Upravljanje u prostoru stanja
(State-space control)

MAT. PODSJETNIK:

$$\left.\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{aligned}\right\} \text{ LTI sustav}$$

$$\begin{aligned}x(k+1) &= Ax(k) + Bu(k)\\ y(k) &= Cx(k) + Du(k)\end{aligned}$$

$$\left.\begin{aligned}x &\in R^n\\ u &\in R^m\\ y &\in R^2\end{aligned}\right\} \text{vektori} \qquad \begin{aligned}t &\in R\\ k &\in Z\end{aligned}$$

$$G(s) = C(sI_n - A)^{-1} \cdot B + D$$

$$Y(s) = G(s) \cdot U(s)$$

$$A \in R^{n \times n}$$

$$\boxed{A \cdot v = \lambda v}$$

$v \rightarrow$ SVOJSTVENI VEKTOR (EIGEN VECTOR)

$\lambda \in C$

$\lambda \rightarrow$ EIGENVALUE

$$\left.\begin{aligned}&\lambda_1, \dots \lambda_n\\ &v_1, \dots v_n\end{aligned}\right\} \text{ jedna matrica imat } n \text{ njih}$$

$$\boxed{\det(A) = \prod_{i=1}^{n} \lambda_i}$$

karakteristični polinom matrice:

$$\boxed{\Delta(\lambda; A) := \det(\lambda I_n - A)}$$ polinom po varijabli λ

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots a_1\lambda + a_0$$

$$\boxed{\Delta(\lambda; A) = 0} \Rightarrow \lambda_i$$ SVOJSTVENE VRIJEDNOSTI

KARAKTERISTIČNA JEDNADŽBA

sustav je stabilan ako:

$$Re\{\lambda_i\} < 0$$

KONTINUIRAN SUSTAV

$$|\lambda_i| < 1$$

DISKRETNI SUSTAV

Cayley - Hamiltonov teorem

Svaka matrica $A \in R^{n \times n}$ zadovoljava svoju karakterističnu jednadžbu

$$\boxed{\Delta(\lambda; A)\Big|_{\lambda = A} = 0_{n \times n}}$$

$$A^n + a_{n-1}A^{n-1} + \dots a_1 A + a_0 I_n = 0$$

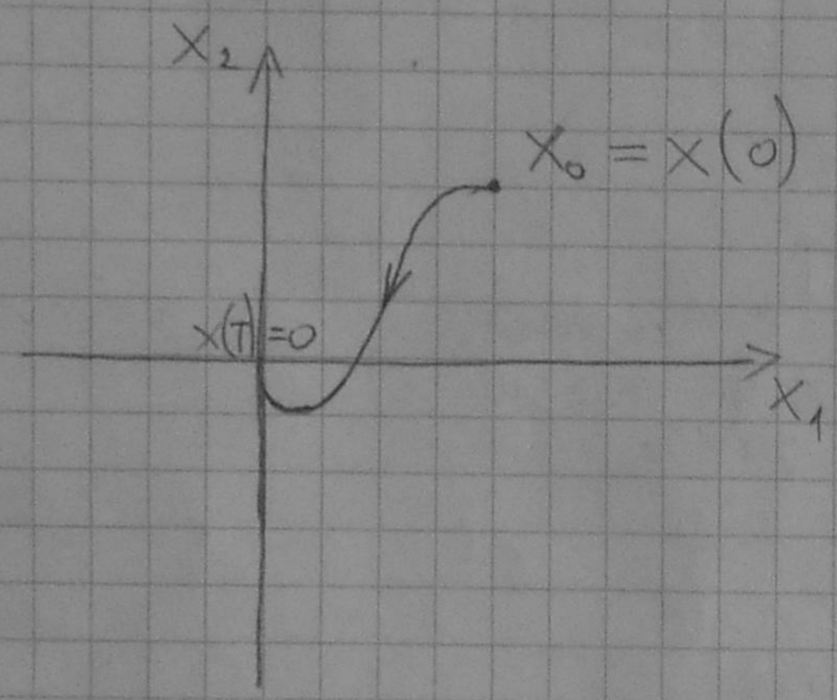
A^n je linearna funkcija prijašnjih vrijednosti

Upravljivost i osmotrivost

DEFINICIJA: Upravljivost (eng. controllability)

$$\forall x_0 \in R^n \ \exists u(t; x_0) : x(0) = x_0, \ x(T) = 0, \ T < \infty$$

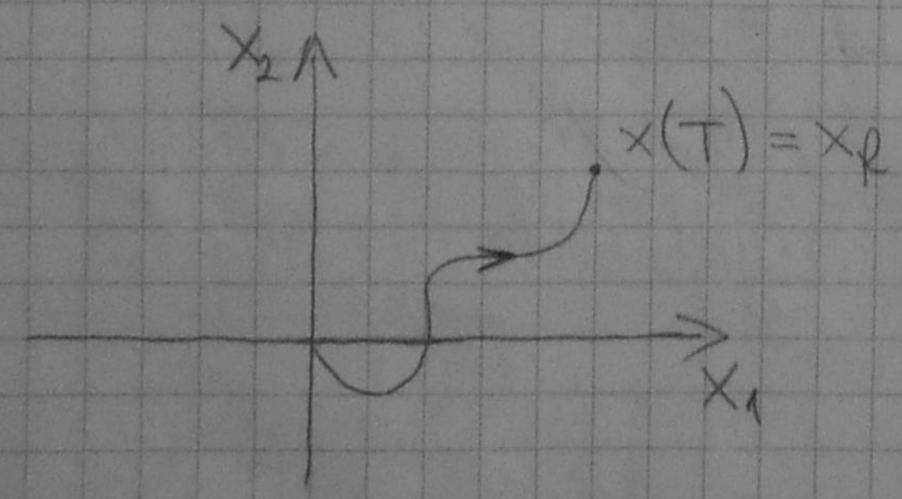
- sustav je potpuno upravljiv ako za svako stanje $x_0 \in R^n$ postoji upravljački signal $u(t; x_0)$ koji u konačnom vremenu T može dovesti sustav iz početnog stanja u ishodište



DEFINICIJA: Dohvatljivost (eng. reachability)

$$\forall x_f \in R^n \ \exists u(t, x_f) : x(0) = 0, \ x(T) = x_f, \ T < \infty$$

- sustav je potpuno dohvatljiv ako za svaki x_f postoji upravljački signal koji može dovesti sustav iz ishodišta u krajnje stanje x_f u konačnom vremenu



Lema:

- Za vremenski kontinuirane sustave UPRAVLJIVOST je ekvivalentna DOHVATLJIVOSTI

- Za vremenski diskretne LTI sustave:

DOHVATLJIVOST $\underset{\text{implikacija}}{\Longrightarrow}$ UPRAVLJIVOST

Pr. $\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u(k)$

$\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} x_2(0) + u(0) \\ 0 \end{bmatrix}$ => za $u(0) = -x_2(0)$ sustav postaje upravljiv

ova komponenta je uvijek 0 pa sustav nije dohvatljiv jer ne može doći u bilo koju mjesto

$$\boxed{A \text{ je IDEMPOTENTNA matrica} \Rightarrow A^i = 0}$$

↳ sustav je upravljiv

$$\left.\begin{aligned} x(k+1) &= Ax(k) + B\cancel{u(k)}^{\,0} \\ x(k+1) &= Ax(k) \\ x(k) &= A^k x(0) \end{aligned}\right\} \text{ VRIJEDI SAMO ZA DISKRETNE}$$

kontinuirani sustav prepušten sam sebi nikad neće doći u ishodište!!!

→ zbog istitravajuće e^{-t}

Test dohvatljivosti

$(A, B) \quad \mathcal{C} \in R^{n \times (n \cdot m)}$

$\mathcal{C} := [B \quad AB \quad A^2B \ldots A^{n-1}B]$

1)- sustav je potpuno dohvatljiv akko je matrica dohvatljivosti punog ranga $\boxed{\text{rang}(\mathcal{C}) = n} \iff \det(\mathcal{C}\mathcal{C}^T) \neq 0$

2)- mod sustava vezan uz

$$\text{rang}\,[(A - \lambda I_n) \quad B] = n$$

- sustav je dohvatljiv ako su dohvatljivi

za $m = 1, 2 \Rightarrow \text{rang}(\mathcal{C}) = n \iff \det(\mathcal{C}) \neq 0$

Test upravljivosti diskretnih sustava

$(A, B) \qquad \text{rang}(\mathcal{C}) = \text{rang}[\mathcal{C} \; A^n]$

DEFINICIJA: Ustabiljivost (stabilizability)

- sustav je ustabiljiv akko su svi njegovi nestabilni modovi dohvatljivi

Primjer:

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\mathcal{C} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad r(\mathcal{C}) = 1$

$\det(\mathcal{C}) = 0 \Rightarrow \text{rang}(\mathcal{C}) \neq 2$

- nije dohvatljiv sustav

$\Delta(\lambda; A) = 0 = \det(\lambda I_2 - A) = \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$

$= \det\left(\begin{bmatrix} \lambda & -1 \\ 0 & \lambda \end{bmatrix}\right) = \lambda^2$

$\lambda^2 = 0 \Rightarrow \boxed{\lambda_{1,2} = 0}$

$\text{rang}\left[(A - \lambda_i \cdot I_2) \quad B\right] = \text{rang}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \underline{1}$

- mod $\lambda = 0$ nije dohvatljiv

2) L71 kont.

$A = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

DZ!

3) L71

$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \quad B \begin{bmatrix} 1 \\ b \end{bmatrix} \quad b \in \mathbb{R}$

za koji b sustav nije dohvatljiv

Osmotrivost i obnovljivost

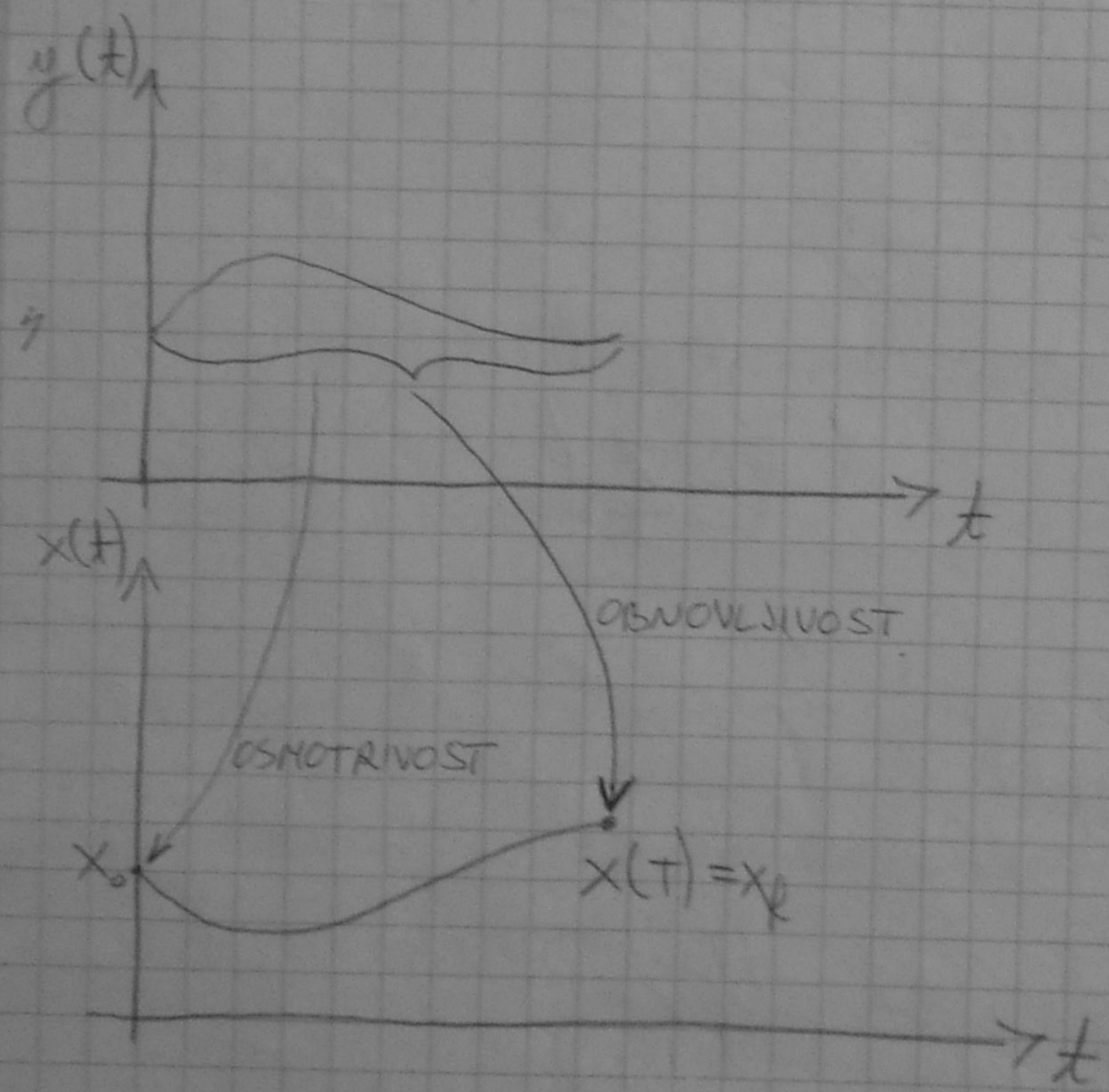
EX

DEF. observability

- sustav je potpuno osmotriv ako za bilo koji $x(0)=x_0$ postoji konačno vrijeme T takvo da je poznavanje $y(T)$, $t\in[0,T]$ dovoljno da se jednoznačno odredi x_0

DEF. Reconstructability

- sustav je potpuno obnovljiv ako za bilo koji $x(T)=x_f$ postoji konačno vrijeme T takvo da je poznavanje $y(T)$, $t\in[0,T]$ dovoljno da se jednoznačno odredi $x(T)$



kontinuirani

OSM. $\Leftrightarrow$ OBNOV.

diskretni

OSM. $\Rightarrow$ OBNOV.

Test osmotrivosti

(A, C) $\quad \mathcal{O} \in \mathbb{R}^{(n \cdot g) \times n}$

$$\mathcal{O} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- sustav je potpuno osmotriv akko je matrica $\mathcal{O}$ osmotrivosti punog ranga

$$\operatorname{rang}(\mathcal{O}) = n \iff \det(\mathcal{O}^T \mathcal{O}) \neq 0$$

- mod sustava je osmotriv akko

$$\operatorname{rang}\begin{bmatrix} A - \lambda I_n \\ C \end{bmatrix} = n$$

- sustav je potpuno osmotriv ako je osmotriv za sve modove

Test obnovljivosti

(A, C) $\quad \operatorname{rang}(\mathcal{O}) = \operatorname{rang}\left(\begin{bmatrix} \mathcal{O} \\ A^n \end{bmatrix}\right)$

DEF. Detektabilnost (eng. detectability)

- sustav je detektabilan akko su svi nestabilni modovi osmotrivi

Primjer:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} \quad B \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = [0]$$

$$\Delta(\lambda; A) = \det\begin{bmatrix} \lambda - 1 & 0 \\ -2 & \lambda - 3 \end{bmatrix} = (\lambda - 1)(\lambda + 3)$$

$$\lambda^2 + 2\lambda - 3 = 0 \qquad \left| \begin{array}{l} \lambda = 1 \\ \lambda = -3 \end{array} \right.$$

$\lambda_2 = 1 \Rightarrow$ nestabilan mod

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \qquad r(\mathcal{O}) = 1 \qquad \operatorname{rang}\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) = 2$$

Dualni sustav

(A, B, C, D)

$-(\bar{A}, \bar{B}, \bar{C}, \bar{D})$

$$\left.\begin{aligned} \bar{A} &= A^T \\ \bar{B} &= C^T \\ \bar{C} &= B^T \\ \bar{D} &= D^T \end{aligned}\right\} \text{DUALNI SUSTAV}$$

- primarni sustav je dohvatljiv ako je dualni osmotriv
  ostvarljiv

DOKAZ:

SVOJSTVO

$$\boxed{\operatorname{rang}(M) = \operatorname{rang}(M^T)}$$

SVOJSTVO

$$(M_1 \cdot M_2)^T = M_2^T M_1^T$$

$$\operatorname{rang}(\bar{O}) = \operatorname{rang}\begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \bar{C}\bar{A}^2 \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{bmatrix} = \operatorname{rang}\begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix} \cancel{= \operatorname{rang}}$$

$$= \operatorname{rang}([B \; AB \; \dots \; A^{n-1}B]) = \operatorname{rang}(\mathcal{C})$$

Transformacija varijabli stanja (KANONIČKI OBLICI)

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

$\tilde{x} = M \cdot x \quad \det(M) \neq 0 \Leftrightarrow \exists M^{-1}$

$\tilde{x}$ ↓ nova varijabla stanja koja je linearna transformacija

$$x = M^{-1}\tilde{x}$$

- dva sustava su ekvivalentna ako postoji M

$$\tilde{x} = \tilde{A}\tilde{x}(k) + \tilde{B}u(k)$$
$$y = \tilde{C}\tilde{x}(k) + \tilde{D}u(k)$$

$$\tilde{A} = M \cdot A \cdot M^{-1}$$
$$\tilde{B} = M \cdot B$$
$$\tilde{C} = C \cdot M^{-1}$$
$$\tilde{D} = D$$

Teorem

- sva svojstva sustava ne mijenjaju se promjenom varijabli stanja (linearnom transformacijom matricom M)

npr. DOHVATLJIVOST

$$\text{rang } \tilde{C} = \text{rang}\left[\tilde{B} \ \tilde{A}\tilde{B} \ \dots \ \tilde{A}^{n-1}\tilde{B}\right]$$

$$= \text{rang}\left[MB \ \ MAM^{-1}MB \ \dots \ MA^{n-1}M^{-1}MB\right]$$

$$= \text{rang}\left[MB \ \ MAB \ \dots \ MA^{n-1}B\right]$$

$$= \text{rang}\left\{M \cdot \left[B \ AB \ \dots \ A^{n-1}B\right]\right\}$$

$= \text{rang } \mathcal{C}$ => jer je M punog ranga pa se množenjem ne mijenja rang

Kanonički upravljivi oblik

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

$u \in \mathbb{R}$ => samo 1 ulaz

$x \in \mathbb{R}^n$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 \dots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -a_1 & -a_2 & -a_3 & \dots & & -a_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\Delta(\lambda; A) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots a_1\lambda + a_0$$

- postoji matrica M koja sustav prevodi u ovaj oblik

$$M = \left(\mathcal{C} \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} & 1 \\ a_2 & a_3 & \dots & 1 & 0 \\ \vdots & & & & \\ a_{n-1} & 0 & & 0 & 0 \\ 1 & 0 & & 0 & 0 \end{bmatrix} \right)^{-1}$$

Kanonický ovládací tvar

$u \in R \Rightarrow$ jedna vstup

$x \in R^n$

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -d_0 \\ 1 & 0 & 0 & \dots & 0 & -d_1 \\ 0 & 1 & 0 & \dots & 0 & -d_2 \\ & \vdots & & & & \\ 0 & 0 & \dots & & 1 & -d_{n-1} \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} d_1 & d_2 & \dots & d_{n-1} & 1 \\ d_2 & d_3 & \dots & 1 & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix} \cdot \mathcal{O}$$

Příklad:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

$$\Delta(\lambda; A) = \det(\lambda I_3 - A) = \lambda^3 \underbrace{-4}_{d_2} \lambda^2 \underset{d_1}{+5}\lambda \underset{d_0}{-2}$$

$$\mathcal{C} = \begin{bmatrix} 0 & -2 & -4 \\ 1 & 1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} 1 & -2 & 0 \\ -1 & -4 & 0 \\ -5 & -8 & -6 \end{bmatrix}$$

$\operatorname{rang} \mathcal{C} = \operatorname{rang} \mathcal{O} = 3$

Upravený tvar

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad M = \frac{1}{2} \begin{bmatrix} 0 & -1 & -1 \\ -1 & -2 & -2 \\ -3 & -4 & -6 \end{bmatrix}$$

$$M = \begin{bmatrix} 4 & -2 & -6 \\ -5 & 4 & 0 \\ 1 & -2 & 0 \end{bmatrix}$$

Primjer (KALMANOVA DEKOMPOZICIJA)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -2 & -2 \\ 3 & -3 & -2 & -2 \\ 2 & 0 & -4 & -2 \\ 1 & 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 7 \\ 6 \\ 4 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 0 \cdot u$$

$$\Delta(\lambda; A) = (\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)$$

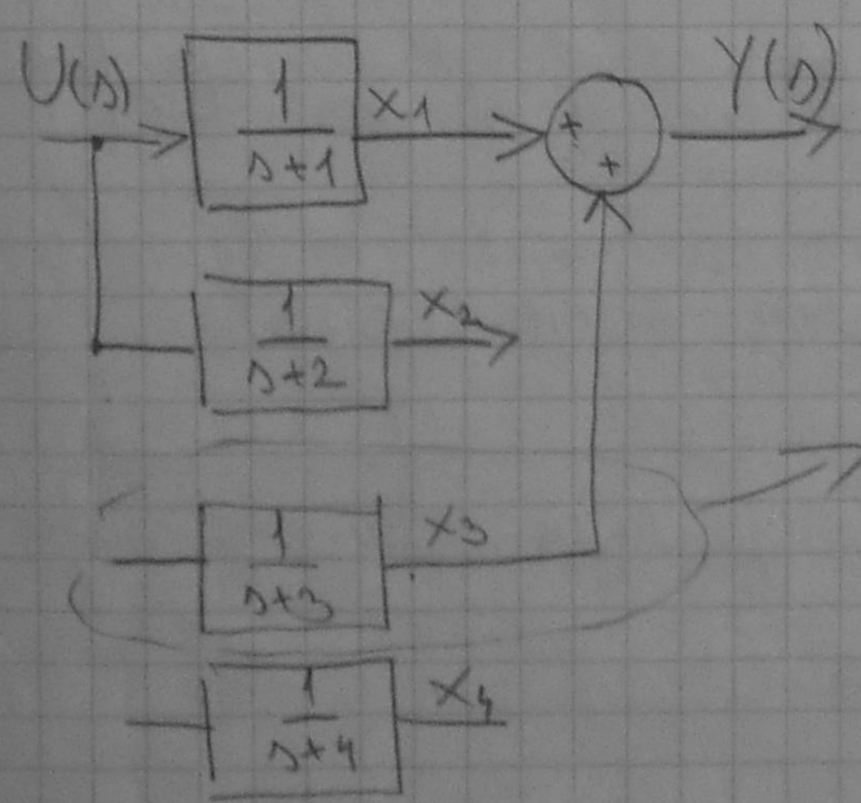
$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D =$$

$$= \frac{5s+8}{(s+1)(s+2)}$$ → došlo je do kraćenja polova i nula

$\frac{1}{s+1}$ je problem napisano i to je pogrešno

$$\tilde{A} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad M = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

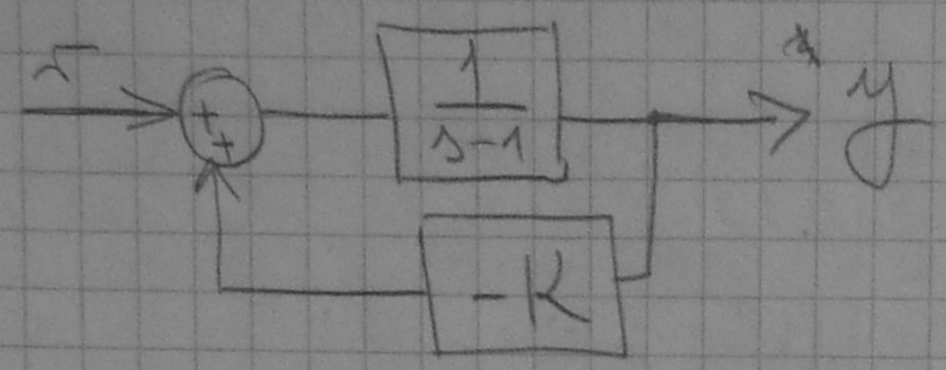
$$\tilde{C} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \quad \tilde{D} = 0$$



na njih se ne može utjecati
- ne vidi se u prijenosnoj funkciji

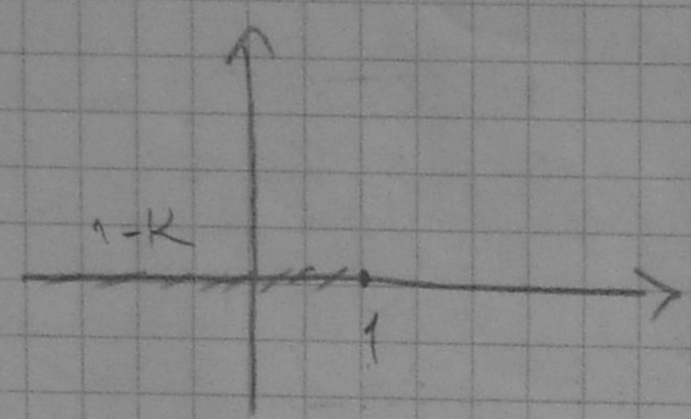
Uvod u regulaciju u prostoru stanja

1) SUSTAV I. REDA



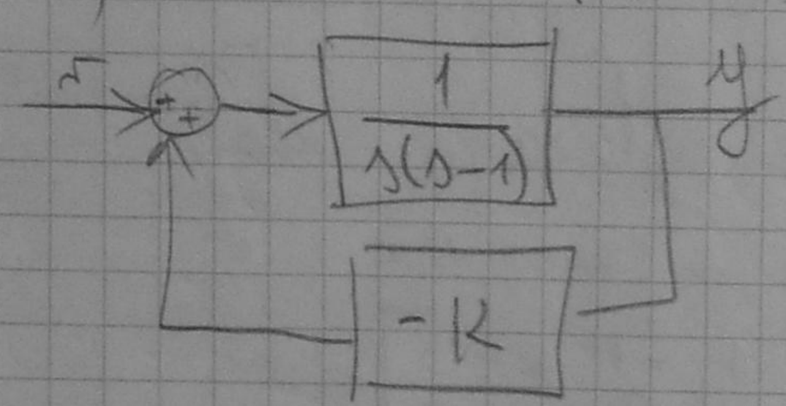
$$G(s) = \frac{\frac{1}{s-1}}{1+\frac{K}{s-1}} = = \frac{1}{s+K-1}$$

$$s_p = 1-K \Rightarrow POL$$



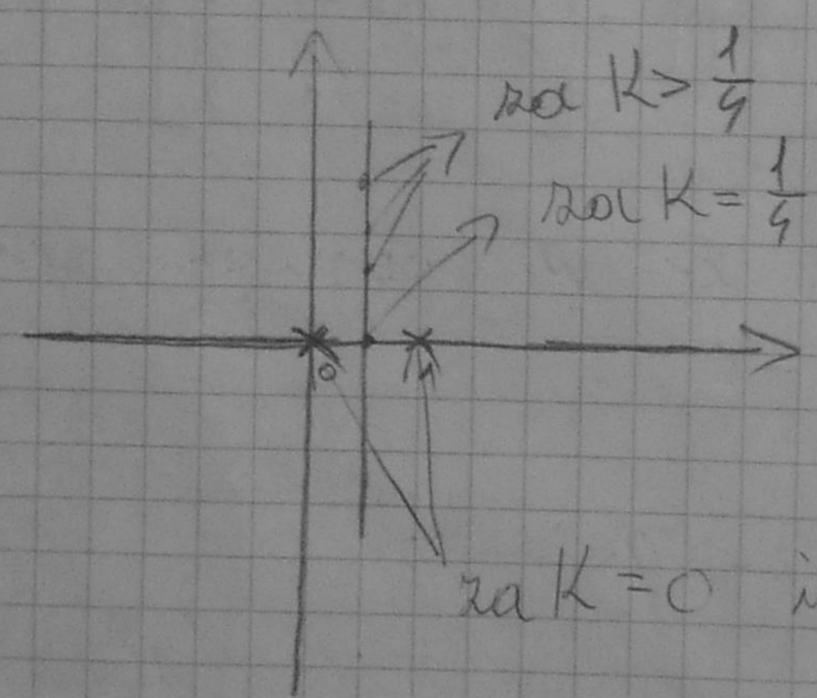
POL $K>0$

2) SUSTAV II. REDA



$$G(s) = \frac{1}{s^2-s+K}$$

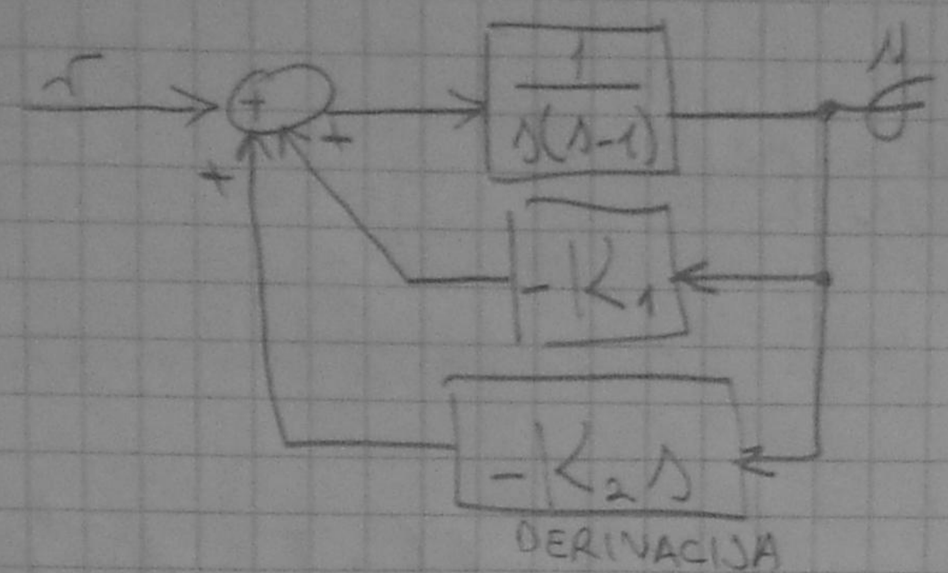
POLOVI: $s_{1,2} = \frac{1 \pm \sqrt{1-4K}}{2}$



krivulja mjesta korijena (eng. ROOT-LOCUS)

- ne može se stabilizirati pojačanjem izlaznog signala

STABILIZACIJA



$$G(s) = \frac{1}{s^2 - s + K_2 s + K_1}$$

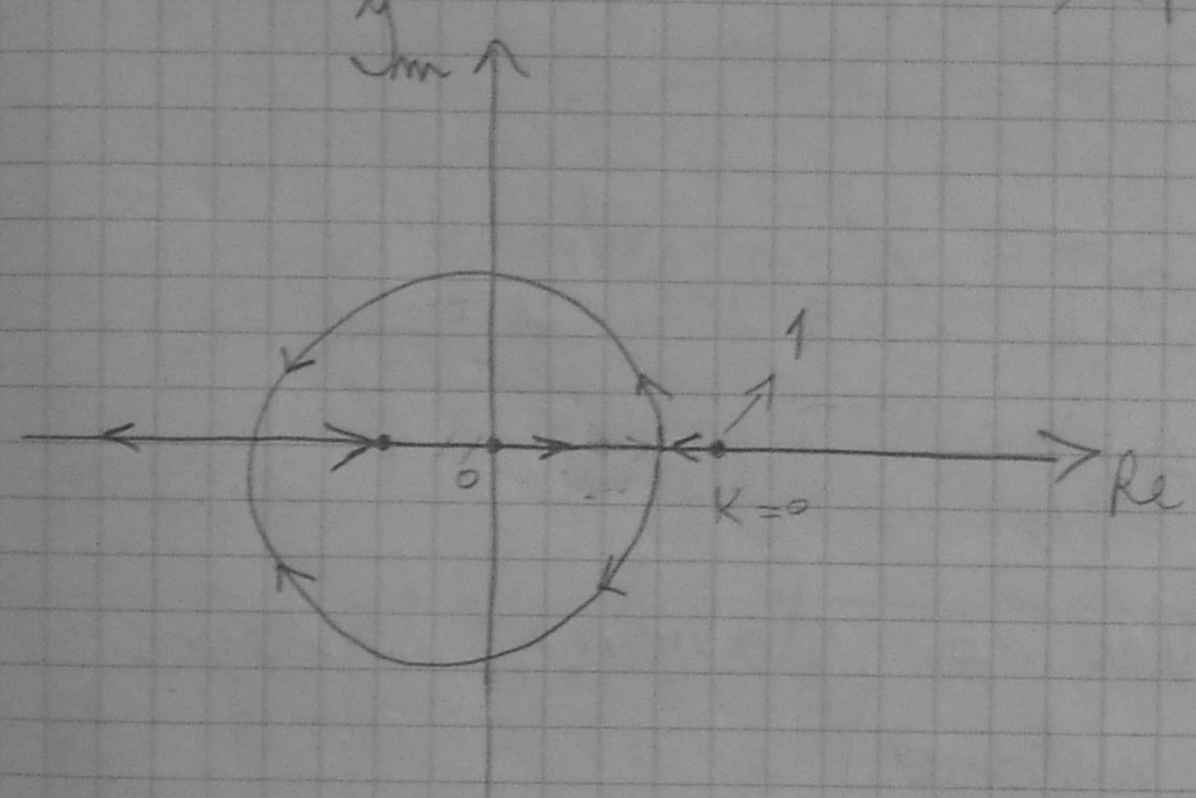
$$s^2 + (K_2 - 1)s + K_1 = 0$$

$$s_{p1,2} = \frac{1 - K_2 \pm \sqrt{(K_2-1)^2 - 4K_1}}{2}$$

npr. može se reći

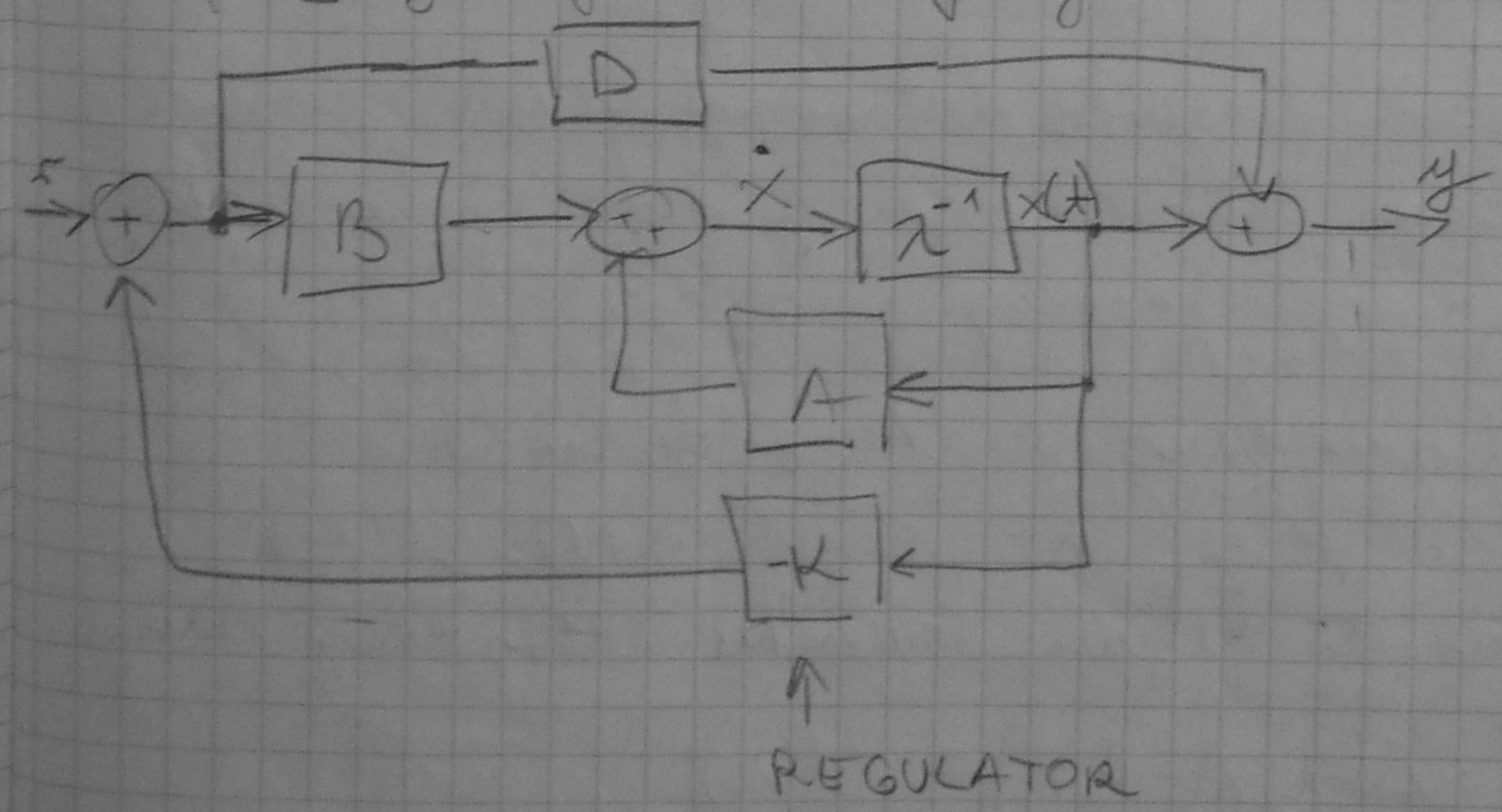
$$\frac{K_1}{K_2} = 1$$

$$\Longrightarrow s_{p1,2} = \frac{1 - K_2 \pm \sqrt{K_2^2 - 6K_2 + 1}}{2}$$



$\frac{1}{1} + \frac{3}{1} - 1 = 0$

-> bolje je umjesto derivacije y koristiti varijable stanja



$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\boxed{u(t) = -Kx(t) + r(t)}$$ -> UPRAVLJAČKI ZAKON

