1) $\dot{x} = -y - x^3 - xy^2$
$\dot{y} = x - y^3 - x^2 y$ $\quad \dot{V}(x,y) = ?$

$V(x,y) = x^2 + y^2$

$$\dot{V}(x,y) = \frac{dV(x,y)}{dt} = \frac{\partial V(x,y)}{\partial x}\frac{dx}{dt} + \frac{\partial V(x,y)}{\partial y}\frac{dy}{dt} = V_x' f + V_y' g$$

$$\dot{V}(x,y) = 2x(-y - x^3 - xy^2) + 2y(x - y^3 - x^2y) = \cancel{-2xy} - 2x^4 - 2x^2y^2 \cancel{+2xy} - 2y^4 - 2x^2y^2 =$$
$$= -2(x^4 + 2x^2y^2 + y^4) = -2(x^2+y^2)^2$$

funkcija je negativno definitna, ishodište je asimptotski stabilno

2) $\dot{x} = -y + x^3 + xy^2$
$\dot{y} = x + y^3 + x^2 y$ $\quad \dot{V}(x,y) = ?$

$V(x,y) = -x^2 - y^2$

$$\dot{V}(x,y) = -2x(-y + x^3 + xy^2) - 2y(x + y^3 + x^2y) = \cancel{2xy} - 2x^4 - 2x^2y^2 \cancel{-2xy} - 2y^4 - 2x^2y^2 =$$
$$= -2(x^4 + 2x^2y^2 + y^4) = -2(x^2+y^2)^2$$

funkcija je negativno definitna, ishodište je asimptotski stabilno

3) $\dot{x} = -x + y + xy$
$\dot{y} = x - y - x^2 - y^3$ $\quad \dot{V}(x,y) = ?$

$V(x,y) = x^2 + y^2$

$$\dot{V}(x,y) = 2x(-x + y + xy) + 2y(x - y - x^2 - y^3) = -2x^2 + 2xy \cancel{+2x^2y} + 2xy - 2y^2$$
$$\cancel{-2x^2y} - 2y^4 = -2(x^2 - 2xy + y^2) - 2y^4 = -2(x-y)^2 - 2y^4$$

funkcija je

4) $\dot{x} = -x - 2y + xy^2$
$\dot{y} = 3x - 3y + y^3$ $\quad \dot{V}(x,y)$

$V(x,y) = ax^2 + by^2$

$$\dot{V}(x,y) = 2ax(-x - 2y + xy^2) + 2by(3x - 3y + y^3) = -2ax^2 - 4axy + 2ax^2y^2 +$$
$$+ 6bxy - 6by^2 + 2by^4 = -2ax^2 + 2ax^2y^2 - 6by^2 + 2by^4 - 2(2a - 3b)xy$$
$$= \left[\begin{array}{ll} 2a - 3b = 0 & b = 2 \\ a = \frac{3}{2}b & a = 3 \end{array}\right] = -6x^2 + 6x^2y^2 - 12y^2 + 6by^4$$

5) $\dot{x} = -y - xy$ $\quad \dot{V}(x,y) = ?$
$\dot{y} = x + x^2$

$V(x,y) = x^2 + y^2$

$\dot{V}(x,y) = 2x(-y-xy) + 2y(x+x^2) = \cancel{-2xy} \cancel{-2x^2y} \cancel{+2xy} \cancel{+2x^2y} = 0$

funkcija

6) $\dot{x} = -x^4 + 5\mu x^2 - 4\mu^2$

a) bifurkacija sedlo – čvor za $\mu = 0$

b) singulariteti

$-x^4 + 5\mu x^2 - 4\mu^2 = 0$

$x_{1,2}^2 = \dfrac{-5\mu \pm \sqrt{25\mu^2 - 16\mu^2}}{-2} = \dfrac{-5\mu \pm 3\mu}{-2}$ $\quad x_1^2 = \mu \;,\; x_2^2 = 4\mu$

$x_{1,2} = \pm\sqrt{\mu}$
$x_{3,4} = \pm 2\sqrt{\mu}$

$\dfrac{\partial f(x,\mu)}{\partial x}\Big|_{x=\pm\sqrt{\mu}} = -4x^3 + 10\mu x\Big|_{x=\pm\sqrt{\mu}} = \mp 4\mu\sqrt{\mu} \pm 10\mu\sqrt{\mu} =$
$= \pm 6\mu\sqrt{\mu}$

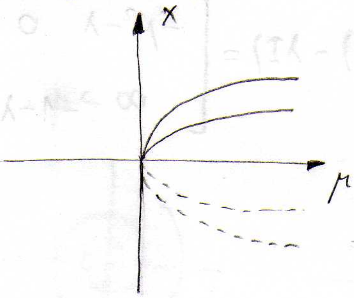
$\dfrac{\partial f(x,\mu)}{\partial x}\Big|_{x=\pm 2\sqrt{\mu}} = \mp 24\mu\sqrt{\mu} \pm 20\mu\sqrt{\mu} = \mp 4\mu\sqrt{\mu}$

