2.MI – 2006.

by Vedax/Rock n Rolla

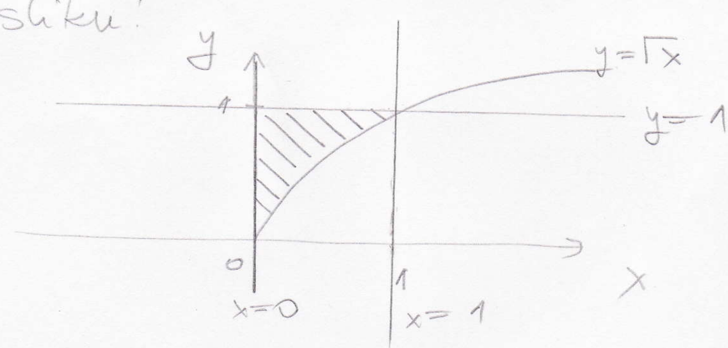
(1.) Skicirajte područje integracije, a zatim promijenite poredak integracije i izračunajte

$$\int_0^1 dx \int_{\sqrt{x}}^1 \sin\left(\frac{y^3+1}{2}\right) dy.$$

Rješenje:

U zadanom integralu su granice:
- za $x \rightarrow$ od 0 do 1 $\rightarrow$ $x_D = 0$, $x_G = 1$
- za $y \rightarrow$ od $\sqrt{x}$ do 1 $\rightarrow$ $y_D = \sqrt{x}$, $y_G = 1$

Nacrtajmo sliku:



Sada moramo najprije uzeti fiksne granice za y. Iz slike vidimo da y ide od 0 do 1.
Za x nam je donja granica 0 a gornja $y=\sqrt{x} \Rightarrow x=y^2$.
Naš integral prelazi u:

$$\int_0^1 dy \int_0^{y^2} \sin\left(\frac{y^3+1}{2}\right) dx = \int_0^1 \sin\left(\frac{y^3+1}{2}\right) dy \int_0^{y^2} dx =$$

$$= \int_0^1 \sin\left(\frac{y^3+1}{2}\right) \cdot \left(x\Big|_0^{y^2}\right) dy = \int_0^1 y^2 \sin\left(\frac{y^3+1}{2}\right) dy = \left| \begin{array}{l} y^3+1=t \,/\,' \\ 3y^2 dy = dt \\ t_D = 1 \\ t_G = 2 \end{array} \right| =$$

$$= \int_1^2 \sin\left(\frac{t}{2}\right) \cdot \frac{dt}{3} = \frac{1}{3}\int_1^2 \sin\left(\frac{t}{2}\right) dt = -\frac{2}{3}\cos\left(\frac{t}{2}\right)\Big|_1^2 =$$

$$= -\frac{2}{3}\left(\cos 1 - \cos\frac{1}{2}\right) = \boxed{\frac{2}{3}\left(\cos\frac{1}{2} - \cos 1\right)}$$

(2.) Izračunajte pomoću dvostrukog integrala površinu lika omeđenog krivuljama $x = 4y - y^2$ i $y = x$.

Rješenje:

Nacrtajmo najprije krivulje da bismo mogli odrediti granice dvostrukog integrala. Također odredimo gdje se sijeku ove dvije krivulje.

1) $x = 4y - y^2$

$y = x$

$y = 4y - y^2$

$y^2 - 3y = 0$

$y(y-3) = 0$

$$\boxed{\begin{array}{l} y_1 = 0 \Rightarrow x_1 = 0 \\ y_2 = 3 \Rightarrow x_2 = 3 \end{array}}$$

