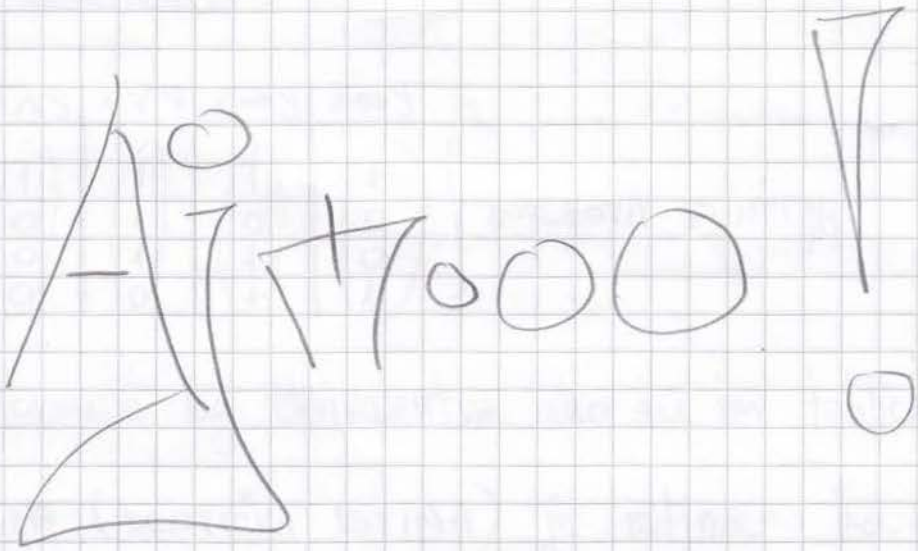
petak: 10-12 h i 14-16 h

konzultacije: četvrtak 12-14 h → D182 (ZPM)

petkom od 12 do 13 h i od 13 do 14 h → auditorne → školske



1. MATEMATIČKA LOGIKA. KOMPLEKSNI BROJEVI

1.1 LOGIKA SUDOVA

Z.) $\pi > e$ 1 → istina

"e je lepši od π" 0 → laž

→ operacije: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$

↳ implikacija ↳ ekvivalencija → ≡

→ tablica istinitosti:

x y	$x \wedge y$	$x \vee y$	$x \Rightarrow y$	$x \Leftrightarrow y$
1 1	1	1	1	1
1 0	0	1	0	0
0 1	0	1	1	0
0 0	0	0	1	1

} BOOLEOVA FUNKCIJA

Dvije su formule log. ekvivalentne ako su im tablice istinosti jednake.

TAUTOLOGIJA (IDENTIČKI ISTINA) je istina za svaki izbor vrijednosti.

KONTRADIKCIJA - je laž za svaki izbor vrijednosti.

1. MI. 2007. - 1) Dokaži da je tautologija $(x \wedge (x \Rightarrow y)) \Rightarrow y$.

x y	$(x \wedge (x \Rightarrow y))$	$\Rightarrow y$
1 1	1 1 1 1 1	1 1
1 0	1 0 1 0 0	1 0
0 1	0 0 0 1 1	1 1
0 0	0 0 0 1 0	1 0

→ sve 1 ⇒ tautologija

ili da su bile sve 0 → kontradikc

• PRAVILA ZA SUDOVE:

$\neg(\neg X) \equiv X$

$\neg(X \wedge Y) \equiv \neg X \vee \neg Y$ } DE MORGANOVE FORMULE

$\neg(X \vee Y) \equiv \neg X \wedge \neg Y$

$\neg(X \Rightarrow Y) \equiv X \wedge \neg Y$

$X \Rightarrow Y \equiv \neg X \vee Y$

$(X \Rightarrow Y) \equiv (\neg Y \Rightarrow) \neg X$ } OBRAT PO KONTRAPOZICIJI

→ dokazi se rade tablicom istine!

ZZV1-2) Pojednostavi: $\neg(\neg A \wedge (B \vee \neg C))$

↳ zadaci za vježbu, 1. knjižica 2. zad.

$\equiv A \vee \neg(B \vee \neg C) \equiv A \vee (\neg B \wedge C)$

De Morgan $\neg(\neg A) \equiv A$

1.2. PREDIKATI

Pr.) $x^2 > \pi$ nije sud, NE ZNAMO x, ako uvrstimo konkretan x onda je sud

↳ PREDIKAT

→ Oblik $P(x_1, \dots, x_n)$ koji za svaki izbor elem. x iz nekog univerzalnog skupa U postaje sud zovemo n-MJESNI PREDIKAT.

Pr. 2)

a) $U = \mathbb{R}$

$x^2 + y^2 \geq 0$ → uvijek → $\underline{1}$ (T) ↳ true

b) $U = \mathbb{N}$

$x^2 + y^2 + z^2 < 3$ → nikada → $\underline{0}$ (F) ← za $U = \mathbb{N}_0$ → postoji slučaj kada je istina

• KVANTIFIKATORI:

$\forall x$ ..... univerzalni – za svaki x vrijedi $P(x)$

$\exists x$ ..... egzistencijalni – postoji x za koji vrijedi $P(x)$

Pr. 2)

b) $\mathcal{U} = \mathbb{N}$

$\exists x \exists y \exists z \ (x^2+y^2+z^2<3)$ F

$\mathcal{U} = \mathbb{N}_0$

$\exists x \exists y \exists z \ (x^2+y^2+z^2<3)$ T

Pr. 3) $\mathcal{U} = \mathbb{R}$

$\forall x \exists y \ (x+y=y)$ F

$\exists x \forall y \ (x+y=y)$ T (→ kada je $x=0$!!)

1.DZ – 3) Ispitaj istinitosti:

$A \equiv (\forall x \in \mathbb{R})(\exists y \in \mathbb{R}) \ (y>x)$ T

$B \equiv (\exists y \in \mathbb{R})(\forall x \in \mathbb{R}) \ (y>x)$ F

• PRAVILA NEGACIJE (DE MORGAN ZA PREDIKATE):

$\neg(\forall x P(x)) \equiv \exists x \neg P(x)$

$\neg(\exists x (Px)) \equiv \forall x \neg P(x)$

ZZV1 – 4)

$A \equiv (\forall x \in \mathbb{R}^+)(\exists y \in \mathbb{R}^+) \ (y \geq x^2 \wedge y \leq x^3)$ F (npr $x=\frac{1}{2}$!!!) ≤ 1 !

$\neg A \equiv (\exists x \in \mathbb{R}^+)(\forall y \in \mathbb{R}^+) \ (y < x^2 \vee y > x^3)$ T

D.Z. ZZV1 → 1, 2, 3, 4 → 1.DZ.

1.3. SKUPOVI I PRESLIKAVANJA

Pr.)

$S = \{x \mid x^2 < 12\}$, $U = \mathbb{N}$

$\downarrow$

$S = \{1, 2, 3\}$

- operacije: $\overline{S} = \{x \mid x \notin S\}$ KOMPLEMENT

$S_1 \cup S_2 = \{x \mid x \in S_1 \vee x \in S_2\}$ UNIJA

$S_1 \cap S_2 = \{x \mid x \in S_1 \wedge x \in S_2\}$ PRESJEK

$S_1 \setminus S_2 = \{x \mid x \in S_1 \wedge x \notin S_2\}$

$S_1 \subseteq S_2$ PODSKUP

1. DZ-5.) $A = \{x \in \mathbb{N} \mid (\exists y \in \mathbb{N})\ x + y = 5\} = \{1, 2, 3, 4\}$

D.Z. $B = ?$

KARTEZIJEV PRODUKT:

$$\boxed{S_1 \times S_2 = \{(x, y) \mid x \in S_1 \wedge y \in S_2\}}$$

Pr.)

$A = \{1, 2, 3\}$

$B = \{1, 2\}$

SVE MOGUĆE KOMBINACIJE

$A \times B = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$

•• PRESLIKAVANJA ••

def.

$f: S_1 \longrightarrow S_2$ je preslikavanje iz S_1 u S_2 ako je svakom $x \in S_1$ pridružen jedinstveni $y \in S_2$.

$y = f(x)$

$D(f) = S_1$ DOMENA, područje definicije

$Im(f) \subseteq S_2$ $\quad Im(f) = f(S_1)$ KODOMENA, slika

def.

a) Preslikavanje $f: S_1 \longrightarrow S_2$ je INJEKCIJA AKO RAZLIČITI ORIGINALI IMAJU RAZLIČITE SLIKE.

$$(\forall x_1, x_2 \in S_1)(x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2))$$

b) Preslikavanje $f: S_1 \longrightarrow S_2$ je SURJEKCIJA ako $f(S_1) = S_2$, tj.

$$(\forall y \in S_2)(\exists x \in S_1)(f(x) = y)$$

c) Preslikavanje $f: S_1 \longrightarrow S_2$ je BIJEKCIJA ako je i injekcija i surjekcija.

def.

$f: S_1 \longrightarrow S_2$, $X \subseteq S_1$. Tada $f|X$ zovemo RESTRIKCIJA preslikavanja f iz X u S_2.

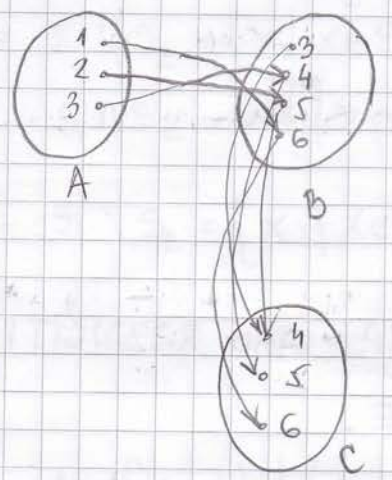
1. DZ - 6a)

$A = \{1, 2, 3\}$

$B = \{3, 4, 5, 6\}$

$C = \{4, 5, 6\}$

Postoji li surjekcija iz A u B ?



INJEKCIJA (DA) → iz jednog iz A ide u samo jedan iz B

SURJEKCIJA (NE) → 3 u B je ostao, ne postoji za sve

iz B → C : INJEKCIJA (NE)

SURJEKCIJA (DA)

kada ima više originala NE MOŽE biti injekcija

→ restrikcijom mičemo „onaj koji nam ne odgovara"

def.

$f: S_1 \longrightarrow S_2$

$g: S_2 \longrightarrow S_3$

<u>KOMPOZICIJA</u> $g \circ f$ je preslikavanje $S_1 \longrightarrow S_3$, takvo da $(\forall x \in S_1)\ (g \circ f)(x) = g(f(x))$.

def. Neka je $f: S_1 \longrightarrow S_2$. Preslikavanje $g: S_2 \longrightarrow S_1$ se zove <u>INVERZNO PRESLIKAVANJE</u> ako $(\forall x \in S_1)\ g(f(x)) = x$ i $(\forall y \in S_2)\ f(g(y)) = y$.

TM: Za $f: S_1 \to S_2$ postoji $f^{-1}: S_2 \to S_1$ ako i samo ako je f bijekcija.

- OMEĐENOST SKUPOVA ..

def. Za podskup $S \subseteq \mathbb{R}$ kažemo da je OMEĐEN ODOZGO (ODOZDO) ako postoji $M \in \mathbb{R}$ ($m \in \mathbb{R}$) tako da je $(\forall x \in S)\ x \leq M$ ($x \geq m$).

def. SUPREMUM skupa S ($\sup S$) je NAJMANJA GORNJA MEĐA.
INFIMUM skupa S ($\inf S$) je NAJVEĆA DONJA MEĐA.

$\sup S$ zovemo MAKSIMUM ako se $\sup S$ nalazi unutar S.
$\inf S$ zovemo MINIMUM ako se $\inf S$ nalazi unutar S.

Z.)
$A = [0, 1\rangle$
$\inf A = 0$
$\min A = 0$
$\sup A = 1$
max NE POSTOJI !

- Skup je OMEĐEN ako je omeđen odozgo i odozdo.

1.4. MATEMATIČKA INDUKCIJA

Dana je tvrdnja $T(n)$, $n \geq n_0$.

1. BAZA : $T(n_0)$
2. PRETPOSTAVI DA $T(n)$ VRIJEDI
3. KORAK INDUKCIJE $T(n+1)$

1.M1-2006-2) Mat. ind. dokaži:
=ZZV1-7.b)

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} \quad \text{za sve } n \in \mathbb{N}.$$

$$T(n): \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

1. BAZA : $\underline{n=1}$

$$\frac{1}{1\cdot 2} = \frac{1}{1+1}$$

$$\frac{1}{2} = \frac{1}{2} \quad \checkmark \text{ O.K.}$$

2. PRETPOSTAVIMO DA TVRDNJA $T(n)$ VRIJEDI za neki $n \in \mathbb{N}$

3. — " — ZA $T(n+1)$ VRIJEDI za $n+1$

$$\underbrace{\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)}}_{\text{pretpostavka}} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$$

$$\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^{\cancel{2}}}{\cancel{(n+1)}(n+2)} = \frac{n+1}{n+2} \quad //$$

$\Downarrow$

1. DZ - 13.)

$$\prod_{i=1}^{n} \cos\left(\frac{x}{2^i}\right) = \frac{\sin x}{2^n \sin\left(\frac{x}{2^n}\right)} \quad \text{za } \forall n \in \mathbb{N} \text{ i } \forall x \in \mathbb{R}.$$

$$\cos\frac{x}{2} \cdot \cos\frac{x}{2^2} \cdot \cos\frac{x}{2^3} \cdot \ldots \cdot \cos\frac{x}{2^n} = \frac{\sin x}{2^n \sin\left(\frac{x}{2^n}\right)}$$

1. BAZA : $\underline{n=1}$

$$\cos\frac{x}{2} = \frac{\sin x}{2^1 \sin\left(\frac{x}{2^1}\right)}$$

$$\cos\frac{x}{2} = \frac{\cancel{2\sin\frac{x}{2}} \cdot \cos\frac{x}{2}}{\cancel{2\sin\frac{x}{2}}}$$

$$\cos\frac{x}{2} = \cos\frac{x}{2} \quad \checkmark \text{ o.k.}$$

2. pretpostavimo da tvrdnja vrijedi za n

3. -|- za $n+1$

$$\underbrace{\cos\frac{x}{2} \cdot \cos\frac{x}{2^2} \cdot \ldots \cos\frac{x}{2^n}} \cdot \cos\frac{x}{2^{n+1}} = \frac{\sin x}{2^{n+1} \sin\frac{x}{2^{n+1}}}$$

$$\frac{\sin x}{2^n \cdot \sin\frac{x}{2^n}} \cdot \cos\frac{x}{2^{n+1}} = \frac{\sin x}{2^n \cdot 2\sin\frac{x}{2^{n+1}} \cdot \cancel{\cos\frac{x}{2^{n+1}}}} \cdot \cancel{\cos\frac{x}{2^{n+1}}} =$$

pobrisati arg

$$= \frac{\sin x}{2^{n+1} \sin\frac{x}{2^{n+1}}} \quad //$$

$\parallel$

zzv1-7d) Mat. ind. dokaži:

$$3^n (n!)^2 < (2n)! \ , \ \underline{\underline{n \geq 5}} .$$

1. baza : $\underline{n=5}$

$$3^5 (5!)^2 < 10! \quad \checkmark \text{ o.k.}$$

2. pretpostavimo da tvrdnja vrijedi za n

3. -1- za $n+1$

$$(n+1)! = n!(n+1) = \underbrace{1 \cdot 2 \cdot \ldots \cdot n}_{n!} \cdot (n+1)$$

$$\underline{3^{n+1} ((n+1)!)^2 < (2n+2)!}$$

rješavamo

$$3^n \cdot 3 \cdot (n!)^2 (n+1)^2 \overset{k=0}{<} (2n)! \, 3(n+1)^2 < (2n)! \, (2n+1)(2n+2)$$

pretp.

$$< 3n^2 + \cancel{6n} + 3 < 4n^2 + \cancel{6n} + 2$$

$$\underline{\underline{1 < n^2}}$$

DA jer $\underline{\underline{n \geq 5}}$! $\Downarrow$

* PODSJETNIK:

→ BINOMNI KOEFICIJENTI:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k!} \quad // \quad k = 0, 1, \ldots, n$$

$$\binom{n}{0} = 1$$

→ BINOMNA FORMULA:

$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k}$$

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \ldots b^n$$

DZ-15.) Mat. ind. dokaži da je zbroj kubova tri uzastopna prirodna broja djeljiv s 9.

1. baza:

$$1^3+2^3+3^3 = 36 \mid 9 \quad \checkmark \text{ o.k.}$$

2. pretpostavimo da

$$9 \mid n^3+(n+1)^3+(n+2)^3 = 9\cdot k$$

$$(n+1)^3+(n+2)^3 = 9k - n^3$$

3.) -|- za $n+1$

$$\underbrace{(n+1)^3+(n+2)^3}+(n+3)^3 =$$

$$9k - n^3 + n^3 + 3n^2\cdot 3 + 3n\cdot 9 + 27 =$$

$$= 9(k+n^2+3n+3)$$

$$= 9\cdot \underline{\underline{l}}$$

Π WOOHOO !!

→ u pon. u 17^{h} predati test srednjoškolskog gradiva!

1.5. KOMPLEKSNI BROJEVI

$\sqrt{-1} = i \qquad i^2 = -1$

$$1 = \sqrt{1} = \sqrt{(-1)\cdot(-1)} = \sqrt{-1}\cdot\sqrt{-1} = i\cdot i = i^2 = -1$$

$\pm\sqrt{A}$

???

$z = x + iy$

$$\boxed{z\cdot\bar{z} = |z|^2} \qquad |z| = \sqrt{x^2+y^2}$$

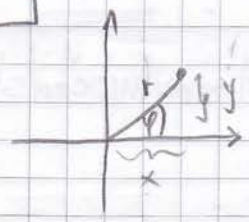
$|z_1 + z_2| \leq |z_1| + |z_2| \longrightarrow$ NEJEDNAKOST TROKUTA

•• TRIGONOMETRIJSKI OBLIK:

$$\boxed{z = r(\cos\varphi + i\sin\varphi) = r\ \mathrm{cis}\ \varphi}$$

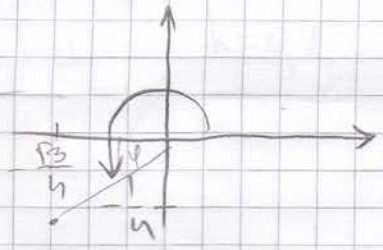
$r = |z|$

$\arg z = \varphi$ (argument)



$\mathrm{tg}\,\varphi = \dfrac{y}{x}$

Pr.) $z = -\frac{1}{4} - \frac{\sqrt{3}}{4}i$



$r = \sqrt{\left(-\frac{1}{4}\right)^2 + \left(-\frac{\sqrt{3}}{4}\right)^2} = \frac{1}{2}$

$\mathrm{tg}\,\varphi = \dfrac{-\frac{\sqrt{3}}{4}}{-\frac{1}{4}} = \sqrt{3} \longrightarrow \boxed{\varphi = 240^\circ} = \pi + \frac{\pi}{3}$

→ SVOJSTVA:

$$\boxed{z_1 \cdot z_2 = r_1 \cdot r_2 (\cos(\varphi_1+\varphi_2) + i \sin(\varphi_1+\varphi_2)) = r_1 \cdot r_2 \operatorname{cis}(\varphi_1+\varphi_2)}$$

$$\boxed{z_1 / z_2 = r_1 / r_2 \operatorname{cis}(\varphi_1-\varphi_2)}$$

$$\boxed{z^n = r^n \operatorname{cis}(n\varphi)}$$

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

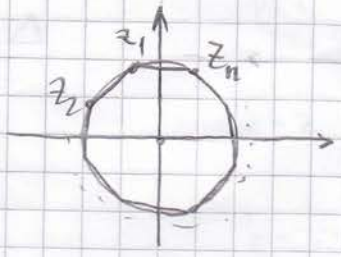
$$|z^n| = |z|^n$$

$$\boxed{\begin{aligned} \arg(z_1 \cdot z_2) &= \arg z_1 + \arg z_2 \\ \arg(z_1 / z_2) &= \arg z_1 - \arg z_2 \\ \arg(z^n) &= n \arg z \end{aligned}}$$

→ KORJENOVANJE:

$$\boxed{\sqrt[n]{z} = \sqrt[n]{r} \operatorname{cis} \frac{\varphi + 2k\pi}{n}, \quad k = 0, 1, \dots, n-1}$$

PRAVILAN n-TEROKUT



DZ-16c)

$$z^3 = (\sqrt{3}-i)^5(1+\sqrt{3}i)$$

UVIJEK PRETVORI U TRIG. OBLIK!!

$r = \sqrt{3+1} = 2$

$tg\,\varphi = \frac{-1}{\sqrt{3}} = \frac{-\sqrt{3}}{3} \rightarrow \varphi = \frac{5\pi}{6} + \pi$

$\boxed{\varphi = \frac{11\pi}{6}}$

$r = 2$

$tg\,\varphi = \frac{\sqrt{3}}{1} \rightarrow \varphi = \frac{\pi}{3}$

$$z^3 = \left(2\,cis\,\frac{11\pi}{6}\right)^5\left(2\,cis\,\frac{\pi}{3}\right)$$

$$z^3 = \left(2^5\,cis\,5\cdot\frac{11\pi}{6}\right)\cdot\left(2\,cis\,\frac{\pi}{3}\right)$$

$$z^3 = 64\,cis\left(\frac{55\pi}{6}+\frac{\pi}{3}\right)$$

$z^3 = 64\,cis\,\frac{57\pi}{6} \longrightarrow$ UVIJEK MORA BITI IZMEĐU 0 i 2π !!

$z^3 = 64\,cis\left(8\pi + \frac{\cancel{3}^{1}\pi}{\cancel{6}_{2}}\right) \Longrightarrow$ IZLUČI MAX PARNIH π ($57:6 = 8$ MAX PARNO)

$\boxed{z^3 = 64\,cis\,\frac{3\pi}{2}}$ NE! $64\,cis\left(\cancel{8\pi+\frac{3\pi}{2}}\right)$ NE!

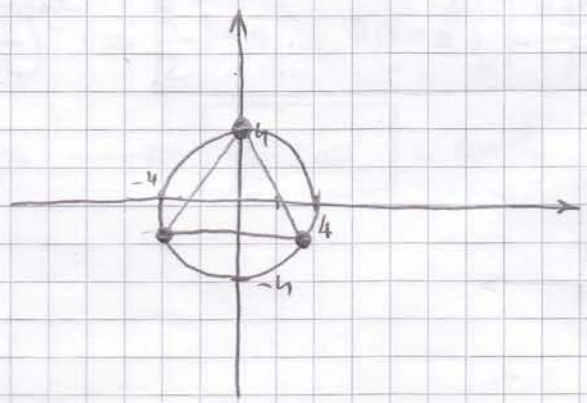
$$z = \sqrt[3]{64}\,cis\,\frac{\frac{3\pi}{2}+2k\pi}{3},\ k = 0,1,2$$

$k=0$ $\quad z_0 = 4\,cis\,\frac{\pi}{2} = 4\cdot(0+i\cdot 1) = 4i = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$k=1$ $\quad z_1 = 4\,cis\,\frac{7\pi}{6} = 4\cdot\left(\frac{-\sqrt{3}}{2}+\frac{1}{2}i\right) = -2\sqrt{3}-2i$

$k=2$ $\quad z_2 = 4\,cis\,\frac{11\pi}{6} = 4\cdot\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) = 2\sqrt{3}-2i$

NAUČI TABLICU ZA $\frac{\pi}{3}$, $\frac{\pi}{6}$, $\frac{\pi}{4}$ sin, cos, tg!!!!



JEDNAKOSTRANIČNI △

1. MI - 2006 - 3) Naći sve z koji zadovoljavaju:

$$\left(z+\frac{1}{2}i\right)^3 = i, \quad \operatorname{Re} z < 0. \qquad \Big/ \sqrt[3]{\ }$$

$$z+\frac{1}{2}i = \sqrt[3]{i} \rightarrow \text{SVEST NA TRIG. OBLIK!}$$

$$= \sqrt[3]{1 \operatorname{cis}\frac{\pi}{2}}$$

$$i \rightarrow 1i \Rightarrow \underline{\underline{|i|=1}}$$

$$\varphi = \frac{\pi}{2}$$

$$z+\frac{1}{2}i = 1 \operatorname{cis}\frac{\frac{\pi}{2}+2k\pi}{3}, \quad k=0,1,2$$

$$z_0 = \operatorname{cis}\frac{\pi}{6} - \frac{1}{2}i = \frac{\sqrt{3}}{2} \quad \text{NE}$$

$$\boxed{z_1 = \operatorname{cis}\frac{5\pi}{6} - \frac{1}{2}i = -\frac{\sqrt{3}}{2}} \quad \text{DA RJEŠENJE}$$

$$z_2 = \operatorname{cis}\frac{3\pi}{2} - \frac{1}{2}i = -\frac{3}{2}i \quad \text{NE}$$

$$|-\pi| = \pi$$

$$z = \pi \operatorname{cis}\pi$$

1. DZ - 18)

$$z^6+z^4+z^2+1=0$$

$$z^4(z^2+1)+(z^2+1)=0$$

$$(z^2+1)(z^4+1)=0$$

$$z^2+1=0 \quad \text{ILI} \quad z^4+1=0$$

$$z^2=-1 \rightarrow \sqrt[2]{-1} \text{ PA TRIG.}$$

PAZI!

$$\boxed{z_{1,2} = \pm i}$$

$$z^4=-1$$

$$z = \sqrt[4]{-1} = \sqrt[4]{1 \operatorname{cis}\pi} = \sqrt[4]{1} \operatorname{cis}\frac{\pi+2k\pi}{4}, \quad k=0,1,2,3$$

$k=0$: $z_0 = \operatorname{cis}\frac{\pi}{4}$

$k=1$: $z_1 = \operatorname{cis}\frac{3\pi}{4}$

$k=2$: $z_2 = \operatorname{cis}\frac{5\pi}{4}$

$k=3$: $z_3 = \operatorname{cis}\frac{7\pi}{4}$

1.ZZV-10g)

$$z^3+z^{-3}=i$$

$$z^3=t$$

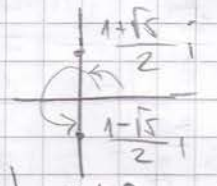
$$t+\frac{1}{t}=i \quad | \; t$$

$$t^2-ti+1=0$$

$$t_{1,2}=\frac{i\pm\sqrt{-1-4}}{2}=\frac{i\pm i\sqrt{5}}{2} \Rightarrow t_1=\left(\frac{1+\sqrt{5}}{2}\right)i$$

$$t_2=\frac{1-\sqrt{5}}{2}i$$

$$z^3=\frac{1+\sqrt{5}}{2}i$$



$$z=\sqrt[3]{\frac{1+\sqrt{5}}{2}i}=\sqrt[3]{\frac{1+\sqrt{5}}{2}\operatorname{cis}\frac{\pi}{2}}=\sqrt[3]{\frac{1+\sqrt{5}}{2}}\cdot\operatorname{cis}\frac{\frac{\pi}{2}+2k\pi}{3}, \quad k=0,1,2$$

$z_0 \ldots$
$z_1 \ldots$
$z_2 \ldots$

$$z=\sqrt[3]{\frac{1-\sqrt{5}}{2}i}=\sqrt[3]{\frac{|1-\sqrt{5}|}{2}\operatorname{cis}\frac{3\pi}{2}}=\sqrt[3]{\frac{|1-\sqrt{5}|}{2}}\cdot\operatorname{cis}\frac{\frac{3\pi}{2}+2k\pi}{3}, \quad k=0,1,2$$

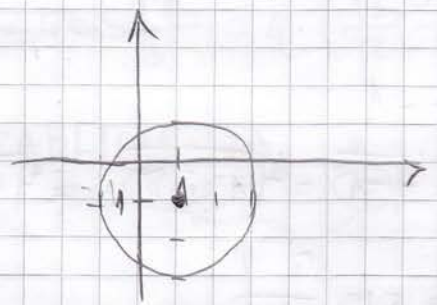
$z_0 \ldots$
$z_1 \ldots$
$z_2 \ldots$

1.ZZV-8) Skiciraj a) $|z-\underbrace{1+i}_{z_1}|=2$

1.DZ-17

1.M1..

$$|z-z_1|$$



1.ZZV-8) Skiciraj: a) $|z-1+i| = 2$

$(x-1)^2+(y+1)^2=4$

b) $|z+1|+|z-1|=3$

$\frac{4x^2}{9}+\frac{4y^2}{5}=1$

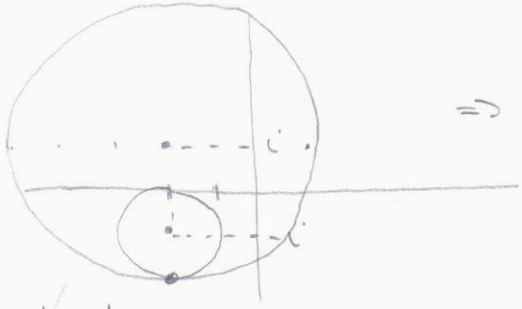
c) $\arg(z^6)=\frac{3\pi}{2}$

$z=\frac{\pi}{4}+\frac{k\pi}{3}$

→ pogledaj sada prethodni zadatak!

1.MI-2008-2) $|z+2+i|=1$

$|z+2-i|=3$



$\Rightarrow z=-2-2i$

$$|L|: \begin{array}{l} x^2+4x+4+y^2+2y+1=1 \\ -\ x^2+4x+4+y^2-2y+1=9 \\ \hline 4y=-8 \end{array}$$

$y=-2 \rightarrow x=-2$

1.ZZV-10 e) $z^5(1+i)=\bar{z}$

$|z|^5\cdot\sqrt{2}=|z|$

$r(r^4\sqrt{2}-1)=0$

$r_1=0 \qquad r^4=\frac{1}{\sqrt{2}}$

$r_2=\frac{1}{\sqrt[8]{2}}$

$5\arg z+\arg(1+i)=-\arg z+2k\pi$

$6\arg z=-\frac{\pi}{4}+2k\pi$

$\arg z=-\frac{\pi}{24}+\frac{k\pi}{3}, \ k=1,2,\dots,6$

Dekanski-2011-1) $|z|=1, \ |z^4+1|=1, \ \operatorname{Re} z>0$

$r=1 \Rightarrow |1^4\operatorname{cis}4\varphi+1|=1$

$(\cos4\varphi+1)^2+\sin^2 4\varphi=1$

$2\cos4\varphi=-1$

$\cos4\varphi=-\frac{1}{2}$

$4\varphi=\frac{2\pi}{3}+2k\pi \qquad 4\varphi=\frac{4\pi}{3}+2k\pi$

$\varphi=\frac{\pi}{6}+\frac{k\pi}{2} \qquad \varphi=\frac{\pi}{3}+\frac{k\pi}{2}$

$z_1=\operatorname{cis}\frac{\pi}{6} \qquad z_3=\operatorname{cis}\frac{\pi}{3}$

$z_2=\operatorname{cis}\frac{5\pi}{3} \qquad z_4=\operatorname{cis}\frac{11\pi}{6}$

* VJEŽBA *

1.) $B \equiv (\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(x^2 = 2xy \Rightarrow x = 2y)$

$C \equiv (\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(x < y \Rightarrow x^2 < y^2)$

Negirajte. Koji je istinit B ili $\neg B$, C ili $\neg C$?

$\neg B \equiv (\exists x \in \mathbb{R}(\exists y \in \mathbb{R})(x^2 = 2xy \wedge x \neq 2y)$ T

$\neg C \equiv (\exists x \in \mathbb{R})(\exists y \in \mathbb{R})(x < y \wedge x^2 \geq y^2)$ T

B npr. $x = 0$, $y = 0$

$0^2 = \underbrace{2 \cdot 0 \cdot 1}_{T} \Rightarrow 0 = \underbrace{2 \cdot 1}_{NE}$ F $\Rightarrow \neg B = T$

C npr. $x = -1$, $y = 1$

$\underbrace{-1 < 1}_{T} \Rightarrow \underbrace{1 < 1}_{F}$ F $\Rightarrow \neg C = T$

2.) Neka je f realna funk. realne varijable s $D(f)$. Zadan je sud

$A \equiv (\forall x_1 \in D(f))(\forall x_2 \in D(f))(x_1 < x_2 \Rightarrow f(x_1) < f(x_2))$ F

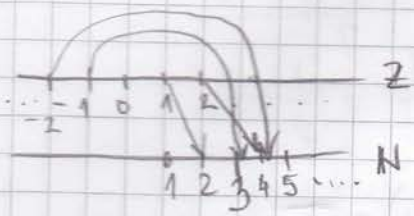
$\neg A = ?$

$\neg A \equiv (\exists x_1 \in D(f))(\exists x_2 \in D(f))(x_1 < x_2 \wedge f(x_1) \geq f(x_2))$ T

npr. $f(x) = \sin x$ $\quad x_1 = \frac{\pi}{4}$, $x_2 = \frac{3\pi}{4}$ $\quad \underbrace{\frac{\pi}{4} < \frac{3\pi}{4}}_{T}$ $\quad \underbrace{f\left(\frac{\pi}{4}\right) = f\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}}_{T}$

3.) Postoji li bijekcija $f: \mathbb{Z} \to \mathbb{N}$? Obrazloži!

$$f(x) = \begin{cases} 2x, & x > 0 \\ -2x+1, & x \leq 0 \end{cases}$$



1. f INJEKCIJA :

$f: S_1 \to S_2 \quad D(f) \subseteq S_2$

$$(\forall x_1, x_2 \in S_1)\big((x_1 \neq x_2) \Longrightarrow f(x_1) \neq f(x_2)\big)$$

→ obrat po kontrapoziciji:

$$\boxed{f(x_1) = f(x_2) \Longrightarrow x_1 = x_2}$$

$f(x) = f(y)$

→ tada obje strane trebaju biti iste parnosti, tj.

$f(x) = 2x$ i $f(y) = -2y+1$ ili

npr. $\underset{\text{nep.}}{5} = \underset{\text{par.}}{2}$ NE

$\underset{\text{par.}}{2} = \underset{\text{par.}}{4}$ DA

$(1^\circ)\ 2x = 2y \Longrightarrow x = y$

$(2^\circ)\ -2x+1 = -2y+1 \Longrightarrow x = y$

⇝ to je dokaz INJEKTIVNOSTI

2. f SURJEKCIJA:

→ ako je $y \in \mathbb{N}$ paran onda je $f(\frac{y}{2}) = y$.

→ ako je $y \in \mathbb{N}$ neparan onda je $f(\frac{y-1}{2}) = y$

↳ f je SURJEKCIJA

4.) Mat. ind. dokaži:

1.DZ 14) $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} - 1, \ \forall n \in \mathbb{N}$

1. $\underline{n=1}$

D... $\frac{1}{\sqrt{1}} = \frac{1}{1}$

L.... $2\sqrt{1} - 1 = 1$

$L \leq D$ ✓ o.k.

2. pretpostavimo da za NEKI $n \in \mathbb{N}$ vrijedi

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2n - 1$$

3. —|— i za $n+1$

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}} \overset{\text{pretp.}}{\leq} 2\sqrt{n} - 1 + \frac{1}{\sqrt{n+1}}$$

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n+1} - 1$$

želimo pokazati da je $2\sqrt{n} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1} \quad | \ \sqrt{n+1}$

$\Leftrightarrow 2\sqrt{n} \cdot \sqrt{n+1} + 1 \leq 2(n+1)$

$\Leftrightarrow 2\sqrt{n^2+n} + 1 \leq 2(n+1)$

$\Leftrightarrow 2\sqrt{n^2+n} \leq 2n+1 \quad |^2$

$\Leftrightarrow 4(n^2+n) \leq 4n^2+4n+1$

$\Leftrightarrow \cancel{4n^2} + \cancel{4n} \leq \cancel{4n^2} + \cancel{4n} + 1$

$\Leftrightarrow 0 \leq 1$

Π

2. Realne funkcije realne varijable

2.1. Svojstva realnih funkcija

$f: \mathbb{R} \to \mathbb{R}$, $y = f(x)$ zakon pridruživanja

domena $D(f)$ = skup svih $x \in \mathbb{R}$ za koje je definiran $f(x)$

slika $\mathrm{Im}(f) = \{ y \in \mathbb{R} \mid y = f(x),\ x \in D(f) \}$

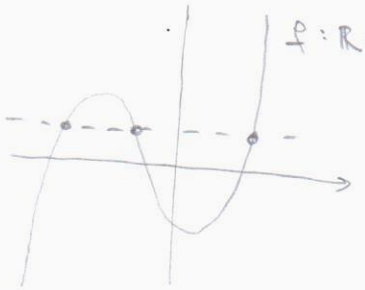
graf $G(f) = \{ (x,y) \in \mathbb{R}^2 \mid x \in D(f),\ y = f(x) \}$

- Podsjetimo se: funkcija je bijekcija ako je surjekcija i injekcija

funkcija f ima inverz f^{-1} ako i samo ako je bijekcija

$$\left.\begin{array}{l} f: A \to B \\ f^{-1}: B \to A \end{array}\right\} \begin{array}{l} D(f) = \mathrm{Im}(f^{-1}) \\ \mathrm{Im}(f) = D(f^{-1}) \end{array}$$

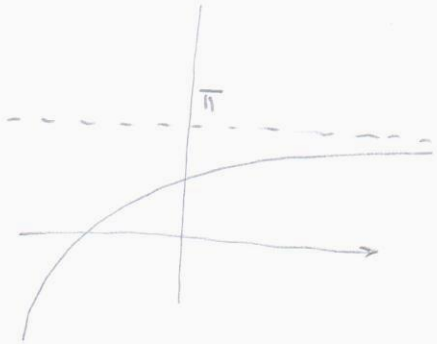
Odredivanje sa grafa:



$f: \mathbb{R} \to \mathbb{R}$

surj. ✓

inj. nije

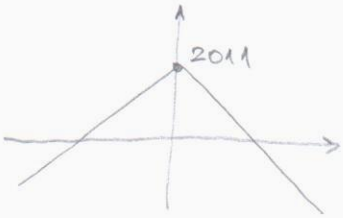


$f: \mathbb{R} \to \mathbb{R}$

surj nije

inj. ✓

1. ZZV - 6) $f: \mathbb{R} \to (-\infty, 2011]$, $f(x) = 2011 - |x| = \begin{cases} 2011 - x, & x > 0 \\ 2011 + x, & x < 0 \end{cases}$



surj. ✓

inj. nije

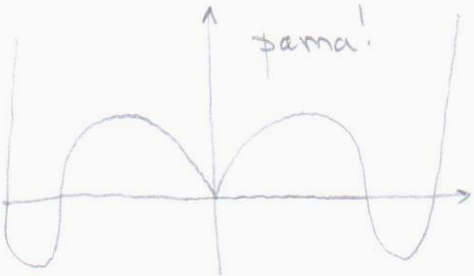
ALI restrikcija $f: (-\infty, 0] \to (-\infty, 2011]$ je bijekcija!

- Parnost i neparnost

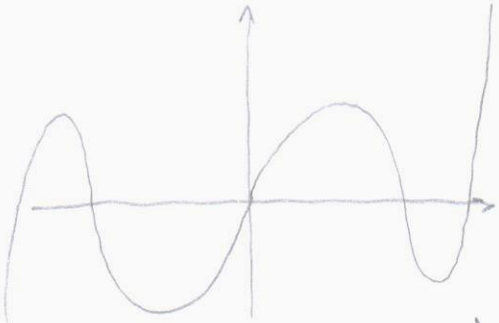
- def. $f(x)$ je parna ako je $f(-x) = f(x)$, za $\forall x \in D(f)$

$f(x)$ je neparna ako je $f(-x) = -f(x)$, za $\forall x \in D(f)$

Odredivanje sa grafa:



→ simetrično s obzirom na os y



neparna!

→ centralno simetrično s obzirom na ishodište

Primjeti: mora prolaziti ishodištem!

2. ZZV - 23) $f(x) = \ln\left(\frac{1-x}{1+x}\right)$

$f(-x) = \ln\left(\frac{1+x}{1-x}\right) = -\ln\left(\frac{1-x}{1+x}\right) = -f(x)$ neparna!

22) $f(x) = e^x - 1$

$f(-x) = e^{-x} - 1 = ?$ ni parna ni neparna!

18.) $f(x) = \sin\left(3x + \frac{\pi}{2}\right)$

$f(-x) = \sin\left(-3x + \frac{\pi}{2}\right) = -\sin\left(3x - \frac{\pi}{2}\right) = -\sin\left(3x + \frac{\pi}{2} - \pi\right) = \sin\left(3x + \frac{\pi}{2}\right)$

parna!

- u kompoziciji su parne funkcije jače od neparnih :)

• Periodičnost

- def. $f(x)$ je periodična ako postoji $T > 0$ t.d. $f(x+T) = f(x)$, $\forall x \in D(f)$
  Najmanji takav T zovemo temeljni period funkcije $f(x)$.
- kompozicija s injektivnom funkcijom čuva period (npr. $f(x) = e^{\operatorname{tg} x}$)

• Monotonost

- def. funkcija $f(x)$ je monotono rastuća na intervalu $[a,b]$ ako
  $(\forall x_1, x_2 \in [a,b])\ (x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2))$
  funkcija $f(x)$ je monotono padajuća na intervalu $[a,b]$ ako
  $(\forall x_1, x_2 \in [a,b])\ (x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2))$
- u kompoziciji se rast i pad ponašaju kao minus i plus
  (npr. padajuća od padajuće je rastuća funkcija)

→ elementarne:

1.) POLINOMI

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \qquad a_n \in R \qquad D(f) = R$

⇒ ako je $a_n \neq 0$ kažemo da je f polinom n-stupnja

→ npr. $f(x) = 3x^4 - \frac{1}{2}x^3 + x^2 - 17x + 7$ polinom 4. stupnja

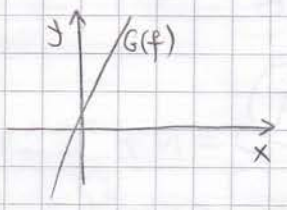
→ posebno: potencije: $f(x) = x^n$, $n \in N$

konstanta: $(n=0)$ $f(x) = c$ $\quad D(f) = R$

$Im(f) = \{c\}$

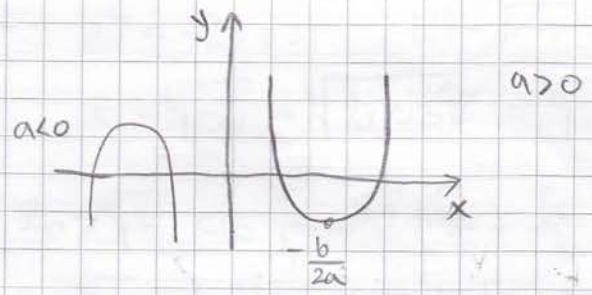
$(n=1)$ $f(x) = ax + b$ (polinom 1. stupnja)

$a \neq 0$



$Im(f) = R$

$(n=2)$ $f(x) = ax^2 + bx + c$, $a \neq 0$



nultočke: $f(x) = 0$

$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

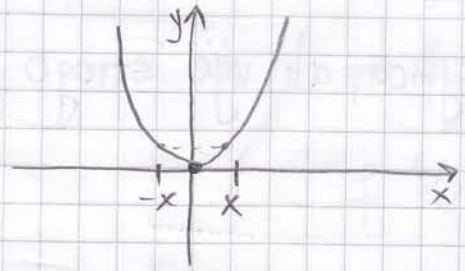
→ ako je $b^2 - 4ac > 0$

ako je $D < 0$ nema nultočk

(n proizvoljno)

$f(x) = x^n \qquad n \in N$

→ za n = paran

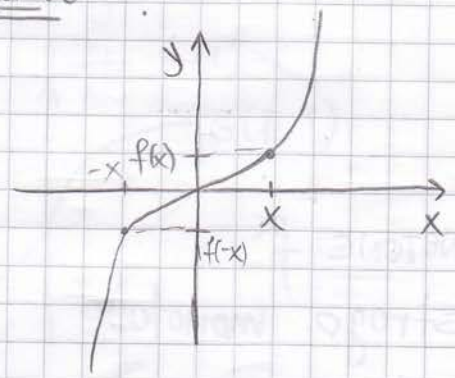


→ simetrične s os-y

→ nisu injekcije

→ $Im(f) = [0, +\infty\rangle$

→ za n neparan



→ simetričan s obzirom na ISHODIŠTE ⇒ CENTRALNOSIMETR.

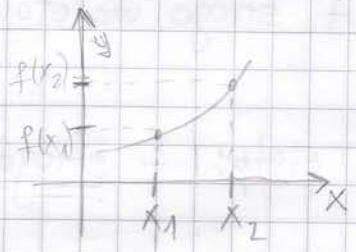
•• SVOJSTVA FUNKCIJE ••

- rast, pad

Def.

$f: I \to R, \quad I \subseteq R$

$\langle a, b\rangle \subseteq I$



f je strogo rastuća na $\langle a, b\rangle$ ako $\forall x_1, x_2 \in \langle a, b\rangle$

$x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$

$(\leq)$ bez strogo

padajuća:

$x_1 < x_2 \Longrightarrow f(x_1) > f(x_2)$

$(\geq)$

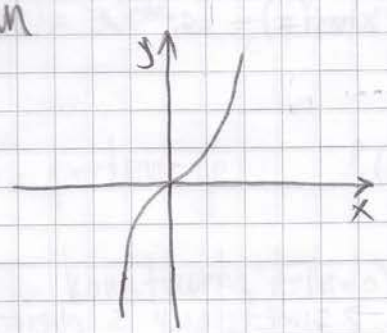
Pr.) $f(x) = x^n$

→ n paran

strogo → padajuća na $\langle -\infty, 0 \rangle$
strogo → rastuća na $\langle 0, +\infty \rangle$

→ konstanta je i rastuća i padajuća, ali nije strogo!

→ n neparan



→ raste na $\mathbb{R}$ (strogo)

⇒ rastuće i padajuće f → MONOTONE f

-|- -|- strogo → strogo monotone

Važna posljedica stroge monotonosti je INJEKTIVNOST:

T: f strogo monotona na $\langle a,b \rangle$ ⇒ f injektivna na $\langle a,b \rangle$

Dokaz: pretp. f strogo rastuća (ako nije strogo onda ne vrijedi)

$x_1 \neq x_2 \Longrightarrow x_1 < x_2$ ili $x_2 < x_1$

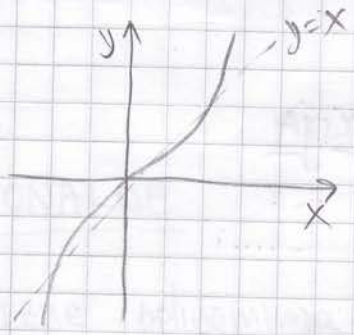
$x_1 < x_2 \Longrightarrow f(x_1) < f(x_2) \Longrightarrow f(x_1) \neq f(x_2)$

ILI $x_2 < x_1 \Longrightarrow f(x_2) < f(x_1) \Longrightarrow f(x_2) \neq f(x_1)$

Dakle, $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$ f injektivna!

analogno f strogo padajuće!

Primjena: $f(x) = x^n$, n neparan $f: \mathbb{R} \to \mathbb{R}$



f strogo raste $\Rightarrow$ injekcija

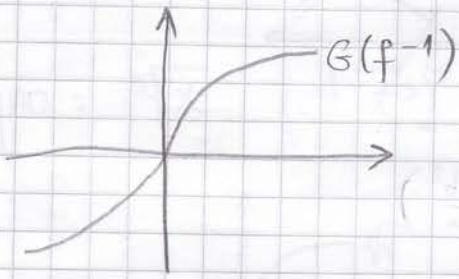
f je i surjekcija $\Rightarrow$ $\mathbb{R}$ je $\mathrm{Im}(f)$

$\Rightarrow f$ je bijekcija $\Rightarrow$ POSTOJI INVERZ

$f^{-1}(x) = \sqrt[n]{x}$, $f^{-1}: \mathbb{R} \to \mathbb{R}$

(za n neparan)

f^{-1} y-pridružuje x



→ Ako je f PERIODIČKA, a g INJEKCIJA tada je $f(g(x))$ PERIODIČKA!
npr. e^{tgx}

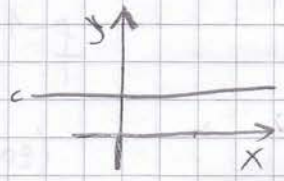
2.2. PREGLED ELEMENTARNIH FUNKCIJA

→ u petak od 14^{h} = → 16^{h}

• POLINOMI

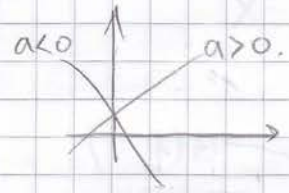
→ konstanta: $f(x) = c$

$f: \mathbb{R} \to \{c\}$

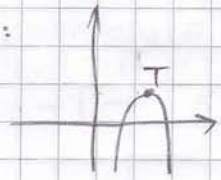


→ linearna: $f(x) = ax + b$

$f: \mathbb{R} \to \mathbb{R}$



→ kvadratna:



$T\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

$f: \mathbb{R} \to \left[\frac{4ac-b^2}{4a}, +\infty\right\rangle \quad a>0$

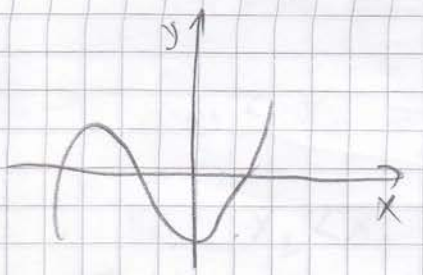
ILI

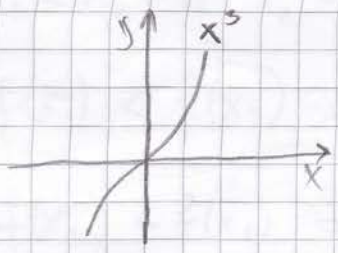
$f: \mathbb{R} \to \left\langle -\infty, \frac{4ac-b^2}{4a}\right] \quad a<0$

→ kubna:

$f(x) = ax^3 + bx^2 + cx + d$

$f: \mathbb{R} \to \mathbb{R}$





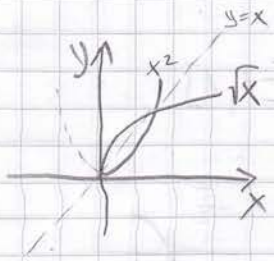
→ općenito n-tog stupnja: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

$f: \mathbb{R} \to \mathbb{R}$

• IRACIONALNE

→ inverzne polinomima

→ restrikcijom kvadratna

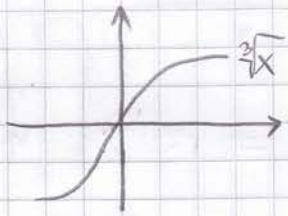


$f: [0, \infty\rangle \to [0, +\infty\rangle$

I SVE PARNE POTENCIJE

INVERZ: $f^{-1} = \sqrt{x}$

⇒ analogno: $\sqrt[3]{x}$ (od x^3) : $\mathbb{R} \to \mathbb{R}$

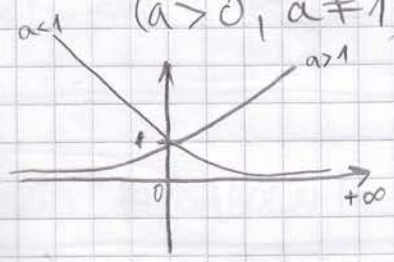


I SVE NEPARNE POTENCIJE

• EKSPONENCIJALNA I LOGARITAMSKA

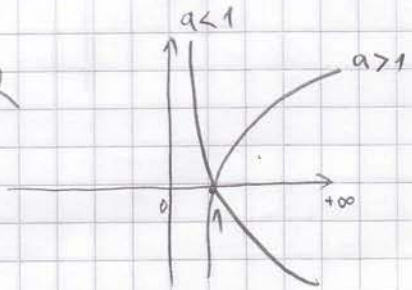
- eksp: $f(x) = a^x$: $\mathbb{R} \to \langle 0, +\infty\rangle$

$(a > 0,\ a \neq 1)$



→ INVERZ: $f(x) = \log_a x$: $\langle 0, +\infty\rangle \to \mathbb{R}$

$(a > 0,\ a \neq 1)$



$a^{\log_a x} = x$, $\log_a a^x = x$

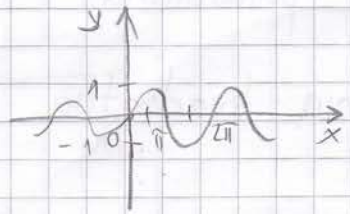
→ 1. splošne potence: $f(x) = x^c$ (npr. $x^{\sqrt{2}}$)

$$\boxed{f(x) = e^{c\ln x} = x^c} \quad : \langle 0, +\infty\rangle \longrightarrow \langle 0, +\infty\rangle$$

npr. $f(x) = x^{\pi} \rightarrow e^{\pi \ln x}$

• TRIGONOMETRIJSKE:

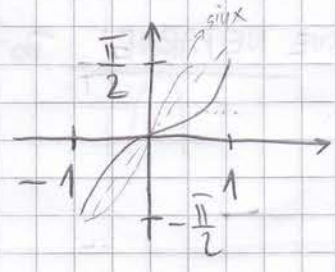
→ $f(x) = \sin x \quad : \mathbb{R} \longrightarrow [-1, 1]$



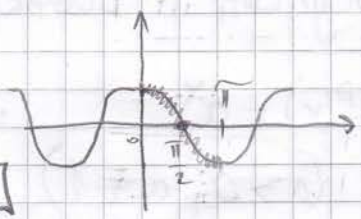
⇒ restrikcija za inverz:

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$

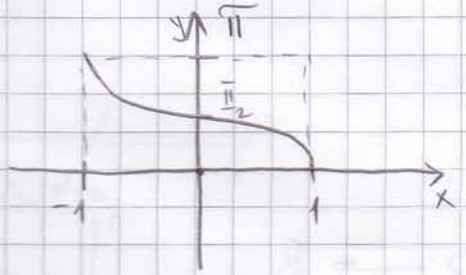
$$f(x) = \arcsin x \quad : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



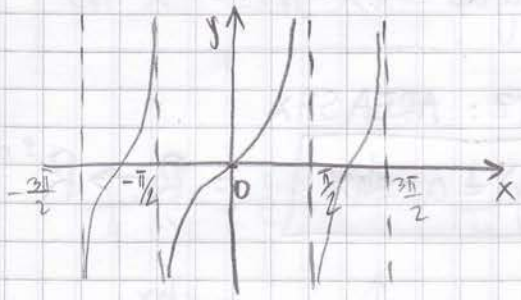
→ $f(x) = \cos x \quad : \mathbb{R} \longrightarrow [-1, 1]$



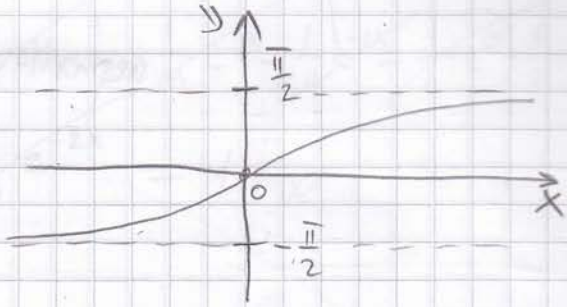
inverz: $f(x) = \arccos x \quad : [-1, 1] \rightarrow [0, \pi]$



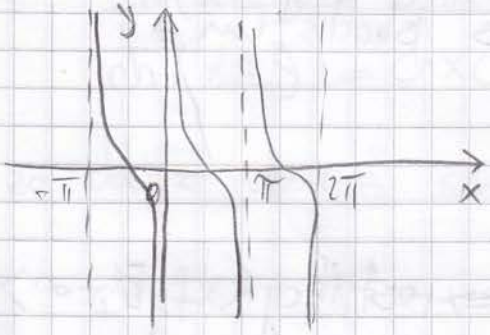
$\longrightarrow$ tangens: $f(x) = \operatorname{tg} x = \frac{\sin x}{\cos x}$: $\mathbb{R} \setminus \{\frac{\pi}{2} + k\pi\} \longrightarrow \mathbb{R}$



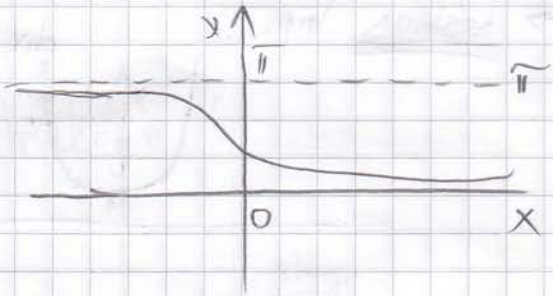
inverz: $f(x) = \operatorname{arctg} x$: $\mathbb{R} \to \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$



$\longrightarrow$ kotangens: $f(x) = \operatorname{ctg} x = \frac{\cos x}{\sin x}$: $\mathbb{R} \setminus \{k\pi\} \longrightarrow \mathbb{R}$



inverz: $f(x) = \operatorname{arcctg} x$: $\mathbb{R} \to \langle 0, \pi \rangle$



- HIPERBOLNE I AREA:

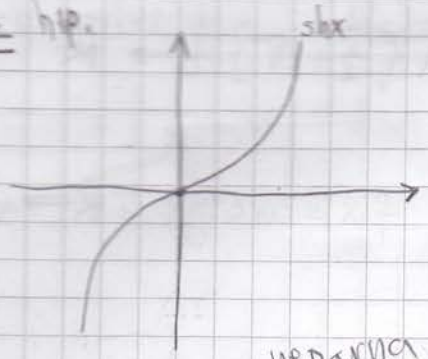
1) SINUS HIPERBOLNI

$$\boxed{f(x) = \operatorname{sh} x = \frac{e^x - e^{-x}}{2}} \quad : \mathbb{R} \longrightarrow \mathbb{R}$$

2) KOSINUS HIPERBOLNI

$$\boxed{f(x) = \operatorname{ch} x = \frac{e^x + e^{-x}}{2}} \quad : \mathbb{R} \longrightarrow [1, \infty\rangle$$

→ <u>sinus</u> hip.

