) = 1} \end{aligned} \;\Rightarrow\; \begin{aligned} &I_1(s)(s+1) = I_2(s) \\ &\underline{I_2(s)(s^2+1) = 1 + I_1(s)} \end{aligned}$$

Nakon malo računanja:

$$\left[I_1(s)(s+1)\right](s^2+1) = 1 + I_1(s); \quad I_1(s)\left[(s+1)(s^2+1) - 1\right] = 1; \quad I_1(s)\left[s^3 + s + s^2 + 1 - 1\right] = 1$$

$$I_1(s)(s^3 + s^2 + s) = 1; \Rightarrow \boxed{I_R(s) = I_1(s) = \frac{1}{s(s^2+s+1)}}.$$

Povratak u vremensku domenu (rastav na parcijalne razlomke):

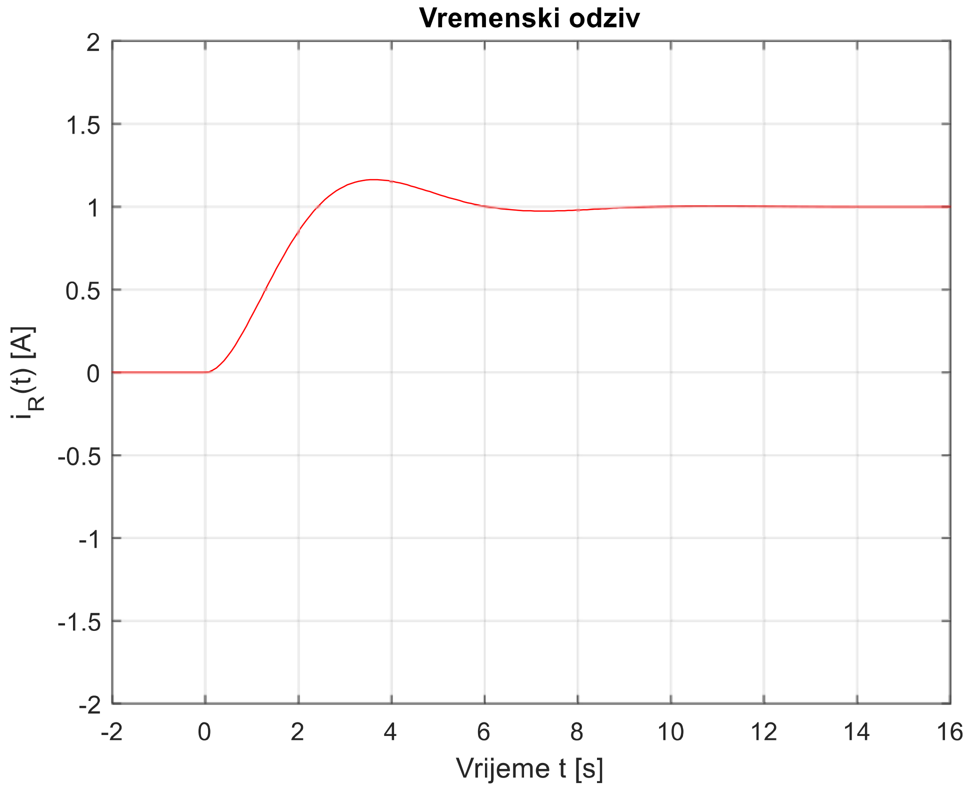
$$I_R(s) = \frac{1}{s(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1} = \frac{A(s^2+s+1) + (Bs+C)s}{s(s^2+s+1)} = \frac{(A+B)s^2 + (A+C)s + A}{s(s^2+s+1)}$$

$$\Rightarrow \underline{A+B=0; A+C=0; A=1} \Rightarrow \underline{A=1; B=-1; C=-1}$$

$$\Rightarrow I_R(s) = \frac{1}{s} - \frac{s+1}{s^2+s+1} = \frac{1}{s} - \frac{s+1}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{1}{s} - \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

Pa je konačno:

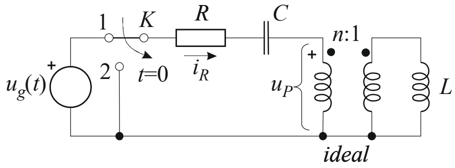
$$\boxed{i_R(t) = 0 \quad \text{za} \quad t<0} \text{ i } \boxed{i_R(t) = \left[1 - e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]\cdot S(t), \quad \text{za} \quad t \geq 0}$$


Vremenski odziv
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
0
2
4
6
8
10
12
14
16
$i_R(t)$ [A]
Vrijeme t [s]

(3 boda)

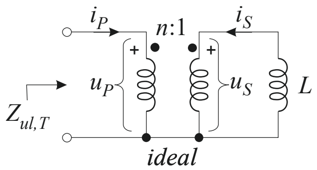
PISMENI ISPIT IZ ELEKTRIČNIH KRUGOVA 2019-2020 – Rješenja

1. U trenutku t=0 preklapa se sklopka K iz položaja 1 u položaj 2 u mreži prikazanoj slikom. Odredite valni oblik struj $i_R(t)$ kroz otpor R za $t \geq 0$ ako je zadano: $u_g(t)=2\cos(2t+\pi)$; $-\infty<t<\infty$ (sinusoidalno stacionarno stanje), te R=1, n=1/2, C=1, L=3. (Uputa: nadomjestiti transformator i induktivitet L ekvivalentnim dvopolom.)



Rješenje:

Najprije izračunajmo ulaznu impedanciju u transformator:

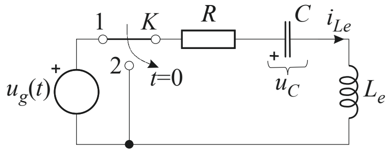


$$u_P(t) = n \cdot u_S(t)$$

$$i_P(t) = -\frac{1}{n} \cdot i_S(t)$$

$$Z_{ul,T}(s) = \frac{U_P(s)}{I_P(s)} = \frac{n \cdot U_S(s)}{-\frac{1}{n} \cdot I_S(s)} = -n^2 \cdot \frac{-I_S \cdot sL}{I_S} = sn^2 L = sL_e\,;\; L_e = n^2 L$$

Mrežu možemo prikazati kao:



a) za $t<0$ računamo početne uvjete (pomoću fazora):

Za sinusni generator vrijedi: $u_g(t) = 2\cos(2t+\pi); -\infty < t < \infty \;\Rightarrow\; U_g(j\omega) = 2 \cdot e^{j\cdot\pi}$

$$I_{L_e}(j\omega) = \frac{U_g(j\omega)}{R + \frac{1}{j\omega C} + j\omega L_e} = U_g(j\omega) \cdot \frac{j\omega C}{j\omega RC + 1 - \omega^2 L_e C} = \ldots = \sqrt{2} \cdot e^{j\cdot\frac{3\pi}{4}}$$

$$U_C(j\omega) = I_{L_e}(j\omega) \cdot \frac{1}{j\omega C} = U_g(j\omega) \cdot \frac{1}{j\omega RC + 1 - \omega^2 L_e C} = \ldots = \frac{\sqrt{2}}{2} \cdot e^{j\cdot\frac{\pi}{4}}$$

U vremenskoj domeni:

$$i_{L_e}(t) = \sqrt{2} \cdot \cos\left(2t + \frac{3\pi}{4}\right)$$

$$u_C(t) = \frac{\sqrt{2}}{2} \cdot \cos\left(2t + \frac{\pi}{4}\right)$$

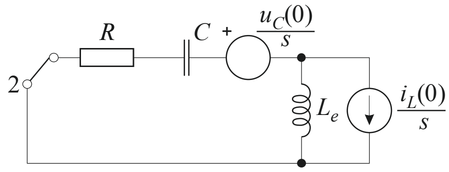
U trenutku t=0 vrijedi (početni uvjeti):

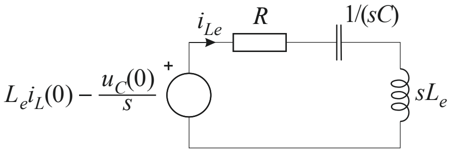
$$i_{L_e}(0) = \sqrt{2} \cdot \cos\left(\frac{3\pi}{4}\right) = -1$$ **(1 bod)**

$$u_C(0) = \frac{\sqrt{2}}{2} \cdot \cos\left(\frac{\pi}{4}\right) = \frac{1}{2}$$ **(1 bod)**

b) za $t \geq 0$ primjena L-transformacije:

$$I_{L_e}(s) = \frac{L_e i_{L_e}(0) - \dfrac{u_C(0)}{s}}{R + \dfrac{1}{sC} + sL_e} = \ldots = -\frac{s + \dfrac{2}{3}}{s^2 + \dfrac{4}{3}s + \dfrac{4}{3}}$$





(2 boda)

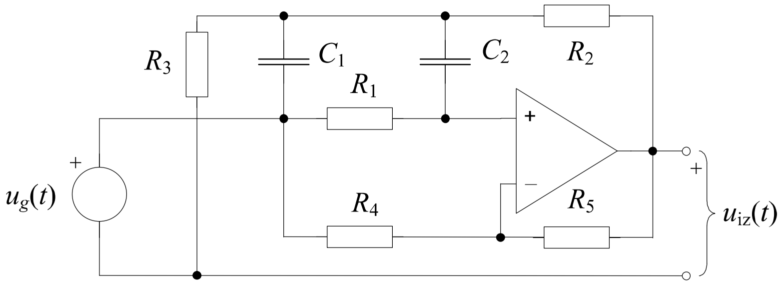
$I_R(s) = I_{L_e}(s)$

Povratak u vremensku domenu: $i_R(t) = -e^{\frac{-2t}{3}} \cdot \cos\left(\frac{2\sqrt{2}}{3}t\right) \cdot S(t)$ **(1 bod)**

PRIJENOSNE FUNKCIJE

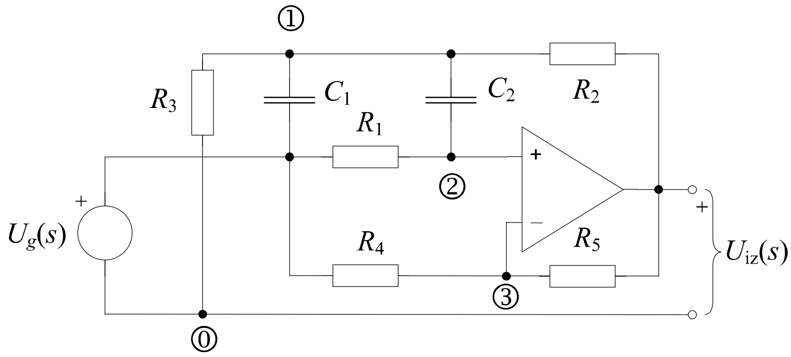
ZAVRŠNI ISPIT IZ ELEKTRIČNIH KRUGOVA 2013-2014 – Rješenja

1. Za električni krug prikazan slikom izračunati naponsku prijenosnu funkciju $H(s)=U_{iz}(s)/U_g(s)$ metodom napona čvorova. Zadane su normalizirane vrijednosti elemenata $R_1=1$, $R_2=1$, $R_3=1$, $R_4=1$, $R_5=1$, $C_1=1$, $C_2=1$. Operacijsko pojačalo je idealno. Početni uvjeti: $u_{C1}(0)=u_{C2}(0)=0$. U jednadžbe čvorišta uvrstiti odmah vrijednosti elemenata. Izračunati valni oblik napona $u_{iz}(t)$ kao odziv, ako je zadana pobuda (stacionarni sinusni signal) $u_g(t)=\sin(\sqrt{2}t)$; $-\infty<t<+\infty$.



Rješenje:

Primjenom Laplaceove transformacije dobivamo slijedeći električni krug. Postavimo jednadžbe čvorišta:



$$1)\ U_1\left(sC_1+sC_2+\frac{1}{R_2}+\frac{1}{R_3}\right)-U_2sC_2=U_g(s)sC_1+U_{iz}(s)\frac{1}{R_2}$$

$$2)\ -U_1sC_2+U_2\left(sC_2+\frac{1}{R_1}\right)=\frac{U_g(s)}{R_1};$$

$$3)\ U_3\left(\frac{1}{R_4}+\frac{1}{R_5}\right)=\frac{U_g(s)}{R_4}+\frac{U_{iz}(s)}{R_5};$$

$$4)\ U_{iz}(s)=A\cdot\left[U_2(s)-U_3(s)\right]\ \Rightarrow\ U_2(s)-U_3(s)=\frac{U_{iz}(s)}{A};$$

$$A\to\infty\quad\Rightarrow\quad U_2(s)=U_3(s)$$

(2 boda)

$$3),\ 4)\Rightarrow U_2=U_3=\frac{R_5}{R_4+R_5}U_g(s)+\frac{R_4}{R_4+R_5}U_{iz}(s)$$

Uvedimo oznaku α, i uvrstimo vrijednosti elemenata:

$$\alpha=\frac{R_5}{R_4+R_5}=\frac{1}{2};\quad(1-\alpha)=\frac{R_4}{R_4+R_5}=\frac{1}{2}$$

$$\Rightarrow U_2=U_3=\alpha U_g(s)+(1-\alpha)U_{iz}(s)=\frac{U_g(s)}{2}+\frac{U_{iz}(s)}{2}$$

$$2)\Rightarrow\; U_1=U_2\left(1+\frac{1}{sC_2R_1}\right)-U_g(s)\frac{1}{sC_2R_1}=U_2\left(1+\frac{1}{s}\right)-U_g(s)\frac{1}{s}\;\rightarrow 1)\Rightarrow$$

$$\left[U_2\left(1+\frac{1}{sC_2R_1}\right)-U_g(s)\frac{1}{sC_2R_1}\right]\left(sC_1+sC_2+\frac{1}{R_2}+\frac{1}{R_3}\right)-U_2sC_2=U_g(s)sC_1+U_{iz}(s)\frac{1}{R_2}$$

$\Rightarrow$

$$\left[U_2\left(1+\frac{1}{s}\right)-U_g(s)\frac{1}{s}\right](2s+2)-U_2s=U_g(s)s+U_{iz}(s)$$

$$2\left[U_2+\frac{U_2}{s}-U_g(s)\frac{1}{s}\right](s+1)-U_2s=U_g(s)s+U_{iz}(s)$$

$$2U_2s+2U_2-2U_g(s)+2U_2+2\frac{U_2}{s}-2\frac{U_g(s)}{s}-U_2s=U_g(s)s+U_{iz}(s)$$

$$U_2s+4U_2+\frac{2}{s}U_2=2U_g(s)+U_g(s)s+\frac{2}{s}U_g(s)+U_{iz}(s)$$

$$\left(\frac{U_g(s)}{2}+\frac{U_{iz}(s)}{2}\right)\left(s+4+\frac{2}{s}\right)=U_g(s)\left(2+s+\frac{2}{s}\right)+U_{iz}(s)\;\Big/\cdot 2s$$

$$\left(U_g(s)+U_{iz}(s)\right)\left(s^2+4s+2\right)=U_g(s)\left(2s^2+4s+4\right)+U_{iz}(s)\cdot 2s$$

$$U_g(s)\left(s^2+4s+2\right)+U_{iz}(s)\left(s^2+4s+2\right)=U_g(s)\left(2s^2+4s+4\right)+U_{iz}(s)\cdot 2s$$

$$U_{iz}(s)\left(s^2+2s+2\right)=U_g(s)\left(s^2+2\right)$$

$$H(s)=\frac{U_{iz}(s)}{U_g(s)}=\frac{s^2+2}{s^2+2s+2}$$ **(2 boda)**

Pobuda je svevremenska sinusoida: $u_g(t)=\sin(\sqrt{2}t)$; $-\infty<t<+\infty$.

Pridruženi fazor napona je $U_g(j\omega)=1\angle 0°$.

Frekvencijska karakteristika naponske prijenosne funkcije je:

$$H(j\omega)=H(s)\big|_{s=j\omega}=\frac{U_{iz}(j\omega)}{U_g(j\omega)}=\frac{2-\omega^2}{2j\omega+(2-\omega^2)}$$

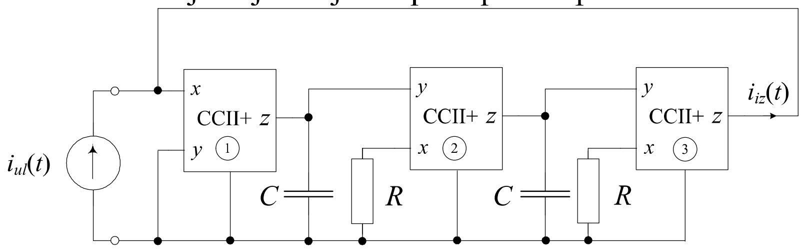
Na frekvenciji signala: $\omega_1=\sqrt{2}$ je amplitudno frekvencijska karakteristika jednaka:

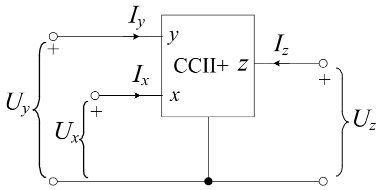
$$\left|H(j\omega)\right|\Big|_{\omega^2=\omega_1^2=2}=\left.\frac{\left|2-\omega^2\right|}{\sqrt{4\omega^2+(2-\omega^2)^2}}\right|_{\omega^2=\omega_1^2=2}=\frac{|2-2|}{\sqrt{8+(2-2)^2}}=0$$

$$\left|U_{iz}(j\omega)\right|=\left|H(j\omega_1)\right|\cdot\left|U_g(j\omega)\right|=0\cdot 1\angle 0°\;\Rightarrow\; u_{iz}(t)=0$$

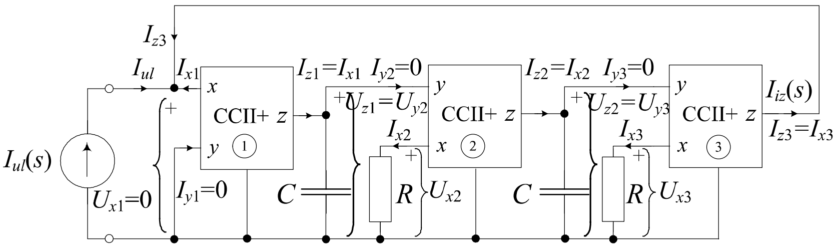
Zbog nule u prijenosnoj funkciji na frekvenciji ulaznog signala izlazni signal je jednak nuli. **(1 bod)**

4. Za električni krug prikazan slikom izračunati strujnu prijenosnu funkciju $H_i(s)=I_{iz}(s)/I_{ul}(s)$. Zadane su normalizirane vrijednosti elemenata $R=1$, $C=1$. Ako je zadana pobuda $i_{ul}(t)=S(t)$ izračunati i skicirati valni oblik struje $i_{iz}(t)$ za $t>0$ kao odziv. Za pozitivni strujni prijenosnik druge generacije (CCII+) vrijede slijedeće definicijske jednadžbe: $U_x=U_y$, $I_y=0$, $I_z=I_x$ uz referentna usmjerenja struja i napona prilaza prikazana na slici.





Rješenje:



Za prvi CCII+ vrijedi:

$$I_{x1}=-I_{ul}-I_{z3};\ U_{x1}=U_{y1}=U_{ul}=0,\ I_{y1}=0,\ I_{z1}=I_{x1}\quad \textbf{(1 bod)}$$

Za drugi CCII+ vrijedi:

$$U_{z1}=U_{y2}=I_{z1}\frac{1}{sC},\ I_{x2}=\frac{U_{x2}}{R}=\frac{U_{y2}}{R};\ I_{y2}=0,\ I_{z2}=I_{x2}\quad \textbf{(1 bod)}$$

Za treći CCII+ vrijedi:

$$U_{z2}=U_{y3}=I_{z2}\frac{1}{sC},\ I_{x3}=\frac{U_{x3}}{R}=\frac{U_{y3}}{R};\ I_{y3}=0,\ I_{iz}=I_{z3}=I_{x3}\quad \textbf{(1 bod)}$$

Nakon malo sređivanja:

$$I_{iz}=I_{z3}$$

$$I_{z3}=I_{x3}=\frac{U_{x3}}{R}=\frac{U_{y3}}{R}=\frac{1}{sRC}I_{z2}$$

$$I_{z2}=I_{x2}=\frac{U_{x2}}{R}=\frac{U_{y2}}{R}=\frac{1}{sRC}I_{z1}$$

$$I_{iz}=I_{z3}=\frac{1}{sRC}I_{z2}=\left(\frac{1}{sRC}\right)^2 I_{z1}\quad\Rightarrow\quad I_{z1}=(sRC)^2 I_{iz}$$

$$I_{z1}=I_{x1}=-I_{ul}-I_{z3}=-I_{ul}-\left(\frac{1}{sRC}\right)^2 I_{z1}\quad\Rightarrow\quad I_{z1}\cdot\left(1+\frac{1}{(sRC)^2}\right)=-I_{ul}$$

$$I_{iz}\cdot(sRC)^2\cdot\left(1+\frac{1}{(sRC)^2}\right)=-I_{ul}\quad\Rightarrow\quad \underline{H_i(s)=\frac{I_{iz}(s)}{I_{ul}(s)}=-\frac{1}{1+(sRC)^2}}$$

Uz uvrštene vrijednosti elemenata:

$$\underline{H_i(s)=\frac{I_{iz}(s)}{I_{ul}(s)}=-\frac{1}{s^2+1}}\quad \textbf{(1 bod)}$$

Za zadanu pobudu: $i_{ul}(t) = S(t) \Rightarrow I_{ul}(s) = \frac{1}{s}$

Uz uvrštene vrijednosti elemenata:

$$I_{iz}(s) = H_i(s) \cdot I_{ul}(s) = -\frac{1}{s} \cdot \frac{1}{s^2+1}$$

Rastav na parcijalne razlomke:

$$\frac{1}{s} \cdot \frac{1}{s^2+1} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{A(s^2+1)+s(Bs+C)}{s(s^2+1)} = \frac{(A+B)s^2+Cs+A}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

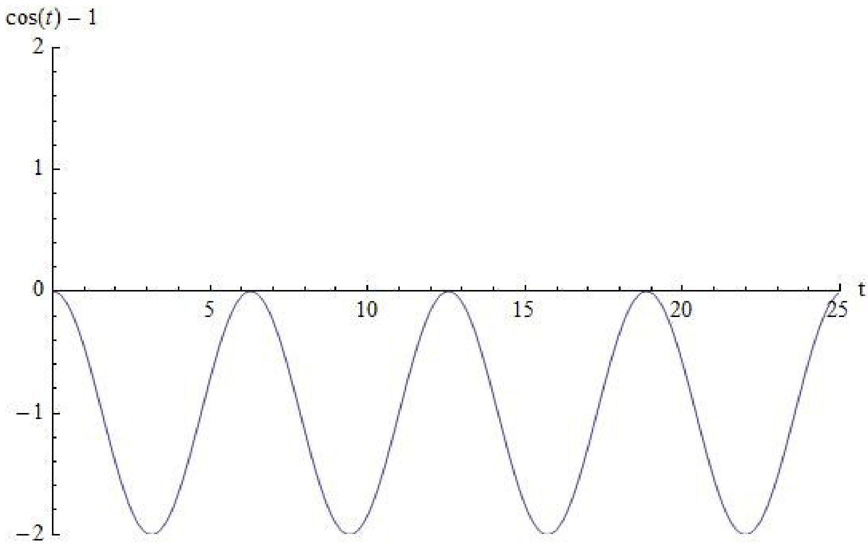
$$A+B=0 \Rightarrow B=-A=-1$$

$$C=0$$

$$A=1$$

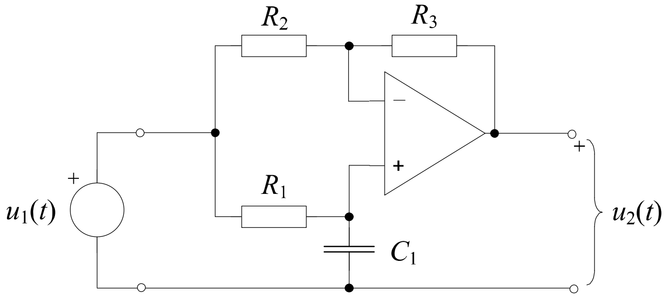
$$I_{iz}(s) = -\frac{1}{s} + \frac{s}{s^2+1} \quad \Rightarrow \quad \underline{i_{iz}(t) = [-1+\cos(t)] \cdot S(t)}$$ **(1 bod)**

Skica odziva $i_{iz}(t)$



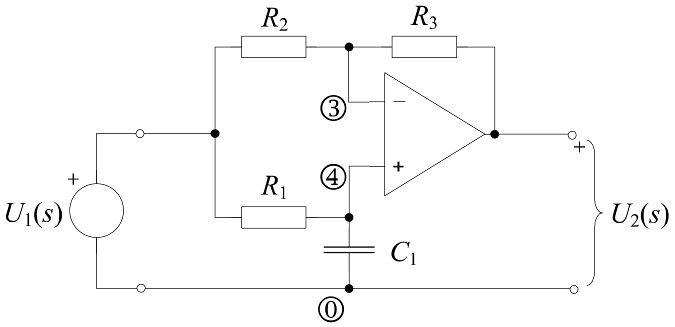
PISMENI ISPIT IZ ELEKTRIČNIH KRUGOVA 2013-2014 – Rješenja

1. Za električni krug prikazan slikom izračunati naponsku prijenosnu funkciju $H(s)=U_2(s)/U_1(s)$ metodom napona čvorova. Zadane su normalizirane vrijednosti elemenata: $R_1=1/2$, $R_2=1/4$, $R_3=1/2$, $C_1=1$. Operacijsko pojačalo je idealno. Početni uvjet $u_{C1}(0)=0$. Izračunati amplitudno-frekvencijsku i fazno-frekvencijsku karakteristiku. Odrediti valni oblik napona $u_2(t)$ kao odziv, ako je zadana pobuda (stacionarni sinusni signal) $u_1(t)=\cos(2t+45°)$; $-\infty<t<+\infty$.



Rješenje:

Primjenom Laplaceove transformacije dobivamo slijedeći električni krug. Postavimo jednadžbe čvorišta:



$$1)\ U_3(s)\left(\frac{1}{R_2}+\frac{1}{R_3}\right)=U_1(s)\frac{1}{R_2}+U_2(s)\frac{1}{R_3}\ ;\Rightarrow\ U_3(s)=U_1(s)\frac{R_3}{R_2+R_3}+U_2(s)\frac{R_2}{R_2+R_3}$$

$$2)\ U_4(s)\left(\frac{1}{R_1}+sC_1\right)=U_1(s)\frac{1}{R_1}\ \Rightarrow\ U_4(s)=U_1(s)\frac{1}{1+sR_1C_1}$$

$$3)\ U_{iz}(s)=A\cdot\left[U_4(s)-U_3(s)\right]\Rightarrow U_4(s)-U_3(s)=\frac{U_{iz}(s)}{A};A\to\infty\ \Rightarrow U_3(s)=U_4(s)$$

$$1)\ ,\ 2)\ ,\ 3)\Rightarrow\ U_1(s)\frac{R_3}{R_2+R_3}+U_2(s)\frac{R_2}{R_2+R_3}=U_1(s)\frac{1}{1+sR_1C_1}$$

$$U_2(s)\frac{R_2}{R_2+R_3}=U_1(s)\frac{1}{1+sR_1C_1}-U_1(s)\frac{R_3}{R_2+R_3}\Big/\frac{R_2+R_3}{R_2}$$

$$U_2(s)=U_1(s)\frac{R_2+R_3}{R_2(1+sR_1C_1)}-U_1(s)\frac{R_3}{R_2}$$

$$H(s)=\frac{U_2(s)}{U_1(s)}=\frac{R_2+R_3}{R_2(1+sR_1C_1)}-\frac{R_3}{R_2}=\frac{R_2+R_3-R_3(1+sR_1C_1)}{R_2(1+sR_1C_1)}$$

$$H(s)=\frac{U_2(s)}{U_1(s)}=\frac{R_2-sR_1R_3C_1}{R_2(1+sR_1C_1)}=\frac{1-sC_1R_1R_3/R_2}{1+sR_1C_1}=\frac{\frac{1}{R_1C_1}-\frac{R_3}{R_2}s}{\frac{1}{R_1C_1}+s}=-\frac{R_3}{R_2}\cdot\frac{s-\frac{R_2}{C_1R_1R_3}}{s+\frac{1}{C_1R_1}}$$

Uz uvrštene vrijednosti elemenata R_1=1/2, R_2=1/4, R_3=1/2, C_1=1 dobivamo:

$$H(s)=\frac{U_2(s)}{U_1(s)}=-\frac{R_3}{R_2}\cdot\frac{s-\frac{R_2}{C_1R_1R_3}}{s+\frac{1}{C_1R_1}}=-\frac{\frac{1}{2}}{\frac{1}{4}}\cdot\frac{s-\frac{(1/4)}{1(1/2)(1/2)}}{s+\frac{1}{1(1/2)}}=-2\cdot\frac{s-1}{s+2}$$

(3 boda)

Pobuda je svevremenska sinusoida (stacionarni sinusni signal):

$u_1(t)=\cos(2t+45°)$; $-\infty<t<+\infty$.

Pridruženi fazor napona je $U_1(j\omega)=1\angle 0°$.

Frekvencijska karakteristika naponske prijenosne funkcije je:

$$H(j\omega)=H(s)\big|_{s=j\omega}=\frac{U_2(j\omega)}{U_1(j\omega)}=\frac{N(j\omega)}{D(j\omega)}=-2\cdot\frac{j\omega-1}{j\omega+2}=\frac{2-j2\omega}{2+j\omega}$$

Amplitudno-frekvencijska karakteristika glasi:

$$|H(j\omega)|=\frac{|N(j\omega)|}{|D(j\omega)|}=2\frac{\sqrt{\omega^2+1}}{\sqrt{\omega^2+4}}$$

Na frekvenciji signala: $\omega_1=2$ [rad/s] amplitudno-frekvencijska karakteristika je jednaka:

$$|H(j\omega)|\big|_{\omega^2=\omega_1^2=4}=2\frac{\sqrt{\omega^2+1}}{\sqrt{\omega^2+4}}\Bigg|_{\omega^2=\omega_1^2=4}=2\frac{\sqrt{4+1}}{\sqrt{4+4}}=\frac{2\sqrt{5}}{2\sqrt{2}}=\frac{\sqrt{5}}{\sqrt{2}}=1{,}5811388$$

Fazno-frekvencijska karakteristika glasi:

$$\varphi(\omega)=\arctan\frac{\mathrm{Im}[N(j\omega)]}{\mathrm{Re}[N(j\omega)]}-\arctan\frac{\mathrm{Im}[D(j\omega)]}{\mathrm{Re}[D(j\omega)]}=\arctan\left(\frac{-2\omega}{2}\right)-\arctan\left(\frac{\omega}{2}\right)$$

$$\varphi(\omega)=-\arctan(\omega)-\arctan\left(\frac{\omega}{2}\right)$$

Na frekvenciji signala: $\omega_1=2$ [rad/s] je fazno-frekvencijska karakteristika je jednaka:

$$\varphi(\omega_1)=-\arctan(2)-\arctan(1)=-63{,}43°-45°=-108{,}43°$$

Provjera kuta: Frekvencijska karakteristika na frekvenciji signala $\omega_1=2$ [rad/s]

$$H(j\omega)\big|_{\omega=\omega_1=2}=\frac{2-j2\omega}{2+j\omega}\bigg|_{\omega=\omega_1=2}=\frac{2-j4}{2+j2}=\frac{1-j2}{1+j}\cdot\frac{1-j}{1-j}=\frac{1-j2-j-2}{2}=\frac{-1-3j}{2}$$

Kompleksni broj se nalazi u III kvadrantu.

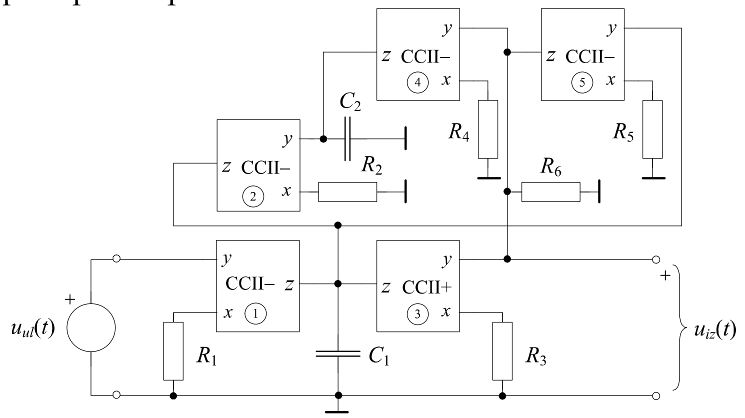
Stoga je signal na izlazu:

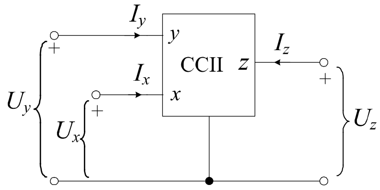
$$|U_2(j\omega)|=|H(j\omega_1)|\cdot|U_1(j\omega)|=1{,}5811388\cdot 1=1{,}5811388$$

$$\varphi_{uk}=\varphi_{sig}+\varphi(\omega_1)=45°-108{,}43°=-63{,}43°$$

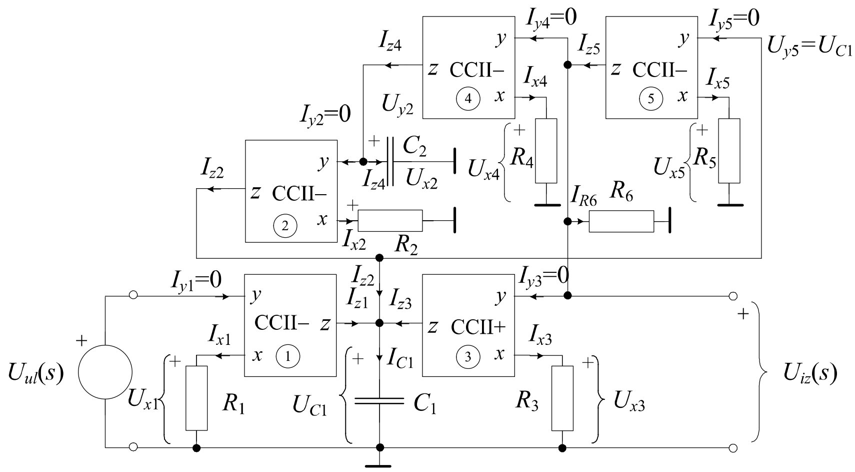
$$\Rightarrow \underline{u_2(t)=1{,}5811388\cdot\cos(2t-63{,}43°)[\mathrm{V}]}$$ **(2 boda)**

4. Za električni krug prikazan slikom izračunati naponsku prijenosnu funkciju $H(s)=U_{iz}(s)/U_{ul}(s)$ kao funkciju varijable „s" i elemenata (R_i i C_j). Zatim uvrstiti normalizirane vrijednosti elemenata: $R_1=1$, $R_2=1$, $R_3=1$, $R_4=1$, $R_5=1$, $R_6=1$, $C_1=1$, $C_2=1$. Za pozitivni strujni prijenosnik druge generacije (CCII+) vrijede slijedeće definicijske jednadžbe: $U_x=U_y$, $I_y=0$, $I_z=I_x$, a za negativni (CCII−) vrijedi $U_x=U_y$, $I_y=0$, $I_z=-I_x$, uz referentna usmjerenja struja i napona prilaza prikazana na slici.





Rješenje:



Za prvi CCII− vrijedi:

$$U_{x1}=U_{y1}=U_{ul},\ I_{y1}=0,\ I_{z1}=-I_{x1},\ I_{x1}=\frac{U_{x1}}{R_1}\Rightarrow \underline{I_{z1}=-\frac{U_{ul}}{R_1}}$$

$$U_{C1}=U_{z1}=U_{z2}=U_{z3}=U_{y5}\Rightarrow \underline{U_{C1}=I_{C1}\frac{1}{sC_1}},\ \underline{I_{C1}=I_{z1}+I_{z2}+I_{z3}}$$

Za drugi CCII− vrijedi:

$$U_{x2}=U_{y2},\ I_{y2}=0,\ I_{z2}=-I_{x2},\ I_{x2}=\frac{U_{x2}}{R_2},\ U_{y2}=I_{z4}\frac{1}{sC_2}\Rightarrow \underline{I_{z2}=-I_{z4}\frac{1}{sR_2C_2}}$$

Za treći CCII+ vrijedi:

$$U_{x3}=U_{y3}=U_{iz},\ I_{x3}=\frac{U_{x3}}{R_3},\ I_{y3}=0,\ I_{z3}=I_{x3}\Rightarrow \underline{I_{z3}=\frac{U_{iz}}{R_3}}$$

Za četvrti CCII− vrijedi:

$$U_{x4}=U_{y4}=U_{iz},\ I_{x4}=\frac{U_{x4}}{R_4},\ I_{y4}=0,\ I_{z4}=-I_{x4}\Rightarrow \underline{I_{z4}=-\frac{U_{iz}}{R_4}}$$

Za peti CCII− vrijedi:

$$U_{x5}=U_{y5}=U_{C1}\,,\ I_{x5}=\frac{U_{x5}}{R_5}\,,\ I_{y5}=0\,,\ I_{R6}=I_{z5}=-I_{x5}\,,\ U_{iz}=I_{R6}R_6\ \Rightarrow\ \underline{U_{iz}=-U_{C1}\frac{R_6}{R_5}}$$

(3 boda)

Nakon malo sređivanja:

$$U_{C1}=\left(I_{z1}+I_{z2}+I_{z3}\right)\cdot\frac{1}{sC_1}=\left(-\frac{U_{ul}}{R_1}-I_{z4}\frac{1}{sR_2C_2}+\frac{U_{iz}}{R_3}\right)\cdot\frac{1}{sC_1}$$

$$U_{iz}=-\left(-\frac{U_{ul}}{R_1}+\frac{U_{iz}}{R_4}\frac{1}{sR_2C_2}+\frac{U_{iz}}{R_3}\right)\cdot\frac{1}{sC_1}\cdot\frac{R_6}{R_5}$$

$$U_{iz}\cdot sC_1\frac{R_5}{R_6}=\frac{U_{ul}}{R_1}-\frac{U_{iz}}{R_4}\frac{1}{sR_2C_2}-\frac{U_{iz}}{R_3}$$

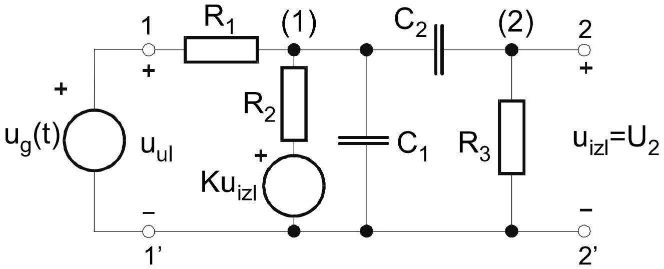
$$U_{iz}\cdot\left(sC_1\frac{R_5}{R_6}+\frac{1}{sR_2R_4C_2}+\frac{1}{R_3}\right)=\frac{U_{ul}}{R_1}\Big/\cdot s\frac{R_6}{C_1R_5}$$

$$H(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{s\dfrac{1}{R_1C_1}\dfrac{R_6}{R_5}}{s^2+s\dfrac{1}{R_3C_1}\dfrac{R_6}{R_5}+\dfrac{R_6}{C_1C_2R_2R_4R_5}}$$ **(1 bod)**

Uz uvrštene vrijednosti elemenata:

$$H(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{s}{s^2+s+1}$$ **(1 bod)**

4. Odrediti prijenosnu funkciju $T(s)=U_{iz}(s)/U_{ul}(s)$ za mrežu prikazanu slikom (koristiti metodu čvorova ili metodu petlji). Izračunati omjer amplituda te razliku u fazi napona na ulazu i izlazu mreže ako je zadano: napon generatora na ulazu $u_g(t)$= 10 sin t, i normirane vrijednosti elemenata R_1=R_2=2, R_3=1, C_1=C_2=1 i K=2.



<u>Rješenje:</u> Metoda čvorišta:

$$(1)\qquad U_1\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_1+sC_2\right)-U_2sC_2=\frac{U_g}{R_1}+\frac{KU_{iz}}{R_2}$$

$$(2)\qquad -U_1sC_2+U_2\left(sC_2+\frac{1}{R_3}\right)=0$$

$$(2)\Rightarrow U_1=U_2\left(1+\frac{1}{sR_3C_2}\right)\rightarrow(1)$$

$$(1)\qquad U_2\left[\left(1+\frac{1}{sR_3C_2}\right)\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_1+sC_2\right)-sC_2-\frac{K}{R_2}\right]=\frac{U_g}{R_1}$$

$$U_2\left[\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_1+sC_2\right)+\frac{1}{sR_1R_3C_2}+\frac{1}{sR_2R_3C_2}+\frac{C_1}{R_3C_2}+\frac{1}{R_3}-sC_2-K\frac{1}{R_2}\right]=\frac{U_g}{R_1}\,/\,R_1R_2R_3C_2s$$

$$U_2\left[sR_2R_3C_2+sR_1R_3C_2+s^2C_1C_2R_1R_2R_3+R_2+R_1+sR_1R_2C_1+sR_1R_2C_2-sKR_1R_3C_2\right]=U_gR_2R_3C_2s$$

Prijenosna funkcija napona:

$$T(s)=\frac{U_2}{U_g}=\frac{sR_2R_3C_2}{s^2C_1C_2R_1R_2R_3+s\left[R_2R_3C_2+R_1R_2(C_1+C_2)+R_1R_3C_2(1-K)\right]+R_1+R_2}$$

Uz uvrštene vrijednosti:

$$T(s)=\frac{s\cdot 2}{s^2 4+s\left[2+8-2\right]+4}=\frac{1}{4}\cdot\frac{2\cdot s}{s^2+2s+1}$$

Signal: $\omega_g=1\quad U_g=10\angle 0^o$

$$\left|T(j\omega)\right|_{\omega=1}=\frac{1}{2}\frac{|j\omega|}{\left|-\omega^2+2j\omega+1\right|}=\frac{1}{2}\frac{\omega}{\sqrt{(1-\omega^2)^2+(2\omega)^2}}$$

$$\left|T(j\omega)\right|_{\omega=1}=\frac{1}{2}\frac{1}{\sqrt{0+2^2}}=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$$

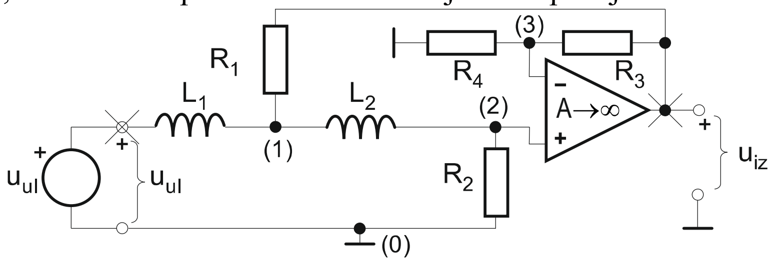
$$T(j\omega)=\frac{1}{2}\cdot\frac{\left[(1-\omega^2)-j(2\omega)\right]j\omega}{(1-\omega^2)^2+(2\omega)^2}=\frac{1}{2}\cdot\frac{2\omega^3+j\omega(1-\omega^2)}{(1-\omega^2)^2+4\omega^2}$$

$$\varphi(\omega)=\arctan\left[\frac{\mathrm{Im}\{T(j\omega)\}}{\mathrm{Re}\{T(j\omega)\}}\right]=\arctan\left[\frac{\omega(1-\omega^2)}{2\omega^3}\right]=\arctan\left[\frac{1-\omega^2}{2\omega^2}\right]$$

$$\varphi(1)=\arctan\left(\frac{1-1}{2\cdot 1}\right)=\arctan(0)=0^o$$

Odgovor: Omjer amplituda iz-ul signala je: 1:4. Razlika u fazi iz-ul signala je: 0

4. Naći naponsku prijenosnu funkciju $T(s)=U_{iz}(s)/U_{ul}(s)$ za električni krug na slici. Nacrtati raspored nula i polova u s-ravnini, izračunati i nacrtati amplitudno-frekvencijsku karakteristiku $|T(j\omega)|$, ako su zadane normalizirane vrijednosti elemenata: $L_1=1/2$, $L_2=2$, $R_1=R_2=R_3=R_4=1$, $A\to\infty$. (Uputa: zadatak riješiti metodom čvorišta za čvorišta koja su označena na slici, a čvorišta naponskih izvora za koja se ne pišu jednadžbe su prekrižena.)



Rješenje:

(1) $$U_1\left(\frac{1}{sL_1}+\frac{1}{sL_2}+\frac{1}{R_1}\right)-U_2\frac{1}{sL_2}=U_{iz}\frac{1}{R_1}+U_{ul}\frac{1}{sL_1}$$

(2) $$-U_1\frac{1}{sL_2}+U_2\left(\frac{1}{R_2}+\frac{1}{sL_2}\right)=0$$

(3) $$U_3\left(\frac{1}{R_3}+\frac{1}{R_4}\right)=\frac{U_{iz}}{R_3}$$

(4) $$U_{iz}=A(U_2-U_3);\qquad A\to\infty\Rightarrow U_2=U_3$$

$$(3),(4)\Rightarrow U_{iz}=U_3\left(1+\frac{R_3}{R_4}\right)=U_2\left(1+\frac{R_3}{R_4}\right)=\mu\cdot U_2;\quad \mu=1+\frac{R_3}{R_4}$$

$$(2)\Rightarrow U_1=U_2\left(1+\frac{sL_2}{R_2}\right)\quad (2)\to(1)$$

$$U_2\left(1+\frac{sL_2}{R_2}\right)\left(\frac{1}{sL_1}+\frac{1}{sL_2}+\frac{1}{R_1}\right)-U_2\frac{1}{sL_2}=U_{iz}\frac{1}{R_1}+U_{ul}\frac{1}{sL_1}$$

$$U_2\left(\frac{1}{sL_1}+\frac{1}{sL_2}+\frac{1}{R_1}+\frac{sL_2}{R_2sL_1}+\frac{1}{R_2}+\frac{sL_2}{R_1R_2}-\frac{1}{sL_2}\right)=\mu U_2\frac{1}{R_1}+U_{ul}\frac{1}{sL_1}$$

$$\frac{U_{iz}}{\mu}\left(\frac{1}{sL_1}+\frac{1-\mu}{R_1}+\frac{sL_2}{R_2sL_1}+\frac{1}{R_2}+\frac{sL_2}{R_1R_2}\right)=U_{ul}\frac{1}{sL_1}\bigg/\mu R_1R_2sL_1$$

$$U_{iz}\left[R_1R_2+(1-\mu)R_2sL_1+sL_2R_1+R_1sL_1+sL_1sL_2\right]=\mu U_{ul}R_1R_2$$

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{\mu R_1R_2}{R_1R_2+s\left[(1-\mu)R_2L_1+R_1(L_1+L_2)\right]+s^2L_1L_2}$$

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\mu\cdot\frac{\dfrac{R_1R_2}{L_1L_2}}{s^2+s\left[(1-\mu)\dfrac{R_2}{L_2}+R_1\left(\dfrac{1}{L_1}+\dfrac{1}{L_2}\right)\right]+\dfrac{R_1R_2}{L_1L_2}}$$

Uz uvrštene vrijednosti elemenata: $L_1=1/2$, $L_2=2$, $R_1=R_2=R_3=R_4=1$

$$\mu=1+\frac{R_3}{R_4}=2$$

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{2}{1+s\left[(1-2)\frac{1}{2}+\left(\frac{1}{2}+2\right)\right]+s^2}=\frac{2}{1+s\left[-\frac{1}{2}+\frac{1}{2}+2\right]+s^2}$$

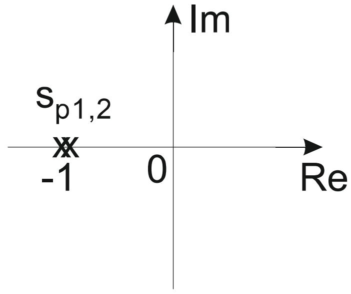
$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{2}{1+2s+s^2}=\frac{2}{(s+1)^2}$$

Prijenosna funkcija ima negativne, realne polove (višestruki pol) i dvije nule u beskonačnosti:

-polovi: $1+2s+s^2=0 \Rightarrow s_{p1,2}=-1$

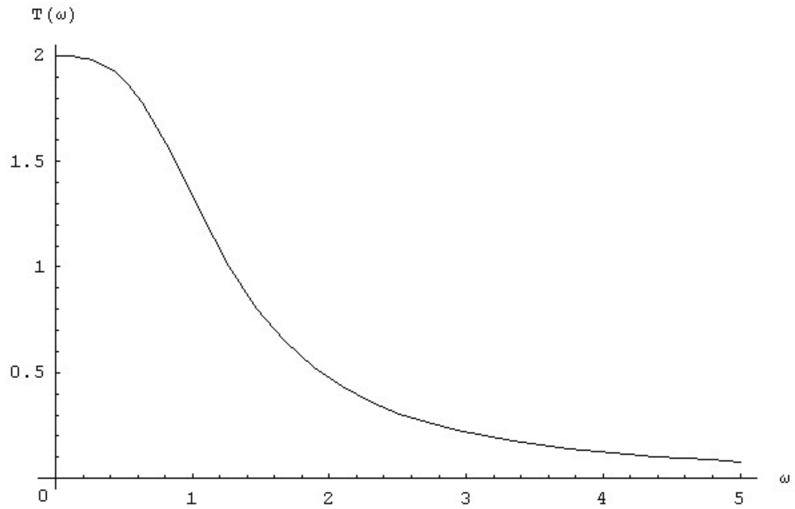
-nule: $s_{o1,2}=\infty$

Raspored polova u s-ravnini:

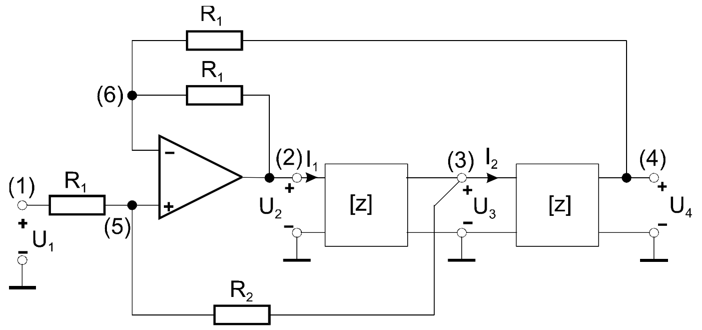


A-F karakteristika: $|T(j\omega)|=2\left|\frac{1}{-\omega^2+2j\omega+1}\right|=\frac{2}{\sqrt{(1-\omega^2)^2+(2\omega)^2}}=\frac{2}{1+\omega^2}$

za $\omega=0 \Rightarrow |T(j\omega)|=2$



4. Za mrežu prikazanu slikom naći prijenosnu funkciju $T(s)=U_4(s)/U_1(s)$ ako je zadano $R_1=1;\ R_2=0{,}414;\ A\to\infty;\ C=1$. Za pobudu $u_1(t)=10\cos(0{,}9\,t)$ naći valni oblik napona $u_4(t)$.

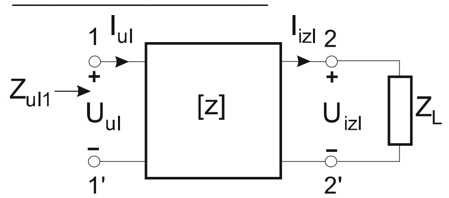


$$[z]=\begin{bmatrix} R_1 & 0 \\ -\dfrac{1}{sC} & 0 \end{bmatrix}$$

Rješenje: a) najprije odredimo naponsku prijenosnu funkciju i ulaznu impedanciju četveropola (koji je zadan [z] ili [a] parametrima):

[z]-parametri:

$$\underline{\begin{aligned} U_{ul} &= z_{11}I_{ul} - z_{12}I_{izl} \\ U_{izl} &= z_{21}I_{ul} - z_{22}I_{izl} \end{aligned}}$$



$$H_u(s)=\frac{U_{izl}}{U_{ul}}=\frac{Z_L z_{21}}{\Delta z + z_{11}Z_L}=\left|Z_L=\infty\right|=\frac{z_{21}}{z_{11}} \text{ ili } H_u(s)=\frac{U_{izl}}{U_{ul}}=\frac{Z_L z_{21}}{\Delta z + z_{11}Z_L}=\begin{vmatrix} z_{12}=0 \\ z_{22}=0 \end{vmatrix}=\frac{z_{21}}{z_{11}};$$

gdje su $\Delta z = z_{22}z_{11} - z_{12}z_{21}$ i $U_{izl}=Z_L\cdot I_{izl}$.

$$Z_{ul1}(s)=\frac{U_{ul}}{I_{ul}}=z_{11}-\frac{z_{12}z_{21}}{z_{22}+Z_L}=\left|Z_L=\infty\right|=z_{11} \text{ ili } Z_{ul1}(s)=\frac{U_{ul}}{I_{ul}}=z_{11}-\frac{z_{12}z_{21}}{z_{22}+Z_L}=\begin{vmatrix} z_{12}=0 \\ z_{22}=0 \end{vmatrix}=z_{11}$$

Moguće je četveropole zadati i prijenosnim [a] parametrima:

$$\underline{\begin{aligned} U_1 &= A\cdot U_2 + B\cdot I_2 \\ I_1 &= C\cdot U_2 + D\cdot I_2 \end{aligned}}$$

$$A=\frac{z_{11}}{z_{21}},B=\frac{z_{11}z_{22}}{z_{21}}-z_{12},C=\frac{1}{z_{21}},D=\frac{z_{22}}{z_{21}} \quad\Rightarrow\quad [a]=\begin{bmatrix} -sR_1C & 0 \\ -sC & 0 \end{bmatrix}$$

$$H_u(s)=\frac{U_{izl}}{U_{ul}}=\frac{Z_L}{AZ_L+B}=\left|Z_L=\infty \text{ ili } \begin{matrix} B=0 \\ D=0 \end{matrix}\right|=\frac{1}{A}=\frac{z_{21}}{z_{11}}$$

odn. $$Z_{ul1}(s)=\frac{U_{ul}}{I_{ul}}=\frac{AZ_L+B}{CZ_L+D}=\left|Z_L=\infty \text{ ili } \begin{matrix} B=0 \\ D=0 \end{matrix}\right|=\frac{A}{C}=z_{11}$$

Stoga je prijenosna funkcija: $H_u(s)=\dfrac{U_{izl}}{U_{ul}}=\dfrac{z_{21}}{z_{11}}=\dfrac{-\frac{1}{sC}}{R_1}=-\dfrac{1}{s}$,

a ulazna impedancija: $Z_{ul1}(s)=\dfrac{U_{ul}}{I_{ul}}=z_{11}=R_1$

Postavimo jednadžbe čvorišta:

(5) $$U_5\left(\frac{1}{R_1}+\frac{1}{R_2}\right)-U_3\frac{1}{R_2}=\frac{U_1}{R_1}$$

(6) $$U_6\left(\frac{1}{R_1}+\frac{1}{R_1}\right)-U_2\frac{1}{R_1}-U_4\frac{1}{R_1}=0$$

(2) $$U_2=A(U_5-U_6)\Rightarrow U_5-U_6=\frac{U_2}{A};A\to\infty\Rightarrow U_5=U_6$$

(3) $$U_3=-\frac{1}{sR_1C}U_2$$

(4) $$U_4=-\frac{1}{sR_1C}U_3$$

Nakon malo sređivanja i uvrštavanja slijedi:

$$\begin{aligned}&U_5\cdot 3{,}4154-U_3\cdot 2{,}4154=U_1\\&U_6\cdot 2-U_2-U_4=0\\&U_3=-\frac{1}{s}U_2\\&U_4=-\frac{1}{s}U_3\end{aligned}$$

$$\begin{aligned}&U_5\cdot 3{,}4154-U_3\cdot 2{,}4154=U_1\\&2U_5-U_2-U_4=0\\&U_2=-sU_3\\&U_3=-sU_4\end{aligned}$$

$$\begin{aligned}&U_5\cdot 3{,}4154-U_3\cdot 2{,}4154=U_1\\&2U_5+s\cdot U_3-U_4=0\\&U_3=-sU_4\end{aligned}$$

$$\begin{aligned}&U_5\cdot 3{,}4154+U_4\cdot s\cdot 2{,}4154=U_1\\&2U_5-U_4\cdot s^2-U_4=0\Rightarrow U_5=\frac{1}{2}(s^2+1)U_4\end{aligned}$$

$$\left[\frac{1}{2}(s^2+1)\cdot 3{,}4154+s\cdot 2{,}4154\right]U_4=U_1$$

$$T(s)=\frac{U_4(s)}{U_1(s)}=\frac{1}{1{,}7077s^2+1{,}7077+s\cdot 2{,}4154}$$

$$\boxed{T(s)=\frac{U_4(s)}{U_1(s)}=\frac{0{,}5858}{s^2+1{,}4142s+1}}$$ uz uvršteno $s=j\omega$ slijedi

$$T(j\omega)=\frac{0{,}5858}{(1-\omega^2)+j\sqrt{2}\omega}=\frac{0{,}5858}{(1-\omega^2)^2+2\omega^2}\left[(1-\omega^2)-j\sqrt{2}\omega\right]$$

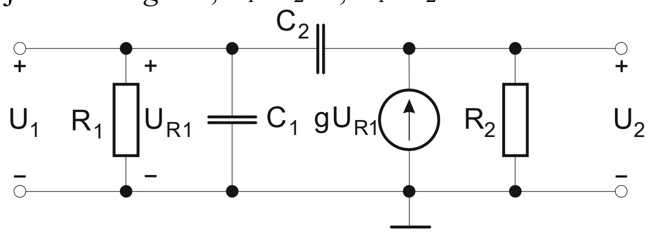
$$\left|T(j\omega)\right|=\left.\frac{0{,}5858}{\sqrt{(1-\omega^2)^2+(\sqrt{2}\cdot\omega)^2}}\right|_{\omega=0,9}=\frac{0{,}5858}{\sqrt{(1-0{,}9^2)^2+2\cdot 0{,}9^2}}=0{,}455204$$

$$\angle T(j\omega)=\arctan\frac{\mathrm{Im}\{T(j\omega)\}}{\mathrm{Re}\{T(j\omega)\}}=\arctan\left.\frac{-\sqrt{2}\omega}{1-\omega^2}\right|_{\omega=0,9}=\arctan\frac{-1{,}2728}{0{,}19}=-1{,}4226\text{ rad}=-81{,}5°$$

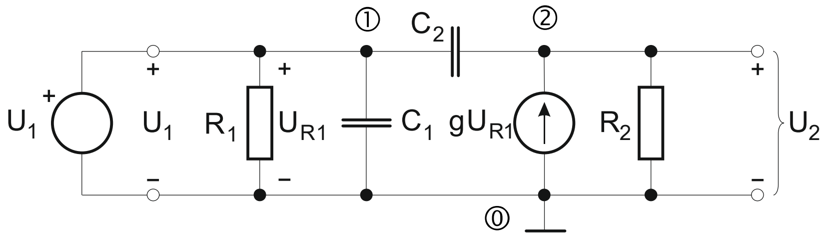
$$u_1(t)=10\cos(0{,}9t)\quad\Rightarrow\quad U_1(j\omega)=10\angle 0^0;\quad \omega=0{,}9\text{ [rad/s]}$$

$$U_4(j\omega)=U_1(j\omega)\cdot T(j\omega)=10\angle 0°\cdot 0{,}455204\angle -81.5°\quad\Rightarrow\quad \underline{u_4(t)=4{,}552\cos(0{,}9t-81{,}5°)}$$

3. Odrediti prijenosnu funkciju $T(s)=U_2(s)/U_1(s)$ za mrežu prikazanu slikom. Nacrtati tok funkcije $|T(j\omega)|$ ako je zadano: $g=10$, $R_1=R_2=1$, $C_1=C_2=1$.



Rješenje:



Naponi čvorišta:

(1) ne piše se jednadžba;

(2) $-U_1 sC_2 + U_2\left(\dfrac{1}{R_2}+sC_2\right) = gU_1$ **(1 bod)**

(2) $\Rightarrow U_2\left(\dfrac{1}{R_2}+sC_2\right) = U_1(g+sC_2) \Rightarrow T(s)=\dfrac{U_2(s)}{U_1(s)}=\dfrac{g+sC_2}{1/R_2+sC_2}$ **(1 bod)**

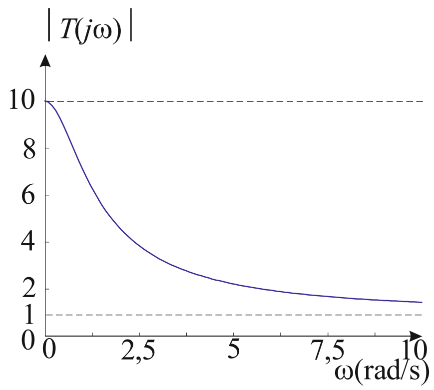
Uz uvrštene vrijednosti elemenata i nakon malo sređivanja slijedi $T(s)=\dfrac{U_2(s)}{U_1(s)}=\dfrac{s+10}{s+1}$

(1 bod)

Amplitudno-frekvencijska (A-F) karakteristika

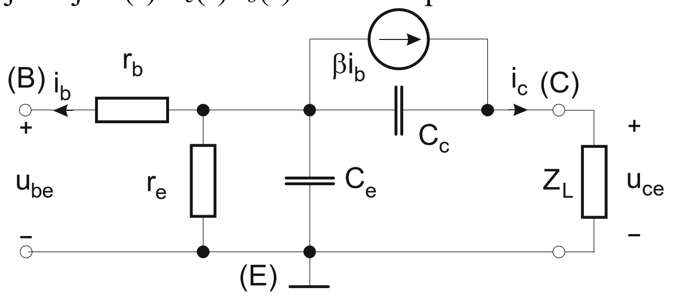
$s=j\omega \quad \Rightarrow T(j\omega)=\dfrac{U_2(j\omega)}{U_1(j\omega)}=\dfrac{j\omega+10}{j\omega+1} \Rightarrow |T(j\omega)|=\dfrac{|U_2(j\omega)|}{|U_1(j\omega)|}=\dfrac{\sqrt{\omega^2+10^2}}{\sqrt{\omega^2+1}}$ **(1 bod)**

Karakteristične točke A-F karakteristike $|T(j0)|=10$; $|T(j\infty)|=1$.

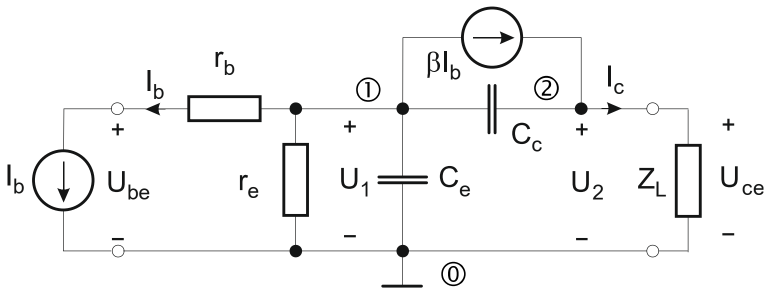


(1 bod)

4. Odrediti strujno pojačanje $T(s)=I_c(s)/I_b(s)$ za mrežu prikazanu slikom.



Rješenje:



Naponi čvorišta:

$$(1)\ U_1\left(\frac{1}{r_e}+sC_e+sC_c\right)-U_2sC_c=-\beta I_b-I_b\,;\ U_2=Z_LI_c$$

$$(2)\ -U_1sC_c+U_2\left(\frac{1}{Z_L}+sC_c\right)=\beta I_b\,;\ \textbf{(2 boda)}$$

$$(1)\Rightarrow U_1\left(\frac{1}{r_e}+sC_e+sC_c\right)-I_cZ_LsC_c=-I_b(\beta+1)$$

$$\Rightarrow U_1=\frac{-I_b(\beta+1)+I_cZ_LsC_c}{1/r_e+sC_e+sC_c}\rightarrow(2)\Rightarrow\frac{I_b(\beta+1)-I_cZ_LsC_c}{1/r_e+sC_e+sC_c}\cdot sC_c+Z_LI_c\left(\frac{1}{Z_L}+sC_c\right)=\beta I_b$$

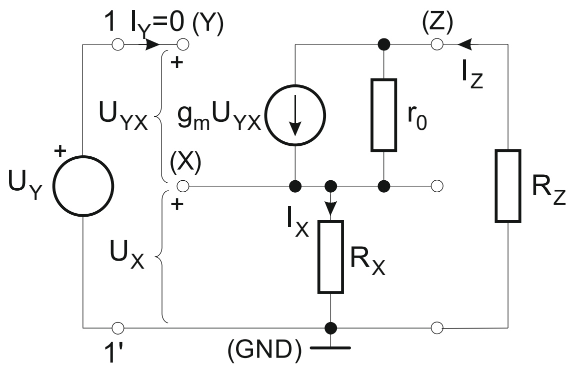
$$I_c\left[\frac{-Z_LsC_c}{1/r_e+sC_e+sC_c}\cdot sC_c+(1+sZ_LC_c)\right]=I_b\left[\beta-\frac{(\beta+1)\cdot sC_c}{1/r_e+sC_e+sC_c}\right]$$

$$T(s)=\frac{I_c(s)}{I_b(s)}=\frac{\beta-\dfrac{(\beta+1)\cdot sC_c}{1/r_e+sC_e+sC_c}}{\dfrac{-Z_LsC_c}{1/r_e+sC_e+sC_c}\cdot sC_c+(1+sZ_LC_c)}=\frac{\beta(1/r_e+sC_e+sC_c)-(\beta+1)sC_c}{-Z_L(sC_c)^2+(1+Z_LsC_c)(1/r_e+sC_e+sC_c)}$$

$$=\frac{\beta(1/r_e+sC_e)+\beta sC_c-\beta sC_c-sC_c}{-Z_L(sC_c)^2+(1+Z_LsC_c)(1/r_e+sC_e)+sC_c+Z_L(sC_c)^2}$$

$$T(s)=\frac{I_c(s)}{I_b(s)}=\frac{\beta(1/r_e+sC_e)-sC_c}{(1+Z_LsC_c)(1/r_e+sC_e)+sC_c}\quad\textbf{(3 boda)}$$

3. Za mrežu prikazanu slikom odrediti naponsko pojačanje $\alpha = U_X/U_Y$ i strujno pojačanje $\beta = I_Z/I_X$. Zadano je: $g_m = 2\cdot10^{-4}$ [A/V], $r_0 = 2\cdot10^5[\Omega]$, $R_X = 10^4[\Omega]$, $R_Z = 10^2[\Omega]$.