$\mu < 0$

$\mu = 0$

$\mu > 0$

c) bifurkacijski dijagram

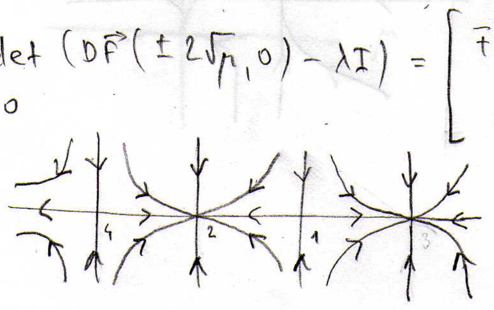


7) $\dot{x} = -x^4 + 5\mu x^2 - 4\mu^2 = 0$
$\dot{y} = -y$

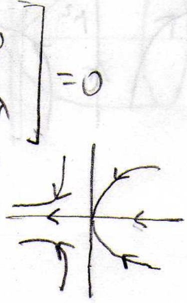
b) $D\vec{F}(x,y) = \begin{bmatrix} -4x^3 + 10\mu x & 0 \\ 0 & -1 \end{bmatrix}$, $\det(D\vec{F}(\pm\sqrt{\mu},0) - \lambda I) = \begin{bmatrix} \pm 6\mu\sqrt{\mu} - \lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} = 0$

$\det(D\vec{F}(\pm 2\sqrt{\mu},0) - \lambda I) = \begin{bmatrix} \mp 4\mu\sqrt{\mu} - \lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} = 0$

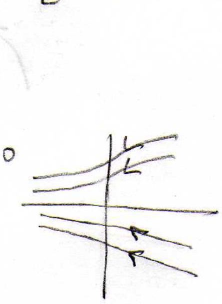
$\mu > 0$



$\mu = 0$



$\mu < 0$



8) $\dot{x} = x^2 - x\mu^2$

a) transkritična bifurkacija

b) singulariteti

$x^2 - x\mu^2 = 0$

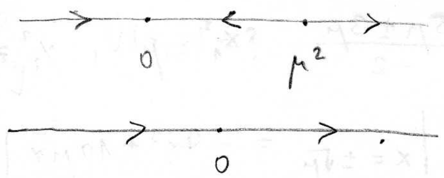
$x(x-\mu^2) = 0$

$x_1 = 0$ $\qquad \left.\dfrac{\partial f(x,\mu)}{\partial x}\right|_{x=0} = \left. 2x - \mu^2 \right|_{x=0} = -\mu^2$

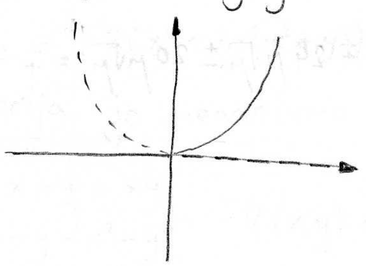
$x_2 = \mu^2$ $\qquad \left.\dfrac{\partial f(x,\mu)}{\partial x}\right|_{x=\mu^2} = \mu^2$

$\mu \neq 0$

$\mu = 0$



c) bifurkacijski dijagram



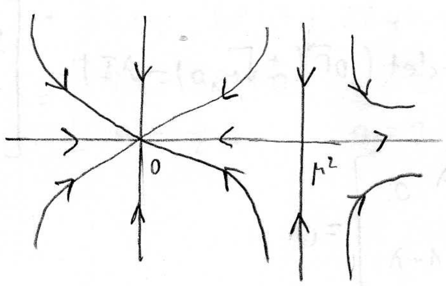
9) $\dot{x} = x^2 - x\mu^2$

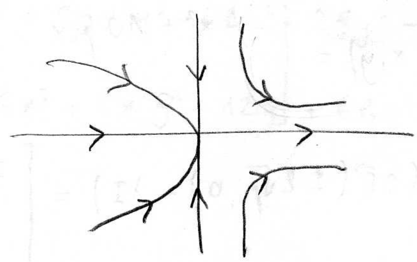
$\dot{y} = -y$

b) $DF(x,y) = \begin{bmatrix} 2x-\mu^2 & 0 \\ 0 & -1 \end{bmatrix}$, $\det(DF(0,0) - \lambda I) = \begin{bmatrix} -\mu^2-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} = 0$

$\det(DF(\mu^2,0) - \lambda I) = \begin{bmatrix} \mu^2-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} = 0$

$\mu \neq 0$ $\qquad\qquad \mu = 0$





10) $\dot{x} = \mu x - y - x\sqrt{x^2+y^2}$

$\dot{y} = x + \mu y - y\sqrt{x^2+y^2}$

a) $r = ?\,,\ \varphi = ?$

$x = r\cos\varphi \qquad \dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi$

$y = r\sin\varphi \qquad \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$

$$\left.\begin{array}{l}\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi = \mu r\cos\varphi - r\sin\varphi - r\cos\varphi\cdot r \quad /\cdot\cos\varphi \\ \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi = r\cos\varphi + \mu r\sin\varphi - r\sin\varphi\cdot r \quad /\cdot\sin\varphi\end{array}\right\} + \quad \left.\begin{array}{l}/\cdot(-\sin\varphi) \\ /\cdot\cos\varphi\end{array}\right\} +$$

$\dot{r} = \mu r - r^2 = r(\mu - r)$

$\cancel{r}\dot{\varphi} = \cancel{r} \Rightarrow \dot{\varphi} = 1$

b) $\nabla\vec{F}(x,y) = ? \qquad \mu > 0$

$$\nabla\vec{F}(x,y) = \mu - \sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}} + \mu - \sqrt{x^2+y^2} - \frac{y^2}{\sqrt{x^2+y^2}}$$

