Odziv sustava na jedinični impuls nazivamo impulsni odziv:

$$f(\delta[n]) = h[n]$$

Budući da je sustav vremenski stalan, odziv na pomaknuti impuls daje pomaknuti impulsni odziv:

$$f(\delta[n-k]) = h[n-k]$$

Odavde dobivamo izraz za odziv sustava:

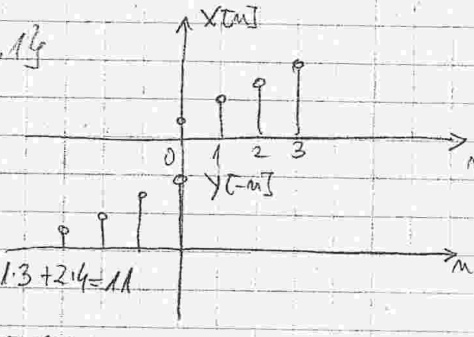
$$y[n] = \sum_{k=-\infty}^{\infty} u[k]h[n-k]$$

Ovakva sumacija naziva se KONVOLUCIJA signala $u[n]$ i $h[n]$ i općenito za bilo koja dva signala $x[n]$ i $y[n]$, konvolucija definirana na sljedeći način:

$$x[n] * y[n] = \sum_{-\infty}^{\infty} x[k]\cdot y[n-k]$$

oznaka konvolucije

b) $x[n] = \{\underline{1}, 2, 3, 4\}$, $y[n] = \{\underline{4}, 3, 2, 1\}$



$x[n] * y[n] = z[n]$

$z[0] = \sum_{k=0}^{0} x[0]y[0-0] = 4$

$z[1] = \sum_{k=0}^{1} x[k]\cdot y[n-k] = x[0]y[1] + x[1]y[0] = 1\cdot 3 + 2\cdot 4 = 11$

$z[2] = \sum_{k=0}^{2} x[k]y[n-k] = x[0]y[2] + x[1]y[1] + x[2]y[0] = 1\cdot 2 + 2\cdot 3 + 3\cdot 4 = 20$

$z[3] = \sum_{k=0}^{3} x[k]y[n-k] = x[0]y[3] + x[1]y[2] + x[2]y[1] + x[3]y[0] = 1\cdot 1 + 2\cdot 2 + 3\cdot 3 + 4\cdot 4 = 30$

$z[4] = \sum_{k=1}^{3} x[k]\cdot y[n-k] = x[1]y[3] + x[2]y[2] + x[3]y[1] = 2\cdot 1 + 3\cdot 2 + 4\cdot 3 = 20$

$z[5] = \sum_{k=2}^{3} x[k]y[n-k] = x[2]y[3] + x[3]y[2] = 3\cdot 1 + 4\cdot 2 = 11$

$h[6] = \sum_{k=3}^{3} x[k]\cdot y[n-k] = x[3]y[3] = 4\cdot 1 = 4$

$z[n] = \{\underline{4}, 11, 20, 30, 20, 11, 4\}$

PROVJERA:

	1	2	3	4
4	4	8	12	16
3	3	6	9	12
2	2	4	6	8
1	1	2	3	4

DULJINA KONVOLUCIJSKE SUMACIJE:

$$L_z = L_x + L_y - 1 = 4 + 4 - 1 = 7$$

5) TEOREM O KONVOLUCIJI ZA VREMENSKI DISKRETNU FT:

vrijedi: $\mathcal{F}(x[n] * y[n]) \circ\!\!-\!\!\bullet\ \mathcal{F}(x[n])\mathcal{F}(y[n])$

budući da je $\mathcal{F}(x[n])$ i $\mathcal{F}(y[n])$ zapravo $\mathcal{Z}$-transformacija za $z = e^{j\omega}$ onda vrijedi i sljedeće:

$\Longrightarrow$

$$x[n] * y[n] \circ\!\!-\!\!\bullet X[z] \cdot Y[z]$$

$$\text{tj. } x[n] * y[n] = \mathcal{Z}^{-1}[X[z] \cdot Y[z]]$$

$\Rightarrow$ konvolucija u vremenskoj domeni odgovara množenju u domeni transformacije

$$x_1[n] = \begin{cases} 2^{-n}, & n \geq 0 \\ 0, & \text{inače} \end{cases} \qquad x_2[n] = \begin{cases} 3^{-n}, & n \geq 0 \\ 0, & \text{inače} \end{cases}$$

$$\mathrm{DTFT}[a^n \mu[n]] = \frac{1}{1-ae^{-j\omega}} \; ; |a|<1$$

$$x_1[n] = 2^{-n}\mu[n] = \left(\frac{1}{2}\right)^n \mu[n] \Rightarrow X_1(e^{j\omega}) = \frac{1}{1-\left(\frac{1}{2}\right)e^{-j\omega}} = \frac{1}{1-\frac{1}{2}(e^{j\omega})^{-1}}$$

$-$ uz $e^{j\omega} = z$;

$$X_1[z] = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$x_2[n] = 3^{-n}\mu[n] = \left(\frac{1}{3}\right)^n \mu[n] \Rightarrow X_2(e^{j\omega}) = \frac{1}{1-\left(\frac{1}{3}\right)(e^{j\omega})^{-1}}$$

$-$ uz $e^{j\omega} = z$;

$$X_2[z] = \frac{1}{1-\frac{1}{3}z^{-1}}$$

$$x_1[n] * x_2[n] = \mathcal{Z}^{-1}(X_1[z] \cdot X_2[z])$$

$$X_1[z] \cdot X_2[z] = \frac{1}{1-\frac{1}{2}z^{-1}} \cdot \frac{1}{1-\frac{1}{3}z^{-1}} = \frac{z}{z-\frac{1}{2}} \cdot \frac{z}{z-\frac{1}{3}} = \frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{3})}$$

$$X_1[z] \cdot X_2[z] = \frac{Az}{z-\frac{1}{2}} + \frac{Bz}{z-\frac{1}{3}} = \frac{Az^2 - \frac{1}{3}Az + Bz^2 - \frac{1}{2}Bz}{(z-\frac{1}{2})(z-\frac{1}{3})} = \frac{(A+B)z^2 + (-\frac{1}{3}A - \frac{1}{2}B)z}{(z-\frac{1}{2})(z-\frac{1}{3})}$$

$$A + B = 1 \Rightarrow A = 1 - B \qquad \longrightarrow A = 1 - B = 1-(-2) = 3$$

$$-\frac{1}{3}A - \frac{1}{2}B = 0 \Rightarrow -\frac{1}{3}A = \frac{1}{2}B \,/\cdot 6$$

$$-2A = 3B$$

$$-2(1-B) = 3B$$

$$-2 + 2B = 3B$$

$$-2 = B$$

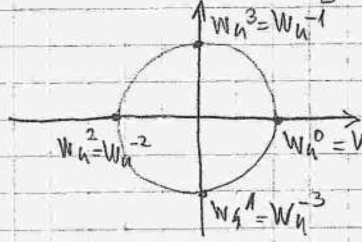
$$X_1[z] \cdot X_2[z] = \frac{3z}{z-\frac{1}{2}} + \frac{-2z}{z-\frac{1}{3}}$$

$\Rightarrow$ Inverznom Z-transformacijom dobivamo:

$$x_1[n] * x_2[n] = 3(2)^{-n}\mu[n] - 2(3)^{-n}\mu[n] = 3\left(\frac{1}{2}\right)^n \mu[n] - 2\left(\frac{1}{3}\right)^n \mu[n]$$

$$y_c[n] = IDFT_4[Y_c[k]] = \frac{1}{4}\left[2\cdot W_4^{-0n} + (4-2W_4^2)W_4^{-1n} + 2\cdot W_4^{-2n} + (4-2W_4^2)W_4^{-3n}\right]$$

$$y_c[0] = \frac{1}{4}\left[2 + 4 - 2W_4^2 + 2 + 4 - 2W_4^2\right] = \frac{1}{4}\left[12 - 2(-1) - 2(-1)\right] = \frac{1}{4}\cdot 16 = 4$$



$$y_c[1] = \frac{1}{4}\left[2\cdot W_4^0 + (4-2W_4^2)\cdot W_4^{-1} + 2\cdot W_4^{-2} + (4-2W_4^2)W_4^{-3}\right]$$

$$= \frac{1}{4}\left[2 + 4\cdot W_4^{-1} - 2W_4^1 + 2W_4^2 + 4\cdot W_4^{-3} - 2W_4^{-1}\right]$$

$W_4^2 = \cos(-\pi) - j\sin(-\pi)$
$= -1$

$$= \frac{1}{4}\left[\cancel{2} + 4\cdot W_4^{-1} - 2W_4^1 + \cancel{2(-1)} + 4\cdot W_4^1 - 2W_4^{-1}\right]$$

$$= \frac{1}{4}\left[W_4^{-1}(4-2) + W_4^1(4-2)\right] =$$

$$= \frac{1}{4}\left[2\left[W_4^{-1} + W_4^1\right]\right] = \frac{1}{2}\left[\cos\frac{\pi}{2} + j\sin\frac{\pi}{2} + \cos\left(-\frac{\pi}{2}\right) + j\sin\left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left[\underbrace{j - j}_{0}\right] = 0$$

$$y_c[2] = \frac{1}{4}\left[2 + (4-2W_4^2)\cdot W_4^{-2} + 2\cdot W_4^{-4} + (4-2W_4^2)\cdot W_4^{-6}\right] =$$

$$= \frac{1}{4}\left[2 + 4\cdot W_4^{-2} - 2 + 2\cdot W_4^{-4} + 4\cdot W_4^{-6} - 2\cdot W_4^{-4}\right] =$$

$$= \frac{1}{4}\left[\cancel{2} + 4\cdot W_4^2 \cancel{-2} + \cancel{2} + 4\cdot W_4^2 \cancel{-2}\right] = \cancel{\frac{1}{4}}\left[\cancel{8}\cdot W_4^2\right] = 2\cdot(W_4^2) = 2\cdot(-1) = -2$$

$$y_c[3] = \frac{1}{4}\left[2 + (4-2W_4^2)\cdot W_4^{-3} + 2\cdot W_4^{-6} + (4-2W_4^2)\cdot W_4^{-9}\right] =$$

$$= \frac{1}{4}\left[2 + 4\cdot W_4^1 - 2\cdot W_4^{-1} + 2\cdot W_4^{-2} + 4\cdot W_4^{-1} - 2\cdot W_4^{-3}\right] =$$

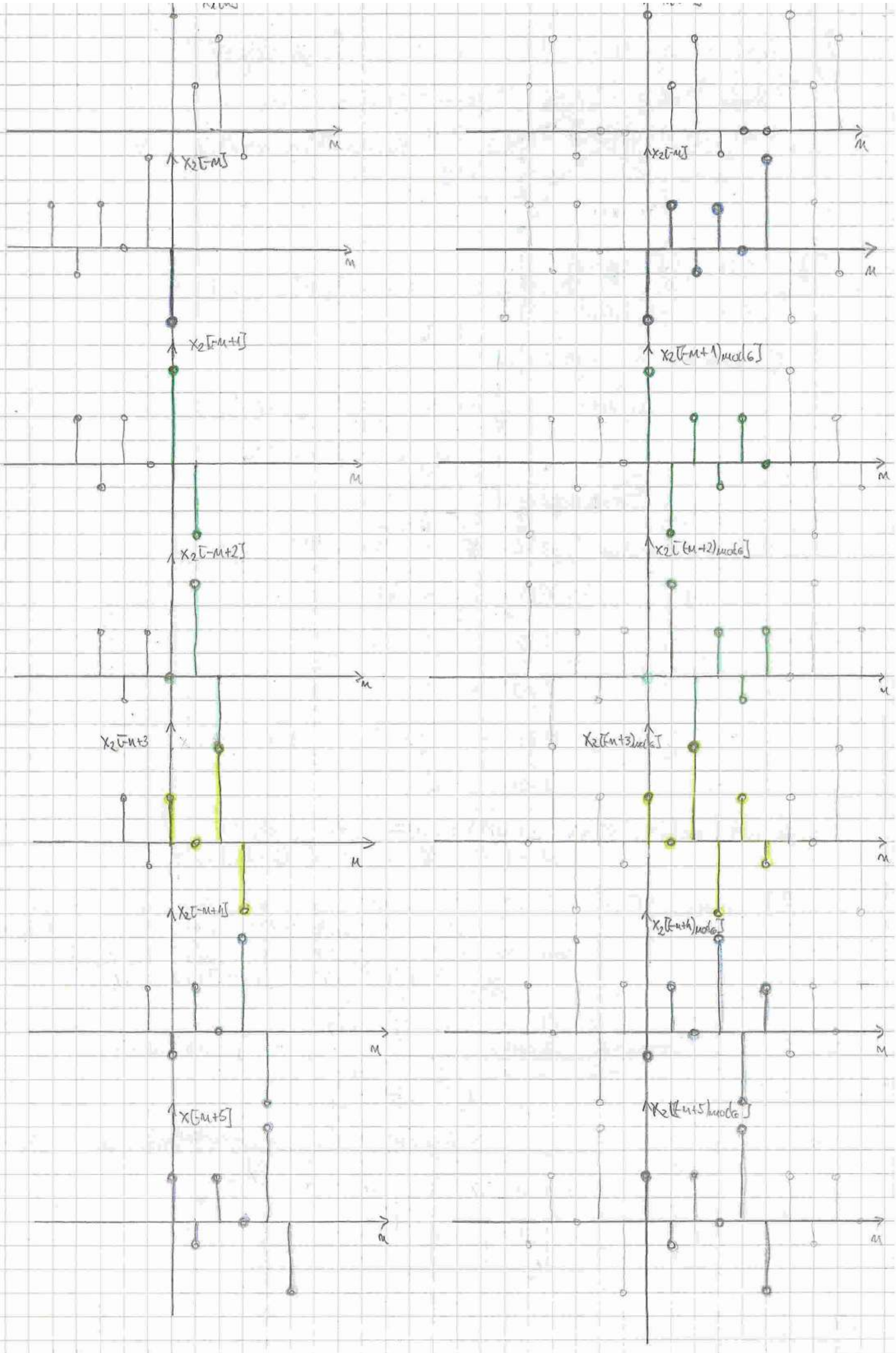
$$= \frac{1}{4}\left[2 + 4\cdot W_4^1 - 2\cdot W_4^3 + 2W_4^2 + 4\cdot W_4^3 - 2\cdot W_4^1\right] =$$

$$= \frac{1}{4}\left[2 + 2\cdot W_4^1 + 2W_4^3 - 2\right] = \frac{1}{4}\left[2\left(W_4^1 + W_4^3\right)\right] = \frac{1}{2}\left[\cos\left(-\frac{\pi}{2}\right) + j\sin\left(-\frac{\pi}{2}\right)\right.$$

$$\left. + \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right)\right] = \frac{1}{2}\left[-j + j\right] = 0$$

CIRKULARNA KONVOLUCIJA:

$$y_c[n] = \{4, 0, -2, 0\}$$



x2[-n]
x2[-n+1]
x2[-n+2]
x2[-n+3
x2[-n+4]
x[-n+5]
x2[-n]
x2[(-n+1)mod6]
x2[(-n+2)mod6]
x2[(-n+3)mod6]
x2[(-n+4)mod6]
x2[(-n+5)mod6]

$$\cancel{X_c[k]\cdot W_6^{-mk}}$$

$$\cancel{X_c[k] = -17+23\cdot W_6^{k} - 6W_6^{2k} + 29W_6^{3k} - 5W_6^{4k} + 16W_6^{5k}}$$

$$\cancel{-17W_6^{-mk} + 23\cdot W_6^{k(1-m)} - 6W_6^{k(2-m)} + 29W_6^{k(3-m)} - 5W_6^{k(4-m)} + 16W_6^{(5-m)k}}$$