<u>Rješenje:</u>

a) Naponsko pojačanje: $\alpha = \dfrac{U_X}{U_Y}$

(1) $I_Z(r_0 + R_X + R_Z) - g_m U_{YX} r_0 = 0$;

(2) $U_X = I_X R_X$, $U_{YX} = U_Y - U_X$, $I_Z = I_X$, $I_Y = 0$ **(2 boda)**

$$(1) \Rightarrow \frac{U_X}{R_X}(r_0 + R_X + R_Z) = g_m(U_Y - U_X)r_0 \Rightarrow$$

$$U_X(r_0 + R_X + R_Z) = g_m U_Y R_X r_0 - g_m U_X R_X r_0$$

$$U_X(r_0 + R_X(1 + g_m r_0) + R_Z) = g_m U_Y R_X r_0$$

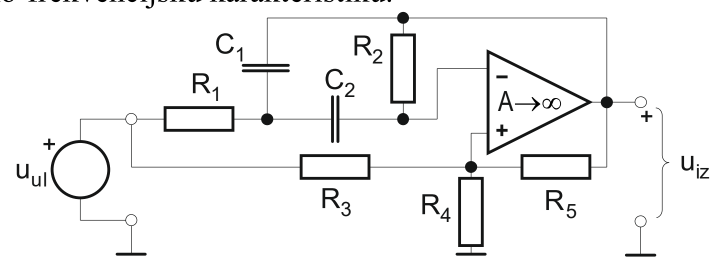
$$\alpha = \frac{U_X}{U_Y} = \frac{g_m R_X r_0}{r_0 + R_X(1 + g_m r_0) + R_Z} = 0{,}6563$$ **(2 boda)**

b) Strujno pojačanje: $\beta = \dfrac{I_Z}{I_X}$

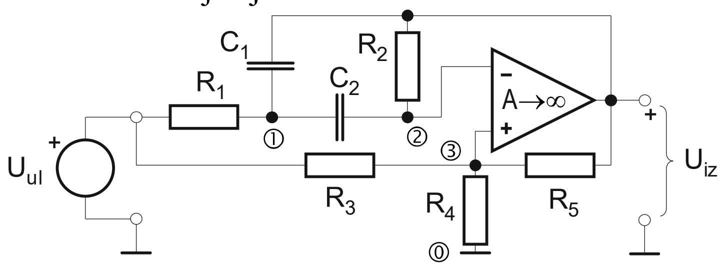
$I_Z = I_X \Rightarrow \beta = \dfrac{I_Z}{I_X} = 1$ **(1 bod)**

FILTRI

4. Zadan je aktivni-RC električni filtar prikazan slikom s normaliziranim vrijednostima elemenata C_1=1, C_2=1, R_1=1, R_2=1, te R_3=R_4=R_5=1. a) Izračunati njegovu naponsku prijenosnu funkciju $T(s)=U_{iz}(s)/U_{ul}(s)$. O kojem se tipu filtra radi (NP, VP, PP ili PB)? b) Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati vrijednost parametara k, ω_p, q_p. c) Prikazati raspored polova i nula u kompleksnoj ravnini. d) Nacrtati amplitudno-frekvencijsku karakteristiku.



Rješenje: Laplaceova transformacija i jednadžbe čvorišta:



Metoda napona čvorišta:

$$(1)\ U_1\left(\frac{1}{R_1}+sC_1+sC_2\right)-U_2sC_2=U_{iz}sC_1+U_{ul}\frac{1}{R_1}\Big/\cdot R_1$$

$$(2)\ -U_1sC_2+U_2\left(\frac{1}{R_2}+sC_2\right)=U_{iz}\frac{1}{R_2}\Big/\cdot R_2$$

$$(3)\ U_3\left(\frac{1}{R_3}+\frac{1}{R_4}+\frac{1}{R_5}\right)=U_{ul}\frac{1}{R_3}+U_{iz}\frac{1}{R_5}\Big/\cdot R_3R_4R_5$$

$$(4)\ A(U_3-U_2)=U_{iz}\ \Rightarrow\ U_3=U_2\ \ (A\to\infty)$$

$$(1)\ U_1(1+sR_1C_1+sR_1C_2)-U_2sR_1C_2=U_{iz}sR_1C_1+U_{ul}$$

$$(2)\ -U_1sR_2C_2+U_2(1+sR_2C_2)=U_{iz}$$

$$(3)\ U_3(R_4R_5+R_3R_5+R_3R_4)=U_{ul}R_4R_5+U_{iz}R_3R_4$$

$$(4)\ U_3=U_2$$

$$(2)\Rightarrow\ U_1=U_2\left(\frac{1}{sR_2C_2}+1\right)-\frac{1}{sR_2C_2}U_{iz}\ \to (1)\Rightarrow$$

$$\left[U_2\left(\frac{1}{sR_2C_2}+1\right)-\frac{1}{sR_2C_2}U_{iz}\right](1+sR_1C_1+sR_1C_2)-U_2sR_1C_2=U_{iz}sR_1C_1+U_{ul}$$

$$U_2\left(\frac{1}{sR_2C_2}+1\right)(1+sR_1C_1+sR_1C_2)-U_2sR_1C_2=$$

$$=\frac{1}{sR_2C_2}U_{iz}(1+sR_1C_1+sR_1C_2)+U_{iz}sR_1C_1+U_{ul}\Big/\cdot sR_2C_2$$

$$U_2(sR_2C_2+1)(1+sR_1C_1+sR_1C_2)-U_2sR_1C_2sR_2C_2=$$

$$=U_{iz}(1+sR_1C_1+sR_1C_2)+U_{iz}sR_1C_1sR_2C_2+U_{ul}sR_2C_2$$

$$U_2\left(1+sR_2C_2+sR_2C_2sR_1C_1+sR_1C_1+sR_1C_2\right)=$$
$$=U_{iz}\left(1+sR_1C_1+sR_1C_2+sR_1C_1sR_2C_2\right)+U_{ul}sR_2C_2$$

$$(3)\Rightarrow U_3=\frac{R_4R_5}{R_4R_5+R_3R_5+R_3R_4}U_{ul}+\frac{R_3R_4}{R_4R_5+R_3R_5+R_3R_4}U_{iz}=\alpha U_{ul}+\beta U_{iz}$$

$$(4)\Rightarrow U_3=U_2$$

$$\left(\alpha U_{ul}+\beta U_{iz}\right)\left(1+sR_2C_2+sR_2C_2sR_1C_1+sR_1C_1+sR_1C_2\right)=$$
$$=U_{iz}\left(1+sR_1C_1+sR_1C_2+sR_1C_1sR_2C_2\right)+U_{ul}sR_2C_2$$

$$\alpha U_{ul}\left(1+sR_2C_2+sR_2C_2sR_1C_1+sR_1C_1+sR_1C_2\right)-U_{ul}sR_2C_2=$$
$$=-\beta U_{iz}\left(1+sR_2C_2+sR_2C_2sR_1C_1+sR_1C_1+sR_1C_2\right)+U_{iz}\left(1+sR_1C_1+sR_1C_2+sR_1C_1sR_2C_2\right)$$

$$U_{ul}\left[1+sR_2C_2\left(1-\frac{1}{\alpha}\right)+sR_2C_2sR_1C_1+sR_1C_1+sR_1C_2\right]=$$
$$=\frac{1-\beta}{\alpha}U_{iz}\left(1+s\frac{-\beta}{1-\beta}R_2C_2+sR_2C_2sR_1C_1+sR_1C_1+sR_1C_2\right)$$

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{\alpha}{1-\beta}\cdot\frac{1+s\left[R_1\left(C_1+C_2\right)-\frac{1-\alpha}{\alpha}R_2C_2\right]+s^2R_2C_2R_1C_1}{1+s\left[R_1\left(C_1+C_2\right)-\frac{\beta}{1-\beta}R_2C_2\right]+s^2R_1C_1R_2C_2}$$

Uz uvrštene vrijednosti elemenata C_1=1, C_2=1, R_1=1, R_2=1, R_3=R_4=R_5=1:

$$\Rightarrow \alpha=1/;\beta=1/3.$$

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{1/3}{2/3}\cdot\frac{1+s\left[1\cdot(1+1)-(2/3)/(1/3)\cdot 1\cdot 1\right]+s^2}{1+s\left[1\cdot(1+1)-(1/3)/(2/3)\cdot 1\cdot 1\right]+s^2}=0{,}5\cdot\frac{1+s^2}{1+1{,}5\cdot s+s^2}$$ **(2 boda)**

b) Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati parametre k, ω_p, q_p.

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=k\cdot\frac{s^2+\omega_p^2}{s^2+(\omega_p/q_p)s+\omega_p^2}\quad\Rightarrow\quad \omega_p=1,\quad q_p=\frac{2}{3}=0{,}667,\quad k=\frac{1}{2}$$ **(1 bod)**

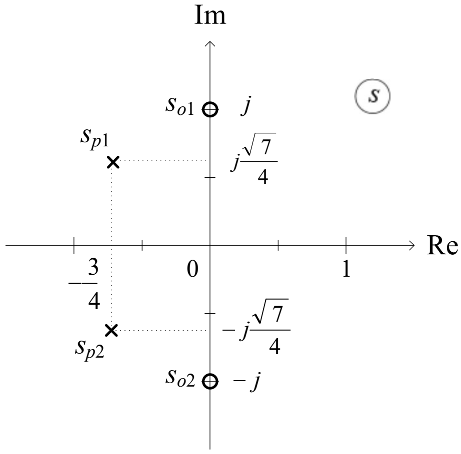
-o kojem se tipu filtra radi (NP, VP, PP ili PB)? $\Rightarrow$ PB

c) raspored polova i nula u kompleksnoj ravnini: **(1 bod)**

$$T(s)=0{,}5\frac{s^2+1}{s^2+1{,}5s+1}$$

nule $\quad s^2+1=0 \quad\Rightarrow\quad s_{o1,2}=\pm j$

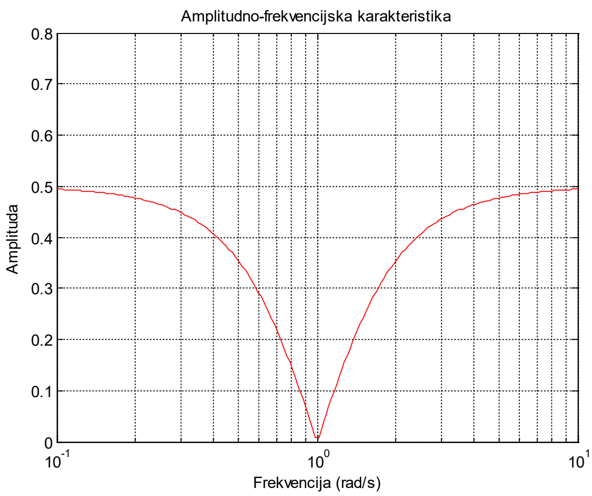
polovi $\quad s^2+\frac{3}{2}s+1=0 \quad\Rightarrow\quad s_{p1,2}=-\frac{3}{4}\pm\sqrt{\frac{9}{16}-1}=-\frac{3}{4}\pm j\frac{\sqrt{7}}{4}=-0{,}75\pm j0{,}6614$



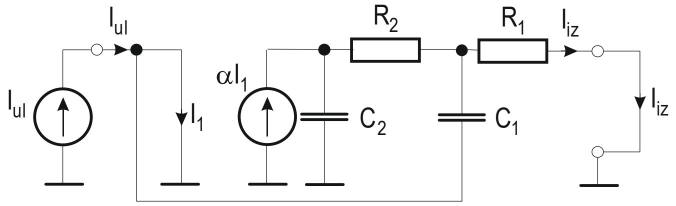
d) amplitudno-frekvencijska karakteristika: **(1 bod)**

$s = j\omega \Rightarrow$

$$T(j\omega) = \frac{1}{2}\frac{-\omega^2 + 1}{-\omega^2 + j\frac{3}{2}\omega + 1} \qquad \Rightarrow \qquad |T(j\omega)| = \frac{1}{2}\frac{|1-\omega^2|}{\sqrt{(1-\omega^2)^2 + \left(\frac{3}{2}\omega\right)^2}}$$

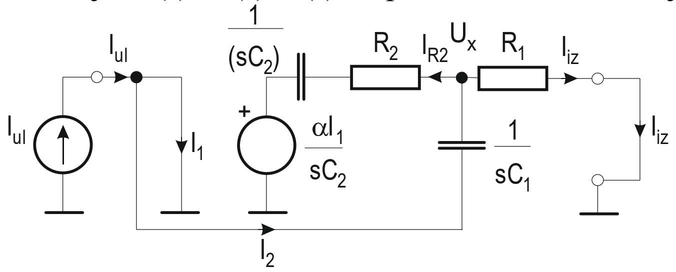


4. Za četveropol prikazan slikom s normaliziranim vrijednostima elemenata: $R_1=1/\sqrt{2}$, $R_2=\sqrt{2}$, $C_1=1$, $C_2=1$, $\alpha=2$: a) izračunati strujnu prijenosnu funkciju $H_i(s)=I_{iz}(s)/I_{ul}(s)$. O kojem se tipu prijenosne funkcije radi (NP, VP, PP ili PB)? b) Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati vrijednost parametara k, ω_p, q_p. c) Prikazati raspored polova i nula u kompleksnoj ravnini. d) Nacrtati amplitudno-frekvencijsku karakteristiku.



Rješenje:

a) Strujna prijenosnu funkciju $H_i(s)=I_{iz}(s)/I_{ul}(s)$. Laplaceova transformacija:



$$(1) \qquad I_{ul} = I_1 + I_2 \qquad \Rightarrow I_1 = I_{ul} - I_2\,;\;\; I_2 = I_{R2} + I_{iz}$$

$$(2) \qquad I_2 = \frac{U_x - \alpha \cdot I_1 \cdot \dfrac{1}{sC_2}}{R_2 + \dfrac{1}{sC_2}} + I_{iz} \quad \Bigg/ \cdot \left(R_2 + \frac{1}{sC_2}\right)$$

$$\left.\begin{aligned} &(3) \qquad I_{iz} = \frac{U_x}{R_1} \qquad \Rightarrow U_x = R_1 \cdot I_{iz} \\ &(4) \qquad I_2 = -U_x \cdot sC_1 \end{aligned}\right\} \Rightarrow I_2 = -I_{iz} \cdot sR_1C_1$$

$$(2) \Rightarrow I_2\left(R_2 + \frac{1}{sC_2}\right) = U_x - \alpha \cdot I_1 \cdot \frac{1}{sC_2} + I_{iz} \cdot \left(R_2 + \frac{1}{sC_2}\right)$$

$$-I_{iz} \cdot sR_1C_1\left(R_2 + \frac{1}{sC_2}\right) = R_1 \cdot I_{iz} - \alpha \cdot (I_{ul} - I_2)\frac{1}{sC_2} + I_{iz}\left(R_2 + \frac{1}{sC_2}\right) \Bigg/ \cdot sC_2$$

$$-I_{iz} \cdot sR_1C_1(sR_2C_2 + 1) = sR_1C_2 \cdot I_{iz} - \alpha \cdot I_{ul} + \alpha \cdot I_2 + I_{iz}(sR_2C_2 + 1)$$

$$-I_{iz}(s^2R_1C_1R_2C_2 + sR_1C_1) = I_{iz} \cdot sR_1C_2 - I_{iz} \cdot \alpha \cdot sR_1C_1 - \alpha \cdot I_{ul} + I_{iz}(sR_2C_2 + 1)$$

$$-I_{iz}(s) \cdot \left[sR_1C_1 - \alpha \cdot R_1C_1 + sR_2C_2 + 1 + s^2R_1C_1R_2C_2 + sR_1C_1\right] = -\alpha \cdot I_{ul}(s)$$

$$H_i(s) = \frac{I_{iz}(s)}{I_{ul}(s)} = \frac{\alpha}{s^2R_1C_1R_2C_2 + s\left[R_1C_2 + R_2C_2 + R_1C_1 \cdot (1-\alpha)\right] + 1}$$

$$H_i(s) = \frac{I_{iz}(s)}{I_{ul}(s)} = \frac{2}{s^2 + \left[\dfrac{1}{\sqrt{2}} + \sqrt{2} + \dfrac{1}{\sqrt{2}}(1-2)\right] \cdot s + 1} = \frac{2}{s^2 + \sqrt{2} \cdot s + 1}$$ **(2 boda)**

b) Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati parametre k, ω_p, q_p.

$$H_i(s) = \frac{I_{iz}(s)}{I_{ul}(s)} = k \cdot \frac{\omega_p^2}{s^2 + \frac{\omega_p}{q_p} s + \omega_p^2} \quad \Rightarrow \quad \omega_p = 1; \quad q_p = \frac{1}{\sqrt{2}} = 0{,}707; \quad k = 2.$$ **(1 bod)**

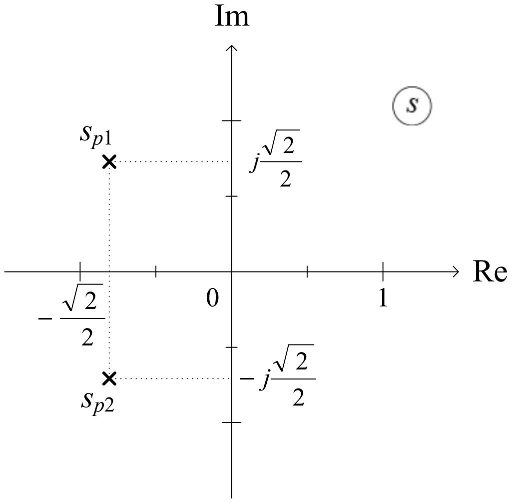
-o kojem se tipu prijenosne funkcije radi (NP, VP, PP ili PB)? $\Rightarrow$ NP

c) Raspored polova i nula u kompleksnoj ravnini: **(1 bod)**

Polovi: $s^2 + \sqrt{2} \cdot s + 1 = 0$

$$s_{p1,2} = \frac{-\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{-\sqrt{2} \pm j\sqrt{2}}{2} = -\frac{\sqrt{2}}{2} \pm j\frac{\sqrt{2}}{2} = -0{,}707 \pm j0{,}707 \, .$$

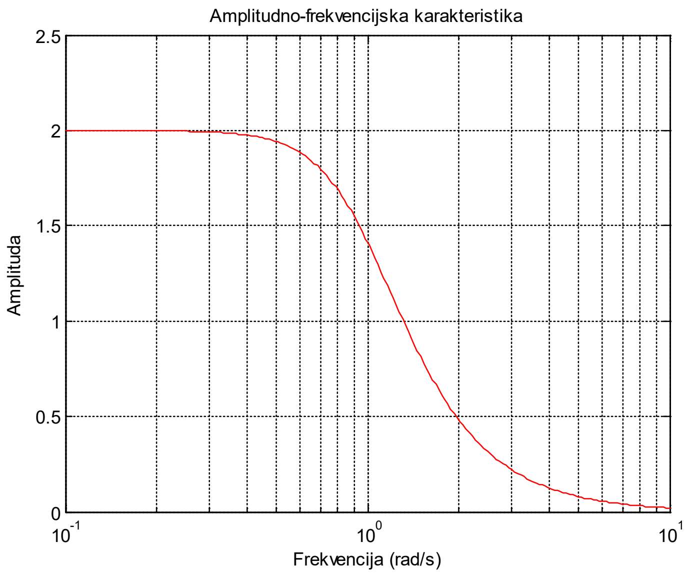
Nule: dvije u beskonačnosti:



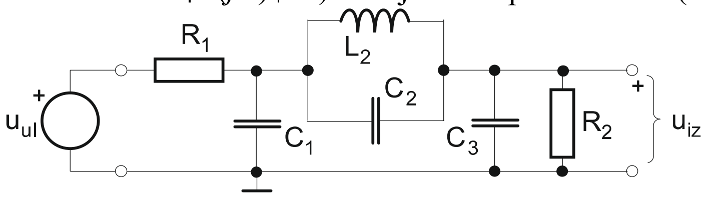
d) Amplitudno-frekvencijska karakteristika: **(1 bod)**

$$s = j\omega \;\Rightarrow\; H_i(j\omega) = 2 \cdot \frac{1}{-\omega^2 + j\sqrt{2}\omega + 1}$$

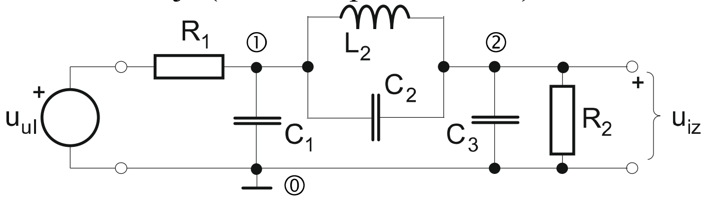
$$|H_i(j\omega)| = 2 \cdot \frac{1}{\sqrt{(1-\omega^2)^2 + (\sqrt{2}\omega)^2}} = 2 \cdot \frac{1}{\sqrt{\omega^4 - 2\omega^2 + 1 + 2\omega^2}} = 2 \cdot \frac{1}{\sqrt{\omega^4 + 1}}$$



3. Za električni krug prikazan slikom odrediti: a) naponsku prijenosnu funkciju $H(s)=U_{iz}(s)/U_{ul}(s)$ kao funkciju varijable „s“ i elemenata (R_i, L_i i C_i). b) Uvrstiti normalizirane vrijednosti elemenata: $R_1=1$, $R_2=1$, $C_1=1$, $C_2=1$, $C_3=1$, $L_2=1$. c) Prikazati raspored polova i nula u kompleksnoj ravnini. d) Izračunati i skicirati amplitudno-frekvencijsku karakteristiku $|T(j\omega)|$. e) O kojem se tipu filtra radi (NP, VP, PP ili PB)?



Rješenje: a) Naponska prijenosna funkcija (metoda napona čvorišta):



$$(1)\quad U_1\left(\frac{1}{R_1}+sC_1+sC_2+\frac{1}{sL_2}\right)-U_2\left(sC_2+\frac{1}{sL_2}\right)=U_{ul}\frac{1}{R_1}$$

$$(2)\quad -U_1\left(sC_2+\frac{1}{sL_2}\right)+U_2\left(sC_2+\frac{1}{sL_2}+sC_3+\frac{1}{R_2}\right)=0$$

$$U_2=U_{iz}$$

$$(2)\Rightarrow U_1=U_2\frac{sC_2+\frac{1}{sL_2}+sC_3+\frac{1}{R_2}}{sC_2+\frac{1}{sL_2}}\rightarrow(1)\Rightarrow$$

$$U_2\frac{sC_2+\frac{1}{sL_2}+sC_3+\frac{1}{R_2}}{sC_2+\frac{1}{sL_2}}\left(\frac{1}{R_1}+sC_1+sC_2+\frac{1}{sL_2}\right)-U_2\left(sC_2+\frac{1}{sL_2}\right)=U_{ul}\frac{1}{R_1}\Bigg/\cdot\left(sC_2+\frac{1}{sL_2}\right)$$

$$U_2\left[s(C_2+C_3)+\frac{1}{sL_2}+\frac{1}{R_2}\right]\left[s(C_1+C_2)+\frac{1}{sL_2}+\frac{1}{R_1}\right]-U_2\left(sC_2+\frac{1}{sL_2}\right)^2=U_{ul}\frac{1}{R_1}\left(sC_2+\frac{1}{sL_2}\right)$$

$$U_{iz}\left[s^2(C_1+C_2)(C_2+C_3)+\frac{C_1+C_2}{L_2}+s\frac{C_1+C_2}{R_2}+\frac{C_2+C_3}{L_2}+\frac{1}{s^2L_2^2}+\frac{1}{sL_2R_2}+s\frac{C_2+C_3}{R_1}+\right.$$

$$\left.+\frac{1}{sL_2R_1}+\frac{1}{R_1R_2}-s^2C_2^2-2\frac{C_2}{L_2}-\frac{1}{s^2L_2^2}\right]=U_{ul}\frac{1}{R_1}\left(sC_2+\frac{1}{sL_2}\right)$$

$$U_{iz}\left[s^2(C_1C_2+C_1C_3+C_2C_3)+\frac{C_1+C_3}{L_2}+s\frac{C_1+C_2}{R_2}+\frac{1}{sL_2R_2}+s\frac{C_2+C_3}{R_1}+\frac{1}{sL_2R_1}+\frac{1}{R_1R_2}\right]=$$

$$=U_{ul}\frac{1}{R_1}\left(sC_2+\frac{1}{sL_2}\right)\Bigg/\cdot sL_2$$

$$U_{iz}\left[s^3L_2(C_1C_2+C_1C_3+C_2C_3)+s(C_1+C_3)+s^2L_2\frac{C_1+C_2}{R_2}+\frac{1}{R_2}+s^2L_2\frac{C_2+C_3}{R_1}+\frac{1}{R_1}+\frac{sL_2}{R_1R_2}\right]=$$

$$=U_{ul}\frac{1}{R_1}\left(s^2L_2C_2+1\right)$$

$$T(s)=\frac{\frac{1}{R_1}\left(s^2L_2C_2+1\right)}{s^3L_2(C_1C_2+C_1C_3+C_2C_3)+s^2L_2\left(\frac{C_1+C_2}{R_2}+\frac{C_2+C_3}{R_1}\right)+s\left(\frac{L_2}{R_1R_2}+C_1+C_3\right)+\frac{1}{R_1}+\frac{1}{R_2}}$$

Dobiveni izraz se može još urediti i prikazati na razne načine:

$$T(s)=\frac{R_2\left(s^2L_2C_2+1\right)}{s^3L_2R_1R_2(C_1C_2+C_1C_3+C_2C_3)+s^2L_2\left[R_1(C_1+C_2)+R_2(C_2+C_3)\right]+s\left[L_2+R_1R_2(C_1+C_3)\right]+R_1+R_2}\quad \text{ili}$$

$$T(s)=\frac{\frac{1}{L_2R_1(C_1C_2+C_1C_3+C_2C_3)}\left(s^2L_2C_2+1\right)}{s^3+s^2\frac{R_1(C_1+C_2)+R_2(C_2+C_3)}{R_1R_2(C_1C_2+C_1C_3+C_2C_3)}+s\frac{L_2+R_1R_2(C_1+C_3)}{L_2R_1R_2(C_1C_2+C_1C_3+C_2C_3)}+\frac{R_1+R_2}{L_2R_1R_2(C_1C_2+C_1C_3+C_2C_3)}},\quad \text{itd.}$$

b) Uvrštene normalizirane vrijednosti elemenata: R_1=1, R_2=1, C_1=1, C_2=1, C_3=1, L_2=1:

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{s^2+1}{3s^3+4s^2+3s+2}\quad \text{a) + b) (3 boda)}$$

c) Raspored polova i nula u kompleksnoj ravnini:

-polovi: $3s^3+4s^2+3s+2=3\left(s^3+\frac{4}{3}s^2+s+\frac{2}{3}\right)=0$

Pretpostavimo vrijednost pola: $s_{p1}=-1$

$$s_{p1}=-1\Rightarrow 3(-1)^3+4(-1)^2+3(-1)+2=0\Rightarrow -3+4-3+2=0\ \text{DA}$$

Podijelimo:

$$\left(3s^3+4s^2+3s+2\right):(s+1)=3s^2+\frac{s^2+3s+2}{s+1}$$
$$\underline{-3s^3-3s^2}$$
$$=s^2+3s+2$$

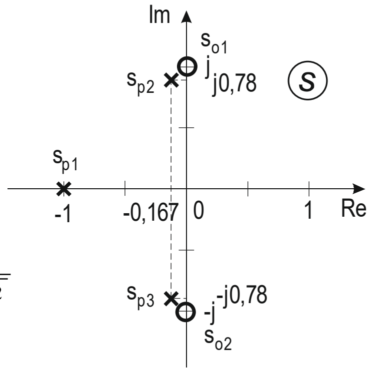
$$\left(s^2+3s+2\right):(s+1)=s+\frac{2s+2}{s+1}$$
$$\underline{-s^2-s}$$
$$=2s+2$$

$$\left(2s+2\right):(s+1)=2$$
$$\underline{-2s-2}$$
$$=0$$

Polovi: $3s^3+4s^2+3s+2=(s+1)(3s^2+s+2)=0$

$$3s^2+s+2=0\ \Rightarrow\ s_{p2,3}=\frac{-1\pm\sqrt{1-24}}{2\cdot 3}=\frac{-1\pm j\sqrt{23}}{6}=-0{,}16667\pm j0{,}799305$$

-nule: $s^2+1=0\ \Rightarrow s_{o1,2}=\pm j$ (1 bod)



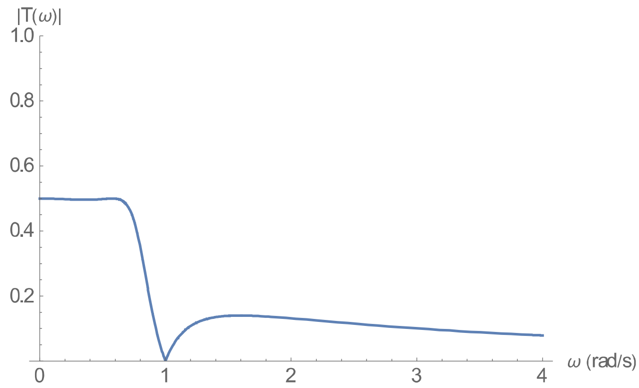
d) amplitudno-frekvencijska karakteristika:

$$T(j\omega)=\frac{-\omega^2+1}{-3j\omega^3-4\omega^2+3j\omega+2}\ \Rightarrow\qquad \left|T(j\omega)\right|=\frac{\left|1-\omega^2\right|}{\sqrt{\left(2-4\omega^2\right)^2+\left(3\omega-3\omega^3\right)^2}}$$

Ako uvrstimo tri karakteristične točke:

s=0 u $T(s)\Rightarrow\ T(0)=\left|T(j\omega)\right|_{\omega=0}=\frac{1}{2}$; s=1 $\Rightarrow\ T(1)=\left|T(j\omega)\right|_{\omega=1}=0$; s=→∞ u $T(s)\Rightarrow\ T(\infty)=\left|T(j\omega)\right|_{\omega\to\infty}=0$

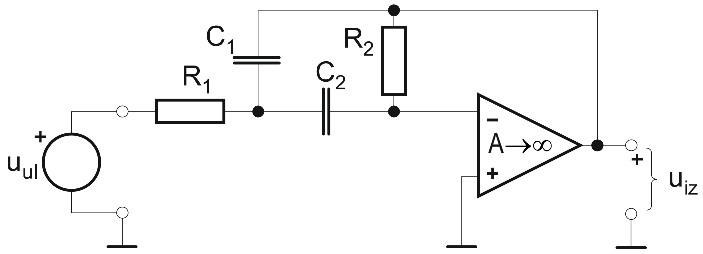
skica



e) O kojem se tipu filtra radi (NP, VP, PP ili PB)?

Točna su dva odgovora: NP- niski propust (NP Notch) ili PB-pojasna brana d) + e) (1 bod)

4. Za pojasno-propusni električni filtar prikazan slikom zadana je naponska prijenosna funkcija $T(s)=U_{iz}(s)/U_{ul}(s)$. a) Usporedbom s općim oblikom PP prijenosne funkcije filtra 2. stupnja odrediti parametre k, ω_p, q_p kao funkcije elemenata filtra. b) Ako su zadane normalizirane vrijednosti parametara ω_p=1 i q_p=0,7071068 te ako je $C_1=C_2=1$, izračunati normalizirane vrijednosti otpora R_1 i R_2 i pojačanje k. c) Prikazati raspored polova i nula u kompleksnoj ravnini. d) Izračunati i skicirati amplitudno-frekvencijsku karakteristiku $|T(j\omega)|$. e) Izračunati denormirane elemente filtra za frekvenciju normalizacije $\omega_0=10^3$ rad/s i otpor R_0=1kΩ.



$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=-\frac{sR_2C_2}{s^2R_1C_1R_2C_2+s\left(R_1C_1+R_1C_2\right)+1}$$

<u>Rješenje:</u>

a) Parametri

Napišemo prijenosnu funkciju tako da je koeficijent uz najveću potenciju od s (s^2) jediničan.

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=-\frac{s\dfrac{1}{R_1C_1}}{s^2+s\dfrac{R_1C_1+R_1C_2}{R_1C_1R_2C_2}+\dfrac{1}{R_1C_1R_2C_2}}$$

Usporedba s općim oblikom:

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=k\cdot\frac{\dfrac{\omega_p}{q_p}s}{s^2+\dfrac{\omega_p}{q_p}s+\omega_p^2}$$

$$\omega_p^2=\frac{1}{R_1R_2C_1C_2}\Rightarrow\omega_p=\frac{1}{\sqrt{R_1R_2C_1C_2}}$$

$$\frac{\omega_p}{q_p}=\frac{R_1(C_1+C_2)}{R_1R_2C_1C_2}\Rightarrow q_p=\omega_p\frac{R_1R_2C_1C_2}{R_1(C_1+C_2)}=\frac{1}{\sqrt{R_1R_2C_1C_2}}\cdot\frac{R_1R_2C_1C_2}{R_1(C_1+C_2)}=\frac{\sqrt{R_1R_2C_1C_2}}{R_1(C_1+C_2)}$$

$$k\cdot\frac{\omega_p}{q_p}=\frac{1}{R_1C_1}\Rightarrow k=\frac{1}{R_1C_1}\cdot\frac{q_p}{\omega_p}=\frac{1}{R_1C_1}\cdot\frac{R_1C_1R_2C_2}{R_1(C_1+C_2)}=\frac{R_2C_2}{R_1(C_1+C_2)}$$

(1 bod)

b) proračun: u proračunu smo pretpostavili $C_1=C_2=C$ pa će izrazi za ω_p, q_p i k iz točke *a*) poprimiti jednostavniji oblik:

$$\omega_p=\frac{1}{C\sqrt{R_1R_2}},\quad q_p=\frac{1}{2}\sqrt{\frac{R_2}{R_1}},\quad k=\frac{R_2}{2R_1},$$

Izračun jednadžbi iz uvjeta za ω_p i q_p:

Iz $q_p=\frac{1}{\sqrt{2}}$ slijedi $\frac{1}{\sqrt{2}}=\frac{1}{2}\sqrt{\frac{R_2}{R_1}}\Rightarrow\frac{2}{\sqrt{2}}=\sqrt{\frac{R_2}{R_1}}\Rightarrow 2=\frac{R_2}{R_1}\Rightarrow R_2=2R_1$

Iz ω_p=1 slijedi $1=\frac{1}{C\sqrt{R_1R_2}}\Rightarrow C\sqrt{R_1R_2}=1\Rightarrow C\sqrt{2R_1^2}=1\Rightarrow R_1=\frac{1}{C\sqrt{2}}$

Proračun elemenata: <u>Uz odabir C=1 je C_1=1, C_2=1 i računamo:</u>

$$R_1=\frac{1}{\sqrt{2}},\ R_2=\frac{2}{\sqrt{2}}=\sqrt{2}.$$

Pojačanje u području propuštanja iznosi: $k = \dfrac{R_2}{2R_1} = \dfrac{\sqrt{2}}{2\dfrac{1}{\sqrt{2}}} = \dfrac{2}{2} = 1$

(1 bod)

c) raspored polova i nula u kompleksnoj ravnini:

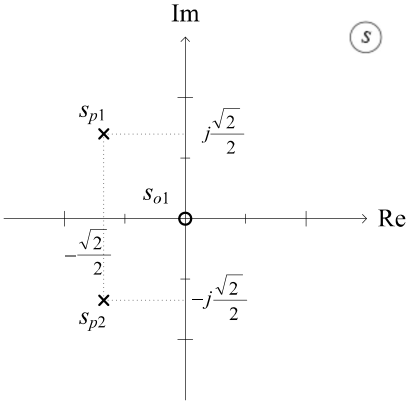
$$H_{PP}(s) = k\frac{(\omega_p / q_p)s}{s^2 + (\omega_p / q_p)s + \omega_p^2} = \frac{\sqrt{2}s}{s^2 + \sqrt{2}s + 1}$$

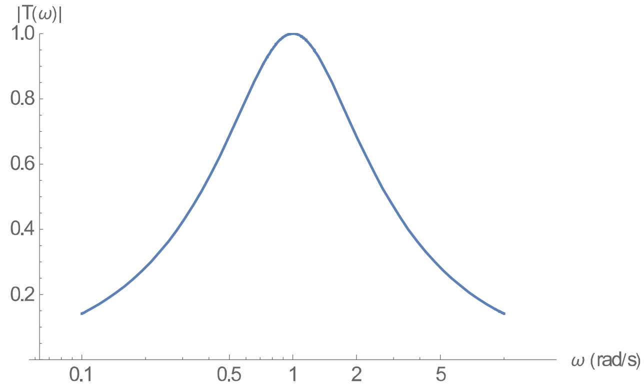
nule $\quad s_{o1} = 0,\ s_{o2} = \infty$

polovi $\ s^2 + \sqrt{2} \cdot s + 1 = 0 \quad \Rightarrow \quad s_{p1,2} = -\dfrac{\sqrt{2}}{2} \pm \sqrt{\dfrac{2}{4} - 1} = -\dfrac{\sqrt{2}}{2} \pm j\dfrac{\sqrt{2}}{2}$ (1 bod)

d) amplitudno-frekvencijska karakteristika:

$$T(j\omega) = \frac{-\omega \cdot \sqrt{2}}{-\omega^2 + j\omega \cdot \sqrt{2} + 1} \quad \Rightarrow \quad |T(j\omega)| = \frac{\left|\sqrt{2} \cdot \omega\right|}{\sqrt{\left(1-\omega^2\right)^2 + \left(\omega \cdot \sqrt{2}\right)^2}} = \frac{\sqrt{2} \cdot \omega}{\sqrt{1+\omega^4}}$$ (1 bod)



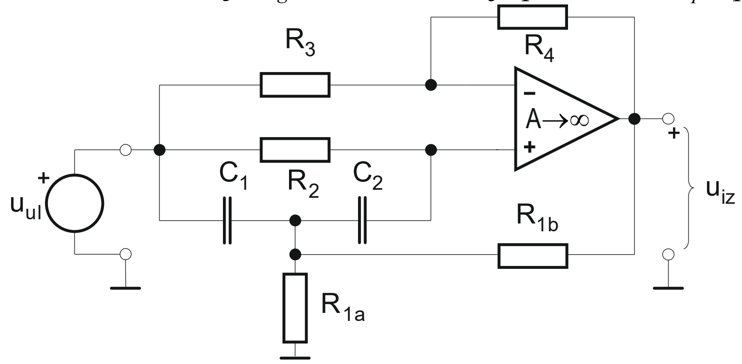


e) Denormalizacija elemenata po $\omega_0=10^3$ rad/s i $R_0=1\text{k}\Omega$:

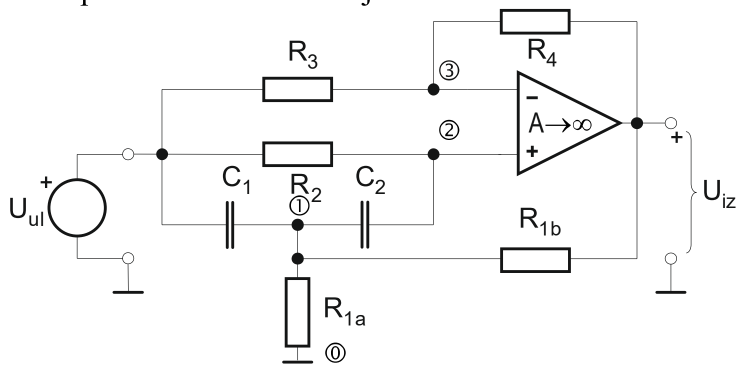
normalizirani elementi	izrazi za denormalizaciju	denormalizirani elementi
$R_1 = 1/\sqrt{2}$	$R = R_0 \cdot R_n;$	$R_1 = R_0 \cdot 0{,}7071 = 707{,}1\Omega$
$R_2 = \sqrt{2}$	$C = \dfrac{C_n}{\omega_0 \cdot R_0};$	$R_2 = R_0 \cdot 1{,}4142 = 1414{,}2\Omega$
$C_1 = 1$	$L = \dfrac{L_n \cdot R_0}{\omega_0}$	$C_1 = \dfrac{1}{\omega_0 R_0} \cdot 1 = C_0 \cdot 1 = 1\mu\text{F}$
$C_2 = 1$		$C_2 = C_0 \cdot 1 = \mu\text{F}$

(1 bod)

4. Zadan je aktivni-RC električni filtar prikazan slikom s normaliziranim vrijednostima elemenata $C_1=C_2=1$; $R_{1a}=3$; $R_{1b}=1{,}5$; $R_2=1$, te $R_3=1$; $R_4=2$. Odrediti: a) njegovu naponsku prijenosnu funkciju $T(s)=U_{iz}(s)/U_{ul}(s)$. O kojem se tipu filtra radi (NP, VP, PP ili PB)? b) Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati vrijednost parametara k, ω_p, ω_z, q_p. c) Prikazati raspored polova i nula u kompleksnoj ravnini. d) Nacrtati amplitudno-frekvencijsku karakteristiku. e) Koliko iznose širina pojasa propuštanja/gušenja B, te gornja i donja granična frekvencija ω_g i ω_d kao funkcije parametara ω_p i q_p ?



Rješenje: Primjenom Laplaceove transformacije**:**



Metoda napona čvorišta:

$$(1)\ U_1\left(\frac{1}{R_1}+sC_1+sC_2\right)-U_2sC_2=U_{iz}\frac{1}{R_{1b}}+U_{ul}sC_1 \Big/ \cdot R_1$$

$$(2)\ -U_1sC_2+U_2\left(\frac{1}{R_2}+sC_2\right)=U_{ul}\frac{1}{R_2}\Big/ : sC_2$$

$$(3)\ U_3\left(\frac{1}{R_3}+\frac{1}{R_4}\right)=U_{ul}\frac{1}{R_3}+U_{iz}\frac{1}{R_4}\Big/ \cdot R_3R_4$$

$$(3)\ \underline{A(U_2-U_3)=U_{iz}\ \Rightarrow\ U_2=U_3\ \ (A\to\infty)}$$

$$(1)\ U_1(1+sR_1C_1+sR_1C_2)-U_2sR_1C_2=U_{iz}\frac{R_1}{R_{1b}}+U_{ul}sR_1C_1$$

$$(2)\ -U_1+U_2\left(\frac{1}{sR_2C_2}+1\right)=U_{ul}\frac{1}{sR_2C_2}$$

$$(3)\ \underline{U_2(R_3+R_4)=U_{ul}R_4+U_{iz}R_3}$$

Uz uvrštene vrijednosti elemenata $C_1=C_2=1$; $R_{1a}=3$; $R_{1b}=3/2=1{,}5$; $R_2=1$, te $R_3=1$; $R_4=2$;

$$R_1=\frac{R_{1a}R_{1b}}{R_{1a}+R_{1b}}=\frac{3\cdot 3/2}{3+3/2}=1$$

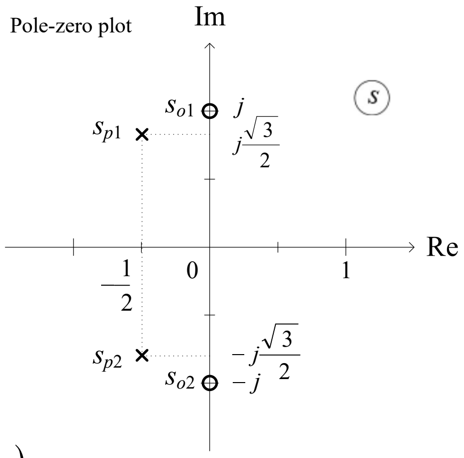
$$(1)\ U_1(1+s+s)-U_2 s=U_{iz}\frac{2}{3}+U_{ul}s$$

$$(2)\ -U_1+U_2\left(\frac{1}{s}+1\right)=U_{ul}\frac{1}{s}$$

$$(3)\ \underline{U_2(1+2)=2U_{ul}+1U_{iz}}$$

$$(2)\Rightarrow U_1=-U_{ul}\frac{1}{s}+U_2\left(\frac{1}{s}+1\right)$$

$$(3)\Rightarrow U_2=2/3U_{ul}+1/3U_{iz}$$



Malo računanja: (2), (3) $\rightarrow$ (1) $\Rightarrow$

$$\left[U_2\left(\frac{1}{s}+1\right)-U_{ul}\frac{1}{s}\right](1+2s)-U_2 s=U_{iz}\frac{2}{3}+U_{ul}s\Big/\cdot s$$

$$[U_2(s+1)-U_{ul}](1+2s)-U_2 s^2=2/3U_{iz}s+U_{ul}s^2$$

$$U_2(s+1)(1+2s)-U_2 s^2=2/3U_{iz}s+U_{ul}s^2+U_{ul}(1+2s)$$

$$U_2(s^2+3s+1)=2/3U_{iz}s+U_{ul}(s^2+2s+1)$$

$$[2/3U_{ul}+1/3U_{iz}](s^2+3s+1)=2/3U_{iz}s+U_{ul}(s^2+2s+1)$$

$$U_{iz}\left[s^2+2s+s\left(1-\frac{2/3}{1/3}\right)+1\right]=U_{ul}\left[s^2+2s+s(1-3)+1\right]\ \Rightarrow\ T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{1+s^2}{1+s+s^2}$$

b) Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati parametre k, ω_p, ω_z, q_p.

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=k\cdot\frac{s^2+\omega_z^2}{s^2+(\omega_p/q_p)\cdot s+\omega_p^2}$$

$$\Rightarrow\quad \omega_p=\omega_z=1,\quad q_p=1,\quad k=1$$

-o kojem se tipu filtra radi (NP, VP, PP ili PB)?

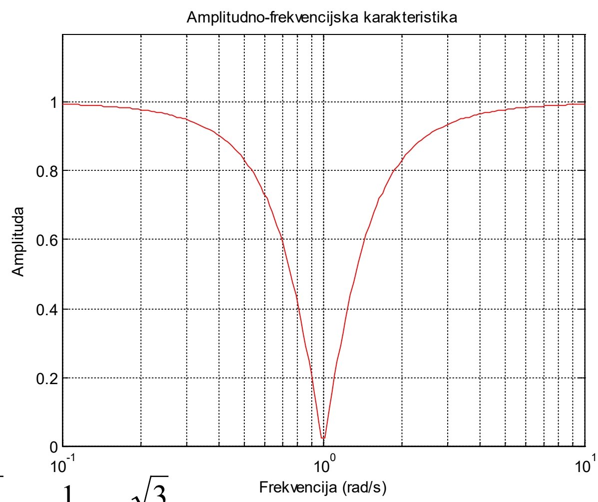
$\Rightarrow$ PB

a) + b) (3 boda)

c) raspored polova i nula u kompleksnoj ravnini:

$$T(s)=\frac{s^2+1}{s^2+s+1}$$

- nule: $\quad s^2+1=0 \quad\Rightarrow\quad s_{o1,2}=\pm j$



- polovi: $\quad s^2+s+1=0 \quad\Rightarrow\quad s_{p1,2}=-\frac{1}{2}\pm\sqrt{\frac{1}{4}-1}=-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}=-0{,}5\pm j0{,}866$

d) amplitudno-frekvencijska karakteristika:

$$s{=}j\omega\ \Rightarrow\ T(j\omega)=\frac{-\omega^2+1}{-\omega^2+j\omega+1}\ \Rightarrow\ |T(j\omega)|=\frac{|1-\omega^2|}{\sqrt{(1-\omega^2)^2+(\omega)^2}}=\frac{|1-\omega^2|}{\sqrt{1-\omega^2+\omega^4}}$$

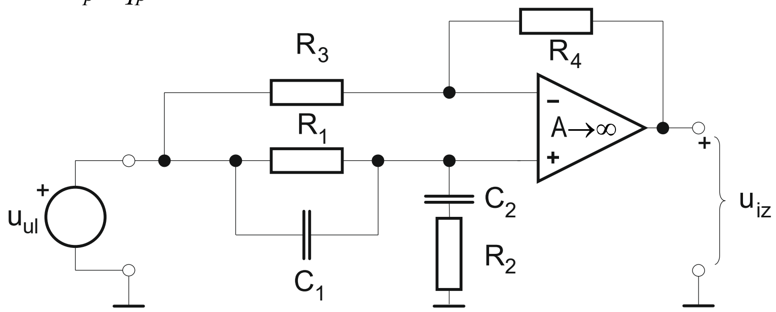
e) Širina pojasa gušenja $B=\omega_p/q_p$=1 [rad/s]

Gornja i donja granična frekvencija pojasa gušenja: $\omega_{g,d}=\omega_p\sqrt{1+\frac{1}{4q_p^2}}\pm\frac{\omega_p}{2q_p}=1\sqrt{1+\frac{1}{4}}\pm\frac{1}{2}=$

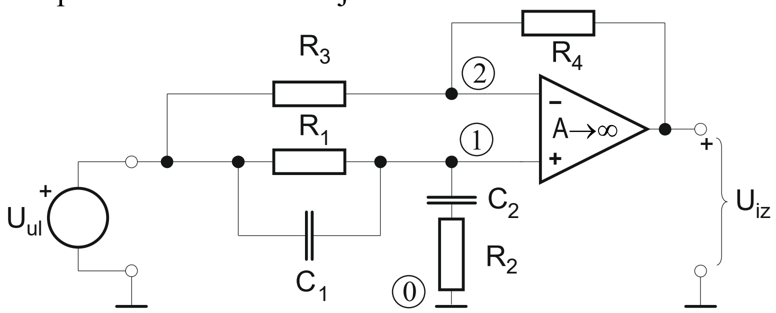
$$=\frac{\sqrt{5}}{2}\pm\frac{1}{2};\ \omega_g=\frac{\sqrt{5}+1}{2}=1{,}618;\ \omega_d=\frac{\sqrt{5}-1}{2}=0{,}618\,[\text{rad/s}]\ \Rightarrow B{=}\omega_g{-}\omega_d=1\ [\text{rad/s}]$$

c) + d) + e) (2 boda)

4. Zadan je aktivni-RC električni filtar prikazan slikom s normaliziranim vrijednostima elemenata $C_1=C_2=1$; $R_1=R_2=1$ te $R_3=1$; $R_4=2$. Odrediti: a) naponsku prijenosnu funkciju $T(s)=U_{iz}(s)/U_{ul}(s)$. O kojem se tipu filtra radi (NP, VP, PP ili PB)? b) Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati vrijednost parametara k, ω_p, ω_z, q_p. c) Prikazati raspored polova i nula u kompleksnoj ravnini. d) Nacrtati amplitudno-frekvencijsku karakteristiku $|T(j\omega)|$. e) Koliko iznose širina pojasa propuštanja/gušenja B, te gornja i donja granična frekvencija ω_g i ω_d kao funkcije parametara ω_p i q_p ?



Rješenje: Primjenom Laplaceove transformacije**:**



Metoda napona čvorišta:

$$(1)\ U_1\left(\frac{1}{R_1}+sC_1+\frac{1}{R_2+\dfrac{1}{sC_2}}\right)=U_{ul}\left(\frac{1}{R_1}+sC_1\right)\Big/\cdot R_1$$

$$(2)\ U_2\left(\frac{1}{R_3}+\frac{1}{R_4}\right)=U_{ul}\frac{1}{R_3}+U_{iz}\frac{1}{R_4}\Big/\cdot R_3R_4$$

$$(3)\ \underline{A(U_1-U_2)=U_{iz}\ \Rightarrow\ U_1=U_2\ \ (A\to\infty)}$$

$$(1)\ U_1\left(1+sR_1C_1+R_1\frac{sC_2}{sR_2C_2+1}\right)=U_{ul}(1+sR_1C_1)\Big/\cdot(sR_2C_2+1)$$

$$(2)\ \underline{U_1(R_3+R_4)=U_{ul}R_4+U_{iz}R_3/:(R_3+R_4)}$$

$$(1)\ U_1\left[(sR_1C_1+1)(sR_2C_2+1)+sR_1C_2\right]=U_{ul}(sR_1C_1+1)(sR_2C_2+1)$$

$$(2)\ \underline{U_1=U_{ul}\frac{R_4}{R_3+R_4}+U_{iz}\frac{R_3}{R_3+R_4}}$$

Uvedimo oznaku:

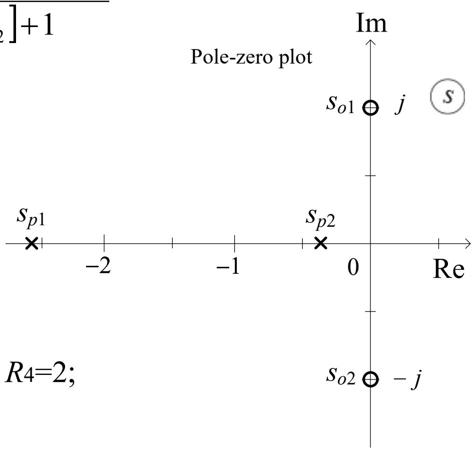
$$\beta=\frac{R_3}{R_3+R_4};\ 1-\beta=\frac{R_4}{R_3+R_4};\ (2)\ \Rightarrow\ U_1=(1-\beta)U_{ul}+\beta U_{iz}$$

Malo računanja:

$$(2)\to(1)\Rightarrow$$

$$\left[(1-\beta)U_{ul}+\beta U_{iz}\right]\cdot\left[(sR_1C_1+1)(sR_2C_2+1)+sR_1C_2\right]=U_{ul}(sR_1C_1+1)(sR_2C_2+1)$$

$$(1-\beta)U_{ul}\cdot(sR_1C_1+1)(sR_2C_2+1)+(1-\beta)U_{ul}\cdot sR_1C_2-U_{ul}(sR_1C_1+1)(sR_2C_2+1)=$$
$$=-\beta U_{iz}(sR_1C_1+1)(sR_2C_2+1)-\beta U_{iz}sR_1C_2$$
$$U_{ul}\cdot\left[-\beta(sR_1C_1+1)(sR_2C_2+1)+(1-\beta)sR_1C_2\right]=-\beta U_{iz}\cdot\left[(sR_1C_1+1)(sR_2C_2+1)+sR_1C_2\right]$$
$$/:(-\beta)$$
$$U_{ul}\cdot\left[(sR_1C_1+1)(sR_2C_2+1)+\frac{1-\beta}{-\beta}sR_1C_2\right]=U_{iz}\left[(sR_1C_1+1)(sR_2C_2+1)+sR_1C_2\right]$$
$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{s^2R_1C_1R_2C_2+s\left[R_1C_1+R_2C_2+R_1C_2\left(1-\frac{1}{\beta}\right)\right]+1}{s^2R_1C_1R_2C_2+s\left[R_1C_1+R_2C_2+R_1C_2\right]+1}$$



Vratimo natrag oznaku: $1-\frac{1}{\beta}=1-\frac{R_3+R_4}{R_3}=-\frac{R_4}{R_3}$

Konačno je:

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{s^2+s\frac{R_1C_1+R_2C_2-R_1C_2R_4/R_3}{R_1C_1R_2C_2}+\frac{1}{R_1C_1R_2C_2}}{s^2+s\frac{R_1C_1+R_2C_2+R_1C_2}{R_1C_1R_2C_2}+\frac{1}{R_1C_1R_2C_2}}$$

Uz uvrštene vrijednosti elemenata $C_1=C_2=1$; $R_1=R_2=1$ te $R_3=1$; $R_4=2$;

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{s^2+1}{s^2+3s+1}$$

b) Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati parametre k, ω_p, ω_z, q_p.

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=k\cdot\frac{s^2+\omega_z^2}{s^2+(\omega_p/q_p)\cdot s+\omega_p^2}\Rightarrow\begin{array}{l}\omega_p=\omega_z=1,\\ q_p=1/3,\ k=1\end{array}$$

-o kojem se tipu filtra radi (NP, VP, PP ili PB)? $\Rightarrow$ PB

a) + b) (3 boda)

c) raspored polova i nula u kompleksnoj ravnini:

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{s^2+1}{s^2+3s+1}$$

- nule: $s^2+1=0 \quad\Rightarrow\quad s_{o1,2}=\pm j$

- polovi: $s^2+3s+1=0\ \Rightarrow\ s_{p1,2}=-\frac{3\pm\sqrt{5}}{2}$; $s_{p1}=-2{,}61803; s_{p2}=-0{,}381966$

d) amplitudno-frekvencijska karakteristika:

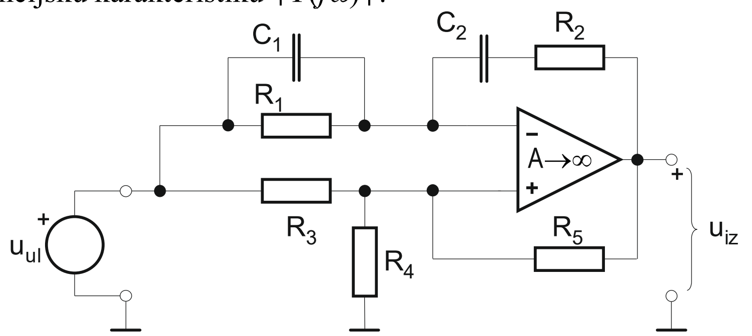
$$s{=}j\omega\ \Rightarrow\ T(j\omega)=\frac{-\omega^2+1}{-\omega^2+j3\omega+1}\quad\Rightarrow\ |T(j\omega)|=\frac{|1-\omega^2|}{\sqrt{(1-\omega^2)^2+(3\omega)^2}}$$

e) Širina pojasa gušenja $B=\omega_p/q_p$=3 [rad/s]. Koristi se isti izraz kao za širinu pojasa propuštanja. Gornja i donja granična frekvencija pojasa gušenja su:

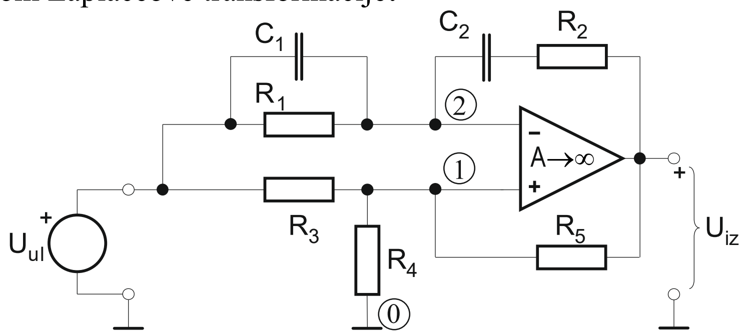
$$\omega_{g,d}=\omega_p\sqrt{1+\frac{1}{4q_p^2}}\pm\frac{\omega_p}{2q_p}\ \Rightarrow\ \omega_g=\sqrt{1+\frac{9}{4}}+\frac{3}{2}=3{,}30278;\ \omega_d=\sqrt{1+\frac{9}{4}}-\frac{3}{2}=0{,}30278\ \text{[rad/s]}$$

$\Rightarrow B=\omega_g-\omega_d=3$ [rad/s] **c) + d) + e) (2 boda)**

4. Zadan je aktivni-RC električni filtar prikazan slikom s normaliziranim vrijednostima elemenata $C_1=C_2=1$; $R_1=R_2=1$ te $R_3=1$; $R_4=2$; $R_5=1$. Odrediti: a) naponsku prijenosnu funkciju $T(s)=U_{iz}(s)/U_{ul}(s)$. O kojem se tipu filtra radi (NP, VP, PP ili PB)? b) Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati vrijednost parametara k, ω_p, q_p, ω_z, q_z. c) Prikazati raspored polova i nula u kompleksnoj ravnini. d) Nacrtati amplitudno-frekvencijsku karakteristiku $\left|T(j\omega)\right|$.



Rješenje: Primjenom Laplaceove transformacije:



Metoda napona čvorišta:

$$(1)\ U_1\left(\frac{1}{R_3}+\frac{1}{R_4}+\frac{1}{R_5}\right)=U_{ul}\frac{1}{R_3}+U_{iz}\frac{1}{R_5}\Big/\cdot R_3R_4R_5$$

$$(2)\ U_2\left(\frac{1}{R_1}+sC_1+\frac{1}{R_2+\frac{1}{sC_2}}\right)=U_{ul}\left(\frac{1}{R_1}+sC_1\right)+U_{iz}\left(\frac{1}{R_2+\frac{1}{sC_2}}\right)\Big/\cdot R_1$$

$$(3)\ A(U_1-U_2)=U_{iz}\ \Rightarrow\ U_1=U_2\ \ (A\to\infty)$$

$$(1)\ U_1(R_3R_4+R_3R_5+R_4R_5)=U_{ul}R_4R_5+U_{iz}R_3R_4\Big/:(R_3R_4+R_3R_5+R_4R_5)$$

$$(2)\ U_2\left(1+sR_1C_1+R_1\frac{sC_2}{sR_2C_2+1}\right)=U_{ul}(1+sR_1C_1)+U_{iz}R_1\frac{sC_2}{sR_2C_2+1}\Big/\cdot(sR_2C_2+1)$$

$$(1)\ U_1=U_{ul}\frac{R_4R_5}{R_3R_4+R_3R_5+R_4R_5}+U_{iz}\frac{R_3R_4}{R_3R_4+R_3R_5+R_4R_5}$$

$$(2)\ U_1\left[(sR_1C_1+1)(sR_2C_2+1)+sR_1C_2\right]=U_{ul}(sR_1C_1+1)(sR_2C_2+1)+U_{iz}sR_1C_2$$

Uvedimo oznaku:

$$\frac{R_0}{R_3}=\frac{R_4R_5}{R_3R_4+R_3R_5+R_4R_5};\ \frac{R_0}{R_5}=\frac{R_3R_4}{R_3R_4+R_3R_5+R_4R_5};\ R_0=R_3\|R_4\|R_5=\frac{R_3R_4R_5}{R_3R_4+R_3R_5+R_4R_5};$$

$$(1) \;\Rightarrow\; U_1 = U_{ul}\frac{R_0}{R_3} + U_{iz}\frac{R_0}{R_5}$$

Malo računanja: $(1) \rightarrow (2) \Rightarrow$

$$\left[U_{ul}\frac{R_0}{R_3} + U_{iz}\frac{R_0}{R_5}\right]\cdot\left[(sR_1C_1+1)(sR_2C_2+1)+sR_1C_2\right] = U_{ul}(sR_1C_1+1)(sR_2C_2+1)+U_{iz}sR_1C_2$$

$$\left(\frac{R_0}{R_3}-1\right)U_{ul}\cdot(sR_1C_1+1)(sR_2C_2+1)+\frac{R_0}{R_3}U_{ul}\cdot sR_1C_2 = -\frac{R_0}{R_5}U_{iz}(sR_1C_1+1)(sR_2C_2+1)+\left(1-\frac{R_0}{R_5}\right)U_{iz}sR_1C_2$$

$$-\left(\frac{R_0}{R_4}+\frac{R_0}{R_5}\right)U_{ul}(sR_1C_1+1)(sR_2C_2+1)+\frac{R_0}{R_3}U_{ul}sR_1C_2 = -\frac{R_0}{R_5}U_{iz}(sR_1C_1+1)(sR_2C_2+1)+\left(\frac{R_0}{R_3}+\frac{R_0}{R_4}\right)U_{iz}sR_1C_2$$

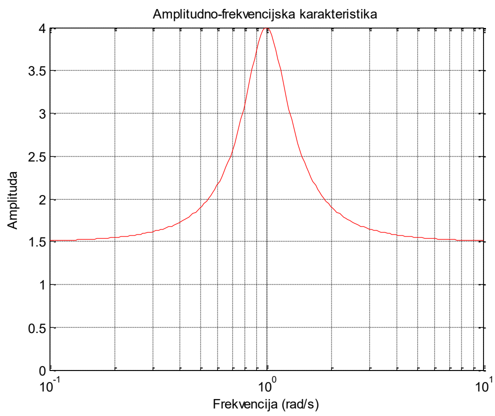
$$\Big/\cdot\left(-\frac{R_5}{R_0}\right)$$

$$\left(1+\frac{R_5}{R_4}\right)U_{ul}\cdot(sR_1C_1+1)(sR_2C_2+1)-\frac{R_5}{R_3}U_{ul}\cdot sR_1C_2 = U_{iz}(sR_1C_1+1)(sR_2C_2+1)-\left(\frac{R_5}{R_3}+\frac{R_5}{R_4}\right)U_{iz}sR_1C_2$$

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\left(1+\frac{R_5}{R_4}\right)\frac{s^2R_1C_1R_2C_2+s\left[R_1C_1+R_2C_2-R_1C_2\Big/\left(\frac{R_3}{R_4}+\frac{R_3}{R_5}\right)\right]+1}{s^2R_1C_1R_2C_2+s\left[R_1C_1+R_2C_2-\left(\frac{R_5}{R_3}+\frac{R_5}{R_4}\right)R_1C_2\right]+1}$$

Uz uvrštene vrijednosti elemenata C_1=C_2=1; R_1=R_2=1 te R_3=1; R_4=2; R_5=1;

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{3s^2+4s+3}{2s^2+s+2}=\frac{3}{2}\cdot\frac{s^2+\frac{4}{3}s+1}{s^2+\frac{1}{2}s+1}$$



b) Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati parametre k, ω_p, q_p, ω_z, q_z.