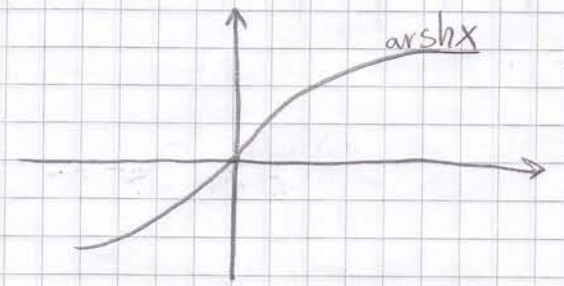


neparna

→ bijekcija

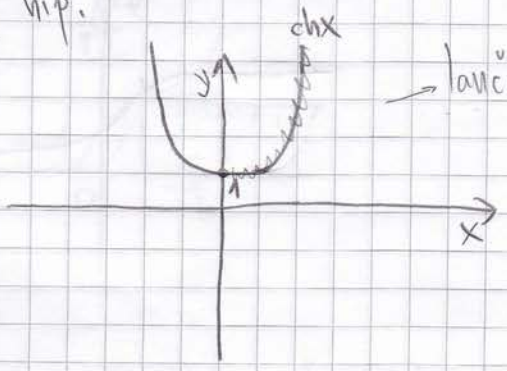
⇒ inverz: AREA SHX

$\boxed{f(x) = \text{arsh}x}$: $\mathbb{R} \to \mathbb{R}$



!! → neparne jer su simetrične s ishodištem

→ <u>kosinus</u> hip.



→ lančanica

↓

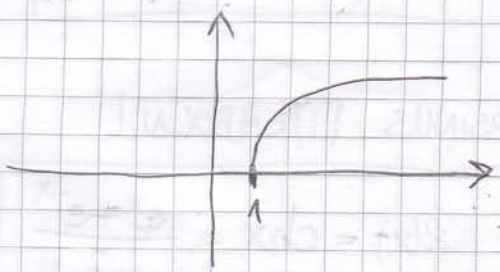
paran!

⇒ restrikcija $f|[0,+\infty\rangle$

→ INVERZ: AREA CHX

$\boxed{f(x) = \text{arch}x}$:

$f : [1,+\infty\rangle \to [0,+\infty\rangle$



ništa!

• OSNOVNI IDENTITET:

$$\boxed{ch^2x - sh^2x = 1}$$

Dokaz:

$$\frac{1}{4}(e^x+e^{-x})^2 - \frac{1}{4}(e^x-e^{-x})^2 = \cancel{\frac{1}{4}e^{2x}}+\frac{1}{2}+\cancel{\frac{1}{4}e^{2x}}-\cancel{\frac{1}{4}e^{-2x}}-\cancel{\frac{1}{4}e^{2x}}+\frac{1}{2}-\cancel{\frac{1}{4}e^{-2x}} = 1 \checkmark$$

→ VRIJEDI (skoro) SVE ISTO:

$$sh(2x) = 2\,shx\,chx$$

$$sh(x\pm y) = shx\,chy \pm chx\,shy$$

$$ch(x\pm y) = chx\,chy \pm shx\,shy \leftarrow \text{DRUKČIJE}$$

za dokaz ubacis samo e-ove ....

$$ch(2x) = ch^2x + sh^2x \leftarrow \text{drukčije}$$

- EKSPLICITAN ZAPIS:

$$y = \frac{1}{2}(e^x - e^{-x})$$

$$x \Leftrightarrow y$$

$$x = \frac{1}{2}(e^y - e^{-y}) \quad \Big| \cdot 2e^y$$

$$2xe^y = e^{2y} - 1$$

$$e^{2y} - 2xe^y - 1 = 0 \qquad \underline{e^y = t}$$

$$t^2 - 2xt - 1 = 0$$

$$t_{12} = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

zbog

$$t_{12} = x \pm \sqrt{x^2+1} = e^y$$

$$e^y = x + \sqrt{x^2+1} \quad \Big| \ln$$

$$y = \ln(x+\sqrt{x^2+1})$$

$$\boxed{arshx = \ln(x+\sqrt{x^2+1})}$$

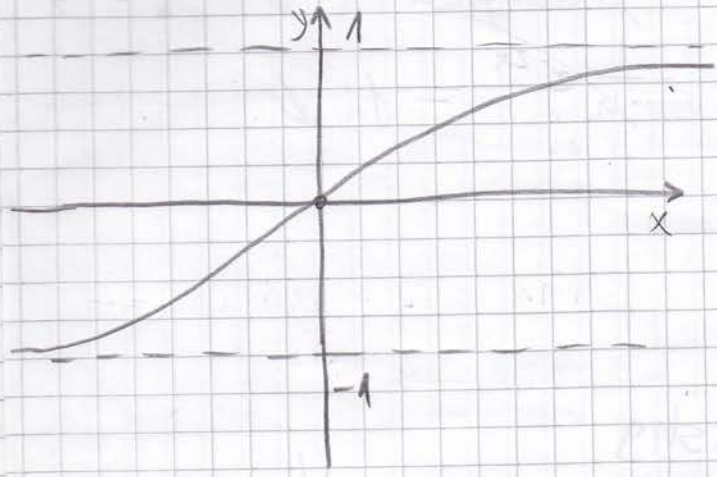
$$\Downarrow$$

$$\boxed{archx = \ln(x+\sqrt{x^2-1})}$$

→ TANGENS HIPERBOLNI:

$$\boxed{f(x) = \operatorname{th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}} : \mathbb{R} \longrightarrow \langle -1, 1 \rangle$$

NIKAD NIJE NULA!

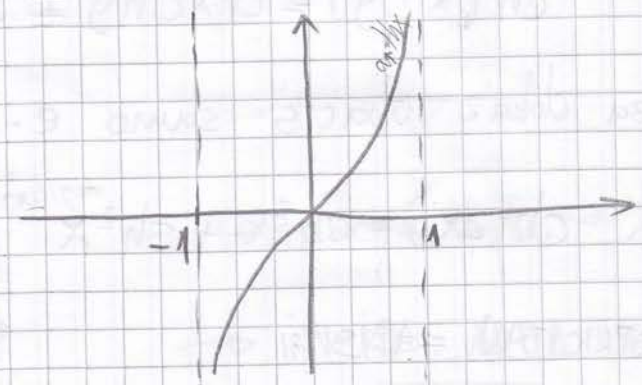


→ neparna

→ bijekcija je

⇒ inverz: AREA THX

$$\boxed{f(x) = \operatorname{arth} x : \langle -1, 1 \rangle \to \mathbb{R}}$$

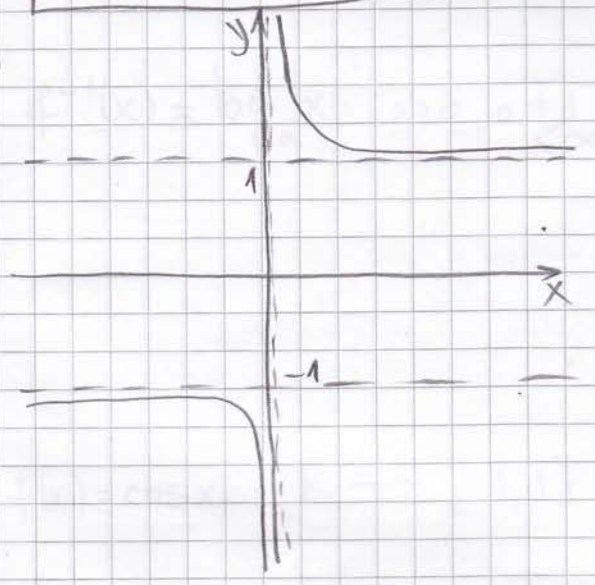


⇒ inverz čuva monotonost i parnost/neparnost

$$\boxed{\operatorname{arth} x = \frac{1}{2} \ln \frac{1+x}{1-x}}$$

→ KOTANGENS HIPERBOLNI:

$$\boxed{f(x) = \operatorname{cth} x = \frac{\operatorname{ch} x}{\operatorname{sh} x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}} \quad : \mathbb{R} \setminus \{0\} \longrightarrow \langle -\infty, -1 \rangle \cup \langle 1, +\infty \rangle$$



→ neparna

→ bijekcija

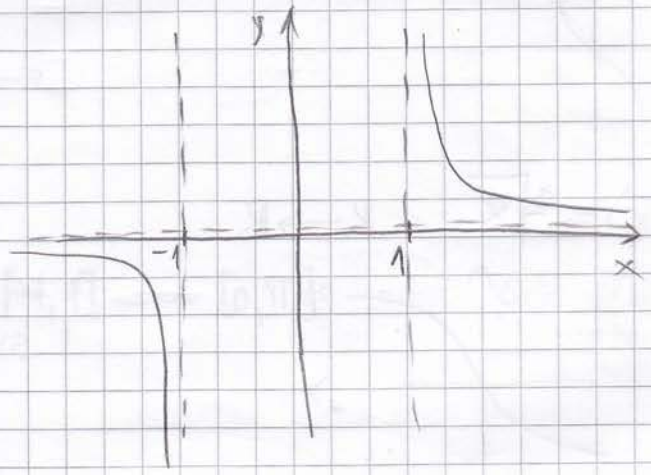
⇒ inverz: AREACTHX

$f(x) = \operatorname{arcth} x \; : \langle -\infty, -1 \rangle \cup \langle 1, +\infty \rangle$

$\longrightarrow \mathbb{R} \setminus \{0\}$

$\boxed{\mathbb{R}}$

nisu periodične



$$\boxed{\operatorname{arcth} x = \frac{1}{2} \ln \frac{x+1}{x-1}}$$

rnost / nepar

2.DZ.-19) $4 \operatorname{sh}(\ln x) = x \quad x > 0$

$4\left(\frac{1}{2}\left(e^{\ln x} - e^{-\ln x}\right)\right) = x$

$4\left(\frac{1}{2}\left(x - \frac{1}{x}\right)\right) = x$

$2\left(x - \frac{1}{x}\right) = x \quad | \cdot x \quad (x > 0)$

$2x^2 - 2 = x^2$

$x^2 = 2$

$\boxed{x = \pm\sqrt{2}}$

$x = \sqrt{2}$

Naći jednadžbu parabole čija je os simetrije paralelna sa osi y.

$T(-1,-2)$ i prolazi točkom $A(0,2)$. Nacrtati, nultočke.

$f(x)=ax^2+bx+c \quad , \quad a,b,c\in\mathbb{R}, \; a\neq 0$

$f(0)=a\cdot 0+b\cdot 0+c=2 \longrightarrow \boxed{c=2}$

$x_T=-\frac{b}{2a} \qquad y_T=\frac{4ac-b^2}{4a}$

$-1=\frac{-b}{2a} \Longrightarrow \underline{b=2a}$

$-2=\frac{8a-b^2}{4a} \Longrightarrow -8a=8a-b^2$

$-16a=-b^2$

$-16a=-(2a)^2$

$-16a=-4a^2$

$4a^2-16a=0$

$4a(a-4)=0$

NE $a=0 \qquad a-4=0$

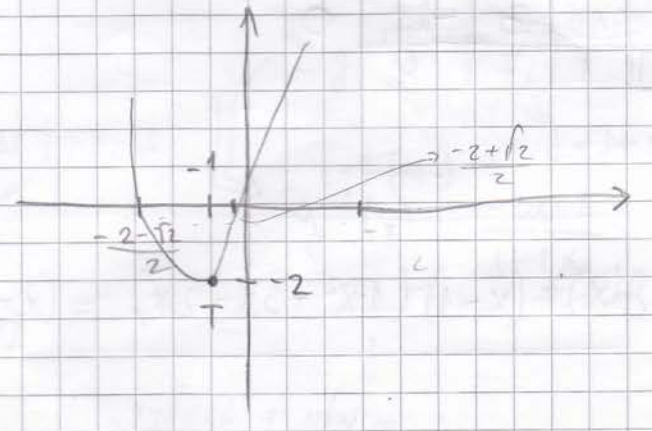
$\boxed{a=4} \longrightarrow \boxed{b=8}$

$f(x)=4x^2+8x+2$

$4x^2+8x+2=0$

$x_{1,2}=\frac{-8\pm\sqrt{64-32}}{8}$

$x_{1,2}=\frac{-2\pm\sqrt{2}}{2}$



2.) $D(f) = ?$

$$f(x) = \sqrt{x^3 + 2x^2 - 31x + 28}$$

$$x^3 + 2x^2 - 31x + 28 \geq 0$$

TEOREM: Ako je racionalan broj $\alpha = \frac{p}{q}$ korijen jednadžbe

$a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = 0$, $a_0 \neq 0$ s cijelobrojnim koeficijentima, a $M(p,q) = 1$ (skraćen do kraja), onda je p djelitelj slobodnog člana (a_n), a q djelitelj vodećeg člana (a_0)

$$g(x) = x^3 + 2x^2 - 31x + 28$$

$p = \pm 1, \pm 2, \pm 4, \pm 7, \pm 14, \pm 28$ → djelitelji

→ moguće nultočke (djeljivo s q)

djelitelji $q = \pm 1$

→ HORNEROV ALGORITAM:

koeficijenti uz x

	1	2	−31	28 (slobodan)
npr. za p → $\boxed{1}$	1	3	−28	0
$\boxed{4}$	1	7	0	

prepišeš

slobodan

$1 \cdot 1 + 2 = 3$

$(-31) \cdot 1 + 3$

$28 \cdot 1 + (-28)$

$$g(x) = (x-1)(1 \cdot x^2 + 3x - 28) = (x-1)(x-4)(x+7)$$

po redu iz tablice

$= (x-1)(x-4)(x+7) \geq 0$

	$\langle -\infty, -7 \rangle$	-7	$\langle -7, 1 \rangle$	1	$\langle 1, 4 \rangle$	4	$\langle 4, +\infty \rangle$
$x-1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x+7$	$-$	0	$+$	$+$	$+$	$+$	$+$
A	$-$	0	$+$	0	$-$	0	$+$

$$\boxed{D(f) = [-7, 1] \cup [4, +\infty\rangle}$$

3.) $f(x) = 3x^2 + 2x - 5$

$g(x) = \frac{x+1}{2x-3}$

a)

$D(g^{-1} \circ f) = ?$

$(g \circ g^{-1})(x) = x$

$g(g^{-1}(x)) = x$

$\frac{g^{-1}(x) + 1}{2g^{-1}(x) - 3} = \frac{x}{1}$

$\frac{x}{1} = \frac{y+1}{2y-3}$

$y + 1 = 2xy - 3x$

$y(1 - 2x) = -3x - 1$

$y = \frac{-3x-1}{1-2x} = g^{-1}(x)$

$\Rightarrow g^{-1}(x) = \frac{-3x-1}{1-2x}$

$g^{-1}(f(x)) = g^{-1}(3x^2 + 2x - 5)$

$= \frac{-3(3x^2+2x-5) - 1}{1 - 2(3x^2+2x+5)} = \frac{-9x^2 - 6x + 14}{-6x^2 - 4x + 11}$

$-6x^2 - 4x + 11 \neq 0$

$x_{1,2} = \frac{-2 \pm \sqrt{70}}{6}$

$$\boxed{D(f) = \mathbb{R} \setminus \left\{ \frac{-2+\sqrt{70}}{6}, \frac{-2-\sqrt{70}}{6} \right\}}$$

2.3. SLOŽENE FUNKCIJE

40.) $f(x) = \dfrac{2^{x-1}+1}{2^{x+1}-1}$, $D(f)$, $Im(f) = ?$

$2^{x+1} - 1 \neq 0$

$2^{x+1} \neq 1$

$x+1 \neq 0$

$\underline{x \neq -1}$

$\boxed{D(f) = \mathbb{R} \setminus \{-1\}}$

$\dfrac{2^{y-1}+1}{2^{y+1}-1} = x \quad | \cdot (2^{y+1}-1)$

$2 \times 2^y - x = 2^y \cdot \frac{1}{2} + 1$

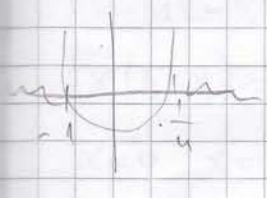
$2^y \left(2x - \frac{1}{2}\right) = x + 1$

$2^y = \dfrac{x+1}{2x-\frac{1}{2}} \quad | \log_2$

$2x - \frac{1}{2} = 0$

$2x = \frac{1}{2} \quad / \cdot 2$

$x = \frac{1}{4}$



$y = \log_2 \dfrac{x+1}{2x-\frac{1}{2}} = f^{-1}$

$\dfrac{x+1}{2x-\frac{1}{2}} > 0$

$\boxed{D(f^{-1}) = Im(f) = \langle -\infty, -1 \rangle \cup \langle \frac{1}{4}, +\infty \rangle}$

23.7)

5.) $f(x) = \arcsin \frac{1}{\sqrt{x-1}}$ i $\underline{x \neq 1}$ i $x-1 \geq 0$

$\underline{x \geq 1}$

$-1 \leq \frac{1}{\sqrt{x-1}} \leq 1$

$\frac{1}{\sqrt{x-1}} \geq -1 \quad | \cdot \sqrt{x-1}$ (uvijek ≥ 0

$1 \geq -\sqrt{x-1}$

$\sqrt{x-1} \geq -1$ NE KVADRIRATI!

UVIJEK!!

$\frac{1}{\sqrt{x-1}} \leq 1 \quad | \cdot \sqrt{x-1}$ (o.k.) ≥ 0 uvijek

$1 \leq \sqrt{x-1} \quad |^2$ MOŽE JER JE SVE POZITIVNO

$1 \leq x-1$

$\underline{x \geq 2}$

$$\boxed{D(f) = [2, +\infty\rangle}$$

invers:

$\arcsin\ [-1,1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

$y = \arcsin \frac{1}{\sqrt{x-1}}$

$x = \arcsin \frac{1}{\sqrt{y-1}} \quad | \sin$

$x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$!!

$\frac{1}{+}$ = UVIJEK +

$\Rightarrow \sin x > 0$ UVIJEK

$\sin x = \frac{1}{\sqrt{y-1}} \quad |^2$

UVIJEK +

$\sin x > 0$!!

$\hookrightarrow x > 0$

$D(f^{-1}) = [0, \frac{\pi}{2}]$

$\sin^2 x = \frac{1}{y-1}$

PAZI! NE!!

$f^{-1}(x) = y = \frac{1}{\sin^2 x} + 1 \Rightarrow \cancel{D(f^{-1}) = \mathbb{R} \setminus \{k\pi\}} = \mathrm{Im}(f)$

$\mathbb{R} \setminus \{0, \pi, 2\pi, \dots\}$

?

]?

$$\boxed{D(f^{-1}) = \mathrm{Im}(f) = \langle 0, \frac{\pi}{2} \rangle}$$

$$f(x) = (x^2-5)^{-\frac{1}{3}} + \sqrt{\operatorname{arcctg} x - \frac{\pi}{6}}$$

$\sqrt[3]{\ }$ je $\mathbb{R}$!

$$\operatorname{arcctg} x - \frac{\pi}{6} \geq 0 \longrightarrow \operatorname{arcctg} x \geq \frac{\pi}{6}$$

! monotono padajuća mijenja znak neparnu ctg

$$\frac{1}{\sqrt[3]{x^2-5}} \longrightarrow x^2-5 \neq 0$$

$\downarrow$ $\mathbb{R}$

$$x^2 \neq 5$$

$$x \neq \pm\sqrt{5}$$

$$x \leq \operatorname{ctg}\frac{\pi}{6}$$

$$\boxed{x \leq \sqrt{3}}$$

DJELUJEMO UVIJEK SA INVERZNOM!

$$\boxed{D(f) = \langle -\infty, \sqrt{3}] \setminus \{-\sqrt{5}\}}$$

2.22V-38) $f(x) = \ln(4x-x^2)\sqrt{\ln(2\sin(2\pi x))}$, $D(f) = ?$

e je rastuća! NE MIJENJA ZNAK

$4x - x^2 > 0$ i $\ln(2\sin(2\pi x)) \geq 0$ / e i $2\sin(2\pi x) \geq 0$

$x(4-x) = 0$

$x = 0 \quad x = 4$

$2\sin(2\pi x) \geq 1 \quad /:2$

POKRIVA PA GA NE MORAMO RIJEŠAVATI.

$\sin(2\pi x) \geq \frac{1}{2}$

$2\pi x \in \left[\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi\right] \quad /:2\pi$

$x \in \langle 0, 4 \rangle$

$x \in \left[\frac{1}{12} + k, \frac{5}{12} + k\right] \quad k \in \mathbb{Z}$

$k=0 \quad k=1 \quad k=2 \quad k=3$

$0 \quad \frac{1}{12} \quad \frac{5}{12} \quad \frac{13}{12} \quad \frac{17}{12} \quad \frac{25}{12} \quad \frac{29}{12} \quad \frac{37}{12} \quad \frac{41}{12} \quad 4$

$\cap$

$$\boxed{D(f) = \left[\frac{1}{12}, \frac{5}{12}\right] \cup \left[\frac{13}{12}, \frac{17}{12}\right] \cup \left[\frac{25}{12}, \frac{29}{12}\right] \cup \left[\frac{37}{12}, \frac{41}{12}\right]}$$

$\sqrt{-1} = i \longrightarrow i^2 = -1$

$\sqrt{12} = \sqrt{3} \cdot \sqrt{4}$ $\quad \sqrt{12} = \sqrt{3 \cdot 4} = \sqrt{3} \cdot \sqrt{4}$

$1 = (-1) \cdot (-1)$

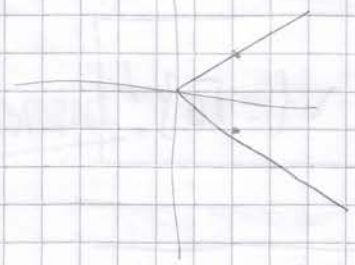
$1 = \sqrt{1} = \sqrt{(-1)\cdot(-1)} = \sqrt{-1} \cdot \sqrt{-1} = i \cdot i = i^2 = -1$

$\boxed{\sqrt{1} = \pm 1}$

$\sqrt{t} =$

$x^2 = 4$

$x = \pm 2$



2.D.Z

16.) $\cos(2 \overset{x}{\arccos x}) = 2x$

$2\cos^2(\arccos x) - 1 = 2x$

$2 \cdot x^2 - 1 = 2x$

$2x^2 - 2x - 1 = 0$

$\cos 2x = \cos^2 x - \overset{1-\cos^2 x}{\sin^2 x} = 2\cos^2 x - 1$

$\cos^2 x - 1 + \cos^2 x$

$\arccos : [-1, 1] \longrightarrow [0, \pi]$

$x_{1,2} = \frac{2 \pm \sqrt{4+8}}{4} = \frac{2 \pm \sqrt{12}}{4} = \frac{2 \pm 2\sqrt{3}}{4} = \frac{\cancel{2}(1 \pm \sqrt{3})}{\cancel{4}} = \frac{1 \pm \sqrt{3}}{2}$

$x_1 = \frac{1+\sqrt{3}}{2}$ $\quad \boxed{x_2 = \frac{1-\sqrt{3}}{2}}$

$x \in [-1, 1]$

$\cancel{x_1 = 1{,}36}$ $\quad x_2 = -0{,}36$ ✓

1)

$T(-1,3)$

$A(0,1)$

$f(x)=ax^2+bx+c \Rightarrow f(0)=\underline{\underline{c=1}}$

$x_T=\frac{-b}{2a}$ $\qquad f(-1)=a-b+1=3 \Rightarrow \underline{a=b+2}$

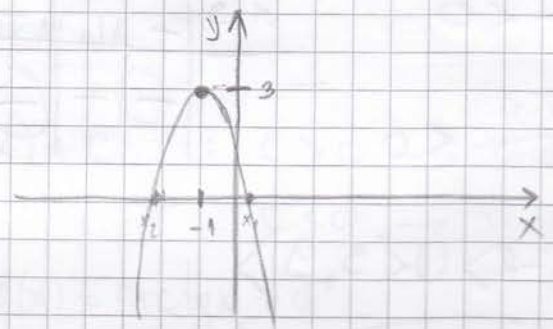
$x_T=\frac{-b}{2b+4}=-1 \rightarrow -\frac{b}{2a}=-1$ $\qquad 3=\frac{4a-4a^2}{4a}$

$\underline{b=2a}$ $\qquad 4a-4a^2=12a$

$b=2b+4$ $\qquad -4a^2=8a$

$-b=4$ $\qquad a^2=-2a$

$\underline{b=-4} \rightarrow \underline{a=-2} \Rightarrow \underline{\underline{f(x)=-2x^2-4x+1}}$ $\qquad \underline{a=-2}$



$x_{1,2}=\frac{4\pm\sqrt{16+8}}{-4}=\frac{4\pm2\sqrt{6}}{-4}=\frac{\overset{1}{\cancel{2}}(2\pm\sqrt{6})}{-\underset{2}{\cancel{4}}}$

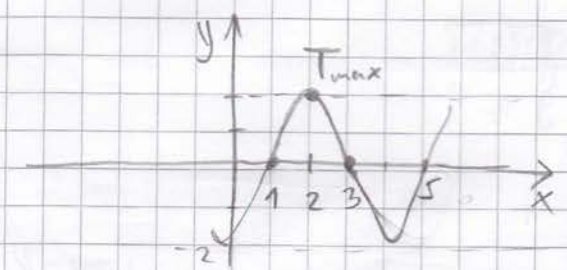
$x_1=\frac{-2+\sqrt{6}}{2}\approx 0,22$

$x_2=\frac{-2-\sqrt{6}}{2}\approx -2,22$

2.) $f(x)=a\sin[\omega(x+c)]$ $\qquad (1,0)\ (3,0)\ T_{max}(2,2)$

$f(x)=2\sin\left[\frac{2\pi}{5}(x-1)\right]$

$\boxed{f(x)=2\sin\left(\frac{2\pi}{5}x-\frac{2\pi}{5}\right)}$



$T=\frac{2\pi}{\omega}$

$\omega=\frac{2\pi}{T}$

$\omega=\frac{2\pi}{5}$

3.) a) $y=2(x-1)^2$ $\qquad$ b) $y=2(x-1)^2+2$ $\qquad$ c) $y=2(-x-1)^2+2=\underline{\underline{2(x+1)^2+2}}$

4.) a) $y=(x+3)^2+x$ $\qquad$ b) $y=(x+3)^2+x-1$ $\qquad$ d) $y=-[(x+3)^2+x-1]$

$=x^2+7x+9$ $\qquad$ $=x^2+7x-8$ $\qquad$ $=-x^2-7x-8$

5.) a) $y=\frac{1}{x+2}+1$ $\qquad$ b) $y=-\frac{1}{x+2}-1$

Zad.) $f(x)=\sqrt{\operatorname{arth}\frac{x^2-15}{x-9}}$

rastuća, ne mijenja se znak

$\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{1-1}{1+1}=0$

$\operatorname{arth}\left(\frac{x^2-15}{x-9}\right)\geq 0 \;/\; \operatorname{th}, \quad -1<\frac{x^2-15}{x-9}<1 \quad i \quad x-9\neq 0$

$x\neq 9$

$\frac{x^2-15}{x-9}\geq 0 \qquad 0\leq\frac{x^2-15}{x-9}<1$

SKRATI POSAO

$\pm\sqrt{15}$

$\frac{x^2-15}{x-9}\geq 0$

9

$x^2-15\geq 0 \quad i \quad x^2-15\leq 0$

$x-9>0 \qquad x-9<0$

$\frac{x^2-15}{x-9}-1<0$

$\frac{x^2-x-6}{x-9}<0$

$\frac{(x+2)(x-3)}{x-9}<0$

	$-\infty$	-2	3	9
$x+2$	$-$	0 $+$	$+$	
$x-3$	$-$	$-$	0 $+$	
$x-9$	$-$	$-$	$-$	0 $+$
	$-$	$+$	$-$	$+$

$x\in\langle -\infty,-2\rangle\cup\langle 3,9\rangle$

$-\sqrt{15}$ $\sqrt{9}$

$x\in\langle$

$x>9$

$$\boxed{D(f)=[-\sqrt{15},-2\rangle\cup\langle 3,\sqrt{15}]}$$

$-\sqrt{15}$ 0 3 $\sqrt{15}$ 9

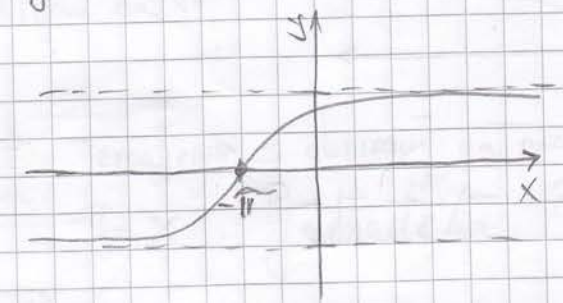
2.4. TRANSFORMACIJE FUNKCIJA

1) HORIZONTALNA TRANSLACIJA

$f(x-c)$: $c > 0 \rightarrow$ UDESNO

$c < 0 \rightarrow$ ULIJEVO

npr. $f(x) = \operatorname{arctg}(x+\pi)$

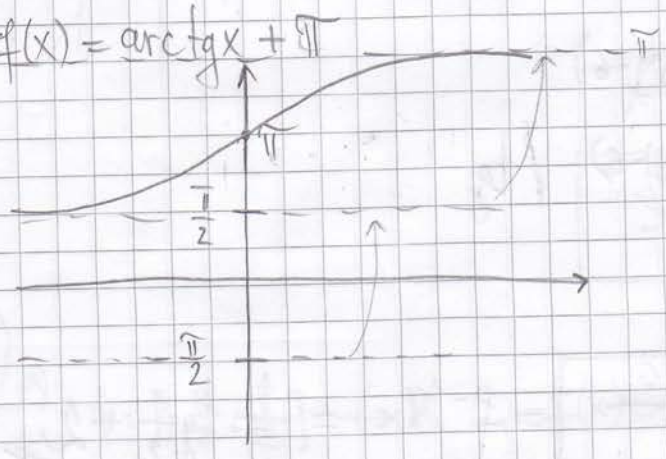


2) VERTIKALNA

$y = f(x) + c$: $c > 0 \rightarrow$ GORE

$c < 0 \rightarrow$ DOLJE

Pr.) $f(x) = \operatorname{arctg} x + \pi$



3) SKALIRANJE u SMJERU y-osi

$y = c \cdot f(x)$ $c = -1 \rightarrow$ zrcaljenje s obzirom x-os $f(x)$

4) -|- -|- x-osi

$y = f(c \cdot x)$ $c < 1 \rightarrow$ RASTEZANJE

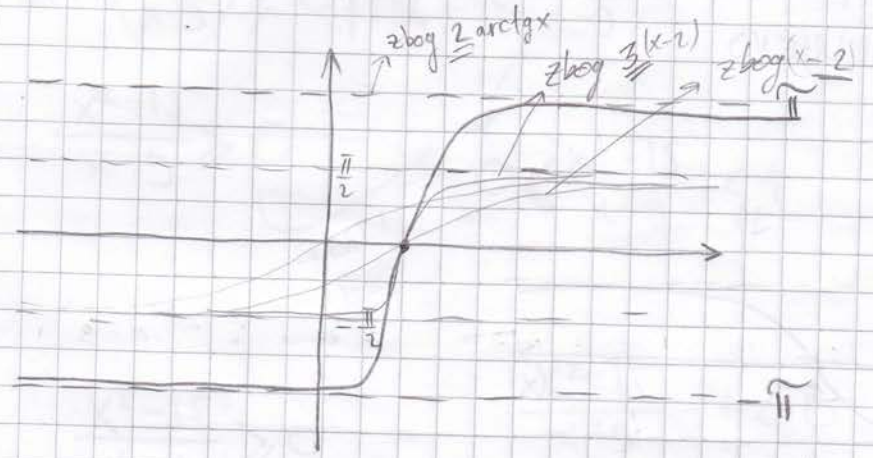
$c > 1 \rightarrow$ STEZANJE

$f(-x)$ $c = -1 \rightarrow$ zrcaljenje s obzirom na y-os

1.M1-2010.-2)

$$f(x) = 2\,\mathrm{arctg}(3x-6) = 2\,\mathrm{arctg}\,3(x-2)$$

a) $D(f)$, $\mathrm{Im}(f)$, $G(f)$ = ?



$D(f) = \mathbb{R}$ $\quad \mathrm{Im}(f) = \langle -\pi, \pi \rangle$

b) je li bijekcija? DA

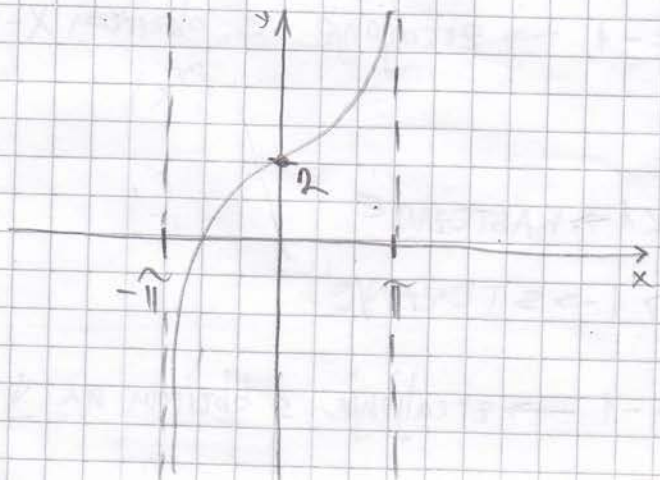
$f^{-1}(x) = ?$

$$x = 2\,\mathrm{arctg}(3y-6)$$

$$\frac{x}{2} = \mathrm{arctg}(3y-6) \quad / \mathrm{tg}$$

$$\mathrm{tg}\frac{x}{2} = 3y-6$$

$$\boxed{y = \frac{\mathrm{tg}\frac{x}{2}+6}{3}} = f^{-1}(x) = \frac{1}{3}\,\mathrm{tg}\frac{x}{2} + 2$$



c) $D(f^{-1}) = \langle -\pi, \pi \rangle$

$\mathrm{Im}(f^{-1}) = \mathbb{R}$

2.DZ. -5.)

$y=\frac{1}{x} \xleftarrow{2}$, $\uparrow 1$, zrcaljenje s obzirom na os y

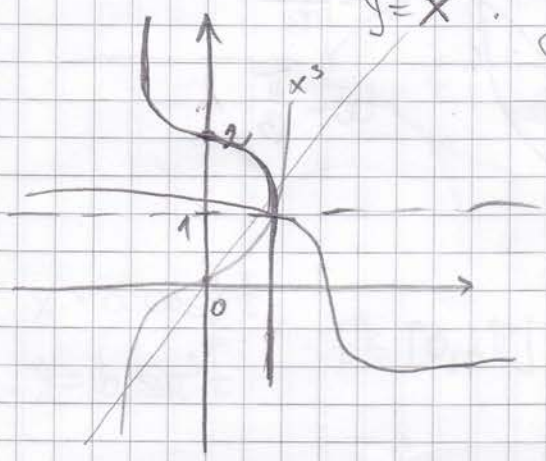
Jednadžba ?

$$y=\frac{1}{-x+2}+1$$

zrcaljenje s obzirom na y

1.MI-2009-3)

$y=x^3$ zrcaljenje s obzirom na pravac $y=1$, a zatim s obzirom $y=x$. Jednadžba ?



$y=(-x)^3+2$

$y=-x^3+2$

s obzirom na $y=x \rightarrow$ NAĆI INVERZ!

$x=-y^3+2$

$\Downarrow$

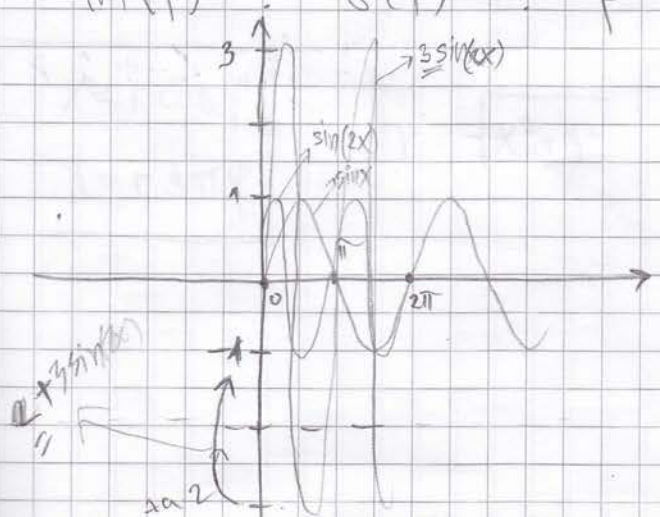
$$\boxed{y=\sqrt[3]{2-x} = f^{-1}(x)}$$

2.DZ-11.)

a) $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\longrightarrow \mathbb{R}$, $f(x)=2+3\sin(2x)$

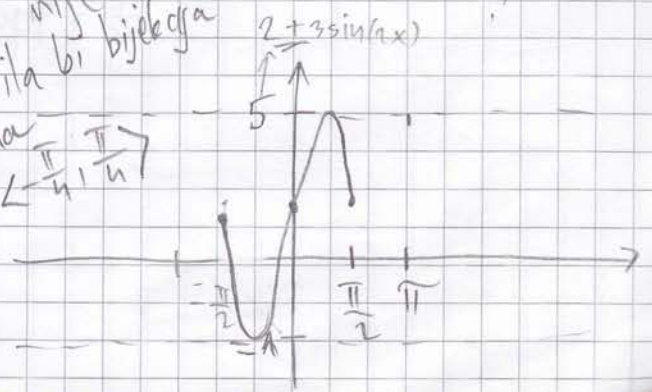
$Im(f)=?$ $G(f)=?$ $f^{-1}(x)=?$

UVIJEK IZNUTRA PREMA VAN!



nije bijekcija

bila bi bijekcija na $\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$



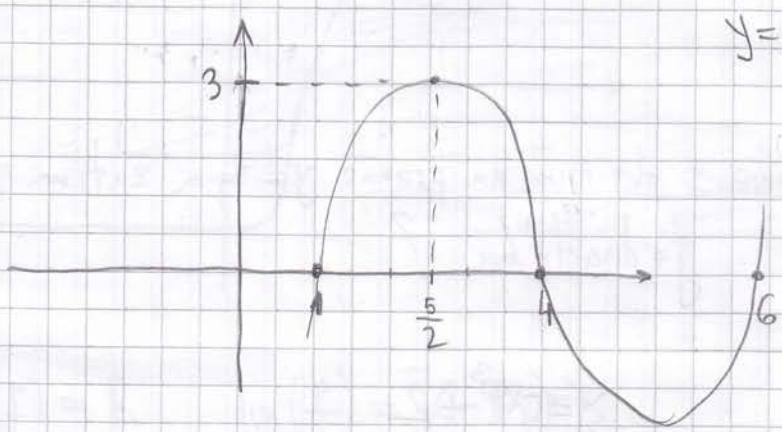
2.DZ-1) $T_{max}\left(\frac{5}{2}, 3\right)$, $\underbrace{(1,0), (4,0)}_{\text{susjedne nultočke}}$

jednadžba sinusoide?

$y = A\sin[\omega(x+\varphi)]$

slika, udesno

$y = 3\sin\left[\frac{\pi}{3}(x-1)\right]$



$T = \frac{2\pi}{\omega}$

$T = 6$!

SLIKA!

$\omega = \frac{2\pi}{6}$

$\omega = \frac{\pi}{3}$

$$\boxed{y = 3\sin\left(\frac{\pi}{3}x - \frac{\pi}{3}\right)}$$

2.5. PARAMETARSKI ZAPIS KRIVULJE

$y = f(x) \implies \begin{array}{l} x = x(t) \\ y = y(t) \end{array} \quad t \in \mathbb{R}$

Pr.

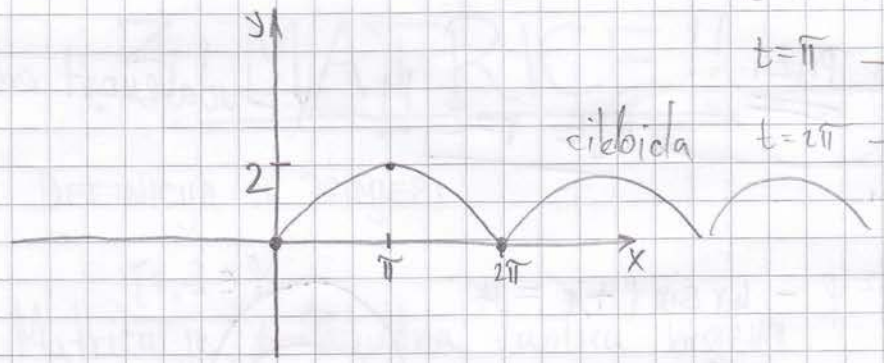
$x = t - \sin t$

$y = 1 - \cos t \qquad t \in [0, +\infty\rangle$

$t = 0 \to x = 0,\ y = 0$

$t = \pi \to x = \pi,\ y = 2$

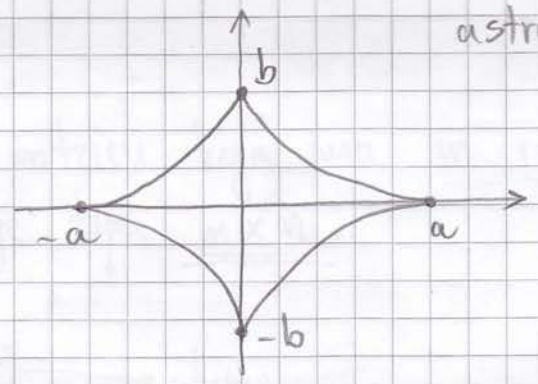
$t = 2\pi \to x = 2\pi \quad y = 0$



Pr.2.)

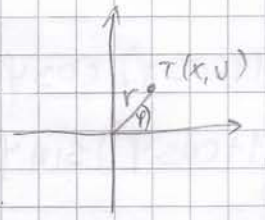
$x = a\cos^3 t$

$y = b\sin^3 t \qquad t \in [0, 2\pi]$



→ POLARNE KOORDINATE:

$$\boxed{\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array}} \qquad r = \sqrt{x^2 + y^2} \qquad \operatorname{tg}\varphi = \frac{y}{x}$$

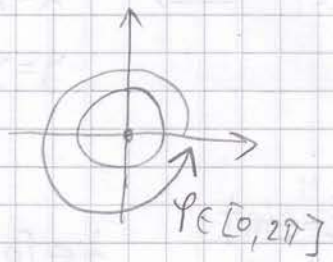


Pr.) • KRUŽNICA:

$$x^2+y^2=2 \longrightarrow r=\sqrt{2}$$

$$\boxed{\begin{aligned} x&=\sqrt{2}\cos\varphi \\ y&=\sqrt{2}\sin\varphi \end{aligned}}, \quad \varphi\in[0,2\pi]$$

→ kod kružnice



• POMAKNUTA KRUŽNICA:

$$x^2+(y-2)^2=4$$

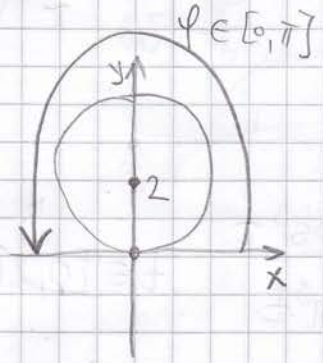
PAZI!!

→ NIJE $r=2$!! r – udaljenost od ishodišta

$$r^2\cos^2\varphi+r^2\sin^2\varphi-4r\sin\varphi+\cancel{4}=\cancel{4}$$

$$r^{\cancel{2}}=4\cancel{r}\sin\varphi \quad |:r$$

$$\boxed{r=4\sin\varphi}$$

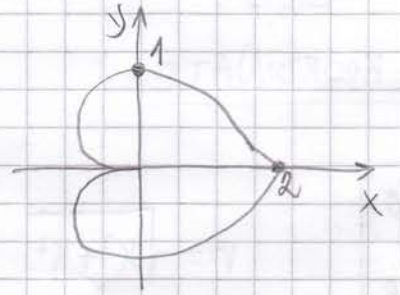


$$\boxed{\begin{aligned} x&=4\sin\varphi\cos\varphi \\ y&=4\sin\varphi\sin\varphi \end{aligned}} \quad \varphi\in[0,\pi]$$

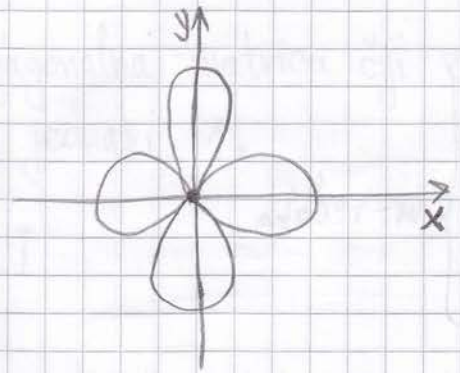
Pr.) • KARDIOIDA:

$$r=1+\cos\varphi \qquad \varphi\in[0,2\pi]$$

$$x=(1+\cos\varphi)\cos\varphi$$
$$y=(1+\cos\varphi)\sin\varphi$$



Pr.) $r = \cos^2 2\varphi$



3. MATRICE

3.1. DEFINICIJA I PRIMJERI

def. Matrica je pravokutna tablica brojeva.

$$A = \begin{bmatrix} 3 & 1 & 5 \\ 6 & 7 & 8 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = B$$

2×3 → STUPCI

↓

2 REDCI

3×1

→ Za matricu koja ima m redaka i n stupaca kažemo da je tipa $m \times n$.

$$\boxed{(A)_{ij} = a_{ij}}$$

→ stupac

redak

Pr.) $a_{23} = 8$ (iz gornje A)

$b_{21} = 2$

$$A = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{n\text{-stupaca}} \Big\} m\text{-redaka}$$

→ tipa $m \times n$

→ Matrice A i B su jednake ako su istog tipa i vrijedi $a_{ij} = b_{ij}$ za svaki i i j.

• OSNOVNI TIPOVI MATRICA:

1. KVADRATNA MATRICA – ima jednak broj redaka i stupaca ($m = n$).

→ Matrica je REDA 2. (... reda n)

2. NUL-MATRICA – na svim elementima ima nulu. ($a_{ij} = 0$ za $\forall i, j$).

→ označavamo je boldano $\mathbf{0}$.

3. DIJAGONALNA – svi elementi nula van glavne dijagonale.

→ Mora biti kvadratna.

→ oznaka: $A = \mathrm{diag}(a_{11}, \dots, a_{mn})$

Pr.)

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ → je dijagonalna, i kvadratna, i nul

JEDINIČNA MATRICA – dijagonalna matrica čiji elementi na dijagonali su jednaki 1.

→ označavamo je sa I

$$I = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{4} \Big\} 4 \longrightarrow \text{jedinič. matrica } \underline{\underline{\text{REDA } 4}}$$

$\rightarrow 4 \times 4$

→ KRONECKEROV SIMBOL: $I = \underline{\underline{(\delta_{ij})}}$

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \rightarrow \text{dijagonala}$$

TROKUTASTA MATRICA

→ Kvadratna matrica čiji su svi elementi ispod ili iznad glavne dijagonale jednaki nuli.

$$\begin{bmatrix} 1 & 5 & 6 \\ 0 & 0 & 7 \\ 0 & 0 & 2 \end{bmatrix} \longrightarrow \text{GORNJA TROKUTASTA}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 4 & 1 & 3 \end{bmatrix} \longrightarrow \text{DONJA TROKUTASTA}$$

→ Dijagonalna matrica je i gornje i donje trokutasta.

6. TRANSPONIRANA

→ Matrica koju dobivamo zamjenom redaka i stupaca.

$$(A^T)_{ij} = (A)_{ji} \qquad m \times n \xrightarrow{T} n \times m$$

Pr.)

$$\underset{2\times 3}{\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}} \longrightarrow \underset{3\times 2}{\begin{bmatrix} 3 & 1 \\ 1 & 5 \\ 4 & 9 \end{bmatrix}}$$

PRIMJETI:

i) $(A^T)^T = A$

ii) ako je D dijag., $D^T = D$ → simetrična $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

iii) ako je A gornja trokutasta tada je A^T donja trokutasta, a dijagonala se neće promijeniti.

7. SIMETRIČNA

→ matrica koja se transponiranjem ne mijenja !!!

→ Kvadratna matrica za koju vrijedi da je $A^T = A$.

→ Uoči: Sve DIJAGONALNE MAT. SU SIMETRIČNE!

$$A = \begin{bmatrix} 5 & -1 & 0 \\ -1 & 3 & 5 \\ 0 & 5 & 4 \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} 5 & -1 & 0 \\ -1 & 3 & 5 \\ 0 & 5 & 4 \end{bmatrix}$$

⇒ primjeti: Ako su ELEMENTI SIMETRIČNI S OBZIROM NA GLAVNU DIJAGONALU.

8. VEKTOR MATRICE

→ matrice koje IMAJU SAMO JEDAN REDAK (vektor redak) $(1 \times n)$ ili samo jedan stupac (vektor stupac matrica) $(n \times 1)$.

$$B = \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} \rightarrow \text{vektor stupac}$$

3×1

Primjeti: $\begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}^T \longrightarrow \begin{bmatrix} 2 & 6 & 5 \end{bmatrix}$

$3 \times 1 \qquad 1 \times 3$

3.2. OPERACIJE S MATRICAMA

a 1) ZBRAJANJE

$\begin{bmatrix} 5 \\ 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 \\ 4 & 2 \end{bmatrix}$ = NE POSTOJI ! ⇒ NE MOŽEMO ZBROJITI SVE MATRICE

→ Možemo ih zbrajati samo ako su ISTOG TIPA, tada zbrojimo elemente na istim mjestima.

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij}$$

Pr)

$$\begin{bmatrix} -1 & 5 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$$

2) MNOŽENJE MATRICE SKALAROM

$$(\lambda \cdot A)_{ij} = \lambda \cdot (A)_{ij} \quad , \lambda \in \mathbb{R}$$

Pr) $3 \cdot \begin{bmatrix} 2 & 6 \\ -1 & 0 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 18 \\ -3 & 0 \\ 15 & 6 \end{bmatrix}$

$0 \cdot A = \mathbf{0}$ → nul matrica

$1 \cdot A = A$

3) ODUZIMANJE: $A - B = A + (-1) \cdot B$

4) DIJELJENJE MATRICA NE POSTOJI ☹

5) MNOŽENJE MATRICA

- def. umnožak vektor-retka i vektor-stupca iste duljine

$$\boxed{\underset{1\times n}{[a_1\ a_2 \cdots a_n]} \cdot \underset{n\times 1}{\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}$$

→ da bi postojao općeniti umnožak dviju matrica matrice moraju biti ULANČANE → br. stupaca prve mora biti jednak broju redaka druge

def. Neka je A tipa $m \times n$, a B tipa $n \times p$. Tada je def. umnožak AB tipa $m \times p$ čiji je opći element umnožak i-tog retka od A i j-tog stupca od B, tj:

$$\boxed{(AB)_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \cdot \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}}$$

Pr.) a)

$1 \cdot 3 + 2 \cdot 1$ $\qquad$ $1 \cdot 1 + 2 \cdot 5$

$$AB = \underset{4\times 2}{\begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 5 & 1 \\ 2 & 0 \end{bmatrix}} \cdot \underset{2\times 3}{\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}} = \underset{4\times 3}{\begin{bmatrix} 5 & 11 & 22 \\ 0 & 14 & 23 \\ 16 & 10 & 29 \\ 6 & 2 & 8 \end{bmatrix}}$$

$$BA = \underset{2\times 3}{\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}} \cdot \underset{4\times 2}{\begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 5 & 1 \\ 2 & 0 \end{bmatrix}}$$

$\downarrow$

NE POSTOJI!

ako postoji AB ne mora postojati $B \cdot A$!!

b)

$$\underset{1\times 3}{\begin{bmatrix} 3 & 1 & 4 \end{bmatrix}} \cdot \underset{3\times 1}{\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}} = \underset{1\times 1}{\begin{bmatrix} 5 \end{bmatrix}} = 5$$

$$\underset{3\times 1}{\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}} \cdot \underset{1\times 3}{\begin{bmatrix} 3 & 1 & 4 \end{bmatrix}} = \underset{3\times 3}{\begin{bmatrix} -3 & -1 & -4 \\ 0 & 0 & 0 \\ 6 & 2 & 8 \end{bmatrix}}$$

$\downarrow$

NE MORA BITI ISTOG TIPA!

c)

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 7 & -7 \\ 7 & -7 \end{bmatrix} = 7 \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\downarrow$

NE MORAJU BITI ISTE AKO SU ISTOG TIPA!

!! $\Rightarrow$ NE VRIJEDI KOMUTATIVNOST MNOŽENJA! $AB \neq BA$!!!

• SVOJSTVA MATRIČNOG MNOŽENJA

→ Ako je $A \cdot B = 0$ onda niti A niti B NE MORA BITI NULA. (c)

1) ASOCIJATIVNOST

$$\boxed{(AB)C = A(BC)} \neq BCA$$ → samo ako su ulančane sve 3

2) DISTRIBUTIVNOST

$$\boxed{(A+B)C = AC + BC}$$ → također ako su ulančane; bitan redosl.

3) MNOŽENJE JEDINIČNOM MAT.

$$\boxed{A \cdot I = I \cdot A = A}$$

I → 1 samo na dijagonali

→ za kvadratne matrice

4) TRANSPONIRANJE

$$\boxed{(AB)^T = B^T A^T}$$

Mat. indukcijom se dokaže:

$$(A_1 \cdots A_n)^T = A_n^T \cdots A_1^T$$

5) MATRIČNE POTENCIJE

→ Ako je A kvadratna, definiramo $\boxed{A^2 = A \cdot A}$

→ Dogovorno, $\boxed{A^0 = I}$

→ Također, $\boxed{A^n = \underbrace{A \cdot A \cdots A}_{n \text{ matrica}}}$

→ Vrijedi, $\boxed{A^p \cdot A^q = A^{p+q}}$

$$\boxed{(A^p)^q = A^{p \cdot q}}$$

→ Ako je $f(x) = a_n x^n + \dots + a_1 x + a_0$ polinom n-tog stupnja, tada se <u>matrični polinom</u> definira kao:

$$\boxed{f(A) = a_n \cdot A^n + \dots + a_1 \cdot A + a_0 \cdot I}$$

Dokaz:

$$[(A+B)C]_{ik} = \sum_{j=1}^{n} (A+B)_{ij} c_{jk} = \sum_{j=1}^{n} (\underset{\text{brj}}{a_{ij}} + \underset{\text{brj}}{b_{ij}}) \cdot c_{jk} = \sum_{j=1}^{n} a_{ij} c_{jk} + \sum_{j=1}^{n} b_{ij} c_{jk} =$$

$$= (AC)_{ik} + (BC)_{ik} \quad \Downarrow$$

provjera ulančanosti: A je $m \times n$, B $n \times p$ $\Rightarrow AB = m \times p$

$\Rightarrow (AB)^T \to p \times m$

B^T je $p \times n$, A^T $n \times m$ $\Rightarrow B^T A^T$ $p \times m$ ✓

$$[(AB)^T]_{ik} = (AB)_{ki} = \sum_{j=1}^{n} a_{kj} b_{ji} = \sum_{j=1}^{n} (A^T)_{jk} (B^T)_{ij} = \sum_{j=1}^{n} (B^T)_{ij} (A^T)_{jk} =$$

$$= [B^T A^T]_{ik} \quad \Downarrow$$

3.DZ-15) Ako su A i B simetrične, dokaži da je AB+BA također simetrično.

$\underline{A^T = A}$ $\underline{B^T = B}$

vrijedi komutativnost

$$(AB+BA)^T = (AB)^T + (BA)^T = B^TA^T + A^TB^T = BA + AB = AB+BA$$

3.DZ-11) Dokažite da A i B komutiraju ako i samo ako vrijedi formula za razliku kvadrata.

PAZI!

$$(A+B)^2 \neq A^2 + 2AB + B^2 \quad !!$$

$$(A+B)(A+B) = A^2 + \underbrace{AB+BA}_{\text{NIJE ISTO! NE VRIJEDI KOMUTATIVNOST}} + B^2.$$

$(\Leftrightarrow)$

$$A^2 - B^2 = (A-B)(A+B)$$

1. smjer

$(\Rightarrow)$ pretpostavka: $AB = BA$ (komutiraju)

$$(A-B)(A+B) \underset{\text{distributivnost}}{=} (A-B)A + (A-B)B \underset{\text{distr.}}{=} A^2 - \cancel{BA} + \cancel{AB} - B^2 \underset{\text{komutat.}}{=} A^2 - B^2$$

2. smjer pretpostavka: vrijedi formula

$(\Leftarrow)$ $(A-B)(A+B) = A^2 + AB - BA - B^2 = A^2 - B^2$

$\hookrightarrow AB - BA = 0$

$\underline{AB = BA} \longrightarrow$ komutiraju A i B !!

$(\Leftrightarrow)$ $\Downarrow$

3.DZ-10) Izračunaj:

$$\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}^n, \quad n \in \mathbb{N}.$$

$$A = \begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3\lambda & 4 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 3\lambda & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 7\lambda & 8 \end{bmatrix}$$

$$A^n = \begin{bmatrix} 1 & 0 \\ (2^n-1)\lambda & 2^n \end{bmatrix}$$

OBAVEZNO DOKAZATI INDUKCIJOM !

UVIJEK !