2) $y = x$

x	-1	0	1	2	3
y	-1	0	1	2	3

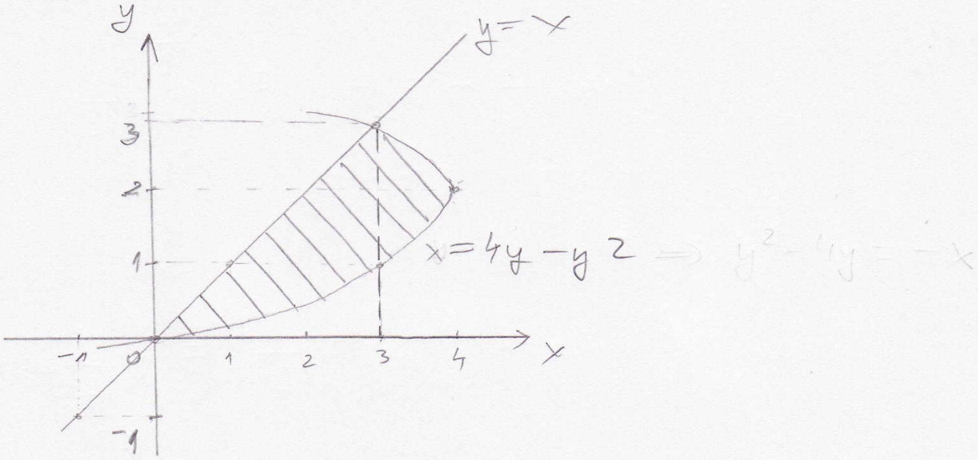
3) $x = 4y - y^2$

y	0	1	2	3
x	0	3	4	3

$x = 4y - y^2 \Rightarrow y^2 - 4y + x = 0$

$_1y_2 = 2 \pm \sqrt{4 - x}$

Slika:



Ovaj zadatak možemo riješiti na 2 načina!

1. način: Uzmemo fiksne granice po x.

Onda integral moramo podijeliti na 2 dijela. Prvi u kojem x ide od 0 do 3, a y od $2-\sqrt{4-x}$ do x, i drugi u kojemu x ide od 3 do 4, a y od $2-\sqrt{4-x}$ do $2+\sqrt{4-x}$.

$$P=\int_0^3 dx\int_{2-\sqrt{4-x}}^{x} dy+\int_3^4 dx\int_{2-\sqrt{4-x}}^{2+\sqrt{4-x}} dy=\int_0^3\left(x-2+\sqrt{4-x}\right)dx+$$

$$+\int_3^4 2\sqrt{4-x}\,dx=\int_0^3 x\,dx-2\int_0^3 dx+\underbrace{\int_0^3\sqrt{4-x}\,dx}_{I_1}+\underbrace{\int_3^4 2\sqrt{4-x}\,dx}_{I_2}$$

Uzimamo supstituciju: $4-x=t \Rightarrow -dx=dt \Rightarrow dx=-dt$.

Granice za t: $t_D=4$, $t_G=1 \Rightarrow$ za I_1

$t_D=1$, $t_G=0 \Rightarrow$ za I_2

$$P=\frac{x^2}{2}\Big|_0^3-2x\Big|_0^3-\int_4^1\sqrt{t}\,dt-2\int_1^0\sqrt{t}\,dt=\frac{9}{2}-6-\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\Big|_4^1-2\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\Big|_1^0=$$

$$=\frac{9}{2}-\frac{12}{2}-\frac{2}{3}\left(1-4^{\frac{3}{2}}\right)-\frac{4}{3}(0-1)=$$

$$=-\frac{3}{2}-\frac{2}{3}+\frac{16}{3}+\frac{4}{3}=\frac{18}{3}-\frac{3}{2}=\frac{36-9}{6}=\frac{\cancel{27}^{\,9}}{\cancel{6}_{\,2}}=\boxed{\frac{9}{2}}$$

2.način: (lakši način) Uzimamo fiksne granice za y.

Onda nam y ide od 0 do 3, a x nam ide od y do $4y-y^2$.

$$P = \int_0^3 dy \int_y^{4y-y^2} dx = \int_0^3 (4y-y^2-y)\,dy = 3\int_0^3 y\,dy - \int_0^3 y^2\,dy =$$

$$= 3\frac{y^2}{2}\Big|_0^3 - \frac{y^3}{3}\Big|_0^3 = \frac{27}{2} - \frac{27}{3} = \frac{81-54}{6} = \frac{\cancel{27}^{\,9}}{\cancel{6}_{\,2}} = \boxed{\frac{9}{2}}$$

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3. Prijelazom na polarne koordinate izračunajte integral