Dinamika otvorenog kruga

$$\Delta(\lambda; A) = \det(\lambda I_n - A) \qquad G(s) = C(\lambda I_n - A)^{-1}B + D$$

Dinamika zatvorenog kruga

$$\dot{x} = (A - BK)x(t) + Br(t)$$
$$y = (C - DK)x + Dr(t$$

$$G(s) = (C - DK)(\lambda I - A + BK)^{-1}B + D$$

$$\Delta(\lambda; A_{CL}) = \det(\lambda I - A_{CL})$$

- upravljački signal ne utiče na stabilnost sustava (stabilnost je svojstvo sustava)

Teorem (regulator po varijablama stanja)

- neka je željena dinamika zatvorenog kruga opisana karakterističnim polinomom n-tog reda s koeficijentima

$$\alpha(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \alpha_{n-2}\lambda^{n-2} + \alpha_1\lambda + \alpha_0 \; ; \; \alpha \in R$$

- ako je otvoreni krug potpuno dohvatljiv onda je regulatorom po varijablama stanja moguće ostvariti željenu dinamiku zatvorenog kruga

$$\exists K \in R^{m \times n} : \Delta(\lambda; A - BK) = \alpha(\lambda)$$

Primjer:

$$A=\begin{bmatrix}1&0&2\\1&2&1\\0&0&1\end{bmatrix}\quad B=\begin{bmatrix}0\\1\\-1\end{bmatrix}\quad C=\begin{bmatrix}1&-2&0\end{bmatrix}\quad D=0$$

$$u(t)=-Kx(t)\qquad K\in\mathbb{R}^{1\times 3}$$

$$\alpha(s)=s^3+6s^2+11s+6 \Rightarrow \text{ovo želimo}$$

$$\Delta(s;A)=\det(sI-A)=\det\begin{bmatrix}s-1&0&-2\\-1&s-2&-1\\0&0&s-1\end{bmatrix}=s^3-4s^2+5s-2$$

$$G(s)=\frac{-2s^2+4s+4}{s^3-4s^2+5s-2}=\frac{-2(s-1-\sqrt{3})(s-1+\sqrt{3})}{(s-1)^2(s-2)}$$

provjera dostatljivosti:

$$\operatorname{rang}(\mathcal{C})=\operatorname{rang}\begin{bmatrix}0&-2&-4\\1&1&-1\\-1&-1&-1\end{bmatrix}=3 \Rightarrow \text{DOMALJIV}$$

OTVORENI KRUG

$$K=\begin{bmatrix}K_1&K_2&K_3\end{bmatrix}$$

$$\Delta(s;A_{cl})=\det\begin{bmatrix}s-1&0&-2\\K_1-1&K_2+s-2&K_3-1\\-K_1&-K_2&s-K_3-1\end{bmatrix}=\underbrace{s^3+(K_2-K_3-4)s^2+(-2K_1-3K_2+3K_3+5)s+4K_1-2K_3-2}_{\alpha(s)}$$

RIJEŠITI SUSTAV

$$\hookrightarrow K=\begin{bmatrix}-18&-30&-40\end{bmatrix}$$

$$G_{cl}(s)=\frac{-2s^2+4s+4}{s^3+6s^2+11s+6}$$

u MATLABU:

~~Skl~~

~~M Matlab = ctrb(A,B)~~

~~rank(Matlab, 10^{-4})~~ ~~Block~~

DeltaA = poly(A)

roots(DeltaA) <=> eigen(A)

P = [-1 -2 -3] -> ŽELJENI POLOVI

K = Pole(A,B,P) <=> $\boxed{K = \text{place}(A,B,P)}$

?

OBIČNA FUNKCIJA

Primjer: prethodni primjer u kanonički upravljivi oblik

$$M = \begin{bmatrix} 0 & -0.5 & -0.5 \\ -0.5 & -1 & -1 \\ -1.5 & -2 & -3 \end{bmatrix} \qquad \tilde{x} = M \cdot x$$

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

?

$$A_{cl} = A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0-k_1 & -a_1-k_2 & -a_2-k_3 \end{bmatrix}$$

$$\Delta(s; A_{cl}) = s^3 + (a_2+k_3)s^2 + (a_1+k_2)s + (a_0+k_1)$$

$$K = [\alpha_0 - a_0 \quad \alpha_1 - a_1 \quad \alpha_2 - a_2]$$

- osmotrivost se gubi kraćenjem polova i nula
  - može doći do gubitka osmotrivosti kod regulatora u prostoru stanja

Ackermanova formula

za SISO dokvantljive sustave možemo napraviti regulator u prostoru stanja:

$$K = [0 \ \dots \ 0 \ 1] \cdot \mathcal{C}^{-1} \cdot \alpha(A)$$

$$\alpha(A) = A^n + \alpha_{n-1} A^{n-1} + \dots + \alpha_1 A + \alpha_0 I_n$$

- u Matlabu

$K = \text{acker}(A, B, p)$ → polovi zatvorenog kruga

REGULATOR PO IZLAZU	REGULATOR PO VARIJABLAMA STANJA
+ mjeriti izlaz	- mjeriti sve varijable stanja (neke se eventualno mogu računati)
- red zatvorenog kruga je veći od otvorenog (dodatna dinamika)	+ regulator 0. reda (nema dodatne dinamike; koristi se postojeća stanja
- ne kontroliramo sve polove	+ kontroliramo položaj svih polova zatvorenog kruga

Vremenski optimalno upravljanje
(Dead beat control)

- odnosi se samo za DISKRETNE sustave

$$\boxed{u(t) = -K_{DB} \cdot x(t)} \Rightarrow \text{OBLIK REGULATORA} \left\{ \begin{array}{l} \text{povratna veza} \\ \text{regulatora u trenutku skoka} \end{array} \right\}$$

- ako sustav ima dinamiku $\alpha(z)$ onda se nazivna

DEAD BEAT

$\alpha(z) = z^n \Leftrightarrow z_{0,i} = 0 \Rightarrow$ svojni polovi zatvorenog kruga

$$\Delta(z; A_{CL}) = \Delta(z; A - BK_{DB}) = z^n$$

$$A_{CL}^n = 0$$

$$x(k+1) = (A - BK_{DB})\, x(k)$$

$$x(k) = (A - BK_{DB})^k \cdot x(0) = A_{CL}^n \cdot x(0) = \underline{0}$$

Primjer: DLTI

$$A = \begin{bmatrix} 0.54 & 1 \\ -0.5184 & 0.54 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0.73 & 0 \end{bmatrix} \quad D = 0$$

1) Dohvatljivost, osmotrivost $\quad x[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

2) $u(t) = -kx(t) + r \quad \alpha(z) = z^2 - 0.6z + 0.25, \quad r(t) = 0.5 \cdot S(t-20)$

$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

3) $u(t) = -K_{DB}\, x(t) \quad \alpha(z) = z^2$

$$\Delta(z; A - BK_{DB}) =$$

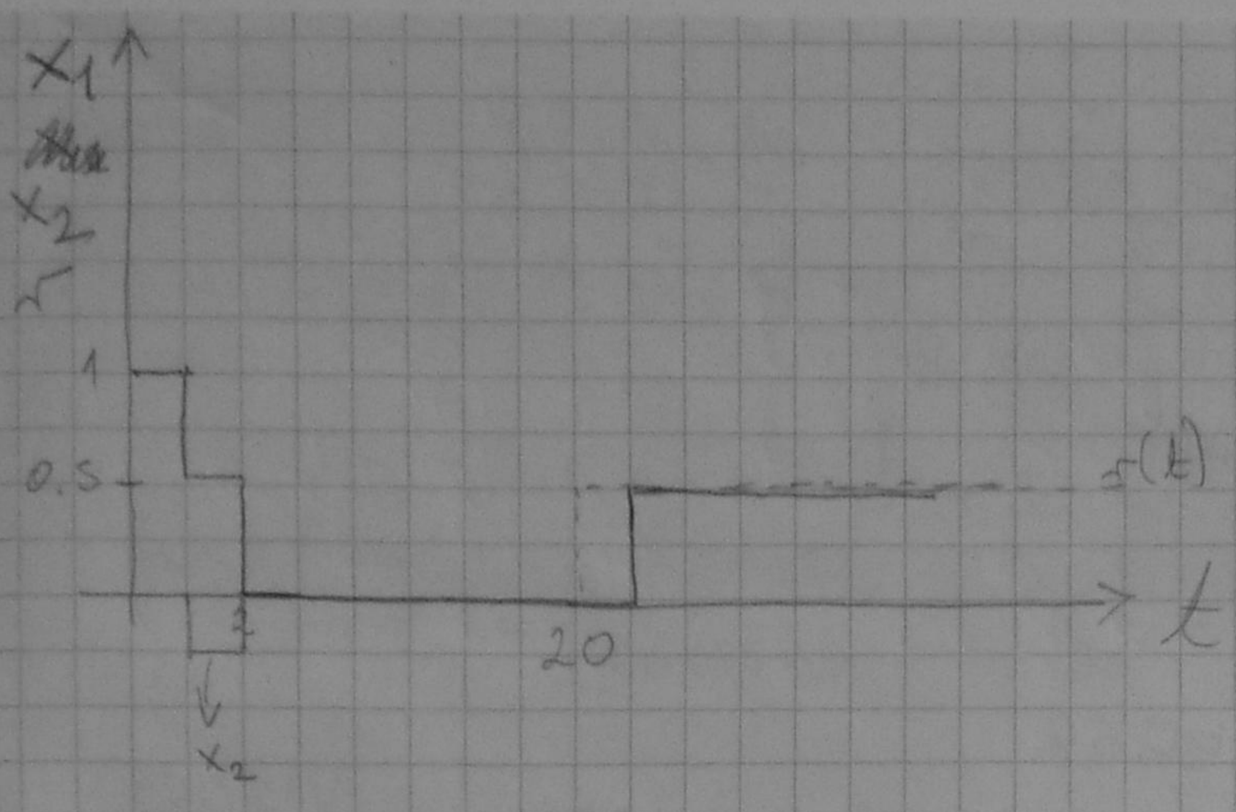
$z^2 \Rightarrow 1 = 1$

$z^1 \Rightarrow 0 = k_2 - 1.08$

$z^0 \Rightarrow 0 = k_1 - 0.54k_2 + 0.81$

$$K_{DB} = \begin{bmatrix} -0.2268 & 1.08 \end{bmatrix}$$

$$G_{CL} = \frac{0.73}{z^2}$$



Izbor svojstvenih vrijednosti zatvorenog kruga

1) SVOĐENJE NA DRUGI RED
   - makniti sve polove osim 2 dominantna daleko

2) OPTIMALNO UPRAVLJANJE

3) PROJEKTIRANJE PO PROTOTIPU

$$\frac{Y}{R} = \frac{1}{s^n + d_n s^{n-1} + \dots 1}$$

ITAE KRITERIJ: $\min \int_0^\infty t \cdot |e(t)| \, dt$

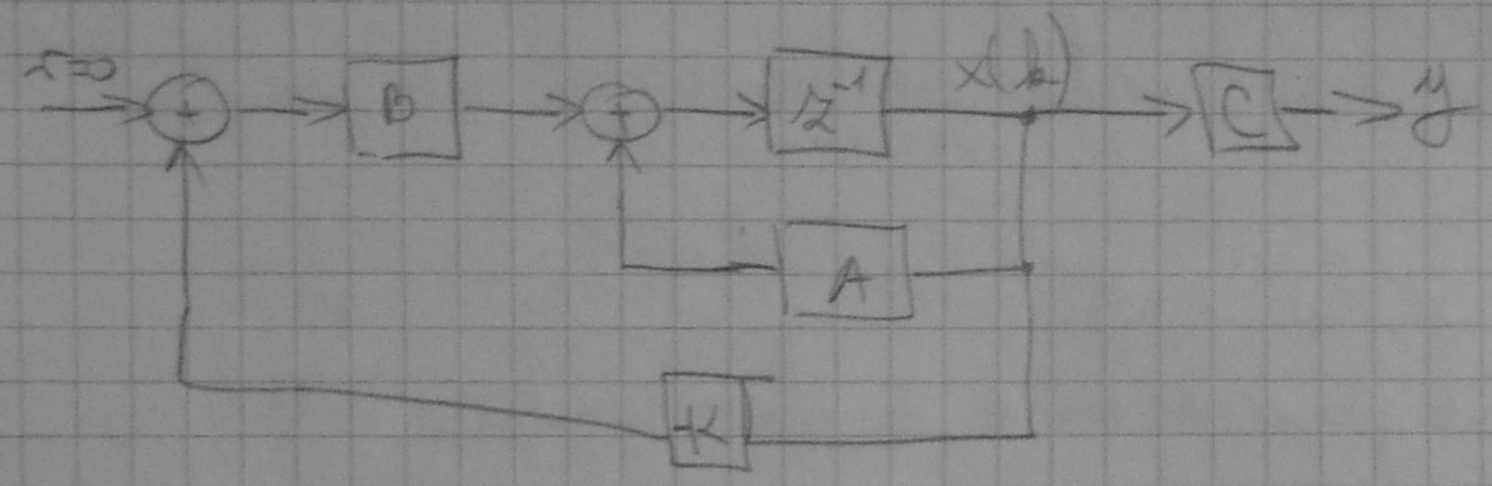
Pr. DLTI

$$A = \begin{bmatrix} 0.79 & 0 \\ 0.176 & 0.857 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$D = 0$$

$$B = \begin{bmatrix} 0.281 \\ 0.0296 \end{bmatrix}$$

1) $\alpha(z) = z^2 - 1.55z + 0.64$



$z_d = 0.775 \pm j\,0.1984$ $\qquad K = [K_1 \; K_2]$

$\Delta(z; A - BK) =$

$$\mathcal{C} = [B \;\; AB] = \begin{bmatrix} 0.281 & 0.2220 \\ 0.029 & 0.0748 \end{bmatrix}$$

Ackerman: $K = [0 \;\; 1]\,\mathcal{C}^{-1} \cdot \alpha(A)$

$$K = [0 \;\; 1] \begin{bmatrix} 0.281 & 0.029 \\ 0.2220 & 0.0748 \end{bmatrix} \cdot \left(A^2 - 1.55A + 0.64\, I_2\right)$$

$$= [0.2508 \;\; 0.8962]$$

b)

neka je $r \neq 0$ $\qquad \bar{x} = \begin{bmatrix} x \\ x_I \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} A & 0 \\ -C & 1 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \quad \bar{C} = [C \;\; 0]$$

$$\bar{x}(k+1) = \bar{A}\,\bar{x}(k) + B\,u(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(k)$$

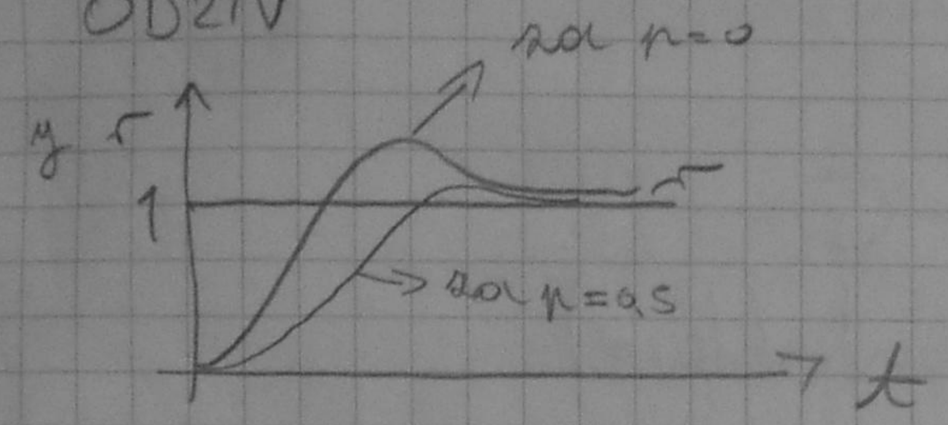
$$y(k) = \bar{C}\,\bar{x}(k)$$

$$\bar{\alpha}(z) = \alpha(z)(z-p) \qquad u = -\bar{K}\cdot\bar{x}$$

$$\bar{K} = [0\ 0\ 1]\cdot\bar{\mathcal{C}}^{-1}\bar{\alpha}(\bar{A}) \qquad \bar{\mathcal{C}} = \begin{bmatrix} 0.2810 & 0.2220 & 0.1754 \\ 0.0236 & 0.0748 & 0.1032 \\ 0 & -0.0296 & -0.1044 \end{bmatrix}$$

$$\bar{K} = \begin{bmatrix} k_1 \\ k_2 \\ k_I \end{bmatrix}^T = \begin{bmatrix} -3.0280p + 3.2788 \\ -5.038p + 5.9341 \\ 1.6166p - 1.6166 \end{bmatrix}$$

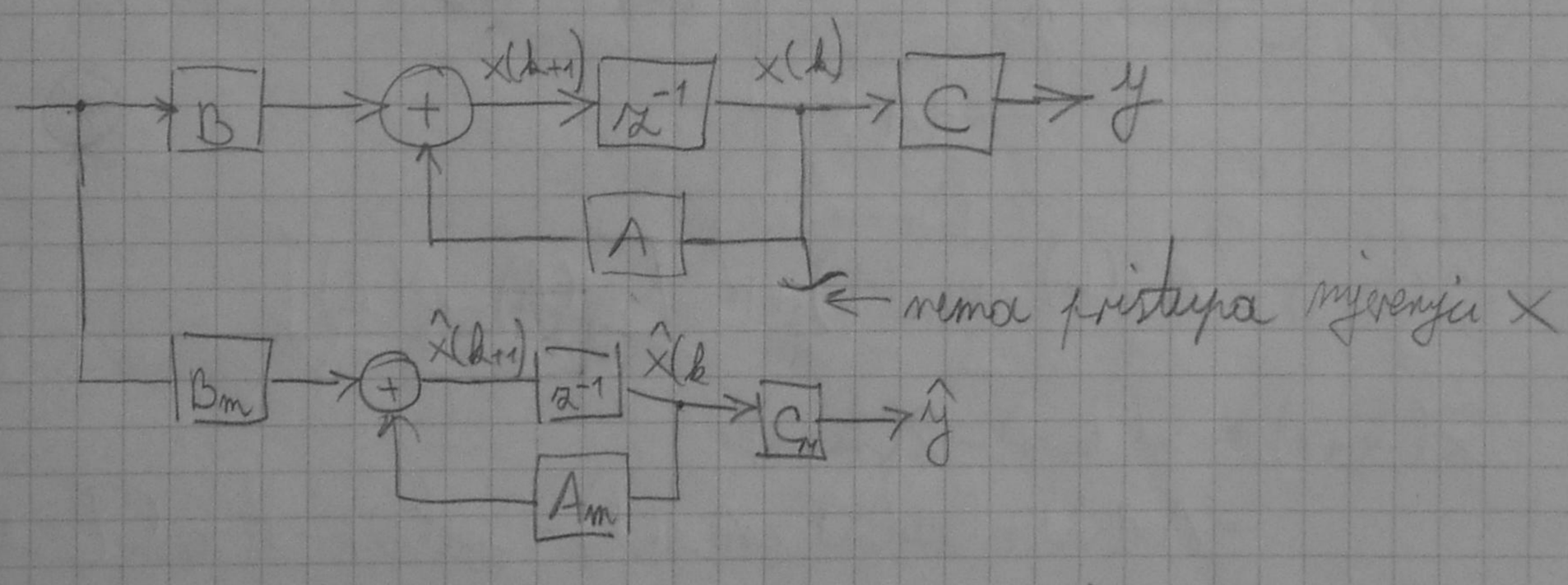
ODZIV



$G_R(1) = 1$

2) Rekonstruktor (observer) punog reda

DLTI



$$x(k+1) = Ax(k) + Bu(k) \qquad \hat{x}(k+1) = A_m\hat{x}(k) + B_m u(k)$$

$$y(k) = Cx(k) \qquad \hat{y}(k) = C_m\hat{x}(k)$$

IDEALNI MODEL : $A = A_m$, $B = B_m$, $C = C_m$

$\tilde{x}(k) := x(k) - \hat{x}(k) \Rightarrow$ pogreška rekonstrukcije

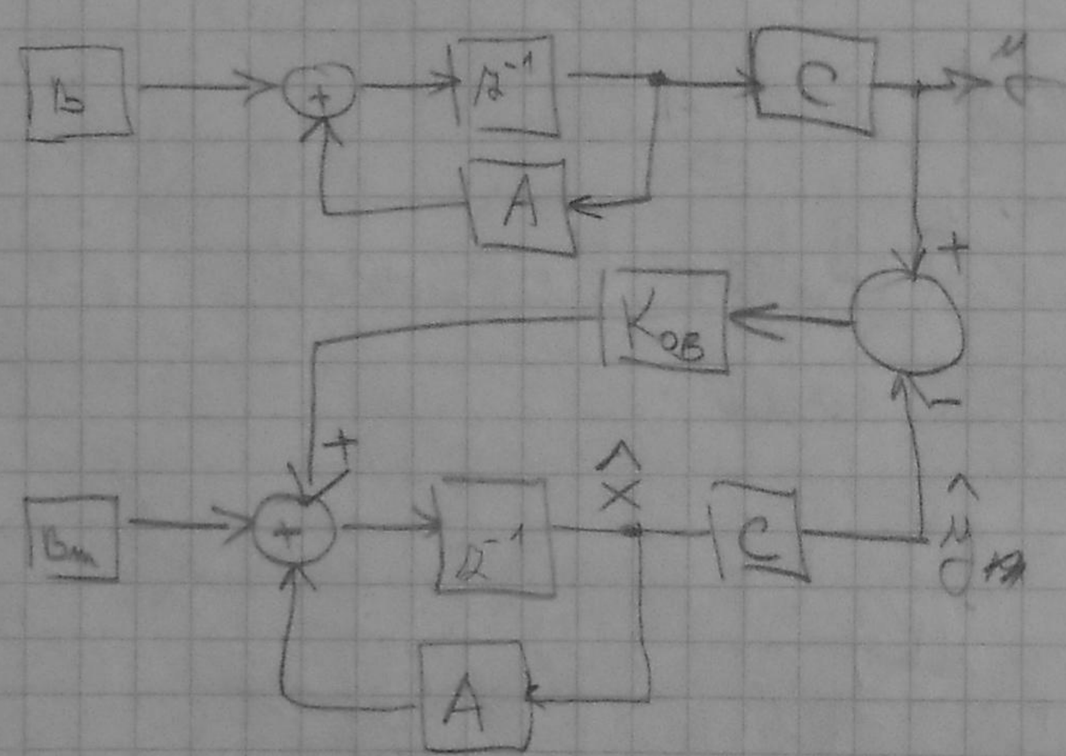
$$\tilde{x}(k+1) = Ax(k) + Bu(k) - A\hat{x}(k) - Bu(k)$$
$$= A\left[x(k) - \hat{x}(k)\right]$$
$$= A\,\tilde{x}(k)$$

$$\tilde{x}(k) = A^k \tilde{x}(0)$$

$$A \to \text{stabilno} \Rightarrow \lim_{k\to\infty} A^k = 0$$

– ŠTO AKO DJELUJE POREMEĆAJ

→ postoji poremećaj (pogriješio konstruktor)

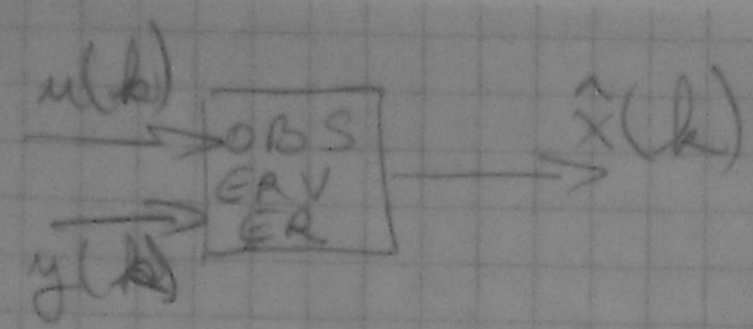


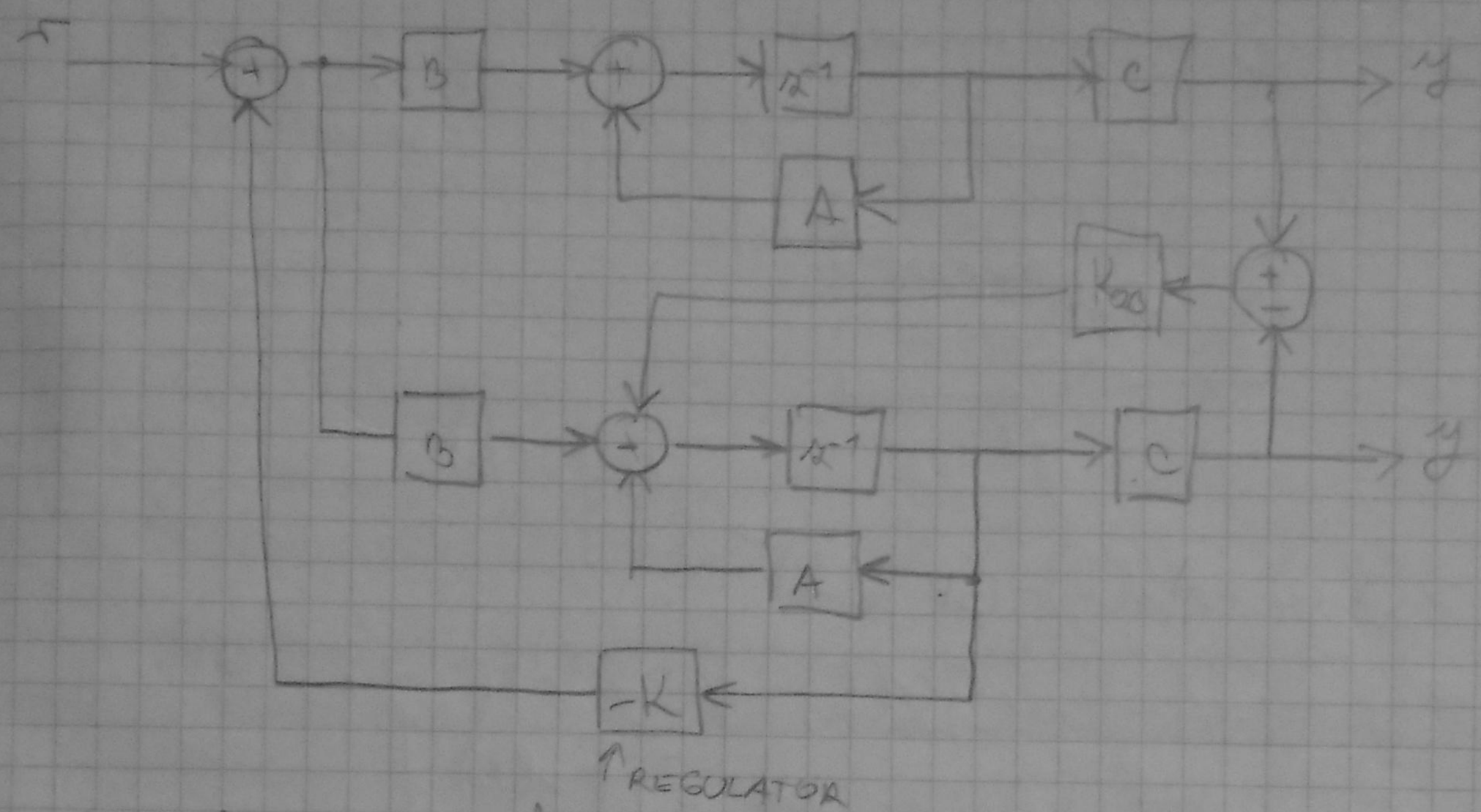
$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + K_{OB}\left[y(k) - \hat{y}(k)\right]$$

$$\tilde{x}(k+1) = x(k+1) - \hat{x}(k+1)$$
$$= Ax(k) + \cancel{Bu(k)} - A\hat{x}(k) - \cancel{Bu(k)} - K_{OB}C\left[x(k) - \hat{x}(k)\right]$$
$$= (A - K_{OB}C)\left[x(k) - \hat{x}(k)\right]$$

$$\tilde{x}(k+1) = (A - K_{OB}C)\,\tilde{x}(k)$$

$\lim_{k\to\infty} \tilde{x}(k) = 0$ ako je $A - K_{OB}C$ stabilna





$$\tilde{x}(k+1) = \underbrace{(A - K_{OB}C)}_{A_{OB}} \cdot \tilde{x}(k)$$

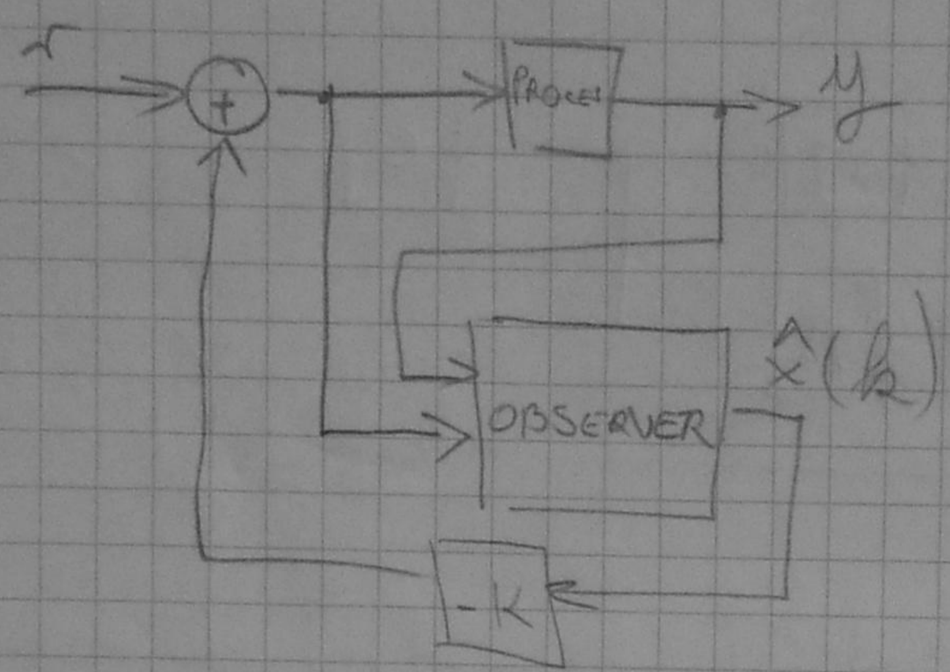
$$\Delta(z; A - K_{OB} \cdot C) = \Delta(z; A_{OB}) = \det(z I_n - A_{OB})$$

$$\Delta(z; A_{OB}) = \Delta(z; A_{OB}^T) = \det(z I_n - A^T + C^T K_{OB}^T)$$

$$K_{OB}^T = [0 \dots 1] \cdot \mathcal{C}_{OB}^{-1} \cdot \alpha_{OB}(A_{OB}^T)$$

$$\mathcal{C}_{OB} = [C^T \; A^T C^T \dots]$$

Separacijsko načelo



$$x(k+1) = Ax(k) - BK\cdot\hat{x}(k) = x(k) - \tilde{x}(k)$$

$$\tilde{x}(k+1) = (A - K_{OB}C)\tilde{x}(k)$$

$$\begin{bmatrix} x(k+1) \\ \tilde{x}(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A-BK & BK \\ 0 & A-K_{OB}C \end{bmatrix}}_{\text{MATRICA ZATV. KRUGA}} \begin{bmatrix} x(k) \\ \tilde{x}(k) \end{bmatrix}$$

$$\Delta(z; A_{cl}) = |zI_{2n} - A_{cl}|$$

$$= \underbrace{|z\cdot I - (A-BK)|}_{\text{DINAMIKA PROCESA + REGULATORA}} \cdot \underbrace{|zI - (A - K_{OB}\cdot C)|}_{\text{DINAMIKA OBSERVERA}}$$

