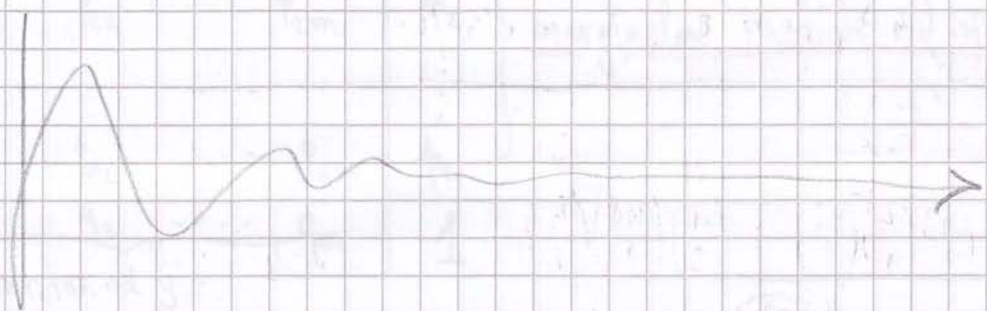
II CIKLUS



7. SLUČAJNI VEKTORI

7.1. DISKRETNI SLUČAJNI VEKTORI

⇝ to je 3.2. IZ PRVE KNJIGE!

⇝ SUTRA OBAVEZNO AUDITORNE → DVOSTRUKI INTEGRALI!

- razdioba SLUČAJNOG VEKTORA (X, Y) se definira s

$$\boxed{p_{ij} = P(X = x_i, Y = y_i)}$$

X/Y	y_1	y_2	......	y_m	
x_1	p_{11}	p_{12}	......	p_{1m}	p_1
x_2	p_{21}	p_{22}	......	p_{2m}	p_2
$\vdots$	$\vdots$				$\vdots$
x_n	p_{n1}	p_{n2}	......	p_{nm}	p_n
	q_1	q_2	.....	q_m	1

$p_1, \dots, p_n$ – marginalne razdiobe od X

$q_1, \dots, q_m$ – mar. raz. od Y

$$\sum_i \sum_j p_{ij} = 1 \qquad p_{ij} \geq 0$$

⇝ MARGINALNE RAZDIOBE = sume u nekom retku ili stupcu

$$X \sim \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix} \qquad Y \sim \begin{pmatrix} y_1 & \dots & y_m \\ q_1 & \dots & q_m \end{pmatrix}$$

... X i Y su NEZAVISNI ako i samo ako $p_{ij} = p_i \cdot q_j$ za $\forall i,j$.

1.MI-10-5.) Zadana je razdioba vektora.

X|Y	0	1	
-1	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{10}{24}$
0	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{7}{24}$
1	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{7}{24}$
	$\frac{13}{24}$	$\frac{11}{24}$	1

marg. razdioba od X

marg. od Y

UVIJEK ISTO PITAJU:

a) marginalne razdiobe

→ napisano sa strane pored zadane razdiobe

b) ispitajte nezavisnost varijabli

⇒ NISU nezav. jer npr. $\frac{1}{4} \neq \frac{10}{24} \cdot \frac{13}{24}$

$p_{ij} \neq p_i \cdot q_j$

c) ILKO UUUUU II

$P(X \geq 0 \mid Y = 1) = ?$

$P(x \geq 0 \mid y=1) = \cancel{\frac{1}{8} + \frac{1}{6}}$

PAZI! IAKO TO IZGLEDA SUPER LOGIČNO TO **NE** VRIJEDI!!

⇝ **Po FORMULI:**

$$\overset{P(A|B)}{P(x \geq 0 \mid y=1)} \underset{\text{UVJETNA VJEROJATNOST}}{=} \frac{\overset{P(A \cap B)}{P(x \geq 0, y=1)}}{\underset{P(B)}{P(y=1)}} = \frac{\frac{1}{8}+\frac{1}{6}}{\frac{11}{24}} = \frac{7}{11}$$

d) Odredite razdiobu od vektora (z, w) ako je $z = x+y$, $w = x \cdot y$.

SKORO UVIJEK NA ISPITU, SVAKE GODINE

$z \downarrow \mid w \rightarrow$	-1	0	1
-1	0	$\frac{1}{4}$	0
0	$\frac{1}{6}$	$\frac{1}{6}$	0
1	0	$\frac{1}{4}$	0
2	0	0	$\frac{1}{6}$

$\frac{1}{8} + \frac{1}{8}$

popunjavamo tako da gledamo kada će biti $w = x \cdot y = 1$
⇒ samo kada je $x = 1$ i $y = 1$
⇒ tada je $z = 2$
→ iz prve tablice za $x = 1$ i $y = 1$ je vjerojatnost $\frac{1}{6}$

⇝ SVE OSTALO 0!

$w = -1 \Rightarrow x = -1, y = 1 \Rightarrow z = 0$ → 1. tablica →

$z = -1 \Rightarrow x = -1, y = 0 \Rightarrow w = 0$

OBRNI

$z=0 \Rightarrow x=0,\ y=0 \longrightarrow w=0$

(za $x=-1,\ y=1$ SMO VEĆ POKRILI TAJ SLUČAJ!)

$z=1 \Rightarrow \left.\begin{array}{l} x=0,\ y=1 \\ x=1,\ y=0 \end{array}\right\} \Rightarrow w=0$ ⇝ $\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$ u tablicu!

... MEĐUSOBNI ODNOS DVIJE VARIJABLE ...

⇝ 1. KNJIGA 103. – 105. str.

⇝ „KORELIRANOST" PODATAKA ...

def. KOVARIJACIJSKI MOMENT od X i Y se definira kao

$$\mathrm{cov}(X,Y) = E\left[(X-m_X)(Y-m_Y)\right]$$

$$\boxed{\mathrm{cov}(X,Y) = E(X\cdot Y) - E(X)\cdot E(Y)}$$

KOEFICIJENT KORELACIJE:

$$\boxed{r(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{D(X)\,D(Y)}}}$$

! **Uočimo:**

Ako su X i Y nezavisni tada je $cov(X,Y)=0$, tj.

