1.6.2010.

1. MASS Diferencijalne jednadžbe

1. jednadžbe sa separiranim varijablama

$$f(y)\,dy = g(x)\,dx$$

2. homogene diferencijalne jednadžbe

$$y' = f\left(\frac{y}{x}\right) \qquad z = \frac{y}{x}$$

provjera da li je fja. homog. – vrijedi: $M(tx, ty) = t^2 M(x, y)$

3. jednadžbe koje se svode na homogene

$$y' = f\left(\frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}\right)$$

$$\left.\begin{aligned} a_1 x + b_1 y + c_1 &= 0 \\ a_2 x + b_2 y + c_2 &= 0 \end{aligned}\right\} \overset{1}{S(x_0, y_0)} \qquad \Big\}^{2} \text{ paralelni} \quad z = a_1 x + b_1 y$$

$$u = x - x_0$$
$$v = y - y_0$$
$$\frac{dy}{dx} = \frac{dv}{du}$$

4. lin. dif. jedn. 1. reda

$$y' + p(x) \cdot y = q(x)$$

1. hom. jed. $\rightarrow$ $y' + p(x) \cdot y = 0$

2. varijacija konst. $C = C(x)$

5. Bernoullijeva

$$y' + p(x) y = q(x) \cdot y^n$$
$$z = y^{1-n}$$

6. Riccatijeva

$$y' + p(x) y + q(x) y^2 = f(x)$$
$$y = y_1 + z$$

7. Egzaktne

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

uvjet: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ $\qquad$ rj: $u(x,y) = C$ – rješavamo na 2 načina

$$u(x,y) = \int_{x_0}^{x} P(x,y)\,dx + \int_{y_0}^{y} Q(x_0,y)\,dy$$

$$u(x,y) = \int_{x_0}^{x} P(x,y_0)\,dy + \int_{y_0}^{y} Q(x,y)\,dx$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

Eulerov multipl.

$$\ln(\mu(x)) = \int \frac{1}{Q}\left(P_y' - Q_x'\right)dx$$

$$\ln(\mu(y)) = \int \frac{1}{P}\left(P_y' - Q_x'\right)dy$$

8. Dif. jedn. familije krivulja

$$y = P(x, c_1, \dots, c_n)$$

$$\Phi(x, y, c_1, \dots c_n) = 0$$

uzastopno deriv. dok se ne riješimo konstanti

9. Polje smjerova

$$y' = f(x,y) \qquad f(x,y) = c$$

10. Ortogonalne i izogonalne trajektorije

(1) $\Phi(x,y,c) = 0$ – zadano

(2) sječe (1) pod nekim kutem α

$\alpha = \frac{\pi}{2} \Rightarrow$ ortogon. traj. $\quad y_1 = -\frac{1}{y_2'}$

$\alpha \neq \frac{\pi}{2} \Rightarrow$ izogonalne traj. $\quad tg\,\alpha = \frac{y_1' - y_2'}{1 + y_1' y_2'}$ $\Big\}$ (*)

a) iz (1) $\rightarrow$ $F(x,y,y') = 0$

b) iskoristite (*) $\Rightarrow$ $y_1' = \varphi(y_2')$

c) $F(x, y, \varphi(y_2')) = 0$

1. Nađi dif. jedn. familije krivulja.

$$y = e^{cx} \;/\ln$$ — deriv. dok ne izvučemo konstantu

$$\ln y = cx \;/'$$

→ jer y ovisi o x

$$\frac{1}{y} \cdot y' = c$$

$$\boxed{\ln y = \frac{1}{y} \cdot y' \cdot x}$$

2. isti text

$$y^2 + cx = x^2 \;/'$$

$$2y \cdot y' + c = 2x \rightarrow c = 2x - 2yy'$$

$$\boxed{y^2 + (2x - 2yy') \cdot x = x^2}$$

3.

$$y = c_1 e^{2x} + c_2 e^{-x} \;/'$$

$$y' = 2c_1 e^{2x} - c_2 e^{-x} \;/'$$

$$y'' = 4c_1 e^{2x} + c_2 e^{-x}$$

$$y' + y'' = 6c_1 e^{2x}$$

$$c_1 = \frac{y' + y''}{6e^{2x}}$$

1 2 2

$$y' = 2 \cdot \frac{y' + y''}{\cancel{6}_3 \cancel{e^{2x}}} \cdot \cancel{e^{2x}} - c_2 e^{-x}$$

$$y' = \frac{y' + y''}{3} - c_2 e^{-x} \Rightarrow \underline{c_2} = \left(\frac{y' + y''}{3} - y'\right) e^{x}$$

$$y = c_1 e^{2x} + c_2 e^{-x} = \frac{y' + y''}{6e^{2x}} \cdot e^{2x} + \left(\frac{y' + y''}{3} - y'\right) e^{x} \cdot e^{-x}$$