Vremenski kontinuirani sustavi

CLTI

$$\dot{\hat{x}} = A\hat{x} + Bu + K_{OB}(y - \hat{y})$$

$$\tilde{x}(t) = x(t) - \hat{x}(t)$$

$$\dot{\tilde{x}} = (A - K_{OB}C)\tilde{x}(t)$$

SVE JE ISTO

Pr. $A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ $D = 0$

$$\alpha(s) = s^2 + 4\omega s + 4\omega^2 \Rightarrow K = \begin{bmatrix} 3\omega^2 & 4\omega \end{bmatrix}$$

$$\alpha_{OB}(s) = (s + 10\omega)^2 = s^2 + 20\omega s + 100\omega^2$$

$$\Delta(s; A_{OB}) = \Delta(s; A - K_{OB}C)$$

$$K_{OB} = \begin{bmatrix} 20\omega \\ 99\omega^2 \end{bmatrix}$$

SLSU

ITAE kriterij

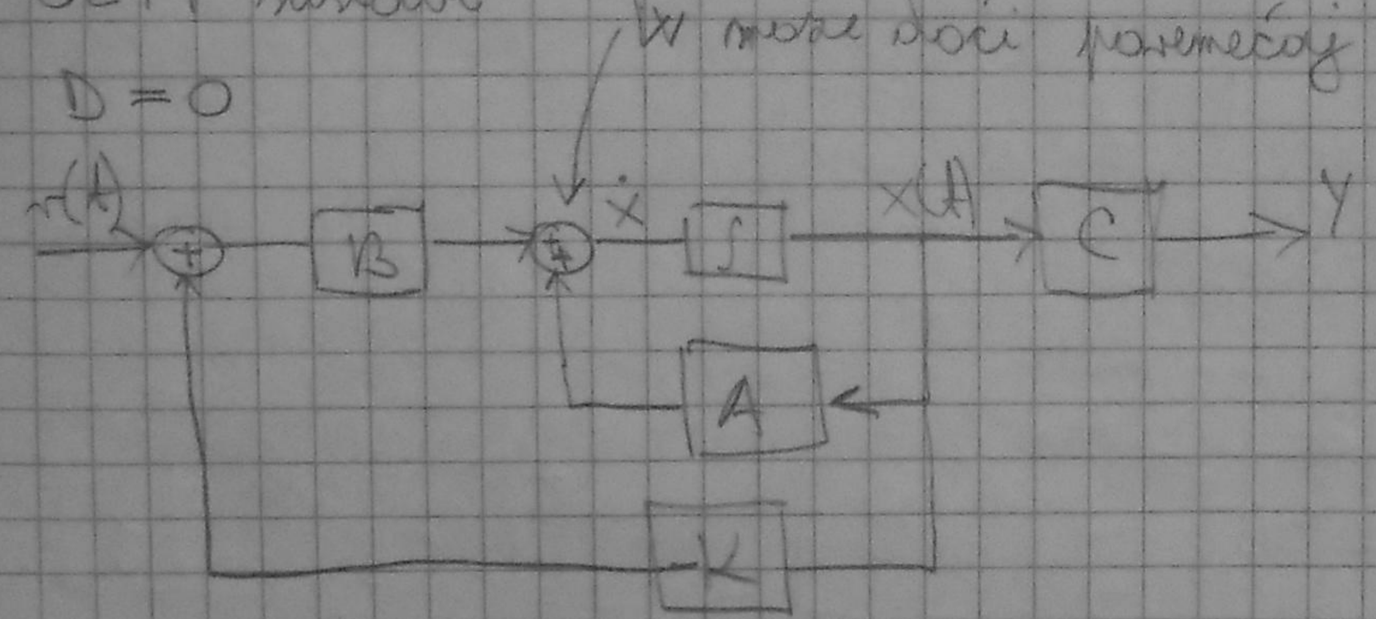
$$\min \int_0^\infty t\,|e(t)|\,dt \qquad \frac{Y(s)}{R(s)} = \frac{1}{a_n s^n + a_{n-1} s^{n-1} + \dots a_0}$$

red sustava	a_0	a_1	a_2	a_3	a_4	a_5
2	1	1.505	1	0	0	0
3	1	2.172	1.783	1	0	0
4	1	2.648	3.347	1.953	1	0
5	1	3.257	4.675	4.439	2.058	1

Slijeđenje reference $r(t) \neq 0$

CLTI sustavi

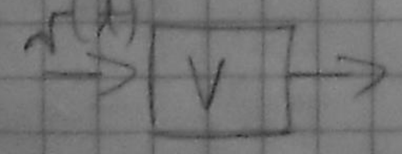
$D = 0$



$$\frac{Y(s)}{R} = G_R = C(sI - A + BK)^{-1} B$$

$$y(\infty) = \lim_{s \to 0} s \cdot G_r(s) \cdot \frac{1}{s} = G_r(0) \longrightarrow \text{uz } r(t) = S(t)$$

$\Rightarrow$ zato se umjesto ulaza $r(t)$ dovodi preko bloka V (prefiltar) da bi izlaz bio jednak ulazu



$$V = \frac{1}{G_r(0)} \Longrightarrow G_{R_{NOVI}}(0) = 1$$

$\rightarrow$ ne rješava problem poremećaja

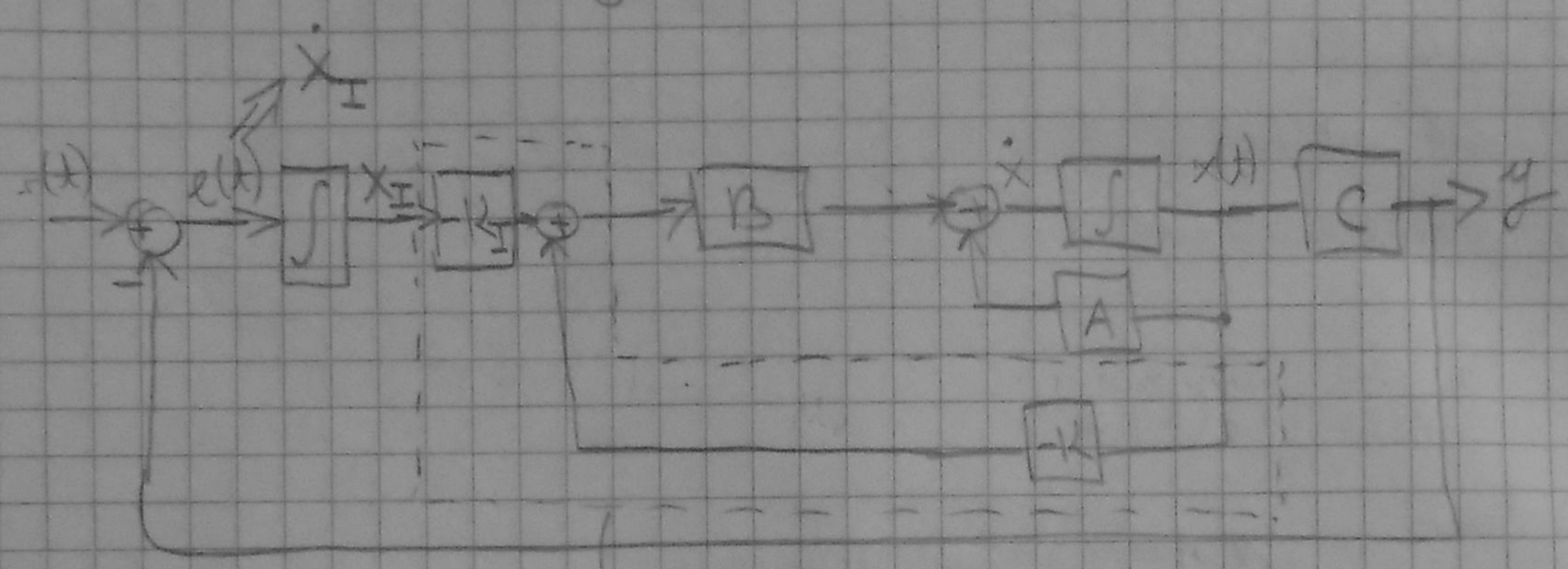
$$\dot{x}(t) = Ax(t) + B\left[V r(t) - K x(t)\right] + w(t)$$

$$\dot{x} = (A - BK)x + \begin{bmatrix} BV & I_n \end{bmatrix} \begin{bmatrix} r(t) \\ w(t) \end{bmatrix}$$

$$y = Cx$$

Uvedimo integrator

$$\dot{x}_I = e(t) = r(t) - y(t) = r(t) - Cx(t)$$



regulator u prostoru stanja

$$\bar{x} = \begin{bmatrix} x \\ x_I \end{bmatrix} \qquad \dot{\bar{x}} = \begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x \\ x_I \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\bar{B}} \cdot u - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot r(t)$$

$$u = \begin{bmatrix} -K & -K_I \end{bmatrix} \cdot \begin{bmatrix} x \\ x_I \end{bmatrix}$$

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$

$$u = -\bar{K}\bar{x}$$

$$y(\infty) = r(\infty)$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix}$$

u STACIONARNOM STANJU NEMA OSCILACIJA

$\Rightarrow$ u stacionarnom stanju $e = 0$ jer je $\dot{x}_I = 0 \Rightarrow e = 0$

POGLEDAJ SHEMU

Př. $G(s) = \frac{1}{s+3} = \frac{Y(s)}{U(s)}$ $A = -3$ $B = 1$ $C = 1$ $D = 0$

$\alpha(s) = s^2 + 6s + 25 = s^2 + 2\xi\omega_n + \omega_n^2 \Rightarrow \frac{Y(s)}{R(s)}$

$$\bar{A} = \begin{bmatrix} -3 & 0 \\ -1 & 0 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \bar{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\bar{K} = \begin{bmatrix} K & K_I \end{bmatrix} \qquad u = -\bar{K} \cdot \bar{x}$$

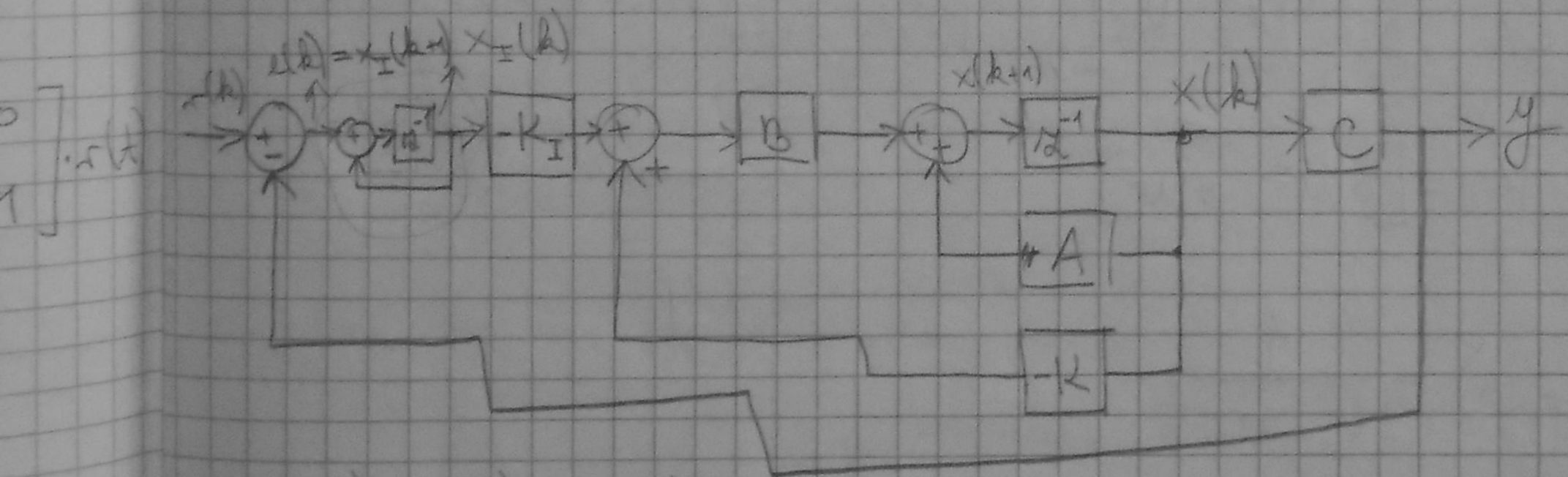
$$\Delta(s; \bar{A} - \bar{B}\bar{K}) = \dots = s^2 + (3+K)s - K_I \Rightarrow \boxed{K_I = -25,\ K = 2}$$

$\Rightarrow$ Ito ako dodamo poremećaj

$\rightarrow$ sustav u stacionarnom stanju prati referencu

Integrator u diskretnoj domeni

$$x_I(k+1) = x_I(k) + e(k)$$



$$e(k) = r(k) - y(k)$$
$$= r(k) - Cx(k)$$

$$\bar{x}(k+1) = \begin{bmatrix} A & 0 \\ -C & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \cdot u - \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(k)$$

$$y(k) = \begin{bmatrix} C & 0 \end{bmatrix} \cdot \begin{bmatrix} x(k) \\ x_I(k) \end{bmatrix} + 0 \cdot u$$

Př. CLTI, SISO

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = 0 \quad \omega > 0$$

$$u_R(t) = -Kx(t) \qquad K \in R^{1\times 2}$$

$$\alpha(s) = s^2 + 4\omega s + 4\omega^2$$

$$u(t) = u_R(t) + r(t); \qquad r(t) = S(t-5)$$

$$x(0) = \begin{bmatrix} 0.3 \\ -0.5 \end{bmatrix}$$

$$\Delta(s; A) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} s & -1 \\ \omega^2 & s \end{bmatrix}\right) = s^2 + \omega^2$$

$$\operatorname{rang}(\mathcal{C}) = [B \;\; AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2 \quad \Rightarrow \text{uvažujeme vstup}$$

$$\Delta(s; A_{CL}) = \Delta(s; A - BK) \qquad K = [K_1 \;\; K_2]$$

$$\det\left\{\begin{bmatrix} s & -1 \\ \omega^2 & s \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix}\right\} = \begin{vmatrix} s & -1 \\ \omega^2 + K_1 & s + K_2 \end{vmatrix} = s^2 + K_2 s + \omega^2 + K_1$$

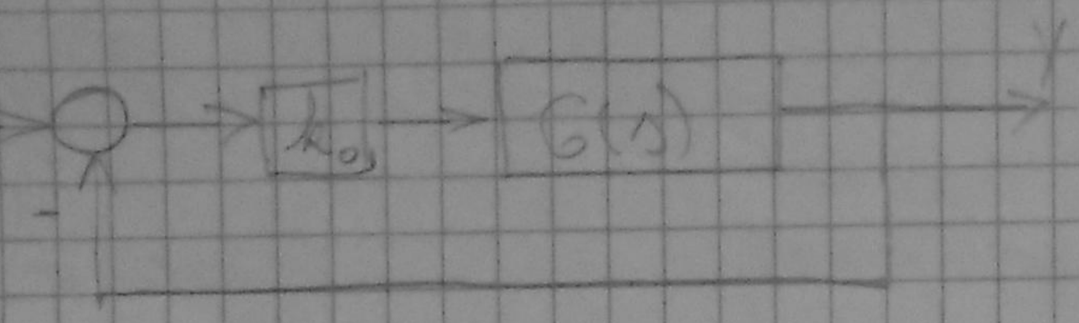
$$K = [3\omega^2 \;\; 4\omega]$$

$$G_r(s) = \frac{1}{(s+2\omega)^2}$$

KRIVULJA MJESTA KORIJENA

(root-locus)

Analiza CLTI sustava:



$$G(s) = \frac{b(s)}{A(s)} = \frac{s^m + b_{m-1}s^{m-1} + \dots + s\cdot b_1 + b_0}{s^n + a_{n-1}s^{n-1} + \dots + s a_1 + a_0} = \frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{j=1}^{n}(s-p_j)}$$

MONIČKI POLINOM

$$G_0(s) = k_0 \cdot G(s)$$

OTVORENOG KRUGA

KARAKTERISTIČNA JEDNADŽBA

$$1 + k_0 \cdot G(s) = 0$$

$$k_0 > 0 \Rightarrow G(s) = -\frac{1}{k_0}$$

DEF:

Krivulja mjesta korijena sustava je skup svih kompleksnih brojeva $s = s(k_0)$, $k_0 \geq 0$ koji zadovoljavaju jednadžbu $G(s) = -\frac{1}{k_0}$

$|G(s)| = \frac{1}{k_0}$ $\qquad$ $\varphi(s) = \arg[G(s)] = \pi + 2k\pi, \; k \in \mathbb{Z}$

AMPLITUDNI UVJET $\qquad$ FAZNI UVJET

→ Krivulja mjesta korijena se crta iz faznog uvjeta.

$$G_0(s) = \frac{K_0}{s(s+1)} \qquad K_0 > 0$$

$$G_0(s) = k_0 \cdot G(s) = k_0 \cdot \frac{1}{(s-p_1)(s-p_2)} \; ; \quad k_0 = K_0, \; p_1 = 0, \; p_2 = -1$$

$$G_R(s) = \frac{G_0(s)}{1+G_0(s)} = \frac{K_0}{s^2+s+K_0}$$

$$s_{1,2} = \frac{-1 \pm \sqrt{1-4K_0}}{2}$$ ⇒ POLOVI ZATVORENOG KRUGA

a) $K_0 = 0$

$s_1 = 0 \quad s_2 = -1$

c) $K_0 > \frac{1}{4}$

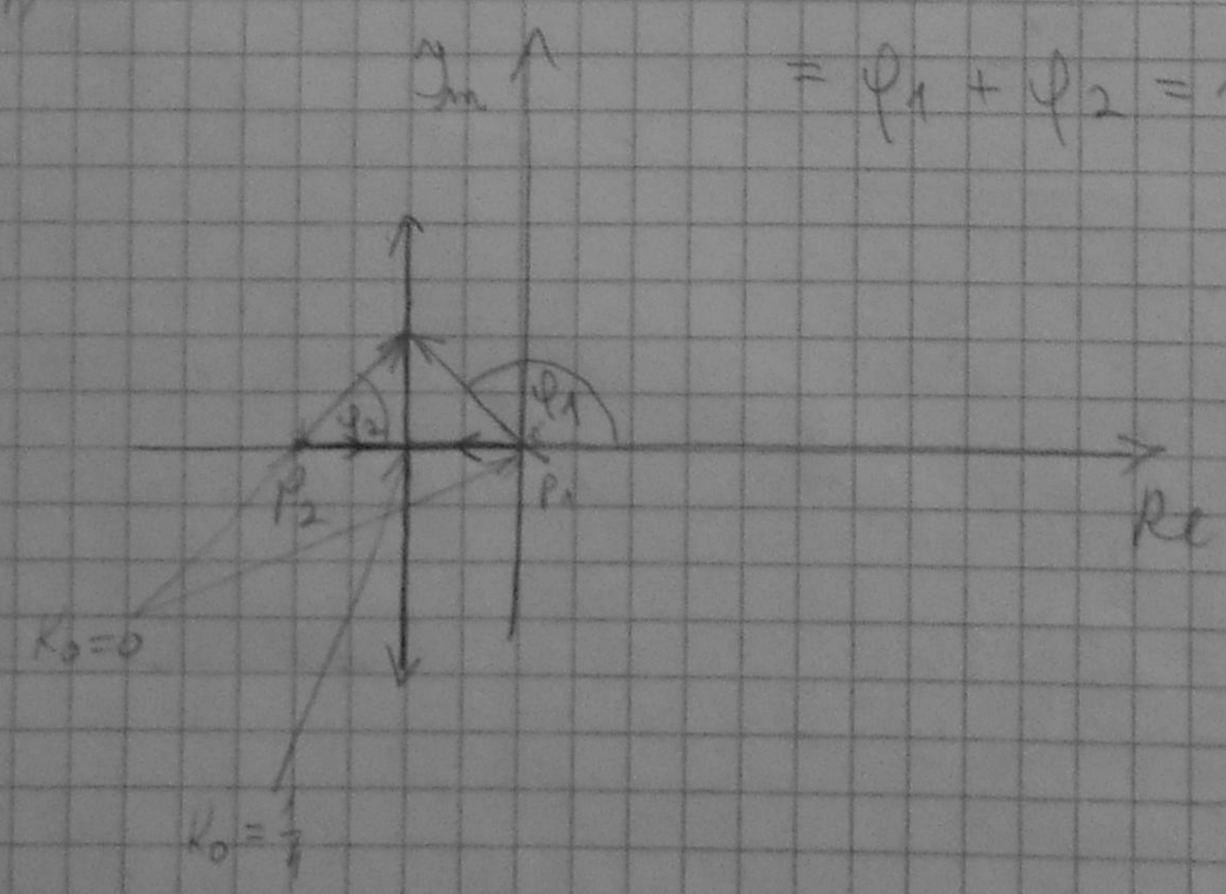
$s_1 = \overline{s_2} = -\frac{1}{2} \pm j\,\omega(k_0)$

b) $K_0 = \frac{1}{4}$

$s_1, s_2 \in \mathbb{R} \in [-1, 0]$

$$\varphi(s) = -\arg(s) - \arg(s+1) = 180^\circ$$

$$= \varphi_1 + \varphi_2 = 180^\circ$$



$$G_0(s) = k_0 \cdot \frac{\prod_{i=1}^{m} |s - z_i|}{\prod_{j=1}^{n} |s - p_j|} \cdot e^{j\left[\sum_{i=1}^{m} \varphi_{z_i}(s) - \sum_{j=1}^{n} \varphi_{p_j}(s)\right]}$$

$$\varphi_{z_i}(s) = \arg(s - z_i)$$

$$\varphi_{p_j}(s) = \arg(s - p_j)$$

amplitudni uvjet

$$k_0 = \frac{\prod |s - p_j|}{\prod |s - z_i|}$$

fazni uvjet

$$\sum \varphi_{z_i}(s) - \sum \varphi_{p_j}(s) = 180^\circ$$

Pravila za crtanje

1) KMK je simetrična u odnosu na realnu os

2) KMK nastaju se iz n grana koje počinju u polovima otvorenog regulacijskog kruga p_j

kad $k \to \infty$ m grana završava u z_i, a n-m grana teži u beskonačnost

$$n \geq m$$

Dokaz: $k_0 \prod_{i=1}^{m} (s - z_i) + \prod_{j=1}^{n} (s - p_j) = 0$

kad $k_0 = 0 \Rightarrow \prod_{j=1}^{n} (s - p_j) = 0$

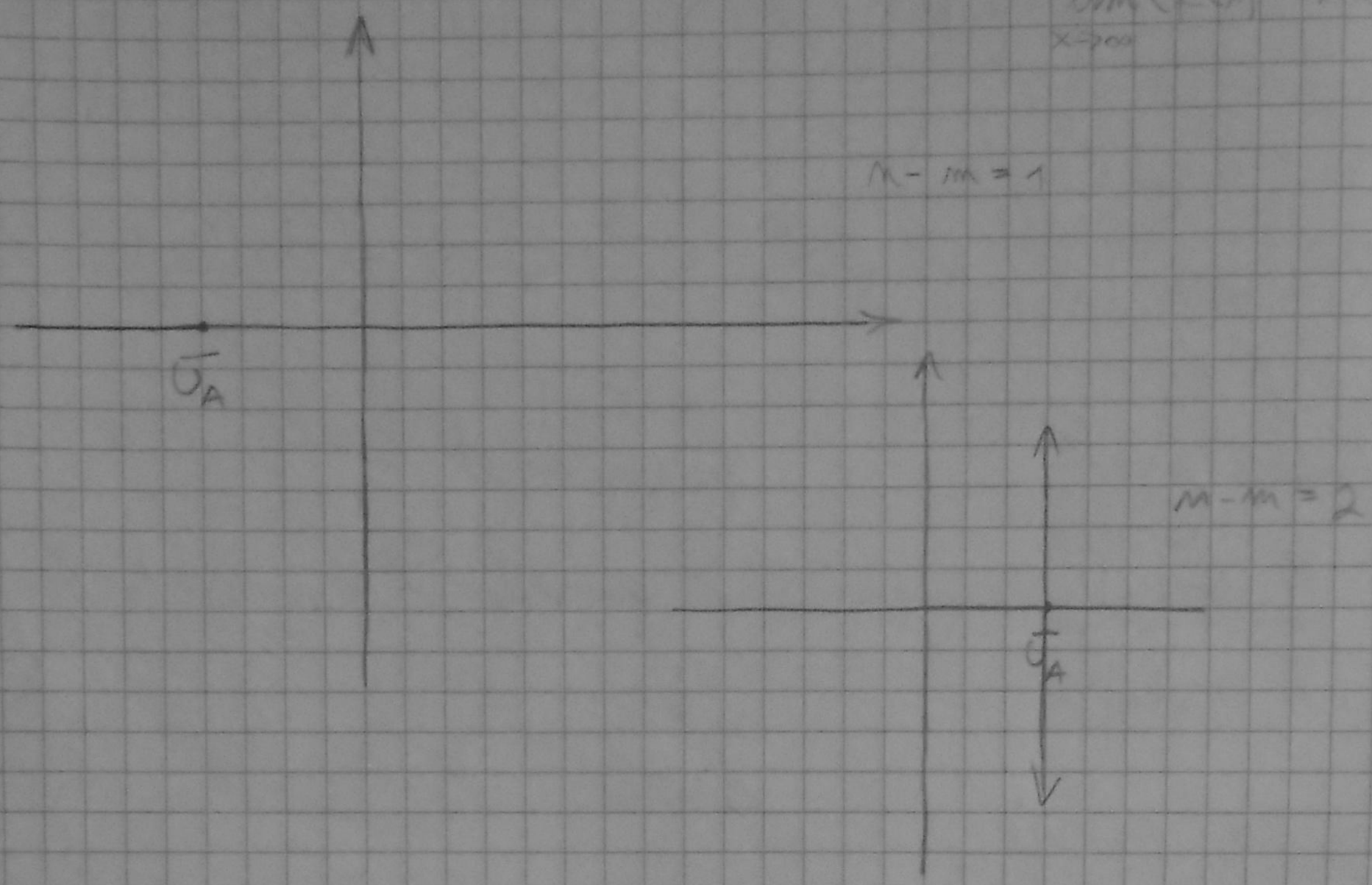
kad $k_0 = \infty \Rightarrow \prod_{i=1}^{m} (s - z_i) = 0$

3) Asimptote n-m grana su pravci s kutem $\alpha_k = \arg s = \frac{180^\circ(2k+1)}{n-m}$, $k = 0 \ldots n-m-1$

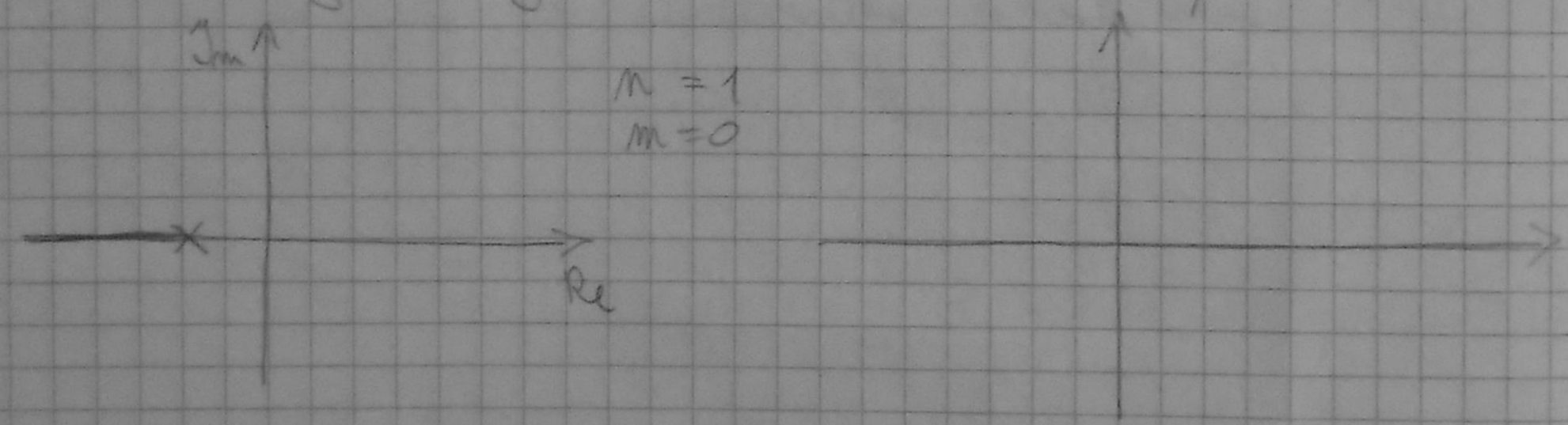
- sve pravci kreću iz iste točke (TEŽIŠTE KORIJENA) $T(\sigma_A, j0)$

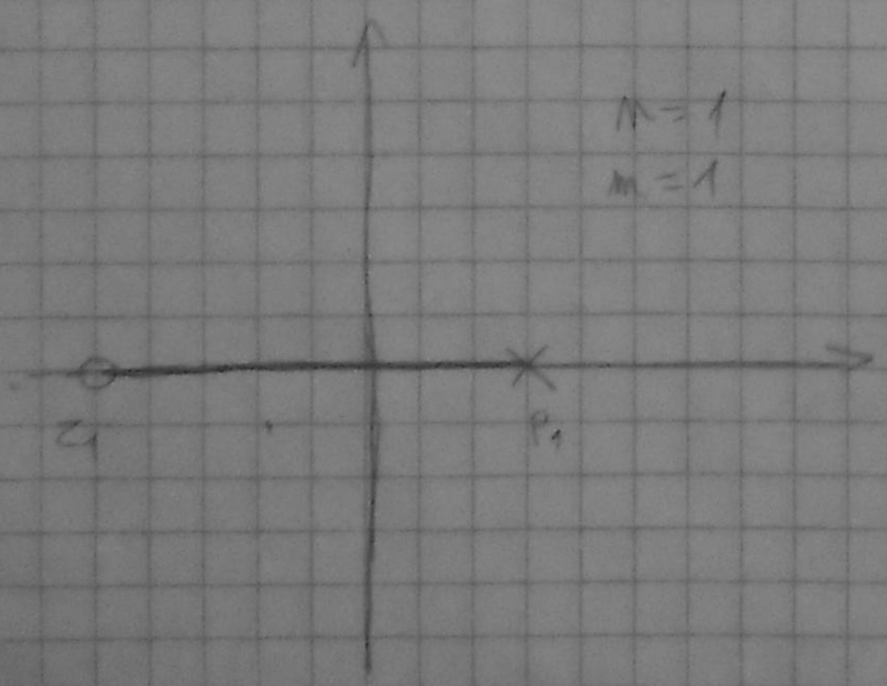
gdje je $\sigma_A = \frac{1}{n-m}\left[\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i\right]$

$\lim_{x\to\infty}(f(x) - kx - l) = 0$



4. Proizvoljna točka na realnoj osi je točka grafa ukoliko broj realnih nula i polova na desnoj strani tog broja dijeli od te točke neparan





5) Neka je $n_G \geq 0$ broj točaka grananja na grani KMK, $n_S \geq 0$ neka je broj točaka sjedinjenja na grani KMK

a) ako grana KMK leži između 2 polja drugog oblika vrijedi $n_G = 1 + n_S$

b) ako je grana KMK leži unutar bunja drugog oblika je $n_S = 1 + n_G$

c) za sve ostale grane vrijedi $n_G = n_S$

grananje / sjedinjenje : $\sum_{j=1}^{n} \frac{1}{s - s_j} = \sum_{i=1}^{m} \frac{1}{s - s_{\infty i}}$

ili

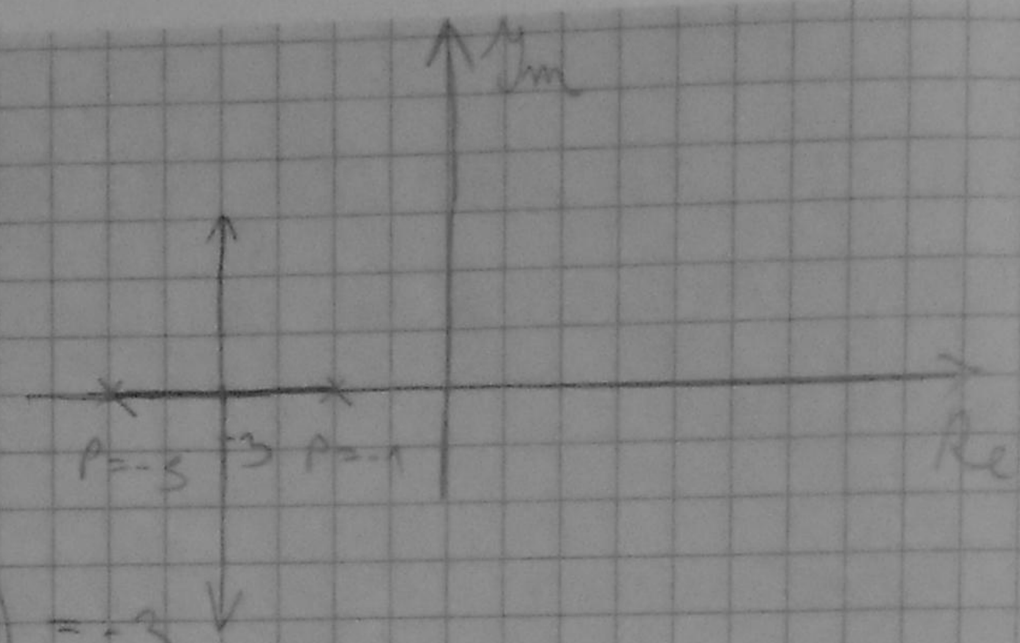
$$G(s) = \frac{B(s)}{A(s)} \qquad A(s) \cdot \frac{dB(s)}{ds} - \frac{dA(s)}{ds} \cdot B(s) = 0$$

RJEŠENJA SU KANDIDATI ZA TOČKE GRANANJA / SJEDINJENJA

~~A M MA~~

Primjer

2 pola otvorenog kruga



$$\sigma_A = \frac{1}{n-m}\left(\sum p_j - \sum z_i\right) = -3$$

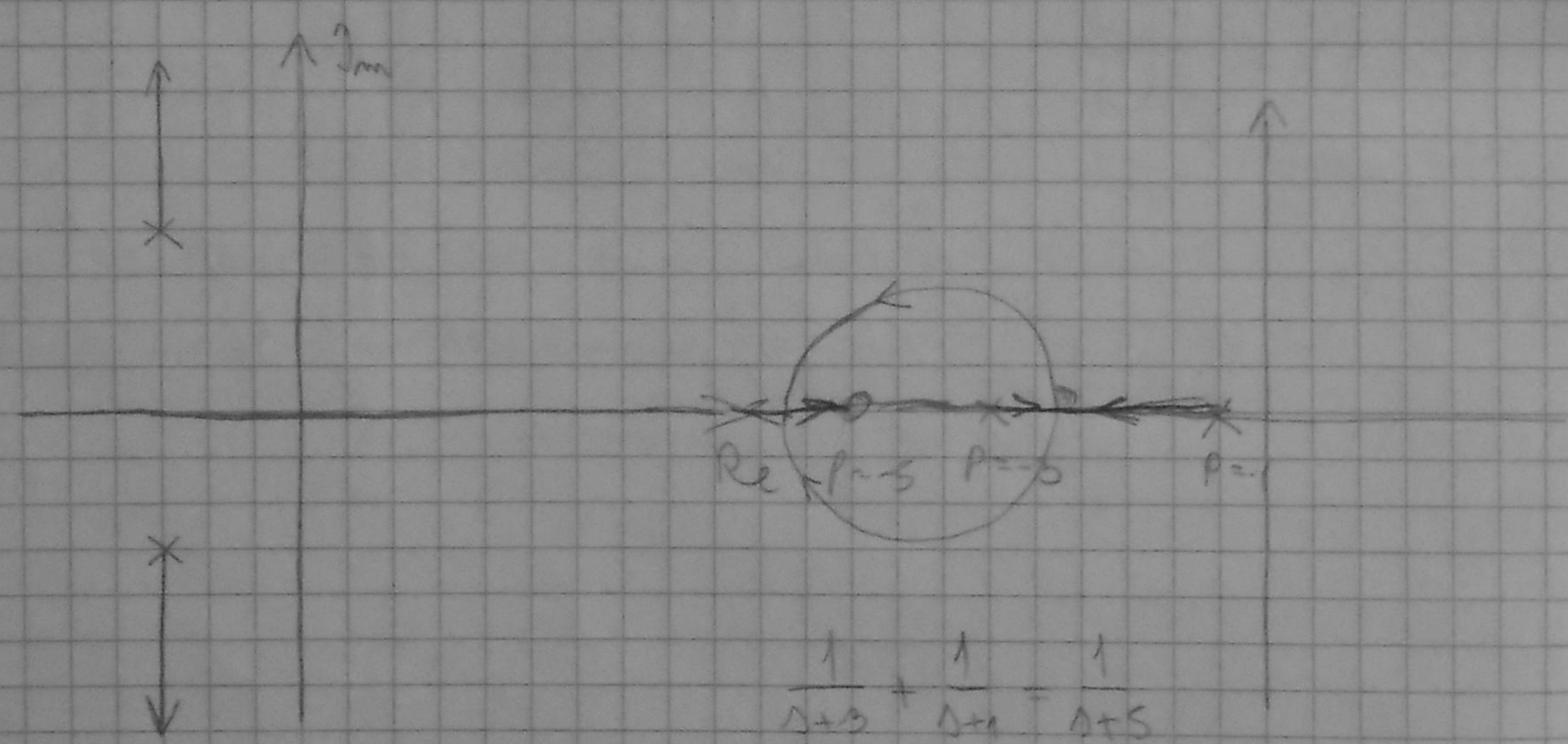
TOČKA GRANANJA

$$\frac{1}{s+1} + \frac{1}{s+5} = 0$$

$$s+5 = -s-1$$

$$2s = -6$$

$$\boxed{s = -3}$$



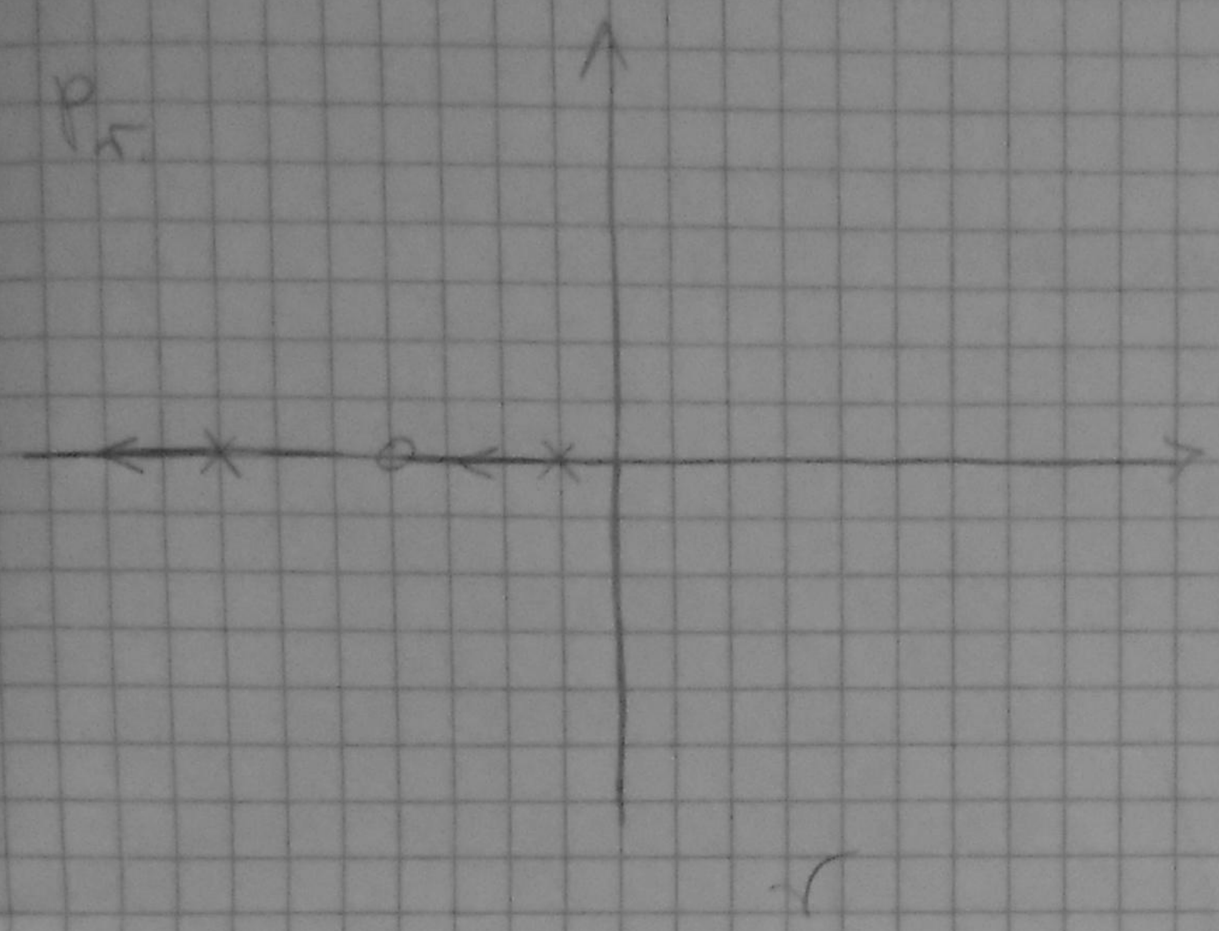
$$\frac{1}{s+3} + \frac{1}{s+1} = \frac{1}{s+5}$$

$$(s+5)(s+1) + (s+3)(s+5) = (s+3)(s+1)$$

$$s^2+6s+5+s^2+8s+15 = s^2+4s+3$$

$$s^2+10s-17=0$$

Pr.