$r(X,Y)=0$, tj. X i Y NISU KORELIRANI !

OBRAT NE VRIJEDI !!

$\boxed{TM}$ „DISPERZIJA ZBROJA"

$$D\left(\sum_i X_i\right) = \sum_i D(X_i) + 2\sum_{i,j} cov(X_i, X_j)$$

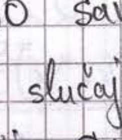
PODSJETNIK: $E\left(\sum X_i\right) = \sum E(X_i)$

DOKAZ: trivić ☺

! Uoči:

Ako su X_i nezavisni : $D\left(\sum_i X_i\right) = \sum_i D(X_i)$.

Zad.) Od 400 studenata koji su slušali SiS i ViS 220 su položili oba predmeta, 90 samo ViS, a 30 samo SiS. Neka je X je 1 ako je slučajno odabran student položio ViS, a Y je 1 ako je položio SiS. Odredite koeficijent korelacije SiS-a i ViS-a.

 → SiS-a

x/y	0	1	
0	$\frac{60}{400}$ ($400-(90+30+220)$)	$\frac{30}{400}$	$\frac{90}{400}$
1	$\frac{90}{400}$	$\frac{220}{400}$	$\frac{310}{400}$
	$\frac{150}{400}$	$\frac{250}{400}$	1

→ ZAVISNI SU !

$$E(x) = 0 \cdot \frac{90}{400} + 1 \cdot \frac{310}{400} = 0.775$$

$$E(y) = 0 \cdot \frac{150}{400} + 1 \cdot \frac{250}{400} = 0.625$$

$$D(x) = 0^2 \cdot \frac{90}{400} + 1^2 \cdot \frac{310}{400} - 0.775^2 = 0.174$$

$$D(y) = 0^2 \cdot \frac{150}{400} + 1^2 \cdot \frac{250}{400} - 0.625^2 = 0.234$$

$$r(x,y) = \frac{E(x \cdot y) - E(x)E(y)}{\sqrt{D(x)D(y)}}$$

!

$$E(x \cdot y) = \sum_{i,j} x_i y_j p_{ij} = \underbrace{0 \cdot 0 \cdot \frac{60}{400}}_{=0} + \underbrace{0 \cdot 1 \cdot \frac{30}{400}}_{=0} + \underbrace{1 \cdot 0 \cdot \frac{90}{400}}_{=0} + 1 \cdot 1 \cdot \frac{220}{400} = 0{,}55$$

$$\Rightarrow r(x,y) = \frac{0.55 - 0.775 \cdot 0.625}{\sqrt{0.174 \cdot 0.234}} = 0.325$$

... CENTRIRANJE I NORMIRANJE VARIJABLI ...

- ako znamo razdiobu od X lako znamo i od $X - a$ (minus)

$$E(X-a) = E(X) - a$$

$$D(X-a) = D(X)$$

⇝ s obzirom da se $D(X)$ NE MIJENJA, ne mijenja se ni $cov(X,Y)$, a samim time ne mijenja se ni $r(X,Y)$

⇝ mi ćemo odabrati $a = m_X$, za takvu varijablu

$$\dot{X} = X - m_X$$ kažemo da je CENTRIRANA!

⇝ $$\boxed{X^* = \frac{X - m_X}{\sigma_X}}$$ kažemo da je **NORMIRANA** SLUČAJNA VARIJABLA

$$E(X^*) = 0 \quad , \quad D(X^*) = 1$$

⇝ centriranje ne mijenja koeficijent relacije $r(X,Y)$, ali ni normiranje ne mijenja $r(X,Y)$!

⇝ zato je ovaj postupak jako bitan!

DOKAZ: (BIO JEDNOM)

$$r(X^*, Y^*) = \frac{E(X^* \cdot Y^*) - \overbrace{E(X^*)}^{0}\,\overbrace{E(Y^*)}^{0}}{\underbrace{\sigma(X^*)}_{=1}\,\underbrace{\sigma(Y^*)}_{=1}} = E\left(\frac{X - m_X}{\sigma_X} \cdot \frac{Y - m_Y}{\sigma_Y}\right)$$

$$= \frac{cov(X,Y)}{\sigma_X\,\sigma_Y} = r(X,Y)$$

JEDINA TEORIJA IZ OVOG CIKLUSA !

TM

Vrijedi $|r(x,y)| \leq 1$. Jednakost vrijedi ako i samo ako je

$y = aX + b$ za neke a i b.

Dokaz:

→ znamo: $r(x^*, y^*) = r(x, y)$

$$D(x^* \pm y^*) = D(x^*) + D(y^*) \pm 2\,\mathrm{cov}(x^*, y^*) =$$

$$= 2\left[1 \pm r(x,y)\right] \geq 0$$

$$\Rightarrow |r(x,y)| \leq 1$$

// TO JE SVE OD TEORIJE //

2.DZ - 10.) Bacamo dvije kocke. X je minimum, a Y maksimum brojeva na kockama. Odredite koeficijent korelacije među ovim varijablama.

1,2 i 2,1

X\Y	1	2	3	4	5	6
1	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
2	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
3	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
4	0	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
5	0	0	0	0	$\frac{1}{36}$	$\frac{2}{36}$
6	0	0	0	0	0	$\frac{1}{36}$

ČIM IMATE NULU U TABLICI ZNATE DA SU ZAVISNI!

PROVJERITI ZA DZ NA KALK.

