PETAK → INTEGRALI, neki ekstremi ... graf neće - predugo!

12.1. INTEGRIRANJE RACIONALNIH FUNKCIJA

PZ1-06-4.)

$$\int \frac{x^2}{x^2+x+1}dx = \int \frac{x^2+x+1-x-1}{x^2+x+1}dx = \int dx + \int \frac{-x-1}{x^2+x+1}dx = x + \int \frac{-x-1}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$= \left| \begin{array}{l} x+\frac{1}{2}=t \\ dx=dt \\ x=t-\frac{1}{2} \end{array} \right| = x + \int \frac{-t-\frac{1}{2}}{t^2+\frac{3}{4}}dt = x + \int \frac{-t}{\underbrace{t^2+\frac{3}{4}}_{u}}dt - \frac{1}{2}\int \frac{dt}{t^2+\frac{3}{4}} =$$

$$= x - \frac{1}{2}\ln|u| - \frac{1}{2}\frac{1}{\frac{\sqrt{3}}{2}}\operatorname{arctg}\frac{t}{\frac{\sqrt{3}}{2}} + C =$$

$$= x - \frac{1}{2}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C$$

12.DZ-2.)

$$\int \frac{x^3 dx}{x^2+2x+4}$$ → UVIJEK KADA JE STUPANJ BROJNIKA VEĆI OD STUPNJA NAZIVNIKA

PODIJELITI POLINOME !!!

$$x^3 : (x^2+2x+4) = x-2$$

$$\begin{array}{l} -x^3 \mp 2x^2 \mp 4x \\ \hline \quad -2x^2-4x \\ \quad \pm 2x^2 \pm 4x \pm 8 \\ \hline \qquad\qquad 8 \end{array}$$

8 OSTATAK!

$$= \int \left(x-2 + \frac{8}{\underbrace{x^2+2x+4}_{(x+1)^2+3}}\right)dx = \frac{x^2}{2} - 2x + 8\cdot\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x+1}{\sqrt{3}} + C$$

$$\int \frac{P_n(x)}{Q_2(x)}\,dx = \begin{cases} \text{svedemo na potpuni kvadrat + supstitucija, ako je } n<2 \\ \text{dodamo i oduzmemo da svedemo na 1. slučaj, ako je } n=2 \\ \text{dijelimo polinome i opet vidi 1. slučaj, ako je } n>2 \end{cases}$$

→ ako je u nazivniku $Q_m(x)$, za $m > 2$:

1. FAKTORIZACIJA NAZIVNIKA (prikazati u obliku umnoška)

2. RASTAV NA PARCIJALNE RAZLOMKE

→ poanta: dobiti ove „jednostavnije" integrale koje znamo

$$\frac{3}{(x^2+x-2)(x^2+x+2)} =$$

$x_{1,2} = \begin{matrix} 1 \\ -2 \end{matrix}$ $x_{1,2} = \in \mathbb{C}$ → NEMA REALNE FAKTORIZACIJE

!! UVIJEK POLINOM 1. STUPNJA

$$= \frac{3}{(x-1)(x+2)(x^2+x+2)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{Cx+D}{x^2+x+2}$$

$$\frac{1}{(x^2-4)(x^3+8)} = \frac{1}{(x-2)(x+2)^2(x^2-2x+4)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{Dx+E}{x^2-2x+4}$$

$(x-2)(x+2)(x+2)(x^2-2x+4)$

→ SA SVAKOM POTENCIJOM SE POJAVLJUJE

VJEROJATNO U ISPITU

5.)

$$\int \frac{dx}{x^3+x^2} = \int \frac{-1}{x}\,dx + \int \frac{1}{x^2}\,dx + \int \frac{1}{x+1}\,dx = -\ln|x| - \frac{1}{x} + \ln|x+1| + C$$

$$\frac{1}{(x+1)x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} = \frac{Ax(x+1) + B(x+1) + Cx^2}{x^2(x+1)}$$

$$Ax(x+1) + B(x+1) + Cx^2 = 1$$

→ UVRSTITI NULTOČKE : $x=0 \rightarrow \boxed{B=1}$

$x=-1 \rightarrow \boxed{C=1}$

neki prikladan x, npr. $x=1 \rightarrow 2A+2x+1=1$

$\boxed{A=-1}$

Z1-10-4.) $\int \frac{7x+1}{\underbrace{2x^3-x^2+4x-2}_{x^2(2x-1)+2(2x-1)}}dx =$

$\longrightarrow$ DA NIJE BILO NAMJEŠTENO POGODILI BI JEDNU NULTOČKU $\rightarrow$ DJELITELJ SLOBODNOG ČLANA ... (5. TJEDAN)

$$\frac{7x+1}{(2x-1)(x^2+2)} = \frac{A}{2x-1} + \frac{Bx+C}{x^2+2}$$

$$A(x^2+2)+(Bx+C)(2x-1) = 7x+1$$

$x=\frac{1}{2}$: $\frac{9}{4}A = \frac{9}{2}$

$\boxed{A=2}$

neki proizvoljan, najljepše : $x=0 \longrightarrow 4-C=1$

$\boxed{C=3}$

$x=1 \longrightarrow 6+B+3=8$

$\boxed{B=-1}$

$$=\int\frac{2}{2x-1}dx + \int\frac{-x+3}{x^2+2}dx = \dots = 2\cdot\frac{1}{2}\ln|2x-1| - \frac{1}{2}\ln|x^2+2| + \frac{3}{\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}} + C$$