~~????~~

$$X_c[k] = -17W_6^{0} + 23\cdot W_6^{k} - 6W_6^{2k} + 29W_6^{3k} - 5W_6^{4k} + 16\cdot W_6^{5k}$$

$$e^{j\frac{2\pi}{N}mk} = W_6^{mk}$$

$$e^{j\omega} = z$$
$$e^{-j\omega} = z^{-1}$$
$$\left(e^{j\frac{\pi}{3}k}\right)^{-m} \Rightarrow z = e^{j\frac{\pi}{3}k}$$

$$X_c[k] = -17\cdot\left(e^{j\frac{\pi}{3}k}\right)^{0} + 23\cdot\left(e^{j\frac{\pi}{3}k}\right)^{-1} - 6\left(e^{j\frac{\pi}{3}k}\right)^{-2} + 29\left(e^{j\frac{\pi}{3}k}\right)^{-3}$$
$$-5\cdot\left(e^{j\frac{\pi}{3}k}\right)^{-4} + 16\cdot\left(e^{j\frac{\pi}{3}k}\right)^{-5}$$
$$= -17z^{0} + 23\cdot z^{-1} - 6\cdot z^{-2} + 29\cdot z^{-3} - 5\cdot z^{-4} + 16\cdot z^{-5}$$

$$\Rightarrow x_c[m] = \{\underline{-17}, 23, -6, 29, -5, 16\}$$

! uvijek je lakše $\Rightarrow$ direktno iz $X_c[k]$ očitano koeficijenta

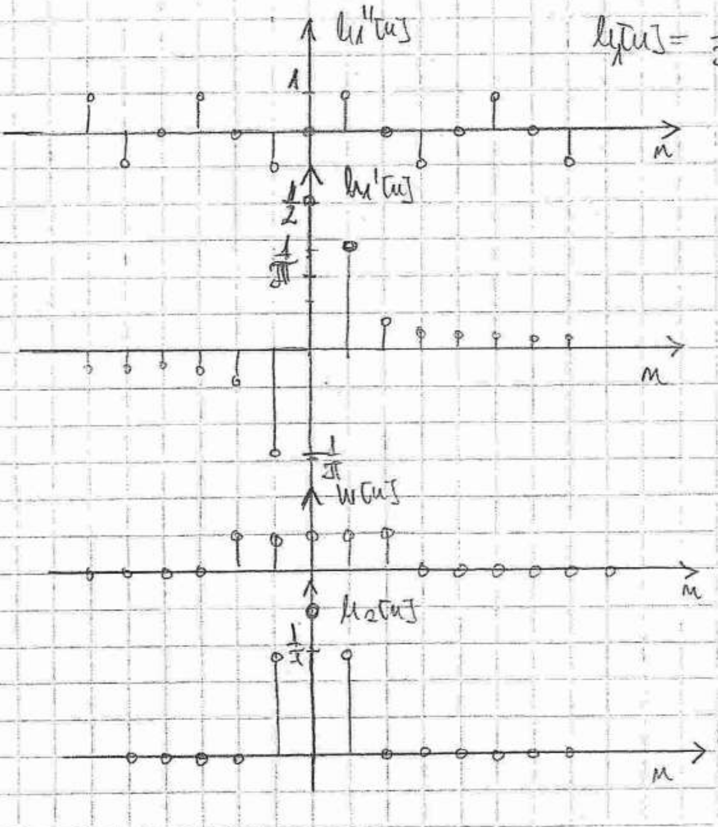
16) C_N ... matrica DCT transformacije

C_N^{-1} ... matrica IDCT transformacije

za matricu $C_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$ pokazujemo da vrijedi: $C_3\cdot C_3^T = C_3^T\cdot C_3 = I$

$$C_3\cdot C_3^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}\cdot\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = C_3^T\cdot C_3 \Rightarrow$$

tada vrijedi: $C_3^{-1} = C_3^T$



$$h_1[n] = \frac{1}{\pi n}\sin\left(\frac{\pi}{2}n\right) = h_1'[n]\cdot h_1''[n]$$

$$h_1'[n] = \frac{1}{\pi n}, \quad h_1''[n] = \sin\left(\frac{\pi}{2}n\right)$$

$$h_1[n]\Big|_0 = \lim_{n\to 0}\frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n} \underset{\text{L'Hopital}}{=} \lim_{n\to 0}\frac{\cos\left(\frac{\pi}{2}n\right)\cdot\frac{\pi}{2}}{\pi} = \frac{1}{2}$$

$$h_2[n] = h_1[n]\cdot w[n] = \left\{\frac{1}{\pi}, \underline{\frac{1}{2}}, \frac{1}{\pi}\right\}$$

$h_2'[n]$ ... kauzalna varijanta impulsnog odziva pomaknutog [illegible]
→ pomak za jedno mjesto udesno

$$h_2'[n] = \left\{\underline{\frac{1}{\pi}}, \frac{1}{2}, \frac{1}{\pi}\right\}$$

$$H_2(e^{j\omega}) = \sum_{n=0}^{2} h_2'[n]\, e^{-j\omega n}$$

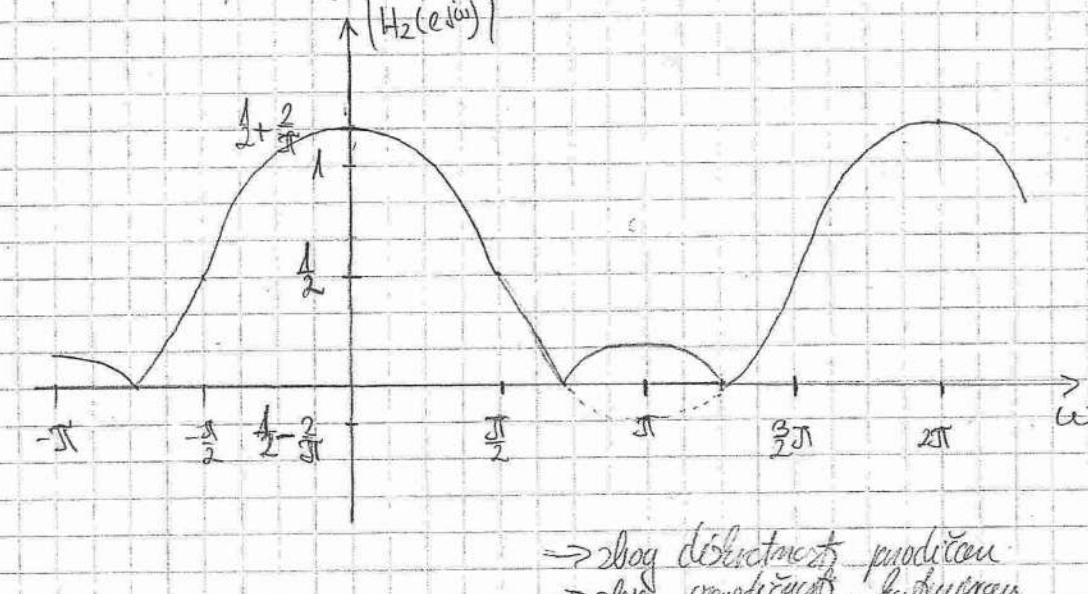
$$= \frac{1}{\pi}\cdot e^{0} + \frac{1}{2}\cdot e^{-j\omega} + \frac{1}{\pi}\cdot e^{-2j\omega}$$

$$= \frac{1}{\pi}e^{-j\omega}\left(e^{j\omega} + e^{-j\omega}\right) + \frac{e^{-j\omega}}{2}$$

$$= e^{-j\omega}\left[\frac{1}{\pi}\left(2\cos(\omega)\right) + \frac{1}{2}\right]$$

$$\left|H_2(e^{j\omega})\right| = \left|e^{-j\omega}\right|\left|\frac{1}{2} + \frac{1}{\pi}(2\cos\omega)\right|$$

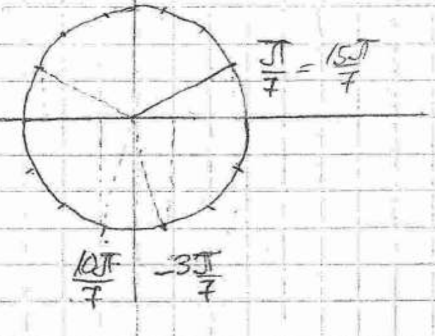
Skica amplitudnog spektra:



→ zbog diskretnosti periodičan
→ zbog aperiodičnosti kontinuiran

$$X[2,2] = \sqrt{\frac{4}{7}} \cdot x[2] \cdot 1 \cdot \cos\left(\frac{10\pi}{7}\right) = \sqrt{\frac{4}{7}} \cdot x[2] \cdot (-1) \cdot \cos\left(-\frac{3\pi}{7}\right) = -\sqrt{\frac{4}{7}}\, x[2] \cdot \cos\left(\frac{3\pi}{7}\right)$$

$$X[2,3] = \sqrt{\frac{4}{7}} \cdot x[3] \cdot \frac{1}{\sqrt{2}} \cos\left(\frac{14\pi}{7}\right) = \sqrt{\frac{2}{7}}\, x[3]$$



$$X[3,0] = \sqrt{\frac{4 - 2\delta[3]}{2\cdot 4 - 1}} \cdot x[0] \cdot \frac{1}{\sqrt{1+\delta[0-4+1]}} \cdot \cos\left(\frac{(2\cdot 0+1)\cdot 3\pi}{2\cdot 4-1}\right)$$

$$= \sqrt{\frac{4}{7}} \cdot x[0] \cdot 1 \cdot \cos\left(\frac{3\pi}{7}\right) = \frac{2}{\sqrt{7}} \cdot x[0] \cdot \cos\left(\frac{3\pi}{7}\right)$$

$$X[3,1] = \sqrt{\frac{4}{7}}\, x[1] \cdot \cos\left(\frac{9\pi}{7}\right) = -\frac{2}{\sqrt{7}}\, x[1] \cdot \cos\left(\frac{2\pi}{7}\right)$$

$$X[3,2] = \sqrt{\frac{4}{7}}\, x[2] \cdot \cos\left(\frac{15\pi}{7}\right) = \frac{2}{\sqrt{7}}\, x[2] \cdot \cos\left(\frac{\pi}{7}\right)$$

$$X[3,3] = \sqrt{\frac{4}{7}}\, x[3] \cdot \frac{1}{\sqrt{2}} \cdot \cos\left(\frac{21\pi}{7}\right) = \sqrt{\frac{2}{7}}\, x[3] \cdot \cos(3\pi) = -\sqrt{\frac{2}{7}}\, x[3]$$

$$DCT_{TM} = \begin{bmatrix} \sqrt{\frac{2}{7}} & \sqrt{\frac{2}{7}} & \sqrt{\frac{2}{7}} & \frac{1}{\sqrt{7}} \\ \frac{2}{\sqrt{7}}\cos\left(\frac{\pi}{7}\right) & \frac{2}{\sqrt{7}}\cos\left(\frac{3\pi}{7}\right) & \frac{2}{\sqrt{7}}\cos\left(\frac{5\pi}{7}\right) & -\sqrt{\frac{2}{7}} \\ \frac{2}{\sqrt{7}}\cos\left(\frac{2\pi}{7}\right) & -\frac{2}{\sqrt{7}}\cos\left(\frac{\pi}{7}\right) & -\frac{2}{\sqrt{7}}\cos\left(\frac{3\pi}{7}\right) & \sqrt{\frac{2}{7}} \\ \frac{2}{\sqrt{7}}\cos\left(\frac{3\pi}{7}\right) & -\frac{2}{\sqrt{7}}\cos\left(\frac{2\pi}{7}\right) & \frac{2}{\sqrt{7}}\cos\left(\frac{\pi}{7}\right) & -\sqrt{\frac{2}{7}} \end{bmatrix}$$

Budući da ulazni signal ima samo 1 uzorak različit od 0, i iznos vrijednosti je 1, DCT tog signala jednaka je prvom stupcu DCT transformacijske matrice:

$$X[k] = DCT_{TM} \cdot x[n] = \left\{ \underline{\sqrt{\frac{2}{7}}} \quad \frac{2}{\sqrt{7}}\cos\left(\frac{\pi}{7}\right) \quad \frac{2}{\sqrt{7}}\cos\left(\frac{2\pi}{7}\right) \quad \frac{2}{\sqrt{7}}\cos\left(\frac{3\pi}{7}\right) \right\}$$

$$= \{ \underline{0{,}535} \quad 0{,}681 \quad 0{,}471 \quad 0{,}168 \}$$

15)

Kosinusna transformacija niza duljine N=4:

$$X[k] = DCT_4[x[n]] = DCT\text{-}II\,(N=4) = \sqrt{\frac{2-\delta[k]}{4}} \sum_{n=0}^{3} x[n] \cdot \cos\left(\frac{(2n+1)k\pi}{8}\right)$$

Inverzna kosinusna transformacija koeficijenata duljine N=4:

$$IDCT_4[X[k]] = DCT\text{-}III = x[n] = \sqrt{\frac{1}{2}} \sum_{k=0}^{3} X[k] \cdot \frac{1}{\sqrt{1+\delta[k]}} \cos\left(\frac{(2n+1)k\pi}{8}\right)$$

$$X[k] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} & \frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} & -\frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} & -\frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} & -\frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} & \frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} & -\frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}}\sqrt{2+\sqrt{2}} \\ \frac{1}{2\sqrt{2}}\sqrt{2-\sqrt{2}} \\ -\frac{1}{2\sqrt{2}}\sqrt{2-\sqrt{2}} \\ -\frac{1}{2\sqrt{2}}\sqrt{2+\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{4\sqrt{2}}\sqrt{2+\sqrt{2}} + \frac{1}{4\sqrt{2}}\sqrt{2-\sqrt{2}} - \frac{1}{4\sqrt{2}}\sqrt{2-\sqrt{2}} - \frac{1}{4\sqrt{2}}\sqrt{2+\sqrt{2}} \\ \frac{1}{8}(2+\sqrt{2}) + \frac{1}{8}(2-\sqrt{2}) + \frac{1}{8}(2-\sqrt{2}) + \frac{1}{8}(2+\sqrt{2}) \\ \frac{1}{4\sqrt{2}}\sqrt{2+\sqrt{2}} - \frac{1}{4\sqrt{2}}\sqrt{2-\sqrt{2}} + \frac{1}{4\sqrt{2}}\sqrt{2-\sqrt{2}} - \frac{1}{4\sqrt{2}}\sqrt{2+\sqrt{2}} \\ \frac{1}{8}(4-2) \quad -\frac{1}{8}(4-2) \quad -\frac{1}{8}(4-2) \quad \frac{1}{8}(4-2) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- budući da je matrica transformacije ortogonalna $\Rightarrow$ matrica inverzne transformacije je transponirana matrica transformacije

$IDCT_4$:

$$x[n] = IDCT_4^{-1}\, X[k] \quad ; \quad IDCT_4^{-1} = DCT^{T}$$

$$x[n] = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} & \frac{1}{2} & \frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} \\ \frac{1}{2} & \frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} & -\frac{1}{2} & -\frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} \\ \frac{1}{2} & -\frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} & -\frac{1}{2} & \frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} \\ \frac{1}{2} & -\frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} & \frac{1}{2} & -\frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} \\ \frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} \\ -\frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} \\ -\frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{2}}\sqrt{2+\sqrt{2}} \\ \frac{1}{2\sqrt{2}}\sqrt{2-\sqrt{2}} \\ -\frac{1}{2\sqrt{2}}\sqrt{2-\sqrt{2}} \\ -\frac{1}{2\sqrt{2}}\sqrt{2+\sqrt{2}} \end{bmatrix} = \left\{ \frac{1}{2}\sqrt{1+\frac{\sqrt{2}}{2}}, \frac{1}{2}\sqrt{1-\frac{\sqrt{2}}{2}}, -\frac{1}{2}\sqrt{1-\frac{\sqrt{2}}{2}}, -\frac{1}{2}\sqrt{2+\frac{\sqrt{2}}{2}} \right\}$$

b) Matrica kosinusne transformacije jednaka je za $N=4$ bez obzira o kojem signalu se radi $\Rightarrow$ možemo koristiti DCT i IDCT matrice iz prethodnog zadatka

$$x[n] = \{ \underline{0}, 0, 0, 1 \}$$

DCT_4:

$$X[k] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} & \frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} & -\frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} & -\frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} & -\frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} & \frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} & -\frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} \\ -\frac{1}{2} \\ \frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} \end{bmatrix}$$

$$\mathrm{IDCT}_4: \quad x[n] = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} & \frac{1}{2} & \frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} \\ \frac{1}{2} & \frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} & -\frac{1}{2} & -\frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} \\ \frac{1}{2} & -\frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} & -\frac{1}{2} & \frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} \\ \frac{1}{2} & -\frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} & \frac{1}{2} & -\frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{2}}{4}\sqrt{2-\sqrt{2}} \\ -\frac{1}{2} \\ \frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \cancel{\frac{1}{2}\frac{1}{2}} + \frac{\sqrt{2}}{4}\cdot\left(-\frac{\sqrt{2}}{4}\right)(4-2) + \cancel{\left(-\frac{1}{2}\right)\frac{1}{2}} + \frac{\sqrt{2}}{4}\left(\frac{\sqrt{2}}{4}\right)\cdot(4-2) \\ \frac{1}{4} + \left(-\frac{2}{16}\right)(2-\sqrt{2}) + \frac{1}{4} + \left(-\frac{2}{16}\right)(2+\sqrt{2}) \\ \frac{1}{4} + \frac{2}{16}(2-\sqrt{2}) + \frac{1}{4} + \frac{2}{16}(2+\sqrt{2}) \\ \frac{1}{4} + \frac{2}{16}(4-2) - \frac{1}{4} - \frac{2}{16}(4-2) \end{bmatrix} \begin{matrix} \to 0 \\ \to 0 \\ \to \underbrace{\frac{2}{4} + \frac{2}{16}(2-\cancel{\sqrt{2}}+2+\cancel{\sqrt{2}})}_{1} \\ \to 0 \end{matrix}$$

$$\Rightarrow x[n] = \{0, 0, 1, 0\}$$

$$\Longrightarrow \quad \begin{aligned} X[k] &= \mathrm{DCT}_4[x[n]] \\ x[n] &= \mathrm{IDCT}_4[X[k]] \end{aligned}$$