6. Pravilo (Ulazni i izlazni kutevi)

$$\varphi^{I}_{p_j,k} = \frac{1}{r_{p_j}}\left\{-\sum_{\substack{r=1\\ r\neq j}}^{n} \varphi_{p_r}(p_j) + \sum_{\mu=1}^{m} \varphi_{z_\mu}(p_j) + 180^\circ(2k+1)\right\}$$

VIŠESTRUKOST POLA OTVORENOG KRUGA

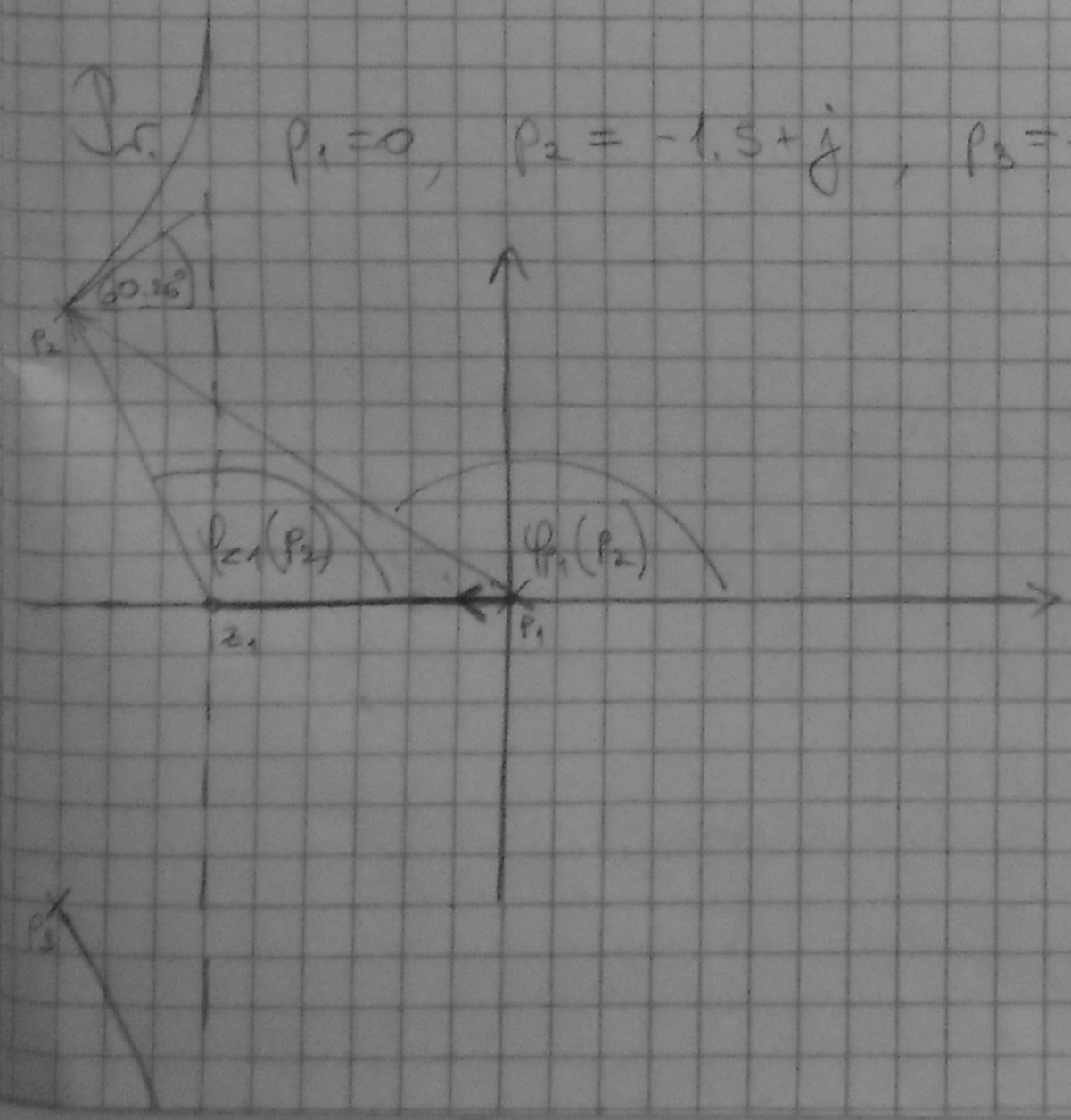
$k = 0, \dots, r_{p_j} - 1$

ulazni

$$\varphi^{U}_{z_i,k} = \frac{1}{r_{z_i}}\left\{-\sum_{\substack{\mu=1\\ \mu\neq i}}^{m} \varphi_{z_\mu}(z_i) + \sum_{\nu=1}^{n} \varphi_{p_\nu}(z_i) + 180^\circ(2k+1)\right\}$$

$k = 0, \dots, r_{z_i} - 1$

Pr. $p_1 = 0$, $p_2 = -1.5 + j$, $p_3 = -1.5 - j$, $z_1 = -1$



$\sigma_A = -1$

TOČKA GRANANJA

$\varphi^{I}_{2} = 60.26^\circ$

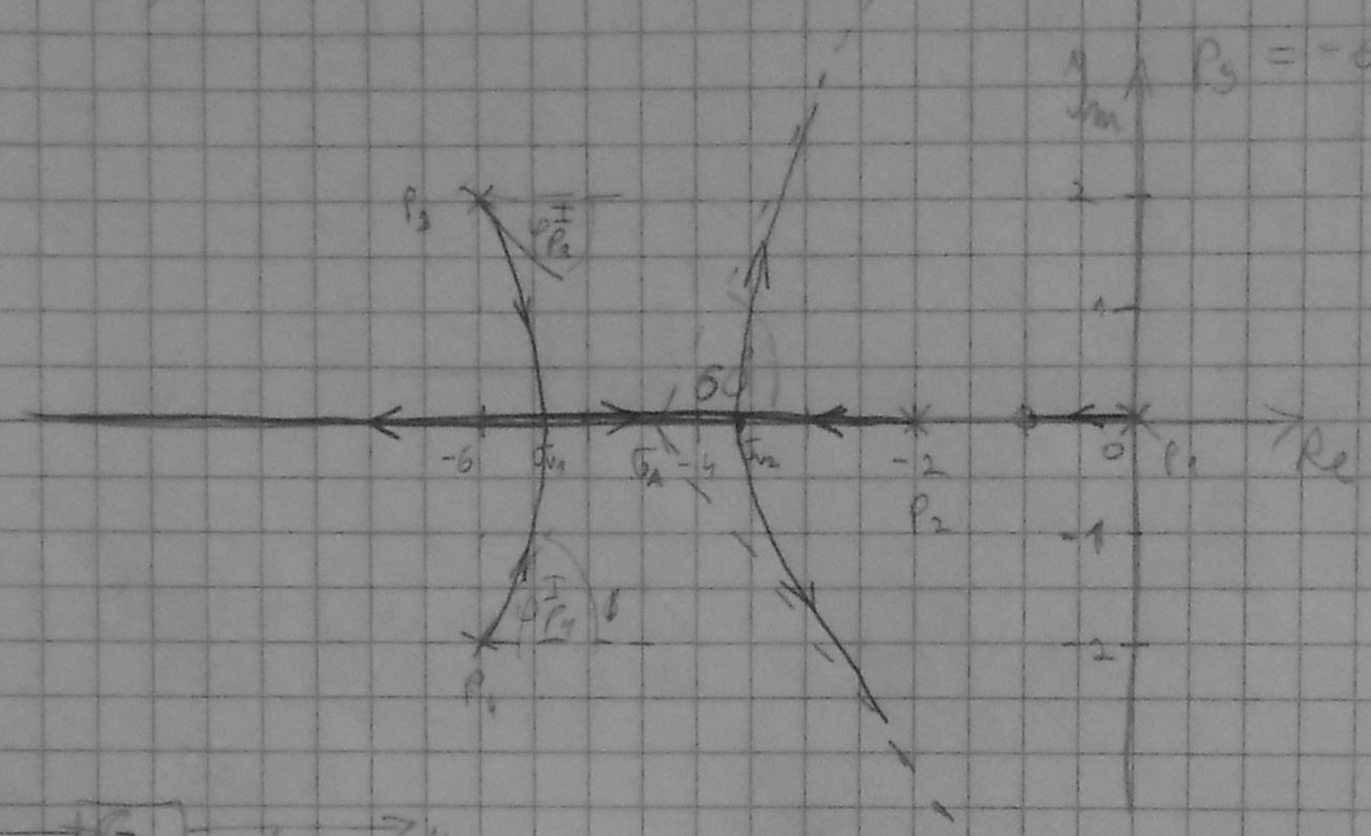
Primjer:

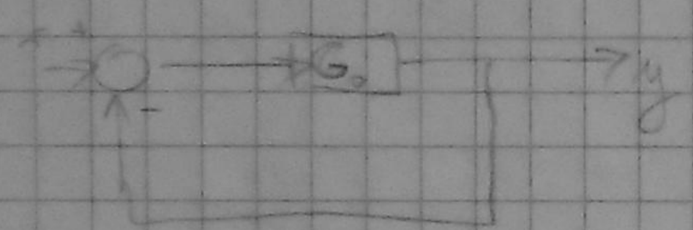
$$G_0(s) = \frac{k_0(s+1)}{s(s+2)(s^2+12s+40)}, \quad k_0 > 0$$

ODREDI TOČNU PRESJEKA Z.18 $\Rightarrow k_0 =$

$m = 1$ nula, $n = 4$ pola $\quad z_1 = -1, \; p_1 = 0, \; p_2 = -2$

$p_3 = -6 + 2j \quad p_4 = -6 - 2j$





$n - m = 3 \Rightarrow 3$ ASIMPTOTE

$$\sigma_A = \frac{1}{n-m} \cdot \left(\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i \right)$$

$$= \frac{1}{3} \cdot (-14 + 1) = -\frac{13}{3} = -4.33$$

SJEDINJENJE/GRANANJE

$\alpha_1 = 60^\circ, \; \alpha_2 = 180^\circ, \; \alpha_3 = 300^\circ$

$$\sum_{j=1}^{n} \frac{1}{s - p_j} = \sum_{i=1}^{m} \frac{1}{s - z_i}$$

$$\frac{1}{s} + \frac{1}{s+2} + \frac{1}{s^2+12s+40} = \frac{1}{s+1} \cdots$$

$s_1 = -5.471$

$s_2 = -3.678$

$s_{3,4} = -0.759 \pm j0.866 \Rightarrow$ SIGURNO NIJE RJEŠENJE

IZLAZNI KUTEVI

$$\varphi_{p_3}^{I} = -\operatorname{arg}(p_3 - p_1) - \operatorname{arg}(p_3 - p_2) - \operatorname{arg}(p_3 - p_4) + \operatorname{arg}(p_3 - z_1) + 180^\circ(2k+1)$$

$$= -153.4^\circ - 161.6^\circ - 90^\circ + 158.2^\circ + 180^\circ(2k+1)$$

$$= -66.8^\circ$$

$\Rightarrow$ AMPLITUDNI UVJET

$$k_0 = \frac{\prod_{j=1}^{m} |s - p_j|}{\prod_{i=1}^{m} |s - z_i|} = \frac{7.2 \cdot 7.4 \cdot 7.9 \cdot 11.1}{7.25} = 644.4$$

KARAKTERISTIČNA JEDNADŽBA

$$s^4 + 14s^3 + 64s^2 + (k_0 + 80)s + k_0 = 0$$

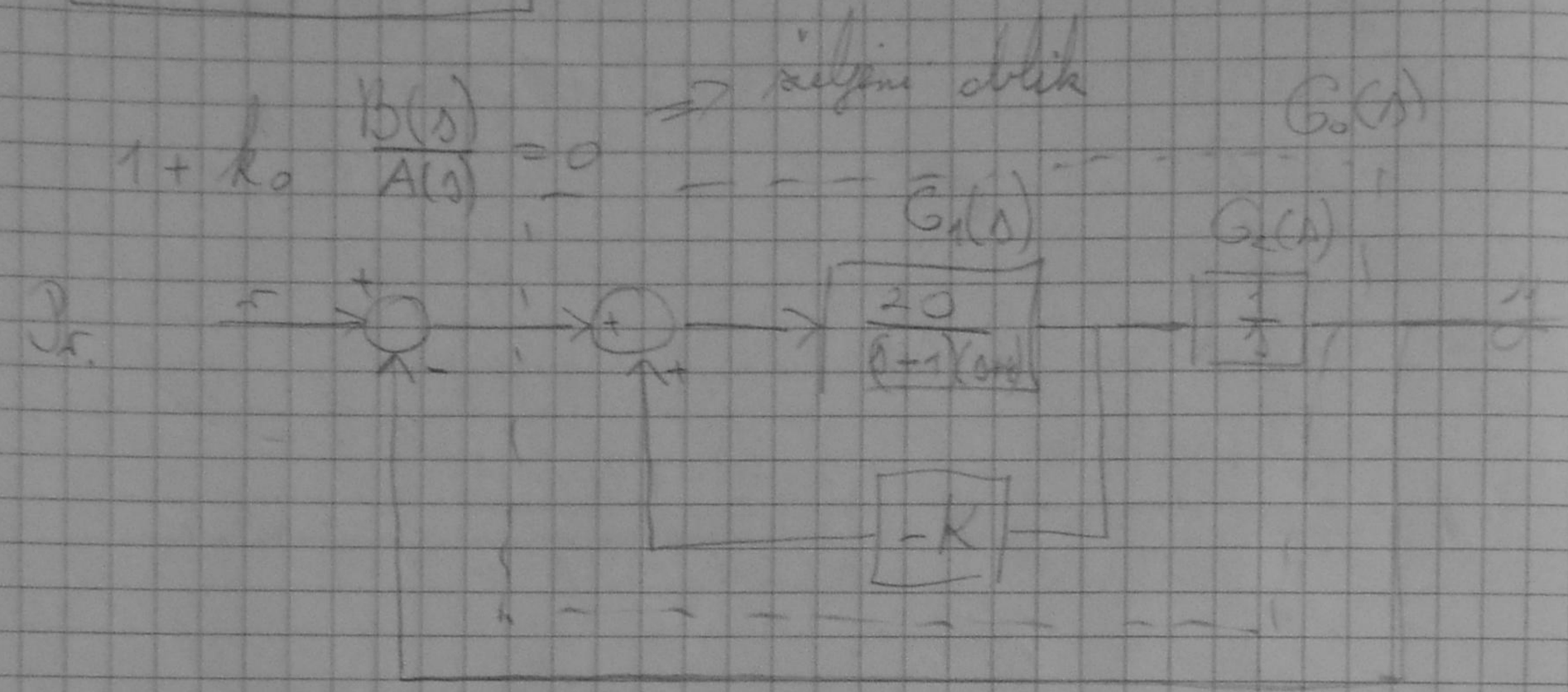
$k_0 = k_{KRITIČNO}$ $\qquad s = j\omega_{KR}$

$$\boxed{\begin{aligned} &\omega^4 - 64\omega + k_{KR} = 0 \\ &(-14\omega^2 + (k_{KR} + 80))\omega_{KR} = 0 \end{aligned}} \quad \begin{aligned} &Re \\ &Im \end{aligned}$$

Linijska KMK za proizvoljni parametar regulacijskog kruga

$$\boxed{1 + G_0(s, \theta) = 0}$$ KARAKTER JEDNADŽBA

$$1 + k_0 \frac{B(s)}{A(s)} = 0$$ → željeni oblik



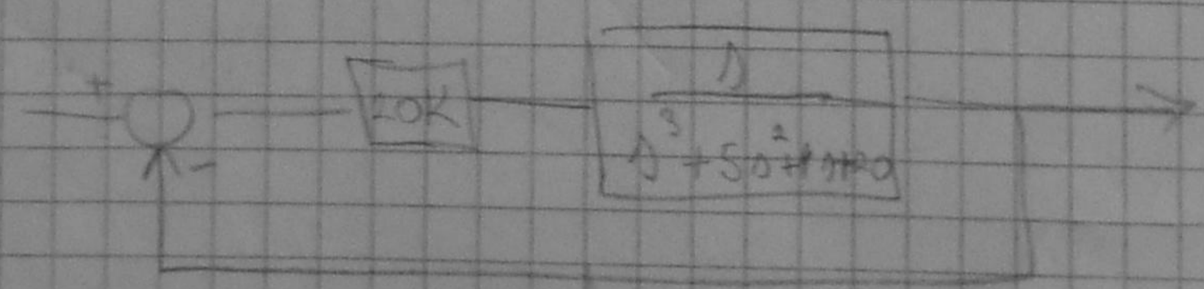
$$G_0(s) = \frac{20}{s^3 + 5s^2 + (20K + 4)s}$$

$$G_k(s) = \underbrace{\frac{20}{s^3 + 5s^2 + (20K + 4)s + 20}}_{\text{KARAKTER JEDNADŽBA}}$$

$$s^3 + 5s^2 + 4s + 20 + 20Ks = 0 \quad /:$$

$$1 + 20K \cdot \frac{s}{s^3 + 5s^2 + 4s + 20}$$

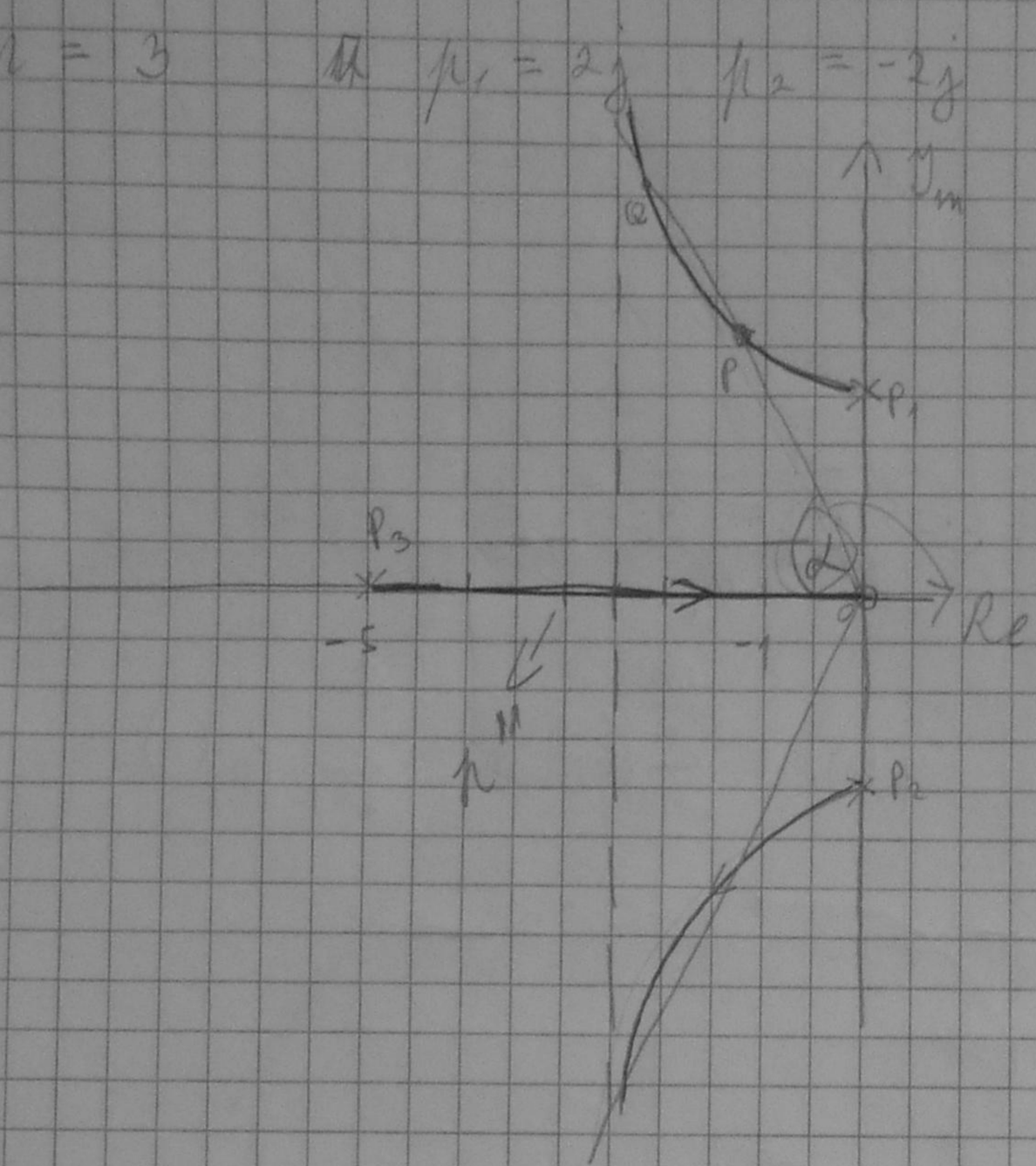
$$k_0 = 20K$$



ovaj sustav ima isti karakter jednadžbe

$m = 1 \qquad z_1 = 0$

$n = 3 \qquad p_1 = 2j \quad p_2 = -2j \quad p_3 = -5$



GRANANJE

$$\frac{1}{s-2j} + \frac{1}{s+2j} + \frac{1}{s+5} = \frac{1}{s}$$

$$\sigma_A = \frac{1}{2}[-5] = -\frac{5}{2}$$

$$\frac{Im}{Re} = \frac{\omega}{\sigma} = \operatorname{tg}\varphi$$

$(1+\operatorname{tg}\varphi)\sigma$

$\xi = 0.4 \qquad \alpha = \arccos\xi = 66.42^\circ$

S DINAMIKOM DRUGOG REDA

$$P(-1.05, +j2.41) \Rightarrow k_0 = \frac{|s^3+5s^2+4s+20|}{|s|}$$

$$Q(-2.16, +j4.94) \Rightarrow k_{0Q}$$

$$K_p = \frac{k_{0P}}{20} = 0.45 \Rightarrow s_3 = -2.89$$

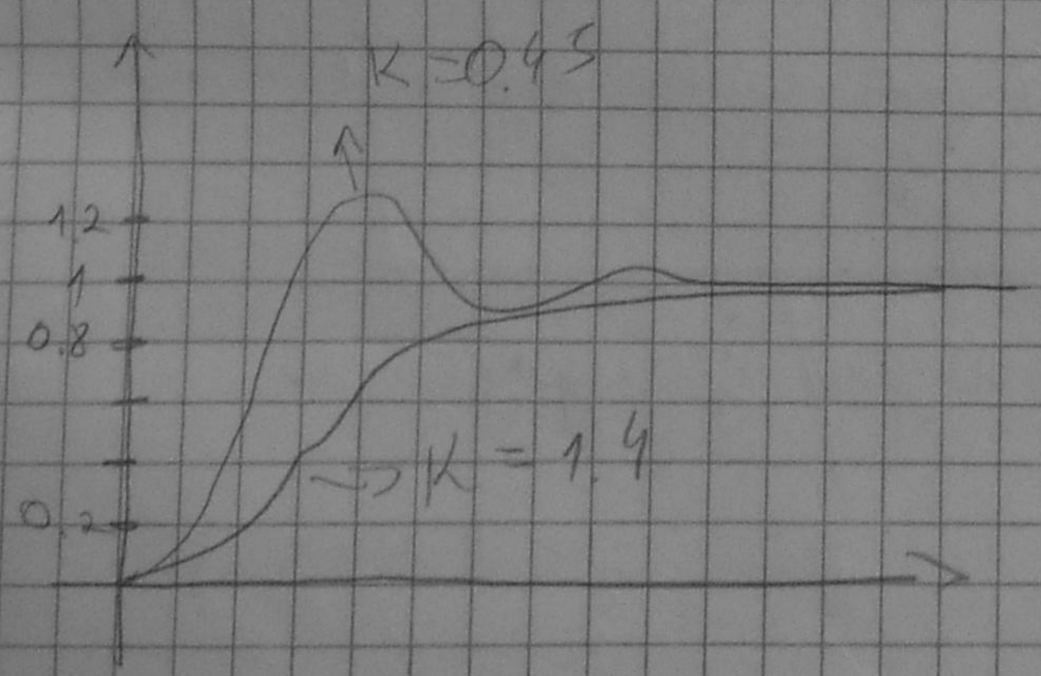
$\frac{84}{100}$ $\frac{16}{100}$

42

21

$\frac{21}{10}$

ODZIV



$$\frac{Im}{Re} = \operatorname{tg}\varphi \qquad \cos = \frac{1}{\sqrt{1-\cos^2}}$$

$$\boxed{y_m = \operatorname{tg}\varphi \cdot Re}$$

$$\operatorname{tg}^2 + 1 = \frac{1}{\cos}$$

$$\operatorname{tg} = \sqrt{\frac{1}{\cos} - 1}$$

$\frac{10}{4} = 1$

2.1

KMK za $k_0 < 0$

AMPLITUDNI UVJET

A.U.

$$|G(s)| = -\frac{1}{k_0}$$

FAZNI UVJET

$$\varphi(s) = \arg\left[G(s)\right] = 360^\circ \cdot k, \quad k \in \mathbb{Z}$$

$$k_0 = -\frac{\prod_{j=1}^{n} |s - p_j|}{\prod_{i=1}^{m} |s - z_i|}, \quad \varphi(s) = \sum_{i=1}^{m} \varphi_{z_i}(s) - \sum_{j=1}^{n} \varphi_{p_j}(s) = 360^\circ \cdot k$$

Pravila za crtanje

1) ISTO

2) ISTO, ali za $k_0 \to -\infty$

3) $n-m$ je broj asimptota, a kutevi asimptota $= \frac{360^\circ k}{n-m}$

$$\sigma_A = \frac{1}{n-m}\left[\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i\right]$$

TOČKE GRANANJE ASIMPTOTA

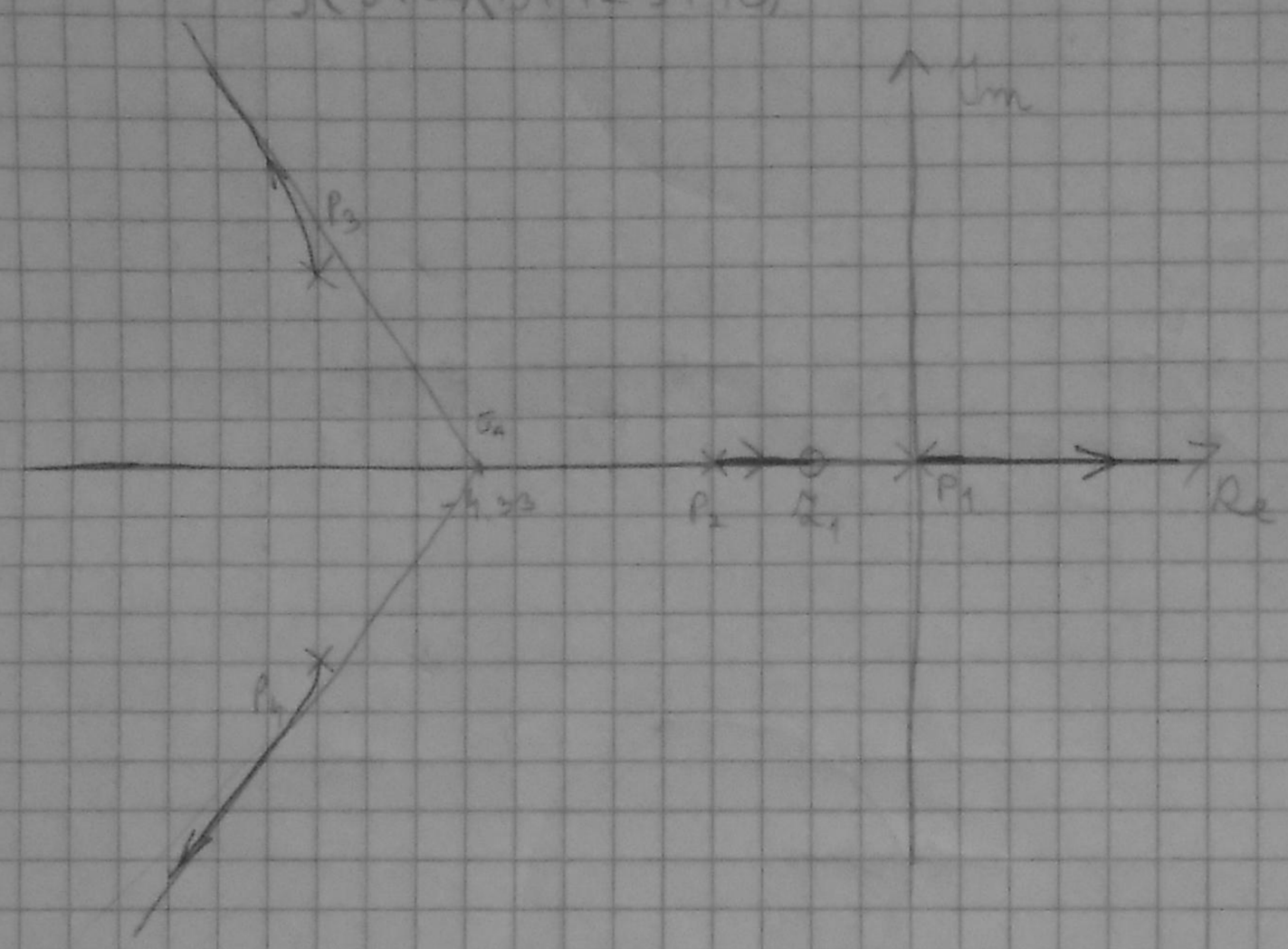
4) točka SER, koja je broj nula s desna na dijelu te točke parom, je dio KMK

