Fizika 1

Grupa 1.03

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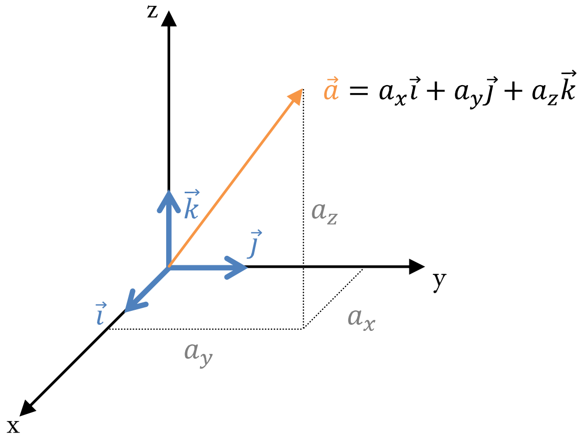
Predavanja 2010.

SADRŽAJ

FIZIKALNE VELIČINE: -**SKALARI** (imaju samo iznos ~ temperatura, tlak, gustoća...)
-**VEKTORI** (imaju iznos i smjer ~ brzina, akceleracija, struja...)
-TENZORI

VEKTORI

PRAVOKUTNI KOORDINATNI SUSTAV (desno orijentirani)



$\vec{\imath}, \vec{\jmath}, \vec{k}$ - jedinični vektori
a_x, a_y, a_z - komponente vektora a

ZBRAJANJE VEKTORA

$$\vec{a} + \vec{b} = (a_x + b_x)\vec{\imath} + (a_y + b_y)\vec{\jmath} + (a_z + b_z)\vec{k}$$

MNOŽENJE VEKTORA SKALAROM

$$\lambda\vec{a} = \lambda a_x\vec{\imath} + \lambda a_y\vec{\jmath} + \lambda a_z\vec{k}$$

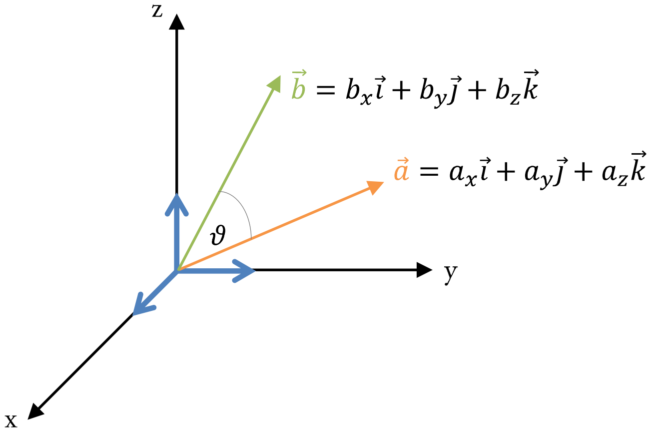
skalar

SKALARNO MNOŽENJE VEKTORA

$$\vec{a} \cdot \vec{b} \equiv \underbrace{a_x b_x + a_y b_y + a_z b_z}_{\text{skalar}}$$

$$\vec{a} \cdot \vec{a} = a_x^2 + a_y^2 + a_z^2$$

$$a = |\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_x^2 + a_y^2 + a_z^2}$$



$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|cos\,\vartheta$

theta, kut među vektorima

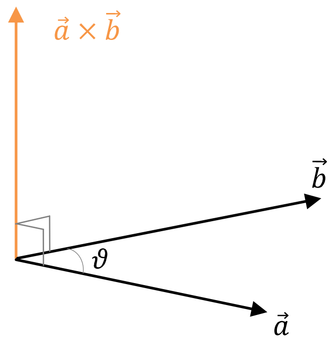
$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + 2|\vec{a}||\vec{b}|cos\,\vartheta + |\vec{b}|^2$

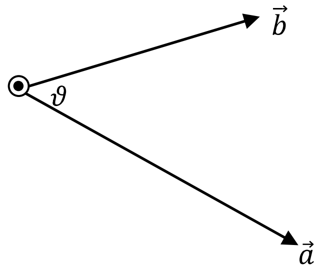
$|\vec{a} + \vec{a}|^2 = 4|\vec{a}|^2 = |\vec{a}|^2 + 2|\vec{a}||\vec{a}|\underbrace{cos\,\vartheta}_{=1} + |\vec{a}|^2$ ■

$|\vec{a} - \vec{a}|^2 = 0 = |\vec{a}|^2 + 2|\vec{a}||\vec{a}|\underbrace{cos\,\vartheta}_{=-1} + |\vec{a}|^2$ ■

VEKTORSKI PRODUKT

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$



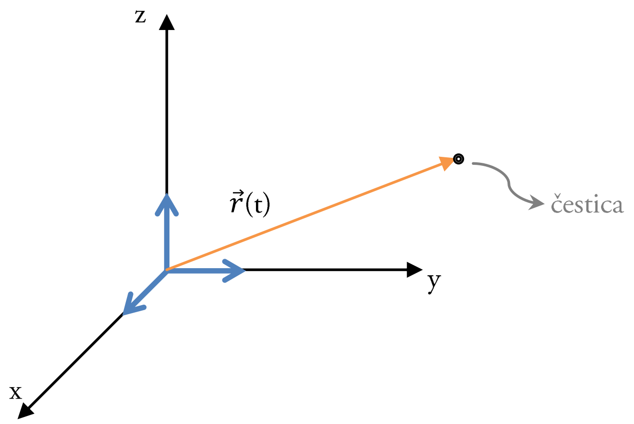


⊗ - „u ploču“
⊙ - prema nama

$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\,sin\,\vartheta$

KINEMATIKA TOČKE



POLOŽAJ:

$$\vec{\boldsymbol{r}}(\boldsymbol{t}) = \boldsymbol{x}(\boldsymbol{t})\vec{\boldsymbol{\imath}} + \boldsymbol{y}(\boldsymbol{t})\vec{\boldsymbol{\jmath}} + \boldsymbol{z}(\boldsymbol{t})\vec{\boldsymbol{k}}$$

BRZINA:

$$\vec{\boldsymbol{v}}(\boldsymbol{t}) = \frac{\boldsymbol{d}}{\boldsymbol{dt}}\vec{\boldsymbol{r}}(\boldsymbol{t})$$

- vremenska derivacija vektora koji opisuje položaj

$$= \frac{d}{dt}x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k} = \frac{dx(t)}{dt}\vec{\imath} + \frac{dy(t)}{dt}\vec{\jmath} + \frac{dz(t)}{dt}\vec{k}$$

$$= v_x(t)\vec{\imath} + v_y(t)\vec{\jmath} + v_z(t)\vec{k}$$

$$\vec{v} \equiv \dot{\vec{r}} \quad \frac{d}{dt}$$

$$v_x\vec{\imath} + v_y\vec{\jmath} + v_z\vec{k} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k}$$

INTEGRALNI ZAPIS:

$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t) \quad /\cdot dt$$

$$d\vec{r}(t) = \vec{v}(t)dt \qquad \int_{t_0}^{t_1}$$

$$\int_{t_0}^{t_1} d\vec{r} = \int_{t_0}^{t_1} \vec{v}(t)\, dt$$

$$\vec{r}(t_1) - \vec{r}(t_0) = \int_{t_0}^{t_1} \vec{v}(t)\, dt$$

$t_1 \to t,\ t \to t'$

$$\vec{\boldsymbol{r}}(\boldsymbol{t}) - \vec{\boldsymbol{r}}(\boldsymbol{t_0}) = \int_{t_0}^{t} \vec{\boldsymbol{v}}(\boldsymbol{t'})\, \boldsymbol{dt'}$$

„početni uvjet“ brzina!

AKCELERACIJA:

$$\boldsymbol{\vec{a}(t) \equiv \frac{d}{dt}\vec{v}(t)}$$

- derivacija brzine po vremenu

$$= \frac{d^2}{dt^2}\vec{v}(t)$$

$$\vec{a}(t) = a_x(t)\vec{\imath} + a_y(t)\vec{\jmath} + a_z(t)\vec{k}$$

$$= \frac{d}{dt}v_x(t)\vec{\imath} + \frac{d}{dt}v_y(t)\vec{\jmath} + \frac{d}{dt}v_z(t)\vec{k} = \frac{d^2x(t)}{dt^2}\vec{\imath} + \frac{d^2y(t)}{dt^2}\vec{\jmath} + \frac{dz(t)}{dt}\vec{k}$$

$$\vec{a} = a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k} = \dot{v}_x\vec{\imath} + \dot{v}_y\vec{\jmath} + \dot{v}_z\vec{k}$$

$$= \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k}$$

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) \quad / \cdot dt$$

$$d\vec{v}(t) = \vec{a}(t)dt \qquad \int_{t_0}^{t_1}$$

$$\int_{t_0}^{t_1} d\vec{v} = \int_{t_0}^{t_1} \vec{a}(t)\, dt$$

$$\vec{v}(t_1) - \vec{v}(t_0) = \int_{t_0}^{t_1} \vec{a}(t)\, dt$$

$t_1 \to t,\ \ t \to t'$

$$\boldsymbol{\vec{v}(t) = v(t_0) + \int_{t_0}^{t} \vec{a}(t')\, dt'}$$

PRIMJER 1 – HARMONIČKO TITRANJE

položaj čestice na x-osi:

$x(t) = Acos\ (\omega t)$

$A, \omega \to konstante$

$v(t) = ?, a(t) = ?$

$x(t) = A \cdot cos\ (\omega t + \Phi)$

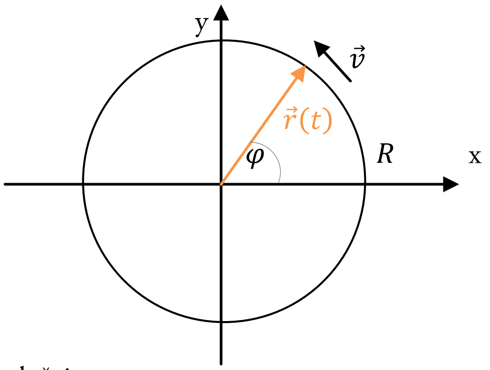
amplituda frekvencija fazni pomak

$$v(t) = \frac{d}{dt}x(t) = \frac{d}{dt}\big(Acos\ (\omega t + \Phi)\big) = \boldsymbol{-A\omega \cdot sin\ (\omega t + \Phi)}$$

$$a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}\big(-A\omega \cdot sin\ (\omega t + \Phi)\big) = -\omega^2 A \cdot \cos(\omega t + \Phi)$$

$$= \boldsymbol{-\omega^2 x(t)}$$

PRIMJER 2 – GIBANJE PO KRUŽNICI POLUMJERA R BRZINOM STALNOG IZNOSA v



položaj:
$\vec{r}(t) = Rcos(\varphi)\vec{\imath} + Rsin(\varphi)\vec{\jmath}$

φ – raste linearno u vremenu
$\varphi = \omega t$

$$\vec{r}(t) = R\big(\vec{\imath}cos(\omega t) + \vec{\jmath}sin(\omega t)\big)$$

$$\vec{v}(t) = R\left(\vec{\imath}\big(-\omega sin(\omega t)\big) + \vec{\jmath}\big(\omega cos(\omega t)\big)\right) = -\omega R\big(\vec{\imath}sin(\omega t) - \vec{\jmath}cos(\omega t)\big)$$

$$\vec{a}(t) = -\omega^2 R\big(\vec{\imath}cos(\omega t) + \vec{\jmath}sin(\omega t)\big) = -\omega^2\vec{r}(t)$$

$$\omega T = 2\pi$$

$$v = \frac{2R\pi}{T} = \frac{2R\pi}{\frac{2\pi}{\omega}} = \omega R \rightarrow \ \omega = \frac{v}{R}$$

$$|\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{\left[-\omega R\big(\vec{\imath}sin(\omega t) - \vec{\jmath}cos(\omega t)\big)\right]\cdot\left[-\omega R\big(\vec{\imath}sin(\omega t) - \vec{\jmath}cos(\omega t)\big)\right]}$$
$$= \sqrt{(-\omega R)^2\cdot\big(\vec{\imath}sin(\omega t) - \vec{\jmath}cos(\omega t)\big)\cdot\big(\vec{\imath}sin(\omega t) - \vec{\jmath}cos(\omega t)\big)}$$
$$= \sqrt{(-\omega R)^2\cdot[(\vec{\imath}\cdot\vec{\imath})\sin^2(\omega t) - (\vec{\imath}\cdot\vec{\jmath})\,sin(\omega t)\cos(\omega t) - (\vec{\jmath}\cdot\vec{\imath})\,sin(\omega t)\cos(\omega t) + (\vec{\jmath}\cdot\vec{\jmath})\cos^2(\omega t)}$$
$$= \omega R\sqrt{\sin^2(\omega t) + \cos^2(\omega t)} = \boldsymbol{\omega R} \quad \blacksquare$$

EKSPERIMENT- SLOBODNI PAD

s/m	t/s			$\bar{t}/s$
0.2000	0.2019	0.2019		0.2019
0.8000	0.4037	0.4037		0.4037
1.8000	0.6062	0.6060	0.6066	0.6063

$$s(t) = At^2$$

$$A = \frac{\sum_{i=1}^{N} S_i t_i^2}{\sum_{i=1}^{N} t_i{}^4}$$

$$s(t) = A' t^n$$

$$g = \bar{g} \pm \sigma_g \qquad \boldsymbol{\bar{g} = \frac{2\bar{s}}{\bar{t}^2}}$$

$$s = (1.8000 \pm 0.0005)$$

$$t = (0.6063 \pm 0.0001)$$

ZADATAK – HORVAT – 1.1

-pješak; pola vremena hoda v_1=2 km/h, pola preostalog puta trči 7 km/h, a drugu polovicu preostalog puta v_3=5 km/h. Izračunaj srednju brzinu gibanja tog pješaka!

$\bar{v}$ =?

$$\overbrace{\underbrace{-\,-\,-\,-\,-\,-\,-\,-\,-}_{s_1}\underbrace{-\,-\,-\,-\,-\,-\,-}_{s_2}\underbrace{-\,-\,-\,-\,-\,-\,-}_{s_3}}^{s}$$

$s_1 = v_1 t_1 \qquad s_2 = v_2 t_2 \qquad s_3 = v_3 t_3$

$t_1 = t_2 + t_3$

$s_2 = s_3$

$$\bar{v} = \frac{s}{t} = \frac{s_1 + s_2 + s_3}{t_1 + t_2 + t_3}$$

$$t_2 = \frac{v_3 t_3}{v_2}$$

$$t_3 = \frac{v_2 t_2}{v_3}$$

$$t_2 + t_3 = \frac{v_3 t_3}{v_2} + \frac{v_2 t_2}{v_3}$$

$$t_3 = t_1 - t_2 \rightarrow t_2 + t_1 - t_2 = \frac{v_3 (t_1 - t_2)}{v_2} + \frac{v_2 t_2}{v_3}$$

$$\left(1 - \frac{v_3}{v_2}\right) t_1 = \left(\frac{v_2}{v_3} - \frac{v_3}{v_2}\right) t_2 \qquad /\cdot v_2 v_3 \qquad (\ldots)$$

$$t_3 = \frac{v_2}{v_2 + v_3} t_1$$

$$\bar{v} = \frac{s_1 + s_2 + s_3}{t_1 + t_2 + t_3} = \frac{v_1 t_1 + v_2 t_2 + v_3 t_3}{t_1 + t_2 + t_3} = \frac{v_1 t_1 + \frac{v_2 v_3 t_1}{v_2 + v_3} + \frac{v_2 v_3 t_1}{v_2 + v_3}}{2 t_1} = \frac{v_1}{2} + \frac{v_2 v_3}{v_2 + v_3} = 1 + \frac{35}{12} = \frac{47}{12}$$

ZADATAK – „ELEKTRIJADA“ – 1

Čestica se giba duž x-osi tako da joj se brzina mijenja kao $v(x) = \frac{A}{B}(1 + B(x)$ gdje su A i B konstante,a x je koordinata čestice. Odredi ubrzanje pri položaju $x = \frac{1}{B}$!

$$a(x) = \frac{d}{dt}v(x) = \frac{d}{dt}v\big(x(t)\big) = \frac{dv}{dx}\frac{dx}{dt} = \left[\frac{d}{dx}\left(\frac{A}{B}\big(1 + B(x)\big)\right)\right]\left[\frac{A}{B}\big(1 + B(x)\big)\right]$$

$$= \frac{A}{B}B\frac{A}{B}(1 + Bx) = \frac{A^2}{B}(1 + Bx)$$

$$a\left(\frac{1}{B}\right) = \frac{A^2}{B}\left(1 + B\frac{1}{B}\right) = \frac{2A^2}{B}$$

ZADATAK

Brzina čestice koja se giba duž x-osi je $v(x) = v_0 e^{-x/b}$. Pri tom su v_0 i b konstante. Odredi položaj kao funkciju vremena ako je poznato da je u početnom trenutku $t_0=0$, $x(t_0)=0$!

$$v = \frac{dx}{dt} = v_0 e^{-\left(\frac{x}{b}\right)} \qquad /\cdot e^{\left(\frac{x}{b}\right)}dt$$

$$e^{\frac{x}{b}}dx = v_0 dt \qquad \int_{"0"}^{"1"}$$

$$\int_{x_0}^{x_1} e^{\frac{x}{b}}\,dx = \int_{t_0}^{t_1} v_0 dt$$

$$be^{\frac{x}{b}}\Big|_{x_0}^{x_1} = v_0 t\Big|_{t_0}^{t_1}$$

$$b\left(e^{\frac{x_1}{b}} - e^{\frac{x_0}{b}}\right) = v_0(t_1 - t_0)$$

$$b\left(e^{\frac{x_1}{b}} - 1\right) = v_0 t_1$$

konačno stanje „poopćujemo“

$$t_1 \to t, x_1 \to x(t)$$

$$b\left(e^{\frac{x(t)}{b}} - 1\right) = v_0 t$$

$$e^{\frac{x(t)}{b}} = \frac{v_0 t}{b} + 1$$

$$x(t) = b \cdot ln\left(\frac{v_0 t}{b} + 1\right)$$

NEWTONOVI AKSIOMI

Čestica **m** – ima samo masu, nema ni oblik, ni dimenzije, ni orijentaciju

KOLIČINA GIBANJA

$\vec{p} = m\vec{v}$

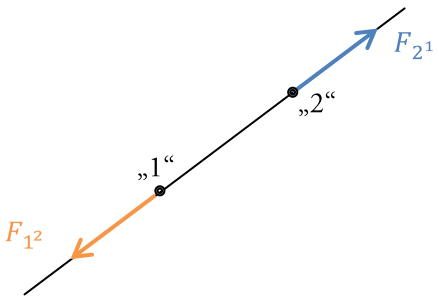
masa brzina

PRVI AKSIOM – Kad na česticu ne djeluje sila ona ostaje u stanju mirovanja ili jednolikog gibanja po pravcu
- vrijedi u inercijalnim sustavima

DRUGI AKSIOM – Vremenska promjena količine gibanja čestice je razmjerna sili koja na česticu djeluje

$\frac{d\vec{p}}{dt} = \vec{F}$

TREĆI AKSIOM – Ako tijelo/čestica djeluje na drugu česticu silom, tada druga čestica djeluje na prvu silom istog iznosa, ali suprotnog smjera. Te sile leže na istom pravcu (pravcu koji prolazi dvjema česticama).