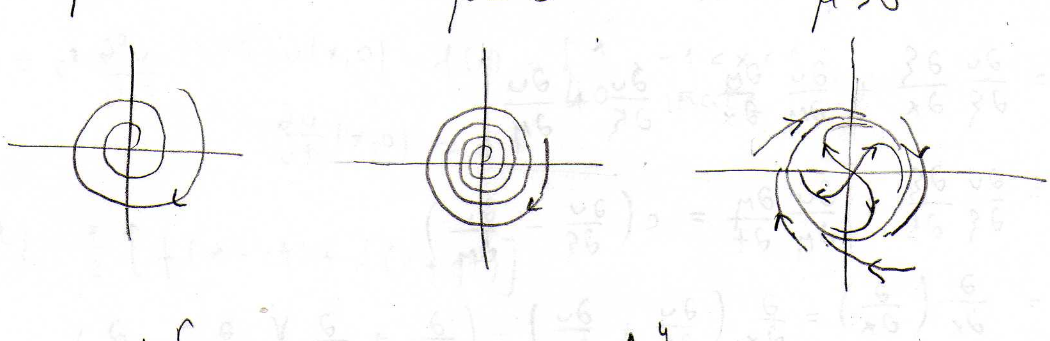
$$= 2\mu - 2\sqrt{x^2+y^2} - \frac{(x^2+y^2)}{\sqrt{x^2+y^2}} = 2\mu - 3\sqrt{x^2+y^2}$$

$$\int_0^{2\pi} \nabla\vec{F}(\gamma(t))\,dt = \int_0^{2\pi}(2\mu-3)\,dt = (2\mu-3)t\Big|_0^{2\pi} = (2\mu-3)2\pi < 0$$

c) $\nabla\vec{F}(x,y) = ? \quad \mu \le 0$

d)

$\mu < 0$ $\qquad$ $\mu = 0$ $\qquad$ $\mu > 0$



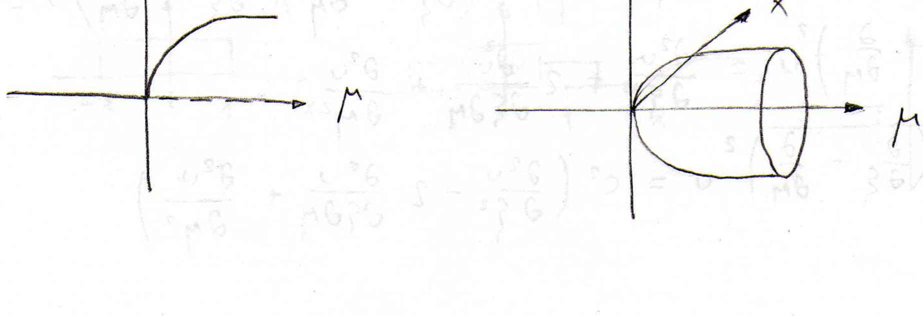
$0 < r < \mu$

$\dot{r} > 0$

$\mu < r < \infty$

$\dot{r} < 0$

e)



11) a) $\dfrac{\partial^2 u(x,y)}{\partial x^2} + \dfrac{\partial^2 u(x,y)}{\partial x \partial y} = 0$

$a = 1, \quad b = 1 \qquad b^2 - 4ac = 1^2 - 4 \cdot 1 \cdot 0 = 1$ hiperbolička

b) $\dfrac{\partial^2 u(x,t)}{\partial t^2} = 2 \dfrac{\partial^2 u(x,t)}{\partial x^2} - \dfrac{\partial u(x,t)}{\partial x} + u(x,t)$

$a = 2, \quad c = -1 \qquad b^2 - 4ac = 0^2 - 4 \cdot 2 \cdot (-1) = 4$ hiperbolička

c) $\dfrac{\partial^2 u(x,y)}{\partial x^2} - \dfrac{\partial^2 u(x,y)}{\partial y^2} = \cos x$

$a = 1, \quad c = -1 \qquad b^2 - 4ac = 0^2 - 4 \cdot 1 \cdot (-1) = 4$ hiperbolička

12) $y''(x) + \lambda y(x) = 0, \quad 0 < x < 1, \quad y(0) = 0, \quad y'(1) = 0$

$r^2 + \lambda = 0 \Rightarrow r_{1,2} = \pm\sqrt{-\lambda}$

$y(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} \Rightarrow y'(x) = \sqrt{-\lambda}\, C_1 e^{\sqrt{-\lambda}x} - \sqrt{-\lambda}\, C_2 e^{\sqrt{-\lambda}x}$

I $\lambda < 0$

$$\left.\begin{array}{l} y(0) = C_1 + C_2 = 0 \\ y'(1) = \sqrt{-\lambda}\, C_1 e^{\sqrt{-\lambda}} - \sqrt{-\lambda}\, C_2 e^{-\sqrt{-\lambda}} = 0 \end{array}\right\} \begin{array}{l} C_1 = 0 \\ C_2 = 0 \end{array} \quad y(x) \equiv 0$$

II $\lambda = 0$

$y(x) = C_1 + C_2 x \Rightarrow y'(x) = C_2$

$y(0) = C_1 = 0$

$y'(1) = C_2 = 0$

III $y > 0$

$r^2 - \lambda = 0 \Rightarrow r_{1,2} = \pm\sqrt{\lambda}$

$y(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) \Rightarrow y'(x) = -\sqrt{\lambda}\, C_1 \sin(\sqrt{\lambda}x) + \sqrt{\lambda}\, C_2 \cos(\sqrt{\lambda}x)$

$y(0) = C_1 = 0$

$y'(1) = -\sqrt{\lambda}\, C_1 \sin\sqrt{\lambda} + \sqrt{\lambda}\, C_2 \cos\sqrt{\lambda} = \sqrt{\lambda}\, C_2 \cos\sqrt{\lambda} = 0, \qquad C_2 \neq 0$