12.DZ-6.) $\int_0^1 \frac{dx}{(x^2+1)^3} =$

$$I_n = \int\frac{dx}{(x^2+1)^n} = \int\frac{x^2+1-x^2}{(x^2+1)^n}dx = \underbrace{\int\frac{dx}{(x^2+1)^{n-1}}}_{I_{n-1}} - \int\frac{x^2}{(x^2+1)^n}dx = \left| \begin{array}{ll} u=x & dv=\frac{x}{(x^2+1)^n} \\ du=dx & v=\frac{1}{2}\dots \end{array}\right.$$

$x\cdot x$ (u, dv)

$$= I_{n-1} + \frac{x}{2(n-1)(x^2+1)^{n-1}} - \frac{1}{2(n-1)}\underbrace{\int\frac{dx}{(x^2+1)^{n-1}}}_{I_{n-1}}$$

$$\boxed{I_n = \frac{2n-3}{2n-2}I_{n-1} + \frac{x}{2(n-1)(x^2+1)^{n-1}}} \quad \boxed{\text{za } n\geq 2} \;!!$$

$n=3$

$$I_3 = \frac{3}{4}I_2 + \frac{x}{4(x^2+1)^2}\Big|_0^1 = \frac{3}{4}\overbrace{\left(\frac{1}{2}I_1 + \frac{x}{2(x^2+1)}\Big|_0^1\right)}^{I_2} + \frac{1}{16} =$$

$$I_1 = \int\frac{dx}{(x^2+1)^1} = \operatorname{arctg} x$$

$$\frac{3}{8}\operatorname{arctg}x\Big|_0^1+\frac{3}{16}+\frac{1}{16}=\frac{3\pi}{32}+\frac{1}{4}$$

→ DOBRO NAUČITI OVE SA IZVODOM REKURZIVNE FORMULE
by Burek :)

9-5)

$$\int\frac{dx}{x\ln^3x+4x\ln x}=\left|\begin{array}{l}\ln x=t\\ \frac{1}{x}\,dx=dt\\ dx=x\,dt\end{array}\right|=\int\frac{\cancel{x}\,dt}{\cancel{x}(t^3+4t)}=$$

NAJRUŽNIJE

$$\frac{1}{t(t^2+4)}=\frac{A}{t}+\frac{Bt+C}{t^2+4}$$

$$A(t^2+4)+(Bt+C)t=1$$

$t=0:\ 4A=1 \longrightarrow \boxed{A=\frac{1}{4}}$

$t=1:\ \frac{5}{4}+B+\cancel{C}=1$

$t=-1:\ \frac{5}{4}+B-\cancel{C}=1$

$+$

$\boxed{B=\frac{-1}{4}}$ $\boxed{C=0}$

$$\int\frac{\frac{1}{4}}{t}\,dt+\int\frac{-\frac{1}{4}t}{\underbrace{t^2+4}_{u}}\,dt=\frac{1}{4}\ln|t|-\frac{1}{8}\ln|t^2+4|+C=$$

$$=\frac{1}{4}\ln|\ln x|-\frac{1}{8}\ln|\ln^2x+4|+C$$

→ do srijede prvih 7 zad. od 12.

NAPOMENA:

12. DZ-7.)

$$\int_0^1\frac{dx}{(x+1)(x^2+x+1)^2}=$$

$$\frac{A}{x+1}+\frac{Bx+C}{x^2+x+1}+\underbrace{\frac{Dx+E}{(x^2+x+1)^2}}_{\text{rekurzivnom}}\ \ldots$$

12.2. INTEGRIRANJE TRIGONOMETRIJSKIH FUNKCIJA

PRIKLADNE SUPSTITUCIJE

npr. $\sin x = t$, $\cos x = t$ $\longrightarrow$ pogledaj 11. dz $\int \frac{\sin x}{\sqrt{\cos x}}dx = \left| \begin{matrix} \cos x = t \\ -\sin x dx = dt \end{matrix} \right|$ ...