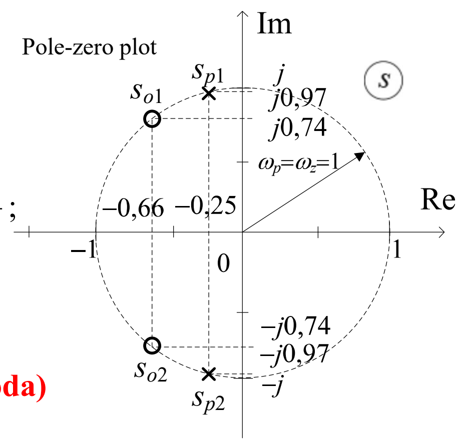
$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=k\cdot\frac{s^2+(\omega_z/q_z)\cdot s+\omega_z^2}{s^2+(\omega_p/q_p)\cdot s+\omega_p^2}$$

$\Rightarrow\; \omega_p=\omega_z=1,\;\; q_p=2,\; q_z=3/4,\quad k=3/2.$ **a) + b) (3 boda)**

c) raspored polova i nula u kompleksnoj ravnini:

- nule: $\quad 3s^2+4s+3=0 \;\;\Rightarrow\;\; s_{o1,2}=\dfrac{-2\pm j\sqrt{5}}{3}=-0{,}667\pm j0{,}7453;$

- polovi: $\;2s^2+s+2=0 \;\;\Rightarrow\;\; s_{p1,2}=-\dfrac{1\pm j\sqrt{15}}{4}=-0{,}25\pm j0{,}9682.$



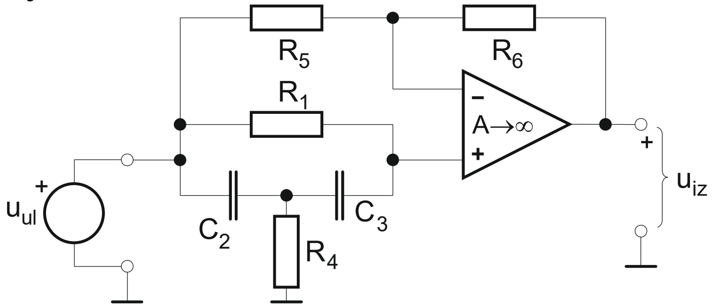
d) amplitudno-frekvencijska karakteristika:

$$s{=}j\omega \;\Rightarrow\; T(j\omega)=\frac{-3\omega^2+j4\omega+3}{-2\omega^2+j\omega+2} \;\;\Rightarrow\;\; |T(j\omega)|=\frac{\sqrt{(3-3\omega^2)^2+(4\omega)^2}}{\sqrt{(2-2\omega^2)^2+(\omega)^2}};$$

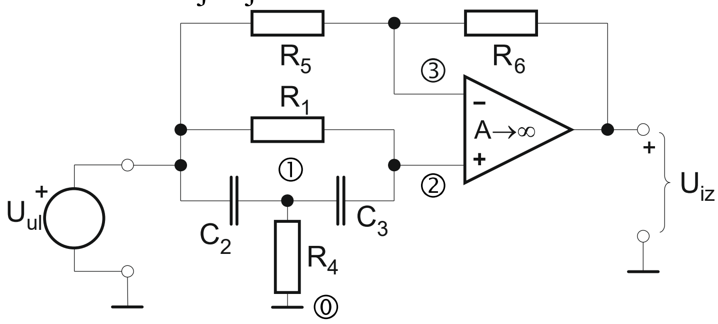
$$T_{\min}=T(0)=T(\infty)=k=3/2;\;\; T_{\max}=|T(j\omega_p)|=k\frac{q_p}{q_z}=4.$$

-o kojem se tipu filtra radi (NP, VP, PP ili PB)? $\Rightarrow$ PP **c) + d) (2 boda)**

4. Zadan je aktivni-RC električni filtar prikazan slikom s normaliziranim vrijednostima elemenata R_1=1, C_2=1, C_3=1, R_4=1, te R_5=1, R_6=2. a) Izračunati njegovu naponsku prijenosnu funkciju $T(s)=U_{iz}(s)/U_{ul}(s)$. O kojem se tipu filtra radi (NP, VP, PP ili PB)? b) Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati vrijednost parametara k, ω_p, ω_z, q_p. c) Prikazati raspored polova i nula u kompleksnoj ravnini. d) Nacrtati amplitudno-frekvencijsku karakteristiku.



Rješenje: Laplaceova transformacija i jednadžbe čvorišta:



Metoda napona čvorišta:

$$(1)\ U_1\left(sC_2+sC_3+\frac{1}{R_4}\right)-U_2sC_3=U_{ul}sC_2 \Big/\cdot R_4$$

$$(2)\ -U_1sC_3+U_2\left(\frac{1}{R_1}+sC_3\right)=U_{ul}\frac{1}{R_1}\Big/:sC_3$$

$$(3)\ U_3\left(\frac{1}{R_5}+\frac{1}{R_6}\right)=U_{ul}\frac{1}{R_5}+U_{iz}\frac{1}{R_6}\Big/\cdot R_5R_6$$

$$(4)\ \underline{A(U_2-U_3)=U_{iz}\ \Rightarrow\ U_2=U_3\ \ (A\to\infty)}$$

Slijedi postepeno računanje korak po korak

$(3)\Rightarrow U_3(R_5+R_6)=U_{ul}R_6+U_{iz}R_5 \Rightarrow U_3=U_{ul}\dfrac{R_6}{R_5+R_6}+U_{iz}\dfrac{R_5}{R_5+R_6}$;Uz oznaku $\alpha=\dfrac{R_5}{R_5+R_6}\Rightarrow$

$U_3=U_{ul}(1-\alpha)+U_{iz}\alpha$; Zajedno sa (4) $\Rightarrow U_2=U_3=U_{ul}(1-\alpha)+U_{iz}\alpha$

$$(2)\Rightarrow U_1=U_2\left(\frac{1}{sC_3R_1}+1\right)-U_{ul}\frac{1}{sC_3R_1}$$

$(1)\Rightarrow U_1(sR_4C_2+sR_4C_3+1)-U_2R_4sC_3=U_{ul}R_4sC_2$; (2) $\to$ (1) (rješavamo se U_1) $\Rightarrow$

$$\left[U_2\left(\frac{1}{sC_3R_1}+1\right)-U_{ul}\frac{1}{sC_3R_1}\right](sR_4C_2+sR_4C_3+1)-U_2R_4sC_3=U_{ul}R_4sC_2$$

$$U_2\left(\frac{1}{sC_3R_1}+1\right)(sR_4C_2+sR_4C_3+1)-U_2R_4sC_3=U_{ul}\frac{1}{sC_3R_1}(sR_4C_2+sR_4C_3+1)+U_{ul}R_4sC_2\Big/\cdot sC_3R_1$$

$$U_2(sC_3R_1+1)(sR_4C_2+sR_4C_3+1)-U_2R_1R_4s^2C_3^2=U_{ul}(s^2R_1C_2C_3R_4+sR_4C_2+sR_4C_3+1)$$

$$U_2(s^2R_1C_2C_3R_4+sC_3R_1+sR_4C_2+sR_4C_3+1)=U_{ul}(s^2R_1C_2C_3R_4+sR_4C_2+sR_4C_3+1)$$

$$[U_{ul}(1-\alpha)+U_{iz}\alpha](s^2R_1C_2C_3R_4+sC_3R_1+sR_4C_2+sR_4C_3+1)=U_{ul}(s^2R_1C_2C_3R_4+sR_4C_2+sR_4C_3+1)$$

$$U_{iz}\alpha(s^2R_1C_2C_3R_4+sC_3R_1+sR_4C_2+sR_4C_3+1)=$$

$$U_{ul}(s^2R_1C_2C_3R_4+sR_4C_2+sR_4C_3+1)-U_{ul}(1-\alpha)(s^2R_1C_2C_3R_4+sC_3R_1+sR_4C_2+sR_4C_3+1)$$

$$U_{iz}\alpha(s^2R_1C_2C_3R_4+sC_3R_1+sR_4C_2+sR_4C_3+1)=$$

$$U_{ul}(s^2R_1C_2C_3R_4+sR_4C_2+sR_4C_3+1)-U_{ul}(s^2R_1C_2C_3R_4+sC_3R_1+sR_4C_2+sR_4C_3+1)+$$

$$+U_{ul}\alpha(s^2R_1C_2C_3R_4+sC_3R_1+sR_4C_2+sR_4C_3+1)$$

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{s^2R_1C_2C_3R_4+s\left[C_3R_1\left(1-\dfrac{1}{\alpha}\right)+R_4C_2+R_4C_3\right]+1}{s^2R_1C_2C_3R_4+s(C_3R_1+R_4C_2+R_4C_3)+1}$$

Konačno je:

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{s^2+s\dfrac{C_3R_1(1-1/\alpha)+R_4C_2+R_4C_3}{R_1C_2C_3R_4}+\dfrac{1}{R_1C_2C_3R_4}}{s^2+s\dfrac{C_3R_1+R_4C_2+R_4C_3}{R_1C_2C_3R_4}+\dfrac{1}{R_1C_2C_3R_4}}=\frac{s^2+1}{s^2+3s+1}\quad \textbf{(2 boda)}$$

Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati parametre k, ω_p, q_p, ω_z.

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=k\cdot\frac{s^2+\omega_z^2}{s^2+(\omega_p/q_p)\cdot s+\omega_p^2}\quad\Rightarrow\quad \omega_p=\omega_z=1,\quad q_p=\frac{1}{3},\quad k=1$$

-o kojem se tipu filtra radi (NP, VP, PP ili PB)? $\Rightarrow$ PB **(1 bod)**

Raspored polova i nula u kompleksnoj ravnini: **(1 bod)**

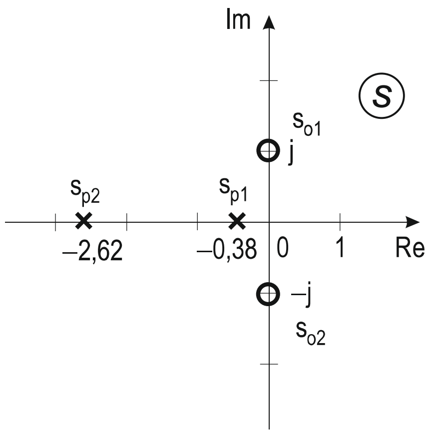
Nule $\quad s^2+1=0\ \Rightarrow\ s_{o1,2}=\pm j$

Polovi $s^2+3s+1=0\Rightarrow s_{p1,2}=-\dfrac{3}{2}\pm\sqrt{\dfrac{9}{4}-1}=-\dfrac{3}{2}\pm\dfrac{\sqrt{5}}{2}$; $s_{p1}=-0{,}381966$; $s_{p2}=-2{,}61803$.

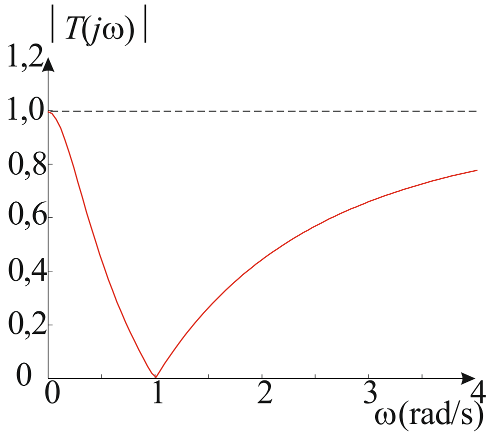
Amplitudno-frekvencijska (A-F) karakteristika **(1 bod)**

$$s=j\omega\Rightarrow\qquad T(j\omega)=\frac{-\omega^2+1}{-\omega^2+3\cdot j\omega+1}\qquad\Rightarrow\qquad |T(j\omega)|=\frac{|1-\omega^2|}{\sqrt{(1-\omega^2)^2+(3\cdot\omega)^2}}$$

Karakteristične točke A-F karakteristike $|T(j0)|=1$; $|T(j1)|=0$; $|T(j\infty)|=1$.

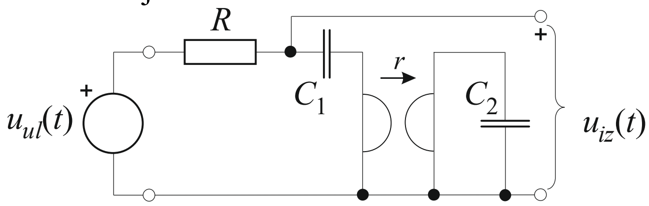


Raspored nula i polova u *s*-ravnini



Amplitudno frekvencijska karakteristika

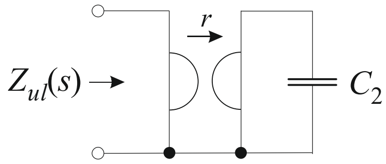
4. Zadan je aktivni-RC električni filtar prikazan slikom s normaliziranim vrijednostima elemenata R_1=1, C_1=1, C_2=1/4, te parametrom giratora r=2. a) Izračunati njegovu naponsku prijenosnu funkciju $T(s)=U_{iz}(s)/U_{ul}(s)$. O kojem se tipu filtra radi (NP, VP, PP ili PB)? b) Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati vrijednost parametara k, ω_p, ω_z, q_p. c) Prikazati raspored polova i nula u kompleksnoj ravnini. d) Nacrtati amplitudno-frekvencijsku karakteristiku.



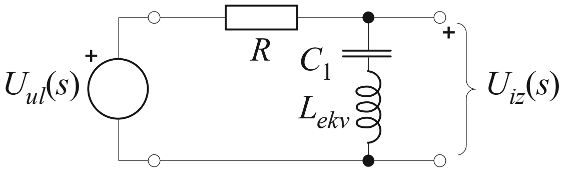
Rješenje:

Ulazna impedancija u girator zaključen kapacitetom C_2 glasi:

$$Z_{ul}(s)=\frac{r^2}{1/(sC_2)}=r^2sC_2=sL_{ekv}\Rightarrow L_{ekv}=r^2C_2=1$$



Laplaceova transformacija:



Serijska kombinacija L_{ekv} i C_1 ima impedanciju

$$Z(s)=\frac{1}{sC_1}+sL_{ekv}=\frac{1+s^2L_{ekv}C_1}{sC_1}$$

Naponska prijenosna funkcija se da lako izračunati ako se promatrani električni filtar promatra kao naponsko dijelilo i ona za navedeni električni krug glasi:

$$H(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{Z(s)}{R+Z(s)}=\frac{\dfrac{1+s^2L_{ekv}C_1}{sC_1}}{R+\dfrac{1+s^2L_{ekv}C_1}{sC_1}}=\frac{1+s^2L_{ekv}C_1}{s^2L_{ekv}C+RsC_1+1}=\frac{s^2+\dfrac{1}{L_{ekv}C_1}}{s^2+s\dfrac{R}{L_{ekv}}+\dfrac{1}{L_{ekv}C_1}}$$

$$H(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{s^2+\dfrac{1}{L_{ekv}C_1}}{s^2+s\dfrac{R}{L_{ekv}}+\dfrac{1}{L_{ekv}C_1}}=\frac{s^2+\dfrac{1}{r^2C_1C_2}}{s^2+s\dfrac{R}{r^2C_2}+\dfrac{1}{r^2C_1C_2}}=k\cdot\frac{s^2+\omega_z^2}{s^2+\dfrac{\omega_p}{q_p}s+\omega_p^2}$$

$$H(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{s^2+1}{s^2+s+1}$$

-o kojem se tipu filtra radi (NP, VP, PP ili PB)? $\Rightarrow$ PB

(2 boda)

Odatle slijede parametri:

$$\omega_p = \omega_z = \frac{1}{\sqrt{L_{ekv}C_1}} = \frac{1}{\sqrt{r^2C_1C_2}} = 1,\ q_p = R\sqrt{\frac{C_1}{L_{ekv}}} = R\sqrt{\frac{C_1}{r^2C_2}} = 1,\ k = 1.$$

(1 bod)

Raspored polova i nula u kompleksnoj ravnini:

Nule $\quad s^2+1=0 \ \Rightarrow \ s_{o1,2} = \pm j$

Polovi $\quad s^2+s+1=0 \Rightarrow s_{p1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4}-1} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} = -0{,}5 \pm 0{,}866\,;$

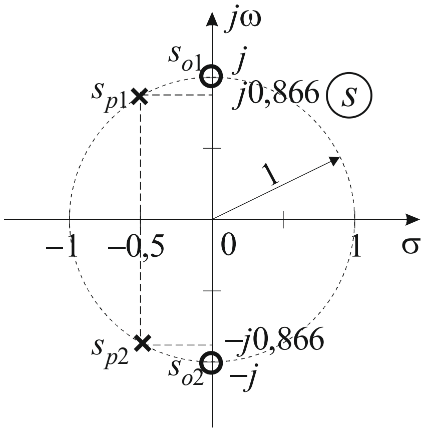
(1 bod)

Amplitudno-frekvencijska (A-F) karakteristika

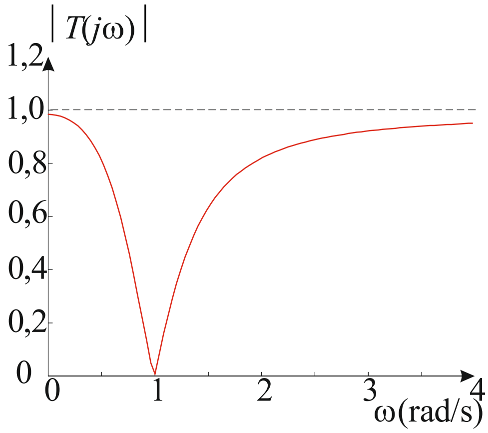
$$s=j\omega \Rightarrow \qquad T(j\omega) = \frac{-\omega^2+1}{-\omega^2+j\omega+1} \qquad \Rightarrow \qquad |T(j\omega)| = \frac{|1-\omega^2|}{\sqrt{(1-\omega^2)^2+\omega^2}}$$

(1 bod)

Karakteristične točke A-F karakteristike $|T(j0)|=1;\ \ |T(j1)|=0;\ \ |T(j\infty)|=1.$

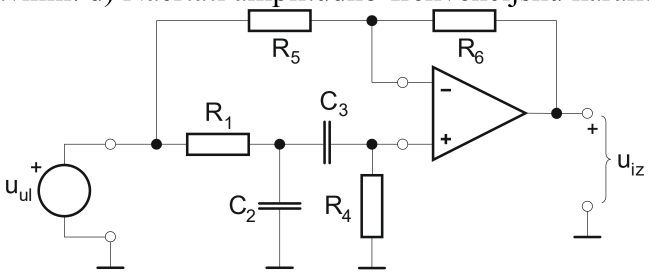


Raspored nula i polova u s-ravnini

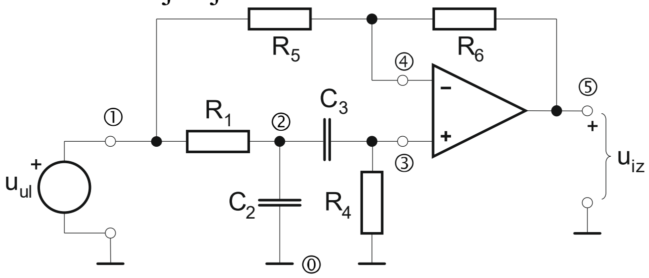


Amplitudno frekvencijska karakteristika

4. Zadan je aktivni-RC električni filtar prikazan slikom s normaliziranim vrijednostima elemenata R_1=1, C_2=1, C_3=1, R_4=1, te R_5=2, R_6=1. Operacijsko pojačalo je idealno ($A\to\infty$). a) Izračunati njegovu naponsku prijenosnu funkciju $T(s)=U_{iz}(s)/U_{ul}(s)$. O kojem se tipu filtra radi (NP, VP, PP ili PB)? b) Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati vrijednost parametara k, ω_p, ω_z, q_p. c) Prikazati raspored polova i nula u kompleksnoj ravnini. d) Nacrtati amplitudno-frekvencijsku karakteristiku.



Rješenje: Laplaceova transformacija i jednadžbe čvorišta:



Metoda napona čvorišta:

$$(1)\ -U_1\frac{1}{R_1}+U_2\left(sC_2+sC_3+\frac{1}{R_1}\right)-U_3sC_3=0\ \Big/\cdot R_1$$

$$(2)\ -U_2sC_3+U_3\left(\frac{1}{R_4}+sC_3\right)=0\ \Big/: sC_3$$

$$(3)\ U_4\left(\frac{1}{R_5}+\frac{1}{R_6}\right)=U_1\frac{1}{R_5}+U_5\frac{1}{R_6}\ \Big/\cdot R_5R_6$$

$$(4)\ \underline{A(U_3-U_4)=U_5\ \Rightarrow\ U_3=U_4\ \ (A\to\infty)}$$

Slijedi postepeno računanje korak po korak uz $U_{ul}=U_1;\ \ U_{iz}=U_5$

$$(3)\Rightarrow\ U_4\left(R_5+R_6\right)=U_1R_6+U_5R_5\ \Rightarrow\ U_4=U_1\frac{R_6}{R_5+R_6}+U_5\frac{R_5}{R_5+R_6}$$

Uz oznaku $\alpha=\dfrac{R_5}{R_5+R_6}\ \Rightarrow\ U_4=U_1(1-\alpha)+U_5\alpha$, i s (4) $\Rightarrow\ U_3=U_4=U_1(1-\alpha)+U_5\alpha$

$$(2)\Rightarrow\ U_2=U_3\left(\frac{1}{sC_3R_4}+1\right)$$

$$(1)\Rightarrow\ U_2\left(sR_1C_2+sR_1C_3+1\right)-U_3R_1sC_3=U_1$$

(2)→(1) rješavamo se $U_2 \Rightarrow U_3\left(\dfrac{1}{sC_3R_4}+1\right)\left(sR_1C_2+sR_1C_3+1\right)-U_3R_1sC_3=U_1\ \Big/\cdot sC_3R_4$

$$U_3\left(1+sC_3R_4\right)\left(sR_1C_2+sR_1C_3+1\right)-U_3R_1sC_3sC_3R_4=U_1sC_3R_4$$

$$U_3\left(R_1sC_2+R_1sC_3+1+R_1sC_2sC_3R_4+R_1sC_3sC_3R_4+sC_3R_4-R_1sC_3sC_3R_4\right)=U_1sC_3R_4$$

$$U_3\left(s^2R_1C_2C_3R_4+R_1sC_2+R_1sC_3+sC_3R_4+1\right)=U_1sC_3R_4$$

$$\left[U_1(1-\alpha)+U_5\alpha\right]\left(s^2R_1C_2C_3R_4+R_1sC_2+R_1sC_3+sC_3R_4+1\right)=U_1sC_3R_4$$

$$U_1(1-\alpha)\left(s^2R_1C_2C_3R_4+R_1sC_2+R_1sC_3+sC_3R_4+1\right)-U_1sC_3R_4=$$

$$=-U_5\alpha\left(s^2R_1C_2C_3R_4+R_1sC_2+R_1sC_3+sC_3R_4+1\right)$$

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{U_5(s)}{U_1(s)}=\frac{(1-\alpha)\left(s^2R_1C_2C_3R_4+R_1sC_2+R_1sC_3+sC_3R_4+1\right)-sC_3R_4}{-\alpha\left(s^2R_1C_2C_3R_4+R_1sC_2+R_1sC_3+sC_3R_4+1\right)}$$

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=-\frac{1-\alpha}{\alpha}\cdot\frac{s^2R_1C_2C_3R_4+R_1sC_2+R_1sC_3+sC_3R_4+1-sC_3R_4/(1-\alpha)}{s^2R_1C_2C_3R_4+R_1sC_2+R_1sC_3+sC_3R_4+1}$$

Konačno je:

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=-\frac{1-\alpha}{\alpha}\cdot\frac{s^2+\dfrac{R_1C_2+R_1C_3-\dfrac{\alpha}{1-\alpha}C_3R_4}{R_1C_2C_3R_4}s+\dfrac{1}{R_1C_2C_3R_4}}{s^2+\dfrac{R_1C_2+R_1C_3+C_3R_4}{R_1C_2C_3R_4}s+\dfrac{1}{R_1C_2C_3R_4}}=-\frac{1}{2}\cdot\frac{s^2+1}{s^2+3s+1}\quad\text{(2 boda)}$$

$$\alpha=\frac{R_5}{R_5+R_6}=\frac{2}{3}\Rightarrow-\frac{1-\alpha}{\alpha}=-\left(1-\frac{2}{3}\right)\frac{3}{2}=-\frac{1}{3}\cdot\frac{3}{2}=-\frac{1}{2}$$

Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati parametre k, ω_p, q_p, ω_z.

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=k\cdot\frac{s^2+\omega_z^2}{s^2+(\omega_p/q_p)\cdot s+\omega_p^2}\quad\Rightarrow\quad\omega_p=\omega_z=1,\ \ q_p=\frac{1}{3},\ \ k=-\frac{1}{2}$$

-o kojem se tipu filtra radi (NP, VP, PP ili PB)? $\Rightarrow$ PB (1 bod)

Raspored polova i nula u kompleksnoj ravnini: (1 bod)

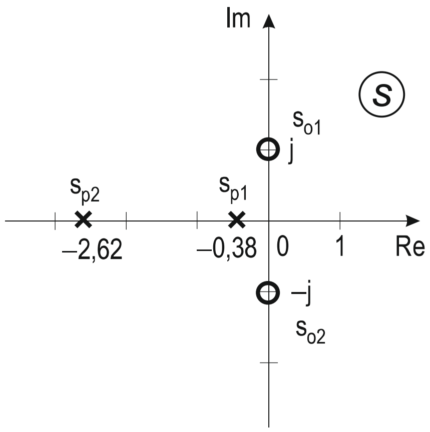
Nule $\quad s^2+1=0\ \Rightarrow\ s_{o1,2}=\pm j$

Polovi $s^2+3s+1=0\Rightarrow s_{p1,2}=-\frac{3}{2}\pm\sqrt{\frac{9}{4}-1}=-\frac{3}{2}\pm\frac{\sqrt{5}}{2}$; $s_{p1}=-0{,}381966$; $s_{p2}=-2{,}61803$.

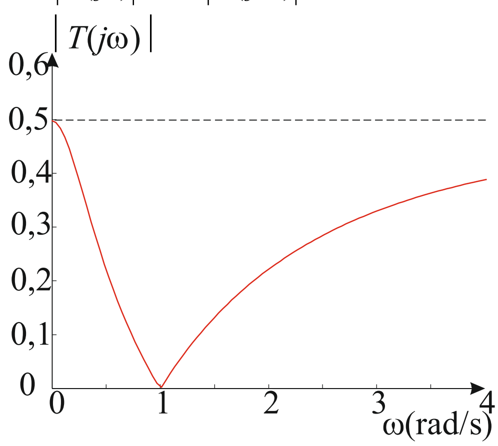
Amplitudno-frekvencijska (A-F) karakteristika (1 bod)

$$s=j\omega\Rightarrow\qquad T(j\omega)=-\frac{1}{2}\cdot\frac{-\omega^2+1}{-\omega^2+3\cdot j\omega+1}\quad\Rightarrow\quad\left|T(j\omega)\right|=\frac{1}{2}\cdot\frac{\left|1-\omega^2\right|}{\sqrt{\left(1-\omega^2\right)^2+(3\cdot\omega)^2}}$$

Karakteristične točke A-F karakteristike $|T(j0)|=1/2$; $|T(j1)|=0$; $|T(j\infty)|=1/2$.

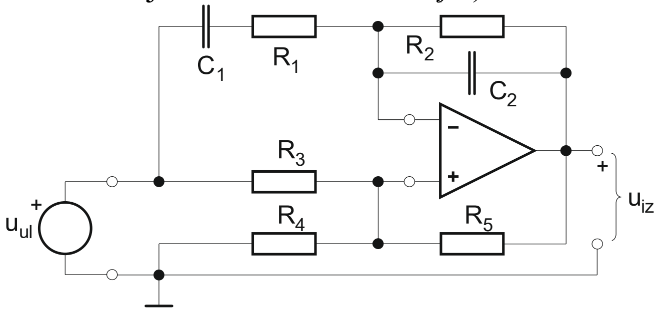


Raspored nula i polova u s-ravnini

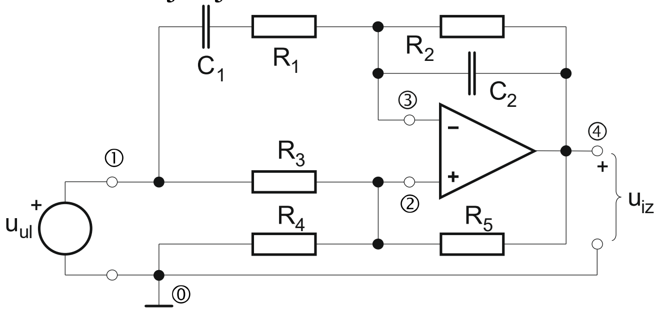


Amplitudno frekvencijska karakteristika

4. Zadan je aktivni-RC električni filtar prikazan slikom s normaliziranim vrijednostima elemenata R_1=1, C_1=1, R_2=1, C_2=1, R_3=1, R_4=1, R_5=1. a) Izračunati njegovu naponsku prijenosnu funkciju $T(s)=U_{iz}(s)/U_{ul}(s)$. O kojem se tipu filtra radi (NP, VP, PP ili PB)? b) Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati vrijednost parametara k, ω_p, ω_z, q_p. c) Prikazati raspored polova i nula u kompleksnoj ravnini. d) Nacrtati amplitudno-frekvencijsku karakteristiku $|T(j\omega)|$.



Rješenje: Laplaceova transformacija i jednadžbe čvorišta:



$$(1)\ -U_1 \frac{1}{R_1+1/(sC_1)} + U_3\left(\frac{1}{R_1+1/(sC_1)}+\frac{1}{R_2}+sC_2\right) - U_4\left(\frac{1}{R_2}+sC_2\right)=0 \Big/ \cdot\left(R_1+\frac{1}{sC_1}\right)$$

$$(2)\ U_2\left(\frac{1}{R_3}+\frac{1}{R_4}+\frac{1}{R_5}\right)=U_1\frac{1}{R_3}+U_4\frac{1}{R_5}\Big/\cdot R_3R_4R_5$$

$$(3)\ \underline{A(U_2-U_3)=U_4 \ \Rightarrow\ U_2=U_3\ \ (A\to\infty)}$$

Slijedi postepeno računanje korak po korak uz $U_{ul}=U_1;\ U_{iz}=U_4$

$$(2)\Rightarrow U_2\left(R_4R_5+R_3R_5+R_3R_4\right)=U_1R_4R_5+U_4R_4R_6\ \Rightarrow$$

$$U_2=U_1\frac{R_4R_5}{R_4R_5+R_3R_5+R_3R_4}+U_4\frac{R_4R_6}{R_4R_5+R_3R_5+R_3R_4}$$

Uz oznake $\alpha=\dfrac{R_4R_5}{R_4R_5+R_3R_5+R_3R_4}$; $\beta=\dfrac{R_4R_6}{R_4R_5+R_3R_5+R_3R_4}$; $\gamma=\dfrac{R_5R_6}{R_4R_5+R_3R_5+R_3R_4}$;

$\alpha+\beta+\gamma=1,\ \ 0<\alpha,\beta,\gamma<1$

$\Rightarrow U_2=\alpha U_1+\beta U_4$ i zajedno sa (3) $\Rightarrow U_2=U_3=\alpha U_1+\beta U_4$

$$(1)\Rightarrow U_1=U_3\left[1+\left(\frac{1}{R_2}+sC_2\right)\left(R_1+\frac{1}{sC_1}\right)\right]-U_4\left(\frac{1}{R_2}+sC_2\right)\left(R_1+\frac{1}{sC_1}\right)\Big/\cdot R_2sC_1$$

$$U_1R_2sC_1=U_3\left[R_2sC_1+\left(1+sR_1C_1\right)\left(1+sR_2C_2\right)\right]-U_4\left(1+sR_1C_1\right)\left(1+sR_2C_2\right)$$

(2) $\to$ (1) (rješavamo se U_3) $\Rightarrow$

$$U_1R_2sC_1=\left(\alpha U_1+\beta U_4\right)\left[R_2sC_1+\left(1+sR_1C_1\right)\left(1+sR_2C_2\right)\right]-U_4\left(1+sR_1C_1\right)\left(1+sR_2C_2\right)$$

$$U_1R_2sC_1-\alpha U_1R_2sC_1-\alpha U_1\left(1+sR_1C_1\right)\left(1+sR_2C_2\right)=\beta U_4R_2sC_1+(\beta-1)U_4\left(1+sR_1C_1\right)\left(1+sR_2C_2\right)$$

$$-\alpha U_1\left[1+sR_1C_1+sR_2C_2+\frac{\alpha-1}{\alpha}sR_2C_1+s^2R_1C_1R_2C_2\right]=$$

$$=(\beta-1)U_4\left[1+sR_1C_1+sR_2C_2+\frac{\beta}{\beta-1}sR_2C_1+s^2R_1C_1R_2C_2\right]$$

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{U_4(s)}{U_1(s)}=\frac{\alpha}{1-\beta}\,\frac{1+sR_1C_1+sR_2C_2-\frac{1-\alpha}{\alpha}sR_2C_1+s^2R_1C_1R_2C_2}{1+sR_1C_1+sR_2C_2-\frac{\beta}{1-\beta}sR_2C_1+s^2R_1C_1R_2C_2}$$

Konačno je: **(2 boda)**

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{\alpha}{1-\beta}\,\frac{s^2+s\frac{R_1C_1+R_2C_2-\left[(1-\alpha)/\alpha\right]R_2C_1}{R_1C_1R_2C_2}+\frac{1}{R_1C_1R_2C_2}}{s^2+s\frac{R_1C_1+R_2C_2-\left[\beta/(1-\beta)\right]R_2C_1}{R_1C_1R_2C_2}+\frac{1}{R_1C_1R_2C_2}}=\frac{1}{2}\cdot\frac{s^2+1}{s^2+\frac{3}{2}s+1}$$

$$\alpha=\frac{R_4R_5}{R_4R_5+R_3R_5+R_3R_4}=\frac{1}{3}\ ;\ \beta=\frac{R_4R_6}{R_4R_5+R_3R_5+R_3R_4}=\frac{1}{3}$$

$$\Rightarrow\ -\frac{1-\alpha}{\alpha}=-\left(1-\frac{1}{3}\right)3=-\frac{2}{3}\cdot3=-2;\ -\frac{\beta}{1-\beta}=-\frac{1}{3}\Big/\left(1-\frac{1}{3}\right)=-\frac{1}{3}\Big/\frac{2}{3}=-\frac{1}{3}\cdot\frac{3}{2}=-\frac{1}{2}$$

Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati parametre k, ω_p, q_p, ω_z.

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=k\cdot\frac{s^2+\omega_z^2}{s^2+\left(\omega_p/q_p\right)\cdot s+\omega_p^2}\quad\Rightarrow\quad\omega_p=\omega_z=1,\ \ q_p=\frac{2}{3},\ \ k=\frac{1}{2}$$

-o kojem se tipu filtra radi (NP, VP, PP ili PB)? ⇒ PB **(1 bod)**

Raspored polova i nula u kompleksnoj ravnini: **(1 bod)**

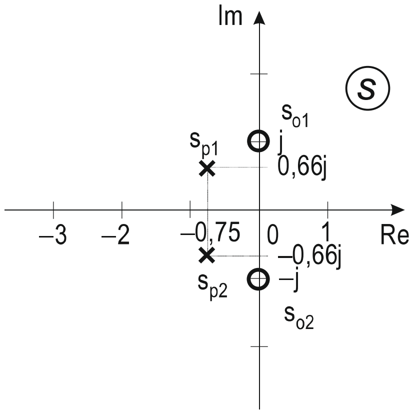
Nule $\quad s^2+1=0\ \Rightarrow\ s_{o1,2}=\pm j$

Polovi $\ s^2+\frac{3}{2}s+1=0\Rightarrow\ s_{p1,2}=-\frac{3}{4}\pm\sqrt{\frac{9}{16}-1}=-\frac{3}{4}\pm j\frac{\sqrt{7}}{4}\ ;\ s_{p1,2}=-0,75\pm j0,661438.$

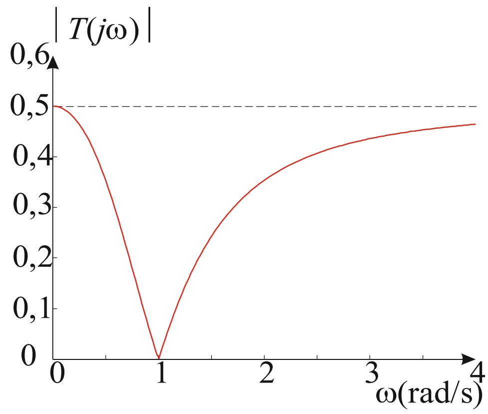
Amplitudno-frekvencijska (A-F) karakteristika **(1 bod)**

$$s=j\omega\Rightarrow\qquad T(j\omega)=0,5\frac{-\omega^2+1}{-\omega^2+1,5\cdot j\omega+1}\ \Rightarrow\left|T(j\omega)\right|=0,5\frac{\left|1-\omega^2\right|}{\sqrt{\left(1-\omega^2\right)^2+\left(1,5\cdot\omega\right)^2}}$$

Karakteristične točke A-F karakteristike $\left|T(j0)\right|=1;\ \ \left|T(j1)\right|=0;\ \ \left|T(j\infty)\right|=1.$

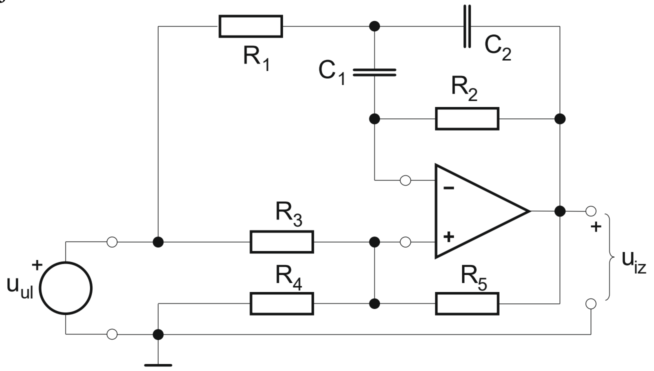


Raspored nula i polova u s-ravnini

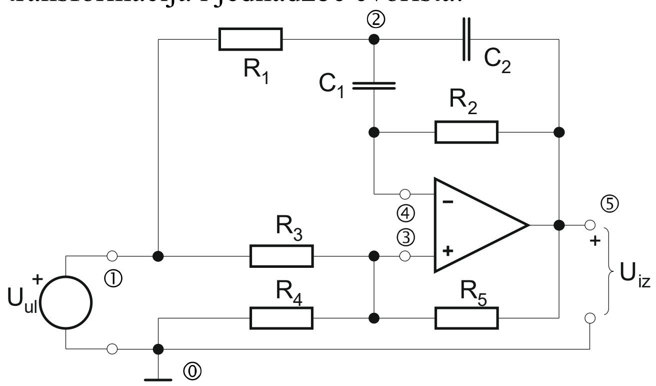


Amplitudno frekvencijska karakteristika

4. Zadan je aktivni-RC električni filtar prikazan slikom s normaliziranim vrijednostima elemenata $C_1=C_2=1$, $R_1=R_2=R_3=R_4=R_5=1$. a) Izračunati njegovu naponsku prijenosnu funkciju $T(s)=U_{iz}(s)/U_{ul}(s)$. O kojem se tipu filtra radi (NP, VP, PP ili PB)? b) Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati vrijednost parametara k, ω_p, ω_z, q_p. c) Prikazati raspored polova i nula u kompleksnoj ravnini. d) Nacrtati amplitudno-frekvencijsku karakteristiku.



Rješenje: Laplaceova transformacija i jednadžbe čvorišta:



Metoda napona čvorišta:

$$(1)\ -U_1\frac{1}{R_1}+U_2\left(sC_1+sC_2+\frac{1}{R_1}\right)-U_4sC_1-U_5sC_2=0\ \Big/\cdot R_1$$

$$(2)\ -U_2sC_1+U_4\left(\frac{1}{R_2}+sC_1\right)-U_5\frac{1}{R_2}=0\ \Big/ :sC_1$$

$$(3)\ U_3\left(\frac{1}{R_3}+\frac{1}{R_4}+\frac{1}{R_5}\right)=U_1\frac{1}{R_3}+U_5\frac{1}{R_5}\ \Big/\cdot R_3R_4R_5$$

$$(4)\ \underline{A(U_3-U_4)=U_5\ \Rightarrow\ U_3=U_4\ \ (A\to\infty)}\ \text{(1 bod)}$$

Slijedi postepeno računanje korak po korak uz $U_{ul}=U_1;\ U_{iz}=U_5$

$(3)\Rightarrow U_3\left(R_3R_5+R_3R_4+R_4R_5\right)=U_1R_4R_5+U_5R_3R_4$, zajedno sa (4) $\Rightarrow$

$$U_3=U_4=U_1\frac{R_4R_5}{R_3R_4+R_3R_5+R_4R_5}+U_5\frac{R_3R_4}{R_3R_4+R_3R_5+R_4R_5}\text{, uz oznake}$$

$$\alpha=\frac{R_4R_5}{R_3R_4+R_3R_5+R_4R_5};\beta=\frac{R_3R_4}{R_3R_4+R_3R_5+R_4R_5};\gamma=\frac{R_3R_5}{R_3R_4+R_3R_5+R_4R_5};\alpha+\beta+\gamma=1$$

$(3), (4)\Rightarrow \underline{U_4=\alpha U_1+\beta U_5}$

$$(2) \Rightarrow\ U_2 = U_4\left(\frac{1}{sC_1R_2}+1\right) - U_5\frac{1}{sC_1R_2}$$

$$(1) \Rightarrow\ U_1 = U_2\left(sR_1C_1 + sR_1C_2 + 1\right) - U_4 sR_1C_1 - U_5 sR_1C_2$$

(2) → (1) (rješavamo se U_2) ⇒

$$U_1 = \left[U_4\left(\frac{1}{sC_1R_2}+1\right) - U_5\frac{1}{sC_1R_2}\right]\left(sR_1C_1 + sR_1C_2 + 1\right) - U_4 sR_1C_1 - U_5 sR_1C_2 \Big/ \cdot\, sC_1R_2$$

$$U_1 sC_1R_2 = \left[U_4\left(sC_1R_2 + 1\right) - U_5\right]\left(sR_1C_1 + sR_1C_2 + 1\right) - U_4 sC_1R_2 sR_1C_1 - U_5 sC_1R_2 sR_1C_2$$

$$U_1 sC_1R_2 = U_4\left(sC_1R_2 + 1\right)\left(sR_1C_1 + sR_1C_2 + 1\right) - U_4 sC_1R_2 sR_1C_1 - U_5\left(sR_1C_1 + sR_1C_2 + 1\right) - U_5 sC_1R_2 sR_1C_2$$

$$U_1 sC_1R_2 = U_4\left(s^2R_1R_2C_1^2 + s^2R_1R_2C_2C_1 + sR_2C_1 + sR_1C_1 + sR_1C_2 + 1 - s^2R_1R_2C_1^2\right)$$
$$-U_5\left(s^2R_1R_2C_2C_1 + sR_1C_1 + sR_1C_2 + 1\right)$$

$$U_1 sR_2C_1 = \left(\alpha U_1 + \beta U_5\right)\left(s^2R_1R_2C_2C_1 + sR_1C_1 + sR_1C_2 + sR_2C_1 + 1\right) - U_5\left(s^2R_1R_2C_2C_1 + sR_1C_1 + sR_1C_2 + 1\right)$$

$$U_1 sR_2C_1 = \alpha U_1\left(s^2R_1R_2C_2C_1 + sR_1C_1 + sR_1C_2 + sR_2C_1 + 1\right)$$
$$+\beta U_5\left(s^2R_1R_2C_2C_1 + sR_1C_1 + sR_1C_2 + 1\right) + \beta U_5 sR_2C_1 +$$
$$-U_5\left(s^2R_1R_2C_2C_1 + sR_1C_1 + sR_1C_2 + 1\right)$$

$$\alpha U_1\left(s^2R_1R_2C_2C_1 + sR_1C_1 + sR_1C_2 + sR_2C_1 - \frac{sR_2C_1}{\alpha} + 1\right) =$$
$$= (1-\beta)U_5\left(s^2R_1R_2C_2C_1 + sR_1C_1 + sR_1C_2 - \frac{\beta}{1-\beta}sR_2C_1 + 1\right)$$

$$T(s) = \frac{U_{iz}(s)}{U_{ul}(s)} = \frac{U_5(s)}{U_1(s)} = \frac{\alpha}{1-\beta}\cdot\frac{s^2R_1R_2C_1C_2 + sR_1C_1 + sR_1C_2 + sR_2C_1\left(1-\frac{1}{\alpha}\right) + 1}{s^2R_1R_2C_1C_2 + sR_1C_1 + sR_1C_2 - \frac{\beta}{1-\beta}sR_2C_1 + 1}$$

Konačno je:

$$T(s) = \frac{U_{iz}(s)}{U_{ul}(s)} = \frac{\alpha}{1-\beta}\cdot\frac{s^2 + \dfrac{R_1\left(C_1+C_2\right) + R_2C_1\left(1-\frac{1}{\alpha}\right)}{R_1R_2C_1C_2}s + \dfrac{1}{R_1R_2C_1C_2}}{s^2 + \dfrac{R_1\left(C_1+C_2\right) - \frac{\beta}{1-\beta}R_2C_1}{R_1R_2C_1C_2}s + \dfrac{1}{R_1R_2C_1C_2}}$$

Uz uvrštene vrijednosti elemenata:

$$T(s) = \frac{1}{2}\cdot\frac{s^2+1}{s^2 + (3/2)\cdot s + 1} \quad \text{(2 boda)}$$

Usporedbom s odgovarajućim općim oblikom prijenosne funkcije izračunati parametre k, ω_p, q_p, ω_z.

$$T(s) = \frac{U_{iz}(s)}{U_{ul}(s)} = k\cdot\frac{s^2+\omega_z^2}{s^2 + \left(\omega_p/q_p\right)\cdot s + \omega_p^2} \quad \Rightarrow \quad \omega_p = \omega_z = 1,\ \ q_p = \frac{2}{3},\ \ k = \frac{1}{2}$$

-o kojem se tipu filtra radi (NP, VP, PP ili PB)? ⇒ PB

Raspored polova i nula u kompleksnoj ravnini: (1 bod)

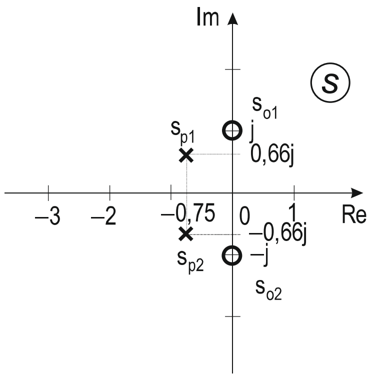
Nule $\qquad s^2+1=0 \;\Rightarrow\; s_{o1,2}=\pm j$

Polovi $\qquad s^2+\frac{3}{2}s+1=0 \Rightarrow s_{p1,2}=-\frac{3}{4}\pm\sqrt{\frac{9}{16}-1}=-\frac{3}{4}\pm j\frac{\sqrt{7}}{4}=-0,75\pm j0,661438$

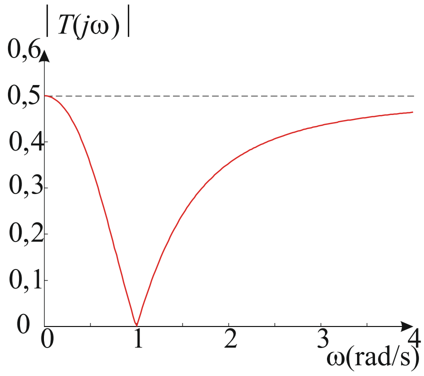
Amplitudno-frekvencijska (A-F) karakteristika (1 bod)

$$s=j\omega\Rightarrow \qquad T(j\omega)=-\frac{1}{2}\cdot\frac{-\omega^2+1}{-\omega^2+\frac{3}{2}\cdot j\omega+1} \;\Rightarrow\; |T(j\omega)|=\frac{1}{2}\cdot\frac{\left|1-\omega^2\right|}{\sqrt{\left(1-\omega^2\right)^2+\left(\frac{3}{2}\cdot\omega\right)^2}}$$

Karakteristične točke A-F karakteristike $|T(j0)|=1/2;\;\; |T(j1)|=0;\;\; |T(j\infty)|=1/2.$

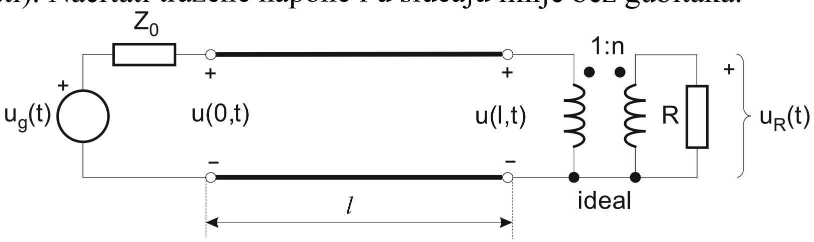


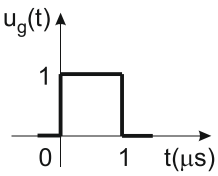
Raspored nula i polova u s-ravnini



Amplitudno frekvencijska karakteristika

5. Zadana je linija s primarnim parametrima R=0,5Ω/km, L=10μH/km, G=2S/km, C=40μF/km, i duljinom l=1km. Na liniju je spojen generator $u_g(t)$ s unutarnjim otporom jednakim zrcalnoj impedanciji linije Z_0 i valnim oblikom prema slici. Koliki mora biti n idealnog transformatora da bi linija bila prilagođena na izlazu po zrcalnoj impedanciji Z_0? a) Odrediti i nacrtati valne oblike napona na izlazu linije $u(l, t)$ i $u_R(t)$ na otporniku R=50Ω. b) kako glase naponi $u(l, t)$ i $u_R(t)$, ako je linija bez gubitaka (primarni parametri: R=0 i G=0, L i C su isti). Nacrtati tražene napone i u slučaju linije bez gubitaka.





<u>Rješenje:</u>

a) $\frac{R}{L}=\frac{G}{C}\Rightarrow\frac{0,5[\Omega/\text{km}]}{10[\mu\text{H/km}]}=\frac{2[\text{S/km}]}{40[\mu\text{F/km}]}$ linija bez distorzije

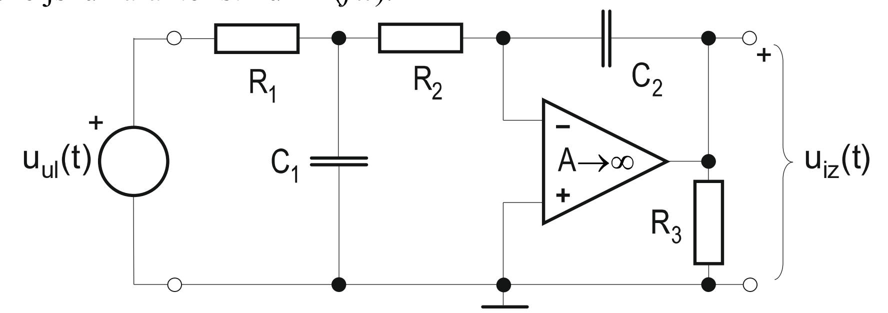
$$Z_0=\sqrt{\frac{L}{C}}=\sqrt{\frac{10\cdot10^{-6}}{40\cdot10^{-6}}}=\frac{1}{2}$$

$$Z_2=\frac{R}{n^2}=Z_0=\frac{1}{2}\Rightarrow n^2=\frac{R}{Z_2}=2\cdot50=100\Rightarrow n=10$$

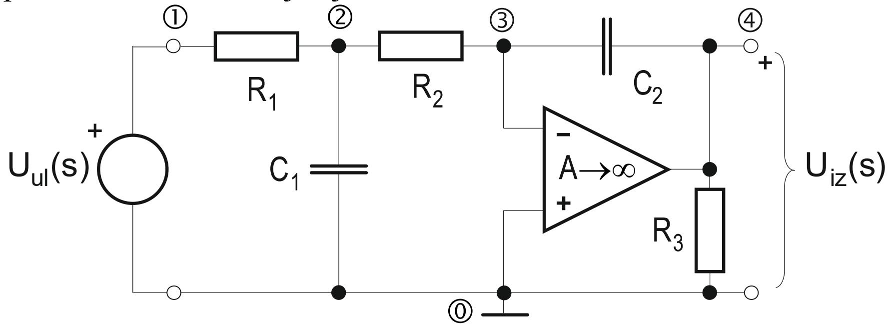
$\gamma=\sqrt{RG}+s\sqrt{LC}=1+20\cdot10^{-6}s$ (1 bod)

$$l=1[\text{km}] \;\Rightarrow\; g=\gamma\cdot l=1+20\cdot10^{-6}s=1+Ts;\; T=20[\mu\text{s}]$$

4. Zadan je aktivni-RC električni filtar prikazan slikom s normaliziranim vrijednostima elemenata R_1=1/2, C_1=2, R_2=1/2, R_3=2, C_2=2. Operacijsko pojačalo je idealno. a) Izračunati njegovu naponsku prijenosnu funkciju $T(s)=U_{iz}(s)/U_{ul}(s)$. b) Prikazati raspored polova i nula u kompleksnoj ravnini. c) Nacrtati amplitudno-frekvencijsku karakteristiku $|T(j\omega)|$. d) Nacrtati fazno-frekvencijsku karakteristiku $\angle T(j\omega)$.



Rješenje: Laplaceova transformacija i jednadžbe čvorišta:



Jednadžbe čvorišta

$$(1)\ -U_1\frac{1}{R_1}+U_2\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_1\right)-U_3\frac{1}{R_2}=0$$

$$(2)\ -U_2\frac{1}{R_2}+U_3\left(\frac{1}{R_2}+sC_2\right)-U_4 sC_2=0$$

$$(3)\ \underline{U_3=0}\ \text{prividni kratki spoj}$$

$$(1)\ U_1=U_2\cdot R_1\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_1\right)\Rightarrow U_1=\left(-U_4\cdot sR_2C_2\right)\cdot R_1\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_1\right)$$

$$(2)\ \underline{U_2=-U_4\cdot sR_2C_2}$$

te uz $U_1=U_{ul}$; $U_4=U_{iz}$

$$U_{ul}=-U_{iz}\cdot sR_2C_2\cdot\left(1+\frac{R_1}{R_2}+sR_1C_1\right)$$

$$U_{ul}=-U_{iz}\cdot\left(sR_2C_2+sR_1C_2+sR_2C_2\cdot sR_1C_1\right)$$

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=-\frac{1}{s\left(R_1C_2+R_2C_2\right)+s^2R_1C_1R_2C_2}$$

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=-\frac{\dfrac{1}{R_1C_1R_2C_2}}{s^2+s\dfrac{R_1C_2+R_2C_2}{R_1C_1R_2C_2}}=-\frac{\dfrac{1}{R_1C_1R_2C_2}}{s\cdot\left(s+\dfrac{R_1C_2+R_2C_2}{R_1C_1R_2C_2}\right)}$$ **(2 boda)**

Uvrstimo vrijednosti: R_1=1/2, C_1=2, R_2=1/2, C_2=2.

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=-\frac{1}{s(s+2)}$$

Raspored polova i nula u kompleksnoj ravnini: **(1 bod)**

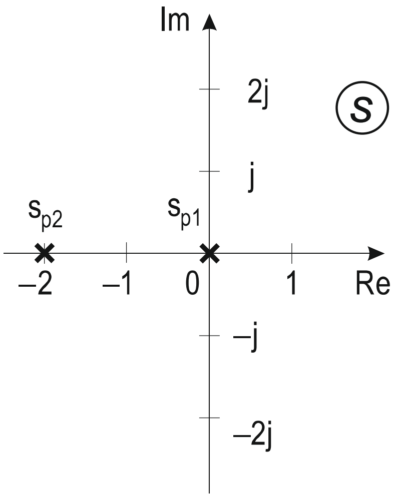
Nule dvije nule u neizmjerno

Polovi $s(s+1)=0 \Rightarrow s_{p1}=0; s_{p2}=-2$

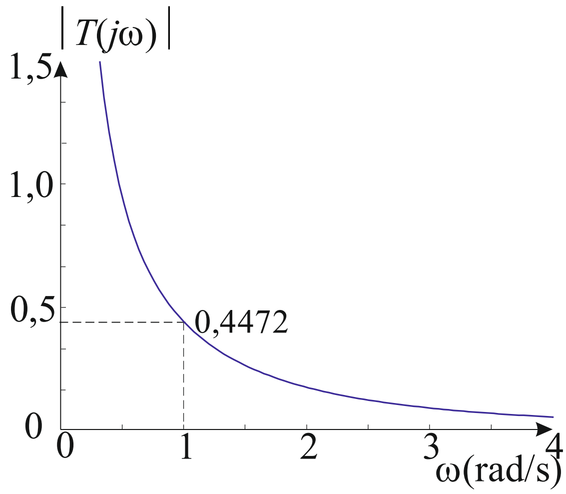
Amplitudno-frekvencijska (A-F) karakteristika **(1 bod)**

$$s{=}j\omega \Rightarrow \qquad T(j\omega)=-\frac{1}{j\omega(j\omega+2)} \Rightarrow |T(j\omega)|=\frac{1}{\omega\sqrt{\omega^2+4}}$$

Karakteristične točke A-F karakteristike $|T(j0)|=\infty;\ |T(j1)|=\frac{1}{\sqrt{5}}=0,4472;\ |T(j\infty)|=0.$



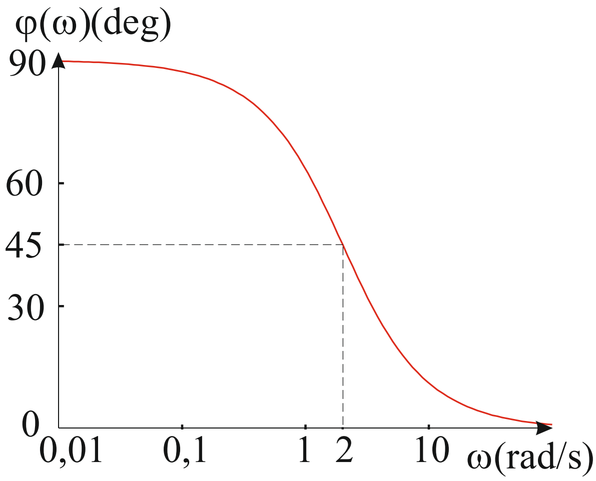
Raspored nula i polova u *s*-ravnini



Amplitudno frekvencijska karakteristika

Fazno-frekvencijska (F-F) karakteristika

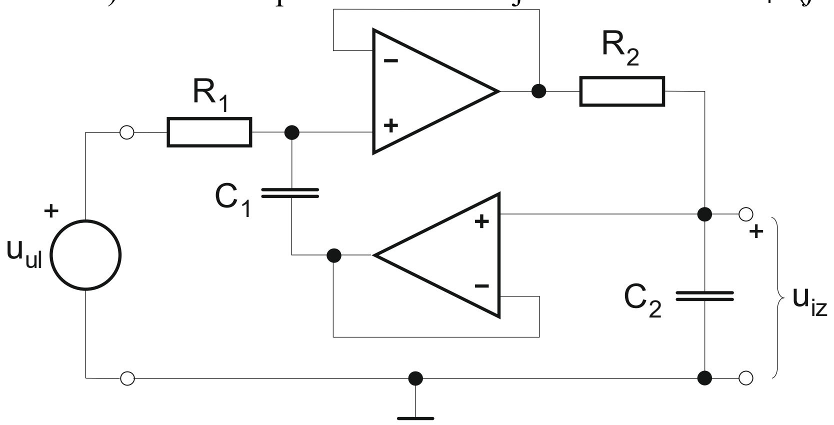
$$\varphi(\omega)=\angle T(j\omega)=\angle-\frac{1}{j\omega(j\omega+2)}=\angle(-1)-\angle(j\omega)-\angle(j\omega+2)=\pi-\frac{\pi}{2}-\arctan\frac{\omega}{2}=\frac{\pi}{2}-\arctan\frac{\omega}{2}$$



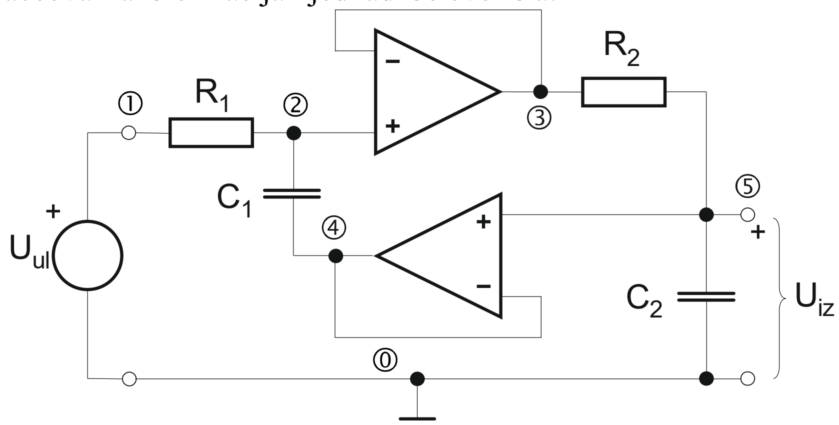
Fazno frekvencijska karakteristika

(1 bod)

4. Zadan je aktivni-RC električni filtar prikazan slikom s normaliziranim vrijednostima elemenata R_1=1/2, C_1=2, R_2=1, C_2=1. a) Izračunati njegovu naponsku prijenosnu funkciju $T(s)=U_{iz}(s)/U_{ul}(s)$. Uvrstiti odmah elemente u proračun. b) Prikazati raspored polova i nula u kompleksnoj ravnini. c) Nacrtati amplitudno-frekvencijsku karakteristiku $|T(j\omega)|$.



Rješenje: Laplaceova transformacija i jednadžbe čvorišta:



Jednadžbe čvorišta

$$(1)\ -U_1\frac{1}{R_1}+U_2\left(\frac{1}{R_1}+sC_1\right)-U_4 sC_1=0$$

$$(2)\ U_3=U_2$$

$$(3)\ U_4=U_5$$

$$(4)\ U_5\left(\frac{1}{R_2}+sC_2\right)-U_3\frac{1}{R_2}=0$$

$$(1)\ -U_1\frac{1}{R_1}+U_2\left(\frac{1}{R_1}+sC_1\right)-U_5 sC_1=0$$

$$(4)\ U_5\left(\frac{1}{R_2}+sC_2\right)-U_2\frac{1}{R_2}=0 \Rightarrow U_5\left(1+sR_2C_2\right)=U_2 \rightarrow (1)$$

$$-U_1\frac{1}{R_1}+U_5\left(1+sR_2C_2\right)\left(\frac{1}{R_1}+sC_1\right)-U_5 sC_1=0$$

$$U_5\left(1+sR_2C_2\right)\left(1+sR_1C_1\right)-U_5 sR_1C_1=U_1$$

$$U_5\left[1+sR_1C_1+sR_2C_2+s^2R_1C_1R_2C_2-sR_1C_1\right]=U_1$$

$$T(s)=\frac{U_5(s)}{U_1(s)}=\frac{1}{1+sR_2C_2+s^2R_1C_1R_2C_2}$$

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{U_5(s)}{U_1(s)}=\frac{\frac{1}{R_1C_1R_2C_2}}{s^2+s\frac{1}{R_1C_1}+\frac{1}{R_1C_1R_2C_2}}$$

Uvrstimo vrijednosti: R_1=1/2, C_1=2, R_2=1, C_2=1.

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{U_5(s)}{U_1(s)}=\frac{1}{s^2+s+1}$$ **(2 boda)**

Raspored polova i nula u kompleksnoj ravnini: **(1 bod)**

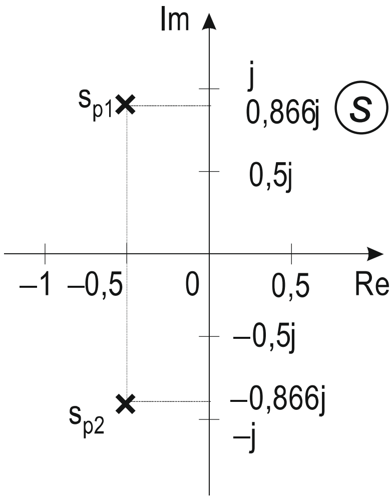
Nule dvije nule u neizmjerno

Polovi $s^2+s+1=0 \Rightarrow s_{p1,2}=-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}=-0,5\pm j0,866$

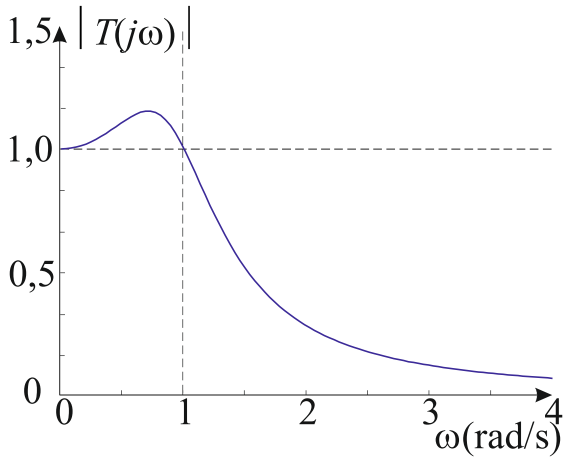
Amplitudno-frekvencijska (A-F) karakteristika **(1 bod)**

$s=j\omega\Rightarrow \qquad T(j\omega)=\frac{1}{-\omega^2+j\omega+1} \Rightarrow |T(j\omega)|=\frac{1}{\sqrt{(1-\omega^2)^2+\omega^2}}$

Karakteristične točke A-F karakteristike $|T(j0)|=1;\ |T(j1)|=1;\ |T(j\infty)|=0.$

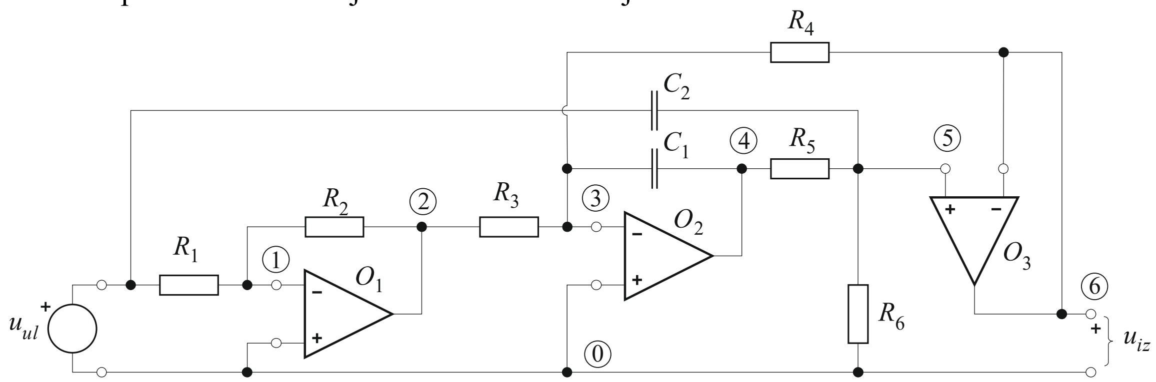


Raspored nula i polova u *s*-ravnini
(1 bod)



Amplitudno frekvencijska karakteristika

4. Zadan je aktivni-RC električni filtar prikazan slikom s normaliziranim vrijednostima elemenata $C_1=C_2=2$, $R_1=R_2=1$, $R_3=1/9$, $R_4=1/4$, $R_5=R_6=1$. a) Izračunati njegovu naponsku prijenosnu funkciju $T(s)=U_{iz}(s)/U_{ul}(s)$. O kojem se tipu filtra radi (NP, VP, PP ili PB)? b) Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja izračunati vrijednost parametara k, ω_p, ω_z, q_p. c) Prikazati raspored polova i nula u kompleksnoj ravnini. d) Nacrtati amplitudno-frekvencijsku i fazno-frekvencijsku karakteristiku.