$\cos\sqrt{\lambda} = 0 \Rightarrow \sqrt{\lambda} = n\pi \Rightarrow \lambda_n = (n\pi)^2 \qquad n = 1, 2, \ldots$

$y_n(x) = C_n \cos(n\pi x)$

13) $y''(x) + \lambda y(x) = 0, \quad 0 < x < 1, \quad y'(0) = 0, \quad y(1) = 0$

$r^2 + \lambda = 0 \Rightarrow r_{1,2} = \pm\sqrt{-\lambda}$

$y(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} \Rightarrow y'(x) = \sqrt{-\lambda}C_1 e^{\sqrt{-\lambda}x} - \sqrt{-\lambda}C_2 e^{-\sqrt{-\lambda}x}$

I $\lambda < 0$

$$\left.\begin{aligned} y'(0) &= \sqrt{-\lambda}C_1 - \sqrt{-\lambda}C_2 = 0 \\ y(1) &= C_1 e^{\sqrt{-\lambda}} + C_2 e^{-\sqrt{-\lambda}} = 0 \end{aligned}\right\} \begin{aligned} C_1 &= 0 \\ C_2 &= 0 \end{aligned} \quad y(x) \equiv 0$$

II $\lambda = 0$

$y(x) = C_1 + C_2 x \Rightarrow y'(x) = C_2$

$y'(0) = C_2 = 0$

$y(1) = C_2 = 0$

III $\lambda > 0$

$r^2 - \lambda = 0 \Rightarrow r_{1,2} = \pm\sqrt{\lambda}$

$y(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x) \Rightarrow y'(x) = -\sqrt{\lambda}C_1\sin(\sqrt{\lambda}x) + \sqrt{\lambda}C_2\cos(\sqrt{\lambda}x)$

$y'(0) = \sqrt{\lambda}C_2 = 0 \Rightarrow C_2 = 0$

$y(1) = C_1\cos\sqrt{\lambda} + C_2\sin\sqrt{\lambda} = C_1\cos\sqrt{\lambda} = 0, \quad C_1 \neq 0$

$\cos\sqrt{\lambda} = 0 \Rightarrow \sqrt{\lambda} = m\pi \Rightarrow \lambda_m = (m\pi)^2, \quad m = 1, 2, \dots$

$y_m = C_m\cos(m\pi x)$

14) $u(x,t) = e^{-t}(A\sin x + B\cos x), \quad \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2}$

$$\begin{aligned} -e^{-t}(A\sin x + B\cos x) &= \frac{\partial}{\partial x}\left[e^{-t}(A\cos x - B\sin x)\right] \\ &= e^{-t}(-A\sin x - B\cos x) \\ &= -e^{-t}(A\sin x + B\cos x) \quad \checkmark \end{aligned}$$

15) $u(x,t) = e^{-\lambda^2\alpha^2 t}(A\sin\lambda x + B\cos\lambda x), \quad \dfrac{\partial u(x,t)}{\partial t} = \alpha^2\dfrac{\partial^2 u(x,t)}{\partial x^2}$

$$\begin{aligned} -\lambda^2\alpha^2 e^{-\lambda^2\alpha^2 t}(A\sin\lambda x + B\cos\lambda x) &= \alpha^2\frac{\partial}{\partial x}\left[e^{-\lambda^2\alpha^2 t}(A\lambda\cos\lambda x - B\lambda\sin\lambda x)\right] \\ &= \alpha^2 \cdot e^{-\lambda^2\alpha^2 t} \cdot (-A\lambda^2\cos\lambda x - B\lambda^2\sin\lambda x) \\ &= -\lambda^2\alpha^2 e^{-\lambda^2\alpha^2 t}(A\cos\lambda x + B\sin\lambda x) \quad \checkmark \end{aligned}$$

16) a) $$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x\in(0,1), \ t>0$$

$u(0,t) = u(1,t) = 0, \quad t>0$

$u(x,0) = 2\sin(4\pi x) + 7\sin(6\pi x) - 8\sin(9\pi x)$

$u(x,t) = X(x)\,T(t)$

$X(x)T'(t) = X''(x)T(t)$

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$X''(x) + \lambda X(x) = 0 \qquad u(0,t) = X(0)T(t) = 0 \qquad\qquad X(0) = 0$

$T'(t) + \lambda T(t) = 0 \qquad u(1,t) = X(1)T(t) = 0 \qquad T(t)\neq 0 \qquad X(1) = 0 \qquad$ S-L

$\lambda_n = (n\pi)^2, \qquad X_n(x) = \sin(n\pi x)$

$$\frac{dT(t)}{dt} = -\lambda T(t) \Rightarrow \frac{dT(t)}{T} = -\lambda dt \ \Big|\int \Rightarrow \ln T(t) = -\lambda t + \ln C \Rightarrow$$

$\Rightarrow T(t) = Ce^{-\lambda_n t} \Rightarrow T_n(t) = C_n e^{-(n\pi)^2 t}$

$u_n(x,t) = X_n(x)T_n(t) = C_n e^{-(n\pi)^2 t}\sin(n\pi x) \qquad n = 1,2,\dots$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} C_n e^{-(n\pi)^2 t}\sin(n\pi x)$$

$$u(x,0) = \sum_{n=1}^{\infty} u_n(x,0) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)$$

$n=4 \Rightarrow C_4 = 2, \quad n=6 \Rightarrow C_6 = 7, \quad n=9 \Rightarrow C_9 = -8$