$\int R(\sin x, \cos x)dx$ = UNIVERZALNA SUPSTITUCIJA (UVIJEK USPJEVA)

$$\boxed{t = \operatorname{tg}\frac{x}{2}}$$

$\hookrightarrow dx = \frac{2dt}{1+t^2}$ $\Longrightarrow$ $\boxed{\sin x = \frac{2t}{1+t^2}}$

$\boxed{\cos x = \frac{1-t^2}{1+t^2}}$

$\int R(\sin^2 x, \cos^2 x)\,dx = \Bigg|$ $\boxed{t = \operatorname{tg} x}$ $\hookrightarrow dx = \frac{dt}{1+t^2}$ $\longrightarrow$ $\boxed{\sin x = \frac{t}{\sqrt{1+t^2}}}$ $\boxed{\cos x = \frac{1}{\sqrt{1+t^2}}}$ $\Bigg|$

$\rightarrow$ knjižica, str. 19 $\longrightarrow$ ZNATI, sve isto sa hiperbolnim!

12. DZ – 12.)

$$\int \frac{dx}{\cos x + 2\sin x + 3} = \{\text{UNIVERZALNA SUPS.}\} =$$

$$= \int \frac{\frac{2dt}{1+t^2}}{\frac{1-t^2}{1+t^2} + 2\cdot\frac{2t}{1+t^2} + 3} = \int \frac{\cancel{\frac{2dt}{1+t^2}}}{\frac{1-t^2+4t+3+3t^2}{\cancel{1+t^2}}} = \int \frac{\cancel{2}dt}{\underbrace{2t^2+4t+4}_{\cancel{2}(t^2+2t+2)}} = \int \frac{dt}{t^2+2t+2} =$$

$$= \int \frac{dt}{(t+1)^2+1} = \operatorname{arctg}(t+1) + C = \operatorname{arctg}\left(\operatorname{tg}\frac{x}{2}+1\right) + C$$

NE ZABORAVI GRANICE !!

PZI – 09 – 4.)

$$\int_0^{\frac{\pi}{3}} \frac{dx}{1+8\cos^2 x} \overset{t=\operatorname{tg}x}{=} \int_0^{\sqrt{3}} \frac{\frac{dt}{1+t^2}}{1+8\frac{1}{1+t^2}} = \int_0^{\sqrt{3}} \frac{\cancel{\frac{dt}{1+t^2}}}{\frac{1+8+t^2}{\cancel{1+t^2}}} = \int_0^{\sqrt{3}} \frac{dt}{t^2+9} = \frac{1}{3}\operatorname{arctg}\left(\frac{t}{3}\right)$$

$$= \frac{\pi}{18}$$

!!!

HIPERBOLNI

12. DZ – 17.)

$$\int \frac{dx}{\operatorname{ch}^4 x} \overset{t=\operatorname{th}x}{=} \int \frac{\frac{dt}{1-t^2}}{\left(\frac{1}{1-t^2}\right)^2} = \int (1-t^2)\,dt = t - \frac{t^3}{3} + C = \operatorname{th}x - \frac{(\operatorname{th}x)^3}{3} + C$$

12. DZ – 13.)

!!! OBIČNA SUPST.

$$\int \frac{dx}{\sin^2 x \cos x} = \left| \begin{array}{l} \sin x = t \\ \cos x\,dx = dt \\ dx = \frac{dt}{\cos x} \end{array} \right| = \int \frac{dt}{t^2 \underbrace{\cos^2 x}_{1-\sin^2 x}} = \int \frac{dt}{t^2-t^4} = \ldots$$

DZ

$$\frac{1}{t^2(1-t^2)} = \frac{1}{t^2(1-t)(1+t)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{1+t} + \frac{D}{1-t} \ldots$$

$$\ldots 7) \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos x\,dx}{\sin^3 x - 2\sin^2 x + \sin x} = \left| \begin{array}{l} \sin x = t \\ \cos x\,dx = dt \\ x = \frac{\pi}{6} \to \alpha = \frac{1}{2} \\ x = \frac{\pi}{4} \to \beta = \frac{\sqrt{2}}{2} \end{array} \right| =$$

$$= \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \frac{dt}{\underbrace{t^3 - 2t + t}_{t(t^2-2t+1)}} =$$

$$\frac{1}{t(t-1)^2} = \frac{A}{t} + \frac{B}{t-1} + \frac{C}{(t-1)^2}$$

$$A(t-1)^2 + Bt(t-1) + Ct = 1$$

$t=0$: $\boxed{A=1}$

$t=1$: $\boxed{C=1}$

$t=7$: $36 + 42B + 7 = 1$

$\boxed{B=-1}$

$$= \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \left(\frac{1}{t} + \frac{-1}{t-1} + \frac{1}{(t-1)^2} \right) dt = \left[\ln|t| - \ln|t-1| + \frac{(t-1)^{-1}}{-1} \right] \Big|_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} = \ldots =$$

$$= \ldots = \frac{2-2\sqrt{2}}{\sqrt{2}-2} + \ln \frac{\sqrt{2}}{2-\sqrt{2}}$$

UMNOŠCI SINUSA I KOSINUSA

...) općenito

$$\int \sin x \underbrace{\cos 2x \sin 2x}_{\frac{1}{2}\sin 4x} dx = \frac{1}{2} \int \underbrace{\sin x \sin 4x}_{\text{FORMULA PRETVORBE}} dx = \frac{1}{2} \int \frac{1}{2} \left[\cos(-3x) - \cos(5x) \right] dx =$$

$$= \frac{1}{4} \left(\frac{1}{3} \sin 3x - \frac{1}{5} \sin 5x \right) + C$$

ii) neparne potencije $\longrightarrow$ $\sin x = t$, $\cos x = t$ (onaj koji nije u ...)

