# 1. domaća zadaća
# Modeliranje dinamičkih sustava

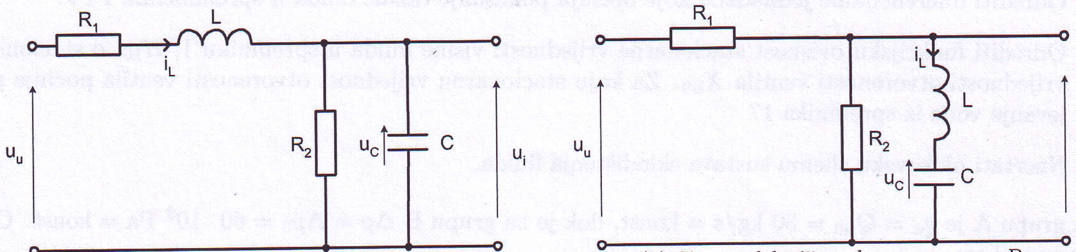
## 1. Uvod

Modeli sustava općenito predstavljaju podlogu za analizu i sintezu sustava upravljanja i zasnivaju se na teoretskoj analizi (fizikalni modeli) te na identifikaciji procesa (empirijski modeli). Fizikalni modeli sustava temelje se na općem principu konzervacije (održanju ravnoteže) uz korištenje konstitutivnih relacija (zakona) na temelju kojih se odvijaju promjene u procesu (zakoni elektrotehnike, mehanike, termodinamike, mehanike fluida i sl.). Nasuprot tome, empirijski modeli u pravilu ne zahtijevaju uvid u fizikalnost pojava koje se modelira, nego samo odgovarajući skup ulazno-izlaznih mjernih podataka na temelju kojeg se odabire prikladna struktura modela sustava i parametri modela. Cilj je ove domaće zadaće doći do fizikalnog modela sustava teoretskom analizom za neke od tipičnih primjera iz prakse.

## Priprema za vježbu

### Zadatak 1

Na Slici 1(a) prikazana je shema električnog kruga za grupu A, a na Slici 1(b) za grupu B. Potrebno je:




(a) Shema električnog kruga za grupu A. (b) Shema električnog kruga za grupu B.

Slika 1: Električni krugovi.

a) Napisati diferencijalnu jednadžbu koja opisuje ovisnost izlaznog napona $u_i(t)$ o ulaznom naponu $u_u(t)$. Jednadžbu treba svesti na oblik u kojem je koeficijent uz $u_i(t)$ jednak 1.

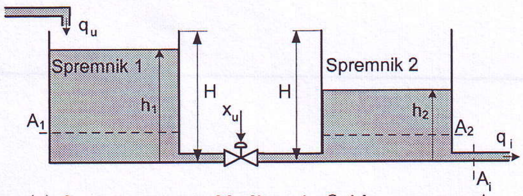
b) Odrediti matrice $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ i $\mathbf{D}$ iz zapisa sustava po varijablama stanja

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u},$$

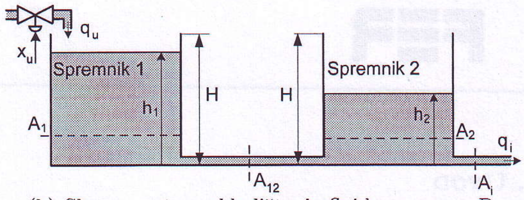
pri čemu su vektori stanja, ulaza i izlaza definirani kao: $\mathbf{x} = [u_C \; i_L]^T$, $\mathbf{u} = [u_u]$, $\mathbf{y} = [u_i]$.

## ZADATAK 2

Na Slici 2(a) prikazana je principna shema sustava skladištenja fluida za grupu A, a na Slici 2(b) za grupu B. Razina fluida u spremnicima regulira se promjenom otvorenosti ventila $x_u(t)$ [%] koja može poprimiti




(a) Shema sustava skladištenja fluida za grupu A.




(b) Shema sustava skladištenja fluida za grupu B.