NEWTONOVA JEDNADŽBA GIBANJA

$$\frac{d}{dt}\vec{p} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = \boldsymbol{m\vec{a}} = \boldsymbol{\vec{F}}$$

$\vec{F}(\vec{r},t) = \vec{F}_0 \rightarrow konst.$

prvi korak: BRZINA

$$\frac{d\vec{p}}{dt} = \overrightarrow{F_0} \rightarrow \frac{d\vec{v}}{dt} = \frac{\overrightarrow{F_0}}{m} \qquad / \cdot dt$$

$$d\vec{v} = \frac{\overrightarrow{F_0}}{m}dt \qquad \int_{poč}^{kon}$$

$$\int_{poč}^{kon} d\vec{v} = \int_{poč}^{kon} \frac{\overrightarrow{F_0}}{m}dt$$

$$\vec{v}\,|_{poč}^{kon} = \frac{\overrightarrow{F_0}}{m}t\,|_{poč}^{kon}$$

$$\vec{v}_{kon} - \vec{v}_{poč} = \frac{\overrightarrow{F_0}}{m}(t_{kon} - t_{poč})$$

$t_{poč} \rightarrow t_0, \qquad t_{kon} \rightarrow t$

$$\vec{v}(t) - \vec{v}(t_0) = \frac{\overrightarrow{F_0}}{m}(t - t_0)$$

$$\vec{v}(t) = \frac{\overrightarrow{F_0}}{m}(t - t_0) + \vec{v}(t_0)$$

početni uvjet

drugi korak: POLOŽAJ

$$\frac{d\vec{r}}{dt} = \vec{v} \qquad / \cdot dt$$

$$d\vec{r} = \vec{v}dt$$

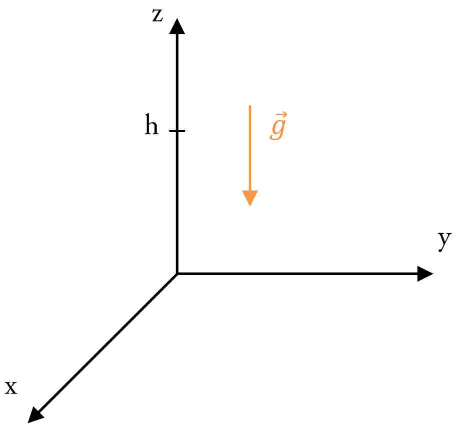
$$\int_{poč}^{kon} d\vec{r} = \int_{poč}^{kon} \vec{v}dt$$

$$\vec{r}(t_1) - \vec{r}(t_0) = \int_{t_0}^{t} \vec{v}(t')dt' = \int_{t_0}^{t} \left[\vec{v}(t_0) + \frac{\overrightarrow{F_0}}{m}(t' - t_0)\right] dt'$$

$$= \vec{v}(t_0)(t' - t_0) + \frac{\overrightarrow{F_0}}{2m}(t' - t_0)^2$$

$$\vec{r}(t) = \vec{r}(t_0) + \vec{v}(t_0)(t - t_0) + \frac{\overrightarrow{F_0}}{2m}(t - t_0)^2$$

PRIMJER – SLOBODNI PAD



sila $\overrightarrow{F_0} = -mg\vec{k}$

početni uvjeti u $t=t_0=0$

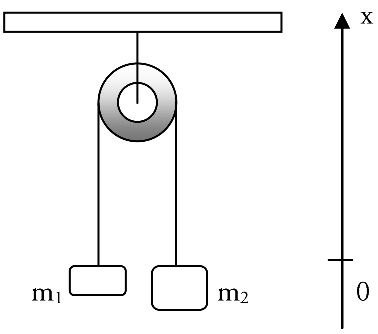
$\vec{v}(t_0) = \vec{v}(0) = \overrightarrow{v_0} = 0$

$\vec{r}(t_0) = \overrightarrow{r_0} = h\vec{k}$

$\vec{v}(t) = -g\vec{k}t$

$\vec{r}(t) = (h - \frac{g}{2}t^2)\vec{k}$

PRIMJER – PADOSTROJ



$\overrightarrow{r_1}(t) = x_1(t)\vec{\imath}$
$\overrightarrow{r_2}(t) = x_2(t)\vec{\imath}$

$x_1(t) = -x_2(t)$

$\dot{x}_1(t) = -\dot{x}_2(t)$
$\ddot{x}_1(t) = -\ddot{x}_2(t)$

JEDNADŽBA GIBANJA

$m_1\ddot{x}_1 = T - m_1 g$
$m_2\ddot{x}_2 = T - m_2 g$

napetost niti

$m_1\ddot{x}_1 - m_2\ddot{x}_2 = (m_2 - m_1)g$

$\ddot{x}_1 = -\ddot{x}_2 = \frac{m_2 - m_1}{m_2 + m_1}g$

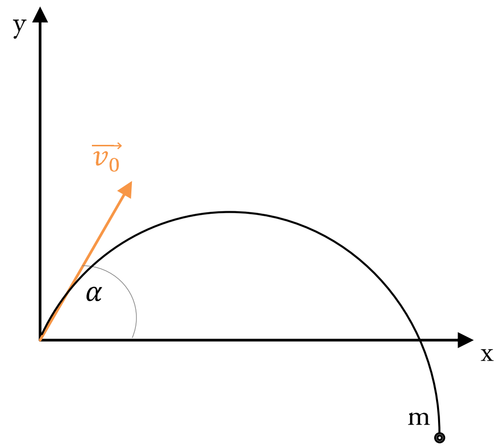
ZADATAK – PADOSTROJ

m_1=400 g
m_2=402 g
iz mirovanja
t_1-t_0=6.4 s
$x_1(t_1)$ –$x_1(t_0)$=0.5 m
izračunaj g !

$$x_1(t_1) - x_1(t_0) = \frac{a_x}{2}(t_1 - t_0)^2$$

$$a_x = \frac{m_2 - m_1}{m_2 + m_1} g \rightarrow g = \frac{m_2 + m_1}{m_2 - m_1} \cdot \frac{2[x_1(t_1) - x_1(t_0)]}{(t_1 - t_0)^2} = \frac{400 + 402}{402 - 400} \cdot \frac{(2 \cdot 0.5)}{(6.4)^2} = \mathbf{9.79}\, \frac{\boldsymbol{m}}{\boldsymbol{s^2}}$$

KOSI HITAC



početni uvjeti u t=t_0=0

$\vec{v}(t_0) = \vec{v}(0) = \vec{v_0} = v_0(\cos\alpha\, \vec{\imath} + \sin\alpha \vec{\jmath})$
$\vec{r}(t_0) = \vec{r}(0) = \vec{r_0} = 0$

sila:
$\vec{F_0} = -mg\vec{\jmath}$

$$\vec{v}(t) = \frac{\vec{F_0}}{m}(t - t_0) + \vec{v}(t_0)$$

$$\vec{r}(t) = \vec{r}(t_0) + \vec{v}(t_0)(t - t_0) + \frac{\vec{F_0}}{2m}(t - t_0)^2$$

uvrštavanje:

$$\rightarrow \vec{v}(t) = v_0(\cos\alpha\, \vec{\imath} + \sin\alpha \vec{\jmath}) - gt\vec{\jmath}$$
$$\rightarrow \vec{r}(t) = v_0(\cos\alpha\, \vec{\imath} + \sin\alpha \vec{\jmath})t - \frac{g}{2}t^2\vec{\jmath}$$

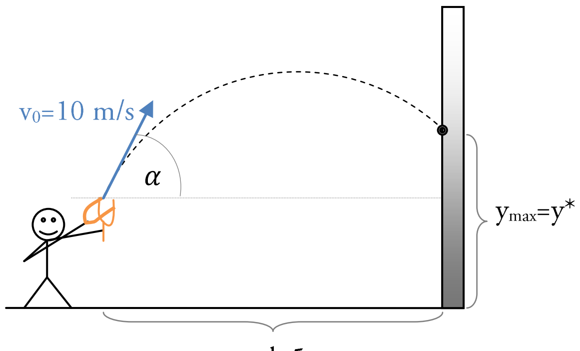
po komponentama:

$$v_x(t) = v_0 \cos\alpha$$
$$v_y(t) = v_0 \sin\alpha - gt$$
$$x(t) = v_0 \cos\alpha \cdot t$$
$$y(t) = v_0 \sin\alpha \cdot t - \frac{g}{2}t^2$$

$$y = v_0 \sin\alpha \frac{x}{v_0 cos\alpha} - \frac{g}{2}\left(\frac{x}{v_0 cos\alpha}\right)^2 = x \cdot tg\, \alpha - \frac{g}{2}\frac{x^2}{v_0^2 \cos^2\alpha}$$

$\boldsymbol{y(x) = ux - \frac{gx^2}{2v_0^2}(1 + u^2)}$ gdje je u=tg α

ZADATAK – DJEČAK S PRAĆKOM



Pod kojim kutem dječak treba iz praćke ispucati kamen brzinom 10 m/s ako stoji na 5 m od zida a da taj kamen pogodi što višu točku zida?

$y(x) = ux - \frac{gx^2}{2v_0^2}(1+u^2)$ gdje je u=tg α

tražimo max funkcije y*

$$0 = \frac{dy^*}{du} = \frac{d}{du}\left(ud - \frac{gd^2}{2v_0^2}(1+u)^2\right) = d - \frac{gd^2}{2v_0^2}2u = d - \frac{gd^2}{v_0^2}u$$

$$\rightarrow u = \frac{v_0^2}{gd} = \frac{100}{9.81 \cdot 5} = \frac{20}{9.81} = 2.038$$

$$arc\ tg\ u = 63.87°$$

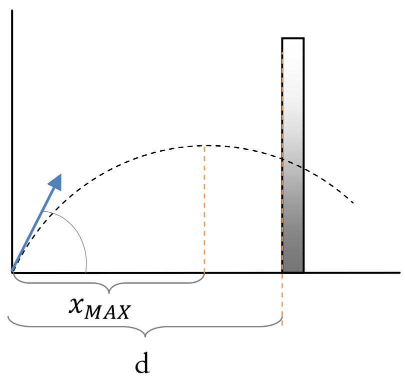
tražimo tjeme parabole

$$y(x) = u^* x - \frac{gx^2}{2v_0^2}(1+(u^*)^2) = \frac{v_0^2}{gd}x - \frac{gx^2}{2v_0^2}\left(1+\frac{v_0^4}{g^2d^2}\right) = \frac{v_0^2}{gd}x - \frac{gx^2}{2v_0^2} - \frac{v_0^2 g x^2}{2g^2d^2} \qquad y = ax^2 + bx + x$$

$$x_{MAX} = -\frac{b}{2a} \rightarrow tjeme\ parabole = \frac{\frac{v_0^2}{gd}}{2\left(\frac{g}{2v_0^2}+\frac{v_0^2}{2gd^2}\right)} = \frac{\frac{v_0^2}{gd}}{\frac{2v_0^2}{gd}\left(\frac{g^2d}{2v_0^4}+\frac{1}{2d}\right)} = \frac{1}{2\left(\frac{g^2d}{2v_0^4}+\frac{1}{2d}\right)} = \frac{1}{\frac{g^2d}{v_0^4}+\frac{1}{d}}$$

$$= \frac{d}{1+\frac{g^2d^2}{v_0^4}}$$

$\boldsymbol{x_{MAX}} < d$ a to je ono što smo dokazivali, da će kamen dosegnuti najvišu točku zida ako mu je tjeme parabole po kojoj leti ispred zida ■



Rješavanje Newtonove jednadžbe gibanja za silu razmjernu brzini

$$\boldsymbol{\vec{F} = -\gamma\vec{v}}$$ → brzina

sila konst.

uvrštavanje:

$$\frac{d\vec{p}}{dt} = \vec{F}$$

$$m \cdot \dot{\vec{v}} = -\gamma\vec{v}$$

x komponenta –

$$m \cdot \dot{v_x} = -\gamma v_x$$

$$m\frac{dv_x}{dt} = -\gamma v_x \qquad /\cdot \frac{dt}{mv_x}$$

$$\frac{dv_x}{v_x} = -\frac{dt\gamma}{m} \qquad \int_{poč}^{kon}$$

$$\ln v_x(kon) - \ln v_x(poč) = -\frac{\gamma}{m}\big(t(kon) - t(poč)\big)$$

v(t) $v(t_0)$ t t_0

$$\boldsymbol{v_x(t) = v_x(t_0) \cdot e^{-\frac{\gamma}{m}(t-t_0)}}$$

položaj:

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$d\vec{r} = \vec{v}dt$$

$$d\vec{r} = \vec{v}_0 e^{-\frac{\gamma}{m}\cdot t}\, dt \qquad \int_{poč}^{kon}$$

$$\vec{r}_k - \vec{r}_p = \vec{v}_0\left(-\frac{m}{\gamma}\right)e^{-\frac{\gamma}{m}t}\ \Big|_{poč}^{kon}$$

r(t) r_0

$$\vec{r}(t) = \vec{r}_0 + \overrightarrow{v_0}\frac{m}{\gamma}\left(1 - e^{-\frac{\gamma}{m}t}\right)$$

$$\vec{r}_\infty = \lim_{t\to\infty} r(t) = \vec{r}_0 + \overrightarrow{v_0}\frac{m}{\gamma}(1-0) = \vec{r}_0 + \overrightarrow{v_0}\frac{m}{\gamma}$$

Rješavanje Newtonove jednadžbe gibanja za konstantnu silu uz otpor razmjeran brzini ($\alpha \vec{v}$)

brzina:

$$m\dot{\vec{v}} = \vec{F_0} - \gamma\vec{v}$$

$$m\frac{d\vec{v_x}}{dt} = \vec{F_0}_x - \gamma\vec{v_x}$$

$$\frac{d\vec{v_x}}{\frac{\vec{F_0}_x}{m} - \frac{\gamma\vec{v_x}}{m}} = dt \qquad \int_{poč}^{kon} \qquad\qquad (\ldots)$$

$$\vec{v}(t) = \vec{v_0}e^{-\frac{\gamma}{m}(t-t_0)} + \frac{\vec{F_0}}{\gamma}\left(1 - e^{-\frac{\gamma}{m}(t-t_0)}\right)$$

položaj:

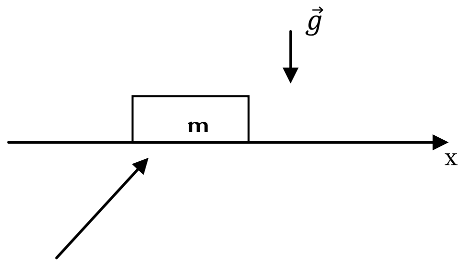
$$\vec{r} = \vec{v}dt \qquad \int_{poč}^{kon}$$

$$\vec{r}(t) = \vec{r_0} + \int_0^t \vec{v}(t')dt' = \qquad\qquad (\ldots)$$

$$\vec{r}(t) = \vec{r_0} + \frac{\vec{F_0}}{\gamma}(t - t_0) + \left(\vec{v_0} - \frac{\vec{F_0}}{\gamma}\right)\frac{m}{\gamma}\left(1 - e^{-\frac{\gamma}{m}(t-t_0)}\right)$$

Trenje

- na vodoravnoj podlozi miruje tijelo mase m

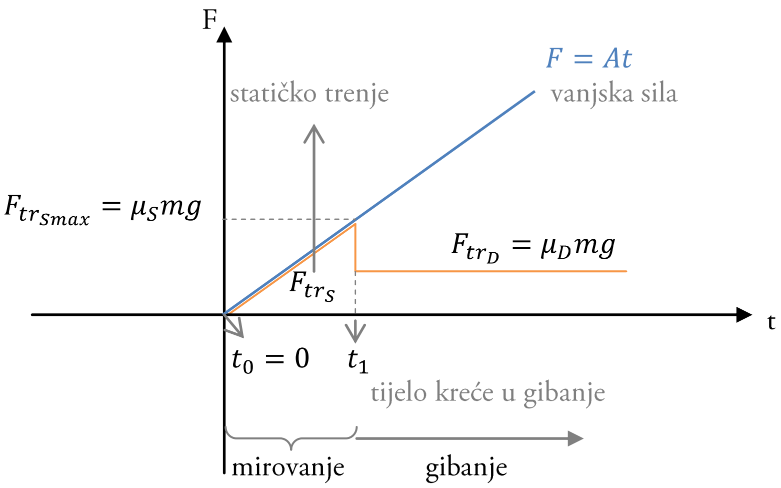


$\mu_D < \mu_S$

$t = t_0 = 0$

$F = At$

$x(t) =?\ za\ t > t_0$



mirovanje

$t_0 < t < t\_1$

$\mu_S mg = At_1 \to \boldsymbol{t_1} = \frac{\boldsymbol{\mu_S mg}}{\boldsymbol{A}}$