Rješenje: Primjenom Laplaceove transformacije:

Metoda napona čvorišta: **(1 bod)**

$$(1)\ U_1\left(\frac{1}{R_1}+\frac{1}{R_2}\right)-U_2\frac{1}{R_2}=U_{ul}\frac{1}{R_1}$$

$$(2)\ U_3\left(\frac{1}{R_3}+\frac{1}{R_4}+sC_1\right)-U_6\frac{1}{R_4}=U_2\frac{1}{R_3}+U_4sC_1$$

$$(3)\ U_5\left(sC_2+\frac{1}{R_5}+\frac{1}{R_6}\right)=sC_2U_{ul}+U_4\frac{1}{R_5}$$

$$(4)\ U_{iz}=U_5=U_6$$

$$U_1=U_3=0$$

Slijedi postepeno računanje korak po korak:

$$(1)\qquad U_2=-\frac{R_2}{R_1}U_{ul}$$

$$(2)\qquad -U_{iz}\frac{1}{R_4}=-\frac{R_2}{R_1}\frac{1}{R_3}U_{ul}+U_4sC_1\Rightarrow U_4=\frac{R_2}{sR_1R_3C_1}U_{ul}-\frac{1}{sR_4C_1}U_{iz}$$

$$(3)\qquad U_{iz}\left(sC_2+\frac{1}{R_5}+\frac{1}{R_6}\right)=sC_2U_{ul}+\frac{1}{R_5}\left(\frac{R_2}{sR_1R_3C_1}U_{ul}-\frac{1}{sR_4C_1}U_{iz}\right)$$

$$U_{iz}\left(sC_2+\frac{1}{R_5}+\frac{1}{R_6}+\frac{1}{sR_4R_5C_1}\right)=U_{ul}\left(sC_2+\frac{R_2}{sR_1R_3R_5C_1}\right)$$

Konačno je: **(1 bod)**

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{s^2+\dfrac{R_2}{R_1R_3R_5C_1C_2}}{s^2+s\dfrac{1}{C_2}\dfrac{R_5+R_6}{R_5R_6}+\dfrac{1}{R_4R_5C_1C_2}}=\frac{s^2+2,25}{s^2+s+1}$$

Usporedba s općim oblikom:

$$T_{PB}(s)=k\cdot\frac{s^2+\omega_z^2}{s^2+(\omega_p/q_p)s+\omega_p^2}\ \rightarrow\text{Pojasna brana}$$

Parametri: **(1 bod)**

$$k=1$$

$$\omega_p=\sqrt{\frac{1}{R_4R_5C_1C_2}}=1$$

$$\omega_z=\sqrt{\frac{R_2}{R_1R_3R_5C_1C_2}}=\frac{3}{2}$$

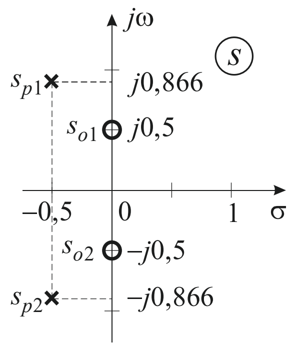
$$\frac{\omega_p}{q_p}=\frac{1}{C_2}\frac{R_5+R_6}{R_5R_6}\ \Rightarrow\ q_p=\sqrt{\frac{R_5C_2}{R_4C_1}}\frac{R_6}{R_5+R_6}=1$$

Raspored nula i polova u s-ravnini: **(1 bod)**

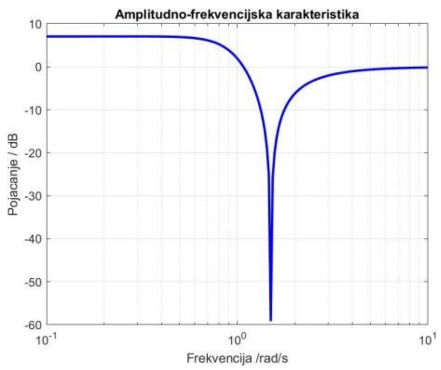
Nule: $s^2+2,25=0\rightarrow s_{o1,2}=\pm j\dfrac{3}{2}=\pm j0,5$

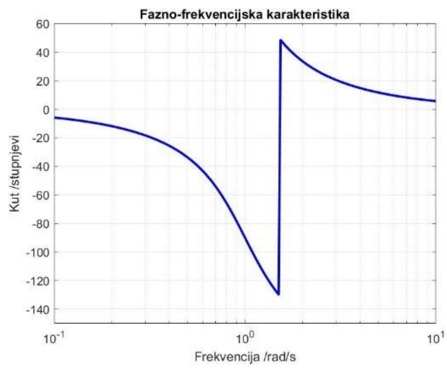
Polovi

$$s^2+s+1=0\rightarrow s_{p1,2}=\frac{-1\pm\sqrt{1-4}}{2}=-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}=-0,5\pm j0,866$$



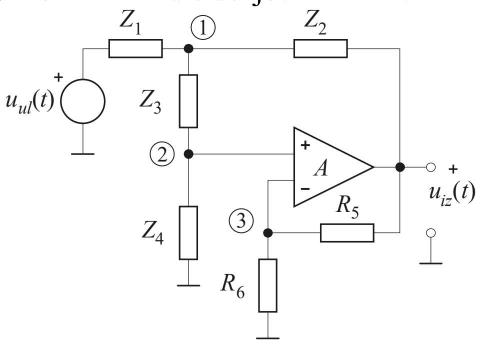
Skica A-F i F-F karakteristike: **(1 bod)**





4. Za aktivni električni filtar prikazan slikom odrediti i izračunati:
a) naponsku prijenosnu funkciju $T(s)$. Operacijsko pojačalo je idealno: $A\to\infty$.
b) prijenosne funkcije $T_1(s)$ i $T_2(s)$ ako impedancije zamijenite elementima kako je to zadano u Tablici. Odrediti o kojim se tipovima filtara radi u slučajevima 1. i 2 (NP, VP, PP, PB).
c) izračunati i prikazati rasporede polova i nula u kompleksnoj ravnini (opisati u čemu je razlika), te skicirati amplitudno-frekvencijske karakteristike tih filtra u slučajevima 1 i 2.

	Filtar 1	Filtar 2
$Z_1 = Z_3 =$	$R = 1/2$	$C = 1/2$
$Z_2 = Z_4 =$	$C = 1$	$R = 1$
$R_5 = R_6 =$	$R_0 = 1$	$R_0 = 1$



Rješenje:

$$(1)\quad U_1\cdot\left(\frac{1}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}\right)-U_2\cdot\left(\frac{1}{Z_3}\right)-U_{ul}\cdot\left(\frac{1}{Z_1}\right)-U_{iz}\cdot\left(\frac{1}{Z_2}\right)=0$$

$$(2)\quad U_2\cdot\left(\frac{1}{Z_3}+\frac{1}{Z_4}\right)-U_1\cdot\left(\frac{1}{Z_3}\right)=0$$

$$(3)\quad U_3\cdot\left(\frac{1}{R_5}+\frac{1}{R_6}\right)-U_{iz}\cdot\left(\frac{1}{R_5}\right)=0$$

(1 bod)

Rješavanjem sustava jednadžbi:

$$(3)\quad (R_5=R_6)\to U_3=\frac{U_{iz}}{2}$$

$$(2)\quad (U_2=U_3)\to U_1=\frac{Z_3+Z_4}{2Z_4}\cdot U_{iz}$$

$$(1)\quad U_{iz}\cdot\left[\frac{Z_3+Z_4}{2Z_4}\cdot\left(\frac{1}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}\right)-\frac{1}{2Z_3}-\frac{1}{Z_2}\right]=U_{ul}\cdot\left(\frac{1}{Z_1}\right)$$

Opći oblik prijenosne funkcije: **(1 bod)**

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{2Z_2Z_3Z_4}{(Z_3+Z_4)(Z_1Z_2+Z_1Z_3+Z_2Z_3)-Z_1Z_2Z_4-2Z_1Z_3Z_4}$$

Za Parametre 1) vrijedi:

$Z_1 = Z_3 = 1/2;\ Z_2 = Z_4 = 1/s$

$$T_1(s) = \frac{U_{iz1}(s)}{U_{ul}(s)} = \ldots = 2 \cdot \frac{4}{s^2 + 2s + 4}$$

$\rightarrow$ NP karakteristika

Za Parametre 2) vrijedi:

$Z_1 = Z_3 = 2/s;\ Z_2 = Z_4 = 1$

$$T_2(s) = \frac{U_{iz2}(s)}{U_{ul}(s)} = \ldots = 2 \cdot \frac{s^2}{s^2 + 2s + 4}$$

$\rightarrow$ VP karakteristika

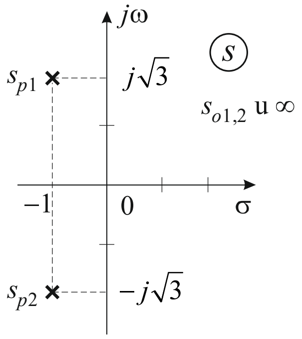
(1 bod)

Raspored nula i polova u s-ravnini:

NP

Nule: $\lim\limits_{s \to \infty} T_1(s) = 0 \rightarrow s_{o1,2} = \infty$

Polovi: $s^2 + 2s + 4 = 0 \rightarrow s_{p1,2} = -1 \pm j\sqrt{3}$

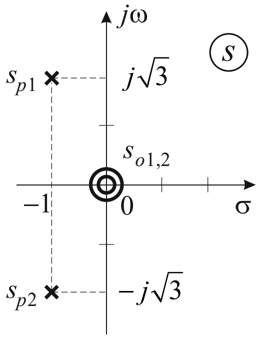


VP

Nule: $s^2 = 0 \rightarrow s_{o1,2} = 0$

Polovi: $s^2 + 2s + 4 = 0 \rightarrow s_{p1,2} = -1 \pm j\sqrt{3}$

(polovi su isti kao kod NP)



(1 bod)

Amplitudno-frekvencijske karakteristike: $s \to j\omega$

NP

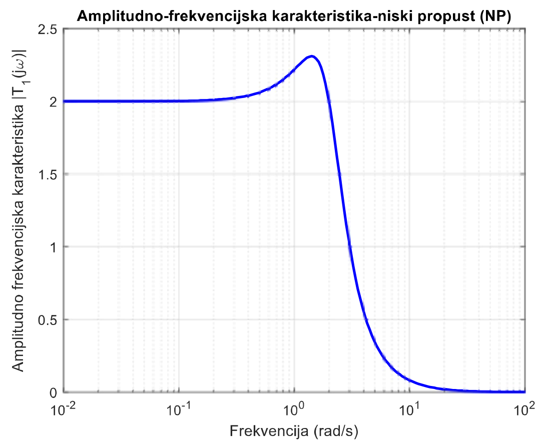
$$T_1(j\omega) = \frac{U_{iz1}(j\omega)}{U_{ul}(j\omega)} = 2 \cdot \frac{4}{-\omega^2 + 2j\omega + 4}$$

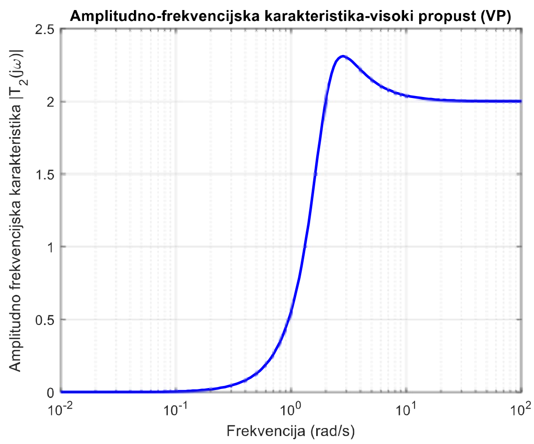
$$|T_1(j\omega)| = 2 \cdot \frac{4}{\sqrt{\left(4-\omega^2\right)^2 + (2\omega)^2}}$$

VP

$$T_2(j\omega) = \frac{U_{iz2}(j\omega)}{U_{ul}(j\omega)} = 2 \cdot \frac{-\omega^2}{-\omega^2 + 2j\omega + 4}$$

$$|T_2(j\omega)| = 2 \cdot \frac{|\omega^2|}{\sqrt{\left(4-\omega^2\right)^2 + (2\omega)^2}}$$

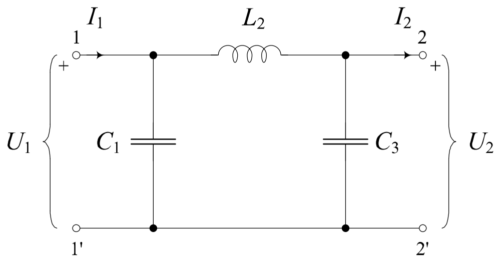


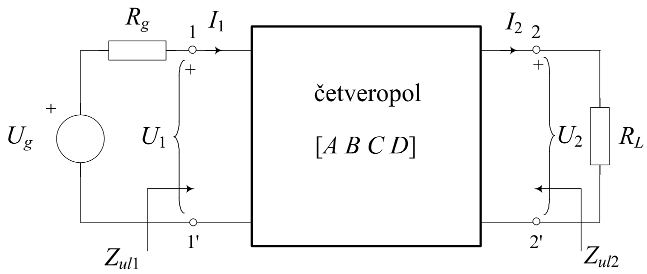


(1 bod)

ČETVEROPOLI

3. Za Π-četveropol prikazan lijevom slikom izračunati prijenosne a-parametre. a) Napisati parametre A, B, C i D pomoću C_1, L_2 i C_3 te uvrstiti normirane vrijednosti elemenata: C_1=1, L_2=2, C_3=1. Četveropol je spojen u električni krug na desnoj slici. b) Ako je izlazni prilaz (2–2') zaključen otporom R_L=1 pomoću a-parametara izračunati: ulaznu impedanciju $Z_{ul1}(s)=U_1(s)/I_1(s)$ gledano sa priključnica 1–1'; c) ako je R_g=1 izračunati izlaznu impedanciju $Z_{ul2}(s)=-U_2(s)/I_2(s)$ gledano sa priključnica 2–2'; i d) izračunati prijenosnu funkciju napona $H(s)=U_2(s)/U_g(s)$.



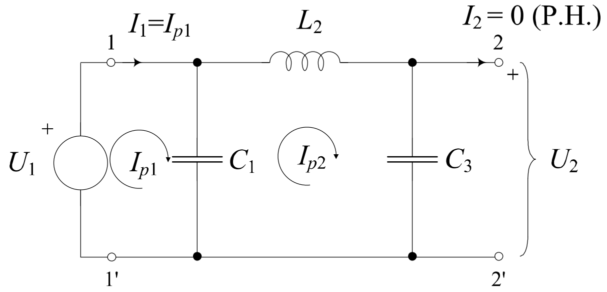


Rješenje:

a) [a]-parametri:

$$U_1 = A \cdot U_2 + B \cdot I_2$$
$$I_1 = C \cdot U_2 + D \cdot I_2$$

$$I_2 = 0 \qquad A = \left.\frac{U_1}{U_2}\right|_{I_2=0}; \quad C = \left.\frac{I_1}{U_2}\right|_{I_2=0}$$



$$\text{(1)}\ \ U_1 = \left(I_{p1} - I_{p2}\right)\frac{1}{sC_1} = I_{p2}\left(sL_2 + \frac{1}{sC_3}\right)$$
$$\text{(2)}\ \ U_2 = I_{p2}\frac{1}{sC_3} \Rightarrow I_{p2} = sC_3U_2$$

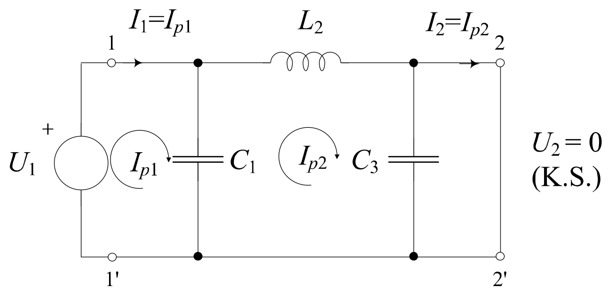
$$\Rightarrow\ A = \frac{U_1}{U_2} = \frac{I_{p2}\left(sL_2 + \frac{1}{sC_3}\right)}{I_{p2}\frac{1}{sC_3}} = s^2L_2C_3 + 1;$$

$$I_{p1} = I_{p2}\left(s^2L_2C_1 + \frac{C_1}{C_3} + 1\right) = sC_3U_2\left(s^2L_2C_1 + \frac{C_1}{C_3} + 1\right) = U_2\left(s^3L_2C_1C_3 + sC_1 + sC_3\right)$$

$$C = \frac{I_1}{U_2} = s^3L_2C_1C_3 + sC_1 + sC_3;$$

A i C **(1 bod)**

$$U_2 = 0 \qquad B = \left.\frac{U_1}{I_2}\right|_{U_2=0}; \quad D = \left.\frac{I_1}{I_2}\right|_{U_2=0}$$



$$(1)\ U_1 = (I_{p1} - I_{p2})\frac{1}{sC_1}$$

$$(2)\ U_1 = I_{p2} sL_2 \Rightarrow I_{p2} = \frac{U_1}{sL_2}$$

$$U_1 = \left(I_{p1} - \frac{U_1}{sL_2}\right)\frac{1}{sC_1} \Rightarrow U_1\left(1 + \frac{1}{s^2 C_1 L_2}\right) = I_{p1}\frac{1}{sC_1} \Rightarrow I_{p1} = U_1\left(sC_1 + \frac{1}{sL_2}\right)$$

$$\Rightarrow\ B = \frac{U_1}{I_2} = sL_2;\quad D = \frac{I_1}{I_2} = \frac{U_1\left(sC_1 + \frac{1}{sL_2}\right)}{\frac{U_1}{sL_2}} = s^2 C_1 L_2 + 1;$$

B i D **(1 bod)**

Ovo su $[a]$-parametri četveropola:

$$[a] = \begin{bmatrix} A & B \\ C & D \end{bmatrix};\quad [a] = \begin{bmatrix} s^2 L_2 C_3 + 1 & s^3 L_2 C_1 C_3 + sC_1 + sC_3 \\ sL_2 & s^2 C_1 L_2 + 1 \end{bmatrix} = \begin{bmatrix} 2s^2 + 1 & 2s^3 + 2s \\ 2s & 2s^2 + 1 \end{bmatrix}$$

b) Ulazna impedancija u četveropol:

Konačno iz jednadžbi

$$\begin{aligned} U_1 &= A \cdot U_2 + B \cdot I_2 \\ I_1 &= C \cdot U_2 + D \cdot I_2 \end{aligned},\quad R_L = \frac{U_2}{I_2},\quad U_g = I_1 R_g + U_1$$

slijede:

$$Z_{ul1}(s) = \frac{U_1}{I_1} = \frac{AU_2 + BI_2}{CU_2 + DI_2} = \frac{A\frac{U_2}{I_2} + B}{C\frac{U_2}{I_2} + D} = \frac{AR_L + B}{CR_L + D};\quad R_L = \frac{U_2}{I_2}$$

$$Z_{ul1}(s) = \frac{(2s^2+1)\cdot 1 + (2s^3 + 2s)}{(2s)\cdot 1 + (2s^2 + 1)} = \frac{2s^3 + 2s^2 + 2s + 1}{2s^2 + 2s + 1}$$ **(1 bod)**

c) Izlazna impedancija iz četveropola: pošto je četveropol recipročan det($[a]$)=$AD-BC$=1 i simetričan (A=D) vrijedi:

$$Z_{ul2}(s)\big|_{U_g=0} = -\frac{U_2}{I_2} = \frac{DU_1 - BI_1}{CU_1 - AI_1} = \frac{D\frac{U_1}{I_1} - B}{C\frac{U_1}{I_1} - A} = \frac{DR_g + B}{CR_g + A};\quad R_g = -\frac{U_1}{I_1}$$

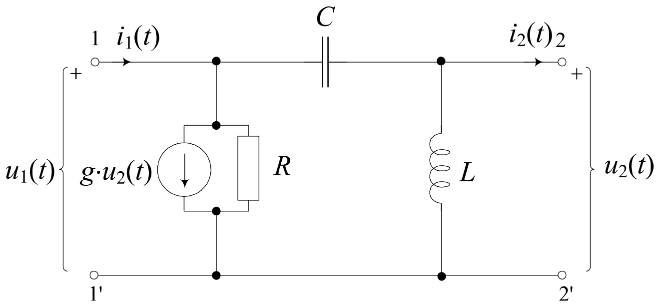
$$\Rightarrow Z_{ul2}(s) = Z_{ul1}(s) = \frac{2s^3 + 2s^2 + 2s + 1}{2s^2 + 2s + 1}$$ **(1 bod)**

d) Prijenosna funkcija napona:

$$U_g = I_1 R_g + U_1 = \left(CU_2 + D\frac{U_2}{R_L}\right)R_g + AU_2 + B\frac{U_2}{R_L} \ \Rightarrow\ H(s) = \frac{U_2}{U_g} = \frac{R_L}{AR_L + B + R_g(CR_L + D)}$$

$$H(s) = \frac{1}{(2s^2+1)\cdot 1 + 2s^3 + 2s + 1\cdot\left[(2s)\cdot 1 + 2s^2 + 1\right]} = \frac{\frac{1}{2}}{s^3 + 2s^2 + 2s + 1}$$ **(1 bod)**

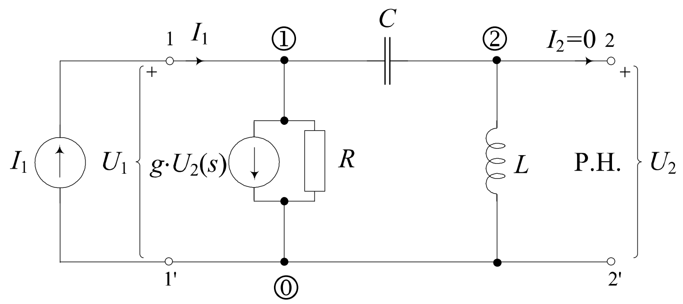
3. Za Π-četveropol prikazan slikom izračunati z-parametre. Napisati: a) parametre z_{11}, z_{21}, z_{12} i z_{22} (izraziti z-parametre pomoću R, L, C i g). b) Izračunati z-parametre uz uvrštene vrijednosti elemenata: R=1, L=1, C=1 i g=2. c) Pomoću z-parametara izračunati strujnu prijenosnu funkciju četveropola $H_i(s)=I_2(s)/I_1(s)$ ako je izlazni prilaz (2–2') zaključen otporom R_L=1; d) Izračunati ulaznu impedanciju u četveropol $Z_{ul1}(s)=U_1(s)/I_1(s)$ ako je $R_L=\infty$. e) Da li je četveropol recipročan, simetričan? Obrazložiti.



Rješenje: Najjednostavnije je izračunati $[z]$-parametre pomoću napona čvorova (pretpostavimo strujne izvore I_1 i I_2).

a) $[z]$-parametri, definicijske (naponske) jednadžbe četveropola:
$$U_1 = z_{11}I_1 - z_{12}I_2$$
$$U_2 = z_{21}I_1 - z_{22}I_2$$

I_2=0 (na prilazu 2–2' prazni hod) ⇒ parametri z_{11} i z_{21}:



$$(1)\ U_1\cdot\left(\frac{1}{R}+sC\right)-U_2\cdot sC = I_1 - g\cdot U_2 \Rightarrow U_1\cdot\left(\frac{1}{R}+sC\right)-U_2\cdot(sC-g)=I_1$$

$$(2)\ -U_1\cdot sC + U_2\cdot\left(sC+\frac{1}{sL}\right)=0 \Rightarrow U_2 = U_1\cdot\frac{s^2LC}{s^2LC+1} \Rightarrow U_1 = U_2\cdot\frac{s^2LC+1}{s^2LC}$$

$$(2)\to(1)\Rightarrow U_1\cdot\left(\frac{1}{R}+sC\right)-U_1\cdot\frac{s^2LC}{s^2LC+1}\cdot(sC-g)=I_1$$

$$z_{11}=\left.\frac{U_1}{I_1}\right|_{I_2=0}=\frac{1}{\left(\frac{1}{R}+sC\right)-\frac{s^2LC}{s^2LC+1}\cdot(sC-g)};$$

Nakon malo sređivanja:

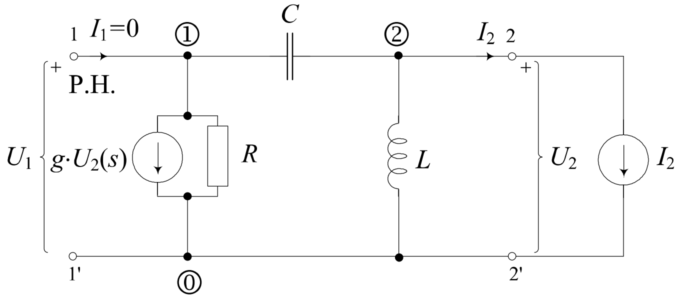
$$z_{11}=\frac{1}{(sRC+1)/R-s^2LC\cdot(sC-g)/(s^2LC+1)}=\frac{R(s^2LC+1)}{(sRC+1)(s^2LC+1)-s^2RLC\cdot(sC-g)}$$

$$z_{11}=\frac{R(s^2LC+1)}{s^3RLC^2+s^2LC+sRC+1-s^3RLC^2+s^2gRLC}=\frac{R(s^2LC+1)}{s^2LC(1+gR)+sRC+1}$$

$$(2)\to(1)\Rightarrow U_2\cdot\frac{s^2LC+1}{s^2LC}\cdot\left(\frac{1}{R}+sC\right)-U_2\cdot(sC-g)=I_1$$

$$z_{21}=\left.\frac{U_2}{I_1}\right|_{I_2=0}=\frac{1}{\frac{s^2LC+1}{s^2LC}\cdot\left(\frac{1}{R}+sC\right)-(sC-g)}=\frac{s^2RLC}{s^2LC(1+gR)+sRC+1}$$

$\underline{I_1=0}$ (na prilazu 1–1' prazni hod) $\Rightarrow$ parametri z_{12} i z_{22}:



$$(1)\ U_1\cdot\left(\frac{1}{R}+sC\right)-U_2\cdot sC=-g\cdot U_2\Rightarrow U_1\cdot\left(\frac{1}{R}+sC\right)-U_2\cdot(sC-g)=0$$

$$(2)\ -U_1\cdot sC+U_2\cdot(sC+1/sL)=-I_2$$

$$(1)\Rightarrow U_1=U_2\cdot\frac{sC-g}{1/R+sC}\Rightarrow U_2=U_1\cdot\frac{1/R+sC}{sC-g}\rightarrow(2)-U_1\cdot sC+U_1\cdot\frac{1/R+sC}{sC-g}\cdot\left(sC+\frac{1}{sL}\right)=-I_2$$

$$z_{12}=-\left.\frac{U_1}{I_2}\right|_{I_1=0}=\frac{1}{-sC+\frac{1/R+sC}{sC-g}\cdot\left(sC+\frac{1}{sL}\right)}=\frac{sRL(sC-g)}{s^2LC(1+gR)+sRC+1}$$

$$(1)\rightarrow(2)\Rightarrow -U_2\cdot\frac{sC-g}{1/R+sC}\cdot sC+U_2\cdot\left(sC+\frac{1}{sL}\right)=-I_2$$

$$z_{22}=-\left.\frac{U_2}{I_2}\right|_{I_1=0}=\frac{1}{-\frac{sC-g}{1/R+sC}\cdot sC+\left(sC+\frac{1}{sL}\right)}=\frac{sL(sRC+1)}{s^2LC(1+gR)+sRC+1}$$

$$[z]=\begin{bmatrix}z_{11} & -z_{12}\\ z_{21} & -z_{22}\end{bmatrix}=\frac{1}{s^2LC(1+gR)+sRC+1}\begin{bmatrix}R(s^2LC+1) & -sRL(sC-g)\\ s^2RLC & -sL(sRC+1)\end{bmatrix}$$ **(1 bod)**

b) $[z]$-parametri uz uvrštene vrijednosti elemenata: R=1, L=1, C=1 i g=2. **(1 bod)**

$$[z]=\begin{bmatrix}z_{11} & -z_{12}\\ z_{21} & -z_{22}\end{bmatrix}=\frac{1}{3s^2+s+1}\begin{bmatrix}s^2+1 & -s(s-2)\\ s^2 & -s(s+1)\end{bmatrix}$$

c) Prijenosna funkcija struje ako je R_L=1:

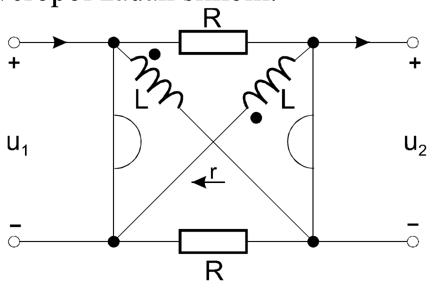
$$H_i(s)=\frac{I_2}{I_1}=\frac{z_{21}}{Z_L+z_{22}}=\frac{s^2/(3s^2+s+1)}{1+s(s+1)/(3s^2+s+1)}=\frac{s^2}{4s^2+2s+1}$$ **(1 bod)**

d) Ulazna impedancija u četveropol ako je $R_L=\infty$ (odn. prazni hod na izlazu ili izlaz „neopterećen“):

$$Z_{ul1}(s)=\frac{U_1}{I_1}=z_{11}-\frac{z_{12}z_{21}}{Z_L+z_{22}}=(Z_L=\infty)=z_{11}=\frac{s^2+1}{3s^2+s+1}$$ **(1 bod)**

e) Četveropol nije recipročan jer ne vrijedi z_{21}=z_{12}, odn. $z_{21}=\frac{s^2}{3s^2+s+1}\neq\frac{s^2+2s}{3s^2+s+1}=z_{12}$, i nije simetričan jer ne vrijedi z_{11}=z_{22}, odn. $z_{11}=\frac{s^2+1}{3s^2+s+1}\neq\frac{s^2+s}{3s^2+s+1}=z_{22}$. **(1 bod)**

3. Odrediti *y*-parametre za četveropol zadan slikom.



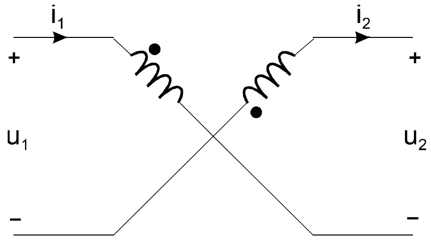
Rješenje: Strujne jednadžbe četveropola:

$$I_1 = y_{11}U_1 - y_{12}U_2 \qquad \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & -y_{12} \\ y_{21} & -y_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$I_2 = y_{21}U_1 - y_{22}U_2$$

Paralelni spoj 3 četveropola:

1. četveropol



$$U_1 = I_1 sL - U_2 + I_1 sL - I_1 2sM$$

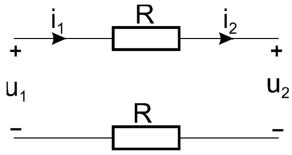
$$U_1 = -I_2 sL - U_2 - I_2 sL + I_2 2sM$$

$$I_1 = \frac{1}{2s(L-M)}U_1 + \frac{1}{2s(L-M)}U_2$$

$$I_2 = -\frac{1}{2s(L-M)}U_1 - \frac{1}{2s(L-M)}U_2$$

$$[\mathrm{y}]_I = \begin{bmatrix} \dfrac{1}{2s(L-M)} & \dfrac{1}{2s(L-M)} \\ -\dfrac{1}{2s(L-M)} & -\dfrac{1}{2s(L-M)} \end{bmatrix}$$

2. četveropol



$$U_1 = I_1 R + U_2 + I_1 R$$

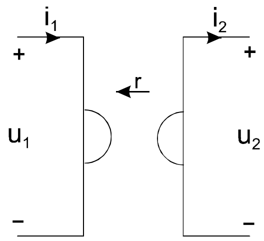
$$U_1 = I_2 R + U_2 + I_2 R$$

$$I_1 = \frac{1}{2R}U_1 - \frac{1}{2R}U_2$$

$$I_2 = \frac{1}{2R}U_1 - \frac{1}{2R}U_2$$

$$[\mathrm{y}]_{II} = \begin{bmatrix} \dfrac{1}{2R} & -\dfrac{1}{2R} \\ \dfrac{1}{2R} & -\dfrac{1}{2R} \end{bmatrix}$$

3. četveropol



$$U_2 = rI_1$$
$$U_1 = rI_2$$

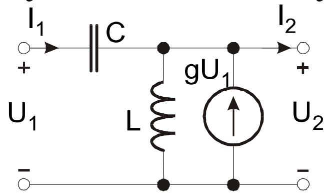
$$I_1 = 0U_1 + \frac{1}{r}U_2$$
$$I_2 = \frac{1}{r}U_1 + 0U_2$$

$$[\mathrm{y}]_{III} = \begin{bmatrix} 0 & \frac{1}{r} \\ \frac{1}{r} & 0 \end{bmatrix}$$

Ukupni y-parametri:

$$[y] = [y]_I + [y]_{II} + [y]_{III} = \begin{bmatrix} \frac{1}{2s(L-M)} + \frac{1}{2R} & \frac{1}{2s(L-M)} - \frac{1}{2R} + \frac{1}{r} \\ -\frac{1}{2s(L-M)} + \frac{1}{2R} + \frac{1}{r} & -\frac{1}{2s(L-M)} - \frac{1}{2R} \end{bmatrix}$$

3. Za četveropol prikazan slikom izračunati prijenosne [a] parametre i iz njih [y]-parametre. Da li je četveropol recipročan i da li je simetričan? Zadano je g=1, C=1, L=2.

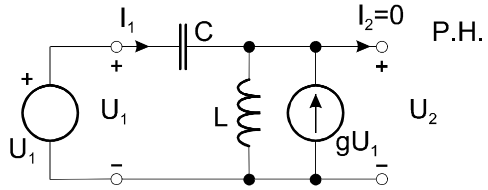


Rješenje:

$$U_1 = AU_2 + BI_2$$
$$\underline{I_1 = CU_2 + DI_2}$$

$$\underline{I_2 = 0} \Rightarrow A, C$$

$$A = \left.\frac{U_1}{U_2}\right|_{I_2=0}, \quad C = \left.\frac{I_1}{U_2}\right|_{I_2=0}$$



$$U_2\frac{1}{sL} = \underbrace{\frac{U_1 - U_2}{\frac{1}{sC}}}_{=I_1} + gU_1, \quad \Rightarrow \quad U_2\left(\frac{1}{sL} + sC\right) = U_1 sC + gU_1$$

$$U_2\left(\frac{1}{sL} + sC\right) = U_1(g + sC) \Rightarrow U_1 = \frac{\frac{1}{sL} + sC}{g + sC}U_2$$

$$A = \left.\frac{U_1}{U_2}\right|_{I_2=0} = \frac{\frac{1}{sL} + sC}{g + sC}$$

$$U_2\frac{1}{sL} = I_1 + g\overbrace{\frac{\frac{1}{sL} + sC}{g + sC}U_2}^{U_1}, \quad \Rightarrow \quad U_2\left(\frac{1}{sL} - \frac{g\frac{1}{sL} + gsC}{g + sC}\right) = I_1$$

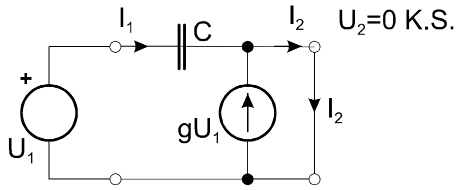
$$U_2\frac{\frac{1}{sL}g + \frac{1}{sL}sC - g\frac{1}{sL} - gsC}{g + sC} = I_1$$

$$U_2\frac{\frac{C}{L} - gsC}{g + sC} = I_1$$

$$C = \left.\frac{I_1}{U_2}\right|_{I_2=0} = \frac{\frac{C}{L} - gsC}{g + sC}$$

$\underline{U_2 = 0} \Rightarrow B, D$

$$B = \left.\frac{U_1}{I_2}\right|_{U_2=0}, \qquad D = \left.\frac{I_1}{I_2}\right|_{U_2=0}$$



$$I_2 = gU_1 + U_1 sC = U_1(g + sC) \Rightarrow U_1 = \frac{I_2}{g + sC}$$

$$B = \left.\frac{U_1}{I_2}\right|_{U_2=0} = \frac{1}{g + sC}$$

$$I_2 = \frac{gI_2}{g + sC} + I_1$$

$$I_2\left(1 - \frac{g}{g + sC}\right) = I_1$$

$$I_2 \frac{g + sC - g}{g + sC} = I_1$$

$$D = \left.\frac{I_1}{I_2}\right|_{U_2=0} = \frac{sC}{g + sC}$$

$$[a_1] = \frac{1}{g + sC}\begin{bmatrix} \frac{1}{sL} + sC & 1 \\ \frac{C}{L} - gsC & sC \end{bmatrix} = \frac{1}{1+s}\begin{bmatrix} \frac{1}{2s} + s & 1 \\ \frac{1}{2} - s & s \end{bmatrix}$$

[y]-parametri:

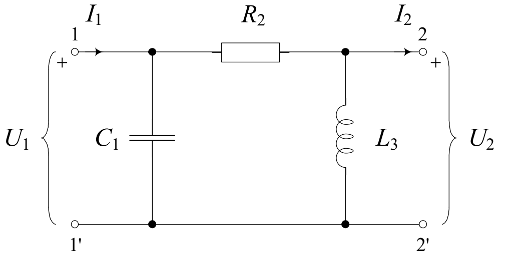
$$y_{11} = \frac{D}{B} = sC, \qquad y_{21} = \frac{1}{B} = g + sC$$

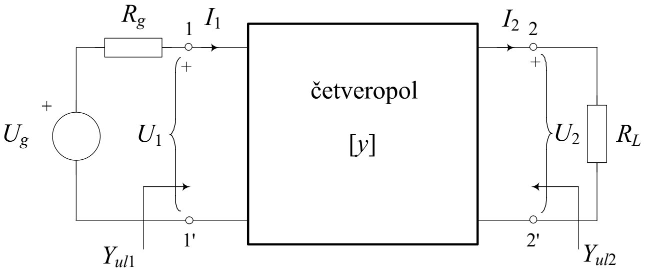
$$y_{12} = \frac{AD - BC}{B} = \frac{\left(\frac{1}{sL} + sC\right)sC - \frac{sC}{sL} + gsC}{(g + sC)^2 \frac{1}{g + sC}} = \frac{\frac{sC}{sL} + s^2C^2 - \frac{sC}{sL} + gsC}{g + sC} = \frac{(g + sC)sC}{g + sC} = sC$$

$$y_{22} = \frac{A}{B} = \frac{1}{sL} + sC$$

$$[\mathrm{y}] = \begin{bmatrix} sC & -sC \\ g + sC & -\left(\frac{1}{sL} + sC\right) \end{bmatrix} = \begin{bmatrix} s & -s \\ 1 + s & -\left(\frac{1}{2s} + s\right) \end{bmatrix}$$

3. Za Π-četveropol prikazan lijevom slikom izračunati y-parametre. a) Napisati y-parametre pomoću C_1, R_2 i L_3 te uvrstiti normirane vrijednosti elemenata: C_1=1, R_2=1/2, L_3=1. Četveropol je spojen u električni krug na desnoj slici. b) Ako je izlazni prilaz (2–2') zaključen otporom R_L=1 pomoću y-parametara izračunati: ulaznu admitanciju $Y_{ul1}(s)=I_1(s)/U_1(s)$ gledano sa priključnica 1–1'; c) ako je R_g=1 izračunati izlaznu admitanciju $Y_{ul2}(s)=-I_2(s)/U_2(s)$ gledano sa priključnica 2–2'; i d) izračunati prijenosnu funkciju napona $H(s)=U_2(s)/U_g(s)$.



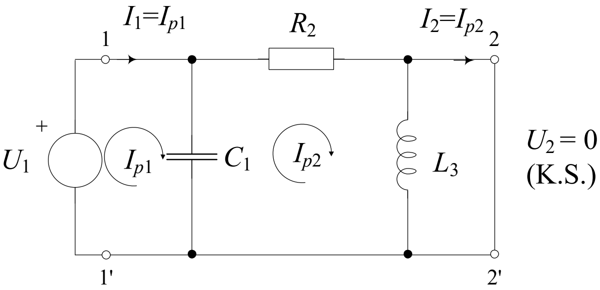


Rješenje:

a) [y]-parametri:

$$I_1 = y_{11} \cdot U_1 - y_{12} \cdot U_2$$
$$I_2 = y_{21} \cdot U_1 - y_{22} \cdot U_2$$

$$U_2 = 0 \qquad y_{11} = \left.\frac{I_1}{U_1}\right|_{U_2=0} \; ; \quad y_{21} = \left.\frac{I_2}{U_1}\right|_{U_2=0}$$



$$(1)\; U_1 = I_{p1}\frac{1}{sC_1} - I_{p2}\frac{1}{sC_1}$$

$$(2)\; 0 = -I_{p1}\frac{1}{sC_1} + I_{p2}\left(\frac{1}{sC_1} + R_2\right)$$

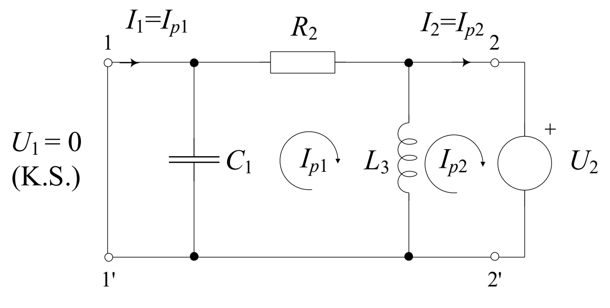
$$(2) \Rightarrow \; I_{p1} = I_{p2}\left(1 + sC_1R_2\right) \; \rightarrow \; (1) \; \Rightarrow$$

$$U_1 = I_{p2}\left(1 + sC_1R_2\right)\frac{1}{sC_1} - I_{p2}\frac{1}{sC_1} = I_{p2}\left[R_2 + \frac{1}{sC_1} - \frac{1}{sC_1}\right] = I_{p2}R_2 \; ;$$

$$U_1 = I_{p1}\frac{1}{sC_1} - \frac{I_{p1}}{1 + sC_1R_2}\cdot\frac{1}{sC_1} = I_{p1}\left[\frac{1}{sC_1} - \frac{1}{1 + sC_1R_2}\cdot\frac{1}{sC_1}\right] = I_{p1}\frac{R_2}{1 + sC_1R_2} = I_{p1}\frac{1}{1/R_2 + sC_1} \; ;$$

$$y_{11} = \left.\frac{I_1}{U_1}\right|_{U_2=0} = \frac{1}{R_2} + sC_1; \quad y_{21} = \left.\frac{I_2}{U_1}\right|_{U_2=0} = \frac{1}{R_2}$$

$$U_1 = 0 \qquad y_{12} = -\left.\frac{I_1}{U_2}\right|_{U_1=0} \; ; \; y_{22} = -\left.\frac{I_2}{U_2}\right|_{U_1=0}$$



$$\begin{aligned}&(1)\ \ 0 = I_{p1}(sL_3 + R_2) - I_{p2}sL_3\\&(2)\ \ -U_2 = -I_{p1}sL_3 + I_{p2}sL_3\end{aligned} \qquad (1)\ \Rightarrow\ I_{p2} = I_{p1}\left(\frac{R_2 + sL_3}{sL_3}\right) \ \to (2)\ \Rightarrow$$

$$-U_2 = -I_{p1}sL_3 + I_{p1}\left(\frac{R_2 + sL_3}{sL_3}\right)sL_3 = I_{p1}R_2$$

$$-U_2 = -I_{p2}\frac{sL_3}{R_2 + sL_3}sL_3 + I_{p2}sL_3 = I_{p2}\left[sL_3 - \frac{sL_3}{R_2 + sL_3}sL_3\right] = I_{p2}\frac{R_2 sL_3}{R_2 + sL_3}$$

$$y_{12} = -\frac{I_1}{U_2}\bigg|_{U_1=0} = \frac{1}{R_2};\ y_{22} = -\frac{I_2}{U_2}\bigg|_{U_1=0} = \frac{1}{R_2} + \frac{1}{sL_3}$$

$$[y] = \begin{bmatrix} y_{11} & -y_{12} \\ y_{21} & -y_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{R_2} + sC_1 & -\frac{1}{R_2} \\ \frac{1}{R_2} & -\left(\frac{1}{R_2} + \frac{1}{sL_3}\right) \end{bmatrix} = \begin{bmatrix} 2+s & -2 \\ 2 & -\left(2 + \frac{1}{s}\right) \end{bmatrix}$$ **(2 boda)**

b) Ulazna admitancija u četveropol:

$$Y_{ul1}(s) = \frac{I_1}{U_1} = y_{11} - \frac{y_{12}y_{21}}{Y_L + y_{22}};\qquad R_L = \frac{1}{Y_L} = \frac{U_2}{I_2}$$

$$= 2 + s - \frac{2\cdot 2}{1 + 2 + 1/s} = 2 + s - \frac{4s}{3s+1} = \frac{(s+2)(3s+1) - 4s}{3s+1} = \frac{3s^2 + 3s + 2}{3s+1}$$

(1 bod)

c) Izlazna admitancija iz četveropola:

$$Y_{ul2}(s) = -\frac{I_2}{U_2} = y_{22} - \frac{y_{12}y_{21}}{Y_g + y_{11}};\quad R_g = \frac{1}{Y_g} = -\frac{U_1}{I_1}$$

$$= 2 + \frac{1}{s} - \frac{2\cdot 2}{1 + 2 + s} = \frac{2s+1}{s} - \frac{4}{s+3} = \frac{(2s+1)(s+3) - 4s}{s(s+3)} = \frac{2s^2 + 3s + 3}{s^2 + 3s};$$

(1 bod)

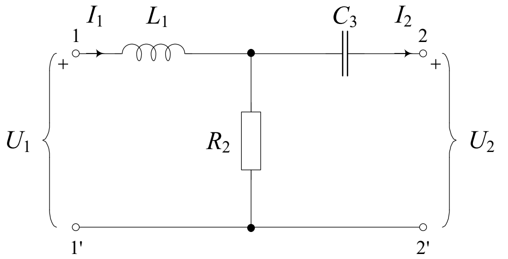
d) Prijenosna funkcija napona: uz $Y_g = 1/R_g = 1; Y_L = 1/R_L = 1$ slijedi:

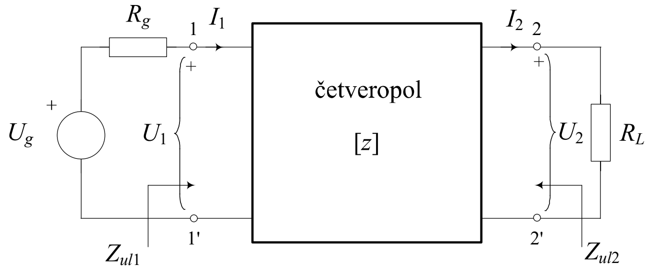
$$H(s) = \frac{U_2}{U_g} = \frac{y_{21}Y_g}{(y_{11} + Y_g)(y_{22} + Y_L) - y_{12}y_{21}} = \frac{2\cdot 1}{(2 + s + 1)(2 + 1/s + 1) - 4} = \frac{2}{(3+s)(3+1/s) - 4}$$

$$H(s) = \frac{U_2}{U_g} = \frac{2}{3/s + 3s + 6} = \frac{2}{3}\cdot\frac{1}{1/s + s + 2} = \frac{2}{3}\cdot\frac{s}{s^2 + 2s + 1} = \frac{2}{3}\cdot\frac{s}{(s+1)^2}$$

(1 bod)

3. Za T-četveropol prikazan lijevom slikom izračunati z-parametre. a) Napisati z-parametre pomoću C_1, R_2 i L_3 te uvrstiti normirane vrijednosti elemenata: L_1=1, R_2=2, C_3=1. Četveropol je spojen u električni krug na desnoj slici. b) Ako je izlazni prilaz (2–2') zaključen otporom R_L=1 pomoću z-parametara izračunati: ulaznu impedanciju $Z_{ul1}(s)=U_1(s)/I_1(s)$ gledano sa priključnica 1–1'; c) ako je R_g=1 izračunati izlaznu impedanciju $Z_{ul2}(s)=-U_2(s)/I_2(s)$ gledano sa priključnica 2–2'; i d) izračunati prijenosnu funkciju napona $H(s)=U_2(s)/U_g(s)$.



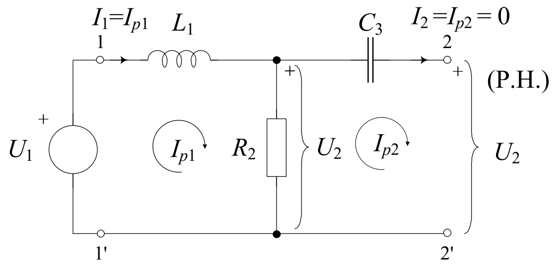


Rješenje:

a) [z]-parametri:

$$U_1 = z_{11} \cdot I_1 - z_{12} \cdot I_2$$
$$U_2 = z_{21} \cdot Z_1 - z_{22} \cdot I_2$$

$$I_2 = 0 \quad z_{11} = \left.\frac{U_1}{I_1}\right|_{I_2=0} \quad ; \quad z_{21} = \left.\frac{U_2}{I_1}\right|_{I_2=0}$$

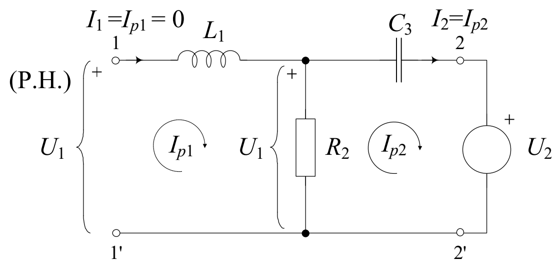


(1) $U_1 = I_{p1}(sL_1 + R_2)$

(2) $U_2 = I_{p1} R_2$

$$(1) \Rightarrow z_{11} = \left.\frac{U_1}{I_1}\right|_{I_2=0} = sL_1 + R_2 \qquad (2) \Rightarrow z_{21} = \left.\frac{U_2}{I_1}\right|_{I_2=0} = R_2$$

$$I_1 = 0 \qquad z_{12} = -\left.\frac{U_1}{I_2}\right|_{I_1=0} \; ; \; z_{22} = -\left.\frac{U_2}{I_2}\right|_{I_1=0}$$



(1) $U_1 = -I_{p2} R_2$

(2) $U_2 = -I_{p2}\left(R_2 + \dfrac{1}{sC_3}\right)$

$$(1) \Rightarrow z_{12} = \left.\frac{U_1}{I_2}\right|_{I_1=0} = R_2 \qquad (2) \Rightarrow z_{22} = -\left.\frac{U_2}{I_2}\right|_{I_1=0} = R_2 + \frac{1}{sC_3}$$

$$[z] = \begin{bmatrix} z_{11} & -z_{12} \\ z_{21} & -z_{22} \end{bmatrix} = \begin{bmatrix} sL_1 + R_2 & -R_2 \\ R_2 & -\left(R_2 + \dfrac{1}{sC_3}\right) \end{bmatrix} = \begin{bmatrix} 2+s & -2 \\ 2 & -\left(2 + \dfrac{1}{s}\right) \end{bmatrix}$$ **(2 boda)**

b) Ulazna impedancija u četveropol:

$$Z_{ul1}(s)=\frac{U_1}{I_1}=z_{11}-\frac{z_{12}z_{21}}{Z_L+z_{22}}\,;\qquad R_L=\frac{U_2}{I_2}$$

$$=2+s-\frac{2\cdot 2}{1+2+1/s}=2+s-\frac{4s}{3s+1}=\frac{(s+2)(3s+1)-4s}{3s+1}=\frac{3s^2+3s+2}{3s+1}$$

(1 bod)

c) Izlazna impedancija iz četveropola:

$$Z_{ul2}(s)=-\frac{U_2}{I_2}=z_{22}-\frac{z_{12}z_{21}}{Z_g+z_{11}}\,;\quad U_g=0;\quad R_g=-\frac{U_1}{I_1}$$

$$=2+\frac{1}{s}-\frac{2\cdot 2}{1+2+s}=\frac{2s+1}{s}-\frac{4}{s+3}=\frac{(2s+1)(s+3)-4s}{s(s+3)}=\frac{2s^2+3s+3}{s^2+3s}\,;$$

(1 bod)

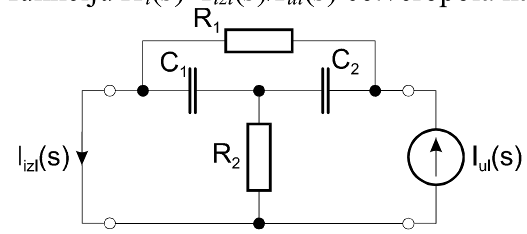
d) Prijenosna funkcija napona: uz $Z_g=R_g=1;\;\; Z_L=R_L=1$ slijedi:

$$H(s)=\frac{U_2}{U_g}=\frac{z_{21}Z_g}{(z_{11}+Z_g)(z_{22}+Z_L)-z_{12}z_{21}}=\frac{2\cdot 1}{(2+s+1)(2+1/s+1)-4}=\frac{2}{(3+s)(3+1/s)-4}$$

$$H(s)=\frac{U_2}{U_g}=\frac{2}{3/s+3s+6}=\frac{2}{3}\cdot\frac{1}{1/s+s+2}=\frac{2}{3}\cdot\frac{s}{s^2+2s+1}=\frac{2}{3}\cdot\frac{s}{(s+1)^2}$$

(1 bod)

3. Naći strujnu prijenosnu funkciju $H_i(s)=I_{izl}(s)/I_{ul}(s)$ četveropola na slici.



Rješenje: Jednadžbe čvorišta: (zadatak se može riješiti i pomoću [y]-parametara)

$$(1)\quad U_1\cdot\left(\frac{1}{R_2}+sC_1+sC_2\right)-U_2\cdot sC_2-U_3\cdot sC_1=0$$

$$(2)\quad -U_1\cdot sC_2+U_2\cdot\left(\frac{1}{R_1}+sC_2\right)-U_3\cdot\frac{1}{R_1}=I_{ul}(s)$$

$$(3)\quad -U_1\cdot sC_1-U_2\cdot\frac{1}{R_1}+U_3\cdot\left(\frac{1}{R_1}+sC_1\right)=-I_{izl}(s)$$

$U_3=0$

$$(1)\quad U_1\cdot\left(\frac{1}{R_2}+sC_1+sC_2\right)-U_2\cdot sC_2=0$$

$$(2)\quad -U_1\cdot sC_2+U_2\cdot\left(\frac{1}{R_1}+sC_2\right)=I_{ul}(s)$$

$$(3)\quad U_1\cdot sC_1+U_2\cdot\frac{1}{R_1}=I_{izl}(s)$$

$(1),(3)\quad\Rightarrow\quad U_1,U_2\quad\rightarrow\quad(2)$

$$\Delta=\begin{vmatrix}\frac{1}{R_2}+sC_1+sC_2 & -sC_2\\ sC_1 & \frac{1}{R_1}\end{vmatrix}=\frac{1}{R_1}\cdot\left(\frac{1}{R_2}+sC_1+sC_2\right)+s^2C_1C_2$$

$$\Delta_1=\begin{vmatrix}0 & -sC_2\\ I_{izl} & \frac{1}{R_1}\end{vmatrix}=sC_2\cdot I_{izl};\qquad \Delta_2=\begin{vmatrix}\frac{1}{R_2}+sC_1+sC_2 & 0\\ sC_1 & I_{izl}\end{vmatrix}=\left(\frac{1}{R_2}+sC_1+sC_2\right)\cdot I_{izl}$$

$$U_1=\frac{\Delta_1}{\Delta};\quad U_2=\frac{\Delta_2}{\Delta}\quad\rightarrow\quad(2):\quad -\frac{\Delta_1}{\Delta}\cdot sC_2+\frac{\Delta_2}{\Delta}\cdot\left(\frac{1}{R_1}+sC_2\right)=I_{ul}(s)$$

$$I_{ul}=\frac{-(sC_2)^2+\left(\frac{1}{R_2}+sC_1+sC_2\right)\cdot\left(\frac{1}{R_1}+sC_2\right)}{\Delta}\cdot I_{izl}$$

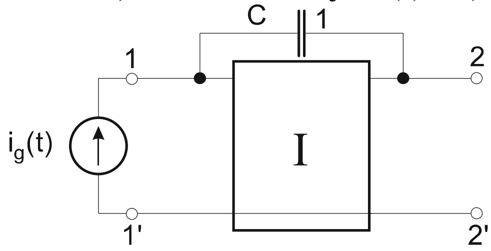
$$H_i(s)=\frac{I_{izl}(s)}{I_{ul}(s)}=\frac{\frac{1}{R_1\cdot R_2}+s\frac{C_1}{R_1}+s\frac{C_2}{R_1}+s^2C_1C_2}{-(sC_2)^2+\frac{1}{R_1\cdot R_2}+s\frac{C_1}{R_1}+s\frac{C_2}{R_1}+s\frac{C_2}{R_2}+s^2C_1C_2+(sC_2)^2}\Bigg/\cdot\ \frac{R_1R_2}{R_1R_2}$$

$$H_i(s)=\frac{s^2R_1R_2C_1C_2+s(C_1R_2+C_2R_2)+1}{s^2R_1R_2C_1C_2+s(C_1R_2+C_2R_2+R_1C_2)+1}$$

ZAVRŠNI ISPIT IZ ELEKTRIČNIH KRUGOVA 2015-2016 – Rješenja

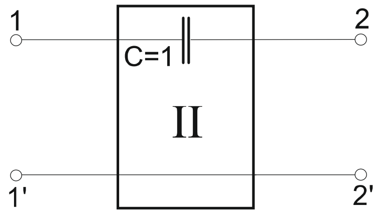
1. Mreža na slici predstavlja paralelni spoj dvaju četveropola. Ako je jedan od njih četveropol I: *a*) nacrtati drugi četveropol. *b*) odrediti *y*-parametre drugog četveropola; *c*) odrediti *y*-parametre paralelne kombinacije dva četveropola. Obzirom na 2–2' odrediti: *d*) Nortonovu struju $I_N(s)$; i e) Nortonovu admitanciju $Y_N(s)$.

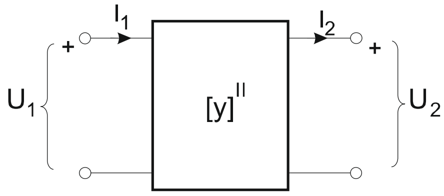
$$[z]=\begin{bmatrix} 2s+1 & -2s \\ 2s & -2s \end{bmatrix}$$



Rješenje:

a) drugi četveropol II:





(1 bod)

b) *y*-parametri drugog četveropola. Naponske jednadžbe za četveropol uz referentne oznake struja i napona:

$$I_1 = y_{11}\cdot U_1 - y_{12}\cdot U_2$$
$$I_2 = y_{21}\cdot U_1 - y_{22}\cdot U_2$$

$U_2 = 0, I_2 = I_1 \Rightarrow$

$y_{11} = I_1 / U_1 = sC = s$

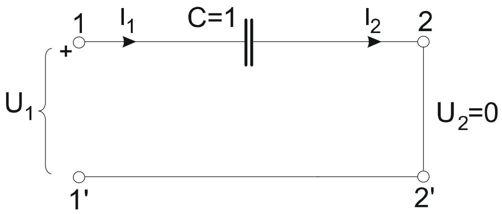
$y_{21} = I_2 / U_1 = sC = s$

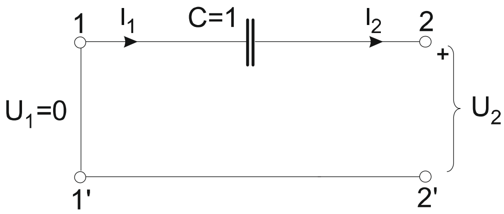
$U_1 = 0, I_2 = I_1 \Rightarrow$

$y_{12} = -I_1 / U_2 = sC = s$

$y_{22} = -I_2 / U_2 = sC = s$

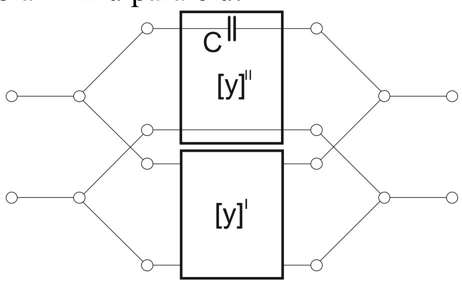
$$\Rightarrow [y]^{II} = \begin{bmatrix} y_{11} & -y_{12} \\ y_{21} & -y_{22} \end{bmatrix} = \begin{bmatrix} s & -s \\ s & -s \end{bmatrix}$$





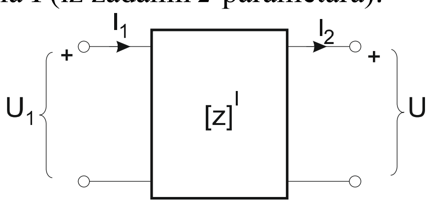
(1 bod)

c) Možemo promatrati dva četveropola I i II u paralelu:



Najprije nađimo *y*-parametre četveropola I (iz zadanih *z*-parametara):

$$U_1 = z_{11}\cdot I_1 - z_{12}\cdot I_2$$
$$U_2 = z_{21}\cdot I_1 - z_{22}\cdot I_2$$

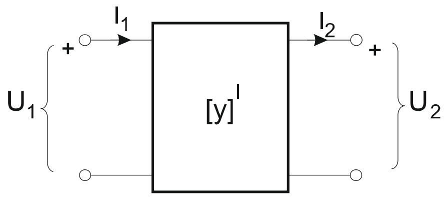


$$U_1 = (2s+1)\cdot I_1 - 2s\cdot I_2$$
$$U_2 = 2s\cdot I_1 - 2s\cdot I_2 \Rightarrow I_2 = I_1 - \frac{1}{2s}U_2$$

$$U_1 = (2s+1)\cdot I_1 - 2s\cdot\left(I_1 - \frac{1}{2s}\cdot U_2\right)$$
$$U_1 = (2s+1)\cdot I_1 - 2s\cdot I_1 + U_2$$

$$U_1 = 1\Omega\cdot I_1 + U_2$$
$$I_1 = \frac{1}{1\Omega}U_1 - \frac{1}{1\Omega}U_2$$

$$I_2 = \frac{1}{1\Omega}U_1 - \frac{1}{1\Omega}U_2 - \frac{1}{2s}\cdot U_2$$
$$I_2 = \frac{1}{1\Omega}U_1 - U_2\left(1+\frac{1}{2s}\right)$$



$$[y]^I = \begin{bmatrix} 1 & -1 \\ 1 & -\left(1+\dfrac{1}{2s}\right) \end{bmatrix}$$

Ili na drugi način: $[y] = [z]^{-1}$

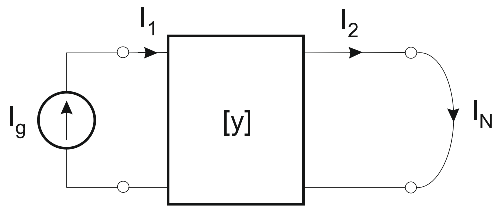
$$y_{11} = \frac{z_{22}}{|\Delta z|} = \frac{2s}{2s} = 1;\quad y_{12} = \frac{z_{12}}{|\Delta z|} = \frac{2s}{2s} = 1;\quad y_{21} = \frac{z_{21}}{|\Delta z|} = \frac{2s}{2s} = 1;\quad y_{22} = \frac{z_{11}}{|\Delta z|} = \frac{2s+1}{2s} = 1+\frac{1}{2s}$$

$|\Delta z| = (2s+1)\cdot 2s - 4s^2 = 2s$ (Dobije se isti rezultat.)