5) kandidati za točke grananja/sjedinjenja su iste

6) $\varphi_{sk}^{I} =$ ISTO $+ 360^\circ \cdot k$

Pr. RMR $k_0 < 0$

$$G_0(s) = \frac{k_0(s+1)}{s(s+2)(s^2+12s+40)}$$

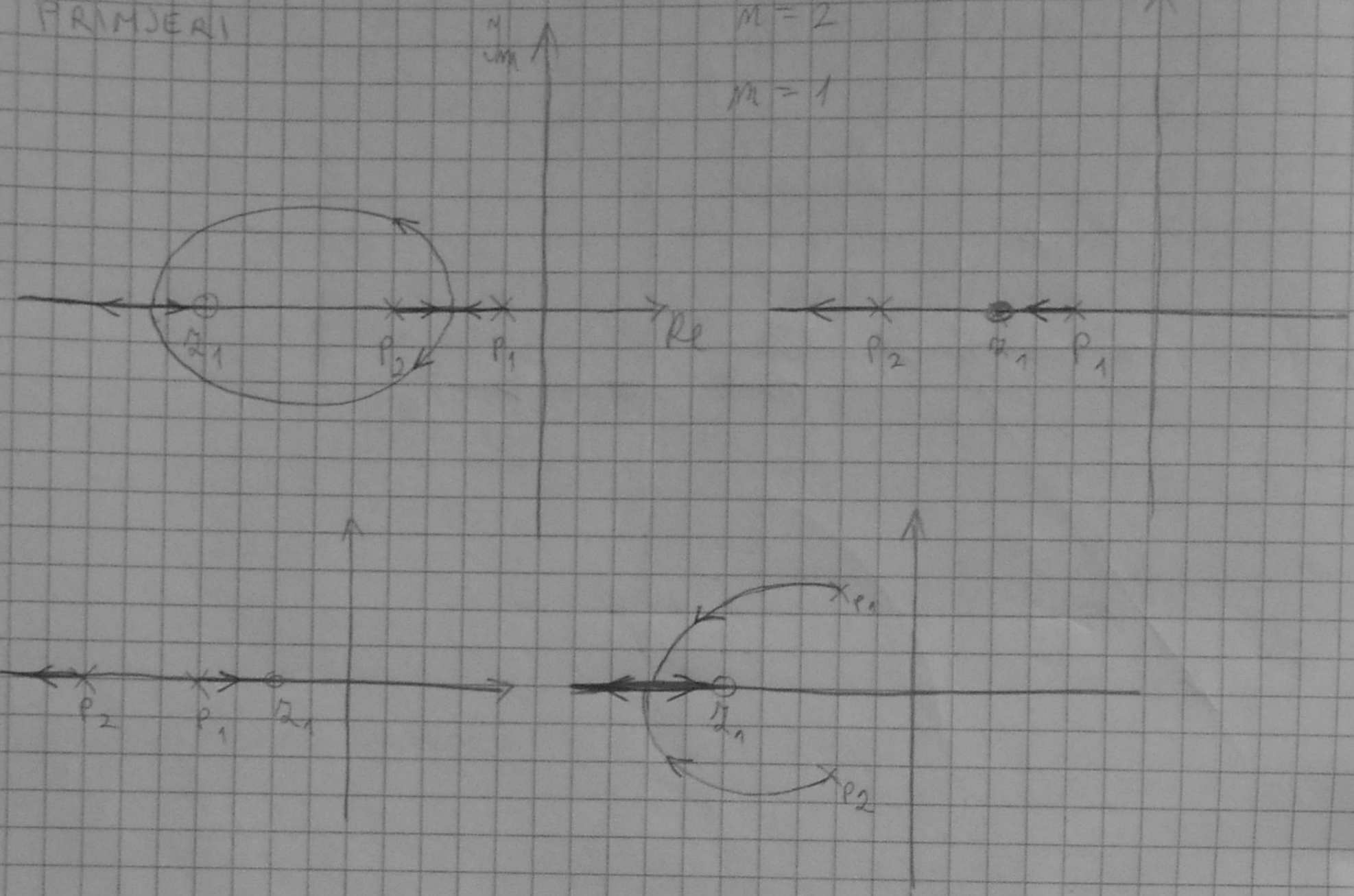


Sinteza pomoću KMK

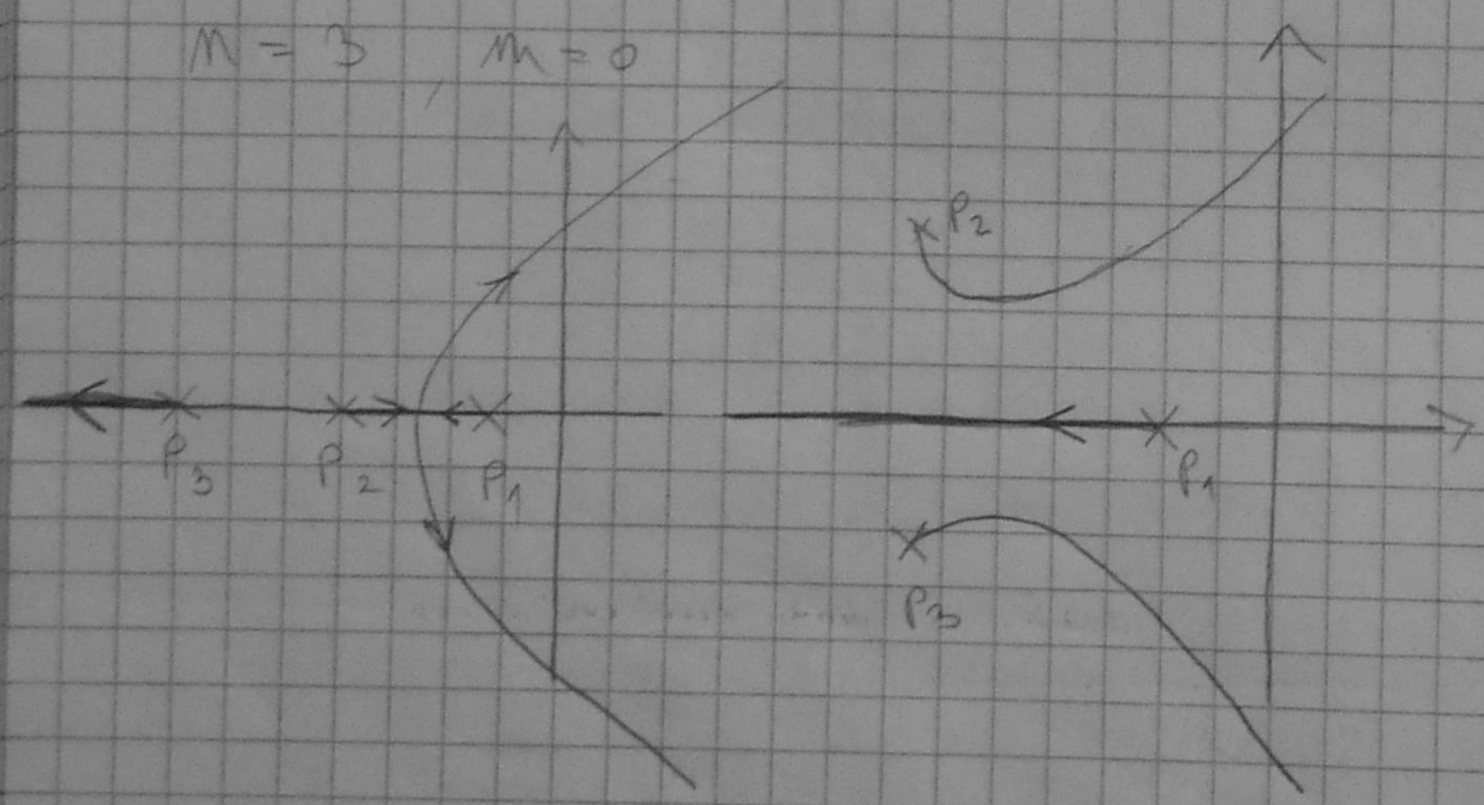
PRIMJERI

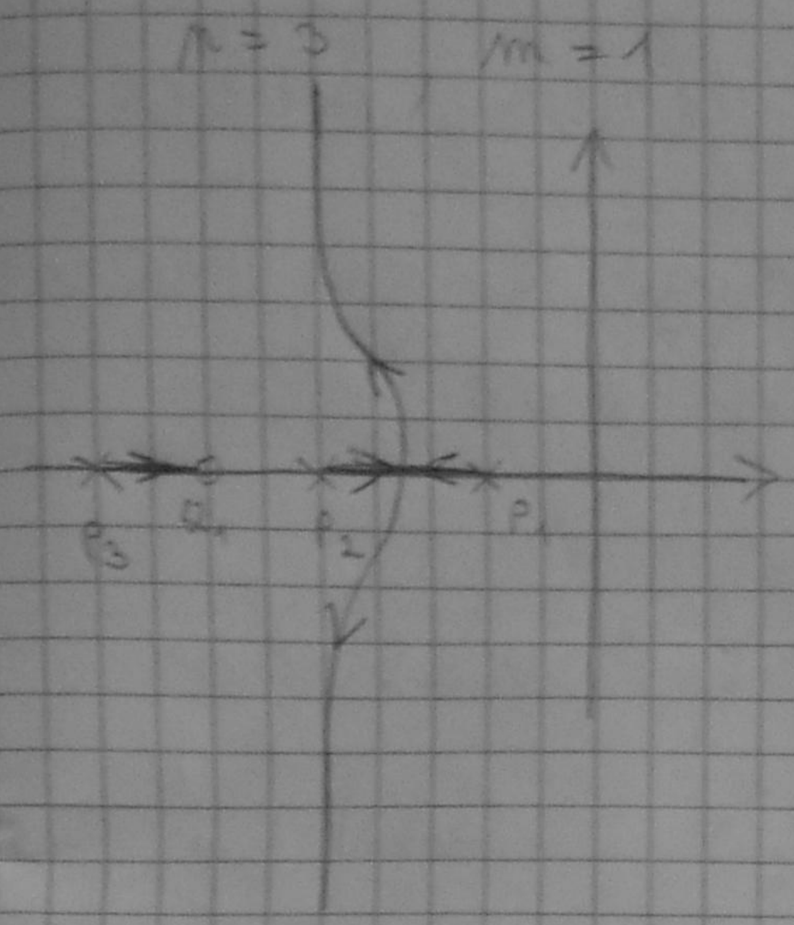
$n = 2$

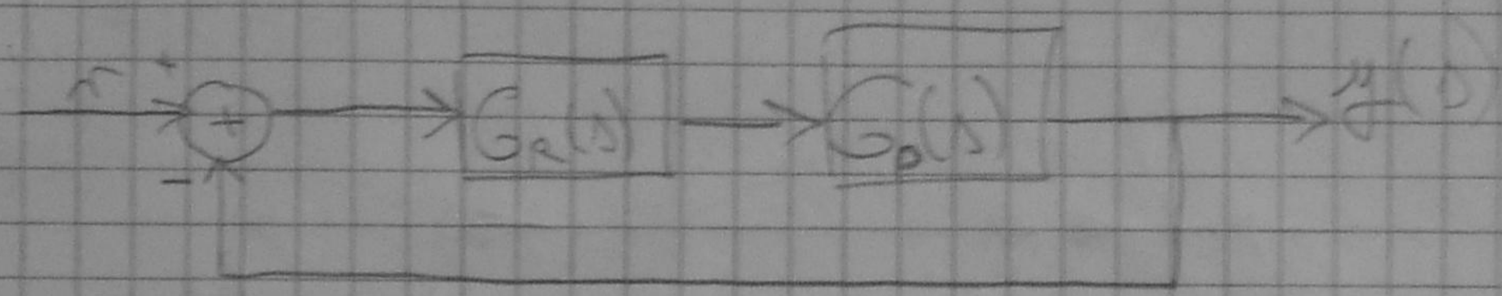
$m = 1$



$n = 3$, $m = 0$







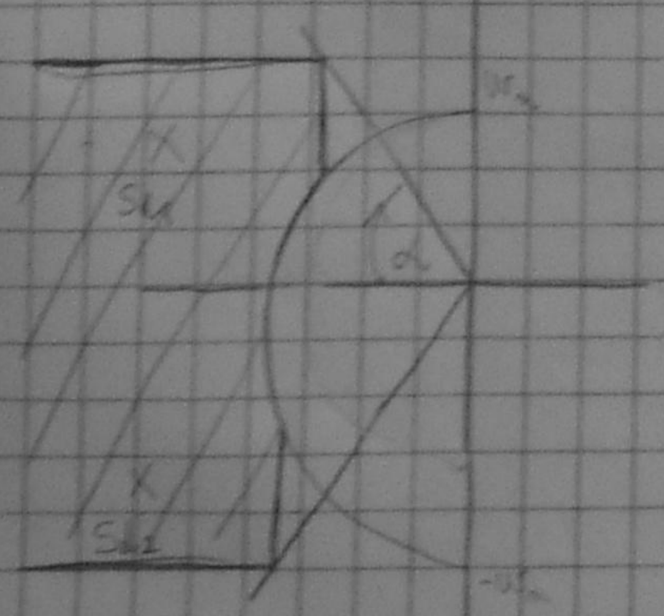
$$G_0(s) = G_R(s) \cdot G_P(s)$$

$$G_r(s) = \frac{G_0(s)}{1+G_0(s)}$$

ŽELIMO: $G_m = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$

POSTUPAK PROJEKTIRANJA

1) u s ravnini nađemo željno vladanje zatvorenog kruga (mjerne karakteristike mjerne kruga mali, mjerne mjerenje max. moguće) => kriterij



2) Prikažemo KMK fiksnog dijela regulacijskog kruga

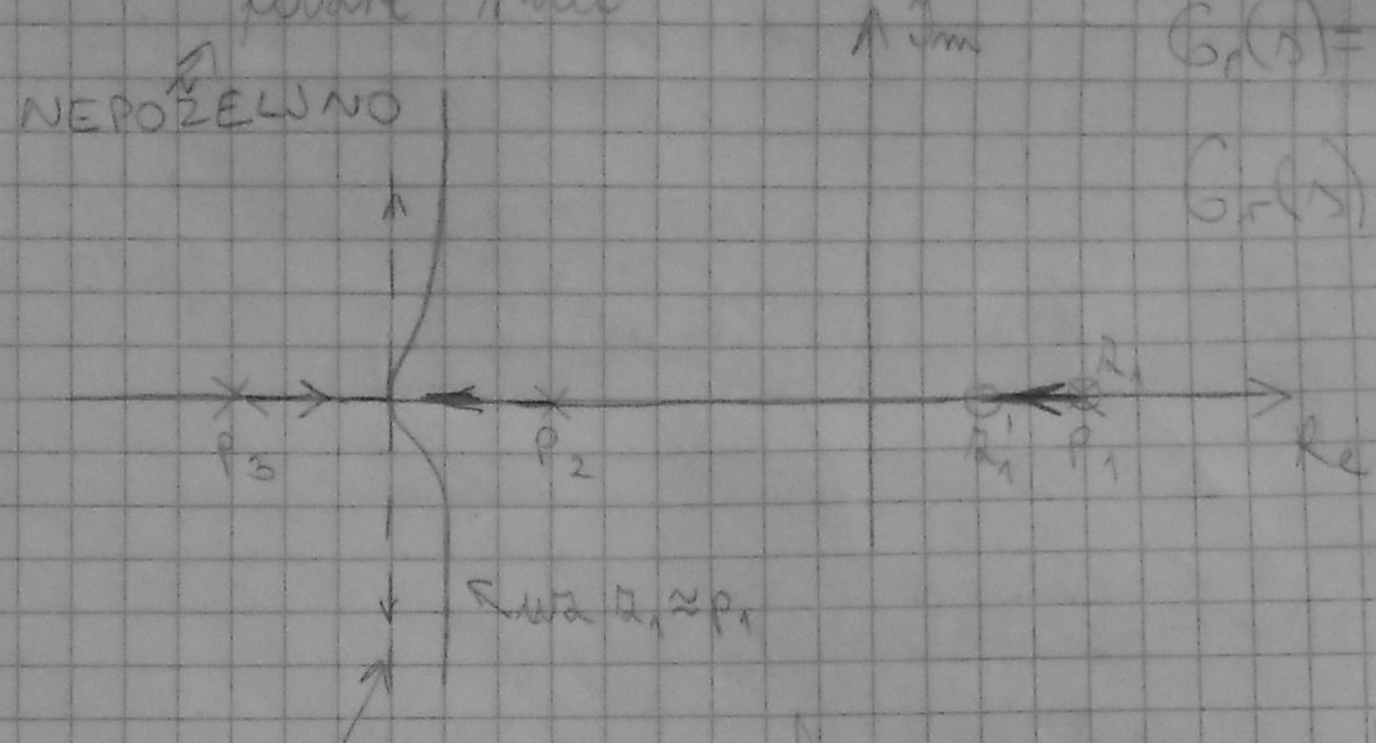
3) Dodavanjem nula i polova otvorenog kruga (REGULATOR) oblikujemo KMK da ona prolazi kroz željeno područje

DIGRESIJA: kraćenje nestabilnog pola i neminimalno fazne nule

$$G_r(s) = \frac{1}{(s-p_1)(s-p_2)}$$

$$G_r(s) = \frac{s-z_1}{s-p_3}$$

ali $\boxed{z_1 = p_1}$



, ali $z_1 \approx p_1$ nije kao u prakši postoji nestabilan pol

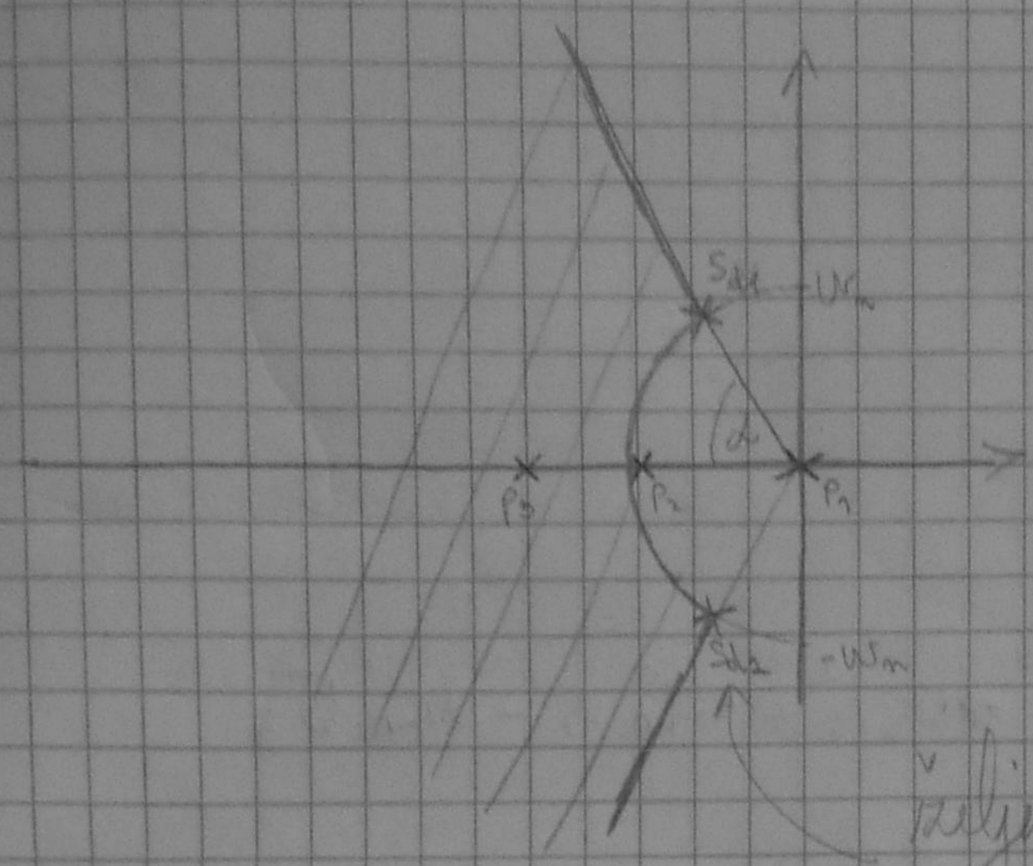
Primjer: $G_p = \dfrac{1}{s(s+3)(s+5)}$, $G_r = ?$

$\sigma_m \leq 16\%$, $t_{a,5\%} \leq 0.6\ s$

$\sigma_m \Rightarrow \xi = 0.5$ $\boxed{t_r = \dfrac{1.8}{\omega_n}}$ $\boxed{t_{a,5\%} = \dfrac{1.8\,9}{\omega_n}}$

$\omega_n \geq 3.1$

$\alpha \leq \arccos \xi = \underline{60^\circ}$

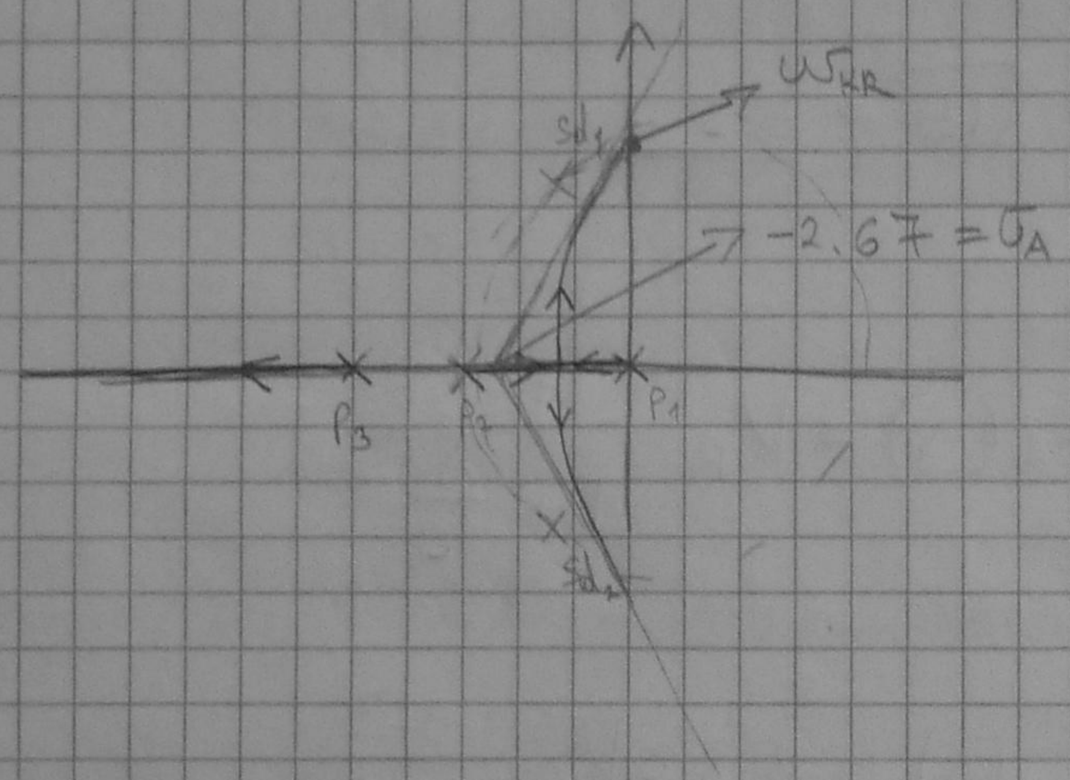


$$G_0(s) = G_R(s) \cdot G_p(s)$$

$\uparrow$ = 1 za kratkog spoja

Ako želimo dvojni polovi npr. za $\omega_n = 3{,}1$ $\xi = 60$

$$s_{d1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$



$\sigma_{KR} = -1{,}2$

za koji K_{OKR} dobijem ω_{KR}?

$$G_0(s) = \frac{K_0}{s(s+3)(s+5)}$$

$$G_R(s) = \frac{K_0}{s^3 + 8s^2 + 15s + K_0}$$

$$\Delta(s) = \underset{a_3}{s^3} + \underset{a_2}{8s^2} + \underset{a_1}{15s} + K_0$$

HURWITZ

1) $a_i > 0 \Rightarrow K_0 > 0$

2) $D_1 = a_1 > 0$

3) $D_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0$

$a_1 a_2 - a_0 a_3 > 0$

$120 - K_0 > 0$

$K_0 < 120$

Za $K_0 = 0 \Rightarrow p_1 = 0$

Za $K_0 = 120 \Rightarrow$ RUBNI SLUČAJ $K_0 = K_{OKR}$

$$\Delta(s) = (s - s_1)(s - s_2)(s - s_3)$$

$$s_{1KR} = j\omega_{KR}$$

$$s_{2KR} = -j\omega_{KR}$$

$$s_{3KR} = \sigma_3$$

$$\Delta(s) = (s^2 + \omega_{KR}^2)(s - \sigma_3) = s^3 - \sigma_3 s^2 + \omega_{KR}^2 s - \omega_{KR}^2 \sigma_3$$

$$\Delta(s) = s^3 + 8s^2 + 15s + \underset{\substack{\uparrow \\ 120}}{K_0}$$

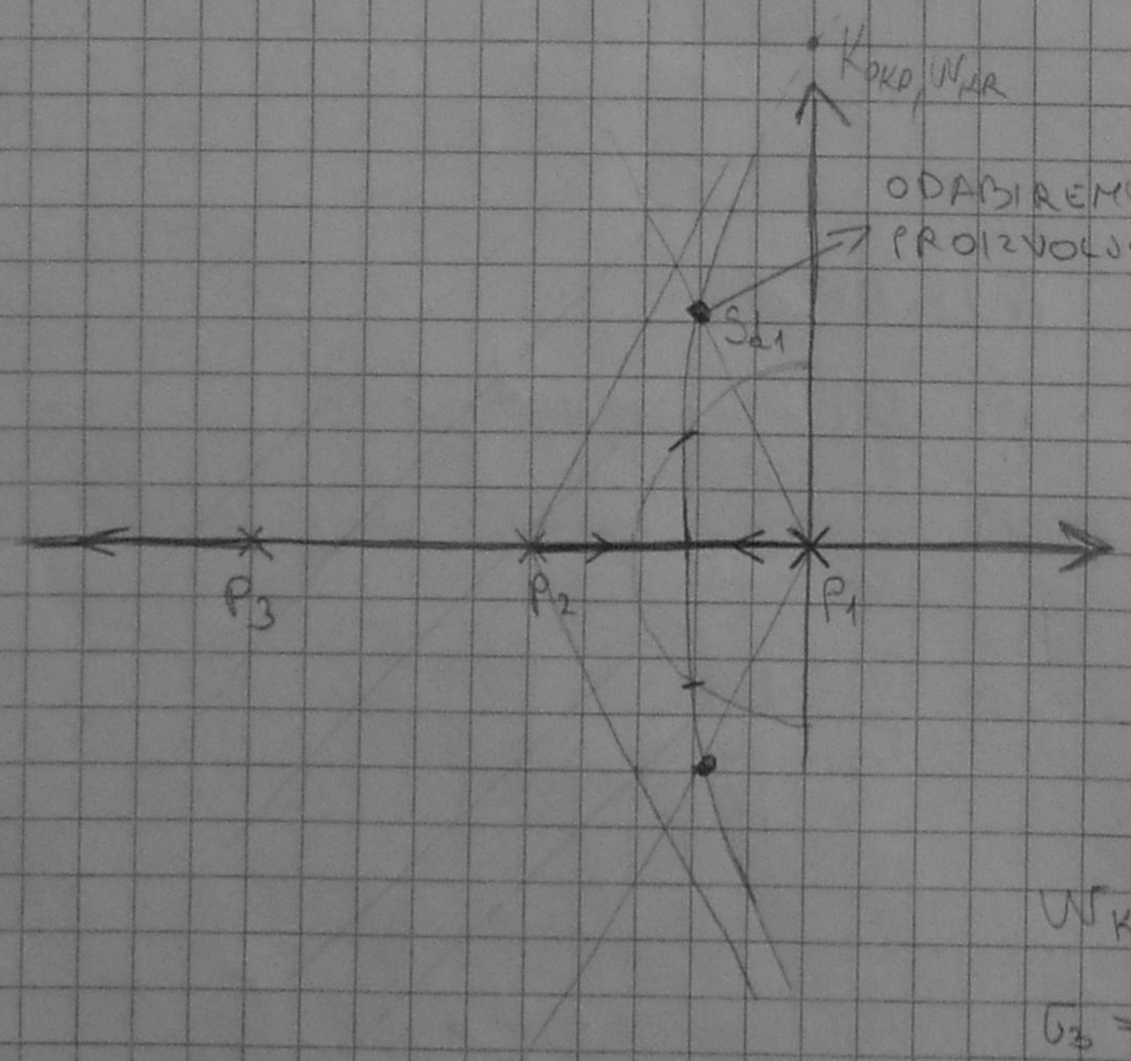
IZ JEDNAČI

$$\sigma_3 = -8, \quad \omega_{KR} = \sqrt{15} = \underline{3.87}$$

! moramo regulator $G_R(s) = K_R \dfrac{1 + \frac{s}{3}}{1 + \frac{s}{10}}$

LEAD regulator

$$G_R(s) = \underbrace{3.33 K_R}_{K_0} \frac{s+3}{s+10}$$



$$G_0(s) = \frac{3.33 K_R}{s(s+5)(s+10)}$$

$$\sigma_A = -5$$

$$\sigma_V = -2.1$$

$$\omega_{KR} = \sqrt{50} = 7.07$$

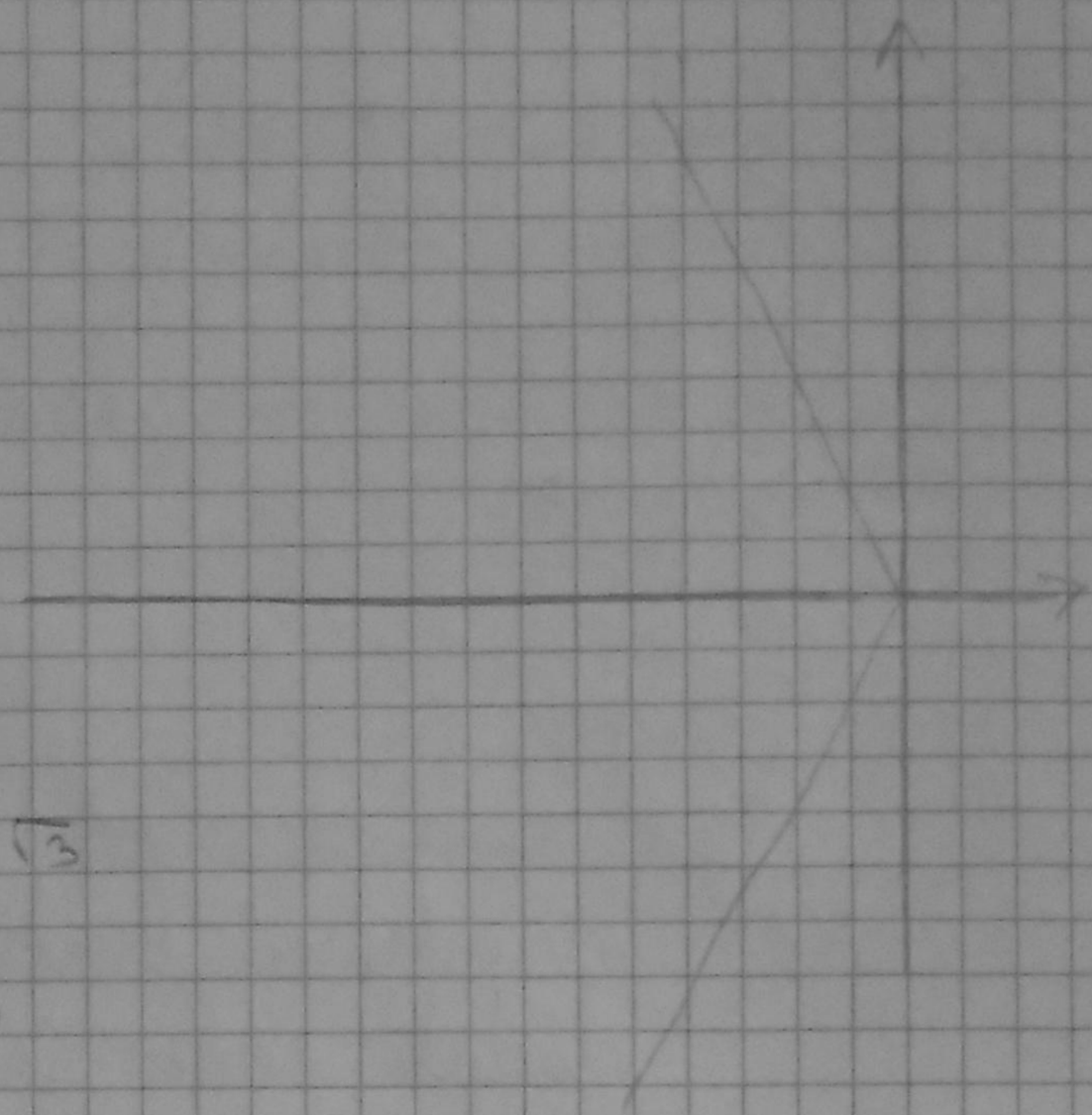
$$\sigma_3 = -15$$

$$K_{RKR} = 225$$

FAZNI UVJET

$$\operatorname{tg}\varphi = \frac{\omega_m}{R_e}$$

$$\varphi = \operatorname{arctg}\frac{\omega_m}{R_e} =$$



$$s_{d1} = \sigma + j\omega$$

$$\frac{\omega}{\sigma} = \sqrt{3} \Rightarrow \omega = -\sigma\sqrt{3}$$

$$-\varphi_1 - \varphi_2 - \varphi_3 = -180^\circ$$

$$\varphi_1 = 120^\circ$$

$$\varphi_2 + \varphi_3 = 60^\circ$$

$$\sigma = -\frac{5}{3}$$

$$\omega = \pm\frac{5}{\sqrt{3}}$$

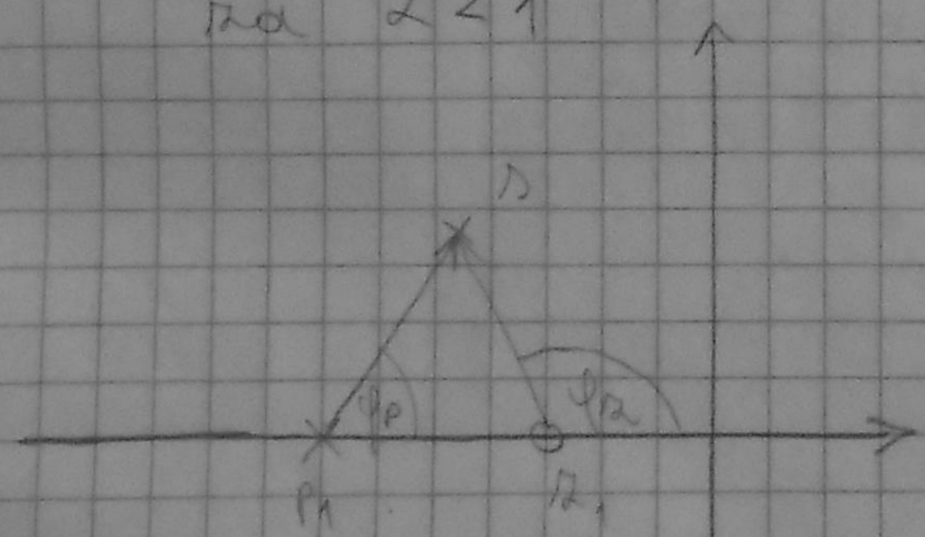
- Korištenje LEAD/LAG kompenzatora

$$G_R(s) = K_R \cdot \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} = K_R \cdot \alpha \cdot \frac{sT + 1}{\alpha T s + 1}$$

za $\alpha < 1 \Rightarrow$ LEAD

za $\alpha > 1 \Rightarrow$ LAG

za $\alpha < 1$

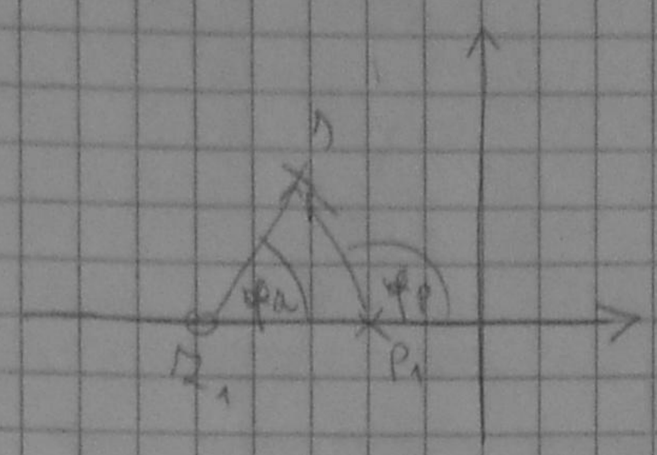


DOPRINOS KOMPENZATORA FAZNOM UVJETU

$$\phi = \varphi_z - \varphi_p = \phi(s)$$

POZITIVAN BROJ

za $\alpha > 1$



$$\phi(s) = \varphi_z - \varphi_p < 0$$

NEGATIVNO

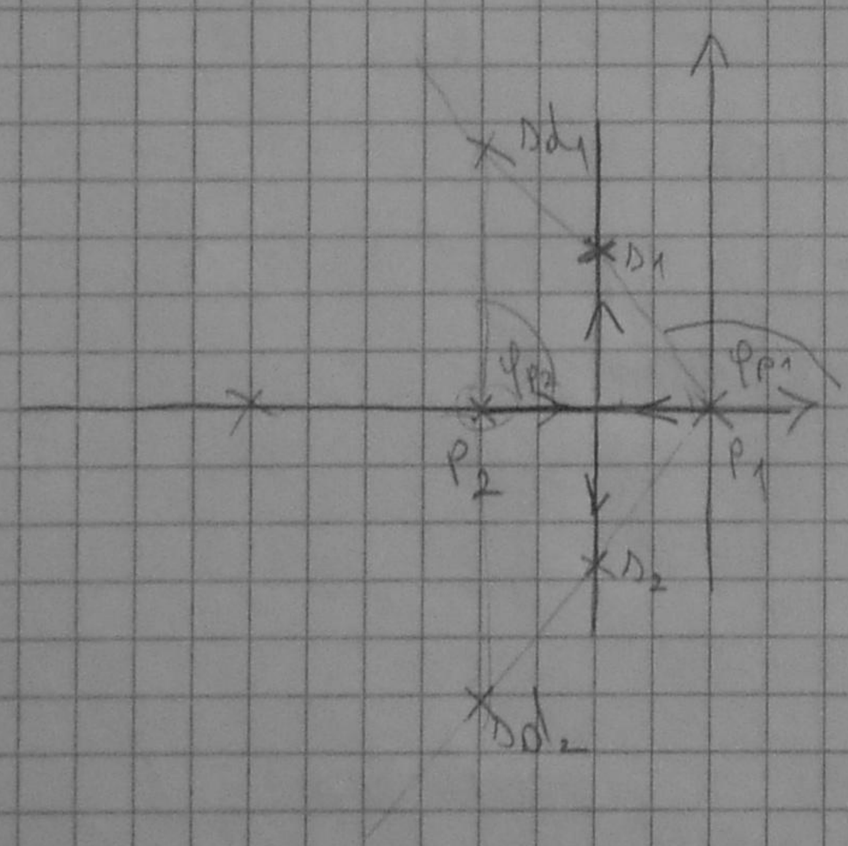
3. KORAK: nedostajući iznos faze u faznom mjestu dodati preko LEAD/LAG kompenzatora