Dokaz:

1) baza : $\underline{n=1}$

$$A^1 = \begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}$$

2) pretpostavimo da vrijedi $A^n = \begin{bmatrix} 1 & 0 \\ (2^n-1)\lambda & 2^n \end{bmatrix}$

3) — | — za $n+1$

$$A^{n+1} = \begin{bmatrix} 1 & 0 \\ (2^{n+1}-1)\lambda & 2^{n+1} \end{bmatrix}$$

$$A^{n+1} = \underbrace{A^n}_{\text{pretp.}} \cdot A = \begin{bmatrix} 1 & 0 \\ (2^n-1)\lambda & 2^n \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \lambda & 2^n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (2^{n+1}-1)\lambda & 2^{n+1} \end{bmatrix}$$

1.MI-2005-5.) $A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 3 \\ -1 & 2 \end{bmatrix}$. Riješi jednadžbu:

$$(X+A)^T = AB$$ X mora biti 2×2

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\right)^T = \begin{bmatrix} 2 & -1 \\ -1 & 11 \end{bmatrix}$$

$$\begin{bmatrix} a+1 & c+3 \\ b-2 & d+1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 11 \end{bmatrix}$$

$a+1 = 2 \rightarrow a = 1$

$b-2 = -1 \rightarrow b = 1$

$c+3 = -1 \rightarrow c = -4$

$d+1 = 11 \rightarrow d = 10$

$$X = \begin{bmatrix} 1 & 1 \\ -4 & 10 \end{bmatrix}$$

3.DZ-3)

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot X = \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

3×3 $\qquad$ 3×2

$\Rightarrow 3\times 2$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} c & d \\ a+e & b+f \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} c=3 \\ d=5 \end{matrix} \quad \begin{matrix} a+e=0 \\ \underline{e=-a} \end{matrix} \quad \begin{matrix} b+f=1 \\ \underline{f=1-b} \end{matrix}$$

PAZI!

DA JE BILO 1 $\Rightarrow 0 \neq 1 \rightarrow$ NEMA RJ.

SVI SE BROJEVI MORAJU SLAGATI!

$$X = \begin{bmatrix} a & b \\ 3 & 5 \\ -a & 1-b \end{bmatrix}$$

$\rightarrow$ zapiši rješenje SA ŠTO MANJE NEPOZNANICA

$a, b \in \mathbb{R}$

3.DZ-7)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underline{X \cdot A = A \cdot X}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\begin{bmatrix} \cancel{a} & \cancel{a+2b} & \cancel{b+c} \\ \cancel{d} & \cancel{d+2e} & \cancel{e+f} \\ \cancel{g} & \cancel{g+2h} & \cancel{h+i} \end{bmatrix} = \begin{bmatrix} a+d & b+e & c+f \\ \cancel{2d+g} & \cancel{2e+h} & \cancel{2f+i} \\ \cancel{g} & \cancel{h} & \cancel{i} \end{bmatrix}$$

$\Rightarrow$ $a = a+d$ $\quad$ $d+2e = 2e+h$ $\quad$ $d = 2d+g$ $\quad$ $a+2b = b+e$

$\boxed{d=0}$ $\quad$ $\boxed{h=0}$ $\quad$ $\boxed{g=0}$ $\quad$ $\boxed{e=a+b}$

$b+c = e+f$ $\quad$ $e+f = 2f+i$

$\boxed{f=b}$ $\quad$ $\underline{i = e-f}$

$i = a+b-b$

$\boxed{i=a}$

$$X = \begin{bmatrix} a & b & c \\ \cancel{0} & a+b & b \\ \cancel{0} & \cancel{0} & a \end{bmatrix} \quad a, b, c \in \mathbb{R}$$

gornja $\triangle$

PET: ∞ 10-12, PIŠEMO U 13

3.3. DETERMINANTE

→ svakoj kvadratnoj matrici pridružujemo jedan broj koji zovemo determinanta, a označavamo ga sa $\det A$ ili $|A|$.

- def. se induktivno:

$n=1$: $A=[a_{11}] \longrightarrow \det A = a_{11}$

PAZI! : $A=[-1]$

$|A| = -1$

$n=2$: $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \longrightarrow \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\cdot a_{22} - a_{12}\cdot a_{21}$

npr. $\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} = 3-4=-1$

$n=3$: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\cdot\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Pr.)

$$\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \\ 2 & 6 & 5 \end{bmatrix} = 3\cdot\begin{vmatrix} 5 & 9 \\ 6 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 5 \\ 2 & 5 \end{vmatrix} + 4\begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} = 3\cdot(-29) - 13 + 4\cdot(-4) = -90$$

→ SARRUSOVO PRAVILO:

→ samo za $n=3$:

$$\begin{array}{ccc|cc} \overset{+}{1} & \overset{+}{2} & \overset{+}{3} & \overset{-}{4} & \overset{-}{5} \\ \end{array}$$

$$\left[\begin{array}{ccc|cc} 3 & 1 & 4 & 3 & 1 \\ 1 & 5 & 9 & 1 & 5 \\ 2 & 6 & 5 & 2 & 6 \end{array}\right] \overset{\overset{-}{6}}{} = 75+18+24-40-162-5=-90$$

3.3. Determinante

→ svakoj kvadratnoj matrici pridružujemo jedan broj = determinanta, kojeg označavamo sa $\det A$ ili $|A|$; definirat ćemo ga induktivno ovisno o redu od A

- za $n=1$: $A=[a_{11}] \Rightarrow \det A = |a_{11}| = a_{11}$ (Pazi: $|-1| = -1$, nije aps. vrijednost!)
- za $n=2$: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} a_{21}$.
- za $n=3$:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Formula se lako pamti: Sarrusovo pravilo (samo za $n=3$!)

Npr.)

$$\begin{array}{ccc|cc} + & + & + & & \\ 3 & 1 & 4 & 3 & 1 \\ 1 & 5 & 9 & 1 & 5 \\ 2 & 6 & 5 & 2 & 6 \\ - & - & - & & \end{array} = 75 + 18 + 24 - 40 - 162 - 5 = -90$$

- za $n \geq 4$? na sličan način!

→ determinantu matrice koja se dobije brisanjem i-tog retka i j-tog stupca nazivamo MINORA: M_{ij}

→ minoru zajedno s predznakom nazivamo algebarski komplement $A_{ij} = (-1)^{i+j} M_{ij}$

Npr.) $\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$

No svejedno po kojem retku ili stupcu razvijamo ⇒ uvijek isti rezultat!!

Npr.) po 2. stupcu dobivamo: $-6\begin{vmatrix} 3 & 4 \\ 1 & 9 \end{vmatrix} + 5\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 9 \\ 2 & 5 \end{vmatrix} = -6 \cdot 23 + 5 \cdot 7 - 1 \cdot (-13) = -90$

- Laplaceov razvoj: $\boxed{\det A = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{i=1}^{n} a_{ij} A_{ij}}$ po bilo kojem i-tom retku ili j-tom stupcu!

Pr.)

$$\begin{vmatrix} 2 & 2 & 3 & 1 \\ -3 & 3 & 1 & 5 \\ 0 & 0 & 3 & 0 \\ -1 & 0 & 0 & 2 \end{vmatrix} = 3 \begin{vmatrix} 2 & 2 & 1 \\ -3 & 3 & 5 \\ -1 & 0 & 2 \end{vmatrix} = 3\left[-1(10-3) + 2(6+6)\right] = 51$$

za ujeziku po nekom drugom retku/stupcu!

Računanje: $c \cdot n!$ operacija → jako složeno, npr. za $n=5$: 10^{-4} s (u 1 s 10^6 operacija)

$n=10$: 9,8 s

$n=25$: 10^{12} godina

$n=100$: 10^{144} godina

→ uz korištenje svojstava može se svesti na $2n^3/3$ operacija (za $n=25$: 0.01 s)

• Svojstva determinanti:

1. Ako matrica A ima redak ili stupac sastavljen samo od nula, onda je $\det A = 0$.
2. Ako je matrica A trokutasta, tada je $\det A = a_{11} a_{22} \cdots a_{nn}$ (umnožak na dijagonali)
3. Ako matrica A ima dva jednaka retka ili stupca, $\det A = 0$.

Dokaz: Indukcijom: 1. baza: $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$

2. pretp. da vrijedi za determinante reda n

3. korak: neka je A reda n+1 i neka su i-ti i j-ti red isti.

Razvijmo po nekom retku k $(k \neq i, k \neq j)$

$$\det A = \sum_{l=1}^{n} (-1)^{k+l} a_{kl} M_{kl} \rightarrow M_{kl} \text{ je determinanta}$$

reda n koja ima dva ista retka, po pretp. $= 0$ //

4. Transponiranje ne mijenja vrijednost determinante: $\det A = \det A^T$
(indukcijom, slično kao prije)

5. Determinanta se množi sa skalarom tako da se bilo koji redak (stupac) pomnoži sa skalarom.

Dokaz: $\lambda \det A = \lambda \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} (\lambda a_{ij}) A_{ij}$ (trivić)

Pr.)
$$\begin{vmatrix} 3 & 8 & -2 \\ 5 & 20 & -15 \\ 3 & 4 & -1 \end{vmatrix} = 5 \begin{vmatrix} 3 & 8 & -2 \\ 1 & 4 & -3 \\ 3 & 4 & -1 \end{vmatrix} = 20 \begin{vmatrix} 3 & 2 & -2 \\ 1 & 1 & -3 \\ 3 & 1 & -1 \end{vmatrix}$$

6. Rastave li se elementi nekog retka na zbroj dva elementa, determinanta je jednaka zbroju dviju odgovarajućih determinanata.

Pr.)
$$\begin{vmatrix} x+2 & 3x-3 & -x \\ 2 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} x & 3x & -x \\ 2 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -3 & 0 \\ 2 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

Dokaz: $\det A = \sum_{j=1}^{n} (a'_{ij} + a''_{ij}) A_{ij} = \sum_{j=1}^{n} a'_{ij} A_{ij} + \sum_{j=1}^{n} a''_{ij} A_{ij} = \det A' + \det A''$

7. Ako zamijenimo dva retka (stupca), determinanta mijenja predznak.

Dokaz:
$$0 = \begin{vmatrix} \vdots \\ a_i + a_j \\ \vdots \\ a_i + a_j \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ a_i \\ \vdots \\ a_i \\ \vdots \end{vmatrix} + \begin{vmatrix} \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \end{vmatrix} + \begin{vmatrix} \vdots \\ a_j \\ \vdots \\ a_i \\ \vdots \end{vmatrix} + \begin{vmatrix} \vdots \\ a_j \\ \vdots \\ a_j \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ a_j \\ \vdots \\ a_i \\ \vdots \end{vmatrix} + \begin{vmatrix} \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \end{vmatrix} = 0$$

(ostali retci isti, i i j želimo zamijeniti) //

8. Ako nekom retku dodamo neki drugi redak pomnožen skalarom, vrijednost determinante ostane ista.

Dokaz:
$$\begin{vmatrix} \vdots \\ a_i \\ \vdots \\ a_j + \lambda a_i \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \end{vmatrix} + \lambda \begin{vmatrix} \vdots \\ a_i \\ \vdots \\ a_i \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \end{vmatrix}$$ //

9. Binet-Cauchy teorem: $\det AB = \det A \cdot \det B$

Dokaz: upiši PMF!

1.MI-2010-4.) Ako za kvadratnu matricu reda n vrijedi $A^2 = A^T$, koje vrijednosti može poprimiti $\det A$?

$$\det A^2 = \det A^T$$
$$\det A \det A = \det A$$
$$\det A = 0,\ \det A = 1$$

(nul matrica, jedinična matrica)

1.MI-2005-6.)

$$\begin{vmatrix} 2 & 2 & 2 & 2 & 2 \\ -1 & 2 & 2 & 2 & 2 \\ -1 & -1 & 2 & 2 & 2 \\ -1 & -1 & -1 & 2 & 2 \\ -1 & -1 & -1 & -1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 & 3 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix} = 2 \cdot 3^4 = 162$$

3.ZZV-13.)

$$\begin{vmatrix} 3 & 2 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 & 2 \\ 2 & 2 & 3 & 2 & 2 \\ 2 & 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 2 & 3 \end{vmatrix} \begin{matrix} \\ -1r \\ -1r = \\ -1r \\ -1r \end{matrix} \begin{vmatrix} 3 & 2 & 2 & 2 & 2 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 11 & 2 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 11 \quad \det A^3 = 11^3$$

1.MI-2008-4.)

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 2 & 2 & 2 \\ 1 & 2 & -2 & 2 & 2 \\ 1 & 2 & 2 & -2 & 2 \\ 1 & 2 & 2 & 2 & -2 \end{vmatrix} \begin{matrix} \\ -1r \\ -1r \\ -1r \\ -1r \end{matrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & 1 & 1 \\ 0 & 1 & -3 & 1 & 1 \\ 0 & 1 & 1 & -3 & 1 \\ 0 & 1 & 1 & 1 & -3 \end{vmatrix} = 1 \begin{vmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{vmatrix} \overset{\text{zbog svih prvom!}}{=} \begin{vmatrix} 0 & 0 & 0 & 0 \\ - & - & - & \\ - & - & - & \\ - & - & - & \end{vmatrix} = 0$$

(zadnji!)

3.ZZV-12.) Determinanta reda 6 zadana s $a_{ij} = \max\{i,j\}$

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 2 & 3 & 4 & 5 & 6 \\ 3 & 3 & 3 & 4 & 5 & 6 \\ 4 & 4 & 4 & 4 & 5 & 6 \\ 5 & 5 & 5 & 5 & 5 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{vmatrix} \begin{matrix} -2r \\ -3r \\ -4r \\ = \\ \\ \\ \end{matrix} \begin{vmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{vmatrix} = -6$$

(poopćiti na n → na drugoj strani)

3.DZ-19.)

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & x & x & x \\ 1 & x & 0 & x & x \\ 1 & x & x & 0 & x \\ 1 & x & x & x & 0 \end{vmatrix} \begin{matrix} \\ \\ \\ -3r \\ -4r \end{matrix} = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -x & x & 0 & 0 \\ 0 & 0 & -x & x & 0 \\ 0 & 0 & 0 & -x & x \\ 1 & x & x & x & 0 \end{vmatrix} = \begin{vmatrix} 0 & 4 & 3 & 2 & 1 \\ 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & x \\ 1 & 3x & 2x & x & 0 \end{vmatrix} = 1 \cdot 4x^3 = 32$$

$$x^3 = 8 \rightarrow x = 2$$

3.DZ-17.)

$$\begin{vmatrix} 2 & 1 & 1 & -1 & 1 \\ 1 & -2 & 1 & 2 & -1 \\ -1 & 1 & 2 & -2 & 1 \\ 1 & -1 & 1 & -2 & 1 \\ 2 & -1 & 1 & 2 & -1 \end{vmatrix} \begin{matrix} \\ \\ +2r = \\ +3r \\ -1r \end{matrix} \begin{vmatrix} 2 & 1 & 1 & -1 & 1 \\ 1 & -2 & 1 & 2 & -1 \\ 0 & -1 & 3 & 0 & 0 \\ 0 & 0 & 3 & -4 & 2 \\ 0 & -2 & 0 & 3 & -2 \end{vmatrix} \begin{matrix} -2 \cdot 2r + \\ \\ = \\ \\ \\ \end{matrix} \begin{vmatrix} 0 & 5 & -1 & -5 & 3 \\ 1 & -2 & 1 & 2 & -1 \\ 0 & -1 & 3 & 0 & 0 \\ 0 & 0 & 3 & -4 & 2 \\ 0 & -2 & 0 & 3 & -2 \end{vmatrix} \begin{matrix} +3 \cdot 1r \\ \\ = - \\ \\ \\ \end{matrix} \begin{vmatrix} 5 & -1 & -5 & 3 \\ -1 & 3 & 0 & 0 \\ 0 & 3 & -4 & 2 \\ -2 & 0 & 3 & -2 \end{vmatrix} \begin{matrix} +5 \cdot 2r \\ \\ \\ -2 \cdot 2r \end{matrix}$$

$$= - \begin{vmatrix} 0 & 14 & -5 & 3 \\ -1 & 3 & 0 & 0 \\ 0 & 3 & -4 & 2 \\ 0 & -6 & 3 & -2 \end{vmatrix} = (+)(-1) \begin{vmatrix} 14 & -5 & 3 \\ 3 & -4 & 2 \\ -6 & 3 & -2 \end{vmatrix} = (-1)(112 + 60 + 27 - 72 - 84 - 30)$$

$$= -13$$

uZas!

Zad.

a)

$$\begin{vmatrix} 2 & 4 & 6 & \cdots & 2n \\ -2 & 0 & 6 & \cdots & 2n \\ -2 & -4 & 0 & \cdots & 2n \\ \vdots & & & & \\ -2 & -4 & -6 & \cdots & 0 \end{vmatrix} \begin{matrix} \\ +1r \\ +1r \\ \\ +1r \end{matrix} = \begin{vmatrix} 2 & 4 & 6 & \cdots & 2n \\ 0 & 4 & 12 & \cdots & 4n \\ 0 & 0 & 6 & \cdots & 4n \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 2n \end{vmatrix} \text{ (prvo izluči 2)} = 2\cdot 4\cdot 6 \cdots 2n = 2^n\, n!$$

b) $a_{ij} = \max\{i,j\}$

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 3 & & n \\ 3 & 3 & 3 & & n \\ & & & & \\ n & n & n & \cdots & n \end{vmatrix} \begin{matrix} -2r \\ -3r \\ -4r \\ \\ \\ \end{matrix} = \begin{vmatrix} -1 & 0 & 0 & & 0 \\ -1 & -1 & 0 & \cdots & 0 \\ -1 & -1 & -1 & \cdots & 0 \\ \vdots & & & & \\ n & n & n & \cdots & n \end{vmatrix} = n(-1)^{n-1}$$

c) $a_{ij} = |i-j|+1$

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 1 & 2 & \cdots & n-1 \\ 3 & 2 & 1 & \cdots & n-2 \\ \vdots & & & & \\ n-1 & n-2 & n-3 & \cdots & 1 \\ n & n-1 & n-2 & \cdots & 1 \end{vmatrix} \begin{matrix} \\ -1r \\ -2r \\ \\ \\ -(n-1)r \end{matrix} = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & -1 & \cdots & -1 \\ 1 & 1 & 1 & -1 \cdots & -1 \\ \vdots & & & & \vdots \\ 1 & 1 & 1 & \cdots 1 & -1 \end{vmatrix} = \begin{vmatrix} n+1 & n+2 & n+3 & \cdots & n \\ 0 & -2 & -2 & & -1 \\ 0 & 0 & -2 & & -1 \\ 0 & 0 & 0 & \ddots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & -1 \end{vmatrix} = -(n+1)(-2)^{n-2}$$

d)

$$\begin{vmatrix} 2 & 2 & 2 & \cdots & 2 \\ 2 & 3 & 2 & \cdots & 2 \\ 2 & 2 & 4 & \cdots & 2 \\ \vdots & & & & \\ 2 & 2 & 2 & \cdots & n+1 \end{vmatrix} \begin{matrix} \\ -1r \\ -1r \\ \\ -1r \end{matrix} = \begin{vmatrix} 2 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & n-1 \end{vmatrix} = 2(n-1)!$$

$$a_{ij} = \begin{cases} i+1, & i=j \\ 2, & i \neq j \end{cases}$$

4. RANG I INVERZ MATRICE

4.1. INVERZNA MATRICA

→ pogledajmo matričnu jednadžbu

$$AX = B$$

→ uvodimo A' t.d. $A' \cdot A = I$

$$A'AX = A'B$$

$$\boxed{X = A'B}$$

$A' \longrightarrow$ INVERZ MATRICE A

→ Neka je $a \in M_n$ (skup svih kvadratnih matrica reda n). Matrica A^{-1} za koju vrijedi

$$\boxed{A^{-1} \cdot A = A \cdot A^{-1} = I}$$

se naziva INVERZ MATRICE A.

→ Za matricu A kažemo da je <u>REGULARNA</u> ako postoji njen inverz A^{-1}. Inače kažemo da je A <u>SINGULARNA</u>.

$\boxed{\text{TM}}$ Ako su A i B regularne matrice tada je umnožak AB regularna matrica i vrijedi da je

$$\boxed{(AB)^{-1} = B^{-1} \cdot A^{-1}}$$

dokaz:

$$(AB)(B^{-1}A^{-1}) \underset{\text{asocijativnost}}{=} A(\overbrace{BB^{-1}}^{I})A^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = (B^{-1}B)(A^{-1}A) = I$$

<u>Primjer</u>: lako se pokaže: $\boxed{(A_1 A_2 \dots A_n)^{-1} = A_n^{-1} \dots A_2^{-1} A_1^{-1}}$

$$AA^{-1} = I$$

$$\det(AA^{-1}) = \det(I)$$

$$\det A^{-1} \cdot \det A^{-1} = 1$$

$$\boxed{\det A^{-1} = \frac{1}{\det A}} \Longrightarrow \det A \neq 0$$

INVERZ POSTOJI AKO JE

$\underline{\underline{\det A \neq 0}}$!!!

$$\sum_{j=1}^{n} a_{jk} A_{jk} = \det A$$

$$\sum_{j=1}^{n} a_{jk} A_{ji} = \begin{cases} \det A, & i=k \\ 0, & i \neq k \end{cases} \quad \Big/ : \det A$$

$$\frac{1}{\det A} \sum_{j=1}^{n} a_{jk} A_{ji} = \delta_{ik}$$

$\to 0$ ako su isti

$\to 1$ ako su različiti

$$\underbrace{\frac{1}{\det A} \tilde{A}^T}_{A^{-1}} \cdot A = I \qquad \tilde{A}^T \to \text{adjunkta matrica}$$

$\boxed{\text{TM}}$ „CRAMEROVO PRAVILO"

→ Neka je $A \in M_n$. A je regularna ako i samo ako je $\det A \neq 0$. Tada je inverz jednak:

$$A^{-1} = \frac{1}{\det A} \cdot \tilde{A}^T = \frac{1}{\det A} \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & & \vdots \\ A_{n1} & \cdots & \cdots & A_{nn} \end{bmatrix}^{\sim}$$

Primjer:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ 1 & 3 & 4 \end{bmatrix} \longrightarrow \det A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ 1 & 3 & 4 \end{vmatrix} = \cdots = 2$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} +\begin{vmatrix} -1 & -1 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 9 & 7 \\ 1 & 1 & -1 \\ 1 & 7 & -5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 9 & 1 & 7 \\ 7 & -1 & -5 \end{bmatrix}$$

1.MI-2010-5)

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & c & -c \\ -1 & -c & 2c \end{bmatrix}$$

a) za koje $c \in \mathbb{R}$ je matrica regularna?

$$\det A = \begin{vmatrix} 1 & 1 & -1 \\ 1 & c & -c \\ -1 & -c & 2c \end{vmatrix} = c^2 - c \neq 0$$

$$c(c-1) \neq 0$$

$$0 \qquad 1$$

A^{-1} postoji $c \in \mathbb{R} \setminus \{0, 1\}$

b) za najmanji $c \in \mathbb{N}$ za koji A^{-1} postoji izračunaj A^{-1}

$c = 2$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 4 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Podsjetimo se:

$$A^{-1} = \frac{1}{\det A} \cdot \tilde{A}^T = \frac{1}{\det A} \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{1n} & \cdots & A_{nn} \end{bmatrix}$$

Primjer:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow A^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

KORISNO

Pr.)

$$\begin{bmatrix} \pi & 2 \\ e & 3 \end{bmatrix}^{-1} = \frac{1}{3\pi - 2e} \begin{bmatrix} 3 & -2 \\ -e & \pi \end{bmatrix}$$

→ PRIMJENA NA MATRIČNE JEDNADŽBE:

$$XA = B \quad \Big| \cdot A^{-1}$$

$$\underline{X = B \cdot A^{-1}}$$

PROVJERITI POSTOJI LI INVERZ!!

1MI-2009-6) $\underline{AX^{-1}B = BA}$ $\quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

POSTOJI LI INVERZ ??? DA

PAZ!! NIJE ISTO JE LI S LIJEVE ILI S DESNE STRANE!

$$A^{-1} \Big/ \; AX^{-1}B = BA$$

$$X^{-1}B = A^{-1}BA \quad \Big| \; B^{-1} \rightarrow \text{PROVJERI JEL' IMA INVERZ}$$

$$X^{-1} = A^{-1}BAB^{-1} \quad \Big/^{-1}$$

$$\underline{X = BA^{-1}B^{-1}A}$$

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \frac{1}{1} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

4. DZ – 15.)

$$(X^{-1}A)^{-1} = I + BX \qquad A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix}$$

$2 \cdot 3 = -2 \cdot (-3)$

$\det B = 0$

INVERZ NE POSTOJI !!

$$A^{-1}X = I + BX$$

$$A^{-1}X - BX = I$$

PAZI !!

NE SMIJEMO SA B^{-1}

$$\cancel{X(A^{-1} - B) = I}$$

NE !! NIJE KOMUTATIVNO !

$$\Updownarrow$$

$$(A^{-1} - B)^{-1} \Big/ (A^{-1} - B)X = I$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$$

$$A^{-1} - B = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \rightarrow \text{postoji inverz!}$$

$$X = (A^{-1} - B)^{-1}$$

$$X = \frac{1}{-10}\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

4.2. ELEMENTARNE TRANSFORMACIJE I ELEMENTARNE MATRICE

- def. Elem. transf. matrice su:

1. ZAMJENA DVA RETKA
2. MNOŽENJE RETKA SKALAROM RAZLIČITIM OD 0
3. DODAVANJE NEKOG RETKA POMNOŽENOG SKALAROM DRUGOM RETKU

def. REDUCIRANI OBLIK matrice se definira pravilima:

1. prvi ne-nulti element (stožer) u svakom retku mora biti 1, a ostali elementi u tom stupcu stožera moraju biti 0
2. svi redci koji sadrže samo nul-elemente moraju biti ispod ostalih redaka koji imaju elemente različite od nule
3. svaki naredni stožer mora biti DESNO od prethodnog !

Pr.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ Supeeer !! :)

→ Reducirani oblik matrice označavamo: $\underline{\underline{A_r}}$

Zad.) Svedi na reducirani oblik:

$$A=\begin{bmatrix}0&-1&3&0\\-2&5&-4&-5\\2&-1&-8&5\end{bmatrix}\sim\underbrace{\begin{bmatrix}2&-1&-8&5\\-2&5&-4&-5\\0&-1&3&0\end{bmatrix}}_{A_1}\cdot\tfrac{1}{2}\sim\underbrace{\begin{bmatrix}1&-\frac{1}{2}&-4&5/2\\-2&5&-4&-5\\0&-1&3&0\end{bmatrix}}_{A_2}+2\cdot1\cdot r\sim\underbrace{\begin{bmatrix}1&-1/2&-4&5/2\\0&-4&-12&0\\0&-1&3&0\end{bmatrix}}_{A_3}\cdot\tfrac{1}{4}\sim$$

$$\sim\underbrace{\begin{bmatrix}1&-1/2&-4&5/2\\0&1&-3&0\\0&-1&3&0\end{bmatrix}}_{A_4,\,A_5}\begin{matrix}+\frac{1}{2}\cdot2r\\ \\+2r\end{matrix}\sim\underbrace{\begin{bmatrix}1&0&-11/2&5/2\\0&1&-3&0\\0&0&0&0\end{bmatrix}}_{A_r}=A_r$$

TM Matrice A i A_r su EKVIVALENTNE matrice po redcima ($\sim$ je relacija ekvivalencije po redcima).

Def. Elemen. matrice su jedinične matrice nad kojima vršimo elementarne transformacije.

1) zamjena 2 redka

$$\begin{bmatrix}1&&&&\\&\ddots&&0&\\&&0&1&\\&1&0&\ddots&\\&&&&1\end{bmatrix}$$

2) množenje redka skalarom

$$\begin{bmatrix}1&&&&\\&\ddots&&0&\\&&\lambda&&\\&0&&\ddots&\\&&&&1\end{bmatrix}$$

3) dodavanje redka

$$\begin{bmatrix}1&&&&\\&\ddots&&0&\\&&1&&\\&0&&\ddots&\\&&\lambda&&1\end{bmatrix}\lambda$$

— za prethodni primjer:

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_5 = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$E_1 \cdot A = A_1$$
$$E_2 \cdot A_1 = A_2$$
$$E_3 \cdot A_2 = A_3$$
$$E_4 \cdot A_3 = A_4$$
$$E_5 \cdot A_4 = A_5$$
$$E_6 \cdot A_5 = A_r$$

$$\Longrightarrow \quad A_r = E_6 E_5 E_4 E_3 E_2 E_1 A$$

$$\boxed{A_r = E_r \cdot \ldots \cdot E_1 A}$$

$$\boxed{E_1^{-1} E_2^{-1} \cdot \ldots \cdot E_r^{-1} \cdot A_r = A}$$

Lema: Element. matrice su regularne. (Dakle, postoji inverz!)

$\Downarrow$

$\underline{\det \neq 0}$

Lema: Ako je A regularna, a B njoj ekvivalentna tada je i B regularna.

4.3. RANG MATRICE

def. Rang matrice je broj nenultih redaka u reduciranom obliku.

→ za prethodni primjer $\underline{r(A) = 2}$

4.DZ.-2) Odredi rang matrica:

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 1 & 1 & 0 & 3 & 2 \\ 4 & 1 & -3 & 3 & 9 \\ -1 & 1 & 2 & 3 & 5 \\ -1 & 0 & 1 & 0 & 3 \end{bmatrix} \begin{matrix} \\ -1\cdot r \\ -4\cdot r \\ +1\cdot r \\ +1\cdot r \end{matrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & +1 & 3 & -1 \\ 0 & 1 & 1 & 3 & -3 \\ 0 & 1 & 1 & 3 & 8 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix} \begin{matrix} \\ \\ -2\cdot r \\ -2\cdot r \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix} \begin{matrix} \\ \\ \cdot -1/2 \\ \\ \\ \end{matrix} \sim$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix} \begin{matrix} -3\cdot 3\cdot r \\ +3\cdot r \\ \\ -9\cdot 3\cdot r \\ -6\cdot 3\cdot r \end{matrix} \sim \begin{bmatrix} \boxed{1} & 0 & -1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{r(A) = 3}$$

AUDITORNE – 4. TJEDAN

Zad.) Neka su A, B, X matrice reda n koje su regularne i za njih vrijedi: $A \cdot X^{-1} \cdot B^{-1} = B \cdot A^{-1}$. Izrazi X pomoću A i B i njihovih inverza, te $\det X$ pomoću $\det A$ i $\det B$ ako je $\det A = 6$ i $\det B = 9$.

$$A X^{-1} B^{-1} = B \cdot A^{-1} \quad / B$$

$$A \cdot X^{-1} \underbrace{B^{-1} \cdot B}_{I} = B \cdot A^{-1} \cdot B$$

$$A X^{-1} \cdot I = B A^{-1} \cdot B$$

$$A^{-1} / \quad A X^{-1} = B A^{-1} B$$

$$X^{-1} = A^{-1} B \cdot A^{-1} \cdot B \quad /(\ )^{-1}$$

$$\boxed{X = B^{-1} A B^{-1} A}$$

BINET-CAUCHY:

$$\det(AB) = \det A \cdot \det B$$

$$X = B^{-1} A B^{-1} A \quad / \det$$

$$\det X = \det(B^{-1} A \cdot B^{-1} A)$$

$$= \det(B^{-1} A) \cdot \det(B^{-1} A)$$

$$\det X = \det(B^{-1}) \cdot \det(A) \cdot \det(B^{-1}) \cdot \det(A)$$

$$\det X = \det(B^{-1}) \cdot \det(B^{-1}) \cdot \det(A) \cdot \det(A)$$

$$B \cdot B^{-1} = I \quad / \det$$

$$\det(B \cdot B^{-1}) = \det I = 1$$

$$\det B \cdot \det B^{-1} = 1 \quad / : \det B$$

$$\boxed{\det(B^{-1}) = \frac{1}{\ }}$$

$$\det X = \frac{1}{\det B}\cdot\frac{1}{\det B}\cdot\det A\cdot\det A = \left(\frac{\det A}{\det B}\right)^2 = \frac{4}{9}$$

Zad.

$$\begin{vmatrix} 2 & 1 & 8 & 4 \\ 2 & 2 & 8 & 4 \\ 4 & 4 & 8 & 4 \\ 8 & 8 & 8 & 8 \end{vmatrix} = 64 \begin{vmatrix} 2 & 1 & 8 & 4 \\ 1 & 1 & 4 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} \begin{matrix} \\ \cdot(-1)+3.r \\ \cdot(-1)+2.r \\ \cdot(-2)+1.r \end{matrix} = 64 \begin{vmatrix} 0 & -1 & 6 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -64 \begin{vmatrix} -1 & 6 & 2 \\ 0 & 3 & 1 \\ 0 & 1 & 0 \end{vmatrix} =$$

$$= 64 \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} = 64\cdot(-1) = -64$$

zzv-3. b) Za zadanu matricu A izračunajte A^n.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^4 = A^3\cdot A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^5 = \dots \begin{bmatrix} 1 & 5 & 10 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$a_1 = 0$

$a_2 = 1$] 1

$a_3 = 3$] 2

$a_4 = 6$] 3

$a_5 = 10$] 4

$$a_n = a_{n-1} + (n-1)$$

$$\Downarrow$$

$$\boxed{a_{n+1} = a_n + n}$$

$a_4 = a_3 + 3$

$a_5 = a_4 + 4$

$a_3 = a_2 + 2$

$\cancel{a_1} = 0$

$\cancel{a_2} = \cancel{a_1} + 1$

$\cancel{a_3} = \cancel{a_2} + 2$

$\vdots$

$\cancel{a_{n-1}} = \cancel{a_{n-2}} + (n-2)$

$a_n = \cancel{a_{n-1}} + n - 1$

$$a_n = 0 + 1 + 2 + \dots + (n-2) + (n-1) = \frac{(n-1)n}{2}$$

$$A^n = \begin{bmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$$

<u>dokaz:</u>

1) $A^1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

2) za $A^n =$

3) za $n+1$

$$A^{n+1} = \begin{bmatrix} 1 & n+1 & \frac{(n+1)n}{2} \\ 0 & 1 & n+1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{n+1} = \overset{\text{pretp}}{A^n} \cdot A$$

BITNO!! ODE BOD!

$$= \begin{bmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+1 & n + \frac{n(n-1)}{2} \\ 0 & 1 & n+1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+1 & \frac{n(n+1)}{2} \\ 0 & 1 & n+1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\rightarrow$ Po principu mat. ind. tvrdnja vrijedi za $\forall\, n \in \mathbb{N}$.

4.ZZV–4.) Ako je $a \neq 0$ dobažite da rang matrice

$$\begin{bmatrix} a & 0 & b & a \\ c & b & 0 & c \\ 0 & -a & c & 0 \end{bmatrix} : a \rightarrow \text{MOŽE}$$

ne ovisi o izboru parametara a, b, c.

$$\begin{bmatrix} 1 & 0 & \frac{b}{a} & 1 \\ c & b & 0 & c \\ 0 & -a & c & 0 \end{bmatrix} -c \cdot 1.r \sim \begin{bmatrix} 1 & 0 & \frac{b}{a} & 1 \\ 0 & b & -\frac{cb}{a} & 0 \\ 0 & -a & c & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{b}{a} & 1 \\ 0 & -a & -c & 0 \\ 0 & b & -\frac{cb}{a} & 0 \end{bmatrix} :(-a) \sim \begin{bmatrix} 1 & 0 & \frac{b}{a} & 1 \\ 0 & 1 & -\frac{c}{a} & 0 \\ 0 & b & -\frac{cb}{a} & 0 \end{bmatrix} -b \cdot 2$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{b}{a} & 1 \\ 0 & 1 & -\frac{c}{a} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \underline{r(A) = 2} \rightarrow \underline{\text{UVIJEK}}!$$

4.DZ–4.) U ovisnosti o parametru λ odredi rang matrice.

$$\begin{bmatrix} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & 1 \\ 1 & 1 & \lambda & 1 \\ 1 & 1 & 1 & \lambda \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & \lambda \\ 1 & \lambda & 1 & 1 \\ 1 & 1 & \lambda & 1 \\ \lambda & 1 & 1 & 1 \end{bmatrix} \begin{matrix} \\ -1.r \\ -1.r \\ -\lambda \cdot 1.r \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & \lambda \\ 0 & \lambda-1 & 0 & 1-\lambda \\ 0 & 0 & \lambda-1 & 1-\lambda \\ 0 & 1-\lambda & 1-\lambda & 1-\lambda^2 \end{bmatrix} \cdot \frac{1}{\lambda-1}$$

UVJET: $\underline{\lambda \neq 1}$

UVIJEK NJEGA

1. slučaj: $\lambda = 1 \rightarrow$ NE SMIJEM MNOŽITI

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{uvrstiti u zadnju}$$

$\underline{r(A) = 1} \Rightarrow \boxed{\text{za } \lambda = 1 \rightarrow \underline{r(A) = 1}}$

2. slučaj: $\lambda \neq 1$ (smijem množiti)

$$\begin{bmatrix} 1 & 1 & 1 & \lambda \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & -1-\lambda \end{bmatrix} \begin{matrix} -2.r \\ \\ \\ +2.r \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 & \lambda+1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -2-\lambda \end{bmatrix} \begin{matrix} -3.r \\ \\ \\ +3.r \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & \lambda \\ 0 & \lambda-1 & 0 & 1-\lambda \\ 0 & 0 & \lambda-1 & 1-\lambda \\ 0 & 1-\lambda & 1-\lambda & 1-\lambda^2 \end{bmatrix} \begin{matrix} \cdot \frac{1}{\lambda-1} \\ \cdot \frac{1}{\lambda-1} \\ \cdot \frac{1}{\lambda-1} \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \lambda+2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -3-\lambda \end{bmatrix} \cdot \frac{1}{-3-\lambda} \rightarrow \underline{\lambda \neq -3}$$

3. slučaj: $\lambda = -3$

(u zadnju vrstiti)

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{reducirana forma}$$

$\underline{r(A) = 3}$ $\rightarrow$ za $\lambda = 3 \Rightarrow \underline{r(A) = 3}$

4. slučaj: $\lambda \neq -3$ i $\lambda \neq 1$ PAZI! NE ZABORAVI!

(smijemo dijeliti)

$$\begin{bmatrix} 1 & 0 & 0 & \lambda+2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} -(\lambda+2)\cdot 4.r \\ +4.r \\ +4.r \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{reducirana}$$

$\underline{r(A) = 4}$

Rješenje:

za $\lambda = 1$: $r(A) = 1$

za $\lambda = -3$: $r(A) = 3$

za $\lambda \in \mathbb{R} \setminus \{-3, 1\}$: $r(A) = 4$

Lema: Kvadratna matrica ima PUNI rang (rang = dimenziji) ako i samo ako je reducirana forma jednaka jediničnoj matrici. $\boxed{A_r = I}$

$$A_r = \underbrace{E_r \cdots E_1}_{A^{-1}} \cdot A = I$$

$$A^{-1} = E_r \cdots E_1$$

TM Kvadratna matrica je regularna (inverz ima) ako i samo ako je $\underline{\underline{r(A)=n.}}$

ALGORITAM ZA NALAŽENJE INVERZA:

(GAUSS-JORDAN-OV)

$$\left[A \,\vdots\, I\right] \sim \left[E_1 A \mid E_1\right] \sim \dots \sim \left[A_r \,\vdots\, E_r \dots E_1\right] = \left[I \,\vdots\, A^{-1}\right]$$

Zad.) Naći inverz:

$$\begin{bmatrix} 2 & 5 & -5 & 2 \\ 0 & 1 & -1 & -1 \\ 2 & 4 & -4 & 2 \\ 1 & -1 & 3 & 1 \end{bmatrix} \sim \left[\begin{array}{cccc|cccc} 2 & 5 & -5 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 4 & -4 & 2 & 0 & 0 & 1 & 0 \\ 1 & -1 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 4 & -4 & 2 & 0 & 0 & 1 & 0 \\ 2 & 5 & -5 & 2 & 1 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ -2\cdot 1.r \\ -2\cdot 1.r \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 6 & -10 & 0 & 0 & 0 & 1 & -2 \\ 0 & 7 & -11 & 0 & 1 & 0 & 0 & -2 \end{array}\right] \begin{array}{l} +2.r \\ \\ +6\cdot 2.r \\ +7\cdot 2.r \end{array} \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 6 & 0 & -6 & 1 & -2 \\ 0 & 0 & -4 & 7 & 1 & -7 & 0 & -2 \end{array}\right] \begin{array}{l} \\ \\ :(-4) \\ \\ \end{array} \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3/2 & 0 & 3/2 & -1/4 & 1/2 \\ 0 & 0 & -4 & 7 & 1 & -7 & 0 & -2 \end{array}\right] \begin{array}{l} -2\cdot 3.r \\ +3.r \\ \sim \\ +4\cdot 3.r \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 3 & 0 & -2 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & -5/2 & 0 & 5/2 & -1/4 & 1/2 \\ 0 & 0 & 1 & -3/2 & 0 & 3/2 & -1/4 & 1/2 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \end{array}\right] \begin{array}{l} -3\cdot 4.r \\ +5/2\cdot 4.r \\ +3/2\cdot 4.r \\ \\ \end{array} \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -3 & 1 & 7/2 & 0 \\ 0 & 1 & 0 & 0 & 5/2 & 0 & -1/4 & 1/2 \\ 0 & 0 & 1 & 0 & 3/2 & 0 & -7/4 & 1/2 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \end{array}\right]$$

$$\underbrace{\qquad\qquad}_{I} \quad \underbrace{\qquad\qquad}_{A^{-1}}$$

$$A^{-1} = \begin{bmatrix} -3 & 1 & 7/2 & 0 \\ 5/2 & 0 & -1/4 & 1/2 \\ 3/2 & 0 & -7/4 & 1/2 \\ 1 & -1 & -1 & 0 \end{bmatrix}$$

Provjeri rješenje!

$$\boxed{A \cdot A^{-1} = I}$$

IZMNOŽITI I DOBITI I

4.4. LINEARNA NEZAVISNOST VEKTORA

def. Linearna kombinacija vektora $a_1, \dots, a_k$ je vektor oblika
$\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_k a_k$, $\lambda_i \in \mathbb{R}$.

Skup svih linearnih kombinacija nazivamo prostorom razapetim vektorima $a_1 \dots a_k$: $L(a_1 \dots a_k)$

$a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad L(a_1, a_2) = \{ x \mid x = \lambda_1 a_1 + \lambda_2 a_2,\ \lambda_1, \lambda_2 \in \mathbb{R} \}$

$a_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 + \lambda_2 \\ \lambda_1 \end{bmatrix}$$

$$x_1 = \lambda_1 + \lambda_2$$
$$x_2 = \lambda_1$$

→ prikaži $\begin{bmatrix} 3 \\ 14 \end{bmatrix}$ kao linear. komb. ovih dva vektora:

$$\left.\begin{aligned} x_1 &= \lambda_1 + \lambda_2 = 3 \\ x_2 &= \lambda_1 = 14 \end{aligned}\right\} \lambda_2 = -11$$

$$\begin{bmatrix} 3 \\ 14 \end{bmatrix} = 14 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 11 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(ml-2009-8.)

$$a_1 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ -2 \\ \lambda \end{bmatrix}.$$ Odredi λ takav da je b linearna kombinacija od a_1 i a_2.

$$b = \lambda_1 a_1 + \lambda_2 a_2$$

$$\begin{bmatrix} 7 \\ -2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 2\lambda_1 \\ 3\lambda_1 \\ 5\lambda_1 \end{bmatrix} + \begin{bmatrix} \lambda_2 \\ 4\lambda_2 \\ 3\lambda_2 \end{bmatrix}$$

$$7 = 2\lambda_1 + \lambda_2$$

$$-2 = 3\lambda_1 + 4\lambda_2$$

$$\lambda = 5\lambda_1 + 3\lambda_2$$

$$\Downarrow$$

$$\lambda_1 = 6 \qquad \lambda_2 = -5$$

$$\boxed{\lambda = 15}$$

$$b = 6a_1 - 5a_2$$

def. Vektori $a_1 \dots a_k$ su linearno nezavisni ako iz jednadžbe $\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_k a_k = 0$ slijedi da su svi $\lambda_1 = 0 \dots \lambda_k = 0$

Linearna kombinacija iščezava samo na trivijalan način.

def. Vektori su linearno zavisni ako nisu nezavisni.

Pr.) $a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Provjeri jesu li linearno nezavisni?!

nul-matrica

$$\lambda_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \lambda_2 \\ 0 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 + \lambda_2 = 0$$

$$\left.\begin{bmatrix} \lambda_1 = 0 \\ \lambda_2 = 0 \end{bmatrix}\right\} \text{LINEARNO NEZAVISNI}$$

$$0 + 0 = 0$$

• PRINCIPIJELNO:

$$A_1 = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 & -2 \\ 0 & 1 & -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Bigg\} \text{NE-NUL RETCI SU LINEARNO NEZAVISNI VEKTORI !!}$$

$$\lambda_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \\ -2 \end{bmatrix} + \lambda_2 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \lambda_3 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 4 \end{bmatrix} = 0$$

$\vdots$

$$\left.\begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{matrix}\right\} \text{Dokazali !}$$

TM ① Elementarne transformacije NE MJENJAJU BROJ LINEARNO NEZAVISNIH REDAKA MATRICE.

TM ② Rang matrice je jednak BROJU LINEARNO NEZAVISNIH REDAKA / STUPACA

TM ③ Broj lin. nezavisnih redaka jednak je broju linearno nezavisnih stupaca!

4.DZ-17.) Jesu li vektori $\begin{bmatrix}1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}2\\1\\3\\7\end{bmatrix}, \begin{bmatrix}4\\1\\5\\1\end{bmatrix}$ linearn. nezavisni? Objasni!

I) način (po definiciji)

$$\lambda_1 \begin{bmatrix}1\\0\\1\\2\end{bmatrix} + \lambda_2 \begin{bmatrix}2\\1\\3\\7\end{bmatrix} + \lambda_3 \begin{bmatrix}4\\1\\5\\1\end{bmatrix} = 0$$

, ako su $\lambda_1, \lambda_2, \lambda_3 = 0$ onda su nez.

$$\begin{aligned}
\lambda_1 + 2\lambda_2 + 4\lambda_3 &= 0\\
\lambda_2 + \lambda_3 &= 0\\
\lambda_1 + 3\lambda_2 + 5\lambda_3 &= 0\\
2\lambda_1 + 7\lambda_2 + \lambda_3 &= 0
\end{aligned}$$

nakon računa ...

$\boxed{\lambda_1 = 0}$ $\boxed{\lambda_2 = 0}$ $\boxed{\lambda_2 = 0}$ $\boxed{\lambda_3 = 0}$

II) način (po TM)

$$\begin{bmatrix}1 & 2 & 4\\0 & 1 & 1\\1 & 3 & 5\\2 & 7 & 1\end{bmatrix} \begin{matrix}\\ \\ -1\cdot r\\ -2\cdot r\end{matrix} \sim \begin{bmatrix}1 & 2 & 4\\0 & 1 & 1\\0 & 1 & 1\\0 & 3 & -7\end{bmatrix} \begin{matrix}-2\cdot r\\ \\ -2\cdot r\\ -3\cdot 2\cdot r\end{matrix} \sim \begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 0\\0 & 0 & -10\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

Rang matrice jednak je broju linearno NEZAVISNIH vektora. (TM②) $\Longrightarrow$ lin. nezavisni

$r(A) = 3$ = broj vektora s početka

$r(A) = 3 =$ br. nez.

4.DZ - 20.) Za koji $a \in \mathbb{R}$ su vektori $\begin{bmatrix}1\\2\\3\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\3\\1\end{bmatrix}, \begin{bmatrix}4\\3\\9\\a\end{bmatrix}$ lin. nezavisni?

$$\begin{bmatrix}1&2&4\\2&-1&3\\3&3&9\\1&1&a\end{bmatrix}\begin{matrix}\\-2\cdot1\cdot r\\-3\cdot1\cdot r\\-1\cdot r\end{matrix}\sim\begin{bmatrix}1&2&4\\0&-5&-5\\0&-3&-3\\0&-1&a-4\end{bmatrix}\begin{matrix}\\:(-5)\\:(-3)\\ \\ \end{matrix}\sim\begin{bmatrix}1&2&4\\0&1&1\\0&1&1\\0&-1&a-4\end{bmatrix}\begin{matrix}-2\cdot2r\\ \\-2\cdot r\\+2\cdot r\end{matrix}\sim\begin{bmatrix}1&0&2\\0&1&1\\0&0&0\\0&0&a-3\end{bmatrix}\begin{matrix}\\ \\ \\:a-3\end{matrix}$$

I slučaj: $\underline{a=3}$ (ne smijem dijeliti)

$$\begin{bmatrix}1&0&2\\0&1&1\\0&0&0\\0&0&0\end{bmatrix}$$

$\underline{r(A)=2} \longrightarrow$ linearno ZAVISNI (rang nije jednak broju vektora)

II slučaj: $a \neq 3$

$$\begin{bmatrix}1&0&2\\0&1&1\\0&0&1\\0&0&0\end{bmatrix}$$

$\underline{r(A)=3} \rightarrow$ <u>lin. nezavisni</u>

$\longrightarrow$ tj. za $a \neq 3$ su lin. nezavisni!

5. LINEARNI SUSTAVI

5.1. GAUSSOVA METODA ELIMINACIJE

5.DZ-9.) Odredi polinom 4. stupnja za koji vrijedi

$$P(0)=1,\ P(1)=3,\ P(-1)=3,\ P(2)=21,\ P(-2)=21.$$

$$P(x)=ax^4+bx^3+cx^2+dx+e$$

$$\boxed{P(0)=e=1}$$

$$P(1)=a+b+c+d+1=3$$
$$P(-1)=a-b+c-d+1=3$$
$$P(2)=16a+8b+4c+2d+1=21$$
$$P(-2)=16a-8b+4c-2d+1=21$$

$$\left.\begin{array}{l} a+b+c+d=2 \\ a-b+c-d=2 \end{array}\right\} 2a+2c=4$$

$$\left.\begin{array}{l} 16a+8b+4c+2d=20 \\ 16a-8b+4c-2d=20 \end{array}\right\} 32a+8c=40$$

$$\left.\begin{array}{l} a+c=2 \\ -4a-c=-5 \end{array}\right\}$$

$$-3a=-3$$

$$\left.\begin{array}{l} \boxed{a=1} \\ \boxed{c=1} \end{array}\right\} \boxed{b=d=0}$$

$$\boxed{P(x)=x^4+x^2+1}$$

→ m jednadžbi sa n nepoznanica:

$$\left.\begin{array}{l} \overbrace{a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n}^{n \text{ nepoznanica}}=b_1 \\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n=b_2 \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+\dots+a_{mn}x_1=b_m \end{array}\right\} m \text{ jednadžbi}$$

→ zapišemo matrično: $\boxed{A\cdot x=b}$

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$ matrica sustava

$$X=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ vektor nepoznanica

$$b=\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$ desna strana

→ Kako riješiti? 1. seljački (srednja škola → vidi 4. DZ-9))

2. inverznom matricom A^{-1} / $Ax=b$

$$x=A^{-1}b$$

ako je $\det A \neq 0$!!

3. GAUSSOVA METODA ELIMINACIJE: $[A \vdots b] \sim \cdots \sim [A_r \vdots b']$

Zad.)

$$2x_1 - x_2 + 5x_3 = 0$$
$$3x_1 + x_3 = 2$$
$$x_1 + 2x_2 - x_3 = -4$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 5 & 0 \\ 3 & 0 & 1 & 2 \\ \underline{1} & +2 & -1 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & +2 & -1 & -4 \\ 3 & 0 & 1 & 2 \\ 2 & -1 & 5 & 0 \end{array}\right] \begin{array}{l} \\ -3\cdot 1.r \\ -2\cdot 1.r \end{array} \sim \left[\begin{array}{ccc|c} 1 & +2 & -1 & -4 \\ 0 & -6 & 4 & 14 \\ 0 & -5 & 7 & 8 \end{array}\right] \begin{array}{l} \\ \cdot\frac{1}{6} \\ \\ \end{array} \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & +2 & -1 & -4 \\ 0 & 1 & -2/3 & -7/3 \\ 0 & -5 & 7 & 8 \end{array}\right] \begin{array}{l} -2\cdot 2.r \\ \\ +5\cdot 2.r \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 2/3 \\ 0 & 1 & -2/3 & -7/3 \\ 0 & 0 & 11/3 & -11/3 \end{array}\right] \cdot 3/11 \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 2/3 \\ 0 & 1 & -2/3 & -7/3 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} -1/3\cdot 3.r \\ +2/3\cdot 3.r \\ \sim \end{array}$$

$$\sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right] \end{array} \longrightarrow \boxed{\begin{array}{l} x_1 = 1 \\ x_2 = -3 \\ x_3 = -1 \end{array}}$$

— važna pitanja:

→ kada uopće postoji rješenje sustava? „POSTOJANOST"

→ koliko rješenja sustav može imati? „JEDNOZNAČNOST"

5.2. HOMOGENI SUSTAVI

→ Sustav oblika $A \cdot X = \mathbf{0}$

⇒ uvijek ima trivijalno rješenje $\mathbf{x}=\mathbf{0}$, no je li to jedino rj.?

TM Sustav $Ax=0$ ima jedinstveno rješenje $x=0$ ako i samo ako je A regularna matrica.

Dokaz:

$$x_1 a_1 + x_2 a_2 + \dots + x_1 a_1 = \mathbf{0} \Rightarrow x_1, x_2 \dots x_n = 0 \Rightarrow \underline{r(A)=n}$$

⇒ samo ako su vektori linearno nezavisni, A je regularna tj. ako je rang jednak n (br. vektora)

ALGORITAM:

1. Sustav zapišemo u obliku $[A \vdots 0]$

2. Gaussovom metodom (elem. transf.) svedemo na $[A_r \vdots 0]$

a) ako je $r(A_r) = n$ (n = broj stupaca, redaka) $\longrightarrow$ ima 1 rješenje $x = 0$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

b) ako je $r(A_r) < n$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\longrightarrow$ onda sustav ima beskonačno rješenja

- stupci sa stožerom = vezani stupci / varijable = $r(A_r)$
- stupci bez stožera = slobodni stupci / varijable

$\longrightarrow$ Dimenzija rješenja: broj slobodnih stupaca $\underline{\underline{(n-r)}}$

Zad.) Riješi sustav:

$$\begin{aligned} x+y+z &= 0 \\ 3x+y-z &= 0 \\ 2x+y \quad &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 1 & -1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} \\ -3\cdot 1.r \\ -2\cdot 1.r \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -4 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \begin{array}{l} \\ :(-2) \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \begin{array}{l} -2\cdot r \\ \sim \\ +2\cdot r \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} //$$

$x - z = 0$

$y + 2z = 0$

$z = \lambda \longrightarrow$ slobodna

$x = z = \lambda$

$y = -2z = -2\lambda$

$z = \lambda$

Rješenje: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda \\ -2\lambda \\ \lambda \end{bmatrix} = \lambda$

Zad.)

$$x_1 + 2x_2 + x_3 + 2x_4 + 3x_5 = 0$$
$$2x_1 + 4x_2 + 3x_3 + 3x_4 + 7x_5 = 0$$
$$2x_1 + 4x_2 + x_3 + 5x_4 + 5x_5 = 0$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 3 & 7 & 0 \\ 2 & 4 & 1 & 5 & 5 & 0 \end{array}\right] \begin{array}{c} \\ -2\cdot 1.r \\ -2\cdot 1.r \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 \end{array}\right] \begin{array}{c} -2\cdot 1.r \\ \\ +2\cdot 1.r \end{array} \sim$$

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \sim \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

slobodan stupac

slobodni stupci

$$x_1 + 2x_2 + 3x_4 + 2x_5 = 0$$
$$x_3 - x_4 + x_5 = 0$$

$$x_2 = \alpha$$
$$x_4 = \beta$$
$$x_5 = \gamma$$

$$x_1 = -2\alpha - 3\beta - 2\gamma$$

$$x_3 = \beta - \gamma$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2\alpha - 3\beta - 2\gamma \\ \alpha \\ \beta - \gamma \\ \beta \\ \gamma \end{bmatrix} = \alpha \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \gamma \cdot \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R}$$

(3-dimenzionalno rješenje)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \lambda \in \mathbb{R}$$

1-dimenzionalno rješenje

5.3. NEHOMOGENI SUSTAVI

→ sustav oblika $Ax=b$

$\boxed{TM}$ Sustav $Ax=b$ <u>ima</u> rješenje ako i samo ako je $r(A)=r(A:b)$

$\boxed{TM}$ Opće rješenje nehomogenog sustava je oblika

$$\boxed{X = X_h + X_p}$$

gdje je X_h rješenje pripadnog homogenog sustava $AX=0$, a X_p je neko partikularno rješenje nehomogenog dijela.

Dokaz: (BITAN ZA ISPIT !!)