Uj. $\Rightarrow y = e^{2x}(c_1 \sin 3x + c_2 \cos 3x)$

4. Rj. sl. dif. jed.

$$y' = (y-1)x$$

$$\frac{dy}{dx} = (y-1)x$$

$$\frac{dy}{y-1} = x\,dx \quad / \int$$

$$\ln|y-1| = \frac{x^2}{2} + c \quad / e$$

$$y-1 = e^{\frac{x^2}{2}} \cdot c \qquad \boxed{y = e^{\frac{x^2}{2}} \cdot c + 1}$$

5. $2y' + \frac{y}{x} = 0$

$$2\frac{dy}{dx} = -\frac{y}{x}$$

$$\frac{dy}{y} = -\frac{1}{2}\frac{dx}{x} \quad / \int$$

$$\ln y = -\frac{1}{2}\ln x + c \quad = -\frac{1}{2}\ln|c|$$

$$\ln y = -\frac{1}{2}\ln|cx| = \ln|cx^{-\frac{1}{2}}| \quad \Rightarrow \quad c^{-\frac{1}{2}} \Rightarrow c$$

$$\boxed{y = \frac{c}{\sqrt{x}}}$$

Vj.: $2x \cdot \sqrt{1-y^2} = y' \cdot (1+x^2)$

6. Provjeri je li homogena:

$$M(x,y) = \frac{x}{2} + y + \frac{2xy}{x-y} \qquad M(tx,ty) = t^2 M(x,y)$$

$$M(tx,ty) = \frac{tx}{2} + ty + \frac{2tx\cdot ty}{tx-ty} =$$

$$= t^1\left(\frac{x}{2} + y + \frac{2xy}{x-y}\right) \Rightarrow \text{homogena } 1 \text{ stupnja}$$

7. $\frac{dy}{dx} = \frac{y^2-x^2}{x^2} = \frac{y^2}{x^2} - 1 \Rightarrow$ homogena

$z = \frac{y}{x} \cdot x$ $\rightarrow$ ovisi o x

$y = z \cdot x$

$y' = z + z'x$

$z'x + z = z^2 + 1$

$z'x = z^2 - z + 1$

$\frac{dz}{dx} \cdot x = z^2 - z - 1$

$\frac{dz}{z^2 - z - 1} = \frac{dx}{x}$

$\frac{dz}{(z-\frac{1}{2})^2 - \frac{5}{4}} = \frac{dx}{x} \quad / \int$

$\int \frac{dz}{(z-\frac{1}{2})^2 - (\frac{\sqrt{5}}{2})^2} = \ln x + \ln c = \ln|cx|$

supst.
$\left| z - \frac{1}{2} = t \qquad dz = dt \right|$

$\int \frac{dt}{t^2 - (\frac{\sqrt{5}}{2})^2} = \ln cx$

$\frac{1}{2 \cdot \frac{\sqrt{5}}{2}} \ln \left| \frac{z - \frac{\sqrt{5}}{2}}{z + \frac{\sqrt{5}}{2}} \right| = \ln cx$

8. $(x+y)\frac{dy}{dx} = 4x - 3y \quad /:x$ $\qquad \frac{y}{x} = z$

$(1+\frac{y}{x})\frac{dy}{dx} = 4 - 3\frac{y}{x}$ $\qquad y' = z'x + z$

$z'x + z = \frac{4-3z}{1+z}$

$z'x = \frac{4 - 3z - z - z^2}{1+z}$

$\frac{dz}{dx} x = \frac{-z^2 - 4z + 4}{1+z}$

$-\frac{z+1}{z^2+4z-4} dz = \frac{dx}{x} \quad / \int$

$$z^2+4z-4=0$$

$$z_{1,2}=-2\pm\sqrt{4+4}=-2\pm 2\sqrt{2}$$

$$-\frac{(z+1)\,dz}{(z-(-2-2\sqrt{2}))\,(z-(-2+2\sqrt{2}))}=\ln Cx$$

$$-\frac{z+1}{(z+2+2\sqrt{2})\,(z+2-2\sqrt{2})}=\frac{A}{z+2+2\sqrt{2}}+\frac{B}{z+2-2\sqrt{2}}$$

$$z+1=Az+(2-\sqrt{2})A+Bz+B(2+2\sqrt{2})$$

$$1=A+B$$

$$1=(2-\sqrt{2})A+(2+2\sqrt{2})B$$

→ naći, integrirati i to je rj.