Sada slijede y-parametri dva četveropola u paralelu:

$$[y] = [y]^I + [y]^{II} = \begin{bmatrix} 1 & -1 \\ 1 & -\left(1+\dfrac{1}{2s}\right) \end{bmatrix} + \begin{bmatrix} s & -s \\ s & -s \end{bmatrix} = \begin{bmatrix} 1+s & -(1+s) \\ 1+s & -\left(1+s+\dfrac{1}{2s}\right) \end{bmatrix}$$ (1 bod)

d) Nortonova struja $I_N(s)$ iz y-parametara:



$$I_1 = y_{11}\cdot U_1 - y_{12}\cdot U_2$$
$$I_2 = y_{21}\cdot U_1 - y_{22}\cdot U_2$$

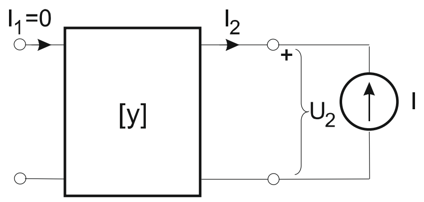
$i_g(t)=\delta(t) \;\Rightarrow\; I_g(s) = 1,\; I_g = I_1;\; I_N = I_2$

$U_2 = 0$ kratki spoj na priključnicama 2–2' kroz koje teče Nortonova struja

$$\begin{aligned} I_1 &= y_{11}\cdot U_1 \\ I_2 &= y_{21}\cdot U_1 \end{aligned} \quad \frac{I_2}{I_1} = \frac{y_{21}}{y_{11}} \;\Rightarrow\; I_2 = \frac{y_{21}}{y_{11}}\cdot I_1 = \frac{1+s}{1+s}\cdot 1 = 1 \;\Rightarrow\; I_N(s) = 1$$ (1 bod)

Drugi način je upotrijebiti izraz : $H_i(s) = \dfrac{I_2}{I_1} = \dfrac{Y_L y_{21}}{\Delta y + y_{11}Y_L}\Bigg|_{Y_L=\infty} = \dfrac{y_{21}}{y_{11}}$ što će dati isti rezultat.

e) Nortonova admitancija $Y_N(s)$ iz y-parametara :



$I_1 = 0$ prazni hod spoj na priključnicama 1–1' jer je isključen strujni izvor

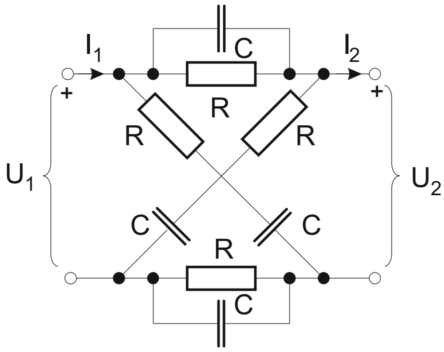
$$I_1 = -I_2;\quad Y_N = \frac{I}{U_2} = \frac{-I_2}{U_2};\;\Rightarrow\; \begin{aligned} 0 &= y_{11}\cdot U_1 - y_{12}\cdot U_2 \\ -I &= y_{21}\cdot U_1 - y_{22}\cdot U_2 \end{aligned} \quad \Rightarrow y_{11}\cdot U_1 = y_{12}\cdot U_2 \Rightarrow U_1 = (y_{12}/y_{11})\cdot U_2$$

$$\Rightarrow -I = y_{21}\cdot\frac{y_{12}}{y_{11}}U_2 - y_{22}\cdot U_2 = \frac{y_{12}y_{21} - y_{11}y_{22}}{y_{11}}\cdot U_2 \Rightarrow Y_N = \frac{I}{U_2} = \frac{y_{11}y_{22} - y_{12}y_{21}}{y_{11}} = \frac{\Delta y}{y_{11}};\; \Delta y = y_{11}y_{22} - y_{12}y_{21}$$

$$\Delta y = (1+s)\left(1+s+\frac{1}{2s}\right) - (1+s)^2 = (1+s)^2 + \frac{1+s}{2s} - (1+s)^2 = \frac{1+s}{2s};\quad Y_N(s) = \frac{\Delta y}{y_{11}} = \frac{(1+s)/(2s)}{1+s} = \frac{1}{2s}$$ (1 bod)

Drugi način je upotrijebiti izraz : $Y_{ul2}(s) = y_{22} - \dfrac{y_{12}y_{21}}{y_{11}+Y_1}\Bigg|_{Y_1=0} = \dfrac{y_{11}y_{22} - y_{12}y_{21}}{y_{11}}$ što će dati isti rezultat.

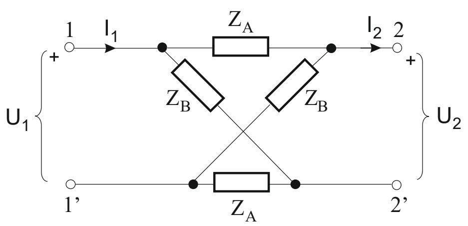
3. Za X-četveropol prikazan slikom izračunati a) prijenosne *a*-parametre, ako su zadane normalizirane vrijednosti elemenata R=1, C=1. b) Iz prijenosnih *a*-parametara izračunati *z*-parametre. c) četveropolu pridružiti ekvivalentni T-četveropol. d) Koliko iznose zrcalni parametri četveropola Z_{C1}, Z_{C2} i g ? e) Da li je četveropol: recipročan, simetričan ? Obrazložiti odgovore.

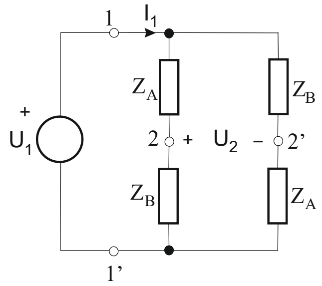


Rješenje:

a) a - parametri $\begin{array}{l} U_1 = A\cdot U_2 + B\cdot I_2 \\ I_1 = C\cdot U_2 + D\cdot I_2 \end{array} \Rightarrow$

$\underline{I_2 = 0}$ $\quad A = \left.\dfrac{U_1}{U_2}\right|_{I_2=0} \qquad C = \left.\dfrac{I_1}{U_2}\right|_{I_2=0}$





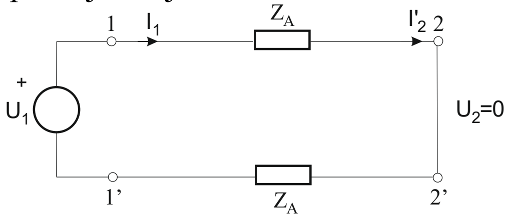
$$I_1 = \frac{U_1}{(Z_A + Z_B)/2} = \frac{2\cdot U_1}{Z_A + Z_B}; \Rightarrow U_1 = I_1\cdot\frac{Z_A + Z_B}{2}$$

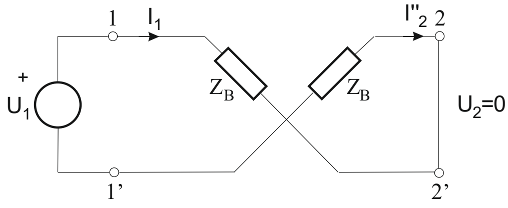
$$U_2 = U_1\cdot\frac{Z_B}{Z_A + Z_B} - U_1\cdot\frac{Z_A}{Z_A + Z_B} = U_1\cdot\frac{Z_B - Z_A}{Z_A + Z_B} \quad\Rightarrow\quad A = \frac{U_1}{U_2} = \frac{Z_A + Z_B}{Z_B - Z_A}$$

$$\frac{U_1}{U_2} = \frac{I_1\cdot(Z_A + Z_B)/2}{U_2} = \frac{Z_A + Z_B}{Z_B - Z_A} \Big/ : \frac{Z_A + Z_B}{2} \quad\Rightarrow\quad C = \frac{I_1}{U_2} = \frac{2}{Z_B - Z_A}$$

$\underline{U_2 = 0}$ $\quad B = \left.\dfrac{U_1}{I_2}\right|_{U_2=0} \qquad D = \left.\dfrac{I_1}{I_2}\right|_{U_2=0}$

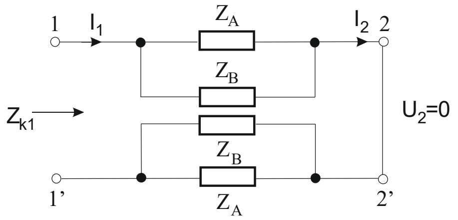
Superpozicija struja:





$$I_2 = I'_2 + I''_2 = \frac{U_1}{2\cdot Z_A} - \frac{U_1}{2\cdot Z_B} = \frac{Z_B - Z_A}{2\cdot Z_A Z_B}\cdot U_1 \quad\Rightarrow\quad B = \frac{U_1}{I_2} = 2\cdot\frac{Z_A Z_B}{Z_B - Z_A}$$

Ulazna impedancija na kratko Z_{k1}:



$$Z_{ul1}=Z_{k1}=2\frac{Z_AZ_B}{Z_A+Z_B};\quad U_1=I_1\cdot Z_{ul1}=I_1\cdot 2\frac{Z_AZ_B}{Z_A+Z_B}$$

$$\frac{U_1}{I_2}=\frac{I_1\cdot 2\cdot Z_AZ_B/(Z_A+Z_B)}{I_2}=2\cdot\frac{Z_AZ_B}{Z_B-Z_A}\Big/:\left(2\cdot\frac{Z_AZ_B}{Z_A+Z_B}\right)\ \Rightarrow\ D=\frac{I_1}{I_2}=\frac{Z_A+Z_B}{Z_B-Z_A}$$

$$[a]=\frac{1}{Z_B-Z_A}\cdot\begin{bmatrix}Z_A+Z_B & 2Z_AZ_B\\ 2 & Z_A+Z_B\end{bmatrix}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}$$

Uz vrijednosti elemenata: $Z_A=\dfrac{R\cdot 1/(sC)}{R+1/(sC)}=\dfrac{R}{1+sRC}=\dfrac{1}{1+s}$; $Z_B=R+\dfrac{1}{sC}=1+\dfrac{1}{s}=\dfrac{s+1}{s}$

$$Z_A+Z_B=\frac{1}{1+s}+\frac{s+1}{s}=\frac{s^2+3s+1}{s(s+1)};\quad Z_B-Z_A=\frac{s+1}{s}-\frac{1}{1+s}=\frac{(s+1)^2-s}{s(1+s)}=\frac{s^2+s+1}{s(1+s)}$$

$$[a]=\frac{s(s+1)}{s^2+s+1}\cdot\begin{bmatrix}\dfrac{s^2+3s+1}{s(s+1)} & \dfrac{2}{s}\\ 2 & \dfrac{s^2+3s+1}{s(s+1)}\end{bmatrix}=\begin{bmatrix}\dfrac{s^2+3s+1}{s^2+s+1} & \dfrac{2(s+1)}{s^2+s+1}\\ \dfrac{2s(s+1)}{s^2+s+1} & \dfrac{s^2+3s+1}{s^2+s+1}\end{bmatrix}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}$$ (1 bod)

b) Slijede z-parametri iz a-parametara: (1 bod)

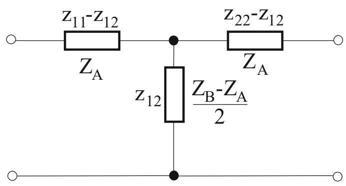
$$z_{11}=\frac{A}{C};\ z_{12}=\frac{AD-BC}{C}=\frac{1}{C};\ z_{21}=\frac{1}{C};\ z_{22}=\frac{D}{C};\ \det(a)=1\Rightarrow z_{12}=z_{21};\ A=D\Rightarrow z_{11}=z_{22};$$

$$z_{11}=z_{22}=\frac{A}{C}=\frac{Z_A+Z_B}{2};\ z_{12}=z_{21}=\frac{1}{C}=\frac{Z_B-Z_A}{2}$$

$$[z]=\frac{1}{2}\cdot\begin{bmatrix}Z_A+Z_B & -(Z_B-Z_A)\\ Z_B-Z_A & -(Z_A+Z_B)\end{bmatrix}=\frac{1}{2s(s+1)}\cdot\begin{bmatrix}s^2+3s+1 & -(s^2+s+1)\\ s^2+s+1 & -(s^2+3s+1)\end{bmatrix}=\begin{bmatrix}z_{11} & -z_{12}\\ z_{21} & -z_{22}\end{bmatrix}$$

c) Nadomjesni T – spoj: (1 bod)

$$z_{11}-z_{12}=z_{22}-z_{12}=\frac{Z_A+Z_B}{2}-\frac{Z_B-Z_A}{2}=Z_A=\frac{1}{1+s};\qquad z_{12}=z_{21}=\frac{Z_B-Z_A}{2}=\frac{s^2+s+1}{2s(s+1)}$$



d) Zrcalni parametri: Z_C, g: (1 bod)

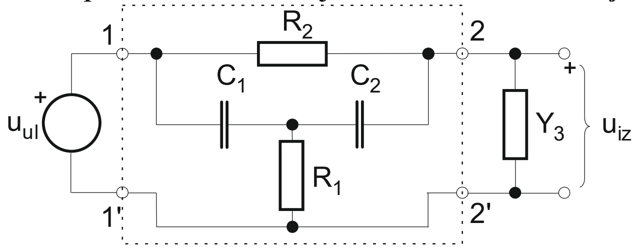
$$Z_{p1}=z_{11}=\frac{Z_A+Z_B}{2};\quad Z_{k1}=\frac{1}{y_{11}}=2\frac{Z_AZ_B}{Z_A+Z_B}$$

$$Z_C=Z_{C1}=Z_{C2}=\sqrt{Z_{k1}\cdot Z_{p1}}=\sqrt{Z_{k2}\cdot Z_{p2}}=\sqrt{Z_A\cdot Z_B}=\sqrt{1/(s+1)\cdot(s+1)/s}=1/\sqrt{s};$$

$$th\,g=\sqrt{\frac{Z_{k1}}{Z_{p1}}}=\frac{2\sqrt{Z_A\cdot Z_B}}{Z_A+Z_B}=\frac{2\sqrt{1/(s+1)\cdot(s+1)/s}}{1/(s+1)+(s+1)/s}=\frac{2}{\sqrt{s}}\cdot\frac{s(s+1)}{s+(s+1)^2}=2\sqrt{s}\cdot\frac{s+1}{s^2+3s+1}$$

e) Očigledno je da vrijedi: $A=D$, det(a)=AD–BC=1, odn. z_{11}=z_{22} i z_{12}=z_{21}; mreža je simetrična i recipročna. Također vrijedi: Z_C=Z_{C1}=Z_{C2} (simetričnost). (1 bod)

3. Za četveropol prikazan slikom s obzirom na polove 2–2' izračunati: a) y-parametre, ako su zadane normalizirane vrijednosti elemenata $R_1=R_2=1$, $C_1=C_2=2$. b) Iz y-parametara izračunati prijenosnu funkciju napona $T(s)=U_{iz}(s)/U_{ul}(s)$ ako je kao admitancija Y_3 spojen kapacitet $C_3=3$. c) Izračunati istim postupkom prijenosnu funkciju napona $T(s)=U_{iz}(s)/U_{ul}(s)$ ako je kao admitancija Y_3 spojen otpor $R_3=1/3$. d) Nacrtati raspored nula i polova u slučajevima b) i c). e) Iz rasporeda nula i polova kvalitativno skicirati amplitudno-frekvencijsku karakteristiku $|T(j\omega)|$ u slučajevima b) i c).



Rješenje: Nađimo najprije y-parametre četveropola. Ukupni četveropol se može promatrati kao kombinacija dva četveropola u paralelu.

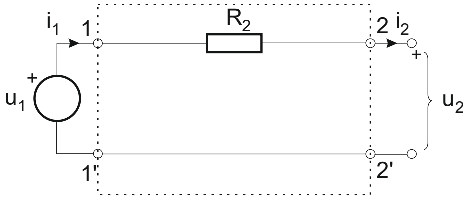
a) Za četveropol uz referentne oznake struja i napona slijede y-parametri:

Strujne jednadžbe četveropola

$$I_1 = y_{11}\cdot U_1 - y_{12}\cdot U_2$$
$$I_2 = y_{21}\cdot U_1 - y_{22}\cdot U_2$$

Naponske jednadžbe četveropola

$$U_1 = z_{11}\cdot I_1 - z_{12}\cdot I_2$$
$$U_2 = z_{21}\cdot I_1 - z_{22}\cdot I_2$$

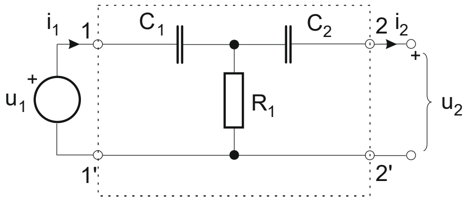


$U_2 = 0, I_2 = I_1 \Rightarrow$ $\quad$ $U_1 = 0, I_2 = I_1 \Rightarrow$

$y_{11} = I_1/U_1 = 1/R_2$ $\quad$ $y_{12} = -I_1/U_2 = 1/R_2$

$y_{21} = I_2/U_1 = 1/R_2$ $\quad$ $y_{22} = -I_2/U_2 = 1/R_2$

$$\Rightarrow [y]^I = \begin{bmatrix} y_{11} & -y_{12} \\ y_{21} & -y_{22} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$



$I_2 = 0, U_2 = U_{R1} = I_1R_1 \Rightarrow$ $\quad$ $I_1 = 0, U_1 = U_{R1} = -I_2R_1 \Rightarrow$

$z_{11} = U_1/I_1 = R_1 + 1/sC_1$ $\quad$ $z_{12} = -U_1/I_2 = R_1$

$z_{21} = U_2/I_1 = R_1$ $\quad$ $z_{22} = -U_2/I_2 = R_1 + 1/sC_2$

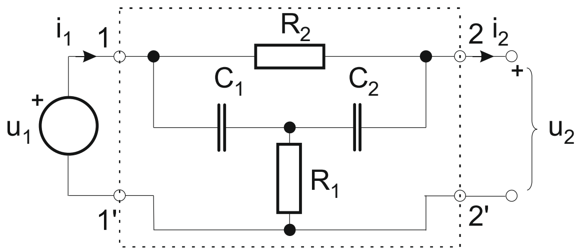
$$\Rightarrow [z]^{II} = \begin{bmatrix} z_{11} & -z_{12} \\ z_{21} & -z_{22} \end{bmatrix} = \begin{bmatrix} 1+1/(2s) & -1 \\ 1 & -[1+1/(2s)] \end{bmatrix}$$

$[y] = [z]^{-1}$;

$$y_{11} = \frac{z_{22}}{|\Delta z|} = \frac{s(s+1/2)}{s+1/4} = \frac{4s^2+2s}{4s+1};\ y_{12} = \frac{z_{12}}{|\Delta z|} = \frac{s^2}{s+1/4} = \frac{4s^2}{4s+1};\ y_{21} = \frac{z_{21}}{|\Delta z|} = y_{12};$$

$$y_{22} = \frac{z_{11}}{|\Delta z|} = y_{11};\ |\Delta z| = [1+1/(2s)]^2 - 1 = [1+1/s+1/(4s^2)] - 1 = 1/s + 1/(4s^2) = (s+1/4)/s^2$$

Ukupni četveropol:



$$[y]=[y]^{I}+[y]^{II}=\begin{bmatrix} \frac{4s^2+2s}{4s+1}+1 & -\left(\frac{4s^2}{4s+1}+1\right) \\ \frac{4s^2}{4s+1}+1 & -\left(\frac{4s^2+2s}{4s+1}+1\right) \end{bmatrix}=\begin{bmatrix} \frac{4s^2+6s+1}{4s+1} & -\frac{4s^2+4s+1}{4s+1} \\ \frac{4s^2+4s+1}{4s+1} & -\frac{4s^2+6s+1}{4s+1} \end{bmatrix}$$ **(2 boda)**

b) Prijenosna funkcija napona sa zaključenjem $Y_3(s)=\frac{1}{R_3}=3$

$$H(s)=\frac{U_2(s)}{U_1(s)}=\frac{y_{21}}{y_{22}+Y_3}$$

$$H(s)=\frac{U_2(s)}{U_1(s)}=\frac{\frac{4s^2}{4s+1}+1}{\frac{4s^2+2s}{4s+1}+1+3}=\frac{4s^2+4s+1}{4s^2+2s+4(4s+1)}=\frac{4s^2+4s+1}{4s^2+18s+4}=\frac{s^2+s+1/4}{s^2+(9/2)s+1}$$

Vidljivo je $H(0)=1/4;\ H(\infty)=1$

d1) raspored nula i polova za slučaj b):

Nule: $4s^2+4s+1=(2s+1)^2=0 \Rightarrow s_{o1,2}=-1/2$

Polovi: $s^2+(9/2)s+1=0 \Rightarrow s_{p1,2}=-\frac{9}{4}\pm j\sqrt{\frac{81}{16}-\frac{16}{16}}=-\frac{9\pm\sqrt{65}}{4}$

ili $s_{p1}=-4{,}26556; s_{p2}=-0{,}234436$ **(b) + d1) =1 bod)**

c) Prijenosna funkcija napona sa zaključenjem $Y_3(s)=sC_3=3s$

$$H(s)=\frac{U_2(s)}{U_1(s)}=\frac{\frac{4s^2}{4s+1}+1}{\frac{4s^2+2s}{4s+1}+1+3s}=\frac{4s^2+4s+1}{4s^2+2s+(4s+1)(3s+1)}=\frac{4s^2+4s+1}{16s^2+9s+1}=\frac{1}{4}\cdot\frac{s^2+s+\frac{1}{4}}{s^2+\frac{9}{16}s+\frac{1}{16}}$$ Vidljivo je $H(0)=1;\ H(\infty)=1/4$

d2) raspored nula i polova za slučaj c):

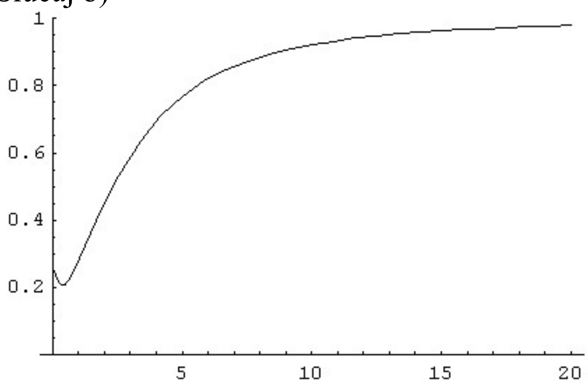
Nule: $4s^2+4s+1=(2s+1)^2=0 \Rightarrow s_{o1,2}=-1/2$

Polovi: $s^2+\frac{9}{16}s+\frac{1}{16}=0 \Rightarrow s_{p1,2}=-\frac{9}{32}\pm\sqrt{\left(\frac{9}{32}\right)^2-\frac{1}{16}}=-\frac{9\pm\sqrt{17}}{32}$

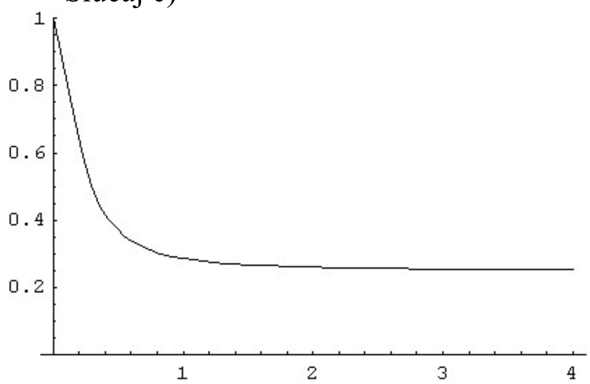
ili $s_{p1}=-0{,}4101; s_{p2}=-0{,}1524$ **(c) + d2) =1 bod)**

e) amplitudno-frekvencijske karakteristike **(1 bod)**

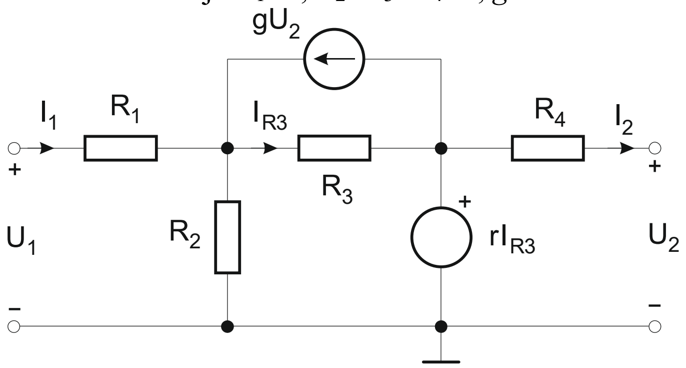
Slučaj b)



Slučaj c)



3. Za četveropol prikazan slikom izračunati z-parametre. Odrediti parametar r tako da zadani četveropol bude simetričan. Zadano je R_1=2, R_2=R_3=R_4=1, g=2.



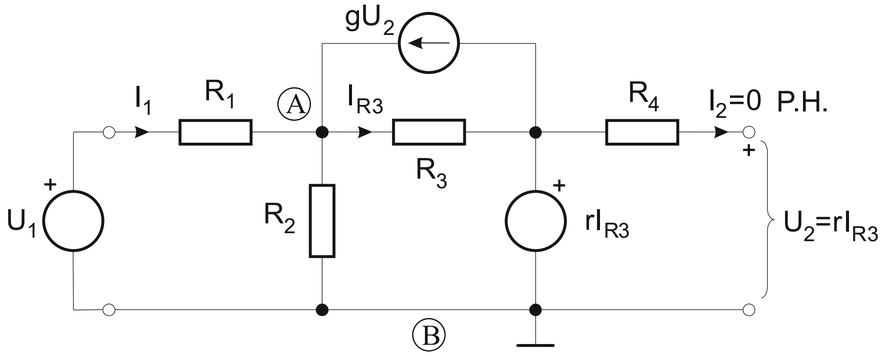
Rješenje: [z]-parametri:

$$U_1 = z_{11}\cdot I_1 - z_{12}\cdot I_2$$
$$U_2 = z_{21}\cdot Z_1 - z_{22}\cdot I_2$$

a) $I_2 = 0$

$$z_{11} = \left.\frac{U_1}{I_1}\right|_{I_2=0}$$

$$z_{21} = \left.\frac{U_2}{I_1}\right|_{I_2=0}$$



$$U_A\left(\frac{1}{R_1}+\frac{1}{R_2}\right) = -I_{R3} + gU_2 + \frac{U_1}{R_1};\quad U_2 = r\cdot I_{R3};\quad I_{R3} = \frac{U_A - r\cdot I_{R3}}{R_3};\quad I_1 = \frac{U_1 - U_A}{R_1}$$

$$\Rightarrow R_3 I_{R3} = U_A - r\cdot I_{R3} \Rightarrow (R_3+r)I_{R3} = U_A \Rightarrow I_{R3} = \frac{1}{R_3+r}U_A \Rightarrow U_2 = r\cdot I_{R3} = \frac{r}{R_3+r}U_A$$

$$U_A\left(\frac{1}{R_1}+\frac{1}{R_2}\right) = -I_{R3}\cdot(1-gr) + \frac{U_1}{R_1} = -\frac{1}{R_3+r}U_A\cdot(1-gr) + \frac{U_1}{R_1} \Rightarrow$$

$$U_A\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1-gr}{R_3+r}\right) = \frac{U_1}{R_1}\Big/\cdot R_1R_2(R_3+r);\, U_A\left[(R_1+R_2)(R_3+r)+R_1R_2(1-gr)\right] = U_1R_2(R_3+r)$$

$$\frac{U_A}{U_1} = \frac{R_2(R_3+r)}{(R_1+R_2)(R_3+r)+R_1R_2(1-gr)};\; U_2 = \frac{rU_A}{R_3+r} = \frac{r}{R_3+r}\frac{R_2(R_3+r)}{(R_1+R_2)(R_3+r)+R_1R_2(1-gr)}U_1$$

$$I_1 = \frac{U_1}{R_1}\left(1-\frac{U_A}{U_1}\right) = \frac{U_1}{R_1}\left[1-\frac{R_2(R_3+r)}{(R_1+R_2)(R_3+r)+R_1R_2(1-gr)}\right] = \frac{(R_3+r)+R_2(1-gr)}{(R_1+R_2)(R_3+r)+R_1R_2(1-gr)}\cdot U_1$$

$$\frac{U_2}{U_1} = \frac{rR_2}{(R_1+R_2)(R_3+r)+R_1R_2(1-gr)}$$

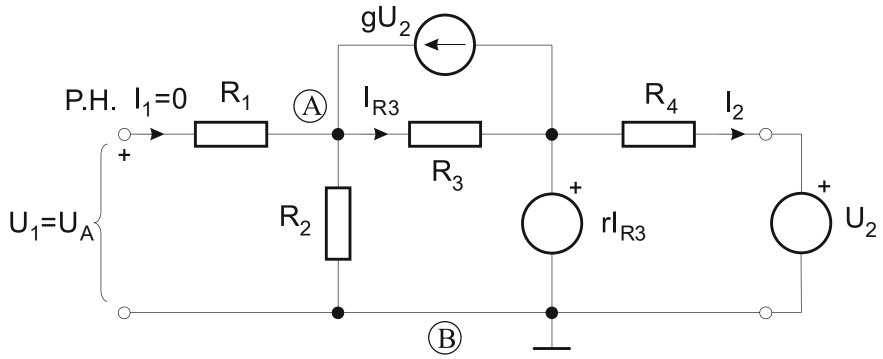
$$z_{11} = \frac{U_1}{I_1} = \frac{(R_1+R_2)(R_3+r)+R_1R_2(1-gr)}{(R_3+r)+R_2(1-gr)} = \frac{5-r}{2-r}$$

$$z_{21} = \frac{U_2}{I_1} = \frac{U_1}{I_1}\cdot\frac{U_2}{U_1} = \frac{rR_2}{(R_3+r)+R_2(1-gr)} = \frac{r}{2-r}$$ **(2 boda)**

b) $I_1=0$

$$z_{12}=-\frac{U_1}{I_2}\bigg|_{I_1=0}$$

$$z_{22}=-\frac{U_2}{I_2}\bigg|_{I_1=0}$$



$$U_A\frac{1}{R_2}=-I_{R3}+gU_2;\ U_1=U_A;\ I_{R3}=\frac{U_A-r\cdot I_{R3}}{R_3};\ \ I_2=\frac{r\cdot I_{R3}-U_2}{R_4}$$

$$I_{R3}=\frac{1}{R_3+r}U_A;\ U_2=r\cdot I_{R3}-I_2R_4\ \Rightarrow\ U_2=\frac{r}{R_3+r}U_1-I_2R_4$$

$$U_A\frac{1}{R_2}=-\frac{1}{R_3+r}U_A+g\cdot U_2\ \Rightarrow\ U_A\left(\frac{1}{R_2}+\frac{1}{R_3+r}\right)=g\cdot U_2\ \Rightarrow\ \frac{U_1}{U_2}=\frac{g}{\dfrac{1}{R_2}+\dfrac{1}{R_3+r}}=\frac{gR_2(R_3+r)}{R_2+R_3+r}$$

$$\Rightarrow U_2=\frac{r}{R_3+r}\cdot\frac{gR_2(R_3+r)}{R_2+R_3+r}U_2-I_2R_4=\frac{rgR_2}{R_2+R_3+r}U_2-I_2R_4\ \Rightarrow$$

$$U_2\left(1-\frac{rgR_2}{R_2+R_3+r}\right)=-I_2R_4\ \Rightarrow\ U_2\frac{R_2+R_3+r-rgR_2}{R_2+R_3+r}=-I_2R_4\ \Rightarrow$$

$$z_{22}=-\frac{U_2}{I_2}=\frac{(R_2+R_3+r)R_4}{(R_2+R_3)+R_2(1-rg)}=\frac{2+r}{2-r}$$

$$z_{12}=-\frac{U_1}{I_2}=-\frac{U_2}{I_2}\cdot\frac{U_1}{U_2}=\frac{(R_2+R_3+r)R_4}{(R_3+r)+R_2(1-rg)}\cdot\frac{gR_2(R_3+r)}{R_2+R_3+r}=\frac{gR_2(R_3+r)R_4}{(R_3+r)+R_2(1-rg)}=\frac{2+2r}{2-r}$$

Zadatak se puno brže rješava ako se odmah uvrste vrijednosti elemenata. Slijede z-parametri:

$$[z]=\begin{bmatrix} z_{11} & -z_{12}\\ z_{21} & -z_{22}\end{bmatrix}=\frac{1}{2-r}\begin{bmatrix}5-r & -(2+2r)\\ r & -(2+r)\end{bmatrix}$$ **(2 boda)**

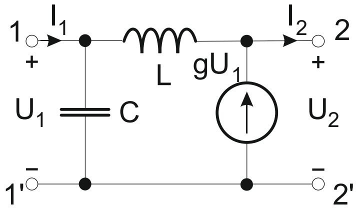
Uvjet recipročnosti: $z_{12}=z_{21}\ \Rightarrow\ 2+2r=r\ \Rightarrow\ r=-2$
(Minus predznak parametra r se realizira obrnutim polaritetom ovisnog izvora.)

Uvjet simetričnosti: $z_{11}=z_{22}\ \Rightarrow\ 5-r=2+r\ \Rightarrow \boxed{r=\frac{3}{2}}$
Četveropol ne može istovremeno biti i recipročan i simetričan.

Za simetričan četveropol slijede z-parametri:

$$[z]=\begin{bmatrix}7 & -10\\ 3 & -7\end{bmatrix}$$ **(1 bod)**

3. Za četveropol prikazan slikom izračunati prijenosne [a]-parametre i napisati matricu [a]-parametara. Pomoću poznatih [a]-parametara izračunati naponsku prijenosnu funkciju četveropola $T(s)=U_2(s)/U_1(s)$ te ulaznu impedanciju $Z_{ul1}(s)=U_1(s)/I_1(s)$ ako je na izlaznom prilazu (2–2') spojen otpor R. Da li je četveropol: a) recipročan, b) simetričan ? Obrazložiti odgovor. Zadane su normalizirane vrijednosti elemenata L=1, C=1 i parametar g=1.



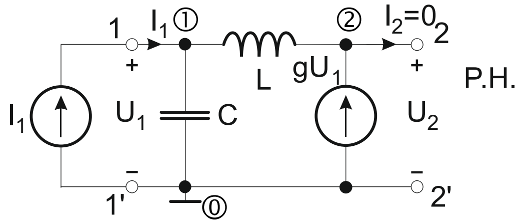
Rješenje:

[a]-parametri:

$$U_1 = A \cdot U_2 + B \cdot I_2$$

$$I_1 = C \cdot U_2 + D \cdot I_2$$

$I_2 = 0$ $\quad A = \left.\dfrac{U_1}{U_2}\right|_{I_2=0} \qquad C = \left.\dfrac{I_1}{U_2}\right|_{I_2=0}$



$$(1)\ U_1\left(\frac{1}{sL}+sC\right)-U_2\frac{1}{sL}=I_1$$

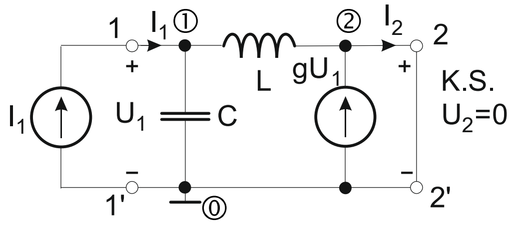
$$(2)\ -U_1\frac{1}{sL}+U_2\frac{1}{sL}=gU_1$$

$$(2) \Rightarrow \left(g+\frac{1}{sL}\right)U_1=\frac{1}{sL}U_2 \Rightarrow U_1=\frac{1}{gsL+1}U_2 \Rightarrow A=\frac{U_1}{U_2}=\frac{1}{gsL+1}$$

$$(2) \to (1) \Rightarrow U_2\frac{1}{gsL+1}\left(\frac{1}{sL}+sC\right)-U_2\frac{1}{sL}=I_1 \Rightarrow C=\frac{I_1}{U_2}=\frac{1}{gsL+1}\left(\frac{1}{sL}+sC\right)-\frac{1}{sL}$$

$$C=\frac{\left(\frac{1}{sL}+sC\right)-\frac{1}{sL}(gsL+1)}{gsL+1}=\frac{sC-g}{gsL+1}$$

$U_2 = 0$ $\quad B = \left.\dfrac{U_1}{I_2}\right|_{U_2=0} \qquad D = \left.\dfrac{I_1}{I_2}\right|_{U_2=0}$



$$(1)\ U_1\left(\frac{1}{sL}+sC\right)-U_2\frac{1}{sL}=I_1$$

$$(2)\ -U_1\frac{1}{sL}+U_2\frac{1}{sL}=gU_1-I_2$$

$$U_2=0\ \ (1)\ \Rightarrow\ U_1\left(\frac{1}{sL}+sC\right)=I_1$$

$$(2)\Rightarrow\ -U_1\frac{1}{sL}=gU_1-I_2\ \Rightarrow\ I_2=U_1\left(g+\frac{1}{sL}\right)\ \Rightarrow\ B=\frac{U_1}{I_2}=\frac{1}{g+1/(sL)}=\frac{sL}{gsL+1}$$

$$(2)\rightarrow(1)\Rightarrow I_2\frac{sL}{gsL+1}\left(\frac{1}{sL}+sC\right)=I_1\ \Rightarrow\ D=\frac{I_1}{I_2}=\frac{sL}{gsL+1}\left(\frac{1}{sL}+sC\right)=\frac{1+s^2LC}{gsL+1}$$

$$[a]=\frac{1}{gsL+1}\begin{bmatrix}1 & sL\\ sC-g & s^2LC+1\end{bmatrix}=\frac{1}{s+1}\begin{bmatrix}1 & s\\ s-1 & 1+s^2\end{bmatrix}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}$$

(3 boda)

Prijenosna funkcija napona:

$$T(s)=\frac{U_2(s)}{U_1(s)}=\frac{Z_2}{AZ_2+B}=\frac{s+1}{s+1}=1$$

Ulazna impedancija u četveropol zaključen s otporom R:

$$Z_{ul1}(s)=\frac{U_1(s)}{I_1(s)}=\frac{AZ_2+B}{CZ_2+D}=\frac{s+1}{s-1+s^2+1}=\frac{s+1}{s(s+1)}=\frac{1}{s}$$

(1 bod)

Odgovori na pitanja:

a) Četveropol nije električki recipročan jer sadrži naponsko ovisni strujni izvor i vrijedi da je:

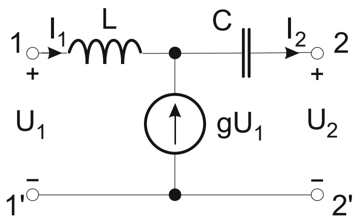
$$|a|=AD-BC=\frac{1+s^2-s(s-1)}{(s+1)^2}=\frac{1+s}{(s+1)^2}=\frac{1}{s+1}\neq 1;$$

b) Četveropol nije električki simetričan jer se parametri A i D razlikuju:

$$A=\frac{1}{s+1}\neq D=\frac{s^2+1}{s+1}$$

(1 bod)

3. Za četveropol prikazan slikom izračunati prijenosne [*a*]-parametre i napisati matricu [*a*]-parametara. Pomoću poznatih [*a*]-parametara izračunati naponsku prijenosnu funkciju četveropola $T(s)=U_2(s)/U_1(s)$ te ulaznu impedanciju $Z_{ul1}(s)=U_1(s)/I_1(s)$ ako je na izlaznom prilazu (2–2') spojen otpor *R*. Da li je četveropol: *a*) recipročan, *b*) simetričan ? Obrazložiti odgovor. Zadane su normalizirane vrijednosti elemenata L=1, C=1 i parametar g=1.



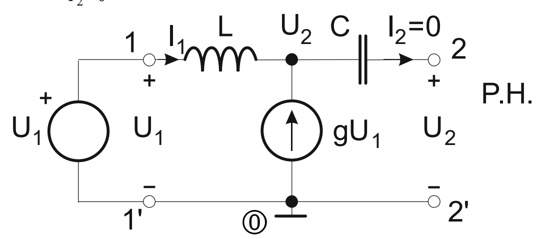
Rješenje:

[*a*]-parametri:

$$U_1 = A \cdot U_2 + B \cdot I_2$$

$$I_1 = C \cdot U_2 + D \cdot I_2$$

$I_2 = 0$ $\quad A = \left.\frac{U_1}{U_2}\right|_{I_2=0} \qquad C = \left.\frac{I_1}{U_2}\right|_{I_2=0}$



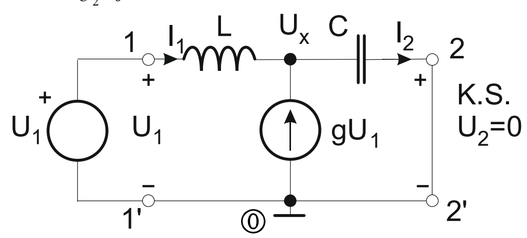
(1) $\dfrac{U_2 - U_1}{sL} = gU_1$

(2) $gU_1 = -I_1$

$$(1) \Rightarrow \frac{U_2}{sL} = \left(\frac{1}{sL} + g\right)U_1 \Rightarrow U_2 = (1 + gsL)U_1 \Rightarrow A = \frac{U_1}{U_2} = \frac{1}{gsL+1}$$

$$(2) \to (1) \Rightarrow U_2 = (1 + gsL)\frac{-I_1}{g} \Rightarrow C = \frac{I_1}{U_2} = \frac{-g}{gsL+1}$$

$U_2 = 0$ $\quad B = \left.\frac{U_1}{I_2}\right|_{U_2=0} \qquad D = \left.\frac{I_1}{I_2}\right|_{U_2=0}$



(1) $U_x\left(\dfrac{1}{sL} + sC\right) - U_2 sC = gU_1 + \dfrac{U_1}{sL}$;

(2) $-U_x sC + U_2 sC = -I_2$

$$U_2=0\ \ (1)\ \Rightarrow\ U_x\left(\frac{1}{sL}+sC\right)=\left(g+\frac{1}{sL}\right)U_1$$

$$(2)\Rightarrow\ U_x sC=I_2\ \Rightarrow\ \frac{I_2}{sC}\left(\frac{1}{sL}+sC\right)=\left(g+\frac{1}{sL}\right)U_1\ \Rightarrow\ B=\frac{U_1}{I_2}=\frac{1+1/(s^2LC)}{g+1/(sL)}=\frac{sL+1/(sC)}{gsL+1}$$

$$(2)\rightarrow(1)\Rightarrow\frac{U_1-U_x}{sL}=I_1\Rightarrow U_1-\frac{I_2}{sC}=I_1 sL\Rightarrow\frac{sL+1/(sC)}{gsL+1}I_2-\frac{1}{sC}I_2=I_1 sL$$

$$\Rightarrow\ D=\frac{I_1}{I_2}=\frac{1+1/(s^2LC)}{gsL+1}-\frac{1}{s^2LC}=\frac{1+1/(s^2LC)-1/(s^2LC)-g/(sC)}{gsL+1}=\frac{1-g/(sC)}{gsL+1}$$

$$[a]=\frac{1}{gsL+1}\begin{bmatrix}1 & sL+1/(sC)\\ -g & 1-g/(sC)\end{bmatrix}=\frac{1}{s+1}\begin{bmatrix}1 & s+1/s\\ -1 & 1-1/s\end{bmatrix}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}$$

(3 boda)

Prijenosna funkcija napona:

$$T(s)=\frac{U_2(s)}{U_1(s)}=\frac{Z_2}{AZ_2+B}=\frac{s+1}{1+s+1/s}=\frac{s(s+1)}{s^2+s+1}$$

Ulazna impedancija u četveropol zaključen s otporom R:

$$Z_{ul1}(s)=\frac{U_1(s)}{I_1(s)}=\frac{AZ_2+B}{CZ_2+D}=\frac{1+s+1/s}{-1+1-1/s}=\frac{1+s+1/s}{-1/s}=-\left(s^2+s+1\right)$$

(1 bod)

Odgovori na pitanja:

a) Četveropol nije električki recipročan jer sadrži naponsko ovisni strujni izvor i vrijedi da je:

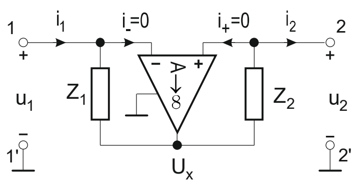
$$|a|=AD-BC=\frac{1-1/s+s+1/s}{(s+1)^2}=\frac{1+s}{(s+1)^2}=\frac{1}{s+1}\neq 1;$$

b) Četveropol nije električki simetričan jer se parametri A i D razlikuju:

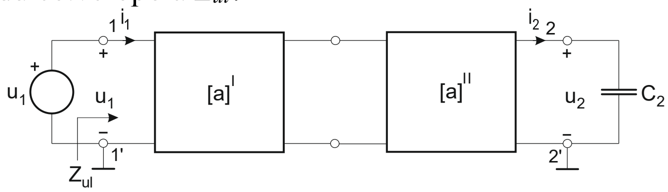
$$A=\frac{1}{s+1}\neq D=\frac{1-1/s}{s+1}$$

(1 bod)

3. Za četveropol koji sadrži operacijsko pojačalo, prikazan slikom 1, izračunati prijenosne [a]-parametre. Zatim pomoću [a]-parametara izračunati ulaznu impedanciju Z_{ul} u kaskadu koja je prikazana na slici 2 i koja je sastavljena od dva ista četveropola sa slike 1. Na izlazu kaskade četveropola nalazi se kapacitet C_2=1/2. Zadano je $Z_1=R=1$, $Z_2=1/(sC)=1/s$. Kojem elementu odgovara ulazna impedancija u kaskadu četveropola Z_{ul}?



Slika 1



Slika 2

<u>Rješenje:</u>

Prijenosni a-parametri:

$$\underline{\begin{aligned}U_1 &= A\cdot U_2 + B\cdot I_2\\ I_1 &= C\cdot U_2 + D\cdot I_2\end{aligned}} \qquad A=\left.\frac{U_1}{U_2}\right|_{I_2=0} \qquad B=\left.\frac{U_1}{I_2}\right|_{U_2=0} \qquad C=\left.\frac{I_1}{U_2}\right|_{I_2=0} \qquad D=\left.\frac{I_1}{I_2}\right|_{U_2=0}$$

Iz električne sheme uslijed "prividnog kratkog spoja" na ulazu u operacijsko pojačalo vrijedi:

$$U_X=-A(U_1-U_2)/:A \;\Rightarrow\; \frac{U_X}{A}=U_2-U_1 \,/\, A\to\infty \;\Rightarrow\; U_2=U_1 .$$

Može se napisati (jer su ulazne struje i_-, i_+ u operacijsko pojačalo jednake nuli):

$$I_1=-\frac{U_X}{Z_1},\; I_2=\frac{U_X}{Z_2} \;\Rightarrow\; U_X=-I_1\cdot Z_1=I_2\cdot Z_2 \Rightarrow I_1=-\frac{Z_2}{Z_1}\cdot I_2$$

Pa je:

$$A=\left.\frac{U_1}{U_2}\right|_{I_2=0}=\frac{U_2}{U_2}=1,\; B=\left.\frac{U_1}{I_2}\right|_{U_2=0}=\frac{U_2}{I_2}=0,\; C=\left.\frac{I_1}{U_2}\right|_{I_2=0}=\left.\frac{-\left(Z_2/Z_1\right)\cdot I_2}{U_2}\right|_{I_2=0}=0,$$

$$D=\left.\frac{I_1}{I_2}\right|_{U_2=0}=-\frac{Z_2}{Z_1} \qquad [a]=\begin{bmatrix}A & B\\ C & D\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & -\dfrac{Z_2}{Z_1}\end{bmatrix}.$$ (2 boda)

Uobičajeno je za konvertore impedancija da su prijenosni parametri B i C jednaki nuli.
Za dva četveropola u kaskadi vrijedi (1 bod)

$$[a]=[a]^I\cdot[a]^{II}=\begin{bmatrix}1 & 0\\ 0 & -\dfrac{Z_2}{Z_1}\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\ 0 & -\dfrac{Z_2}{Z_1}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & \left(\dfrac{Z_2}{Z_1}\right)^2\end{bmatrix}$$, te uz Z_1= 1, Z_2=1/s je $[a]=\begin{bmatrix}1 & 0\\ 0 & \dfrac{1}{s^2}\end{bmatrix}$

Na izlazu četveropola (na slici 2) nalazi se impedancija opterećenja, za koju vrijedi:

$$Z_{opt}=\frac{U_2}{I_2}$$

Ulazna impedancija Z_{ul} u četveropol koji je definiran [a]-parametrima i zaključen impedancijom opterećenja Z_{opt} na izlazu glasi: (1 bod)

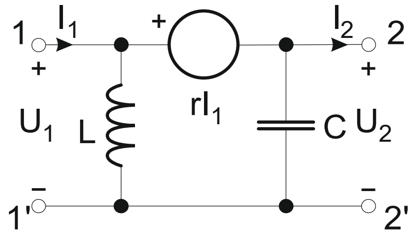
$$Z_{ul}=\frac{U_1}{I_1}=\frac{A\cdot U_2+B\cdot I_2}{C\cdot U_2+D\cdot I_2}=\frac{A\cdot\left(U_2/I_2\right)+B}{C\cdot\left(U_2/I_2\right)+D}=\left.\frac{A\cdot Z_{opt}+B}{C\cdot Z_{opt}+D}\right|_{B=0,C=0}=\frac{A}{D}\cdot Z_{opt}$$

U našem zadatku je uz Z_1= 1, Z_2=1/s, i uz Z_{opt}=2/s

$$Z_{ul}=\frac{A}{D}\cdot Z_{opt}=\frac{1}{\left(Z_2/Z_1\right)^2}\cdot Z_{opt}=\left(\frac{Z_1}{Z_2}\right)^2\cdot Z_{opt}=s^2\cdot\frac{2}{s}=2s=L_{ekv}s$$

Na ulazu u četveropol je vidljiv ekvivalentni induktivitet L_{ekv}=2. (1 bod)

3. Za četveropol prikazan slikom izračunati prijenosne [a]-parametre i napisati matricu [a]-parametara. Pomoću poznatih [a]-parametara izračunati [y]-parametre. Pomoću [y]-parametara izračunati naponsku prijenosnu funkciju četveropola $T(s)=U_2(s)/U_1(s)$ te ulaznu impedanciju $Z_{ul1}(s)=U_1(s)/I_1(s)$ ako je na izlaznom prilazu (2–2') spojen otpor R. Da li je četveropol: a) recipročan, b) simetričan ? Obrazložiti odgovor. Zadane su normalizirane vrijednosti elemenata L=1, C=1, R=1 i parametar r=1.



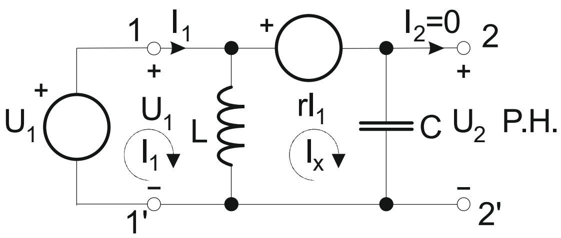
Rješenje:

[a]-parametri:

$U_1 = A \cdot U_2 + B \cdot I_2$

$I_1 = C \cdot U_2 + D \cdot I_2$

$I_2 = 0 \quad A = \left.\frac{U_1}{U_2}\right|_{I_2=0} \quad C = \left.\frac{I_1}{U_2}\right|_{I_2=0}$



(1) $U_1 = \left(I_1 - I_x\right) sL$

(2) $U_1 = U_2 + rI_1$

(3) $U_2 = I_x \frac{1}{sC}$

(2) $\Rightarrow \; I_1 = \frac{U_1 - U_2}{r}$

(3) $\Rightarrow \; I_x = sCU_2 \rightarrow (1) \Rightarrow \; U_1 = \left(\frac{U_1 - U_2}{r} - sCU_2\right) sL$

$U_1 - \frac{U_1}{r} sL = -U_2 \frac{sL}{r} - sCsLU_2 \Big/ \cdot r$

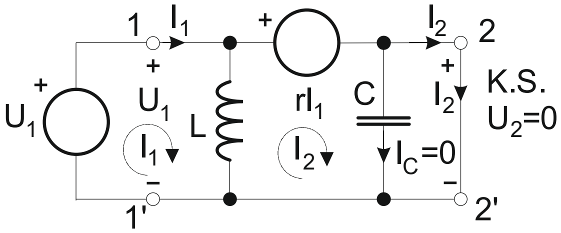
$U_1 (sL - r) = U_2 \left(sL + rsCsL\right)$

$$A = \left.\frac{U_1}{U_2}\right|_{I_2=0} = \frac{sL(1+rsC)}{sL - r} = \frac{s(1+s)}{s-1}$$

$$C = \left.\frac{I_1}{U_2}\right|_{I_2=0} = \frac{U_1 - U_2}{rU_2} = \frac{1}{r}\left(\frac{U_1}{U_2} - 1\right) = \frac{1}{r}\left(\frac{sL(1+rsC)}{sL - r} - 1\right)$$

$$C = \frac{1}{r}\left(\frac{sL + rsCsL - sL + r}{sL - r}\right) = \frac{1 + sCsL}{sL - r} = \frac{1+s^2}{s-1}$$

$\underline{U_2 = 0}$ $\quad B = \left.\dfrac{U_1}{I_2}\right|_{U_2=0} \quad D = \left.\dfrac{I_1}{I_2}\right|_{U_2=0}$



(1) $U_1 = (I_1 - I_2)sL$

(2) $\underline{U_1 = rI_1}$

$$rI_1 = I_1 sL - I_2 sL \Rightarrow I_2 sL = I_1(sL - r) \quad D = \left.\frac{I_1}{I_2}\right|_{U_2=0} = \frac{sL}{sL-r} = \frac{s}{s-1}$$

$$B = \left.\frac{U_1}{I_2}\right|_{U_2=0} = \frac{rI_1}{I_2} = \frac{rsL}{sL-r} = \frac{s}{s-1}$$

$$[a] = \frac{1}{sL-r}\begin{bmatrix} sL(1+rsC) & rsL \\ 1+sCsL & sL \end{bmatrix} = \frac{1}{s-1}\begin{bmatrix} s(1+s) & s \\ 1+s^2 & s \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

(2 boda)

[y]-parametri:

$$y_{11} = \frac{D}{B} = \frac{sL}{rsL} = \frac{1}{r} = 1, \qquad y_{21} = \frac{1}{B} = \frac{sL-r}{rsL} = \frac{s-1}{s}$$

$$y_{12} = \frac{AD-BC}{B} = \frac{\dfrac{sL(1+rsC)sL}{(sL-r)^2} - \dfrac{rsL(1+sCsL)}{(sL-r)^2}}{\dfrac{rsL}{sL-r}} = \frac{sL(1+rsC)sL - rsL(1+sCsL)}{rsL(sL-r)} = \frac{sL(sL-r)}{rsL(sL-r)} = \frac{1}{r} = 1$$

$$y_{22} = \frac{A}{B} = \frac{sL(1+rsC)}{rsL} = \frac{1+rsC}{r} = 1+s$$

$$[\mathrm{y}] = \begin{bmatrix} \dfrac{1}{r} & -\dfrac{1}{r} \\ \dfrac{sL-r}{rsL} & -\dfrac{1+rsC}{r} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ \dfrac{s-1}{s} & -(s+1) \end{bmatrix} = \begin{bmatrix} y_{11} & -y_{12} \\ y_{21} & -y_{22} \end{bmatrix}$$ **(1 bod)**

Prijenosna funkcija napona:

$$T(s) = \frac{U_2(s)}{U_1(s)} = \frac{y_{21}}{Y_2 + y_{22}} = \frac{1-1/s}{1+s+1} = \frac{1-1/s}{s+2}$$

Ulazna impedancija u četveropol zaključen s otporom R: **(1 bod)**

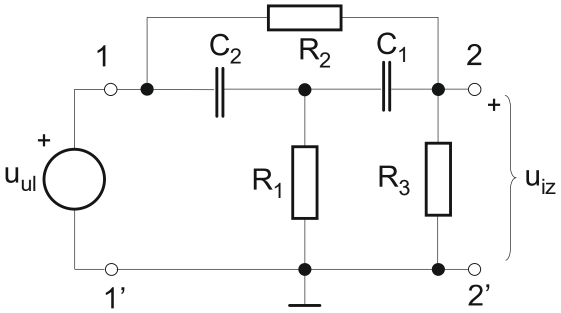
$$Y_{ul1}(s) = \frac{I_1(s)}{U_1(s)} = y_{11} - \frac{y_{12}y_{21}}{Y_2 + y_{22}} = 1 - \frac{1-\frac{1}{s}}{s+2} = \frac{s+2-1+\frac{1}{s}}{s+2} = \frac{s^2+s+1}{s(s+2)} \Rightarrow Z_{ul1}(s) = \frac{1}{Y_{ul1}(s)} = \frac{s(s+2)}{s^2+s+1}$$

Odgovori na pitanja:

a) Četveropol nije električki recipročan jer sadrži strujno ovisni naponski izvor i vrijedi: $y_{12} \neq y_{21}$.

b) Četveropol nije električki simetričan jer se parametri y_{11} i y_{22} razlikuju: $y_{11} \neq y_{22}$. **(1 bod)**

3. Za četveropol prikazan slikom izračunati [y]-parametre i napisati matricu [y]-parametara. Pomoću poznatih [y]-parametara izračunati naponsku prijenosnu funkciju četveropola $T(s)=U_{iz}(s)/U_{ul}(s)$. Da li je četveropol: *a*) recipročan, *b*) simetričan ? Obrazložiti odgovor. Zadane su normalizirane vrijednosti elemenata $R_1=R_2=R_3=1$, $C_1=C_2=1$.

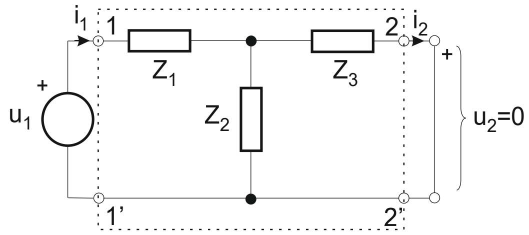


Rješenje:

[y]-parametri:

$$I_1 = y_{11}\cdot U_1 - y_{12}\cdot U_2$$

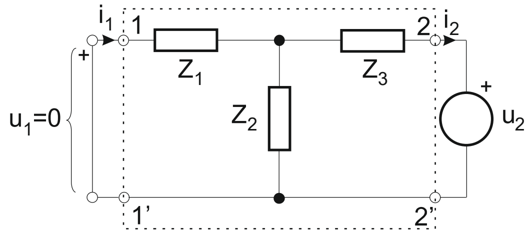
$$I_2 = y_{21}\cdot U_1 - y_{22}\cdot U_2$$



$$U_2 = 0,\quad I_2 = I_1\frac{Z_2}{Z_2+Z_3} \Rightarrow$$

$$y_{11} = \frac{I_1}{U_1} = \left(Z_1 + \frac{Z_2Z_3}{Z_2+Z_3}\right)^{-1} = \frac{Z_2+Z_3}{Z_1Z_2+Z_1Z_3+Z_2Z_3}$$

$$y_{21} = \frac{I_2}{U_1} = \frac{I_2}{I_1}\cdot\frac{I_1}{U_1} = \frac{Z_2}{Z_1Z_2+Z_1Z_3+Z_2Z_3}$$



$$U_1 = 0,\quad I_1 = \frac{Z_2}{Z_1+Z_2}I_2 \Rightarrow$$

$$y_{12} = -\frac{I_1}{U_2} = -\frac{I_1}{I_2}\cdot\frac{I_2}{U_2} = \frac{Z_2}{Z_1Z_2+Z_1Z_3+Z_2Z_3}$$

$$y_{22} = -\frac{I_2}{U_2} = \left(Z_3 + \frac{Z_1Z_2}{Z_1+Z_2}\right)^{-1} = \frac{Z_1+Z_2}{Z_1Z_2+Z_1Z_3+Z_2Z_3}$$

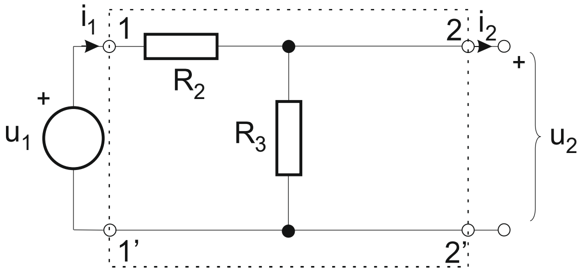
$$[y] = \begin{bmatrix} y_{11} & -y_{12} \\ y_{21} & -y_{22} \end{bmatrix} = \frac{1}{Z_1Z_2+Z_1Z_3+Z_2Z_3}\begin{bmatrix} Z_2+Z_3 & -Z_2 \\ Z_2 & -(Z_1+Z_2) \end{bmatrix}$$ **(1 bod)**

Prvi četveropol:

$$Z_1 = \frac{1}{sC_2}; Z_2 = R_1; Z_3 = \frac{1}{sC_1}$$

$$[y]^I = \frac{1}{\dfrac{R_1}{sC_2}+\dfrac{1}{sC_1sC_2}+\dfrac{R_1}{sC_1}}\begin{bmatrix} R_1+\dfrac{1}{sC_1} & -R_1 \\ R_1 & -\left(R_1+\dfrac{1}{sC_2}\right) \end{bmatrix} = \frac{1}{\dfrac{2}{s}+\dfrac{1}{s^2}}\begin{bmatrix} 1+\dfrac{1}{s} & -1 \\ 1 & -\left(1+\dfrac{1}{s}\right) \end{bmatrix}$$ **(1 bod)**

Drugi četveropol:



$Z_1 = R_2; Z_2 = R_3; Z_3 = 0$

$$[y]^{II} = \frac{1}{R_2 R_3}\begin{bmatrix} R_3 & -R_3 \\ R_3 & -(R_2+R_3) \end{bmatrix} = \begin{bmatrix} 1/R_2 & -1/R_2 \\ 1/R_2 & -(1/R_2+1/R_3) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$$ **(1 bod)**

Ukupni četveropol:

$$[y] = [y]^I + [y]^{II} = \frac{s^2}{2s+1}\begin{bmatrix} \frac{s+1}{s} & -1 \\ 1 & -\frac{s+1}{s} \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} \frac{s(s+1)}{2s+1}+1 & -\frac{s^2}{2s+1}-1 \\ \frac{s^2}{2s+1}+1 & -\frac{s(s+1)}{2s+1}-2 \end{bmatrix}$$

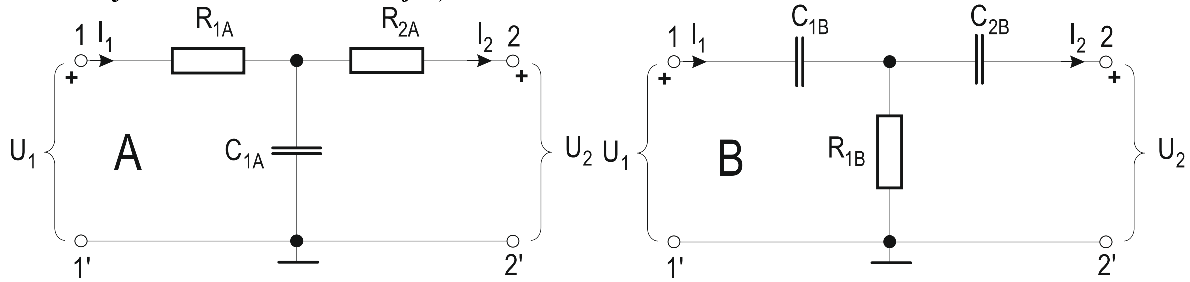
$$[y] = \begin{bmatrix} \frac{s^2+3s+1}{2s+1} & -\frac{s^2+2s+1}{2s+1} \\ \frac{s^2+2s+1}{2s+1} & -\frac{s^2+5s+2}{2s+1} \end{bmatrix}$$

Da li je četveropol: *a*) recipročan: DA, *b*) simetričan: NE **(1 bod)**

Prijenosna funkcija napona:

$$T(s) = \frac{U_2}{U_1} = \frac{y_{21}}{y_{22}} = \frac{\frac{s^2}{2s+1}+1}{\frac{s(s+1)}{2s+1}+2} = \frac{s^2+2s+1}{s(s+1)+4s+2} = \frac{s^2+2s+1}{s^2+5s+2} = \frac{(s+1)^2}{s^2+5s+2}$$ **(1 bod)**

3. Za četveropole A i B prikazane slikom izračunati: a) y-parametre, ako su zadane normalizirane vrijednosti elemenata $R_{1A}=R_{2A}=1$, $C_{1A}=2$ i $R_{1B}=1/2$, $C_{1B}=C_{2B}=1$. b) Iz pojedinih y-parametara izračunati ukupne y-parametare paralelne kombinacije dva četveropola (dvostruki T četveropol). c) Iz y-parametara izračunati prijenosnu funkciju napona $T(s)=U_{iz}(s)/U_{ul}(s)$ dvostrukog T četveropola ako je izlaz u praznom hodu. d) Nacrtati raspored nula i polova u s ravnini. e) Iz rasporeda nula i polova kvalitativno skicirati amplitudno-frekvencijsku karakteristiku $\left|T(j\omega)\right|$.