$\Longleftrightarrow$ $\Longrightarrow$ Neka je x rješenje sustava $Ax=b$, i neka je X_p neko partik. rješenje

$$A\cdot(x-X_p) = Ax - \underbrace{Ax_p}_{\text{isto rj.}} = b-b = 0$$

$$X_h = x - X_p \Rightarrow \boxed{X = X_h + X_p}$$

$\Longleftarrow$ Neka je $x = X_p + X_h$. Tada je rješenje x rješenje od $Ax=b$

$$Ax = A\cdot(X_p+X_h) = AX_p + AX_h = b+0 = b$$

ALGORITAM ZA RJEŠAVANJE:

1. sustav zapišemo kao $[A \vdots b]$
2. svedemo ga na $[A_r \vdots b']$ Gaussovom metodom
3. i) ako je $r(A) < r(A:b) \Longrightarrow$ sustav nema rješenja
   ii) ako je $r(A) = r(A:b) < n$ (broj stupaca) $\Longrightarrow$ sustav ima beskonačno rješenja (vezane, slobodne varijable ....)

iii) ako je $r(A) = r(A:b) = n \Longrightarrow$ sustav ima jedinstveno rješenje

Zad.)

b)

$$3x_1 - x_2 + 5x_3 - x_4 = 3$$
$$x_1 - 2x_2 + 3x_3 + 2x_4 = 1$$
$$2x_1 + x_2 + 2x_3 - 3x_4 = 4$$

$$\left[\begin{array}{cccc|c} 3 & -1 & 5 & -1 & 3 \\ 1 & -2 & 3 & 2 & 1 \\ 2 & 1 & 2 & -3 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 3 & 2 & 1 \\ 3 & -1 & 5 & -1 & 3 \\ 2 & 1 & 2 & -3 & 4 \end{array}\right] \begin{array}{l} \\ -3\cdot 1.r \\ -2\cdot 1.r \end{array} \sim \begin{array}{c} \\ \\ -2.r \end{array} \left[\begin{array}{cccc|c} 1 & -2 & 3 & 2 & 1 \\ 0 & 5 & -4 & -7 & 0 \\ 0 & 5 & -4 & -7 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 3 & 2 & 1 \\ 0 & 5 & -4 & -7 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

$\underline{0 \neq 2} \longrightarrow$ Sustav nema rješenja!

k.
ješenje

Zad2.)

$$x_1 + x_2 - x_3 - 3x_4 + 4x_5 = 2$$
$$3x_1 + x_2 - x_3 - x_4 = 2$$
$$\vdots$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & -3 & 4 & 2 \\ 3 & 1 & -1 & -1 & 0 & 2 \\ 9 & 1 & -2 & -1 & -2 & 5 \\ 1 & -1 & 0 & -1 & 2 & 1 \end{array}\right] \begin{array}{l} \\ -3\cdot 1.r \\ -9\cdot 1.r \\ -1.r \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 1 & -1 & -3 & 4 & 2 \\ 0 & -2 & 2 & 8 & -12 & -4 \\ 0 & -8 & 7 & 26 & -38 & -13 \\ 0 & -2 & 1 & 2 & -2 & -1 \end{array}\right] \begin{array}{l} \\ |:(-2) \\ \\ \\ \end{array} \sim$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & -1 & -3 & 4 & 2 \\ 0 & 1 & -1 & -4 & 6 & 2 \\ 0 & -8 & 7 & 26 & -38 & -13 \\ 0 & -2 & 1 & 2 & -2 & -1 \end{array}\right] \begin{array}{l} -2.r \\ \\ +8\cdot 2.r \\ +2\cdot 2.r \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & -1 & -4 & 6 & 2 \\ 0 & 0 & -1 & -6 & 10 & 3 \\ 0 & 0 & -1 & -6 & 10 & 3 \end{array}\right] \begin{array}{l} \\ \\ \cdot(-1) \\ \\ \end{array} \sim$$

$$\left[\begin{array}{ccccc|c} \overbrace{1 \quad 0 \quad 0}^{\text{vezane}} & \overbrace{1 \quad -2}^{\text{slobodne}} & 0 \\ 0 \quad 1 \quad 1 \quad 0 & 2 \quad -4 & -1 \\ 0 \quad 0 \quad 1 & 6 \quad -10 & -3 \\ 0 \quad 0 \quad 0 & 0 \quad 0 & 0 \end{array}\right] \longrightarrow \text{beskonačno rješenja}$$

na

dimenzija $= \underset{\substack{5-3 \\ \downarrow \\ \text{br. stupaca}}}{n-r} = 2$

$$x_1 + x_4 - 2x_5 = 0 \implies x_1 = -\alpha + 2\beta$$
$$x_2 + 2x_4 - 4x_5 = -1 \implies x_2 = -1 - 2\alpha + 4\beta$$
$$x_3 + 6x_4 - 10x_5 = -3 \implies x_3 = -3 - 6\alpha + 10\beta$$

$$x_4 = \alpha$$
$$x_5 = \beta$$

Rj:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\alpha + 2\beta \\ -1 - 2\alpha + 4\beta \\ -3 - 6\alpha + 10\beta \\ \alpha \\ \beta \end{bmatrix} = \underbrace{\overset{\text{KONSTANTE}}{\begin{bmatrix} 0 \\ -1 \\ -3 \\ 0 \\ 0 \end{bmatrix}}}_{x_p} + \underbrace{\alpha \begin{bmatrix} -1 \\ -2 \\ -6 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 4 \\ 10 \\ 0 \\ 1 \end{bmatrix}}_{x_h} \qquad \alpha, \beta \in \mathbb{R}$$

1.MI-2010-6.) U ovisnosti $\lambda \in \mathbb{R}$ riješi sustav:

$$\lambda x + y = 0$$
$$\lambda y + z = \lambda$$
$$x + \lambda z = \lambda^2$$

$$\left[\begin{array}{ccc|c} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & \lambda \\ 1 & 0 & \lambda & \lambda^2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \lambda & \lambda^2 \\ 0 & \lambda & 1 & \lambda \\ \lambda & 1 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ -\lambda \cdot 1.r \end{array} \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & \lambda & \lambda^2 \\ 0 & \lambda & 1 & \lambda \\ 0 & 1 & -\lambda^2 & -\lambda^3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \lambda & \lambda^2 \\ 0 & 1 & -\lambda^2 & -\lambda^3 \\ 0 & \lambda & 1 & \lambda \end{array}\right] \begin{array}{l} \\ \\ -\lambda \cdot 2.r \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & \lambda & \lambda^2 \\ 0 & 1 & -\lambda^2 & -\lambda^3 \\ 0 & 0 & 1+\lambda^3 & \lambda+\lambda^4 \end{array}\right] \cdot \frac{1}{1+\lambda^3}$$

$$1+\lambda^3 \neq 0 \rightarrow \lambda^3 \neq -1 \,/\sqrt[3]{\ } \rightarrow \underline{\lambda \neq -1}$$

1. slučaj: $\underline{\lambda = -1}$ (ne smijemo množiti)

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & +1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \rightarrow x - z = 1 \Rightarrow x = 1 + t \\ \rightarrow y - z = 1 \Rightarrow y = 1 + t \\ \quad\quad z = t \quad\quad\quad z = t \end{array}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

2. slučaj: $\lambda \neq -1$ (smijemo množiti)

$$\left[\begin{array}{ccc|c} 1 & 0 & \lambda & \lambda^2 \\ 0 & 1 & -\lambda^2 & -\lambda^3 \\ 0 & 0 & 1 & \lambda \end{array}\right] \begin{array}{l} -\lambda \cdot 3.r \\ +\lambda^2 \cdot 3r \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda \end{array}\right] \quad \boxed{\begin{array}{l} x = 0 \\ y = 0 \\ z = \lambda \end{array}}$$

NE ZAPISUJE SE PREKO MATRICE!

SAMO KAD IMA BESKONAČNO RJ.

1.M1 - 2007 - 8)

$$x+y+z=0$$
$$x+(\lambda+1)y+2z=1$$
$$x+2y+(4\lambda+1)z=2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & \lambda+1 & 2 & 1 \\ 1 & 2 & 4\lambda+1 & 2 \end{array}\right] \begin{array}{l} \\ -1.r \\ -1.r \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & \lambda & 1 & 1 \\ 0 & 1 & 4\lambda & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4\lambda & 2 \\ 0 & \lambda & 1 & 1 \end{array}\right] \begin{array}{l} -2r \\ \\ -\lambda r \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1-4\lambda & -2 \\ 0 & 1 & 4\lambda & 2 \\ 0 & 0 & 1-4\lambda^2 & 1-2\lambda \end{array}\right] : (1-4\lambda^2) \longrightarrow \lambda^2 \neq \frac{1}{4}$$

$$\lambda^2 \neq \pm\frac{1}{2}$$

1. slučaj: $\lambda = -\frac{1}{2}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$0 \neq 2 \longrightarrow$ SUSTAV <u>NEMA</u> rješenja!

2. slučaj: $\lambda = \frac{1}{2}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{l} x-z=-2 \Rightarrow x=-2+\check{c} \\ y+2z=2 \Rightarrow y=2-2\check{c} \\ z=\check{c} \end{array}$$

$\downarrow$ beskonačno rj.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} + \check{c}\cdot\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \check{c} \in \mathbb{R}$$

3. slučaj: $\lambda \neq \pm \frac{1}{2}$ (smijemo dijeliti)

$$\left[\begin{array}{ccc|c} 1 & 0 & 1-4\lambda & -2 \\ 0 & 1 & 4\lambda & 2 \\ 0 & 0 & 1 & \frac{1}{2\lambda+1} \end{array}\right] \begin{array}{l} -(1+4\lambda)\cdot 3.r \\ -4\lambda\cdot 3.r \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{3}{1+2\lambda} \\ 0 & 1 & 0 & \frac{2}{1+2\lambda} \\ 0 & 0 & 1 & \frac{1}{1+2\lambda} \end{array}\right]$$

sustav ima jedinstveno rješenje

$$\boxed{x = \frac{-3}{1+2\lambda}}$$

$$\boxed{y = \frac{2}{1+2\lambda}}$$

$$\boxed{z = \frac{1}{1+2\lambda}}$$

JEDINSTVENA RJEŠENJA, NEMA MATRICE U RJEŠENJU !!

5.4. CRAMEROVO PRAVILO

→ samo za kvadratne matrice, n jednadžbi i n nepoznanica

$\boxed{TM}$ Neka je A kvadratna matrica. Sustav $Ax=b$ će imati točno jedno rješenje ako i samo ako je A REGULARNA MATRICA.

Dokaz: SAM! KNJIŽICA!

Podsjetimo se: $A^{-1} / \; Ax=b$ $\qquad x=$

$$\underline{x = A^{-1} \cdot b}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

druga oznaka

$$x_i = \sum_{j=1}^{n} (A^{-1})_{ij} b_j = \sum_{j=1}^{n} \frac{A_{ji}}{\det A} b_j = \frac{\sum_{j=1}^{n} A_{ji} b_j}{\det A} = \frac{\det A_i}{\det A} = \frac{D_i}{D} \quad //$$

$\boxed{TM}$ (Cramer)

Svaka komponenta x_i sustava $Ax=b$ (gdje je A regularna) se može dobiti kao razlomak u čijem nazivniku je determinanta od A, a u brojniku determinanta matrice u kojoj je <u>i-ti</u> stupac zamijenjen sa b.

Dokaz: vidi gore.

1. M1-2007-7) Cramerovim pravilom riješi sustav:

$$\begin{aligned} 2x + 3y - 2z &= 3 \\ x + 4y + 2z &= 5 \\ 3x + 7y - 4z &= 0 \end{aligned}$$

D.Z. riješi Gaussom.

$$D = \begin{vmatrix} 2 & 3 & -2 \\ 1 & 4 & 2 \\ 3 & 7 & -4 \end{vmatrix} \begin{matrix} 2 & 3 \\ 1 & 4 \\ 3 & 7 \end{matrix} = -32 + 18 - 14 + 24 + 12 = -20$$

$= 0$ NEMA JEDINST. RJEŠENJE

ZAMIJENITI STUPAC OD x-a SA b !!

$$D_x = \begin{vmatrix} 3 & 3 & -2 \\ 5 & 4 & 2 \\ 0 & 7 & -4 \end{vmatrix} = \cdots \text{Sarrus} \cdots = -100$$

$$D_y = \begin{vmatrix} 2 & 3 & -2 \\ 1 & 5 & 2 \\ 3 & 0 & -4 \end{vmatrix} = \cdots \text{Sarrus} \cdots = 20$$

$$D_z = \begin{vmatrix} 2 & 3 & 3 \\ 1 & 4 & 5 \\ 3 & 7 & 0 \end{vmatrix} = \cdots \text{Sarrus} = -40$$

$$x = \frac{D_x}{D} = \frac{-100}{-20} = 5$$

$$y = \frac{D_y}{D} = \frac{20}{-20} = -1$$

$$z = \frac{D_z}{D} = \frac{-40}{-20} = 2$$

→ jedino u slučaju kada imamo beskonačno rješenja pišemo ih u vektorskom obliku

$$\left[\begin{array}{ccc|c} 1 & & & a \\ & 1 & & b \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$0=0$

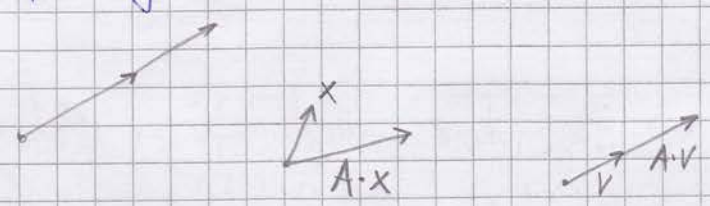
5.5. SVOJSTVENE VRIJEDNOSTI I SVOJSTVENI VEKTOR

def. Vektor $\boxed{v \neq 0}$!! zovemo SVOJSTVENI (VLASTITI) vektor matrice A ako postoji $\lambda \in \mathbb{R}$ takav da je

$$\boxed{\underset{n\times n}{A} \cdot \underset{n\times 1}{v} = \lambda \cdot \underset{n\times 1}{v}}$$

, a λ nazivamo SVOJSTVENU VRIJEDNOST.

→ geometrijska interpretacija:



→ primjetimo:

i) Ako je v svojstveni vektor tada je i $\alpha \cdot v$ svojstveni vektor

ii) Linearna kombinacija svojstvenih vektora (od iste λ) je također svojstveni vektor od λ

→ kako ih računamo?

$A \cdot v = \lambda \cdot v$

$0 = \lambda v - Av$

$0 = (\lambda \cdot I - A) v$

→ mora se izlučiti i I jer ne može broj (λ) minus matrica

→ uvijek se dodaje I kada se izlučuje!

→ budući da je $v \neq 0$ matrica $(\lambda \cdot I - A)$ mora vrijediti da je $\boxed{\det(\lambda \cdot I - A) = 0}$ → NE SMIJE BITI REGULARNA

→ karakteristični polinom matrice A (oznaka χ (kapa)):

$$\boxed{\chi(\lambda) = \det(\lambda I - A)}$$

→ karakteristična jednadžba:

$$\boxed{\chi(\lambda) = 0}$$

→ nultočke karakterističnog polinoma su SVOJSTVENE VRIJEDNOSTI

1.MI-2007-9) Odredi sve svojstvene vrijednosti i vektore matrice A.

$A = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & 3 \\ -1 & \lambda - 5 \end{bmatrix}$$

$$\chi(\lambda) = \begin{vmatrix} \lambda - 1 & 3 \\ -1 & \lambda - 5 \end{vmatrix} = \lambda^2 - 6\lambda + 8 = 0$$

$\boxed{\lambda_1 = 2}$ $\boxed{\lambda_2 = 4}$ → svojstvene vrijednosti!

$(\lambda I - A)V = 0$

1.) $\underline{\lambda = 2}$

→ uvrstiti $\lambda = 2$ u $\lambda I - A = \begin{bmatrix} \lambda-1 & 3 \\ -1 & \lambda-5 \end{bmatrix}$

$$\Longrightarrow \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ -1 & -3 & 0 \end{array}\right] +1.r \sim \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow x_1 + 3x_2 = 0$$

$$\boxed{x_2 = t}$$

$$\boxed{x_1 = -3t}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

npr. $\underline{t = 7}$

$$V_1 = 7 \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -21 \\ 7 \end{bmatrix}$$

2.) $\underline{\lambda = 4}$

$$\begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ -1 & -1 & 0 \end{array}\right] |:3 \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad x_1 + x_2 = 0$$

$$\boxed{x_2 = t}$$

$$\boxed{x_1 = -t}$$

$$V_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ -t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

npr. $\underline{t = 77}$ $\quad V_2 = \begin{bmatrix} 77 \\ -77 \end{bmatrix}$

Zad.) Odredi svojstvene vektore i vrijednosti:

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 4 & 4 & -1 \end{bmatrix} \longrightarrow$$ PROMIJENIŠ PREDZNAK I DODAŠ λ PO DIJAGONALI!

$$\begin{vmatrix} \lambda-4 & -1 & 1 \\ -2 & \lambda-5 & 2 \\ -4 & -4 & \lambda+1 \end{vmatrix} = \dots = \lambda^3 - 8\lambda^2 + 21\lambda - 18 = 0$$

$p = \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

pogađati, uvrštavati sve po redu, $1, -1, 2, -2 \dots$

$\longrightarrow$

$\underline{\lambda = 2}$ je jedno rješenje

$$(\lambda^3 - 8\lambda^2 + 21\lambda - 18) : (\lambda - 2) = \lambda^2 - 6\lambda + 9$$

$$\begin{array}{l} -\lambda^3 + 2\lambda^2 \\ \hline -6\lambda^2 + 21\lambda - 18 \\ +6\lambda^2 \pm 12\lambda \\ \hline 9\lambda - 18 \\ 9\lambda - 18 \\ \hline 0 \end{array}$$

$$(\lambda^2 - 6\lambda + 9)(\lambda - 2) = (\lambda - 3)(\lambda - 2)$$

$$\lambda_{12} = \frac{6 \pm \sqrt{36 - 36}}{2}$$

$$\boxed{\lambda = 3}$$

1.) $\underline{\lambda = 2}$

$$\begin{bmatrix} -2 & -1 & 1 \\ -2 & -3 & 2 \\ -4 & -4 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ -2 & -3 & 2 & 0 \\ -4 & -4 & 3 & 0 \end{array}\right] \begin{array}{l} :(-2) \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ -2 & -3 & 2 & 0 \\ -4 & -4 & 3 & 0 \end{array}\right] \begin{array}{l} \\ +2\cdot 1.r \\ +4\cdot 1.r \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right] \begin{array}{l} \\ :(-2) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \end{array}\right] \begin{array}{l} -2\cdot 2.r \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/4 & 0 \\ 0 & 1 & -1/2 & 0 \end{array}\right]$$

$$x_1 + \frac{x_2}{2} - \frac{x_3}{4} = 0 \longrightarrow \boxed{x_1 = \frac{t}{2} - \frac{t}{4} = \frac{t}{4}}$$

$$x_2 - \frac{x_3}{2} = 0 \longrightarrow \boxed{x_2 = \frac{t}{2}}$$

$$\boxed{x_3 = t}$$

$$v_1 = \begin{bmatrix} t/4 \\ t/2 \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 1/4 \\ 1/2 \\ 1 \end{bmatrix} //$$

npr. $\underline{t = 7}$ $\quad v_1 = \begin{bmatrix} 7/4 \\ 7/2 \\ 7 \end{bmatrix} //$

2.) $\underline{\lambda = 3}$

$$\begin{bmatrix} -1 & -1 & 1 \\ -2 & -2 & 2 \\ -4 & -4 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ -2 & -2 & 2 & 0 \\ -4 & -4 & 4 & 0 \end{array}\right] \cdot (-1) \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -2 & -2 & 2 & 0 \\ -4 & -4 & 4 & 0 \end{array}\right] \begin{array}{l} \\ +2 \cdot 1.r \\ +4 \cdot 1.r \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + x_2 - x_3 = 0$$

$$\boxed{x_2 = \alpha} \qquad \boxed{x_1 = \beta - \alpha}$$

$$\boxed{x_3 = \beta}$$

PAZI !!

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \beta - \alpha \\ \alpha \\ \beta \end{bmatrix} = \beta \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} //$$

2. SVOJSTVENI VEKTOR

npr.

1 SVOJSTVENI VEKTOR

npr. $\alpha = 1$

$$v_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} //$$

npr. $\beta = 1$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} //$$

6. NIZOVI

6.1. DEFINICIJA I OSNOVNI POJMOVI

def. Funkcija $a: \mathbb{N} \to S$ se zove niz u skupu S.
Ako je $S = \mathbb{R}$, "a" zovemo niz realnih brojeva (a_n).

Pr. 1)

	omeđen odozgo	o. odozdo	omeđen
1.) $a_n = \frac{1}{2n-1} \to 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots$	+ (1)	+ (0)	+
2.) $a_n = (-1)^n \frac{n-1}{n} \to 0, -\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, \frac{-4}{5} \dots$	+ (1)	+ (-1)	+
3.) $a_n = 2^{n-1} \to 1, 2, 4, 8, 16, \dots$	−	+ (1)	−
4.) $a_n = (-1)^{n+1} \sqrt{n} \to 1, -\sqrt{2}, \sqrt{3}, -\sqrt{4}, \sqrt{5} \dots$	−	−	−

def. Niz realnih brojeva (b_n) je podniz niza (a_n) ako postoji strogo rastuća funkcija $f: \mathbb{N} \to \mathbb{N}$ tako da je $b_n = a_{f(n)}$.

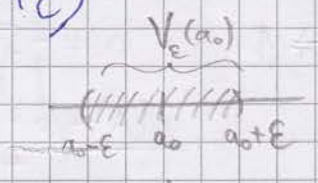
def. Kažemo da je niz a_n OMEĐEN (odozgo) odozdo ako je skup vrijednosti niz $\{a_n\}$ omeđen odozgo ili odozdo.

→ Niz je OMEĐEN ako je OMEĐEN i ODOZGO i ODOZDO. !!

def. Neka je $\varepsilon > 0$ (kao neki mali br., npr. 0,001). Interval oblika

$$V_\varepsilon(a_0) = \{a \in \mathbb{R} : |a - a_0| < \varepsilon\} = (a_0 - \varepsilon, a_0 + \varepsilon)$$

se naziva <u>ε okolina</u>, (ε-okoliš) broja a_0.



Okolina broja a_0 je bilokoji skup za koji postoji ε-okolina.
(širi skup unutar kojeg postoji ε-okolina)

AUDITORNE

Zad.) Odredite rang matrice A u ovisnosti o parametru $a \in \mathbb{R}$.

$$A = \begin{bmatrix} a & 1 & 1 & 1 & 1 & 1 \\ 1 & a & 1 & 1 & 1 & 1 \\ 1 & 1 & a & 1 & 1 & 1 \\ 1 & 1 & 1 & a & 1 & 1 \\ 1 & 1 & 1 & 1 & a & 1 \\ 1 & 1 & 1 & 1 & 1 & a \end{bmatrix} \xrightarrow{+2.r+3.r+4.r+5.r+6.r} \sim \begin{bmatrix} a+5 & a+5 & a+5 & a+5 & a+5 & a+5 \\ 1 & a & 1 & 1 & 1 & 1 \\ 1 & 1 & a & 1 & 1 & 1 \\ 1 & 1 & 1 & a & 1 & 1 \\ 1 & 1 & 1 & 1 & a & 1 \\ 1 & 1 & 1 & 1 & 1 & a \end{bmatrix} \Big| \cdot \frac{1}{a+5}$$

1) ako je:

$a+5=0$

$\underline{a=-5}$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -5 & 1 & 1 & 1 & 1 \\ 1 & 1 & -5 & 1 & 1 & 1 \\ 1 & 1 & 1 & -5 & 1 & 1 \\ 1 & 1 & 1 & 1 & -5 & 1 \\ 1 & 1 & 1 & 1 & 1 & -5 \end{bmatrix} \qquad \underline{\underline{r(A) = 5}}$$

2) ako je $a+5 \neq 0$, $a \neq -5$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a & 1 & 1 & 1 & 1 \\ 1 & 1 & a & 1 & 1 & 1 \\ 1 & 1 & 1 & a & 1 & 1 \\ 1 & 1 & 1 & 1 & a & 1 \\ 1 & 1 & 1 & 1 & 1 & a \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & a-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a-1 \end{bmatrix} \Big| \cdot \frac{1}{a-1}$$

3) $\underline{a=1}$

$\underline{r(A)=1}$ } punog ranga

4) $a \neq 1$ i $a \neq -5$

$\underline{r(A)=6}$

Rješenje: za $a=-5$ $r(A)=5$

za $a=1$ $r(A)=1$

za $a \neq -5, 1$ $r(A)=6$

Zad.) $\lambda \in \mathbb{R}$, diskutirati rješenja sustava:

$$\begin{aligned} \lambda x_1 + x_2 + x_3 + x_4 &= \lambda \\ x_1 + \lambda x_2 + x_3 + x_4 &= \lambda \\ x_1 + x_2 + \lambda x_3 + x_4 &= 1 \\ x_1 + x_2 + x_3 + \lambda x_4 &= 1 \end{aligned}$$

$$\left[\begin{array}{cccc|c} \lambda & 1 & 1 & 1 & \lambda \\ 1 & \lambda & 1 & 1 & \lambda \\ 1 & 1 & \lambda & 1 & 1 \\ 1 & 1 & 1 & \lambda & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & \lambda & 1 \\ 1 & \lambda & 1 & 1 & \lambda \\ 1 & 1 & \lambda & 1 & 1 \\ \lambda & 1 & 1 & 1 & \lambda \end{array}\right] \begin{array}{l} |\cdot(-1) \\ \\ \\ |\cdot(-\lambda) \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & \lambda & 1 \\ 0 & \lambda-1 & 0 & 1-\lambda & \lambda-1 \\ 0 & 0 & \lambda-1 & 1-\lambda & 0 \\ 0 & 1-\lambda & 1-\lambda & 1-\lambda^2 & 0 \end{array}\right] \begin{array}{l} \\ |\cdot\frac{1}{\lambda-1} \\ |\cdot\frac{1}{\lambda-1} \\ |\cdot\frac{1}{\lambda-1} \end{array}$$

1.) $\underline{\lambda=1}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_1 &= 1 - x_2 - x_3 - x_4 \end{aligned}$$

$$\left.\begin{aligned} x_2 &= t \\ x_3 &= r \\ x_4 &= q \end{aligned}\right\} \in \mathbb{R} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1-t-r-q \\ t \\ r \\ q \end{bmatrix} =$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + q \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

2.) $\underline{\lambda \neq 1}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & \lambda & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & -1-\lambda & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & \lambda+1 & 0 \\ 0 & 1 & 0 & -1 & +1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -2-\lambda & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & \lambda+2 & 0 \\ 0 & 1 & 0 & -1 & +1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -3-\lambda & 1 \end{array}\right] \sim$$

3.) $-3-\lambda = 0$

$\underline{\lambda = -3}$ $\longrightarrow$ sustav nema rješenja! $\boxed{0 \neq 1}$!

4.) za $\lambda \neq -3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & \lambda+2 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1/-3-\lambda \end{array}\right]$$

$$x_1 \qquad + x_4(\lambda+2) = 0$$

$$x_2 \qquad - x_4 = 1 \longrightarrow x_2 = x_4 - 1$$

$$x_3 - x_4 = 0 \rightarrow x_3 = x_4$$

$$\boxed{x_4 = \frac{1}{-3-\lambda}}$$

$$\boxed{x_2 = \frac{1}{-3-\lambda} - 1}$$

Zad.) Odredi inverz

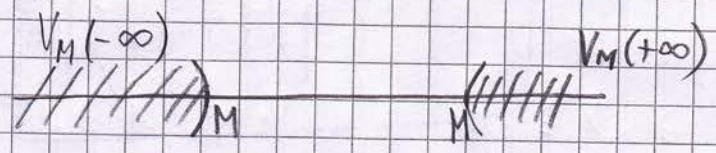
$$\left[\begin{array}{cccc|cccc} 0 & 0 & 9 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 11 & 5 & 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 2 & 0 & 0 & 1 & 0 \\ 7 & 5 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 7 & 5 & 2 & 5 & 0 & 0 & 0 & 1 \\ 3 & 2 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 11 & 5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 4 & 1 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} -2\cdot 2.r \\ \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & -2 & 1 \\ 3 & 2 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 11 & 5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 4 & 1 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ -3\cdot 1.r \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & -1 & 1 & -1 & 0 & 0 & -5 & -3 \\ 0 & 0 & 11 & 5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 4 & 1 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} -2.r \\ \cdot(-1) \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 0 & -7 & -2 \\ 0 & 1 & -1 & 1 & 0 & 0 & 5 & 3 \\ 0 & 0 & 11 & 5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 4 & 1 & 0 & 0 & 0 \end{array}\right]$$

		omeđen	GOMILIŠTA	LIMES
i)	$a_n = \frac{1}{2n-1}$: $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7} \dots$	DA	0	0
ii)	$a_n = (-1)^n \frac{n-1}{n}$: $0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4} \dots$	DA	$1, -1$	NE
iii)	$a_n = 2^{n-1}$: $1, 2, 4, 8, 16 \dots$	samo odozdo	$+\infty$	NE
iv)	$a_n = (-1)^{n+1} \sqrt{n}$: $1, \sqrt{2}, \sqrt{3}, -\sqrt{4} \dots$	NE	$-\infty, +\infty$	NE

def. Skup $V_M(+\infty) = \{a \in \mathbb{R} : a > M\}$ se naziva M-okolina točke $+\infty$

$V_M(-\infty) = \{a \in \mathbb{R} : a < M\}$ se naziva M-okolina točke $-\infty$



def. Broj A zove se gomilište niza a_n ako se unutar svake okoline od A nalazi beskonačno mnogo članova niza.

TM Svaki omeđen niz realnih brojeva ima barem jedno gomilište.

def. LIMES SUPERIOR = najveće gomilište niza ($\lim\sup a_n$)

LIMES INFERIOR = najmanje gomilište niza ($\lim\inf a_n$)

6.2. KONVERGENCIJA I LIMES!

→ intuitivno, ako niz IMA ISKLJUČIVO JEDNO GOMILIŠTE tada to gomilište nazivamo LIMES

→ razlika od gomilišta je što je limes jedinstven, samo jedan!!

VAŽNO ZA TEST SIGURNO!!

def. Niz brojeva (a_n) konvergira [teži] k broju L ako za svaki $\varepsilon > 0$ postoji $n_0 \in \mathbb{N}$ takav da za svaki $n \geq n_0$ vrijedi

$$|a_n - L| < \varepsilon.$$

Tada taj broj L zovemo LIMES NIZA I PIŠEMO

$$L = \lim_{n\to\infty} (a_n), \text{ ili kraće } a_n \to L.$$

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)(\underbrace{|a_n - L| < \varepsilon}_{a_n \in (L-\varepsilon,\, L+\varepsilon)}).$$

→ KRITERIJ KONVERGENCIJE:

broj L je limes niza (a_n) ako se izvan svake okoline od L nalazi KONAČNO mnogo članova tog niza.

Zad.) Po definiciji izračunaj:

a) $\lim\limits_{n\to\infty} \frac{1}{n} = ?$

Neka je $\varepsilon > 0$. Tada postoji $n_0 \in \mathbb{N}$ takav da je $n_0 > \frac{1}{\varepsilon}$. Tada i za sve $n \geq n_0$ je također $n > \frac{1}{\varepsilon}$. Iz definicij limesa $|a_n - L| < \varepsilon$?

$$\frac{1}{n} < \varepsilon$$

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon \checkmark \quad \text{tj. } \lim_{n\to\infty} \frac{1}{n} = 0.$$

ŠK: OD 12 DO 13^{h} ZA NAS !

TM „OSNOVNI TEOREMI O LIMESU NIZA"

1. Svaki konvergentan niz je omeđen. (Obrat ne vrijedi !!) npr. primjer sa 2 gomilišta

2. Konvergentan niz ima samo jedan limes.

3. Ako $a_n \to L$, tada i njegov podniz $b_n \to L$.

4. Ako postoji $n_0 \in \mathbb{N}$ takav da je $a_n \leq b_n$ za sve $n \geq n_0$, tada je $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$.

5. „SENDVIČ TEOREM"

Ako postoji $n_0 \in \mathbb{N}$ takav da je $a_n \leq c_n \leq b_n$ za sve $n \geq n_0$ i ako je $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = L$ tada je i $\lim\limits_{n\to\infty} c_n = L$.

TM (PRAVILA RAČUNANJA S LIMESIMA)

Ako su (a_n) i (b_n) konvergentni nizovi tada:

a) $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

b) $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

c) $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, ako je $\lim b_n \neq 0$ $(b_n \neq 0)$

Dokaz: (VAŽNO ZA ISPIT !!)

a) Neka je $\varepsilon > 0$. Tada postoji $n_1 \in \mathbb{N}$ i $n_2 \in \mathbb{N}$ tada za sve $n \geq n_1$, $n \geq n_2$ vrijedi $|a_n - a| < \frac{\varepsilon}{2}$ i $|b_n - b| < \frac{\varepsilon}{2}$.

namješten fucking dokaz !!

Neka je $n_3 = \max(n_1, n_2)$ tada za sve $n \geq n_3$ vrijedi ovo od prije i

nejednakost trokuta

$$|(a_n+b_n) - (a+b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

b) Budući da je a_n konvergentan tada je omeđen, tj. postoji M_1 takav da je $|a_n| < M_1$. Neka je $M = \max\{M_1, |b|\}$ i neka je $\varepsilon > 0$.

Postoji $n_1 \in \mathbb{N}$ i $n_2 \in \mathbb{N}$ tako da za sve $n > n_1$ i $n \geq n_2$ vrijedi

namješteno → nastrebati

$|a_n - a| < \frac{\varepsilon}{2M}$ i $|b_n - b| < \frac{\varepsilon}{2M}$. Neka je $n_3 = \max\{n_1, n_2\}$

Tada za sve $n \geq n_3$ vrijedi ovo od prije i

$$|a_n b_n - ab| = |a_n b_n - a_n \cdot b + a_n \cdot b - ab| = |a_n(b_n - b) + b(a_n - a)|$$

$$\leq |a_n||b_n - b| + |b||a_n - a| < \cancel{M} \cdot \frac{\varepsilon}{2M} + \cancel{M} \cdot \frac{\varepsilon}{2\cancel{M}} = \varepsilon$$

- def.- Niz brojeva (a_n) divergira ka $+\infty$ i pišemo $\lim_{n\to\infty} a_n = +\infty$, ako za svaki $M > 0$ postoji $n_0 \in \mathbb{N}$ takav da za sve $n \geq n_0$ vrijedi $a_n > M$.

= || – || – || – ... ka $-\infty$ –||– –||– $-\infty$ ...

... ako –||– $m < 0$ –||– –||– $a_n < m$.

TM

TM „Operacije s divergentnim ~~redovima~~ nizovima"

1. Neka su nizovi (a_n) i (b_n) konvergentni tako da $a_n \to a > 0$, $b_n \to 0$. Tada $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = +\infty$.

2. Neka $a_n \to +\infty$, $b_n \to +\infty$, $c_n \to c$. Tada

a) $\lim\limits_{n\to\infty} (a_n + b_n) = +\infty$

b) $\lim\limits_{n\to\infty} (a_n + c_n) = +\infty$

c) $\lim\limits_{n\to\infty} (a_n \cdot b_n) = +\infty$

d) $\lim\limits_{n\to\infty} (a_n \cdot c_n) = +\infty$

e) $\lim\limits_{n\to\infty} \frac{c_n}{a_n} = 0$

TM

Dokaz: trivić!

→ Odredeni oblici:

$\infty + \infty = \infty$

$\infty \cdot \infty = \infty$

$\infty + c = \infty$

$\infty \cdot c = \infty$

$\frac{\pm c}{0} = \pm\infty$ $\quad \frac{c}{\infty} = 0$

$\frac{\infty}{0} = 0$ $\quad \frac{0}{\infty} = 0$

→ Neodredeni oblici:

$\infty - \infty$

$\frac{\infty}{\infty}$ $\quad \frac{0}{0}$

∞^0

0^0

1^∞

Zad.)

$$\lim_{n\to\infty} \frac{n(n^2+1)}{\frac{2}{n}+1} = \left(\frac{\infty \cdot (\infty^2+1)}{\frac{2}{\infty}+1} = \frac{\infty \cdot \infty}{0+1} = \frac{\infty}{1} \right) = \infty$$

Zad.) $\lim\limits_{n\to\infty}\left(n^2-\frac{n^4+n^2}{n^2+4}\right)=\left(\infty-\frac{\infty}{\infty}\right)=\lim\limits_{n\to\infty}\frac{\cancel{n^4}+4n^2-\cancel{n^4}-n^2}{n^2+4}\,\frac{/:n^2}{/:n^2}\lim\limits_{n\to\infty}\frac{3}{1+\underbrace{\frac{4}{n^2}}_{\to 0}}=3$ //

Zad.) $\lim\limits_{n\to\infty}\frac{n+1}{n+\underbrace{\sqrt{n^3+1}}_{n^{3/2}}}\,\frac{/:n^{3/2}}{/:n^{3/2}}=\lim\limits_{n\to\infty}\frac{\underbrace{\frac{1}{n^{1/2}}}_{\to 0}+\underbrace{\frac{1}{n^{3/2}}}_{\to 0}}{\underbrace{\frac{1}{n^{1/2}}}_{\to 0}+\sqrt{1+\underbrace{\frac{1}{n^3}}_{\to 0}}}=\frac{0}{0+1}=0$ //

$\frac{1}{n^{3/2}}\sqrt{n^3+1}=$

$=\sqrt{\left(\frac{1}{n^{3/2}}\right)^2(n^3+1)}=$

$=\sqrt{1+\frac{1}{n^3}}$

Zad.) $\lim\limits_{n\to\infty}\frac{\sqrt{n^4+n^3+1}-n^2}{n+1}\cdot\frac{\sqrt{n^4+n^3+1}+n^2}{\sqrt{n^4+n^3+1}+n^2}=\lim\limits_{n\to\infty}\frac{\cancel{n^4}+n^3+1-\cancel{n^4}\;/:n^3}{\underbrace{(n+1)}_{n}\underbrace{(\sqrt{n^4+n^3+1}+n^2)}_{2n^2}\;/:n^3}=$

$n\cdot 2n^2=2n^3$

$\left\{=\lim\limits_{n\to\infty}\frac{1+\frac{1}{n^3}}{\cdot\;\cdot}\right\}\sim\lim\limits_{n\to\infty}\frac{n^3}{2n^3}=\frac{1}{2}$ //

NAJVEČE POTENCIJE

Z.) $\lim\limits_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt[3]{n^3+n^2}\right)$ → preko kubova $a^3-b^3=(a-b)(a^2+ab+b^2)$

$a \qquad b$

$=\lim\limits_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt[3]{n^3+n^2}\right)\cdot\left(\frac{\left(n^2+n+\sqrt{n^2+n}\cdot\sqrt[3]{n^3+n^2}+\sqrt[3]{(n^3+n^2)^2}\right)}{\left(n^2+n+\sqrt{n^2+n}\cdot\sqrt[3]{n^3+n^2}+\sqrt[3]{(n^3+n^2)^2}\right)}\right)=$

$=\lim\limits_{n\to\infty}\frac{\sqrt{(n^2+n)^3}-(n^3+n^2)}{n^2+n+\sqrt{n^2+n}\cdot\sqrt[3]{n^3+n^2}+\sqrt[3]{(n^3+n^2)^2}}\cdot\frac{\sqrt{(n^2+n)^3}+(n^3+n^2)}{\sqrt{(n^2+n)^3}+(n^3+n^2)}=$

$3n^5 \quad \cancel{n^6} - \cancel{n^6} - 2n^5 \sim \cancel{n^5}$

$=\lim\limits_{n\to\infty}\frac{(n^2+n)^3-(n^3+n^2)^2}{\left(n^2+n+\sqrt{n^2+n}\cdot\sqrt[3]{n^3+n^2}+\sqrt[3]{(n^3+n^2)^2}\right)\cdot\left(\sqrt{(n^2+n)^3}+n^3+n^2\right)}=\frac{1}{6}$ //

$\sim(n^2+n\cdot n+n^2)\cdot(n^3+n^3)\sim(3n^2)\cdot(2n^3)\sim 6n^5$

$$*\ (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\ *$$

6.DZ.-6) U ovisnosti o parametru a izračunaj limes: $(a \in \mathbb{R})$

$$\boxed{\binom{n}{k} = \frac{n!}{(n-k)!\,k!}}$$

$$\lim_{n\to\infty}\left(n^2 - \frac{(n+a)^4}{(n+3)^2}\right) = \lim_{n\to\infty} \frac{n^4+6n^3+9n^2 - \sum_{k=0}^{4}\binom{4}{k} n^{4-k} a^k}{n^2+6n+9} =$$

$$= \lim_{n\to\infty} \frac{n^4+6n^3+9n^2 - \left(n^4 + \binom{4}{1} n^3 a + \binom{4}{2} n^2 a^2 + \dots\right)}{\sim n^2}$$

$$\binom{n}{0} = \frac{n!}{n!\,0!} = 1$$

$$\binom{n}{1} = \frac{n!}{(n-1)!\,1!} = \frac{(n-1)!\cdot n}{(n-1)!\cdot 1} = n$$

$$\binom{4}{2} = \frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 2\cdot 1\cdot 2} = 6$$

1. slučaj: $6n^3 - 4n^3 a = 0$ → ONDA ĆE SE TREĆE POTENCIJE POKRATITI!!

$$\boxed{a = \frac{3}{2}} \qquad \sim \lim_{n\to\infty} \frac{9n^2 - 6n^2\cdot\frac{9}{4}}{\sim n^2} = \lim_{n\to\infty} \frac{-\frac{9}{2}n^2}{n^2} = -\frac{9}{2}$$

2. slučaj: $a > \frac{3}{2}$ → DESNO JE VEĆA UZ n^3

$$\sim \lim_{n\to\infty} \frac{-n^3}{n^2} = -\infty$$

3. slučaj: $a < \frac{3}{2}$ → DESNO JE MANJE UZ n^3

$$\sim \lim_{n\to\infty} \frac{n^3}{n^2} = +\infty$$

6.DZ-7.) a) Po definiciji dokaži da je $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

$\lim_{n\to\infty} \underbrace{\sin(n)}_{(-1,1)}$ = NE POSTOJI !

definicija: $\forall n > n_0,\ n > \frac{1}{\varepsilon} \to \frac{1}{n} < \varepsilon$

$$|a_n - L| < \varepsilon$$

$$\left|\frac{\sin}{n} - 0\right| < \varepsilon$$

$$\frac{\overbrace{|\sin(n)|}^{(0,1)}}{|n|} \leq \frac{1}{n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon}.$$

→ Neka je $\varepsilon > 0$ proizvoljan. Tada postoji $n_0 \in \mathbb{N}$ takav da je $n_0 > \frac{1}{\varepsilon}$

⟹ da za svaki $n \geq n_0$ također je $n > \frac{1}{\varepsilon}$. o.k.

b) $$\lim_{n\to\infty} \frac{n\cdot\sin(n) \;/:n^2}{n^2+n+1 \;/:n^2} = \lim_{n\to\infty} \frac{\overbrace{\frac{\sin n}{n}}^{\to 0}}{1+\underbrace{\frac{1}{n}}_{\to 0}+\underbrace{\frac{1}{n^2}}_{\to 0}} = \frac{0}{1} = 0$$

c) ZZV-2.) U ovisnosti o $k\in\mathbb{N}$:

$$\lim_{n\to\infty} \frac{n\sin n \;/:n^2}{n^k+1 \;/:n^2} = \lim_{n\to\infty} \frac{\overbrace{\frac{\sin n}{n}}^{\to 0}}{n^{k-2}+\underbrace{\frac{1}{n^2}}_{\to 0}} = \begin{cases} \text{ne postoji}, & k=1 \\ 0, & \text{za } k=2 \\ \frac{0}{\infty}=0, & \text{za } k=3,4,5\ldots \end{cases}$$

PRIMJEDBA:

$$\lim_{n\to\infty} \frac{an^k\ldots}{bn^l\ldots} = \begin{cases} \infty, & \text{za } k>l \\ \frac{a}{b}, & \text{za } k=l \\ 0, & \text{za } k<l \end{cases}$$

Zad.) Izračunaj $\lim_{n\to\infty} q^n$, za $|q|<1$. ($1, q, q^2, q^3 \ldots$ geom. niz)

1) za $q=0$

$$\lim_{n\to\infty} 0 = 0$$

2) $q\in(0,1)$: tada $\frac{1}{q}>1$, tj. postoji $h>0$ takav da je $\frac{1}{q}=1+h$

tada

$$q^n = \frac{1}{(1+h)^n} \leq \overbrace{\frac{1}{1+n\cdot h}}^{\text{BERNOULLIJEVA}} < \frac{1}{n\cdot h}$$

Dakle, za sve $n\in\mathbb{N}$ vrijedi $0<q^n<\frac{1}{n\cdot h}$ pa po sendvič teoremu je $\lim_{n\to\infty} q^n = 0$.

Za $q\in(-1,0)$ analogno ovome

MAT → 2A 2. TJEDNA U PETAK U 9⁰⁰

2MI-2007-4.) b) Izračunaj: $\lim\limits_{n\to\infty} \frac{5^{n+1}+3\cdot 4^n}{5^{n-1}+2\cdot 3^n} \begin{matrix} /:5^n \\ /:5^n \end{matrix} = \lim\limits_{n\to\infty} \frac{5+3\cdot\left(\frac{4}{5}\right)^n}{\frac{1}{5}+2\cdot\left(\frac{3}{5}\right)^n} = 25$ //

$\left(\frac{4}{5}\right)^n \to 0$, $\left(\frac{3}{5}\right)^n \to 0$ → po prethodnom zadatku

2.MI-2008-1) a) $\lim\limits_{n\to\infty}\left(1-\frac{2}{3}+\left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^3+\dots+(-1)^n\left(\frac{2}{3}\right)^n\right)$

suma geom. niza: $\boxed{1+q+q^2+\dots q^n = \frac{1-q^{n+1}}{1-q}}$

po // manje od 1, dokazat!

$\Longrightarrow q=-\frac{2}{3} \longrightarrow \lim\limits_{n\to\infty}\left(\frac{1-\left(-\frac{2}{3}\right)^{n+1}}{1+\frac{2}{3}}\right) = \frac{1}{\frac{5}{3}} = \frac{3}{5}$

b) $\lim\limits_{n\to\infty}\frac{3+3^2+3^3+\dots+3^n}{1+3^{n+1}} = \lim\limits_{n\to\infty}\frac{3(1+3+3^2+\dots+3^{n-1})}{1+3^{n+1}} = \lim\limits_{n\to\infty}\frac{3\cdot\frac{1-3^n}{1-3}}{1+3^{n+1}} =$

$= -\frac{3}{2}\lim\limits_{n\to\infty}\frac{1-3^n}{1+3^{n+1}} \begin{matrix} /:3^n \\ /:3^n \end{matrix} = -\frac{3}{2}\cdot\lim\limits_{n\to\infty}\frac{\frac{1}{3^n}-1}{\frac{1}{3^n}+3} = -\frac{3}{2}\cdot\frac{-1}{3} = \frac{1}{2}$ //

$\frac{1}{3^n}\to 0$

6.DZ-10.) $\lim\limits_{n\to\infty}\frac{1+2+3+\dots+2n}{n^2+1} = \lim\limits_{n\to\infty}\frac{\frac{2n(2n+1)}{2}}{n^2+1} \sim \lim\limits_{n\to\infty}\frac{2n^2}{n^2} = 2$ //

$\frac{n(n+1)}{2}$ GAUSS

! UVIJEK PREKO TOGA ZAPIŠI

6.ZZV-18.) $\lim\limits_{n\to\infty}\frac{1+4+7+\dots+(3n+1)}{n^2+1} = \lim\limits_{n\to\infty}\frac{\sum\limits_{k=0}^{n}(3k+1)}{n^2+1} =$

PAZI! k ide od 0!

$= \lim\limits_{n\to\infty}\frac{3\sum\limits_{k=0}^{n}k+\sum\limits_{k=0}^{n}1}{n^2+1} = \lim\limits_{n\to\infty}\frac{3\cdot\frac{n(n+1)}{2}+n+1}{n^2+1} \sim \lim\limits_{n\to\infty}\frac{\frac{3}{2}n^2}{n^2} = \frac{3}{2}$ //

Zad.) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = (1^\infty) = e$ //

$\lim\limits_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$ //

$= \lim\limits_{n\to\infty}\left[\left(1+\frac{1}{\frac{n}{a}}\right)^{\frac{n}{a}}\right]^a$

$$\lim_{n\to\infty}\left(\frac{n^2-2+2}{n^2-2}\right)^{n^2} = \lim_{n\to\infty}\left(1+\frac{2}{n^2-2}\right)^{n^2-2+2} = \underbrace{\lim_{n\to\infty}\left(1+\frac{2}{n^2-2}\right)^{n^2-2}}_{e^2}\cdot\left(1+\frac{2}{\underset{\infty}{n^2-2}}\right)^2 =$$

$$= e^2\cdot(1+0)^2 = e^2$$

o.k.

Zad.) $$\lim_{n\to\infty}\left(\frac{3n-1}{3n+2}\right)^n = \lim_{n\to\infty}\left(\frac{3n+2-3}{3n+2}\right)^n = \lim_{n\to\infty}\left[\underbrace{\left(1+\frac{-3}{3n+2}\right)^{3n+2}}_{e^{-3}}\right]^{\frac{n}{3n+2}} =$$

$$= \left[e^{-3}\right]^{\lim_{n\to\infty}\frac{n\,/:n}{3n+2\,/:n}} = \left(e^{-3}\right)^{\frac{1}{3}} = e^{-1}$$

6.3. MONOTONI NIZOVI

def. Kažemo da je niz a_n rastući ako za $n>m$ vrijedi $a_n \geq a_m$, a padajući ako za $n>m$ vrijedi $a_n \leq a_m$.

→ Niz a_n je monoton ako je padajući ili rastući, a strogo monoton ako je strogo rastući ili padajući (stroga nejednakost $a_n > a_m$..)

$\boxed{\text{TM}}$ (VAŽNO ZA ISPIT!!)

Ako je niz monoton i omeđen onda je konvergentan (tj. ima limes).

Dokaz: (ZA ISPIT, ZNATI!!)

→ pretpostavimo da je a_n rastući (analogno se dokazuje padajući)

Skup vrijednosti od a_n je omeđen pa ima supremum, označimo ga sa L. Za svaki $\varepsilon>0$, broj $L-\varepsilon$ nije supremum, pa postoji $n_0\in\mathbb{N}$ t.d. je $a_{n_0} > L-\varepsilon$. Budući je niz rastući tada za sve $n>n_0$ vrijedi $a_n > L-\varepsilon$.

⇒ izvan svake ε-okoline od L se nalazi konačno mnogo članova tog niz pa prema tome $L=\lim_{n\to\infty} a_n$. ✓

$\boxed{TM}$ monoton i omeđen $\longrightarrow$ konvergentan!

Primjedba:

1. niz može biti monoton tek nakon nekog člana, tj. nije bitno što se događa s prvih nekoliko članova
2. obrat $\boxed{TM}$ NE VRIJEDI!! ako niz konvergira ne mora biti monoton!

npr. $a_n = \frac{(-1)^n}{n}$ ..... $-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4} \dots$

6.DZ-19.)

$$a_1 = 1$$
$$a_{n+1} = 3 - \frac{1}{a_n + 3}$$

Dokaži da je konvergentan i nađi mu limes!

i) MONOTON

$\rightarrow$ pretpostavljam da je rastući $a_{n+1} > a_n$

INDUKCIJA!

1.) baza

$a_2 > a_1$ ✓

2.) pretpostavimo da vrijedi $a_{n+1} > a_n$

3.) korak $(n+1)$

$$\cancel{3} - \frac{1}{a_{n+1}+3} > \cancel{3} - \frac{1}{a_n+3} \quad / \cdot (-1) \quad / \cdot (-1) \rightarrow \text{obrne } <$$

$\hookrightarrow$ obrne

$$a_{n+1} + \cancel{3} > a_n + \cancel{3}$$

$$\underline{\underline{a_{n+1} > a_n}} \quad ✓$$

ii) OMEĐEN (sa 3)

1. baza

$a_1 < 3$ ✓

2. pretp. da vrijedi

3. korak indukcije NE

$$\cancel{a_{n+1} < 3}$$

$$\cancel{3 - \frac{1}{a_n+3} < 3}$$

$$\cancel{a_n+3 > 0}$$

→ omeđenost: očito! zato što svaki novi član dobivamo tako da od 3 oduzimamo pozitivan broj pa je omeđen odozgo s 3.

$a_n > -3$?? omeđenost: očito! ⇓ NE DOKAZUJE SE INDUKC.

→ Dokazali smo da je monoton i omeđen → znači da konvergira i postoji limes.

$$L = \lim_{n\to\infty} a_n$$

$$a_{n+1} = 3 - \frac{1}{a_n+3} \quad \Big| \lim_{n\to\infty}$$

$$L = 3 - \frac{1}{L+3} \quad \Big| \cdot (L+3)$$

$$3L^2 + \cancel{3L} = \cancel{3L} + 9 - 1$$

$$L^2 = 8$$

$$L = \pm 2\sqrt{2}$$ → UZIMAMO + JER JE NIZ RASTUĆI!

$$\boxed{L = 2\sqrt{2}}$$

→ DA JE BIO PADAJUĆI UZELI BI −

6.DZ-18.)

$$a_n = \underbrace{\sqrt[3]{9\cdot\sqrt[3]{9\cdots\sqrt[3]{9}}}}_{n\text{-korijena}}$$

Dokaži da je konvergentan i nađi limes.

$$\Longrightarrow a_{n+1} = \sqrt[3]{9\cdot a_n}$$

(I) monoton

→ rastući (pretpostavka $a_{n+1} > a_n$)

1.) baza

o.k. $a_1 = \sqrt[3]{9}$

2.) pretpo.

3.) korak indukcije, $n+1$

$$\underline{a_{n+2} > a_{n+1}}$$

$$\sqrt[3]{9\cdot a_{n+1}} > \sqrt[3]{9\cdot a_n} \quad \Big|^3 \quad |:9$$

$$\underline{a_{n+1} > a_n} \quad ✓$$

(II) omeđenost

→ odozgo sa 9

$a_n < 9$

1) baza

$\sqrt[3]{9} < 9$ ✓

2) ...

3) $a_{n+1} < 9$

$$\sqrt[3]{9\cdot a_n} < 9 \quad \Big|^3$$

$$9\cdot a_n < 9^3 \quad |:9$$

$$a_n < 81 \quad ✓ \quad \cancel{\text{ako je mam}}$$

→ niz konvergira, pa postoji limes

$$a_{n+1} = \sqrt[3]{9 \cdot a_n} \quad \Big| \lim_{n\to\infty}$$

$$L = \sqrt[3]{9 \cdot L} \quad \Big|^3$$

$$L^3 - 9L = 0$$

$$L_1 = 0 \quad L_2 = -3 \quad \boxed{L_3 = 3}$$ zato što je niz rastući, a kreće od pozitivnih brojeva!

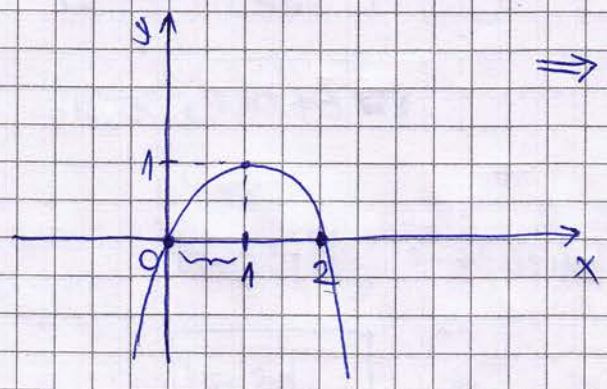
6. DZ-20.)

$$a_1 = x$$

$$a_{n+1} = 2a_n - a_n^2$$

Dokaži da a_n konvergira samo za $x \in \langle 0, 2 \rangle$ s limesom 1.

$$a_2 = f(x) = 2x - x^2$$



⟹ niz je omeđen s 1 za $x \in \langle 0, 2 \rangle$

→ niz je monotono rastuć jer su članovi niza (osim prvog) između 0 i 1, a za $x \in \langle 0, 1 \rangle$ funkcija $f(x) = 2x - x^2$ je rastuća

⟹ niz konvergira i ima limes

$$a_{n+1} = 2a_n - a_n^2 \quad \Big| \lim_{n\to\infty}$$

$$L = 2L - L^2$$

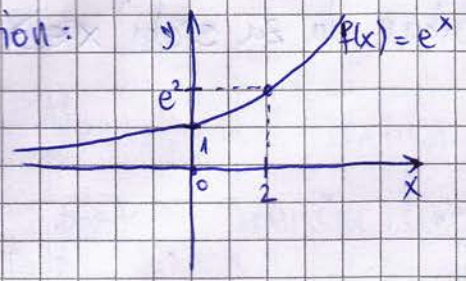
$$L^2 - L = 0$$

$$L = 0 \quad \boxed{L = 1}$$ → jer je rastući!

7. LIMES FUNKCIJE

7.1. DEFINICIJE I RAČUNANJE LIMESA

– motivation:



$\lim\limits_{x\to\infty} e^x = +\infty$ $\qquad$ $\lim\limits_{x\to 0} e^x = 1$

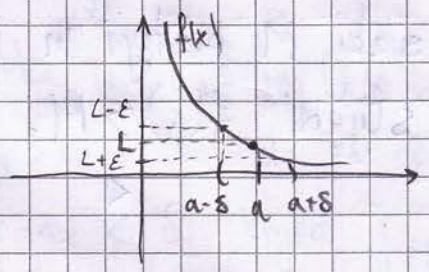
$\lim\limits_{x\to -\infty} e^x = 0$ $\qquad$ $\lim\limits_{x\to 2} e^x = e^2$

-<u>def.</u> (VAŽNE DEF. ZA ISPIT!!)

„KONAČAN LIMES U KONAČNOSTI"

$$\lim_{x\to a} f(x) = L$$

→ Za broj L kažemo da je limes funkcije f u točki $x = a \in \mathbb{R}$ ako za svaki $\varepsilon > 0$ postoji $\delta > 0$ takav da za svaki $x \in D(f)$ iz $|x-a| < \delta$ slijedi $|f(x) - L| < \varepsilon$.

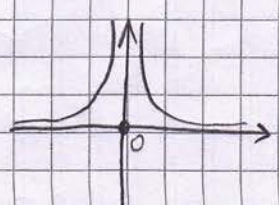


-<u>def.</u> „BESKONAČAN LIMES U KONAČNOSTI"

$$\lim_{x\to a} f(x) = \pm\infty$$

→ Limes funkcije f u točki $x = a$ je jednak $+\infty$ ($-\infty$) ako za svaki M postoji $\delta > 0$ takav da za svaki $x \in D(f)$ iz $|x-a| < \delta$ slijedi $f(x) > M$ ($<$).