$E(x) = \cdots = 2.52778 \quad \rightarrow 1 \cdot \frac{1+5\cdot 2}{36} + 2 \cdot \frac{1+4\cdot 2}{36} + 3 \cdot \frac{1+3\cdot 2}{36} + \cdots + 6 \cdot \frac{1}{36} = \frac{91}{36}$

$E(y) = \cdots = 4.47222 \quad \rightarrow 1 \cdot \frac{1}{36} + 2 \cdot \frac{3}{36} + 3 \cdot \frac{4+1}{36} + \cdots + 6 \cdot \frac{5\cdot 2+1}{36}$

$D(x) = 1.97 \quad \rightarrow 1^2 \cdot \frac{11}{36} + 2^2 \cdot \frac{9}{36} + 3^2 \cdot \frac{7}{36} + \cdots + 6^2 \cdot \frac{1}{36} - 2.52778^2 = 1{,}97$

$D(y) = 1.97$

$r(x,y) = \frac{cov(x,y)}{\sqrt{D(x)D(y)}} = \frac{E(x\cdot y) - E(x)E(y)}{\sqrt{D(x)D(y)}}$

$\Rightarrow r(x,y) = 0.479$

$E(x\cdot y) = \frac{1}{36} + \frac{4}{36} + \frac{6}{36} + \frac{8}{36} + \frac{10}{36} + \frac{12}{36} + \frac{4}{36} + \frac{12}{36} + \frac{16}{36} +$

$0.4798029\ldots$

$+ \frac{20}{36} + \frac{24}{36} + \frac{9}{36} + \frac{24}{36} + \frac{30}{36} + \frac{36}{36} + \frac{16}{36} + \cdots = \frac{441}{36}$

Zad.) Bacamo 2 kocke. Neka je X apsolutna vrijednost razlike brojeva na kockama, a Y je manji od 2 broja koji su pali, inače nula (ako su isti). Izračunajte koef. korelacije varjabli X i Y.

x|y	0	1	2	3	4	5	
0	1,1 2,2 3,3 6,6 $\frac{6}{36}$	0	0	0	0	0	$\frac{6}{36}$
1	0	1,2 ili 2,1 $\frac{2}{36}$	2,3 ili 3,2 $\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$ 5,6 6,5	$\frac{10}{36}$
2	0	1,3 3,1 $\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$ 4,6 6,4	0	$\frac{8}{36}$
3	0	1,4 4,1 $\frac{2}{36}$	$\frac{2}{36}$ 5,2 2,5	$\frac{2}{36}$ 6,3 3,6	0	0	$\frac{6}{36}$
4	0	5,1 1,5 $\frac{2}{36}$	$\frac{2}{36}$ 6,2 2,6	0	0	0	$\frac{4}{36}$
5	0	6,1 1,6 $\frac{2}{36}$	0	0	0	0	$\frac{2}{36}$
	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	1

$$r(X,Y) = \frac{cov(X,Y)}{\sqrt{D(x)D(y)}} = \frac{E(x\cdot y) - E(x)E(y)}{\sqrt{D(x)D(y)}}$$

$$E(x) = E(y) = 0 \cdot \frac{6}{36} + 1 \cdot \frac{10}{36} + \dots = 1.94$$

$$D(x) = D(y) = \left(0^2 \cdot \frac{6}{36} + 1^2 \cdot \frac{10}{36} + 2^2 \cdot \frac{8}{36} + \dots\right) - (\underbrace{1.94}_{E(x)})^2$$

$$E(XY) = \sum x_i y_j p_{ij} = 3.889$$

$$\Rightarrow r(x,y) = 0.061$$

7.2. NEPREKINUTI SLUČAJNI VEKTORI

def.

n-dimenzionalni slučajni vektor je uređena n-torka slučajnih varijabli:

$$X = (X_1, X_2, \dots, X_n)$$

FUNKCIJA RAZDIOBE slučajnog vektora se definira s

$$F(X_1, \dots, X_n) = P(X_1 < x_1, X_2 < x_2, \dots, X_n - x_n)$$

⇝ mi ćemo (NAŽALOST) raditi samo 2D slučaj (X, Y)

$$\boxed{F(x,y) = P(X < x, Y < y)}$$

KLASIKA

def.

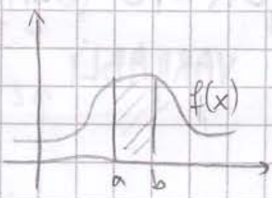
Za slučajni vektor kažemo da je **NEPREKINUT** ako postoji nenegativna funkcija $f : \mathbb{R}^2 \longrightarrow [0, +\infty\rangle$ takva da je

$$\boxed{F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dx\,dy}$$

⇝ $f(x,y)$ nazivamo **FUNKCIJOM GUSTOĆE** i vrijedi

$$\boxed{f(x,y) = \frac{\partial^2 F}{\partial x \partial y}}$$

* u 1D:



$P = \int_a^b f(x)\,dx$ $\quad P(a<x<b)$

KAKO RAČUNAMO VJEROJATNOST U 2D ??

$$P(X<x,\ Y<y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dx\,dy$$

$$\Rightarrow P(x_1 < X < x_2,\ y_1 < Y < y_2) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dx\,dy$$

→ 

PRAVOKUTNIK

$$\boxed{P\left[(X,Y)\in G\right] = \iint_G f(x,y)\,dx\,dy}$$

→ OPĆENITO, AKO NIJE

PRAVOKUTNIK!

Očito:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = 1$$

⇝ ako je poznata razdioba od (X,Y) kako odrediti marginalnu razdiobu od X i od Y?

$$F_X(x) = P(X<x) = P(X<x, -\infty<Y<+\infty)$$ → margin. razd. od X

$$= \int_{-\infty}^{x}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = \int_{-\infty}^{x} f_X(x)\,dx$$

⇒ $$\boxed{f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy}$$

UVIJEK PO SUPROTNOJ VARIJABLI

⇒ $$\boxed{f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx}$$

MARGINALNE GUSTOĆE

„NEMA ZADATKA BEZ NJIH"

TM X i Y su nezavisni ako i samo ako

$$\boxed{f(x,y) = f_X(x)\cdot f_Y(y)}$$

⇝ JAKO BITAN! SKORO U SVAKOM ZADATKU ĆEMO GA KORISTITI

Dokaz:

$\Rightarrow$

$$F(x,y) = P(X<x,\, Y<y) \overset{\text{nez.}}{=} P(X<x)\cdot P(Y<y) =$$

$$= F_X(x)\cdot F_Y(y) \;\Big|\; \frac{\partial x}{\partial y} \;\{\text{da dobijemo gustoću}\}$$