$F_{tr_{Smax}}$

početni uvjeti: $t = t_1 = \frac{\mu_S mg}{A}, x_1 = 0, v_1 = 0$

jednadžba gibanja za $t > t_1$

$$m\dot{\vec{v}} = -\mu_D mg + At$$

$$\frac{dv}{dt} = -\mu_D g + \frac{A}{m} t$$

$$dv = \left(-\mu_D g + \frac{A}{m} t\right) dt \qquad \int_{poč}^{kon}$$

$$\int_{poč}^{kon} dv = \int_{poč}^{kon} \left(-\mu_D g + \frac{A}{m} t\right) dt$$

$$\overrightarrow{v_k} - \overrightarrow{v_p} = \int_{t_1}^{t} \left(-\mu_D g + \frac{A}{m} t'\right) dt'$$

$$\vec{v}(t) - \vec{v}(t_1) = -\mu_D g(t - t_1) + \frac{A}{2m}(t - t_1)^2$$

=0

$$\vec{v}(t) = -\mu_D g(t - t_1) + \frac{A}{2m}(t - t_1)^2$$

$$x(t) = -\frac{\mu_D g}{2}(t - t_1)^2 + \frac{A}{6m}(t - t_1)^3$$

Veza između promjene količine gibanja i impulsa sile

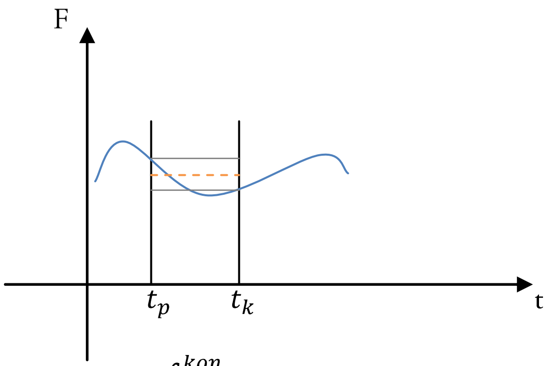
$\frac{d\vec{p}}{dt} = \vec{F} \to vremenska\ derivacija\ količine\ gibanja$

$$d\vec{p} = \vec{F} dt \qquad \int_{poč}^{kon}$$

$$\overrightarrow{p_K} - \overrightarrow{p_P} = \int_{poč}^{kon} \vec{F} dt$$

$\Delta\vec{p} = \vec{F}\Delta t \longrightarrow$ impuls sile

promjena količine gibanja

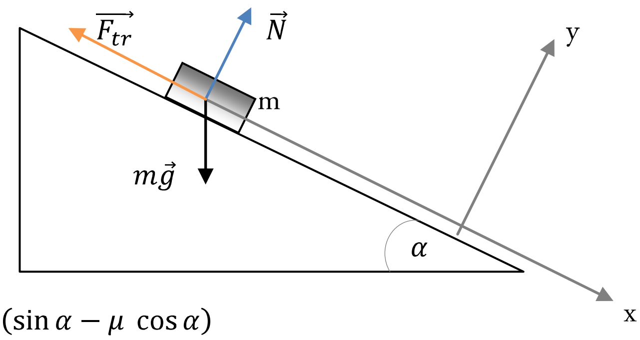


$$\int_{poč}^{kon} \vec{F} dt \simeq \vec{F}\Delta t$$

PRIMJER – KOSINA

$\alpha, \mu_D = \mu$

$a = ?$



$m\ddot{y} = -mg\cos\alpha + N$
$N = mg\cos\alpha$

$m\ddot{x} = mg\sin\alpha - \mu N = mg\sin\alpha - \mu\, mg\cos\alpha = mg(\sin\alpha - \mu\ \cos\alpha)$

kružno gibanje:

$$\boldsymbol{F_C = \frac{mv^2}{R} = m\omega^2 R}$$

-centripetalna sila

položaj:

$\vec{r} = R(\cos\Phi\,\vec{\imath} + \sin\Phi\,\vec{\jmath}), \quad \Phi = \Phi(\mathrm{t})$

$$\vec{v} = \frac{d\vec{r}}{dt} = R\big(-\dot{\Phi}\sin\Phi\,\vec{\imath} + \dot{\Phi}\cos\Phi\,\vec{\jmath}\big) = R\dot{\Phi}\,(-\sin\Phi\,\vec{\imath} + \cos\Phi\,\vec{\jmath})$$

$$\vec{a} = R\ddot{\Phi}(-\sin\Phi\,\vec{\imath} + \cos\Phi\,\vec{\jmath}\,) + R\dot{\Phi}\,\big(-\cos\Phi\dot{\Phi}\,\vec{\imath} + \sin\Phi\dot{\Phi}\,\vec{\jmath}\big)$$

$$= \boldsymbol{\frac{\ddot{\Phi}}{\dot{\Phi}}\vec{v} - \dot{\Phi}^2\vec{r}}$$

komponenta tangencijalna na putanju i komponenta okomita na putanju

važno!
$\vec{r} \perp \vec{v} \rightarrow (\vec{r}\cdot\vec{v}) = 0$

$\vec{r}\cdot\vec{v} = r_x v_x + r_y v_y + r_z v_z = -R\cos\Phi\, R\dot{\Phi}\sin\Phi + \ R\sin\Phi\, R\dot{\Phi}\cos\Phi = 0$

$$\vec{a} = \frac{\ddot{\Phi}}{\dot{\Phi}}v\hat{\vec{v}} - \dot{\Phi}^2\mathrm{R}\hat{\vec{r}}\,, \qquad v = R\dot{\Phi} \rightarrow\ \vec{a} = R\ddot{\Phi}\hat{\vec{v}} - R\dot{\Phi}^2\hat{\vec{r}}\ \equiv \vec{a}_{tang} - \vec{a}_{rad}$$

$\alpha = \ddot{\Phi} \rightarrow$ kutna akceleracija
$\omega = \dot{\Phi} \rightarrow$ kutna brzina

NEWTONOVA JEDNADŽBA

$$\vec{F} = m\vec{a} = \vec{F}_{tang} + \vec{F}_{rad}$$

$\hat{\vec{v}}\, m\alpha R \qquad -\hat{\vec{r}}\, m\omega^2 R = -\hat{\vec{r}}\frac{v^2}{R}m$

RAD

Sila $\vec{F}$ djeluje duž puta $d\vec{s}$

diferencijal obavljenog rada:

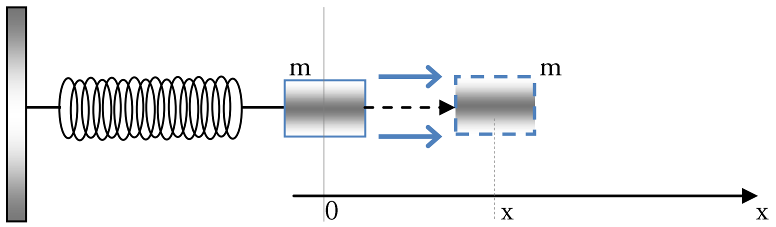
$$dW = \vec{F}d\vec{s}$$

primjeri: „W=Fs“

-sila paralelna s putanjom
-sila je stalna

$$W = \int \vec{F}d\vec{s} = \int F\cos\overset{0}{\vartheta}\ ds = \int F\,ds = F\int ds = F\cdot s$$

PRIMJER – OPRUGA

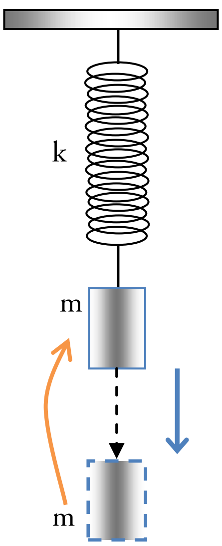


$\boldsymbol{\vec{F} = -kx}$ pomak koji ćemo napraviti

konstanta opruge

rad pri rastezanju opruge za x:

$$W = \int \vec{F}d\vec{s} = \int_0^x (kx')dx' = \frac{kx^2}{2}$$



$$W = \int \vec{F}d\vec{s} = \int_0^x (-kx')(-dx') = \frac{kx^2}{2} \ \blacksquare$$

KINETIČKA ENERGIJA

Neka sila $\vec{F}$ djeluje na tjelo mase m, gibanje duž x-osi

$$\vec{F} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\frac{d\vec{v}}{dx}\frac{dx}{dt} = m\vec{v}\frac{d\vec{v}}{dx}$$

$$W = \int \vec{F}dx = \int mv\frac{dv}{dx}dx = \int mv\,dv = \frac{mv^2}{2} \equiv E_{KIN}$$

$$W_{1\to 2} = \int_1^2 F\,dx = \frac{mv_2^2}{2} - \frac{mv_1^2}{2}$$

Teorem o radu i kinetičkoj energiji:

(obavljen rad)=(promjena kinetičke energije)
$\Delta W = \Delta E_{KIN}$

SNAGA

mjera rada obavljenog u jedinici vremena

$$P = \frac{dW}{dt} = \frac{d}{dt}\vec{F}d\vec{s} = \vec{F}\frac{d\vec{s}}{dt} = \vec{F}\vec{v}$$

PRIMJER

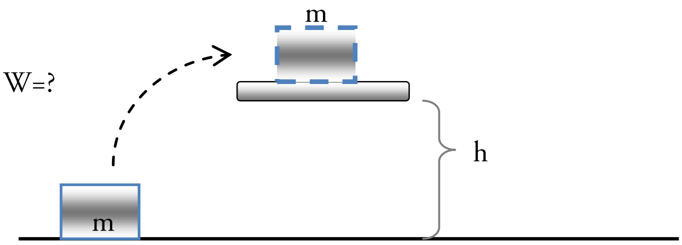
auto kreće iz mirovanja ($\vec{v}_0 = 0$); $m = 1T = 1000\,kg$; za vrijeme $\Delta t = t_1 - t_0 = 15\,s$ postiže vrzinu $v_1 = 100\frac{km}{h} = 27.777\frac{m}{s}$; kolika je prosječna snaga?

$$\bar{P} = \frac{\Delta W}{\Delta t} = \frac{\Delta E_{KIN}}{\Delta t} = \frac{\frac{m}{2}(v_1^2 - v_0^2)}{\Delta t} = \frac{\frac{m}{2}v_1^2}{\Delta t} = \frac{500 \cdot 771.6}{15} = 25.72\,kW$$

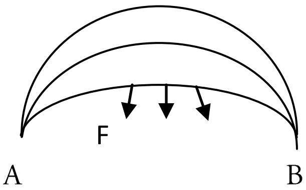
Potencijalna energija

PRIMJER – HOMOGENO POLJE GRAVITACIJSKE SILE

W=?



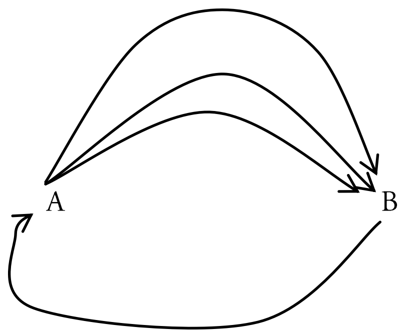
$$W = mgh = mg\sin\alpha\frac{h}{\sin\alpha}$$



Sila je „konzervativna“!

$$W_{A\to B} = \int_A^B \vec{F}\,d\vec{s}$$

sve ovisi o izboru putanje A – B

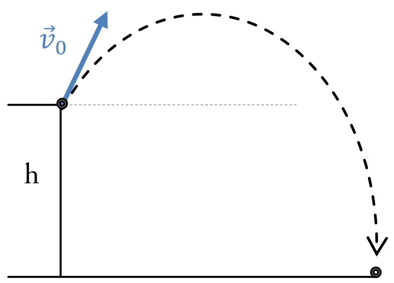


$$W_{B\to A} = -W_{A\to B}$$

$$\oint \vec{F}\, d\vec{s} = 0$$

integral zatvorene putanje

Zadatak



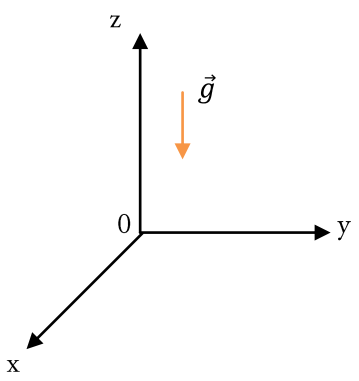
-**potencijalna en.** – skalarna funkcija u polju konzervativne sile; njena vrijednost je jednaka radu potrebnom da česticu dovedemo u neku točku

$$U(B) - U(A) = -\int_A^B \vec{F}\, d\vec{s}$$

$$U(B) = U(A) - \underbrace{\int_A^B \vec{F}\, d\vec{s}}_{\text{rad } A \to B}$$

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}\, d\vec{r}'$$

Primjer – homogena gravitacija



$$\vec{F} = -mg\vec{k}$$

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} (-mg\vec{k})d\vec{r} = -\int_{\vec{r}_0}^{\vec{r}} (-mg)dz$$

$$= mg\int_{\vec{r}_0}^{\vec{r}} dz = \boldsymbol{mgz}$$

$\vec{F}(\vec{r})$ ... *konzervativna sila*
$U(\vec{r})$ ... *potencijalna en.*

$$U(\vec{r}) = U(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}')d\vec{r}'$$

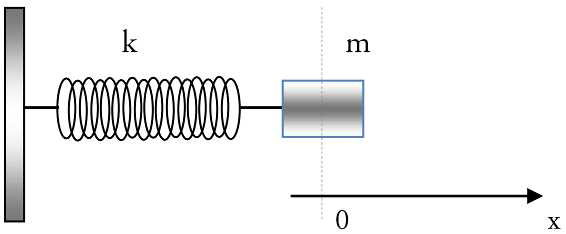
$$\vec{F} = -\frac{\partial U}{\partial x}\vec{\imath} - \frac{\partial U}{\partial y}\vec{\jmath} - \frac{\partial U}{\partial z}\vec{k} = -\frac{\partial U}{\partial \vec{r}} = -\vec{\nabla}U$$

Primjer

$U = mgz$

$$\vec{F} = -\frac{\partial U}{\partial x}\vec{\imath} - \frac{\partial U}{\partial y}\vec{\jmath} - \frac{\partial U}{\partial z}\vec{k} = -(mg)\vec{k}$$

Primjer – opruga



$\vec{F} = -kx$

$$U(x) = U(0) - \int_0^x \vec{F}(x')\,dx' = \frac{1}{2}kx^2$$

$$F(x) = -\frac{\partial}{\partial x}U(x) = -\frac{\partial}{\partial x}\frac{1}{2}kx^2 = -kx$$

Primjer

$$U = -\frac{k}{x}$$

sila:

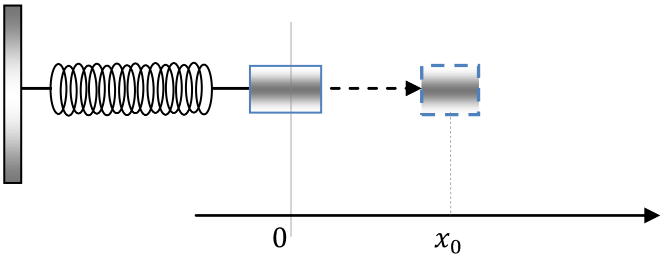
$$F(x) = -\frac{\partial U}{\partial x} = -\frac{\partial}{\partial x}\left(-\frac{k}{x}\right) = -\frac{k}{x^2}$$

Zakon očuvanja mehaničke energije

$\Delta W_{KONZ} = \Delta E_{KIN} = -\Delta(U)$

$\rightarrow \Delta(E_{KIN} + U) = 0$

Primjer



početni otklon $x = x_0$ $v = v_0 = 0$
brzina pri x=0=?

$$E = E_{KIN} + E_{POT} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$
$$\Delta E = 0$$

$$E_0 = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2 = \frac{1}{2}kx_0^2$$

$$E_1 = \frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_1^2$$

$$E = E_0 = E_1$$

$$\frac{1}{2}kx_0^2 = \frac{1}{2}mv_1^2$$

$$v_1 = x_0\sqrt{\frac{k}{m}}$$

Zakon očuvanja količine gibanja sustava čestica

$$\underline{\vec{P}} = \sum_{i=1}^{N} \vec{P_i}$$

$$\frac{d}{dt}\underline{\vec{P}} = \frac{d}{dt}\sum_{i=1}^{N} \vec{P_i} = \sum_{i=1}^{N} \frac{d\vec{P_i}}{dt} = \sum \vec{F_i}$$

$$\vec{F_i} = \vec{F}_{i\,EXT} + \sum_{j \neq i} \vec{F}_{ij}$$

sila kojom j-ta čestica djeluje na i-tu

vanjska sila na i-tu česticu

$$\vec{F}_{ij} = -\vec{F}_{ji}$$

$$\frac{d}{dt}\vec{P} = (\dots) = \sum_i \left(\vec{F}_{i_{EXT}} + \sum_{j \neq i} \vec{F}_{i_j} \right) = \sum_i \vec{F}_{i_{EXT}} + \sum_i \sum_{j \neq i} \vec{F}_{i_j} = \sum_i \vec{F}_{i_{EXT}}$$

ako su $\vec{F}_{i_{EXT}} = 0$ P=konst

ENERGIJA

$$E = \frac{mv_0^2}{2} = mgh + F_{TR}\frac{h}{\sin\alpha} = \frac{mv_2^2}{2} + 2F_{TR}\frac{h}{\sin\alpha}$$

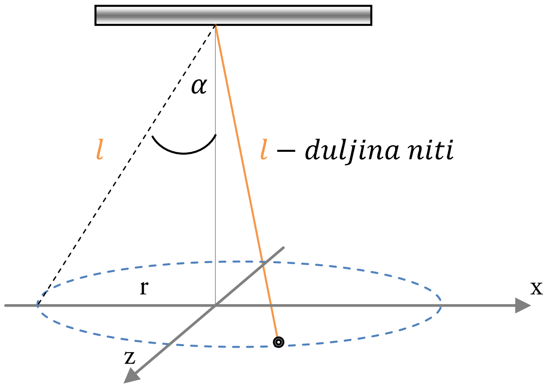
$$h = \frac{mv_0^2}{2}\left(mg + \frac{F_{TR}}{\sin\alpha}\right)^{-1}$$

$$\frac{mv_2^2}{2} = \frac{mv_0^2}{2} - \frac{2F_{TR}}{\sin\alpha}h = \frac{mv_0^2}{2} - \frac{2F_{TR}}{\sin\alpha}\frac{mv_0^2}{2}\left(mg + \frac{F_{TR}}{\sin\alpha}\right)^{-1}$$

$$\frac{v_2^2}{v_0^2} = 1 - \frac{2\mu mg\cos\alpha}{\sin\alpha}\left(mg + \frac{\mu mg\cos\alpha}{\sin\alpha}\right)^{-1} = 1 - \frac{2\mu}{\tan\alpha + \mu} = \frac{\tan\alpha - \mu}{\tan\alpha + \mu}$$

$$v_2 = v_0\sqrt{\frac{\tan\alpha - \mu}{\tan\alpha + \mu}}$$

PRIMJER – STOŽASTO NJIHALO



općenito:

$$m\vec{a} = m(\ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}) = \vec{F} + m\vec{g} = -m\omega\vec{r}$$

u trenutku z=0:

$$\vec{T} + mg = T(\sin\alpha\,\vec{i} + \cos\alpha\,\vec{j}) - mg\vec{j} = (T\sin\alpha)\vec{i} + (T\cos\alpha)\vec{j} - m\omega^2\vec{r} = -m\omega^2 R(-\vec{i}) = m\omega^2 R\vec{i}$$

$$T\cos\alpha = mg$$
$$T\sin\alpha = m\omega^2 R$$
$$\tan\alpha = \frac{\omega^2 R}{g} \to \omega^2 = \frac{g}{R}\tan\alpha = \frac{g}{l\sin\alpha}\cdot\frac{\sin\alpha}{\cos\alpha} = \frac{g}{l}\cdot\frac{1}{\cos\alpha}$$

Zadatak – raketa

Ako raketa kreće iz mirovanja, kolika će biti njena brzina u trenutku u kojem će se njena masa smanjiti na polovicu početne mase?