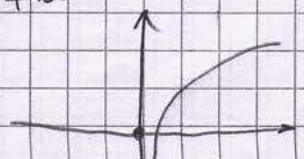
npr. $f(x) = \frac{1}{x^2}$



$\lim\limits_{x\to 0} \frac{1}{x^2} = +\infty$

npr. $f(x) = \ln x$

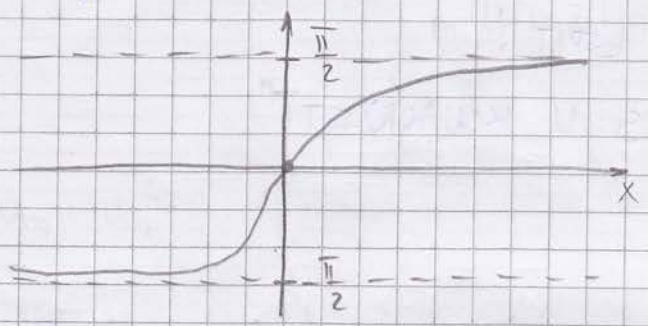
$\lim\limits_{x\to 0} \ln x =$



- def. „KONAČAN LIMES U BESKONAČNOSTI"

→ Za realan broj L kažemo da je limes funkcije f u "točki" $x = +\infty$ ($-\infty$) ako za svaki $\varepsilon > 0$ postoji $M > 0$ ($m < 0$) takav da za svaki $x \in D(f)$ iz $x > M$ ($x < m$) slijedi $|f(x) - L| < \varepsilon$.

$$\lim_{x \to \pm\infty} f(x) = L$$



$$\lim_{x \to +\infty} \operatorname{arctg} x = +\frac{\pi}{2}$$

$$\lim_{x \to -\infty} \operatorname{arctg} x = -\frac{\pi}{2}$$

- def. „BESKONAČAN LIMES U BESKONAČNOSTI"

$$\lim_{x \to \pm\infty} f(x) = \pm\infty$$

→ Limes funkcije f u $x = +\infty$ ($-\infty$) je $+\infty$ ($-\infty$) ako za svaki M postoji m takav da za svaki $x \in D(f)$ iz $x > m$ ($<$) slijedi $f(x) > M$ ($<$).

$$\lim_{x \to +\infty} \operatorname{ch} x = +\infty$$

$$\lim_{x \to -\infty} \operatorname{ch} x = +\infty$$

$\boxed{TM}$ Broj L je limes funkcije $f(x)$ u točki $x=a$ ako i samo ako za svaki niz $(x_n) \in D(f)$ koji $x_n \to a$ vrijedi $\lim\limits_{x\to a} f(x_n) = L$.

Dokaz: PMF!

Važnost! VRIJEDE ISTA PRAVILA KAO ZA LIMES <u>NIZA</u>!

$\boxed{TM}$ Neka postoje konačni limesi funkcija f i g u točki $x=a$.

Tada:

$$\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a}(g(x))}$$

$$\boxed{\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} f(x)^{\lim\limits_{x\to a} g(x)}}$$

→ <u>Dokaze</u> znati!

⇓

~ Neka je $L_1 = \lim\limits_{x\to a}(f(x))$, $L_2 = \lim\limits_{x\to a}(g(x))$. Za svaki $\varepsilon > 0$ postoji $\delta_1 > 0$, $\delta_2 > 0$ takav da za sve $x \in D(f)$ ako je $|x-a| < \delta_1$ i $|x-a| < \delta_2$ vrijedi $|f(x) - L_1| < \frac{\varepsilon}{2}$ i $|g(x) - L_2| < \frac{\varepsilon}{2}$. To vrijedi za $|x-a| < \delta$ gdje je $\delta = \min(\delta_1, \delta_2)$.

$\lim\limits_{x\to a}(f(x) + g(x)) = L_1 + L_2$ stoga

$$|(f(x) + g(x) - (L_1 + L_2)| = |(f(x) - L_1) + (g(x) - L_2)| \leq |f(x) - L_1| + |g(x) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

⇓

Zad.)

a) $\lim\limits_{x\to-\infty}\frac{3x^2-5x+1}{4x^2+x+2}=\frac{3}{4}$ (isto kao $x\to+\infty$)

PAZI! $\sqrt{x^2}=|x|$

b) $\lim\limits_{x\to+\infty}\frac{x+1\ /:x}{\sqrt{4x^2+3}\ /:x}=\lim\limits_{x\to+\infty}\frac{1+\frac{1}{x}}{+\sqrt{4+\frac{3}{x^2}}}=\frac{1}{2}$

$\lim\limits_{x\to-\infty}\frac{x+1\ /:x}{\sqrt{4x^2+3}\ /:x}=\lim\limits_{x\to-\infty}\frac{1+\frac{1}{x}}{-\sqrt{4+\frac{3}{x^2}}}=-\frac{1}{2}$

c) $\lim\limits_{x\to+\infty}\frac{2x+1\ /:x}{\sqrt[3]{x^3+x+1}\ /:x}=\lim\limits_{x\to+\infty}\frac{2+\frac{1}{x}}{\sqrt[3]{1+\frac{1}{x^2}+\frac{1}{x^3}}}=2$

$\lim\limits_{x\to-\infty}\frac{2x+1}{\sqrt[3]{x^3+x+1}}=2$ JER KOD $\sqrt[3]{x^3}$ NEMA APSOLUTNO!

7.DZ-5) $\lim\limits_{x\to-\infty}\left(\sqrt[4]{x^4+x^3+x^2+x+1}+x\right)=(\infty-\infty)$ PRIMJETI!

AKO JE $x\to+\infty$ RJEŠENJE BI BILO $+\infty+\infty=\infty$!

(ZA $\infty-\infty$ JE NEODREĐENO)

$=\lim\limits_{x\to-\infty}\left(\sqrt[4]{x^4+x^3+x^2+x+1}+x\right)\cdot\frac{\sqrt[4]{x^4+x^3+x^2+x+1}-x}{\sqrt[4]{x^4+x^3+x^2+x+1}-x}=$

$=\lim\limits_{x\to-\infty}\frac{\sqrt{x^4+x^3+x^2+x+1}-x^2}{\sqrt[4]{x^4+x^3+x^2+x+1}-x}\cdot\frac{\sqrt{x^4+x^3+x^2+x+1}+x^2}{\sqrt{x^4+x^3+x^2+x+1}+x^2}=$

$=\lim\limits_{x\to-\infty}\frac{\cancel{x^4}+x^3+x^2+x+1-\cancel{x^4}}{\left(\sqrt[4]{x^4+x^3+x^2+x+1}-x\right)\left(\sqrt{x^4+x^3+x^2+x+1}+x^2\right)}\sim x^3=\frac{-1}{4}$

$\sim -4x^3$

$\cancel{(x+x)}\cdot(2x^2)=-4x^3$

$(-x-x)$

KOD SVIH PARNIH KORJENA KADA $x\to-\infty$ IZLUČI SE MINUS VAN!

PAZ!

$\sqrt{x^4}=x^2$ UVIJEK +!

BEZ OBZIRA NA PARNI KORJEN $|x^2$ UVIJEK +

Dekanski-11-1) $\lim\limits_{x\to+\infty} \frac{x\,\mathrm{arctg}\,x \;/:x}{x+\sqrt{x^2+x+1}\;/:x} = \lim\limits_{x\to+\infty} \frac{\mathrm{arctg}\,x}{1+\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$ //

$\lim\limits_{x\to-\infty} \frac{x\,\mathrm{arctg}\;/:x}{x+\sqrt{x^2+x+1}\;/:x} = \lim\limits_{x\to-\infty} \frac{\mathrm{arctg}\,x}{1-\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}} = \frac{-\frac{\pi}{2}}{0} = -\infty$ //

7. ZZV-6) $\lim\limits_{x\to+\infty} (\mathrm{arch}\,x - \ln x) = (\infty-\infty) =$

$= \lim\limits_{x\to+\infty} \left(\ln(x+\sqrt{x^2-1}) - \ln x\right) = \lim\limits_{x\to+\infty} \ln \frac{x+\sqrt{x^2-1}\;/:x}{x\;/:x} = \ln 2$ //

Zad.) $\lim\limits_{x\to2} \frac{x^4-8x}{x^2+x-6} = \left(\frac{0}{0}\right) = \lim\limits_{x\to2} \frac{x(x^3-8)}{\ldots} = \lim\limits_{x\to2} \frac{x\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x+3)} =$

$= \lim\limits_{x\to2} \frac{24}{5} = \frac{24}{5}$ //

Zad.) $\lim\limits_{x\to1} \frac{1-\sqrt{x}}{1-\sqrt[3]{x}} = \left| \begin{array}{l} x=t^6 \\ t\to1 \end{array} \right| = \lim\limits_{t\to1} \frac{1-t^3}{1-t^2} = \lim\limits_{t\to1} \frac{\cancel{(1-t)}(1+t+t^2)}{\cancel{(1-t)}(1+t)} = \frac{3}{2}$ //

*7. DZ-5.) PREKO SUPSTITUCIJE

$\ldots \left| \begin{array}{l} x=-t \\ t\to+\infty \end{array} \right| = \lim\limits_{t\to+\infty} \left(\sqrt[4]{t^4-t^3+t^2-t+1} - t\right) = \ldots$ ZNAMO, ZA D.Z. RIJEŠI

• ⟶ NEKI POZNATI LIMESI:

(I.) $$\boxed{\lim_{x\to\pm\infty}\left(1+\frac{a}{x}\right)^x = e^a}$$

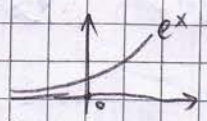
(1^∞)

Zad.) $$\lim_{x\to+\infty}\left(\frac{2x+\overset{1+2}{3}}{2x+1}\right)^x = \lim_{x\to+\infty}\left(1+\frac{2}{2x+1}\right)^x = \lim_{x\to\infty}\left[\left(1+\frac{2}{2x+1}\right)^{2x+1}\right]^{\frac{x}{2x+1}} =$$

$$= \left(e^2\right)^{\lim_{x\to\infty}\frac{x}{2x+1}} = \left(e^2\right)^{1/2} = e$$

7.DZ-14.) $$\lim_{x\to\infty}\left(\frac{x+1-1}{x+1}\right)^{x^2\,(1^\infty)} = \lim_{x\to\infty}\left[\left(1+\frac{-1}{x+1}\right)^{x+1}\right]^{\frac{x^2}{x+1}} = \left(e^{-1}\right)^{\lim_{x\to\infty}\frac{x^2}{x+1}} = \left(e^{-1}\right)^{\infty} = e^{-\infty} =$$

$$= 0$$



2.MI-2009-2.) $$\lim_{x\to\infty}\left(\frac{3x+1}{x+1}\right)^{\frac{x+1}{2x}} = \cancel{\lim} = 3^{\frac{1}{2}} = \sqrt{3}$$

$$\lim_{x\to\infty}\left(\frac{3x+1}{x+1}\right)^{\lim_{x\to\infty}\frac{x+1}{2x}} = 3^{\frac{1}{2}}$$

!! PROVJERI OBLIK !!! NEMOJ ĆI NAMJEŠTAVATI NA e!

(II.) $$\boxed{\lim_{x\to\infty}\frac{\sin x}{x} = 0}$$

$\sin x = \text{const.}$ $\quad x\to\infty \quad \frac{\text{const.}}{\infty} = 0$

PAZI (NIJE ZBRAJANJE PA $\frac{2}{x^2}$ ...)

7.ZZV-18.) $$\lim_{x\to\infty}\frac{(2x+1)\sin x}{3x^2+1}\,\Big|_{:x^2} = \lim_{x\to\infty}\frac{\left(2+\frac{1}{x}\right)\cdot\frac{\sin x}{x}}{3+\frac{1}{x^2}} = \frac{0}{3} = 0$$

Zad.) $$\lim_{x\to 0}\frac{\ln(x+1)}{x} = \left(\frac{0}{0}\right) = \lim_{x\to\infty}\frac{1}{x}\cdot\ln(1+x) = \lim_{x\to 0}\left(\ln(1+x)\right)^{\frac{1}{x}} = \left|\begin{array}{l} x=\frac{1}{t} \\ t\to\infty \end{array}\quad t=\frac{1}{x}\right|$$

$$= \lim_{t\to\infty}\left(\ln\left(1+\frac{1}{t}\right)^t\right) = \ln e = 1$$

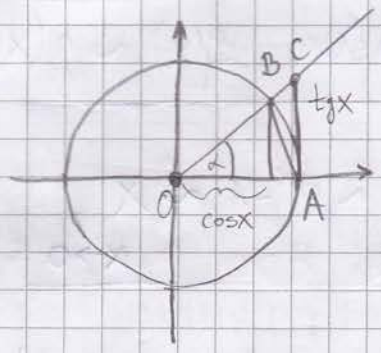
lim MOŽE UĆI POD ln!

Zad.) $\lim\limits_{x\to 0} \frac{e^x-1}{x} = \left(\frac{0}{0}\right) = \left| \begin{array}{l} x = \ln(t+1) \\ t\to 0 \end{array} \right| = \lim\limits_{t\to 0} \frac{t}{\ln(t+1)} = \{\text{zad. prije}\} = \frac{1}{1} \neq 1$

III.

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

Dokaz: (ZNATI ZA ISPIT!!)



$\triangle OAB$, $\angle OAB$, $\triangle OAC$

kružni isječak

$P_{\triangle OAB} < P_{\angle OAB} < P_{\triangle OAC}$

$\frac{1}{2}\cdot 1\cdot \sin x < \frac{1}{2}\cdot 1\cdot x < \frac{1}{2}\cdot 1\cdot tg x \quad |\cdot 2$

$\sin x < x < \frac{\sin x}{\cos x} \quad |:\sin x$ (za $x \in \langle 0, \frac{\pi}{2}\rangle$)

$1 < \frac{x}{\sin x} < \frac{1}{\cos x} \quad |\lim\limits_{x\to 0}$

$\lim\limits_{x\to 0} \frac{1}{\cos x} = \frac{1}{1} = 1$

$1 < \lim\limits_{x\to 0} \frac{x}{\sin x} < 1$

→ po sendvič teoremu $\lim\limits_{x\to 0} \frac{x}{\sin x}$ je također 1

→ ZBOG PARNOSTI FUNKCIJE sin VRIJEDI I ZA $x \in \langle -\frac{\pi}{2}, 0\rangle$

$\frac{\sin x}{x} = \frac{\sin(-x)}{-x} = \frac{-\sin x}{-x} = \frac{\sin x}{x}$

Zad.) $\lim\limits_{x\to 0} \frac{tg 2x}{x} = \left(\frac{0}{0}\right) = \lim\limits_{x\to 0} \frac{2\cdot \sin 2x}{\underbrace{\cos 2x}_{1}\cdot x\cdot 2} = \frac{2}{1} = 2$

→ FINTA!!

AKO DODAŠ DOLJE 2 ONDA MORAŠ I GORE!

Zad.) $\lim\limits_{x\to 0} \frac{\arcsin x}{x} = \left| \begin{array}{l} x = \sin t \\ t\to 0 \end{array} \right| =$ UVIJEK INVERZNU FUNKCIJU DA SE PONIŠTE!!

$= \lim\limits_{t\to 0} \frac{t}{\sin t} = 1$

→ def. Kažemo da su $f(x)$ i $g(x)$ neizmjerno male veličine istog reda kada $x \to a$ ako je $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a}(g(x)) = 0$ i ako je

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = C \in R^+}$$

Posebno, ako je $C=1$ kažemo da su $f(x)$ i $g(x)$ <u>ekvivalentne</u> neizmjerno male veličine istog reda i pišemo $f(x) \sim g(x)$.

$$\boxed{\sin x \sim \operatorname{tg} x \sim \arcsin x \sim \operatorname{arctg} x \sim \ln(1+x) \sim e^x - 1 \sim x, \quad x\to 0}$$

2.MI-2009-3.) $\lim\limits_{x\to 0} \frac{\operatorname{tg}^2(3x)}{x\ln(1+2x)} \overset{\sim 3x}{=} \left(\frac{0}{0}\right) = \lim\limits_{x\to 0} \frac{(3x)^2}{x\cdot 2x} = \lim\limits_{x\to 0} \frac{9x^2}{2x^2} = \frac{9}{2}$ //

2.MI-2006-2.) $\lim\limits_{x\to+\infty} \left(\sqrt{x^4-x^3} - x^2\right)\cdot\left(\operatorname{arctg}\left(\frac{1}{x}\right)\right) =$ $\to 0$

TA FUNKCIJA MORA TEŽITI U 0 DA BI BILO EKVIVALENT

$= \lim\limits_{x\to+\infty} \frac{\cancel{x^4}-x^3-\cancel{x^4}}{\sqrt{x^4-x^3}+x^2}\cdot\frac{1}{x} = \frac{-x^2}{2x^2} = -\frac{1}{2}$ //

UVIJEK PROVJERI!

$\frac{e^0}{2} = \frac{1}{2}$ NE MOŽEMO ZAMJENITI!

ZZV-17.) $\lim\limits_{x\to 0} \frac{x}{e^{2x}-1} \arcsin\frac{e^{-x}}{2} = \left(\frac{0}{0}\right) = \lim\limits_{x\to 0} \frac{\cancel{x}}{2\cancel{x}}\cdot\frac{e^{-x}}{2}$ NE !. OVA FUNK. POD. ARCSIN MORA $\to 0$

$= \lim\limits_{x\to 0} \frac{1}{2}\cdot\arcsin\frac{1}{2} = \frac{1}{2}\cdot\frac{\pi}{6} = \frac{\pi}{12}$ //

7.DZ-17.) $\lim\limits_{x\to\infty} \frac{x\sqrt{x}}{x+2}\ln\frac{x}{x+1} = \left(\frac{\infty}{\infty}\right)$: $\ln\left(\frac{x+1-1}{x+1}\right) = \ln\left(1+\frac{-1}{x+1}\right)$

$\frac{1}{\infty} = 0$

$\frac{-1}{x+1} \to 0$

O.K.!!

$= \lim\limits_{x\to\infty} \frac{x\sqrt{x}}{x+2}\cdot\frac{-1}{x+1} = \lim\limits_{x\to\infty} \frac{\sim -x^{3/2}}{\sim x^2} = 0$ //

Dekanski - 2011. - 1.) (ILKO <3)

$$\lim_{x\to\infty}\frac{\ln(\operatorname{ch}(2x))}{x} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty}\frac{\ln\left(\frac{\overset{\nearrow\infty}{e^{2x}}+\overset{\nearrow 0}{e^{-2x}}}{2}\right)}{x} =$$

$\ln(1+x)\sim x$

$$= \lim_{x\to\infty}\frac{\ln\left(\frac{\frac{1}{e^{-2x}}+e^{-2x}}{2}\right)}{x} = \lim_{x\to\infty}\frac{\ln\left(\frac{1+e^{-4x}}{2e^{-2x}}\right)}{x} = \lim_{x\to\infty}\frac{\ln(1+\overset{\nearrow 0}{\overset{e^{-4x}}{e^{-4x}}})-\ln(2e^{-2x})}{x}$$

$$= \lim_{x\to\infty}\frac{e^{-4x}-\ln 2-(-2x)}{x} = \lim_{x\to\infty}\left(\frac{\overset{\nearrow 0}{e^{-4x}}}{x}-\frac{\overset{\nearrow 0}{\ln 2}}{x}+2\right) = 2$$

ZZV-28.) $\lim_{x\to 3}(x-2)^{\frac{x}{x-3}} = (1^\infty)$

UVIJEK KADA IMAMO "$x^{\text{na } x}$", A NE TEŽI $x\to\infty$ DA SE MOŽE NAMJESTITI NA e, KORISTIMO TRIK:

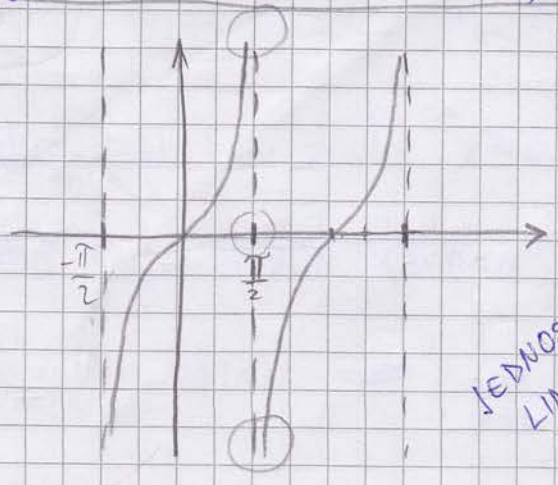
$$\boxed{x^y = e^{y\ln x}}$$

$$= \lim_{x\to 3} e^{\frac{x}{x-3}\ln(x-2)} = \lim_{x\to 3} e^{\frac{x}{\cancel{x-3}}\cdot\overset{\nearrow 0}{\ln(1+x-3)}\cdot\cancel{(x-3)}} = \lim_{x\to 3} e^{x} = e^3$$

2.MI-2007-6.) $\lim_{x\to 0}(1+\sin x)^{\operatorname{ctg}x} = (1^\infty) = \lim_{x\to 0} e^{\frac{\cos x}{\sin x}\ln(1+\overset{\nearrow 0}{\sin x})} = \lim_{x\to 0} e^{\frac{\cos x}{\cancel{\sin x}}\cdot\cancel{\sin x}}$

$$= \lim_{x\to 0} e^{\cos x} = e^{\cos 0} = e^1 = e$$

~~Zad.~~ $\lim_{x\to\frac{\pi}{2}} \operatorname{tg}x =$



→ LIMES NE POSTOJI!

ZATO SE UVODE LIM S LIJEVA I DESNA:

JEDNOSTRANI LIMESI

$\lim_{x\to\frac{\pi}{2}^-} \operatorname{tg}x = +\infty$ (S LIJEVA)

$\lim_{x\to\frac{\pi}{2}^+} \operatorname{tg}x = -\infty$ (S DESNA)

7.2. JEDNOSTRANI LIMESI

def.

Za broj L kažemo da je DESNI limes funkcije f u točki $x=a$ i pišemo $\lim\limits_{x\to a+} f(x) = L$ ako za svaki $\varepsilon>0$ postoji $\delta>0$ takav da za svaki $x\in D(f)$ iz $x\in(a, a+\delta)$ slijedi $|f(x)-L|<\varepsilon$.

def.

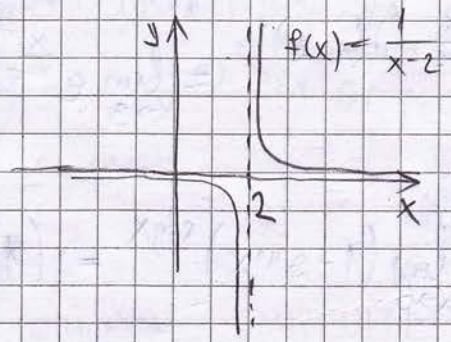
Za broj L kažemo da je LIJEVI limes funkcije f u točki $x=a$ i pišemo $\lim\limits_{x\to a-} f(x) = L$ ako za svaki $\varepsilon>0$ postoji $\delta>0$ takav da za $\forall x\in D(f)$ iz $x\in(a-\delta, a)$ slijedi $|f(x)-L|<\varepsilon$.

Napomena: analogno je definiranju beskonačnih limesa.

Pr.) $\lim\limits_{x\to 2+} \frac{1}{x-2} = \left(\frac{1}{0^+}\right) = +\infty$ //

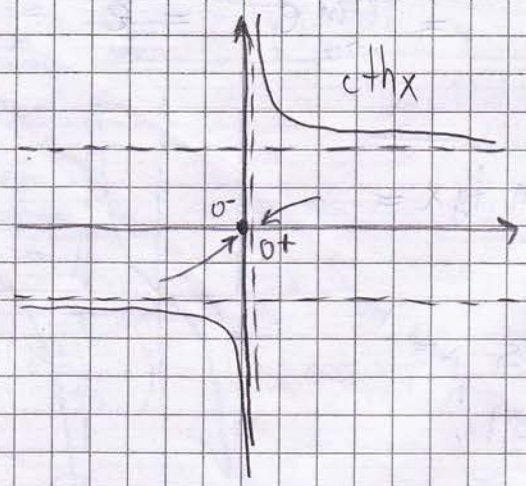
$+\infty$ ili $-\infty$??

$\lim\limits_{x\to 2-} \frac{1}{x-2} = \left(\frac{1}{0^-}\right) = -\infty$ //



Pr.) $\lim\limits_{x\to 0+} \operatorname{cth} x = +\infty$

$\lim\limits_{x\to 0-} \operatorname{cth} x = -\infty$



$x_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2}$

Zad.) $\lim\limits_{x\to 1^+} \frac{x+1}{x^2+x-2} = \lim\limits_{x\to 1^+} \frac{x+1}{(x-1)(x+2)} = \left(\frac{2}{0^+\cdot 3} = \frac{2}{0^+}\right) = +\infty$ //

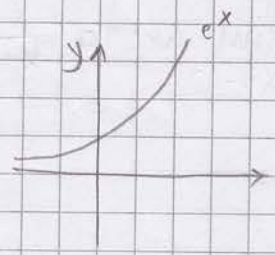
iz uvećih br. dolazimo

$\lim\limits_{x\to 1^-} \frac{x+1}{x^2+x-2} = \lim\limits_{x\to 1^-} \frac{x+1}{(x-1)(x+2)} = \left(\frac{2}{0^-\cdot 3}\right) = -\infty$ //

$\lim\limits_{x\to 2^+} \frac{x+1}{(x-1)(x+2)} = \left(\frac{-1}{-3\cdot 0^+}\right) = +\infty$ // PRIBLIŽAVAMO SE NULI S DESNA

$\lim\limits_{x\to -2^-} \frac{x+1}{(x-1)(x+2)} = \left(\frac{-1}{-3\cdot 0^-}\right) = -\infty$ // — II — S LIJEVA

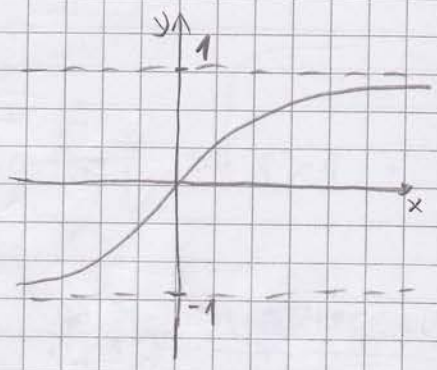
Zad.) $\lim\limits_{x\to 2^+} e^{\frac{2+x}{2-x}} = \left(e^{\frac{4}{0^-}}\right) = e^{-\infty} = 0$ //



$\lim\limits_{x\to 2^-} e^{\frac{2+x}{2-x}} = \left(e^{\frac{4}{0^+}}\right) = e^{+\infty} = +\infty$ //

7. DZ-11.) $\lim\limits_{x\to 0^+} th\left(\frac{1}{x^2-x}\right) = \lim\limits_{x\to 0^+} th\left(\frac{1}{x(x-1)}\right) =$

$= \left(th \frac{1}{\underbrace{0^+\cdot(-1)}_{0^-}} = th(-\infty)\right) = -1$ //



$\lim\limits_{x\to 0^-} th\left(\frac{1}{x(x-1)}\right) = \left(th \frac{1}{\underbrace{0^-\cdot(-1)}_{0^+}} = th(+\infty)\right) = 1$ //

$\boxed{TM}$ Limes funkcije $f(x)$ u točki $x=a$ postoji ako i samo ako postoje oba jednostrana limesa i JEDNAKI SU, tj.

$\lim\limits_{x\to a^+}(f(x)) = \lim\limits_{x\to a^-}(f(x)) \Longrightarrow \lim\limits_{x\to a} f(x)$ //

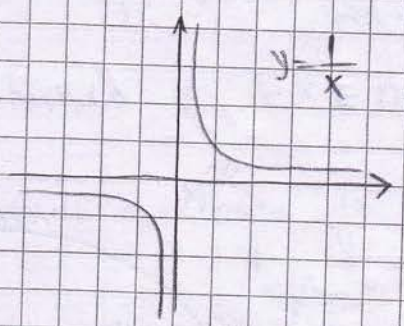
Dokaz: Trivić!

7. ZZV – 10.) Postoji li limes $\lim\limits_{x\to -1} \operatorname{arctg}\frac{x}{x+1}$?

$$\lim_{x\to -1^+} \operatorname{arctg}\frac{x}{x+1} = \left(\operatorname{arctg}\frac{-1}{0^+} = -\infty\right) = -\frac{\pi}{2}$$

$$\lim_{x\to -1^-} \operatorname{arctg}\frac{x}{x+1} = \left(\operatorname{arctg}\frac{-1}{0^-} = +\infty\right) = \frac{\pi}{2}$$

$\rightarrow \lim\limits_{x\to -1^+} \operatorname{arctg}\frac{x}{x+1} \neq \lim\limits_{x\to -1^-} \operatorname{arctg}\frac{x}{x+1} \rightarrow$ Ne postoji limes.



$$\lim_{x\to 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

ZA ISPIT → ČETI OD 12⁰⁰ – 14⁰⁰ KONZ.

6. zad. , 2 matrice

u svakom zadatku pod a) definicija
b) riješi

↓ sve zadaci koji nisu bili

vektori, svojstvene vrijednosti, inverz

→ sigurno nešto sa funk. bijekcija, surjekc.

→ 2 limesa (limes niza, limes sa dokazom monot. omeđenost)

→ kompleksni br. iz prve, bijek, surj.

→ neprekinutost ...

7.3. NEPREKINUTOST FUNKCIJE

def. Funkcija $f(x)$ je neprekinuta u točki $x=a \in \mathbb{R}$ ako je $a \in D(f)$ i ako $\forall \varepsilon > 0$, $\exists \delta > 0$ tako da za sve $x \in D(f)$ iz $|x-a| < \delta$ slijedi $|f(x)-f(a)| < \varepsilon$. (ISTO KAO DEF. LIMESA U TOČKI)

$\boxed{\text{TM}}$ Funkcija $f(x)$ je neprekinuta u točki $x=a$ ako i samo ako postoji limes funkcije u točki a, tj.

$$\boxed{\lim_{x\to a} f(x) = f(a)}$$

def. Funkcija je neprekinuta na skupu I ako je neprekinuta u svim točkama tog skupa.

7.DZ - 18.) Odredi parametar a t.d. je $f(x) = a\frac{x^2-x}{x^2-1}$ za $x>1$ i $f(x) = x^2+x$, $x \leq 1$ neprekinuta.

TOČKA PREKIDA

DA BI BILA NEPREKINUTA

→ primjeti: $f(1) = 1+1 = 2$ (s lijeve, za $x \leq 1$)

$$\lim_{x\to 1^+} a\frac{x^2-x}{x^2-1} = \lim_{x\to 1^+} a\frac{x\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = \lim_{x\to 1^+} a\cdot\frac{1}{2} = 2 \to \boxed{a=4}$$

2.MI - 2008 - 3.) $f(x) = \begin{cases} e^{\frac{1}{x-1}}, & x<1 \\ ax-1, & x \geq 1 \end{cases}$ Odredi a takav da je f neprekinuta.

$f(1) = a-1$, s desne strane, $x \geq 1$

lijeva strana → $\lim_{x\to 1^-} e^{\frac{1}{x-1}} = \left(e^{\frac{1}{0^-}} = e^{-\infty}\right) = 0$ → $0 = a-1 \Rightarrow \boxed{a=1}$

DA BI BILA NEPREKINUTA

prošlogodišnji

7. DZ. – 20.) Odredi a i b t.d. fja $f(x)=\dfrac{e^{\frac{1}{x}}+a}{be^{\frac{1}{x}}+2}$, $x\neq 0$ i $f(0)=1$ bude neprekinuta.

problematična točka

DA BUDE NEPREK.

$$\lim_{x\to 0^-}\frac{e^{\frac{1}{x}}+a}{be^{\frac{1}{x}}+2}=\left(\frac{e^{-\infty}+a}{be^{-\infty}+2}\right)=\frac{a}{2}=1 \longrightarrow \boxed{a=2}$$

$$\lim_{x\to 0^+}\frac{e^{\frac{1}{x}}+\overset{a}{\cancel{2}}\,/:e^{\frac{1}{x}}}{be^{\frac{1}{x}}+2\,/:e^{\frac{1}{x}}}=\left(\frac{\infty}{\infty}\right)=$$

VIJEK KADA IMAMO $\frac{\infty}{\infty}$ DJELIMO S MAX POTENCIJOM!!

$$=\lim_{x\to 0^+}\frac{1+\frac{a}{e^{\frac{1}{x}}}}{b+\frac{2}{e^{\frac{1}{x}}}}=\frac{1}{b}=1\longrightarrow\boxed{b=1}$$

$\frac{a}{e^{\frac{1}{x}}}\to 0$, $\frac{2}{e^{\frac{1}{x}}}\to 0$

da bude neprek.

2. MI – 4.)
$$f(x)=\begin{cases}\frac{1}{2}+\frac{\sin(ax)}{x}, & x<0\\ b, & x=0\\ \frac{\operatorname{sh}x}{x}, & x>0\end{cases}$$

Odredi a i b t.d. da f bude neprekinuta.

problematična $\boxed{x=0}$

$$\lim_{x\to 0^-}\left(\frac{1}{2}+\frac{\sin(ax)}{x}\right)=\lim_{x\to 0^-}\left(\frac{1}{2}+\frac{a\cancel{x}}{\cancel{x}}\right)=\frac{1}{2}+a=b$$

$\frac{\sin(ax)}{x}\to 0$, $\sim ax$

mora biti = f. u 0.

$$\lim_{x\to 0^+}\frac{\operatorname{sh}x}{x}=\lim_{x\to 0^+}\frac{\frac{1}{2}(e^{x}-e^{-x})\,/:e^{-x}}{x\,/:e^{-x}}=\lim_{x\to 0^+}\frac{e^{2x}-1}{2xe^{x}}=$$

$e^{2x}-1\to 0$, $\sim 2x$

$$=\lim_{x\to 0^+}\frac{\cancel{2x}}{\cancel{2x}e^{x}}=\frac{1}{e^{0}}=1=b\longrightarrow\boxed{b=1}$$

→ AUDITORNE ←

Zad 1. Sva gomilišta i limes ako postoji naći:

$$a_n = (-1)^{n+1} \frac{3n^2+2n+1}{n^2+2n+3} \quad : \frac{n^2}{n^2}$$

$$b_n = (-1)^n \cdot \frac{2n+3}{n^2+1} \quad : \frac{n^2}{n^2}$$

$$c_n = (-1)^{n+1} \cdot \frac{n^2+1}{2n+3} \quad : \frac{n^2}{n^2}$$

$$a_n = (-1)^{n+1} \frac{3+\overbrace{\frac{2}{n}+\frac{1}{n^2}}^{\to 0}}{1+\underbrace{\frac{2}{n}+\frac{3}{n^2}}_{\to 0}}$$

→ za n paran "$\lim a_n = -3$"
→ za n neparan "$\lim a_n = 3$"
} gomilišta ↓ nema limesa

$$b_n = (-1)^n \cdot \frac{\overbrace{\frac{2}{n}+\frac{3}{n^2}}^{\to 0}}{1+\underbrace{\frac{1}{n^2}}_{\to 0}} = 0 = \lim b_n$$

$$c_n = (-1)^{n+1} \frac{1+\underbrace{\frac{1}{n^2}}_{\to 0}}{\underbrace{\frac{2}{n}+\frac{3}{n^2}}_{\to 0}}$$

→ za n paran "$\lim c_n = -\infty$"
→ za n neparan "$\lim c_n = +\infty$"
} GOMILIŠTA

→ kada konvergira u $\pm\infty$ kažemo da konvergira u $\overline{\mathbb{R}}$

Zad. 2.

$$a_n = \underbrace{\sqrt{20+\sqrt{20+\sqrt{20 \cdots +\sqrt{20}}}}}_{n \text{ korijena}}$$

tvrdnja: a_n konvergentan! → monoton, → omeđen

a) $a_{n+1} = \sqrt{20+a_n}$

$a_1 = \sqrt{20}$

$a_2 = \sqrt{20+\sqrt{20}}$

NE MIJENJA SE

$20+\sqrt{20} \geq 20 \quad |\sqrt{\ }$ (rastuća)

$\sqrt{20+\sqrt{20}} \geq \sqrt{20}$

$\boxed{a_2 \geq a_1}$ → tvrdimo da je niz rastući

1.) baza: $a_2 \geq a_1$ (pokazali)

2.) pretpostavka: $a_n \leq a_{n+1}$ za <u>neki</u> $n \in \mathbb{N}$

3.) korak ind.

$$a_{n+1} \leq a_{n+2}$$

$$a_{n+2} = \sqrt{20 + \underbrace{a_{n+1}}_{\geq a_n}} \geq \sqrt{20 + a_n} = a_{n+1}$$

VAŽNA REČ!!! iz pretp.

Po principu matematičke indukcije tvrdnja vrijedi za $\forall n \in \mathbb{N}$.

b) $a_n \leq 5 \quad \forall n \in \mathbb{N}$

baza: $a_1 = \sqrt{20} \leq 5$ DA

$20 \leq 25 \quad | \sqrt{\ }$

$\sqrt{20} \leq 5$ o.k. ✓

2.) pretpost. $a_n \leq 5$ za <u>neki</u> $n \in \mathbb{N}$

3.) korak:

$$a_{n+1} \leq 5$$

$$a_{n+1} = \sqrt{20 + \underbrace{a_n}_{\leq 5}} \leq \sqrt{20+5} = \sqrt{25} = 5$$

Po principu mat. ind. vrijedi za sve $n \in \mathbb{N}$.

→ Sada računati limes:

poslije obrisati

$a_{n+1} = \sqrt{20 + a_n} \quad | \lim$

$L = \sqrt{20 + L} \quad |^2$

$L^2 - 20 - L = 0$

$\boxed{L_1 = 5} \quad \cancel{L_2 = -4}$ NE

↓

članovi pozitivni

Zad. 3.

$$\lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+x+1}+x} \cdot \operatorname{arctg} x = \lim_{x\to+\infty} \frac{2x+1 \;/\cdot x}{\sqrt{x^2+x+1}+x \;/\cdot x} \cdot \lim_{x\to\infty} \operatorname{arctg} x =$$

$$= \lim_{x\to\infty} \frac{2+\frac{1}{x}}{\sqrt{1+\underbrace{\frac{1}{x}}_{0}+\underbrace{\frac{1}{x^2}}_{0}}+1} \cdot \operatorname{arctg}\left(\lim_{x\to\infty}(x)\right) = 1 \cdot \frac{\pi}{2} = \frac{\pi}{2}$$

$$\lim_{x\to-\infty} \frac{2x+1}{\sqrt{x^2+x+1}+x} \cdot \operatorname{arctg} x = \left| \begin{matrix} -t=x \\ t\to+\infty \end{matrix} \right| = \lim_{t\to+\infty} \frac{-2t+1}{\sqrt{t^2+t+1}-t} \cdot \operatorname{arctg}(-t) =$$

$$= \lim_{t\to\infty} \frac{(-2t+1)\left(\sqrt{t^2-t+1}\right)+t \;/\cdot t}{\cancel{t^2}-t+1-\cancel{t^2} \;/\cdot t} \cdot \operatorname{arctg}(-t) = \frac{1}{\ldots} = -\infty \;//$$

Zad. 4.

$$\lim_{x\to\infty} \frac{x^2+2}{2x+1} \cdot \operatorname{arctg}\underbrace{\frac{1}{x}}_{\to 0 \;\sim \frac{1}{x}} = \lim_{x\to\infty} \frac{x^2+2}{2x+1} \cdot \frac{1}{x} \overset{\sim x^2}{\underset{\sim 2x^2}{=}} \frac{1}{2} \;//$$

$$\lim_{x\to 0} \frac{e^x-1}{x} = 1$$

Zad. 5.

$$\lim_{x\to 0} \frac{x}{e^{2x}-1} \cdot \operatorname{arctg}(e^{-x}) = \lim_{x\to 0} \frac{x}{e^{2x}-1} \cdot \lim_{x\to 0} \operatorname{arctg}(e^{-x}) =$$

$$= \lim_{x\to 0} \underbrace{\frac{2\cdot x}{(e^{2x}-1)\cdot 2}}_{\to 1} \cdot \operatorname{arctg}\left(\lim_{x\to 0} e^{-x}\right) = \frac{1}{2} \cdot \operatorname{arctg} 1 = \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8}$$

$$\boxed{x-1 \sim \ln x}$$

Zad. 6.

$$\lim_{x\to 1} x^{\frac{1}{x-1}} = \lim_{x\to 1} e^{\frac{1}{x-1}\cdot \ln(x)} = \lim_{x\to 1} e^{\underbrace{\frac{\ln(x)}{x-1}}_{\to 1 \;\sim x}} =$$

$$= \lim_{x\to 1} e^1 = e \;//$$

Zad. 7. Odredi $a, b \in \mathbb{R}$ da je f neprekinuta.

$$f(x) = \begin{cases} 1 + \operatorname{arctg}\frac{1}{x}, & x < 0 \\ 0, & x = 0 \\ b + \operatorname{arctg}\frac{1}{x}, & x > 0 \end{cases}$$

problematična točka: 0

neprekinuta:

(I) funkcija MORA biti definirana u točki

(II) limesi slijeva i desna moraju biti isti

→ Da bi fja bila neprekinuta u točki a treba vrijediti:

1.) f mora biti definirana u a

2.) $\lim\limits_{x \to a} f(x)$ mora biti konačan

$\to a^+$ $\to a^-$ moraju biti isti

3.) limes mora biti jednak funkcijskoj vrijednosti u $x = a$.

$$\lim_{x \to 0^-} f(x) = f(0) = 0$$

$$\lim_{x \to 0^+} f(x) = f(0) = 0$$

$$\lim_{x \to 0^-} \left(a + \operatorname{arctg}\frac{1}{x}\right) = 0$$

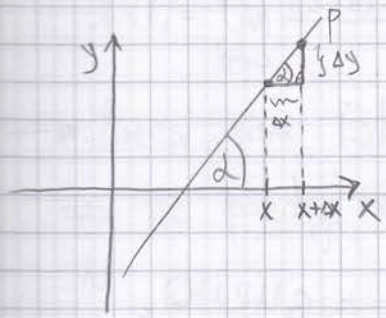
$$a + \left(-\frac{\pi}{2}\right) = 0 \to \boxed{a = \frac{\pi}{2}}$$

$$\lim_{x \to 0^+} \left(b + \operatorname{arctg}\frac{1}{x}\right) = 0$$

$$\boxed{b = -\frac{\pi}{2}}$$

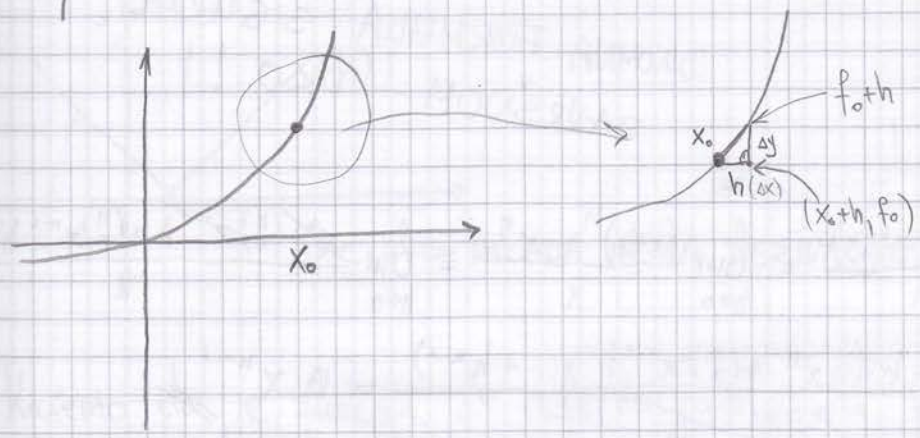
8. DIFERENCIJALNI RAČUN

8.1. MOTIVACIJA I DEFINICIJA DERIVACIJE



$p \ldots y = kx + l$

$k = \operatorname{tg}\alpha = \frac{\Delta y}{\Delta x}$ „brzina promjene y ovisno o x"



Def. Derivaciju funkcije f u točki x_0 definiramo sa

$$\boxed{f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

→ još koristimo oznaku $f'(x_0) = \frac{df(x_0)}{dx}$

$df(x) = f'(x) \cdot dx$ zovemo diferencijal funkcije f

$\Delta f(x) = f(x+\Delta x) - f(x)$ zovemo prirast funkcije f

$$\boxed{f'(x) = \lim_{h \to 0} \frac{\Delta f(x)}{\Delta x} = \frac{df(x)}{dx}}$$

1.) $f(x)=c$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{c-c}{h}=\lim_{h\to 0}\frac{0}{h}=0$$

$$\boxed{c'=0}$$

2.) $f(x)=x^n,\ n\in\mathbb{N}$

$$\boxed{f'(x)=nx^{n-1}}$$

BINOMNA FORMULA SIGURNO U ZAVRŠNOM - ILKO !!

!!! VAŽNO ZA ISPIT : !!

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}=\lim_{h\to 0}\frac{\cancel{x^n}+\binom{n}{1}x^{n-1}h+\binom{n}{2}x^{n-2}h^2+\dots+h^n-\cancel{x^n}}{h}$$

$$=\left(\frac{0}{0}\right)=\lim_{h\to 0}\frac{\cancel{h}\cdot\left(nx^{n-1}+\binom{n}{2}x^{n-2}h+\dots+h^{n-1}\right)}{\cancel{h}}=nx^{n-1}$$

2.MI-2009-4) $f(x)=\sqrt[3]{x}\longrightarrow(x^{1/3})'=\frac{1}{3}x^{\frac{1}{3}-1}=\frac{1}{3}x^{-\frac{2}{3}}=\frac{1}{3\sqrt[3]{x^2}}$ → vrijedi i za $n\in\mathbb{R}$ WOW !!

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt[3]{(x+h)}-\sqrt[3]{x}}{h}=\left(\text{ZA DZ. }\sqrt[6]{x}\right)=$$

$$=\lim_{h\to 0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}\cdot\frac{\sqrt[3]{(x+h)^2}+\sqrt[3]{(x+h)\cdot x}+\sqrt[3]{x^2}}{-\,\|\,-}=$$

$$=\lim_{h\to 0}\frac{\cancel{x+h-x}}{\cancel{h}\cdot\left(\sqrt[3]{(x+h)^2}+\sqrt[3]{(x+h)\cdot x}+\sqrt[3]{x^2}\right)}=\lim_{h\to 0}\frac{1}{3\sqrt[3]{x^2}}=\frac{1}{3\sqrt[3]{x^2}}$$

ZAVRŠNI → 8 ZAD PO 5 BODOVA IZ DRUGOG DIJELA

2 ZAD PO 5 BODOVA IZ PRVOG DIJELA (vjerojatno matrice) } 50

= 10 ZAD × 5 BOD , 2.5 h ISPIT

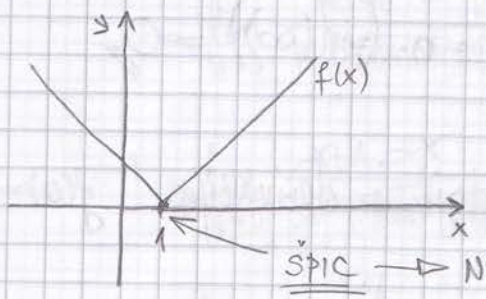
IM-08-4.)

$f(x) = |x-1|$. Postoji li $f'(1)$?

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{|1+h-1|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h} = \begin{cases} \lim\limits_{h\to 0^+} \frac{h}{h} = 1 \\ \lim\limits_{h\to 0^-} \frac{-h}{h} = -1 \end{cases}$$

→ limesi nisu isti ⟹ ne postoji limes $h\to 0$

⟹ ne postoji ni derivacija



ŠPIC → NIKADA NEMA DERIVACIJE U TOJ TOČKI !!!

def. Kažemo da je funkcija f diferencijabilna u točki x ako postoji derivacija funkcije u x.
Još kažemo da je f GLATKA u x.

⟹ !! Ako je funkcija neprekinuta ne mora značiti da je diferencijabil.

$\boxed{TM}$ Ako je funkcija diferencijabilna u točki x onda je neprekinuta u toj točki.

!!! VAŽNO ZA ISPIT !!

Dokaz:

$$\lim_{h\to 0}[f(x+h)-f(x)] =$$

$$= \lim_{h\to 0} \frac{h\cdot[f(x+h)-f(x)]}{h} = 0\cdot f'(x) = 0$$

$f'(x)$ → postoji po pretpostavci TM

* podsjetnik neprekinutosti:

$$\lim_{x\to a} f(x) = f(a)$$

↳ $\lim\limits_{h\to 0} f(x+h) = f(x)$ pa je f neprekinuta !!

NAPOMENA: OBRAT $\boxed{TM}$ NE VRIJEDI !!! (Rečenica prije, ako ima špic)

npr. $f(x) = |x|$ nije diferenc. u $x=0$

8.DZ-7.) a) Postoji li $f'(0)$ za $f(x) = x \sin\frac{1}{x}$, $x \neq 0$, $f(0) = 0$?

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\cancel{h} \sin\frac{1}{h} - 0}{\cancel{h}} = \lim_{h\to 0} \sin\frac{1}{h} = (\sin(\infty)) \Rightarrow \text{LIMES}$$

$\to [-1,1]$

NE POSTOJI PA NE POSTOJI $f'(0)$!

b) $g(x) = x^2 \sin\frac{1}{x}$, $x \neq 0$, $g(0) = 0$

0 · nešto omeđeno

$$g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{h^{\cancel{2}} \sin\frac{1}{h} - 0}{\cancel{h}} = 0 \cdot \underbrace{(\sin(\infty))}_{[-1,1]} = 0$$

$\to$ postoji limes, jednak je 0, pa postoji i derivacija ! $g'(0) = 0$.

$\boxed{\text{TM}}$ Vrijedi: 1) $(c \cdot f(x)) = c \cdot f'(x)$

2) $(f(x) \pm g(x))' = f'(x) \pm g'(x)$

3) $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g(x)$

4) $$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Dokazi: (VAŽNI ZA ISPIT !!!)

1. ubaciti u definiciju $\to$ TRIVIC

2. $$(f(x) \pm g(x))' = \lim_{h\to 0} \frac{f(x+h) \pm g(x+h) - f(x) - g(x)}{h} =$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \pm \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(x) \pm g'(x)$$

3. $$(f(x) \cdot g(x))' = \lim_{h\to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x) - f(x) \cdot g(x+h) + f(x) \cdot g(x+h)}{h} =$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot g(h+x) + \lim_{h\to 0} f(x) \cdot \frac{g(x+h) - g(x)}{h} = f'(x) \cdot g(x) - f(x) \cdot g'(x)$$

4. DOMA ! zad. za TR: a) $f(x) = (3x^2+1)\sin x$ b) $f(x) = \frac{3x+1}{5\cos x}$

VJEŽBA:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \frac{d f(x_0)}{dx} \longrightarrow \boxed{d f(x) = f'(x)\cdot dx}$$

Iz mj Burić:

$$f(x) = \sqrt[6]{x}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt[6]{(x+h)} - \sqrt[6]{x}}{h} \cdot \frac{\left(\sqrt[6]{(x+h)^2} + \sqrt[6]{(x+h)\cdot x} + \sqrt[6]{(x)^2}\right)}{\left(\sqrt[6]{(x+h)^2} + \sqrt[6]{(x+h)\cdot x} + \sqrt[6]{(x)^2}\right)}$$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)} - \sqrt{x}}{h\cdot\left(\sqrt[6]{(x+h)^2} + \sqrt[6]{(x+h)\cdot x} + \sqrt[6]{x^2}\right)} \cdot \frac{\sqrt{(x+h)} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} =$$

$$= \lim_{h\to 0} \frac{\cancel{x}+\cancel{h}-\cancel{x}}{\cancel{h}\cdot\left(\sqrt[6]{(x+h)^2} + \sqrt[6]{(x+h)\cdot x} + \sqrt[6]{x^2}\right)\left(\sqrt{x+h} + \sqrt{x}\right)} =$$

$$= \lim_{h\to 0} \frac{1}{\left(\sqrt[3]{x} + \sqrt[3]{x} + \sqrt[3]{x}\right)\left(2\sqrt{x}\right)} = \frac{1}{3\underbrace{\sqrt[3]{x}}_{x^{1/3}} \cdot 2\underbrace{\sqrt{x}}_{x^{1/2}}} = \frac{1}{6\sqrt[6]{x^5}}$$

$$\frac{1}{3}+\frac{1}{2} = \frac{2+3}{6} = \frac{5}{6}$$

provjera:

$$f(x) = x^{1/6}$$

$$f'(x) = \frac{1}{6}\cdot x^{\frac{1}{6}-1} = \frac{1}{6\sqrt[6]{x^5}} = \frac{1}{6}\cdot x^{-\frac{5}{6}}$$

o.k.

Ako je funkcija diferencijabilna onda je neprekinuta u tj točki.
Obrat ne mora vrijediti!
Ako je funkcija neprekinuta u točki ne mora značiti da je i diferencijabilna u toj točki!

Dokaz:

* podsjetnik neprekinutosti: $\lim_{x\to a} f(x) = f(a)$

$$\lim_{h\to 0}\left[f(x+h) - f(x)\right] = \lim_{h\to 0} \frac{h\cdot\left[f(x+h)-f(x)\right]}{h} = \lim_{h\to 0} h\cdot f'(x) = 0\cdot f'(x) = 0$$

po pretpostavci je diferenc.

$\longrightarrow \lim_{h\to 0} f(x+h) = f(x)$ → neprekinuta

Dokazi:

2.) $$(f(x) \pm g(x))' = \lim_{h\to 0} \frac{f(x+h) \pm g(x+h) - f(x) - g(x)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \pm \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) \pm g'(x)$$

3.) $$(f(x)\cdot g(x))' = \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x) - f(x)\cdot g(x+h) + f(x)\cdot g(x+h)}{h} =$$

$$= \lim_{h\to 0} \underbrace{\frac{f(x+h) - f(x)}{h}}_{f'(x)} \cdot g(x+h) + \lim_{h\to 0} \underbrace{\frac{g(x+h) - g(x)}{h}}_{g'(x)} \cdot f(x) =$$

$$= f'(x)\cdot g(x+0) + g'(x)\cdot f(x) = f'(x)\cdot g(x) + g'(x)\cdot f(x)$$

BUREK DZ:

1.) $f(x) = (3x^2+1)\sin x$

$f'(x) = 6x\cdot \sin x + (3x^2+1)\cdot \cos x$

$$\boxed{\begin{aligned} (\sin x)' &= \cos x \\ (\cos x)' &= -\sin x \end{aligned}}$$

2.) $f(x) = \frac{5x+1}{4\cos x}$

$$f'(x) = \frac{5\cdot 4\cos x + (5x+1)4\cdot \sin x}{16\cos^2 x} = \frac{20\cos x + 4(5x+1)\sin x}{16\cos^2 x}$$

$f(x)$	$f'(x)$
c	0
x	1
x^n	nx^{n-1}
e^x	e^x
a^x	$a^x \ln a$
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x \ln a}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\operatorname{tg} x$	$\frac{1}{\cos^2 x}$
$\operatorname{ctg} x$	$\frac{-1}{\sin^2 x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$
$\operatorname{arctg} x$	$\frac{1}{x^2+1}$
$\operatorname{arcctg} x$	$\frac{-1}{1+x^2}$
$\operatorname{sh} x$	$\operatorname{ch} x$
$\operatorname{ch} x$	$\operatorname{sh} x$
$\operatorname{th} x$	$\frac{1}{\operatorname{ch}^2 x}$
$\operatorname{cth} x$	$\frac{-1}{\operatorname{sh}^2 x}$
$\operatorname{arsh} x$	$\frac{1}{\sqrt{1+x^2}}$
$\operatorname{arch} x$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{arth} x$	$\frac{1}{1-x^2}$
$\operatorname{arcth} x$	$\frac{1}{1-x^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\frac{1}{x}$	$-\frac{1}{x^2}$

DOKAZI:

1) $f(x) = \sin x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0} \frac{2\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right)}{h} \overset{!!}{=} \boxed{\sin\alpha - \sin\beta}$$

$$= \lim_{h\to 0} \frac{\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right)}{\frac{h}{2}} = \cos(x+0) = \cos x$$

$\frac{\sin\frac{h}{2}}{\frac{h}{2}} \to 1$

2) $f(x) = \cos x$

$f'(x) =$ analogno

3) $f(x) = \sin(x^2) \Rightarrow f'(x) = \cos(x^2)\cdot 2x$

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h)^2 - \sin(x)^2}{h} = \left| \begin{array}{l} u = x^2 \\ \Delta u = (x+h)^2 - x^2 \\ \Delta u \to 0 \\ (x+h)^2 = x^2 + \Delta u = u + \Delta u \end{array} \right| =$$

$$= \lim_{\Delta u\to 0} \frac{\sin(u+\Delta u) - \sin u}{h} \cdot \underbrace{\frac{(x+h)^2 - x^2}{(x+h)^2 - x^2}}_{\Delta u} =$$

$$= \underbrace{\lim_{\Delta u\to 0} \frac{\sin(u+\Delta u) - \sin u}{\Delta u}}_{\cos u} \cdot \underbrace{\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}}_{2x} = \cos x^2 \cdot 2x$$

MOŽE DOĆI NA ISPITU!!

BUREK PREPORUKA! HE HE

TM Neka je $f\circ g$ definirana u točki x, te neka je g diferencijabilna u x, a f diferencijabilna u $g(x)$. Tada $f\circ g$ ima derivaciju u x i vrijedi:

$$\boxed{(f\circ g)'(x) = f'(g(x))\cdot g'(x)}$$

DOKAZ:

$$(f\circ g)'(x) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \cdot \frac{g(x+h)-g(x)}{g(x+h)-g(x)} = \left| \begin{array}{l} g(x+h) = u \\ g(x+h) - g(x) = \Delta u \\ \Delta u \to 0 \end{array} \right| =$$

Pretp. da postoji okoliš od x takav da za sve točke iz okoliša je $g(a) \neq g(x)$

$$= \lim_{\Delta u\to 0} \frac{f(u+\Delta u) - f(u)}{\Delta u} \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(u)\cdot g'(x) = f'(g(x+h))\cdot g'(x)$$

$\overset{\to 0}{h}$

TT

Primjedba! „LANČANO PRAVILO"

npr. $$(f\circ g\circ h)'(x) = f'(g(h(x)))\cdot g'(h(x))\cdot h'(x)$$!