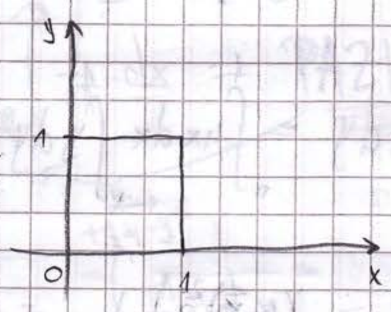
$\Leftarrow$

$$P(X\in A,\, Y\in B) = \iint_{AB} f(x,y)\,dx\,dy = \iint f_X(x)\cdot f_Y(y)\,dx\,dy =$$

$$= \int_A f_X(x)\,dx \int_B f_Y(y)\,dy = P(X\in A)\cdot P(Y\in B)$$

2MI – 07 – 6.) Neka je slučajni vektor zadan sa $f(x,y) = Cxy$, $0<x<1$, $0<y<1$. (Pretpostavlja se nula inače.)

a) $C = ?$



$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = 1$$

$$\int_0^1\int_0^1 Cxy\,dx\,dy = C\int_0^1 x\,dx \int_0^1 y\,dy = \frac{1}{4}C$$

$$\Rightarrow \frac{1}{4}C = 1 \Rightarrow \boxed{C=4} \quad (1\text{ bod})$$

$$\Rightarrow f(x,y) = 4xy$$

b) ispitaj nezavisnost i odredi marginalne gustoće

$$f_X(x) = \int_0^1 4xy\,dy = 4x\cdot\frac{y^2}{2}\Big|_0^1 = 2x$$ → NE MOŽEŠ IMAT Y-ONE !!

$$f_Y(y) = \int_0^1 4xy\,dx = 2y$$

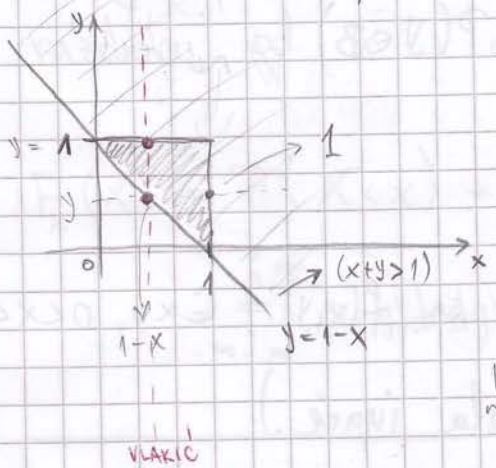
$\Rightarrow f(x,y) = f(x) \cdot f(y) = 2x \cdot 2y = 4xy$ //

$\Rightarrow$ X i Y SU NEZAVISNI ! (2 boda)

c) $P(X+Y>1) = ?$

PAZI !!!

$P(X+Y>1) = \iint_G f(x,y)\,dx\,dy = \cancel{\frac{1}{2}}$ NE ! NIJE POVRŠINA !



GUSTOĆA NIJE KONSTANTNA I SVUGDJE ISTA, DA JE SAMO 4 ONDA BI BILA $\frac{1}{2}$!

INTUITIVNO: MORA BITI VEĆE OD $\frac{1}{2}$ JER GUSTOĆA RASTE S PORASTOM X i Y !

$$P(X+Y>1) = \iint_G f(x,y)\,dx\,dy = \int_0^1 dx \int_{1-x}^{1} 4xy\,dy = \int_0^1 4x\,dx \int_{1-x}^{1} y\,dy =$$

$$= \int_0^1 4x\,dx \left.\frac{y^2}{2}\right|_{1-x}^{1} = 2\int_0^1 x\left(\frac{1}{2} - \frac{(1-x)^2}{2}\right)dx =$$

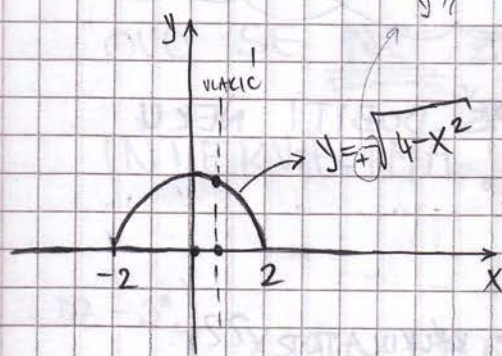
$$= \dots = \frac{5}{6} \;//$$

2.MI – 08 – 5.) $f(x,y) = C$, $x^2+y^2 \le 4$, $y \ge 0$

kružnica, $r=2$

$y \ge 0$

a) $C = ?$



NE ! PAZI !!

$$\iint f(x,y)\,dx\,dy = \int_{-2}^{2} dx \int_{0}^{\cancel{2}} c\,dy = \int_{-2}^{2} dx \int_{0}^{\sqrt{4-x^2}} c\,dy$$

$$= c\int_{-2}^{2} dx \int_{0}^{\sqrt{4-x^2}} dy = c\int_{-2}^{2} \sqrt{4-x^2}\,dx = \left| x = 2\sin t \right|$$

$\frac{r^2\pi}{2}$ → POVRŠINA → JEDNOSTRUKI INTEGRAL!

$$= 2\pi C = 1 \Rightarrow \boxed{C = \frac{1}{2\pi}}$$

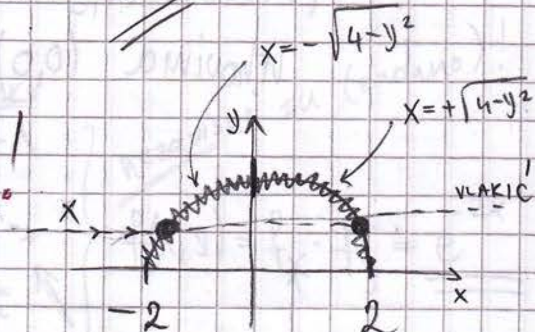
$\Rightarrow f(x,y) = \frac{1}{2\pi}$

b) marginalne gustoće ?