Slika 2: Sustavi skladištenja fluida.

vrijednosti između 0% i 100%. Karakteristika ventila opisana je izrazom

$$q(t) = A_v\sqrt{\rho}\sqrt{2\Delta p}\cdot\frac{x_u}{100\%},$$

pri čemu je:

$x_u$ – otvorenost ventila [%],
$A_v$ – poprečni presjek potpuno otvorenog ventila [m$^2$],
$\Delta p$ – razlika tlakova na krajevima ventila [Pa],
$\rho$ – gustoća fluida [kg/m$^3$],
$q$ – maseni protok kroz ventil [kg/s].

Potrebno je:

a) Odrediti diferencijalne jednadžbe koje opisuju ponašanje razine fluida u spremnicima 1 i 2.

b) Odrediti funkcijsku ovisnost stacionarne vrijednosti visine fluida u spremniku 1, $H_{10}$, o stacionarnoj vrijednosti otvorenosti ventila $X_{u0}$. Za koju stacionarnu vrijednost otvorenosti ventila počinje prelijevanje vode iz spremnika 1?

c) Nacrtati blokovsku shemu sustava skladištenja fluida.

Za grupu A je $q_u = Q_{u0} = 30$ kg/s = konst, dok je za grupu B $\Delta p = \Delta p_0 = 60\cdot 10^3$ Pa = konst. Ostali parametri su:

$A_1 = 5$ m$^2$ – površina poprečnog presjeka spremnika 1,
$A_2 = 5$ m$^2$ – površina poprečnog presjeka spremnika 2,
$A_{12} = 0.01$ m$^2$ – površina poprečnog presjeka spojne cijevi između spremnika,
$A_i = 0.01$ m$^2$ – površina poprečnog presjeka izlazne cijevi drugog spremnika,
$\rho = 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3}$ – gustoća fluida,
$A_v = 0.008$ m$^2$ – poprečni presjek potpuno otvorenog ventila,
$H = 5$ m – visina spremnika 1 i spremnika 2,
$g = 9.81\ \frac{\mathrm{m}}{\mathrm{s}^2}$ – ubrzanje sile teže.

Kod računanja izlaznih protoka iz spremnika može se uzeti da je $A_{12} \ll A_1$ i $A_i \ll A_2$.

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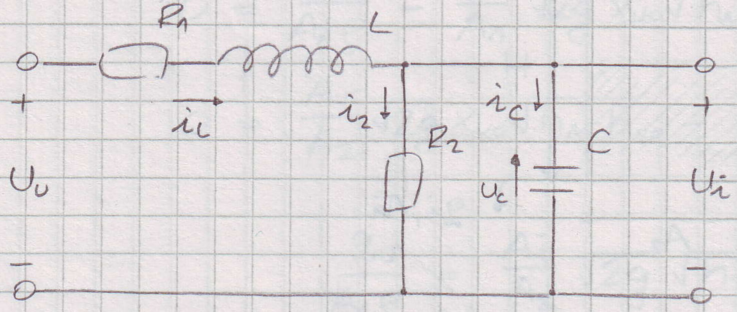
• ZADATAK 1




(a) $$U_v = i_L R_1 + L\frac{di_L}{dt} + U_i \rightarrow (i_L)' = -\frac{R_1}{L} i_L - \frac{1}{L}U_c + \frac{1}{L}U_v$$

$$U_i = U_C = U_{R2}$$

$$U_C = \frac{1}{C}\int_0^t i_C(\tau)\,d\tau \rightarrow i_C = C\dot{U}_C, \quad i_C = C\dot{U}_i$$

$$U_{R2} = i_2 R_2 \rightarrow i_2 = \frac{1}{R_2}U_{R2}, \quad i_2 = \frac{1}{R_2}U_i$$

$$i_L = i_C + i_2 \rightarrow \dot{U}_C = \frac{1}{C} i_L - \frac{1}{R_2C}U_C$$

$$i_L = C\dot{U}_i + \frac{1}{R_2}U_i, \quad (i_L)' = C\ddot{U}_i + \frac{1}{R_2}\dot{U}_i$$

$$U_v = CR_1\dot{U}_i + \frac{R_1}{R_2}U_i + CL\ddot{U}_i + \frac{L}{R_2}\dot{U}_i + U_i$$

$$U_v = CL\ddot{U}_i + \left(CR_1 + \frac{L}{R_2}\right)\dot{U}_i + \left(1+\frac{R_1}{R_2}\right)U_i$$