Pr.) $$f(x) = (x^3+\pi)^{314} \longrightarrow f'(x) = 314\,(x^3+\pi)^{313}\cdot 3x^2$$

Pr.2) $$f(x) = \frac{1}{x^5+\sin(3x+14)} = -\frac{1}{(x^5+\sin(3x+14))^2}\cdot(5x^4+\cos(3x+14))\cdot 3$$

$$* \left(\frac{1}{x}\right)' = (x^{-1})' = -\frac{1}{x^2} *$$

- DOKAZ DERIVACIJE KVOCIJENTA:

NE ZABORAVI!

$$\left(\frac{f}{g}\right)' = \left(f\cdot\frac{1}{g}\right)' = f'\cdot\frac{1}{g} + f\cdot\left(\frac{1}{g}\right)' = f'\cdot\frac{1}{g} + f\cdot\frac{-1}{g^2}\cdot g' =$$

$$= \frac{f'g - fg'}{g^2}$$

- DERIVIRANJE INVERZNE FUNKCIJE:

$$(f\circ f^{-1})(y) = y \quad \Big|' \text{ (po } y)$$

$$f'(f^{-1}(y))\cdot(f^{-1}(y))' = 1$$

$$\boxed{(f^{-1}(y)) = \frac{1}{f'(f^{-1}(y))}}$$

$$y' = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{x'} \Longrightarrow x'(y) = \frac{1}{y'(x)} \qquad \begin{array}{l} y = f(x) \\ x = f^{-1}(y) \end{array}$$

3.) $f(x) = \arcsin x$

$$f'(x) = (\arcsin x)' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

arcsin x

$f(x) = \operatorname{tg} x = \frac{\sin x}{\cos x}$

$$f'(x) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} \quad //$$

$f(x) = \operatorname{arctg} x$

$$\left.\begin{array}{l} y = \operatorname{arctg} x \\ x = \operatorname{tg} y \end{array}\right\} (\operatorname{arctg} x)' = \frac{1}{\frac{1}{\cos^2 y}} = \frac{1}{\frac{\sin^2 y + \cos^2 y}{\cos^2 y}} = \frac{1}{\operatorname{tg}^2 y + 1} = \frac{1}{x^2+1} \quad //$$

$f(x) = \ln x$

$$f'(x) = \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h} = \lim_{h \to 0} \frac{\ln \frac{x+h}{x}}{h} = \lim_{h \to 0} \frac{\ln(1 + \overset{\to 0}{\frac{h}{x}}) \sim \frac{h}{x}}{h} = \lim_{h \to 0} \frac{\frac{\cancel{h}}{x}}{\frac{\cancel{h}}{1}} = \frac{1}{x} \quad //$$

$f(x) = e^x$

$$\left.\begin{array}{l} y = e^x \\ x = \ln y \end{array}\right\} (e^x)' = \frac{1}{\frac{1}{y}} = y = e^x$$

$f(x) = \log_a x = \frac{\ln x}{\ln a}$

$$f'(x) = \frac{1}{x \ln a} \quad //$$

$f(x) = \operatorname{sh} x$

$$(\operatorname{sh} x)' = \left(\frac{1}{2}(e^x - e^{-x})\right)' = \frac{1}{2}\left(e^x - e^{-x} \cdot (-x)'\right) = \frac{1}{2}(e^x + e^{-x}) = \operatorname{ch} x \quad //$$

!!

$f(x) = \operatorname{arsh} x$

$$\left.\begin{array}{l} y = \operatorname{arsh} x \\ x = \operatorname{sh} y \end{array}\right\} (\operatorname{arsh} x)' = \frac{1}{\operatorname{ch} y} = \frac{1}{\sqrt{1 + \operatorname{sh}^2 y}} = \frac{1}{\sqrt{1+x^2}} \quad //$$

• FUNKCIJA OPĆE POTENCIJE : $f(x) = x^c$, $c \in \mathbb{R}$

$$f(x) = e^{c \ln x}$$

$$f'(x) = e^{c \ln x} \cdot c \frac{1}{x} = x^c \cdot \frac{c}{x} = c \cdot x^{c-1}$$

2. MI-2010.-4.) $y = x \cdot \operatorname{arctg}(e^{-x^2})$

NE ZABORAVI !!!

$$y' = 1 \cdot \operatorname{arctg}(e^{-x^2}) + x \cdot \frac{1}{1+(e^{-x^2})^2} \cdot e^{-x^2} \cdot (-2x)$$

Zad.) $f(x) = \ln\left(\operatorname{sh}\left(\frac{\pi}{2}x\right)\right)$, $f''(x) = ?$

$$f'(x) = \frac{1}{\operatorname{sh}(\frac{\pi}{2}x)} \cdot \operatorname{ch}\left(\frac{\pi}{2}x\right) \cdot \frac{\pi}{2} = \frac{\pi}{2} \cdot \operatorname{cth}\left(\frac{\pi}{2}x\right)$$

$$f''(x) = \frac{\pi}{2} \cdot \frac{-1}{\operatorname{sh}^2(\frac{\pi}{2}x)} \cdot \frac{\pi}{2} = \frac{-\pi^2}{4 \operatorname{sh}^2(\frac{\pi}{2}x)}$$

$$f''(1) = \frac{-\pi^2}{4 \operatorname{sh}^2(\frac{\pi}{2})}$$ $\longrightarrow$ ako imaš $f''(1)$ PRVO DVODERIVIRAJ PA UVRSTI 1 !! !

8ZZV-25.) $f(x) = \ln(ax+b)$, $f^{(n)}(x) = ?$

$$f'(x) = \frac{1}{ax+b} \cdot a$$

$$f''(x) = \frac{-a}{(ax+b)^2} \cdot a = \frac{-a^2}{(ax+b)^2}$$ * prema $\frac{1}{x} = \frac{-1}{x^2} \cdot x'$

$$f'''(x) = \left\{\left(\frac{1}{x}\right)^2 = (x^{-1})^2 = x^{-2} = \frac{1}{x^2} = -2x^{-3} = \frac{-2}{x^3}\right\} = \frac{2a^3}{(ax+b)^3}$$

$1 = 1$
$2 = 1 \cdot 2$
$6 = 1 \cdot 2 \cdot 3$

$$f^{IV}(x) = \frac{-6a^4}{(ax+b)^4}$$

$$\boxed{f^{(n)}(x) = (-1)^{n+1} \cdot \frac{(n-1)!\, a^n}{(ax+b)^n}}$$

DOKAZATI IND. !

1. baza $n=1$

$$f'(x) = \frac{a}{(ax+b)} \quad \checkmark$$

2. pretpostavka da je $f^{(n)}(x) = (-1)^{n+1} \frac{(n-1)!\,a^n}{(ax+b)^n}$ za neki $n \in \mathbb{N}$

3. korak indukcije, za $n+1$

$$f^{(n+1)}(x) = \frac{(-1)^{n+1} \cdot (n-1)! \cdot a^n \cdot (-n) \cdot a}{(ax+b)^{n+1}} = \frac{(-1)^{n+2}\, n!\, a^{n+1}}{(ax+b)^{n+1}} \quad \Leftrightarrow$$

$$\overset{\text{pretp.}}{(f^{n}(x))'}$$

$$f(x) = \underbrace{(1+x)^{\sin 2x}}_{!!\ \text{UVIJEK}\ e^{y \ln x}\ !!} = e^{\sin 2x \ln(1+x)}$$

PAZI !!

UVIJEK !

$x^y = e^{y \ln x}$

$$f'(x) = e^{\sin 2x \ln(1+x)} \cdot \left[\cos 2x \cdot 2 \cdot \ln(1+x) + \sin 2x \cdot \frac{1}{1+x} \cdot 1\right]$$

AUDITORNE VJEŽBE

$$\boxed{(\sqrt{x})' = \frac{1}{2\sqrt{x}}}$$

1) $f(x) = \sqrt[3]{x} + \sqrt[4]{x} + \sqrt[6]{x}$

$f'(1) = ?$

PRVO DERIVIRATI PA UVRSTITI 1 !!

$$f'(x) = \left(x^{\frac{1}{3}}\right)' + \left(x^{\frac{1}{4}}\right)' + \left(x^{\frac{1}{6}}\right)' =$$

$$= \frac{1}{3} x^{-\frac{2}{3}} + \frac{1}{4} x^{-\frac{3}{4}} + \frac{1}{6} x^{-\frac{5}{6}}$$

$$f'(1) = \frac{1}{3} \cdot 1^{-\frac{2}{3}} + \frac{1}{4} \cdot 1^{-\frac{3}{4}} + \frac{1}{6} \cdot 1^{-\frac{5}{6}} = \frac{1}{3} + \frac{1}{4} + \frac{1}{6} = \frac{3}{4}$$

2) $f(x) = x \cdot \sin x + \operatorname{tg} x$

$$f'(x) = (x \sin x)' + (\operatorname{tg} x)' =$$

$$= 1 \cdot \sin x + x \cdot \cos x + \frac{1}{\cos^2 x}$$

3.) $f(x) = \dfrac{\log_{10} x}{x}$

$f'(x) = ?$

$f(x) = [\log_{10} x] \cdot x^{-1}$

$f'(x) = (\log_{10} x)' \cdot x^{-1} + \log_{10} x \cdot (x^{-1})' =$

$= \dfrac{1}{x \ln 10} \cdot \dfrac{1}{x} + \log_{10} x \cdot \left(-\dfrac{1}{x^2}\right)$

4.) $f(x) = \dfrac{x \operatorname{arctg} x}{x^2+1}$

$f'(1) = ?$

$f'(x) = \dfrac{(x \operatorname{arctg} x)' \cdot (x^2+1) - (x \cdot \operatorname{arctg} x) \cdot (x^2+1)'}{(x^2+1)^2} =$

$= \dfrac{\left(\operatorname{arctg} x + x \cdot \frac{1}{1+x^2}\right)(x^2+1) - 2x^2 \operatorname{arctg} x}{(x^2+1)^2} =$

$= \dfrac{(x^2+1)\operatorname{arctg} x + x - 2x^2 \operatorname{arctg} x}{(x^2+1)^2} =$

$= \dfrac{(1-x^2)\operatorname{arctg} x + x}{(x^2+1)^2}$

$f'(1) = \dfrac{1}{2^2} = \dfrac{1}{4}$ //

$f(x) = 2^{\frac{1}{x}}$

$f'(x) = ?$

$$\boxed{(a^x)' = a^x \ln a} = a^x \ln a \cdot (x)'$$

PAZI !!

$$f'(x) = 2^{\frac{1}{x}} \cdot \ln 2 \cdot \left(\frac{1}{x}\right)'$$

$$= 2^{\frac{1}{x}} \ln 2 \cdot \frac{-1}{x^2}$$

$$f'(x) = -\frac{2^{\frac{1}{x}}}{x^2} \cdot \ln 2$$

$f(x) = x \cdot e^{-x} \cdot \sin(\pi x)$

$$(f \cdot g \cdot h)' = (f \cdot g)' h + (f \cdot g) \cdot h'$$
$$= (f' \cdot g + f \cdot g') h + f \cdot g \cdot h'$$
$$= f' \cdot g \cdot h + f \cdot g' \cdot h + f \cdot g \cdot h'$$

$$= e^{-x} \cdot \sin(\pi x) + x \cdot e^{-x} \cdot \overset{-x}{(-x)} \cdot \sin(\pi x) +$$
$$+ x \cdot e^{-x} \cdot \cos(\pi x) \cdot \pi$$
$$= (e^{-x} - x e^{-x}) \cdot \sin(\pi x) + x \cdot e^{-x} \cdot \pi \cdot \cos(\pi x)$$

$$f(x) = \frac{x^2}{\sqrt{x^4+1}}$$

PAZI !

$$f'(x) = \frac{2x \cdot \sqrt{x^4+1} - x^2 \cdot \frac{1}{2} \cdot (x^4+1)^{-\frac{1}{2}} \cdot (x^4+1)'}{x^4+1} =$$

$$= \frac{2x \cdot \sqrt{x^4+1} - x^2 \cdot \frac{1}{\cancel{2}} \cdot \frac{1}{\sqrt{x^4+1}} \cdot \cancel{4}^{2} x^3}{x^4+1} = \frac{2x \cdot \sqrt{x^4+1} - \frac{2x^5}{\sqrt{x^4+1}}}{x^4+1}$$

$$= \frac{\frac{2x \cdot \sqrt{x^4+1} \cdot \sqrt{x^4+1} - 2x^5}{\sqrt{x^4+1}}}{x^4+1} = \frac{\frac{\overset{2x^5+2x}{2x \cdot (x^4+1)} \cancel{-2x^5}}{\sqrt{x^4+1}}}{\frac{x^4+1}{1}} = \frac{2x}{\sqrt{x^4+1} \cdot (x^4+1)} =$$

$$(x^4+1)^{\frac{1}{2}} \cdot (x^4+1)^1 = (x^4+1)^{\frac{3}{2}}$$

$$= \frac{2x}{\sqrt{(x^4+1)^2 (x^4+1)}} = \frac{2x}{\sqrt{(x^4+1)^3}}$$

8.) $f(x) = sh^4(10x)$

$$f'(x) = 4sh^3(10x) \cdot (sh(10x))' =$$
$$= 4sh^3(10x) \cdot (ch(10x) \cdot (10x)' \cdot (x'))$$
$$= 4sh^3 10x \cdot ch\, 10x \cdot 10$$
$$f'(x) = 40\, sh^3(10x) \cdot ch(10x) \quad //$$

9.) $f(x) = e^{-\cos^2(5x)}$

$$f'(x) = e^{-\cos^2(5x)} \cdot (-\cos^2(5x))' =$$
$$= -e^{-\cos^2(5x)} \cdot (2\cos(5x) \cdot (\cos(5x))' =$$
$$= -e^{-\cos^2(5x)} \cdot (2\cos(5x) \cdot (-\sin(5x) \cdot (5x)') =$$
$$= \boxed{2}\, e^{-\cos^2 5x} \cdot \boxed{\cos 5x \cdot \sin 5x} \cdot 5 =$$
$$= 5e^{-\cos^2 5x} \cdot \sin 10x \quad //$$

10.) $f(x) = \underbrace{x^x}_{g(x)} + \underbrace{\left(\frac{x}{x+1}\right)^x}_{h(x)}$

$f'(1) = ?$

$$f'(x) = g'(x) + h'(x)$$
$$f'(x) = x^x(\ln x + 1) + \left(\frac{x}{x+1}\right)^x \left(\ln\frac{x}{x+1} + \frac{1}{x+1}\right)$$
$$f'(1) = 1 + \frac{1}{2}\left(\ln\frac{1}{2} + \frac{1}{2}\right)$$
$$f'(1) = 1 + \frac{1}{2}\ln\frac{1}{2} + \frac{1}{4} =$$
$$= \frac{5}{4} + \frac{1}{2}\ln\frac{1}{2}$$

$$g(x) = x^x$$
$$g(x) = e^{x\ln x} \quad /'$$
$$g'(x) = e^{x\ln x} \cdot (x\ln x)' =$$
$$= e^{x\ln x} \cdot \left(\ln x + x \cdot \frac{1}{x}\right) =$$
$$= e^{x\ln x} \cdot (\ln x + 1) = x^x \cdot (\ln x + 1) \quad //$$

$$h(x) = e^{x\ln\frac{x}{x+1}} \quad /'$$
$$h'(x) = e^{x\ln\frac{x}{x+1}} \cdot \left(x\ln\frac{x}{x+1}\right)' =$$
$$= e^{x\ln\frac{x}{x+1}} \cdot \left(\ln\frac{x}{x+1} + x \cdot \left(\frac{1}{\frac{x}{x+1}}\right) \cdot \left(\frac{x}{x+1}\right)'\right)$$
$$= e^{x\ln\frac{x}{x+1}} \cdot \left(\ln\frac{x}{x+1} + (x+1) \cdot \frac{x+1-x}{(x+1)^2}\right) =$$
$$= e^{x\ln\frac{x}{x+1}} \cdot \left(\ln\frac{x}{x+1} + \frac{1}{x+1}\right) = \left(\frac{x}{x+1}\right)^x \cdot (\ln$$

D.Z.

$f(-2) = ?$

$f(x) = \arcsin^2\left(\frac{1}{x}\right)$

$f(x)' = x \cdot \ln(x \cdot \mathrm{ch}\, x)$

$f'' = ?$

$f(x) = \cos(\mathrm{arctg}\, x)$

DERIVIRAJ SRCE:

$x = 16\sin^3 t$

$y = 13\cos t - 5\cos 2t - 2\cos 3t - \cos 4t$

$\heartsuit' = ?$ u $\tilde{\pi}$

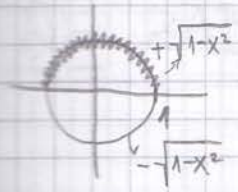
8.3. IMPLICITNO I PARAMETARSKI ZADANE FUNKCIJE

$f(x,y) = 0$

$x^2 + y^2 = 1 \longrightarrow$ kružnica

$\longrightarrow y^2 = 1 - x^2$

$y = \pm\sqrt{1-x^2}$



IMPLICITNO DERIVIRANJE:

$x^2 + y^2 = 1 \quad |'$

$2x + 2y \cdot y' = 0 \longrightarrow$ svugdje gdje je y još y' !!!

$$\boxed{y' = \frac{-x}{y}}$$

SVA MUDROST:

16.) $x \cdot y^2 + e^{x+y} = 2 \quad |'$

y', y'' u $T(1,-1) = ?$

$1 \cdot y^2 + x \cdot 2y \cdot y' + e^{x+y} \cdot (1 + y') = 0$

$y'(x \cdot 2y + e^{x+y}) = -y^2 - e^{x+y}$

$$y' = \frac{-y^2 - e^{x+y}}{2xy + e^{x+y}}$$

$$y'\Big|_{T(1,-1)} = 2$$

uvrstiti

kao umnožak

$$y'' = \frac{\left(-2y\cdot y' - e^{x+y}\cdot(1+y')\right)\cdot\left(2xy+e^{x+y}\right) - \left(-y^2-e^{x+y}\right)\cdot\left(2y+2xy'+e^{x+y}(1+y')\right)}{\left(2xy+e^{x+y}\right)^2}$$

uvrstiti

$$y''\Big|_{T(1,-1)} = 9$$

2.MI-2008-6.)

$$x^y + e^{x-1} = \sqrt{y} + 1 \quad /' \qquad x^y = e^{y\ln x}$$

$$e^{y\ln x}\cdot\left(y'\cdot\ln x + y\cdot\frac{1}{x}\right) + e^{x-1}\cdot 1 = \frac{1}{2\sqrt{y}}\cdot y'$$

$$y'\left(x^y\ln x - \frac{1}{2\sqrt{y}}\right) = -e^{x-1} - x^y\cdot\frac{y}{x}$$

$$y' = \frac{-e^{x-1} - x^y\cdot\frac{y}{x}}{x^y\ln x - \frac{1}{2\sqrt{y}}}$$

$$y'\Big|_{T(1,1)} = 4$$

* PARAMETARSKI ZADANE FUNKCIJE:

10.

$$\left.\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}\right\}$$

$$y' = \frac{dy}{dx}\cdot\frac{dt}{dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}}$$

$$\boxed{y' = \frac{\dot{y}}{\dot{x}}}$$

$\dot{y}$ = derivacija y

$$y'' = \frac{dy'}{dx}\cdot\frac{dt}{dt} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{\ddot{y}-\dot{y}\ddot{x}}{(\dot{x})^2}}{\dot{x}} = \frac{\ddot{y}\dot{x}-\dot{y}\ddot{x}}{(\dot{x})^3}$$

PAZI

$$\boxed{y'' = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{(\dot{x})^3}}$$

Zad-zad-7.)

$$x = t + \ln t$$
$$y = t^2 + \ln 2t \qquad y', y'' \text{ za } t=1 \rightarrow ?$$

$$y' = \frac{2t + \frac{1}{2t}\cdot 2}{1+\frac{1}{t}} = \frac{2t^2+1}{t+1} \qquad \text{za } t=1 \rightarrow y'\Big|_{t=1} = \frac{3}{2} \;//$$

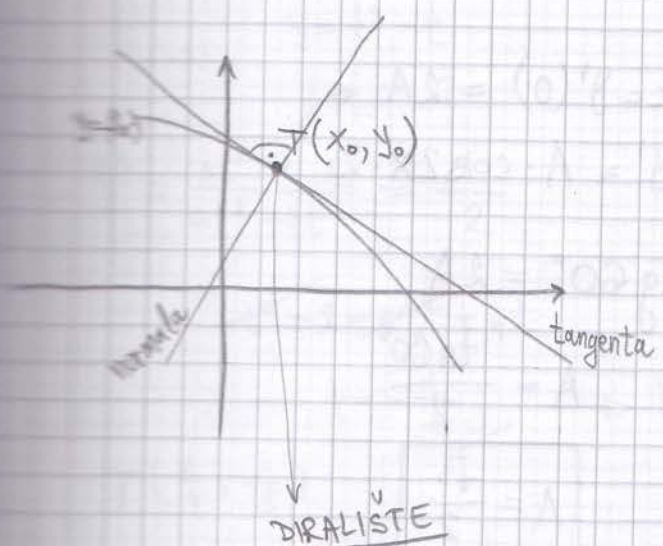
$y'' =$ (nemo preko formule sa točkicama, nego još jednom y')

$$y'' = \frac{\frac{4t(t+1) - (2t^2+1)}{(t+1)^2}}{1+\frac{1}{t}}$$

$1+\frac{1}{t} \rightarrow \dot{x} = x$ derivirano !! PAZI !!

NE ZABORAVI !!

8.4. TANGENTA I NORMALA



$$\boxed{k = f'(x_0)}$$

k..... koeficijent smjera tangente u točki x_0

$$y - y_0 = k(x - x_0)$$

$$\boxed{y - y_0 = f'(x_0)(x - x_0)} \quad \text{.... tangenta}$$

$$\boxed{y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)} \quad \text{..... normala}$$

8.DZ.-9.) $y = x \cdot \sqrt[3]{x-1}$

Odredi tangentu u točki s apcisom u: a) 0 b) $\frac{3}{4}$ c) 1

$T(0,0)$ $T\left(\frac{3}{4}, -\frac{3}{4\sqrt[3]{4}}\right)$ $T(1,0)$

$y' = 1 \cdot \sqrt[3]{x-1} + x \cdot \frac{1}{3}(x-1)^{-\frac{2}{3}} \cdot 1$

a) za $x=0$

$y - 0 = -1 \cdot (x - 0)$

$\boxed{y = -x}$

$T(0,0)$, $y' = \sqrt[3]{x-1} + x \cdot (x-1)^{-\frac{2}{3}}$

b) $T\left(\frac{3}{4}, -\frac{3}{4\sqrt[3]{4}}\right)$ → uvrštavamo u y'

$y + \frac{3}{4\sqrt[3]{4}} = 0 \cdot \left(x - \frac{3}{4}\right)$

$\boxed{y = -\frac{3}{4\sqrt[3]{4}}}$

c) $T(1,0)$

$y' = \frac{1}{\sqrt[3]{(x-1)^3}} = \frac{1}{0} = \infty \rightarrow k = "\infty"$

$y - 0 = "\infty" \cdot (x-1)$

$\Rightarrow \boxed{x = 1}$

2.MI-06-5.) $y = A \sin 2x$, $A = ?$

$\alpha = 60°$

$k = y'(0) = 2A$

$y' = A \cdot \cos 2x \cdot 2$

$\operatorname{tg} 60° = 2A$

$A = \frac{\operatorname{tg} 60°}{2}$

$\boxed{A = \frac{\sqrt{3}}{2}}$

2012-R.) $a = ?$ da se diraju $f_1(x) = \ln x$ i $f_2(x) = ax^2$

→ moraju imati isti koeficijent smjera

→ diralište: $T(x_0, y_0)$, parametar $a = ?$

→ $\underline{\ln x_0 = a x_0^2}$

$$\underbrace{f_1'(x) = \frac{1}{x} \qquad f_2'(x) = a \cdot 2x}$$

$\frac{1}{x_0} = 2a x_0$ → koef. isti u toj točki x_0

→ $a = \frac{1}{2x_0^2}$

→ $\ln x_0 = \frac{1}{2\cancel{x_0^2}} \cdot \cancel{x_0^2}$

$\ln x_0 = \frac{1}{2}$

$x_0 = e^{\frac{1}{2}}$ → $x_0 = \sqrt{e}$ → $\boxed{a = \frac{1}{2e}}$

2011-2008-5.)

$$\left.\begin{aligned} x &= 2t^3 + 2t^2 \\ y &= t^3 - t \end{aligned}\right\} \quad t \parallel p \ldots\ldots x - 2y - 1 = 0$$

$t \ldots ?$

$-2y = 1 - x$

$y = \frac{x}{2} - \frac{1}{2}$ → $k = \frac{1}{2}$

$$y' = \frac{3t^2 - 1}{6t^2 + 4t} = \frac{1}{2}$$

$\cancel{6t^2} - 2 = \cancel{6t^2} + 4t$

$t = -\frac{1}{2}$ → koordinate dirališta: $x = 2 \cdot \left(-\frac{1}{2}\right)^3 + 2 \cdot \left(-\frac{1}{2}\right)^2$

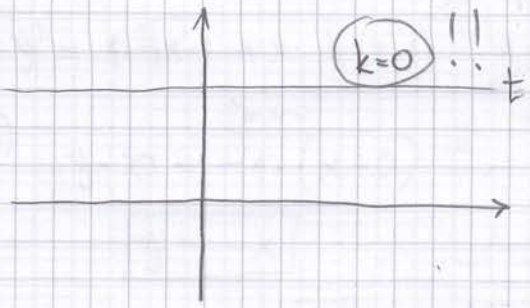
$y = \left(-\frac{1}{2}\right)^3 + \frac{1}{2}$

$T\left(\frac{1}{4}, \frac{3}{8}\right)$

$y - \frac{3}{8} = \frac{1}{2}\left(x - \frac{1}{4}\right)$

$$\boxed{y = \frac{1}{2}x + \frac{1}{4}}$$

2.MI-2010-5.) Odredi sve točke na krivulji $3x^2y-2y^3+2x^3=2$ u kojima je tangenta okomita na y os.



$\underline{k=0} \rightarrow k=f'(x)$

$3x^2y-2y^3+2x^3=2 \quad /'$

$6xy+3x^2y'-6y^2\cdot y'+6x^2=0$

$y'(3x^2-6y^2)=-6xy-6x^2$

$$y'=\frac{-6x^2-6xy}{3x^2-6y^2}=0$$

$-6x^2-6xy=0 \quad /:(6)$

$x^2+xy=0$

$x(x+y)=0$

$\boxed{x=0} \quad \boxed{y=-x}$

I $\underline{x=0}$

$0-2y^3+0=2$

$y^3=-1$

$\boxed{y=-1}$

$\boxed{T_1(0,-1)}$

II $y=-x$

$-3x^3+2x^3+2x^3=2$

$x^3=2$

$x=\sqrt[3]{2} \rightarrow y=-\sqrt[3]{2}$

$\boxed{T_2(\sqrt[3]{2},-\sqrt[3]{2})}$

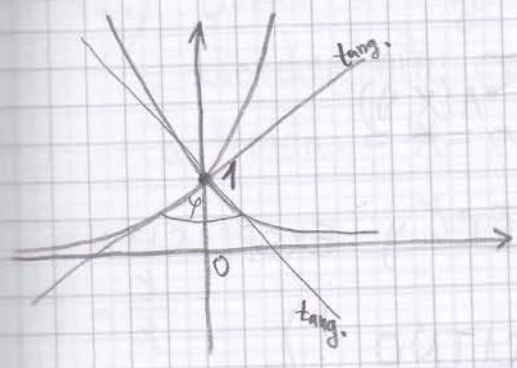
NEĆE DOĆI LAKO !!!

8.DZ-20.)

$y = e^{2x}$

$y = e^{-3x}$

kut između tih krivulja?



$D(0,1)$ $\quad e^{2x} = e^{-3x} \quad /\ln$

$2x = -3x$

$\boxed{x=0} \rightarrow \boxed{y=1}$ (sa slike

→ kut između krivulja je kut između tangenti u točki u kojoj se sijeku !!!

$y' = e^{2x} \cdot 2$

$y' = e^{-3x} \cdot (-3)$ } u točki $D(0,1)$ → $k_2 = 2$, $k_1 = -3$ za k_2 uvijek veći k

$$\boxed{tg\,\varphi = \frac{k_2 - k_1}{1 + k_1 k_2}}$$

$$tg\,\varphi = \frac{2+3}{1-6} = -1 \longrightarrow \boxed{\varphi = \frac{3\pi}{4}}$$

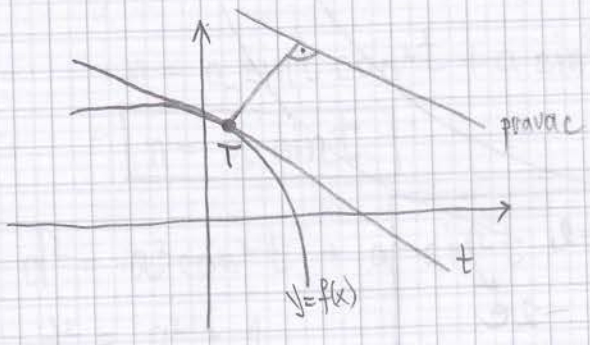
→ UVIJEK NA $[0, \pi]$

8.DZ-14.) Odredi točku na paraboli $y = x^2 + x - 2$ koja je najbliže pravcu $y = 5x - 10$

$y' = 2x + 1 = 5$

$\boxed{x=2} \rightarrow \boxed{y=4}$

$T(2,4)$



udaljenost dvije točke:

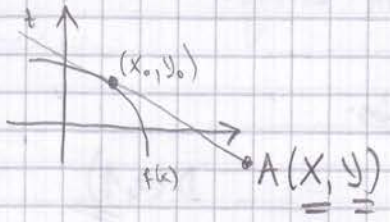
$$\boxed{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

→ da nas je tražio točku na pravcu

– odredili bi normalu i našli presjek normale i pravca

Zad.) Iz točke $A(-1,0)$ povučene su tangente na krivulju $2\ln\frac{y}{x}+x=0$. Odredi njihove jednadžbe.

→ to nije točka dirališta !



→ PAZI !! UVIJEK POGLEDAJ MOŽEŠ LI y EKSPLICITNO IZVUĆI !! (IAKO IZGLEDA IMPLICITNO !)

$$2\ln\frac{y}{x}+x=0$$

$$\ln\frac{y}{x}=-\frac{x}{2} \quad \Big| \, e$$

$$\frac{y}{x}=e^{-\frac{x}{2}}$$

$A(-1,0)$ $\qquad y=x\cdot e^{-\frac{x}{2}} \longrightarrow \underbrace{y_0=x_0 e^{-\frac{x_0}{2}}}$

$$y'=e^{-\frac{x}{2}}+x\cdot e^{-\frac{x}{2}}\cdot\frac{-1}{2}=e^{-\frac{x}{2}}\left(1-\frac{x}{2}\right)$$

$$t\ldots \quad \underline{\underline{y}}-y_0=e^{-\frac{x_0}{2}}\left(1-\frac{x_0}{2}\right)(\underline{\underline{x}}-x_0)$$

UVRŠTAVAMO y, x, a ne x_0, y_0

$$-y_0=e^{-\frac{x_0}{2}}\left(1-\frac{x_0}{2}\right)(-1-x_0)$$

$$x_0\cancel{e^{-\frac{x_0}{2}}}=\cancel{e^{-\frac{x_0}{2}}}\left(1-\frac{x_0}{2}\right)(-1-x_0)$$

$$x_0=1+x_0-\frac{x_0}{2}-\frac{x_0^2}{2}$$

$$x_0^2+x_0-2=0$$

$$\boxed{\begin{aligned}x_1&=1\\ y_1&=e^{-\frac{1}{2}}\end{aligned}} \qquad \boxed{\begin{aligned}x_2&=-2\\ y_2&=-2e\end{aligned}}$$

$$y - e^{-\frac{1}{2}} = e^{-\frac{1}{2}} \cdot \frac{1}{2}(x-1)$$

$$y + 2e = e \cdot 2(x+2)$$

12.) Dokaži da osječak tangente na astroidu između koordinatnih osi ima konstantnu duljinu.

astroida: $x = a\cos^3 t$

$y = a\sin^3 t$

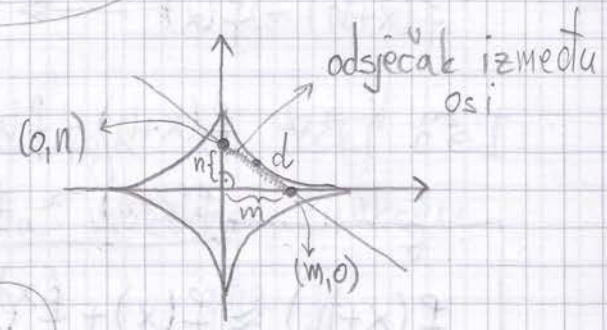
$$y' = \frac{\dot{y}}{\dot{x}}$$

$$y' = \frac{\cancel{a} \cdot \cancel{3}\sin^2 t \cdot \cancel{\cos t}}{\cancel{a} \cdot \cancel{3}\cos^2 t \cdot (-\cancel{\sin t})} = -\frac{\sin t}{\cos t} = -\operatorname{tg} t$$

$$y - y_0 = -\operatorname{tg} t\,(x - x_0)$$

točke na krivulji (diralište)

$$y - a\sin^3 t = -\operatorname{tg} t\,(x - a\cos^3 t)$$



za $(m,0)$:

$$0 - a\sin^3 t = -\operatorname{tg} t\,(m - a\cos^3 t) \quad / : \operatorname{tg} t \quad a\cos^3 t$$

$$d^2 = n^2 + m^2$$

$$\boxed{m = a\cos t}$$

za $(0,n)$:

$$n - a\sin^3 t = -\operatorname{tg} t\,(0 - a\cos^3 t)$$

$$n = a\sin t\cos^2 t + a\sin^3 t = a\sin t\,(\underbrace{\cos^2 t + \sin^2 t}_{1})$$

$$\boxed{n = a\sin t}$$

$$d^2 = a^2\cos^2 t + a^2\sin^2 t$$

$$d^2 = a^2 \cdot 1$$

$\boxed{d = a}$ $\longrightarrow$ ne ovisi o koordinati dirališta!

9. PRIMJENE DIFERENCIJALNOG RAČUNA

9.1. Aproksimacija Taylorovom formulom

2.MI-2009-6.) a) Odredi aproksimaciju broja $\sqrt{35}$ koristeći se diferencijalom.

$\sqrt{35} = ?$

$\sqrt{35} = 5.91608\ldots$

PODSJETIMO SE:

$$\boxed{df(x) = f'(x) \cdot dx}$$

$f(x+h) - f(x) \longrightarrow$ PRIRAST FUNKCIJE

za $h \to 0$ $\quad f(x+h) - f(x) \approx df(x)$

$f(x+h) \approx f(x) + f'(x) \cdot h$

$$\boxed{f(x+h) \approx f(x) + f'(x) \cdot h}$$

x..... najbliži br. kojem možemo izvaditi $\sqrt{\ }$

$x = 36$

$h = -1 \rightarrow$ razlika do "pravog" broja $36 \rightarrow 35 \longrightarrow \underline{-1 = h}$

$f(x) = \sqrt{x} \longrightarrow f'(x) = \frac{1}{2\sqrt{x}}$

$\sqrt{35} = f(36-1) \approx f(36) + f'(36) \cdot (-1) \approx 6 - \frac{1}{12} \approx 5{,}9166 \rightarrow$ točno na 3 decimale

$\rightarrow$ bez kalka

$$f(x) \approx f(c) + f'(c)\cdot(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$\longrightarrow$ TAYLOROV POLINOM n-tog STUPNJA OKO TOČKE $x=c$.

$\sqrt{35}$, aproksimiraj Taylorovim polinomom 2. stupnja!

$c=36$,

$$f''(x) = \frac{1}{2\sqrt{x}} \longrightarrow f''(x) = \left(\frac{1}{2}x^{-\frac{1}{2}}\right)' = -\frac{1}{4}x^{-\frac{3}{2}}$$

$$\sqrt{35} \approx 6 + \frac{1}{12}\cdot(-1) + \frac{-\frac{1}{4\cdot 6^3}}{2!}\cdot(-1)^2$$ $\longrightarrow$ do tuda u ispitu

$\approx 5{,}91608$ $\longrightarrow$ točno na 5 decimala

TM „TAYLOROV TEOREM"

Neka je zadana funkcija $f: I \to \mathbb{R}$ koja u točki $c \in I$ ima sve derivacije do reda $n+1$. Tada možemo funkciju zapisati kao $\boxed{f(x) = T_n(x) + R_n(x)}$ gdje je

$$\boxed{T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k}$$ $\longrightarrow$ TAYLOROV polinom n-tog stup.

$$\boxed{R_n(x) = \frac{f^{(n+1)}(x_1)}{(n+1)!}(x-c)^{n+1}}$$ $x_1 \in (x,c)$ ili (c,x)

$\longrightarrow$ OSTATAK u LANGRANGOVOM OBLIKU

c.... točka oko koje se razvija

2.M1-2009-7.) Napiši $f(x)=e^x$ u obliku Taylorovog polinoma oko točke $x=0$, pritom dokaži da je $\lim_{n\to\infty} R_n(x)=0$.

$\boxed{c=0}$ $f(x)=e^x \longrightarrow f(0)=1$

$f'(x)=e^x \longrightarrow f'(0)=1$

$f''(x)=e^x \longrightarrow f''(0)=1$

$\vdots$

$f^{(n)}(x)=e^x \qquad f^{(n)}(0)=1$

$$T_n(x)=\sum_{k=0}^{n}\frac{x^k}{k!} \rightarrow c=0$$

dokaži: $\lim_{n\to\infty} R_n(x)=0$

$$R_n(x)=\frac{e^{x_1}}{(n+1)!}\cdot x^{n+1} \quad \text{ili } x_1\in(x,0) \text{ ili } x_1\in(0,x)$$

$$\lim_{n\to\infty}\frac{e^{x_1}\cdot x^{n+1}}{(n+1)!}=0$$ jer faktorijele brže teže u ∞

def. TAYLOROV polinom oko točke $x=0$ se zove MACLAURINOV POLINOM.

Zad.) a) $f(x)=\sin x$, razvij oko $x=0$.

$f(x)=\sin x$

$f'(x)=\cos x$

$f''(x)=-\sin x$

$f'''(x)=-\cos x$

$f^{IV}(x)=\sin x$

$\vdots$

$f^{4k}(x)=\sin x \longrightarrow$ svaka 4. je $\sin x \longrightarrow f^{4k}(0)=0$

$f^{4k+1}(x)=\cos x \longrightarrow f^{4k+1}(0)=1$

$f^{4k+2}(x)=-\sin x \longrightarrow f^{4k+2}(0)=0$

$f^{4k+3}(x)=-\cos x \rightarrow f^{4k+3}(0)=-1$ ne treba mat. ind.

→ kako se mijenjaju predznaci 1 ili −1

$$T_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

samo neparni članovi "preživi" → $f^{(2k+1)} = 1$ ili -1, ostali $\underline{\underline{0}}$!!

$$T_n(x) = \underset{k=0}{x} - \underset{k=1}{\frac{x^3}{3!}} + \underset{k=2}{\frac{x^5}{5!}} - \underset{k=3}{\frac{x^7}{7!}} + \dots$$

$$R_{2n+1}(x) = \frac{f^{(2n+2)}(x_1)}{(2n+2)!} \cdot x^{2n+2}$$

zbog pot. kada uvrstiš $\underline{\underline{k=n}}$

razvoj $\cos x$

$$\cos x = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} + R_{2n}(x)$$ } D.Z.

koji stupanj T_n treba uzeti da bi $\sin x \in \langle 0, \frac{\pi}{4} \rangle$ izračunali s točnošću 10^{-3} ?

$0 \to c$, $\frac{\pi}{4} \to x = \frac{\pi}{4}$

$|R_{2n+1}(x)| \leq 10^{-3}$ → jel greška može biti $\pm$

$$\left| \frac{\overbrace{f^{(2n+2)}(x_1)}^{<1}}{(2n+2)!} \cdot \left(\frac{\pi}{4}\right)^{2n+2} \right| \leq \frac{\left(\frac{\pi}{4}\right)^{2n+2}}{(2n+2)!} \leq 10^{-3}$$

uvrštavati : $n=0$ → nije $\leq 10^{-3}$

$n=1$ → nije $\leq 10^{-3}$

$\boxed{n=2}$ → JE $\leq 10^{-3}$ ✓

dakle : $\boxed{T_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}}$ nam daje $\sin x$ aproksimativno s točnošću 10^{-3}

9. DZ-10.) Prikaži $f(x) = x^3 + 2x^2 - x + 1$ kao polinom u varijabli $(x-1)$.

!! $\boxed{c=1}$

$\boxed{c=1}$

$$f'(x) = 3x^2 + 4x - 1 \rightarrow f'(1) = 6$$
$$f''(x) = 6x + 4 \rightarrow f''(1) = 10$$
$$f'''(x) = 6 \rightarrow f'''(1) = 6$$

$$f(x) = 3 + \frac{6}{1!}(x-1) + \frac{10}{2!}(x-1)^2 + \frac{6}{3!}(x-3)$$

$$\boxed{f(x) = 3 + 6(x-1) + 5(x-1)^2 + (x-1)^3}$$

9. ZZV-2.) $f(x) = \ln\frac{1+x}{1-x}$ napiši u formi $T_6(x) + R_6(x)$ oko $c = 0$.

!! $f(x) = \ln(1+x) - \ln(1-x)$

$\downarrow$

$$f'(x) = \frac{1}{1+x} - \frac{1}{1-x}\cdot(-1) = \frac{1}{1+x} + \frac{1}{1-x} \rightarrow 2$$

$$f''(x) = \frac{-1}{(x+1)^2} + \frac{1}{(1-x)^2} \rightarrow 0$$

$$f'''(x) = \frac{2}{(x+1)^3} + \frac{2}{(1-x)^3} \rightarrow 4$$

$$f^{IV}(x) = \frac{-6}{(1+x)^4} + \frac{6}{(1-x)^4} \rightarrow 0$$

$$f^{V}(x) = \frac{24}{(1+x)^5} + \frac{24}{(1-x)^5} \rightarrow 48$$

$$f^{VI}(x) = \frac{-120}{(1+x)^6} + \frac{120}{(1-x)^6} \rightarrow 0$$

$$\ln\frac{1+x}{1-x} = 2x + \frac{4}{3!}x^3 + \frac{48}{5!}x^5 + R_6(x)$$

$$= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + R_6(x)$$

još jednom derivirati $f^{VI}(x)$

$$R_6(x) = \left(\frac{720}{(1+x_1)^7} + \frac{720}{(1-x_1)^7}\right)\frac{x^7}{7!}$$

9.2. L'HOSPITALOVO PRAVILO

„L'Hospitalov teorem"

Neka su f i g diferencijabilne na (a,b), $g'(x) \neq 0$. Ako za $x_0 \in (a,b)$ vrijedi $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = 0$ ili $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = \pm\infty$ i ako postoji $\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)}$, tada je $\boxed{\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)} = \lim\limits_{x\to x_0} \frac{f(x)}{g(x)}}$.

NAPOMENA: Teorem vrijedi i kada $x \to \infty$!

ZI-6.) $$\lim_{x\to\frac{\pi}{4}} \frac{\ln \operatorname{tg} x}{\cos 2x} = \left(\frac{0}{0}\right) \overset{\text{L'Hospital}}{=} \lim_{x\to\frac{\pi}{4}} \frac{\frac{1}{\operatorname{tg} x}\cdot\frac{1}{\cos^2 x}}{-\sin 2x\cdot 2} = \frac{1\cdot\frac{1}{\frac{2}{4}}}{-2} = -1$$

LM-09-P.) $$\lim_{x\to\frac{\pi}{2}} \frac{\operatorname{tg} 5x}{\operatorname{tg} 3x} =$$ PAZI ! kada $x \to 0$!! onda možemo $\frac{\sim 5x}{\sim 3x}$

$$= \left(\frac{\infty}{\infty}\right) \overset{L'H.}{=} \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\cos^2 5x}\cdot 5}{\frac{1}{\cos^2 3x}\cdot 3} = \lim_{x\to\frac{\pi}{2}} \frac{\cos^2 3x}{\cos^2 5x}\cdot\frac{5}{3} = \frac{5}{3}\cdot\left(\frac{0}{0}\right)$$ JOŠ JEDNOM L'H !

$$= \frac{\cancel{5}}{\cancel{3}}\cdot\lim_{x\to\frac{\pi}{2}} \frac{\cancel{2}\cos 3x\cdot(-\sin 3x)\cdot\cancel{3}}{\cancel{2}\cos 5x\cdot(-\sin 5x)\cdot\cancel{5}} = \left(\frac{0}{0}\right) =$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos 3x(-\sin 3x)}{\cos 5x(-\sin 5x)} = \lim_{x\to\frac{\pi}{2}} \frac{-\sin 3x\cdot 3\cdot(-\sin 3x) + \cos 3x\cdot(-\cos 3x)\cdot 3}{-\sin 5x\cdot 5\,(-\sin 5x) + \cos 5x\cdot(-\cos 5x)\cdot 5} = \frac{3}{5}$$

ZZV-4.) $$\lim_{x\to\infty} \frac{\ln^n x}{x} = \left(\frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to\infty} \frac{n\ln^{n-1} x\cdot\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{n\ln^{n-1} x}{x} = \lim_{x\to\infty} \frac{n(n-1)\ln^{n-2} x}{x}$$

$$= \dots = \lim_{x\to\infty} \frac{\sim n!}{x} = 0$$

2.MI-09-8.) $\lim\limits_{x\to\infty} x\cdot \operatorname{arcctg}(2x) = (\infty\cdot 0)$!! NE MOŽE ODMAH L'H. !

$$\lim_{x\to\infty}\frac{\operatorname{arcctg}(2x)}{\frac{1}{x}} = \left(\frac{0}{0}\right) \overset{L'H}{=} \text{DA}! = \lim_{x\to\infty}\frac{\frac{-1}{1+4x^2}\cdot 2}{-\frac{1}{x^2}} =$$

$$= \lim_{x\to\infty}\frac{2x^2}{1+4x^2} = \frac{\sim 2x^2}{\sim 4x^2} = \frac{1}{2}$$

9.DZ-15.) $\lim\limits_{x\to-\infty}\left(x e^{-\frac{1}{x^2}} - x\right) = (-\infty+\infty)$ = NE MOŽE ODMAH L'H

NAMJESTITI NA RAZLOMAK !

$$= \lim_{x\to-\infty}\left(x\cdot\left(e^{-\frac{1}{x^2}}-1\right)\right) = \lim_{x\to-\infty}\frac{e^{-\frac{1}{x^2}}-1}{\frac{1}{x}} = \left(\frac{0}{0}\right) \overset{L'H.}{=} \lim_{x\to-\infty}\frac{e^{-\frac{1}{x^2}}\cdot\left(-\frac{-2}{x^3}\right)}{-\frac{1}{x^2}} =$$

$$= \lim_{x\to-\infty}\frac{2e^{-\frac{1}{x^2}}}{-x} = \frac{2}{\infty} = 0$$

9.DZ-14.) $$\lim_{x\to-\infty}\frac{\ln \operatorname{ch}(x+3)}{x} = \left(\frac{\infty}{-\infty}\right) \overset{L'H.}{=} \lim_{x\to-\infty}\frac{\frac{1}{\operatorname{ch}(x+3)}\cdot \operatorname{sh}(x+3)\cdot 1}{1} =$$

$$= \lim_{x\to-\infty}\frac{\operatorname{sh}(x+3)}{\operatorname{ch}(x+3)} = \lim_{x\to-\infty}\operatorname{th}(x+3) = \operatorname{th}(-\infty) = -1$$

$$\left(\frac{\infty}{\infty}\right) \overset{L'H.}{=} \lim_{x\to-\infty}\frac{\operatorname{ch}(x+3)}{\operatorname{sh}(x+3)} = \left(\frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to-\infty}\frac{\operatorname{sh}(x+3)}{\operatorname{ch}(x+3)}$$

L'H. NE MORA UVIJEK DOVESTI DO RJEŠENJA !!

2.MI-10.-7.) U ovisnosti o parametru izračunaj:

$$\lim_{x\to 0^+}\frac{x\cdot\cos x + a\sin x}{x^3} = \left(\frac{0}{0}\right) \overset{L'H.}{=} \lim_{x\to 0^+}\frac{\cos x - x\sin x + a\cos x}{3x^2} =$$

$\cos x(1+a)$

(I) za $a=-1$:

$$\lim_{x\to 0^+}\frac{-x\sin x}{3x^2} = -\frac{1}{3}$$

I za $a > -1$:

$$\lim_{x\to 0^+} \frac{\cos\overbrace{(a+1)}^{>0}}{3x^2} = +\infty$$

II za $a < -1$:

$$-\infty$$

-11.) $\lim_{x\to\infty} \frac{x+\sin x}{x+\cos x}$ može li se upotrijebiti L'Hospital ?

→ $\sin\infty$ i $\cos\infty$ NE POSTOJE ⟹ LIMES NE POSTOJI

→ L'Hospital: „Ako postoji limes..." → NE MOŽEMO L'H.

~ sjedno izračunajte limes:

$$\lim_{x\to\infty} \frac{x+\sin x}{x+\cos x} \cdot \frac{x}{x} = \lim_{x\to\infty} \frac{1+\overbrace{\frac{\sin x}{x}}^{\to 0}}{1+\underbrace{\frac{\cos x}{x}}_{\to 0}} = 1$$

-16.) $\lim_{x\to\infty} x^{\frac{1}{x}} = (\infty^0) =$

$$= \lim_{x\to\infty} e^{\frac{1}{x}\ln x} = e^{\lim_{x\to\infty}\frac{1}{x}\ln x} \overset{L'H}{=} e^{\lim_{x\to\infty}\frac{\frac{1}{x}}{1}} = e^0 = 1$$

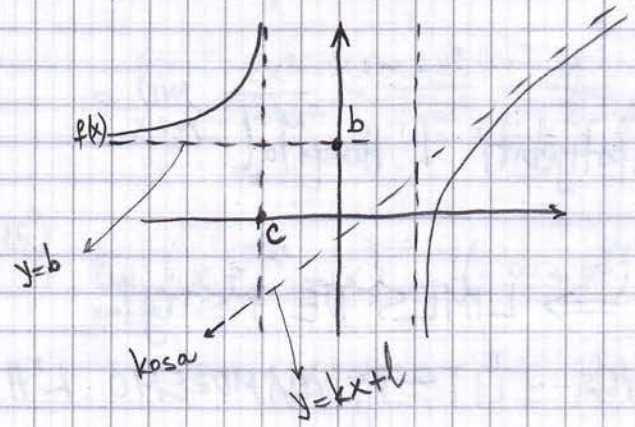
-06-10.) $\lim_{x\to\pi} \cos(\pi-x)^{\frac{1}{(\pi-x)^2}} = (1^\infty) =$

$$= e^{\lim_{x\to\pi}\frac{1}{(\pi-x)^2}\ln\cos(\pi-x)} \overset{L'H}{=} e^{\lim_{x\to\pi}\frac{\frac{1}{\cos(\pi-x)}\cdot(-\sin(\pi-x))\cdot\cancel{(-1)}}{(2\pi-2x)\cdot\cancel{(-1)}}} =$$

$$= e^{\lim_{x\to\pi}\frac{-\overbrace{tg(\pi-x)}^{\to 0}}{2\pi-2x}} = \frac{-\overbrace{(\pi-x)}^{\sim\pi-x}}{2(\pi-x)} = e^{-\frac{1}{2}}$$

9.3 ASIMPTOTE

→ pravac čija udaljenost od funkcije teži nuli



def. Pravac $x=c$ je VERTIKALNA ASIMPTOTA ako vrijedi $\lim\limits_{x\to c^-} f(x) = \pm\infty$ ili $\lim\limits_{x\to c^+} f(x) = \pm\infty$.

def. Pravac $y=b$ je HORIZONTALNA ASIMPTOTA ako vrijedi $\lim\limits_{x\to\pm\infty} f(x) = b$.

Za $x\to\infty$ kažemo DESNA HA

Za $x\to-\infty$ kažemo LIJEVA HA

def.

Pravac $y=kl+l$ je KOSA ASIMPTOTA ako vrijedi:

$$\lim_{x\to\pm\infty} (f(x) - kx - l) = 0$$

$$\boxed{k = \lim_{x\to\pm\infty} \frac{f(x)}{x}}$$

$$\boxed{l = \lim_{x\to\pm\infty} (f(x) - kx)}$$

zad. 26.)

$$y = \frac{x^3+2x^2+1}{2x^2-x} \Rightarrow \left.\begin{array}{l} 2x^2-x \neq 0 \\ x(2x-1) \neq 0 \\ x \neq 0 \ , \ x \neq \frac{1}{2} \end{array}\right\} D(f) = \mathbb{R} \setminus \left\{0, \frac{1}{2}\right\}$$

VERTIKALNE ASIMP.:

TREBA ODREDITI LIMES I SLIJEVA I SDESNA

$$\left.\begin{array}{l} \lim\limits_{x \to 0^+} \frac{x^3+2x^2+1}{x(2x-1)} = \frac{1}{0^-} = -\infty \\ \lim\limits_{x \to 0^-} \frac{x^3+2x^2+1}{x(2x-1)} = \frac{1}{0^+} = +\infty \end{array}\right\} \boxed{x=0} \text{ JEST VERT. AS. po def.}$$

$$\left.\begin{array}{l} \lim\limits_{x \to \frac{1}{2}^+} \frac{x^3+2x^2+1}{x(2x-1)} = \frac{\frac{1}{8}+\frac{1}{2}+1}{\frac{1}{2} \cdot 0^+} = +\infty \\ \lim\limits_{x \to \frac{1}{2}^-} \frac{x^3+2x^2+1}{x(2x-1)} = \frac{\frac{1}{8}+\frac{1}{2}+1}{\frac{1}{2} \cdot 0^-} = -\infty \end{array}\right\} \boxed{x=\frac{1}{2}} \text{ JEST VER. AS.}$$

HORIZONTALNE ASIM.:

$$\lim_{x \to \pm\infty} f(x) = \infty \longrightarrow \text{NEMA HA.}$$

KOSA ASIM.:

$$k = \lim_{x \to \pm\infty} \frac{x^3+2x^2+1}{2x^3-x^2} \begin{array}{l} \sim x^3 \\ \sim 2x^3 \end{array} = \frac{1}{2} \longrightarrow$$ i lijeva i desna, $\pm\infty$ NE UTJEČE

INAČE PROVJERITI OBA SLUČAJA!

$$l = \lim_{x \to \pm\infty} \left(\frac{x^3+2x^2+1}{2x^2-x} - \frac{1}{2}x \right) =$$

$$= \lim_{x \to \pm\infty} \frac{2x^3+4x^2+2-2x^3+x^2}{4x^2-2x} = \frac{5}{4} \longrightarrow$$ opet ne ovisi o $\pm\infty$

$$\longrightarrow \boxed{y = \frac{1}{2}x + \frac{5}{4}}$$ JE KOSA ASIMP.

2.M1-08'-9.)

$$y = e^{\frac{1}{x^2-4x}} \longrightarrow \left.\begin{array}{l} x^2-4x \neq 0 \\ x(x-4) \neq 0 \end{array}\right\} D(f) = \mathbb{R} \setminus \{0,4\}$$

1° VA:

$$\lim_{x\to 0^+} e^{\frac{1}{x(x-4)}} = e^{\frac{1}{0^-}} = e^{-\infty} = 0 \longrightarrow \underline{\underline{\text{NIJE}}} \text{ VA s desne !}$$

$$\lim_{x\to 0^-} e^{\frac{1}{x(x-4)}} = e^{\frac{1}{0^+}} = e^{+\infty} = +\infty \Rightarrow x=0 \text{ je LIJEVA VA !}$$

$$\lim_{x\to 4^+} e^{\frac{1}{x(x-4)}} = e^{\frac{1}{0^+}} = e^{+\infty} = +\infty \rightarrow x=4 \text{ je DESNA VA !}$$

$$\lim_{x\to 4^-} e^{\frac{1}{x(x-4)}} = e^{\frac{1}{0^-}} = e^{-\infty} = 0 \rightarrow \text{NIJE VA s LIJEVE !}$$

2° HA:

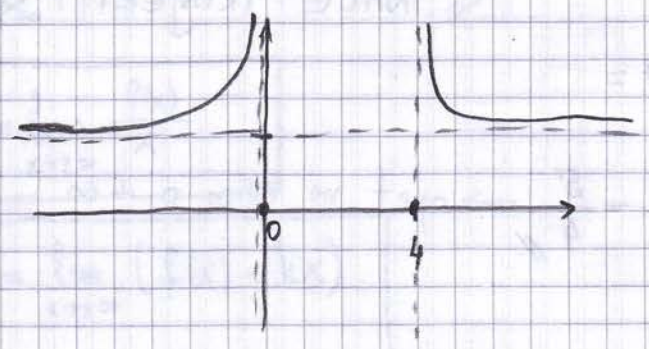
$$\lim_{x\to \pm\infty} e^{\frac{1}{x^2-4x}} = e^{\frac{1}{\infty}} = e^{0} = \underline{\underline{1}} \rightarrow \text{nema veze je li } x \to \infty$$

↳ s obje strane

$\longrightarrow y = 1$ je HA !