12.ZZV-2.)

jer će sinus svesti na parnu!

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^6 x \, dx = \left| \begin{array}{l} \cos x = t \\ -\sin x \, dx = dt \\ x=0 \rightarrow t=1 \\ x=\frac{\pi}{2} \rightarrow t=0 \end{array} \right| =$$

$$= -\int_1^0 \underbrace{\sin^2 x}_{1-t^2} t^6 dt = -\int_1^0 (t^6 - t^8) dt = \left(-\frac{t^7}{7} + \frac{t^9}{9}\right) \Big|_1^0 = \frac{2}{63}$$

iii) samo parne potencije : $\sin^2 x = \frac{1-\cos 2x}{2}$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

ZZV-1.KO-40.)

$$\int \sin^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) \, dx =$$

$$= \frac{1}{4}x - \frac{2}{4}\cdot\frac{1}{2}\sin 2x + \frac{1}{4}\int \frac{1+\cos 4x}{2} dx =$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C$$

12.DZ-15.)

$$\int_0^{\frac{\pi}{2}} \sin^8 x \, dx = \cancel{\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2x}{2}\right)^4 dx} = \text{NE} \; !!!$$

REKURZIVNA :

$$I_n = \int \sin^n x \, dx = \int \underbrace{\sin^{n-1} x}_{u} \cdot \underbrace{\sin x \, dx}_{dv} = \left| \begin{array}{ll} u = \sin^{n-1} x & dv = \sin x \, dx \\ du = (n-1)\sin^{n-2} x \cdot \cos x \, dx & v = -\cos x \end{array} \right|$$

$$= -\cos x \sin^{n-1} x + \int (n-1) \sin^{n-2} x \underbrace{\cos^2 x}_{1-\sin^2 x} dx =$$

$$I_n = -\cos x \sin^{n-1} x + (n-1)\underbrace{\int \sin^{n-2} x \, dx}_{I_{n-2}} - (n-1)\underbrace{\int \sin^n x \, dx}_{I_n}$$

$$I_n(1+n-1) = -\cos x \sin^{n-1} x + (n-1) I_{n-2} \quad | : n$$

$$\boxed{I_n = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n} \cdot I_{n-2}}$$

$$= \left(-\frac{1}{8}\overset{0}{\widetilde{\cos x}}\sin^7 x\right)\Big|_0^{\frac{\pi}{2}} + \frac{7}{8} I_6 = \frac{7}{8}\left[\left(-\frac{1}{6}\overset{0}{\widetilde{\cos x}}\sin^5 x\right)\Big|_0^{\frac{\pi}{2}} + \frac{5}{6} I_4\right] =$$

$$= \frac{35}{48}\left[\left(-\frac{1}{4}\cos x \sin^3 x\right)\Big|_0^{\frac{\pi}{2}} + \frac{3}{4} I_2\right] = \frac{35}{64}\left[\left(-\frac{1}{2}\cos x \sin x\right)\Big|_0^{\frac{\pi}{2}} + \frac{1}{2} I_0\right] =$$

$$= \frac{35}{128} x \Big|_0^{\frac{\pi}{2}} = \frac{35\pi}{256} \quad //$$

Z-18.)

$$\int sh^2 x \, ch^4 x \, dx =$$

HIPERBOLNE FORMULE

I. način: $\int \left(\frac{ch 2x - 1}{2}\right)\left(\frac{ch 2x + 1}{2}\right)^2 dx = \dots\dots = DZ \ \widetilde{\text{II}}$

II. način: $\int \left(\frac{e^x - e^{-x}}{2}\right)^2 \left(\frac{e^x + e^{-x}}{2}\right)^4 dx = \dots\dots = DZ \ \widetilde{\text{II}}$

III. način: $\left\{ \underline{\underline{ch^2 x - sh^2 x = 1}} \right\}$

$$\int (ch^2 x - 1)\, ch^4 x \, dx = \int ch^6 x \, dx - \int ch^4 x \, dx =$$

IV. način: ....

$$I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n} sh x \, ch^{n-1} x \qquad I_n = \int ch^n x \, dx$$

$$I_6 - I_4 = \dots\dots =$$

VJEŽBA ZA ŠKOLSKU by BUREK

1.) $$\int_0^{16}\frac{dx}{(1+\sqrt[4]{x})^2} = \left|\begin{array}{l} 1+\sqrt[4]{x}=t \\ \frac{1}{4}x^{-\frac{3}{4}}dx=dt \\ dx=4x^{\frac{3}{4}}dt \\ x=0\to\alpha=1 \\ x=3\to\beta=3 \\ \sqrt[4]{x}=t-1 \end{array}\right| = \int_1^3\frac{4x^{\frac{3}{4}}dt}{t^2} = \int_1^3\frac{4(t-1)^3dt}{t^2} =$$