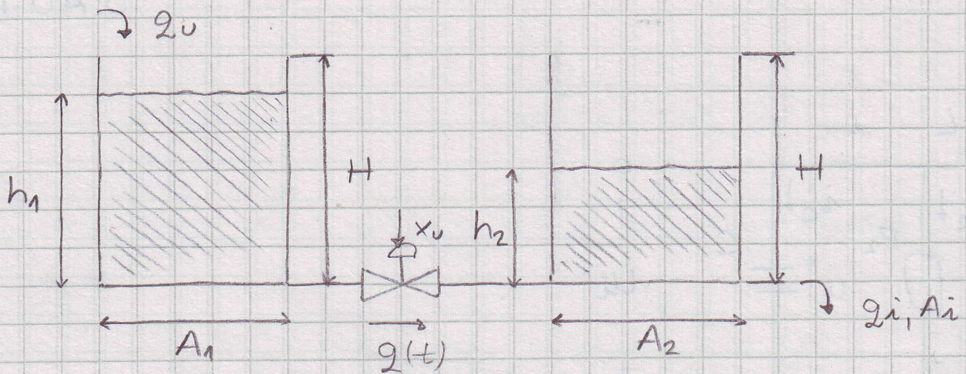
$$\frac{R_2LC}{R_1+R_2}\ddot{U}_i + \frac{L+R_1R_2C}{R_1+R_2}\dot{U}_i + U_i = \frac{R_2}{R_1+R_2}U_v$$

(b) $x = [U_c \;\; i_L]^T, \quad u = [U_v], \quad y = [U_i]$

$$\begin{bmatrix} \dot{U}_c \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_2C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_1}{L} \end{bmatrix} \begin{bmatrix} U_c \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} [U_v]$$

$$[U_i] = [0 \;\; 1] \begin{bmatrix} U_c \\ i_L \end{bmatrix} + [0][U_v]$$

• ZADATAK 2




$$q(t) = A_v \sqrt{\rho} \sqrt{2\Delta p}\, x_v$$

$x_v$ – OTVORENOST VENTILA

$\Delta p$ – RAZLIKA TLAKOVA NA KRAJEVIMA VENTILA

(a) (1) $q_u(t) - q(t) = A_1 \rho \dfrac{dh_1}{dt}$

$$q_u(t) - A_v \sqrt{\rho} \sqrt{2\Delta p}\, x_v = A_1 \rho \frac{dh_1}{dt}$$

$\Delta p = p_1 - p_2$ $\qquad$ $p_a$ – ATMOSFERSKI TLAK

$$\left.\begin{aligned} p_1 &= p_a + \rho g h_1 \\ p_2 &= p_a + \rho g h_2 \end{aligned}\right\} \Delta p = \rho g (h_1 - h_2)$$

$$q_u(t) - A_v \rho \sqrt{2g} \sqrt{h_1 - h_2}\, x_v = A_1 \rho \frac{dh_1}{dt}$$

• $\dfrac{dh_1}{dt} = \dfrac{q_u}{A_1 \rho} - \dfrac{A_v}{A_1} \sqrt{2g}\, x_v \sqrt{h_1 - h_2}$

(2) $q(t) - q_i(t) = A_2 \rho \dfrac{dh_2}{dt}$

$$q_i(t) = A_i \rho \sqrt{2g} \sqrt{h_2}$$

$$A_v \rho \sqrt{2g} \sqrt{h_1 - h_2}\, x_v - A_i \rho \sqrt{2g} \sqrt{h_2} = A_2 \rho \frac{dh_2}{dt}$$