$$y': \quad x\cdot\frac{dy}{dx}=3y+\frac{y^2}{x}$$

9. Jedn. koje se svode na hom

$$\frac{dy}{dx}=\frac{x-3y+1}{x+y}$$

$$x-3y+1=0$$
$$x+y=0 \quad /\cdot(-1)$$

$$-4y=-1$$
$$y_0=\frac{1}{4} \qquad x_0=-\frac{1}{4}$$

$$u=x-x_0=x+\frac{1}{4}$$
$$v=y-y_0=y-\frac{1}{4}$$

$$\boxed{\begin{aligned}x&=u-\tfrac{1}{4}\\ y&=v+\tfrac{1}{4}\end{aligned}}$$

$$\frac{dy}{dx}=\frac{dv}{du}=\frac{u-\frac{1}{4}-3(v+\frac{1}{4})+1}{u-\frac{1}{4}+v+\frac{1}{4}}=\frac{u-3v}{u+v}$$

$$\frac{dv}{du}=\frac{1-3\frac{v}{u}}{1+\frac{v}{u}} \qquad z=\frac{v}{u}$$

$$z'u+z=v'$$

$$z'u+z=\frac{1-3z}{1+z}$$

$$\frac{dz}{du}\cdot u=\frac{1-3z-z-z^2}{1+z}$$

$$\frac{dz}{du}\cdot u=\frac{1-4z-z^2}{z+1}$$

$$\frac{z+1}{1-4z-z^2}\,dz=\frac{du}{u} \quad \Big/ \int$$

$z^2+4z-1=0$

$z_{1,2}=-2\pm\sqrt{5}$

rastaviti preko parc. razlom. i onda integriraš i onda još vrati supst.

10. linear. jedn. 1. reda

$y'+y=e^x$

$y'+p(x)\cdot y=g(x)$

1. korak $\quad y'+y=0$

$\frac{dy}{dx}=-y$

$\frac{dy}{y}=-dx \quad /\int$

$\ln y=-x+c$

$y=e^{-x}\cdot c$

2. korak $\quad c=c(x)$

$y=c(x)\cdot e^{-x}$ vrati se u poč. jedn.

$(c(x)\cdot e^{-x})'+c(x)\cdot e^{-x}=e^x$

$c'(x)\cdot e^{-x}-\cancel{c(x)\cdot e^{-x}}+\cancel{c(x)\cdot e^{-x}}=e^x$

$c'(x)e^{-x}=e^x \,/e^x$

$c'(x)=e^{2x} \quad /\int$

$c(x)=\int e^{2x}=\frac{1}{2}e^{2x}+c$

$y=e^{-x}\cdot\left(\frac{1}{2}e^{2x}+c\right)$

V/. 1) $x\cdot y'-2y=x^2\cdot\ln x$

2) $y'+tg x\cdot y=2\cos 2x$

11. Bernoullijeva jed.

$$xy' + 3y - x^4y^2 = 0 \quad / :x$$

$$y' + \frac{3}{x}\cdot y - x^3y^2 = 0$$

$$y' + \underset{p(x)}{\frac{3}{x}}\, y = \underset{q(x)}{x^3}\, y^{2} \quad n \quad / :y^2 \qquad \text{zamj.} \quad z = y^{1-n} = y^{1-2} = y^{-1}$$

$$z = y^{-1} \; /'$$

$$z' = -y^{-1-1}\cdot y' = -y^{-2}\cdot y'$$

$$y' = -z'\cdot y^2$$

$$\frac{y'}{y^2} + \frac{3}{x}\cdot\frac{1}{y} = x^3$$

$$-z' + \frac{3}{x}\cdot z = x^3 \quad \Rightarrow \underline{\text{lin. dif. jed. 1. reda}} \; !!$$