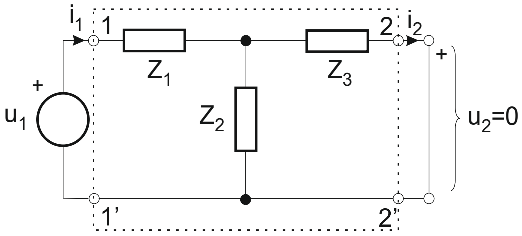


<u>Rješenje:</u> Nađimo najprije y-parametre četveropola. Ukupni četveropol se može promatrati kao kombinacija dva četveropola u paralelu.

a) $[y]$-parametri:

$I_1 = y_{11}\cdot U_1 - y_{12}\cdot U_2$

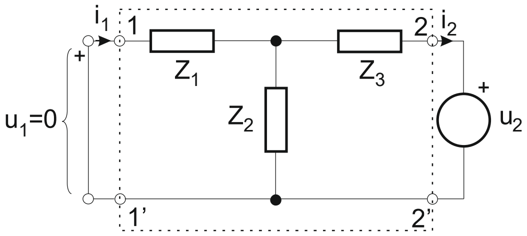
$\underline{I_2 = y_{21}\cdot U_1 - y_{22}\cdot U_2}$



$$U_2=0,\quad I_2 = I_1\frac{Z_2}{Z_2+Z_3}\Rightarrow$$

$$y_{11}=\frac{I_1}{U_1}=\left(Z_1+\frac{Z_2Z_3}{Z_2+Z_3}\right)^{-1}=\frac{Z_2+Z_3}{Z_1Z_2+Z_1Z_3+Z_2Z_3}$$

$$y_{21}=\frac{I_2}{U_1}=\frac{I_2}{I_1}\cdot\frac{I_1}{U_1}=\frac{Z_2}{Z_1Z_2+Z_1Z_3+Z_2Z_3}$$



$$U_1=0,\quad I_1=\frac{Z_2}{Z_1+Z_2}I_2\Rightarrow$$

$$y_{12}=-\frac{I_1}{U_2}=-\frac{I_1}{I_2}\cdot\frac{I_2}{U_2}=\frac{Z_2}{Z_1Z_2+Z_1Z_3+Z_2Z_3}$$

$$y_{22}=-\frac{I_2}{U_2}=\left(Z_3+\frac{Z_1Z_2}{Z_1+Z_2}\right)^{-1}=\frac{Z_1+Z_2}{Z_1Z_2+Z_1Z_3+Z_2Z_3}$$

$$[y]=\begin{bmatrix} y_{11} & -y_{12}\\ y_{21} & -y_{22}\end{bmatrix}=\frac{1}{Z_1Z_2+Z_1Z_3+Z_2Z_3}\begin{bmatrix} Z_2+Z_3 & -Z_2\\ Z_2 & -(Z_1+Z_2)\end{bmatrix}$$ **(1 bod)**

Prvi četveropol: $Z_1=R_{1A}; Z_2=\dfrac{1}{sC_{1A}}; Z_3=R_{2A}$

$$[y]^I=\frac{1}{\dfrac{R_{1A}}{sC_{1A}}+R_{1A}R_{2A}+\dfrac{R_{2A}}{sC_{1A}}}\begin{bmatrix} R_{2A}+\dfrac{1}{sC_{1A}} & -\dfrac{1}{sC_{1A}}\\ \dfrac{1}{sC_{1A}} & -\left(R_{1A}+\dfrac{1}{sC_{1A}}\right)\end{bmatrix}=\frac{1}{\dfrac{1}{s}+1}\begin{bmatrix} 1+\dfrac{1}{2s} & -\dfrac{1}{2s}\\ \dfrac{1}{2s} & -\left(1+\dfrac{1}{2s}\right)\end{bmatrix}$$

$$[y]^I = \begin{bmatrix} \dfrac{2s+1}{2(s+1)} & -\dfrac{1}{2(s+1)} \\ \dfrac{1}{2(s+1)} & -\dfrac{2s+1}{2(s+1)} \end{bmatrix}$$ **(1 bod)**

Drugi četveropol: $Z_1 = \dfrac{1}{sC_{1B}}; Z_2 = R_{1B}; Z_3 = \dfrac{1}{sC_{2B}}$

$$[y]^{II} = \frac{1}{\dfrac{R_{1B}}{sC_{1B}} + \dfrac{1}{s^2 C_{1B} C_{2B}} + \dfrac{R_{1B}}{sC_{2B}}} \begin{bmatrix} R_{1B} + \dfrac{1}{sC_{2B}} & -R_{1B} \\ R_{1B} & -\left(R_{1B} + \dfrac{1}{sC_{1B}}\right) \end{bmatrix} = \frac{1}{\dfrac{1}{s}\left(1+\dfrac{1}{s}\right)} \begin{bmatrix} \dfrac{1}{2}+\dfrac{1}{s} & -\dfrac{1}{2} \\ \dfrac{1}{2} & -\left(\dfrac{1}{2}+\dfrac{1}{s}\right) \end{bmatrix}$$

$$[y]^{II} = \begin{bmatrix} \dfrac{s(s+2)}{2(s+1)} & -\dfrac{s^2}{2(s+1)} \\ \dfrac{s^2}{2(s+1)} & -\dfrac{s(s+2)}{2(s+1)} \end{bmatrix}$$

(1 bod)

b) Ukupni četveropol:

$$[y] = [y]^I + [y]^{II} = \begin{bmatrix} \dfrac{2s+1}{2(s+1)} & -\dfrac{1}{2(s+1)} \\ \dfrac{1}{2(s+1)} & -\dfrac{2s+1}{2(s+1)} \end{bmatrix} + \begin{bmatrix} \dfrac{s(s+2)}{2(s+1)} & -\dfrac{s^2}{2(s+1)} \\ \dfrac{s^2}{2(s+1)} & -\dfrac{s(s+2)}{2(s+1)} \end{bmatrix} = \begin{bmatrix} \dfrac{s^2+4s+1}{2(s+1)} & -\dfrac{s^2+1}{2(s+1)} \\ \dfrac{s^2+1}{2(s+1)} & -\dfrac{s^2+4s+1}{2(s+1)} \end{bmatrix}$$

c) Prijenosna funkcija napona s praznim hodom na izlazu (nula i pol se pokrate):

$$H(s) = \frac{U_2(s)}{U_1(s)} = \frac{y_{21}}{y_{22}} = \frac{\dfrac{s^2+1}{2(s+1)}}{\dfrac{s^2+4s+1}{2(s+1)}} = \frac{2(s+1)(s^2+1)}{2(s+1)(s^2+4s+1)} = \frac{s^2+1}{s^2+4s+1}$$

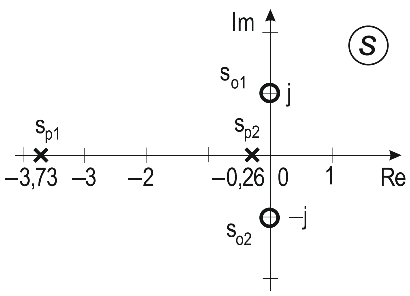
d) raspored nula i polova: **(1 bod)**

Nule: $s^2+1=0 \Rightarrow s_{o1,2} = \pm j$

Polovi: $s^2+4s+1=0 \Rightarrow$

$s_{p1,2} = -2 \pm \sqrt{4-1} = -2 \pm \sqrt{3}$ ili

$s_{p1} = -3{,}73205; s_{p2} = -0{,}267949$

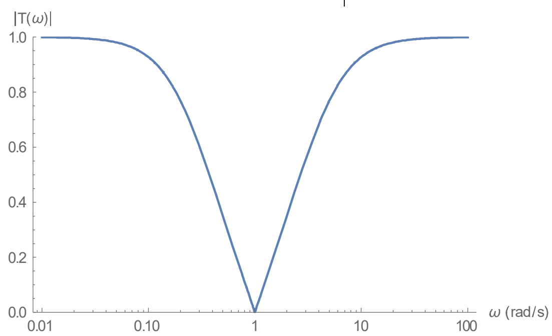


e) amplitudno-frekvencijska karakteristika

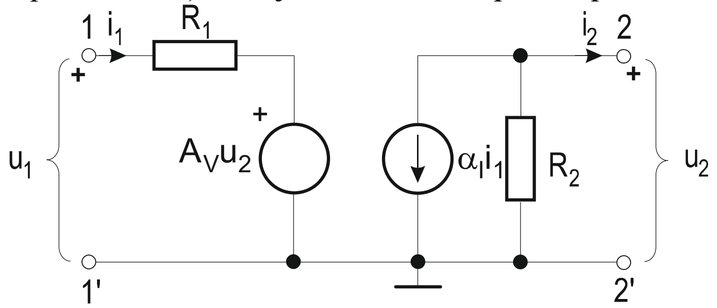
Vidljivo je $H(0)=1;\ H(1)=0; H(\infty)=1;$

$$s{=}j\omega \Rightarrow T(j\omega) = \frac{-\omega^2+1}{-\omega^2+4\cdot j\omega+1} \Rightarrow$$

$$|T(j\omega)| = \frac{|1-\omega^2|}{\sqrt{(1-\omega^2)^2+(4\cdot\omega)^2}}$$ **(1 bod)**



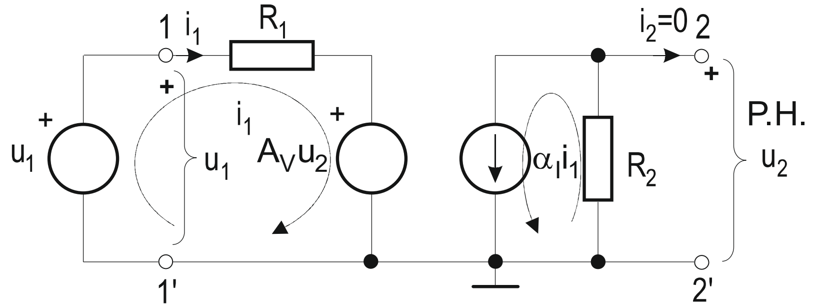
3. Za četveropol prikazan slikom izračunati: *a*) prijenosne *a*-parametre, ako su zadane normalizirane vrijednosti elemenata R_1=1, R_2=2, te parametri A_V=1 i α_I=1/2. *b*) Iz *a*-parametara izračunati hibridne *h*-parametre. *c*) Da li je zadani četveropol recipročan i da li je simetričan?



Rješenje: Nađimo najprije *a*-parametre četveropola.

a) *a*-parametri

$$\begin{aligned} U_1 &= A\cdot U_2 + B\cdot I_2 \\ I_1 &= C\cdot U_2 + D\cdot I_2 \end{aligned} \qquad \underline{I_2 = 0} \qquad A = \left.\frac{U_1}{U_2}\right|_{I_2=0} \qquad C = \left.\frac{I_1}{U_2}\right|_{I_2=0}$$

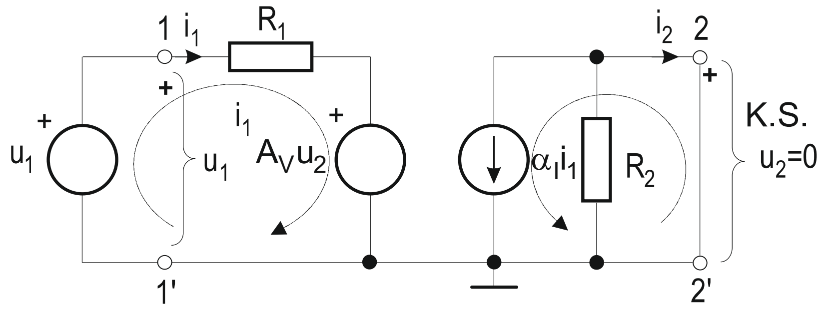


$$U_1 = I_1R_1 + A_V\cdot U_2 \Rightarrow U_1 = I_1\cdot R_1 - A_V\cdot\alpha_I\cdot I_1\cdot R_2 = I_1\cdot(R_1 - A_V\alpha_I R_2)$$

$$\underline{U_2 = -\alpha_I I_1\cdot R_2} \Rightarrow \qquad C = \frac{I_1}{U_2} = -\frac{1}{\alpha_I\cdot R_2}$$

$$A = \frac{U_1}{U_2} = \frac{I_1\cdot(R_1 - A_V\alpha_I R_2)}{-\alpha_I I_1\cdot R_2} = -\frac{R_1 - A_V\alpha_I R_2}{\alpha_I R_2} = A_V - \frac{R_1}{\alpha_I R_2}$$

$$\underline{U_2 = 0} \qquad B = \left.\frac{U_1}{I_2}\right|_{U_2=0} \qquad D = \left.\frac{I_1}{I_2}\right|_{U_2=0}$$



$$U_1 = I_1R_1$$

$$\underline{I_2 = -\alpha_I I_1} \Rightarrow \qquad D = \frac{I_1}{I_2} = -\frac{1}{\alpha_I}$$

$$B = \frac{U_1}{I_2} = \frac{R_1\cdot I_1}{-\alpha_I\cdot I_1} = -\frac{R_1}{\alpha_I}$$

Matrica $[a]=\begin{bmatrix} A & B \\ C & D \end{bmatrix}=\begin{bmatrix} A_V-\dfrac{R_1}{\alpha_I R_2} & -\dfrac{R_1}{\alpha_I} \\ -\dfrac{1}{\alpha_I R_2} & -\dfrac{1}{\alpha_I} \end{bmatrix}=\begin{bmatrix} 0 & -2 \\ -1 & -2 \end{bmatrix}$ **(2 boda)**

$\det(a)=A\cdot D-B\cdot C=-2\neq 1$ četveropol nije recipročan. $A\neq D$ četveropol nije simetričan.

b) Hibridni *h*-parametri

$$U_1=h_{11}I_1+h_{12}U_2$$
$$I_2=h_{21}I_1+h_{22}U_2$$

$\underline{U_2=0}$ $\quad h_{11}=\left.\dfrac{U_1}{I_1}\right|_{U_2=0}\quad h_{21}=\left.\dfrac{I_2}{I_1}\right|_{U_2=0}\quad \underline{I_1=0}\quad h_{12}=\left.\dfrac{U_1}{U_2}\right|_{I_1=0}\quad h_{22}=\left.\dfrac{I_2}{U_2}\right|_{I_1=0}$

Ako malo preuredimo jednadžbe za prijenosne parametre:

$$U_1=A\cdot U_2+B\cdot I_2\Rightarrow U_1=A\cdot U_2+B\cdot\left(\frac{1}{D}\cdot I_1-\frac{C}{D}\cdot U_2\right)\Rightarrow U_1=\frac{B}{D}\cdot I_1+\left(A-\frac{B\cdot C}{D}\right)\cdot U_2$$

$$I_1=C\cdot U_2+D\cdot I_2\Rightarrow I_2=\frac{1}{D}\cdot I_1-\frac{C}{D}\cdot U_2$$

I usporedimo ih sa jednadžbama za hibridne parametre, slijedi:

$$h_{11}=\frac{U_1}{I_1}=\frac{B}{D};\; h_{12}=\frac{U_1}{U_2}=A-\frac{B\cdot C}{D}=\frac{A\cdot D-B\cdot C}{D}=\frac{\det(a)}{D}$$

$$h_{21}=\frac{I_2}{I_1}=\frac{1}{D};\; h_{22}=\frac{I_2}{U_2}=-\frac{C}{D}$$

Uvrstivši *a*-parametre slijedi:

$$h_{11}=\frac{B}{D}=\frac{-\dfrac{R_1}{\alpha_I}}{-\dfrac{1}{\alpha_I}}=R_1$$

$$h_{12}=\frac{AD-BC}{D}=\frac{\left(A_V-\dfrac{R_1}{\alpha_I R_2}\right)\cdot\left(-\dfrac{1}{\alpha_I}\right)-\left(-\dfrac{R_1}{\alpha_I}\right)\cdot\left(-\dfrac{1}{\alpha_I R_2}\right)}{-\dfrac{1}{\alpha_I}}=\left(A_V-\frac{R_1}{\alpha_I R_2}\right)-\left(-\frac{R_1}{\alpha_I R_2}\right)=A_V$$

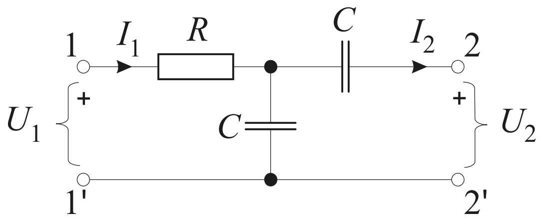
$$h_{21}=\frac{1}{D}=-\alpha_I$$

$$h_{22}=-\frac{C}{D}=\alpha_I\cdot\left(-\frac{1}{\alpha_I\cdot R_2}\right)=-\frac{1}{R_2}$$

Matrica $[h]=\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}=\begin{bmatrix} R_1 & A_V \\ -\alpha_I & -\dfrac{1}{R_2} \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ -\dfrac{1}{2} & -\dfrac{1}{2} \end{bmatrix}$ **(2 boda)**

$h_{12}\neq h_{21}$ četveropol nije recipročan. $\det(h)=h_{11}\cdot h_{22}-h_{12}\cdot h_{21}=0\neq 1$ četveropol nije simetričan. **(1 bod)**

3. Za četveropol prikazan slikom izračunati prijenosne [z]-parametre i napisati matricu [z]-parametara. Odrediti naponsku prijenosnu funkciju četveropola $H_u(s)=U_2/U_1$ ako je prilaz 2–2' ostavljen u praznom hodu. Koliko iznose zrcalni parametri četveropola Z_{C1}, Z_{C2} i $th(g)$? Spojimo li dva četverpola u lanac, kakve moraju biti zrcalne impedancije Z_{C1} i Z_{C2} drugog četveropola da bi lanac bio prilagođen? Zadane su normalizirane vrijednosti elemenata R=1, C=1.

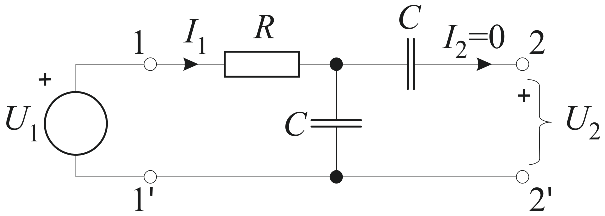


Rješenje:

[z]-parametri: **(1 bod)**

$$U_1 = z_{11}I_1 - z_{12}I_2$$
$$U_2 = z_{21}I_1 - z_{22}I_2$$

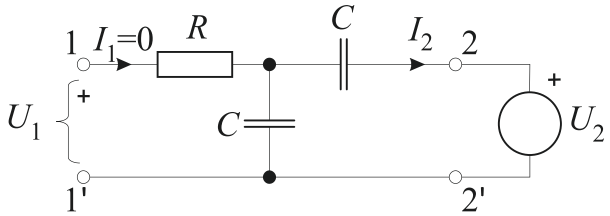
I_2=0 $\Rightarrow$ $z_{11} = \left.\frac{U_1}{I_1}\right|_{I_2=0}$ $\quad z_{21} = \left.\frac{U_2}{I_1}\right|_{I_2=0}$



$$I_1 = \frac{U_1}{R+\frac{1}{sC}} \rightarrow z_{11} = R + \frac{1}{sC}$$

$$U_2 = I_1 \cdot \frac{1}{sC} \rightarrow z_{21} = \frac{1}{sC}$$

I_1=0 $\Rightarrow$ $z_{12} = -\left.\frac{U_1}{I_2}\right|_{I_1=0}$ $\quad z_{22} = -\left.\frac{U_2}{I_2}\right|_{I_1=0}$



$$I_2 = -U_2 \cdot s\frac{C}{2} \rightarrow z_{22} = \frac{2}{sC}$$

$$U_1 = -I_2 \cdot \frac{1}{sC} \rightarrow z_{12} = \frac{1}{sC}$$

$$[z] = \frac{1}{sC} \cdot \begin{bmatrix} 1+sRC & -1 \\ 1 & -2 \end{bmatrix} = \frac{1}{s}\begin{bmatrix} 1+s & -1 \\ 1 & -2 \end{bmatrix}$$

Naponska prijenosna funkcija: **(1 bod)**

$$H_u(s) = \frac{U_2(s)}{U_1(s)} = \frac{Z_T \cdot z_{21}}{z_{11} \cdot (z_{22} + Z_T) - z_{12}z_{21}}$$

$$\rightarrow H_u(s) = \lim_{Z_T \to \infty}\left(\frac{Z_T \cdot z_{21}}{z_{11}\cdot(z_{22}+Z_T) - z_{12}z_{21}}\right) = \frac{z_{21}}{z_{11}} = \frac{\frac{1}{sC}}{R+\frac{1}{sC}} = \frac{1}{1+sRC} = \frac{1}{1+s}$$

Zrcalne impedancije: **(2 boda)**

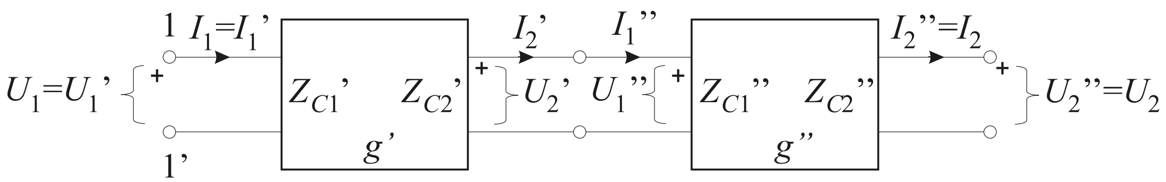
$$[y]=[z]^{-1}=\frac{1}{\frac{1}{s}\cdot\begin{vmatrix}1+s & -1\\ 1 & -2\end{vmatrix}}\cdot\begin{bmatrix}-2 & 1\\ -1 & 1+s\end{bmatrix}=\frac{-s}{2s+1}\cdot\begin{bmatrix}-2 & 1\\ -1 & 1+s\end{bmatrix}$$

$$Z_{C1}=\sqrt{Z_{p1}\cdot Z_{k1}}=\sqrt{z_{11}\cdot\frac{1}{y_{11}}}=\sqrt{\frac{\frac{1+s}{s}}{\frac{2s}{2s+1}}}=\sqrt{\frac{1+s}{s}\cdot\frac{2s+1}{2s}}=\frac{1}{s\sqrt{2}}\cdot\sqrt{(s+1)\cdot(2s+1)}$$

$$Z_{C2}=\sqrt{Z_{p2}\cdot Z_{k2}}=\sqrt{z_{22}\cdot\frac{1}{y_{22}}}=\sqrt{\frac{\frac{2}{s}}{\frac{s(s+1)}{2s+1}}}=\sqrt{\frac{2}{s}\cdot\frac{2s+1}{s(s+1)}}=\frac{\sqrt{2}}{s}\cdot\sqrt{\frac{2s+1}{s+1}}$$

$$tg(g)=\sqrt{\frac{Z_{k1}}{Z_{p1}}}=\sqrt{\frac{\frac{1}{y_{11}}}{z_{11}}}=\sqrt{\frac{1}{z_{11}\cdot y_{11}}}=\sqrt{\frac{1}{\frac{s+1}{s}\cdot\frac{2s}{2s+1}}}=\sqrt{\frac{2s+1}{2(s+1)}}$$

Spoj dva četveropola: **(1 bod)**

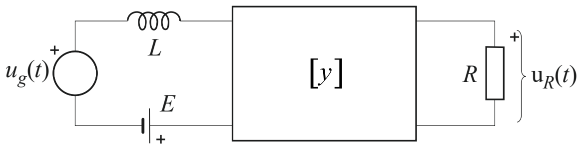


Da bi ova kaskada bila prilagođena mora vijediti: $Z_{C1}"=Z_{C2}'$

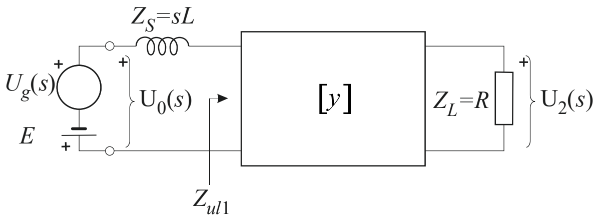
Za $Z_{C2}"$ ne postoji nikakav uvjet u pogledu prilagođenja ova dva četveropola. Prilikom daljnjeg spajanja kaskade potrebno je izabrati $Z_T=Z_{C2}"$.

3. Za mrežu prikazanu slikom odredite i skicirajte napon na izlaznom otporu, $u_R(t)$, ako je zadano: R=1, L=1, E=1 (napon baterije), $u_g(t)$=sin(t); $-\infty<t<\infty$, i četveropol je zadan y-parametrima. Pobuda je u stacionarnom stanju, početni uvjeti su jednaki nula. Koristiti metodu superpozicije.

$$[y]=\begin{bmatrix}3 & -3\\ 4 & -5\end{bmatrix}$$



Rješenje:

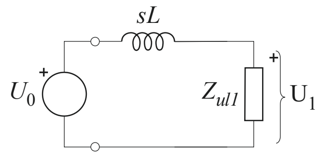


Uz $[y]=\begin{bmatrix}3 & -3\\ 4 & -5\end{bmatrix}=\begin{bmatrix}y_{11} & -y_{12}\\ y_{21} & -y_{22}\end{bmatrix}\Rightarrow y_{11}=3; y_{21}=4; y_{12}=3; y_{22}=5$ slijedi:

Ulazna admitancija četveropola:

$$Y_{ul1}=y_{11}-\frac{y_{12}\cdot y_{21}}{Y_L+y_{22}}=3-\frac{3\cdot 4}{1+5}=1 \quad \textbf{(1 bod)}$$

$$Z_{ul1}=\frac{1}{Y_{ul1}}=1$$



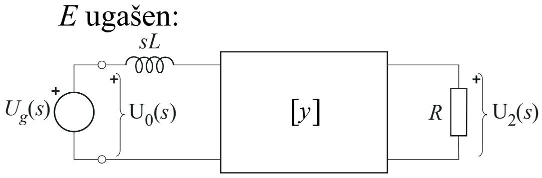
$$\rightarrow U_1=\frac{y_{21}}{Z_s+Z_{ul1}}\cdot U_0=\frac{1}{s+1}\cdot U_0 \quad \textbf{(1 bod)}$$

Prijenosna funkcija četveropola:

$$H(s)=\frac{U_2(s)}{U_1(s)}=\frac{y_{21}}{Y_L+y_{22}}=\frac{4}{1+5}=\frac{2}{3} \quad\Rightarrow\quad \frac{U_2(s)}{U_0(s)}=\frac{U_2(s)}{U_1(s)}\cdot\frac{U_1(s)}{U_0(s)}=\frac{2}{3}\cdot\frac{1}{s+1}$$

Metoda superpozicije:

E ugašen:

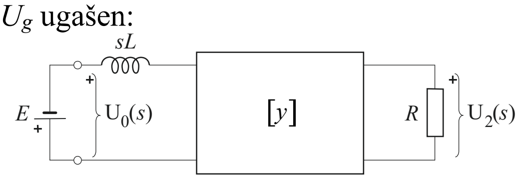


$$U_0(j\omega)=1\cdot e^{j\cdot 0} \text{ za } \omega=1$$

$$U_2(j\omega)=\frac{2}{3}\cdot\frac{1}{j\omega+1}\cdot 1\Big|_{\omega=1}=\frac{1}{3}\cdot(-j+1)=\frac{\sqrt{2}}{3}\cdot e^{-j\cdot\frac{\pi}{4}}$$

$$\underline{u_{R\_1}(t)=\frac{\sqrt{2}}{3}\cdot\sin\left(t-\frac{\pi}{4}\right)} \quad \textbf{(1 bod)}$$

U_g ugašen:



$$U_0(j\omega)=-1\cdot e^{j\cdot 0} \text{ za } \omega=0$$

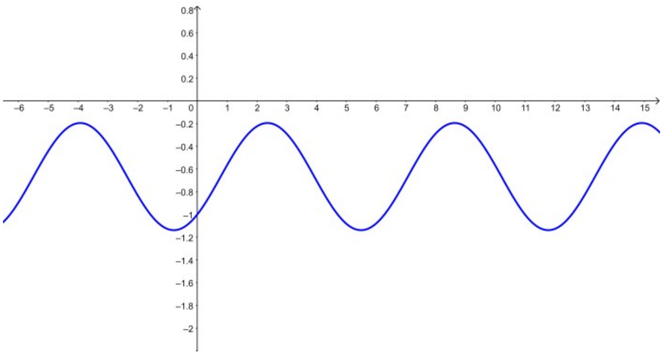
$$U_2(j\omega)=\frac{2}{3}\cdot\frac{1}{j\omega+1}\cdot(-1)\Big|_{\omega=0}=-\frac{2}{3}$$

$$\underline{u_{R\_2}(t)=-\frac{2}{3}} \quad \textbf{(1 bod)}$$

Konačno:

$$\underline{u_R(t) = u_{R\_1}(t) + u_{R\_2}(t) = -\frac{2}{3} + \frac{\sqrt{2}}{3} \cdot \sin\left(t - \frac{\pi}{4}\right)}$$

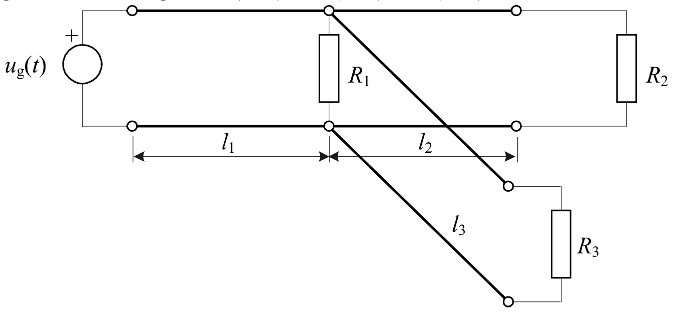
Skica napona na otporu R: **(1 bod)**



LINIJE

5. Tri linije bez gubitaka spojene su prema slici. Zadano je: L=0,2mH/km, C=80nF/km, u_g= 10cos(2.5π10^5 t) V, R_2 = 25 Ω, R_3= 100 Ω, l_1=3λ/4, l_2=λ/4 i l_3=λ/2. Odrediti:

a) valnu impedanciju i koeficijent prijenosa linija;
b) brzinu širenja vala na linijama i duljinu druge i treće linije;
c) otpor R_1 da bi prva linija bila prilagođena na izlazu;
d) faktore refleksije na krajevima druge i treće linije: Γ_{i2} i Γ_{i3};
e) napone na kraju svake linije: $u_1(l_1,t)$, $u_2(l_2,t)$, $u_3(l_3,t)$.



Rješenje:

a) $Z_0=\sqrt{\dfrac{L}{C}}=\sqrt{\dfrac{2\cdot10^{-4}}{8\cdot10^{-8}}}=50\Omega\,;\gamma=j\beta\,;$

$\beta=\omega_0\sqrt{LC}=2.5\cdot\pi\cdot10^5\sqrt{2\cdot10^{-4}\cdot8\cdot10^{-8}}=\pi$ [rad/km] **(1 bod)**

b) $v=\dfrac{\omega}{\beta}=\dfrac{1}{\sqrt{LC}}=\dfrac{1}{\sqrt{2\cdot10^{-4}\cdot8\cdot10^{-8}}}=\dfrac{10^6}{4}=25\cdot10^4$ [km/s]

$\lambda=\dfrac{2\cdot\pi}{\beta}=\dfrac{2\cdot\pi}{\pi}=2\,[\text{km}]$ $\qquad l_2=\dfrac{\lambda}{4}=500$ [m]; $\qquad l_2=\dfrac{\lambda}{2}=1$ [km]; **(1 bod)**

c) $\gamma\cdot l_2=j\cdot\beta\cdot l_2=j\dfrac{\pi}{2}\,;\qquad \gamma\cdot l_3=j\cdot\beta\cdot l_3=j\pi$

$$Z_{ul2}=\frac{R_2ch(\gamma\cdot l_2)+Z_0sh(\gamma\cdot l_2)}{\frac{R_2}{Z_0}sh(\gamma\cdot l_2)+ch(\gamma\cdot l_2)}=\frac{R_2\cos(\beta\cdot l_2)+jZ_0\sin(\beta\cdot l_2)}{j\frac{R_2}{Z_0}\sin(\beta\cdot l_2)+\cos(\beta\cdot l_2)}=\frac{Z_0^2}{R_2}=\frac{2500}{25}=100\,\Omega$$

$$Z_{ul3}=\frac{R_3\cos(\beta\cdot l_3)+jZ_0\sin(\beta\cdot l_3)}{j\frac{R_3}{Z_0}\sin(\beta\cdot l_2)+\cos(\beta\cdot l_3)}=\frac{-R_3}{-1}=R_3=100\,\Omega\quad Z_{ul2}\|Z_{ul3}=50\,\Omega\Rightarrow R_1=\infty$$ **(1 bod)**

d) $\Gamma_{i2}=\dfrac{R_2-Z_0}{R_2+Z_0}=\dfrac{-25}{75}=-\dfrac{1}{3}\qquad \Gamma_{i3}=\dfrac{R_3-Z_0}{R_3+Z_0}=\dfrac{50}{150}=\dfrac{1}{3}$ **(1 bod)**

e) $\gamma\cdot l_1=j\beta\dfrac{3\lambda}{4}=j\beta\dfrac{3\cdot2\pi}{4\cdot\beta}=j\dfrac{3\pi}{2}\,;$

$U_1(l_1)=U(0)\cdot e^{-j\beta l_1}=10\cdot e^{-j3\pi/2}=10j\,;\quad u_1(l_1,t)=10\cos\left(\omega t+\dfrac{\pi}{2}\right)$

$U_2(l_2)=U_1(l_1)\cdot\cos(\beta\cdot l_2)-jU_1(l_1)\cdot\dfrac{Z_0}{Z_{ul2}}\sin(\beta\cdot l_2)=-jU_1(l_1)\cdot\dfrac{Z_0}{Z_{ul2}}=j5\cdot e^{-j\pi/2}=5$

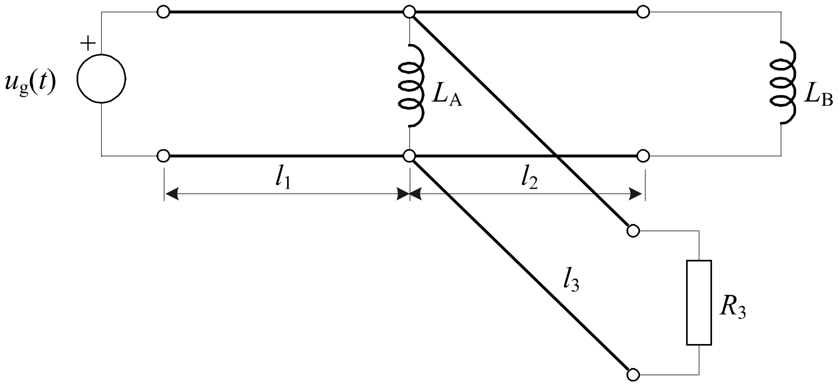
$u_1(l_2,t)=5\cos(\omega t)$

$U_3(l_3)=U(l_1)\cdot\cos(\beta\cdot l_3)-jU(l_1)\cdot\dfrac{Z_0}{Z_{ul3}}\sin(\beta\cdot l_3)=-U(l_1)=-10\cdot e^{-j3\pi/2}=10\cdot e^{-j\pi/2}$

$u_1(l_3,t)=10\cos(\omega t-\pi/2)$ **(1 bod)**

5. Tri linije bez gubitaka spojene su prema slici. Zadano je: L_1=0,3mH/km, C_1=120nF/km, L_2= L_3=281,25μH/km, C_2= C_3=50nF/km, L_A=0,1mH, u_g=10 cos($2\pi 10^5$ t) V, $l_1=3\lambda_1/4$, $l_2=\lambda_2/4$ i $l_3=\lambda_3/4$. Odrediti:

a) valne impedancije i koeficijente prijenosa linija;
b) brzine širenja vala na linijama i duljine linija;
c) induktivitet L_B da bi ukupna impedancija na izlazu prve linije bila realna;
d) otpor R_3 da bi prva linija bila prilagođena na izlazu;
e) napone na kraju svake linije: $u_1(l_1,t)$, $u_2(l_2,t)$, $u_3(l_3,t)$.



Rješenje:

a) $Z_{01}=\sqrt{\dfrac{L_1}{C_1}}=\sqrt{\dfrac{3\cdot10^{-4}}{1{,}2\cdot10^{-7}}}=50\Omega \qquad Z_{02}=Z_{03}=\sqrt{\dfrac{L_2}{C_2}}=\sqrt{\dfrac{2{,}8125\cdot10^{-4}}{5\cdot10^{-8}}}=75\Omega$

$\gamma=j\beta \qquad \beta_1=\omega_0\sqrt{L_1C_1}=2\cdot\pi\cdot10^5\sqrt{3\cdot10^{-4}\cdot12\cdot10^{-8}}=2\cdot\pi\cdot10^5\cdot6\cdot10^{-6}=1{,}2\cdot\pi\ \text{[rad/km]}$

$\beta_2=\beta_3=\omega_0\sqrt{L_2C_2}=2\cdot\pi\cdot10^5\sqrt{2{,}8125\cdot10^{-4}\cdot5\cdot10^{-8}}=2\cdot\pi\cdot10^5\cdot3{,}75\cdot10^{-6}=0{,}75\cdot\pi\ \text{[rad/km]}$

(1 bod)

b) $v_1=\dfrac{1}{\sqrt{L_1C_1}}=\dfrac{1}{\sqrt{3\cdot10^{-4}\cdot12\cdot10^{-8}}}=\dfrac{10^6}{6}=16{,}67\cdot10^4\ \text{[km/s]}$

$v_2=v_3=\dfrac{1}{\sqrt{L_2C_2}}=\dfrac{1}{\sqrt{2{,}8125\cdot10^{-4}\cdot5\cdot10^{-8}}}=\dfrac{10^6}{3{,}75}=26{,}67\cdot10^4\ \text{[km/s]}$

$l_1=\dfrac{3\lambda_1}{4}=\dfrac{3\cdot2\cdot\pi}{4\beta_1}=\dfrac{6\cdot\pi}{4\cdot1{,}2\pi}=1{,}25\,\text{km}\quad l_2=l_3=\dfrac{\lambda_2}{4}=\dfrac{2\pi}{4\beta_2}=\dfrac{2\pi}{4\cdot0{,}75\pi}=666{,}67\ \text{[m]}$; **(1 bod)**

c) $\gamma_1\cdot l_1=j\cdot\beta_1\cdot\dfrac{3\lambda_1}{4}=j\beta_1\dfrac{3\cdot2\pi}{4\beta_1}=j\dfrac{3\pi}{2};\qquad \gamma_2\cdot l_2=\gamma_3\cdot l_3=j\cdot\beta_2\cdot l_2=j\dfrac{\pi}{2}$

$$Z_{ul2}=\frac{Z_2ch(\gamma_2\cdot l_2)+Z_{02}sh(\gamma_2\cdot l_2)}{\dfrac{Z_2}{Z_{02}}sh(\gamma_2\cdot l_2)+ch(\gamma_2\cdot l_2)}=\frac{Z_2\cos(\beta_2\cdot l_2)+jZ_{02}\sin(\beta_2\cdot l_2)}{j\dfrac{Z_2}{Z_{02}}\sin(\beta_2\cdot l_2)+\cos(\beta_2\cdot l_2)}=\frac{Z_{02}^2}{j\omega L_B}\,\Omega$$

$\dfrac{1}{Z_{LA}}+\dfrac{1}{Z_{ul2}}=\dfrac{1}{j\omega L_A}+j\dfrac{\omega L_B}{Z_{02}^2}=0 \qquad L_B=\dfrac{Z_{02}^2}{\omega^2L_A}=\dfrac{75^2}{4\pi^2\cdot10^{10}\cdot1\cdot10^{-4}}=142{,}5\mu\text{H}$ **(1 bod)**

d) $Z_{ul3}=\dfrac{R_3\cos(\beta_3\cdot l_3)+jZ_{03}\sin(\beta_3\cdot l_3)}{j\dfrac{R_3}{Z_{03}}\sin(\beta_3\cdot l_3)+\cos(\beta_3\cdot l_3)}=\dfrac{Z_{03}^2}{R_3}=Z_{01}=50\,\Omega\ \Rightarrow\ R_3=\dfrac{Z_{03}^2}{Z_{01}}=\dfrac{75^2}{50}=112{,}5\Omega$

(1 bod)

e) $U_1(l_1) = U(0) \cdot e^{-j\beta_1 l_1} = 10 \cdot e^{-j3\pi/2} = 10j \qquad u_1(l_1,t) = 10\cos\left(\omega t + \dfrac{\pi}{2}\right)$

$$U_2(0) = U_3(0) = U_1(l_1) = 10j$$

$$U_2(l_2) = U_2(0) \cdot \cos(\beta_2 \cdot l_2) - jU_2(0) \cdot \frac{Z_{02}}{Z_{ul2}} \sin(\beta_2 \cdot l_2) =$$

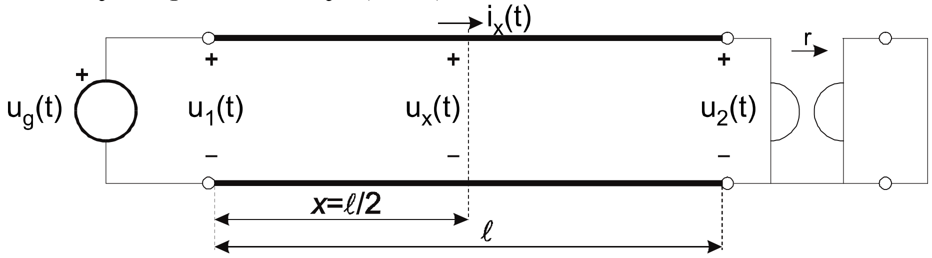
$$= -jU_1(l_1) \cdot \frac{Z_{02}}{-j\omega L_A} = 10j \cdot \frac{75}{2\pi 10^5 \cdot 10^{-4}} = 11{,}94j$$

$$u_2(l_2,t) = 11{,}94\cos\left(\omega t + \frac{\pi}{2}\right)$$

$$U_3(l_3) = U_3(0) \cdot \cos(\beta_3 \cdot l_3) - jU_3(0) \cdot \frac{Z_{03}}{Z_{ul3}} \sin(\beta_3 \cdot l_3) = -jU_1(l_1) \cdot \frac{Z_{03}}{Z_{01}} = 15$$

$u_3(l_3,t) = 15\cos(\omega t)$ **(1 bod)**

5. Zadana je linija bez gubitaka prema slici s primarnim parametrima L=4mH/km i C=8nF/km i duljine $l=\sqrt{2}$/16km. Na ulazu linije spojen je generator sinusnog valnog oblika $u_g(t)$=10 sin 2π 10^6 t , a na izlazu linije girator koji je na kraju kratko spojen. Odrediti valni oblik napona i struje na polovini linije ($x=l/2$).



Rješenje:

$L = 4[\text{mH/km}];\; C = 8[\text{nF/km}] \Rightarrow$

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{4\cdot 10^{-3}}{8\cdot 10^{-9}}} = \frac{\sqrt{2}}{2}\cdot 10^3[\Omega]$$

$$\gamma = j\beta = j\omega\sqrt{LC} = j\omega\sqrt{4\cdot 10^{-3}\cdot 8\cdot 10^{-9}} = j\omega\cdot 4\sqrt{2}\cdot 10^{-6}[\text{/km}]$$

$$g = \gamma l = j\omega_0\cdot\sqrt{LC}\cdot l = j\cdot 2\pi\cdot 10^6\cdot 4\sqrt{2}\cdot 10^{-6}\left[\frac{1}{\text{km}}\right]\cdot\frac{\sqrt{2}}{16}[\text{km}] = j\pi$$

$$\lambda_0 = \frac{2\pi}{\omega_0\sqrt{LC}} = \frac{2\pi}{2\pi\cdot 10^6\cdot 4\sqrt{2}\cdot 10^{-6}} = \frac{\sqrt{2}}{8}[\text{km}] \;\Rightarrow\; l = \frac{\lambda_0}{2}$$

a) na izlazu je prazni hod, tj. $i_2(t) = 0$, a zbog $l = \dfrac{\lambda_0}{2}$ na ulazu je također $i_1(t) = 0$

te je $u_1(t) = u_g(t)$. Također vrijedi:

$$shjx = j\sin x$$
$$chjx = \cos x$$

za $x = l;\quad g = j\pi$

$$shj\pi = j\sin\pi = 0$$
$$ch\, j\pi = \cos\pi = 1$$

Pa je:

$$Z_{ul} = \frac{U_1}{I_1} = \frac{U_2 ch\, g + I_2 Z_0 sh\, g}{U_2\dfrac{sh\, g}{Z_0} + I_2 ch\, g} = \frac{Z_2 ch\, g + Z_0 sh\, g}{\dfrac{Z_2}{Z_0} sh\, g + ch\, g} = Z_2;$$

$Z_2 = \infty$ je ulazni otpor u girator kojemu je izlaz u kratkom spoju

$$Z_{ul} = Z_2 = \infty \quad\Rightarrow\quad I_1 = 0,\; U_1 = U_g$$

b) na mjestu $x = \dfrac{l}{2} = \dfrac{\lambda_0}{4}$ je $u_x(t) = 0$, te još treba izračunati $i_x(t)$

$$g_x = \gamma\frac{l}{2} = j\frac{\pi}{2}$$

$$U(x) = U_1 ch\, \gamma x - I_1 Z_0 sh\, \gamma x$$

$$I(x) = -\frac{U_1}{Z_0} sh\, \gamma x + I_1 ch\, \gamma x$$

$$U\left(x=\frac{l}{2}=\frac{\lambda_0}{4}\right)=U_1 ch\, j\frac{\pi}{2}-I_1 Z_0 sh\, j\frac{\pi}{2}$$

$$I\left(x=\frac{l}{2}=\frac{\lambda_0}{4}\right)=-\frac{U_1}{Z_0} sh\, j\frac{\pi}{2}+I_1 ch\, j\frac{\pi}{2}$$

$$shj\frac{\pi}{2}=j\sin\frac{\pi}{2}=j$$

$$chj\frac{\pi}{2}=\cos\frac{\pi}{2}=0$$

$$U(x)=U_1\cdot 0-I_1 Z_0 j$$

$$I(x)=-\frac{U_1}{Z_0}j+I_1 0$$

$$U_1=10\angle 0°,\quad I_1=0$$

Pa je stoga:

$$U(x)=0$$

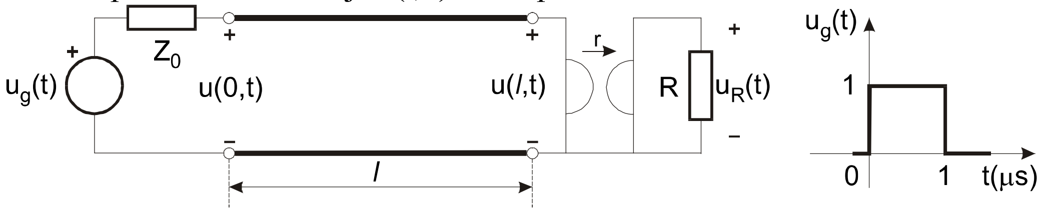
$$I(x)=-\frac{U_1}{Z_0}j=-\frac{10}{\sqrt{2}/2}\cdot 10^{-3}j=-\sqrt{2}\cdot 10^{-2}j=\sqrt{2}\cdot 10^{-2}\angle -180°+90°=\sqrt{2}\cdot 10^{-2}\angle -90°$$

Rješenje:

$$u_x(t)=0$$

$$i_x(t)=\sqrt{2}\cdot 10^{-2}\cdot\sin(2\pi 10^6 t-90°)$$

5. Zadana je linija s primarnim parametrima R=0,5Ω/km, L=10μH/km, G=2S/km, C=40μF/km, i duljinom l=1km. Na liniju je spojen generator $u_g(t)$ s unutarnjim otporom jednakim zrcalnoj impedanciji linije Z_0 i valnim oblikom prema slici. Koliki mora biti r giratora da bi linija bila prilagođena na izlazu po zrcalnoj impedanciji Z_0? Odrediti i nacrtati valne oblike napona na izlazu linije $u(l, t)$ i na otporniku R=50Ω.



Rješenje:

$$Z_0 = \frac{1}{2}$$

$$Z_2 = \frac{r^2}{R} = \frac{1}{2} \Rightarrow r^2 = \frac{1}{2}R = \frac{50}{2} = 25 \Rightarrow r = 5$$

$$\frac{R}{L} = \frac{G}{C} \Rightarrow \frac{0{,}5\Omega}{10\mu H} = \frac{25}{40\mu F} \qquad \text{vod bez distorzije}$$

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{10\cdot 10^{-6}}{40\cdot 10^{-6}}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

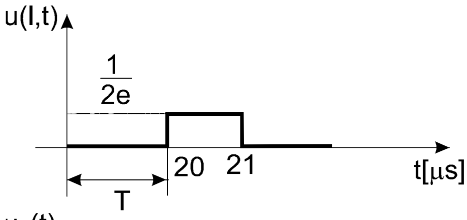
$$\gamma = s\sqrt{LC} + \sqrt{RG} = 20\cdot 10^{-6}s + 1$$

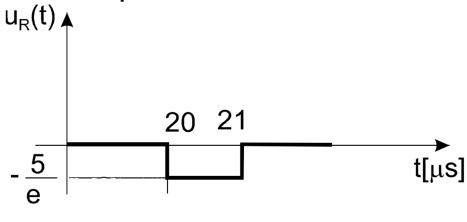
$$g = \gamma \cdot l = 20\cdot 10^{-6}s + 1 \qquad (l = 1km)$$

$$U(l) = U(0)e^{-g} = \frac{U_g}{2}e^{-1}e^{-\underbrace{20\cdot 10^{-6}}_{T\,ka\check{s}njenje}s}$$

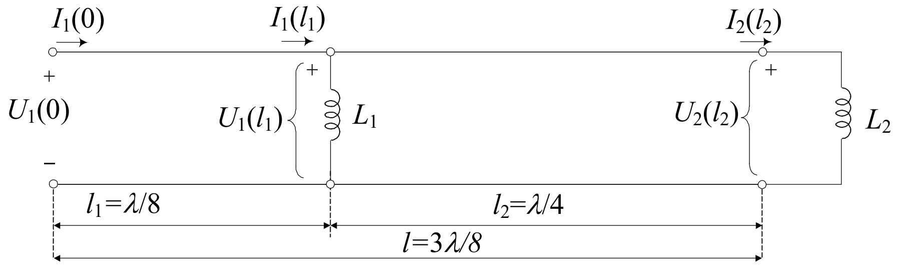
$$I(l) = \frac{U(l)}{Z_0}$$

$$U_R(s) = -rI(l) = \frac{-r}{Z_0}U(l) = \frac{-5}{\frac{1}{2}}U(l) = -10U(l)$$





5. Na ulazu kaskadnoga spoja dviju linija bez gubitaka s istim primarnim prametrima L=450µH/km i C=80nF/km, duljina $l_1=\lambda/8$ i $l_2=\lambda/4$, djeluje napon $u(0,t)=4\cdot\cos(\omega_0\, t)$. Na kraj prve linije priključen je induktivitet L_1=15mH, a na kraj druge L_2=3,75mH. Odrediti: a) karakterističnu impedanciju Z_0 i brzinu propagacije signala po linijama v; b) izraz za ulaznu impedanciju druge linije Z_{ul2}; c) frekvenciju ω_0 za koju je struja na kraju prve linije $i_1(l_1,t)$ jednaka nuli i koeficijent prijenosa γ; d) napon na kraju prve linije $u_1(l_1,t)$ i na kraju druge linije: $u_2(l_2,t)$; e) struju na kraju druge linije $i_2(l_2, t)$ i duljine linija l_1 i l_2.



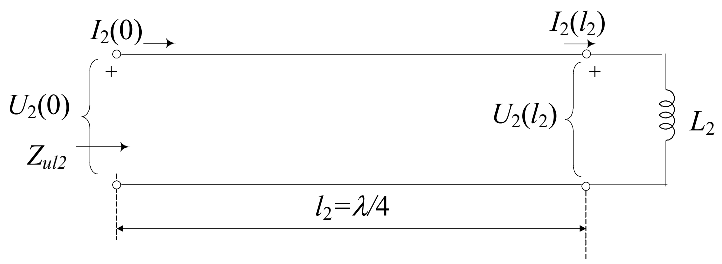
Rješenje: Liniju ćemo analizirati kao dvije linije s istim primarnim parametrima, spojene u kaskadu

a) Karakteristična impedancija Z_0 i brzina propagacije signala v

$$\boxed{Z_0=\sqrt{\frac{L}{C}}=\sqrt{\frac{450\cdot10^{-6}}{80\cdot10^{-9}}}=75\Omega}\ ; \qquad \boxed{v=\frac{1}{\sqrt{LC}}=\frac{1}{6\cdot10^{-6}}=166{,}7\cdot10^{3}\,\text{km/s}};$$ **(1 bod)**

b) Ulazna impedancija druge linije:

$$Z_{ul2}=\frac{U_2(0)}{I_2(0)}=\frac{U_2(l_2)ch(\gamma l_2)+I_2(l_2)Z_0 sh(\gamma l_2)}{\frac{U_2(l_2)}{Z_0}sh(\gamma l_2)+I_2(l_2)Z_0 ch(\gamma l_2)}$$



$$\gamma\cdot l_2=j\beta\cdot l_2=j\beta\frac{\lambda}{4}=j\frac{\pi}{2}\qquad \sin(\beta l_2)=1\qquad \cos(\beta l_2)=0\qquad U_2(l_2)=I_2(l_2)Z_2$$

$$\boxed{Z_{ul2}=\frac{U_2(l_2)\cos(\beta l_2)+jI_2(l_2)Z_0\sin(\beta l_2)}{j\frac{U_2(l_2)}{Z_0}\sin(\beta l_2)+I_2(l_2)Z_0\cos(\beta l_2)}=Z_0^2\frac{I_2(l_2)}{U_2(l_2)}=\frac{Z_0^2}{Z_2}=\frac{Z_0^2}{j\omega L_2}}$$ **(1 bod)**

c) Struja na kraju prve linije: **(1 bod)**

$$I_1(l_1)=U_1(l_1)\left(\frac{1}{j\omega L_1}+\frac{1}{Z_{ul2}}\right)=U_1(l_1)\left(\frac{1}{j\omega L_1}+\frac{j\omega L_2}{Z_0^2}\right)=U_1(l_1)\frac{Z_0^2-\omega_0^2L_1L_2}{j\omega L_1}=0$$

$$Z_0^2-\omega_0^2L_1L_2=0\qquad\Rightarrow\qquad \boxed{\omega_0=\frac{Z_0}{\sqrt{L_2L_1}}=\frac{75}{\sqrt{3{,}75\cdot10^{-3}\cdot15\cdot10^{-3}}}=10^4[\text{rad}/s]}$$

$$\boxed{\gamma=j\beta=j\omega_0\sqrt{LC}=j6\cdot10^{-2}/\text{km}}$$

d) Napon na kraju prve linije:

$$\gamma \cdot l_1 = j\beta \cdot l_1 = j\beta \frac{\lambda}{8} = j\frac{\pi}{4} \Rightarrow \ \sin(\beta l_1) = \cos(\beta l_1) = \frac{1}{\sqrt{2}}$$

$$U_1(l_1) = U_1(0)\cos(\beta l_1) - jI_1(0)Z_0 \sin(\beta l_1)$$

$$Z_{ul1} = \frac{U_1(l_1)\cos(\beta l_1) + jI_1(l_1)Z_0 \sin(\beta l_1)}{j\frac{U_1(l_1)}{Z_0}\sin(\beta l_1) + I_1(l_1)Z_0 \cos(\beta l_1)} = -jZ_0 \frac{\cos(\beta l_1)}{\sin(\beta l_1)} = -jZ_0$$

$$I_1(0) = \frac{U_1(0)}{Z_{ul1}} = \frac{U_1(0)}{-jZ_0} = j\frac{U_1(0)}{Z_0} = j\frac{4}{75} = 53{,}333e^{j\frac{\pi}{2}}[mA]$$

$$U_1(l_1) = U_1(0)\cos(\beta l_1) - j\left(j\frac{U_1(0)}{Z_0}\right)Z_0 \sin(\beta l_1) = U_1(0)(\cos(\beta l_1) + \sin(\beta l_1)) = \sqrt{2} \cdot U_1(0)$$

$$\boxed{u_1(l_1, t) = 4\sqrt{2} \cdot \cos(\omega_0 t)}$$

Napon na kraju druge linije:

$$U_2(l_2) = U_2(0)\cos(\beta l_2) - jI_2(0)Z_0 \sin(\beta l_2)$$

$$I_2(0) = \frac{U_2(0)}{Z_{ul2}} = \frac{U_2(0)}{Z_0^2} j\omega L_2 \qquad \omega L_2 = 10^4 \cdot 3{,}75 \cdot 10^{-3} = 37{,}5\Omega$$

$$U_2(l_2) = U_2(0)\cos(\beta l_2) - j\frac{U_2(0)}{Z_0^2} j\omega L_2 Z_0 \sin(\beta l_2) = \frac{\omega L_2}{Z_0} U_1(l_1) = \frac{1}{2}U_1(l_1) = \frac{\sqrt{2}}{2}U_1(0)$$

$$\boxed{u_2(l_2, t) = 2\sqrt{2} \cdot \cos(\omega_0 t)}$$ **(1 bod)**

e) Struja na kraju druge linije $i_2(l_2, t)$ i duljine linija l_1 i l_2:

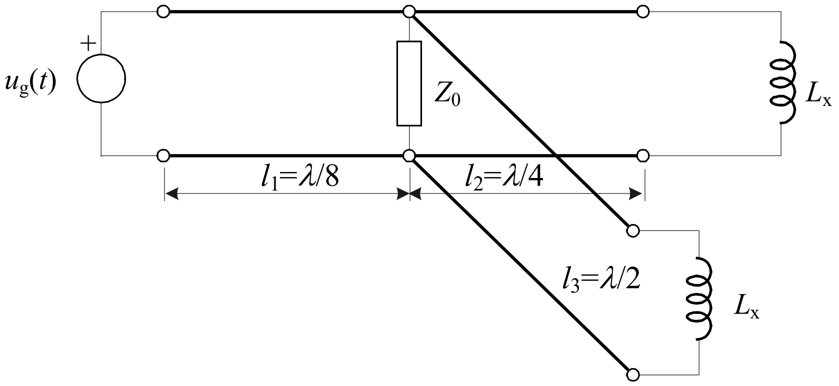
$$I_2(l_2) = -jU_2(0)\frac{\sin(\beta l_2)}{Z_0} + I_2(0)\cos(\beta l_2) = \frac{-jU_2(0)}{Z_0} = -j75{,}42 \cdot 10^{-3}$$

$$\boxed{i_2(l_2, t) = 75{,}42 \cdot \cos\left(\omega_0 t - \frac{\pi}{2}\right)[mA]}$$

$$\boxed{l_1 = \frac{1}{8}\lambda = \frac{2\pi}{8\beta} = \frac{\pi}{4\omega_0\sqrt{LC}} = \frac{\pi}{4 \cdot 10^4 \cdot 6 \cdot 10^{-6}} = \frac{\pi}{24 \cdot 10^{-2}} = 13{,}08[km]}$$

$$\boxed{l_2 = \frac{1}{4}\lambda = \frac{\pi}{2\beta} = \frac{\pi}{2\omega_0\sqrt{LC}} = \frac{\pi}{2 \cdot 10^4 \cdot 6 \cdot 10^{-6}} = \frac{\pi}{12 \cdot 10^{-2}} = 26{,}16[km]}$$ **(1 bod)**

5. Tri linije bez gubitaka s istim primarnim parametrima L=200μH/km i C=80nF/km, spojene su prema slici. Duljine linija su: $l_1=\lambda/8$, $l_2=\lambda/4$ i $l_3=\lambda/2$. Zadano je: $u_g(t)=2\cos\left(1{,}25\cdot\pi\cdot10^5 t\right)$ [V]. Odrediti: a) valnu impedanciju, brzinu širenja vala na linijama i koeficijent prijenosa linija; b) ulazne impedancije druge i treće linije: Z_{ul2} i Z_{ul3}; c) induktivitet L_x, za kojeg je prva linija prilagođena na izlazu, te duljine linija: l_1, l_2 i l_3; d) faktore refleksije na krajevima druge i treće linije: Γ_{i2} i Γ_{i3}; e) napone na kraju svake linije: $u_1(l_1,t)$, $u_2(l_2,t)$, $u_3(l_3,t)$.



Rješenje:

a) $$Z_0=\sqrt{\frac{L}{C}}=\sqrt{\frac{200\cdot10^{-6}}{80\cdot10^{-9}}}=50\Omega$$

$$\gamma=j\beta=j\omega_0\sqrt{LC}=j1{,}25\cdot\pi\cdot10^5\sqrt{2\cdot10^{-4}\cdot8\cdot10^{-8}}=j\frac{\pi}{2}\ \text{[rad/km]}$$

$$v=\frac{\omega_0}{\beta}=\frac{1}{\sqrt{LC}}=\frac{1}{\sqrt{2\cdot10^{-4}\cdot8\cdot10^{-8}}}=\frac{10^6}{4}=25\cdot10^4\ \text{[km/s]}$$

(1 bod)

b) $$\gamma\cdot l_2=j\cdot\beta\cdot l_2=j\frac{\pi}{2}\qquad\qquad \gamma\cdot l_3=j\cdot\beta\cdot l_3=j\pi$$

$$Z_{ul2}=\frac{j\omega_0L_x ch(\gamma\cdot l_2)+Z_0 sh(\gamma\cdot l_2)}{\frac{j\omega_0L_x}{Z_0}sh(\gamma\cdot l_2)+ch(\gamma\cdot l_2)}=\frac{j\omega_0L_x\cos(\beta\cdot l_2)+jZ_0\sin(\beta\cdot l_2)}{j\frac{j\omega_0L_x}{Z_0}\sin(\beta\cdot l_2)+\cos(\beta\cdot l_2)}=\frac{Z_0^2}{j\omega_0L_x}$$

$$Z_{ul3}=\frac{j\omega_0L_x ch(\gamma\cdot l_2)+Z_0 sh(\gamma\cdot l_2)}{\frac{j\omega_0L_x}{Z_0}sh(\gamma\cdot l_2)+ch(\gamma\cdot l_2)}=\frac{j\omega_0L_x\cos(\beta\cdot l_2)+jZ_0\sin(\beta\cdot l_2)}{j\frac{j\omega_0L_x}{Z_0}\sin(\beta\cdot l_2)+\cos(\beta\cdot l_2)}=j\omega_0L_x$$

(1 bod)

c) Ukupna impedancija na izlazu prve linije: $$Z_{uk}=Z_0\,||\,Z_{ul2}\,||\,Z_{ul3}=\frac{1}{Y_0+Y_{ul2}+Y_{ul3}}=\frac{1}{Y_0}$$

$$\Rightarrow\ Y_{ul2}+Y_{ul3}=0\ \Rightarrow\qquad \frac{j\omega_0L_x}{Z_0^2}+\frac{1}{j\omega_0L_x}=0$$

$$\Rightarrow L_x=\frac{Z_0}{\omega_0}=\frac{50}{1{,}25\cdot\pi\cdot10^5}=12{,}73\cdot10^{-6}H$$

$$\lambda=\frac{2\cdot\pi}{\beta}=\frac{2\cdot\pi}{0{,}5\cdot\pi}=4\ \text{[km]}$$

$$l_1=\frac{\lambda}{8}=500\ \text{[m]};\quad l_2=\frac{\lambda}{4}=1\ \text{[km]};\quad l_3=\frac{\lambda}{2}=2\ \text{[km]}$$

(1 bod)

d) $\Gamma_{i2} = \Gamma_{i3} = \dfrac{j\omega_0 L_x - Z_0}{j\omega_0 L_x + Z_0} = \dfrac{j-1}{j+1} = \dfrac{(j-1)^2}{2} = j$

(1 bod)

e) $\gamma \cdot l_1 = j\beta \dfrac{\lambda}{8} = j\beta \dfrac{2\pi}{8 \cdot \beta} = j\dfrac{\pi}{4} \quad U_1(l_1) = U(0) \cdot e^{-j\beta l_1} = 2 \cdot e^{-j\pi/4} \quad u_1(l_1, t) = 2\cos\left(\omega_0 t - \dfrac{\pi}{4}\right)$

$$U_2(l_2) = U_1(l_1) \cdot \cos(\beta \cdot l_2) - jU_1(l_1) \cdot \frac{Z_0}{Z_{ul2}} \sin(\beta \cdot l_2) = -jU_1(l_1) \cdot \frac{j\omega_0 L_2}{Z_0} = 2 \cdot e^{-j\pi/4}$$

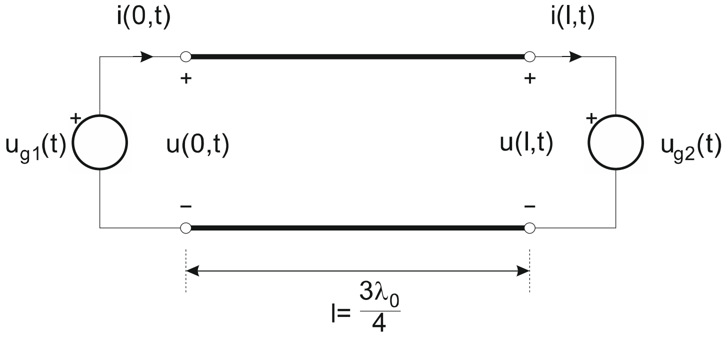
$$u_1(l_2, t) = 2\cos\left(\omega t - \frac{\pi}{4}\right)$$

$$U_3(l_3) = U(l_1) \cdot \cos(\beta \cdot l_3) - jU(l_1) \cdot \frac{Z_0}{Z_{ul3}} \sin(\beta \cdot l_3) = -U(l_1) = -2 \cdot e^{-j\pi/4}$$

$$u_1(l_3, t) = -2\cos(\omega t - \pi/4)$$

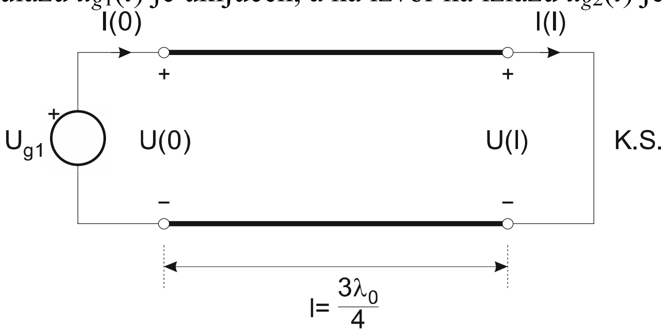
(1 bod)

5. Zadana je linija bez gubitaka s primarnim parametrima $L = 10\,\mu\text{H/km}; C = 40\,\mu\text{F/km}$ i duljine $l = (3/4)\cdot\lambda_0$ na frekvenciji generatora $\omega_0 = 10^5$ rad/s. Na ulaz linije je spojen naponski izvor $u_{g1}(t) = 10\cos(\omega_0 t)$, a na izlaz linije spojen je naponski izvor $u_{g2}(t) = 20\cos(\omega_0 t)$. Treba izračunati valne oblike struje na ulazu linije $i(x=0,t)$ i na izlazu $i(x=l,t)$. Kolika je duljina linije *l*?



<u>Rješenje:</u> Metodom superpozicije:

a) naponski izvor na ulazu $u_{g1}(t)$ je uključen, a na izvor na izlazu $u_{g2}(t)$ je isključen.