• $\dfrac{dh_2}{dt} = \dfrac{A_v}{A_2} \sqrt{2g}\, x_v \sqrt{h_1 - h_2} - \dfrac{A_i}{A_2} \sqrt{2g} \sqrt{h_2}$

(b) $$\frac{dh_1}{dt} = \frac{dh_2}{dt} = 0$$

$$0 = \frac{q_v}{A_1 \rho} - \frac{A_v}{A_1}\sqrt{2g}\, x_{v0}\sqrt{h_{10}-h_{20}}$$

$$0 = \frac{A_v}{A_2}\sqrt{2g}\, x_{v0}\sqrt{h_{10}-h_{20}} - \frac{A_i}{A_2}\sqrt{2g}\,\sqrt{h_{20}}$$

$$\frac{q_v}{\cancel{A_1}\rho} = \frac{A_i}{\cancel{A_2}}\sqrt{2g}\,\sqrt{h_{20}}\,, \quad A_1 = A_2 = 5\ m$$

$$h_{20} = \left(\frac{q_v}{\sqrt{2g}\,A_i}\cdot\frac{1}{\rho}\right)^2 \rightarrow h_{20} = 0.4587\ m$$

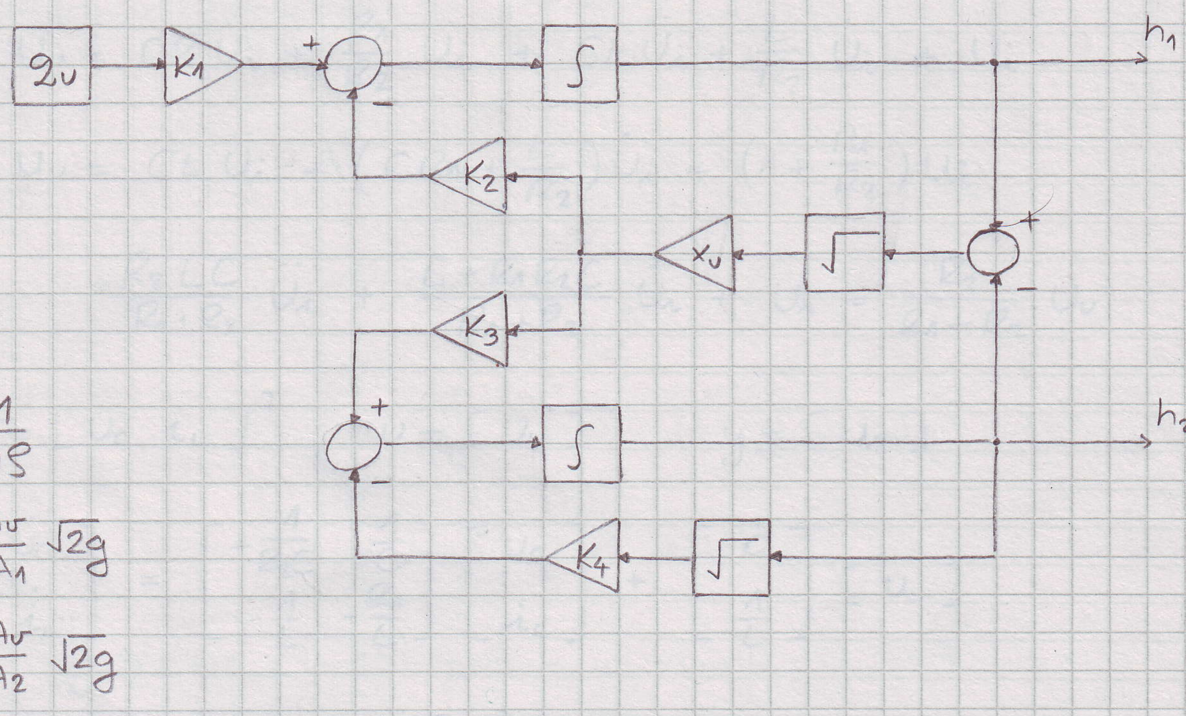
$$\frac{A_v}{\cancel{A_2}}\cancel{\sqrt{2g}}\, x_{v0}\sqrt{h_{10}-h_{20}} = \frac{A_i}{\cancel{A_2}}\cancel{\sqrt{2g}}\,\sqrt{h_{20}}$$