po y-onu gledamo

$$f_X(x) = \int_{0}^{\sqrt{4-x^2}} \frac{1}{2\pi}\,dy = \frac{1}{2\pi}\sqrt{4-x^2}, \quad x \in (-2,2)$$

$$f_Y(y) = \int_{\cancel{-2}}^{\cancel{2}} \frac{1}{2\pi}\,dx = \text{PAZI !! NE !!}$$



$$f_Y(y) = \int_{-\sqrt{4-y^2}}^{+\sqrt{4-y^2}} \frac{1}{2\pi}\,dx = \frac{1}{\pi}\sqrt{4-y^2}, \quad y \in (0,2)$$

$\Rightarrow$ OČITO SU ZAVISNI !

c) $E(y) = ?$

*PODSJETNIK:

→ očekivanje neprek. sluč. VARIJABLE:

$$E(x) = \int_{-\infty}^{+\infty} x \cdot f(x)\, dx$$

$$E(y) = \int_0^2 y \cdot f_y(y)\, dy$$!!!

→ OČEKIVANJE JE BROJ, SJETI SE, MORAŠ DOBITI NEKU VRIJEDNOST !!

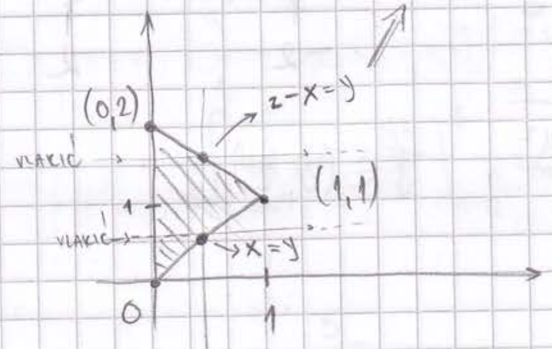
$$E(y) = \int_0^2 y \frac{1}{\pi} \sqrt{4-y^2}\, dy = \left| 4-y^2 = t \right|$$ → KALKULATOR ??

$$= \frac{8}{3\pi}$$

2.MI – 10 – 5.) Vektor (X,Y) ima jednoliku razdiobu na trokutu zadan vrhovima $(0,0)$, $(1,1)$, $(0,2)$. Odredi marginalne gustoće.

*PODSJETNIK: $X \sim U[a,b]$

$$f(x) = \frac{1}{b-a}$$



$f(x,y) = \frac{1}{P}$ → JEDNOLIKA RAZDIOBA !

$$f(x,y) = \frac{1}{\frac{1}{2} \cdot 2 \cdot 1} = 1$$

$$f_X(x) = \int_x^{2-x} 1\, dy = 2 - 2x, \quad x \in (0,1)$$ NE ZABORAVI! NE PRIZNA[JE SE] BEZ TOGA!

$$f_Y(y) = \int_0^y 1\, dx = y \quad \text{za } y \in [0,1]$$

$$f_y(y) = \int_0^{2-y} 1\,dx = 2-y \quad , \quad \text{za } y \in [1,2]$$

PAZI!

OVO SE **NE** ZBRAJA! OSTAVLJA SE NA OVIM INTERVALIMA!

5. DZ - 3*.) X ima eksponencijalnu razdiobu s očekivanjem $\frac{1}{2}$. Y ima jednoliku razdiobu na intervalu [0,2]. Ako su X i Y nezavisni, izračunaj vjerojatnost da Y poprimi manju vrijednost od X.

$$\left.\begin{array}{l} X \sim E(2) \\ Y \sim U[0,2] \end{array}\right\} \text{nez.} \qquad \text{jer } E(\lambda) = \frac{1}{\lambda}$$

$$P(Y \leq X) = ?$$

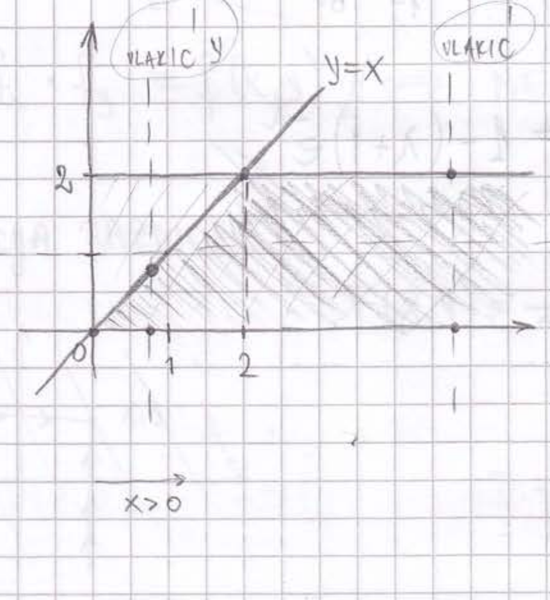
*PODSJETNIK:

$$f_X(x) = 2e^{-2x}, \; x > 0 \quad \leftrightarrow f(x) = \lambda e^{-\lambda}$$

$$f_Y(y) = \frac{1}{2}, \; y \in \langle 0,2 \rangle \quad \leftrightarrow f(x) = \frac{1}{b-a}$$

nezavisni su (zadano)!

$$f(x,y) = f_X \cdot f_Y = e^{-2x}$$



$$P(Y \leq X) = \iint_G f(x,y)\,dx\,dy =$$

$$P(Y \leq X) = \int_0^2 dx \int_0^x e^{-2x}\,dy + \int_2^\infty dx \int_0^2 e^{-2x}\,dy$$

$$P(Y \leq X) = \dots = \frac{1}{4} - \frac{1}{4}e^{-4}$$

ILI → VLAK X →

OKRENEMO PORETDAK:

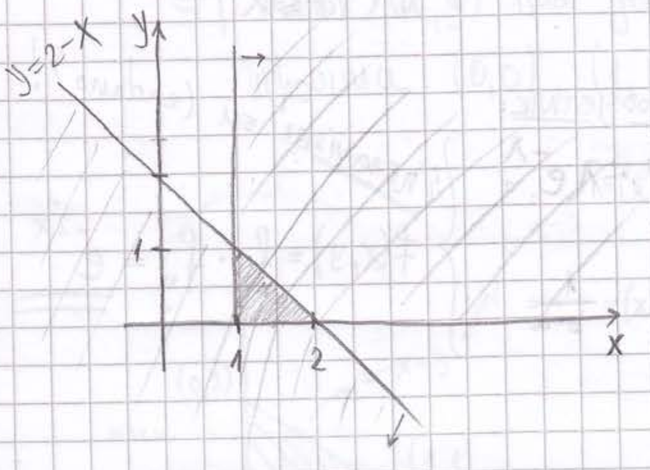
ILI $\displaystyle\int_0^2 dy \int_y^{\infty} e^{-2x}\,dx = \frac{1}{4} - \frac{1}{4}e^{-4}$ //

5.DZ-9.) Neka su X i Y nezavisne sluč. varjable eksponencijalne razdiobe s istim parametrom. Izračunajte vjerojatnost da suma X i Y bude manja od 2 ako znamo da je X veći od 1.

$X, Y \sim E(\lambda)$

$P(X+Y < 2 \mid X > 1) = ?$

$\Rightarrow f(x,y) \underset{\text{nezav.}}{=} f_X \cdot f_Y = \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda y} = \lambda^2 e^{-\lambda(x+y)} \quad x>0,\ y>0$ //



$y < 2-x$

!! PO DEFINICIJI:

$$P(X+Y<2 \mid X>1) = \frac{P(X+Y<2,\ X>1)}{P(X>1)} = \frac{\cancel{\lambda^2}\int_1^2 dx \int_0^{2-x} e^{-\lambda(x+y)}\,dy}{\cancel{\lambda^2}\int_1^{\infty} dx \int_0^{\infty} e^{-\lambda(x+y)}\,dy} = \dots$$

$= 1 - (\lambda+1)e^{-\lambda}$ //

⟶ KAKO ODREDITI FUNKCIJU RAZDIOBE ... ?

2MI-07-5.) Neka je $f(x,y) = \dfrac{C}{x^2+y^2+x^2y^2+1}$, $x, y \in \mathbb{R}$.

Odrediti C, marginalne razdiobe, $F(x,y)$.

$$C\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{x^2+y^2+x^2y^2+1} = C\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{(x^2+1)(y^2+1)} = C\int_{-\infty}^{\infty}\frac{dx}{1+x^2}\int_{-\infty}^{\infty}\frac{dy}{1+y^2} =$$

$$x^2(1+y^2)+y^2+1 = (1+y^2)(1+x^2)$$

$$= C \operatorname{arctg} x \Big|_{-\infty}^{\infty} \cdot \operatorname{arctg} y \Big|_{-\infty}^{\infty} = C\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\cdot\pi = C\cdot\pi^2$$

$$\Rightarrow \underline{C\cdot\pi^2 = 1} \Rightarrow \boxed{C = \frac{1}{\pi^2}}$$

$$f_X(x) = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{dy}{(x^2+1)(y^2+1)} = \frac{1}{\pi^2(x^2+1)}\int_{-\infty}^{\infty}\frac{dy}{y^2+1} = \frac{1}{\pi(x^2+1)} \;//$$

$$f_Y(y) = \frac{1}{\pi(y^2+1)} \;//$$

$$f_X \cdot f_Y = f(x,y) \checkmark \Rightarrow \text{NEZAVISNE!}$$

FUNKCIJA RAZDIOBE:

$$F(x,y) = \frac{1}{\pi^2}\int_{-\infty}^{x}\frac{dx}{1+x^2}\int_{-\infty}^{y}\frac{dy}{1+y^2} =$$

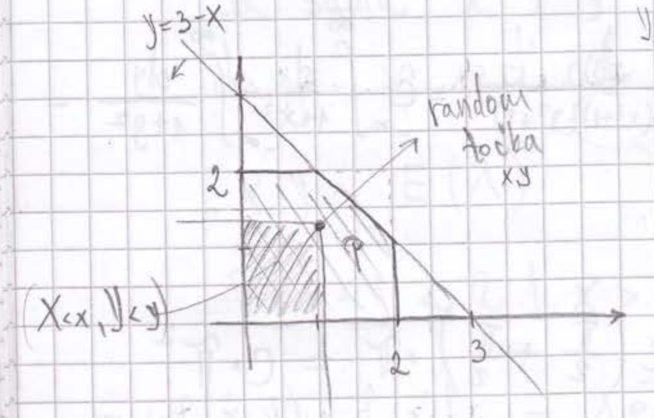
$$= \frac{1}{\pi^2}\left(\operatorname{arctg} x + \frac{\pi}{2}\right)\left(\operatorname{arctg} y + \frac{\pi}{2}\right) \;//$$

ZI-09-10.) Biramo nasreću točku unutar $\Omega = \{x, y \in (0,2), x+y<3\}$. Neka je X x-koordinata sluč. odabrane točke, a Y y-koordinata. Odredi sve ... (marginalne gustoće, razdiobu...

$$\Omega = \{x, y \in (0,2),\ x+y<3\} \qquad (x,y) = ?$$

$y < 3 - x$

$$P = \frac{3\cdot 3}{2} - 2\cdot\frac{1\cdot 1}{2} = \frac{7}{2}$$



$$f(x,y) = \frac{1}{P} = \frac{1}{\frac{7}{2}} = \frac{2}{7} \,/\!/$$

$$F(x,y) = P(X<x, Y<y)$$

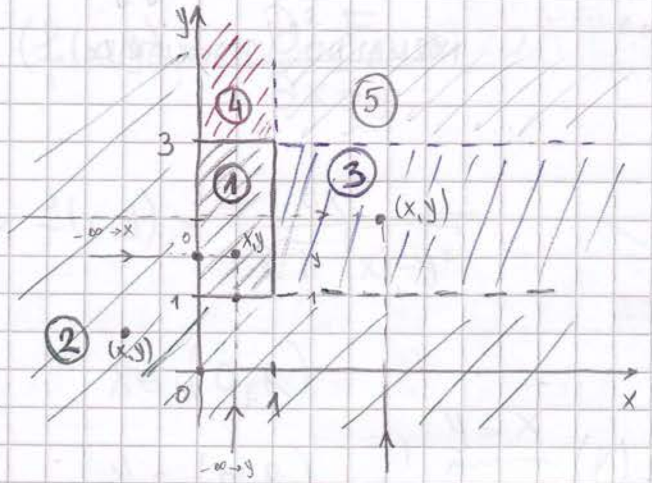
$$= \frac{x\cdot y}{\frac{7}{2}} = \frac{2}{7}xy \,/\!/$$

P dijelog

PROVJERA:
$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y} = \frac{2}{7} \,/\!/$$

$\rightsquigarrow$ MOGLI PREKO POVRŠINA JER SE RADI O JEDNOLIKOJ RAZDIOBI.