$M \rightarrow \frac{1}{2} M$

ukupna količina gibanja - konstantna

$\underline{\vec{P}} = 0$

$dpg = -dpr$

$dmv_g = -mdv_r$

$v_g \frac{dm}{m} = -dv_r \qquad \int_{poč}^{kon}$

$v_g \ln m \ |_{poč}^{kon} = -v_r \ |_{poč}^{kon}$

$v_g \ln \frac{\frac{1}{2} M}{M} = v_{poč} - v_{kon}$

$v_g \ln \frac{1}{2} = -v_{kon}$

$|v_{kon}| = v_g \ln 2$

Zakon gibanja središta mase

$\underline{\vec{P}} = \sum_i \vec{P}_i = \sum_i m_i \, \vec{v}_i$

$\frac{\mathrm{d}}{\mathrm{dt}} \underline{\vec{P}} = \sum_i \vec{F}_{i_{EXT}} = \vec{F}_{EXT}$

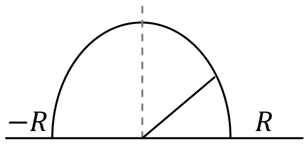
definicija središta mase:

$\vec{R} = \frac{\sum_i m_i \, \vec{r}_i}{\sum_i m_i} = \frac{1}{M} \sum_i m_i \, \vec{r}_i$

$\vec{R} = \frac{1}{M} \int \vec{r} g \; dV = \frac{1}{M} \int \vec{r} \vec{v} \; da = \frac{1}{M} \int \vec{r} \lambda \; dl$

Primjer

žica u svinuta u polukrug



$\vec{R} = ?$

$$Y = \frac{1}{M}\int y\lambda \, dl = \frac{1}{M}\int y \frac{M}{R\pi} \, dl$$

$$\lambda = \frac{M}{R\pi} \qquad dl = R \, d\Phi \qquad \mathrm{y} = \mathrm{R} \sin \Phi$$

$$Y = \frac{1}{M}\int_0^{\pi} (\mathrm{R} \sin \Phi) \frac{M}{R\pi} \, R \, d\Phi = \frac{\mathrm{R}}{\pi}\int_0^{\pi} \sin \Phi \, \mathrm{d}\Phi = \frac{R}{\pi}(-\cos \Phi)\,|_0^{\pi} = \frac{2R}{\pi}$$

Primjer

$$\vec{V} = \vec{R} = \frac{1}{M}\sum_i m_i \vec{r}_i = \frac{1}{M}\sum_i \vec{P}_i = \frac{\vec{P}}{M}$$

$$\underline{\vec{P}} = M\vec{V}$$

$$\underline{\dot{\vec{P}}} = M\dot{\vec{V}} = \vec{F}_{EXT}$$

- jednadžba gibanja za središte mase sustava čestica

Sudari 2 čestice

u svim sudarima očuvana je količina gibanja

$$\underline{\vec{P}} = \sum_i \vec{P}_i = \sum_i \vec{P}_i{}' \rightarrow m_1\overrightarrow{v_1} + m_2\overrightarrow{v_2} = m_1\overrightarrow{v_1}' + m_2\overrightarrow{v_2}'$$

SAVRŠENO ELASTIČNI SUDAR:

$$E_{KIN} = \sum_i \frac{m_i \vec{v}_i^2}{2} = \sum_i \frac{m_i \vec{v'}_i^2}{2}$$

2 čestice:

$$m_1\overrightarrow{v_1} + m_2\overrightarrow{v_2} = m_1\overrightarrow{v_1}' + m_2\overrightarrow{v_2}'$$

$$m_1\overrightarrow{v_1}^2 + m_2\overrightarrow{v_2}^2 = m_1\overrightarrow{v_1}'^2 + m_2\overrightarrow{v_2}'^2$$

pojednostavljeno u 1D:

$$m_1 v_{1X} + m_2 v_{2X} = m_1 v_{1X}' + m_2 v_{2X}'$$

$$m_1 v_{1X}^2 + m_2 v_{2X}^2 = m_1 v_{1X}'^2 + m_2 v_{2X}'^2$$

zadano:

$m_1, m_2; v_{1x}, v_{2x};$ $\qquad v_{1x}', v_{2x}' =?$

$$v_{1x}' = \begin{cases} v_{1x} & - nema\ sudara \\ \frac{m_1 - m_2}{m_1 + m_2} v_{1x} + \frac{2m_2}{m_1 + m_2} v_{2x} \end{cases}$$

$$v_{2x}' = \begin{cases} v_{2x} & - nema\ sudara \\ \frac{m_2 - m_1}{m_1 + m_2} v_{2x} + \frac{2m_1}{m_1 + m_2} v_{1x} \end{cases}$$

općenit (ne/?)elastični sudar:

$$m_1\overrightarrow{v_1} + m_2\overrightarrow{v_2} = m_1\overrightarrow{v_1}' + m_2\overrightarrow{v_2}'$$

$$m_1\overrightarrow{v_1}^2 + m_2\overrightarrow{v_2}^2 = m_1\overrightarrow{v_1}'^2 + m_2\overrightarrow{v_2}'^2 + 2Q$$

savršeno ne-elastični sudar

$$Q \to Q_{MAX}$$

$$\overrightarrow{v_1}' = \overrightarrow{v_2}' = \frac{m_1\overrightarrow{v_1} + m_2\overrightarrow{v_2}}{m_1 + m_2}$$

Zadatak

na križanju se događa savršeni ne-elastični sudar, $m_1 = 1800\ kg$ nalijeće s $\overrightarrow{v_1} = 60\frac{km}{h}$ gibajući se na istok. $m_2 = 2400\ kg; \overrightarrow{v_2} = 30\frac{km}{h}$ i giba se u smjeru sjevera. Kolikom će se brzinom nakon sudara gibati „olupina“?

$$\vec{v}' = \frac{m_1\overrightarrow{v_1} + m_2\overrightarrow{v_2}}{m_1 + m_2} = \frac{m_1\overrightarrow{v_1}\,\vec{\imath} + m_2\overrightarrow{v_2}\,\vec{\jmath}}{m_1 + m_2} = \frac{\overrightarrow{v_1}\,\vec{\imath} + \frac{4}{3}\overrightarrow{v_2}\,\vec{\jmath}}{\frac{7}{3}} = 25.71\vec{\imath} + 17.14\,\vec{\jmath} \to v = 30.9\frac{km}{h}$$

SUSTAV N ČESTICA

-svaka čestica ima masu i položaj

i=1,...,N

ukupna masa

$$M = \sum_{i=1}^{N} m_i$$

središte mase

$$\vec{R} = \frac{1}{M}\sum_i m_i \vec{r_i}$$

brzina središta mase

$$\vec{V} = \dot{\vec{R}} = \frac{1}{M}\sum_i m_i \vec{v_i} = \frac{1}{M}\sum_i \vec{p_i} = \frac{1}{M}\underline{\vec{P}}$$

zbroj količina gibanja

ukupna količina gibanja P određena je kao zbroj količina gibanja p;

$$\underline{\vec{P}} = \sum_i \vec{p}_i = M\vec{V}$$

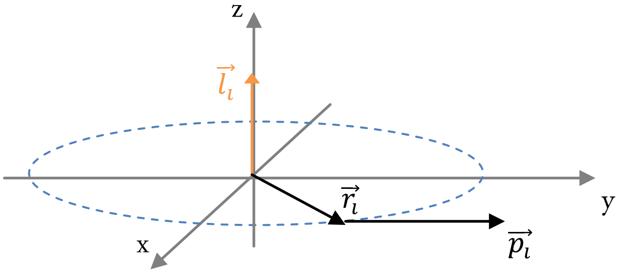
Zakon gibanja središta mase

$$\frac{\mathrm{d}}{\mathrm{dt}}\underline{\vec{P}} = M\dot{\vec{V}} = \sum_i \vec{F}_{iEXT} = \vec{F}_{EXT}$$

$$\underline{\vec{P}} - \overrightarrow{konst.}\ za\ izolirani\ (zatvoreni) sustav$$

Kutna količina gibanja

$$\boldsymbol{\vec{l_i}} = \boldsymbol{\vec{r_i}} \times \boldsymbol{\vec{p_i}}$$



ukupna kutna količina gibanja čestica:

$$\vec{L} = \sum_i \vec{l_i} = \sum_i \vec{r_i} \times \vec{p_i} = \sum_i \vec{r_i} \times (m_i \vec{v_i})$$

$$\frac{d}{dt}\vec{L} = \sum_i \underbrace{\vec{v_i} \times (m_i \vec{v_i})}_{=0} + \sum_i \vec{r_i} \times (m_i \vec{a_i})$$

$$\vec{F_i} = \vec{F_i}_{EXT} + \sum_{j \neq i} \vec{F_{ij}}$$

$$\frac{d}{dt}\vec{L} = \sum_i \underbrace{\vec{r_i} \times \vec{F_i}_{EXT}}_{} + \sum_{i,j \neq i} \vec{r_i} \times \vec{F_{ij}}$$

moment sile koji djeluje na i-tu česticu $\equiv \vec{M_i}$

=0 (III. N. AKSIOM)

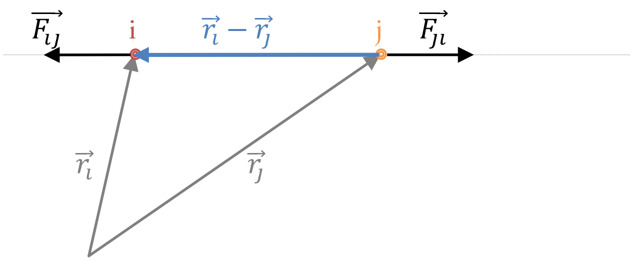
$$\boldsymbol{\frac{d}{dt}\vec{L} = \sum_i \vec{M_i}_{EXT}}$$

- vremenska promjena kutne količine gibanja jednaka je zbroju momenata vanjskih sila

dokaz:

$$\sum_{i,j \neq i} \vec{r_i} \times \vec{F_{ij}} = \sum_{i,j>i} \left(\vec{r_i} \times \vec{F_{ij}} + \vec{r_j} \times \vec{F_{ji}}\right) = \xrightarrow{3.\ aksiom\ (\vec{F_{ij}} = -\vec{F_{ji}})} = \sum_{i,j>i} \left(\vec{r_i} \times \vec{F_{ij}} + \vec{r_j}(-\vec{F_{ij}})\right)$$

$$= \sum_{i,j>i} \left((\vec{r_i} - \vec{r_j}) \times \vec{F_{ij}}\right) = 0 \ \blacksquare$$



$$\vec{l_i} = \vec{r_i} \times \vec{p_i}$$

$$\vec{M_i} = \vec{r_i} \times \vec{F_i}$$

$\vec{p_i}$	-MOMENTUM	-količina gibanja
$\vec{l_i}$	-ANGULAR MOMENTUM	-kutna količina gibanja
$\vec{F_i}$	-FORCE	-sila
$\vec{M_i}$	-TORQUE	-zaokretni moment (moment sile)

KRUTO TIJELO

sustav N čestica takav da

$$r_{ij} = |\vec{r_i} + \vec{r_j}| = konst.$$

- STUPNJEVI SLOBODE:

1 ČESTICA ... f=3 broj stupnjeva slobode

n ČESTICA ... **f=3n**

2 ČESTICE –KRUTO TIJELO ... f=3+2=5
3+ ČEST. – KRUTO TIJELO ... **f**=3+2+1=**6**

STRUKTURA KRUTOG TIJELA
pretpostavimo:

$$\underline{\vec{P}} = 0 = \vec{L}$$

zahtjevamo:

$$\underline{\dot{\vec{P}}} = 0 = \dot{\vec{L}}$$

$$\rightarrow \sum_i \vec{F_\iota}_{EXT} = 0 = \sum_i \vec{M_\iota}_{EXT}$$

TEŽIŠTE

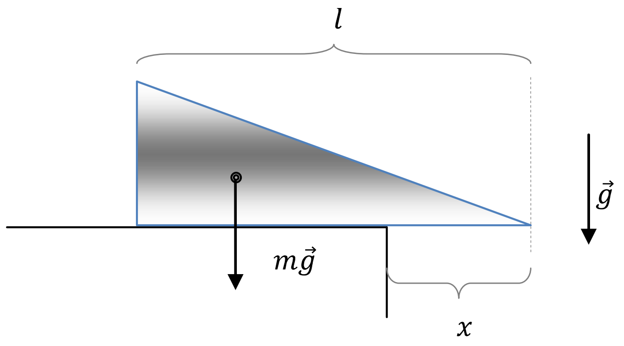
u homogenom polju gravitacijske sile vrijedi:
SREDIŠTE MASE=TEŽIŠTE!

-djelovanje gravitacijske sile na kruto tijelo može se poistovjetiti sa djelovanjem čitave težine tijela u središtu mase (u težištu)

$$\sum_i \vec{G_\iota} = \sum_i m_i \vec{g} = \vec{g} \cdot M$$

$$\sum_i \vec{r_\iota} \times \vec{G_\iota} = \sum_i \vec{r_\iota} \times (m_i \vec{g}) = \left(\sum_i m_i \vec{r_\iota} \right) \times \vec{g} = (M\vec{R}) \times \vec{g} = \vec{R} \times (M\vec{g})$$

ZADATAK – „KLIN“



$x_{MAX} = ?$

-koliko najviše smijemo gurnuti „klin“ prema desno prije nego se prevrne?

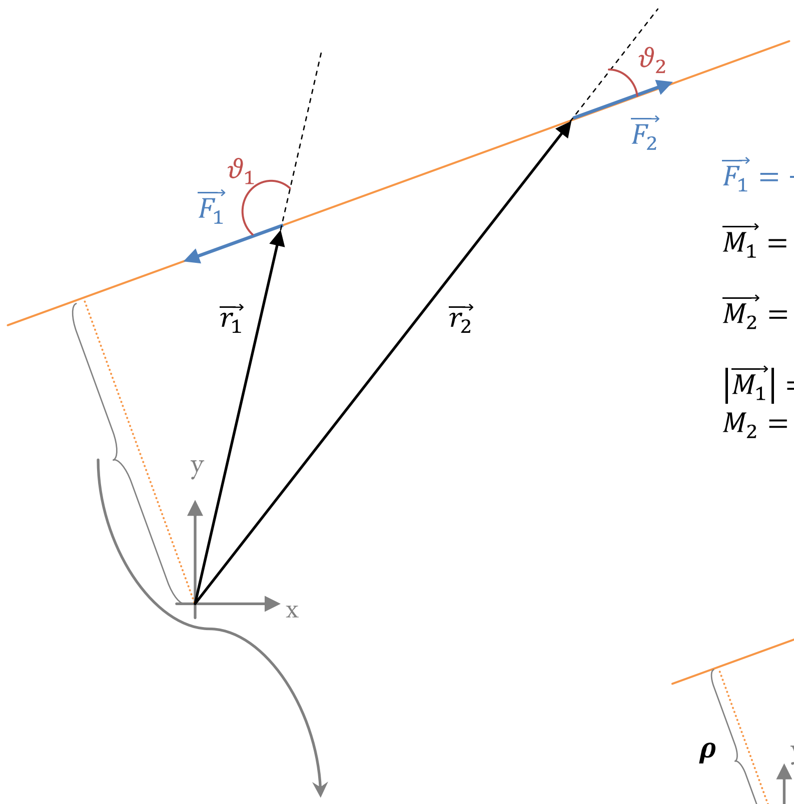
rješenje:

I. uvjet – zbroj svih sila mora biti 0

$$\sum_i \vec{F_i} = 0$$

II. uvjet – zbroj svih momenata mora biti jednak 0

$$\sum_i \vec{M_i} = 0$$



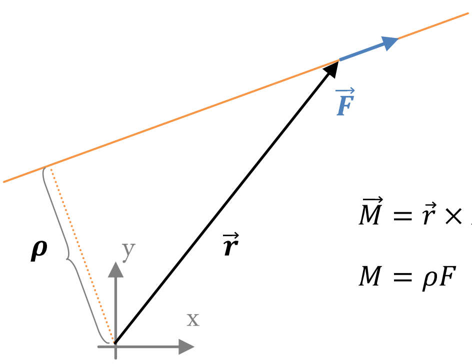
$$r_1 \sin\vartheta_1 = r_2 \sin\vartheta_2$$

$$\vec{F_1} = -\vec{F_2}$$

$$\vec{M_1} = \vec{r_1} \times \vec{F_1}$$

$$\vec{M_2} = \vec{r_2} \times \vec{F_2}$$

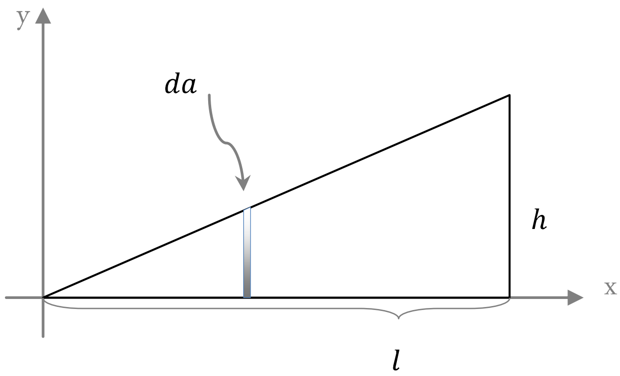
$$\left|\vec{M_1}\right| = M_1 = r_1 F_1 \sin\vartheta_1$$
$$M_2 = r_2 F_2 \sin\vartheta_2$$



$$\vec{M} = \vec{r} \times \vec{F} = \vec{\rho} \times \vec{F}$$

$$M = \rho F$$

- težište trokuta;



$$\vec{R} = \frac{1}{M}\sum_i m_i\vec{r_i}$$

$$x = \frac{1}{M}\sum_i m_i x_i$$

površinska gustoća mase

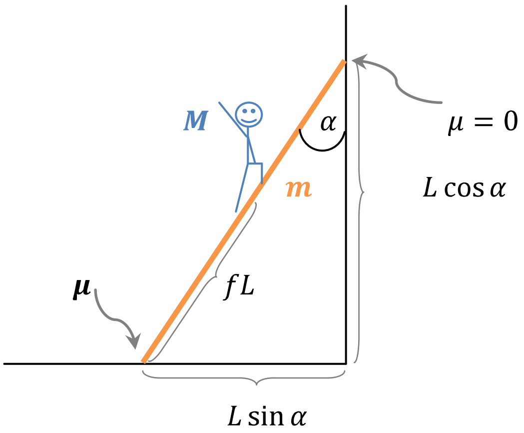
$$x = \frac{1}{M}\int \sigma x \, da$$

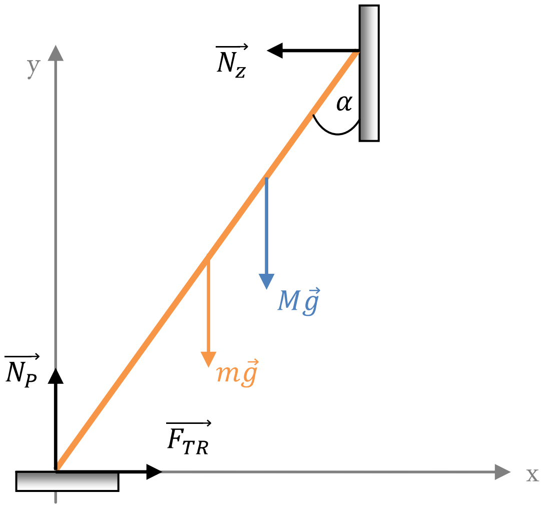
$$\sum_i m_i\vec{r_i} = \sum_i \Delta m_i\vec{r_i} = \int \vec{r}\, dm = \int \vec{r}\rho\, dV = \int \vec{r}\sigma\, da = \int \vec{r}\lambda\, dl$$

$$x = \frac{1}{M}\int \frac{M}{\frac{1}{2}lh} xh\frac{x}{l}\, dx \; = \frac{1}{M}\frac{2Mh}{l^2 h}\int x^2\, dx = \frac{2}{l^2}\int_0^1 x^2\, dx = \frac{2}{l^2}\frac{l^3}{3} = \frac{2}{3}l$$

$$x_{MAX} = \frac{2}{3}l$$

PRIMJER – LJESTVE





M - masa čovjeka
m - masa ljestvi
$\overrightarrow{N_P}$ - sila podloge, $\overrightarrow{N_Z}$ - sila zida
fL - na koji dio ljestvi se čovjek može popesti prije nego se one sruše?