Primjer: $G_p(s) = \dfrac{4}{s(s+2)}$

ZATV. KRUGA

$$\Delta(s) = s^2 + 2s + 4 = 0$$

$$s_{1,2} = -1 \pm j\sqrt{3}$$



$$s_{d_{1,2}} = -2 \pm j2\sqrt{3}$$

ŽELJENA DINAMIKA

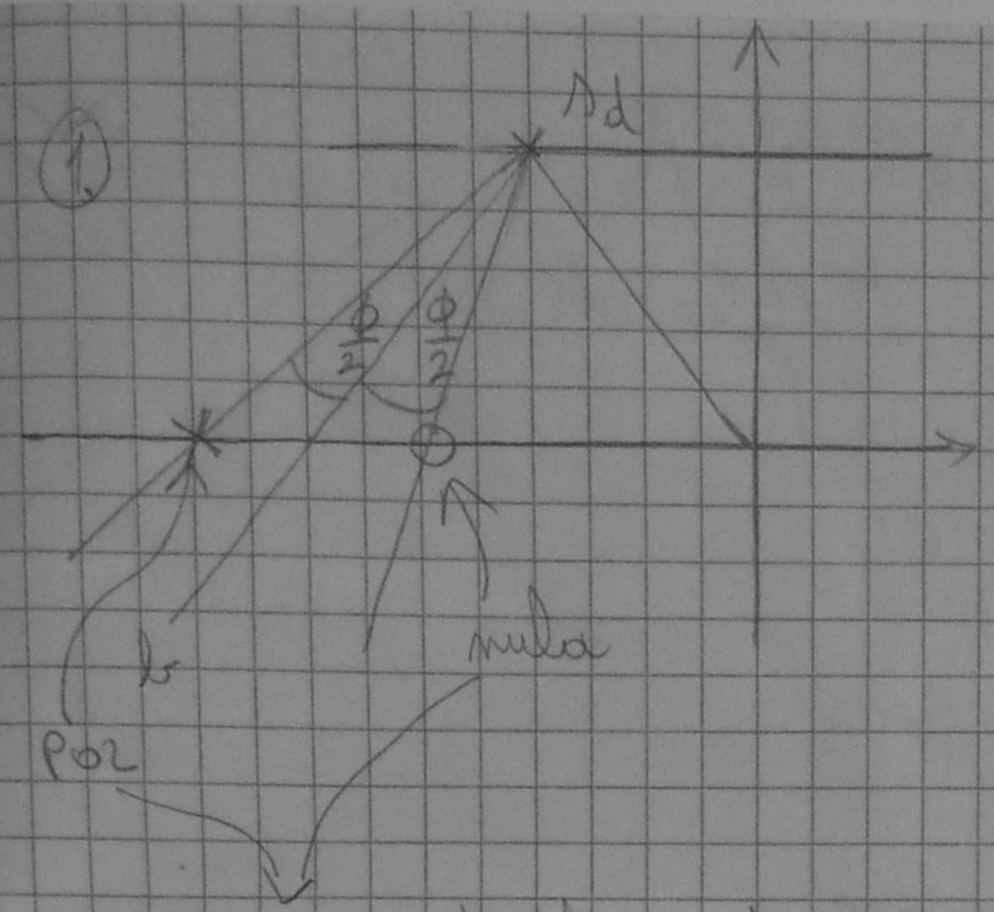
FAZNI UVJET:

$$-\varphi_{p_1} - \varphi_{p_2} + \phi = -180^\circ$$

$$-120^\circ - 90^\circ + \phi = -180^\circ$$

$$\underline{\phi = 30^\circ}$$

$$\boxed{\begin{aligned} p_1 &= -4 \\ z_1 &= -2 \end{aligned}}$$



1. PRAVAC a kroz S_d paralelno s realnom osi

2. povuči simetralu kuta $\measuredangle(a, S_d, O) = b$

3. obje se povuče prema gore s kutovima $\pm\phi/2$

ovisi je li lead ili lag

$\alpha_c = -\frac{1}{T}$ $\qquad$ $\alpha_c = -2.9 \Rightarrow T = 0.345$

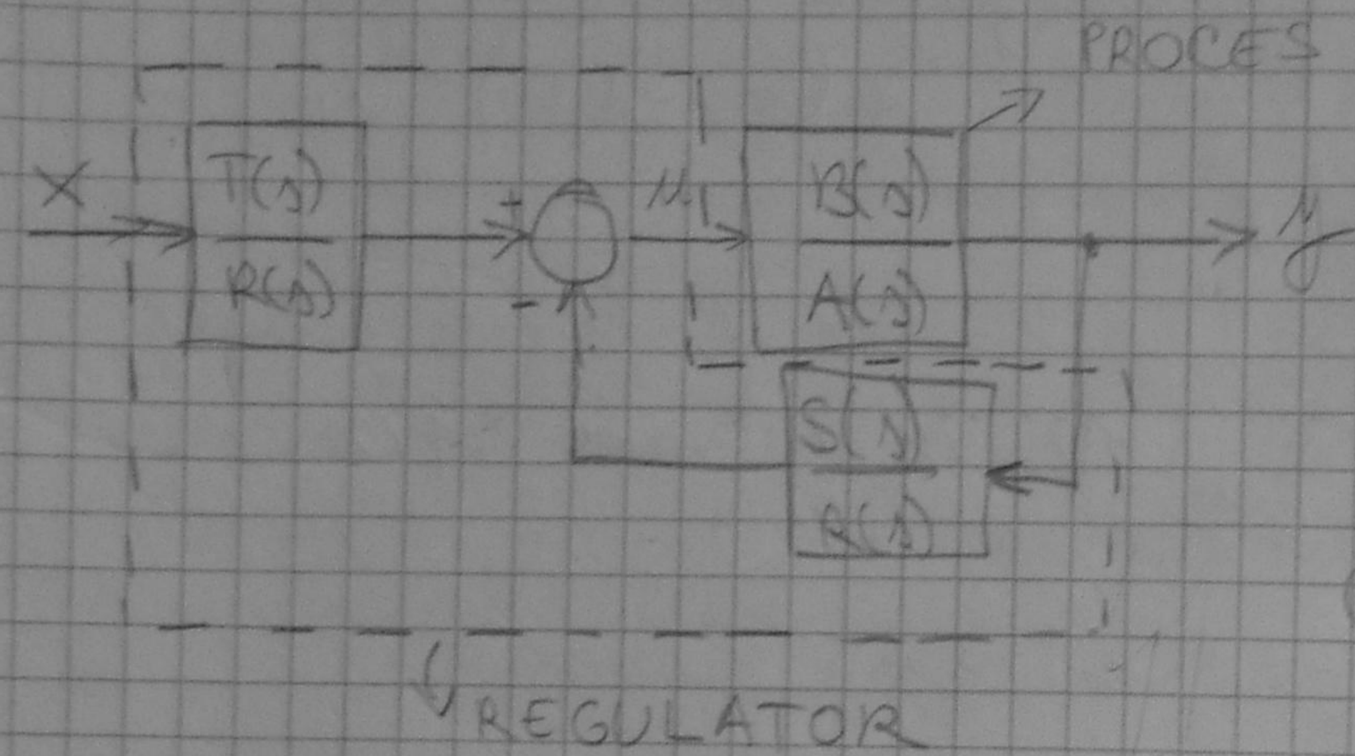
$p_c = -\frac{1}{\alpha T}$ $\qquad$ $p_c = -5.4 \Rightarrow \alpha = 0.537$

$K_R \Rightarrow$ iz amplitudnog uvjeta

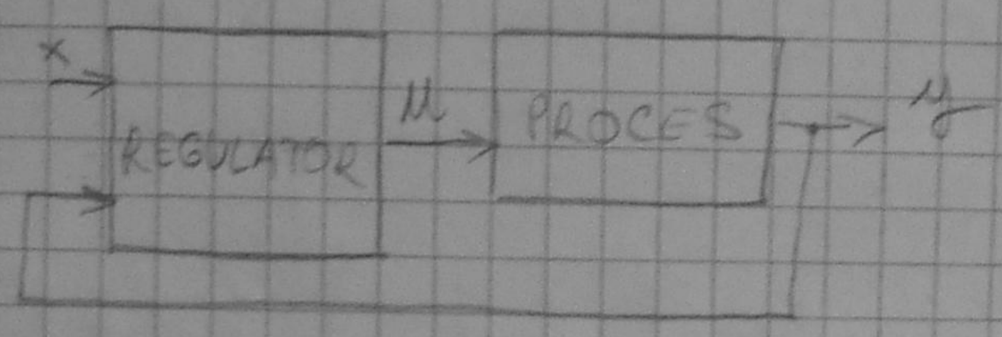
→ ako se radi o kompenzatorima s $\phi > 60°$ onda se projektira $\frac{\phi}{2}$ regulator i podupliraju nule i polovi

SINTEZA REGULATORA POSTAVLJANJEM POLOVA

(pole placement)

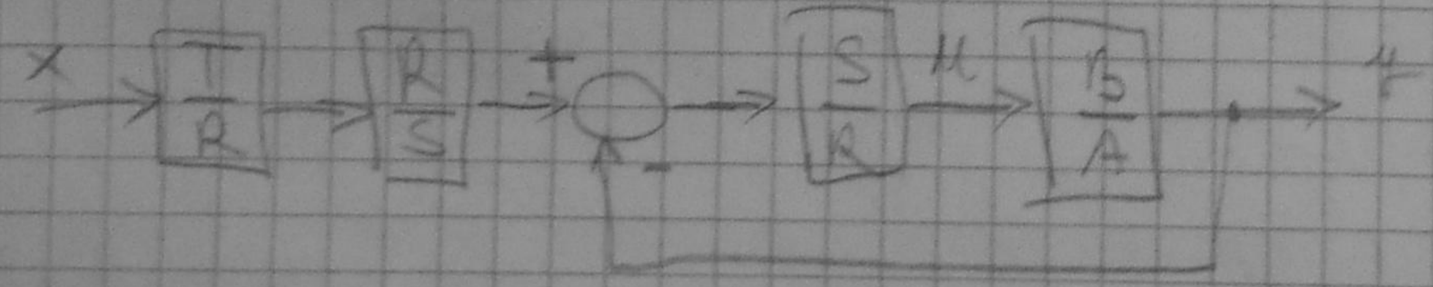


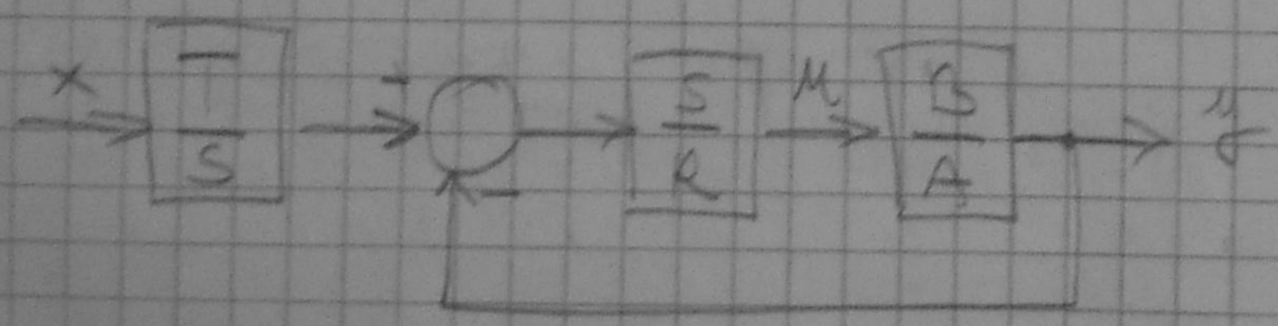
2 DOF
(2 degrees of freedom)



EKVIVALENTNI PRIKAZI:

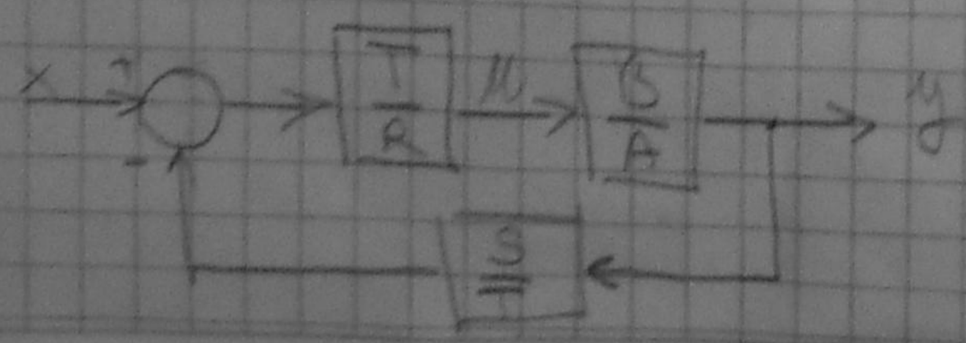
1)





⇒ implementacijska pitanje (treba li osigurati stabilnost i R i S ili samo R(s) da bi regulator bio stabilan)

2)



Mi PRIKAZI možemo istu prijenosnu funkciju

$$\cancel{Y(s)} = \frac{Y(s)}{X(s)} = \frac{B(s)\,T(s)}{A(s)R(s) + B(s)\,S(s)}$$

REGULATOR ČINE 3 POLINOMA: R S T

- može se reći da je R(s) monički polinom bez gubitka općenitosti

Karakteristična jednadžba:

$$A(s)R(s) + B(s)S(s) = 0$$

Željena dinamika (katanga):

$$\underbrace{\boxed{P(s) = A(s)R(s) + B(s)S(s)}}_{P(s)}$$

DIOFANTSKA JEDNADŽBA

Primjer:

$$G(s) = \frac{0.25s + 0.25}{s^2 + 0.6s + 0.05} = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

$$G_R(s) = \frac{S(s)}{R(s)} = \frac{s_1 s + s_0}{s + R_0}$$

$$P(s) = s^3 + 0.7s^2 + 0.12s + 0.01$$

$p_2 \quad p_1 \quad p_0$

$\Rightarrow$ 1) PRVO SE IZABERE ŽELJENA DINAMIKA, ~~[illegible]~~

2) ONDA SE PREMA A(s) ODREDI STUPANJ R(s)

3) RED S(s) se MOŽE izvesti maksimalni

$$\left.\begin{aligned} R_0 + a_1 + b_1 S_1 &= p_2 \\ a_0 + a_1 R_0 + b_1 S_0 &= p_1 \\ a_0 R_0 + b_0 S_0 &= p_0 \end{aligned}\right\}$$

$R_0 = 0.0889$

$S_0 = 0.0222$

$S_1 = 0.0444$

PRIMJER 2:

$$G(s) = \frac{B(s)}{A(s)} = \frac{5s+1}{s^2+4s}$$

$$P(s) = s^3 + 3s^2 + 3s + 2$$

$$G_R(s) = \frac{S_1 s + S_0}{s + R_0}$$

$$G_R(s) = \frac{s+2}{s-2}$$

NESTABILAN REGULATOR

↳ ovo se ne može dogoditi

kod regulatora u prostoru stanja

$$\begin{bmatrix} 0 & 1 & 0 \\ 4 & 5 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_0 \\ S_0 \\ S_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$R_0 = -2$

$S_0 = 2$

$S_1 = 1$

nulovi $P(s)$:

$s_{p1} = -2 \quad s_{p2,3} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

A je monički isto

$\deg(A) = n \qquad \deg(P) = 2n-1$

$\deg(B) = \deg(R) = \deg(S) = n-1$

$$\begin{array}{c} \overbrace{\qquad\qquad\qquad\qquad}^{n-1} \quad \overbrace{\qquad\qquad\qquad\qquad}^{n} \\ \left[\begin{array}{cccc|ccccc} a_0 & 0 & 0 \dots & 0 & b_0 & 0 & 0 \dots & 0 & 0 \\ a_1 & a_0 & 0 \dots & 0 & b_1 & b_0 & 0 \dots & 0 & 0 \\ \vdots & & & & & & & & \\ a_{n-2} & a_{n-3} & a_{n-4} \dots & a_0 & b_{n-2} & b_{n-3} & b_{n-4} \dots & b_0 & 0 \\ a_{n-1} & a_{n-2} & a_{n-3} \dots & a_1 & b_{n-1} & b_{n-2} & b_{n-3} \dots & b_1 & b_0 \\ 1 & a_{n-1} & a_{n-2} \dots & a_2 & 0 & b_{n-1} & b_{n-2} \dots & b_2 & b_1 \\ 0 & 1 & a_{n-1} \dots & a_3 & 0 & 0 & b_{n-1} \dots & b_3 & b_2 \\ \vdots & & \vdots & & \vdots & & & \vdots & \\ 0 & 0 & 0 \dots & a_{n-1} & 0 & 0 & 0 \dots & b_{n-1} & b_{n-2} \\ 0 & 0 & 0 \dots & 1 & 0 & 0 & 0 \dots & 0 & b_{n-1} \end{array}\right] \end{array} \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n-2} \\ s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{n-2} \\ s_{n-1} \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{n-1} \\ p_{n-1} \\ p_n \\ p_{n+1} \\ \vdots \\ p_{2n-3} \\ p_{2n-2} \end{bmatrix} = \begin{bmatrix} \left.\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\} n \\ \left.\begin{matrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-2} \\ a_{n-1} \end{matrix}\right\} n-1 \end{bmatrix}$$

RJEŠENJE DIOFANTSKE JEDNADŽBE

Kompenzacija polova i nula procesa polovima i nulama regulatora:

$A = A^+ \cdot A^-$ $\qquad$ $B = B^+ \cdot B^-$ $\qquad$ + je postojano, ponašanje od A

A^+ i B^+ su monički polinomi

$R = B^+ \cdot \overline{R}$

$S = A^+ \overline{S}$

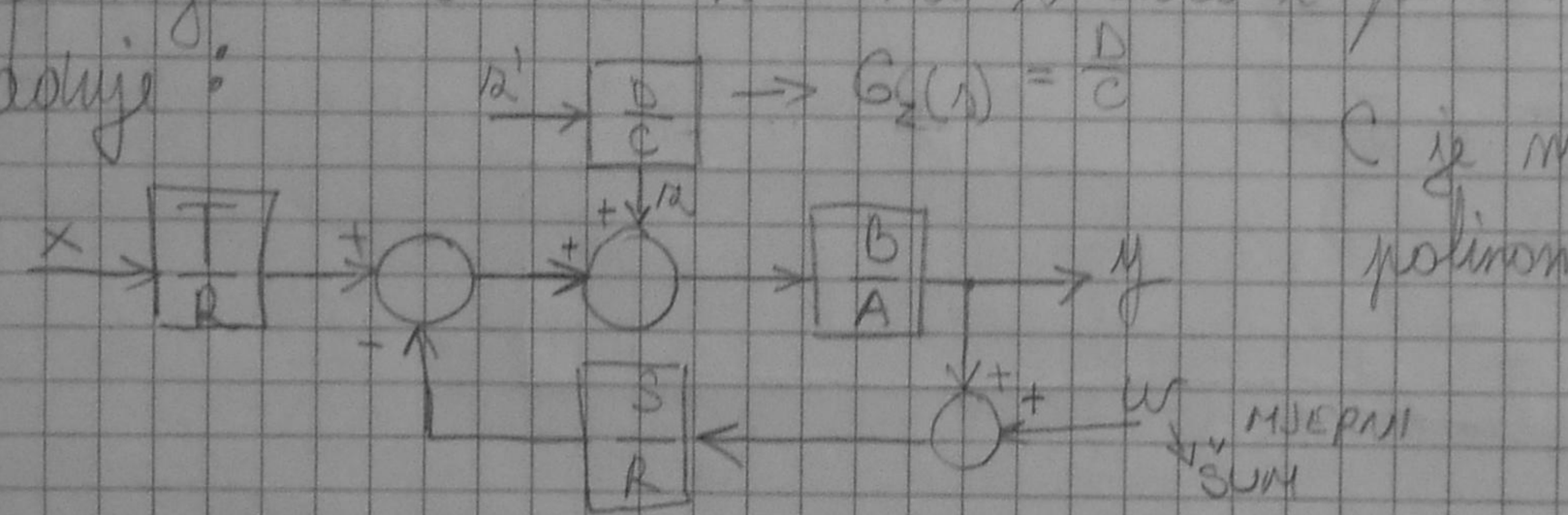
$T = A^+ \cdot \overline{T}$

$$G_z(s) = \frac{BT}{AR+BS} = \frac{B^+ B^- A^+ \overline{T}}{A^+ A^- B^+ \overline{R} + B^+ B^- A^+ \overline{S}} = \frac{B^- \overline{T}}{A^- \overline{R} + B^- \overline{S}}$$

KRAĆENJE STABILNIH POLOVA I NULA

Sinteza regulatora s obzirom na vođenje i poremećaje

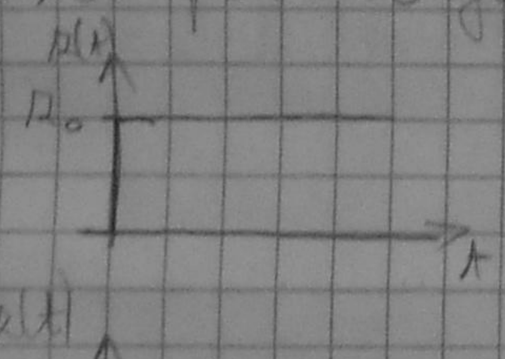
vođenje:



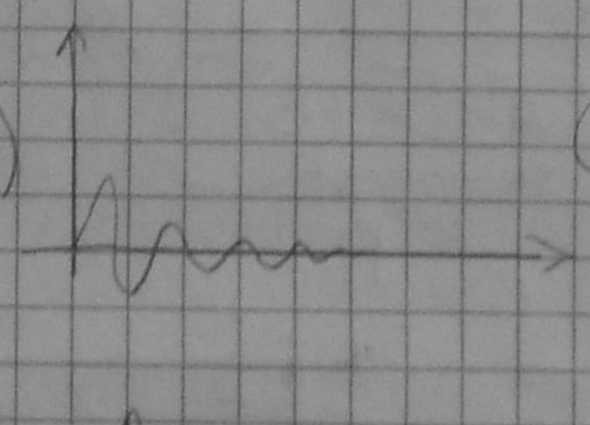
$G_r(s) = \frac{D}{C}$

C je monički polinom

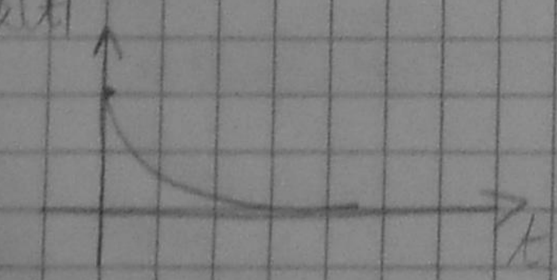
vrste poremećaja:

1) 

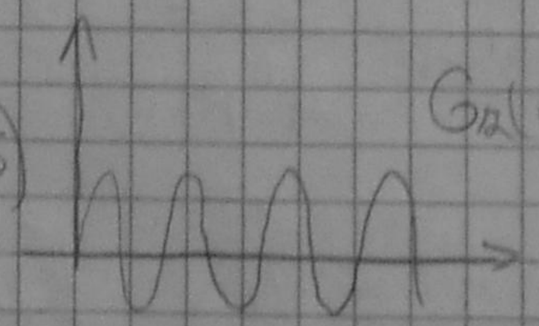
$G_z = \frac{w_0}{s}$

4) 

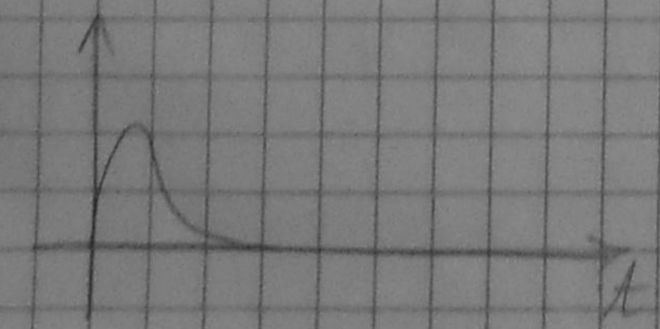
$G_w(s) = \frac{w_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

2) 

$G_w(s) = \frac{w_0}{s + c_0}$

5) 

$G_w(s) = \frac{w_0^2}{s^2 + \omega_w^2}$

3) 

$G_z(s) = \frac{w_0}{(s+s_{21})(s+s_{22})}$

PRIJENOSNA MATRICA

$$Y(s) = \frac{B \cdot T}{AR+BS} X(s) + \frac{BRD}{(AR+BS)C} Z'(s) - \frac{B \cdot S}{AR+BS} W(s)$$

$$U(s) = \frac{A \cdot T}{AR+BS} X(s) - \frac{BSD}{(AR+BS)C} Z'(s) - \frac{AS}{AR+BS} W(s)$$

G_m = MODELSKA FUNKCIJA $= \frac{B_m(s)}{A_m(s)} = \frac{Y(s)}{X(s)}$ → vođenje / vladanje

$G_m(0) = 1$ ⇒ nema pogreške

$B_m = b_{m0} \cdot \overline{B}_m \cdot B^-$ → mora se zadržati (nulativno)

→ bitna dinamika

$b_{m0} = \frac{A_m(0)}{\overline{B}_m(0) \cdot B^-(0)}$ → OSIGURAVA DA NEMA POGREŠKE U STAC. STANJU

DIOFANTSKA JEDNADŽBA

$R = A_m B^+ \cdot C\overline{R}$ $\qquad A^- C\overline{R} + B^- \cdot \overline{S} = P$

$S = A_m A^+ \overline{S}$

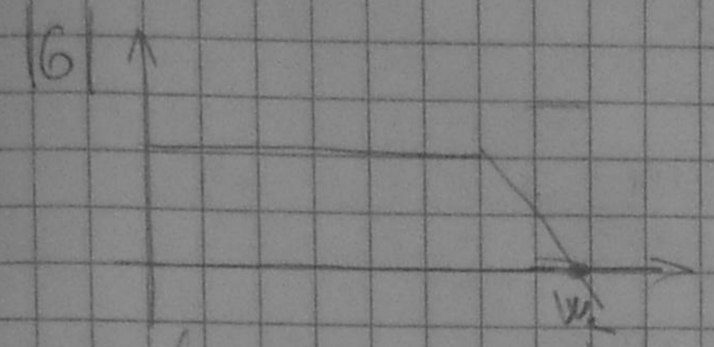
$T = b_{m0} \overline{B}_m A^+ \cdot P$

$$Y(s) = \frac{b_{m0} B^- \overline{B}_m \cdot P}{A_m \underbrace{(A^- C\overline{R} + B^- \overline{S})}_{P}} X(s) + \frac{B^+ B^- C\overline{R} D}{A^+ (A^- C\overline{R} + B^- \overline{S}) C} Z'(s) - \frac{B^- \overline{S}}{A^- C\overline{R} + B^- \overline{S}} W(s)$$

$$Y(s) = \frac{B_m}{A_m} X(s) + \frac{B\overline{R}D}{A^+ P} Z'(s) - \frac{B^- \overline{S}}{P} \cdot W(s)$$

$$U(s) = \frac{b_{m0} A \overline{B}_m}{B^+ A_m} X(s) - \frac{B^- \overline{S} D}{C \cdot P} \cdot Z'(s) - \frac{A\overline{S}}{B^+ P} W(s)$$

BODE

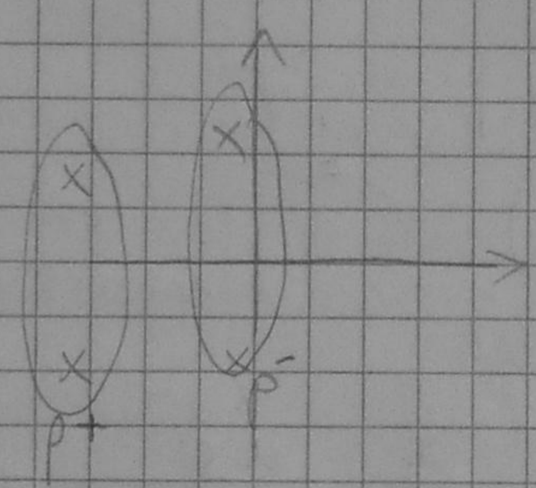


- ako je pol jako lijevo u s ravnini u kojem je presjeca ω_c jako dobro
- zbog toga se može dogoditi da šum koji je visoke frekvencije prođe nesmetano

1) jedan način sinteze:

$$P = P^+ P^-$$

$$A_m = P^-$$



$$R = B^+ C \bar{R}$$

$$S = A^+ \bar{S}$$

$$T = b_{m0} \bar{B}_m A^+ P^+$$

$$Y(s) = \frac{B_m}{P^-} X(s) + \frac{B \bar{R} D}{A^+ P} - \frac{B^- \bar{S}}{P} W(s)$$

Selekcija reda regulatora:

$N(s)$

1) IZ DIOFANTSKE JEDNADŽBE: $A^- C \bar{R} + B^- \bar{S} = P$

$$\deg(A^-) + \deg(C) + \deg(\bar{R}) > \deg(B^-) + \deg(\bar{S})$$

2) $\deg(\bar{R}) + \deg(\bar{S}) + 1 \geq \deg(P)$

3) $\deg(A^-) + \deg(C) + \deg(\bar{R}) = \deg P$

4) $\deg(\bar{S}) \geq \deg(A^-) + \deg(C) - 1$

5) $\deg(R) \geq \deg(S)$

↳ zbog ostvarivosti regulatora

$$\deg B^+ + \deg C + \deg \bar{R} \geq \deg A^+ + \deg \bar{S}$$

$$\deg \bar{R} \geq \deg A^+ - \deg B^+ - \deg C + \deg \bar{S}$$

$$\boxed{\deg \bar{R} \geq \deg A - \deg B^+ - 1}$$

gdje je $\deg \bar{S} \geq \deg A^- + \deg C - 1$

OSJETLJIVOST
(sensitivity)

$$\boxed{S_\delta^M := \frac{d[\ln(M)]}{d[\ln(\delta)]}} = \frac{d[\ln(M)]}{d\delta} \cdot \frac{d\delta}{d[\ln(\delta)]} = \frac{1}{M} \cdot \frac{dM}{d\delta} \cdot \delta$$

M - značajka

δ - parametar

$$\boxed{S_\delta^M = \frac{\delta}{M} \cdot \frac{dM}{d\delta}}$$

2)

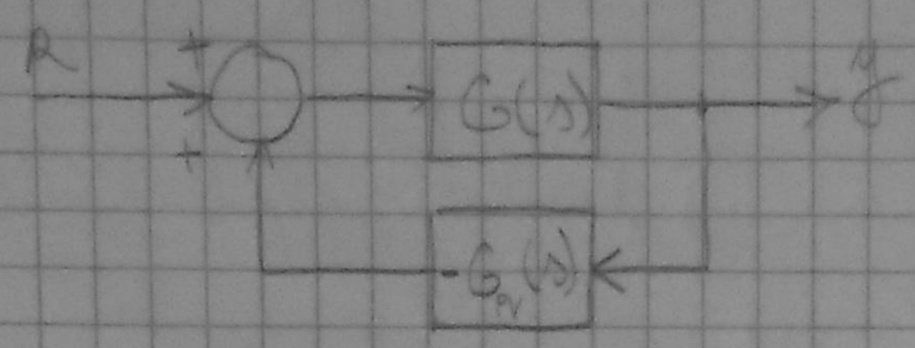
u stacionarnom stanju $\delta = \delta_0$, $M = M_0$

ako se promjena analizira: $\delta = \delta_0 + \Delta\delta$ $M = M_0 + \Delta M$

$$S_\delta^M \approx \frac{\delta_0}{M_0} \cdot \frac{\Delta M}{\Delta \delta} = \frac{\frac{\Delta M}{M_0}}{\frac{\Delta \delta}{\delta_0}}$$ ⇒ približna veličina koja je točna u limesu

$$S_{\delta_i}^M = \frac{\partial(\ln M)}{\partial(\ln \delta_i)}$$ ⇒ za više parametara

Osjetljivost sustava:



$$G_z(s) = \frac{G(s)}{1+G(s)G_{pv}(s)}$$

1) OTVORENI KRUG:

$$G_{pv}(s) = 0$$

$$Y(s) = G(s)R(s)$$

$$\frac{dY(s)}{dG(s)} = R(s)$$

$$S_{G(s)}^{Y(s)} = \frac{G(s)}{Y(s)} \cdot R(s) = 1$$

po definiciji

2) ZATVORENI KRUG S JEDINIČNOM NEG. POV. VEZOM:

$$G_{pv}(s) = 1$$

$$Y(s) = \frac{G(s)}{1+G(s)} \cdot R(s)$$

$$\frac{dY(s)}{dG(s)} = \frac{1+G(s)-G(s)}{[1+G(s)]^2} \cdot R(s)$$

$$S_{G(s)}^{Y(s)} = \frac{G(s)}{Y(s)} \cdot \frac{R(s)}{[1+G(s)]^2}$$

$$= \frac{1+G(s)}{R(s)} \cdot \frac{R(s)}{[1+G(s)]^2}$$

$$\boxed{S_{G(s)}^{Y(s)} = \frac{1}{1+G(s)} = S}$$

SMANJENA OSJETLJIVOST

(zato se koristi povratna veza)