$\longrightarrow$ ISTOVREMENO NE MOGU BITI I HA I KA NA ISTOJ STRANI

$\Longrightarrow$ NEMA KA !



$$y = x \cdot \operatorname{arctg} \frac{x^2}{x+1} \qquad \left.\begin{array}{l} x+1 \neq 0 \\ \underline{x \neq -1} \end{array}\right\} D(f) = \mathbb{R} \setminus \{-1\}$$

$\underline{\text{IT:}}$

$$\left.\begin{array}{l} \lim\limits_{x \to -1^+} x \cdot \operatorname{arctg} \overset{+\infty}{\left(\frac{x^2}{x+1}\right)} = -1 \cdot \frac{\pi}{2} = \frac{-\pi}{2} \\ \lim\limits_{x \to -1^-} x \cdot \operatorname{arctg} \overset{-\infty}{\left(\frac{x^2}{x+1}\right)} = -1 \cdot \frac{-\pi}{2} = \frac{\pi}{2} \end{array}\right\} \underline{\underline{\text{NEMA}}} \text{ VERTIKALNE ASIMP.}$$

$\underline{\text{HA NE POSTOJ}} \Longrightarrow$ $\underline{\text{KOSU}}$ ASIMP.

$$k_1 = \lim_{x \to +\infty} \frac{\cancel{x} \operatorname{arctg} \frac{x^2}{x+1}}{\cancel{x}} \sim x^2 \sim x = \frac{\pi}{2}$$

$$k_2 = \lim_{x \to -\infty} \operatorname{arctg}(-\infty) = -\frac{\pi}{2}$$

$$l_1 = \lim_{x \to +\infty} \left(x \operatorname{arctg} \frac{x^2}{x+1} - \frac{\pi}{2} x \right) = \lim_{x \to +\infty} \left(x \cdot \left(\operatorname{arctg} \frac{x^2}{x+1} - \frac{\pi}{2} \right) \right) =$$

$$= \lim_{x \to +\infty} \frac{\operatorname{arctg} \frac{x^2}{x+1} - \frac{\pi}{2}}{\frac{1}{x}} \overset{L'H}{=} \lim_{x \to +\infty} \frac{\frac{1}{1+\left(\frac{x^2}{x+1}\right)^2} \cdot \frac{2x(x+1) - x^2}{(x+1)^2}}{-\frac{1}{x^2}} = -1$$

$$l_2 = \lim_{x \to -\infty} \cdots\cdots = -1$$

$\rightarrow y = \frac{\pi}{2} x - 1$ DKA

$\rightarrow y = -\frac{\pi}{2} x - 1$ LKA

DOMAĆA ZADAĆA:

1) $f(x) = tg^3 \frac{\pi x}{3} = \left(tg \frac{\pi x}{3}\right)^3$

$$f'(x) = \cancel{3}\left(tg \frac{\pi x}{3}\right)^2 \cdot \frac{1}{\cos^2 \frac{\pi x}{3}} \cdot \frac{\pi}{\cancel{3}} = \frac{\pi \cdot tg^2 \frac{\pi x}{3}}{\cos^2 \frac{\pi x}{3}} = \pi \cdot \underbrace{\frac{\frac{\sin^2 \frac{\pi x}{3}}{\cos^2 \frac{\pi x}{3}}}{\cos^2 \frac{\pi x}{3}}}_{1} = \pi \cdot \frac{\sin^2 \frac{\pi x}{3}}{\cos^4 \frac{\pi x}{3}}$$

2) $f(x) = \ln^2(\sin(5x)) = \left(\ln(\sin(5x))\right)^2$

$$f'(x) = 2 \ln(\sin(5x)) \cdot \frac{1}{\sin(5x)} \cdot \cos 5x \cdot 5 = 10 \ln(\sin(5x)) \cdot ctg(5x)$$

3) $f(x) = x \cdot \cos^2(5x) = x \cdot (\cos(5x))^2$

$$f'(x) = \cos^2(5x) + x \cdot 2\cos(5x) \cdot (-\sin(5x)) \cdot 5 = \cos^2(5x) - 10x \cdot \cos(5x) \cdot \sin(5x)$$

$$= \cos^2(5x) - 5x \cdot \sin(10x)$$

4) $f(x) = \frac{x}{\sqrt{x^2+1}}$

$$f'(x) = \frac{\sqrt{x^2+1} - x \cdot \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x}{x^2+1} = \frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1} = \underbrace{\frac{\frac{x^2+1-x^2}{\sqrt{x^2+1}}}{x^2+1}}_{1} = \frac{1}{(x^2+1)(\sqrt{x^2+1})}$$

$f(x) = arctg \frac{1}{x^2+1}$

$f''(1) = ?$

$$f'(x) = \frac{1}{\left(\frac{1}{x^2+1}\right)^2 + 1} \cdot \frac{-1}{(x^2+1)^2} \cdot 2x = \frac{-2x}{\left(\frac{1}{(x^2+1)^2} + 1\right)(x^2+1)^2} = \frac{-2x}{1 + (x^2+1)^2}$$

$$f''(x) = \frac{-2 \cdot (1 + (x^2+1)^2) + 2x \cdot (2(x^2+1) \cdot 2x)}{(1 + (x^2+1)^2)^2}$$

$$f''(1) = \frac{-2 \cdot (1+4) + 2 \cdot (2 \cdot 2 \cdot 2)}{(1+4)^2} = \frac{-10 + 16}{25} = \frac{6}{25}$$

$y = xe^{-x}$

$$(e^x)^{-1} = -(e^x)^{-2} \cdot e^x = -\frac{1}{\cancel{(e^x)^2}} \cdot \cancel{e^x} = -\frac{1}{e^x} = -e^{-x}$$

$xy' = (1-x)y$

$$y' = e^{-x} + x \cdot e^{-x} \cdot (-1) = e^{-x} - x e^{-x} = e^{-x}(1-x)$$

o.k.

$$x \cdot y' = \underbrace{x \cdot e^{-x}}_{y}(1-x) = y \cdot (1-x)$$

4.) za $\forall x \neq 0$ vrijedi:

$$(\ln|x|)' = \frac{1}{x}$$

1° za $x>0 \longrightarrow (\ln|x|)' = (\ln x)' = \frac{1}{x}$ //

2° za $x<0 \longrightarrow (\ln|x|)' = (\ln(-x))' = \frac{1}{-x} \cdot (-x)' = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$ //

5.)

a) $f(x) = \sin x$

$f^{(n)} = ?$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{IV}(x) = \sin x$

$\vdots$

$$\boxed{\begin{aligned} f^{4k}(x) &= \sin x \\ f^{4k+1}(x) &= \cos x \\ f^{4k+2}(x) &= -\sin x \\ f^{4k+3}(x) &= -\cos x \end{aligned}}$$

$k \in \mathbb{N}_0$

Dokaz mat. ind.? NE!

b) $f(x) = a^x$

$f^{(n)} = ?$

$f'(x) = a^x \cdot \ln a$ c

$f''(x) = \ln a \cdot (a^x)' = \ln a \cdot a^x \ln a = a^x \ln^2 a$

$f'''(x) = a^x \ln^2 a = \ln^2 a \cdot (a^x)' = a^x \ln a \cdot \ln^2 a = a^x \ln^3 a$

c

$\vdots$

$f^k(x) = a^x \ln^k a, \; k \in \mathbb{N}_0$ //

$\underline{k=1}$

$f'(x) = a^x \ln a$ // o.k.

pretpostavimo da tvrdnja $f^k(x) = a^x \ln^k a$ vrijedi za <u>neki</u> $k \in \mathbb{N}$

pokažimo da tvrdnja vrijedi za $(k+1) \in \mathbb{N}$: $f^{k+1}(x) = a^x \ln^{k+1} a$

$f^{k+1}(x) = (f^k(x))' = (a^x \ln^k a)' = \ln^k a \cdot (a^x)' = \ln^k a \cdot a^x \cdot \ln a = a^x \cdot \ln^{k+1} a$

$f^{k+1}(x) = a^x \ln^{k+1} a$ // o.k.

Po principu mat. ind. dokazali smo da tvrdnja vrijedi za $\forall k \in \mathbb{N}$.

NE!

6.) $f'(0)$ postoji ??

$f(x) = \sqrt{1-\sqrt{1-x^2}}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\sqrt{1-\sqrt{1-h^2}}-0}{h} = \lim_{h\to 0} \frac{\sqrt{1-\sqrt{1-h^2}}}{h} \cdot \frac{\sqrt{1+\sqrt{1-h^2}}}{\sqrt{1+\sqrt{1-h^2}}} =$$

$$= \lim_{h\to 0} \frac{\sqrt{1-1+h^2}}{h\left(\sqrt{1+\sqrt{1-h^2}}\right)} = \lim_{h\to 0} \frac{\sqrt{h^2}}{h\cdot\left(\sqrt{1+\sqrt{1-h^2}}\right)}$$

$$\longrightarrow \lim_{h\to 0^+} \frac{\sqrt{h^2}}{h\cdot\sqrt{1+\sqrt{1-h^2}}} = \lim_{h\to 0^+} \frac{+\cancel{h}}{\cancel{h}\cdot\sqrt{1+\sqrt{1-h^2}}} = \frac{1}{\sqrt{2}} \;//$$

$$\longrightarrow \lim_{h\to 0^-} \frac{\sqrt{h^2}}{h\cdot\sqrt{1+\sqrt{1-h^2}}} = \lim_{h\to 0^-} \frac{-\cancel{h}}{\cancel{h}\cdot\sqrt{1+\sqrt{1-h^2}}} = \frac{-1}{\sqrt{2}} \;//$$

$$\Longrightarrow \lim_{h\to 0^+} \frac{\sqrt{h^2}}{h\cdot\sqrt{1+\sqrt{1-h^2}}} \neq \lim_{h\to 0^-} \frac{\sqrt{h^2}}{\sqrt{1+\sqrt{1-h^2}}\cdot h} \Longrightarrow \text{NE POSTOJI } f'(0) \;!$$

7.) $f(x) = x\sin\frac{1}{x}$, $x\neq 0$, $f(0)=0$

a) $f'(0) =$ postoji li ?

$$f'(0) = \lim_{h\to 0} \frac{h\sin\frac{1}{h}-0}{h} = \lim_{h\to 0} \frac{\cancel{h}\sin\frac{1}{h}}{\cancel{h}} = \lim_{h\to 0} \sin\frac{1}{h}$$

$$\lim_{h\to 0^+} \sin\frac{1}{0^+} = \lim_{h\to 0^+} \sin(+\infty) = \text{NE POSTOJI}$$ $[-1,1]$, isto vrijedi za $h\to 0^-$, $\sin(-\infty$

PA NE POSTOJI LIMES $\Longrightarrow$ NE POSTOJI $f'(0)$!

b) $g'(0) =$ postoji li ?

$g(x) = x^2\sin\frac{1}{x}$, $x\neq 0$, $g(0)=0$

$$g'(0) = \lim_{h\to 0} \frac{h^{\cancel{2}}\sin\frac{1}{h}-0}{\cancel{h}} = \lim_{h\to 0} h\cdot\sin\frac{1}{h} = 0\cdot[-1,1] = 0 \;// \longrightarrow \text{postoji limes}$$

$g'(0) = 0$

$$f(x) = \begin{cases} -1+4x-x^2, & x<1 \longrightarrow f(1)=2 \\ ax, & x \geq 1 \longrightarrow f(1)=a \end{cases}$$

mora biti neprekinuta! $a=?$

$\lim_{x\to 1} f(x) = f(a)$

$\lim_{x\to 1^-} (-1+4x-x^2) = 2 \longrightarrow$

$\lim_{x\to 1^+} (f(x)) = \lim_{x\to 1^-} (f(x))$

$\lim_{x\to 1^+} (ax) = a \cdot \lim_{x\to 1^+} (x) = a$

$\boxed{a=2}$

$f(x) = 2x, \; x \geq 1$

$$f'(1) = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{-1+4(1+h)-(1+h)^2-2}{h} = \lim_{h\to 0^-} \frac{\cancel{-1}+\cancel{4}+4h\cancel{-1}-2h-h^2\cancel{-2}}{h} =$$

$$= \lim_{h\to 0^-} \frac{2h-h^2}{h} = \lim_{h\to 0^-} \frac{\cancel{h}(2-h)}{\cancel{h}} = \lim_{h\to 0^-} (2-h) = 2$$

O.K.

$$f'(1) = \lim_{h\to 0^+} \frac{f(1+h)-2}{h} = \lim_{h\to 0^+} \frac{2(1+h)-2}{h} = \lim_{h\to 0^+} \frac{\cancel{2}+2h\cancel{-2}}{\cancel{h}} = 2$$

$f(x) = -1+4x-x^2, \; x<1 \rightarrow f(1)=2$

$f(x) = ax^2, \; x \geq 1 \longrightarrow f(1)=a \longrightarrow g(x) = 2x^2$

$a=2 \longrightarrow g(x) = 2x^2, \; x \geq 1$

$$g'(1) = \lim_{h\to 0^+} \frac{g(1+h)-g(1)}{h} = \lim_{h\to 0^+} \frac{2(1+h)^2-2}{h} = \lim_{h\to 0^+} \frac{2(1+2h+h^2)-2}{h} = \lim_{h\to 0^+} \frac{4h+2h^2}{h} =$$

$$= \lim_{h\to 0^+} \frac{\cancel{h}(4+2h)}{\cancel{h}} = \lim_{h\to 0^+} (4+2h) = 4$$

$$g'(1) = \lim_{h\to 0^-} \frac{-1+4(1+h)-(1+h)^2-2}{h} = \lim_{h\to 0^-} \frac{\cancel{-1}+\cancel{4}+4h\cancel{-1}-2h-h^2\cancel{-2}}{h} = \lim_{h\to 0^-} \frac{2h-h^2}{h} = \frac{\cancel{h}(2-h)}{\cancel{h}}$$

$$= \lim_{h\to 0^-} (2-h) = 2$$

NE POSTOJI!!

9.) $y = x \cdot \sqrt[3]{x-1} = x \cdot (x-1)^{\frac{1}{3}}$

a) $T(0,0)$

$$y' = \sqrt[3]{x-1} + x \cdot \frac{1}{3}(x-1)^{-\frac{2}{3}}$$

$$y - 0 = -1(x-0)$$

$$\boxed{y = -x}$$

b) $T\left(\frac{3}{4}, \frac{-3}{4\sqrt[3]{4}}\right)$ $\qquad \frac{3}{4} \cdot \sqrt[3]{\left(-\frac{1}{4}\right)} = -\frac{1}{\sqrt[3]{4}}$

$$y + \frac{3}{4\sqrt[3]{4}} = -\frac{1}{\sqrt[3]{4}} + \frac{3}{4} \cdot \frac{1}{3}\left(\frac{3}{4} - 1\right)^{-\frac{2}{3}}\Big)\left(x - \frac{3}{4}\right)$$

$$y + \frac{3}{4\sqrt[3]{4}} = \underbrace{\sqrt[3]{\left(-\frac{1}{4}\right)} + \frac{1}{4} \cdot \frac{1}{\sqrt[3]{\left(-\frac{1}{4}\right)^2}}}_{0} \cdot \left(x - \frac{3}{4}\right)$$

$$\boxed{y = -\frac{3}{4\sqrt[3]{4}}}$$

c) $T(1,0)$

$$y - 0 = \frac{1}{3\sqrt[3]{0^2}}(x-1)$$

$$y = \text{"}\infty\text{"}(x-1) \longrightarrow \boxed{x = 1}$$

$y = A \cdot \sin(\omega x)$

$k = 1$

$T = 4\pi$

$A = ?\quad \omega = ?$

$\omega = \frac{2\pi}{T} = \frac{2\pi}{4\pi} = \frac{1}{2}$

$k = y'(0) = 1$

$y' = A \cdot \cos\frac{1}{2}x \cdot \frac{1}{2}$

$\frac{1}{2}A = 1 \quad | \cdot 2$

$\boxed{A = 2}$

$y = x \ldots\ldots t \rightarrow k = 1 \rightarrow k_2 = -1$

$y = \lambda e^x$

$y' = \lambda e^x = 1 \rightarrow \lambda = \frac{1}{e^x} = e^{-x}$

$y = e^{-x} \cdot e^x = 1 = x \rightarrow \boxed{\lambda_1 = e^{-1}}$

$y' = \lambda e^x = -1$

$\lambda = \frac{-1}{e^x} = -e^{-x}$

$y = -e^{-x} \cdot e^x = -1 = x$

$\Rightarrow \boxed{\lambda_2 = -e^1 = -e}$

$f_1(x) = \ln x$

$f_2(x) = ax^2$

$k_1 = k_2 \qquad f_1(x) = f_2(x)$

$\frac{1}{x_0} = \frac{a \cdot 2x_0}{1}$

$1 = 2ax_0^2$

$a = \frac{1}{2x_0^2}$

$\boxed{a = \frac{1}{2e}}$

$\ln x_0 = ax_0^2$

$a = \frac{\ln x_0}{x_0^2} = \frac{1}{2x_0^2}$

$\cancel{x_0^2} = 2\cancel{x_0^2} \cdot \ln x_0$

$\ln x_0 = \frac{1}{2} \quad | e$

$\boxed{x_0 = e^{\frac{1}{2}}}$

g 13.) $M(-2,-3)$

$$y=\frac{1}{4}x^2+\frac{1}{2}x-\frac{3}{4} \longrightarrow y_0=\frac{1}{4}x_0^2+\frac{1}{2}x_0-\frac{3}{4}$$

$$t \ldots \quad \underline{\underline{y}}-y_0=y'(x_0)\,(\underline{\underline{x}}-x_0)$$

M

$$y'=\frac{1}{2}x+\frac{1}{2} \longrightarrow y'(x_0)=\frac{1}{2}x_0+\frac{1}{2}=\frac{1}{2}(x_0+1)$$

$$-3+\frac{1}{4}x_0^2+\frac{1}{2}x_0-\frac{3}{4}=\frac{1}{2}(x_0+1)(-2-x_0) \quad |\cdot 4$$

$$-12+x_0^2+2x_0-3=2(x_0+1)(-2-x_0)$$

$$x_0^2+2x_0-15=2\left(-2x_0-x_0^2-2-x_0\right)$$

$$x_0^2+2x_0^2+2x_0+6x_0-15+4=0$$

$$3x_0^2+8x_0-11=0$$

$$x_{0_{12}}=\frac{-8\pm\sqrt{64+12\cdot 11}}{6}=\frac{-8\pm 14}{6}$$

$$x_{0_1}=1 \qquad x_{0_2}=-\frac{11}{3}$$

$$y_{0_1}=\frac{1}{4}+\frac{1}{2}-\frac{3}{4}=0 \qquad y_{0_2}=\frac{1}{4}\cdot\frac{11^2}{3^2}+\frac{1}{2}\cdot\frac{-11}{3}-\frac{3}{4}=\frac{7}{9}$$

$$y'(x_{0_1})=\underline{\underline{1}} \qquad y'(x_{0_2})=\frac{1}{2}\cdot\frac{-11}{3}+\frac{1}{2}=-\frac{4}{3}$$

$$t_1 \ldots \quad y=x-1 \qquad t_2 \ldots \quad y-\frac{7}{9}=-\frac{4}{3}\left(x+\frac{11}{3}\right)\cdot 9$$

$$y-7=-12\left(x+\frac{11}{3}\right)$$

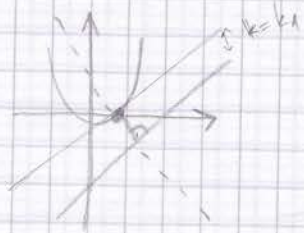
$$y=-12\left(x+\frac{11}{3}\right)+7$$

$$y=-12x-44+7$$

$$t_2 \ldots \quad y=-12x-37$$

$y = x^2 + x - 2$

$y = 5x - 10$



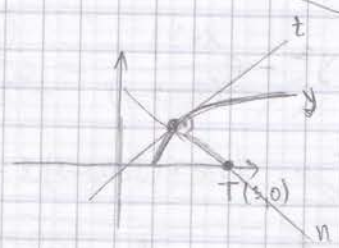
$k_1 = 5$

$y = 2x + 1 = 5 \longrightarrow 2x = 4$

$\boxed{x = 2} \longrightarrow \boxed{y = 4}$ $\quad T(2,4)$

$y = \sqrt{x^2+1}$

$T(3,0)$



$t = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$y - 0 = \frac{-1}{k}(x-3)$

$y' = \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x = k$

$y = \frac{-1}{k}(x-3)$

$y = \sqrt{x^2+1}$

$k = \frac{x}{\sqrt{x^2+1}}$

$\sqrt{x^2+1} = \frac{-1}{\frac{x}{\sqrt{x^2+1}}} \cdot (x-3)$

$\sqrt{x^2+1} = \frac{-\sqrt{x^2+1} \cdot (x-3)}{x}, \quad x \neq 0$

$x \cdot \sqrt{x^2+1} = -(x-3) \cdot \sqrt{x^2+1} \quad |^2$

$x^2 \cdot \cancel{(x^2+1)} = (x-3)^2 \cdot \cancel{(x^2+1)}$

$\cancel{x^2} = \cancel{x^2} - 6x + 9$

$6x = 9 \quad |:6$

$\boxed{x = \frac{3}{2}} \longrightarrow y = \sqrt{\frac{9}{4}+1} = \sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{2}$

$$\boxed{T\left(\frac{3}{2}, \frac{\sqrt{13}}{2}\right)}$$

16.) $T(1,-1)$

$$xy^2 + e^{x+y} = 2$$

$$y', y''\Big|_{T(1,-1)} = ?$$

$$xy^2 + e^{x+y} = 2 \quad /'$$

$$y^2 + x\cdot 2y\cdot y' + e^{x+y}\cdot(1+y') = 0$$

$$y'\cdot\left(2xy + e^{x+y}\right) = -y^2 - e^{x+y}$$

$$y' = \frac{-y^2 - e^{x+y}}{2xy + e^{x+y}}$$

$$y'\Big|_{T(-1,1)} = \frac{-1-1}{-2+1} = \frac{-2}{-1} = 2$$

$$y'' = \frac{\left(-2y\cdot y' - e^{x+y}\cdot(1+y')\right)\cdot\left(2xy + e^{x+y}\right) - \left(-y^2 - e^{x+y}\right)\cdot\left(2(y + x\cdot y') + e^{x+}\right.}{\left(2xy + e^{x+y}\right)^2}$$

$$y''\Big|_{T(-1,1)} = \frac{\left(-2\cdot 2 - 1\cdot(1+2)\right)\cdot(-2+1) - \overbrace{(-1-1)}^{-2}\cdot\left(2\cdot\overbrace{(1-2)}^{-2} + 1\cdot\overbrace{(1+2)}^{3}\right)}{(-2+1)^2}$$

$$= -7\cdot(-1) + 2 = 7 + 2 = 9$$

17.) $T(1,1)$

$$y^x + \sin(x-1) = \sqrt{y} \quad /' \qquad y^x = e^{x\ln y}$$

t..... ? n..... ?

$$e^{x\ln y}\cdot\left(\ln y + x\cdot\frac{1}{y}\cdot y'\right) + \cos(x-1)\cdot 1 - \frac{1}{2\sqrt{y}}\cdot y' = 0$$

$$y'\cdot\left(y^x\cdot\frac{x}{y} - \frac{1}{2\sqrt{y}}\right) = -\cos(x-1) - y^x\cdot\ln y$$

$$y' = \frac{-\left(\cos(x-1) + y^x\cdot\ln y\right)}{\frac{y^x\cdot x}{y} - \frac{1}{2\sqrt{y}}} = \frac{-2y\cdot\sqrt{y}\cdot\left(\cos(x-1)+\right.}{2xy^x\sqrt{y} - y}$$

$$y'\Big|_{T(1,1)} = \frac{-2\cdot(1+0)}{2-1} = -2$$

t.... $y - 1 = -2(x-1) \longrightarrow y = -2x + 2 + 1 = -2x + 3$ n.... $y = \frac{1}{2}x + \frac{1}{2}$

$x = t - \sin t$

$y = 1 - \cos t$

$$y' = \frac{\dot{y}}{\dot{x}} = \frac{\sin t}{1-\cos t}$$

$$y'' = \frac{\left(\frac{\sin t}{1-\cos t}\right)'}{1-\cos t} = \frac{\frac{\cos t\cdot(1-\cos t) - \sin t\cdot\sin t}{(1-\cos t)^2}}{1-\cos t} = \frac{\frac{\cos t - \cos^2 t - \sin^2 t}{(1-\cos t)^2}}{1-\cos t} = \frac{\cos t - (\cos^2 t + \sin^2 t)}{(1-\cos t)^3}$$

$$y'' = \frac{\cos t - 1}{(1-\cos t)^3} = \frac{-\cancel{(1-\cos t)}}{(1-\cos t)^{\cancel{3}\,2}} = \frac{-1}{(1-\cos t)^2}$$

$$y''' = \frac{\left(\frac{-1}{(1-\cos t)^2}\right)'}{1-\cos t} = \frac{\frac{1}{(1-\cos t)^{\cancel{4}\,3}}\cdot 2\cancel{(1-\cos t)}\cdot\sin t}{1-\cos t} = \frac{\frac{2\sin t}{(1-\cos t)^3}}{1-\cos t} = \frac{2\sin t}{(1-\cos t)^4}$$

$x = \sin^2 t - \cos t \longrightarrow x = 1$

$y = \ln\left(\operatorname{tg}\frac{t}{2}\right) \longrightarrow y = 0$

$t = \frac{\pi}{2} \longrightarrow T\,(1,0)$

? n..... ?

$$y' = \frac{\frac{1}{\operatorname{tg}\frac{t}{2}}\cdot\frac{1}{\cos^2\frac{t}{2}}\cdot\frac{1}{2}}{2\sin t\cdot\cos t + \sin t} = \frac{\frac{1}{2\cdot\frac{\sin\frac{t}{2}}{\cancel{\cos\frac{t}{2}}}\cdot\cos^{\cancel{2}}\frac{t}{2}}}{\sin 2t + \sin t} = \frac{1}{\sin t\,(\sin 2t + \sin t)}$$

$$y'(t) = \frac{1}{\sin^2\frac{\pi}{2} + \sin\frac{\pi}{2}\cdot\sin\pi} = \frac{1}{1} = 1$$

$y - 0 = 1\cdot(x-1)$

$y = x - 1$

$y = -x + 1$

20.) $y = e^{2x}$

$y = e^{-3x}$

$\alpha = ?$

$$\boxed{tg\,\alpha = \frac{k_2 - k_1}{1 + k_1 \cdot k_2}}$$

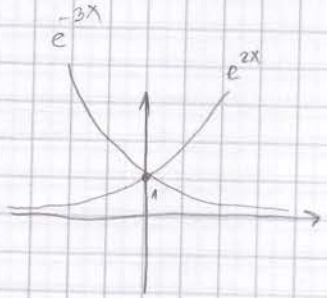
$D(0,1)$

$y' = e^{2x} \cdot 2 \longrightarrow k_2 = 2$

$y' = e^{-3x} \cdot (-3) \longrightarrow k_1 = -3$

$tg\,\alpha = \frac{2+3}{1-6} = -1$

$\alpha = -\frac{\pi}{4} \longrightarrow \boxed{\alpha = \frac{3\pi}{4}}$



$e^{-3x} = e^{2x} \quad | \ln$

$-3x = 2x$

$-5x = 0$

$\boxed{x = 0} \rightarrow \boxed{y = 1}$

a i sa slike $D(0,1)$

NASTAVAK PREDAVANJA

10.) Odredi sve asimptote:

$$f(x) = 2x - 1 - \sqrt{x^2 - x + 1}$$

$D(f)$: $x^2 - x + 1 \geq 0$

$$x_{1,2} = \frac{1 \pm \sqrt{1-4}}{2}$$

$\Big\}$ $D(f) = \mathbb{R}$ $\longrightarrow$ NEMA VERTIKALNIH ASIMP.

$\rightarrow$ AKO NE MOŽEMO ODREDITI IMA LI H.A. ILI K.A PRVO RAČUNAMO KOSU A. !!!

$\longrightarrow$ ako nam ispadne $k = 0 \Rightarrow$ H.A.

$$k = \lim_{x \to +\infty} \frac{2x - 1 - \sqrt{x^2 - x + 1}}{x} = \lim_{x \to +\infty} \left(2 - \frac{1}{x} - \underbrace{\frac{\sqrt{x^2 - x + 1}}{x}}_{1} \right) = 1 = k_1$$

$$k_2 = \lim_{x \to -\infty} \left(2 - \frac{1}{x} - \frac{\sqrt{x^2 - x + 1}}{x} \right) = 3 = k_2$$

$2x - x \rightarrow$ po def. za l...

$$l_1 = \lim_{x \to +\infty} \left(x - 1 - \sqrt{x^2 - x + 1} \right) \cdot \frac{(x-1) + \left(\sqrt{x^2 - x + 1}\right)}{(x-1) + \sqrt{x^2 - x + 1}} =$$

$$= \lim_{x \to +\infty} \frac{\cancel{x^2} - 2x + 1 - \cancel{x^2} + x - 1 \sim -x}{(x-1) + \sqrt{x^2 - x + 1} \sim 2x} = -\frac{1}{2} = l_1$$

$$l_2 = \lim_{x \to -\infty} \left(2x - 1 - \sqrt{x^2 - x + 1} - 3x \right) = \lim_{x \to -\infty} \left(-x - 1 - \sqrt{x^2 - x + 1} \right) = (\infty - \infty)$$

$\sqrt{(-\infty)^2} = \sqrt{\infty} = +\infty$

$$= \frac{x^2 + 2x + 1 - x^2 + x - 1}{-x - 1 + \sqrt{x^2 - x + 1}} = -\frac{3}{2} \Longrightarrow \text{L.K.A}: \boxed{y = 3x - \frac{3}{2}}$$

!!

D.K.A: $\boxed{y = 3x - \frac{1}{2}}$

$x^2 = 9 \quad | \sqrt{\ }$

$x = \pm\sqrt{9}$

$x = \pm 3$

9.zzv. – 25.) $y = \dfrac{(\sqrt{x}+a)^3}{\sqrt{x}+2}$

→ za koji a će fja imati desnu kosu asimptotu?

$D(f) = [0, +\infty\rangle$ // → zato nema LIJEVE KOSE A.

$$k = \lim_{x\to+\infty} \frac{x\sqrt{x} + 3xa + 3\sqrt{x}\cdot a^2 + a^3}{x\sqrt{x} + 2x} = 1$$ → bez obzira na a

$$l = \lim_{x\to+\infty}\left(\frac{(\sqrt{x}+a)^3}{\sqrt{x}+2} - x\right) = \lim_{x\to+\infty} \frac{\cancel{x\sqrt{x}} + 3xa + 3\sqrt{x}\cdot a^2 + a^3 - \cancel{x\sqrt{x}} - 2x}{\sqrt{x}+2} =$$

$$= \lim_{x\to+\infty} \frac{x(3a-2) + 3\sqrt{x}\cdot a^2 + a^3}{\sqrt{x}+2} = \begin{cases} 3\cdot\left(\frac{2}{3}\right)^2, & a = \frac{2}{3} \\ \pm\infty, & a \neq \frac{2}{3} \end{cases}$$

→ za $a = \frac{2}{3}$ postoji D.K.A!

D.K.A... $y = x + \frac{4}{3}$

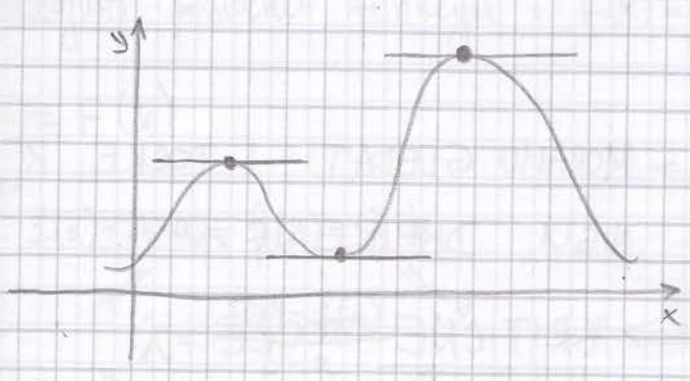
9.4. OSNOVNI TEOREMI DIF. RAČUNA

Za točku $a \in S$ kažemo da je točka maksimuma od funkcije f ako za sve $x \in S$ vrijedi $f(x) \leq f(a)$.

Za točku $a \in S$ kažemo da je točka minimuma od funkcije f ako za sve $x \in S$ vrijedi $f(x) \geq f(a)$.

$$f(a) = \max_{x \in S} f(x) \quad , \quad f(a) = \min_{x \in S} f(x)$$

$\Rightarrow$ jednim imenom: ekstremi funkcije f



$$\boxed{f'(x) = 0}$$

TM „FERMATOV TEOREM"

Neka je I otvoreni interval u $\mathbb{R}$ i neka je $f: I \to \mathbb{R}$ diferencijabilna. Ako je $a \in I$ točka LOKALNOG EKSTREMA, tada je $f'(a) = 0$.

Dokaz: (sa slike)

Neka je a lok. max $\Rightarrow$ tada je $f(x) \leq f(a)$. Ako je $x < a$:

$$\frac{f(x) - f(a)}{x - a} \geq 0 \quad \Big| \lim_{x \to a^-} \quad \text{, ako je } x > a: \quad \frac{f(x) - f(a)}{x - a} \leq 0 \quad \Big| \lim_{x \to a^+}$$

$$f'(a) \geq 0 \qquad\qquad f'(a) \leq 0$$

$$f'(a) = 0 \quad \Updownarrow$$

def.

Za diferencijabilnu funkciju $f: I \to R$ točke koje su rješenje jednadžbe $f'(x) = 0$ zovemo STACIONARNE TOČKE.

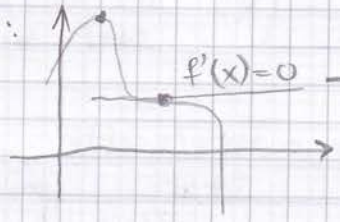
• NAPOMENA:

Obrat Fermatovog teorema NE VRIJEDI.

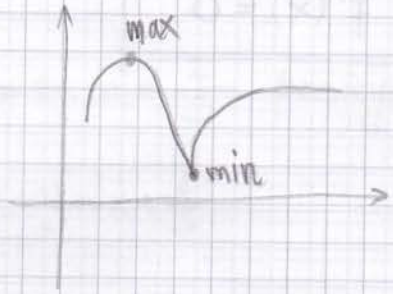
$\Rightarrow$ ako postoji derivacija $= 0$ ne mora imati ekstreme

npr. x^3

ili:



$\to$ a nije ekstrem!



$\to$ MORAMO GLEDATI I TOČKE KOJE NISU DIFIRENC.

$\Rightarrow$ KRITIČNE TOČKE

def. KRITIČNE TOČKE su točke u kojima je $f'(x) = 0$ ili $f'(x)$ NE POSTOJI.

TM „ROLLEOV TEOREM"

Neka je $f: [a,b] \to R$ neprekinuta funkcija i diferencijabilna na (a,b). Ako je $f(a) = f(b)$, tada postoji neka točka takva da je $f'(c) = 0$.

Dokaz:

$\to$ ako je f konstanta, onda je $f'(x) = 0$ za $\forall x$

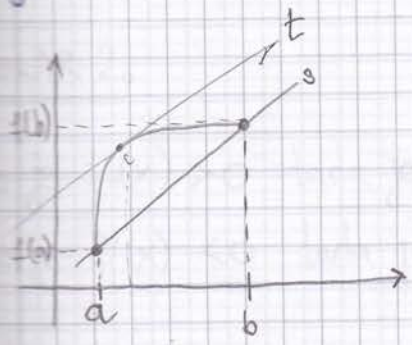
$\to$ ako f nije konstanta, onda postoji min ili max tada po Fermatovom t. $\Rightarrow$ da je $f'(x) = 0$

ISPIT

„LAGRANGEOV TEOREM SREDNJE VRIJEDNOSTI DIF. RAČUNA"

Neka je $f: [a,b] \longrightarrow \mathbb{R}$ neprekinuta i diferencijabilna na (a,b). Tada postoji $c \in (a,b)$ tako da

$$\boxed{f(b) - f(a) = f'(c) \cdot (b-a)}$$



koefic. smjera $s = \dfrac{f(b)-f(a)}{b-a} = f'(c)$

LAGRANGE: UVIJEK POSTOJI $t \parallel s$.

$f'(c) = k \ldots t$

DOKAZ:

definirajmo pomoćnu funkciju: $F(x) = f(x) - \lambda x$ takvu da je

$F(a) = F(b)$

$$\Rightarrow f(a) - \lambda \cdot a = f(b) - \lambda \cdot b$$

$$\lambda = \frac{f(b)-f(a)}{b-a}$$

$\Rightarrow$ po Rollovom TM postoji $c \in (a,b)$ takav da je $F'(c) = 0$.

$$F'(x) = f'(x) - \lambda$$

$$f'(c) = \lambda = \frac{f(b)-f(a)}{b-a}$$

9. DZ - 7.) $f(x) = \arcsin x$, primjeni Lagrangeov teorem srednje vrij. dif. račun. odredi potreban c i geom. interpretiraj.

$$f'(c) = \frac{f(b)-f(a)}{b-a} \qquad f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{1}{\sqrt{1-c^2}} = \frac{f(1)-f(-1)}{1-(-1)} \qquad D(f) = [-1,1]$$

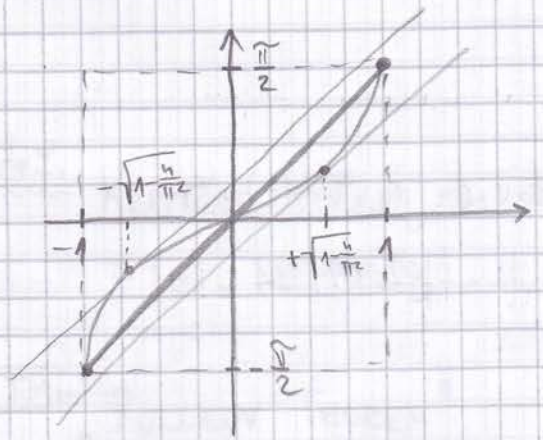
$$\frac{1}{\sqrt{1-c^2}} = \frac{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)}{1+1} = \frac{\pi}{2} \quad \Big|^2$$

$$\frac{1}{1-c^2} = \frac{\pi^2}{4}$$

$$1-c^2 = \frac{4}{\pi^2}$$

$$c^2 = 1-\frac{4}{\pi^2}$$

$$\boxed{c = \pm\sqrt{1-\frac{4}{\pi^2}}}$$



→ prema Lagrangeu postoje tangente paralelne pravcu koji spaja točke

TEOREM 1.

Ako je $f'(x)=0$ za sve x, tada je f konstanta.

Ako je $f'(x)=g'(x)$, tada je $f(x)=g(x)+C$, tj. razlikuju se za konstantu.

TEOREM 2.

Ako je $f'(x)>0$ tada je f STROGO RASTUĆA.

Ako je $f'(x)<0$ tada je f STROGO PADAJUĆA.

Dokaz:

Uzmemo $x_1<x_2$ i neka je $f'(x)>0$

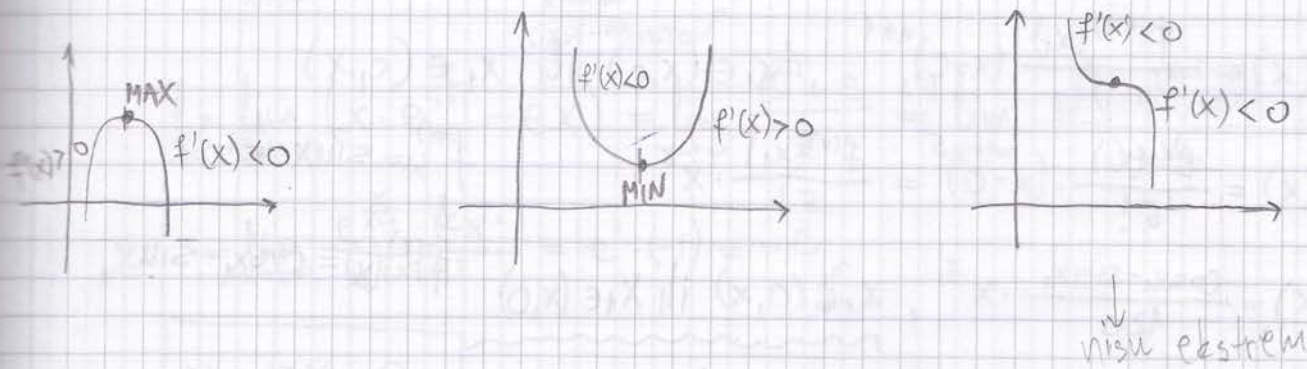
$$f(x_2)-f(x_1)=\underbrace{f'(c)}_{>0}\underbrace{(x_2-x_1)}_{>0}>0 \qquad f(x_2)>f(x_1) \Rightarrow f \text{ raste!}$$

10.1. INTERVALI MONOTONOSTI I EKSTREMI

Intervali monotonosti su intervali na kojima je funkcija rastuća ili padajuća.

→ Krajnje točke intervala su RUBOVI DOMENE ili KRITIČNE TOČKE.

Kako odrediti karakter ekstrema?



→ AUDITORNE ←

2MI. - 2010 - 6.)

$f(x) = \sin x + \cos x$ prikaži u obliku $f(x) = T_4(x) + R_4(x)$ oko točke $c = 0$

Rj.:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

$$c = 0 \longrightarrow T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k$$

$\underline{n=4}$

$$T_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} \cdot x^4 = \frac{f^{(0)}(0)}{0!} \cdot x^0 + \frac{f^{(1)}(0)}{1!} \cdot x^1 + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}}{3}$$

$$+ \frac{f^{(4)}(0)}{4!} \cdot x^4$$

$$T_4(x) = f(0) + \frac{f'(0)}{1!} \cdot x + \frac{f''(0)}{2!} \cdot x^2 + \frac{f'''(0)}{3!} \cdot x^3 + \frac{f^{IV}(0)}{4!} \cdot x^4$$

$f(x) = \sin x + \cos x \longrightarrow f(0) = 1$

$f'(x) = \cos x - \sin x \longrightarrow f'(0) = 1$

$f''(x) = -\sin x - \cos x \longrightarrow f''(0) = -1$

$f'''(x) = -\cos x + \sin x \longrightarrow f'''(0) = -1$

$f^{IV}(x) = \sin x + \cos x \longrightarrow f^{IV}(0) = 1$

$T_4(x) = 1 + \frac{1}{1!}x - \frac{1}{2!}x^2 - \frac{1}{3}x^3 + \frac{1}{4!}x^4$ → možemo ostaviti tako !!

$T_4(x) = 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{8}x^4$ //

$$R_n(x) = \frac{f^{(n+1)}(x_1)}{(n+1)!}(x-c)^{n+1}, \quad x_1 \in (x,c) \text{ ili } x_1 \in (c,x)$$

$$R_4(x) = \frac{f^{V}(x_1)}{5!} \cdot (x-0)^5 = \frac{f^{(V)}(x_1)}{5!} \cdot x^5$$

$f^{(IV)}(x) = \sin x + \cos x$

$\downarrow$

$f^{(V)}(x_1) = \cos x_1 - \sin x_1$

$$R_4(x) = \frac{\cos x_1 - \sin x_1}{120} \cdot x^5, \quad x_1 \in (0,x) \text{ ili } x_1 \in (x,0)$$

PIŠI !!!!

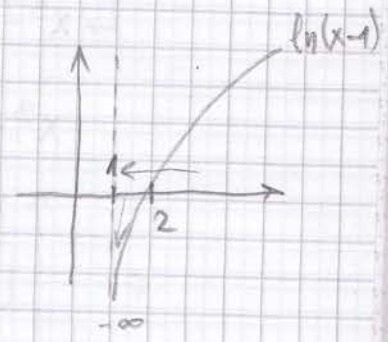
$$x^x = e^{x\ln x}$$

$$\lim_{x\to 1}\frac{x^x - x}{1-x+\ln x} = \left(\frac{0}{0}\right) \overset{L'H}{=} \lim_{x\to 1}\frac{e^{x\ln x}\cdot\left(x\cdot\frac{1}{x}+\ln x\right)-1}{-1+\frac{1}{x}} = \lim_{x\to 1}\frac{x^x\cdot(1+\ln x)-1}{\frac{1}{x}-1} \overset{L'H}{=} \left(\frac{0}{0}\right)$$

o.k.

$$= \lim_{x\to 1}\frac{e^{x\ln x}\cdot(\ln x+1)\cdot(\ln x+1) + e^{x\ln x}\left(\frac{1}{x}-0\right)-0}{0-\frac{1}{x^2}} = \frac{2}{-1} = -2$$

$$\lim_{x\to 1^+}\ln x\cdot\ln(x-1) = (0\cdot(-\infty)) =$$



$$= \lim_{x\to 1^+}\frac{\ln(x-1)}{\frac{1}{\ln x}} = \left(\frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to 1^+}\frac{\frac{1}{x-1}}{-\frac{1}{\ln^2 x}\cdot\frac{1}{x}} =$$

o.k.

$$= \lim_{x\to 1^+}\frac{-x\ln^2 x}{x-1} = \left(\frac{0}{0}\right) \overset{L'H}{=} \lim_{x\to 1^+}\frac{-\ln^2 x - \cancel{x}\cdot 2\ln x\cdot\frac{1}{\cancel{x}}}{1} = \lim_{x\to 1^+}\frac{-\ln^2 x - 2\ln x}{1} = 0$$

Odrediti sve asimptote krivulje: $y = x\cdot e^{\frac{x}{x+1}}$.

D: $D(f) = \mathbb{R}\setminus\{-1\}$

VA:

$$\left.\begin{aligned}\lim_{x\to -1^+} x\cdot e^{\frac{x}{x+1}} &= -1\cdot e^{-\infty} = 0\\ \lim_{x\to -1^-} x\, e^{\frac{x}{x+1}} &= (-1\cdot e^{+\infty}) = -\infty\end{aligned}\right\}\ \underset{\text{(lijeva)}}{x=-1}$$

KA: $y = kx + l$

$$k = \lim_{x\to+\infty}\frac{\cancel{x}\cdot e^{\frac{x}{x+1}}}{\cancel{x}} = \lim_{x\to\pm\infty} e^{\frac{x}{x+1}} = e^1 = e$$

$x\to-\infty$ ISTO

$$l = \lim_{x\to\pm\infty}\left(x\cdot e^{\frac{x}{x+1}} - ex\right) = \lim_{x\to\pm\infty}\frac{e^{\frac{x}{x+1}}-e}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to\pm\infty}\frac{e^{\frac{x}{x+1}}\cdot\frac{x+1-x}{(x+1)^2}}{-\frac{1}{x^2}} =$$

$$= \lim_{x\to\pm\infty}\frac{e^{\frac{x}{x+1}}\cdot(-x^2)}{(x+1)^2} = e^1\cdot(-1) = -e$$

$$\boxed{y = ex - e}$$

2.MI-2009-5.) Jednadžbe tangente i normale na krivulju

$$x^3 - 2e^x(1-y)^2 - y^2 \ln y + 8 = 0$$ u točki s ordinatom 1

Rj.:

$y=1$

$$x^3 - 2e^x(1-1)^2 - \ln 1 + 8 = 0$$

$$x^3 + 8 = 0$$

$$x^3 = -8$$

$$\boxed{x=-2} \quad T(-2,1)$$

$$x^3 - 2e^x(1-y)^2 - y^2 \ln y + 8 = 0 \quad /'$$

$$3x^2 - 2e^x(1-y)^2 - 2e^x \cdot 2 \cdot (1-y) \cdot (-y)' - 2y \cdot y' \cdot \ln y - y^2 \cdot \frac{1}{y} \cdot y' = 0$$

uvrstimo točku: $12 - 0 - 0 - 0 - 1 \cdot y' = 0 \Rightarrow y' = 12$

t..... $y - 1 = 12(x+2)$

$$y = 12(x+2) + 1$$

t..... $y = 12x + 25$

n..... $y = -\frac{1}{12}x -$

DOMAĆA ZADAĆA

$y = 3x^4 - 4x^3 - 12x^2 + 20$

$x \longrightarrow \underline{k=0}$

$12x^3 + 12x^2 - 24x = 0$

$x(12x^2 + 12x - 24) = 0$

$\boxed{x_1 = 0}$

$\downarrow$

$\boxed{y = 20}$

$T_1(0, 20)$

$T_2(-2, -12)$

$T_3(1, 15)$

$12x^2 + 12x - 24 = 0 \quad /:12$

$x^2 + x - 2 = 0$

$x_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$

$\boxed{x_2 = -2}$

$\downarrow$

$y_2 = 3 \cdot 2^4 - 4 \cdot 2^3 - 12 \cdot 2^2 + 20$

$y_2 = 2^2 \cdot (3 \cdot 2^2 - 8 - 12) + 20$

$\boxed{y_2 = -12}$

$\boxed{x_3 = 1}$

$\downarrow$

$y_3 = 3 + 4 - 12 + 20$

$\boxed{y_3 = 15}$

$y^4 = 4x^4 + 6xy \quad /'$

$A(1,2)$

$t \ldots ?$

$4y^3 \cdot y' = 16x^3 + 6 \cdot (y + x \cdot y')$

$y'(4y^3 - 6x) = 16x^3 + 6y$

$y' = \frac{\cancel{16}^{8}x^3 + \cancel{6}^{3}y}{\cancel{4}_{2}y^3 - \cancel{6}_{3}x} = \frac{8x^3 + 3y}{2y^3 - 3x}$ //

$y'\big|_{A(1,2)} = \frac{8+6}{16-3} = \frac{14}{13}$ //

$y - 2 = \frac{14}{13}(x - 1)$

$y = \frac{14}{13}x - \frac{14}{13} + 2$

$\boxed{y = \frac{14}{13}x + \frac{12}{13}} \ldots t$

3.) $T(1,1)$

$$xy^2 + x^4y^3 = 2 \quad /'$$

$$y', y''\Big|_{T(1,1)} = ?$$

$$y^2 + x\cdot 2y\cdot y' + 4x^3\cdot y^3 + x^4\cdot 3y^2\cdot y' = 0$$

$$y'\cdot(2xy + x^4\cdot 3y^2) = -y^2 - 4x^3y^3$$

$$y' = \frac{-y^2 - 4x^3y^3}{2xy + x^4\cdot 3y^2} = \frac{y^{\cancel{2}}(-1-4x^3y)}{\cancel{y}(2x + x^4\cdot 3y)} = \frac{-y - 4x^3y^2}{2x + x^4\cdot 3y}$$

$$y'\Big|_{T(1,1)} = \frac{-1-4}{2+3} = -1$$

$$y'' = \frac{(-y' - 4(3x^2\cdot y^2 + x^3\cdot 2y\cdot y'))(2x + x^4\cdot 3y) - (-y - 4x^3y^2)(2 + 4x^3\cdot 3y + x^4\cdot 3y')}{(2x + x^4\cdot 3y)^2}$$

$$y''\Big|_{T(1,1)} = \frac{(1 - 4(\overbrace{3-2}^{1}))(2+3) - (\overbrace{-1-4}^{5})(\overbrace{2+12-3}^{11})}{(2+3)^2} = \frac{-3\cdot 5 + 55}{25} = \frac{\cancel{40}^{8}}{\cancel{25}_{5}} = \frac{8}{5}$$

4.) $x = a\cos^3 t$

$y = a\sin^3 t$

$$t = \frac{\pi}{4} \nearrow x = a\cdot\left(\frac{\sqrt{2}}{2}\right)^3 = a\cdot\frac{\cancel{2}^{1}\sqrt{2}}{\cancel{8}_{4}} = \frac{a\sqrt{2}}{4} \qquad T\left(\frac{a\sqrt{2}}{4}, \frac{a\sqrt{2}}{4}\right)$$

$$\searrow y = x = \frac{a\sqrt{2}}{4}$$

$t \ldots ?$

$$y' = \frac{\dot{y}}{\dot{x}} = \frac{\cancel{a}\cdot\cancel{3}\sin^{\cancel{2}} t\cdot\cancel{(\cos t)}}{\cancel{a}\cdot\cancel{3}\cos^{\cancel{2}} t\cdot\cancel{(-\sin t)}} = \frac{-\sin t}{\cos t} = -\operatorname{tg} t = -\operatorname{tg}\frac{\pi}{4} = -1 = k$$

$$y - \frac{a\sqrt{2}}{4} = -1\left(x - \frac{a\sqrt{2}}{4}\right)$$

$$y = -x + \frac{a\sqrt{2}}{4} + \frac{a\sqrt{2}}{4}$$

$$\boxed{y = -x + \frac{a\sqrt{2}}{2}} \ldots t$$

$= \frac{3+e^t}{2t}$?

CRIVO

$-2\cdot 3 - 2\cdot e^t$

$$= \frac{\frac{3+e^t}{2t}}{2t} = \frac{\frac{e^t\cdot 2t-(3+e^t)\cdot 2}{(2t)^2}}{2t} = \frac{2e^t\cdot t-2\cdot(3+e^t)}{(2t)^3=8t^3} = \frac{2\cdot e^t(t-1)-6}{8t^3}$$

$$= \frac{\left(\frac{2te^t-2(e^t+3)}{8t^3}\right)'}{2t} = \frac{\frac{(2e^t-2\cdot e^t)8t^3-(2te^t-2(e^t+3))(8\cdot 3t^2)}{64t^6}}{2t} =$$

$$= \frac{-8t^3e^t-24t^2\cdot(2te^t-2e^t-6)}{128t^7} = \frac{-\cancel{8t^2}(te^t+3(2te^t-2e^t-6))}{\cancel{128}\,16\,t^{\cancel{7}5}} =$$

$$= \frac{-te^t-6te^t+6e^t+18}{16t^5} = \frac{-7t\cdot e^t+6e^t+18}{16t^5} = \frac{-7\cdot\cancel{t}e^t}{16t^{\cancel{5}}} + \frac{\overbrace{6e^t+18}^{2\cdot(3e^t+3)}}{16t^5} =$$

$$= \frac{-7e^t}{16t^4} + \frac{3(e^t+3)}{8t^5}$$ NE

$f(x) = \sin x$

$I = [0, \frac{\pi}{2}]$

$c = ?$

$f(b) - f(a) = f'(c)\cdot(b-a)$ LAGRANGE

$f'(x) = \cos x$

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{1-0}{\frac{\pi}{2}-0} = \frac{2}{\pi}$$

$\cos c = \frac{2}{\pi}$

$c = \cos^{-1}\frac{2}{\pi} = \arccos\frac{2}{\pi}$

7.) $f(x) = \arcsin x$

$I = [-1, 1]$

$$f'(c) = \frac{f(b) - f(a)}{b-a} = \frac{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)}{1+1} = \frac{\pi}{2}$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

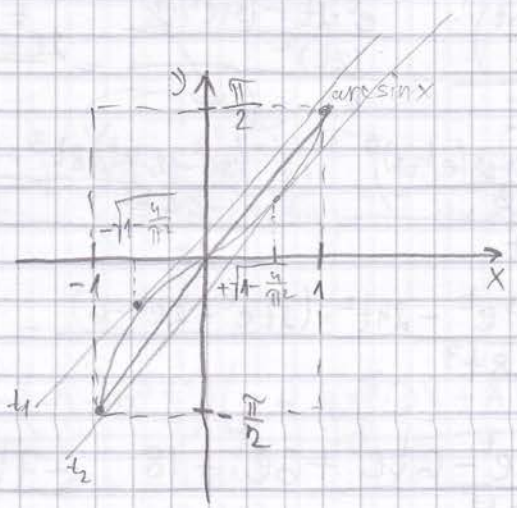
$$\frac{1}{\sqrt{1-c^2}} = \frac{\pi}{2}$$

$$\sqrt{1-c^2} = \frac{2}{\pi} \quad \Big|^2$$

$$1 - c^2 = \frac{4}{\pi^2}$$

$$c^2 = 1 - \frac{4}{\pi^2}$$

$$\boxed{c = \pm\sqrt{1 - \frac{4}{\pi^2}}}$$



8.) $f(x) = \operatorname{ch} x$

$c = 0$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

$$R_n(x) = \frac{f^{n+1}(x_1)}{(n+1)!} \cdot (x-c)^{n+1} \qquad x_1 \in (c, x) \text{ ili } x_1 \in (x, c)$$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{n+1}(x_1)}{(n+1)!} \cdot x^{n+1} \quad \text{za } x_1 \in (0, x) \text{ ili } x_1 \in (x, 0)$$

$$f'(x) = \operatorname{sh} x \rightarrow f^{(2k)}(x) = \operatorname{sh} x \rightarrow \frac{e^x - e^{-x}}{2} = \frac{1-1}{2} = 0$$

$$f''(x) = \operatorname{ch} x \rightarrow f^{2k}(x) = \operatorname{ch} x \rightarrow 1$$

$$f'''(x) = \operatorname{sh} x \rightarrow f^{2k+1}(x) = \operatorname{sh} x$$

KOROLAR 1.

Ako je $f'(x)=0$ za sve x, tada je f konstanta.

Ako je $f'(x)=g'(x)$, tada je $f(x)=g(x)+C$, tj. razlikuju se za konstantu.

KOROLAR 2.

Ako je $f'(x)>0$ tada je f STROGO RASTUĆA.

Ako je $f'(x)<0$ tada je f STROGO PADAJUĆA.

Dokaz:

Uzmemo $x_1<x_2$ i neka je $f'(x)>0$

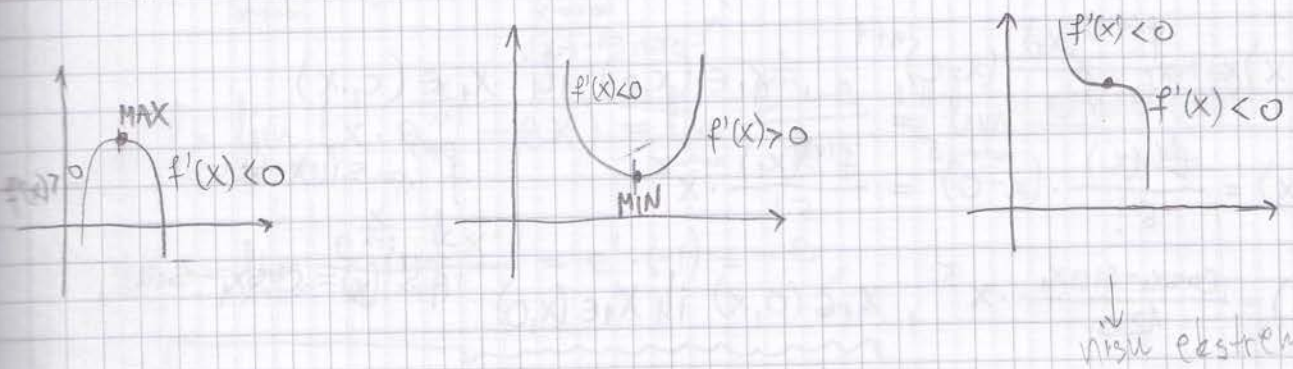
$$f(x_2)-f(x_1)=\underbrace{f'(c)}_{>0}\underbrace{(x_2-x_1)}_{>0}>0 \qquad f(x_2)>f(x_1) \Rightarrow f \text{ raste!}$$

10.1. INTERVALI MONOTONOSTI I EKSTREMI

Intervali monotonosti su intervali na kojima je funkcija rastuća ili padajuća.