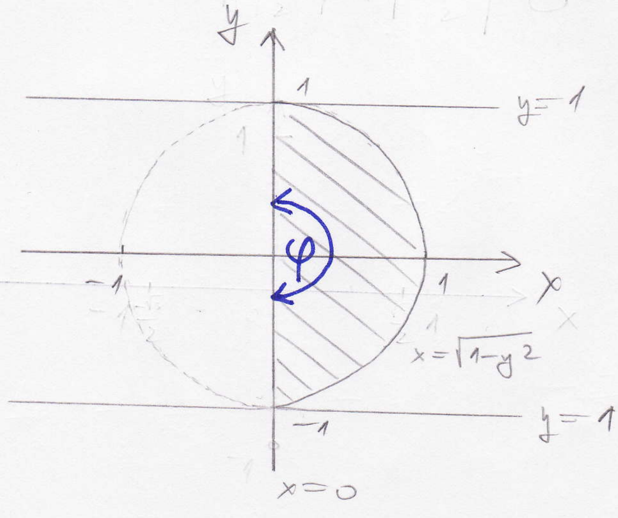
$$\int_{-1}^{1} dy \int_0^{\sqrt{1-y^2}} \sqrt{x^2+y^2}\,dx .$$

Rješenje:

Iz zadanoga integrala vidimo da nam y ide od -1 do 1, a x ide od 0 do $\sqrt{1-y^2}$. Nacrtajmo sliku.

1) $x=\sqrt{1-y^2} \Rightarrow x^2+y^2=1 \Rightarrow$ jedinična kružnica

"desni dio"



Prijelaz na polarne koordinate:

$x = r\cos\varphi$

$y = r\sin\varphi$

$J = r$

Iz slike vidimo da nam kut φ ide od $-\frac{\pi}{2}$ do $\frac{\pi}{2}$.
Radij-vektor r ide od o pa do polukružnice u kojoj poprima najveću vrijednost 1.

Podintegralna f-ja je: $\sqrt{x^2+y^2} = \sqrt{r^2\cos^2\varphi + r^2\sin^2\varphi} = r$

Pa je naš integral jednak:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^1 |J| \cdot r\,dr = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^1 r^2\,dr = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{r^3}{3}\Big|_0^1\right) d\varphi =$$

$$= \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi = \frac{1}{3}\,\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{3}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \boxed{\frac{\pi}{3}}$$

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4. Iskažite i dokažite teorem srednje vrijednosti integralnog računa za dvostruki integral.

<u>Rješenje:</u> Pogledajte u knjižici! :D

(5.) a) Definirajte Jacobijan transformacije

$x = x(u,v)$ i $y = y(u,v)$.

b) Izračunajte Jacobijan transformacije

$$x = \sqrt{\frac{u+v}{2}}, \quad y = \sqrt{\frac{u-v}{2}}$$

c) Izračunajte integral $\iint_D xy\,dx\,dy$, ako je D područje u prvom kvadrantu $(x \geq 0, y \geq 0)$ omeđeno krivuljama

$$x^2+y^2=4, \quad x^2+y^2=9, \quad x^2-y^2=1, \quad x^2-y^2=4$$

Rješenje:

a) $$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

b) $$\frac{\partial x}{\partial u} = \frac{1}{2\sqrt{\frac{u+v}{2}}} \cdot \frac{1}{2} = \frac{\sqrt{2}}{4\sqrt{u+v}}$$

$$\frac{\partial x}{\partial v} = \frac{1}{2\sqrt{\frac{u+v}{2}}} \cdot \frac{1}{2} = \frac{\sqrt{2}}{4\sqrt{u+v}}$$

$$\frac{\partial y}{\partial u} = \frac{1}{2\sqrt{\frac{u-v}{2}}} \cdot \frac{1}{2} = \frac{\sqrt{2}}{4\sqrt{u-v}}$$

$$\frac{\partial y}{\partial v} = \frac{1}{2\sqrt{\frac{u-v}{2}}} \cdot \left(-\frac{1}{2}\right) = -\frac{\sqrt{2}}{4\sqrt{u-v}}$$

$$J = \begin{vmatrix} \frac{\sqrt{2}}{4\sqrt{u+v}} & \frac{\sqrt{2}}{4\sqrt{u+v}} \\ \frac{\sqrt{2}}{4\sqrt{u-v}} & -\frac{\sqrt{2}}{4\sqrt{u-v}} \end{vmatrix} = -\frac{\cancel{2}^{1}}{\cancel{16}_{8}\sqrt{u^2-v^2}} - \frac{\cancel{2}^{1}}{\cancel{16}_{8}\sqrt{u^2-v^2}} =$$

$$= -\frac{\cancel{2}^{1}}{\cancel{8}_{4}\sqrt{u^2-v^2}} = \boxed{-\frac{1}{4\sqrt{u^2-v^2}}}$$

c) Prijeđemo na zamjenu $x=\sqrt{\frac{u+v}{2}}$ i $y=\sqrt{\frac{u-v}{2}}$.