19) $\mathrm{DFT}_N: \; x[n] \,\circledN\, y[n] \;\circ\!\!-\!\!\bullet\; X[k]\,Y[k] \Rightarrow x[n] \,\circledN\, y[n] = \mathrm{IDFT}_N\left[\mathrm{DFT}[x[n]] \cdot \mathrm{DFT}[y[n]]\right]$

$\mathrm{DCT}_N: \; x[n] \,\circledN\, y[n] \neq \mathrm{IDCT}_N\left[\mathrm{DCT}[x[n]] \cdot \mathrm{DCT}[y[n]]\right]$

→ objašnjeno na pismenom

$$X[2] = \left[0\ldots+0\ldots+0\ldots+e^{-j\frac{\pi}{4}\cdot 2\cdot 3}+0\ldots+e^{-j\frac{\pi}{4}\cdot 2\cdot 5}+0\ldots+0\ldots\right] =$$

$$= \cos\left(\tfrac{3}{2}\pi\right) - j\sin\left(\tfrac{3}{2}\pi\right) + \cos\left(\tfrac{5}{2}\pi\right) - j\sin\left(\tfrac{5}{2}\pi\right) =$$

$$= \underbrace{\cos\left(\tfrac{3}{2}\pi\right)}_{0} - j\underbrace{\sin\left(\tfrac{3}{2}\pi\right)}_{-1} + \cos\left(\tfrac{3}{2}\pi\right) + j\underbrace{\sin\left(\tfrac{3}{2}\pi\right)}_{-1} = 0$$

⋮

- dalje ne treba računati jer nam trebaju samo prvih N (a u ovom slučaju N=2)

$$\Rightarrow \text{DCT-II}(x[n]) = X_C[k] = \left\{\frac{1}{\sqrt{2}}, -\frac{\sqrt{2}}{2}\right\}$$

$$\text{DFT}_{4N}(x'[n']) = X_F[k] = \{2, -\sqrt{2}, 0, \ldots\}$$

$$\Rightarrow \text{pomnožimo } X_F[k] \text{ s } \frac{\sqrt{\frac{2-\delta[k]}{N}}}{2} = \frac{\sqrt{\frac{2-\delta[k]}{2}}}{2}$$

$$\frac{\sqrt{\frac{2-\delta[k]}{2}}}{2} \cdot X_F[k] = \left\{\frac{\sqrt{\frac{2-1}{2}}}{2}\cdot 2, \frac{\sqrt{\frac{2-0}{2}}}{2}\cdot(-\sqrt{2}), \frac{\sqrt{\frac{2-0}{2}}}{2}\cdot 0, \ldots\right\}$$

$$= \left\{\frac{1}{\sqrt{2}}, -\frac{\sqrt{2}}{2}, 0, \ldots\right\}$$

- pitanje je još bitno naglasiti da $x[n]$ mora biti realan jer inače ne možemo koristiti DCT

18) DCT-I: $$X_C[k] = \sqrt{\frac{2-\delta[k]-\delta[k-N+1]}{N-1}} \cdot \sum_{n=0}^{N-1} x[n] \cdot \frac{1}{\sqrt{1+\delta[n]+\delta[n-N+1]}} \cdot \cos\left(\frac{nk\pi}{N-1}\right), \; k = 0, \ldots, 2$$

za $n=1$ i $n=N-1$ imamo je $\frac{1}{\sqrt{2}}$ inače je 1

za $k=0$ ili $k=N-1$ imamo $\sqrt{\frac{1}{N-1}}$

$$\text{DFT}_{2N-2}: \; X_F[k] = \sum_{n=0}^{(2N-2)-1} x[n]\cdot e^{-j\left(\frac{2\pi}{2N-2}\cdot nk\right)} = \sum_{n=0}^{(2N-2)-1} x[n]\cdot e^{-\left(\frac{\pi}{N-1}\cdot nk\right)}$$

$\Rightarrow$ treba primijetiti da je za:

$$n=0 \quad x[0]\cdot \underbrace{e^{-j\left(\frac{\pi}{N-1}\cdot 0\cdot k\right)}}_{\to 1} = x[0], \text{ za svaki } k.$$

$$n=N-1 \quad x[N-1]\cdot e^{-j\left(\frac{\pi}{N-1}(N-1)k\right)} = x[N-1]\cdot e^{-j(\pi k)} =$$

$$= x[N-1]\cdot\left[\underbrace{\cos(\pi k)}_{(-1)^k} - \underbrace{j\sin(\pi k)}_{0 \text{ za svaki } k}\right] = (-1)^k\cdot x[N-1]$$

$\Rightarrow$ trebamo jednačiti ... DCT-a i DFT-a. $\Rightarrow$ moramo se riješiti sinusa iz DFT $\Rightarrow$ parno proširenje

$\Rightarrow$ Suma DFT_{4N} zahtjeva 4N uzoraka, pričemu je prvi u $n'=0$ $\Rightarrow$ radi se o pomaku signala, potrebno je prosto periodično proširenje s periodom 4N.

- dakle, ako od ovako dobivenih uzoraka (uzorci umetnuti s) provedemo DFT_{4N} dobit ćemo $X[k]$ (isto 4N uzoraka).

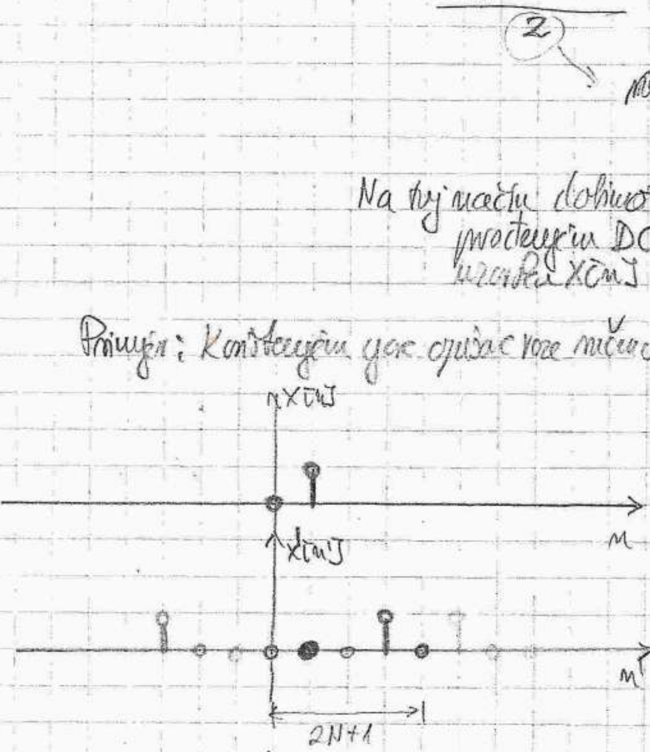
Od dobivenih $X[k]$ uzoraka, treba izdvojiti prvih N te ih pomnožiti izrazom:

$$\frac{\sqrt{\frac{2-\delta[k]}{2}}}{2}$$

množenje DFT-a (jer smo prošli prošireno proširenje) podijeliti s brojem

Na taj način dobivamo koeficijente transformiranih uzoraka koje bismo dobili provedbom DCT-II transformacije nad originalnim uzorcima $x[n]$

Primjer: Konstrukcija gore opisane veze među DCT-II i DFT_{4N} za niz: $x[n]=\{0,1\}$



DCT-II: $X[k]=\sqrt{\frac{2-\delta[k]}{2}}\sum_{n=0}^{1} x[n]\cdot\cos\left(\frac{(2n+1)\cdot k\pi}{4}\right) \quad k=0,1$

$$X[0]=\sqrt{\frac{2-1}{2}}\left[x[0]\cdot\cos\left(\frac{(2\cdot 0+1)\cdot 0\cdot\pi}{4}\right)+x[1]\cdot\cos\left(\frac{(2\cdot 1+1)\cdot 0\cdot\pi}{4}\right)\right]=\frac{1}{\sqrt{2}}$$

$$X[1]=\sqrt{\frac{2-0}{2}}\left[x[0]\cdot\cos(\dots)+x[1]\cdot\cos\left(\frac{(2\cdot 1+1)\cdot\pi}{4}\right)\right]=\sqrt{\frac{2}{2}}\cos\left(\frac{3\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

DFT_{4N}: $X[k]=\sum_{n'=0}^{4N-1} x[n']\,e^{-j\frac{2\pi}{4N}kn'}=\sum_{n'=0}^{7} x[n']\,e^{-j\frac{\pi}{4}kn'} \quad ; k=0,\dots,7$

$$X[0]=\left[0\cdot\ldots+0\cdot\ldots+0\cdot\ldots+1e^{-j\frac{\pi}{4}\cdot 0\cdot 3}+0\cdot\ldots+1\cdot e^{-j\frac{\pi}{4}\cdot 0\cdot 5}+0\cdot\ldots+0\cdot\ldots\right]=2$$

$$X[1]=\left[0\cdot\ldots+0\cdot\ldots+0\cdot\ldots+1\cdot e^{-j\frac{\pi}{4}\cdot 1\cdot 3}+0\cdot\ldots+1\cdot e^{-j\frac{\pi}{4}\cdot 1\cdot 5}+0\cdot\ldots+0\cdot\ldots\right]=$$

$$=\cos\left(\frac{\pi}{4}3\right)-j\sin\left(\frac{\pi}{4}3\right)+\cos\left(\frac{5}{4}\pi\right)-j\sin\left(\frac{5}{4}\pi\right)=$$

$$=\cos\left(\frac{\pi}{4}3\right)-\cancel{j\sin\left(\frac{\pi}{4}3\right)}+\cos\left(\frac{\pi 3}{4}\right)+\cancel{j\sin\left(\frac{3\pi}{4}\right)}=2\cdot\cos\left(\frac{\pi}{4}3\right)=2\left(-\frac{\sqrt{2}}{2}\right)=-\sqrt{2}$$

$X_1[k] = DFT_8 = \sqrt{2} + 4\cdot\cos\left(\frac{\pi}{4}k\right) - 4\cos\left(\frac{3\pi}{4}k\right) + (-1)^k$

računamo za $k = 0, 1, 2, 3, 4$:

$k = 0$: $\cancel{\frac{1}{2}\left(\sqrt{2}+4-4+1\right)\cdot\sqrt{\frac{1}{4}} = \frac{1}{4}\cdot(\sqrt{2}+1)}$ $\quad \frac{1}{2}\sqrt{\frac{1}{4}}\left(\sqrt{2}+1-1+\sqrt{2}\right) = \frac{1}{\sqrt{2}}$

$k = 0$: $\cancel{\frac{1}{2}\left(\sqrt{2}+2\sqrt{2}+2\sqrt{2}-1\right)\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}\left(5\sqrt{2}-1\right) = \frac{5}{4}\cdot\frac{\sqrt{2}}{4} = \frac{5\cdot\sqrt{2}}{4}}$

$\cancel{2,2,2}$ $\quad \frac{1}{2}\cdot\frac{\sqrt{2}}{2}\left(\sqrt{2}+2\sqrt{2}+2\sqrt{2}-\sqrt{2}\right) = \frac{\sqrt{2}}{4}\cdot 4\sqrt{2} = \sqrt{2^2} = 2$

19) DFT_N: $\quad x[n] \circledast_N y[n] \circ\!\!-\!\!\bullet\ X[k]Y[k] \Rightarrow x[n] \circledast_N y[n] = IDFT_N\left[DFT_N[x[n]]\cdot DFT_N[y[n]]\right]$

DCT_N: $\quad x[n] \circledast_N y[n] \neq IDCT_N\left[DCT_N[x[n]]\cdot DCT_N[y[n]]\right]$

→ pokazujemo na primjeru:

$x[n] = \{\underline{2}, 1, 1, 2\}$; $y[n] = \{\underline{1}, -1, -1, 1\}$

DCT_4 možemo računati pomoću matrice transformacije C_4 koja je ista za sve nizove duljine 4 i koje prethodno izračunate:

$$X[k] = \begin{bmatrix} 0{,}5 & 0{,}5 & 0{,}5 & 0{,}5 \\ 0{,}653 & 0{,}271 & -0{,}271 & -0{,}653 \\ 0{,}5 & -0{,}5 & -0{,}5 & 0{,}5 \\ 0{,}271 & -0{,}653 & 0{,}653 & -0{,}271 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad Y[k] = \begin{bmatrix} 0{,}5 & 0{,}5 & 0{,}5 & 0{,}5 \\ 0{,}653 & 0{,}271 & -0{,}271 & -0{,}653 \\ 0{,}5 & -0{,}5 & -0{,}5 & 0{,}5 \\ 0{,}271 & -0{,}653 & 0{,}653 & -0{,}271 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$R[k] = X[k]\cdot Y[k] = \{\underline{3}, 0, 1, 0\}\cdot\{\underline{0}, 0, 2, 0\} = \{\underline{0}, 0, 2, 0\}$

$IDCT_4$ se također može računati pomoću inverzne matrice C_4^{-1}. Budući da je matrica C_4 ortogonalna, inverzna matrica je zapravo transponirana matrica C_4:

$$r[k] = \begin{bmatrix} 0{,}5 & 0{,}653 & 0{,}5 & 0{,}271 \\ 0{,}5 & 0{,}271 & -0{,}5 & -0{,}653 \\ 0{,}5 & -0{,}271 & -0{,}5 & 0{,}653 \\ 0{,}5 & -0{,}653 & 0{,}5 & -0{,}271 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Brži način računanja cirkularne konvolucije:

→ okrećemo drugi signal

2	1	1	2	
1	1	-1	-1	→ 0
-1	1	1	-1	→ -2
-1	-1	1	1	→ 0
1	-1	-1	1	→ 2

⇒ $x[n] \circledast_N y[n] = \{\underline{0}, -2, 0, 2\}$

$IDCT_4\left[DCT_4[x[n]]\cdot DCT_4[y[n]]\right] = \{\underline{1}, -1, -1, 1\}$

Što očito nije jednako

$x[n] = \{1, 2, 0, -2, 1\}$ → za $k = N-1$ i $k = 0$ vrijednost je $\sqrt{\frac{1}{N-1}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$, inače $\frac{\sqrt{2}}{2}$

DCT-I: $$X_c[k] = \sqrt{\frac{2-\delta[k]-\delta[k-N+1]}{N-1}} \cdot \sum_{n=0}^{N-1} x[n] \cdot \frac{1}{\sqrt{1+\delta[n]+\delta[n-N+1]}} \cos\left(\frac{nk\pi}{N-1}\right)$$

→ $\frac{1}{\sqrt{2}}$ samo za $n=0$ i $n=N-1$

→ može se riješiti množenjem signala matricom 5×5:

$$\begin{array}{c c} k \downarrow \; n \rightarrow & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2} & \underbrace{\frac{\sqrt{2}}{2}\cos\left(\frac{\pi}{4}\right)}_{\Downarrow \frac{\sqrt{2}}{2}} & 0 & -\frac{\sqrt{2}}{2}\cos\left(\frac{\pi}{4}\right) & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{2}}{2}\cos\left(\frac{\pi}{4}\right) & 0 & \frac{\sqrt{2}}{2}\cos\left(\frac{\pi}{4}\right) & -\frac{1}{2} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2\sqrt{2}} \end{bmatrix} \end{array}$$

$$X_c[k] = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 2 & 1 & -2 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Parno, simetrično proširenje signala jest: $x'[n] = \{1, 2, 0, -2, 1, -2, 0, 2\}$

Množimo prvi i N-ti član s $\sqrt{2}$ odnosno: $x''[n] = \{\sqrt{2}, 2, 0, -2, \sqrt{2}, -2, 0, 2\}$

→ $(-1)^k$ → $(-1)^k \cdot \sqrt{2}$

$$DFT_{\underset{8}{2N-2}}[x''[n]] = \sqrt{2} \cdot \underbrace{W_8^{0 \cdot k}}_{1} + 2 \cdot W_8^{k} + 2 \cdot W_8^{3k} + \sqrt{2}\, W_8^{4k} - 2 W_8^{5k} + 2 \cdot W_8^{7k}$$