$$A_v x_{v0}\sqrt{h_{10}-h_{20}} = A_i\sqrt{h_{20}}$$

$$\cdot\ h_{20} = h_{10}\,\frac{1}{1+\left(\frac{A_i}{A_v x_{v0}}\right)^2}$$

ZA $h_{10} = 5\ m$, $h_{20} = 0.4587\ m \rightarrow x_{v0} = 0.397$

(c)




$$K_1 = \frac{1}{A_1\rho}$$

$$K_2 = \frac{A_v}{A_1}\sqrt{2g}$$

$$K_3 = \frac{A_v}{A_2}\sqrt{2g}$$

$$K_4 = \frac{A_i}{A_2}\sqrt{2g}$$

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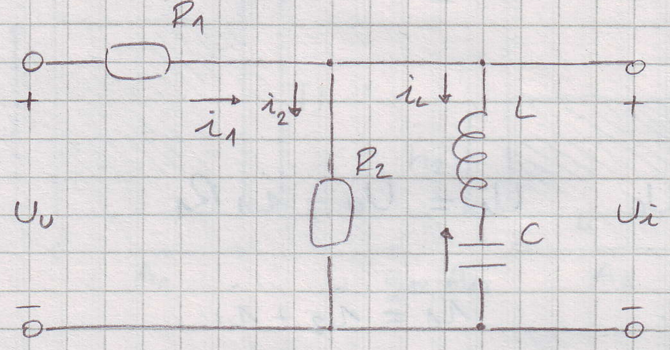
• ZADATAK 1




(a) $U_u = i_1 R_1 + U_i$

$$i_1 = i_2 + i_L \quad , \quad U_i = U_{R2} = U_C + U_L$$

$$i_2 = \frac{1}{R_2} U_i \quad , \quad i_1 = \frac{1}{R_1} U_u - \frac{1}{R_1} U_i$$

$$i_L = \frac{1}{R_1} U_u - \frac{1}{R_1} U_i - \frac{1}{R_2} U_i$$

$$i_L = \frac{1}{R_1} U_u - \frac{R_1 + R_2}{R_1 R_2} U_i$$

$$U_i = \frac{1}{C} \int i_L(\tau) d\tau + L \frac{di_L}{dt} \Big/ \frac{d}{dt}$$

$$\dot{U}_i = \frac{1}{C} i_L + L \frac{d^2 i_L}{dt^2}$$

$$(i_L)' = \frac{1}{R_1} \dot{U}_u - \frac{R_1 + R_2}{R_1 R_2} \dot{U}_i$$

$$(i_L)'' = \frac{1}{R_1} \ddot{U}_u - \frac{R_1 + R_2}{R_1 R_2} \ddot{U}_i$$

$$\dot{U}_i = \frac{1}{C R_1} U_u - \frac{R_1 + R_2}{C R_1 R_2} U_i + \frac{L}{R_1} \ddot{U}_u - \frac{L(R_1 + R_2)}{R_1 R_2} \ddot{U}_i$$

$$LC \ddot{U}_i + \frac{C R_1 R_2}{R_1 + R_2} \dot{U}_i + U_i = \frac{LC R_2}{R_1 + R_2} \ddot{U}_u + \frac{R_2}{R_1 + R_2} U_u$$

(b) $x = [\,U_C \quad i_L\,]^T$ , $u = [\,U_U\,]$ , $y = [\,U_i\,]$

$$U_C = \frac{1}{C}\int i_L(\tau)\,d\tau \rightarrow \quad \cdot\ \dot{U}_C = \frac{1}{C}\, i_L$$

$$U_i = L\frac{di_L}{dt} + U_C$$

$$\frac{di_L}{dt} = \frac{1}{L}U_i - \frac{1}{L}U_C \quad , \quad U_i = U_U - i_1 R_1$$