Granice odredimo iz zadanih jednadžbi (za u i v):

$$x^2 = \frac{u+v}{2} \quad i \quad y^2 = \frac{u-v}{2}$$

$$\boxed{\begin{aligned} x^2+y^2 &= u \\ x^2-y^2 &= v \end{aligned}}$$

- granice za u: od 4 do 9
- granice za v: od 1 do 4

Jacobijan (izračunali smo ga u b) dijelu zadatka):

$$|J| = \left| -\frac{1}{4\sqrt{u^2-v^2}} \right| = \frac{1}{4\sqrt{u^2-v^2}}$$

Pa je naš integral:

$$\int_4^9 du \int_1^4 \sqrt{\frac{u+v}{2}} \cdot \sqrt{\frac{u-v}{2}} \cdot \frac{1}{4\sqrt{u^2-v^2}} \, dv =$$

$$= \int_4^9 du \int_1^4 \frac{\cancel{\sqrt{u^2-v^2}}}{2} \cdot \frac{1}{4\cancel{\sqrt{u^2-v^2}}}\, dv = \frac{1}{8}\int_4^9 3\, du =$$

$$= \frac{3}{8} \cdot 5 = \boxed{\frac{15}{8}}$$

(6.) Izračunajte volumen tijela

$$V = \{(x,y,z) : z^2 \leq x^2+y^2,\ z \geq 0,\ x^2+y^2 \leq 2x\}.$$

Rješenje:

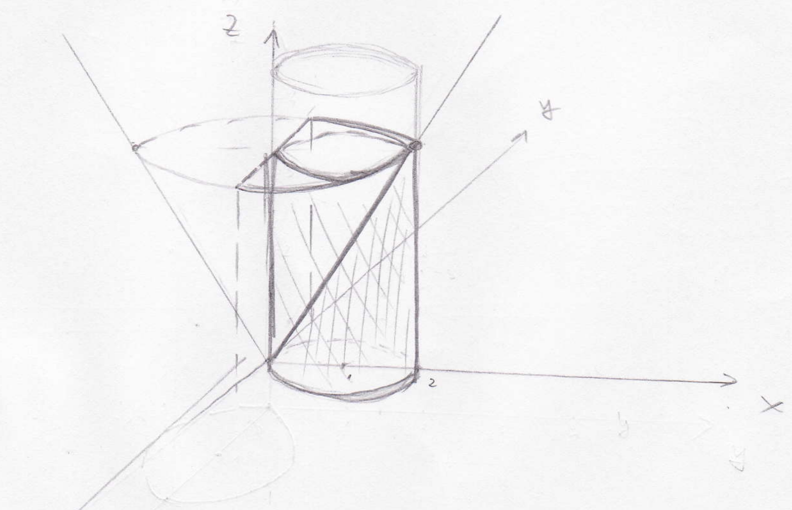
Naše tijelo omeđeno je plohama:

$x^2+y^2 \geq z^2$ $\quad (x^2+y^2=z^2 \rightarrow$ stožac$)$

$z \geq 0$ $\quad (z=0 \Rightarrow$ ravnina $xy)$

$x^2+y^2 \leq 2x$ $\quad ((x-1)^2+y^2=1 \Rightarrow$ valjak$)$

Nacrtajmo skicu:



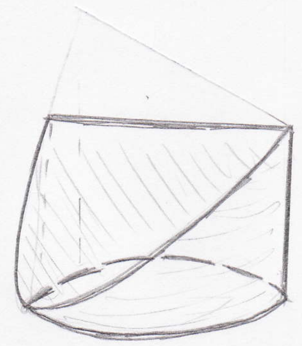
Sjecišta:

$$\left.\begin{aligned} x^2+y^2 &= z^2 \\ x^2+y^2 &= 2x \end{aligned}\right\} \quad 2x = z^2 \longrightarrow$$

x ide od $0 \Rightarrow z=0$

x ide do $2 \Rightarrow z=2$

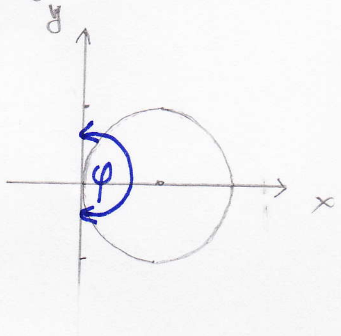
Naše tijelo izgleda ovako (ako se ne varam :D):