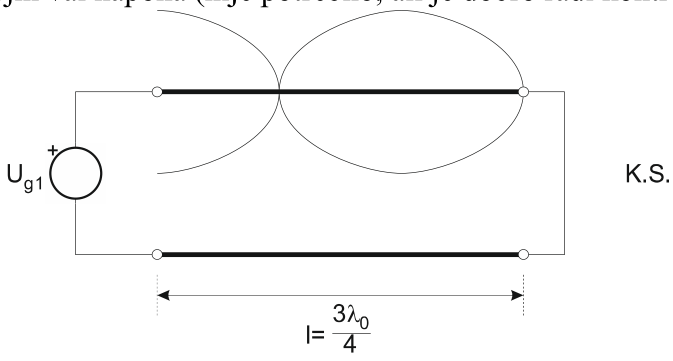
$$U(l) = R_2 \cdot I(l),\ (R_2 = 0,\ \text{KS}) \Rightarrow U(l) = 0;$$

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{10\cdot 10^{-6}}{40\cdot 10^{-6}}} = \frac{1}{2}[\Omega];$$

$$\gamma = \alpha + j\beta;\ \alpha = 0;\ \beta = \omega_0\sqrt{LC}$$

$$g = \gamma \cdot l = j\omega_0\sqrt{LC}\cdot\frac{3}{4}\cdot\frac{2\pi}{\omega_0\sqrt{LC}} = j\frac{3\pi}{2};\ \lambda_0 = \frac{2\pi}{\beta}$$

Možemo skicirati stojni val napona (nije potrebno, ali je dobro radi kontrole):



<u>Najprije izračunajmo *I*(0):</u>

Prijenosne jednadžbe linije:

$$U(0) = U(l)\cdot \text{ch}\, g + I(l)\cdot Z_0 \cdot \text{sh}\, g$$

$$I(0) = U(l)\frac{\text{sh}\, g}{Z_0} + I(l)\cdot \text{ch}\, g$$

Vrijedi da je $\begin{array}{l} \mathrm{ch}(g) = \mathrm{ch}(j\beta \cdot l) = \cos(\beta \cdot l) \\ \mathrm{sh}(g) = \mathrm{sh}(j\beta \cdot l) = j\sin(\beta \cdot l) \end{array}$ $\Rightarrow$ $\begin{array}{l} \mathrm{ch}\left(j\dfrac{3\pi}{2}\right) = \cos\left(\dfrac{3\pi}{2}\right) = 0 \\ \mathrm{sh}\left(j\dfrac{3\pi}{2}\right) = j\sin\left(\dfrac{3\pi}{2}\right) = -j \end{array}$

$$U(0) = -jZ_0 I(l)$$
$$I(0) = \frac{-j}{Z_0}U(l)$$

Odavdje se odmah vidi da je uz $U(l) = 0$ (KS) $\Rightarrow \boxed{I(0) = 0}$

Ili na drugi način:

$$Z_{ul} = \frac{U(0)}{I(0)} = \frac{R_2 \cdot \mathrm{ch}\, g + Z_0 \cdot \mathrm{sh}\, g}{\dfrac{R_2}{Z_0} \cdot \mathrm{sh}\, g + \mathrm{ch}\, g} = \frac{R_2 \cdot \cos\dfrac{3\pi}{2} + Z_0 \cdot j \cdot \sin\dfrac{3\pi}{2}}{\dfrac{R_2}{Z_0} \cdot j \cdot \sin\dfrac{3\pi}{2} + \cos\dfrac{3\pi}{2}} = \frac{Z_0 \cdot (-j)}{(-j) \cdot \dfrac{R_2}{Z_0}} = \frac{Z_0^{\ 2}}{R_2}$$

uz $R_2 = 0$, (KS) $\Rightarrow Z_{ul} = \infty \Rightarrow I(0) = 0$

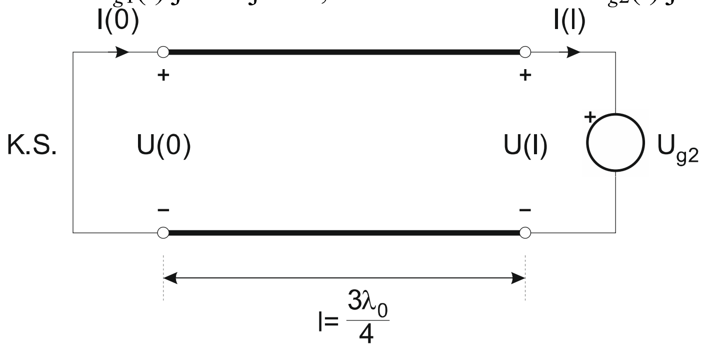
<u>Zatim izračunajmo $I(l)$:</u>

Prijenosne jednadžbe linije:

$$\begin{array}{l} U(l) = U(0) \cdot \mathrm{ch}\, g - I(0) \cdot Z_0 \cdot \mathrm{sh}\, g \\ I(l) = -U(0) \cdot \dfrac{\mathrm{sh}\, g}{Z_0} + I(0) \cdot \mathrm{ch}\, g \end{array} \quad \Rightarrow \quad \begin{array}{l} U(l) = j \cdot I(0) \cdot Z_0 \\ I(l) = j \cdot \dfrac{U(0)}{Z_0} \end{array}$$

Odavdje se odmah vidi da je uz $U_{g1}(j\omega) = 10\angle 0° = U(0)$ $\Rightarrow \boxed{I(l) = 20\angle 90°}$

b) naponski izvor na ulazu $u_{g1}(t)$ je isključen, a na izvor na izlazu $u_{g2}(t)$ je uključen.



$$U_{g2}(j\omega) = 20\angle 0° = U(l)$$
$$U(0) = jI(l) \cdot Z_0 \Rightarrow U(0) = 0 \Rightarrow \boxed{I(l) = 0}$$
$$I(0) = \frac{-j}{Z_0}U(l) \Rightarrow \boxed{I(0) = 40\angle -90°}$$

Zbroj slučajeva a) i b) daje struje:

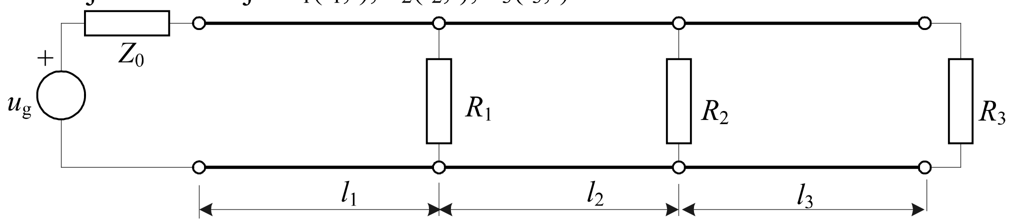
$$i(0,t) = 40\cos(\omega_0 t - 90°)$$
$$i(l,t) = 20\cos(\omega_0 t + 90°)$$

Duljina linije je:

$$l = \frac{3}{4} \cdot \lambda_0 = \frac{3}{4} \cdot \frac{2\pi}{\omega_0\sqrt{LC}} = \frac{2 \cdot 3\pi/4}{10^5 \cdot \sqrt{10 \cdot 10^{-6} \cdot 40 \cdot 10^{-6}}} = \frac{3\pi/2}{10^5 \cdot 20 \cdot 10^{-6}} = \frac{3}{4}\pi[\mathrm{km}]$$

5. Tri linije bez gubitaka spojene su u kaskadu prema slici. Zadano je: L=0,25mH/km, C=100nF/km, u_g=10 cos(2.5π 10^5 t) V, R_2 =150Ω, R_3=25Ω, l_1=3λ/4, l_2=λ/2 i l_3=λ/4. Odrediti:

a) valnu impedanciju i koeficijent prijenosa linija;
b) brzinu širenja vala na linijama i duljinu druge i treće linije;
c) otpor R_1 da bi prva linija bila prilagođena na izlazu;
d) faktore refleksije na krajevima druge i treće linije: Γ_{i2} i Γ_{i3};
e) napone na kraju svake linije: $u_1(l_1$,t), $u_2(l_2$,t), $u_3(l_3$,t).



Rješenje:

a) $Z_0=\sqrt{\frac{L}{C}}=\sqrt{\frac{2,5\cdot 10^{-4}}{10^{-7}}}=50\Omega$ $\gamma=j\beta=j\omega_0\sqrt{LC}=j2,5\cdot\pi\cdot 10^5\sqrt{0,25\cdot 10^{-10}}=\frac{j5}{4}\pi$ [rad/km]

(1 bod)

b) $v=\frac{\omega}{\beta}=\frac{1}{\sqrt{LC}}=\frac{1}{\sqrt{0,25\cdot 10^{-10}}}=2\cdot 10^5$ [km/s] $\qquad \lambda=\frac{2\cdot\pi}{\beta}=\frac{2\cdot\pi}{1,25\cdot\pi}=\frac{8}{5}=1,6$ km

$l_3=\frac{\lambda}{4}=400$ [m]; $\qquad l_2=\frac{\lambda}{2}=800$ [m]; (1 bod)

c) $\gamma\cdot l_1=j\cdot\beta\cdot l_1=j\frac{3\pi}{2}$ $\qquad \gamma\cdot l_2=j\cdot\beta\cdot l_2=j\pi$; $\qquad \gamma\cdot l_3=j\cdot\beta\cdot l_3=j\frac{\pi}{2}$

$$Z_{ul3}=\frac{R_3 ch(\gamma\cdot l_3)+Z_0 sh(\gamma\cdot l_3)}{\frac{R_3}{Z_0}sh(\gamma\cdot l_3)+ch(\gamma\cdot l_3)}=\frac{R_3\cos(\beta\cdot l_3)+jZ_0\sin(\beta\cdot l_3)}{j\frac{R_3}{Z_0}\sin(\beta\cdot l_3)+\cos(\beta\cdot l_3)}=\frac{Z_0^2}{R_3}=\frac{2500}{25}=100\,\Omega$$

$$R_{eq2}=\frac{R_2\cdot Z_{ul3}}{R_2+Z_{ul3}}=\frac{150\cdot 100}{250}=60\,\Omega \qquad Z_{ul2}=\frac{R_{eq2}\cos(\beta\cdot l_2)+jZ_0\sin(\beta\cdot l_2)}{j\frac{R_{eq2}}{Z_0}\sin(\beta\cdot l_2)+\cos(\beta\cdot l_2)}=\frac{-R_{eq2}}{-1}=R_{eq2}=60\,\Omega$$

$$R_{eq1}=\frac{R_1\cdot Z_{ul2}}{R_1+Z_{ul2}}=\frac{R_1\cdot R_{eq2}}{R_1+R_{eq2}}=Z_0=50\,\Omega \qquad\Rightarrow\qquad R_1=\frac{Z_0\cdot R_{eq2}}{R_{eq2}-Z_0}=\frac{50\cdot 60}{10}=300\,\Omega$$

(1 bod)

d) $\Gamma_{i2}=\frac{R_{eq2}-Z_0}{R_{eq2}+Z_0}=\frac{10}{110}=\frac{1}{11}$ $\qquad \Gamma_{i3}=\frac{R_3-Z_0}{R_3+Z_0}=\frac{-25}{75}=-\frac{1}{3}$ (1 bod)

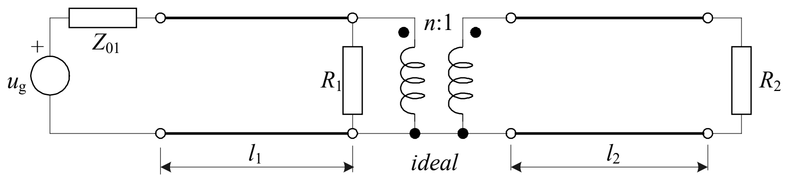
e) $\gamma\cdot l_1=j\beta\frac{3\lambda}{4}=j\frac{3\pi}{2}$ $\qquad U_1(l_1)=U(0)\cdot e^{-j\beta l_1}=5\cdot e^{-j3\pi/2}=5j$ $\quad u_1(l_1,t)=5\cos\left(\omega t+\frac{\pi}{2}\right)$

$$U_2(l_2)=U_1(l_1)\cdot\cos(\beta\cdot l_2)-jU_1(l_1)\cdot\frac{Z_0}{Z_{ul2}}\sin(\beta\cdot l_2)=-U_1(l_1)=-5j \quad u_2(l_2,t)=-5\cos\left(\omega t+\frac{\pi}{2}\right)$$

$$U_3(l_3)=U_2(l_2)\cdot\cos(\beta\cdot l_3)-jU_2(l_2)\cdot\frac{Z_0}{Z_{ul3}}\sin(\beta\cdot l_3)=-jU_2(l_2)\cdot\frac{Z_0}{Z_{ul3}}\sin(\beta\cdot l_3)=-j(-5j)\frac{1}{2}=-\frac{5}{2}$$

$u_1(l_3,t)=-2,5\cdot\cos(\omega t)$ (1 bod)

5. Dvije linije bez gubitaka i idealni transformator spojeni su u kaskadu prema slici. Zadano je: L_1=0,45mH/km i C_1=80nF/km, L_2=0,2mH/km, C_2=80nF/km, u_g=10 cos(2,5π 10^5 t) V, R_1 =300Ω, R_2=100Ω, l_1=3λ_1/4, l_2=λ_2/4. Odrediti: a) valne impedancije Z_{01} i Z_{02}, te koeficijente prijenosa γ_1 i γ_2 linija; b) brzine širenja vala na linijama i duljine l_1 i l_2 linija; c) omjer transformacije n da bi prva linija bila prilagođena na izlazu; d) faktor refleksije na kraju druge Γ_{i2}; i napon na kraju prve linije $u_1(l_1,t)$; e) napone na početku i na kraju druge linije: $u_2(0,t)$, $u_2(l_2,t)$.



Rješenje:

a) $Z_{01}=\sqrt{\dfrac{L_1}{C_1}}=\sqrt{\dfrac{4{,}5\cdot 10^{-4}}{8\cdot 10^{-8}}}=75\Omega \qquad Z_{02}=\sqrt{\dfrac{L_2}{C_2}}=\sqrt{\dfrac{2\cdot 10^{-4}}{8\cdot 10^{-8}}}=50\Omega$ (1 bod)

$$\gamma_1=j\beta_1=j\omega_0\sqrt{L_1C_1}=j2{,}5\cdot\pi\cdot 10^5\sqrt{4{,}5\cdot 10^{-2}\cdot 8\cdot 10^{-8}}=j2{,}5\cdot\pi\cdot 10^5\sqrt{36\cdot 10^{-12}}=j1{,}5\pi\ [\text{rad/km}]$$

$$\gamma_2=j\beta_2=j\omega_0\sqrt{L_2C_2}=j2{,}5\cdot\pi\cdot 10^5\sqrt{2\cdot 10^{-4}\cdot 8\cdot 10^{-8}}=j\pi\ [\text{rad/km}]$$

b) $v_1=\dfrac{\omega}{\beta_1}=\dfrac{1}{\sqrt{L_1C_1}}=\dfrac{1}{\sqrt{36\cdot 10^{-12}}}=166{,}7\cdot 10^3\ [\text{km/s}]$

$$v_2=\frac{\omega}{\beta_2}=\frac{1}{\sqrt{L_2C_2}}=\frac{1}{\sqrt{16\cdot 10^{-12}}}=250\cdot 10^3\ [\text{km/s}]$$

$\lambda_1=\dfrac{2\cdot\pi}{\beta_1}=\dfrac{2\cdot\pi}{1{,}5\cdot\pi}=\dfrac{4}{3}=1{,}33\,\text{km}\quad l_1=\dfrac{3\lambda_1}{4}=1\ [\text{km}];\quad \lambda_2=\dfrac{2\cdot\pi}{\beta_2}=\dfrac{2\cdot\pi}{\pi}=2\,\text{km}\quad l_2=\dfrac{\lambda_2}{4}=0{,}5$ [km]

(1 bod)

c) $\gamma_1\cdot l_1=j\cdot\beta_1\cdot l_1=j\dfrac{3\pi}{2} \qquad \gamma_2\cdot l_2=j\cdot\beta_2\cdot l_2=j\dfrac{\pi}{2}$

$$Z_{ul2}=\frac{R_2ch(\gamma_2\cdot l_2)+Z_{02}sh(\gamma_2\cdot l_2)}{\dfrac{R_2}{Z_{02}}sh(\gamma_2\cdot l_2)+ch(\gamma_2\cdot l_2)}=\frac{R_2\cos(\beta_2\cdot l_2)+jZ_{02}\sin(\beta_2\cdot l_2)}{j\dfrac{R_2}{Z_{02}}\sin(\beta_2\cdot l_2)+\cos(\beta_2\cdot l_2)}=\frac{Z_{02}^2}{R_2}=\frac{2500}{100}=25\,\Omega$$

$$Z_{ulT}=n^2Z_{ul2}=n^2 25\,\Omega\quad \frac{R_1\cdot Z_{ulT}}{R_1+Z_{ulT}}=\frac{R_1\cdot n^2\cdot Z_{ul2}}{R_1+n^2\cdot Z_{ul2}}=75\Omega\quad\Rightarrow n^2=\frac{75\cdot R_1}{(R_1-75)\cdot Z_{ul2}}=\frac{75\cdot 300}{(300-75)\cdot 25}=4;$$

$n=2$ (1 bod)

d) $\Gamma_{i2}=\dfrac{R_2-Z_{02}}{R_2+Z_{02}}=\dfrac{50}{150}=\dfrac{1}{3}$

$U_1(l_1)=U(0)\cdot e^{-j\beta_1 l_1}=5\cdot e^{-j3\pi/2}=5j \qquad u_1(l_1,t)=5\cos(\omega t+\pi/2)$

(1 bod)

e)

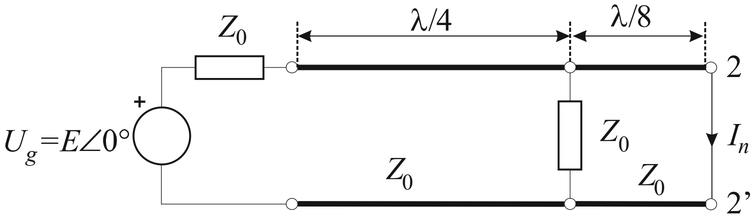
$U_2(0)=U_1(l_1)/n=2{,}5\cdot e^{-j3\pi/2}=2{,}5j \qquad u_2(0,t)=2{,}5\cos(\omega t+\pi/2)$

$$U_2(l_2)=U_2(0)\cdot\cos(\beta_2\cdot l_2)-jU_2(0)\cdot\frac{Z_{02}}{Z_{ul2}}\sin(\beta_2\cdot l_2)=-jU_2(0)\cdot\frac{50}{25}=-j\cdot 2{,}5\cdot j\cdot 2=5$$

$u_2(l_2,t)=5\cdot\cos(\omega t)$

(1 bod)

5. Na slici je zadan sustav linija bez gubitaka istih primarnih parametara. Karakteristične impedancije linija su jednake Z_0. Napon generatora na ulazu je stacionarni sinusni signal $u_g(t)=E\cdot\cos(\omega t)$; $-\infty<t<\infty$ (prilaz 2–2' druge linije je kratko spojen). Odrediti: a) ulaznu impedanciju u drugu liniju; b) ulaznu impedanciju u prvu liniju; c) fazore napona i struje na izlazu prve linije; d) fazor struje ($\mathbf{I}_n$) na izlazu druge linije (napon je $\mathbf{U}_{2-2'}=0$); e) valni oblik struje $i_n(t)$ na izlazu 2–2' u vremenskoj domeni.



Rješenje: Zadatak rješavamo pomoću fazora jer se radi o stacionarnoj sinusnoj pobudi. Ulazna impedancija linije lako se računa iz prijenosnih jednadžbi uz $x=l$:

$$U(0)=U(x)\cdot ch\gamma x+I(x)Z_0 sh\gamma x$$

$$I(0)=\frac{U(x)}{Z_0}sh\gamma x+I(x)ch\gamma x \qquad Z_{ul}=\frac{U(0)}{I(0)}=\frac{Z_2 ch(\gamma l)+Z_0 sh(\gamma l)}{\frac{Z_2}{Z_0}sh(\gamma l)+ch(\gamma l)};\ Z_2=\frac{U(l)}{I(l)}$$

Z_2 je impedancija kojom je linija zaključena na izlazu.

Linija bez gubitaka $Z_0=\sqrt{\frac{L}{C}}$; $\gamma=j\beta=j\omega\sqrt{LC};\alpha=0$; $\lambda=\frac{2\pi}{\beta}$.

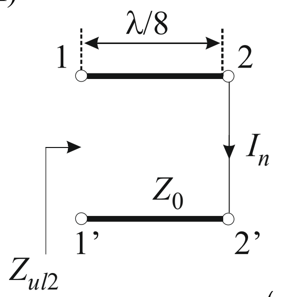
Ako je duljina linije: $l_1=\frac{\lambda}{4}$ tada je $\gamma\cdot l_1=j\beta\frac{\lambda}{4}=j\beta\frac{2\pi}{\beta}\frac{1}{4}=j\frac{\pi}{2}$.

Ako je duljina linije: $l_2=\frac{\lambda}{8}$ tada je $\gamma\cdot l_2=j\beta\frac{\lambda}{8}=j\beta\frac{2\pi}{\beta}\frac{1}{8}=j\frac{\pi}{4}$.

Za $l_1=\lambda/4$: $sh(\gamma\cdot l_1)=sh(j\pi/2)=j\sin(\pi/2)=j$; $ch(\gamma\cdot l_1)=ch(j\pi/2)=\cos(\pi/2)=0$;

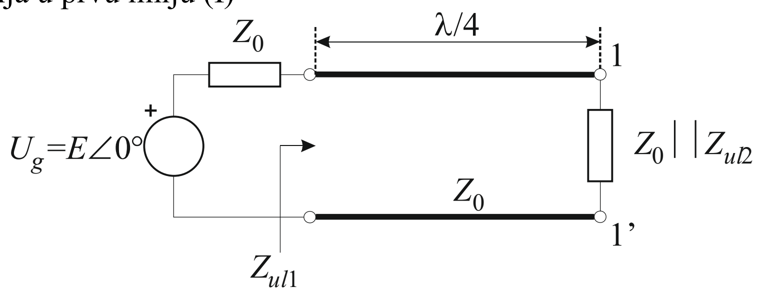
Za $l_2=\frac{\lambda}{8}$: $sh(\gamma\cdot l_2)=sh(j\pi/4)=j\sin(\pi/4)=j\frac{\sqrt{2}}{2}$; $ch(\gamma\cdot l_2)=ch(j\pi/4)=\cos(\pi/4)=\frac{\sqrt{2}}{2}$.

a) Ulazna impedancija u drugu liniju (II)



$$Z_{ul2}=\left.\frac{U^{\mathrm{II}}(0)}{I^{\mathrm{II}}(0)}\right|_{Z_2=0,\ l_2=\lambda/8}=\frac{Z_0 sh(j\pi/4)}{ch(j\pi/4)}=\frac{Z_0 j\sin(\pi/4)}{\cos(\pi/4)}=\frac{Z_0 j(\sqrt{2}/2)}{(\sqrt{2}/2)}=jZ_0.$$ **(1 bod)**

b) Ulazna impedancija u prvu liniju (I)



$$Z_{ul1}=\frac{U(0)}{I(0)}\bigg|_{Z_2=Z_0\mathrm{II}Z_{ul2},\ l_1=\lambda/4}=\frac{Z_2 ch(j\pi/2)+Z_0 sh(j\pi/2)}{(Z_2/Z_0)sh(j\pi/2)+ch(j\pi/2)}=\frac{Z_0^2}{Z_2}=\frac{Z_0^2}{Z_0\mathrm{II}Z_{ul2}}=Z_0(1-j)$$ **(1 bod)**

gdje je

$$Z_0\mathrm{II}Z_{ul2}=\frac{Z_0^2 j}{Z_0(1+j)}=Z_0\frac{j}{1+j}=Z_0\frac{1}{1-j}.$$

Napon i struja na ulazu prve linije:

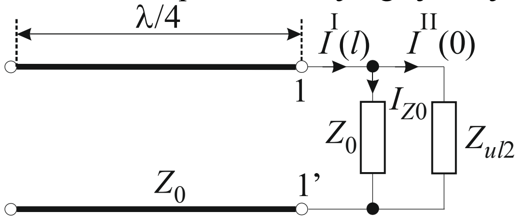
$$U^{\mathrm{I}}(0)=U_g\frac{Z_{ul1}}{Z_0+Z_{ul1}}=U_g\frac{1-j}{2-j};\ I^{\mathrm{I}}(0)=\frac{U_g}{Z_0+Z_{ul1}}=\frac{U_g}{Z_0(2-j)}$$

c) Napon i struja na izlazu prve linije (I) (slijede iz prijenosnih jednadžbi uz $x=l_1$):

$$\begin{aligned}U(x)&=U(0)\cdot ch\gamma x-I(0)Z_0 sh\gamma x\\ I(x)&=-\frac{U(0)}{Z_0}sh\gamma x+I(0)ch\gamma x\end{aligned}\quad\Rightarrow\quad\begin{aligned}U^{\mathrm{I}}(l)&=-I^{\mathrm{I}}(0)Z_0 sh(j\pi/2)=-jZ_0 I^{\mathrm{I}}(0)=U_g\cdot\frac{-j}{2-j}\\ I^{\mathrm{I}}(l)&=-\frac{U^{\mathrm{I}}(0)}{Z_0}sh\,(j\pi/2)=-j\frac{U^{\mathrm{I}}(0)}{Z_0}=U_g\cdot\frac{-(1+j)}{Z_0(2-j)}\end{aligned}$$

(1 bod)

Napon na ulazu u drugu liniju jednak je naponu na izlazu prve linije $U^{\mathrm{II}}(0)=U^{\mathrm{I}}(l)$.
Struja na ulazu u drugu liniju se izračunava pomoću strujnog djelitelja:



$$I^{\mathrm{II}}(0)=\frac{Z_0}{Z_0+Z_{ul2}}\cdot I^{\mathrm{I}}(l)=\frac{Z_0}{Z_0(1+j)}\cdot U_g\frac{-(1+j)}{Z_0(2-j)}=\frac{-U_g}{Z_0(2-j)}.$$

d) Napon i struja na izlazu druge linije (II)

$$U^{\mathrm{II}}(l)=U^{\mathrm{II}}(0)ch(j\pi/4)-I^{\mathrm{II}}(0)Z_0 sh(j\pi/4)=U^{\mathrm{II}}(0)\frac{\sqrt{2}}{2}-jZ_0 I^{\mathrm{II}}(0)\frac{\sqrt{2}}{2}$$

$$I^{\mathrm{II}}(l)=-\frac{U^{\mathrm{II}}(0)}{Z_0}sh\,(j\pi/4)+I^{\mathrm{II}}(0)ch(j\pi/4)=-j\frac{U^{\mathrm{II}}(0)}{Z_0}\frac{\sqrt{2}}{2}-I^{\mathrm{II}}(0)\frac{\sqrt{2}}{2}$$

Uz uvrštene vrijednosti $U^{\mathrm{II}}(0)$ i $I^{\mathrm{II}}(0)$

$$U^{\mathrm{II}}(l)=\frac{\sqrt{2}}{2}U_g\cdot\left[\frac{-j}{2-j}+jZ_0\frac{1}{Z_0(2-j)}\right]=0$$

$$I^{\mathrm{II}}(l)=\frac{\sqrt{2}}{2}U_g\cdot\left[\frac{-1}{Z_0(2-j)}+\frac{-1}{Z_0(2-j)}\right]=\frac{\sqrt{2}}{2}U_g\cdot\frac{-2}{Z_0(2-j)}$$

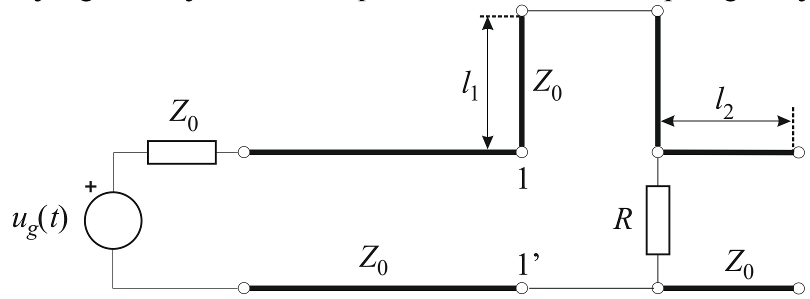
Odnosno

$$I^{\mathrm{II}}(l)=\sqrt{2}\cdot U_g\cdot\frac{-1}{Z_0(2-j)}=\sqrt{2}\cdot U_g\cdot\frac{-j}{Z_0(1+2j)}=\frac{U_g}{Z_0}\cdot\sqrt{\frac{2}{5}}\cdot e^{-\frac{\pi}{2}-\arctan(2)}$$ **(1 bod)**

Konačno je fazor struje $\mathbf{I}_n=I^{\mathrm{II}}(l)=\frac{\sqrt{2}E}{Z_0\sqrt{5}}\angle-\frac{\pi}{2}-\arctan(2)$.

e) U vremenskoj domeni je struja $i_n(t)=\frac{E}{Z_0}\sqrt{\frac{2}{5}}\cos\left[\omega t-\frac{\pi}{2}-\arctan(2)\right]$; $-\infty<t<\infty$. **(1 bod)**

5. Na slici je zadan sustav tri linije bez gubitaka istih primarnih parametara. Napon generatora na ulazu je stacionarni sinusni signal. Karakteristične impedancije linija su jednake Z_0. Da bi se izvršilo prilagođenje otpora R na izlaz glavne linije koristimo se dvjema pomoćnim linijama duljina l_1 i l_2. Odrediti najkraće duljine l_1 i l_2 kako na prilazu 1–1' ne bi bilo refleksije. Zadano je Z_0=50Ω, R=100Ω i valna duljina signala pri frekvenciji generatora je λ=1m. Da li postoji ograničenje na iznos otpornika R da bi ovakvo prilagođenje bilo moguće?



Rješenje:

a) Ulazna impedancija linije iz prijenosnih jednadžbi uz x=l:

$$U(0) = U(x)\cdot ch\gamma x + I(x)Z_0 sh\gamma x$$

$$I(0) = \frac{U(x)}{Z_0} sh\gamma x + I(x) ch\gamma x$$

$$Z_{ul} = \frac{U(0)}{I(0)} = \frac{Z_2 ch(\gamma l) + Z_0 sh(\gamma l)}{\frac{Z_2}{Z_0} sh(\gamma l) + ch(\gamma l)}; \; Z_2 = \frac{U(l)}{I(l)}$$

Z_2 je impedancija kojom je linija zaključena na izlazu.

Linija bez gubitaka $Z_0 = \sqrt{\frac{L}{C}}$; $\gamma = j\beta = j\omega\sqrt{LC}; \alpha = 0$; $\lambda = \frac{2\pi}{\beta}$

Ako je duljina linije: l_1 tada je $g_1 = \gamma\cdot l_1 = j\beta\cdot l_1 = j\frac{2\pi}{\lambda}\cdot l_1 = jB_1$; $B_1 = \beta\cdot l_1 = \frac{2\pi}{\lambda}\cdot l_1$.

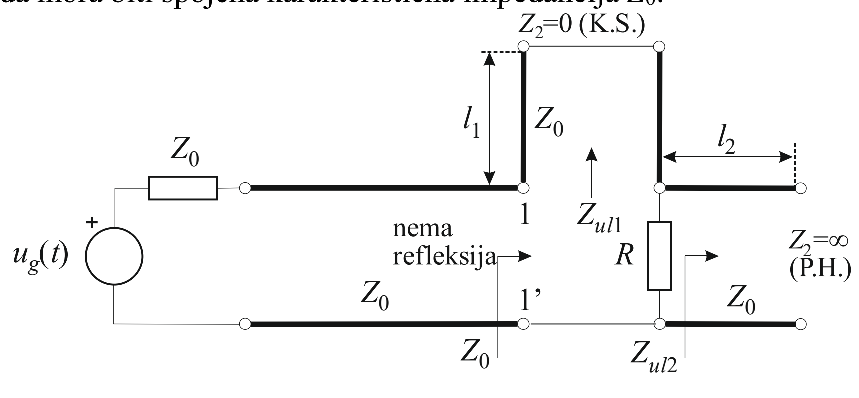
Ako je duljina linije: l_2 tada je $g_2 = \gamma\cdot l_2 = j\beta\cdot l_2 = j\frac{2\pi}{\lambda}\cdot l_2 = jB_2$; $B_2 = \beta\cdot l_2 = \frac{2\pi}{\lambda}\cdot l_2$.

Vrijedi

$$sh(g_i) = sh(\gamma\cdot l_i) = sh\left(j\frac{2\pi}{\lambda}l_i\right) = j\sin\left(\frac{2\pi}{\lambda}l_i\right); \; ch(g_i) = ch(\gamma\cdot l_i) = ch\left(j\frac{2\pi}{\lambda}l_i\right) = \cos\left(\frac{2\pi}{\lambda}l_i\right);$$

za i=1, 2.

Ako je postignuto prilagođenje na izlazu glavne linije (prilaz 1–1') tada nema refleksije. Na prilaz 1–1' tada mora biti spojena karakteristična impedancija Z_0.



Ulazni otpor u prvu pomoćnu liniju (uz kratki spoj na njenom izlazu Z_2=0) je

$$Z_{ul1} = \left.\frac{Z_2 ch(g_1) + Z_0 sh(g_1)}{(Z_2/Z_0)sh(g_1) + ch(g_1)}\right|_{Z_2=0} = Z_0 \frac{sh(g_1)}{ch(g_1)} = Z_0 \frac{sh(j2\pi/\lambda \cdot l_1)}{ch(j2\pi/\lambda \cdot l_1)} =$$

(1 bod)

$$= Z_0 \frac{j\sin(2\pi/\lambda \cdot l_1)}{\cos(2\pi/\lambda \cdot l_1)} = Z_0 j\tan(2\pi/\lambda \cdot l_1) = jZ_0 \tan(B_1)$$

Ulazni otpor u drugu pomoćnu liniju (uz prazni hod na njenom izlazu Z_2=∞) je

$$Z_{ul2} = \left.\frac{ch(g_2) + (Z_0/Z_2)sh(g_2)}{(1/Z_0)sh(g_2) + (1/Z_2)ch(g_2)}\right|_{Z_2=\infty} = Z_0 \frac{ch(g_2)}{sh(g_2)} = Z_0 \frac{ch(j2\pi/\lambda \cdot l_1)}{sh(j2\pi/\lambda \cdot l_1)} =$$

(1 bod)

$$= Z_0 \frac{\cos(2\pi/\lambda \cdot l_2)}{j\sin(2\pi/\lambda \cdot l_2)} = -jZ_0 ctg(2\pi/\lambda \cdot l_2) = -jZ_0 ctg(B_2)$$

Prilagođenje na izlazu glavne linije (prilaz 1–1') vrijedi kad je $Z_0 = Z_{ul1} + R \| Z_{ul2}$, odnosno

$$Z_0 = jZ_0 \tan(B_1) + \frac{R \cdot [-jZ_0 ctg(B_2)]}{R - jZ_0 ctg(B_2)}.$$

Nakon malo računanja

$$Z_0 = jZ_0 \tan(B_1) + \frac{R \cdot Z_0}{Rj\tan(B_2) + Z_0} \Big/ : Z_0 \quad \Rightarrow 1 = j\tan(B_1) + \frac{R_L}{Rj\tan(B_2) + Z_0}$$

$$Rj\tan(B_2) + Z_0 = j\tan(B_1)[Rj\tan(B_2) + Z_0] + R$$

$$jR\tan(B_2) + Z_0 = j\tan(B_1)Z_0 - R\tan(B_1)\tan(B_2) + R$$

Usporedimo Re i Im dijelove:

Re: $Z_0 = R[1 - \tan(B_1)\tan(B_2)] \Rightarrow Z_0 - R = -R\tan(B_1)\tan(B_2) \Rightarrow \tan(B_2) = \dfrac{R - Z_0}{R\tan(B_1)}$

Im: $R\tan(B_2) = \tan(B_1)Z_0 \Rightarrow R\tan(B_2) = \tan(B_1)Z_0 \Rightarrow \dfrac{R - Z_0}{\tan(B_1)} = \tan(B_1)Z_0$

Malo računanja

$$\tan(B_1) = \sqrt{\frac{R}{Z_0} - 1} \Rightarrow \boxed{l_1 = \frac{\lambda}{2\pi}\arctan\sqrt{\frac{R}{Z_0} - 1}}$$

$$\tan(B_2) = \frac{R - Z_0}{R\sqrt{\frac{R}{Z_0} - 1}} \Rightarrow \boxed{l_2 = \frac{\lambda}{2\pi}\arctan\frac{R - Z_0}{R\sqrt{\frac{R}{Z_0} - 1}}}$$

Uz uvrštene vrijednosti elemenata Z_0=50Ω, R=100Ω i λ=1m slijedi

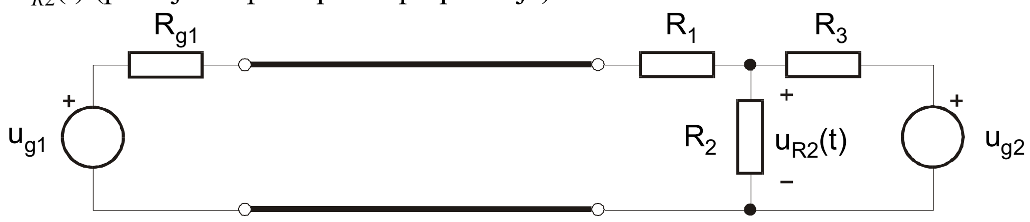
$l_1 = \dfrac{1m}{2\pi}\arctan\sqrt{\dfrac{100}{50} - 1} = \dfrac{1m}{2\pi} \cdot \arctan(1) = \dfrac{1m}{2\pi} \cdot \dfrac{\pi}{4} = \dfrac{1m}{8} = 12{,}5cm$ **(1 bod)**

$l_2 = \dfrac{1m}{2\pi}\arctan\dfrac{100 - 50}{100\sqrt{\dfrac{100}{50} - 1}} = \dfrac{1m}{2\pi} \cdot \arctan\left(\dfrac{1}{2}\right) = \dfrac{1m}{2\pi} \cdot 0{,}463648 = 7{,}37918cm$ **(1 bod)**

Uvjet : $\dfrac{R}{Z_0} - 1 \geq 0 \Rightarrow \boxed{R \geq Z_0}$.

Ograničenje na iznos otpornika R postoji: R≥50Ω. **(1 bod)**

5. Zadana je linija s primarnim parametrima R=0,8Ω/km, G=12,5mS/km, L=1,6μH/km i C=25nF/km. Duljina linije je 10km. Unutrašnji otpori generatora prilagođeni su zrcalnoj impedanciji linije. Zadano je: R_1=4Ω, R_2=6Ω, $u_{g1}(t)$=$u_{g2}(t)$=4$S(t)$. Koliki je R_3? Odrediti i nacrtati $u_{R2}(t)$ (primijeniti postupak superpozicije).



<u>Rješenje:</u> Ispitajmo da li vrijedi $\frac{R}{L}=\frac{G}{C}$ odnosno $\frac{0{,}8}{1{,}6\cdot 10^{-6}}=\frac{12{,}5\cdot 10^{-3}}{25\cdot 10^{-9}}$. Vrijedi! ⇒

To je linija bez distorzije. Računamo sekundarne parametre po pojednostavljenim formulama

$$Z_0=\sqrt{\frac{L}{C}}=\sqrt{\frac{1{,}6\cdot 10^{-6}}{25\cdot 10^{-9}}}=\sqrt{64}=8\Omega$$

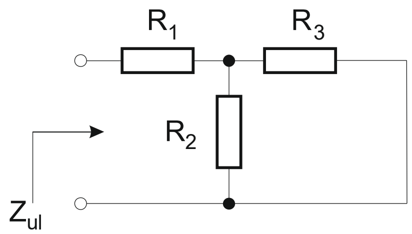
$$\gamma=\sqrt{RG}+s\sqrt{LC}=\sqrt{0{,}8\cdot 12{,}5\cdot 10^{-3}}+s\sqrt{1{,}6\cdot 10^{-6}\cdot 25\cdot 10^{-9}}=0{,}1+s\cdot 0{,}2\cdot 10^{-6}$$

(1 bod)

Metoda superpozicije:

a) izvor $u_{g2}(t)$ isključen

$R_{g1}=8\Omega$ zbog prilagođenja.

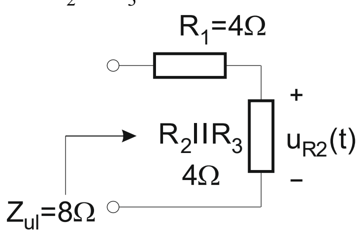


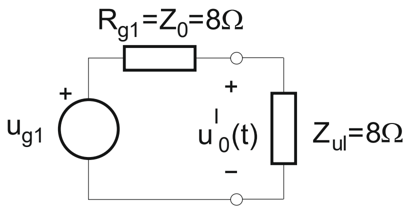
$$Z_{ul}=8\Omega=R_1+\frac{R_2R_3}{R_2+R_3}\;\Rightarrow\; Z_{ul}(R_2+R_3)=(R_2+R_3)R_1+R_2R_3$$

$$\Rightarrow\; Z_{ul}R_2+Z_{ul}R_3=R_1R_2+(R_1+R_2)R_3\;\Rightarrow\; Z_{ul}R_3-(R_1+R_2)R_3=R_1R_2-Z_{ul}R_2\;\Rightarrow$$

$$R_3=\frac{R_1R_2-Z_{ul}R_2}{Z_{ul}-(R_1+R_2)}=\frac{4\cdot 6-8\cdot 6}{8-(4+6)}=\frac{-24}{-2}=12\Omega$$

$$R_2\|R_3=\frac{R_2R_3}{R_2+R_3}=\frac{6\cdot 12}{6+12}=4\Omega$$





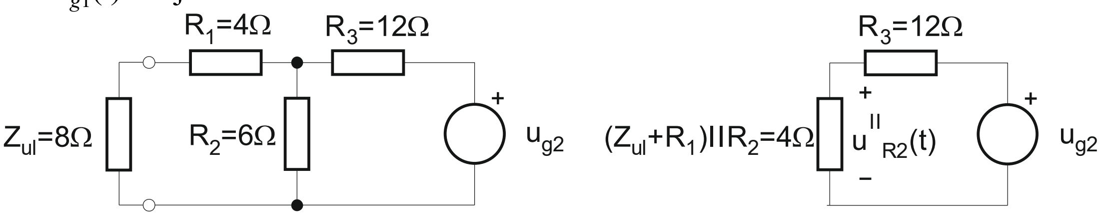
$$u_0^I=\frac{u_{g1}}{2}$$

$$g_1=\gamma\cdot l_1=\left(0{,}1+s\cdot 0{,}2\cdot 10^{-6}\right)\cdot 10=1+s\cdot 2\cdot 10^{-6}$$

$$U_{R2}^I(s)=\frac{U_{g1}}{2}\cdot\frac{1}{2}\cdot e^{-1}\cdot e^{-s\cdot 2\cdot 10^{-6}}=\frac{1}{s}e^{-1}\cdot e^{-s\cdot 2\cdot 10^{-6}}$$

(2 boda)

b) izvor $u_{g1}(t)$ isključen



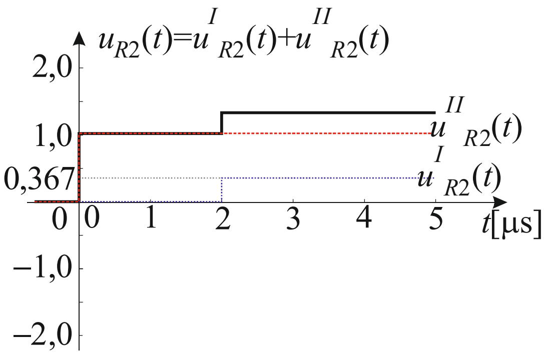
$$U^{II}_{R2}(s) = U_{g2}\cdot\frac{4}{12+4} = U_{g2}\cdot\frac{4}{16} = U_{g2}\cdot\frac{1}{4} = \frac{1}{s}$$

(1 bod)

c) ukupan napon na R_2:

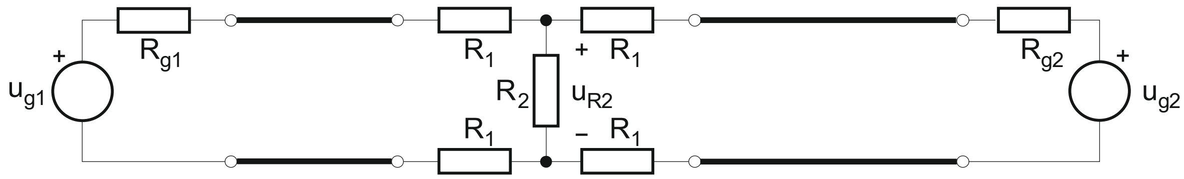
$$U_{R2}(s) = U^{I}_{R2}(s) + U^{II}_{R2}(s) = \frac{1}{s}e^{-1}\cdot e^{-s\cdot 2\cdot 10^{-6}} + \frac{1}{s}$$

$$u_{R2}(t) = u^{I}_{R2}(t) + u^{II}_{R2}(t) = \frac{1}{e}S(t-2\cdot 10^{-6}) + S(t) = 0{,}367879\ \cdot S(t-2\cdot 10^{-6}) + S(t)$$



(1 bod)

5. Zadan je spoj linija s primarnim parametrima R=0,8Ω/km, G=12,5mS/km, L=1,6μH/km i C=25nF/km. Duljina kraće linije je 10km, a dulje 20km. Unutrašnji otpori generatora prilagođeni su zrcalnim impedancijama pripadnih linija. Da li je zadovoljeno prilagođenje linija po zrcalnim impedancijama i na stranama suprotnim od generatora? Obrazloži. Zadano je: R_1=2Ω, R_2=6Ω, $u_{g1}(t)=u_{g2}(t)=4S(t)$. Odrediti i nacrtati $u_{R2}(t)$ (primijeniti postupak superpozicije).



Rješenje: Ispitajmo da li vrijedi $\frac{R}{L}=\frac{G}{C}$ odnosno $\frac{0,8}{1,6\cdot 10^{-6}}=\frac{12,5\cdot 10^{-3}}{25\cdot 10^{-9}}$. Vrijedi! ⇒

To je linija bez distorzije. Računamo sekundarne parametre po pojednostavljenim formulama

$$Z_0=\sqrt{\frac{L}{C}}=\sqrt{\frac{1,6\cdot 10^{-6}}{25\cdot 10^{-9}}}=\sqrt{64}=8\Omega$$

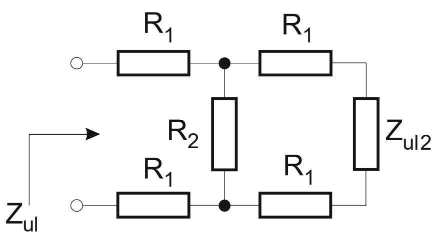
$$\gamma=\sqrt{RG}+s\sqrt{LC}=\sqrt{0,8\cdot 12,5\cdot 10^{-3}}+s\sqrt{1,6\cdot 10^{-6}\cdot 25\cdot 10^{-9}}=0,1+s\cdot 0,2\cdot 10^{-6}$$

(1 bod)

Metoda superpozicije:

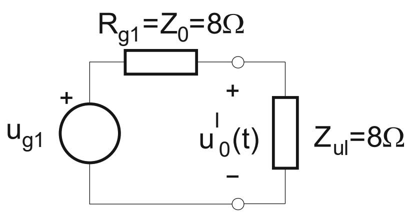
a) izvor $u_{g2}(t)$ isključen (prva linija)

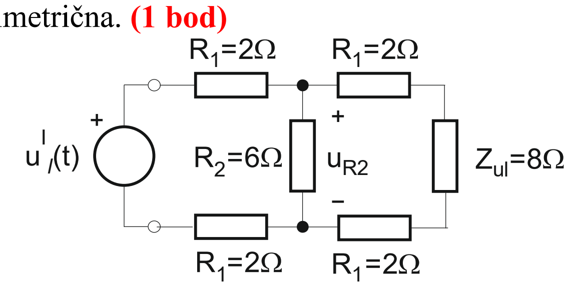
$R_{g1}=8\Omega$ zbog prilagođenja.



$$Z_{ul}=8\Omega=2R_1+\frac{R_2(2R_1+Z_{ul2})}{R_2+2R_1+Z_{ul2}}=4+\frac{6\cdot(4+8)}{6+4+8}=4+\frac{72}{18}=8\Omega$$ prilagođenje je zadovoljeno.

To vrijedi za obje linije jer je mreža sa R_1 i R_2 simetrična. **(1 bod)**





$$u^I_0=\frac{u_{g1}}{2}$$

$$u^I_{R2}=\frac{u^I_l}{2}$$

$$g_1=\gamma\cdot l_1=\left(0,1+s\cdot 0,2\cdot 10^{-6}\right)\cdot 10=1+s\cdot 2\cdot 10^{-6}$$

$$U^I_{R2}(s)=\frac{U_{g1}}{2}\cdot\frac{1}{2}\cdot e^{-1}\cdot e^{-s\cdot 2\cdot 10^{-6}}=\frac{1}{s}e^{-1}\cdot e^{-s\cdot 2\cdot 10^{-6}}$$

To vrijedi za obje linije.

(1 bod)

b) izvor $u_{g1}(t)$ isključen (druga linija)

$$g_2 = \gamma \cdot l_2 = \left(0{,}1 + s \cdot 0{,}2 \cdot 10^{-6}\right) \cdot 20 = 2 + s \cdot 4 \cdot 10^{-6}$$

$$U_{R2}^{II}(s) = \frac{U_{g2}}{2} \cdot \frac{1}{2} \cdot e^{-2} \cdot e^{-s \cdot 4 \cdot 10^{-6}} = \frac{1}{s} e^{-2} \cdot e^{-s \cdot 4 \cdot 10^{-6}}$$

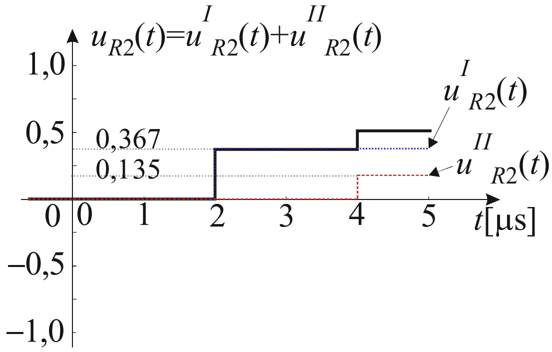
(1 bod)

c) ukupan napon na R_2:

$$U_{R2}(s) = U_{R2}^{I}(s) + U_{R2}^{II}(s) = \frac{1}{s} e^{-1} \cdot e^{-s \cdot 2 \cdot 10^{-6}} + \frac{1}{s} e^{-2} \cdot e^{-s \cdot 4 \cdot 10^{-6}}$$

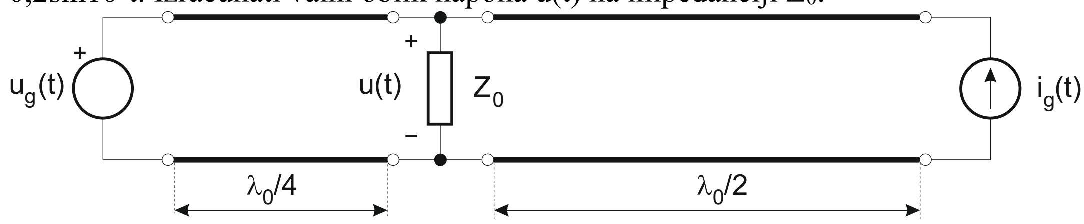
$$u_{R2}(t) = u_{R2}^{I}(t) + u_{R2}^{II}(t) = \frac{1}{e} S(t - 2 \cdot 10^{-6}) + \frac{1}{e^2} S(t - 4 \cdot 10^{-6}) S(t)$$

$$u_{R2}(t) = 0{,}367879 \cdot S(t - 2 \cdot 10^{-6}) + 0{,}135335 \cdot S(t - 4 \cdot 10^{-6})$$



(1 bod)

5. Dvije linije bez gubitaka s primarnim parametrima L=8mH/km i C=800nF/km spojene su prema slici. Na ulaz prve linije duljine $\lambda_0/4$ kod frekvencije $\omega_0=10^5$rad/s priključen je naponski izvor $u_g(t)=10\sin 10^5 t$, a na ulaz druge linije duljine $\lambda_0/2$ strujni izvor $i_g(t)=0{,}2\sin 10^5 t$. Izračunati valni oblik napona u(t) na impedanciji Z_0.



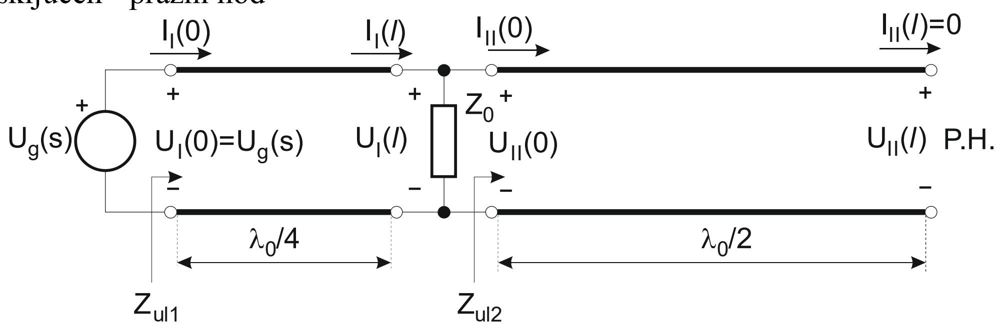
Rješenje:

Linija bez gubitaka $Z_0=\sqrt{\frac{L}{C}}$; $\gamma=j\beta=j\omega_0\sqrt{LC};\alpha=0;\ \lambda_0=\frac{2\pi}{\beta}$.

$Z_0=\sqrt{\frac{8\cdot 10^{-3}}{800\cdot 10^{-9}}}=100\Omega$; $\gamma=j10^5\sqrt{8\cdot 10^{-3}\cdot 800\cdot 10^{-9}}=j\cdot 8/\text{km}$ **(1 bod)**

Metoda superpozicije

a) $i_g(t)$ isključen - prazni hod



Prijenosne jednadžbe linije:

$U(x)=U(0)\cdot ch\,\gamma x-I(0)Z_0 sh\,\gamma x$ ili $U(0)=U(x)\cdot ch\,\gamma x+I(x)Z_0 sh\,\gamma x$

$I(x)=-\frac{U(0)}{Z_0}sh\,\gamma x+I(0)ch\,\gamma x$ $\qquad I(0)=\frac{U(x)}{Z_0}sh\,\gamma x+I(x)ch\,\gamma x$

Ulazna impedancija linije lako se računa iz prijenosnih jednadžbi uz $x=l$:

$$Z_{ul}=\frac{U(0)}{I(0)}=\frac{Z_2 ch(\gamma l)+Z_0 sh(\gamma l)}{\frac{Z_2}{Z_0}sh(\gamma l)+ch(\gamma l)};\ Z_2=\frac{U(l)}{I(l)}$$

Z_2 je impedancija kojom je linija zaključena na izlazu.

Ako je duljina linije: $l_1=\frac{\lambda_0}{4}$ tada je $\gamma\cdot l_1=j\beta\frac{\lambda_0}{4}=j\beta\frac{2\pi}{\beta}\frac{1}{4}=j\frac{\pi}{2}$,

Ako je duljina linije: $l_2=\frac{\lambda_0}{2}$ tada je $\gamma\cdot l_2=j\beta\frac{\lambda_0}{2}=j\beta\frac{2\pi}{\beta}\frac{1}{2}=j\pi$. **(1 bod)**

Za $l_1=\lambda_0/4: sh(\gamma\cdot l_1)=sh(j\pi/2)=j\sin(\pi/2)=j$; $ch(\gamma\cdot l_1)=ch(j\pi/2)=\cos(\pi/2)=0$;

Za $l_2=\lambda_0/2$: $sh(\gamma\cdot l_2)=sh(j\pi)=j\sin(\pi)=0$; $ch(\gamma\cdot l_2)=ch(j\pi)=\cos(\pi)=-1$.

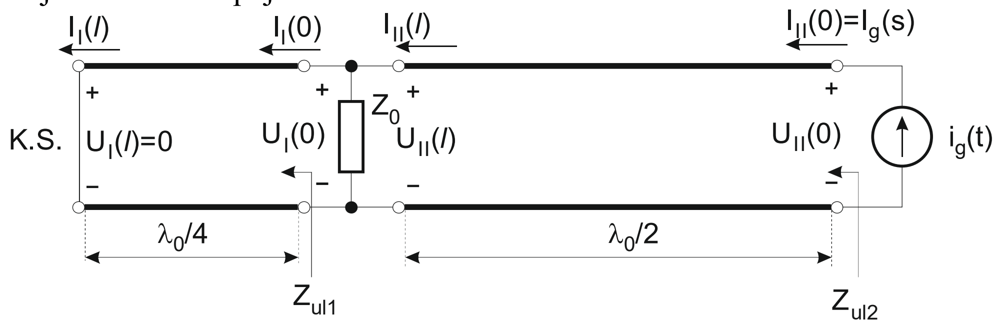
$$Z_{ul2}=\left.\frac{U_{II}(0)}{I_{II}(0)}\right|_{Z_2=\infty,\ l_2=\lambda_0/2}=\frac{U_{II}(l)ch(\gamma l_2)}{\frac{U_{II}(l)}{Z_0}sh(\gamma l_2)}=Z_0\frac{ch(\lambda_0/2)}{sh(\lambda_0/2)}=Z_0\frac{ch(j\pi)}{sh(j\pi)}=Z_0\frac{\cos\pi}{j\sin\pi}=\frac{-Z_0}{0}=\infty$$

$$Z_{ul1} = \left.\frac{U_{\mathrm{I}}(0)}{I_{\mathrm{I}}(0)}\right|_{Z_2=Z_0} = Z_0 \;\Rightarrow\; I_{\mathrm{I}}(0) = \frac{U_{\mathrm{I}}(0)}{Z_0}$$

$$U_I(l_1) = U_I(0)\cdot ch\,\gamma l_1 - I_I(0)\cdot Z_0 sh\,\gamma l_1 = U_I(0)\cdot(ch\,\gamma l_1 - sh\,\gamma l_1) = U_I(0)\cdot e^{-\gamma l_1}$$

$U_I(l_1) = U_I(0)\cdot e^{-j\pi/2} = -jU_I(0) = -jU_g(s)$ **(1 bod)**

b) $u_g(t)$ isključen - kratki spoj



$$Z_{ul1} = \left.\frac{U_{\mathrm{I}}(0)}{I_{\mathrm{I}}(0)}\right|_{Z_2=0,\; l_2=\lambda_0/4} = \frac{I_{\mathrm{I}}(l)Z_0 sh(\gamma l_1)}{I_{\mathrm{I}}(l) ch(\gamma l_1)} = Z_0\frac{sh(\lambda_0/4)}{ch(\lambda_0/4)} = Z_0\frac{sh(j\pi/2)}{ch(j\pi/2)} = Z_0\frac{j\sin\pi/2}{\cos\pi/2} = \frac{jZ_0}{0} = \infty$$

$$Z_{ul1} = \left.\frac{U_{\mathrm{II}}(0)}{I_{\mathrm{II}}(0)}\right|_{Z_2=Z_0} = Z_0 \;\Rightarrow\; U_{\mathrm{II}}(0) = I_{\mathrm{II}}(0)Z_0$$

$$U_{II}(l_2) = U_{II}(0)\cdot ch\,\gamma l_2 - I_{II}(0)\cdot Z_0 sh\,\gamma l_2 = I_{II}(0)\cdot Z_0\cdot(ch\,\gamma l_2 - sh\,\gamma l_2) = I_{II}(0)\cdot Z_0\cdot e^{-\gamma l_2}$$

$U_{II}(l_2) = I_{II}(0)\cdot Z_0\cdot e^{-j\pi} = -I_{II}(0)\cdot Z_0 = -I_g(s)\cdot Z_0$ **(1 bod)**

c) Superpozicija:

$u(t) = 10\sin\left(10^5\cdot t - \frac{\pi}{2}\right) - 20\sin(10^5\cdot t)$ **(1 bod)**

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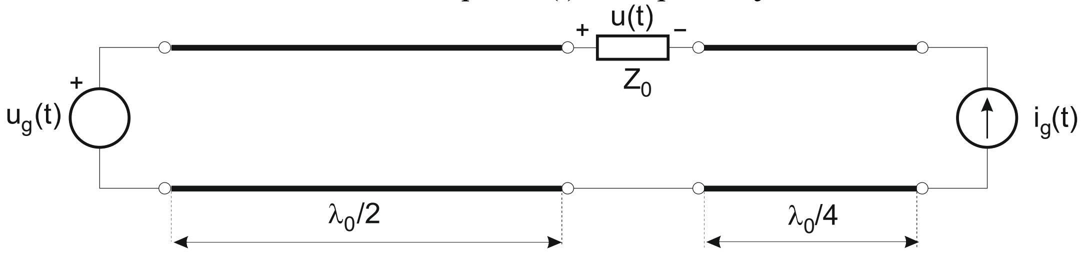
5. Dvije linije bez gubitaka s primarnim parametrima L=8mH/km i C=800nF/km spojene su prema slici. Na ulaz prve linije duljine $\lambda_0/2$ kod frekvencije $\omega_0=10^5$rad/s priključen je naponski izvor $u_g(t)=10\sin 10^5 t$, a na ulaz druge linije duljine $\lambda_0/4$ strujni izvor $i_g(t)=0{,}2\sin 10^5 t$. Izračunati valni oblik napona u(t) na impedanciji Z_0.



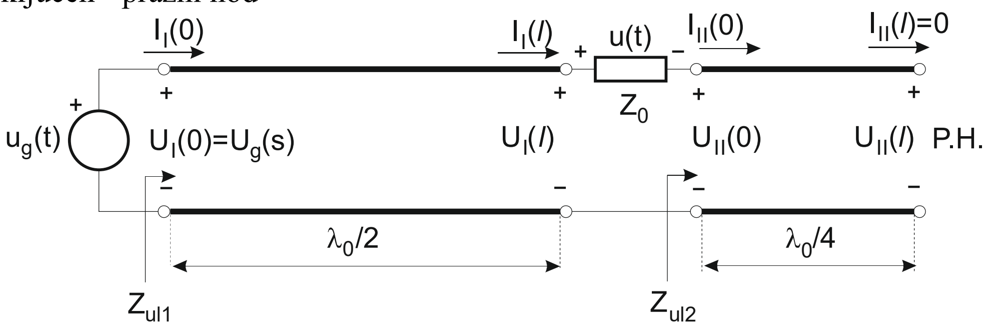
Rješenje:

Linija bez gubitaka $Z_0=\sqrt{\dfrac{L}{C}}$; $\gamma=j\beta=j\omega_0\sqrt{LC};\alpha=0$; $\lambda_0=\dfrac{2\pi}{\beta}$.

$$Z_0=\sqrt{\frac{8\cdot10^{-3}}{800\cdot10^{-9}}}=100\Omega\,;\ \gamma=j10^5\sqrt{8\cdot10^{-3}\cdot800\cdot10^{-9}}=j\cdot8/\text{km}$$ **(1 bod)**

Metoda superpozicije

a) $i_g(t)$ isključen - prazni hod



Prijenosne jednadžbe linije:

$$U(x)=U(0)\cdot ch\gamma x-I(0)Z_0 sh\gamma x \qquad\qquad U(0)=U(x)\cdot ch\gamma x+I(x)Z_0 sh\gamma x$$

$$I(x)=-\frac{U(0)}{Z_0}sh\gamma x+I(0)ch\gamma x \qquad \text{ili} \qquad I(0)=\frac{U(x)}{Z_0}sh\gamma x+I(x)ch\gamma x$$

Ulazna impedancija linije lako se računa iz prijenosnih jednadžbi uz $x=l$:

$$Z_{ul}=\frac{U(0)}{I(0)}=\frac{Z_2 ch(\gamma l)+Z_0 sh(\gamma l)}{\frac{Z_2}{Z_0}sh(\gamma l)+ch(\gamma l)};\ Z_2=\frac{U(l)}{I(l)}$$

Z_2 je impedancija kojom je linija zaključena na izlazu.

Ako je duljina linije: $l_1=\dfrac{\lambda_0}{2}$ tada je $\gamma\cdot l_1=j\beta\dfrac{\lambda_0}{2}=j\beta\dfrac{2\pi}{\beta}\dfrac{1}{2}=j\pi$,

Ako je duljina linije: $l_2=\dfrac{\lambda_0}{4}$ tada je $\gamma\cdot l_2=j\beta\dfrac{\lambda_0}{4}=j\beta\dfrac{2\pi}{\beta}\dfrac{1}{4}=j\dfrac{\pi}{2}$. **(1 bod)**

Za $l_1=\lambda_0/2$: $sh(\gamma\cdot l_1)=sh(j\pi)=j\sin(\pi)=0$; $ch(\gamma\cdot l_1)=ch(j\pi)=\cos(\pi)=-1$.