1. korak $\quad -z' + \frac{3}{x}z = 0$

$$-\frac{dz}{dx} = -\frac{3}{x}\, z$$

$$\frac{dz}{z} = 3\,\frac{dx}{x} \; / \int \qquad (1)$$

$$\ln z = 3\ln Cx$$

$$z = Cx^3$$

2. korak $\quad z = C(x)\,x^3$

$$-(C(x)\,x^3)' + \frac{3}{x}\cdot C(x)\,x^3 = x^3$$

$$-C'(x)\cdot x^3 - \cancel{C(x)\cdot 3x^2} + \cancel{3\,C(x)\cdot x^2} = x^3 \qquad (2)$$

$$-C'(x) = 1$$

$$C(x) = -x + C$$

$$z = (-x + C)\,x^3$$

$$\frac{1}{y} = (C - x)\,x^3$$

$$\boxed{y = \frac{1}{(C-x)\,x^3}}$$

12. Egzaktne

$$\underbrace{(3y^2-12x)}_{P}dx+\underbrace{6xy}_{Q}dy=0$$

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

$$6y=6y \quad \Rightarrow \text{ jedn. je egzaktna}$$

$$u(x,y)=\int_{x_0=0}^{x}P(x,y)\,dx+\int_{y_0=0}^{y}Q(x_0,y)\,dy$$

$$u(x,y)=\int_0^x(3y^2-12x)\,dx+\cancel{\int_0^y 6x_0 y\,dy}=0$$

$$=3y^2\cdot x\Big|_0^x-12\cdot\frac{x^2}{2}\Big|_0^x=\boxed{3y^2x-6x^2=C}$$

13. $$\big((y-1)\cos x-\sin x\big)\underset{\frac{dy}{dx}}{y'}+y\sin x+\cos x=0$$

$$\big((y-1)\cos x-\sin x\big)\frac{dy}{dx}=-(y\sin x+\cos x)$$

$$\big((y-1)\cos x-\sin x\big)\,dy=-(y\sin x+\cos x)\cdot dx$$

$$\underbrace{(y\sin x+\cos x)}_{P}dx+\underbrace{\big((y-1)\cos x-\sin x\big)}_{Q}dy=0$$

$$\left.\begin{array}{l}P'_y=\sin x\\ Q'_x=(1-y)\sin x-\cos x\end{array}\right\}\quad P'_y-Q'_x=\sin x-\sin x+y\sin x+\cos x==y\sin x+\cos x$$

$$F(y)=1 \qquad \frac{1}{P}\cdot(P'_y-Q'_x)=1$$

$$u=\mu(y)$$

$$\ln\mu(y)=-\int\frac{1}{P}(P'_y-Q'_x)\,dy=-\int dy=-y$$

$$\boxed{\mu(y)=e^{-y}} \Rightarrow \text{množimo poč. jedn.}$$

$e^{-y}(y\sin x+\cos x)\,dx + e^{-y}((y-1)\cos x - \sin x)\,dy = 0 \quad /\cdot e^{y}$

dobijemo egzaktnu

$$u(x,y) = \int_0^x \underbrace{e^{-y_0}}_{1}(\underbrace{y_0 \sin x}_{0} + \cos x)\,dx + \int_0^y e^{-y}((y-1)\cos x - \sin x)\,dy$$

$$u(x,y) = \sin x\Big|_0^x + \cos x\int_0^y e^{-y}(y-1)\,dy - \sin x\int_0^y e^{-y}\,dy$$

$$= \sin x + \cos\int_0^y y e^{-y}\,dy - \cos x\int_0^y e^{-y}\,dy + \sin x\, e^{-y}\Big|_0^y$$

parcijalno derivirati

NJ: $(x^2+y-1)\,dx + (3y - \frac{x}{y})\,dy = 0$

14. Polje smjerova

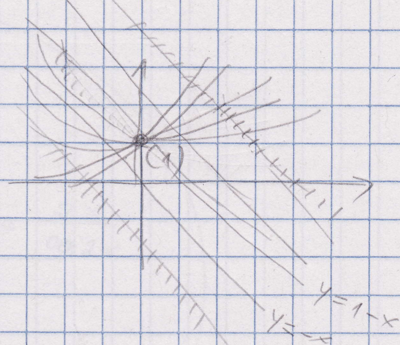
$y' = \underbrace{x+y}_{\text{uvijek } c}$

$x+y=c$

$y=c-x$

$c=0 \quad y=-x$

$c=1 \quad y=1-x$



$y'=x+y$

$\underbrace{y'-y}_{P(x)} = \underbrace{x}_{Q(x)} \Rightarrow$ lin. dif.