Zbog oblika tijela, ovdje je najlakše prijeći na cilindrične koordinate:

$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ z &= z \end{aligned}\right\} J = r$$

Projiciramo na xy ravninu:



φ ide od $-\frac{\pi}{2}$ do $\frac{\pi}{2}$

r ide od 0 pa do jednadžbe valjkaste plohe:

$$(r\cos\varphi - 1)^2 + r^2\sin^2\varphi = 1$$

$$r^2\cos^2\varphi - 2r\cos\varphi + \cancel{1} + r^2\sin^2\varphi = \cancel{1}$$

$$r^2 - 2r\cos\varphi = 0$$

$$r(r - 2\cos\varphi) = 0$$

$$\underline{r = 2\cos\varphi}$$

z nam je omeđen odozdo sa 0 a odozgo s jednadžbom stošca:

$$r^2 = z^2 \Rightarrow \underline{z = r}$$

$$V=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r\,dr \int_0^{r} dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^2\,dr = \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cos^3\varphi\,d\varphi =$$

$$=\frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\cdot\cos^2\varphi\,d\varphi = \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi(1-\sin^2\varphi)\,d\varphi =$$

$$=\frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\,d\varphi - \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\sin^2\varphi\,d\varphi = \begin{bmatrix} \sin\varphi = t \,/' \\ \cos\varphi\,d\varphi = dt \\ t_D = \sin(-\frac{\pi}{2}) = -1 \\ t_G = \sin\frac{\pi}{2} = 1 \end{bmatrix} =$$

$$=\frac{8}{3}\sin\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{8}{3}\int_{-1}^{1} t^2\,dt = \frac{8}{3}\cdot 2 - \frac{8}{3}\cdot\frac{t^3}{3}\Big|_{-1}^{1} =$$

$$=\frac{16}{3} - \frac{8}{9}\cdot 2 = \frac{16}{3} - \frac{16}{9} = \frac{48-16}{9} = \boxed{\frac{32}{9}}$$

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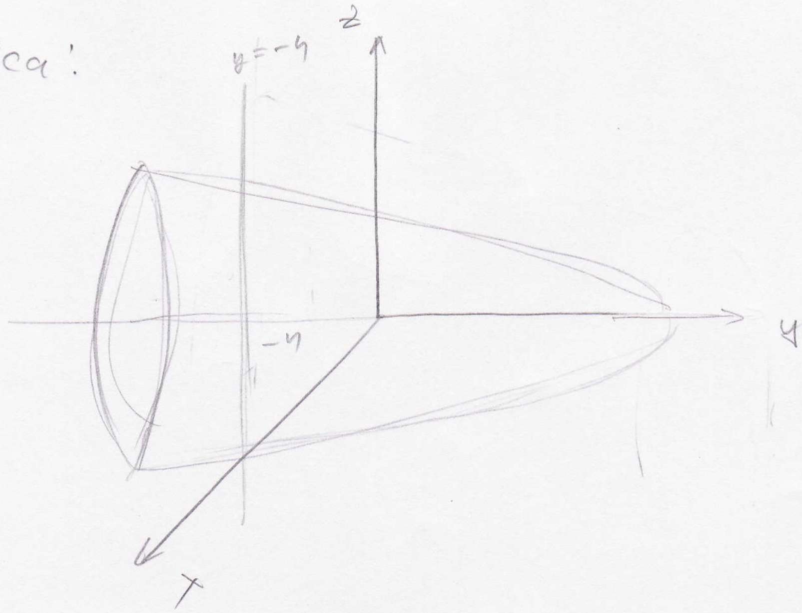
7. Izračunajte $\iiint_V y\,dx\,dy\,dz$ pri čemu je V omeđeno plohom $y=-x^2-3z^2+5$ i ravninom $y=-4$.