$x^{\frac{3}{4}} \to (\sqrt[4]{x})^3$

$$= 4\int_1^3\frac{t^3-3t^2+3t-1}{t^2}dt = 4\int_1^3\left(t-3+\frac{3}{t}-\frac{1}{t^2}\right)dt = 4\left(\frac{t^2}{2}-3t+3\ln t-\frac{t^{-1}}{-1}\right.$$

$$= 12\ln 3-\frac{32}{3}$$

2.) $$\int\frac{x}{\sqrt{4x-x^2}}dx =$$

$$4x-x^2 = -(x^2-4x) = -(\underbrace{x^2-4x+4}_{(x-2)^2}-4) = 4-(x-2)^2$$

$$= \int\frac{x}{\sqrt{4-(x-2)^2}}dx = \left|\begin{array}{l} x-2=t \to x=t+2 \\ dx=dt \end{array}\right| = \int\frac{t+2}{\sqrt{4-t^2}}dt = \int\frac{t}{\sqrt{4-t^2}}dt + \int\frac{2}{\sqrt{4-t^2}}dt =$$

$$\left|\begin{array}{l} 4-t^2=u \\ -2t\,dt=du \end{array}\right|$$

$$= -\frac{1}{2}\int\frac{du}{\sqrt{u}} + 2\arcsin\frac{t}{2}+C =$$

$u^{-\frac{1}{2}}$

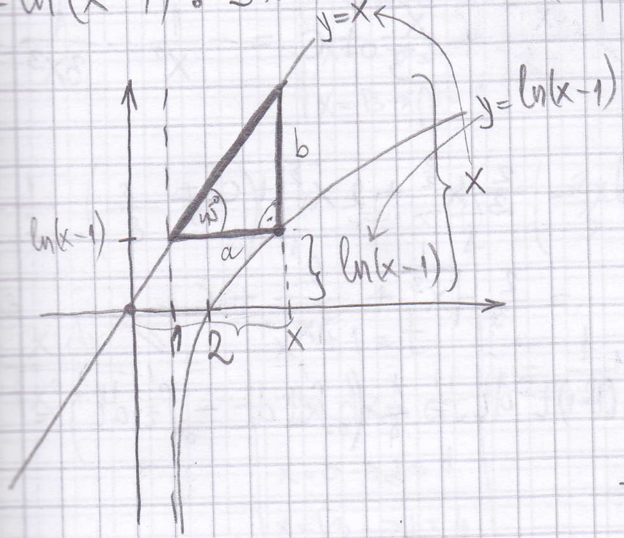
$$= -\frac{1}{\cancel{2}}\cdot\frac{u^{\frac{1}{2}}}{\frac{1}{\cancel{2}}} + 2\arcsin\frac{x-2}{2}+C$$

3.) $$\int_1^{\sqrt{3}}\frac{\mathrm{arctg}\frac{1}{x}}{x^2}dx = \left|\begin{array}{l} \frac{1}{x}=t \\ -\frac{1}{x^2}dx=dt \end{array}\right| = -\int_1^{\frac{1}{\sqrt{3}}}\underbrace{\mathrm{arctg}\,t}_{u}\,\underbrace{dt}_{dv} = \left|\begin{array}{ll} u=\mathrm{arctg}\,t & dv= \\ du=\frac{1}{1+t^2}dt & v= \end{array}\right.$$

$$= -t\,\mathrm{arctg}\,t\Big|_1^{\frac{1}{\sqrt{3}}} + \int_1^{\frac{1}{\sqrt{3}}}\frac{t}{1+t^2}dt = \quad u$$

$$= -\frac{1}{\sqrt{3}}\mathrm{arctg}\frac{1}{\sqrt{3}} - (-\mathrm{arctg}\,1) + \frac{1}{2}\ln|1+t^2|\Big|_1^{\frac{1}{\sqrt{3}}} = \dots$$

…edi pravokutni trokut minimalne površine čija hipotenuza leži na pravcu $y=x$, a vrh nasuprot hipotenuze leži na krivulji $y=\ln(x-1)$. Stranice su mu paralelne sa koordinatnim osima.



$$P=\frac{a\cdot b}{2}$$

$$b=x-\ln(x-1)$$

$y=x$ ide pod $45^\circ \Rightarrow$ jednakokračan $\triangle$

$\underline{a=b}$

$$\hookrightarrow P=\frac{1}{2}\left(x-\ln(x-1)\right)^2$$

$$P'=\frac{1}{\cancel{2}}\cdot \cancel{2}\left(x-\ln(x-1)\right)\cdot\left(1-\frac{1}{x-1}\right)$$

$$P'=\frac{\left(x-\ln(x-1)\right)(x-2)}{x-1}=0$$

$x-2=0$ $\qquad$ $x=\ln(x-1)$ GRAF $=$ NEMAJU PRESJEK!