$u(x,t) = 2e^{-16\pi^2 t}\sin(4\pi x) + 7e^{-36\pi^2 t}\sin(6\pi x) - 8e^{-81\pi^2 t}\sin(9\pi x)$

b) $$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x\in(0,1), \ t>0$$

$u(0,t) = u(1,t) = 0, \quad t>0$

$u(x,0) = x$

$$C_n = 2\int_0^1 u(x,0)\sin(n\pi x)\,dx = 2\int_0^1 x\sin(n\pi x)\,dx = \left[\begin{matrix} u = x & dv = \sin(n\pi x)dx \\ du = dx & v = \frac{-1}{n\pi}\cos(n\pi x)\end{matrix}\right]$$

$$= \frac{2x}{n\pi}\cos(n\pi x)\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \cos(n\pi x)\,dx = \frac{2}{n\pi}\cos(n\pi) + \frac{2}{n^2\pi^2}\sin(n\pi x)\Big|_0^1 =$$

$$= \frac{2}{n\pi}\cos(n\pi) + \frac{2}{n^2\pi^2}\sin(n\pi)$$

$u(x,t) =$

17) a) $\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$, $x \in (0,1)$, $t > 0$

$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0$, $t > 0$

$u(x,0) = 2\cos(2\pi x) + 4\cos(7\pi x) - 3\cos(5\pi x)$

$u(x,t) = X(x)\,T(t)$

$X(x)\,T'(t) = X''(x)\,T(t)$

$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = k$

$X''(x) - kX(x) = 0$ $\quad \frac{\partial u}{\partial x}(0,t) = X'(0)\,T(t) = 0$ $\quad X'(0) = 0$

$T'(t) - kT(t) = 0$ $\quad \frac{\partial u}{\partial x}(1,t) = X'(1)\,T(t) = 0$ $\quad T(t) \neq 0$ $\quad X'(1) = 0$ $\quad$ S-L

$k_n = -(n\pi)^2$ $\quad X_n(x) = \cos(n\pi x)$ $\quad \lambda = 0 \Rightarrow X_n(x) = \frac{C_0}{2}$

$T_n(t) = C_n e^{-(n\pi)^2 t}$

$u_n(x,t) = X_n(x)\,T_n(t) = C_n e^{-(n\pi)^2 t}\cos(n\pi x)$, $n = 1, 2, \ldots$

$u(x,t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} u_n(x,t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n e^{-(n\pi)^2 t}\cos(n\pi x)$

$u(x,0) = \frac{C_0}{2} + \sum_{n=1}^{\infty} u_n(x,0) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(n\pi x)$

$n = 2 \Rightarrow C_2 = 2$, $n = 7 \Rightarrow C_7 = 4$, $n = 5 \Rightarrow C_5 = -3$

$u(x,t) = 2e^{-4\pi^2 t}\cos(2\pi x) - 3e^{-25\pi^2 t}\cos(5\pi x) + 4e^{-49\pi^2 t}\cos(7\pi x)$

b) $\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$, $x \in (0,1)$, $t > 0$

$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0$, $t > 0$

$u(x,0) = x$

$C_n = 2\int_0^1 u(x,0)\cos(n\pi x) = 2\int_0^1 x\cos(n\pi x)\,dx =$

18) $$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \sin(\pi x), \quad x \in (0,1), \; t > 0.$$

$$u(0,t) = u(1,t) = 0, \quad t > 0$$

$$u(x,0) = \sin(4\pi x)$$

$$u_n(x,t) = X_n(x)T_n(t) \Rightarrow u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x) \quad \text{po S-L}$$

$$f(x,t) = f_1(t)X_1(x) + f_2(t)X_2(x) + \ldots + f_n(t)X_n(x) = \sum_{n=0}^{\infty} f_n(t)X_n(x) = \sum_{n=0}^{\infty} f_n(t)\sin(n\pi x)$$

$$X(x)T'(t) = X''(x)T(t) + f(t)X(x)$$

$$\sum_{n=1}^{\infty} T_n'(t)\sin(n\pi x) = -\sum_{n=1}^{\infty}(n\pi)^2 T_n(t)\sin(n\pi x) + \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x)$$

$$\sum_{n=1}^{\infty}\left[T_n'(t) + (n\pi)^2 T_n(t) - f_n(t)\right]\sin(n\pi x) = 0$$

$$T_n'(t) + (n\pi)^2 T_n(t) = f_n(t)$$

$$u(x,0) = \sum_{n=1}^{\infty} T_n(0)\sin(n\pi x)$$

$n=1$ $\quad T_1'(t) + \pi^2 T_1(t) = 1, \quad T_1(0) = 0$

$$\frac{dT_1(t)}{dt} = 1 - \pi^2 T_1(t) \Rightarrow \frac{dT_1(t)}{1-\pi^2 T_1(t)} = dt \quad \left[\begin{array}{l} z(t) = 1-\pi^2 T_1(t) \\ dz(t) = -\pi^2\, dT_1(t) \end{array}\right] \Rightarrow$$

$$\frac{dz(t)}{z(t)} = -\pi^2 dt \;\Big/ \int \Rightarrow \ln z(t) = -\pi^2 t + \ln C_1 \Rightarrow$$

$$\ln(1-\pi^2 T_1(t)) = -\pi^2 t + \ln C_1 \Rightarrow 1 - \pi^2 T_1(t) = C_1 e^{-\pi^2 t} \Rightarrow$$

$$T_1(t) = \frac{1}{\pi^2}(1 - C_1 e^{-\pi^2 t}), \quad T_1(0) = \frac{1}{\pi^2}(1 - C_1) = 0 \Rightarrow C_1 = 1.$$

$$T_1(t) = \frac{1}{\pi^2}(1 - e^{-\pi^2 t})$$

$n=2$ $\quad T_2'(t) + 4\pi^2 T_2(t) = 0, \quad T_2(0) = 0 \quad T_2(t) = C_2 e^{-4\pi^2} = 0, \; C_2 = 0,$