Rješenje:

Skica:



Koristit ćemo poopćene cilindrične koordinate:

$$\left.\begin{aligned} x &= 3r\cos\varphi \\ z &= \sqrt{3}\, r\sin\varphi \\ y &= y \end{aligned}\right\} \quad J = 3\sqrt{3}\,r$$

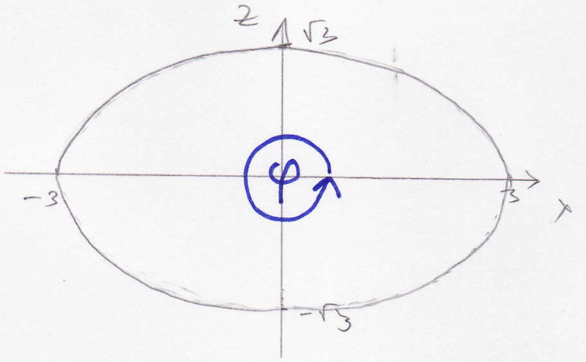
Sjecište od $y=-4$ i $y = -x^2 - 3z^2 + 5$:

$$-4 = -x^2 - 3z^2 + 5$$

$$x^2 + 3z^2 = 9 \;/:9$$

$$\frac{x^2}{3^2} + \frac{z^2}{(\sqrt{3})^2} = 1 \rightarrow \text{elipsa s velikom poluosi } a=3 \text{ i malom poluosi } b=\sqrt{3}$$

Projekcija na xz:



- φ ide od 0 do 2π
- r ide od 0 do jednadžbe rotacijskog paraboloida:

$$y = -9r^2\cos^2\varphi - 9r^2\cos^2\varphi - 5$$

// - max postiže za $y=-4$

$$-9r^2 = -9 \Rightarrow r = 1$$

- y ide od -4 do jednadžbe paraboloida:

$$\boxed{y = 5 - 9r^2}$$

$$\int_0^{2\pi} d\varphi \int_0^3 3\sqrt{3}\, r\, dr \int_{-4}^{5-9r^2} y\, dy = 3\sqrt{3} \int_0^{2\pi} d\varphi \int_0^3 r\left(\frac{25}{2} - \frac{90}{2}r^2 + \frac{81}{2}r^4 - \frac{16}{2}\right) dr =$$

$$= -\frac{27\sqrt{3}}{4} \cdot 2\pi = \boxed{-\frac{27\sqrt{3}\,\pi}{2}}$$

8. Prijelazom na pomaknute sferne koordinate izračunajte $\iiint_V (x^2+y^2)\,dx\,dy\,dz$ ako je

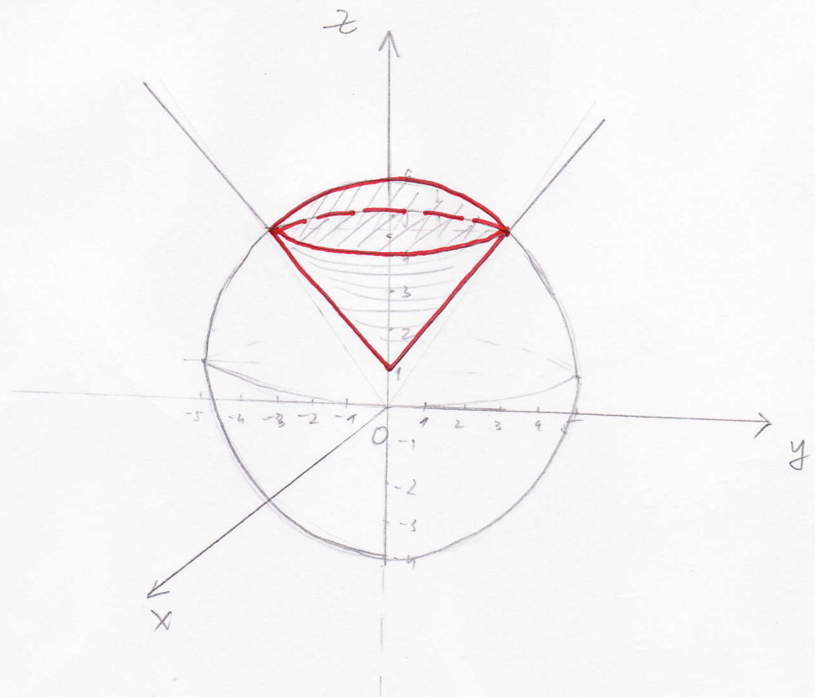
$$V=\{(x,y,z): x^2+y^2+(z-1)^2\leq 25,\ z\geq\sqrt{x^2+y^2}+1\}.$$

<u>Rješenje:</u>

$x^2+y^2+(z-1)^2\leq 5^2$ $\left(x^2+y^2+(z-1)^2=5^2 \rightarrow S(0,0,1),\ r=5\right)$

$z\geq\sqrt{x^2+y^2}+1$ $\left((z-1)^2=x^2+y^2 \rightarrow \text{stožac}\right)$

<u>skica:</u>



<u>Sjecište:</u>

$$(z-1)^2+(z-1)^2=5^2$$

$$2(z-1)^2=25$$

$$2z^2-4z+2-25=0$$

$$2z^2-4z-23=0$$

$$z_{1,2}=\frac{4\pm\sqrt{16-4\cdot2\cdot23}}{4}=\frac{4\pm\sqrt{200}}{4}=1\pm\frac{5\sqrt{2}}{2}$$

Uzimamo $z = 1 + \frac{5\sqrt{2}}{2}$.