$\boxed{x=2}$ $\qquad$ NIKADA

OBAVEZNO DOKAZATI !

= TABLICA !

	1		2	
P'		$-$		$+$
P		↘		↗

minimum WOOHOO !

$$P_{min}=P(2)=\frac{1}{2}(2-\ln 1)^2=2$$

12.3. INTEGRIRANJE IRACIONALNIH FUNKCIJA

→ cilj: ODABRATI SUPSTITUCIJU DA SE RIJEŠIMO KORJENA

1. supstitucija t^n, gdje je n najmanji zajednički višekratnik svih korjena!

$\sqrt[2]{\ }, \sqrt[3]{\ } \to 6$!!

12.DZ-8)

$$\int_0^1 \frac{1+\sqrt{x}}{1+\sqrt[3]{x}}\,dx = \left| \begin{array}{l} x = t^6 \\ dx = 6t^5\,dt \end{array} \right| =$$

$$= \int_0^1 \frac{1+t^3}{1+t^2}\cdot 6t^5\,dt = 6\int_0^1 \frac{t^8+t^5}{t^2+1}\,dt$$

→ U BROJNIKU VEĆI STUPANJ: PODIJELITI POLINOME!

$$(t^8+t^5):(t^2+1) = t^6 - t^4 + t^3 + t^2 - t - 1$$

$$\begin{array}{l}
-\cancel{t^8} \pm t^6 \\
\hline
-\cancel{t^6}+t^5 \\
\pm \cancel{t^6}+t^4 \\
\hline
\cancel{t^5}+t^4 \\
-\cancel{t^5} \pm t^3 \\
\hline
\cancel{t^4}-t^3 \\
-\cancel{t^4} \pm t^2 \\
\hline
-\cancel{t^3}-t^2 \\
\pm \cancel{t^3} \pm t \\
\hline
-\cancel{t^2}+t \\
\pm \cancel{t^2}+1 \\
\hline
t+1
\end{array}$$

$t+1$ → ostatak

$$= 6\int_0^1 \left(t^6 - t^4 + t^3 + t^2 - t - 1 + \frac{t+1}{t^2+1}\right) dt =$$

$$= 6\left(\frac{t^7}{7} - \frac{t^5}{5} + \frac{t^4}{4} + \frac{t^3}{3} - \frac{t^2}{2} - t + \frac{1}{2}\ln|t^2+1| + \operatorname{arctg} t\right)\Big|_0^1 =$$

$$= 6\cdot\left(-\frac{409}{420} + \frac{1}{2}\ln 2 + \frac{\pi}{4}\right)$$

9.) $\displaystyle\int_2^3 \sqrt{\frac{x+1}{x-1}}\,dx =$

ŽELIMO SE RIJEŠITI KORJENA !!

$$\frac{x+1}{x-1} = t^2$$

PRVO SE IZRAZI X preko t-a :

$$x+1 = t^2x - t^2$$

$$x(1-t^2) = -1-t^2$$

$$x = \frac{-1-t^2}{1-t^2} = \frac{t^2+1}{t^2-1}$$

$$dx = \frac{2t(t^2-1)-(t^2+1)\cdot 2t}{(t^2-1)^2}$$

$$dx = \frac{-4t}{(t^2-1)^2}\,dt$$

PAZI !

$$x=2 \longrightarrow \alpha = \sqrt{\frac{2+1}{2-1}} = \sqrt{3}$$

$$x=3 \longrightarrow \beta = \sqrt{\frac{3+1}{3-1}} = \sqrt{2}$$

$$= \int_{\sqrt{3}}^{\sqrt{2}} t\cdot\frac{-4t}{(t^2-1)}\,dt =$$

$$\frac{-4t^2}{(t-1)^2(t+1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2}$$

$\vdots$

$$\boxed{A=B=D=-1}$$

$$\boxed{C=1}$$

$$= \int_{\sqrt{3}}^{\sqrt{2}} \left(\frac{-1}{t-1} - \frac{1}{(t-1)^2} + \frac{1}{t+1} - \frac{1}{(t+1)^2}\right)dt = \left(-\ln|t-1| - \frac{(t-1)^{-1}}{-1} + \ln|t+1| - \frac{(t+1)^{-1}}{-1}\right)\Bigg|_{\sqrt{3}}^{\sqrt{2}} =$$

$$= 2\sqrt{2} - \sqrt{3} + \ln\frac{3+2\sqrt{2}}{2+\sqrt{3}}$$

//

12.DZ-13.)