kritično stanje:

$$\overrightarrow{F_{TR}} = \left(\overrightarrow{F_{TR}}\right)_{MAX} = \mu \overrightarrow{N_P}$$

uvjeti statike:

$$\sum_i \vec{F_i} = 0 \rightarrow \overrightarrow{N_P} = (m+M)\vec{g}$$

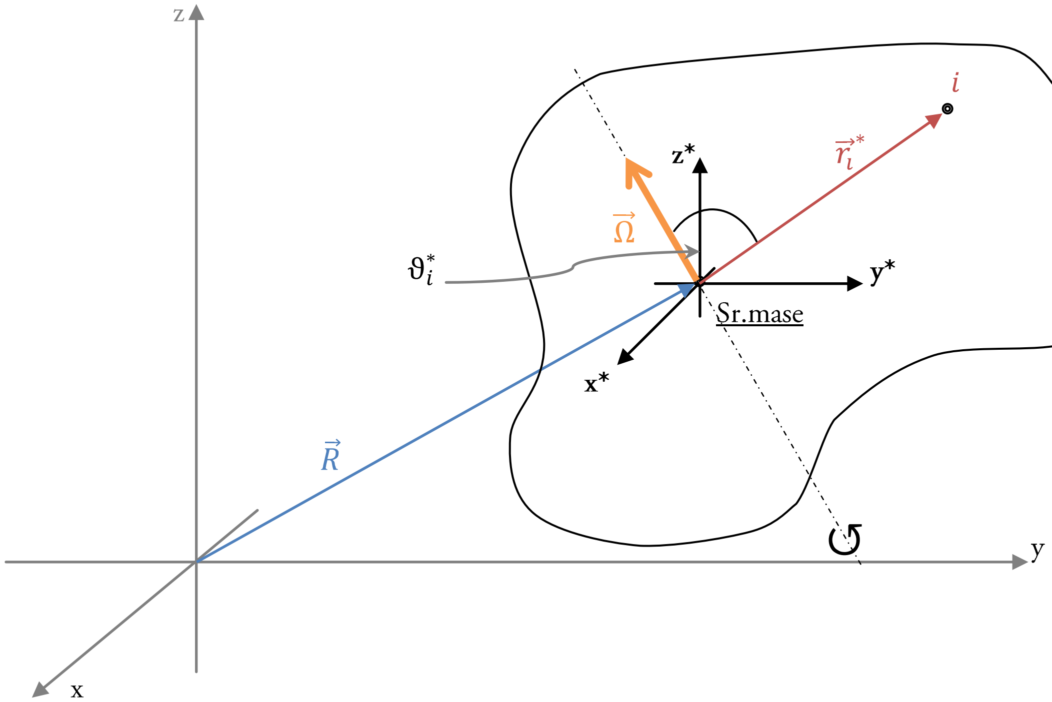
$$\mu \overrightarrow{N_P} = \overrightarrow{N_Z}$$

$$\sum_i \vec{M_i} = 0 \rightarrow \overrightarrow{N_Z} L\cos\alpha - m\vec{g}\frac{L}{2}\sin\alpha - M\vec{g}\, fL\sin\alpha = 0$$

$$\mu(m+M)\vec{g}\, L\cos\alpha - m\vec{g}\frac{L}{2}\sin\alpha - M\vec{g}\, fL\sin\alpha = 0$$

$$\mu(m+M) = \left(\frac{m}{2} + fM\right)\operatorname{tg}\alpha$$

DINAMIKA KRUTOG TIJELA



$\vec{R}$ - položaj središta mase

$\vec{r}_i^*$ - položaj i-te čestice u odnosu na središte mase

$\vec{V} \equiv \frac{d}{dt}\vec{R}$ - brzina središta mase

$\vec{\Omega}$ - kutna brzina rotacije krutog tijela oko osi koja prolazi središtem njegove mase

ϑ_i^* - kut između $\vec{\Omega}$ i $\vec{r}_i^*$

$\vec{v}_i^* = \dot{\vec{r}}_i^* = \vec{\Omega} \times \vec{r}_i^*$

$v_i^* = \Omega\, r_i^* \sin\vartheta_i^*$

$r_i^* \sin\vartheta_i^* \rightarrow \textit{udaljenost od osi}$

$$\sum_i m_i r_i^* = 0$$

$$\sum_i m_i \vec{v}_i^* = 0$$

POLOŽAJ I BRZINA:

$\vec{r}_i = \vec{R} + \vec{r}_i^*$

$\vec{v}_i = \vec{V} + \vec{v}_i^* = \vec{V} + \vec{\Omega} \times \vec{r}_i^*$

Ukupna E_{KIN} tijela koje se giba

$E_{KIN} \equiv T$

$$T = \sum_i T_i = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \sum_i m_i \left(\vec{V} + \vec{v}_i^*\right)^2 = \frac{1}{2} \sum_i m_i \left(v^2 + 2\vec{V}\vec{v}_i^* + \vec{v}_i^{*2}\right)$$

$$= \frac{1}{2} M V^2 + \vec{V} \underbrace{\left(\sum_i m_i \vec{v}_i^*\right)}_{=0} + \underbrace{\sum_i \boldsymbol{\frac{1}{2} m_i v_i^{*2}}}_{T^*}$$

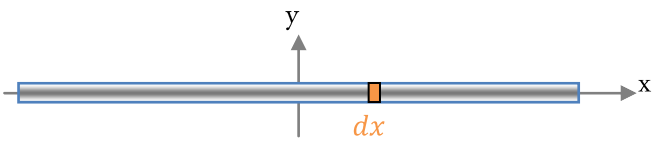
$$T^* = \sum_i \frac{1}{2} m_i v_i^{*2} = \frac{1}{2} \sum_i m_i (\Omega r_i^* \sin \vartheta_i^*)^2 = \frac{1}{2} \sum_i m_i \Omega^2 \mathrm{r}_i^{*2} \sin^2 \vartheta_i^* = \frac{1}{2} \left[\sum_i m_i \underbrace{(r_i^* \sin \vartheta_i^*)^2}\right] \Omega^2$$

$\equiv \rho_{i^*}$ - udaljenost i-te čestice od osi

$$= \frac{1}{2} \underbrace{\left(\sum_i m_i (\rho_{i^*})^2\right)}_{} \Omega^2$$

$\equiv$ I^* - moment tromosti tijela u odnosu na os (kroz središte mase)

Primjeri– štap

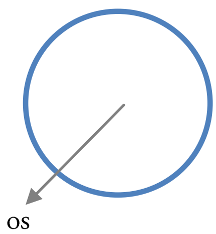


M – masa , L – duljina , λ - linijska gustoća mase $= \frac{M}{L}$

$$I^* = \sum_i m_i (\rho_{i^*})^2 = \int_{-\frac{L}{2}}^{\frac{L}{2}} (\rho_{i^*})^2 \, dm = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \lambda \, dx = \lambda \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \, dx = 2\lambda \int_{0}^{\frac{L}{2}} x^2 \, dx = 2\lambda \frac{\left(\frac{L}{2}\right)^3}{3} = 2 \frac{M}{L} \left(\frac{\frac{L^3}{8}}{3}\right) = \frac{1}{12} M L^2$$

- ista stvar ako imamo ploču, samo tada uvrštavamo njenu površinsku gustoću σ

Primjeri– obruč

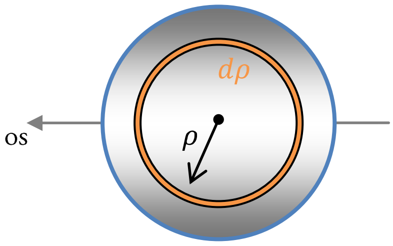


M, R

$$I^* = \sum_i m_i (\rho_{i^*})^2 = I^* = \sum_i m_i R^2 = M R^2$$

- ista stvar ako imamo valjak

PRIMJERI– DISK

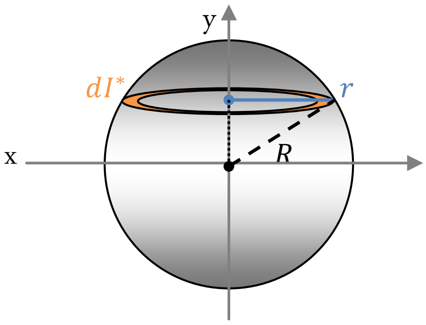


$$M, R, \sigma = \frac{M}{R^2\pi}$$

$$da = 2\rho\pi\, d\rho$$

$$I^* = \int \rho_i^2\, dm = \int_0^R \rho^2 \sigma\, da = \int_0^R \rho^2 \frac{M}{R^2\pi} 2\rho\pi\, d\rho = \frac{2M}{R^2}\int_0^R \rho^3\, d\rho = \frac{2M}{R^2}\left(\frac{R^4}{4}\right) = \frac{MR^2}{2}$$

PRIMJERI– KUGLA



$$M, R, \rho = \frac{M}{\frac{4}{3}R^3\pi}$$

$$dV = r^2\pi\, dy$$

$$R^2 = r^2 + y^2 \rightarrow r^2 = R^2 - y^2 \rightarrow r^4 = (R^2 - y^2)^2$$

$$I^* = \int dI^* = I^* = I^* = \int \frac{1}{2} r^2 dm = I^* = \int \frac{1}{2} r^2 \rho\, dV = \int \frac{1}{2} r^2 \rho\, r^2 \pi\, dy = \frac{1}{2}\rho\pi \int r^4\, dy$$

$$= \frac{1}{2}\rho\pi \int_{-R}^{R} (R^2 - y^2)^2 dy = \rho\pi \int_0^R (R^2 - y^2)^2 dy = \rho\pi \int_0^R (R^4 - 2R^2y^2 + y^4) dy = \rho\pi\left(R^5 - \frac{2}{3}R^5 + \frac{1}{5}R^5\right)$$

$$= \frac{3M}{4R^3\pi}\pi\left(R^5 - \frac{2}{3}R^5 + \frac{1}{5}R^5\right) = \frac{3M}{4}R^2\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{2}{5}MR^2$$

KUTNA KOLIČINA GIBANJA

$$\vec{L} = \sum_i \vec{l_i} = \sum_i \vec{r_i} \times \vec{p_i} = \sum_i \vec{r_i} \times (m_i \vec{v_i}) = \sum_i \vec{r_i} \times \left(m_i \vec{\mathrm{V}} + \mathrm{m_i} \vec{\Omega} \times \vec{r_i}^*\right)$$

$$= \sum_i m_i \vec{r_i} \times \vec{\mathrm{V}} \qquad + \sum_i m_i \left(\vec{r_i} \times \left(\vec{\Omega} \times \vec{r_i}^*\right)\right)$$

$$= \left(\sum_i m_i \vec{r_i}\right) \times \vec{\mathrm{V}} \qquad + (\dots)$$

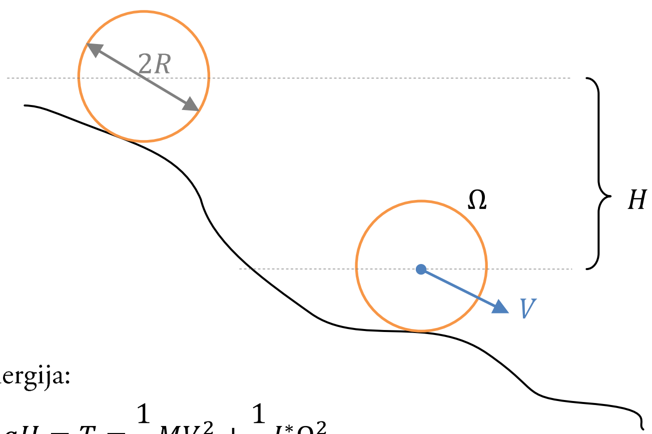
$$= M\vec{R} \times \vec{\mathrm{V}} \qquad + (\dots)$$ bez izvoda

$$= \vec{R} \times \left(M\vec{\mathrm{V}}\right) \qquad + (\dots)$$

$$= \vec{R} \times \underline{P} \qquad + I^* \vec{\Omega} \qquad + (\text{dodatni člnanovi okomiti na } \vec{\Omega})$$

ZADATAK – KOTRLJANJE

Homogena kugla, disk i obruč kreću iz mirovanja u kotrljanje (bez klizanja) niz padinu.
Nađi iznos brzine središta mase nakon visinske razlike H!



energija:

$$MgH = T = \frac{1}{2} MV^2 + \frac{1}{2} I^* \Omega^2$$

uvjet kotrljanja:

$$V = \Omega \mathrm{R} \rightarrow \Omega = \frac{\mathrm{V}}{\mathrm{R}}$$

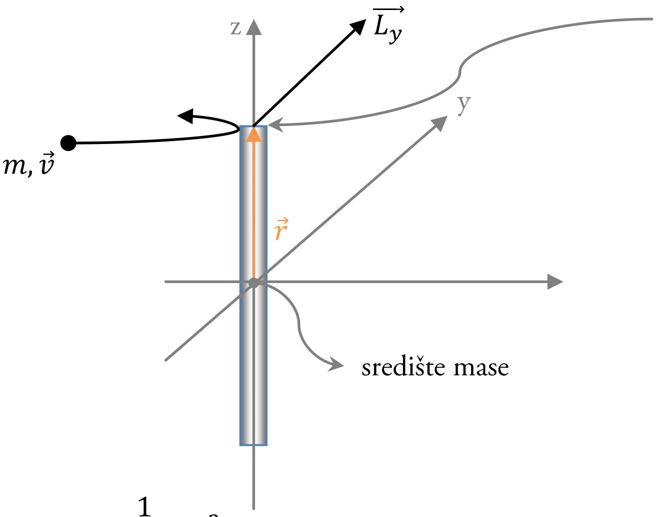
$$MgH = \frac{1}{2} MV^2 + \frac{1}{2} I^* \Omega \left(\frac{\mathrm{V}}{\mathrm{R}}\right)^2 = \frac{1}{2}\left(\mathrm{M} + \frac{\mathrm{I}^*}{\mathrm{R}^2}\right) V^2$$

$$V = \sqrt{\frac{MgH}{\frac{M}{2}\left(1 + \frac{I^*}{MR^2}\right)}} = \sqrt{\frac{2gH}{1 + \frac{I^*}{MR^2}}}$$

$$\frac{I^*}{MR^2} = \begin{cases} 1 \dots obruč \\ \frac{1}{2} \dots disk \\ \frac{2}{5} \dots kugla \end{cases}$$

PRIMJER – ŠTAP I KUGLICA

- štap u svemiru je pogođen kuglicom...



sudar: savršeno elastičan i „centralan“

$$M, b, I^* = \frac{1}{12}Mb^2$$

$\vec{V} = 0, \vec{\Omega} = 0$ $\qquad$ $V_x' = ?, \Omega_y' = ?$

očuvane veličine:

-kinetička energija

$$T = \frac{1}{2}mv_x^2 = \frac{1}{2}mv_x'^2 = \frac{1}{2}MV_x'^2 + \frac{1}{2}b^*\Omega_y'^2$$

-količina gibanja

$$P_x = mv_x = mv_x' + MV_x'$$

-kutna količina gibanja

$$L_y = \frac{b}{2}mv_x = \frac{b}{2}mv_x' + I^*\Omega_y'$$

rješenje 1:

$$v_x = v_x', v_x' = 0 = \Omega_y'$$

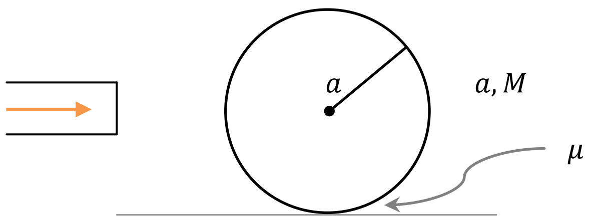
rješenje 2:

$$v_x' = \frac{1 - 4\frac{m}{M}}{1 + 4\frac{m}{M}} v_x$$

$$V_x' = \frac{2\frac{m}{M}}{1 + 4\frac{m}{M}} v_x$$

$$\Omega_y' = \frac{12\frac{m}{M}}{1 + 4\frac{m}{M}} \frac{v_x}{b}$$

PRIMJER – BILJARSKA KUGLA



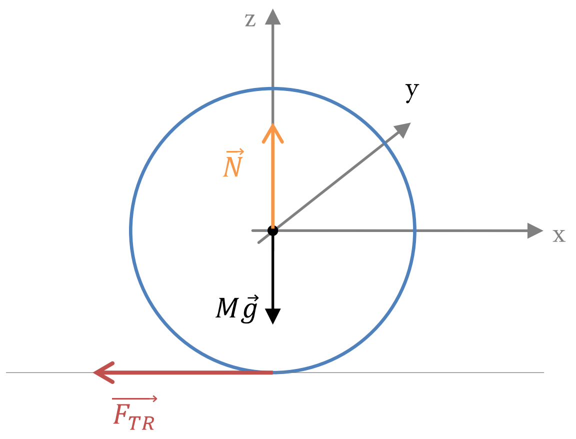
početni uvjet:

$V_0 > 0$
$\Omega_0 = 0$

-kugla je udarena štapom te klizi po podlozi u prvoj fazi; druga faza – kotrljanje. Koliko traje prva faza „struganja“?

općenito:

$$\dot{\vec{P}} = \vec{F}_{EXT}$$
$$\dot{\vec{L}} = \vec{M}_{EXT}$$



$$\overrightarrow{F_{TR}} = -\mu M g\vec{\imath} = \vec{F}_{EXT}$$

$$\vec{M}_{EXT} = \vec{r} \times \vec{F}_{EXT} = \left(-a\vec{k}\right) \times (-\mu M g\vec{\imath}) = a\mu M g\vec{\jmath}$$

jednadžba gibanja:

$$M\dot{V}_x = -\mu M g$$
$$I^*\dot{\Omega}_y = \mu M g a$$

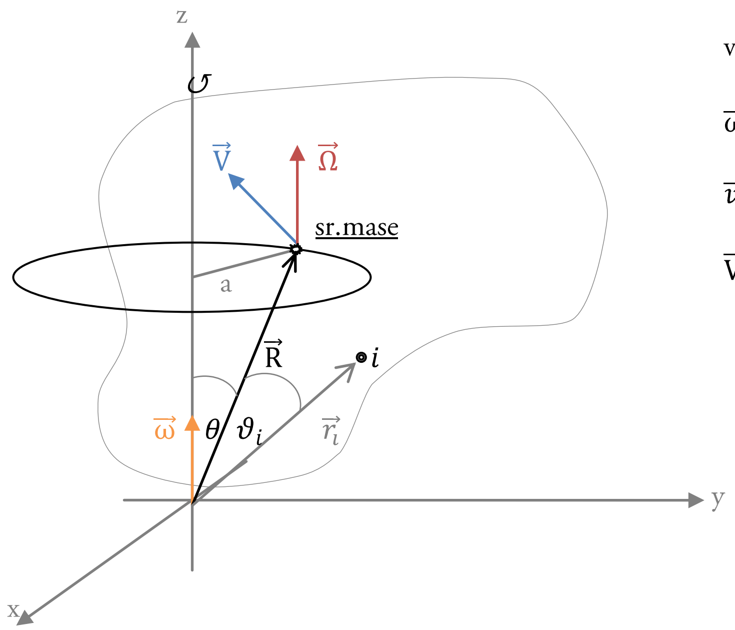
$$\dot{V}_x = -\mu g \rightarrow V_x(t) = V_0 - \mu g t$$

$$\dot{\Omega}_y = \frac{5}{2}\frac{\mu g}{a} \rightarrow \Omega_y(t) = \frac{5}{2}\frac{\mu g}{a}t$$

prva faza završava kada je ispunjen uvjet kotrljanja bez klizanja tj. $V_x = \Omega_y a$

$$\tau = \frac{2V_0}{7\mu g}$$

VRTNJA KRUTOG TIJELA OKO ČVRSTE OSI



važno!

$\vec{\omega} = \vec{\Omega}$

$\vec{v_i} = \vec{\omega} \times \vec{r_i}$

$\vec{V} = \vec{\omega} \times \vec{R}$

KINETIČKA ENERGIJA

$$T = \sum_i T_i = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i (\vec{\omega} \times \vec{r_i})^2 = \sum_i \frac{1}{2} m_i \omega^2 \underbrace{r_i^2 \sin^2 \vartheta_i}$$

ρ_i^2 – udaljenost čestice od osi vrtnje

$$= \frac{1}{2}\left(\sum_i m_i \rho_i^2\right)\omega^2 = \frac{1}{2} I \omega^2$$

I - moment tromosti krutog tijela u odnosu na čvrstu os vrtnje

$$T = \frac{1}{2} M V^2 + \frac{1}{2} I^* \Omega^2 = \frac{1}{2} M (a\omega)^2 + \frac{1}{2} I^* \Omega^2 = \frac{1}{2} (Ma^2 + I^*)\omega^2$$

I

$$I = \sum_i m_i \rho_i^2 = Ma^2 + I^*$$

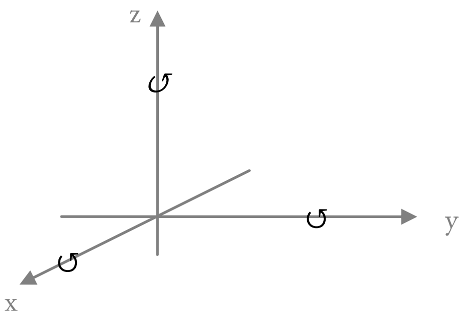
M – ukupna masa
a - udaljenost središta mase od čvrste osi
I^* - moment tromosti u odnosu na os kroz središte mase koja je **paralelna** s osi vrtnje

→ TEOREM O PARALELNIM OSIMA / STEINEROV TEOREM

$\boldsymbol{I = Ma^2 + I^*}$

Teorem o okomitim osima

- za ravninska tijela
- neka je M raspoređena u x,y-ravnini

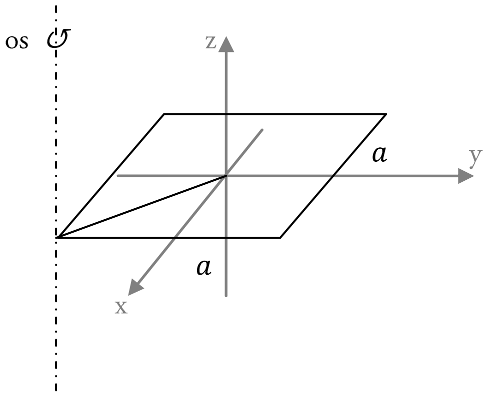


$$\boldsymbol{I_z = I_x + I_y}$$

dokaz

$$I_z = \int (x^2 + y^2) dm = \int x^2 dm + \int y^2 dm = I_x + I_y$$

PRIMJER



Odredi moment tromosti homogenog kvadrata u odnosu na os okomitu na kvadrat, a koja prolazi vrhom kvadrata

$$I_z^* = I_x^* + I_y^* = \frac{1}{12} M a^2 + \frac{1}{12} M a^2 = \frac{1}{6} M a^2$$

$$I_{os} = M \left(\frac{1}{2} a\sqrt{2} \right)^2 + I_z^* = \frac{M}{2} a^2 + \frac{1}{6} M a^2 = \frac{2}{3} M a^2$$

Rad vanjskih sila

$$dW = \sum_i \vec{F}_{i_{EXT}} \cdot d\vec{r_i} = \sum_i \vec{F}_{i_{EXT}} \cdot \vec{v_i} dt$$

- pri rotaciji krutog tijela oko čvrste osi: $\vec{v_i} = \vec{\omega} \times \vec{R}$

$$= \sum_i \vec{F}_{i_{EXT}} \cdot (\vec{\omega} \times \vec{r_i}) dt$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})$$

$$dW = \sum_i \vec{\omega}(\vec{r_i} \times \vec{F}_{i_{EXT}})\, dt = \left(\sum_i \vec{M}_{i_{EXT}} \right) \cdot (\vec{\omega} dt) = \vec{M}_{EXT} \cdot (\vec{\omega} dt) = M_{z_{EXT}} d\Phi$$

Jednadžba gibanja (iz teorema o radu i kin. energiji)

$$dT = d\left(\frac{1}{2} I \omega^2\right) = \frac{1}{2} I \cdot 2\omega\, d\omega = I\omega\, d\omega$$

$$dW = dT$$

$$M_{z_{EXT}} d\Phi = I\omega\, d\omega \qquad / \cdot \frac{1}{dt}$$

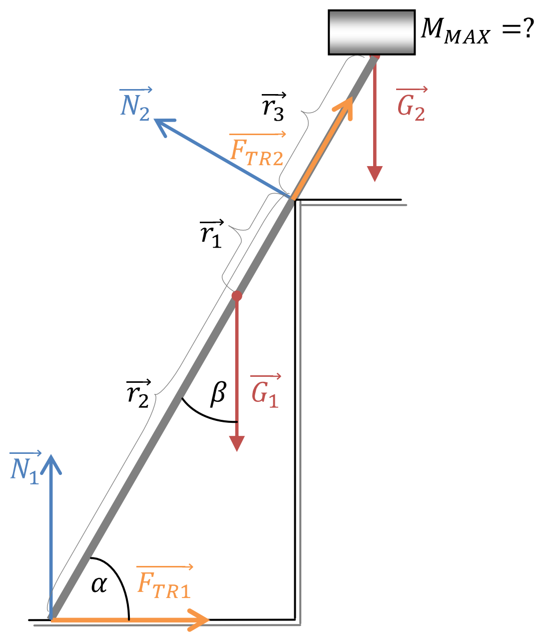
$$M_{z_{EXT}} \frac{d\Phi}{\mathrm{dt}} = I\omega \frac{d\omega}{dt}$$

$$M_{z_{EXT}} \omega = I\omega\alpha$$

$$\boldsymbol{M_{z_{EXT}} = I\alpha}$$

ZADATAK 1

Homogena greda $m = 1\ kg$ duljine l, naslonjena je na zid prema slici; kut nagiba $\alpha = 60°$, koef. trenja $\mu_1 = 1.1$ na tlu a na zidu je $\mu_2 = 0.2$. Koliki smije biti M da sustav ostane u ravnoteži?



$$\sum \vec{F} = 0 \rightarrow \vec{N_1} + \vec{F_{TR1}} + \vec{G_1} + \vec{N_2} + \vec{F_{TR2}} + \vec{G_2}$$

$$\sum \vec{M} = 0$$

$$r_1 = \frac{2}{3}l - \frac{l}{2} = \frac{1}{6}l$$
$$r_2 = \frac{2}{3}l$$
$$r_3 = \frac{1}{3}l$$

$\hat{x}$: $F_{TR1} + F_{TR2}\cos\alpha - N_2 \sin\alpha = 0$

$\hat{y}$: $N_1 + N_2\cos\alpha + F_{TR2}\sin\alpha - G_1 - G_2 = 0$

$$\vec{M_0} = \vec{r_1} \times \vec{G_1} + \vec{r_2} \times \vec{N_1} + \vec{r_2} \times \vec{F_{TR1}} + \vec{r_3} \times \vec{G_2} = 0$$

$$r_1 G_1 \sin\beta\,\hat{k} + r_2 N_1 \sin(90+\alpha)\left(-\hat{k}\right) + r_2 F_{TR1}\cos\beta\,\hat{k} + r_3 G_2 \sin(90+\alpha)\left(-\hat{k}\right) = 0$$

$F_{TR1} = \mu_1 N_1,\quad F_{TR2} = \mu_2 N_2$

$\hat{x}$: $\mu_1 N_1 + \mu_2 N_2 \cos\alpha - N_2 \sin\alpha = 0$

$$\mu_1 N_1 = N_2(\sin\alpha - \mu_2\cos\alpha) \rightarrow N_1 = N_2\frac{(\sin\alpha - \mu_2\cos\alpha)}{\mu_1}$$

$\hat{y}$: $G_2 + G_1 = N_1 + (\cos\alpha + \mu_2\sin\alpha)N_1\dfrac{\mu_1}{(\sin\alpha - \mu_2\cos\alpha)}$

$$G_2 + G_1 = N_1\left(1 + \mu_1\frac{(\cos\alpha + \mu_2\sin\alpha)}{(\sin\alpha - \mu_2\cos\alpha)}\right)$$

$$\frac{1}{6}l\,G_1\sin\beta\,\hat{k} + \frac{2}{3}lN_1\sin(90+\alpha)\left(-\hat{k}\right) + \frac{2}{3}l\mu_1 N_1\cos\beta\,\hat{k} + \frac{1}{3}lG_2\sin(90+\alpha)\left(-\hat{k}\right) = 0$$

$$\frac{1}{6}\,G_1\cos\alpha + \frac{2}{3}\mu_1 N_1\sin\alpha - \frac{2}{3}N_1\cos\,\alpha - \frac{1}{3}G_2\cos\alpha = 0$$

$$G_1 + 4\mu_1 N_1\tan\alpha - 4N_1 - 2G_2 = 0$$

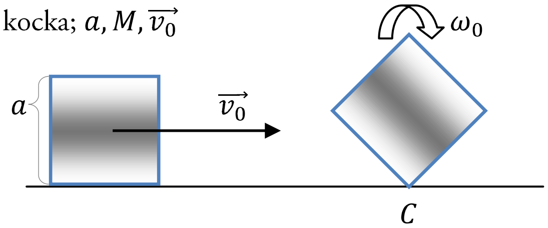
$$G_1 - 2G_2 + 4N_1(\mu_1\tan\alpha - 1) = 0$$

$$G_1 - 2G_2 + \frac{4(G_2 + G_1)}{\left(1 + \mu_1\frac{(\cos\alpha + \mu_2\sin\alpha)}{(\sin\alpha - \mu_2\cos\alpha)}\right)}(\mu_1\tan\alpha - 1) = 0$$

$G_2 = 176.01\ N \rightarrow \boldsymbol{M = 17.94\ kg}$

PRIMJER 4.16

kocka; $a, M, \vec{v_0}$ ω_0 $\vec{v_0}_{MIN}$ da se kocka prevrne



$$\vec{L} = \vec{r} \times \vec{p} \to L = \frac{a}{2} M v_0 = I\omega_0 = \frac{2}{3} Ma^2 \omega_0 \to v_0 = \frac{4}{3} a\omega_0$$

$$\vec{L} = I\vec{\omega}$$

$$\vec{M} = \frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha}$$

$$E_{KIN} = \frac{1}{2} I{\omega_0}^2 = E_P\left(\frac{d}{2} - \frac{a}{2}\right) \to \frac{1}{2} \cdot \frac{2}{3} Ma^2{\omega_0}^2 = mg\frac{a}{2}\left(\sqrt{2} - 1\right)$$

$$v_0 = \sqrt{\frac{8}{3} ga\left(\sqrt{2} - 1\right)}$$

JEDNADŽBA GIBANJA (PREKO KUTNE KOLIČINE GIBANJA)

$$\vec{L} = \vec{R} \times \underline{\vec{P}} + I^*\vec{\Omega} + [\#]\left(\text{dodatni članovi okomiti na } \vec{\Omega}\right) = \vec{R} \times M\left(\vec{\omega} \times \vec{R}\right) + I^*\vec{\Omega} + [\#]$$

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = (\vec{a}\vec{c})\vec{b} - \left(\vec{a}\vec{b}\right)\vec{c}$$

$$\vec{L} = M\vec{R}^2\vec{\omega} - M\left(\vec{R}\vec{\omega}\right)\vec{R} + I^*\vec{\Omega} + [\#]$$

z komponenta:

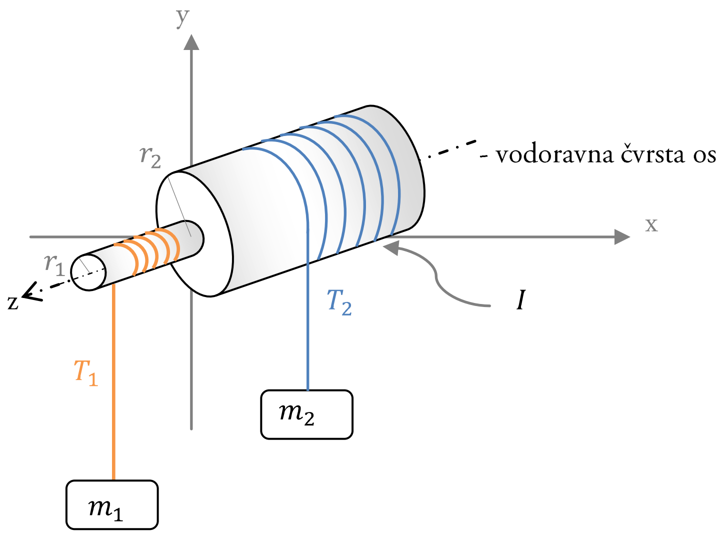
$$L_z = M\vec{R}^2\vec{\omega}[1 - cos^2\theta] + I^*\Omega = M(R\sin\theta)^2\omega + I^*\Omega = Ma^2\omega + I^*\Omega = (Ma^2 + I^*)\omega = I\omega$$

općenito:

$$\frac{d}{dt}\vec{L} = \vec{M}_{EXT}$$

$$\frac{d}{dt}L_z = \frac{d}{dt}(I\omega) = I\frac{d\omega}{dt} = I\alpha = M_{z\,EXT}$$

ZADATAK – „VRETENO“



odredi kutnu akceleraciju valjka!

odredi napetost niti!

$$M_{z_{EXT}} = r_1 T_1 - R_2 T_2 = I\alpha$$

$$m_1 a_1 = T_1 - m_1 \vec{g}$$
$$m_2 a_2 = T_2 - m_2 \vec{g}$$

$$a_1 = -\alpha r_1$$
$$a_2 = \alpha r_2$$

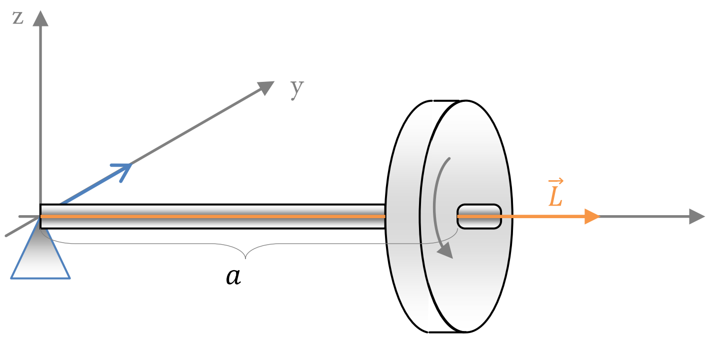
$$T_1 = m_1(a_1 + g)$$
$$T_2 = m_2(a_2 + g)$$

$$m_1(-\alpha r_1 + g) = m_2(\alpha r_2 + g)$$
$$r_1 m_1(-\alpha r_1 + g) - r_2 m_2(\alpha r_2 + g) = \alpha I$$