5. DZ - 1*.) Slučajni vektor ima gustoću $f(x,y) = c(5-x-y)$, $0<x<1$, $1<y<3$. Odredi funkciju razdiobe ovog slučajnog vektora.



$$c \int_0^1 dx \int_1^3 (5-x-y)\,dy = c \cdot 5 = 1$$

$$\Rightarrow \boxed{c = \frac{1}{5}}$$

$$\Rightarrow f(x,y) = \frac{1}{5}(5-x-y)$$

$$F(x,y) = P(X<x,\ Y<y) = \int_{-\infty}^{x} dx \int_{-\infty}^{y} \tfrac{1}{5}(5-x-y)\,dy$$

NE MOŽEMO POVRŠINE JER JE NEKA f gustoće! NIJE JEDNOLIKA RAZDIOBA!

1) $F(x,y) = \frac{1}{5}\int_0^x dx \int_1^y (5-x-y)\,dy = \frac{1}{10}\,x\,(y-1)(9-x-y)$, $x \in (0,1)$, $y \in (1,3)$

UOČI!!

2) $F(x,y) = 0$, $x<0$, $y<1$

3) $F(x,y) = \int_0^1 dx \int_1^y \frac{1}{5}(5-x-y)\,dy = \frac{1}{10}(y-1)(8-y)$, $x>1$, $y \in (1,3)$

4) $F(x,y) = \int_0^x dx \int_1^3 \frac{1}{5}(5-x-y)\,dy = \frac{1}{5}\,x(6-x)$, $x \in (0,1)$, $y>3$

5) $F(x,y) = \int_0^1 dx \int_1^3 \frac{1}{5}(5-x-y)\,dy = 1$, $x>1$, $y>3$

"JOŠ NIKAD NIJE BILO NA ISPITU"

... OČEKIVANJE SLUČAJNOG VEKTORA:

$$E(xy) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f(x,y)\, dx\, dy$$

GRANICE SE POSTAVLJAJU „NORMALNO" (PRAVOKUTNIK....)

TM

„SVOJSTVA OČEKIVANJA"

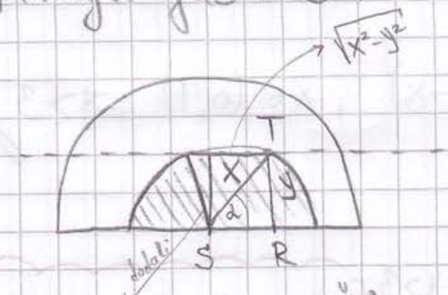
a) $E(sX + tY) = sE(X) + tE(Y)$

b) $E(X \cdot Y) = E(X) \cdot E(Y)$ samo ako su nezavisni!

DOKAZIĆ: $E(X \cdot Y) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} xy\, f_X(x) \cdot f_Y(y)\, dy = \int x f_X(x)\, dx \int y f_Y(y)\, dy =$

$= E(X) \cdot E(Y)$

5.DZ-2.) Biramo točku T unutar kruga polumjera R. Neka je X udaljenost točke T do središta, a Y do polumjera. Odredite funkciju gustoće.



TROKUT + 2 KRUŽNA ISJEČKA

$f(x,y) = ?$

NIJE $f = \frac{1}{P}$!

NIJE JEDNOLIKA

$F(x,y) = P(X < x,\ Y < y)$

$$F(x,y) = \frac{\overbrace{\frac{1}{2}\cdot y\cdot 2\sqrt{x^2-y^2}}^{\text{TROKUT}} + \overbrace{2\cdot\frac{1}{2}\arcsin\left(\frac{y}{x}\right)\cdot x^2}^{P_{k1}}}{\frac{1}{2}R^2\pi}$$

$${}^{*}P_{k1} = \frac{\alpha}{2\pi} r^2 \pi$$

$$P_{k1} = \frac{1}{2}\alpha r^2$$

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

"BUREK: SUMNJAM DA ĆE OVO STAVITI NA ZI, ALI NA ROKU BI MOGLI" $\Downarrow$

$$\Rightarrow f(x,y) = \frac{4x^2}{R^2\pi\sqrt{x^2-y^2}}$$

$\longrightarrow$ prvo po x-u derivirati pa onda po y

$x \in (0, R)$

$y \in (0, R)$, $y \le x$

7.3. UVJETNE RAZDIOBE

motivacija: - biramo $Y \in [0,2]$
- zatim biramo $X \in [Y,2]$
- razdioba od X ?