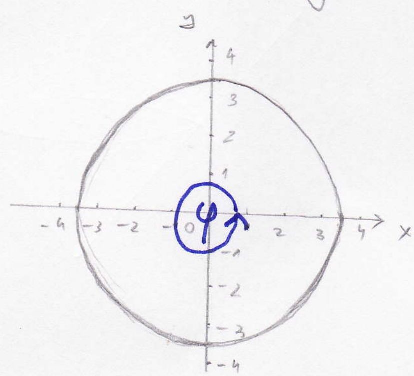
Kada taj z uvrstimo u jednadžbu (ili sfere ili stošca) dobijemo:

$$x^2 + y^2 + \left(\frac{5\sqrt{2}}{2}\right)^2 = 25$$

$$x^2 + y^2 = 25 - \frac{25 \cdot 2}{4} = \frac{50-25}{2} = \frac{25}{2}$$

$$\underline{x^2 + y^2 = \left(\frac{5}{\sqrt{2}}\right)^2 = \left(\frac{5\sqrt{2}}{2}\right)^2}$$

projiciramo u xy da odredimo φ

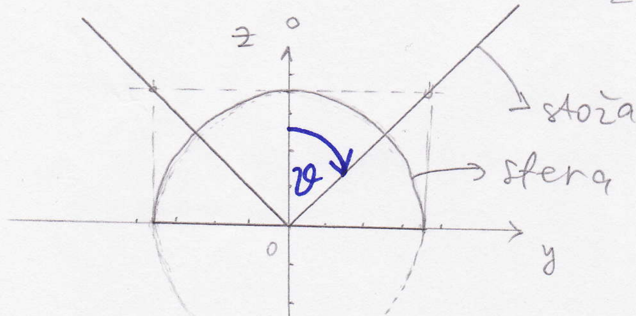


φ ide od 0 do 2π

Moramo uvesti pomaknute sferne koordinate:

$$\left.\begin{aligned} x &= r\sin\vartheta\cos\varphi \\ y &= r\sin\vartheta\sin\varphi \\ z-1 &= r\cos\vartheta \end{aligned}\right\} J = r^2\sin\vartheta$$

Pomaknutim sfernim koordinatama stavljamo sferu i stožac u ishodište, pa imamo ovo:



$z = \frac{5\sqrt{2}}{2} \Rightarrow y^2 + \frac{25}{2} = 25$

$y^2 = \frac{25}{2} \Rightarrow y = \pm\frac{5\sqrt{2}}{2}$

$z = y \Rightarrow \operatorname{tg}\vartheta = 1$

$$\boxed{\vartheta = \frac{\pi}{4}}$$

ϑ ide od 0 do $\frac{\pi}{4}$

r ide od 0 pa do jednadžbe sfere:

$$x^2+y^2+(z-1)^2=25$$

$$\underbrace{r^2\sin^2\vartheta\cos^2\varphi+r^2\sin^2\vartheta\sin^2\varphi+r^2\cos^2\vartheta}=25$$

$$r^2=25$$

$$\underline{\underline{r=5}}$$

podintegralna f-ja je:

$$x^2+y^2=r^2\sin^2\vartheta$$

Pa je integral jednak:

$$\int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{4}}d\vartheta\int_0^5 r^2\sin\vartheta\cdot r^2\sin^2\vartheta\,dr=$$

$$=\int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{4}}\sin^3\vartheta\,d\vartheta\int_0^5 r^4\,dr=\frac{\cancel{3125}^{625}}{\cancel{5}_1}\int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{4}}\sin^3\vartheta\,d\vartheta=$$

$$=625\cdot\frac{1}{12}(8-5\sqrt{2})\int_0^{2\pi}d\varphi=\boxed{\frac{625(8-5\sqrt{2})\pi}{6}}$$