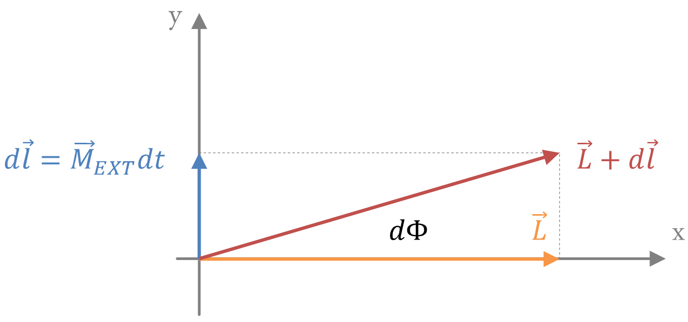
$$\alpha = \frac{m_1 g r_1 - m_2 g r_2}{I + m_1 r_1^2 + m_2 r_2^2}$$

$$T_{1/2} = m_{1/2} g \mp m_{1/2} r_{1/2} \alpha$$

PRECESIJA ZVRKA



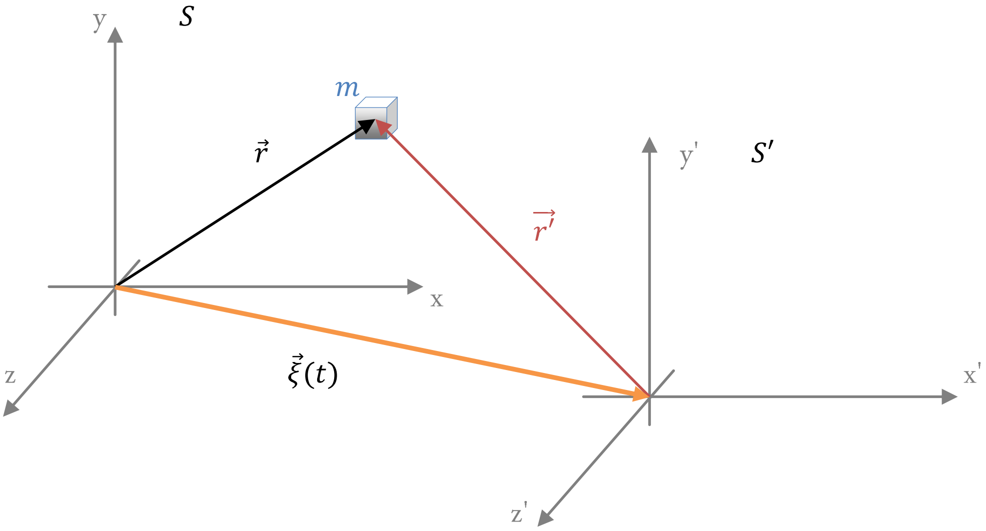
a - udaljenost od uporišta do središta mase pogled odozgo



$M_{ext}dt = L_0 d\Phi$

$$\omega_{prec} = \frac{d\Phi}{\mathrm{dt}} = \frac{\mathrm{M_{EXT}}}{\mathrm{L_0}} = \frac{mga}{I^{*}\omega_{rot.zvrka}}$$

Ne-inercijalni sustavi



sustav $S(x, y, z)$ je inercijalan

$\Leftrightarrow \vec{F} = m\vec{a}$

$\vec{F}$ - stvarna sila

$\vec{a}$ - akceleracija čestice u sustavu S

sustav $S'(x', y', z')$ je pomaknut za $\vec{\xi}(t)$

$\vec{r}$ - položaj u sustavu S

$\vec{r'}$ - položaj u sustavu S'

vrijedi:

$\vec{r} = \vec{\xi} + \overrightarrow{r'}$

za $t \simeq t_0$

$\vec{\xi}(t) = \vec{\xi}(t_0) + \dot{\vec{\xi}}(t_0)(t - t_0) + \frac{1}{2}\ddot{\vec{\xi}}(t - t_0)^2 + \cdots = \vec{\xi}(t_0) + \overrightarrow{v_0}(t - t_0) + \frac{1}{2}\overrightarrow{a_0}(t - t_0)^2 + \cdots$

ako $t_0 = 0$

$\vec{\xi}(t) = \vec{\xi}_0 + \overrightarrow{v_0}t + \frac{1}{2}\overrightarrow{a_0}t^2 + \cdots$

JEDNADŽBA GIBANJA U SUSTAVU S

$\vec{F} = m\vec{a} = m\frac{d^2}{dt^2}\vec{r} = m\frac{d^2}{dt^2}\left(\vec{\xi} + \overrightarrow{r'}\right) = m\left(\ddot{\vec{\xi}} + \ddot{\overrightarrow{r'}}\right) = m\left(\overrightarrow{a_0} + \overrightarrow{a'}\right)$

$\overrightarrow{a_0}$ - akceleracija sustava S' u odnosu na S
$\overrightarrow{a'}$ - akecleracija čestice u S'

JEDNADŽBA GIBANJA U SUSTAVU S' - D'ALAMBERTOV PRINCIP

$m\overrightarrow{a'} = \vec{F} - m\overrightarrow{a_0}$

PRIMJERI:

- auto ubrzava stalnom akceleracijom $\overrightarrow{a_0}$

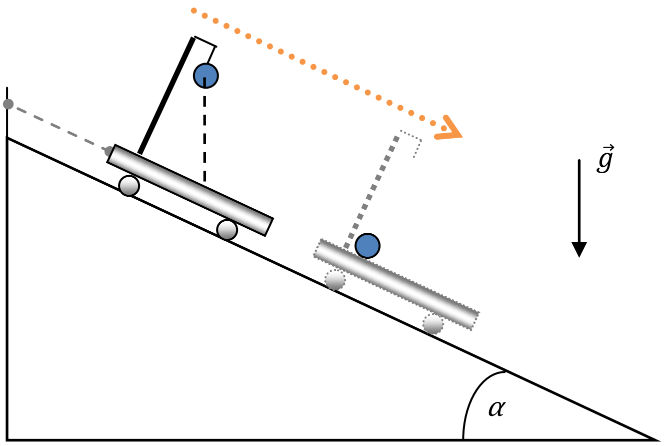
$m\overrightarrow{a'} = \vec{F} - m\overrightarrow{a_0}$
ako mirujemo u autu, onda $\overrightarrow{a'} = 0 \rightarrow \vec{F} = m\overrightarrow{a_0}$
- platforma koja se okreće ukupnom brzinom ω

$\overrightarrow{a_0} = -\omega^2\vec{r}$

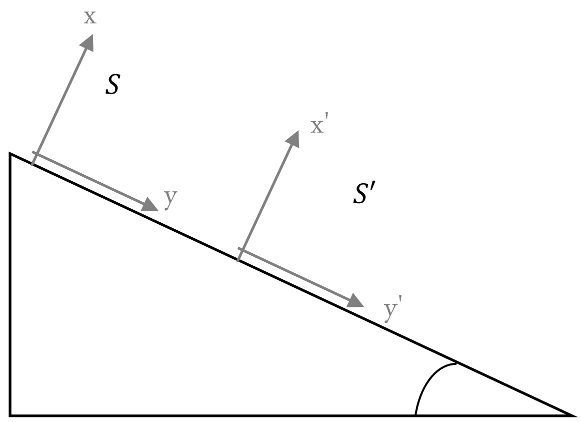
$m\overrightarrow{a'} = \vec{F} - m\overrightarrow{a_0} = \vec{F} + m\omega^2\vec{r}$

ako $\overrightarrow{a'} = 0 \rightarrow \vec{F} = -m\omega^2\vec{r}$

PRIMJER



ako istovremeno pustimo kuglu da pada i pustimo kolica u gibanje niz kosinu, kugla će pasti u podnožje „jarbola“



veza S i S'

$$\vec{\xi} = \frac{1}{2}\overrightarrow{a_0}t^2$$
$$\overrightarrow{a_0} = g\sin\alpha\,\vec{\imath}$$

Jednadžba gibanja u S'

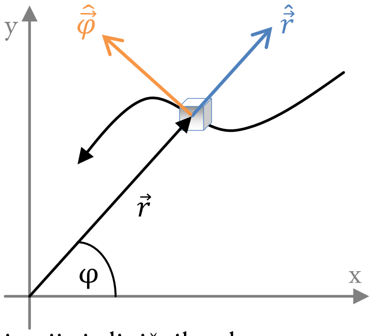
$m\overrightarrow{a'} = \vec{F} - m\overrightarrow{a_0}$ - **D'Alambertov princip**

$$\vec{F} = \vec{G} = mg(\sin\alpha\,\vec{\imath} - \cos\alpha\,\vec{\jmath})$$

$$m\overrightarrow{a'} = mg(\sin\alpha\,\vec{\imath} - \cos\alpha\,\vec{\jmath}) - mg\sin\alpha\,\vec{\imath} = -mg\cos\alpha\,\vec{\jmath}$$

CORIOLISOVA SILA

općenito:



jedinični vektori u polarnom koordinatnom sustavu

derivacije jediničnih vektora:

$$\dot{\hat{\vec{r}}} = \dot{\varphi}\hat{\vec{\varphi}}$$
$$\dot{\hat{\vec{\varphi}}} = -\dot{\varphi}\hat{\vec{r}}$$

položaj:

$$\vec{r} = r\hat{\vec{r}}$$

brzina:

$$\vec{v} = \dot{\vec{r}} = \dot{r}\hat{\vec{r}} + r\dot{\hat{\vec{r}}} = \dot{r}\hat{\vec{r}} + r\dot{\varphi}\hat{\vec{\varphi}}$$

akceleracija:

$$\overrightarrow{a_0} = \dot{\vec{v}} = \ddot{r}\hat{\vec{r}} + \dot{r}\dot{\hat{\vec{r}}} + \dot{r}\dot{\varphi}\hat{\vec{\varphi}} + r\ddot{\varphi}\hat{\vec{\varphi}} + r\dot{\varphi}\dot{\hat{\vec{\varphi}}} = \ddot{r}\hat{\vec{r}} + \dot{r}\dot{\varphi}\hat{\vec{\varphi}} + \dot{r}\dot{\varphi}\hat{\vec{\varphi}} + r\ddot{\varphi}\hat{\vec{\varphi}} + r\dot{\varphi}\left(-\dot{\varphi}\hat{\vec{r}}\right)$$

$$= \hat{\vec{r}}(\ddot{r} - r\dot{\varphi}^2) + \hat{\vec{\varphi}}(2\dot{r}\dot{\varphi} + r\ddot{\varphi}) \qquad (!)$$

posebni slučaj:

$$\left.\begin{array}{l}\dot{\varphi} = \omega = konst. \rightarrow \ddot{\varphi} = 0 \\ \dot{r} = v_r = konst. \rightarrow \ddot{r} = 0\end{array}\right. \rightarrow (!)$$

$$\overrightarrow{a_0} = -r\dot{\varphi}^2\hat{\vec{r}} + 2\dot{r}\dot{\varphi}\hat{\vec{\varphi}} = -r\omega^2\hat{\vec{r}} + 2v_r\omega\hat{\vec{\varphi}}$$

FIKTIVNE SILE (D'A)

$$\overrightarrow{F_{D'A}} = -m\overrightarrow{a_0} = mr\omega^2\hat{\vec{r}} - 2mv_r\omega\hat{\vec{\varphi}}$$

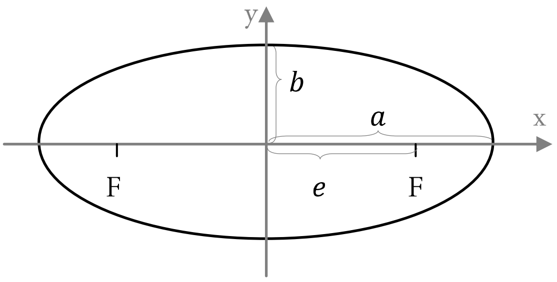
$mr\omega^2\hat{\vec{r}}$ - $\overrightarrow{F_{cp}}$

$2mv_r\omega\hat{\vec{\varphi}}$ - Coriolisova sila

GRAVITACIJA

- Keplerovi zakoni:

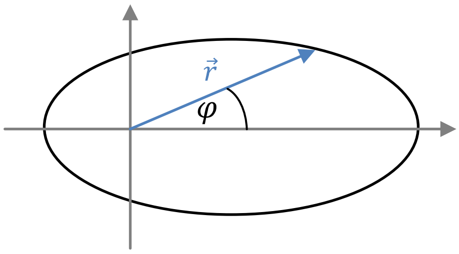
PRVI KEPLEROV TAKON – svi planeti gibaju se po elipsama kojima je u jednom od žarišta Sunce
DRUGI KEPLEROV ZAKON – radijus-vektor Sunce-planet prelazi u jednakim vremenskim razmacima jednake površine
TREĆI KEPLEROV ZAKON – kvadrati ophodnih vremena planeta proporcionalni su kubovima njihovih srednjih udaljenosti od Sunca



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$e=\sqrt{a^2-b^2}\ \ za\ a>b$$

$$\varepsilon=\frac{e}{a}=\frac{\sqrt{a^2-b^2}}{a}$$



$$x=r\cos\varphi$$
$$y=r\sin\varphi$$

elipsa

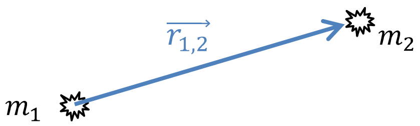
$$r=\frac{a(1-\varepsilon^2)}{1-\varepsilon\cos\varphi}$$

OPĆI ZAKON GRAVITACIJE

-Newton, 1687.

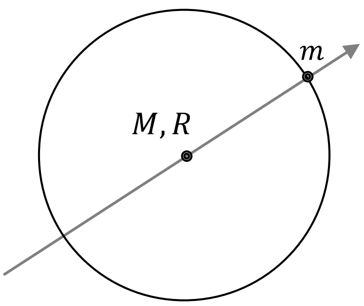
$\overrightarrow{F_{1,2}}$ - sila na m_1

$\overrightarrow{F_{2,1}}$ - sila na m_2



$$G=6.673\cdot10^{-11}\frac{Nm^2}{kg^2}$$

$$\overrightarrow{F_{1,2}}=G\frac{m_1m_2}{\left|\overrightarrow{r_{1,2}}\right|^2}\widehat{\overrightarrow{r_{1,2}}}$$

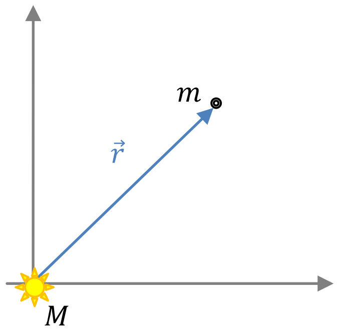


$$\vec{F}=-G\frac{m_1M}{R^2}\vec{k}=-m\left(\frac{GM}{R^2}\right)\vec{k}=-mg\vec{k}$$

$$g\approx10\frac{m}{s^2}=10\frac{N}{kg}$$

GRAVITACIJA KAO „CENTRALNA SILA“

„ $\vec{F}(\vec{r}) = f(r)\hat{\vec{r}}$ “



$M \gg m$

$\vec{R} \approx 0$

<u>gravitacijsko polje:</u>

$$\vec{g}(\vec{r}) = -G_N \frac{M}{r^2}\hat{\vec{r}}$$

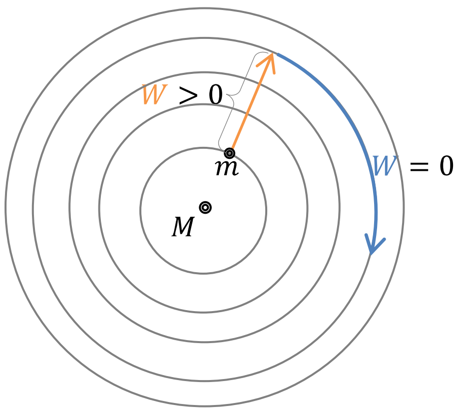
sila na česticu m:

$$\vec{F}(\vec{r}) = m\vec{g}(\vec{r}) = -G\frac{Mm}{r^2}\hat{\vec{r}}$$

- OČUVANJE KUTNE KOLIČINE GIBANJA ČESTICE M U KOORDINATNOM SUSTAVU CENTRALNE SILE

$$\frac{d}{dt}\vec{F} = \vec{M}_{ext} = \vec{r} \times (m\vec{g}) = \vec{r} \times \left(m\left(= -G_N \frac{M}{r^2}\hat{\vec{r}}\right)\right) = 0$$

- SILA JE KONZERVATIVNA



-potencijalna energija u centralnom polju grav. sile:

$$E_{pot}(\vec{r}) = U(\vec{r}) = \int_{\infty}^{\vec{r}} \left(-\vec{F}(\vec{r}')\right) d\vec{r}' = \int_{\infty}^{\vec{r}} \left(G\frac{Mm}{r'^2}\hat{\vec{r}}'\right) d\vec{r}' = \int_{\infty}^{r} G\frac{Mm}{r'^2} dr' = -GmM\int_{r}^{\infty} \frac{dr'}{r'^2} = -GmM\left(-\frac{1}{r'}\right)\Big|_{r}^{\infty}$$

$$= -\frac{GmM}{r}$$

<u>gravitacijski potencijal:</u>

$$E_{pot}(\vec{r}) = m\Phi(\vec{r})$$

$$\boldsymbol{\Phi(\vec{r}) = -G\frac{M}{r}}$$

općenito; i=1,2,3,...