def. Neka je $f(x,y)$ **GUSTOĆA VEKTORA** (X,Y) te neka je poznata razdioba od Y. Tada se uvjetna gustoća od X uz uvjet $Y=y$ definira s

$$\boxed{f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}}$$

ČEŠĆE ĆEMO KORISTITI: $\boxed{f(x,y) = f_{X|Y=y}(x) \cdot f_Y(y)\,dy}$

! Uoči:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y=y}(x) \cdot f_Y(y)\,dy$$

marginalna gustoća od X

def. UVJETNO OČEKIVANJE varijable X koja ovisi o realizacijama od Y je

$$\boxed{E(X|Y=y) = \int_{-\infty}^{+\infty} x \cdot f_{X|Y=y}(x)\,dx}$$

te vrijedi

$$\boxed{E(X) = \int_{-\infty}^{+\infty} E(X|Y=y)\, f_Y(y)\,dy}$$

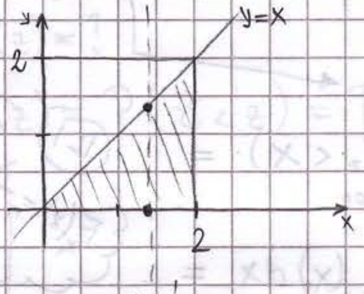
def. Vjerojatnost događaja A koji ovisi o realizacijama od X:

$$\boxed{P(A)=\int_{-\infty}^{+\infty} P(A \mid X=x)\, f_X(x)\, dx}$$

2.MI-07-7.) Biramo $y \in [0,2]$ i zatim $x \in [y,2]$. Odredi gustoću i očekivanje od X.

$$\left.\begin{array}{l} f_Y(y)=\frac{1}{2} \quad (\text{uniformna} - y \in [0,2]) \\ f_{X \mid Y=y}=\frac{1}{2-y} \end{array}\right\} f(x,y)=f_Y(y)\cdot f_{X\mid Y=y}$$

$$\Rightarrow f(x,y)=\frac{1}{2(2-y)} \qquad 0 \le y \le x \le 2$$



$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\, dy=\int_0^x \frac{1}{2(2-y)}\, dy = \cdots$$

$$f_X(x)=\frac{1}{2}\left[\ln 2-\ln(2-x)\right], \quad x \in [0,2]$$

$E(X)=?$

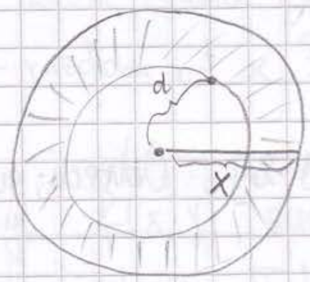
I. NAČIN: $E(X)=\int x\cdot f_X(x)\,dx=\frac{1}{2}\int_0^2 x(\ln 2-\ln(2-x))\,dx=\cdots=\frac{3}{2}$ //

II. NAČIN: (BRŽI i LJEPŠI ?) II ?

$$E(X\mid Y=y)=\int_{-\infty}^{+\infty} x\cdot f_{X\mid Y=y}(x)\,dx=\int_y^2 x\cdot\frac{1}{2-y}\,dx=\frac{1}{2}x^2\frac{1}{2-y}\Big|_y^2=\frac{y+2}{2}$$

$$\Rightarrow E(X)=\int_{-\infty}^{+\infty} E(X\mid Y=y)\, f_Y(y)\,dy=\int_0^2 \frac{1}{2}(y+2)\cdot\frac{1}{2}\,dy=\frac{1}{4}\left(\frac{y^2}{2}+2y\right)\Big|_0^2=\cdots=\frac{3}{2} \quad ///$$

2.MI-11-6.) Radijus kruga je jednoliko distribuirana varijabla na $[1,2]$. Biramo točku T unutar tog kruga. Izračunaj vjerojatnost da je udaljenost $|TS| > \frac{1}{2}$.



$$f_X(x) = \frac{1}{2-1} = 1$$

$\left(\frac{1}{2}\right)^2$

$$P(A|X=x) = \frac{x^2\pi - \frac{1}{4}\pi}{x^2\pi} = 1 - \frac{1}{4x^2}$$

$$P(A) = \int_{-\infty}^{+\infty} P(A|X=x)\, f_X(x)\, dx = \int_1^2 \left(1 - \frac{1}{4x^2}\right)\cdot 1 \cdot dx = \dots = \frac{7}{8}$$

2.MI-09-6.) Biramo X_1 i X_2 iz $[0,1]$. Neka je sluč. var. $X = \max\{X_1, X_2\}$. Zatim biramo Y iz $[0,X]$. Gustoće, očekivanje od Y?

$$f_{X_1}(x) = 1 \qquad f_{X_2}(x) = 1$$

ako je $X_2 < X$ onda je i $X_1 < X$

$0 \quad x_1 \quad x_2 \quad x \quad 1$

$$F_X(x) = P(X < x) = P(\max\{X_1, X_2\} < x) =$$

$$= P(X_1 < x, X_2 < x) = P(X_1 < x) \cdot P(X_2 < x) =$$

NEZAVISNE SU

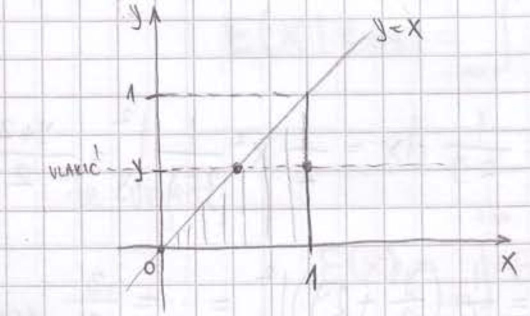
ILI $F_{X_1}(x) = \frac{x-a}{b-a}$

$$= F_{X_1}(x) \cdot F_{X_2}(x) = \int_{-\infty}^{+\infty} f_{X_1}(x)\, dx \cdot \int_{-\infty}^{+\infty} f_{X_2}(x)\, dx =$$

$$= \int_0^x dx \cdot \int_0^x dx = x \cdot x = x^2$$

uniformna na $[0,x]$

$$\Rightarrow f_X(x) = F_X'(x) = 2x \qquad f_{Y|X=x}(y) = \frac{1}{x} \quad y \in [0,x]$$



$$\Rightarrow f(x,y) = f_X(x) \cdot f_{Y|X=x}(y) = 2\cancel{x} \cdot \frac{1}{\cancel{x}} = 2$$

$$\Rightarrow f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\, dx = \int_y^1 2\, dx = 2 - 2y, \ y \in [0,1]$$

$$\Rightarrow E(Y) = \int_{-\infty}^{+\infty} y \cdot f_Y(y)\, dy = \int_0^1 y(2-2y)\, dy = \dots = \frac{1}{3}$$