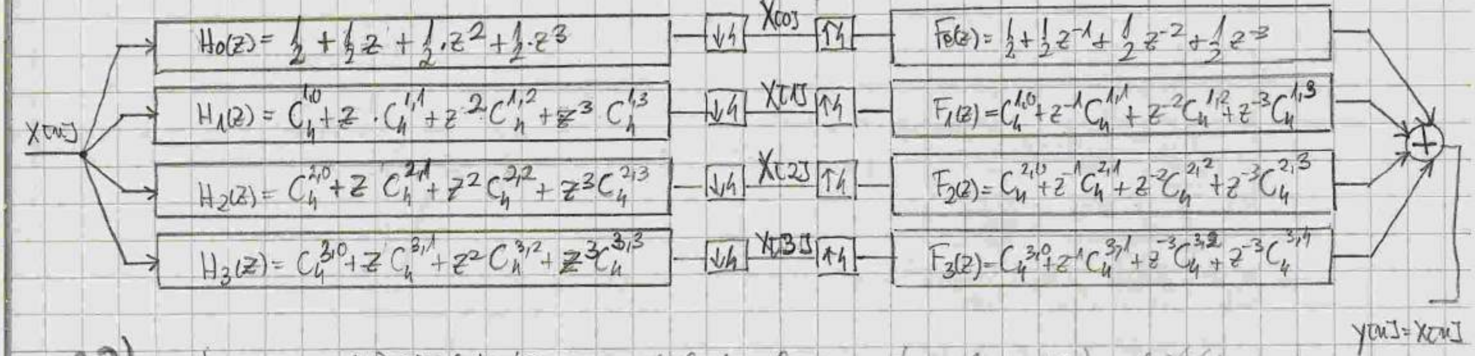
$$W_8^{3k} = e^{-j\frac{2\pi}{8} \cdot 3k} = \cos\left(\frac{3\pi}{4}k\right) - j\sin\left(\frac{3\pi}{4}k\right)$$

$$W_8^{5k} = e^{-j\frac{2\pi}{8} 5k} = \cos\left(\frac{5\pi}{4}k\right) - j\sin\left(\frac{5\pi}{4}k\right) = \cos\left(\frac{3\pi}{4}k\right) + j\sin\left(\frac{3\pi}{4}k\right)$$

$$W_8^{k} = e^{-j\frac{2\pi}{8} k} = \cos\left(\frac{\pi}{4}k\right) - j\sin\left(\frac{\pi}{4}k\right)$$

$$W_8^{7k} = e^{-j\frac{2\pi}{8} 7k} = \cos\left(\frac{\pi}{4}7k\right) - j\sin\left(\frac{\pi}{4}7k\right) = \cos\left(\frac{\pi}{4}k\right) + j\sin\left(\frac{\pi}{4}k\right)$$

FILTARSKI SLOG izgleda ovako:



22) Izrazi za MDCT (Modificiranu diskretnu kosinusnu transformaciju) su sljedeći:

$$X[k] = \sum_{n=0}^{2N-1} x[n] \cdot \cos \frac{(2n+1+N)(2k+1)\pi}{4N}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot \cos \frac{(2n+1+N)(2k+1)\pi}{4N}$$

(inverz je sličan, X[k] se skalira s $\frac{1}{N}$, suma do N-1, zamjena n i k)

- budući da na obradu blokova mogu nastati diskontinuiteti, ključno je preklapanje blokova što je ovdje slučaj

$\Rightarrow$ transformacijske matrice bit će dimenzija $N \times 2N$ odnosno $2N \times N$

- uz to primjenjuje se i vremenski otvor na ulazni niz (ulazni signal) i na rekonstruirani signal (prije MDCT-a i nakon IMDCT)

Vremenski otvor u ovom slučaju je $w[n] = \sin \frac{(2n+1)\pi}{4N}$

- budući da se analizirajući filtar i rekonstruirajući filtar grade na temelju MDCT, odnosno IMDCT matrice, može se množenje sa vremenskim otvorom ugraditi u filtar $\Rightarrow$

- svaki blok analizirajućeg FIR filtra množi se s $w[n]$, a svaki blok rekonstruirajućeg filtra množi se s $2 \cdot w[n]$

- vremenski otvor mora zadovoljavati Princen-Bradley-ev uvjet:

$$w_n^2 + w_{n+N}^2 = 1$$

dodatno, mora je simetričan

$w_k = w_{2N-1-k}$

- taj uvjet osigurava da retci transformacijske matrice pomnoženi s vremenskim otvorom, ostanu i dalje ortonormalni

MDCT transformacijska matrica:

$$w[n] \cdot C_N^{n,k} = w[n] \cdot \cos \frac{(2n+1+N) \cdot (2k+1)\pi}{4N} = \sin \frac{(2n+1)\pi}{4N} \cdot \cos \frac{(2n+1+N) \cdot (2k+1)\pi}{4N}$$

pri čemu su n stupci i mijenjaju se od 0 do 2N-1, a k retci i mijenjaju se od 0 do N-1

$$C_N = \begin{bmatrix} \left(\sin\frac{(2\cdot 0+1)\pi}{4N}\cdot\cos\frac{(2\cdot 0+1+N)(2\cdot 0+1)\pi}{4N}\right) & \cdots & \left(\sin\frac{(2\cdot(2N-1)+1)\pi}{4N}\cos\frac{(2\cdot(2N-1)+1+N)(2\cdot 0+1)\pi}{4N}\right) \\ \vdots & & \vdots \\ \left(\sin\frac{(2\cdot 0+1)\pi}{4N}\cdot\cos\frac{(2\cdot 0+1+N)(2\cdot(N-1)+1)\pi}{4N}\right) & \cdots & \left(\sin\frac{2\cdot(2N-1)+1)\pi}{4N}\cdot\cos\frac{(2(2N-1)+1+N)(2\cdot(N-1)+1)\pi}{4N}\right) \end{bmatrix}$$

$$C_4 = \underbrace{\begin{bmatrix} C_4^{0,0} & C_4^{1,0} & C_4^{2,0} & C_4^{3,0} & C_4^{4,0} & C_4^{5,0} & C_4^{6,0} & C_4^{7,0} \\ C_4^{0,1} & C_4^{1,1} & C_4^{2,1} & C_4^{3,1} & C_4^{4,1} & C_4^{5,1} & C_4^{6,1} & C_4^{7,1} \\ C_4^{0,2} & C_4^{1,2} & C_4^{2,2} & C_4^{3,2} & C_4^{4,2} & C_4^{5,2} & C_4^{6,2} & C_4^{7,2} \\ C_4^{0,3} & C_4^{1,3} & C_4^{2,3} & C_4^{3,3} & C_4^{4,3} & C_4^{5,3} & C_4^{6,3} & C_4^{7,3} \end{bmatrix}}$$

$; \quad C_N^{m,k} = \cos\frac{(2m+1+N)(2k+1)\cdot\pi}{4N}$

$w[n] = \sin\frac{(2n+1)\pi}{4N}$

svaki redak matrice čini koeficijente analizirajućeg filtra, pri čemu se dodatno svaki koeficijent množi prozorskom funkcijom $w[n]$:

$$H_k = w[0]\cdot C_N^{0,k} + w[1]\cdot C_N^{1,k} z + w[2]\cdot C_N^{2,k} z^2 + \ldots\ldots + w[2N-1]\cdot C_N^{2N-1,k} z^{2N-1}$$

- matrica inverzne transformacije C_4^{-1} jednaka je transponiranoj matrici C_4 budući da su matrice ortogonalne
  - stupci matrice C_4^{-1} čine koeficijente filtra (a to su upravo retci matrice C_4)
  - dodatno, zbog izraza za inverznu transformaciju, svaki se koeficijent dijeli s N, a zbog prirode prozora (rekonstruirani signal može se dodatno modificirati prozorskom funkcijom, ili jednako s 2) svaki se koeficijent množi s 2

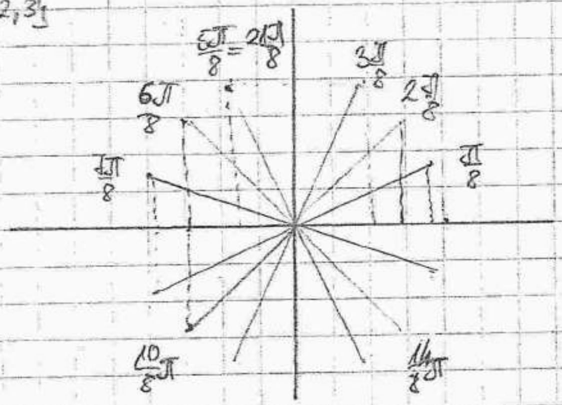
$\Longrightarrow$ koeficijenti rekonstrukcijskog filtra nalaze se u retcima C_4 matrice, pri čemu se svaki još množi s $\frac{2}{N}w[n]$

$$F_k = \frac{2}{N}\left[w[0]\cdot C_N^{0,k} + w[1]\cdot C_N^{1,k}\cdot z^{-1} + w[2] C_N^{2,k} z^{-2} + \ldots + w[2N-1] C_N^{2N-1,k} z^{-(2N-1)}\right]$$

b)

$$X[n] = \left\{ \tfrac{1}{2}\sqrt{1+\tfrac{\sqrt{2}}{2}},\ \tfrac{1}{2}\sqrt{1-\tfrac{\sqrt{2}}{2}},\ -\tfrac{1}{2}\sqrt{1-\tfrac{\sqrt{2}}{2}},\ -\tfrac{1}{2}\sqrt{1+\tfrac{\sqrt{2}}{2}} \right\} = \left\{ \tfrac{1}{2\sqrt{2}}\sqrt{2+\sqrt{2}},\ \tfrac{1}{2\sqrt{2}}\sqrt{2-\sqrt{2}},\ -\tfrac{1}{2\sqrt{2}}\sqrt{2-\sqrt{2}},\ -\tfrac{1}{2\sqrt{2}}\sqrt{2+\sqrt{2}} \right\}$$

$$\sqrt{\frac{2-\delta[k]}{4}} = \begin{cases} \sqrt{\frac{2-1}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}, & k=0 \\ \sqrt{\frac{2-0}{4}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}, & k \in \{1,2,3\} \end{cases}$$



$$X[0,0] = \tfrac{1}{2} x[0] \cdot \cos\left(\frac{(2\cdot 0+1)\cdot 0\cdot \pi}{8}\right) = \tfrac{1}{2}$$

$$X[0,1] = \tfrac{1}{2} x[1] \cdot \cos(0) = \tfrac{1}{2} x[1]$$

$$X[0,2] = \tfrac{1}{2} x[2]$$

$$X[0,3] = \tfrac{1}{2} x[3]$$

$$X[1,0] = \tfrac{1}{\sqrt{2}} \cdot x[0] \cdot \cos\left(\frac{(2\cdot 0+1)\cdot \pi}{8}\right) = \tfrac{\sqrt{2}}{2} \cdot x[0] \cdot \cos\left(\tfrac{\pi}{8}\right)$$

$$X[1,1] = \tfrac{1}{\sqrt{2}} x[1] \cdot \cos\left(\frac{(2\cdot 1+1)\pi}{8}\right) = \tfrac{\sqrt{2}}{2} \cos\left(\tfrac{3\pi}{8}\right) x[1]$$

$$X[1,2] = \tfrac{1}{\sqrt{2}} x[2] \cdot \cos\left(\tfrac{5\pi}{8}\right) = -\tfrac{\sqrt{2}}{2} \cos\left(\tfrac{3\pi}{8}\right) \cdot x[2]$$

$$X[1,3] = \tfrac{1}{\sqrt{2}} x[3] \cdot \cos\left(\tfrac{7\pi}{8}\right) = -\tfrac{\sqrt{2}}{2} \cos\left(\tfrac{\pi}{8}\right) \cdot x[3]$$

$$X[2,0] = \tfrac{1}{\sqrt{2}} \cdot x[0] \cdot \cos\left(\frac{(2\cdot 0+1)\cdot 2\pi}{8}\right) = \tfrac{\sqrt{2}}{2} \cos\left(\tfrac{2\pi}{8}\right) \cdot x[0]$$

$$X[2,1] = \tfrac{1}{\sqrt{2}} \cdot x[1] \cdot \cos\left(\frac{(2\cdot 1+1)\cdot 2\pi}{8}\right) = -\tfrac{\sqrt{2}}{2} \cos\left(\tfrac{2\pi}{8}\right) \cdot x[1]$$

$$X[2,2] = \tfrac{1}{\sqrt{2}} \cdot x[2] \cdot \cos\left(\tfrac{10\pi}{8}\right) = -\tfrac{\sqrt{2}}{2} \cos\left(\tfrac{2\pi}{8}\right) x[2]$$

$$X[2,3] = \tfrac{1}{\sqrt{2}} x[3] \cdot \cos\left(\tfrac{14\pi}{8}\right) = \tfrac{\sqrt{2}}{2} \cos\left(\tfrac{2\pi}{8}\right) x[3]$$

$$X[3,0] = \tfrac{1}{\sqrt{2}} x[0] \cdot \cos\left(\frac{(2\cdot 0+1)\cdot 3\pi}{8}\right) = \tfrac{\sqrt{2}}{2} \cos\left(\tfrac{3\pi}{8}\right) \cdot x[0]$$

$$X[3,1] = \tfrac{1}{\sqrt{2}} x[1] \cdot \cos\left(\frac{(2\cdot 1+1)\cdot 3\pi}{8}\right) = \tfrac{\sqrt{2}}{2} \cdot \cos\left(\tfrac{9\pi}{8}\right) \cdot x[1] = -\tfrac{\sqrt{2}}{2} \cos\left(\tfrac{\pi}{8}\right) x[1]$$

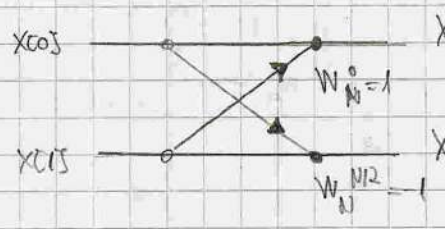
$$X[3,2] = \tfrac{1}{\sqrt{2}} x[2] \cdot \cos\left(\frac{(2\cdot 2+1)\cdot 3\pi}{8}\right) = \tfrac{\sqrt{2}}{2} \cos\left(\tfrac{15\pi}{8}\right) \cdot x[2] = \tfrac{\sqrt{2}}{2} \cos\left(\tfrac{\pi}{8}\right) x[2]$$

$$X[3,3] = \tfrac{1}{\sqrt{2}} x[3] \cdot \cos\left(\tfrac{21\pi}{8}\right) = -\tfrac{\sqrt{2}}{2} \cos\left(\tfrac{3\pi}{8}\right)$$

DCT$_4$:

$$X[k] = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\sqrt{2+\sqrt{2}} & \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\sqrt{2-\sqrt{2}} & -\frac{\sqrt{2}}{2}\frac{1}{2}\sqrt{2-\sqrt{2}} & -\frac{\sqrt{2}}{2}\frac{1}{2}\sqrt{2+\sqrt{2}} \\ \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\sqrt{2-\sqrt{2}} & -\frac{\sqrt{2}}{2}\frac{1}{2}\sqrt{2+\sqrt{2}} & \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\sqrt{2+\sqrt{2}} & -\frac{\sqrt{2}}{2}\cdot\frac{1}{2}\sqrt{2-\sqrt{2}} \end{bmatrix}}_{\text{transformacijska matrica}} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

2) Graf tijeka signala za DFT u duljini dva (DFT_2) je sljedećeg oblika:



ova struktura naziva se leptir

8) $x_1[n] = \{\underline{5}, 2, 4, -1\}$

$x_2[n] = \{\underline{-3}, 4, 0, 2, -1, 2\}$

a) Linearna konvolucija:

DULJINA: duljina L.K. je jednaka $l_1 + l_2 - 1 \Rightarrow$

$$l_{LK} = 4 + 6 - 1 = 9$$

KONVOLUCIJA

	5	2	4	-1
-3	-15	-6	-12	3
4	20	8	16	-4
0	0	0	0	0
2	10	4	8	-2
-1	-5	-2	-4	1
2	10	4	8	-2

$LK = \{\underline{-15}, 14, -4, 29, -5, 16, -2, 9, -2\}$

b) $x_1[n] = \{\underline{5}, 2, 4, -1, 0, 0\}$ ... proširuje se niz $x_1[n]$ nulama na duljinu $N=6$

! cirkularnu konvoluciju moguće je izračunati iz linearne:

$N=6$

	-15	14	-4	29	-5	16
+	-2	9	-2			
	-17	23	-6	29	-5	16

objašnjenje u slici na sljedećoj stranici →

3) $N=2L,\ L\in\mathbb{N} \Rightarrow$ broj uzoraka je paran

DFT_N ... diskretna Fourierova transformacija signala duljine N:

$$X[k]=\sum_{m=0}^{N-1} x[m]\, W_N^{mk}\ ,\quad k=0,\dots,N-1$$

Budući da je broj uzoraka paran, gornju sumaciju moguće je rastaviti na dvije sume, svaka s N/2 članova. Neka prvu sumu čine parni, a drugu neparni uzorci:

$$X[k]=\sum_{m=0}^{N-1} x[m]\cdot W_N^{mk}=\sum_{m=0}^{N-1} x[m]\cdot e^{-j\,2\pi k\cdot\frac{m}{N}} \quad ; \text{pri čemu je } W_N^{mk}=\left(e^{-j\frac{2\pi}{N}}\right)^{mk}$$

rastavljanjem na "parnu" i "neparnu" sumu:

$$X[k]=\underbrace{\sum_{m=0}^{\frac{N}{2}-1}}_{N/2\text{ uzoraka}} x[2m]\left[e^{-j\frac{2\pi}{N}}\right]^{2m\cdot k}+\sum_{m=0}^{\frac{N}{2}-1} \underbrace{x[2m+1]}_{\text{neparni uzorci}}\left(e^{-j\frac{2\pi}{N}}\right)^{(2m+1)\cdot k}=$$

poznato je da vrijedi: $W_N^{2r}=W_{N/2}^{r}\quad \left(\left[e^{-j\frac{2\pi}{N}}\right]^{2r}=\left[e^{-j\frac{2\cdot 2\cdot\pi}{N}}\right]^{r}=\left[e^{-j\frac{2\pi}{\frac{N}{2}}}\right]^{r}\right)$

$$=\sum_{m=0}^{\frac{N}{2}-1} x[2m]\cdot\left[e^{-j\frac{2\pi}{\frac{N}{2}}}\right]^{mk}+\sum_{m=0}^{\frac{N}{2}-1} x[2m+1]\cdot\left[\left(e^{-j\frac{2\pi}{N}}\right)^{2mk}\cdot\left(e^{-j\frac{2\pi}{N}}\right)^{k}\right]=$$

ne ovisi o m pa se može "izlučiti"

$$=\sum_{m=0}^{\frac{N}{2}-1} x[2m]\cdot W_{\frac{N}{2}}^{mk}+e^{-j\frac{2\pi k}{N}}\sum_{m=0}^{\frac{N}{2}-1} x[2m+1]\cdot W_{\frac{N}{2}}^{mk}=\left|\left(e^{-j\frac{2\pi}{N}}\right)^{k}=W_N^{k}\right|$$

$$=\sum_{m=0}^{\frac{N}{2}-1} x[2m]\cdot W_{\frac{N}{2}}^{mk}+W_N^{k}\sum_{m=0}^{\frac{N}{2}-1} x[2m+1]\cdot W_{\frac{N}{2}}^{mk}\qquad (*)$$

napišimo DFT signala $x[2m]$ duljine N/2 prema formuli za DFT:

$$DFT_{\frac{N}{2}}[x[2m]]=\sum_{m=0}^{\frac{N}{2}-1} x[2m]\cdot W_{\frac{N}{2}}^{mk}\qquad (**)$$

novi signal, različit od originalnog, koji djelomično predstavlja uzorke samo i isključivo s indeksima $2m,\ m\in[0,\frac{N}{2}-1]$

napišimo DFT signala $x[2m+1]$ duljine N/2 prema formuli za DFT:

$$DFT_{\frac{N}{2}}[x[2m+1]]=\sum_{m=0}^{\frac{N}{2}-1} x[2m+1]\cdot W_{\frac{N}{2}}^{mk}\qquad (***)$$

(TO JE RAZLOG DA SE U EKSPONENTU W_N^{nk} ne javlja $2n$)

uspoređivanjem $(*)$, $(**)$, $(***)$ vidimo da vrijedi (uz početnu pretpostavku $N=2L$)

$$X[k]=DFT_{2L}[x[n]]=\sum_{m=0}^{N-1} x[m]\cdot W_{2L}^{mk}=\sum_{m=0}^{\frac{N}{2}-1} x[2m]\cdot W_L^{mk}+W_{2L}^{k}\sum_{m=0}^{\frac{N}{2}-1} x[2m+1]\, W_L^{mk}=$$

$$=DFT_L[x[2m]]+W_{2L}^{k}\, DFT_L[x[2m+1]]$$

6) Cirkularna konvolucija:

Rezultat cirkularne konvolucije je jedna perioda konvolucije dvaju periodičnih signala i dobiva se izrazom:

$$y_c[n] = \sum_{i=0}^{N-1} h[i]\,x[(n-i)_{mod\,N}] = h[n] \,Ⓝ\, x[n]$$; N - period oba signala

oznaka cirkularne konvolucije

$n = 0, 1, \dots, N-1$

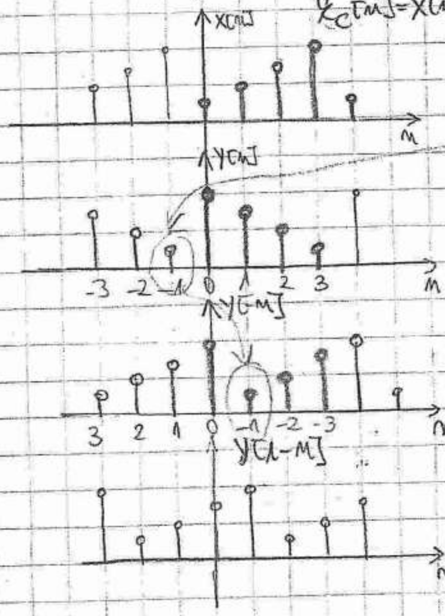
$(n-i) = \alpha N + \beta \qquad \beta \in [0, \dots, N-1],\ \alpha \in \mathbb{Z}$

$(n-i)_{mod\,N} = \beta$

b) $x[n] = \{1, 2, 3, 4\}$

$y[n] = \{4, 3, 2, 1\}$

$k_c[n] = x[n] \,Ⓝ\, y[n]$



$$k_c[0] = \sum_{i=0}^{3} x[i]\,y[(0-i)_{mod4}] = 1\cdot4 + 2\cdot1 + 3\cdot2 + 4\cdot3 = 24$$

$y[(0-0)_{mod4}] = y[0] = 4$

$y[(0-1)_{mod4}] = y[(-1)_{mod4}] = y[3] = 1$

$y[(0-2)_{mod4}] = y[(-2)_{mod4}] = y[2] = 2$

$y[(0-3)_{mod4}] = y[(-3)_{mod4}] = y[1] = 3$

$$k_c[1] = \sum_{i=0}^{3} x[i]\,y[(1-i)_{mod4}] = 1\cdot3 + 2\cdot4 + 3\cdot1 + 4\cdot2 = 22$$

$y[(1-0)_{mod4}] = y[1_{mod4}] = y[1] = 3$

$y[(1-1)_{mod4}] = y[0_{mod4}] = y[0] = 4$

$y[(1-2)_{mod4}] = y[(-1)_{mod4}] = y[3] = 1$

$y[(1-3)_{mod4}] = y[(-1)_{mod4}] = y[2] = 2$

$$k_c[2] = \sum_{i=0}^{3} x[i]\,y[(2-i)_{mod4}] = 1\cdot2 + 2\cdot3 + 3\cdot4 + 4\cdot1 = 24$$

$$k_c[3] = \sum_{i=0}^{3} x[i]\,y[(3-i)_{mod4}] = 1\cdot1 + 2\cdot2 + 3\cdot3 + 4\cdot4 = 30$$

PROVJERA:

izvorni prvi signal

zrcaljeni drugi signal

n				$k_c[n]$
	$x[i]$	=	$\{1, 2, 3, 4\}$	
0	$y[(0-i)_{mod4}]$	=	$\{4, 1, 2, 3\}$	24
1	$y[(1-i)_{mod4}]$	=	$\{3, 4, 1, 2\}$	22
2	$y[(2-i)_{mod4}]$	=	$\{2, 3, 4, 1\}$	24
3	$y[(3-i)_{mod4}]$	=	$\{1, 2, 3, 4\}$	30

rotiram (za po jedno mjesto u desno) zrcaljeni signal

$k_c[n] = \{\underline{24}, 22, 24, 30\}$

7) Važno svojstvo cirkularne konvolucije je sljedeći teorem:

Rezultat množenja $X[k]Y[k]$ u DFT_N domeni odgovara cirkularnoj konvoluciji $x[n] ⓃN y[n]$ u vremenskoj domeni:

$$X[k] = DFT_N[x[n]]$$
$$Y[k] = DFT_N[y[n]]$$

$$Y_c[k] = X[k] \cdot Y[k]$$

$$y_c[n] = IDFT_N[Y_c[k]]$$

$$DFT_N[x[n]] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N}kn}$$
$$k = 0, \dots, N-1$$

$$x[n] = \{2, 1, 0, 1\}$$
$$y[n] = \{2, -1, 0, -1\}$$

$$x[n] ④ y[n] = IDFT_4[Y_c[k]]$$

$$IDFT_N[X[k]] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$
$$n = 0, \dots, N-1$$

$X[k] = DFT_4[x[n]]$:

$$X[k] = 2 \cdot W_4^{0k} + 1 \cdot W_4^{1k} + 0 \cdot W_4^{2k} + 1 W_4^{3k}$$

$Y[k] = DFT_4[y[n]]$:

$$Y[k] = 2 \cdot W_4^{0k} - 1 \cdot W_4^{1k} + 0 \cdot W_4^{2k} - 1 W_4^{3k}$$

$$Y_c[k] = X[k] \cdot Y[k] = (2W_4^{0k} + W_4^{1k} + W_4^{3k}) \cdot (2W_4^{0k} - W_4^{1k} - W_4^{3k})$$
$$= 4W_4^{0k} \cancel{-2W_4^{1k}} \cancel{-2W_4^{3k}} \cancel{+2W_4^{1k}} - W_4^{2k} \cancel{-W_4^{4k}} \cancel{+2W_4^{3k}} \cancel{-W_4^{4k}} - W_4$$
$$= 4W_4^{0k} - W_4^{2k} - W_4^{6k}$$

W_N se nalazi na jediničnoj kružnici nastale dijeljenjem jednog kruga s N

$$\left(|W_N| = |e^{-j\frac{2\pi}{N}}| = \sqrt{\underbrace{\cos^2\left(\frac{2\pi}{N}\right) + \sin^2\left(\frac{2\pi}{N}\right)}} = \sqrt{1} = 1 \right)$$

zbog periodičnosti W_N vrijedi:

$$W_4^{6k} = W_4^{2k}$$

pa je $Y_c[k] = 4W_4^{0k} - W_4^{2k} - W_4^{2k} = 4W_4^{0k} - 2W_4^{2k}$

$k=0$: $Y_c[0] = 4W_4^0 - 2W_4^0 = 4 - 2 = 2$
$k=1$: $Y_c[1] = 4W_4^0 - 2W_4^2 = 4 - 2W_4^2$
$k=2$: $Y_c[2] = 4W_4^0 - 2W_4^4 = 4 - 2W_4^4 = 4 - 2W_4^0 = 4 - 2 = 2$
$k=3$: $Y_c[3] = 4W_4^0 - 2W_4^6 = 4 - 2W_4^6 = 4 - 2W_4^2$

$$Y_c[k] = \{\underline{2}, 4 - 2W_4^2, 2, 4 - 2W_4^2\}$$

9) $x_1[n] = \{\underline{5}, 4, 3, 2, 1\}$
$x_2[n] = \{\underline{1}, 2, 3, 4, 5\}$

$x_1[n] * x_2[n]$:

	5	4	3	2	1
1	5	4	3	2	1
2	10	8	6	4	2
3	15	12	9	6	3
4	20	16	12	8	4
5	25	20	15	10	5

5	4	3	2	1	
1	5	4	3	2	
2	1	5	4	3	
3	2	1	5	4	
4	3	2	1	5	
5	4	3	2	1	
5	20	12	6	2	45
10	4	15	8	3	40
15	8	3	10	4	40
20	12	6	2	5	45
25	16	9	4	1	55

Linearna konvolucija:

$y[n] = \{\underline{5}, 14, 26, 40, 55, 40, 26, 14, 5\} = x_1[n] * x_2[n]$

Cirkularna konvolucija:

$y_c[n] = \{\underline{45}, 40, 40, 45, 55\} = x_1[n] \circledS x_2[n]$

Neka su duljine signala: L_1 i L_2, te su signali kauzalni
Linearna konvolucija tih signala daje niz uzoraka duljine L_1+L_2-1

Uzorke linearne konvolucije moramo odrediti pomoću cirkularne, a tada mora biti zadovoljeno da je duljina cirkularne konvolucije $N \geq L_1+L_2-1$

Za gore zadani zadatak taj N mora biti 9 što znači da signale najprije proširimo s 4 nule, a zatim provedemo cirkularnu konvoluciju, koja će tada biti jednaka linearnoj.

$\Rightarrow N=9$; $x_1[n] = \{\underline{5}, 4, 3, 2, 1, 0, 0, 0, 0\}$
$x_2[n] = \{\underline{1}, 2, 3, 4, 5, 0, 0, 0, 0\}$

5	4	3	2	1	0	0	0	0	
1	0	0	0	0	5	4	3	2	5
2	1	0	0	0	0	5	4	3	14
3	2	1	0	0	0	0	5	4	26
4	3	2	1	0	0	0	0	5	40
5	4	3	2	1	0	0	0	0	55
0	5	4	3	2	1	0	0	0	40
0	0	5	4	3	2	1	0	0	26
0	0	0	5	4	3	2	1	0	14
0	0	0	0	5	4	3	2	1	5

$\Rightarrow x_1[n] \circledNine x_2[n] = x_1[n] * x_2[n] = \{\underline{5}, 14, 26, 40, 55, 40, 26, 14, 5\}$

10) a) PRAVOKUTNI VREMENSKI OTVOR:

Izračunavanje uzoraka u vremenu pravokutnog otvora, vrši se prema formuli:

$$w[k]=1, \quad -\frac{N}{2} \le k \le \frac{N}{2}$$

ŠIRINA GLAVNE LATICE: $\Delta_{ML} = 4\pi/(N+1)$

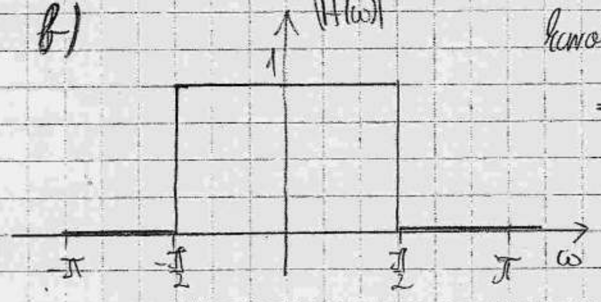
GUŠENJE PRVE BOČNE LATICE: 13,3

ŠIRINA PRIJELAZNOG PODRUČJA: $\Delta\omega = \dfrac{0{,}92\pi}{\frac{N}{2}}$

ŠIRINA GLAVNE LATICE POVEĆAVA se kada se BROJ VREMENSKIH UZORAKA otvora SMANJUJE

(vidljivo iz izraza za širinu glavne latice)

11) b)



karakteristika je aperiodična, kontinuirana

$\Rightarrow$ IDTFT($H(e^{j\omega})$) je onda zbog kontinuiteta neperiodičan (je beskonačan)

$\Rightarrow$ množenje vremenskim otvorom

impulsni odziv: $\mathrm{IDTFT}\left(H(e^{j\omega})\right) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$

amplitudna karakteristika: $H(\omega)=H(e^{j\omega}) = \begin{cases} 1, & -\frac{\pi}{2} \le \omega \le \frac{\pi}{2} \\ 0, & \text{inače} \end{cases}$

(faza je nula)

$$h[n] = \mathrm{IDTFT}[H(\omega)] = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{j\omega n}\, d\omega = \frac{1}{2\pi}\frac{1}{jn} e^{j\omega n}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{\pi n}\cdot \underbrace{\frac{1}{2j}\left(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}\right)}_{\sin\left(\frac{\pi}{2}n\right)}$$

$$= \frac{1}{\pi n}\cdot \sin\left(\frac{\pi}{2}n\right)$$

pravokutni vrem. otvor duljine N raspona N je:

$$w[n] = \begin{cases} 1, & -\frac{N-1}{2} \le n \le \frac{N-1}{2} \\ 0, & \text{inače} \end{cases}$$

$$N=5 \Rightarrow \quad -\frac{N-1}{2} = -\frac{5-1}{2} = -2$$

$$w[n] = \begin{cases} 1, & -2 \le n \le 2 \\ 0, & \text{inače} \end{cases}$$

12) P-udaljenost dviju funkcija $f(\omega)$ i $g(\omega)$ definirana je kao:

$$\left[\int |f(\omega)-g(\omega)|^p d\omega\right]^{\frac{1}{p}}$$

- p-udaljenost idealne amplitudne karakteristike $A_s(\omega)$ i stvarne amplitudne karakteristike $A_s(\omega)$ možemo sada definirati kao:

$$d_p(A_s(\omega), A(\omega)) = \| A_s(\omega) - A(\omega) \| = \left[\int |A_s(\omega)-A(\omega)|^p d\omega\right]^{\frac{1}{p}}$$

- za neke vrijednosti parametra, p-udaljenost ima specifične nazive:

za $p=1$ → Manhattan ili pravokutna udaljenost

za $p=2$ → Euklidska udaljenost

za $p=\infty$ → Čebiševljeva udaljenost

13) b)

$$A(\omega) = \begin{cases} 1, & -\frac{\pi}{4} < \omega < \frac{\pi}{4} \\ 0, & \text{inače} \end{cases}$$

$$a[m] = a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} Ad(\omega)\cdot\cos(\omega m)\,d\omega$$

$$a[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Ad(\omega)\,d\omega$$

$$a[m] = \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos(\omega m)\,d\omega = \frac{1}{\pi m}\sin(\omega m)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{\pi m}\Big[\underbrace{\sin\left(m\frac{\pi}{4}\right) - \sin\left(-\frac{\pi}{4}m\right)}_{\sin\left(\frac{\pi}{4}m\right)}\Big] = \frac{2}{\pi m}\sin\left(\frac{\pi}{4}m\right)$$

$$a[0] = \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1\,d\omega = \frac{1}{2\pi}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{1}{\cancel{2\pi}}\,\frac{\cancel{2\pi}}{4} = \frac{1}{4}$$

$$a[1] = \frac{2}{\pi}\cdot\sin\left(\frac{\pi}{4}\right) = \cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}}\cdot\frac{1}{\pi} = \frac{\sqrt{2}}{\pi}$$

$$a[2] = \frac{\cancel{2}}{\cancel{2}\pi}\cdot\sin\left(2\frac{\pi}{4}\right) = \frac{1}{\pi}$$

$$H(\omega) = e^{-j\frac{5}{2}\omega}\left(a[0] + a[1]\cos(\omega) + a[2]\cos(2\omega)\right)$$

$$a[0] = h\left[\frac{N}{2}\right] \Rightarrow h\left[\frac{4}{2}\right] = h[2] = \frac{1}{4}$$

$$a[m] = 2\cdot h\left[\frac{N}{2}-m\right],\ 1 \le m \le \frac{N}{2} \Rightarrow 2h\left[\frac{4}{2}-m\right] = 2h\left[2-m\right] = a[m]$$

$$2h[2-1] = 2h[1] = \frac{\sqrt{2}}{\pi} \Rightarrow h[1] = \frac{\sqrt{2}}{2\pi}$$

$$2h[2-2] = 2h[0] = \frac{1}{\pi} \Rightarrow h[0] = \frac{1}{2\pi}$$

$$h[n] = \left\{ \underline{\frac{1}{2\pi}}, \frac{\sqrt{2}}{2\pi}, \frac{1}{4} \right\}$$