$T_2(t) = 0$ itd. $\quad n = 3, 5, \ldots$

$n=4$ $\quad T_4'(t) + 16\pi^2 T_4(t) = 0, \quad T_4(0) = 1$

$$\frac{dT_4(t)}{dt} = -16\pi^2 T_4(t) \Rightarrow \frac{dT_4(t)}{T_4(t)} = -16\pi^2 dt \int \Rightarrow \ln T_4(t) = -16\pi^2 t + \ln C_4$$

$$\Rightarrow T_4(t) = C_4 e^{-16\pi^2 t}, \quad T_4(0) = C_4 = 1 \Rightarrow T_4(t) = e^{-16\pi^2 t}$$

$$u(x,t) = \frac{1}{\pi^2}\left[1 - e^{-\pi^2 t}\right]\sin(\pi x) + e^{-16\pi^2 t}\sin(4\pi x)$$

19) $$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \sin(2\pi x), \quad x \in (0,1), \quad t > 0$$

$$u(0,t) = u(1,t) = 0, \quad t > 0$$

$$u(x,0) = \sin(\pi x)$$

$n = 1$ $\quad T_1'(t) + \pi^2 T_1(t) = 0, \quad T_1(0) = 1$

$$\frac{dT_1(t)}{dt} = -\pi^2 T_1(t) \Rightarrow \frac{dT_1(t)}{T_1(t)} = -\pi^2 dt \,\Big|\int \Rightarrow \ln T_1(t) = -\pi^2 t + \ln$$

$$\Rightarrow T_1(t) = C_1 e^{-\pi^2 t}, \quad T_1(0) = C_1 = 1 \Rightarrow T_1(t) = e^{-\pi^2 t}$$

$n = 2$ $\quad T_2'(t) + 4\pi^2 T_2(t) = 1, \quad T_2(t) = 0$

$$\frac{dT_2(t)}{dt} = 1 - 4\pi^2 T_2(t) \Rightarrow \frac{dT_2(t)}{1 - 4\pi^2 T_2(t)} = dt \quad \left[\begin{array}{l} z(t) = 1 - 4\pi^2 T_2(t) \\ dz(t) = -4\pi^2 dT_2(t) \end{array}\right]$$

$$\frac{dz(t)}{z(t)} = -4\pi^2 dt \,\Big|\int \Rightarrow \ln z(t) = -4\pi^2 t + \ln C_2 \Rightarrow$$

$$\ln(1 - 4\pi^2 T_2(t)) = -4\pi^2 t + \ln C_2 \Rightarrow 1 - 4\pi^2 T_2(t) = C_2 e^{-4\pi^2 t} \Rightarrow$$

$$T_2(t) = \frac{1}{4\pi^2}\left(1 - C_2 e^{-4\pi^2 t}\right), \quad T_2(0) = \frac{1}{4\pi^2}(1 - C_2) = 0 \Rightarrow C_2 = 1$$

$$T_2(t) = \frac{1}{4\pi^2}\left(1 - e^{-4\pi^2 t}\right)$$

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x) + \frac{1}{4\pi^2}\left[1 - e^{-4\pi^2 t}\right] \sin(2\pi x)$$

20) a) $$\frac{\partial^2 u(x,y)}{\partial x^2} - 2\frac{\partial^2 u(x,y)}{\partial x \partial y} + \frac{\partial^2 u(x,y)}{\partial y^2} + u(x,y) = 0$$

$$a = 1, \quad b = -2, \quad c = 1, \quad f = 1 \Rightarrow \frac{dx}{dy} = \frac{b}{2a} = \frac{-2}{2 \cdot 1} = -1$$

$$y = -x + C \Rightarrow \xi = y + x$$

$$\eta = y$$

$$C = a\left(\frac{\partial \eta}{\partial x}\right)^2 + b\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + c\left(\frac{\partial \eta}{\partial y}\right)^2 = 1 \cdot 0^2 - 2 \cdot 0 \cdot 1 + 1 \cdot 1^2 = 1, \quad F = f$$

$$\frac{\partial^2 u(x,y)}{\partial \eta^2} + u(x,y) = 0$$

b) $$\frac{\partial^2 u(x,y)}{\partial x^2} - 3\frac{\partial^2 u(x,y)}{\partial x \partial y} + \frac{9}{4}\frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

$$a = 1, \quad b = -3, \quad c = \frac{9}{4} \Rightarrow \frac{dx}{dy} = \frac{b}{2a} = \frac{-3}{2 \cdot 1} = \frac{3}{2}$$

$$y = \frac{3}{2}x + C \Rightarrow \xi = \frac{3}{2}x - y$$

$$\eta = y$$

$$C = 1 \qquad \frac{\partial^2 u(x,y)}{\partial \eta^2} = 0$$

21) $\dfrac{\partial^2 u(x,t)}{\partial t^2} = \dfrac{\partial^2 u(x,t)}{\partial x^2}$, $x \in (0,1)$, $t > 0$