Za $l_2=\lambda_0/4$: $sh(\gamma\cdot l_2)=sh(j\pi/2)=j\sin(\pi/2)=j$; $ch(\gamma\cdot l_2)=ch(j\pi/2)=\cos(\pi/2)=0$;

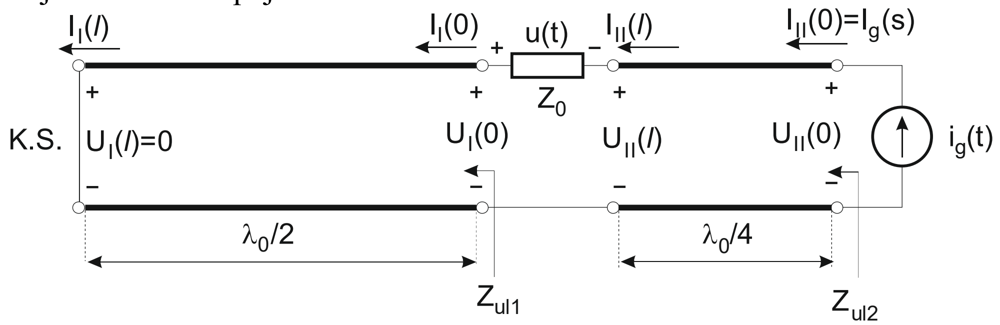
$$Z_{ul2} = \left.\frac{U_{\mathrm{II}}(0)}{I_{\mathrm{II}}(0)}\right|_{Z_2=\infty,\ l_2=\lambda_0/4} = \frac{U_{\mathrm{II}}(l)ch(\gamma l_2)}{\frac{U_{\mathrm{II}}(l)}{Z_0}sh(\gamma l_2)} = Z_0\frac{ch(\lambda_0/4)}{sh(\lambda_0/4)} = Z_0\frac{ch(j\pi/2)}{sh(j\pi/2)} = Z_0\frac{\cos(\pi/2)}{j\sin(\pi/2)} = Z_0\frac{0}{-1} = 0$$

$$Z_{ul1} = \left.\frac{U_{\mathrm{I}}(0)}{I_{\mathrm{I}}(0)}\right|_{Z_2=Z_0} = Z_0 \Rightarrow I_{\mathrm{I}}(0) = \frac{U_{\mathrm{I}}(0)}{Z_0}$$

$$U_I(l_1) = U_I(0)\cdot ch\gamma l_1 - I_I(0)\cdot Z_0 sh\gamma l_1 = U_I(0)\cdot(ch\gamma l_1 - sh\gamma l_1) = U_I(0)\cdot e^{-\gamma l_1}$$

$U_I(l_1) = U_I(0)\cdot e^{-j\pi} = -U_I(0) = -U_g(s)$ **(1 bod)**

b) $u_g(t)$ isključen - kratki spoj



$$Z_{ul1} = \left.\frac{U_{\mathrm{I}}(0)}{I_{\mathrm{I}}(0)}\right|_{Z_2=0,\ l_2=\lambda_0/2} = \frac{I_{\mathrm{I}}(l)Z_0 sh(\gamma l_1)}{I_{\mathrm{I}}(l)ch(\gamma l_1)} = Z_0\frac{sh(\lambda_0/2)}{ch(\lambda_0/2)} = Z_0\frac{sh(j\pi)}{ch(j\pi)} = Z_0\frac{j\sin\pi}{\cos\pi} = Z_0\frac{j0}{-1} = 0$$

$$Z_{ul1} = \left.\frac{U_{\mathrm{II}}(0)}{I_{\mathrm{II}}(0)}\right|_{Z_2=Z_0} = Z_0 \Rightarrow U_{\mathrm{II}}(0) = I_{\mathrm{II}}(0)Z_0$$

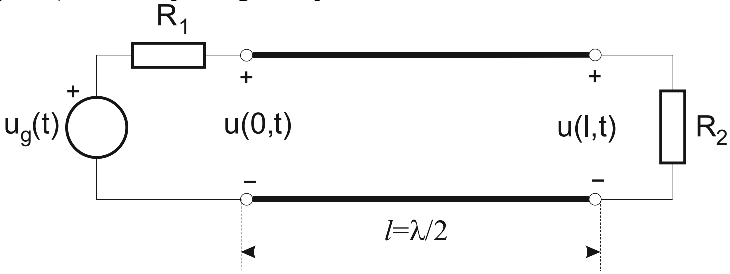
$$U_{II}(l_2) = U_{II}(0)\cdot ch\gamma l_2 - I_{II}(0)\cdot Z_0 sh\gamma l_2 = I_{II}(0)\cdot Z_0\cdot(ch\gamma l_2 - sh\gamma l_2) = I_{II}(0)\cdot Z_0\cdot e^{-\gamma l_2}$$

$U_{II}(l_2) = I_{II}(0)\cdot Z_0\cdot e^{-j\pi/2} = -j\cdot I_{II}(0)\cdot Z_0 = -j\cdot I_g(s)\cdot Z_0$ **(1 bod)**

c) Superpozicija:

$u(t) = -10\sin(10^5\cdot t) + 20\sin\left(10^5\cdot t - \frac{\pi}{2}\right)$ **(1 bod)**

5. Linija bez gubitaka duljine $l=\lambda/2$ na frekvenciji signala, s primarnim parametrima L=0,6 mH/km i C=40 nF/km, zaključena je otporom $R_2=1/\sqrt{3}\cdot 10^3\,[\Omega]$ na izlazu. Na ulazu linije nalazi se naponski generator $u_g(t)=2\cos(10^4\,t)$[V], koji ima unutarnji otpor R_1, jednak karakterističnoj impedanciji linije Z_0. *a*) Koliki je faktor refleksije linije Γ_1 na njenom ulazu i koliki je faktor refleksije Γ_2 na njenom izlazu? *b*) Koliki je napon $u(0,t)$ na ulazu linije? *c*) Kolika je amplituda polaznog, a kolika reflektiranog vala izlazu linije? *d*) Koliki je napon $u(l,t)$ na izlazu linije? *e*) Koliko je duga linija?



Rješenje:

Linija bez gubitaka $\rightarrow\ R=0,\ G=0 \qquad \Rightarrow \qquad Z_0=\sqrt{L/C},\ \gamma=s\sqrt{LC}$

$$Z_0=\sqrt{L/C}=\sqrt{3/2}\cdot 10^2\Omega=50\sqrt{6}=122,474\Omega$$

$$\gamma=s\sqrt{LC}=s\sqrt{24}\cdot 10^{-6}=s2\sqrt{6}\cdot 10^{-6}=s\,4,89898\cdot 10^{-6}\,/\,km$$

a) $Z_1=R_1=Z_0 \Rightarrow \Gamma_1=\dfrac{Z_1-Z_0}{Z_1+Z_0}=0$

- na ulazu u liniju nema refleksije

$$Z_2=R_2\neq Z_0 \Rightarrow \Gamma_2=\frac{Z_2-Z_0}{Z_2+Z_0}=\frac{1/\sqrt{3}\cdot 10^3-\sqrt{3/2}\cdot 10^2}{1/\sqrt{3}\cdot 10^3+\sqrt{3/2}\cdot 10^2}=\frac{10\sqrt{2}-3}{10\sqrt{2}+3}=\frac{11,1}{17,1}=0,65$$

- na izlazu linije postoji refleksija

(1 bod)

b) Napon $u(0,t)$ na ulazu linije

Za liniju duljine $l=\dfrac{\lambda}{2}$ vrijedi: $\gamma=j\omega\sqrt{LC}=j\beta;\ \lambda=\dfrac{2\pi}{\beta} \Rightarrow \gamma l=j\beta\cdot\dfrac{\lambda}{2}=j\beta\cdot\dfrac{1}{2}\cdot\dfrac{2\pi}{\beta}=j\pi$

ili $\beta l=\dfrac{2\pi}{\lambda}\cdot\dfrac{\lambda}{2}=\pi \Rightarrow \gamma l=j\beta\cdot l=j\pi$

Prijenosne jednadžbe linije:

$$U(x)=U(0)\cosh(\gamma x)-I(0)Z_0\sinh(\gamma x) \qquad U(0)=U(x)\cosh(\gamma x)+I(x)Z_0\sinh(\gamma x)$$

$$I(x)=-\frac{U(0)}{Z_0}\sinh(\gamma x)+I(0)\cosh(\gamma x) \quad \text{ili} \quad I(0)=\frac{U(x)}{Z_0}\sinh(\gamma x)+I(x)\cosh(\gamma x)$$

Ulazna impedancija linije lako se računa iz prijenosnih jednadžbi uz $x=l$:

$$Z_{ul}=\frac{U(0)}{I(0)}=\frac{U(l)\cosh\gamma l+I(l)Z_0\sinh\gamma l}{\dfrac{U(l)}{Z_0}\sinh\gamma l+I(l)\cosh\gamma l}=\frac{Z_2\cosh\gamma l+Z_0\sinh\gamma l}{\dfrac{Z_2}{Z_0}\sinh\gamma l+\cosh\gamma l}\ ;\ Z_2=\frac{U(l)}{I(l)}=R_2$$

ili $Z_{ul}=Z_0\dfrac{Z_2\cosh\gamma l+Z_0\sinh\gamma l}{Z_2\sinh\gamma l+Z_0\cosh\gamma l}=Z_0\dfrac{e^{2\gamma l}+\Gamma_2}{e^{2\gamma l}-\Gamma_2} \Rightarrow$

$$Z_{ul}=Z_0\frac{e^{j2\pi}+0,65}{e^{j2\pi}-0,65}=Z_0\frac{1+0,65}{1-0,65}=50\sqrt{6}\,\frac{1,65}{0,35}=577,38\Omega$$

ili $Z_{ul} = Z_0 \dfrac{Z_2 \cosh(j\pi) + Z_0 \sinh(j\pi)}{Z_2 \sinh(j\pi) + Z_0 \cosh(j\pi)} = Z_0 \dfrac{Z_2 \cos(\pi) + Z_0 j \sin(\pi)}{Z_2 j \sin(\pi) + Z_0 \cos(\pi)} = Z_0 \dfrac{-Z_2}{-Z_0} = Z_2 = R_2$

$$U(0) = \frac{Z_{ul}}{Z_{ul} + Z_0} U_g = \frac{577,38}{577,38 + 122,47} \cdot 2 = 1,65[\mathrm{V}]$$

pa je $u(0,t)$=1,65 cos($10^4 t$) [V]

(1 bod)

c) Amplituda polaznog i reflektiranog vala na izlazu linije

$$U_p\left(l = \frac{\lambda}{2}\right) = A_1 e^{-j\beta\frac{\lambda}{2}} = \frac{U(0) + Z_0 I(0)}{2} e^{-j\beta\frac{\lambda}{2}} = \frac{U(0) + Z_0 I(0)}{2} e^{-j\pi} =$$

$$= \frac{U(0) + Z_0 \dfrac{U(0)}{Z_{ul}}}{2} e^{-j\pi} = \frac{U(0)}{2}\left(1 + \frac{Z_0}{Z_{ul}}\right) e^{-j\pi} = \frac{1,65}{2}\left(1 + \frac{122,47}{577,38}\right) e^{-j\pi} = 1 \cdot e^{-j\pi} = -1$$

ili $U_p(l) = U(0) e^{-\gamma l} = \dfrac{U_g}{2} e^{-j\pi} = -\dfrac{U_g}{2} = -1$

(Kada val krene, na ulazu se jedino vidi kao ulazna impedancija: $Z_{ul}=Z_0$.)

$u_p(l,t)$= 1 cos (10^4 t– 180°)= – cos 10^4 t

$$U_r = \Gamma_2 \cdot U_p = \Gamma_2 \cdot A_1 e^{-j\beta_1\frac{\lambda_1}{2}} = 0,65\cdot(-1) = -0,65$$

$u_r(l,t)$= –0,65 (cos 10^4 t)

(1 bod)

d) Napon $u(l, t)$ na izlazu linije je

$u(l,t)=u_p(l,t)+u_r(l,t)= U_p\,(1+\Gamma)$ (cos 10^4 t– 180°)= 1·1,65 (cos 10^4 t– 180°)

$u(l,t)$= –1,65 (cos 10^4 t)

ili $U(l) = U(0)\cos(\beta l) - I(0) Z_0 j \sin(\beta l) = -U(0) = -U_g \dfrac{R_2}{R_1 + R_2} = -1,65$

(Ako se računa napon na izlazu pomoću prijenosnih jednadžbi linije, onda se na ulazu vidi ulazna impedancija: $Z_{ul}=Z_2=R_2$.)

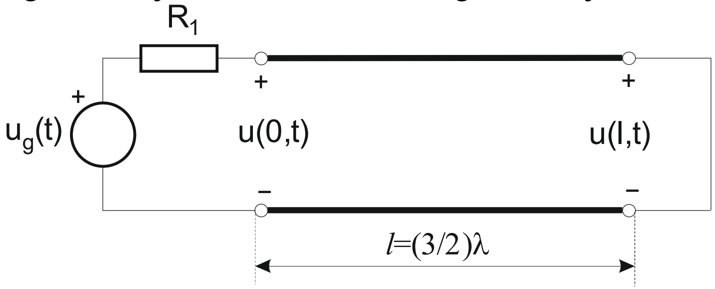
(1 bod)

e) Duljina linije je

$$l = \frac{\lambda}{2};\ \beta = \omega\sqrt{LC};\ \lambda = \frac{2\pi}{\beta} \Rightarrow l = \frac{\pi}{\omega\sqrt{LC}} = \frac{\pi}{10^4\sqrt{24\cdot 10^{-6}}} = 25\sqrt{\frac{2}{3}}\pi = 64,1275[\mathrm{km}]$$

(1 bod)

5. Linija bez gubitaka duljine l=(3/2)·λ na frekvenciji signala, s primarnim parametrima L=2 mH/km i C=6 nF/km, zaključena je s kratkim spojem na izlazu. Na ulazu linije nalazi se naponski generator $u_g(t)=10\cos(10^4 t)$[V], koji ima unutarnji otpor R_1, jednak karakterističnoj impedanciji linije Z_0. *a*) Koliki je faktor refleksije linije Γ_1 na njenom ulazu i koliki je faktor refleksije Γ_2 na njenom izlazu? *b*) Koliki je napon $u(l/2,t)$ na polovini linije? *c*) Kolika je amplituda polaznog vala napona, a kolika reflektiranog vala napona na izlazu linije? *d*) Kolika je amplituda polaznog vala struje, a kolika reflektiranog vala struje na izlazu linije?



Rješenje:

Linija bez gubitaka $\rightarrow$ $R=0$, $G=0$ $\Rightarrow$ $Z_0=\sqrt{L/C}$, $\gamma=s\sqrt{LC}$

$$Z_0=\sqrt{L/C}=\sqrt{2/6}\cdot 10^3\Omega=1/\sqrt{3}\cdot 10^3\Omega=577{,}35\Omega$$

$$\gamma=s\sqrt{LC}=s\sqrt{12}\cdot 10^{-6}=s2\sqrt{3}\cdot 10^{-6}=s\cdot 3{,}4641\cdot 10^{-6}\,/\,\text{km}$$

a) $Z_1=R_1=Z_0 \Rightarrow \Gamma_1=\dfrac{Z_1-Z_0}{Z_1+Z_0}=0$

- na ulazu u liniju nema refleksije

$$Z_2=0\neq Z_0 \Rightarrow \Gamma_2=\frac{Z_2-Z_0}{Z_2+Z_0}=\frac{0-Z_0}{0+Z_0}=-1$$

- na izlazu linije postoji refleksija

(1 bod)

b) Napon $u(l/2,t)$ na polovini linije

Za liniju duljine $l=\dfrac{3}{2}\lambda$ vrijedi: $\gamma=j\omega\sqrt{LC}=j\beta$; $\lambda=\dfrac{2\pi}{\beta}$ $\Rightarrow$

$$\gamma l=j\beta\cdot\frac{\lambda}{2}=j\beta\cdot\frac{3}{2}\cdot\frac{2\pi}{\beta}=j3\pi$$

ili $\beta l=\dfrac{2\pi}{\lambda}\cdot\dfrac{3}{2}\lambda=3\pi \Rightarrow \gamma l=j\beta\cdot l=j3\pi$

Na polovini linije vrijedi $\gamma l/2=j\beta\cdot l/2=j1{,}5\pi$

Prijenosne jednadžbe linije:

$$U(x)=U(0)\cosh(\gamma x)-I(0)Z_0\sinh(\gamma x) \qquad U(0)=U(x)\cosh(\gamma x)+I(x)Z_0\sinh(\gamma x)$$

$$I(x)=-\frac{U(0)}{Z_0}\sinh(\gamma x)+I(0)\cosh(\gamma x) \quad \text{ili} \quad I(0)=\frac{U(x)}{Z_0}\sinh(\gamma x)+I(x)\cosh(\gamma x)$$

$$\cosh(j3\pi)=\cos(3\pi)=\cos(\pi)=-1;\ \sinh(j3\pi)=j\sin(3\pi)=j\sin(\pi)=0$$

Ulazna impedancija linije lako se računa iz prijenosnih jednadžbi uz $x=l$:

$$Z_{ul}=\frac{U(0)}{I(0)}=\frac{U(l)\cosh\gamma l+I(l)Z_0\sinh\gamma l}{\dfrac{U(l)}{Z_0}\sinh\gamma l+I(l)\cosh\gamma l}=Z_0\frac{Z_2\cos(3\pi)+Z_0 j\sin(3\pi)}{Z_2 j\sin(3\pi)+Z_0\cos(3\pi)}=Z_2=0$$

Ako se računa napon na polovini linije pomoću prijenosnih jednadžbi linije, onda se na ulazu vidi ulazna impedancija: Z_{ul}=0.

$$U(0)=0\left[\mathrm{V}\right]$$

ali je

$$I(0)=\frac{U_g}{R_1}=\frac{U_g}{Z_0}=\sqrt{3}\,\frac{10}{10^3}=\sqrt{3}\cdot10^{-2}=17{,}3\left[\mathrm{mA}\right]$$

$$U(l/2)=U(0)\cosh(j3\pi/2)-I(0)Z_0\sinh(j3\pi/2)$$

$$U(l/2)=-I(0)Z_0 j\sin(3\pi/2)=-\frac{U_g}{Z_0}Z_0 j\sin(3\pi/2)=U_g j$$

pa je $u(l/2,t)$=10 cos($10^4 t$+90°) [V]

(2 boda)

c) Amplituda polaznog i reflektiranog vala napona na izlazu linije

Kada val krene, na ulazu se jedino vidi kao ulazna impedancija: $Z_{ul}=Z_0$.

$$U_p(l)=U(0)e^{-\gamma l}=\frac{U_g}{2}e^{-j3\pi}=-\frac{U_g}{2}=-\frac{10}{2}=-5$$

$u_p(l,t)$= 5 cos ($10^4\,t$– 180°)= –5 cos $10^4\,t$

$$U_r=\Gamma_2\cdot U_p=\Gamma_2\cdot A_1 e^{-j\beta\frac{3\lambda}{2}}=(-1)\cdot(-5)=5$$

$u_r(l,t)$= 5 (cos $10^4\,t$)

Napon $u(l, t)$ na izlazu linije je $u(l,t)=u_p(l,t)+u_r(l,t)$=0V

(1 bod)

d) Amplituda polaznog i reflektiranog vala struje na izlazu linije

$$I_p(l)=I(0)e^{-\gamma l}=\frac{U(0)}{Z_0}e^{-\gamma l}=\frac{U_g}{2Z_0}e^{-j3\pi}=-\frac{U_g}{2Z_0}=-\frac{10}{2\cdot1/\sqrt{3}\cdot10^3}=-\frac{\sqrt{3}}{2}\cdot10^{-2}=-8{,}66\left[\mathrm{mA}\right]$$

$i_p(l,t)$= –8,66 cos $10^4\,t$ [mA]

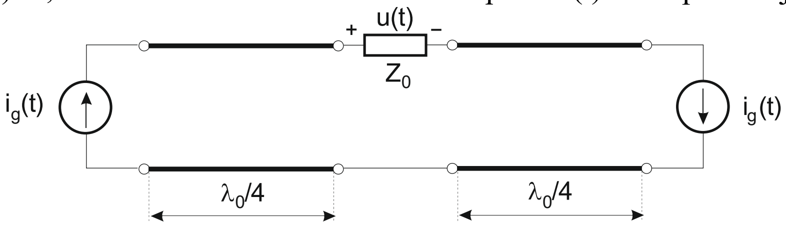
$$I_r=-\Gamma_2\cdot I_p==(1)\cdot(-8{,}66)\ \mathrm{mA}=-8{,}66\ [\mathrm{mA}]$$

$i_r(l,t)$= –8,66 cos $10^4\,t$ [mA]

Struja $i(l, t)$ na izlazu linije je $i(l,t)=i_p(l,t)+i_r(l,t)=I_p\,(1+\Gamma_2)$ = –17,32 (cos $10^4\,t$) [mA]

(1 bod)

5. Dvije linije bez gubitaka duljine $\lambda_0/4$ s primarnim parametrima L=8mH/km i C=800nF/km spojene su prema slici. Na ulazima obiju linija kod frekvencije $\omega_0=10^5$rad/s priključeni su strujni izvori $i_g(t)=0{,}2\sin 10^5 t$. Izračunati valni oblik napona $u(t)$ na impedanciji Z_0.



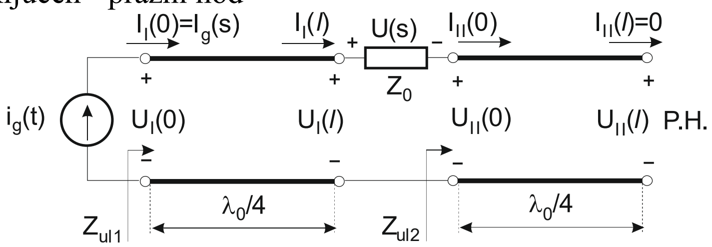
Rješenje:

Linija bez gubitaka $Z_0=\sqrt{\dfrac{L}{C}}$; $\gamma=j\beta=j\omega_0\sqrt{LC};\alpha=0$; $\lambda_0=\dfrac{2\pi}{\beta}$.

$$Z_0=\sqrt{\frac{8\cdot10^{-3}}{800\cdot10^{-9}}}=100\Omega;\ \gamma=j10^5\sqrt{8\cdot10^{-3}\cdot800\cdot10^{-9}}=j\cdot8/\mathrm{km}$$ **(1 bod)**

Metoda superpozicije

a) $i_g(t)$ na izlazu isključen - prazni hod



Prijenosne jednadžbe linije:

$$U(x)=U(0)\cdot ch\gamma x-I(0)Z_0 sh\gamma x \qquad U(0)=U(x)\cdot ch\gamma x+I(x)Z_0 sh\gamma x$$

$$I(x)=-\frac{U(0)}{Z_0}sh\gamma x+I(0)ch\gamma x \quad \text{ili} \quad I(0)=\frac{U(x)}{Z_0}sh\gamma x+I(x)ch\gamma x$$

Ulazna impedancija linije lako se računa iz prijenosnih jednadžbi uz $x=l$:

$$Z_{ul}=\frac{U(0)}{I(0)}=\frac{Z_2 ch(\gamma l)+Z_0 sh(\gamma l)}{\dfrac{Z_2}{Z_0}sh(\gamma l)+ch(\gamma l)};\ Z_2=\frac{U(l)}{I(l)}$$

Z_2 je impedancija kojom je linija zaključena na izlazu.

Ako je duljina linije: $l=\dfrac{\lambda_0}{4}$ tada je $\gamma\cdot l=j\beta\dfrac{\lambda_0}{4}=j\beta\dfrac{2\pi}{\beta}\dfrac{1}{4}=j\dfrac{\pi}{2}$. **(1 bod)**

Za $l=\lambda_0/4$: $sh(\gamma\cdot l)=sh(j\pi/2)=j\sin(\pi/2)=j$; $ch(\gamma\cdot l)=ch(j\pi/2)=\cos(\pi/2)=0$;

$$Z_{ul2}=\left.\frac{U_{II}(0)}{I_{II}(0)}\right|_{Z_2=\infty,\ l=\lambda_0/4}=\frac{U_{II}(l)ch(\gamma l)}{\dfrac{U_{II}(l)}{Z_0}sh(\gamma l)}=Z_0\frac{ch(\lambda_0/4)}{sh(\lambda_0/4)}=Z_0\frac{ch(j\pi/2)}{sh(j\pi/2)}=Z_0\frac{\cos(\pi/2)}{j\sin(\pi/2)}=Z_0\frac{0}{j}=0$$

$$Z_{ul1}=\left.\frac{U_I(0)}{I_I(0)}\right|_{Z_2=Z_0}=Z_0\ \Rightarrow\ U_I(0)=I_I(0)Z_0$$

$$U_I(l)=U_I(0)\cdot ch\gamma l-I_I(0)\cdot Z_0 sh\gamma l=I_I(0)\cdot Z_0\cdot(ch\gamma l-sh\gamma l)=I_I(0)\cdot Z_0\cdot e^{-\gamma l}$$

$$U_I(l)=I_I(0)\cdot Z_0\cdot e^{-j\pi/2}=-j\cdot I_I(0)\cdot Z_0=-j\cdot I_g(s)\cdot Z_0$$

$$\boxed{U_I(l)=-j\cdot I_g(s)\cdot Z_0}$$ **(1 bod)**

b) $i_g(t)$ na ulazu isključen - prazni hod: ista situacija kao pod a) uz suprotan predznak strujnog generatora

$$U_{II}(0) = I_{II}(l) \cdot Z_0 \cdot e^{-j\pi/2} = -j \cdot I_{II}(l) \cdot Z_0 = +j \cdot I_g(s) \cdot Z_0$$

$\boxed{U_{II}(0) = +j \cdot I_g(s) \cdot Z_0}$ **(1 bod)**

c) Superpozicija:

$$u(t) = 20\sin\left(10^5 \cdot t - \frac{\pi}{2}\right) + 20\sin\left(10^5 \cdot t - \frac{\pi}{2}\right) = 40\sin\left(10^5 \cdot t - \frac{\pi}{2}\right)$$ **(1 bod)**

Raspored polova i nula u kompleksnoj ravnini: (1 bod)

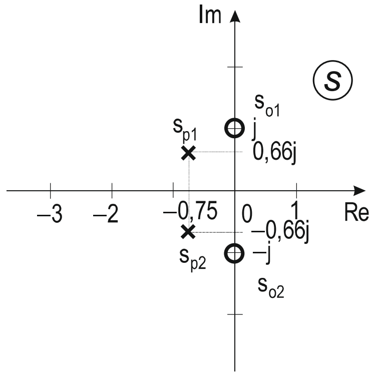
Nule $\qquad s^2+1=0 \Rightarrow s_{o1,2}=\pm j$

Polovi $\qquad s^2+\frac{3}{2}s+1=0 \Rightarrow s_{p1,2}=-\frac{3}{4}\pm\sqrt{\frac{9}{16}-1}=-\frac{3}{4}\pm j\frac{\sqrt{7}}{4}=-0,75\pm j0,661438$

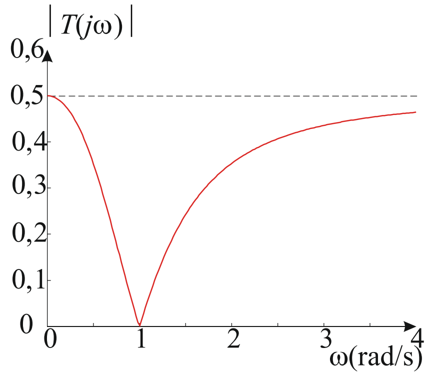
Amplitudno-frekvencijska (A-F) karakteristika (1 bod)

$$s=j\omega \Rightarrow \qquad T(j\omega)=-\frac{1}{2}\cdot\frac{-\omega^2+1}{-\omega^2+\frac{3}{2}\cdot j\omega+1} \Rightarrow \qquad |T(j\omega)|=\frac{1}{2}\cdot\frac{|1-\omega^2|}{\sqrt{\left(1-\omega^2\right)^2+\left(\frac{3}{2}\cdot\omega\right)^2}}$$

Karakteristične točke A-F karakteristike $|T(j0)|=1/2;\ |T(j1)|=0;\ |T(j\infty)|=1/2.$

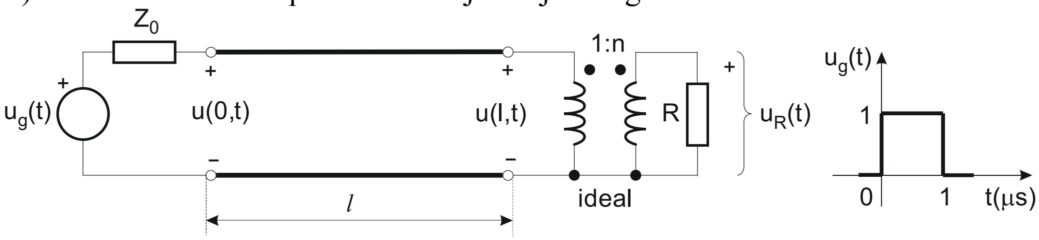


Raspored nula i polova u s-ravnini



Amplitudno frekvencijska karakteristika

5. Zadana je linija s primarnim parametrima R=0,5Ω/km, L=10μH/km, G=2S/km, C=40μF/km, i duljinom l=1km. Na liniju je spojen generator $u_g(t)$ s unutarnjim otporom jednakim zrcalnoj impedanciji linije Z_0 i valnim oblikom prema slici. Koliki mora biti n idealnog transformatora da bi linija bila prilagođena na izlazu po zrcalnoj impedanciji Z_0? a) Odrediti i nacrtati valne oblike napona na izlazu linije $u(l, t)$ i $u_R(t)$ na otporniku R=50Ω. b) kako glase naponi $u(l, t)$ i $u_R(t)$, ako je linija bez gubitaka (primarni parametri: R=0 i G=0, L i C su isti). Nacrtati tražene napone i u slučaju linije bez gubitaka.



Rješenje:

a) $\frac{R}{L}=\frac{G}{C} \Rightarrow \frac{0,5[\Omega/\text{km}]}{10[\mu\text{H/km}]}=\frac{2[\text{S/km}]}{40[\mu\text{F/km}]}$ linija bez distorzije

$$Z_0=\sqrt{\frac{L}{C}}=\sqrt{\frac{10\cdot10^{-6}}{40\cdot10^{-6}}}=\frac{1}{2}$$

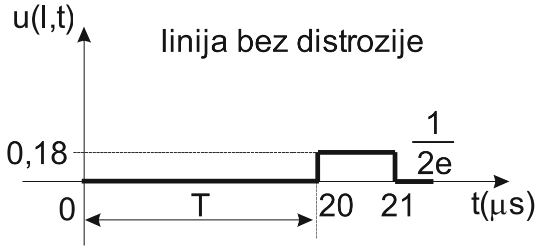
$$Z_2=\frac{R}{n^2}=Z_0=\frac{1}{2}\Rightarrow n^2=\frac{R}{Z_2}=2\cdot50=100\Rightarrow n=10$$

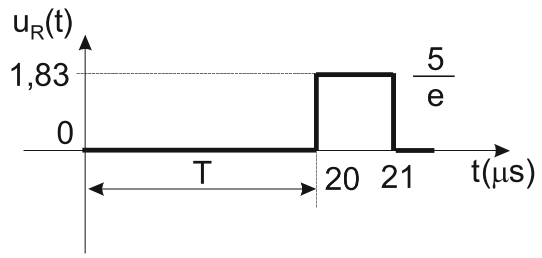
$\gamma=\sqrt{RG}+s\sqrt{LC}=1+20\cdot10^{-6}s$ (1 bod)

$$l=1[\text{km}] \Rightarrow g=\gamma\cdot l=1+20\cdot10^{-6}s=1+Ts;\ T=20[\mu\text{s}]$$

$$U(l) = U(0)e^{-g} = \frac{U_g}{2} e^{-1} e^{-Ts}$$

$$U_R(s) = nU(l) = 10U(l)$$





Valni oblici su ostali isti po obliku kao i pobuda, zakašnjeli su za jednak vremenski iznos *T*, i prigušeni su za iznos 1/e.
(2 boda)

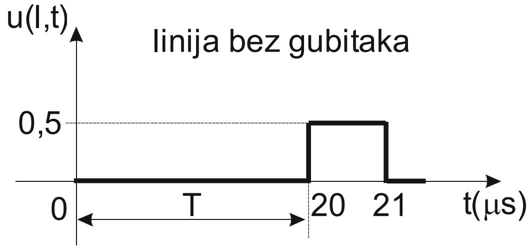
b) *R*=0 i *G*=0 linija bez gubitaka

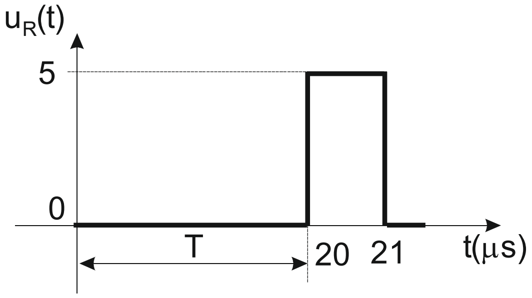
$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{10 \cdot 10^{-6}}{40 \cdot 10^{-6}}} = \frac{1}{2}; \; \gamma = s\sqrt{LC} = 20 \cdot 10^{-6} s$$

$$l = 1[\text{km}] \Rightarrow g = \gamma \cdot l = 20 \cdot 10^{-6} s = Ts; \; T = 20[\mu\text{s}]$$

$$U(l) = U(0)e^{-g} = \frac{U_g}{2} e^{-Ts}$$

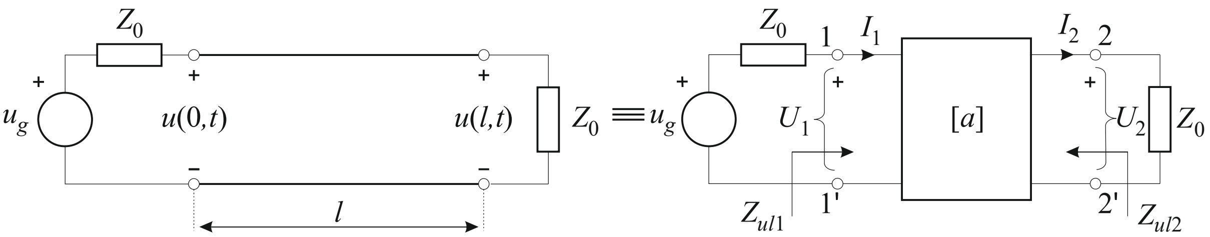
$$U_R(s) = nU(l) = 10U(l)$$





Valni oblici su ostali isti po obliku, zakašnjeli su za jednak vremenski iznos *T*, kao i u slučaju linije bez distorzije, ali nisu prigušeni za iznos 1/e.
(2 boda)

5. Zadana je linija s primarnim parametrima R=0,5Ω/km, L=10μH/km, G=2S/km, C=40μF/km, i duljinom l=1km, prikazana na slici lijevo. Na liniju je spojen generator $u_g(t)$ s unutarnjim otporom jednakim zrcalnoj impedanciji linije Z_0, a na izlaz je spojena impedancija također jednaka Z_0? Na slici desno prikazana je ista linija predstavljena kao četveropol, tako je prijenos signala na obje slike identičan. a) Odrediti prijenosne a-parametre četveropola. b) Pomoću a-parametara izračunati: ulaznu impedanciju u liniju $Z_{ul1}=U_1(s)/I_1(s)$, izlaznu impedanciju $Z_{ul2}=-U_2(s)/I_2(s)$, te naponsku prijenosnu funkciju linije $H(s)=U_2(s)/U_g(s)$. c) Ako je zadana pobuda na ulazu linije kao naponski generator $u_g(t)=2\cos(\omega_g\cdot t)$[V] svevremenska sinusoida ($-\infty<t<\infty$), stacionarno stanje, izračunati valni oblik napona na izlazu $u_2(t)=u(l,t)$. Zadana je frekvencija generatora $\omega_g=(\pi/40)\cdot 10^6$ [rad/s].



Rješenje:

a) $\dfrac{R}{L}=\dfrac{G}{C}\Rightarrow\dfrac{0,5[\Omega/\text{km}]}{10[\mu\text{H/km}]}=\dfrac{2[\text{S/km}]}{40[\mu\text{F/km}]}$ linija bez distorzije

$$Z_0=\sqrt{\frac{L}{C}}=\sqrt{\frac{10\cdot 10^{-6}}{40\cdot 10^{-6}}}=\frac{1}{2}[\Omega]$$

$$\gamma=\sqrt{RG}+s\sqrt{LC}=1+20\cdot 10^{-6}s\;\left[\text{km}^{-1}\right]$$

$$l=1[\text{km}]\Rightarrow \underline{g=\gamma\cdot l=1+20\cdot 10^{-6}s}$$ **(1 bod)**

Nadomjesni (prijenosni) a-parametri linije slijede iz prijenosnih jednadžbi linije.

$[a]$-parametri:

$$U_1=A\cdot U_2+B\cdot I_2$$
$$I_1=C\cdot U_2+D\cdot I_2$$

Prijenosne jednadžbe linije:

$$U(0)=U(x)\cdot ch\gamma x+I(x)Z_0 sh\gamma x$$
$$I(0)=\frac{U(x)}{Z_0}sh\gamma x+I(x)ch\gamma x$$

Ako prijenosne a-parametre izjednačimo s prijenosnim jednadžbama linije uz $x=l$ slijedi:

$$[a]=\begin{bmatrix}A & B\\ C & D\end{bmatrix}=\begin{bmatrix}ch\gamma x & Z_0 sh\gamma x\\ \dfrac{sh\gamma x}{Z_0} & ch\gamma x\end{bmatrix}\Bigg|_{x=l}=\begin{bmatrix}ch\gamma l & Z_0 sh\gamma l\\ \dfrac{sh\gamma l}{Z_0} & ch\gamma l\end{bmatrix}\Rightarrow \begin{array}{l}A=D=ch\gamma l\\ B=Z_0 sh\gamma l\\ C=\dfrac{sh\gamma l}{Z_0}\end{array}$$

Ulazna impedancija u četveropol zaključen s impedancijom $Z_2=Z_0$ na izlazu pomoću a-parametara:

$$Z_{ul1}(s)=\frac{U_1(s)}{I_1(s)}=\frac{AZ_2+B}{CZ_2+D}=\frac{ch(\gamma l)\cdot Z_2+Z_0 sh(\gamma l)}{\left(sh(\gamma l)/Z_0\right)\cdot Z_2+ch(\gamma l)}\Bigg|_{Z_2=Z_0}=\frac{ch(\gamma l)\cdot Z_0+Z_0\cdot sh(\gamma l)}{sh(\gamma l)+ch(\gamma l)}=Z_0=\frac{1}{2}$$

Dobije se isti izraz za ulaznu impedanciju kao da smo upotrijebili formulu dobivenu direktno iz prijenosnih jednadžbi linije:

$$Z_{ul1}=\frac{U(0)}{I(0)}=\frac{Z_2 ch(\gamma l)+Z_0 sh(\gamma l)}{\frac{Z_2}{Z_0}sh(\gamma l)+ch(\gamma l)};\ Z_2=\frac{U(l)}{I(l)}$$

Izlazna impedancija četveropola zaključenog s impedancijom $Z_1=Z_0$ (otpor generatora) na ulazu pomoću *a*-parametara:

$$Z_{ul2}(s)=-\frac{U_2(s)}{I_2(s)}=\frac{DZ_1+B}{CZ_1+A}=\left.\frac{ch(\gamma l)\cdot Z_1+Z_0 sh(\gamma l)}{\left(sh(\gamma l)/Z_0\right)\cdot Z_1+ch(\gamma l)}\right|_{Z_1=Z_0}=\frac{ch(\gamma l)\cdot Z_0+Z_0\cdot sh(\gamma l)}{sh(\gamma l)+ch(\gamma l)}=Z_0=\frac{1}{2}$$

(1 bod)

Prijenosna funkcija napona četveropola zaključenog s impedancijom $Z_2=Z_0$ na izlazu:

$$T(s)=\frac{U_2(s)}{U_1(s)}=\frac{Z_2}{AZ_2+B}=\left.\frac{Z_2}{ch(\gamma l)\cdot Z_2+Z_0 sh(\gamma l)}\right|_{Z_2=Z_0}=\frac{Z_0}{ch(\gamma l)\cdot Z_0+Z_0\cdot sh(\gamma l)}$$

$$=\frac{1}{ch(\gamma l)+sh(\gamma l)}=\frac{1}{\frac{e^{\gamma l}+e^{-\gamma l}}{2}+\frac{e^{\gamma l}-e^{-\gamma l}}{2}}=\frac{1}{e^{\gamma l}}=e^{-\gamma l}$$ **(1 bod)**

c) Ako je zadana pobuda na ulazu linije kao sinusni naponski generator $u_g(t)=2\cos(\omega_g\cdot t)$[V] uz zadanu frekvenciju generatora $\omega_g=(\pi/40)\cdot 10^6$ [rad/s] tada je:

$$U_1(s)=\frac{U_g(s)}{2};\ U_2(s)=U_1(s)\cdot T(s)=\frac{U_g(s)}{2}\cdot e^{-\gamma l}=\frac{U_g(s)}{2}\cdot e^{-(1+20\cdot 10^{-6}s)}$$

Fazor pobudnog napona generatora glasi:

$U_g(j\omega)=2\angle 0^\circ$ uz frekvenciju $\omega_g=(\pi/40)\cdot 10^6$ [rad/s].

Uvrstimo $s=j\omega$ u gornji izraz i dobivamo:

$$U_2(j\omega)=U_1(j\omega)\cdot T(j\omega)=\frac{U_g(j\omega)}{2}\cdot e^{-1}\cdot e^{-20\cdot 10^{-6}j\omega}$$

$$=\left.\frac{2\angle 0^\circ}{2}\cdot e^{-1}\cdot e^{-20\cdot 10^{-6}j\omega}\right|_{\omega=\omega_g}=e^{-1}\cdot e^{-20\cdot 10^{-6}j\frac{\pi}{40}\cdot 10^6}=\frac{1}{e}\cdot e^{-j\frac{\pi}{2}}$$

Stoga je valni oblik signala na izlazu:

$u_2(t)=\frac{1}{e}\cdot\cos\left(\omega_g t-\frac{\pi}{2}\right)=0{,}3678\cdot\cos(\omega_g t-90^\circ)\,[\mathrm{V}]$, gdje je $\omega_g=(\pi/40)\cdot 10^6$ [rad/s].

(2 boda)

Napomena:

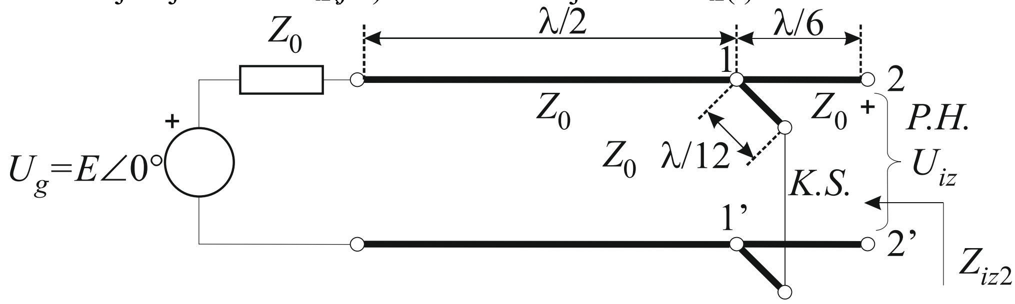
Isti rezultat bi se dobio na pojednostavljen način. Iz prve prijenosne jednadžbe linije slijedi:

$U(0)=U(l)\cdot ch(\gamma l)+I(l)\cdot Z_0 sh(\gamma l)=U(l)\cdot[ch(\gamma l)+sh(\gamma l)]=U(l)\cdot e^{\gamma l}$.

Odatle slijedi: $\boxed{U(l)=U(0)\cdot e^{-\gamma l}=\frac{U_g(s)}{2}\cdot e^{-\gamma l}}$.

Ovaj rezultat je jednostavan jer je linija zaključena s oba kraja njezinom pripadajućom karakterističnom impedancijom Z_0, dakle prilagođena je s oba prilaza.
No mi u našem zadatku želimo provesti cijeli izračun pomoću ekvivalentnih prijenosnih parametre linije, odnosno promatrajući liniju kao četveropol opisan prijenosnim *a*-parametrima.

5. Na slici je zadan sustav linija bez gubitaka istih primarnih parametara. Karakteristične impedancije linija su jednake Z_0, a njihove duljine su označene na slici. Napon generatora na ulazu je stacionarni sinusni signal $u_g(t)=E\cdot\cos(\omega t)$; $-\infty<t<\infty$ (izlaz 2–2' druge linije koja je spojena na prilaz 1–1' je u praznom hodu, a izlaz treće linije je kratko spojen). Odrediti: a) ulazne impedancije u sve tri linije; b) izlaznu impedanciju na izlazu 2–2' Z_{iz2}; c) napon na izlazu 2–2' u frekvencijskoj domeni $\mathbf{U}_{iz}(j\omega)$ i u vremenskoj domeni $u_{iz}(t)$.



Rješenje: Zadatak rješavamo pomoću fazora jer se radi o stacionarnoj sinusnoj pobudi. Ulazna impedancija linije lako se računa iz prijenosnih jednadžbi uz $x=l$:

$$U(0)=U(x)\cdot ch\gamma x+I(x)Z_0 sh\gamma x$$

$$I(0)=\frac{U(x)}{Z_0}sh\gamma x+I(x)ch\gamma x$$

$$Z_{ul}=\frac{U(0)}{I(0)}=\frac{Z_2 ch(\gamma l)+Z_0 sh(\gamma l)}{\frac{Z_2}{Z_0}sh(\gamma l)+ch(\gamma l)};\ Z_2=\frac{U(l)}{I(l)}$$, gdje je

Z_2 impedancija kojom je linija zaključena na izlazu.

Linija bez gubitaka $Z_0=\sqrt{\frac{L}{C}}$; $\gamma=j\beta=j\omega\sqrt{LC};\alpha=0$; $\lambda=\frac{2\pi}{\beta}$.

Ako je duljina linije: $l_1=\frac{\lambda}{2}$ tada je $\gamma\cdot l_1=j\beta\frac{\lambda}{2}=j\beta\frac{2\pi}{\beta}\frac{1}{2}=j\pi$.

Ako je duljina linije: $l_2=\frac{\lambda}{6}$ tada je $\gamma\cdot l_2=j\beta\frac{\lambda}{6}=j\beta\frac{2\pi}{\beta}\frac{1}{6}=j\frac{\pi}{3}$.

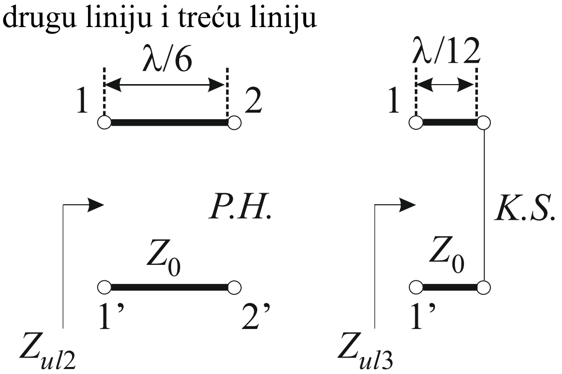
Ako je duljina linije: $l_3=\frac{\lambda}{12}$ tada je $\gamma\cdot l_3=j\beta\frac{\lambda}{12}=j\beta\frac{2\pi}{\beta}\frac{1}{12}=j\frac{\pi}{6}$.

Za $l_1=\frac{\lambda}{2}$: $sh(\gamma\cdot l_1)=sh(j\pi)=j\sin(\pi)=0$; $ch(\gamma\cdot l_1)=ch(j\pi)=\cos(\pi)=-1$;

Za $l_2=\frac{\lambda}{6}$: $sh(\gamma\cdot l_2)=sh\left(j\frac{\pi}{3}\right)=j\sin\left(\frac{\pi}{3}\right)=j\frac{\sqrt{3}}{2}$; $ch(\gamma\cdot l_2)=ch\left(j\frac{\pi}{3}\right)=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$;

Za $l_3=\frac{\lambda}{12}$; $sh(\gamma\cdot l_3)=sh\left(j\frac{\pi}{6}\right)=j\sin\left(\frac{\pi}{6}\right)=j\frac{1}{2}$; $ch(\gamma\cdot l_3)=ch\left(j\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$.

a) Ulazne impedancije u drugu liniju i treću liniju

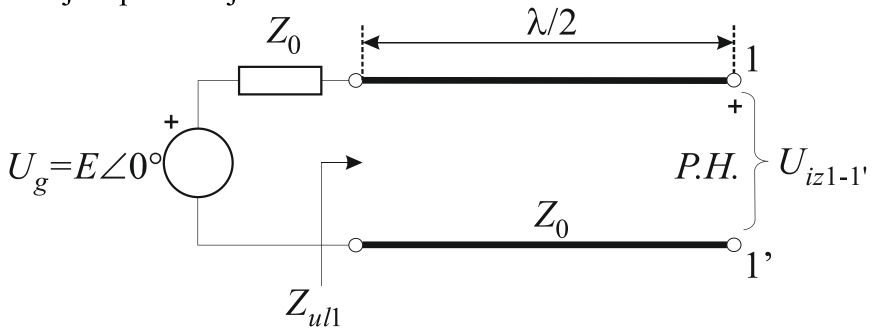


$$Z_{ul2}=\left.\frac{U(0)}{I(0)}\right|_{Z_2=\infty,\ l_1=\lambda/6}=\frac{Z_2 ch\left(j\frac{\pi}{3}\right)+Z_0 sh\left(j\frac{\pi}{3}\right)}{\frac{Z_2}{Z_0}sh\left(j\frac{\pi}{3}\right)+ch\left(j\frac{\pi}{3}\right)}=\frac{\frac{1}{2}+\frac{Z_0}{Z_2}\frac{j\sqrt{3}}{2}}{\frac{1}{Z_0}\frac{j\sqrt{3}}{2}+\frac{1}{Z_2}\frac{1}{2}}=\frac{\frac{1}{2}}{\frac{1}{Z_0}\frac{j\sqrt{3}}{2}}=Z_0\cdot(-j)\cdot\frac{1}{\sqrt{3}}$$

$$Z_{ul3}=\left.\frac{U(0)}{I(0)}\right|_{Z_2=0,\ l_3=\lambda/12}=\frac{Z_0 sh\left(j\frac{\pi}{6}\right)}{ch\left(j\frac{\pi}{6}\right)}=\frac{Z_0 j\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)}=\frac{Z_0 j\left(\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}=Z_0\cdot j\cdot\frac{1}{\sqrt{3}}.$$

Impedancija na mjestu 1–1' je $Z_{1-1'}=Z_{ul2}\|Z_{ul3}=\dfrac{Z_{ul2}\cdot Z_{ul3}}{Z_{ul2}+Z_{ul3}}=\dfrac{Z_0\frac{-j}{\sqrt{3}}\cdot Z_0\frac{j}{\sqrt{3}}}{Z_0\frac{-j}{\sqrt{3}}+Z_0\frac{j}{\sqrt{3}}}=\infty$ **(1 bod)**

Ulazna impedancija u prvu liniju



$$Z_{ul1}=\left.\frac{U(0)}{I(0)}\right|_{Z_2=\infty,\ l_1=\lambda/2}=Z_0\frac{ch(j\pi)}{sh(j\pi)}=Z_0\frac{-1}{0}=\infty$$ **(1 bod)**

b) izlazna impedanciju na izlazu 2–2':

$$Z_{iz2}=\left.\frac{U(l)}{I(l)}\right|_{Z_2=\infty,\ l_1=\lambda/6}=\frac{Z_p ch\left(j\frac{\pi}{3}\right)+Z_0 sh\left(j\frac{\pi}{3}\right)}{\frac{Z_p}{Z_0}sh\left(j\frac{\pi}{3}\right)+ch\left(j\frac{\pi}{3}\right)}=\frac{Z_p\frac{1}{2}+Z_0\frac{j\sqrt{3}}{2}}{\frac{Z_p}{Z_0}\frac{j\sqrt{3}}{2}+\frac{1}{2}};\ \text{gdje je}$$

$$Z_p=Z_{ul3}\|Z_0=\frac{Z_{ul3}\cdot Z_0}{Z_{ul3}+Z_0}=\frac{j\cdot\frac{Z_0}{\sqrt{3}}\cdot Z_0}{j\cdot\frac{Z_0}{\sqrt{3}}+Z_0}=Z_0\cdot\frac{j}{j+\sqrt{3}}=Z_0\cdot\frac{j}{j+\sqrt{3}}\cdot\frac{\left(-j+\sqrt{3}\right)}{\left(-j+\sqrt{3}\right)}=\frac{Z_0}{4}\cdot(1+j\sqrt{3})$$

pa je: $$Z_{iz2}=\left.\frac{U(l)}{I(l)}\right|_{Z_2=\infty,\ l_1=\lambda/6}=\frac{\frac{Z_0}{4}\cdot(1+j\sqrt{3})\frac{1}{2}+Z_0\frac{j\sqrt{3}}{2}}{\frac{1}{Z_0}\cdot\frac{Z_0}{4}\cdot(1+j\sqrt{3})\cdot\frac{j\sqrt{3}}{2}+\frac{1}{2}}=Z_0\frac{(1+j\sqrt{3})+j4\sqrt{3}}{(1+j\sqrt{3})\cdot j\sqrt{3}+4}$$

$$=Z_0\frac{1+j5\sqrt{3}}{1+j\sqrt{3}}=Z_0\frac{1+j5\sqrt{3}}{1+j\sqrt{3}}\cdot\frac{\left(1-j\sqrt{3}\right)}{\left(1-j\sqrt{3}\right)}=Z_0\frac{16+j4\sqrt{3}}{4}=Z_0\left(4+j\sqrt{3}\right)$$ **(1 bod)**

c) napon $\mathbf{U}_{iz}(j\omega)$ na izlazu 2–2':

$$U(0)=U(l_1)\cdot ch\left(\gamma\frac{\lambda}{2}\right)+I(l_1)Z_0 sh\left(\gamma\frac{\lambda}{2}\right)=U(l_1)\cdot\cos(\pi)=-U(l_1)\ \Rightarrow\ \underline{U(0)=-U(l_1)}$$

$$I(0)=\frac{U(l_1)}{Z_0}sh\left(\gamma\frac{\lambda}{2}\right)+I(l_1)ch\left(\gamma\frac{\lambda}{2}\right)=I(l_1)\cdot\cos(\pi)=-I(l_1)\ \Rightarrow\ \underline{I(0)=-I(l_1)=0}$$

$$E=U(0)+I(0)Z_0=U(0)\ \Rightarrow\ \underline{U(l_1)=-U(0)=-E}$$

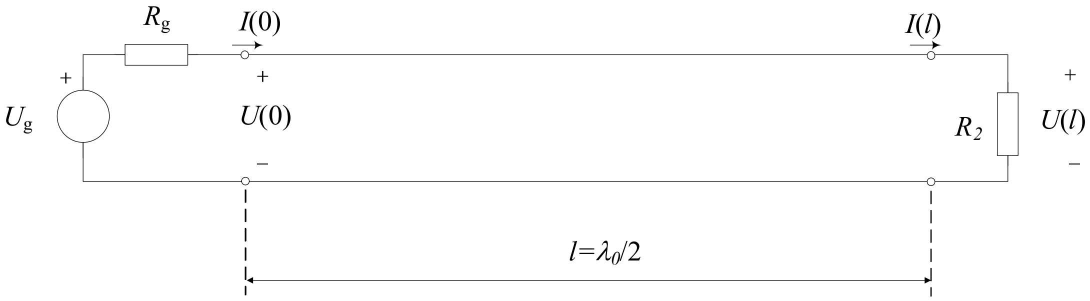
$$U(l_1)=U(l_2)\cdot ch\left(\gamma\frac{\lambda}{6}\right)+I(l_2)Z_0 sh\left(\gamma\frac{\lambda}{6}\right)=U(l_2)\cdot\cos\left(\frac{\pi}{3}\right)=U(l_2)\cdot\frac{1}{2}\ \Rightarrow\ \underline{U(l_2)=2\cdot U(l_1)}$$

$\Rightarrow$ fazor napona na izlazu 2–2’ $\boxed{\mathbf{U}_{iz}(j\omega)=2\cdot U(l_1)=-2\cdot E\angle 0°}$

$\Rightarrow$ valni oblik napona na izlazu 2–2’ $\Rightarrow$ $\underline{u_{iz}(t)=-2\cdot E\cdot\cos(\omega t)}$; $-\infty<t<\infty$. **(2 boda)**

5. Zadana je linija bez gubitaka s L=1,6 mH/km, C=10 nF/km i $l=\lambda_0/2$ kod ω_0. Na ulaz linije priključen je generator napona $u_g(t)$ s unutarnjim otporom R_g, a na kraju linije je otpor R_2=2kΩ. Na frekvenciji $\omega_0=2\cdot10^6$ rad/s ulazna impedancija je prilagođena na Rg.

a) Koliki je Rg?
b) Koliko je duga linija?
c) Odrediti $u(l, t)$ i $i(l, t)$ na toj liniji ako je $u_g(t) = 10\cos(\omega_0 t)$.



Rješenje:

Linija bez gubitaka $\rightarrow$ $R=0$, $G=0$ $\Rightarrow$

$$Z_0=\sqrt{\frac{L}{C}}=\sqrt{\frac{1,6\cdot10^{-3}}{10\cdot10^{-9}}}=\sqrt{0,16\cdot10^{6}}=0,4\cdot10^{3}\Omega \quad\Rightarrow\quad Z_0=400\Omega$$

$\gamma=s\sqrt{LC}$ Stacionarna sinusna pobuda $\rightarrow$ $s=j\omega$ $\Rightarrow$ $\gamma=j\omega\sqrt{LC}=j\beta$

$$\beta=\omega\sqrt{LC}=2\cdot10^6\sqrt{1,6\cdot10^{-3}\cdot10\cdot10^{-9}}=2\cdot10^6\sqrt{16\cdot10^{-12}}=2\cdot10^6\cdot4\cdot10^{-6}=8/\text{km}$$

(1 bod)

a) Koristimo prvi sustav prijenosnih jednadžbi linije

$$U(0)=U(x)\cdot ch(\gamma x)+I(x)Z_0 sh(\gamma x)$$

$$I(0)=\frac{U(x)}{Z_0}sh(\gamma x)+I(x)ch(\gamma x)$$

uz uvrštenu vrijednost $x=l$ te uz zaključenje sa R_2 na izlazu:

$$Z_{ul1}=\frac{U(0)}{I(0)}=\frac{U(l)\cdot ch(\gamma l)+I(l)Z_0 sh(\gamma l)}{\frac{U(l)}{Z_0}sh(\gamma l)+I(l)ch(\gamma l)}=\frac{\frac{U(l)}{I(l)}\cdot ch(\gamma l)+Z_0 sh(\gamma l)}{\frac{U(l)}{I(l)Z_0}sh(\gamma l)+ch(\gamma l)};\quad \frac{U(l)}{I(l)}=R_2$$

$$Z_{ul1}=\frac{U(0)}{I(0)}=\frac{R_2\cdot ch(\gamma l)+Z_0 sh(\gamma l)}{\frac{R_2}{Z_0}sh(\gamma l)+ch(\gamma l)}$$

te uz $l=\frac{\lambda_0}{2}=\frac{2\pi}{\beta}\cdot\frac{1}{2}=\frac{\pi}{\beta}$; $\gamma\cdot l=j\beta\cdot l=j\beta\cdot\frac{\pi}{\beta}=j\pi$ slijedi:

$$Z_{ul1}=\frac{R_2\cdot ch(j\pi)+Z_0 sh(j\pi)}{(R_2/Z_0)\cdot sh(j\pi)+ch(j\pi)}=\frac{R_2\cdot\cos(\pi)+Z_0 j\sin(\pi)}{(R_2/Z_0)\cdot j\sin(\pi)+\cos(\pi)}=\frac{R_2\cdot(-1)}{(-1)}=R_2$$

jer je $\sin(\pi)=0$; $\cos(\pi)=-1$ $\Rightarrow$

$R_g=Z_{ul1}=R_2=2\ [\text{k}\Omega]$ **(1 bod)**

b) $$l=\frac{\lambda_0}{2}=\frac{2\pi}{\beta}\cdot\frac{1}{2}=\frac{\pi}{\omega_0\sqrt{LC}}=\frac{\pi}{2\cdot 10^6\sqrt{1,6\cdot 10^{-3}\cdot 10\cdot 10^{-9}}}=\frac{\pi}{8}=0,3927\text{ km}$$ **(1 bod)**

c) Koristimo drugi sustav prijenosnih jednadžbi linije:

$$U(x)=U(0)\cdot ch(\gamma x)-I(0)Z_0 sh(\gamma x)$$

$$I(x)=-\frac{U(0)}{Z_0}sh(\gamma x)+I(0)ch(\gamma x)$$

uz uvrštenu vrijednost x=l, te uz $l=\frac{\lambda_0}{2}=\frac{2\pi}{\beta}\cdot\frac{1}{2}=\frac{\pi}{\beta}$; $\gamma\cdot l=j\beta\cdot l=j\beta\cdot\frac{\pi}{\beta}=j\pi$

i jer je $\sin(\pi)=0$; $\cos(\pi)=-1$ slijedi:

$$U(l)=U(0)\cdot ch(j\pi)-I(0)Z_0 sh(j\pi)=U(0)\cdot\cos(\pi)-I(0)Z_0 j\sin(\pi)=-U(0)$$

$$I(l)=-\frac{U(0)}{Z_0}\cdot sh(j\pi)-I(0)ch(j\pi)=-\frac{U(0)}{Z_0}\cdot j\sin(j\pi)-I(0)\cos(\pi)=I(0)$$

(1 bod)

$$U(l)=-U(0)=-U_g\frac{Z_{ul1}}{R_g+Z_{ul1}}=-\frac{U_g}{2}=-\frac{10\angle 0^\circ}{2}=-5\angle 0^\circ=5\angle 180^\circ$$

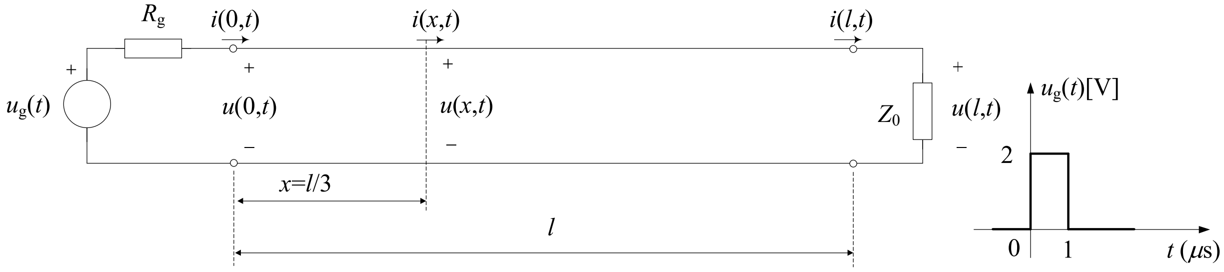
$$\Rightarrow \underline{u(l,t)=5\cos(\omega_0 t-180^\circ)\text{ V}}$$

$$I(l)=I(0)=\frac{U_g}{R_g+Z_{ul1}}=\frac{U_g}{2Z_{ul1}}=\frac{10\angle 0^\circ}{2\cdot 2\cdot 10^3}=\frac{1}{400}\angle 0^\circ=2,5\cdot 10^{-3}\angle 0^\circ$$

$$\Rightarrow \underline{i(l,t)=0,0025\cos(\omega_0 t)\text{ A}=2,5\cos(\omega_0 t)\text{ mA}}$$

(1 bod)

5. Zadana je linija prema slici s primarnim parametrima L=250μH/km, C=100nF/km, R=0, G=0, duljine l=3km. Na liniju je spojen generator napona $u_g(t)$ s unutarnjim otporom R_g jednakim zrcalnoj impedanciji linije Z_0 i valnim oblikom prema slici, a linija je zaključena sa Z_0. Odrediti i nacrtati valni oblik napona $u(x, t)$ na mjestu x=1/3·l od početka linije.



Rješenje:
Ovo je linija bez gubitaka jer je R=0, G=0:

$$Z_0=\sqrt{\frac{L}{C}}=\sqrt{\frac{250\cdot 10^{-6}}{100\cdot 10^{-9}}}=50$$

$$\gamma=s\sqrt{LC}=5\cdot 10^{-6}s[/\,\mathrm{km}]$$

x=1/3·l=1 [km]

$$\gamma\cdot x=5\cdot 10^{-6}s\ \ (x=1km)$$ **(1 bod)**

Ulazna impedancija u liniju:

$$Z_2=\frac{U(l)}{I(l)}=Z_0\Rightarrow \mathrm{Z}_{\mathrm{ul}}=\frac{U(0)}{I(0)}=\mathrm{Z}_0\ \ \text{pa je: } U(0)=\frac{U_g(s)}{2};\ \ I(0)=\frac{U_g(s)}{2\mathrm{Z}_0}$$

$$u_g(t)=2\cdot S(t)-2\cdot S(t-10^{-6})\Rightarrow U_g(s)=\frac{2}{s}-\frac{2}{s}\cdot e^{-s\cdot 10^{-6}}$$ **(1 bod)**

Jer je $U(0,s)=\mathrm{Z}_0 I(0,s)$ za napon na nekom mjestu x na liniji vrijedi:

$$U(x,s)=U(0,s)\cdot ch(\gamma x)-I(0,s)Z_0 sh(\gamma x)=U(0,s)\cdot(ch(\gamma l)-sh(\gamma l))$$

$$U(x,s)=U(0,s)\cdot\left(\frac{e^{\gamma l}+e^{-\gamma l}}{2}-\frac{e^{\gamma l}-e^{-\gamma l}}{2}\right)=U(0,s)\cdot e^{-\gamma x}$$

$$\boxed{U(x,s)=U(0,s)\cdot e^{-\gamma x}}$$ **(1 bod)**

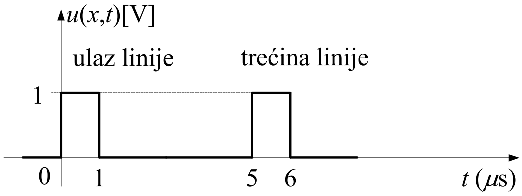
2. način je ako promatramo liniju koja je zaključena svojom valnom impedancijom Z_0, tada se ona ponaša kao beskonačno duga linija. Pritom je ulazni otpor $\mathrm{Z}_{\mathrm{ul}}=\mathrm{Z}_0$, faktor refleksije je $\Gamma_2=(Z_2-\mathrm{Z}_0)/(Z_2+\mathrm{Z}_0)=0$ pa postoji samo polazni val koji glasi: $U_p(x,s)=U(0,s)\cdot e^{-\gamma x}$, a reflektirani val ne postoji jer je $U_p(x,s)=\Gamma_2\cdot U_r(x,s)$, također je $U(0)=U_g(s)/2$. Dobili smo isto rješenje.

$$U(x,s)=\frac{U_g(s)}{2}\cdot e^{-\gamma x}=U(0,s)\cdot e^{-s\cdot 5\cdot 10^{-6}}=\left(\frac{1}{s}-\frac{1}{s}\cdot e^{-s\cdot 10^{-6}}\right)\cdot e^{-s\cdot 5\cdot 10^{-6}}$$

$$U(x,s)=\frac{1}{s}\cdot e^{-s\cdot 5\cdot 10^{-6}}-\frac{1}{s}\cdot e^{-s\cdot(10^{-6}+5\cdot 10^{-6})}$$

$$u\left(x=\frac{l}{3},t\right)=\frac{1}{2}u_g\left(t-5\cdot10^{-6}\right)=S\left(t-5\cdot10^{-6}\right)-S\left(t-6\cdot10^{-6}\right)$$ **(1 bod)**

Za liniju bila bez gubitaka, impuls prolazi linijom, ne mijenja svoj valni oblik niti amplitudu, te kasni za 5μs:



(1 bod)