14) f) $X[k]=\sqrt{\dfrac{4-2\delta[k]}{2N-1}}\sum_{m=0}^{N-1} x[m]\dfrac{1}{\sqrt{1+\delta[m-N+1]}}\cos\left(\dfrac{(2m+1)k\pi}{2N-1}\right)$

→ oblik diskretne kosinusne transformacije

$X[k]=?\ ;\ x[n]=\{1,0,0,0\}\ ,\ N=4$

$$X[0]=\sqrt{\frac{4-2\delta[0]}{2\cdot4-1}}\left[x[0]\cdot\frac{1}{\sqrt{1+\delta[0-4+1]}}\cos\left(\frac{(2\cdot0+1)\cdot0\cdot\pi}{2\cdot4-1}\right)+x[1]\frac{1}{\sqrt{1+\delta[1-4+1]}}\cos\left(\frac{(2\cdot1+1)\cdot0\cdot\pi}{2\cdot4-1}\right)+\right.$$

$$\left.+x[2]\frac{1}{\sqrt{1+\delta[2-4+1]}}\cdot\cos\left(\frac{(2\cdot2+1)\cdot0\cdot\pi}{2\cdot4-1}\right)+x[3]\frac{1}{\sqrt{1+\delta[3-4+1]}}\cos\left(\frac{(3\cdot2+1)\cdot0\cdot\pi}{2\cdot4-1}\right)\right]$$

$$X[1]=\sqrt{\frac{4-2\delta[1]}{2\cdot4-1}}\left[x[0]\cdot\frac{1}{\sqrt{1+\delta[0-4+1]}}\cos\left(\frac{(2\cdot0+1)\cdot1\cdot\pi}{2\cdot4-1}\right)+x[1]\frac{1}{\sqrt{1+\delta[1-4+1]}}\cdot\cos\left(\frac{(2\cdot1+1)\cdot1\cdot\pi}{2\cdot4-1}\right)+\dots\right.$$

$\vdots$

$$\Downarrow$$

član koji je jednak za $X[k,0], X[k,1]$ $X[k,2]$ i $X[k,3]$ pišem se konstanta

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3\\ b_0 & b_1 & b_2 & b_3\\ c_0 & c_1 & c_2 & c_3\\ d_0 & d_1 & d_2 & d_3\end{bmatrix}\begin{bmatrix}x[0]\\x[1]\\x[2]\\x[3]\end{bmatrix}=\begin{bmatrix}a_0x[0]+a_1x[1]+a_2x[2]+a_3x[3]\\ b_0x[0]+\cdot\ \cdot\ \cdot+b_3x[3]\\ c_0x[0]+\cdot\ \cdot\ \cdot+c_3x[3]\\ d_0x[0]+d_1x[1]+d_2x[2]+d_3x[3]\end{bmatrix}=\begin{bmatrix}X[0]\\X[1]\\X[2]\\X[3]\end{bmatrix}$$

$X[k,m]$:

→ pojavljuje se samo za $X[k,3]$ članove i iznosi $\frac{1}{\sqrt{2}}$

$$X[0,0]=\sqrt{\frac{4-2\cdot\delta[0]}{2\cdot4-1}}x[0]\cdot\frac{1}{\sqrt{1+\delta[0-4+1]}}\cdot\cos\left(\frac{(2\cdot0+1)\cdot0\cdot\pi}{2\cdot4-1}\right)=\sqrt{\frac{2}{7}}x[0]\cdot\cos(0)=\sqrt{\frac{2}{7}}x[0]$$

$$X[0,1]=\sqrt{\frac{4-2\delta[0]}{2\cdot4-1}}x[1]\frac{1}{\sqrt{1+\delta[1-4+1]}}\cos\left(\frac{(2\cdot1+1)\cdot0\cdot\pi}{2\cdot4-1}\right)=\sqrt{\frac{2}{7}}\cdot x[1]\cdot\cos(0)=\sqrt{\frac{2}{7}}x[1]$$

$$X[0,2]=\sqrt{\frac{2}{7}}x[2]\cdot1\cdot\cos(0)=\sqrt{\frac{2}{7}}x[2]$$

$$X[0,3]=\sqrt{\frac{2}{7}}\cdot x[3]\cdot\frac{1}{\sqrt{1+\delta[3-4+1]}}\cdot\cos(0)=\sqrt{\frac{2}{7}}\cdot x[3]\cdot\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{7}}x[3]$$

$$X[1,0]=\sqrt{\frac{4-2\delta[1]}{2\cdot4-1}}x[0]\cdot\frac{1}{\sqrt{1+\delta[0-4+1]}}\cdot\cos\left(\frac{(2\cdot0+1)\cdot1\cdot\pi}{2\cdot4-1}\right)=\sqrt{\frac{4}{7}}x[0]\cdot1\cdot\cos\left(\frac{\pi}{7}\right)=\frac{2}{\sqrt{7}}x[0]\cdot\cos\left(\frac{\pi}{7}\right)$$

$$X[1,1]=\sqrt{\frac{4}{7}}\cdot x[1]\cdot1\cdot\cos\left(\frac{(2\cdot1+1)\cdot\pi}{7}\right)=\sqrt{\frac{4}{7}}\cdot x[1]\cdot\cos\left(\frac{3\pi}{7}\right)$$

$$X[1,2]=\sqrt{\frac{4}{7}}x[2]\cdot\cos\left(\frac{5\pi}{7}\right)$$

$$X[1,3]=\sqrt{\frac{4}{7}}\cdot x[3]\cdot\frac{1}{\sqrt{2}}\cdot\cos\left(\frac{7\pi}{7}\right)=\sqrt{\frac{2}{7}}\cdot x[3]\cdot(-1)=-\sqrt{\frac{2}{7}}x[3]$$

$$X[2,0]=\sqrt{\frac{4-2\delta[2]}{2\cdot4-1}}\cdot x[0]\cdot\frac{1}{\sqrt{1+\delta[0-4+1]}}\cdot\cos\left(\frac{(2\cdot0+1)\cdot2\cdot\pi}{2\cdot4-1}\right)=\sqrt{\frac{4}{7}}\cdot x[0]\cdot1\cdot\cos\left(\frac{2\pi}{7}\right)=\sqrt{\frac{4}{7}}\cdot x[0]\cdot\cos\left(\frac{2\pi}{7}\right)$$

$$X[2,1]=\sqrt{\frac{4}{7}}\cdot x[1]\cdot1\cdot\cos\left(\frac{6\pi}{7}\right)=\sqrt{\frac{4}{7}}x[1]\cdot\cos\left(\frac{6\pi}{7}\right)=-\sqrt{\frac{4}{7}}x[1]\cdot\cos\left(\frac{\pi}{7}\right)$$

17) DCT-II: $X[k] = \sqrt{\frac{2-\delta[k]}{N}} \sum_{n=0}^{N-1} x[n] \cdot \cos\left[\frac{(2n+1) k\pi}{2N}\right]$, $k = 0, \dots, N-1$

DFT: $X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N} nk}$, $k = 0, \dots, N-1$

- želimo pokazati da se računanjem DFT u 4N točaka može izračunati DCT u N točaka za neki zadani signal "širine" N:

  - temeljna funkcija DCT-II jest: $\cos\left(\frac{(2n+1)\cdot k\pi}{2N}\right)$
  - temeljna funkcija DFT$_{4N}$ jest: $e^{-j\frac{2\pi}{4N}\cdot nk} = e^{-j\frac{\pi}{2N}\cdot nk}$

    $\Rightarrow$ te je funkcije potrebno na neki način izjednačiti

- budući da su n i k jedini faktori koji utječu na iznose gornjih funkcija, a pri izračunu DFT/DCT-a računamo za neki određeni k (mijenja se k fiksan, a n mijenjamo), faktor kojim manipuliramo izjednačavajući funkcije je n

$\Rightarrow$ DCT-II:

$\cos\left(\frac{(2n+1)k\pi}{2N}\right)$

DFT:

$e^{-j\frac{\pi}{2N}n'k} = \cos\left(\frac{n'\pi k}{2N}\right) - j\sin\left(\frac{n'\pi\cdot k}{2N}\right)$

uz $n' = 2n+1$ dobivamo:

$\cos\left(\frac{(2n+1)\pi\cdot k}{2N}\right) - j\sin\left(\frac{(2n+1)\pi\cdot k}{2N}\right)$

$\Downarrow$

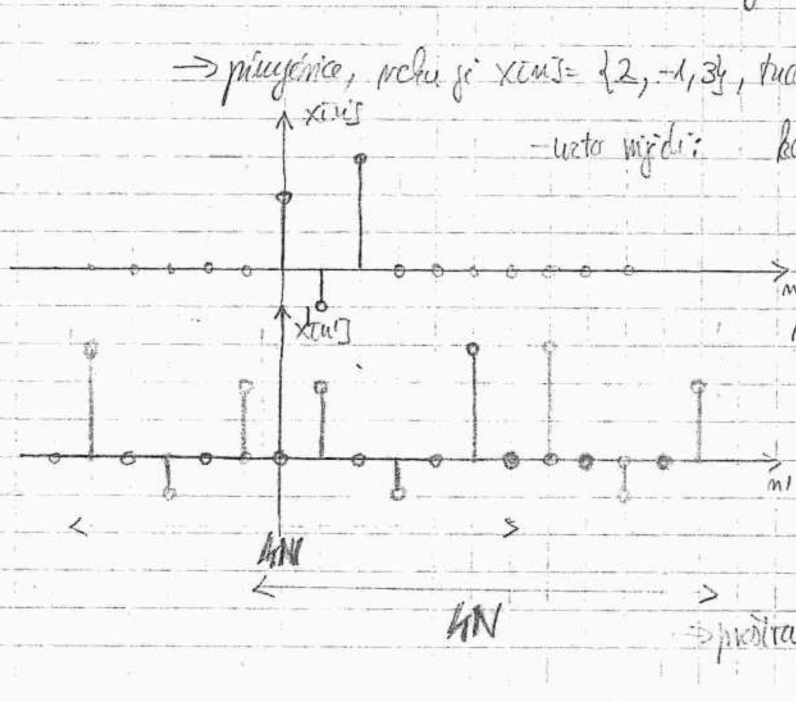
uzimamo puni uzorak redukcije... uzorka:

$n=0$: $n' = 2\cdot 0+1 = 1$
$n=1$: $n' = 2\cdot 1+1 = 3$
$n=2$: $n' = 2\cdot 2+1 = 5$
$\vdots$

$\Rightarrow$ uzorak na mjestu n originalnog signala, za računanje DFT$_{4N}$, treba premjestiti na mjesto $2n+1$

$\rightarrow$ između umetnemo nule *

$\Rightarrow$ primjerice, neka je $x[n] = \{2, -1, 3\}$, tada je proširen signal za računanje ovako izgleda:



- uz to vrijedi: kosinus je parna funkcija:

$\cos\left[-\left(\frac{\pi}{2N} n'k\right)\right] = \cos\left[\left(\frac{\pi}{2N} n'k\right)\right]$

sinus je neparna funkcija:

$\sin\left[-\left(\frac{\pi}{2N} n'k\right)\right] = -\sin\left[\left(\frac{\pi}{2N} n'k\right)\right]$

$\Rightarrow$ ako signal proširimo parno, sinusi će se poništiti, a kosinusi podupLati

$\Rightarrow$ proširuje se prvo tako da se nulti i zadnji (2N+1)-i uzorak ne udvostruče *

20) Iz izraza za DFT: $X[k]=\sum_{n=0}^{N-1} x[n]\cdot e^{-j\frac{2\pi}{N}nk}=\sum_{n=0}^{N-1} x[n]\cdot W_N^{kn}$, $k=1,\dots,N-1$; $W_N^{kn}=(e^{-j\frac{2\pi}{N}}$

vidimo da $X[k]$ možemo izračunati u matričnom obliku, pri čemu koristimo DFT transformacijsku matricu $\mathbf{W}_N$

Isto tako, moguće je IDFT izračunati preko $\mathbf{W}_N^{-1}$ matrice (matrice inverzne diskretne Fourierove transformacije) pri čemu je $\mathbf{W}_N^{-1}=\frac{1}{N}(\mathbf{W}_N^T)^*$ (budući da je $\mathbf{W}_N$ matrica ortogonalna)

- za $N=4$ odavde prikaz:

$$\begin{bmatrix} X[0]\\ X[1]\\ X[2]\\ X[3]\end{bmatrix}=\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1\\ 1 & W_4^{1\cdot1} & W_4^{2\cdot1} & W_4^{3\cdot1}\\ 1 & W_4^{1\cdot2} & W_4^{2\cdot2} & W_4^{3\cdot2}\\ 1 & W_4^{1\cdot3} & W_4^{2\cdot3} & W_4^{3\cdot3}\end{bmatrix}}\begin{bmatrix} x[0]\\ x[1]\\ x[2]\\ x[3]\end{bmatrix} \qquad \begin{bmatrix} x[0]\\ x[1]\\ x[2]\\ x[3]\end{bmatrix}=\frac{1}{N}\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1\\ 1 & W_4^{-1\cdot1} & W_4^{-2\cdot1} & W_4^{-3\cdot1}\\ 1 & W_4^{-1\cdot2} & W_4^{-2\cdot2} & W_4^{-3\cdot2}\\ 1 & W_4^{-1\cdot3} & W_4^{-2\cdot3} & W_4^{-3\cdot3}\end{bmatrix}}\begin{bmatrix} X[0]\\ X[1]\\ X[2]\\ X[3]\end{bmatrix}$$

svaki redak možemo interpretirati kao FIR filtar duljine N:

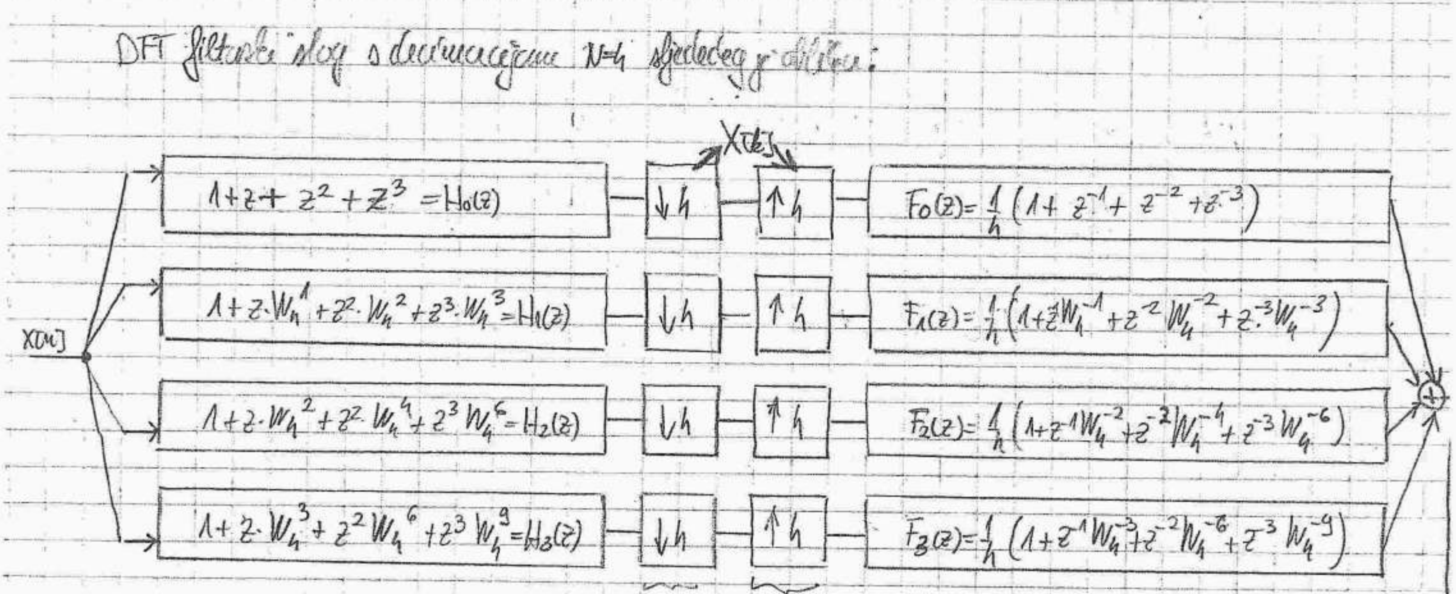
$$H_k(z)=1+z\cdot W_4^{1\cdot k}+z^2\cdot W_4^{2\cdot k}+z^3 W_4^{3k}$$

svaki stupac predstavlja doprinos pojedinog koeficijenta u rekonstrukciji izlaznog signala

ako svakom dodamo $N-1$ nulu i pogledamo to kao dobiveni impulsni odziv FIR filtra čiji su koeficijenti k-ti stupac matrice $\mathbf{W}_N^{-1}$, dobit ćemo doprinos svakog k u izlaznom signalu (x_k)

$$\Rightarrow F_k(z)=\frac{1}{4}\left(1+z^{-1}W_4^{-1k}+z^{-2}W_4^{-2k}+z^{-3}W_4^{-3k}\right)$$