$u(0,t) = u(1,t) = 0$, $t > 0$

$u(x,0) = 2\sin(4\pi x) + 3\sin(5\pi x)$

$\dfrac{\partial u}{\partial t}(x,0) = 2\pi \sin(\pi x) + 6\pi \sin(2\pi x)$

$u(x,t) = X(x)\,T(t)$

$X(x)\,T''(t) = X''(x)\,T(t)$

$\dfrac{T''(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = -\lambda$

$X''(x) - \lambda X(x) = 0$ $\quad u(0,t) = X(0)\,T(t) = 0$ $\quad$ $X(0) = 0$

$T''(t) - \lambda T(t) = 0$ $\quad u(1,t) = X(1)\,T(t) = 0$ $\quad T(t) \neq 0$ $\quad X(1) = 0$ $\quad$ S-L

$\lambda = -\beta^2$ $\quad X(x) = A\sin(\beta x) + B\cos(\beta x)$, $T(t) = C\sin(\beta t) + D\cos(\beta t)$

$u(x,t) = [A\sin(\beta x) + B\cos(\beta x)][C\sin(\beta t) + D\cos(\beta t)]$

$u(0,t) = X(0)\,T(t) = B[C\sin(\beta t) + D\cos(\beta t)] = 0 \Rightarrow B = 0$

$u(1,t) = X(1)\,T(t) = A\sin\beta\,[C\sin(\beta t) + D\cos(\beta t)] = 0 \Rightarrow A\sin\beta = 0$

$\beta = n\pi$, $n = 1,2,\dots$ $\quad u_n(x,t) = X_n(x)\,T_n(t) = \sin(n\pi x)[a_n\sin(n\pi t) + b_n\cos(n\pi t)]$

$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)[a_n \sin(n\pi t) + b_n\cos(n\pi t)]$

$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$ $\quad n = 4 \Rightarrow b_4 = 2$, $n = 5 \Rightarrow b_5 = 3$

$\dfrac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)[a_n n\pi\cos(n\pi t) - b_n n\pi \sin(n\pi t)]$

$\dfrac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} a_n n\pi \sin(n\pi x)$ $\quad n = 1 \Rightarrow a_1 = 2$, $n = 2 \Rightarrow a_2 = 3$

$u(x,t) = 2\sin(\pi t)\sin(\pi x) + 3\sin(2\pi t)\sin(2\pi x) + 2\cos(4\pi t)\sin(4\pi x) +$
$+ 3\cos(5\pi t)\sin(5\pi x)$

22) $\dfrac{\partial^2 u(x,t)}{\partial t^2} = \dfrac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in (0,1), \quad t > 0$

$u(0,t) = u(1,t) = 0, \quad t > 0$

$u(x,0) = \sin(5\pi x) + 2\sin(6\pi x) + 3\sin(8\pi x)$

$\dfrac{\partial u}{\partial t}(x,0) = \pi \sin(\pi x) + 5\pi \sin(2\pi x) + 7\pi \sin(3\pi x)$

$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left[a_n \sin(n\pi t) + b_n \cos(n\pi t)\right]$

$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \qquad n=5 \Rightarrow b_5 = 1, \quad n=6 \Rightarrow b_6 = 2, \quad n=8 \Rightarrow b_8 = 3$

$\dfrac{\partial u}{\partial t}(x,t) = \sum_{n=0}^{\infty} \sin(n\pi x)\left[a_n n\pi \cos(n\pi t) - b_n n\pi \sin(n\pi t)\right]$

$\dfrac{\partial u}{\partial t}(x,t) = \sum_{n=0}^{\infty} a_n n\pi \sin(n\pi x) \qquad n=1 \Rightarrow a_1 = 1, \quad n=2 \Rightarrow a_2 = \frac{5}{2}, \quad n=3 \Rightarrow a_3 = \frac{7}{2}$

$u(x,t) = \sin(\pi t)\sin(\pi x) + \frac{5}{2}\sin(2\pi t)\sin(2\pi x) + \frac{7}{2}\sin(3\pi t)\sin(3\pi x) +$

$+ \cos(5\pi t)\sin(5\pi x) + 2\cos(6\pi t)\sin(6\pi x) + 3\cos(8\pi t)\sin(8\pi x)$

23) $\dfrac{\partial^2 u(x,t)}{\partial t^2} = c^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} \quad t > 0$

$u(x,0) = 0$

$\dfrac{\partial u}{\partial t}(x,0) = \cos(2x)$

$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$ D'ALEMBERTOVA FORMULA

$f(x) = 0, \quad g(x) = \cos(2x)$

$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \cos(2s)\,ds = \frac{1}{4c}\sin(2s)\Big|_{x-ct}^{x+ct} = \frac{1}{4c}\left[\sin[2(x+ct)] - \sin[2(x-ct)]\right]$

24) $\dfrac{\partial^2 u(x,t)}{\partial t^2} = c^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} \quad t > 0$

$u(x,0) = 0$

$\dfrac{\partial u}{\partial t}(x,0) = \sin(3x)$

$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$

$f(x) = 0, \quad g(x) = \sin(3x)$

$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(3s)\,ds = \frac{1}{6c}\left[-\cos(3s)\right]\Big|_{x-ct}^{x+ct} =$

$= \frac{1}{6c}\left[\cos[3(x-ct)] - \cos[3(x+ct)]\right]$

25) $$\frac{\partial^2 u(x,y)}{\partial x^2} + 3\frac{\partial^2 u(x,y)}{\partial x \partial y} - 4\frac{\partial^2 u(x,y)}{\partial y^2} + 2\frac{\partial u(x,y)}{\partial y} - u(x,y) = 5$$