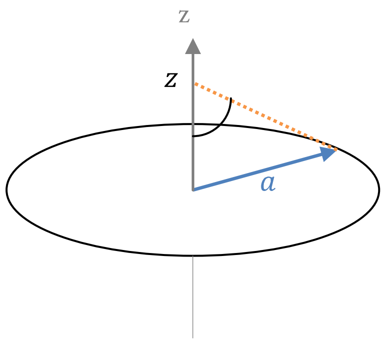
$$\vec{g}(\vec{r}) = G_N \sum_i \frac{m_i}{|\vec{r_i} - \vec{r}|^3}(\vec{r_i} - \vec{r})$$

$$\Phi(\vec{r}) = -G \sum_i \frac{m_i}{|\vec{r_i} - \vec{r}|}$$

$$\vec{F} = -\vec{\nabla} U$$

$$\vec{g}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r}) = -\frac{d}{d\vec{r}}\Phi(\vec{r})$$

PRIMJER – PRSTEN



koliko je gravitacijsko polje uz z-os?

$$g_z(z) = -G\frac{M}{a^2 + z^2}\cos\theta = -G\frac{M}{a^2 + z^2} \cdot \frac{z}{\sqrt{z^2 + a^2}}$$

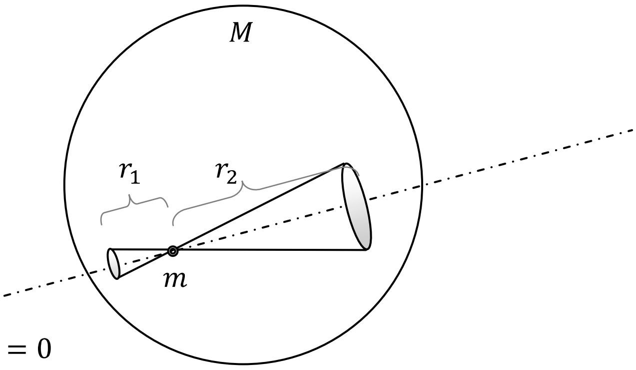
$$= -GM\frac{z}{\sqrt{(z^2 + a^2)^3}}$$

$$\Phi(z) = -G\frac{M}{\sqrt{z^2 + a^2}}$$

$$g_z(z) = -\frac{d}{dz}\Phi(z) = GM\left(-\frac{1}{2}\right)\frac{2z}{\sqrt{(z^2 + a^2)^3}} = -GM\frac{z}{\sqrt{(z^2 + a^2)^3}} \quad \blacksquare$$

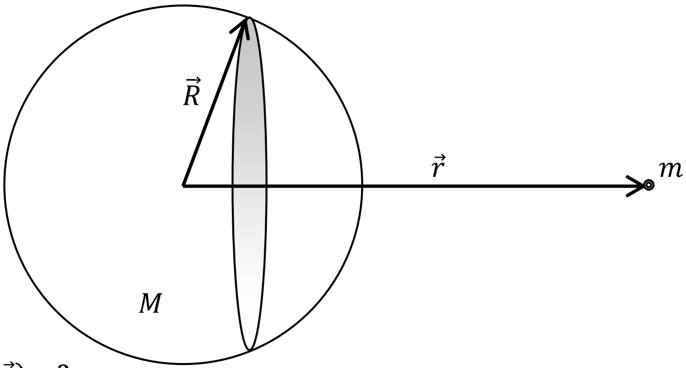
PRIMJER - SFERA

-čestica unutar sfere



$\vec{g} = 0$

- čestica izvan sfere



$\vec{g}(\vec{r}) =?$

$$\Phi(\vec{r}) = -G \int \frac{dM}{|\vec{r} - \vec{R}|}$$

$$|\vec{r} - \vec{R}| = \sqrt{(r^2 - 2Rr\cos\theta + R^2}$$

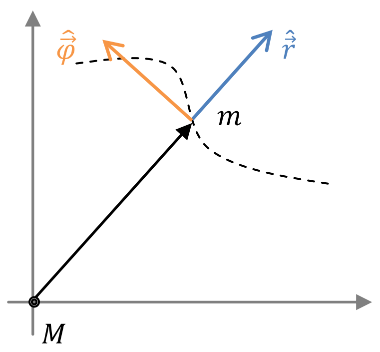
$$dM = \sigma\, da = \frac{M}{4R^2\pi}(2R\sin\theta\,\pi)R\,d\theta = \frac{1}{2}M\sin\theta\,d\theta$$

$$\Phi(\vec{r}) = -G \int \frac{\frac{1}{2}M\sin\theta\,d\theta}{r\sqrt{(1 - \frac{2R}{r}\cos\theta + \left(\frac{R}{r}\right)^2}}$$

$$-\cos\theta = u \qquad \frac{R}{r} = a$$

$$\Phi(\vec{r}) = -G\frac{M}{r}\int_{-1}^{1} \frac{du}{2\sqrt{(1 + 2au + a^2}} = -G\frac{M}{r}$$

Efektivni potencijal



$\vec{r}$ – položaj

$\vec{v} = \dot{\vec{r}} = \dot{r}\hat{\vec{r}} + r\dot{\varphi}\hat{\vec{\varphi}}$

$GMm \equiv k$

$$E = T + U = \frac{1}{2}mv^2 - G\frac{mM}{r} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \frac{k}{r}$$

$$\vec{L} = \vec{r} \times (m\vec{v}) = (\dots) = mr^2\dot{\varphi}\vec{k}$$

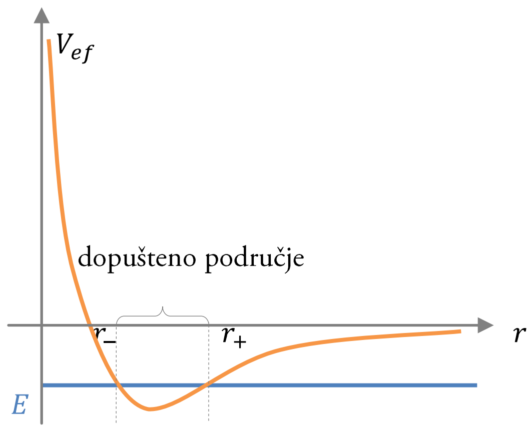
$$L = mr^2\dot{\varphi}$$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{k}{r}$$

$$\frac{1}{2}m\dot{r}^2 \geq 0$$

$$\boldsymbol{\frac{L^2}{2mr^2} - \frac{k}{r} = V(r)}$$

-efektivni potencijal



$$r\pm = -\frac{k}{2E}\left[1 \pm \sqrt{1 + \frac{2EL^2}{mk^2}}\right] = a(1 \pm \varepsilon)$$

$$a = -\frac{k}{2E} \qquad \varepsilon = \sqrt{1 + \frac{2EL^2}{mk^2}}$$

$$L = \sqrt{mka(1 - \varepsilon^2)}$$

DRUGI KEPLEROV ZAKON

$$\Delta S = \frac{1}{2} r(r\Delta\phi) = \frac{r^2}{2}\Delta\phi$$

$$\frac{dS}{dt} = \frac{r^2}{2}\frac{d\phi}{dt} = \frac{L}{2m} = konst.$$

TREĆI KEPLEROV ZAKON

$$S = ab\pi = \frac{dS}{dt}T = \frac{L}{2m}T$$

$$T = \frac{2mab\pi}{L} = \frac{2maa\sqrt{1-\varepsilon^2}\pi}{\sqrt{mka(1-\varepsilon^2)}} = \frac{2ma^2\pi}{\sqrt{mMmGa}} = \frac{2\pi\sqrt{a^3}}{\sqrt{GM}}$$

$$T^2 = \frac{4\pi^2}{GM}a^3$$

Relativistička dinamika

Newton

$$\vec{v} = \frac{\vec{p}}{m} = \frac{1}{m}\int \vec{F}dt$$

$\vec{F}dt$ - proizvoljno veliko $\rightarrow |\vec{v}| > c$ - brzina svj. (???)

Einstein

$$\boldsymbol{\vec{p} = \gamma m \vec{v}}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\beta \equiv \frac{v}{c}$$

PREMA TEOREMU O RADU I KINETIČKOJ ENERGIJI:

$$T|_{poč}^{kon} = W = \int_{poč}^{kon} \vec{F} d\vec{s} = \int_{poč}^{kon} \frac{d\vec{p}}{dt} d\vec{s} = \int_{poč}^{kon} d\vec{p}\vec{v} = \int_{poč}^{kon} d(\vec{p}\vec{v}) - \int_{poč}^{kon} \vec{p} d\vec{v} = \vec{p}\vec{v}|_{poč}^{kon} - \int_{poč}^{kon} \vec{p} d\vec{v}$$

1dim. gibanje

$v_{poč} = 0, \quad v_{kon} = v, \quad T_{kon} = T$

$$T = pv - \int_0^v p' dv' = \frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}} - \int_0^v \frac{mv'}{\sqrt{1 - \frac{v'^2}{c^2}}} dv' = \frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}} + mc^2 \sqrt{1 - \frac{v'^2}{c^2}} \Bigg|_0^v$$

$$= \frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}} + mc^2 \left(\sqrt{1 - \frac{v^2}{c^2}} - 1 \right) = mc^2 \left[\frac{\beta^2}{\sqrt{1 - \beta^2}} + \sqrt{1 - \beta^2} - 1 \right] = mc^2 \left[\frac{\beta^2 + (1 - \beta^2)}{\sqrt{1 - \beta^2}} - 1 \right]$$

$$= mc^2 \left[\frac{1}{\sqrt{1 - \beta^2}} - 1 \right] = \boldsymbol{mc^2(\gamma - 1)}$$

interpretacija:

$$mc^2 \gamma = mc^2 + T$$

$$E = \gamma mc^2 = mc^2 + T$$

$E = \gamma mc^2$ - energija čestice
mc^2 - energija mirovanja čestice
T - kinetička energija čestice

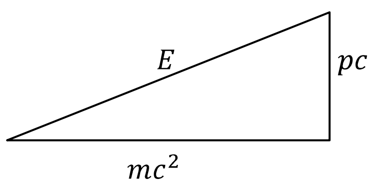
važno: ako je $v \ll c \rightarrow \ \beta \ll 1 \rightarrow$ tada:

$$T = mc^2(\gamma - 1) = mc^2 \left(\frac{1}{\sqrt{1 - \beta^2}} - 1 \right) = mc^2 \left[\left(1 + \frac{1}{2}\beta^2 + \cdots \right) - 1 \right] = mc^2 \left(\frac{1}{2} \left(\frac{v}{c} \right)^2 + \cdots \right) \approx \frac{1}{2} mv^2$$

$$E^2 - (mc^2)^2 = (\gamma mc^2)^2 - (mc^2)^2 = (mc^2)^2(\gamma^2 - 1) = (mc^2)^2 \left(\frac{1}{1 - \beta^2} - 1 \right) = (mc^2)^2 \left(\frac{1 - 1 + \beta^2}{1 - \beta^2} \right)$$

$$= (mc^2)^2 \left(\frac{\beta^2}{1 - \beta^2} \right) = \left(mc^2 \frac{\beta}{\sqrt{1 - \beta^2}} \right)^2 = \left(\frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} c \right)^2 = (pc)^2$$

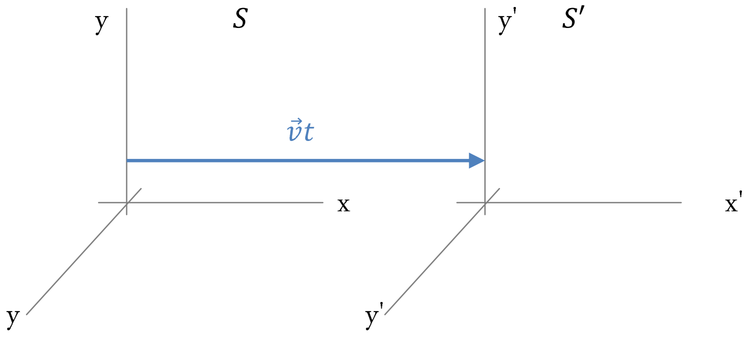
$$\rightarrow \boldsymbol{E^2 = \left(mc^2\right)^2 + (pc)^2}$$



Specijalna teorija relativnosti

- 1905. g
- <u>neovisnos o izboru koordinatnog sustava!</u>

primjer – Galilejeve transformacije (GT) i Newtonova jednadžba



$x = x' + \vec{v}t$

$y = y'$

$z = z'$

$t = t'$

GT: $\vec{r} = \vec{v}t + \vec{r'}$

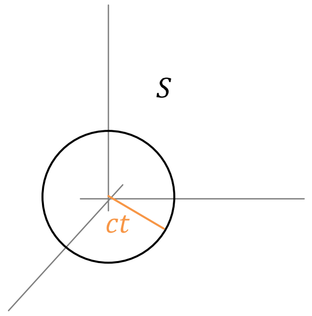
Newtonova jednadžba

$m\ddot{\vec{r}} = \vec{F}$

$m\ddot{\vec{r'}} = \vec{F}$

GT: $\ddot{\vec{r}} = \ddot{\vec{r'}}$ ■

primjer – GT i bljesak



$x^2 + y^2 + z^2 = (ct)^2$

$x = x' + vt$

$y = y'$

$z = z'$

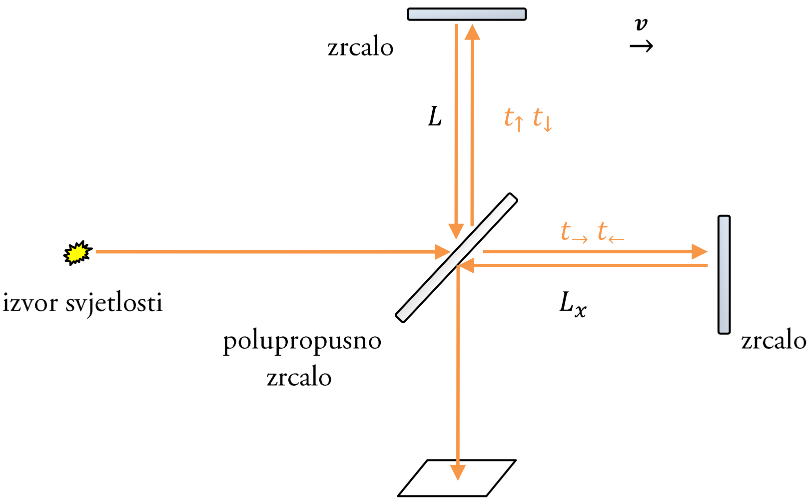
$t = t'$

sustav S'

$(x' + vt')^2 + y'^2 + z'^2 = (ct')^2$

$x'^2 + y'^2 + z'^2 = (ct')^2 - \mathbf{2x'vt'} - (\mathbf{vt'})^{\mathbf{2}}$

Michelson-Morleyev pokus



$t_\rightarrow$: $L + vt_\rightarrow = ct_\rightarrow$

$$t_\rightarrow = \frac{L}{c - v}$$

$$t_\leftarrow = \frac{L}{c + v}$$

$t_\uparrow = t_\downarrow$: $L^2 + (vt_\updownarrow)^2 = (ct_\updownarrow)^2$

$$t_\uparrow = t_\downarrow = \frac{L}{\sqrt{c^2 - v^2}}$$

$$\Delta t = (t_\rightarrow + t_\leftarrow) - (t_\uparrow + t_\downarrow) = \frac{2LC}{c^2 - v^2} - \frac{2L}{\sqrt{c^2 - v^2}}$$

eksperiment daje $\Delta t = 0$ neovisno što napravili, dokaz nemogućnosti širenja eterom

$$\frac{2LC}{c^2\left(1 - \frac{v^2}{c^2}\right)} - \frac{2L}{c\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2L}{c}\frac{1}{1 - \frac{v^2}{c^2}} - \frac{2L}{c}\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

ako

$L_x \rightarrow L_x\sqrt{1 - \frac{v^2}{c^2}}$ - relativistička kontrakcija duljine

Zahtjevi za Lorentzove transformacije:

1. Prelaze u Galilejeve za $v \ll c$
2. Želimo da budu simetrične u osnosu na $\pm v$
3. Želimo da budu linearne funkcijama x, y, z i t
4. Želimo da budu sukladne s drugim Einsteinovim postulatom

IZVODI LORENTZOVIH TRANSFORMACIJA

i-iii)

$$(1)\quad \begin{cases} x' = \gamma(x - vt) \\ x = \gamma(x' + vt') \end{cases}$$

eliminacijom x' slijedi:

$$(2)\quad \begin{cases} t' = \gamma t + \dfrac{\gamma^2 - 1}{\gamma v} x \\ t = \gamma t' - \dfrac{\gamma^2 - 1}{\gamma v} x' \end{cases}$$

za sada γ nepoznat!

iiii)

S: $\quad x = ct$

S': $\quad x' = ct'$

$$\rightarrow (1) \rightarrow \begin{cases} t' = \dfrac{\gamma t}{c}(c - v) \\ t = \dfrac{\gamma t'}{c}(c + v) \end{cases}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Lorentzove transformacije

$$\boldsymbol{x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

$$\boldsymbol{x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

$$\boldsymbol{y' = y, \qquad z' = z}$$

$$\boldsymbol{t' = \frac{t - \frac{v_x}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

$$\boldsymbol{t = \frac{t' + \frac{v_x{}'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

PRIMJER – LT I BLJESAK

$S\colon \quad x^2 + y^2 + z^2 = (ct)^2$

uvrštavanje:

$S'\colon \quad x'^2 + y'^2 + z'^2 = (ct')^2$

TRANSFORMACIJA BRZINE

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} = x' = \gamma(x - vt)$$

$$u_x' = \frac{dx'}{dt'} = \gamma\left(\frac{dx}{dt}\frac{dt}{dt'} - v\frac{dt}{dt'}\right) = \gamma(u_x - v)\frac{dt}{dt'}$$

$$t' = \gamma\left(t - \frac{v_x}{c^2}\right)$$

$$\frac{dt'}{dt} = \gamma\left(1 - \frac{vu_x}{c^2}\right)$$

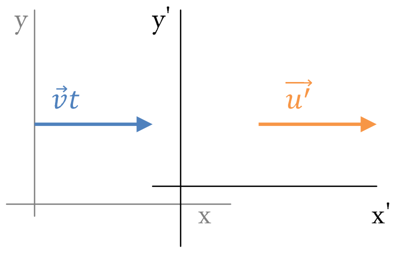
$$u_x' = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

$$y' = y$$

$$u_y' = \frac{dy'}{dt'} = \frac{dy}{dt}\frac{dt}{dt'} = u_y \frac{1}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} = u_y \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{vu_x}{c^2}}$$

ZADATAK

Svemirski brod se giba $v = \frac{1}{2}c$ u odnosu na mirnog promatrača. Putnik u brodu ispali hitac brzine $u_x' = \frac{1}{2}c$ u odnosu na svemirski brod. Brzina hica u odnosu na promatrača =??



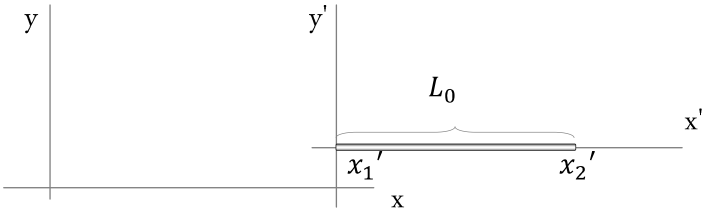
$$u_x = \frac{u_x' + v}{1 + \frac{vu_x'}{c^2}} = \frac{\frac{1}{2}c + \frac{1}{2}c}{1 + \frac{\frac{1}{2}c\,\frac{1}{2}c}{c^2}} = \frac{c}{1 + \frac{\frac{1}{4}c^2}{c^2}} = \frac{c}{\frac{5}{4}} = \boldsymbol{\frac{4}{5}c}$$

ako:

$$u_x' = c$$

$$u_x = \frac{\left(c + \frac{1}{2}c\right)}{1 + \frac{c\frac{1}{2}c}{c^2}} = \frac{\frac{3}{2}c}{\frac{3}{2}c} = \boldsymbol{c}$$

KONTRAKCIJA DULJINE



S': $L_0 = x_2' - x_1{}'$ u trenutku t'

x_2' - kraj
x_1' - početak

S: $L = x_2 - x_1$ u trenutku t

$$L_0 = x_2' - x_1' = \gamma(x_2 - vt) - \gamma(x_1 - vt) = \gamma(x_2 - x_1) = \gamma L$$

$$\boldsymbol{L} = \frac{\boldsymbol{L_0}}{\boldsymbol{\gamma}}$$

zaključno sa predavanjem 05. 05. '10.

-prvi i drugi ciklus