DFT filtarski slog s decimacijom $N=4$ sljedećeg je oblika:



21)

Iz izraza za DTC-II vidimo da se DCT može računati upotrebom pojedinačne matrice C_N. Također, IDCT-II računamo upotrebom inverzne matrice transformacije C_N^{-1}

→ matrica IDCT nastaje transponiranjem matrice DCT (bez ikakvih skaliranja, samo zamjena n i k), budući da je DCT matrica unitarna

$$X[k] = w_k \cdot \sum_{n=0}^{N-1} x[n] \cos \frac{\pi}{N}\left(n+\frac{1}{2}\right)\cdot k$$

$$w_k = \begin{cases} \sqrt{1/N}, & k=0 \\ \sqrt{2/N}, & k\neq 0 \end{cases}$$

$$\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} \sqrt{1/N}\cdot\cos\left(\frac{\pi}{N}\left(0+\frac{1}{2}\right)\cdot 0\right) & \sqrt{1/N}\cdot\cos\left(\frac{\pi}{N}\left(1+\frac{1}{2}\right)\cdot 0\right) & \cdots & \sqrt{1/N}\cdot\cos\left(\frac{\pi}{N}\left(N-1+\frac{1}{2}\right)\cdot 0\right) \\ \sqrt{2/N}\cdot\cos\left(\frac{\pi}{N}\left(0+\frac{1}{2}\right)\cdot 1\right) & \sqrt{2/N}\cdot\cos\left(\frac{\pi}{N}\left(1+\frac{1}{2}\right)\cdot 1\right) & \cdots & \sqrt{2/N}\cos\left(\frac{\pi}{N}\left(N-1+\frac{1}{2}\right)\cdot 1\right) \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \sqrt{2/N}\cos\left(\frac{\pi}{N}\left(0+\frac{1}{2}\right)(N-1)\right) & \sqrt{2/N}\cos\left(\frac{\pi}{N}\left(1+\frac{1}{2}\right)(N-1)\right) & \cdots & \sqrt{2/N}\cos\left(\frac{\pi}{N}\left(N-1+\frac{1}{2}\right)(N-1)\right) \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ \vdots \\ x[N-1] \end{bmatrix}$$

- zamjenom:
$$C_N^{n,k} = \begin{cases} \sqrt{1/N}\cdot\cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)\cdot k\right]; & k=0 \\ \sqrt{2/N}\cdot\cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)\cdot k\right]; & k\neq 0 \end{cases}$$

, dobivamo da su DCT matrica transformacije i IDCT matrica transformacije za $N=4$

$$C_4 = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2}\cos\left(\frac{\pi}{8}\right) & \frac{\sqrt{2}}{2}\cos\left(\frac{3\pi}{8}\right) & -\frac{\sqrt{2}}{2}\cos\left(\frac{3\pi}{8}\right) & -\frac{\sqrt{2}}{2}\cos\left(\frac{\pi}{8}\right) \\ \frac{\sqrt{2}}{2}\cos\left(\frac{2\pi}{8}\right) & -\frac{\sqrt{2}}{2}\cos\left(\frac{2\pi}{8}\right) & -\frac{\sqrt{2}}{2}\cos\left(\frac{2\pi}{8}\right) & \frac{\sqrt{2}}{2}\cos\left(\frac{2\pi}{8}\right) \\ \frac{\sqrt{2}}{2}\cos\left(\frac{3\pi}{8}\right) & -\frac{\sqrt{2}}{2}\cos\left(\frac{\pi}{8}\right) & \frac{\sqrt{2}}{2}\cos\left(\frac{\pi}{8}\right) & -\frac{\sqrt{2}}{2}\cos\left(\frac{3\pi}{8}\right) \end{bmatrix}}$$

Od ovog retka možemo shvatiti analizirajući FIR filter duljine N

$$H_k(z) = C_4^{k,0} + z\cdot C_4^{k,1} + z^2 C_4^{k,2} + z^3\cdot C_4^{k,3}$$

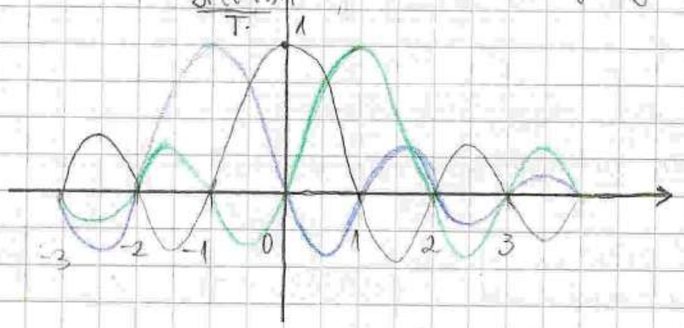
iz svakog stupca IDCT matrice stvaramo rekonstrukcijski filter, međutim, budući da C_4^{-1} matrica nastaje transponiranjem C_4 matrice, k-ti stupac matrice C_4^{-1} odgovara k-tom retku matrice C_4

$$\Rightarrow F_k(z) = C_4^{k,0} + z^{-1}\cdot C_4^{k,1} + z^{-2} C_4^{k,2} + z^{-3} C_4^{k,3}$$

23) Određujemo interpolacijsku funkciju oblika sinc:

normalizirana sinc funkcija je oblika ; $\mathrm{sinc}(t) = \frac{\sin(\pi t)}{(\pi t)}$

$\frac{\sin\left(\frac{\pi(t-iT)}{T}\right)}{\frac{\pi(t-iT)}{T}}$ - normalizirana funkcija ima nultočke u cjelobrojnim točkama t



$$\mathrm{sinc}\left(\frac{t-iT_s}{T_s}\pi\right) = \frac{\sin\left[\left(\frac{t-iT_s}{T_s}\right)\cdot\pi\right]}{\left(\frac{t-iT_s}{T_s}\right)\cdot\pi}$$

pomiče ishodište sinc funkcije (točku koja ima vrijednost 1) u cjelobrojne višekratnike perioda otipkavanja

osigurava da širina vrške latice bude točno jednaka širini intervala otipkavanja => nultočke će past na mjesta drugih uzoraka, a vrijednost 1 na mjestu pojedinog uzorka u koji smo funkciju postavili

za $t = iT_s$ dobiva se vrijednost koju sinc(t) ima u 0

- interpolirani signal bit će jednak:

$$x_{int}(t) = \sum_{i=-\infty}^{\infty} x[i]\,\mathrm{sinc}\left(\pi\cdot\frac{t-iT_s}{T_s}\right) = \sum_{i=-\infty}^{\infty} x[i]\cdot\frac{\sin\left[\pi\left(\frac{t-iT_s}{T_s}\right)\right]}{\pi\left(\frac{t-iT_s}{T_s}\right)}$$

$\cos\left(\frac{x}{3}\right)$ duži 3 puta širina funkcije od $\cos(x)$

a) $x[n] = \{\ldots, 0, 0, 0, \underline{1}, 2, 0, -2, -1, 0, 0, 0, \ldots\}$, $T_s = 1s$

$$(x_{int})_1(t) = \sum_{i=0}^{4} x[i]\cdot\mathrm{sinc}(t-i) = \mathrm{sinc}(t) + 2\,\mathrm{sinc}(t-1) - 2\,\mathrm{sinc}(t-3) - \mathrm{sinc}(t-4)$$

$t_1 = 0{,}5s$: $(x_{int})_1(0{,}5) = \frac{\sin(0{,}5\pi)}{0{,}5\pi} + 2\cdot\frac{\sin(-0{,}5\pi)}{(-0{,}5\pi)} - 2\cdot\frac{\sin(-2{,}5\pi)}{(-2{,}5\pi)} - \frac{\sin(-3{,}5\pi)}{(-3{,}5\pi)} =$

$$= \frac{\sin(0{,}5\cdot 180°)}{0{,}5\cdot 3{,}14} + 2\cdot\frac{\sin(-0{,}5\cdot 180°)}{(-0{,}5\cdot 3{,}14)} + \ldots$$

$$= 0{,}637 + 2\cdot 0{,}637 - 2\cdot 0{,}127 + 0{,}0909 = 1{,}1179$$

$t_2 = 0{,}25s$: $(x_{int})_1(0{,}25) = \frac{\sin(0{,}25\pi)}{0{,}25\pi} + 2\cdot\frac{\sin(-0{,}75\pi)}{-0{,}75\pi} - 2\cdot\frac{\sin(-2{,}75\cdot\pi)}{(-2{,}75\pi)} - \frac{\sin(-3{,}75\cdot\pi)}{(-3{,}75\pi)}$

$$= 0{,}9003 + 2\cdot 0{,}3 - 2\cdot 0{,}08 + 0{,}06 = 1{,}396$$

b) $x[n] = \{\ldots 0, 0, \underline{1}, 0, 2, 1, 1, 0, 1, 1, 0, 0, \ldots\}$

$$(x_{int})_2(t) = \sum_{i=0}^{7} x[i]\cdot\mathrm{sinc}\left(\frac{t-iT_s}{T_s}\right) = \mathrm{sinc}(t) + 2\,\mathrm{sinc}(t-2) + \mathrm{sinc}(t-3) + \mathrm{sinc}(t-4) + \mathrm{sinc}(t-6) + \mathrm{sinc}(t-7)$$

$t_1 = 0{,}5s$: $(x_{int})_2(0{,}5) = \frac{\sin(\pi\, 0{,}5)}{\pi\, 0{,}5} + 2\frac{\sin(\pi(-1{,}5))}{-\pi\, 1{,}5} + \frac{\sin(-2{,}5\pi)}{-2{,}5\cdot\pi} + \frac{\sin(-3{,}5\pi)}{-3{,}5\pi} + \frac{\sin(-5{,}5\pi)}{-5{,}5\pi} + \frac{\sin(-6{,}5\pi)}{-6{,}5\pi}$

$$= 0{,}637 - 2\cdot 0{,}212 + 0{,}127 - 0{,}0909 - 0{,}0579 + 0{,}0489 = 0{,}24$$

$t_2=0{,}25:\ (X_{int})_2(0{,}25)=\frac{\sin(0{,}25\pi)}{0{,}25\pi}+2\frac{\sin(-1{,}75\pi)}{-1{,}75\pi}+\frac{\sin(-2{,}75\pi)}{-2{,}75\pi}+\frac{\sin(-3{,}75\pi)}{-3{,}75\pi}+\frac{\sin(-5{,}75\pi)}{-5{,}75\pi}+\frac{\sin(-6{,}75\pi)}{-6{,}75\pi}$

$=0{,}9003-0{,}2572+0{,}0818-0{,}060-0{,}0391+0{,}0333$

c) $x[n]=\{\ldots,0,0,\underline{1},1,0,1,-1,0,0,-1,0,0,0,\ldots\}$

$(X_{int})_3(t)=\mathrm{sinc}(t)+\mathrm{sinc}(t-1)+\mathrm{sinc}(t-3)-\mathrm{sinc}(t-4)-\mathrm{sinc}(t-7)$

$t_1=0{,}5:\ (X_{int})_1(0{,}5)=\frac{\sin(0{,}5\pi)}{0{,}5\pi}+\frac{\sin(-0{,}5\pi)}{-0{,}5\pi}+\frac{\sin(-2{,}5\pi)}{-2{,}5\pi}-\frac{\sin(-3{,}5\pi)}{-3{,}5\pi}-\frac{\sin(-6{,}5\pi)}{-6{,}5\pi}$

$=0{,}6366+0{,}6366+0{,}1273+0{,}0909-0{,}049=1{,}425$

$t_2=0{,}25\ (X_{int})_2(0{,}25)=\frac{\sin(0{,}25\pi)}{0{,}25\pi}+\frac{\sin(-0{,}75\pi)}{-0{,}75\pi}+\frac{\sin(-2{,}75\pi)}{-2{,}75\pi}-\frac{\sin(-3{,}75\pi)}{-3{,}75\pi}-\frac{\sin(-6{,}75\pi)}{-6{,}75\pi}$

$=0{,}9003+0{,}3001+0{,}0818+0{,}0600-0{,}0333=1{,}3089$

2h) $T_s=1s$

$x[n]=5\delta[n]-6\delta[n-2]+\delta[n-5]-\delta[n-7]=\{\underline{5},0,-6,0,0,1,0,-1\}$

$h_{II}(t)=\mathrm{sinc}\left(\frac{t}{T_s}\right)$ – impulsni odziv idealnog interpolatora

$h_{ZOH}(t)=\mathrm{rect}\left(\frac{t}{T_s}\right)$ – impulsni odziv interpolatora nultog reda (Zero Order Hold)

Idealna interpolacija:

$$(X_{int_{II}})(t)=\sum_{i=-\infty}^{\infty}x[i]\cdot\mathrm{sinc}\left(\frac{t-T_s i}{T_s}\cdot\pi\right)$$

$t_1=0{,}5s\ :\ (X_{int_{II}})(0{,}5)=5\cdot\frac{\sin(0{,}5\pi)}{0{,}5\pi}-6\cdot\frac{\sin(-1{,}5\pi)}{-1{,}5\pi}+\frac{\sin(-4{,}5\pi)}{-4{,}5\pi}-\frac{\sin(-6{,}5\pi)}{-6{,}5\pi}$

$=3{,}1831+1{,}2732+0{,}0707-0{,}0490=4{,}4781$

INTERPOLACIJA FILTROM PRVOG REDA

$(X_{int})_{ZOH}(t)=\sum_{i=-\infty}^{\infty}x[i]\cdot\mathrm{rect}\left(\frac{t-t_s i}{T_s}\right)=5\cdot\mathrm{rect}(t)-6\,\mathrm{rect}(t-2)+1\,\mathrm{rect}(t-5)-\mathrm{rect}(t-7)$

$t_1=0{,}5s:\ (X_{int})_{ZOH}(0{,}5)=5\,\mathrm{rect}(0{,}5)-6\,\mathrm{rect}(-1{,}5)+\mathrm{rect}(-4{,}5)-\mathrm{rect}(-6{,}5\pi)$

$=5\cdot1-6\cdot0+0-0=5$

$$\mathrm{rect}(t)=\begin{cases}1, & -\frac{1}{2}\le t\le\frac{1}{2}\\ 0, & \text{inače}\end{cases}$$

25) $x[n] = \{\underline{1}, 2, 0, -1\}$

$T_S = 1s$
$t_1 = 0{,}5s$

$$tri(t) = \begin{cases} 1-|t| ; & |t| \le 1_s \\ 0 ; & inače \end{cases}$$

Impulsni odziv idealnog interpolatora:

$$h_{II}(t) = sinc\left(\frac{t}{T_S}\right)$$

Impulsni odziv interpolatora prvog reda (FOH):

$$h_{FOH}(t) = tri\left(\frac{t}{T_S}\right)$$

INTERPOLACIJA IDEALNIM FILTROM:

$$(X_{int})_{II}(t) = \sum_{i=-\infty}^{\infty} x[i] \cdot sinc\left(\frac{t-iT_S}{T_S}\right) = 1 \cdot sinc(t) + 2 \cdot sinc(t-1) - sinc(t-3)$$