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx = \left| \begin{array}{l} 1+\sqrt[4]{x} = t^3 \rightarrow t = \sqrt[3]{1+\sqrt[4]{x}} \\ \text{PRVO IZRAZI X !!!} \\ x = (t^3-1)^4 \\ dx = 4(t^3-1)\cdot 3t^2\,dt \end{array} \right| =$$

!!! PAZI!

$$= \int \frac{t}{\cancel{(t^3-1)^2}} 4(t^3-1)^{\cancel{3}} 3t^2\,dt = 12\int (t^6 - t^3)\,dt =$$

$$= 12\frac{t^7}{7} - 12\frac{t^4}{4} + C =$$

$$= \frac{12}{7}\left(\sqrt[3]{1+\sqrt[4]{x}}\right)^7 - 3\left(\sqrt[3]{1+\sqrt[4]{x}}\right)^4 + C$$

12.DZ-10.)

$$\int \frac{dx}{x^4\cdot\sqrt{1+x^2}} = \left| \begin{array}{l} 1+x^2 = t^2 \\ x = \sqrt{t^2-1} \\ dx = \frac{1}{2\sqrt{t^2-1}}\cdot 2t\,dt \\ dx = \frac{t}{\sqrt{t^2-1}}\,dt \end{array} \right| =$$

$$= \int \frac{dt}{(t^2-1)^2\sqrt{t^2-1}}$$ FAIL !!

TRIK:

$$\int \frac{dx}{x^4\cdot\sqrt{1+x^2}} = \int \frac{dx}{x^5\cdot\sqrt{\frac{1}{x^2}+1}} = \left| \begin{array}{l} \frac{1}{x^2}+1 = t^2 \rightarrow t = \sqrt{\frac{1}{x^2}+1} \\ x = \frac{1}{\sqrt{t^2-1}} = (t^2-1)^{-\frac{1}{2}} \\ dx = -\frac{1}{2}(t^2-1)^{-\frac{3}{2}}\cdot 2t\,dt \end{array} \right| = \int \frac{-\frac{1}{\cancel{2}}(t^2-1)^{-\frac{3}{2}}\cdot\cancel{2t}\,dt}{(t^2-1)^{-\frac{5}{2}}\cdot\cancel{t}} =$$

$$= -\int (t^2-1)\,dt = -\frac{t^3}{3} + t + c = -\frac{1}{3}\left(\sqrt{\frac{1}{x^2}+1}\right)^3 + \sqrt{\frac{1}{x^2}+1} + C$$

12.2.1, 12.2.2, 12.2.3 $\longrightarrow$ ZA NADOBUDNE !

TRIGONOMETRIJSKE SUPSTITUCIJE

$$\int\sqrt{a^2-x^2}\,dx = \left|\begin{array}{l} x = a\sin t \\ \vdots \end{array}\right|$$

(A) $\int R(x, \sqrt{a^2-x^2})\,dx \qquad \left| x = a\sin t \right|$

(B) $\int R(x, \sqrt{a^2+x^2})\,dx \qquad \left| x = a\,\underline{\underline{\mathrm{sh}\,t}} \right|$

(C) $\int R(x, \sqrt{x^2-a^2})\,dx \qquad \left| x = a\,\underline{\underline{\mathrm{ch}\,t}} \right|$

PAZI ! a^2

1.)

$$\int\sqrt{1-3x^2}\,dx = \int\left(\sqrt{3}\cdot\sqrt{\tfrac{1}{3}-x^2}\right)dx = \sqrt{3}\int\sqrt{\tfrac{1}{3}-x^2}\,dx = \left|\begin{array}{l} x = \frac{1}{\sqrt{3}}\sin t \\ dx = \frac{1}{\sqrt{3}}\cos t\,dt \end{array}\right|$$

$$= \cancel{\sqrt{3}}\int\sqrt{\tfrac{1}{3}\cos^2 t}\cdot\frac{1}{\cancel{\sqrt{3}}}\cos t\,dt = \frac{1}{\sqrt{3}}\int\cos^2 t\,dt = \frac{1}{\sqrt{3}}\int\frac{1+\cos 2t}{2}\,dt =$$

$$= \frac{1}{2\sqrt{3}}t + \frac{1}{4\sqrt{3}}\sin 2t + C = \frac{1}{2\sqrt{3}}\arcsin(\sqrt{3}x) + \frac{1}{4\sqrt{3}}\sin\left(2\arcsin(\sqrt{3}x)\right) + C$$

DZ-14.)

RIJEŠITI SE sinusom

$$\int_0^{\frac{\pi}{2}}\cos x\sqrt{3\sin^2 x+2\cos^2 x}\,dx = \left|\begin{array}{l} \sin x = t \\ \cos x\,dx = dt \\ x=0 \to \alpha=0 \\ x=\frac{\pi}{2} \to \beta=1 \end{array}\right| =$$