1. korak $\quad y'-y=0$

$\frac{dy}{dx}=y$

$\frac{dy}{y}=dx$

$\ln y = C+x \rightarrow y = C\cdot e^x$

2. korak $\quad y=C(x)\cdot e^x$

$(C(x)e^x)' - C(x)e^x = x$

$C'(x)e^x + \cancel{C(x)e^x} - \cancel{C(x)e^x} = x$

$C'(x) = x\cdot e^{-x} \quad /\int$

$C(x) = \int(x\cdot e^{-x})\,dx = \left| \begin{array}{l} u=x \\ du=dx \\ dv=e^{-x}dx/\int \\ v=-e^{-x} \end{array}\right| = u\cdot v - \int v\,du =$

$= -xe^{-x} + \int e^{-x}dx + C = -xe^{-x} - e^{-x} + C$

$y = (-xe^{-x} - e^{-x} + C)\,e^x \qquad \boxed{y = -x-1+Ce^x}$

15. Ortogonal. traj.

Odredi krivulju koja prolazi točkom $(1,0)$ i siječe svaku krivulju ove familije pod pravim kutom $\quad \alpha = \frac{\pi}{2}$ – ortog.

$y = \text{arc}\,tg\,(cx)$

$$y_1' = -\frac{1}{y_2'}$$

a) $$y = \text{arc}\,tg\,(cx) \,/'$$

$$y' = \frac{1}{1+(cx)^2} \cdot c$$

$$/\,tg$$

$$tg\,y = cx \,/'$$

$$\frac{1}{\cos^2 y} \cdot y' = c$$

$$y = \text{arc}\,tg\left(\frac{1}{\cos^2 y} \cdot y' \cdot x\right) \,/\,tg$$

$$tg\,y = \frac{1}{\cos^2 y} \cdot x \cdot y' \,/\cdot \cos^2 y$$

$$\sin y \cos y = x\,y_1'$$

$$y_1' = \frac{\sin y \cos y}{x}$$

$$y_1' = -\frac{1}{y_2'}$$

$$\frac{\sin y \cos y}{x} = -\frac{1}{y_2'}$$

$$y' = -\frac{x}{\sin y \cos y}$$

$$\frac{dy}{dx} = -\frac{x}{\sin y \cos y}$$

$$\sin y \cos y \, dy = -x\,dx \,/\int$$

$$\int \sin y \cos y \, dy = -\frac{x^2}{2} + c \qquad \left|\begin{array}{l} \sin y = t \,/' \\ \cos y \, dt = dt \end{array}\right.$$

$$\int t \, dt = -\frac{x^2}{2} + c$$

$$\frac{t^2}{2} = -\frac{x^2}{2} + c$$

$$\boxed{\frac{\sin^2 y}{2} = -\frac{x^2}{2} + c} \quad c = \frac{1}{2}$$

$$\frac{0}{2} = -\frac{0}{2} = c = \frac{1}{2}$$

Vj. U svakoj točki krivulje koeficijent smjera tangente $= \sqrt[3]{}$ iz koeficijenta smjera spojnice te točke s ishodištem.

$$k_1 = \sqrt[3]{k_2}$$

16. Cauchyeva zadaća

$$\underbrace{y' - \operatorname{tg}x}_{P(x)}\, y = \underbrace{\frac{2x}{\cos x}}_{Q(x)} \qquad y(0) = 1$$

1) $y' - \operatorname{tg}x \cdot y = 0$

$$\frac{dy}{dx} = \operatorname{tg}x \cdot y$$

$$\frac{dy}{y} = \operatorname{tg}x\,dx \Big/ \int \qquad \frac{\sin x}{\cos x}$$

$$\ln y = -\ln\cos x + \ln c$$

$$y = \frac{c}{\cos x}$$

2) $y = \dfrac{c(x)}{\cos x}$

$$\left(\frac{c(x)}{\operatorname{tg}x}\right)' - \operatorname{tg}x \cdot \frac{c(x)}{\cos x} = \frac{2x}{\cos x}$$

$$\frac{c'(x)\cos x - \cancel{c(x)\,'\cdot\sin x}}{\cos^2 x} - \cancel{c(x)\cdot\frac{\sin(x)}{\cos^2(x)}} = \frac{2x}{\cos 2x}$$

$$c'(x)\cdot\cos x = \frac{2x}{\cos 2x}$$

$$c'(x) = 2x \Big/ \int$$

$$c(x) = x^2 + c$$

$$y = \frac{x^2 + c}{\cos x} \qquad y(0) = 1$$

$$1 = \frac{c}{1} \qquad \boxed{c = 1}$$

$$\boxed{y = \frac{x^2 + 1}{\cos x}}$$