5) ZATVORENI KRUG OPĆENITO

$G_{pv}(s) \neq 0, 1$

$$S_{G(s)}^{Y(s)} = \frac{1}{1+\underbrace{G_{pv}(s)\,G_o(s)}}$$

ako je pojačanje G_o veliko ono je jako neosjetljivo

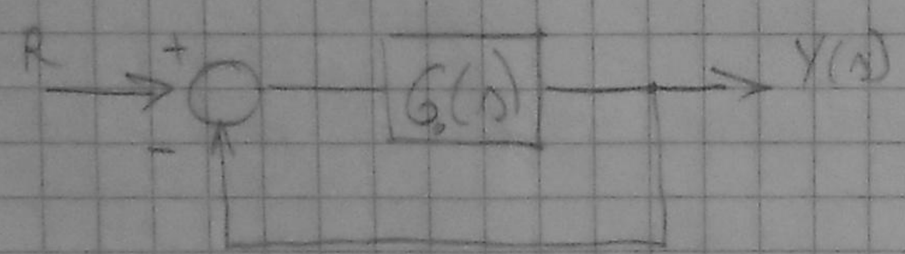
Zatvoreni krug

$$S_{G_{pv}(s)}^{Y(s)} = \frac{G_{pv}(s)}{Y(s)} \cdot \frac{-G^2(s)}{[1+G_{pv}G(s)]^2} \approx -1 \quad \text{ako} \quad |G_{pv}(s)\cdot G(s)| \gg 1$$

$$= \frac{-G(s)\,G_{pv}(s)}{1+G(s)\,G_{pv}(s)}$$

$\Rightarrow$ ako je G_{pv} od pouzdanih komponenata ono nema problema

Primjena osjetljivosti u sintezi:

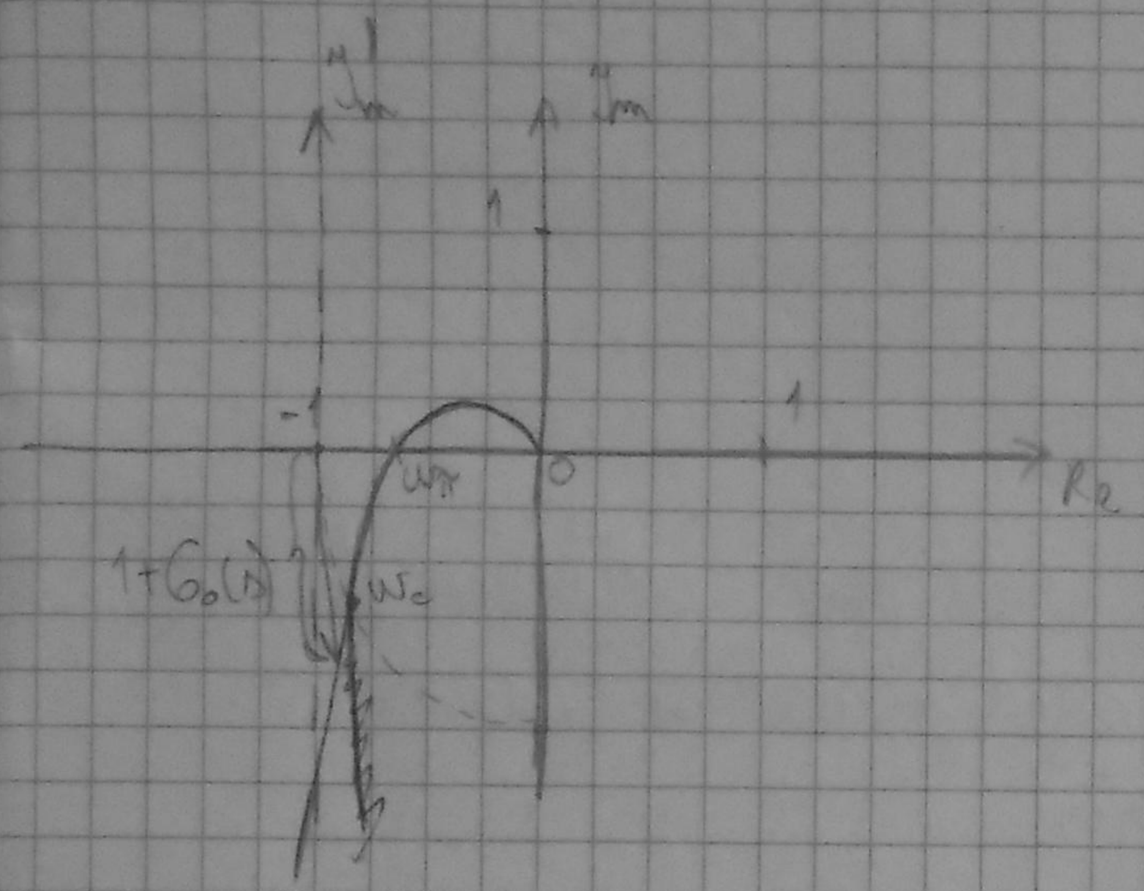


$$S = \frac{1}{1+G_o(s)}$$

DINAMIČKI ČLAN	PROMJENJIVI PARAMETAR δ	FUNKCIJA OSJETLJIVOSTI δ
PT_1		
$G(s) = \frac{K}{1+T(s)}$	pojačanje K	$S_K^{G(s)} = 1$
$G(s) = \frac{K}{s-s_p}$	vremenska konst. T	$S_T^{G(s)} = \frac{-T_o s}{1+T_o s}$
	položaj pola s_p	$S_{s_p} = \frac{s_{po}}{s-s_{po}}$
mrtvo vrijeme		
$G(s) = Ke^{-T_t s}$	pojačanje K	$S_K^{G(s)} = 1$
	po T_t	$S_{T_t}^{G(s)} = -T_{to}\, s$

$$|S(j\omega)| = \left| \frac{1}{1+G_0(j\omega)} \right| < k$$

$$|1+G_0(j\omega)| \geq \frac{1}{k}$$



=> mala točka kružić ograničava dijagram mora se nalaziti lijevo od udaljenosti $\frac{1}{k}$ od točke -1

Ograničenje funkcije osjetljivosti sustava:

BODEOVA FORMULA:

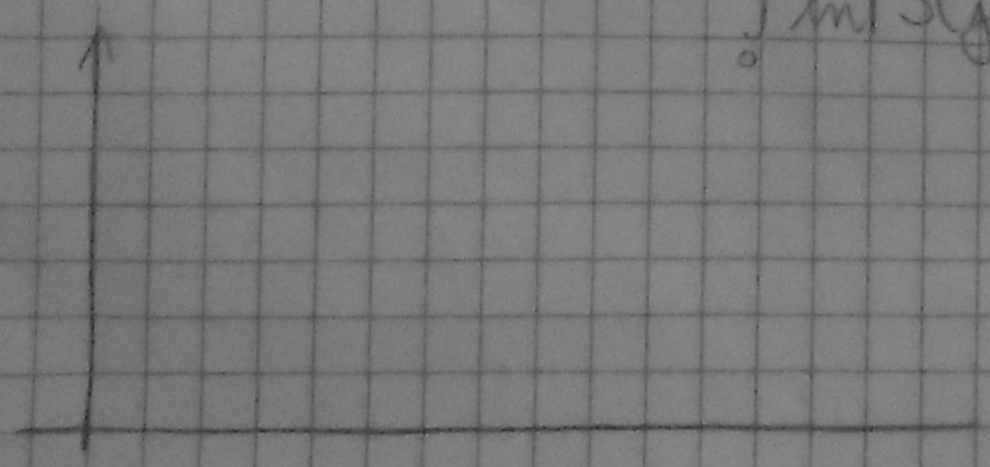
$$\boxed{\int_0^\infty \ln|S(j\omega)|\,d\omega = \pi \cdot \sum_{i=1}^{n_R} s_{R,Pi} - \frac{\pi}{2} \lim_{s\to\infty} s\,G_0(s)}$$

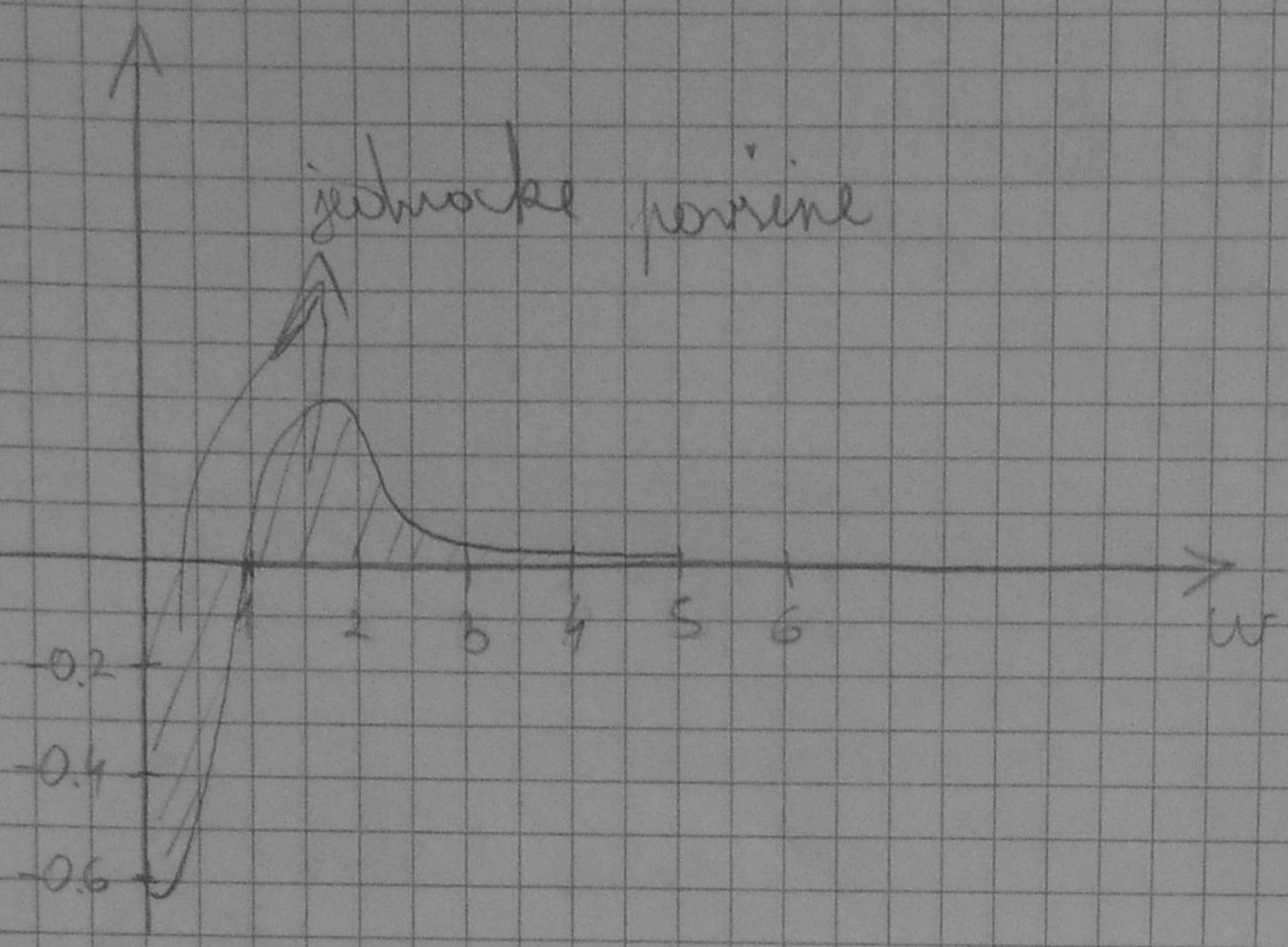
n_R → broj polova u desnoj poluravnini

$s_{R,Pi}$ → jesu ti polovi

↳ ako je $G_0(s)$ stabilno i red brojnika $\leq$ red nazivnika -2

$$G_0 = \frac{1}{s^2+s+1} \qquad \int_0^\infty \ln|S(j\omega)|\,d\omega = 0$$





⇒ ovo govori da ako mijenjamo osjetljivost na nekim frekvencijama, osjetljivost se mora povećati na nekim drugim
(WATER BED EFFECT)

⇒ ugl. se želi osigurati stabilnost na frekvencijama gdje djeluje poremećaj

Primjer: $G_o(s) = \dfrac{K}{s(s+1)}$ $\qquad S_K^{s_p} = \dfrac{K}{s_{pi}} \cdot \dfrac{\partial s_{pi}}{\partial K}$

K.J. $\Rightarrow s^2 + s + K = 0$

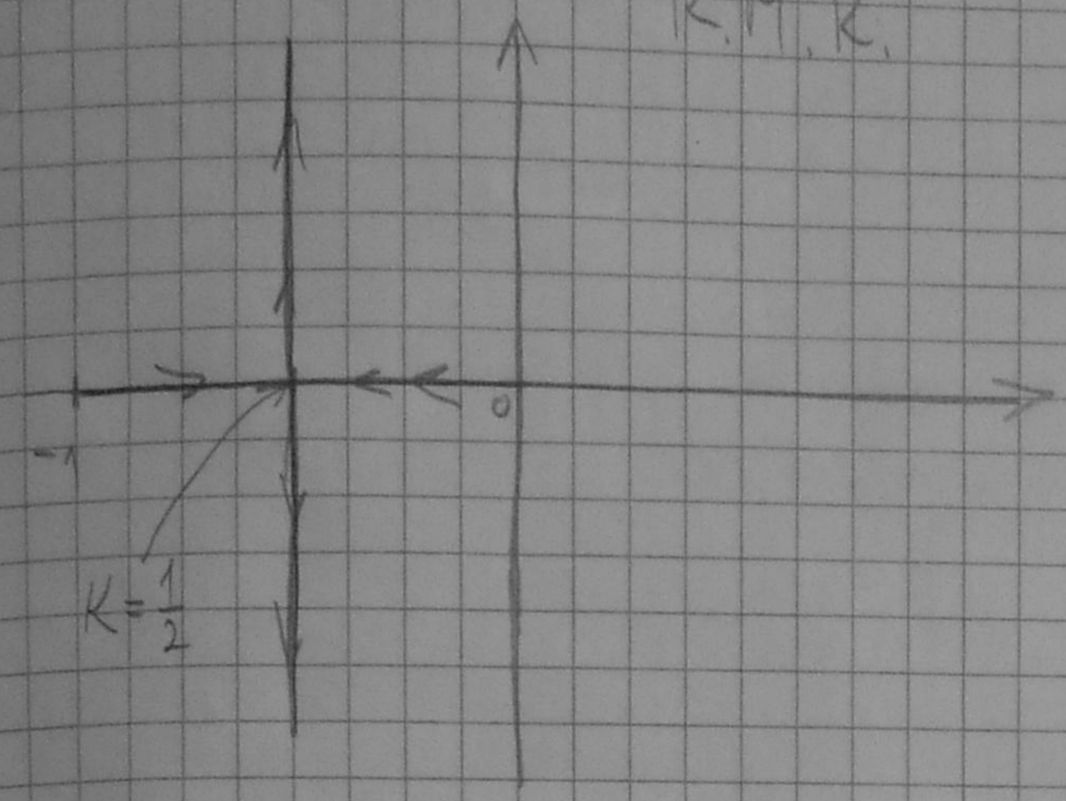
$$s_{p1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4K}$$

$$2s_{p1,2} + 1 = \pm\sqrt{1-4K}$$

$$\frac{\partial s_{pi}}{\partial K} = \mp \frac{1}{\sqrt{1-4K}} = \frac{-1}{2s_{pi}+1}$$

$$S_K^{s_p} = \frac{-s_{pi}(s_{pi}+1)}{s_{pi}} \cdot \frac{-1}{2s_{pi}+1} = \frac{s_{pi}+1}{2s_{pi}+1}$$

K.M.K.



1. $s_{pi} \in [0, \to -\frac{1}{2}]$, $s_{pi} \in \mathbb{R}$

$s_{pi} = \sigma + j0$

$$|S_K^{s_{pi}}| = \left|\frac{\sigma+1}{2\sigma+1}\right|$$

2. $s_{pi} = -\frac{1}{2} - j\omega$, $\omega \in (0, \infty)$

$$|S_K^{s_{pi}}| = \sqrt{\frac{\omega^2+0.25}{4\omega^2}}$$

1) $S_K^{s_{pi}} \to \infty$, $\sigma \to -\frac{1}{2}$

2) $S_K^{s_{pi}} \to \infty$, $\omega \to 0$

Za točke granica KMK vrijedi

$\frac{\partial K}{\partial s} = 0 \Rightarrow$ STACIONARNE TOČKE

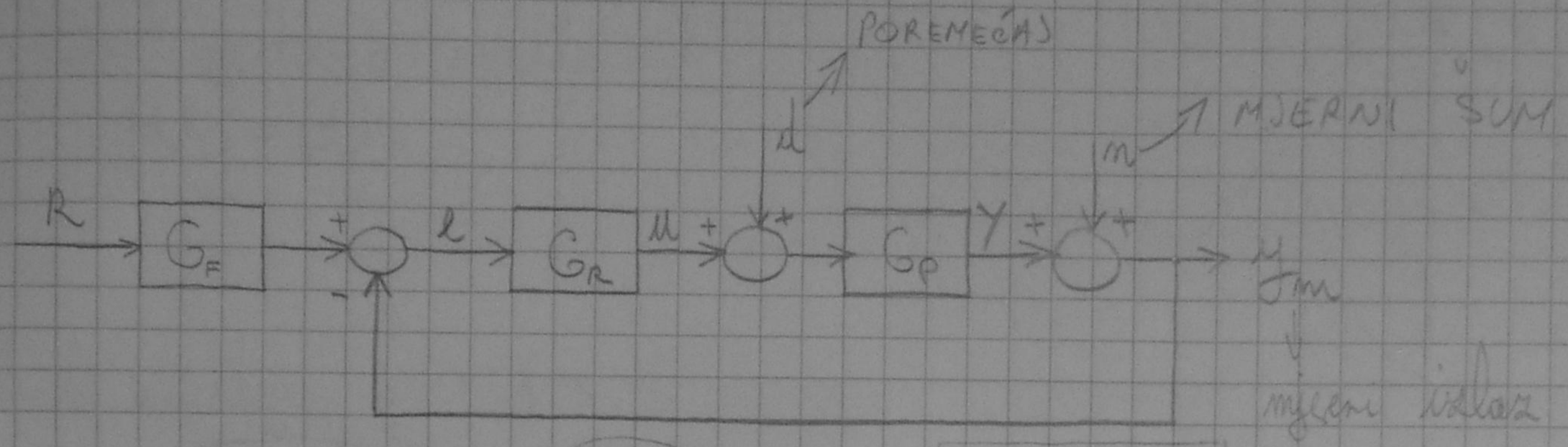
MAKSIMUM je točka granica

MINIMUM je -||- sjedinjenja

Primjer: KMK. $G_0(s) = \frac{k_0}{s(s^2+4s+5)}$ $k_0 > 0$

amplitudni uvjet $k_0 = -\frac{A(s)}{B(s)} = -(s^3+4s^2+5s)$

$\frac{\partial k_0}{\partial s} = 0 = 3s^2+8s+5$



$$Y = \frac{G_p}{1+G_pG_R}D - \frac{G_pG_R}{1+G_pG_R}\cdot N + \frac{G_pG_RG_F}{1+G_pG_R}\cdot R$$

$$U = -\frac{G_pG_R}{1+G_pG_R}D - \frac{G_R}{1+G_pG_R}N + \frac{G_RG_F}{1+G_pG_R}R$$

$$E = -\frac{G_p}{1+G_pG_R}D - \frac{1}{1+G_pG_R}N + \frac{G_F}{1+G_pG_R}R$$

ovo se ugl. ne analizira

$$Y_m = \frac{G_p}{1+G_pG_R}D + \frac{1}{1+G_pG_R}N + \frac{G_pG_RG_F}{1+G_pG_R}R$$

⇒ GANG OF FOUR → 4 najbitnije prijenosne funkcije

GANG OF SIX

Primjer:

$$G_p(s) = \frac{1}{(s+1)(s+0.02)}$$

$$G_R(s) = \frac{s+0.02}{s}$$

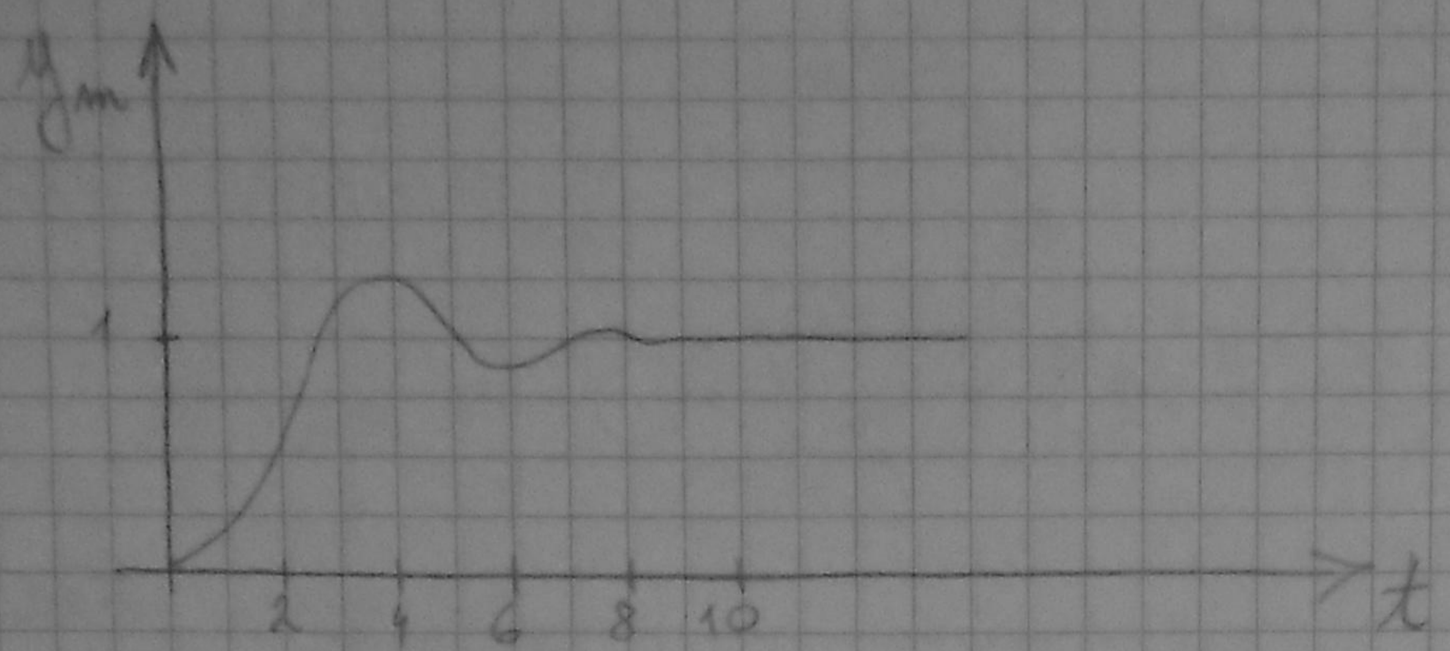
$$G_o(s) = \frac{1}{s(s+1)}$$

1) $$\frac{G_pG_R}{1+G_pG_R} = \frac{1}{s^2+s+1}$$

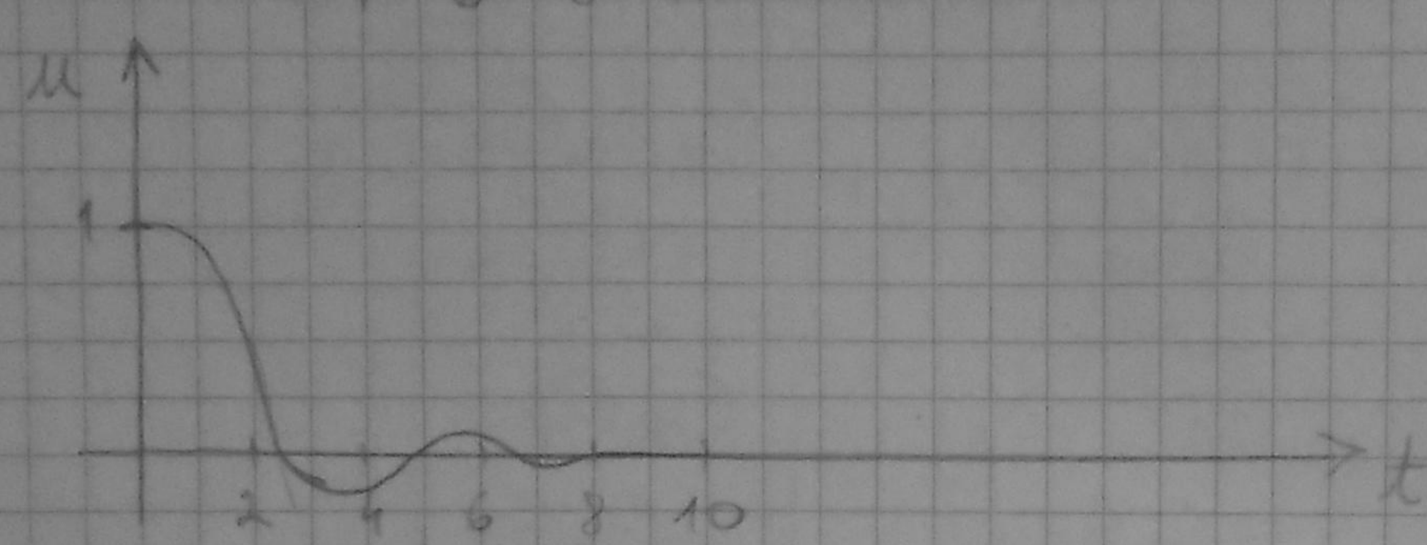
2) $$\frac{G_p}{1+G_pG_R} = \frac{s}{(s+0.02)(s^2+s+1)}$$

3) $$\frac{G_R}{1+G_pG_R} = \frac{(s+0.02)(s+1)}{s^2+s+1}$$

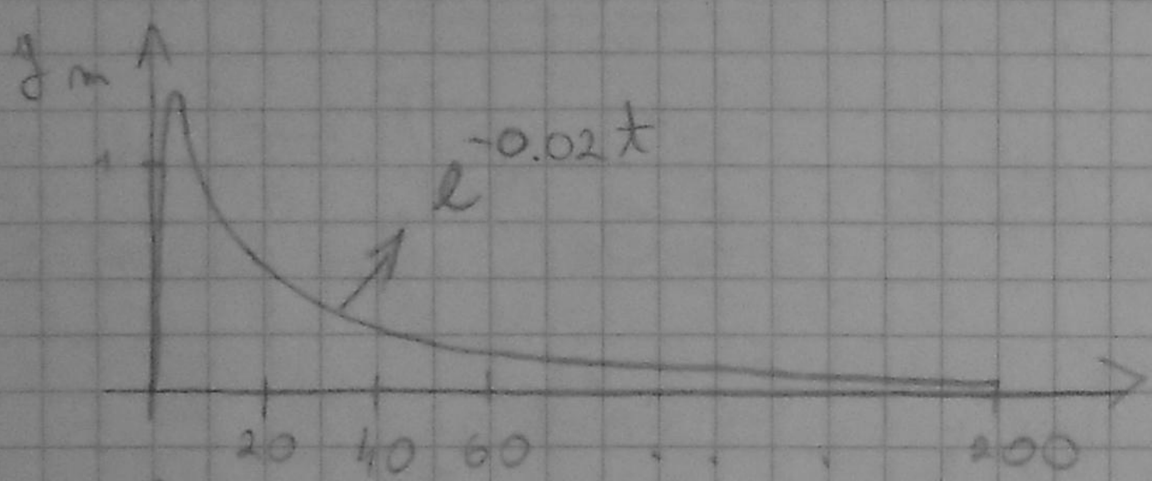
4) $$\frac{1}{1+G_pG_R} = \frac{s(s+1)}{s^2+s+1}$$



$r(t) = S(t)$

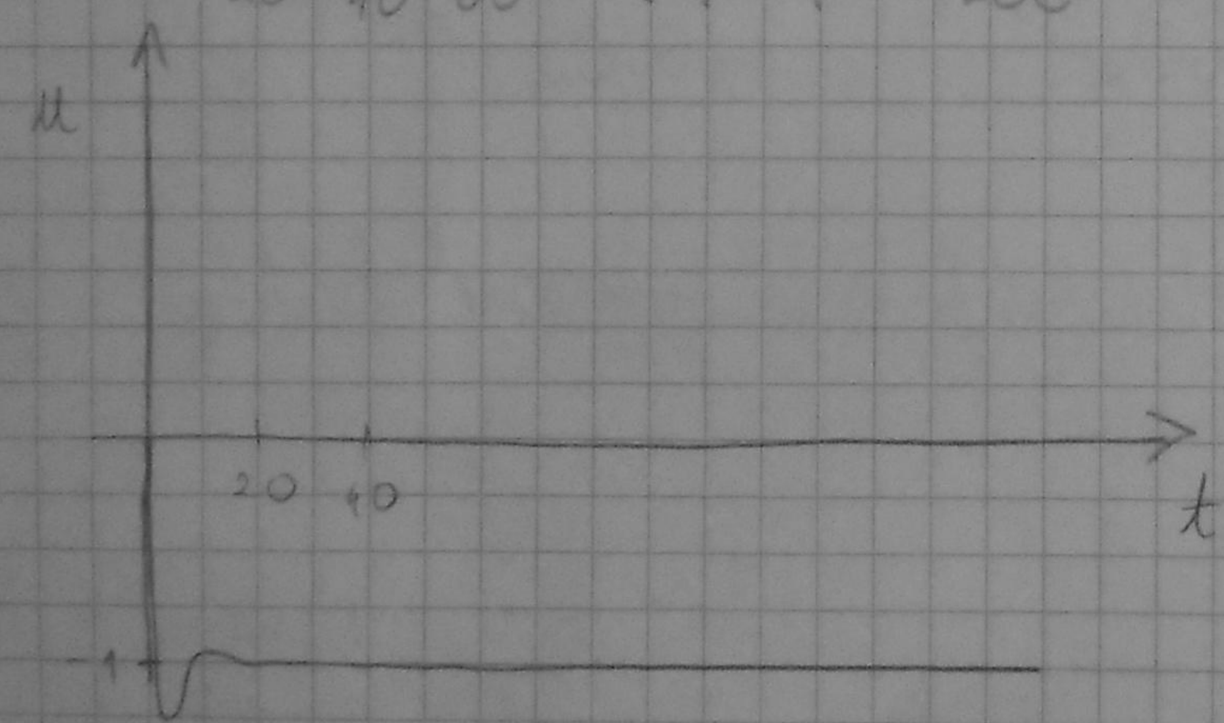


→ projektirano linearno s obzirom na referencu



$d(t) = S(t)$

→ projektirano linearno s obzirom na poremećaj



- jako spor odziv na poremećaj (loše)
→ također treba poraditi na filtriranje šuma

Komplementarna osjetljivost

$$T = \frac{G_0}{1+G_0} \quad , \quad S = \frac{1}{1+G_0}$$

$$S + T = 1$$

OPTIMALNO UPRAVLJANJE

podsjetnik:

$y(x) : R \to R^n$ => skalar u vektor

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1(x)}{\partial x} \\ \frac{\partial y_2(x)}{\partial x} \\ \vdots \\ \frac{\partial y_n(x)}{\partial x} \end{bmatrix} \qquad y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}$$

$y(x) : R^n \to R$

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} & \cdots & \frac{\partial y}{\partial x_n} \end{bmatrix} \qquad \nabla y = \left(\frac{\partial y}{\partial x}\right)^T$$

gradijent

$y(x) : R^n \to R^m$

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & & & \\ \vdots & & \ddots & \\ \frac{\partial y_m}{\partial x_1} & & & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

JACOBIAN MATRICA

$$y(x) = R^n \to R$$

$$\frac{\partial^2 y}{\partial x^2} = \begin{bmatrix} \frac{\partial^2 y}{\partial x_1^2} & \frac{\partial^2 y}{\partial x_1 x_2} & \cdots & \frac{\partial^2 y}{\partial x_1 \partial x_n} \\ \frac{\partial^2 y}{\partial x_2 \partial x_1} & \frac{\partial^2 y}{\partial x_2^2} & \cdots & \frac{\partial^2 y}{\partial x_2 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 y}{\partial x_n \partial x_1} & & \cdots & \frac{\partial^2 y}{\partial x_n^2} \end{bmatrix}$$

HESSIAN MATRICA
(HESSOVA)

- ova matrica je simetrična za sve kontinuirane funkcije

Primjer: $x \in R^n$

1) $f(x) = x$ $\quad \frac{\partial f}{\partial x} = 1$ $\quad \frac{\partial f(x)}{\partial x} = I_n$

2) $g(x) = x^T \cdot a \qquad x \in R^n, \; a^T \in R^n$

skalarna fja. zbog skalarnog umnožka vektora

$$\frac{\partial g(x)}{\partial x} = \frac{\partial (x^T \cdot a)}{\partial x} = \frac{\partial (a^T x)}{\partial x} = a^T \cdot I_n = a^T$$

3) $h(x) = x^T A \quad , A \in R^{n \times m} \quad , x \in R^n \quad h(x) \in R^{1 \times m}$

$$\frac{\partial h}{\partial x} = \frac{\partial (x^T A)}{\partial x} = \frac{\partial}{\partial x}(A^T x) = A^T \cdot I_n = A^T$$

koristi se isti izraz iako je $h(x)$ RETČANI VEKTOR!!

NEKONZISTENTNO!!!