$$i_1 = i_2 + i_L$$

$$i_2 = \frac{1}{R_2}U_i$$

$$i_1 = \frac{1}{R_2}U_i + i_L$$

$$U_i = U_U - \left(\frac{1}{R_2}U_i + i_L\right)R_1$$

$$U_i = U_U - \frac{R_1}{R_2}U_i - R_1 i_L$$

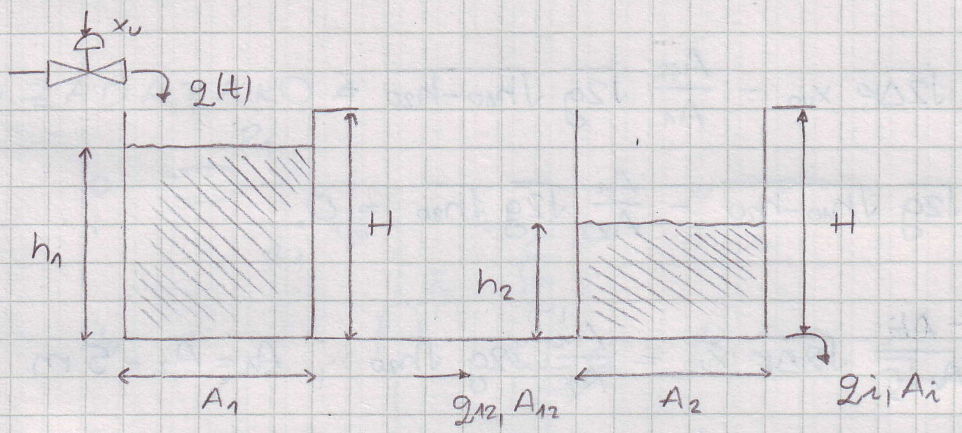
$$\cdot\ U_i = \frac{R_2}{R_1+R_2}U_U - \frac{R_1R_2}{R_1+R_2}i_L$$

$$\cdot\ (i_L)' = \frac{R_2}{L(R_1+R_2)}U_U - \frac{R_1R_2}{L(R_1+R_2)}i_L - \frac{1}{L}U_C$$

$$\begin{bmatrix}\dot{U}_C\\ \dot{i}_L\end{bmatrix} = \begin{bmatrix}0 & \frac{1}{C}\\ -\frac{1}{L} & -\frac{R_1R_2}{L(R_1+R_2)}\end{bmatrix}\begin{bmatrix}U_C\\ i_L\end{bmatrix} + \begin{bmatrix}0\\ \frac{R_2}{L(R_1+R_2)}\end{bmatrix}[\,U_U\,]$$

$$[\,U_i\,] = \left[\,0 \quad -\frac{R_1R_2}{R_1+R_2}\right]\begin{bmatrix}U_C\\ i_L\end{bmatrix} + \left[\frac{R_2}{R_1+R_2}\right][\,U_U\,]$$

· ZADATAK 2




$$q(t) = A_v \sqrt{\rho} \sqrt{2\Delta p}\, x_v$$

$x_v$ – OTVORENOST VENTILA

$\Delta p$ – RAZLIKA TLAKOVA NA KRAJEVIMA VENTILA

(a) (1) $$q(t) - q_{12}(t) = A_1 \rho \frac{dh_1}{dt}$$

$$q_{12}(t) = A_{12} \rho \sqrt{2g} \sqrt{h_1 - h_2}$$

$$A_v \sqrt{\rho} \sqrt{2\Delta p}\, x_v - A_{12} \rho \sqrt{2g} \sqrt{h_1 - h_2} = A_1 \rho \frac{dh_1}{dt}$$

$$\frac{dh_1}{dt} = \frac{A_v}{A_1 \sqrt{\rho}} \sqrt{2\Delta p}\, x_v - \frac{A_{12}}{A_1} \sqrt{2g} \sqrt{h_1 - h_2}$$