$t_1 = 0{,}5s$: $\quad (X_{int})_{II}(0{,}5) = \frac{sin(0{,}5\pi)}{0{,}5\pi} + 2 \cdot \frac{sin(-0{,}5\pi)}{-0{,}5\pi} - \frac{sin(-2{,}5\pi)}{-2{,}5\pi}$

$$= 0{,}6366 + 1{,}2732 - 0{,}1273 = 1{,}7825$$

INTERPOLACIJA FILTROM PRVOG REDA

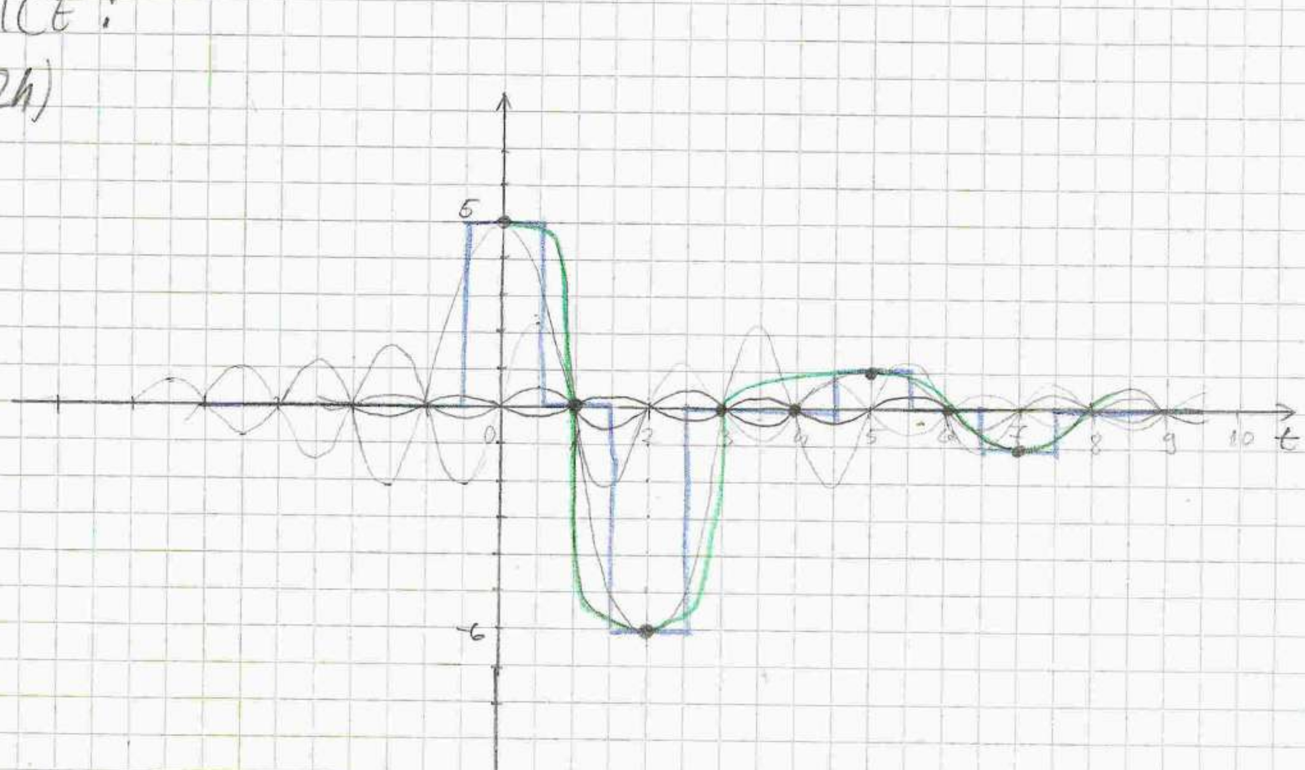
$$(X_{int})_{FOH}(t) = \sum_{i=-\infty}^{\infty} x[i] \cdot tri\left(\frac{t-iT_S}{T_S}\right) = 1 \cdot tri(t) + 2 \cdot tri(t-1) - tri(t-3)$$

$t_1 = 0{,}5s$ $\quad (X_{int})_{FOH}(0{,}5) = tri(0{,}5) + 2 \cdot tri(-0{,}5) - tri(-1{,}5) =$

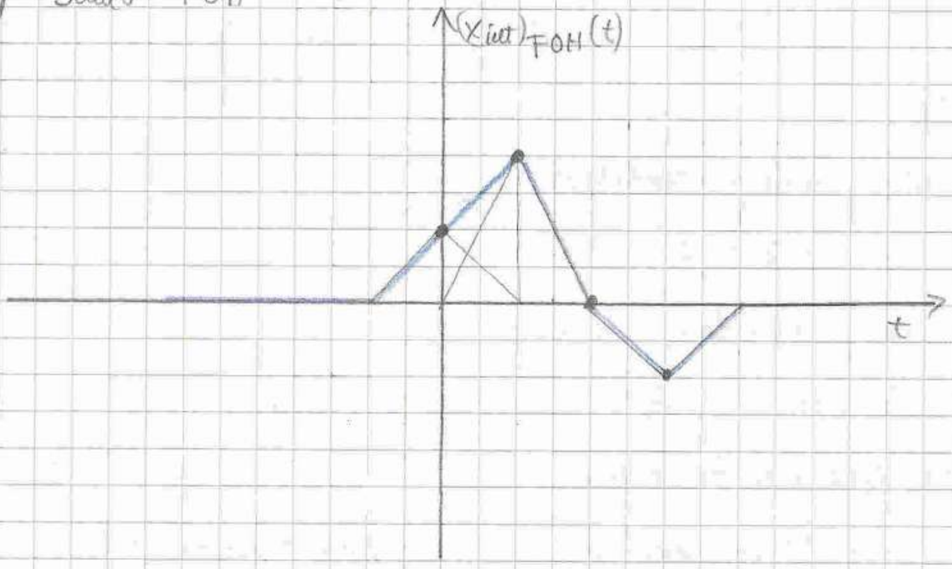
$$= (1-0{,}5) + 2(1-0{,}5) - 0 = 0{,}5 + 2 \cdot 0{,}5 = 1{,}5$$

SKICE:

2h)



25) Same FOH



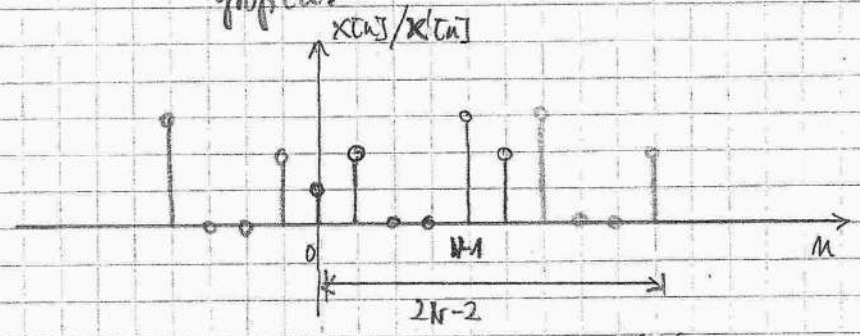
→ funkcija sinus je neparna, pa bi se sinusi za negativne vrijednosti n poništili s vrijednostima za pozitivne vrijednosti n, a vrijednosti kosinusa će se poduplati

⇒ signal na kojem provodimo DFT_{2N-2} mora biti sljedećeg oblika:

- ima 2N-2 uzorka
- 1. i N-ti uzorak nemaju simetričnog para

⇒ radimo parno proširenje pri čemu se prvi i zadnji uzorak ne dupliciraju, a zatim se simetrično signal proširuje od 1-og do (2N-2)-og uzorka

Grafički:



- sad treba uzeti u obzir još i sljedeće:

pri ručnom računanju DCT, za uzorak, x[n] se još dodatno množio $\frac{1}{\sqrt{2}}$ za $n=0$ i $n=N-1$

⇒ prije računanja DFT_{2N-2}, uzorke za $n=0$ i $n=N-1$ treba pomnožiti s $\sqrt{2}$

uzorci za n=0 i n=N-1 nemaju duplikate kosinusa i DFT za njih daje vrijednost /2 množenjem $\sqrt{2}$ dobivamo $\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$ što i dobivamo za uzorke

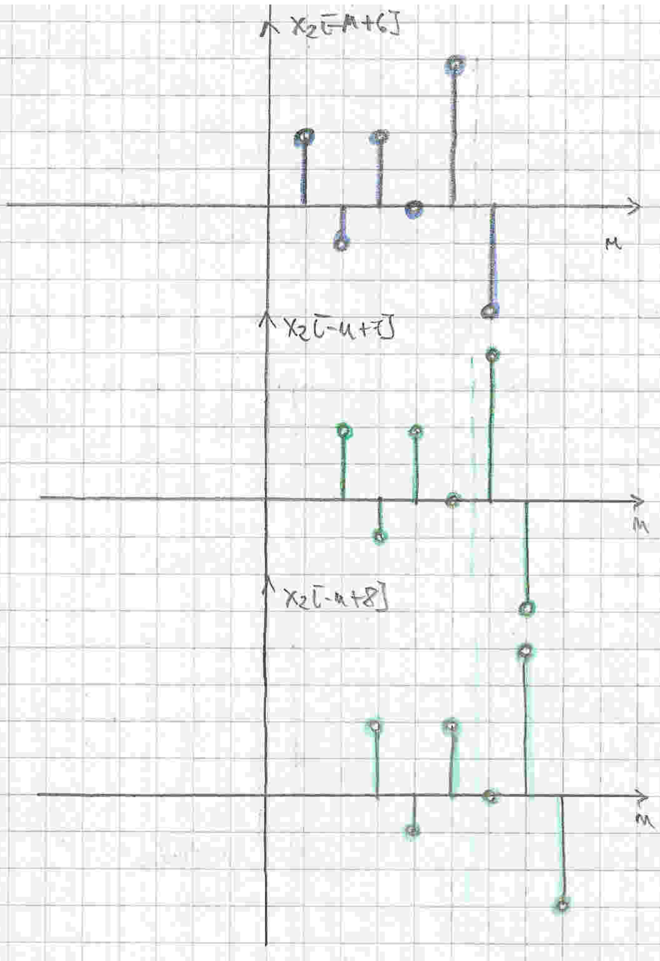
- za računanje DCT-a uvijek ima množenje s $\sqrt{\frac{2-\delta[k]-\delta[k-N+1]}{N-1}}$, pa to treba napraviti i sa DFT_{2N-2}

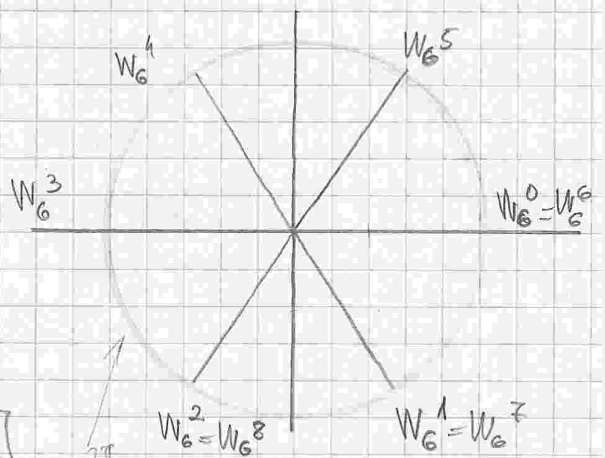
⇒ POTPUNI POSTUPAK je sljedeći:

- neka nam je zadan realni signal $x[n]$
- želimo izračunati $DCT\text{-}I_N(x[n])$ koristeći $DFT_{2N-2}(x'[n])$
- najprije zadani signal proširimo parno oko posljednjeg uzorka tako da posljednji i prvi uzorak ne dupliciramo:

(svejedno je signala, za koje se traži DCT-I, da ga parno proširimo i preko prvog i preko zadnjeg uzorka ⇒ bitno je samo da li parno proširimo tako da prvo oko prvog uzorka izvedemo parno proširenje a zatim periodično proširimo ili obrnuto izvedemo parno proširenje oko zadnjeg uzorka)

- zatim prvi ($n=0$) i N-ti ($n=N-1$) uzorak množimo s $\sqrt{2}$
- nad tako dobivenim signalom provodimo DFT_{2N-2} za $k=0,\dots,2N-3$
- dobivene vrijednosti koeficijenata množimo s $\sqrt{\frac{2-\delta[k]-\delta[k-N+1]}{N-1}}\cdot\frac{1}{2}$ → zato jer DFT podupla vrijednost kosinusa
- izdvojimo prvih N $X_F[k]$ koeficijenata





$\cos(\frac{\pi}{3}) - j\sin(\frac{\pi}{3})$

$0{,}5 - j\frac{\sqrt{3}}{2}$

c) $x_2[n] \circledN x_1'[n]$

$$x[n] \circledN y[n] = IDFT_N\left[DFT_N(x[n]) \cdot DFT_N(y[n])\right]$$

$$DFT_6(x_1'[n]) = \sum_{n=0}^{5} x_1'[n] \cdot W_6^{nk} = \sum_{n=0}^{5} x_1'[n] \cdot e^{-j\frac{\pi}{3}nk} \quad ; k=0,\dots,6$$

$\frac{2\pi}{6}$

$$= 5W_6^{0\cdot k} + 2\cdot W_6^{1\cdot k} + 4\cdot W_6^{2k} - W_6^{3k}$$

$$DFT_6(x_2[n]) = \sum_{n=0}^{5} x_2[n] \cdot W_6^{nk} =$$

$$= -3W_6^{0k} + 4W_6^{1k} + 0\cdot W_6^{2k} + 2W_6^{3k} - W_6^{4k} + 2W_6^{5k}$$

$$X_c[k] = DFT_6(x_1'[n]) \cdot DFT_6(x_2[n]) = (5 + 2W_6^k + 4W_6^{2k} - W_6^{3k})(-3 + 4W_6^k + 2W_6^{3k} - W_6^{4k} + 2W_6^{5k})$$

$$= -15 + 20W_6^k + 10W_6^{3k} - 5W_6^{4k} + 10W_6^{5k} - 6W_6^k + 8W_6^{2k} \cancel{+4W_6^{4k}} - 2W_6^{5k} \cancel{+4W_6^{6k}}$$
$$-12W_6^{2k} + 16W_6^{3k} + 8\cdot W_6^{5k} \cancel{-4W_6^{6k}} + 8W_6^{7k} + 3W_6^{3k} \cancel{-4W_6^{4k}} + 2W_6^{6k} + W_6^{7k} - 2W_6^{8k}$$

$$= -15 + 14W_6^k + 29W_6^{3k} - 5W_6^{4k} + 16W_6^{5k} - 4W_6^{2k} - 2W_6^{6k} + 9W_6^{7k} - 2W_6^{8k}$$

$$= -15 + 14W_6^k + 29W_6^{3k} - 5W_6^{4k} + 16W_6^{5k} - 4W_6^{2k} - 2W_6^0 + 9W_6^k - 2\cdot W_6^2$$

$$= -17 + 23W_6^k - 6W_6^{2k} + 29W_6^{3k} - 5W_6^{4k} + 16W_6^{5k}$$

$$IDFT_6[X_c[k]] = x_c[n] = \frac{1}{6}\sum_{k=0}^{6} \underbrace{X_c[k] \cdot W_6^{-nk}}\quad ; n=0,\dots,6$$

$\longrightarrow$

OBRADBA INFORMACIJA: 2. DOMAĆA ZADAĆA

BRZA FOURIEROVA TRANSFORMACIJA

1) Izravna realizacija DFT-a (diskretne Fourierove transformacije) na računalu ima asimptotsku složenost $O(N^2)$ za N uzoraka signala, budući da DFT pravi preslikavanje niza uzoraka perioda N u niz kompleksnih brojeva perioda N.
Time se određuje da je dimenzija matrice DFT-a $N \times N$, pa je potrebno provesti $N \times N$ kompleksnih množenja i $N \times (N-1)$ kompleksnih zbrajanja za transformaciju N uzoraka

⇒ broj operacija raste s kvadratom duljine signala

Stoga se, često koriste "brže" transformacije, koje koriste svojstva PERIODIČNOSTI i SIMETRIJE kompleksne eksponencijale:

$$W_N^{2r} = W_{N/2}^{r} \qquad W_N^0 = 1$$

$$W_N^{r+\frac{N}{2}} = -W_N^{r} \qquad W_N^{N/2} = -1$$

$$W_N^{r+N} = W_N^{r} \qquad W_N^{N/4} = -j$$

$$W_N^{3\frac{N}{4}} = j$$

FFT (Fast Fourier Transform) je postupak kojim se DFT_N transformacija razlaže u više stupnjeva s po manje točaka
Zbog jednostavnosti, najčešće se odabire broj N koji se može rastaviti na produkt jednakih malih faktora (najčešće dvojke, odnosno najčešće je $N = 2^n$).
Ako se odabere broj N ($N = 2^n$), metoda FFT dobiva naziv korijen-2 metoda
U tom slučaju, izvornu sumu možemo dijeliti n puta i to dvije sume s brojem članova jednakim polovini broja članova prethodne glavne sume.

- svaki od n stupnjeva ima N množenja, a koraka ima $n = \log_2 N$, pa je ukupan broj operacija: $N \log_2 N$, a asimptotska složenost je

$O(n \log_2 n)$

LINEARNA I CIRKULARNA KONVOLUCIJA:

4) Linearna konvolucija:

Odziv nekog LTI (linearnog i vremenski stalnog) sustava na bilo koju pobudu u možemo odrediti kao:

$$y[n] = f(u[n])$$

Budući da je signal $u[n]$ diskretan, možemo ga zapisati pomoću jediničnih impulsa:

$$u[n] = \sum_{-\infty}^{\infty} u[k]\delta[n-k]$$

Sada, odziv sustava možemo zapisati na sljedeći način:

$$y[n] = f\left(\sum_{-\infty}^{\infty} u[k]\cdot\delta[n-k]\right) = \sum_{-\infty}^{\infty} u[k]\cdot f(\delta[n-k])$$

za svaki k je konstantna vrijednost (tj. realni broj)