KVADRATNA FORMA

def. pozitivno semidefinitna funkcija:

$f(x) \geq 0, \forall x \in R^n$, $f(x): R^n \to R$

$f(0_n) = 0$

def: pozitivno definitna funkcija

$f(x) > 0$, $\forall x \notin 0_n$

$f(0_n) = 0$

→ funkcija je negativno semidefinitna ako je

- $f(x)$ pozitivno ~~se~~definitna

→ isto vrijedi za negativno definitnu

def. KVADRATNA FORMA

- Neka je $A \in R^{m \times m}$ onda je skalarna fja.:

$f(x) = x^T A x$ kvadratna forma

i vrijedi : $f(x) = x^T A x = \frac{1}{2} x^T (A + A^T) \cdot x$

ova matrica je simetrična iako A možda nije

- zato x A može pretpostaviti da je simetrična

$f(x) = x^T \cdot A \cdot x \Rightarrow$ kad je pozitivno definitna

$\Rightarrow$ matrica je pozitivno definitna ako je njena kvadratna forma pozitivno definitna

SIMETRIČNA MATRICA $A \in R^{n \times n}$ je pozitivno definitna ako:

1. $A > 0 \Leftrightarrow x^T A x > 0$
2. sve svojstvene vrijednosti $\lambda_i > 0$
3. SYLVESTEROV KRITERIJ
   - sve glavne minore su pozitivne

1. minora
2. minora

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & \cdots & & \\ \vdots & & \ddots & \\ A_{n1} & & & A_{nn} \end{bmatrix}$$

$A \in R^{m \times n} \rightarrow$ ne pozitivna

$A \cdot A^T \in R^{m \times m}$, AA^T je simetrična jer $(AA^T)^T = AA^T$

$A \cdot A^T > 0$

DOKAZ: $x \in R^m$

$x^T (AA^T) \cdot x \geq 0$

$x^T A = y$

$y \cdot y^T = |y|^2 \geq 0$

$A^T A \geq 0$ isto vrijedi

neka je $B = B^T \geq 0$ $\quad B \in R^{n \times n}$

onda:

$$ABA^T \geq 0$$

neka je $A^T \in R^{n \times n}$ i $B = B^T \in R^{n \times n}$

ako je $(A \geq 0 \wedge B > 0) \Rightarrow (A + B \geq 0)$

$(A > 0 \wedge B \geq 0) \Rightarrow (A + B > 0)$

Primjer:

$h(x) = x^T A x$, $A \in R^{n \times n}$

$$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}(x^T A x)$$

$$= \frac{\partial}{\partial x}[x^T \cdot Ax] + \frac{\partial}{\partial x}(x^T \cdot Ax)$$

$$= x^T A^T \cdot I_n + x^T A I_n$$

$$= x^T (A^T + A)$$

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x}\left[x^T (A^T + A)\right]$$

$$= (A^T + A)^T \cdot I_n$$

$$= A + A^T$$

ako vrijedi $f(x) = \frac{1}{2} x^T A x$

$\hookrightarrow$ SIMETRIČNA

$$\frac{\partial^2 h}{\partial x^2} = A$$

Vremenski diskretni linearni kvadratni regulator s konačnim horizontom

$$J(x(0), x(1), \dots, x(N), u(0), u(1), \dots u(N-1)) =$$

KONAČNI HORIZONT

$$x(k+1) = Ax(k) + Bu(k)$$

$$= \frac{1}{2} \cdot x^T(N) \cdot P \cdot x(N) \Rightarrow \text{kvadratna forma simetrična } P$$

$$+ \frac{1}{2} \sum_{k=0}^{N-1} \left[x^T(k) \cdot Q \cdot x(k) + u^T(k) R\, u(k) \right] \qquad P \geq 0$$

uz $x(0) = x_0$

$Q = Q^T \geq 0 \qquad R = R^T > 0 \qquad , x \in R^n, \; u \in R^m$

- funkcija realizacije: pokazuje TRAJEKTORIJU odstupanja
- treba je minimizirati za optimalno upravljanje:

$$J^*(x_0) = \min_{u_0, u_1, \dots u_{N-1}} \left\{ \frac{1}{2} x_N^T P x_N + \frac{1}{2} \sum_{k=0}^{N-1} \left[x_k^T Q x_k + u_k^T R u_k \right] \right\}$$

realizacije odstupanja

uz uvjet: $x_{k+1} = Ax_k + Bu_k \quad , k = 0, 1, \dots N-1$

(DLQR) problem
(discrete linear quadratic regulator)

rješenje:

$u_k^* = -G_k \cdot x_k^* \quad , k = 0, \dots, N-1 \Rightarrow$ upravljački signal je ovog oblika,

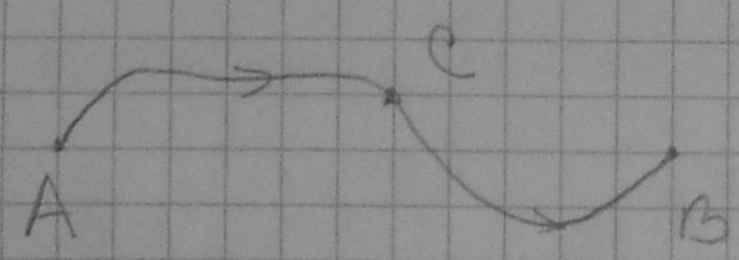
$x_{k+1}^* = Ax_k^* + Bu_k^* \qquad x_0^* = x_0 \qquad G_k$ je vremenski promjenjiv

DINAMIČKO PROGRAMIRANJE:

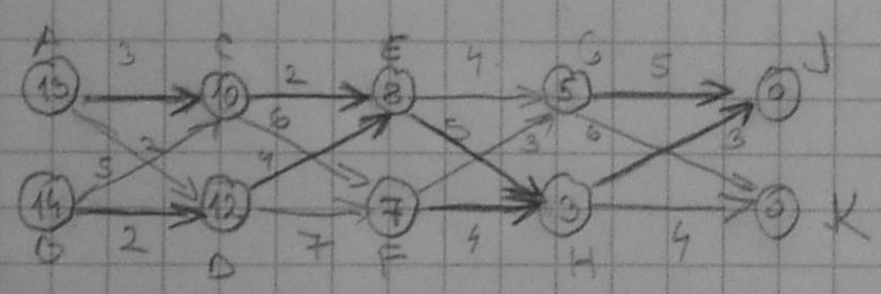
(Bellmanovo načelo optimalnosti)

- optimalna strategija ima svojstvo da neovisno o početnom stanju prostule odluke moraju tvoriti optimalnu strategiju s obzirom na stanje do kojeg se došlo početnim odlukama

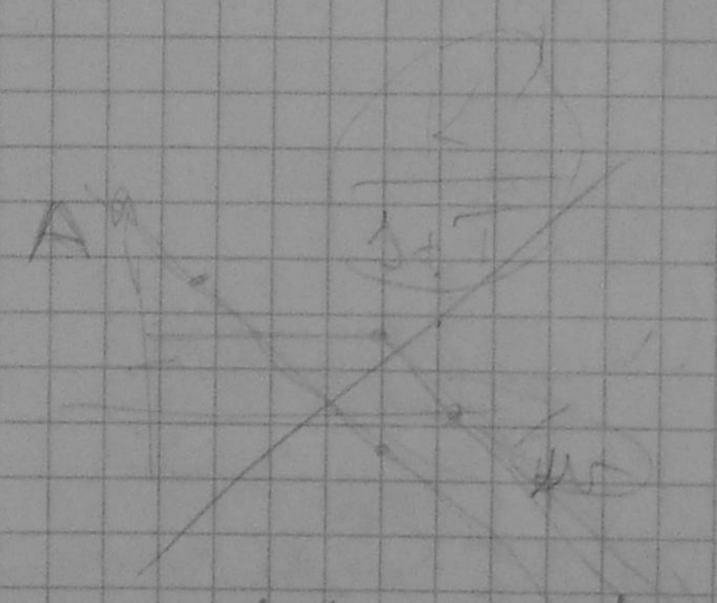
Pr.) traži se optimalni put A – B



- ako kreneš iz C optimalni put je dijelak puta A – B

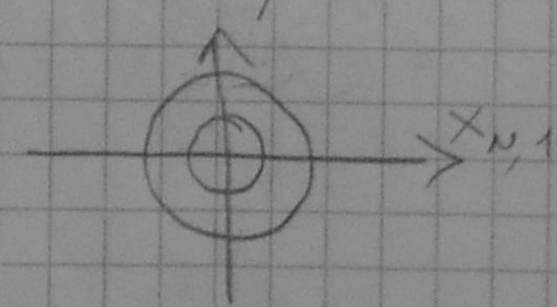


broj operacija $(N-1)M$



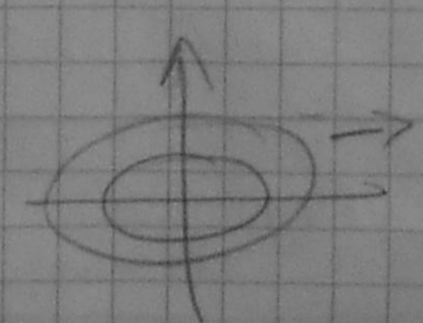
- ali: ponaci mreže čvorovima moraju biti pozitivni

$x^T P x = f(x_N)$, $P = I$ =>



NIVO KRIVULJE $f(x_N)$ KRUŽNICE

$x^T P x = f(x_N)$, $P = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$



ELIPSE

općenito su rotirane elipsoide

Primjena Bellmanovog načela

$$J_i^*(x_i) = \min_{u_i, u_{i+1}, \dots, u_{N-1}} \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} \left[x_k^T Q x_k + u_k^T R u_k \right]$$

$$x_{k+1} = A x_k + B u_k$$

$$i \in \{0, \dots, N-1\}$$

ovo nije fja od u_i

$$J_i^*(x_i) = \min_{u_i} J_{i+1}^*(x_{i+1}) + \frac{1}{2} x_i^T Q x_i + \frac{1}{2} u_i^T R u_i$$

$$x_{i+1} = A x_i + B u_i$$

$$J_i^*(x_i) = \frac{1}{2} x_i^T Q x_i + \min_{u_i} \frac{1}{2} u_i^T R u_i + J_{i+1}^*(A x_i + B u_i)$$

$$J_N^*(x_N) = \frac{1}{2} x_N^T S_N x_N$$

PREPOSTAVKA:

$$J_i^*(x_i) = \frac{1}{2} x_i^T S_i x_i \quad , \quad S_i = S_i^T \geq 0 \quad i = N, \dots, 0$$

$$J_i^*(x_i) = \frac{1}{2} x_i^T Q x_i + \min_{u_i} \frac{1}{2} u_i^T R u_i + \frac{1}{2} (A x_i + B u_i)^T S_{i+1} (A x_i + B u_i)$$

$$= \frac{1}{2} x_i^T (Q x_i + A^T S_{i+1} A) x_i + \min_{u_i} \underbrace{\frac{1}{2} u_i^T \cdot (R + B^T S_{i+1} B) u_i + x_i^T \cdot A^T \cdot S_{i+1} \cdot B \cdot u_i}_{f(u_i) : R^m \to R}$$

POZITIVNO DEFINITNA pa postoji inverz

$$\frac{\partial f(u)}{\partial u_i} = u_i^T \overbrace{(R + B^T S_{i+1} B)} + x_i^T A^T S_{i+1} \cdot B = 0_{1 \times m} \Rightarrow$$

TRAŽENJE EKSTREMA FUNKCIJE

$$\boxed{u_i^* = -(R + B^T S_{i+1} B)^{-1} B^T S_{i+1} A x_i} = -G_i x_i$$

$$J_i^*(x_i) = \frac{1}{2} x_i^T S_i x^T$$

$$\boxed{S_i = Q + A^T S_{i+1} A - A^T S_{i+1} B (R + B^T S_{i+1} B)^{-1} B^T S_{i+1} A}$$

RICCATIJEVA JEDNADŽBA

Postupak rješavanja DLQR:

1) - poznato: $A, B, R, P, Q, N, x(0)$

2) riješi Riccatijevu jednadžbu unazad u vremenu
$$S_N = P$$

3) odredi G_k za $k = 0, 1, \dots, N-1$

4) $x(k+1) = Ax(k) + Bu(k)$ $\qquad u(k) = -G_k x(k)$

$$J_i(x_i) = \frac{1}{2} x_i^T Q x_i + \frac{1}{2}(-G_i x_i)^T R(-G_i x_i) + \frac{1}{2} x_i^T (A - BG_i)^T S_{i+1} (A - BG_i) x_i$$

$$= \frac{1}{2} x_i^T \underbrace{\left[Q + G_i^T R G_i + (A - BG_i)^T S_{i+1} (A - BG_i)\right]}_{S_i \geq 0} x_i$$

JOSEPHOVA FORMA

Lema o inverziji matrice:

$A_{11} \in R^{n \times n}, \ \exists A_{11}^{-1}$

$A_{12} \in R^{n \times m}$

$A_{21} \in R^{m \times n}$

$A_{22} \in R^{m \times m}, \ \exists A_{22}^{-1}$

$$(A_{11}^{-1} + A_{12} A_{22} A_{21})^{-1} =$$

$$= A_{11} - A_{11} A_{12} (A_{21} A_{11} A_{12} + A_{22}^{-1})^{-1} A_{21} A_{11}$$

ako je A_{22} malih dimenzije ovo je puno lakše IZRAČUNAT

4 DRUGI ZAPIS RICCATIJEVE JEDNADŽBE:

$$S_i = Q + A^T S_{i+1} (I_n + B R^{-1} B^T S_{i+1})^{-1} A$$

selekcijom: $I_n = A_{11}$, $B = A_{12}$, $R^{-1} = A_{22}$

$B^T S_{i+1} = A_{21}$

se prevodi u ekvivalentni zapis

Brysonov postupak

1) Specificirani zahtjevi:

$|x_i(k)|_{MAX.\,SPEC}$:= maksimalna dopuštena vrijednost $i = 1, \dots n$ i-te komponente u trenutku $k = 0, \dots N-1$

$|x_i(N)|_{MAX\,SPEC}$:= -||- i-te komponente u $k = N$

$|u_i(k)|_{MAX.\,SPEC}$:= -||- i-te komponente za trenutak $k = 0, \dots N-1$

2) INICIJALIZACIJA (izbor R, P, Q)

$$P = \mathrm{diag}([P_{11} \dots P_{nn}])$$

$$= \begin{bmatrix} P_{11} & 0 & \dots & 0 \\ 0 & P_{12} & \dots & 0 \\ \vdots & & & \\ 0 & \dots & & P_{nn} \end{bmatrix} \qquad P_{ii} = \frac{1}{|x_{N,i}|^2_{MAX.\,SPEC.}}$$

$$R = \mathrm{diag}([R_{11} \dots R_{mm}]) \qquad r_{ii} = \frac{1}{|u_{k,i}|^2_{\substack{MAX.\\SPEC.}}}$$

$$Q = \mathrm{diag}([Q_{11} \dots Q_{nn}]) \qquad q_{ii} = \frac{1}{|x_{k,i}|^2_{\substack{MAX.\\SPEC}}}$$

3. Ponavljati do zadovoljenja specifikacija s trenutnim vrijednostima P, Q, R

DOBIJE SE $\{u_0, \dots, u_{N-1}\}$ $\{x_0, \dots, x_N\}$

ODREDI SE $|x_i(k)|_{\substack{MAX.\\STVARNO}}$ $|u_i(k)|_{\substack{MAX.\\STVARNO}}$ $|x_i(N)|_{\substack{MAX.\\STVARNO}}$

PROVJERI ZADOVOLJAVAJU LI SVI SPECIFIKACIJE

AKO $|\ |_{\substack{MAX\\STVARNO}} \leq |\ |_{\substack{MAX\\SPEC.}}$ KRAJ

INAČE:

4. $$p_{ii} := r_{ii} \cdot \left(\frac{|x_{N,i}|_{\substack{MAX.\\STVARNO}}}{|x_{N,i}|_{\substack{MAX\\SPEC.}}} \right)^2 \qquad i = 1, \dots n$$

isto za q_{ii} i r_{ii} elemente

Primjer: $x_{k+1} = 2x_k + u_k$

$A = 2 \quad B = 1$

naći $J^*(x_0) = \frac{1}{2} x_4^2 \cdot 100 + \frac{1}{2}\sum_{k=0}^{3}(x_k^2 \cdot 20 - u_k^2 \cdot 2)$

$x_0 = x(0) = 100$

$P = 100 \quad Q = 20 \quad R = 2$

$N = 4$

$S_N = S_4 = P = 100$

$S_k = \ldots$ (RICCATI)

$G_k = (B^T S_{k+1} B + R)^{-1} B^T S_{k+1} A$

$S_k = 4 S_{k+1} - \frac{4 S_{k+1}^2}{S_{k+1} + 2} + 20$

$u_k = -G_k x_k$

$S_3 = 400 - \frac{40000}{102} + 20$

COST TO GO

$J_k^*(x_k) = \frac{1}{2} x_k^T S_k x_k$

x	S_k	G_k	u_k	x_k	$J_k^*(x_k)$
4	100	/	/	$9.7 \cdot 10^{-3}$	$4.69 \cdot 10^{-3}$
3	27.843	1.9608	-0.4845	0.2471	$8.5 \cdot 10^{-1}$
2	27.464	1.866	-3.4399	1.8435	$4.67 \cdot 10^{1}$
1	27.457	1.8642	-25.315	13.579	$2.53 \cdot 10^{3}$
0	27.457	1.8642	-186.42	100	$1.37 \cdot 10^{5}$

vrijedi u stacionarnom stanju

- upravljačka akcija u_k su najsnažnije na početku x_0
- ona se upravljanje u otvorenoj petlji jer je x_k procjenjuje modela, a nije mjeren
- kad se x_k mjeri onda je upr. u zat. petlji

ako želimo koristiti rezultate ove analize nakon $k=4$ najbolje je uzeti $S(0)$ jer on gleda najdalje u budućnost: $u_k^* = -G_\infty x_k$

IDEALNI SLUČAJ

APROKSIM. IDEAL. SLUČAJ

Vremenski kontinuirani linearni kvadratni regulator s konačnim horizontom

$$J(X_0) = \min_{u(\cdot)} \left\{ \frac{1}{2} x^T(T) S_T x(T) + \frac{1}{2} \int_0^T \left[x^T(t) Q x(T) + u^T(t) R u(t) \right] d\tau \right\}$$

uz uvjet: $\dot{x}(t) = A x(t) + B u(t)$

$x(0) = x_0$

$S_T = S_T^T \geq 0$ $\qquad x \in R^n$

$Q = Q^T \geq 0$

$R = R^T > 0$

funkcija Hamiltonijan:

$H := H(x, u, p)$

→ dualne varijable (Lagrangeovi multiplikatori)

→ pomoćne varijable stanja

$p \in R^n$

$$H := \frac{1}{2} x^T(T) Q x(T) + \frac{1}{2} u^T(t) R u(t) + p^T \left[A x(t) + B u(t) \right]$$

$$\left.\begin{aligned} \dot{p}(t) &= -\left(\frac{\partial H}{\partial x}\right)^T \\ \left(\frac{\partial H}{\partial u}\right)^T &= 0_m \\ \frac{\partial \dot{x}(t)}{\partial t} &= \left(\frac{\partial H}{\partial p}\right)^T \end{aligned}\right\} \text{uvjeti za optimalno rješenje}$$

$= A x(t) + B u(t)$

$$\dot{p}(t) = -\left(\frac{\partial H}{\partial x}\right)^T = -Q\,x(t) - A^T p(t) \quad \text{uz} \quad p(T) = S_T\,x(T)$$

$$\left(\frac{\partial H}{\partial u}\right)^T = 0_m \Rightarrow u(t) = -R^{-1}B^T p(t)$$

$$\dot{x}(t) = \left(\frac{\partial H}{\partial p}\right)^T = Ax(t) + Bu(t)$$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \Rightarrow$$ sustav koji treba riješiti uz uvjete $x(0) = x_0$, $p(T) = S_T\,x(T)$

$\Rightarrow$ uz pretpostavku: $p(t) = S(t)x(t)$ može se riješiti diferencijalna jednadžba:

$$-\dot{S}(t) - A^T S(t) + S(t)A - S(t)BR^{-1}B^T S(t) + Q$$

$$\text{uz} \quad S(T) = S_T$$

$$u(t) = -R^{-1}B^T p(t) = -R^{-1}B^T S(t)\cdot x(t) = -G(t)\cdot x(t)$$

$$S(t) = S^T(t)$$

$$S(t) \geq 0$$

DLQR s konačnim horizontom P

$$J^*(x_0) = \min_{u_0,\dots,u_{N-1}} \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=0}^{N-1} \left[x_k^T Q x_k + u_k^T R u_k \right]$$

$$x_{k+1} = A x_k + B u_k \qquad Q = Q^T \geq 0$$

$$S_N = S_N^T \geq 0$$

$$R = R^T > 0$$

$$u_k = -(R + B^T S_{N+1} B)^{-1} B^T S_{N+1} A \cdot x_k = -G_k x_k$$

$$S_k = Q + A^T S_{k+1} A - A^T S_{k+1} B (R + B^T S_{k+1} B)^{-1} B^T S_{k+1} A$$

$$k = N-1, \dots, 0$$

$$x(k+1) = a x(k) + b u(k)$$

$$\min x^2(N) \cdot p + \sum_{k=0}^{N-1} x^2(k) \cdot q + u^2(k) \cdot r$$

Primjer

$A = 2, \quad B = 1 \quad x(0) = 100$

1) $Q = 20 \quad R = 2 \quad S_N = 100 \quad N = 4 \rightarrow$ POGLEDAJ RANIJE

2) $Q = 2 \quad R = 20$ -||- -||-

ovo je bolje jer je $x(4) \approx 0$ i bavili smo se ishodištu

2)

k	S_k	G_k	u_k	x_k	J_k
4	100			3.455	$5.97 \cdot 10^2$
3	68.667	1.6667	-17.276	10.366	$3.69 \cdot 10^3$
2	63.955	1.5489	-35.589	22.977	$1.63 \cdot 10^4$
1	62.942	1.5236	-73.475	48.226	$7.32 \cdot 10^4$
0	62.709	1.5177	-151.77	100	$3.14 \cdot 10^5$

DLQR s beskonačnim horizontom

$$J_\infty^*(x_0) = \lim_{N\to\infty} \min_{u_0 \dots u_{N-1}} \frac{1}{2} \sum_{k=0}^{N-1} \left[x_k^T Q x_k + u_k^T R u_k \right]$$

$$u_k = -G_\infty \cdot x_k \quad , \quad G_0 = G_1 = \dots = G_\infty$$

PRETPOSTAVKA

$$\boxed{S_\infty = Q + A^T S_\infty A - A^T S_\infty B (R + B^T S_\infty B)^{-1} B^T S_\infty A}$$

DISKRETNA ALGEBARSKA RIKCATIJEVA JEDNADŽBA (DARE)

$$S_\infty = S_\infty^T \geq 0$$

$$S_\infty \in R^{n \times n}$$

TM: Neka je diskretni sustav takav da je par (A, B) ustabilan, a (A, Q) detektabilan onda vrijedi da DARE ima

- jedinstveno $S_\infty \geq 0$ i
- uz $u_k = -G_\infty x_k$ zatvoreni krug je asimptotski stabilan

je

$$J_\infty^*(x_0) = \frac{1}{2} x_0^T S_\infty x_0$$

CLQR s beskonačnim horizontom

$$\min_u \frac{1}{2}\int_0^\infty \left[x^T(t)\,Q\,x(t) + u^T(t)\,R\,u(t)\right] dt = J(x_0)$$

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ x(0) &= x_0 \end{aligned}\right\} \text{uvjet}$$

$Q = Q^T \geq 0$
$R = R^T > 0$

$$\boxed{u(t) = -Gx(t) = -R^{-1}B^T S\, x(t)}$$ RJEŠENJE

$S \in R^{n\times n}$
$S = S^T \geq 0$

$$\boxed{A^T S + SA^T - SBR^{-1}B^T S + Q = 0_{n\times n}}$$

KONTINUIRANA RICATTIJEVA JEDNADŽBA (CARE)

TM: ako je CLTI sustav takav da je par (A,B) ustabilniv i (A,Q) osmotriv onda:

- CARE ima jedinstveno rješenje $S = S^T \geq 0$
- uz $u(t) = -Gx(t)$ zatvoreni krug je asimptotski stabilan

$$\dot{x} = Ax + Bu = \underbrace{(A - BG)}x(t)$$

$Re\{\lambda_i\} < 0 \Rightarrow$ krug je asimptotski stabilan
pa svojstvene vrijednosti imaju negativan realni dio

Primjer:

$$x_{k+1} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} \cdot x_k + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u_k \qquad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad N = 150$$

SPECIFIKACIJE:

$$|x_k|_{max_spec} = \begin{bmatrix} 10.25 \\ 3 \end{bmatrix} \qquad |x_N|_{max_spec} = \begin{bmatrix} 10^{-3} \\ 10^{-3} \end{bmatrix}$$

$$|u_k|_{max_spec} = 7$$

a) varijanta - proizvoljom izbor početnih P, Q, R

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; i \; P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R = 1$$

• rješenje DLQR, $N = 150$

$$|x_N|_{max_stvarno} = \begin{bmatrix} 7 \cdot 10^{-5} \\ 4.9 \cdot 10^{-5} \end{bmatrix} \qquad |x_k|_{max_stvarno} = \begin{bmatrix} 10.1 \\ \boxed{4.08} \end{bmatrix}$$

$$|u_k|_{max_stvarno} = \boxed{10.857} \longrightarrow \text{NE ZADOVOLJAVA}$$

• UPDATE PO BRYSONU

- nakon 6 iteracija dobiju se dobri rezultati

b) varijanta - izbor P, Q, R po Brysonu

$$Q = \begin{bmatrix} 9.52 \cdot 10^{-3} & 0 \\ 0 & 0.111 \end{bmatrix} \qquad P = \begin{bmatrix} 10^6 & 0 \\ 0 & 10^6 \end{bmatrix} \qquad R = 2.04 \cdot 10^{-2}$$

• rješenje DLQR, $N = 150$

$$|x_k|_{max_stvarno} = \begin{bmatrix} 10.12 \\ 2.208 \end{bmatrix} \qquad |x_N|_{max_stvarno} = \begin{bmatrix} \boxed{1.92 \cdot 10^{-5}} \\ \boxed{1.92 \cdot 10^{-5}} \end{bmatrix}$$

$$|u_k|_{max_stvarno} = \boxed{8.35}$$

• UPDATE PO BRYSONU: - nakon 3 iteracije!!!

$$\Rightarrow |X_k|_{max_stvarno} = \begin{bmatrix} 10.13 \\ 2.8 \end{bmatrix}, \quad |X_N|_{max_stvarno} = \begin{bmatrix} 5.02 \cdot 10^{-7} \\ 5.02 \cdot 10^{-4} \end{bmatrix}$$

$$|u_k|_{max_stvarno} = 6.86$$

Linearno kvadratno slijeđenje reference kod DLTI sustava

DLQT $\Rightarrow$ discrete linear quadratic tracking

$$J = \min_{u_0 \dots u_{N-1}} \frac{1}{2} (y_N - r_N)^T P (y_N - r_N) + \frac{1}{2} \sum_{k=0}^{N-1} \left[(y_k - r_k)^T Q (y_k - r_k) + u_k^T R u_k \right]$$

uz uvjet $\begin{cases} x_{k+1} = A x_k + B u_k \\ y_k = C x_k \end{cases}$

$P = P^T \geq 0$

$R = R^T > 0$

$Q = Q^T \geq 0$

$r \Rightarrow$ referentni signal

TM: optimalni upravljački zakon glasi:

$$u_k = -G_k x_k + \check{G}_k \cdot v_{k+1}$$

$$k = 0, \dots, N-1$$

$v_k \in R^p$

$$G_k = \left[R + B^T S_{k+1} B\right]^{-1} B^T S_{k+1} A \qquad k = 0, \dots, N-1$$

$$G_k^v = \left[R + B^T S_{k+1} B\right]^{-1} B^T$$

$$V_k = C^T Q \cdot r_k + \left[A^T - A^T S_{k+1} B (R + B^T S_{k+1} B)^{-1} B^T\right] V_{k+1}$$

$$k = N-1, \dots, 0$$

$$V_N := C^T P r_N$$

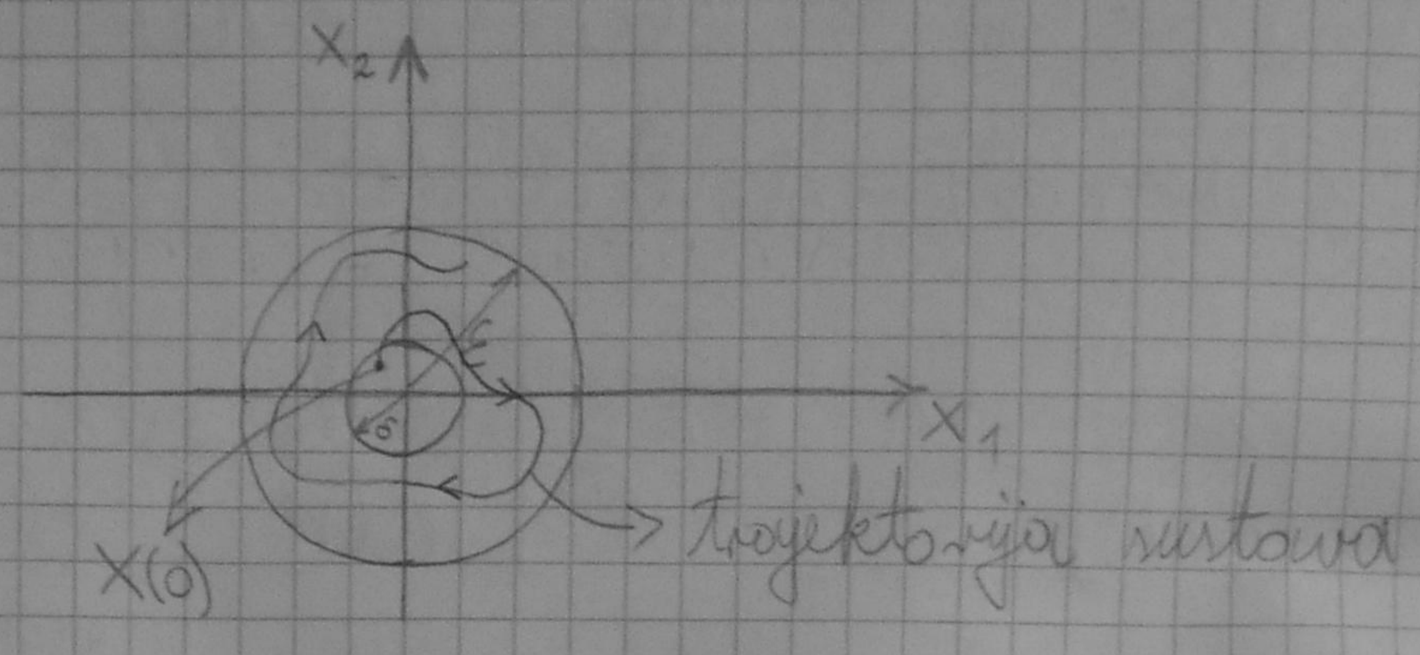
$$S_k = C^T Q C + A^T \left[S_{k+1} - S_{k+1} B (R + B^T S_{k+1} B)^{-1} B^T S_{k+1}\right] A$$

$$k = N-1, \dots, 0$$

$$S_N := C^T P \cdot C$$

Ljapunovljeva stabilnost vremenski kontinuiranih sustava

$\dot{x} = f(x(t))$



STABILNOST

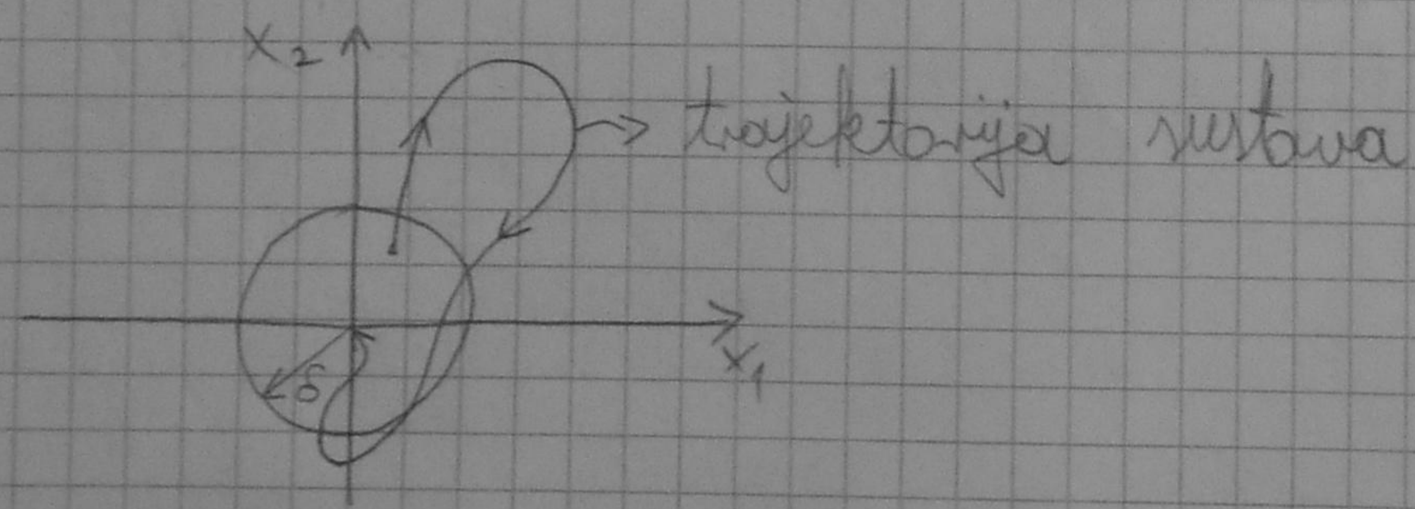
- točka $x=0$ je stabilna ako

$$(\forall \varepsilon > 0)(\exists \delta = \delta(\varepsilon) > 0)(\|x(0)\| < \delta \rightarrow \|x(t)\| < \varepsilon)$$
$$(\forall t \geq 0)$$

ASIMPTOTSKA STABILNOST

- točka $x=0$ je asimptotski stabilna ako je STABILNA i:

$$(\forall \varepsilon > 0)(\exists \delta = \delta(\varepsilon) > 0)(\|x(0)\| < \delta \rightarrow \lim_{t \to \infty} \|x(t)\| = 0)$$



Ljapunovljeva funkcija:

- $V(x) : R^n \rightarrow R$
- $V(x) > 0 \quad \forall x \neq 0_n, \quad V(0_n) = 0$ POZITIVNO DEFINITNA
- $\frac{d}{dt} V(x(t)) \leq 0 \quad \forall x$ NEGATIVNO SEMIDEFINITNA

- ako se za sustav može pronaći Ljapunovljeva fja, onda je $x=0$ stabilna točka

JAČI UVJET II.:

$$\frac{d}{dt}V(x(t)) < 0 \quad \forall x(t) \neq 0 \; ; \; \frac{d}{dt}V(x(t)) = 0 \quad x(t) = 0$$

ASIMPTOTSKA STABILNOST

$$\frac{d}{dt}V(x(t)) = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial V}{\partial x} f(x)$$

- za svaki stabilan sustav postoji Ljapunovljeva funkcija

Primjer: CLTI

$$\dot{x} = Ax(t) \quad , \quad V(x) = x^T P x \qquad P = P^T \in R^{n \times n}$$

- kakav mora biti P ako je $x=0$ asimptotski stabilno

$$V(x) = x^T P x > 0 \quad \forall x \neq 0 \quad \Rightarrow \boxed{P > 0}$$

$$\frac{d}{dt}V(x(t)) = \frac{d}{dt}(x^T P x) = \frac{dx^T}{dt} \cdot P \cdot x + x^T \frac{dPx}{dt} =$$

$$= x^T A^T \cdot P \cdot x + x^T P A x$$

$$= x^T (A^T P + PA) x < 0$$

$$\boxed{(A^T P + PA) < 0}$$

Teorem: $x = Ax(t)$ je asimptotski stabilan

ako postoje $P>0$, $Q>0$ takav da vrijedi

$$\boxed{A^T P + PA + Q = 0_{n \times n}}$$

LJAPUNOVLJEVA JEDNADŽBA

$P = lyap(A, Q)$ → MATLAB naredba

- za linearne sustave može se odabrati bilo koja pozitivno definitna matrica Q

- ovom jednadžbom može se dobiti najbolja inicijalizacija DLQR problema

Ljapunovljeva funkcija

$x(k+1) = f(x(k))$

$V(x): R^n \to R$

- $V(x) > 0 \quad \forall x \neq 0 \qquad V(0) = 0$
- $V(x(k+1)) - V(x(k)) \leq 0 \quad \forall x(k)$

} za ove uvjete $x=0$ je stabilan

- $V(x(k+1)) - V(x(k)) < 0 \quad \forall x(k) \neq 0$
  $V(x) = 0 \quad x = 0$

} ASIMPTOTSKI STABILNO

Primjer DLTI

$x(k+1) = Ax(k) \qquad V(x) = x^T P x$

- $V(x) > 0 \implies \boxed{P > 0}$
- $V(x(k+1)) - V(x(k)) = x^T(k) A^T P A x(k) - x^T(k) P x(k) < 0$

$$\boxed{A^T P A - P < 0}$$

Teorem: $x(k+1) = Ax(k)$ je asimptotski stabilan ako postoje $P > 0$, $Q > 0$ da vrijedi:

$$\boxed{A^T P A - P + Q = 0_{n \times n}}$$

$P = \text{dlyap}(A, Q)$