(2) $$q_{12}(t) - q_i(t) = A_2 \rho \frac{dh_2}{dt}$$

$$q_i(t) = A_i \rho \sqrt{2g} \sqrt{h_2}$$

$$A_{12} \rho \sqrt{2g} \sqrt{h_1 - h_2} - A_i \rho \sqrt{2g} \sqrt{h_2} = A_2 \rho \frac{dh_2}{dt}$$

$$\frac{dh_2}{dt} = \frac{A_{12}}{A_2} \sqrt{2g} \sqrt{h_1 - h_2} - \frac{A_i}{A_2} \sqrt{2g} \sqrt{h_2}$$

(b) $\frac{dh_1}{dt} = \frac{dh_1}{dt} = 0$

$$\frac{A_v}{A_1\sqrt{\rho}}\sqrt{2\Delta p}\,x_{v0} - \frac{A_{12}}{A_1}\sqrt{2g}\sqrt{h_{10}-h_{20}} = 0$$

$$\frac{A_{12}}{A_2}\sqrt{2g}\sqrt{h_{10}-h_{20}} - \frac{A_i}{A_2}\sqrt{2g}\sqrt{h_{20}} = 0$$

$$\frac{A_v}{A_1\sqrt{\rho}}\sqrt{2\Delta p}\,x_{v0} = \frac{A_i}{A_2}\sqrt{2g}\sqrt{h_{20}}\quad,\quad A_1 = A_2 = 5\,m$$

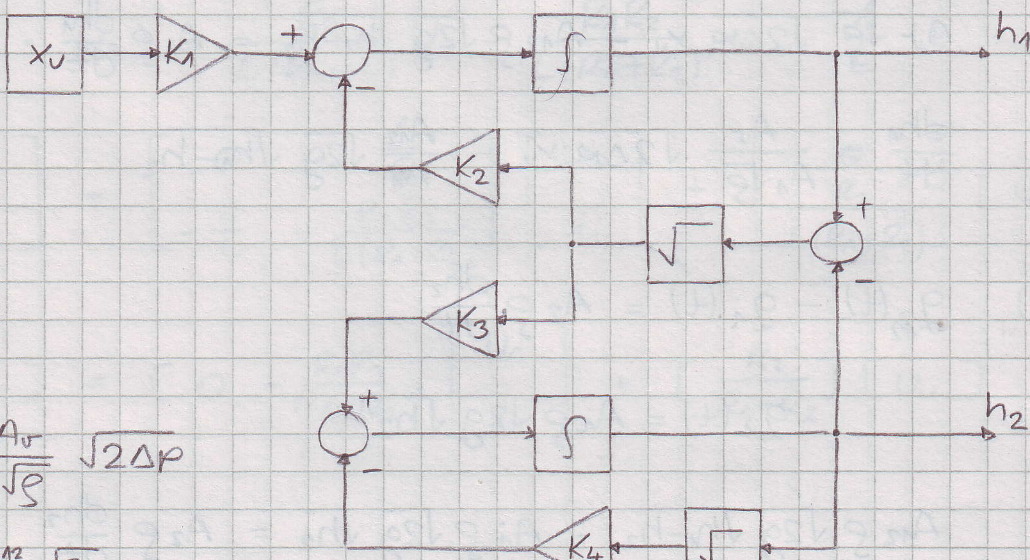
$$\cdot\ h_{20} = \left(\frac{A_v x_{v0}}{A_i}\right)^2 \frac{\Delta p}{\rho g}$$

$$A_{12}\sqrt{2g}\sqrt{h_{10}-h_{20}} = A_i\sqrt{2g}\sqrt{h_{20}}$$

$$\cdot\ h_{10} = \left(1 + \frac{A_{12}^2}{A_i^2}\right) h_{20}$$

ZA $h_{10} = 5\,m \rightarrow h_{20} = 2{,}5\,m\ ,\ x_{v0} = 0.7992$

(c)




$$K_1 = \frac{A_v}{A_1\sqrt{\rho}}\sqrt{2\Delta p}$$

$$K_2 = \frac{A_{12}}{A_1}\sqrt{2g}$$

$$K_3 = \frac{A_{12}}{A_2}\sqrt{2g}$$

$$K_4 = \frac{A_i}{A_2}\sqrt{2g}$$