26) $$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \qquad x \in (0,1),\ y \in (0,2)$$

$$\frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial x}(1,y) = 0 \qquad y \in (0,2)$$

$$u(x,2) = 0, \quad u(x,0) = 2\cos(\pi x) + 3\cos(2\pi x) \quad x \in (0,1)$$

$$u(x,y) = X(x)Y(y)$$

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = k$$

$$X''(x) - kX(x) = 0 \qquad \frac{\partial u}{\partial x}(0,y) = X'(0)Y(y) = 0 \qquad X'(0) = 0$$

$$Y''(y) + kY(y) = 0 \qquad \frac{\partial u}{\partial x}(1,y) = X'(1)Y(y) = 0 \qquad Y(y) \neq 0 \qquad X'(1) = 0$$

$$k_n = -(n\pi)^2 \qquad X_n(x) = a_n\cos(n\pi x), \quad n = 0,1,2,\dots$$

$$Y_0(y) = A_0 + B_0 y$$

$$u(x,2) = Y(2) = 0$$

$$Y_n(y) = A_n\,ch(n\pi y) + B_n\,sh(n\pi y)$$

$$Y_0(2) = A_0 + 2B_0 = 0 \quad \Rightarrow \quad A_0 = -2B_0$$

$$Y_n(2) = A_n\,ch(2n\pi) + B_n\,sh(2n\pi) \quad \Rightarrow \quad A_n = -B_n\,th(2n\pi)$$

$$Y_n(y) = -B_n\,th(2n\pi)\,ch(n\pi y) + B_n\,sh(n\pi y)$$

$$Y_n(y) = \frac{B_n}{ch(2n\pi)}\left[-sh(2n\pi)\,ch(n\pi y) + sh(n\pi y)\,ch(2n\pi)\right)$$

$$Y_n(y) = C_n\,sh(n\pi(y-2))$$

$$u_0(x,y) = X_0(x)Y_0(y) = a_0 B_0 (y-2) = E_0(y-2)$$

$$u_n(x,y) = X_n(x)Y_n(y) = a_n\cos(n\pi x)\,C_n\,sh(n\pi(y-2))$$

$$= E_n\cos(n\pi x)\,sh(n\pi(y-2))$$

$$u(x,y) = E_0(y-2) + \sum_{n=1}^{\infty} E_n \cos(n\pi x)\, sh(n\pi(y-\ell))$$

$$u(x,0) = -2 E_0 + \sum_{n=1}^{\infty} E_n \cos(n\pi x)\, sh(-2n\pi)$$

$$n=1 \Rightarrow E_n = 2 \quad , \quad n=2 \, , \quad E_n = 3$$

$$u(x,t) = 2\cos(\pi x)\, sh(\pi(y-2)) + 3\cos(2\pi x)\, sh(2\pi(y-2))$$

27) $$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \, , \quad x \in (0,1), \quad y \in (0,2)$$

$$\frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial x}(1,y) = 0 \, , \quad y \in (0,2)$$

$$u(x,2) = 0, \quad u(x,0) = x$$

$$u(x,y) = E_0(y-2) + \sum_{n=1}^{\infty} E_n \cos(n\pi x)\, sh(y-2)$$

$$E_0 = -\frac{1}{1\cdot 2} \int_0^1 x\, dx = -\frac{1}{2} \frac{x^2}{2}\Big|_0^1 = -\frac{1}{4}$$

$$E_n = \frac{2}{1 \cdot sh(-2n\pi)} \int_0^1 x\cos(n\pi x)\, dx =$$

28) $$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \qquad (x,y) \in K$$

$$u(x,y) = x^2 + y^2 \qquad (x,y) \in \partial K \quad , \ r = 2$$

$$\frac{2}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} = 0 \qquad r \in (0,2)$$

$$u(2,\phi) = r^2$$

$$u(r,\phi) = R(r)\, T(\phi)$$

$$\frac{2}{r} R'(r) T(\phi) + R''(r) T(\phi) + \frac{1}{r^2} R(r) T''(\phi) = 0$$

$$T(\phi)\left[\frac{2R'(r) + rR''(r)}{\cancel{r}}\right] = -\frac{1}{r^{\cancel{2}}} R(r)\, T''(\phi)$$

$$\frac{r^2 R''(r) + 2r R'(r)}{R(r)} = -\frac{T''(\phi)}{T(\phi)} = \lambda \qquad ?$$

$$r^2 R''(r) + r R'(r) - \lambda R(r) = 0$$

$$T''(\phi) + \lambda T(\phi) = 0$$

$\lambda_n = n^2 \quad T_n(\phi) = A_n \cos(n\phi) + B_n \sin(n\phi)$

$r^2 R''(r) + r R'(r) - n^2 R(r) = 0 \qquad . \; R_n(r) = C_n r^n .$

$u_0(r,\phi) = R_0(r) T_0(\phi) = \frac{a_0}{2}$

$U_n(r,\phi) = R_n(r) T_n(\phi) = C_n r^n (A_n \cos(n\phi) + B_n \sin(n\phi))$

$u(r,\phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (a_n \cos(n\phi) + b_n \sin(n\phi))$

$u(2,\phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{2}\right)^n (a_n \cos(n\phi) + b_n \sin(n\phi))$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} 4 \cos(n\phi)\, d\phi = \frac{2 \cdot 4}{\pi} \; \frac{2 \sin(n\pi)}{n} = 0$

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} 4\, d\phi = \frac{2 \cdot 4}{\cancel{\pi}} \cancel{\pi}$

$= 8 \Rightarrow \frac{a_0}{2} = 4$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} r^2 \sin(n\phi)\, d\phi = 0$

$u(r,\phi) = 4$

29)

30)