$$= \int_0^1\sqrt{2-t^2}\,dt = \{\text{supstitucija (B)}\} =$$

$$= \left|\begin{array}{l} t = \sqrt{2}\,\mathrm{sh}\,u \\ dt = \sqrt{2}\,\mathrm{ch}\,u\,du \\ t=0 \to \alpha=0 \\ t=1 \to \beta=\mathrm{arsh}\frac{1}{\sqrt{2}} \end{array}\right| = \int_0^{\mathrm{arsh}\frac{1}{\sqrt{2}}}\sqrt{\underbrace{2+2\,\mathrm{sh}^2 u}_{\substack{2(1+\mathrm{sh}^2 u) \\ 2\mathrm{ch}^2 u}}}\cdot\sqrt{2}\,\mathrm{ch}\,u\,du = \int_0^{\mathrm{arsh}\frac{1}{\sqrt{2}}}\sqrt{2}\,|\mathrm{ch}\,u|\sqrt{2}\,\mathrm{ch}\,u\,du$$

zbog granica je pozitivno, pa micemo | |

možemo maknuti zbog granica

$$= 2\int_0^{\operatorname{arsh}\frac{1}{\sqrt{2}}} ch^2 u\,du = 2\int_0^{\operatorname{arsh}\frac{1}{\sqrt{2}}} \frac{ch2u+1}{2}\,du = \left(\frac{1}{2}sh2u+u\right)\Big|_0^{\operatorname{arsh}\frac{1}{\sqrt{2}}} =$$

$$= \underbrace{\frac{1}{2}sh\left(2\operatorname{arsh}\frac{1}{\sqrt{2}}\right)}_{\text{WOLFRAM: } \frac{\sqrt{3}}{2}} + \operatorname{arsh}\frac{1}{\sqrt{2}}$$

12. DZ-20.)

$$\int_0^1 x^2\sqrt{x-x^2}\,dx =$$

$$-(x^2-x) = -\left(\left(x-\frac{1}{2}\right)^2-\frac{1}{4}\right)$$

$\sqrt{a^2-x^2}$

$x = a\sin t$

$$= \int_0^1 x^2\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}\,dx = \left| \begin{array}{l} x-\frac{1}{2} = \frac{1}{2}\sin t \\ dx = \frac{1}{2}\cos t\,dt \\ x = \frac{1}{2}\sin t+\frac{1}{2} \\ x=0 \to \alpha = \frac{\pi}{2} \\ x=1 \to \beta = -\frac{\pi}{2} \end{array} \right| =$$

NE SMIJE BIT $\frac{3\pi}{2}$!! U SMJERU PORASTA GLEDATI

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2}\sin t+\frac{1}{2}\right)^2\sqrt{\frac{1}{4}\cos^2 t}\cdot\frac{1}{2}\cos t\,dt =$$

$$= \frac{1}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^2 t+2\sin t+1)\cos^2 t\,dt =$$

$$= \frac{1}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t\cos^2 t\,dt + \frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin t\cos^2 t\,dt + \frac{1}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\,dt =$$

$$\underbrace{(\sin t\cdot\cos t)^2}\qquad \left(\frac{1}{2}\sin 2t\right)^2$$

$$\left| \begin{array}{l} \cos t = u \\ -\sin t\,dt = du \\ x = -\frac{\pi}{2} \to 0 = \alpha \\ x = \frac{\pi}{2} \to \beta = 0 \end{array} \right|$$

$$\frac{1}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\,dt = \frac{\pi}{32}$$

$$\frac{1}{64}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\cos 4t}{2}\,dt = \frac{5\pi}{128}$$

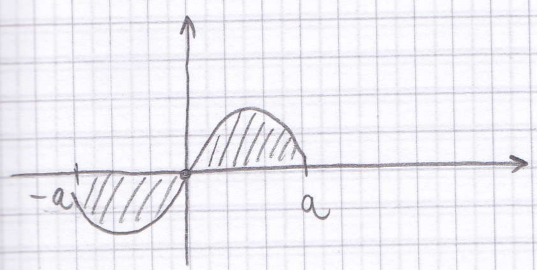
$$-\int_0^0 u^2\,du = 0$$

$$= \frac{\pi}{128}$$

-11.)

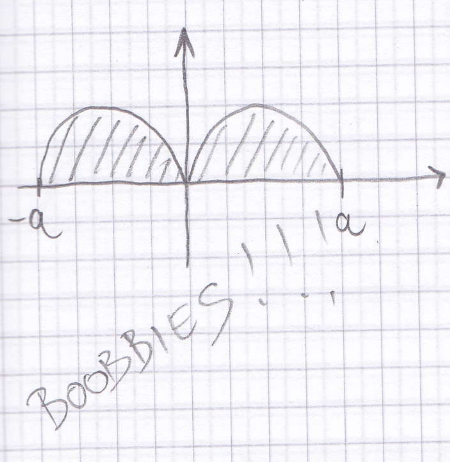
$$\int_{-1}^{1} x\sqrt{\frac{4-x^2+x^4}{4+x^2-x^4}}\,dx = 0$$

→ NEPARNE FUNKCIJE NA SIMETRIČNOM INTERVALU IMAJU VRIJEDNOST INTEGRALA 0 !



$\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$ ako je f neparna!

→ PARNE FUNKCIJE:



$\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\cdot\int_{0}^{a} f(x)\,dx$, ako je f parna!