→ Krajnje točke intervala su RUBOVI DOMENE ili KRITIČNE TOČKE.

Kako odrediti karakter ekstrema?



NASTAVAK PREDAVANJA *

Odredi intervale monotonosti, ekstreme i skiciraj:

a) $f(x) = x^5 - 2x^4 + x^3$

1. KORAK: DOMENA

$D(f) = \mathbb{R}$

2. KORAK: STACIONARNE TOČKE

$f'(x) = 0$

$f'(x) = 5x^4 - 8x^3 + 3x^2 = 0$

$x^2(5x^2 - 8x + 3) = 0$

$\boxed{x_1 = 0}$ $\quad x_{2,3} = \frac{8 \pm \sqrt{64-60}}{10}$

$\boxed{x_2 = 1}$ $\quad \boxed{x_3 = \frac{3}{5}}$

3. KORAK: TABLICA

	$-\infty$		0		$3/5$		1		$+\infty$
$f'(x)$		$+$		$+$	0	$-$	0	$+$	
$f(x)$		$\nearrow$		$\nearrow$	MAX	$\searrow$	MIN	$\nearrow$	

INTERVALI RASTA: $\langle -\infty, \frac{3}{5} \rangle \cup \langle 1, +\infty \rangle$

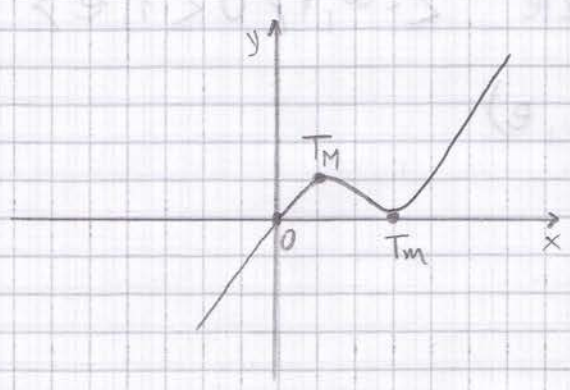
INTERVAL PADA: $\langle 3/5, 1 \rangle$

EKSTREMI:

max $\rightarrow$ $T_M\left(\frac{3}{5}, 0.034\right)$

min $\rightarrow$ $T_m(1, 0)$

4. KORAK: GRAF



b) PZI-06-3)

$$f(x) = \frac{x}{\ln x}$$ → NAJČEŠĆE NA ISPITU, ILI tg ... ZANIMLJIVE f

1.) DOMENA

$\ln x \neq 0$

$\underline{x \neq 1}$ i $\underline{x > 0}$

$D(f) = R^+ \setminus \{1\} = \langle 0, +\infty \rangle \setminus \{1\}$

2.) STAC. TOČKE

$f'(x) = 0$

$$f'(x) = \frac{\ln x - x \cdot \frac{1}{x}}{\ln^2 x} = \frac{\ln x - 1}{(\ln x)^2} = 0$$

$\ln x - 1 = 0$

$\ln x = 1$

$\boxed{x = e}$

3.) TABLICA

NE ZABORAVI V.A. !!!

	0	1	e	$+\infty$
$f'(x)$	−	−	+	
$f(x)$	↘	↘	↗	

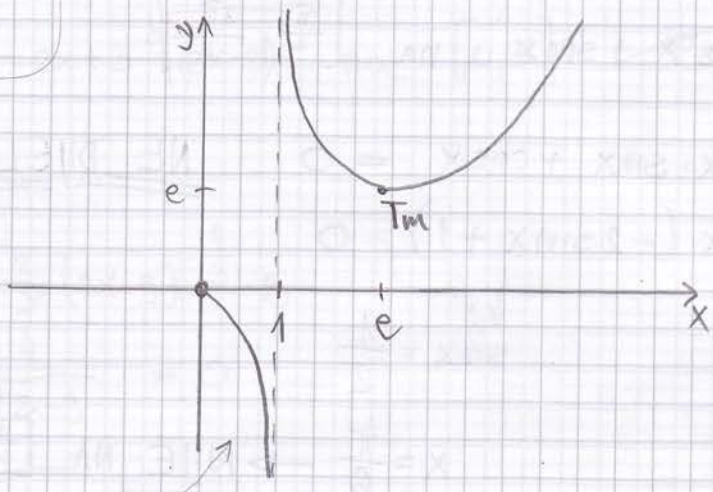
MIN

PAZI : INTERVALI PADA : BEZ 1 !!

dakle $\langle -\infty, 1 \rangle \cup \langle 1, e \rangle$

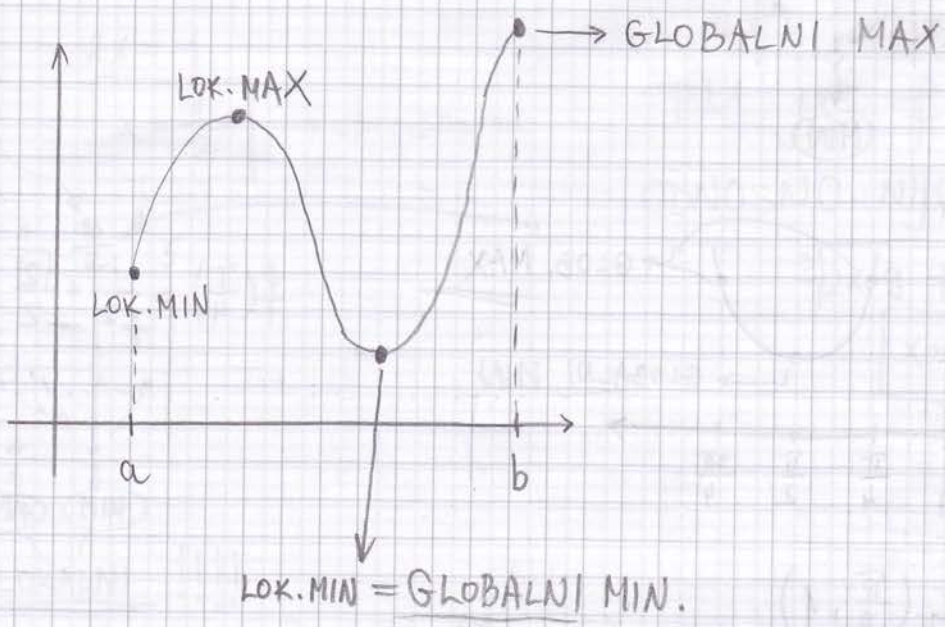
min → $T_m(e, e)$

4. GRAF



$\lim_{x \to 0^+} \frac{x}{\ln x} = 0$ → DA VIDIMO KAKO SE GRAF PONAŠA KADA IDEMO PRETA NULI S DESNA

Odredi lok. i globalne ekstreme f. na INTERVALU:



→ Dakle, trebamo vidjeti kako se ponašamo na RUBOVIMA INTERVALA

Z1-06.-2.) $f(x) = \cos^2 x + \sin x$, na $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

$f'(x) = -2\cos x \cdot \sin x + \cos x = 0$ NE DIJELI S $\cos x$!!

$\cos x \left(-2\sin x + 1\right) = 0$

$\cos x = 0$

$\boxed{x = \frac{\pi}{2}}$

samo na $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

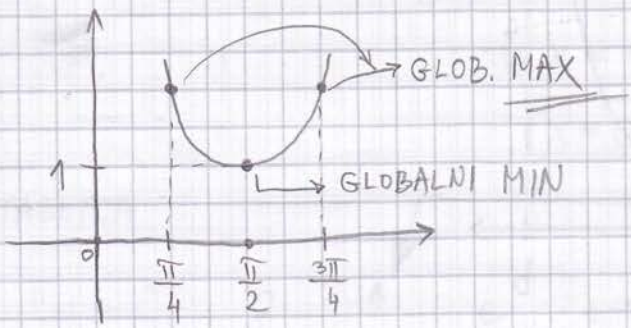
$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$ → NIJE NA $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$!!

NE

	$\frac{\pi}{4}$		$\frac{\pi}{2}$		$\frac{3\pi}{4}$
$f'(x)$		$-$		$+$	
$f(x)$		↘		↗	

MIN



$f\left(\frac{\pi}{4}\right) = \frac{1}{2} + \frac{\sqrt{2}}{2} = f\left(\frac{3\pi}{4}\right)$

$T_m\left(\frac{\pi}{2}, 1\right)$

$T_M\left(\frac{\pi}{4}, \frac{1}{2} + \frac{\sqrt{2}}{2}\right)$ i za $\frac{3\pi}{4}$!!

-09.-13) $f(x) = 2x + 3\cdot\sqrt[3]{(2x-9)^2}$ na $[0,5]$ → OVAKAV TIP VRLO LAKO NA ISPITU !!!

BHHH

→ nema V.A.

$$f'(x) = 2 + \cancel{3}\cdot\frac{2}{\cancel{3}}(2x-9)^{-\frac{1}{3}}\cdot 2$$

$$f'(x) = 2 + \frac{4}{\sqrt[3]{2x-9}} = 0$$

$$\frac{2\cdot\sqrt[3]{2x-9} + 4}{\sqrt[3]{2x-9}} = 0$$

→ NE ZABORAVI !!! IDE U TABLICU, SVE KRITIČNE TOČKE !!! TU JE f ŠILJAK, ODNOSNO NIJE DIF.!

$$\sqrt[3]{2x-9} \neq 0$$

$$\boxed{x \neq \frac{9}{2}}$$

$$\sqrt[3]{2x-9} = -2 \quad |^3$$

$$2x - 9 = -8$$

$$2x = 1$$

$$\boxed{x = \frac{1}{2}}$$ → stacionarna

	0		$\frac{1}{2}$		$\frac{9}{2}$		5
$f'(x)$		+		−		+	
$f(x)$		↗		↘		↗	
			MAX		MIN		

LOKALNI !!!!

$T_M\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) \longrightarrow T_M\left(\frac{1}{2}, 13\right)$ lokalni maximum

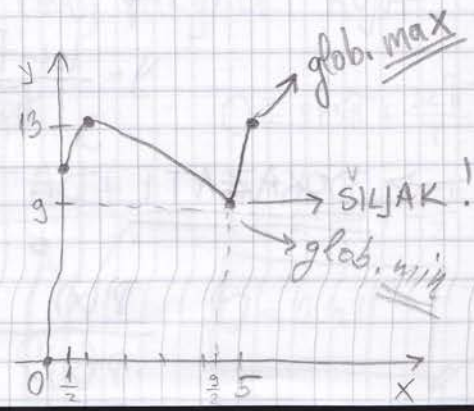
$T_m\left(\frac{9}{2}, f\left(\frac{9}{2}\right)\right) \longrightarrow T_m\left(\frac{9}{2}, 9\right)$ lokalni minimum → ŠILJAK !!!

sada vidjeti je li globalno, RUBOVI:

$f(0) = 9\sqrt[3]{3}$ $\qquad f(5) = 13$

⟹ globalni minimum: $T_m\left(\frac{9}{2}, 9\right)$

globalni maximum: $(5, 13)$



ZADACI S „RJEČIMA" → SIGURNO NA ISPITU !!

→ ZNATI SVE FORMULE ZA OPLOŠJA ....

Z1-10'-1.) Pravokutna livada, ograda 100 kn/m, pokriti podlogom livadu 200 kn/m². Ako imamo 20 000 kn kolike moraju biti dimenzije ove livade da ogradimo maksimalno veliku površinu?

$2x+2y$ ograde

$$\begin{cases} (2x+2y)\cdot 100 \rightarrow \text{novci ograde} \\ x\cdot y\cdot 200 \rightarrow \text{novci za podlogu} \end{cases}$$

$$(2x+2y)\cdot 100 + x\cdot y\cdot 200 = 20\,000$$

$$\vdots$$

$$\underline{x+y+xy=100}$$

$$\rightarrow y = \frac{100-x}{1+x}$$

$$P = x\cdot y$$

$$P = x\cdot\frac{100-x}{1+x} = \frac{100x-x^2}{1+x}$$ → NAĆI MAX OVE FUNKCIJE JER SE TRAŽI max velika površina!

$$P'(x) = \frac{100-x^2-2x}{(1+x)^2} = 0$$

$$x_{1,2} = \frac{2\pm\sqrt{404}}{-2} = \frac{2\pm 2\sqrt{101}}{-2}$$

$$x_1 = -1-\sqrt{101}$$ NE

$$\boxed{x_2 = -1+\sqrt{101}}$$

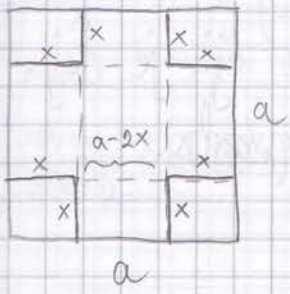
$$y = \frac{101-\sqrt{101}}{\sqrt{101}} = \sqrt{101}-1$$

$$\left.\right\} \underline{x = y = \sqrt{101}-1} \rightarrow \underline{\text{KVADRAT}}$$

→ <u>DOKAZATI</u> DA SE RADI O MAX !!

		$\sqrt{101}-1$	
$P'(x)$	+	0	−
$P(x)$	↗		↘

Karton, kvadrat, a stranica. Želimo napraviti kutiju (otvorenu) maksimalnog volumena.



$$V = \underbrace{(a-2x)^2}_{\text{BAZA}} \cdot \underbrace{x}_{\text{VISINA}}$$

$$V = \overset{\text{konstanta!!}}{a^2}x - 4ax^2 + 4x^3 \quad \Big|'$$

$$V'(x) = a^2 - 8ax + 12x^2 = 0$$

NE $\boxed{x_1 = \frac{1}{2}a}$ $\boxed{x_2 = \frac{1}{6}a}$ DA!!

NEMA SMISLA JER : $x = \frac{a}{2} \rightarrow$ BAZA:

$$a - 2x = a - 2\cdot\frac{a}{2} = 0$$

	0		$\frac{1}{6}a$		$\frac{1}{2}a$
$V'(x)$		$+$		$-$	
$V(x)$		$\nearrow$		$\searrow$	

MAX $\rightarrow$ DOKAZALI!

$$V\left(\frac{1}{6}a\right) = \frac{4}{9}a^2 \cdot \frac{1}{6}a$$

$$\boxed{V\left(\frac{1}{6}a\right) = \frac{2a^3}{27}}$$

2.) Koliko je minimalno oplošje valjka ako mu je volumen 16π?

$$V = 16\pi = B\cdot v = r^2\pi \cdot v \rightarrow v = \frac{16}{r^2}$$

$$O = 2B + P = 2r^2\pi + 2r\pi \cdot v$$

$$O = 2r^2\pi + \frac{32\pi}{r}$$ $\longrightarrow$ DERIVIRATI PO ONOJ VARIJABLI KOJA OSTANE!!!

$$O'(r) = 4r\pi - \frac{32\pi}{r^2} = 0 \quad \Big| \cdot \frac{r^2}{4\pi}$$

$$r^3 - 8 = 0$$

$$r^3 = 8 \longrightarrow \boxed{r = 2}$$

	0		2
$O'(r)$		$-$	$+$
$O(r)$		$\searrow$	$\nearrow$

MIN

$$O = 2r^2\pi + \frac{32\pi}{r}$$

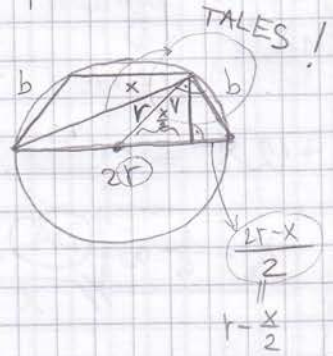
$$O = 2\cdot 4\pi + \frac{32\pi}{2}$$

$$\boxed{O = 24\pi}$$

→ (Z1) → SVI ĆE BITI SA RIJEČIMA ! ?? BUREK ?

→ 6 ZAD, 3 SA DERIVACIJAMA I SVA 3 SA RIJEČIMA !

10.DZ-8.) U kružnicu polumjera r upisan je trapez čija je dulja osnovnica promjer kružnice. Kada će imati maximalnu površinu?



$P=\frac{a+c}{2}\cdot v$

$a=2r$

$c=x$

$P=\frac{2r+x}{2}\cdot v$

ILI $v=\sqrt{r^2-\left(\frac{x}{2}\right)^2}$

$\boxed{v=\sqrt{p\cdot q}} \longrightarrow v=\sqrt{\left(r-\frac{x}{2}\right)\left(r+\frac{x}{2}\right)}$

$$P=\frac{2r+x}{2}\cdot\sqrt{\left(r-\frac{x}{2}\right)\left(r+\frac{x}{2}\right)}=\left(r+\frac{x}{2}\right)\sqrt{r^2-\frac{x^2}{4}}$$

$$P'(x)=\frac{1}{2}\sqrt{r^2-\frac{x^2}{4}}+\left(r+\frac{x}{2}\right)\cdot\frac{1}{2\sqrt{r^2-\frac{x^2}{4}}}\cdot\frac{-2x}{4}=0$$

$$\frac{1}{2}r^2-\frac{1}{8}x^2-\frac{1}{4}rx-\frac{1}{8}x^2=0 \quad |\cdot(-8)$$

$$2x^2+2rx-4r^2=0$$

$$x_{1,2}=\dots$$

$x_1=-2r$ NE

$\boxed{x_2=r}$

	0		r	
$P'(x)$		+		−
$P(x)$		↗	MAX	↘

$$\boxed{P=\frac{3r^2\sqrt{3}}{4}}$$

* NASTAVAK PREDAVANJA *

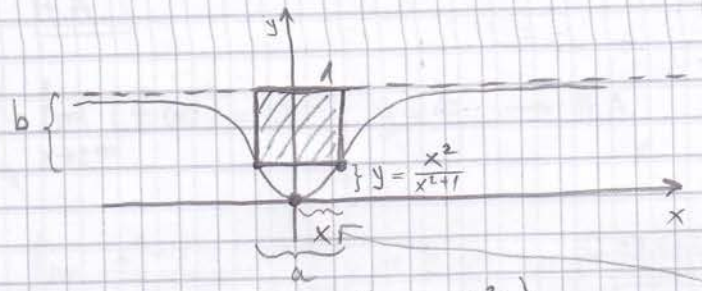
21-09-13.) Naći pravokutnik max površine čija se 2 vrha nalaze na $y=\frac{x^2}{x^2+1}$, a jedna stranica na horizontalnoj asimptoti te krivulje.

H.A.

$$\lim_{x\to\pm\infty}\frac{x^2 \;/:x^2}{x^2+1 \;/:x^2} = 1 \longrightarrow \boxed{y=1} \text{ H.A.}$$

→ parna je ⇒ simetrična, zbog kvadrata

za $x=0 \to y=0$, sve vrijednosti su između 0 i 1 ! kvadrat



$$P(x) = \underbrace{2x}_{a}\cdot\underbrace{\left(1-\frac{x^2}{x^2+1}\right)}_{b}$$

$$P(x) = \frac{2x}{x^2+1} \longrightarrow P'(x) = \frac{2-2x^2}{(x^2+1)^2} = 0$$

$x^2 = 1$

$x = \pm 1$ → zbog slike

$\boxed{x=1}$

	0	1	
$P'(x)$	+		−
$P(x)$	↗		↘

MAX

$$P(1) = \frac{2}{1+1} = 1$$

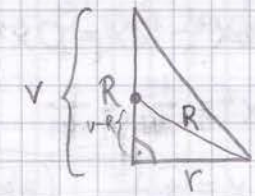
8.) Kugli polumjera R upiši uspravan kružni stožac.

$$V=\frac{1}{3}r^2\pi\cdot v$$

$$V=\frac{\pi}{3}\left(R^2-(v-R)^2\right)\cdot v$$

$$\vdots$$

$$V=\frac{\pi}{3}\left(2v^2R-v^3\right)$$



$$r=\sqrt{R^2-(v-R)^2}$$

$$V'(v)=\frac{\pi}{3}\left(4vR-3v^2\right)=0$$

ZADAN !! R je zadan da je R !!! kečap?

$$\boxed{v=\frac{4R}{3}}$$

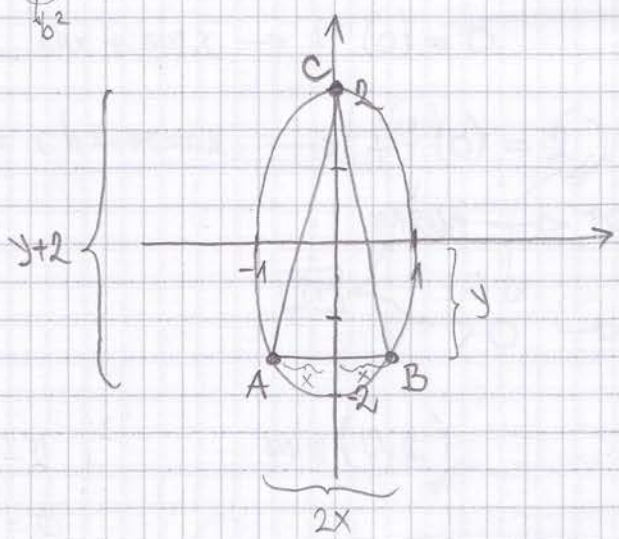
$V'(v)$	$+$	0	$-$
$V(v)$	$\nearrow$	MAX	$\searrow$

$$\boxed{V\left(\frac{4R}{3}\right)=\frac{32R^3\pi}{81}}$$

9.) Jednakokračan trokut ABC upiši u elipsu $4x^2+y^2=4$ tako da mu je vrh $C(0,2)$ i nasuprot AB.

$$\frac{x^2}{1}+\frac{y^2}{4}=1$$

$$y=\sqrt{4-4x^2}$$



$$P=\frac{1}{2}a\cdot v_a=\frac{1}{2}\cdot 2x\cdot(y+2)$$

$$P=x\left(2+\sqrt{4-4x^2}\right)$$

$$P'(x)=2+\sqrt{4-4x^2}+x\cdot\frac{1}{2\sqrt{4-4x^2}}\cdot(-8x)$$

$$\vdots$$

$$P'(x)=\frac{\sqrt{4-4x^2}+2-4x^2}{\frac{1}{2}\sqrt{4-4x^2}}=0$$

$$\sqrt{4-4x^2} = 4x^2-2 \quad |^2$$

$$16x^4 - 12x^2 = 0$$

$$x^2(16x^2-12) = 0$$

$$x^2 = \frac{12}{16} = \frac{3}{4}$$

$$x = \pm\frac{\sqrt{3}}{2} \quad \text{(zbog slike)}$$

	0		$\frac{\sqrt{3}}{2}$	
$P'(x)$		$+$		$-$
$P(x)$		$\nearrow$		$\searrow$

MAX

$$\boxed{P = \frac{3\sqrt{3}}{2}}$$

10.DZ-6.) Dokaži da je $\underbrace{x - \frac{x^2}{2} < \ln(1+x)}_{\text{I}} \underbrace{< x}_{\text{II}}$, za $\forall\, x > 0$.

I $\underbrace{\ln(1+x) - x + \frac{x^2}{2}}_{f(x)} > 0$

$f(0) = 0 \rightarrow$ DOKAZATI DA JE f RASTUĆA! (poslije nule sve veće od nule)

$f'(x) = \frac{1}{1+x} - 1 + x = \frac{x^2}{1+x}$ za $x > 0$ $f'(x)$ UVIJEK JE VEĆE OD 0

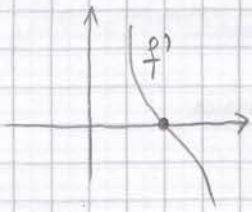
FUNKCIJA JE RASTUĆA KADA JOJ JE DERIVACIJA > 0

II $x - \ln(1+x) > 0$

10.2. KONVEKSNOST I KONKAVNOST

→ koeficijenti se smanjuju → $f'(x)$ pada

nagib pada



$\Rightarrow \boxed{f''(x) < 0}$

Ako je $f'(c)=0$, $f''(c)=0$, ..., $f^{(n-1)}(c)=0$ i ako je $f^{(n)}(c)\neq 0$ tada c NIJE EKSTREM ako je n neparan, a JE EKSTREM ako je n paran i to c je MINIMUM ako je $f^{(n)}(c)>0$, a c je MAXIMUM ako je $f^{(n)}(c)<0$.

Dokaz: direktno iz Taylora ⇓

$f(x) = \operatorname{ch} x + \cos x$. Ekstremi?

$f'(x) = \operatorname{sh} x - \sin x = 0$

$\operatorname{sh} x = \sin x$

$\boxed{x=0}$

$f''(x) = \operatorname{ch} x - \cos x \longrightarrow f''(0) = 0 \longrightarrow$ tražimo prvu, različitu od 0

$f'''(x) = \operatorname{sh} x + \sin x \longrightarrow f'''(0) = 0$

$f^{iv}(x) = \operatorname{ch} x + \cos x \longrightarrow f^{(iv)}(0) = 2$

PARNA → POSTOJI EKSTREM!

$\boxed{TM}$ → >0 → MINIMUM

$\min(0,2)$

$\boxed{\text{ŠK}}$

→ der. po def.

→ Postoji li der. ?

→ implic. i param. →

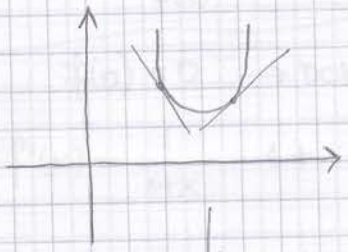
→ tangente, s riječima zadaci, kut između krivulja

→ divljački L'H

→ sve cakice sa asimptotama

def. • Ako je $f''(x) > 0$ na (a,b), prva derivacija raste (raste nagib tangente) i tada kažemo da je funkcija **KONVEKSNA**.

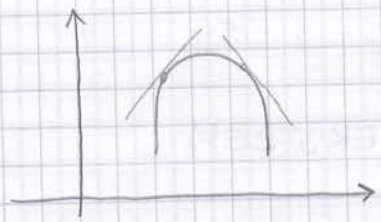
• Ako je $f''(x) < 0$, prva derivacija pada i tada kažemo da je f **KONKAVNA**.



$f''(x) > 0$

↓

KONVEKSNA FUNKCIJA JE **SRETNA** !!! ☺

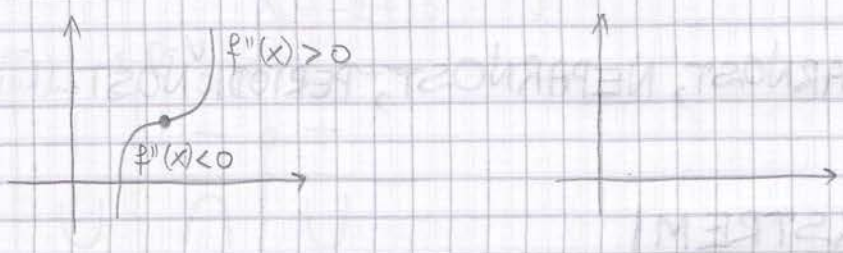


$f''(x) < 0$

↓

KONKAVNA FUNKCIJA JE **TUŽNA** !!! ☹

Točka kod koje dolazi do promjene predznaka druge derivacije zovemo točka **INFLEKSIJE** (pregiba).



ODZ-13B) $f(x) = (1+x^2)e^x \implies D(f) = \mathbb{R}$

$$f'(x) = 2xe^x + (1+x^2)e^x$$

$$f''(x) = 2e^x + 2xe^x + 2xe^x + (1+x^2)e^x$$

$$= e^x(3+4x+x^2) = 0$$

>0 UVIJEK!

$$x^2+4x+3=0$$

$$x_1 = -1$$

$$x_2 = -3$$

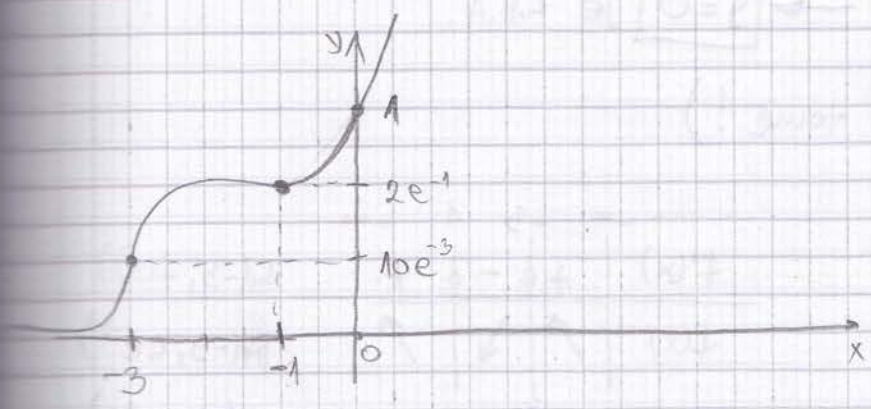
	$-\infty$	-3	-1	$+\infty$
$f''(x)$	+	−	+	
$f(x)$	∪	∩	∪	

T_{i_1} T_{i_2}

UVRSTITI U $f(x)$ IZVORNU FUNKCIJU!!

NE U $f''(x)$ ili $f'(x)$!

$T_{i_1}(-3, 10e^{-3})$ $T_{i_2}(-1, 2e^{-1})$



10.3. KVALITATIVNI GRAF FUNKCIJE

→ sigurno na ispitu !!

1. DOMENA
2. NULTOČKE, PARNOST, NEPARNOST, PERIODIČNOST... (karakteristi
3. ASIMPTOTE
4. RAST, PAD, EKSTREMI
5. KIKI (KONVEKSNOST, KONKAVNOST, INFLEKSIJA)
6. GRAF

z1-'07-3.) $f(x) = (x^2-3)e^x$

1. $D(f) = \mathbb{R}$

2. $x^2 - 3 = 0$

$x^2 = 3 \longrightarrow x_{12} = \pm\sqrt{3}$ // → NULTOČKE

3. V.A. → nema (po $D(f)$)

H.A.

$\lim\limits_{x\to+\infty} (x^2-3)e^x = (\infty\cdot\infty) = +\infty$ → NEMA D.H.A.

$\lim\limits_{x\to-\infty} (x^2-3)e^x = (\infty\cdot 0) = \lim\limits_{x\to-\infty} \dfrac{x^2-3}{\frac{1}{e^x}} = \left(\dfrac{\infty}{\infty}\right)$ DEREMO L'H =

$= \lim\limits_{x\to-\infty} \dfrac{2x}{-e^{-x}}$ = DEREMO JOŠ JEDNOM L'H =

$= \lim\limits_{x\to-\infty} \dfrac{2}{e^{-x}} = \dfrac{2}{\infty} = 0$ // → $\boxed{y=0}$ JE L.H.A.

→ KOSE NEMA (razmisliti o tome !)

4. $f'(x) = e^x(x^2+2x-3) = 0$

$x^2+2x-3=0$

$x_1 = 1$

$x_2 = -3$

	$-\infty$	-3	1	$+\infty$
$f'(x)$	+	0 −	0	+
$f(x)$	↗	↘		↗
		MAX	MIN	

$T_M(-3, f(3))$

$T_M(-3, 6e^{-3})$

$T_m(1, -2e)$

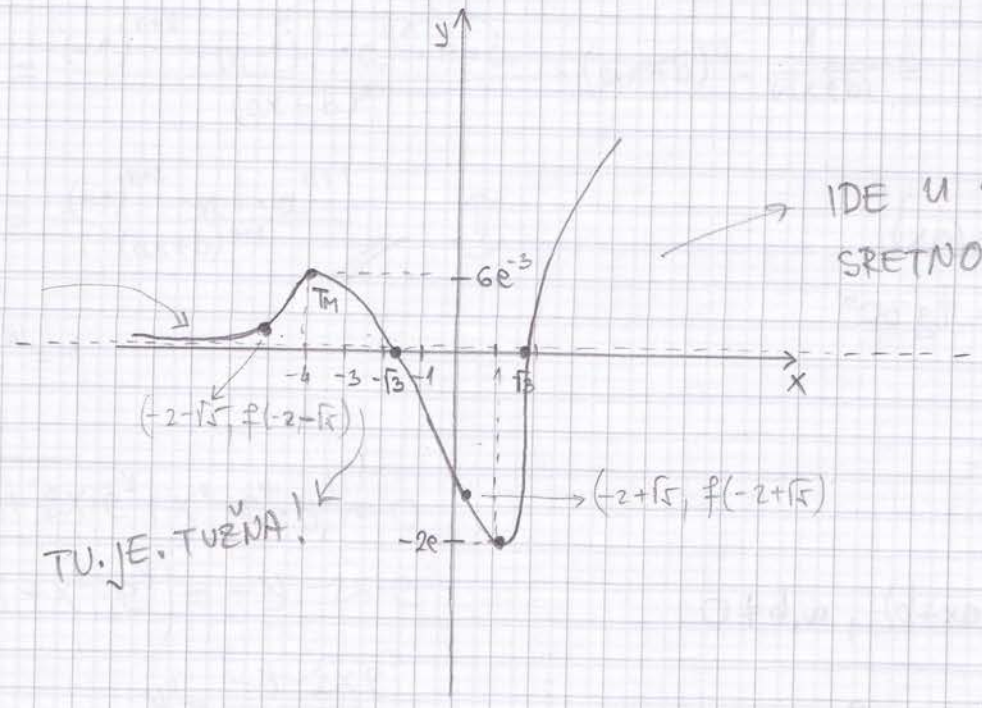
$$f''(x) = e^x(x^2+4x-1) = 0$$

$$x_1 = -2-\sqrt{5}$$

$$x_2 = -2+\sqrt{5}$$

NAJČEŠĆE „RUŽNI" BR. jer je teško namjestiti

	$-\infty$		$-2-\sqrt{5}$		$-2+\sqrt{5}$		$+\infty$
$f''(x)$		$+$	0	$-$	0	$+$	
$f(x)$		$\cup$	T_{i_1}	$\cap$	T_{i_2}	$\cup$	



* NASTAVAK PREDAVANJA *

z1-10-2.) Izračunaj sve i nacrtaj $f(x) = x + 2\,\mathrm{arctg}\,\frac{1}{x}$.

1.) $D(f) = \mathbb{R} \setminus \{0\}$

2.) nultočke → nema smisla, nisu potrebne

3.) asimptote

1° V.A.

$$\lim_{x\to 0^+}\left(x + 2\,\mathrm{arctg}\,\overset{+\infty}{\frac{1}{x}}\right) = \pi$$

$$\lim_{x\to 0^-}\left(x + 2\,\mathrm{arctg}\,\overset{-\infty}{\frac{1}{x}}\right) = -\pi$$

} NEMA V.A (nije $\pm\infty$)

2° H.A

→ nema! (očito da je ∞)

3° K.A

$$k = \lim_{x\to\pm\infty}\left(1 + \frac{2\,\mathrm{arctg}\,\overset{0}{\frac{1}{x}}}{\underset{\infty}{x}}\right) = \lim_{x\to\pm\infty}(1+0) = 1$$

ISTO

$$l = \lim_{x\to+\infty}\left(\cancel{x} + 2\,\mathrm{arctg}\,\overset{0}{\frac{1}{x}} - \cancel{x}\right) = 0$$

ISTO

$y = x$ → kosa i sa lijeve i sa desne

4.) rast, pad, ekstremi

$$f'(x) = 1 + \frac{-1}{1+\left(\frac{1}{x}\right)^2}\cdot\frac{-1}{x^2} = \frac{x^2-1}{x^2+1} = 0$$

$x^2 - 1 = 0$

$x = \pm 1$

	$-\infty$	-1	0	1	
$f'(x)$	+	−	−	+	
$f(x)$	↗	↘	↘	↗	
		M		m	

$T_{MAX}\left(-1, -\frac{\pi}{2}\right)$

$T_{min}\left(1, 1+\frac{\pi}{2}\right)$

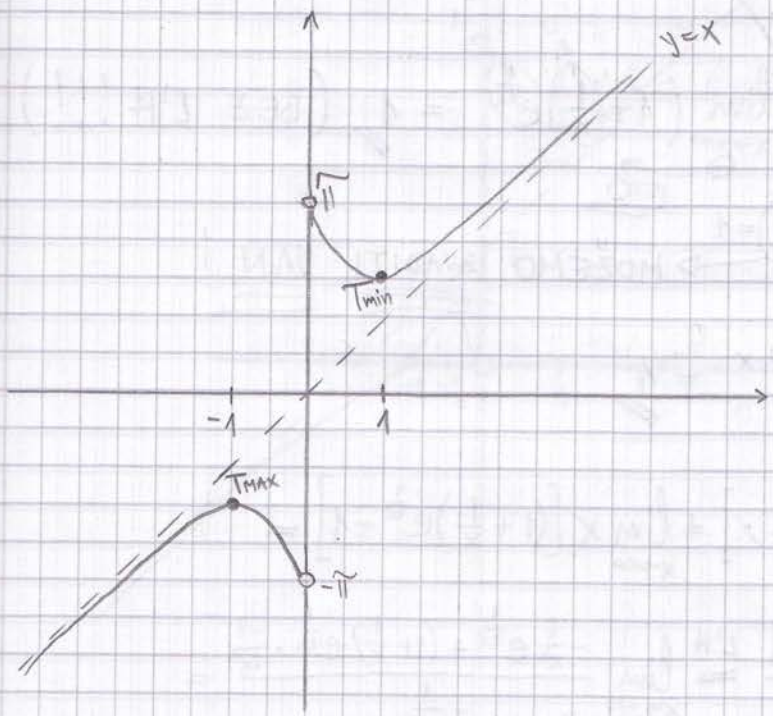
$$f''(x) = \dots = \frac{4x}{(x^2+1)^2} = 0$$

$\boxed{x=0}$ → NEMA INFLEKSIJE !!

	$-\infty$	0	$+\infty$
$f''(x)$	$-$		$+$
$f(x)$	$\cap$		$\cup$

PAZI !!

!! NIJE TOČKA INFLEKSIJE JER NIJE U DOMENI FUNKCIJE !



10.DZ-20.) Izračunaj sve i nacrtaj $f(x)=(x+1)e^{\frac{1}{2x}}$. (bez kiki).

ILKO DOBRE VOLJE ?!

1.) $D(f)=\mathbb{R}\setminus\{0\}$

2.) nultočke:

$\boxed{x=-1}$

3.) asimptote:

1° V.A.

NEĆE TU BITI NIKAKVOG L'HOSPITALA SHOE !! UVRSTI, PROVJERI !!)

$$\lim_{x\to 0^+}(x+1)e^{\overset{+\infty}{\frac{1}{2x}}}=+\infty \quad \text{DA}$$

BAŠ KAO NA KPZ...

$$\lim_{x\to 0^-}(x+1)e^{\overset{-\infty}{\frac{1}{2x}}}=0 \quad \text{NE}$$

$\boxed{x=0} \longrightarrow$ D.V.A. 2? !!

2° H.A

↳ NEMA

TRIK! SJETI SE BEZ L'H !!!

3° K.A.

$$k=\lim_{x\to+\infty}\frac{(x+1)e^{\frac{1}{2x}}}{x}=\lim_{x\to+\infty}\left(1+\overset{0}{\frac{1}{x}}\right)\overset{1}{e^{\frac{1}{2x}}}=1 \quad (\text{BEZ L'H !!!})$$

ISTO

ILI

$$\lim_{x\to+\infty}\frac{(x+1)e^{\frac{1}{2x}}}{x}=1 \longrightarrow \text{MOŽEMO IZVADITI VAN!}$$

$$=\lim_{x\to+\infty}\frac{x+1}{x}\begin{matrix}/:x\\/:x\end{matrix}=1$$

$$l=\lim_{x\to+\infty}\left[(x+1)e^{\frac{1}{2x}}-x\right]=\lim_{x\to+\infty}x\left[\left(1+\frac{1}{x}\right)e^{\frac{1}{2x}}-1\right]=$$

$$=\lim_{x\to+\infty}\frac{\left[\left(1+\frac{1}{x}\right)e^{\frac{1}{2x}}-1\right]}{\frac{1}{x}}\overset{L'H}{=}\lim_{x\to+\infty}\frac{-\frac{1}{x^2}e^{\frac{1}{2x}}+\left(1+\frac{1}{x}\right)\cdot e^{\frac{1}{2x}}\cdot\frac{-1}{2x^2}}{-\frac{1}{x^2}}=$$

ISTO

$$=\frac{1+\frac{1}{2}}{1}=\frac{3}{2}$$

BEZ L'H PAMETNA KOLEGICA !???

$\longrightarrow y=x+\frac{3}{2}$ K.A. s obje strane

$= \{$ KOLEGICA BEZ DERIVIRANJA NARAVNO $\}$

$= e^{\frac{1}{2x}} + (x+1)e^{\frac{1}{2x}} \cdot \frac{-1}{2x^2} = 0 \quad | : e^{\frac{1}{2x}}$ UVIJEK $\neq 0$ PA MOŽEMO !

$1 - \frac{1}{2x} - \frac{1}{2x^2} = 0 \quad | \cdot 2x^2$

$2x^2 - x - 1 = 0$

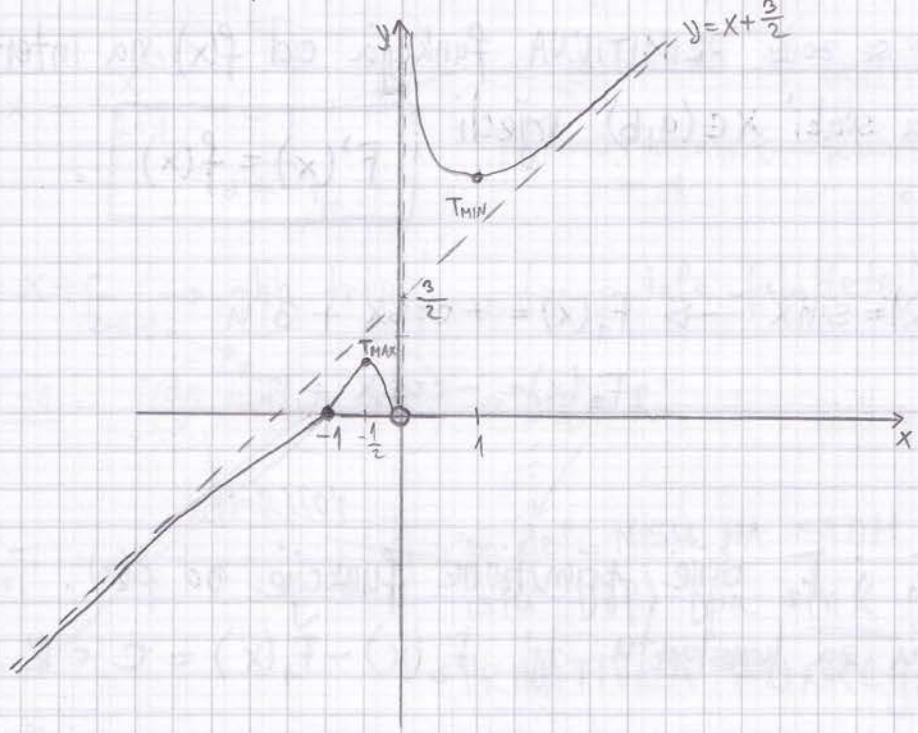
$x_{1,2} = \frac{1 \pm 3}{4}$ $\nearrow x_1 = 1$, $\searrow x_2 = -\frac{1}{2}$

	$-\infty$	$-\frac{1}{2}$	0	1	$+\infty$
$f'(x)$	$+$	$-$	$-$	$+$	
$f(x)$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$	
		MAX		MIN	

$T_{MAX}\left(-\frac{1}{2}, \frac{1}{2}e^{-1}\right)$

$T_{MIN}\left(1, 2e^{\frac{1}{2}}\right)$

$\rightarrow$ ne treba $\tilde{\pi}$ PROKLETI ILKO !!!!



INTEGRALNI RAČUN

11.1. NEODREĐENI INTEGRALI

zzv-5.) Naći funkciju $f(x)$ za koju vrijedi $F'(x) = e^{-x}$ uz uvjet $F(0)=0$.

$\longrightarrow$ DIFERENCIJALNE JEDNADŽBE $\rightarrow$ $F(x) = ?$

$\hookrightarrow$ cijela FIZIKA 2 ....

$F(x) = -e^{-x} + C$ $\longrightarrow$ SKUP svih funkcija različitih za const.

$F(0) = -e^{0} + C = 0$

$\boxed{C = 1}$ $\longrightarrow$ $\boxed{F(x) = -e^{-x} + 1}$

def. ČESTO NA ISPITIMA

Funkcija $F(x)$ se zove PRIMITIVNA funkcija od $f(x)$ na intervalu $[a,b]$ ako za svaki $x \in (a,b)$ vrijedi $\boxed{F'(x) = f(x)}$.

npr.

ako je $f(x) = \sin x$ $\longrightarrow$ $F_1(x) = -\cos x + 314$

$F_2(x) = -\cos x + \pi^{\pi^{\pi^{\cdots}}}$ u p. mat.

$\vdots$

$\boxed{\text{TM}}$ Neka su F_1 i F_2 dvije primitivne funkcije od $f(x)$. Tada se one razlikuju za konstantu, tj. $F_2(x) - F_1(x) = C \in \mathbb{R}$.

Dokaz:

$F_1'(x) = F_2'(x) = f(x)$

$\longrightarrow$ po korolaru La Grangeova TM srednje vrijednosti odmah slijedi da se razlikuju za konstantu $\Updownarrow$

Skup svih primitivnih funkcija od f na (a,b) nazivamo NEODREĐENI INTEGRAL od f i označavamo ga s

$$\int f(x)\,dx = \{F(x)+C,\ C\in\mathbb{R}\}$$

Funkcija za koju postoji neodređeni integral na (a,b) se zove INTEGRABILNA funkcija.

$$\boxed{\int f(x)\,dx = F(x)+C \iff F'(x)=f(x)}$$

$f(x)$ → podintegralna funkcija

dx → DIFERENCIJAL VARIJABLE PO KOJOJ INTEGRIRAMO

$$\boxed{\int x^{\alpha}\cdot dx = \frac{x^{\alpha+1}}{\alpha+1}+C}$$

$\alpha \neq -1$

→ sve suprotno od deriviranja

podjelimo i povećamo za 1 ← pomnožimo i smanjimo za 1

$\int dx = x+C$ → ph trivić.... (za ostale fakultete)

za $\alpha=-1$: $\int \frac{dx}{x} = \ln x + C$ → $D(f)=\mathbb{R}^+$

$D(f)=\mathbb{R}\setminus\{0\}$

→ po def., NISU NA ISTOM intervalu, nemaju istu $D(f)$ pa $\ln x$ NIJE PRIMITIVNA FUNKCIJA OD $\frac{1}{x}$!!!

→ zato definiramo:

$$\boxed{\int \frac{dx}{x} = \ln|x|+C}$$

→ tablica integrala → sve naopako

PRIMJEDBA:

$$\boxed{\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C}$$ na $(-\infty,-a)$ ili $(-a,a)$ ili $(a,+\infty)$

↓

PAMTI NAPAMET!

$$\boxed{\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C}$$ ALI na intervalima $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ....

↳ BITNO KASNIJE ZA ODREĐENE!

→ NEELEMENTARNI INTEGRALI:

npr.) $\int \frac{\sin x}{x} dx$ ili $\int e^{x^2} dx$ → NE ZNAMO!! (sve do 4. godine ...)

• NEPOSREDNO (DIREKTNO) INTEGRIRANJE •

Pr.1) $\int (2x^2 - 4e^x + 5)\, dx = 2\int x^2 dx - 4\int e^x dx + 5\int dx =$

$$= 2\frac{x^3}{3} - 4e^x + 5x + C$$

Zad.) $\int \frac{(x+1)^2}{\sqrt{x}} dx = \int \frac{x^2+2x+1}{\sqrt{x}} dx = \int \left(\frac{x^2}{\sqrt{x}} + \frac{2x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx =$

$$= \int \left(x^{\frac{3}{2}} + 2\cdot x^{1/2} + x^{\frac{1}{2}}\right) dx = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{2\cdot x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$

*

TABLICA INTEGRALA

1) $\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C$

2) $\int a^{x} dx = \frac{a^{x}}{\ln a} + C$

3) $\int \sin x \, dx = -\cos x + C$

4) $\int \cos x \, dx = \sin x + C$

5) $\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$

6) $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$

7) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$

8) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \operatorname{arctg}\left(\frac{x}{a}\right) + C$

9) $\int \operatorname{sh} x \, dx = \operatorname{ch} x + C$, $\int \operatorname{ch} x \, dx = \operatorname{sh} x + C$

10) $\int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x + C$

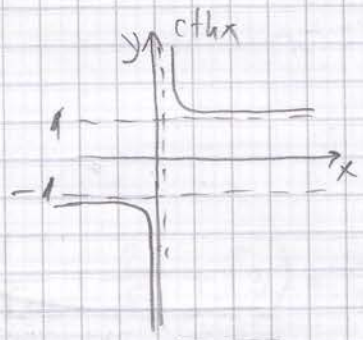
11) $\int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x + C$

12) $\int \frac{dx}{\sqrt{x^2 + a}} = \ln\left|x + \sqrt{x^2 + a}\right| + C$

13) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$

DOMAĆE VJEŽBE ⟵

Izvedi sve i nacrtaj $f(x) = \text{cth}\left(\frac{x^2}{1-x}\right)$.



1) $D(f) = \mathbb{R} \setminus \{0, 1\}$

2) 1° V.A:

$$\lim_{x\to 0^+} \text{cth}\frac{x^2}{1-x} = \text{cth}\left(\lim_{x\to 0^+} \frac{x^2}{1-x}\right) = +\infty \qquad \text{V.A.} \Rightarrow \boxed{x=0}$$

$\ominus$ → ISTO

$$\lim_{x\to 1^-} \text{cth}\frac{x^2}{1-x} = \text{cth}\left(\lim_{x\to 1^-} \frac{x^2}{1-x}\right) = \text{cth}(+\infty) = 1 \quad \text{NE}$$

$$\lim_{x\to 1^+} \text{cth}\frac{x^2}{1-x} = \text{cth}\left(\lim_{x\to 1^+} \frac{x^2}{1-x}\right) = \text{cth}(-\infty) = -1 \quad \text{NE}$$

2° H.A.:

$$\lim_{x\to +\infty} \text{cth}\frac{x^2}{1-x} = \text{cth}\left(\lim_{x\to +\infty} \frac{x^2}{1-x}\right) = \text{cth}\left(\lim \frac{1}{\frac{1}{x^2} - \frac{1}{x}}\right) =$$

$$= \text{cth}(-\infty) = -1 \qquad \boxed{y=-1} \quad \text{D.H.A}$$

$$\lim_{x\to -\infty} \text{cth}\frac{x^2}{1-x} = 1 \qquad \boxed{y=1} \quad \text{L.H.A}$$

3) ekstremi

$$f'(x) = \frac{-1}{\text{sh}^2\left(\frac{x^2}{1-x}\right)} \cdot \frac{2x(1-x) + x^2}{(1-x)^2}$$

$$f'(x) = 0 \iff 2x - 2x^2 + x^2 = 0$$

$$2x - x^2 = 0$$

$$x(2-x) = 0$$

NE $x=0$ → NIJE U $D(f)$ $\qquad \boxed{x=2}$

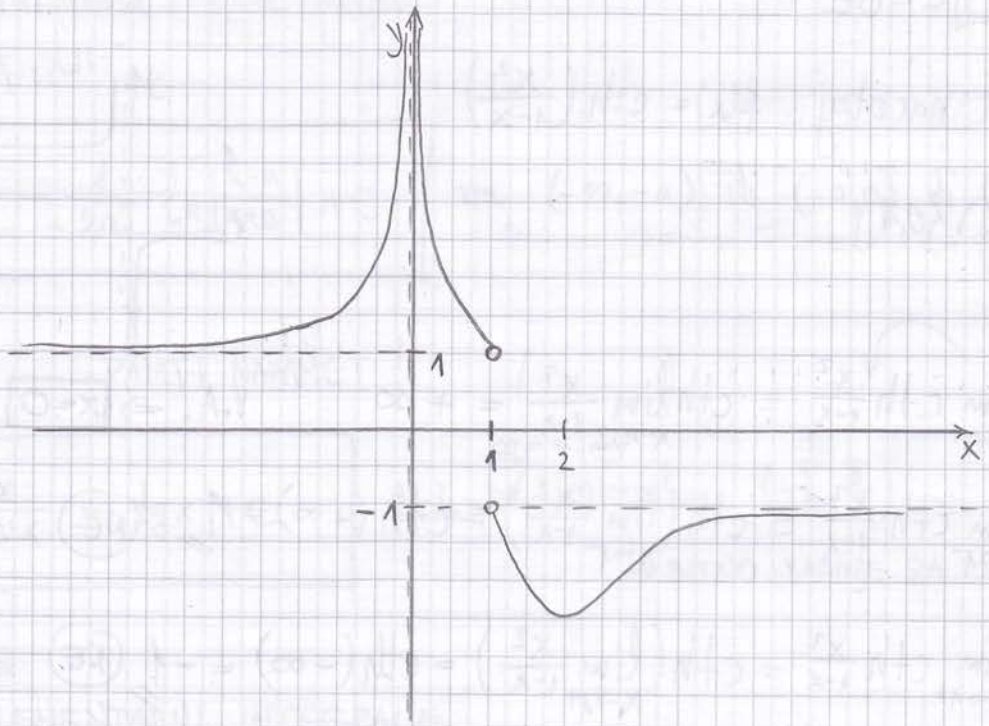
$f(2) = \text{cth}(-4)$

$f(2) < -1$

	$-\infty$	0	1	2	$+\infty$
$f'(x)$	+	−	−	+	
$f(x)$	↗	↘	↘	↗	

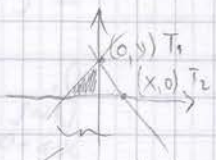
$T_m(2, <-1)$

PAZI! NIJE MAX $\qquad$ MIN



ZZV-E-11) Iz točke na osi ordinata povučene su tangente na $y=1-x^2$ tako da s osi apscisa zatvaraju trokut minimalne površine. Odredite ga.

$$y' = -2x$$



2 trokutića čine veliki trok.

$T_1(0,y)$

$$t_1 \dots \quad y-(1-x_0^2) = -2x_0 \cdot (0-x_0)$$

$$y = x_0^2+1$$

$$t_2 \dots \quad 0-(1-x_0)^2 = -2x_0(x-x_0) \longrightarrow T_2(x,0)$$

$$x = \frac{1}{2x_0} + \frac{1}{2}x_0$$

$$P = \frac{1}{2} \cdot (x_0^2+1)\left(\frac{1}{2x_0} + \frac{1}{2}x_0\right) \cdot 2$$

$$= \frac{1}{2}\left(x_0^3 + 2x_0 + \frac{1}{x_0}\right)$$

$$P' = \frac{1}{2}\left(3x^2 + 2 - \frac{1}{x_0^2}\right)$$

$$P' = 0 \iff \left(3x_0^2 + 2 - \frac{1}{x_0^2}\right) = 0$$

$$3x_0^4 + 2x_0 - 1 = 0$$

$$x_{1,2}^2 = \frac{-2 \pm 4}{6} \quad \begin{cases} \nearrow -1 \text{ NE} \\ \searrow \frac{1}{3} \Rightarrow x = \pm\frac{1}{\sqrt{3}} \end{cases}$$

1. derivirate P'
2. $P''\left(\frac{1}{\sqrt{3}}\right) > 0$

lok. min $= \frac{1}{\sqrt{3}}$

2. način:

	0		$\frac{1}{\sqrt{3}}$	
$P'(x)$		$-$		$+$
$P(x)$		$\searrow$	MIN	$\nearrow$

$$P = \frac{1}{2}\left(\frac{1}{3\sqrt{3}} + \frac{2}{\sqrt{3}} + \sqrt{3}\right) = \frac{8}{3\sqrt{3}}$$

* PREDAVANJA *

„SVOJSTVA NEODREĐENIH INTEGRALA"

a) $\frac{d}{dx}\int f(x)\,dx = \left(\int f(x)\,dx\right)' = f(x)$

b) $\int f'(x)\,dx = f(x) + C$ Dokaz: očito! (po definiciji)

c) $\int a f(x)\,dx = a\int f(x)\,dx$

d) $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$ PAZI!!

$$\int \frac{(\sqrt{x}+1)^2}{x^2}\,dx = \int\left(\frac{1}{x} + 2x^{-\frac{3}{2}} + \frac{1}{x^2}\right)dx = \ln|x| + 2\cdot\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + \frac{x^{-1}}{-1} + C$$

!!

$$\int \frac{x^2}{x^2+5}\,dx = \int \frac{x^2+5-5}{x^2+5}\,dx = \int 1\,dx - 5\cdot\int\frac{dx}{x^2+5} = x - 5\cdot\frac{1}{\sqrt{5}}\,\mathrm{arctg}\,\frac{x}{\sqrt{5}} + C$$

TIPIČNA FINTA!

TABLICA:

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\,\mathrm{arctg}\,\frac{x}{a} + C}$$

TIN IKER ŠARIĆ GAY

Pr.) $\int tg^2x\,dx = \int \frac{\sin^2 x}{\cos^2 x}dx = \int \frac{1-\cos^2 x}{\cos^2 x}dx = \underbrace{\int \frac{1}{\cos^2 x}dx}_{\text{TABLICA! } tg x + C} - \int dx =$

$= tg x - x + C$

Pr.) $\int \frac{\sin x}{tg\frac{x}{2}}dx = \int \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}} = 2\int \cos^2\frac{x}{2}\,dx = 2\int\left(\frac{1}{2}+\frac{1}{2}\cos x\right)dx =$

S MJESECA SE VIDI ? BUREK ?

NAPOKON KORIST OD PODSJETNIKA

$= \int dx + \int \cos x\,dx = x + \sin x + C$

Pr.) $\int (3x+14)^{69}dx = ??$

$t = 3x+14$, $dt = ??$

$\Downarrow$

$\boxed{TM}$ „METODA SUPSTITUCIJE"

Neka je $f(x)$ integrabilna na $[a,b]$, a $\varphi(t)$ neprekinuta diferencijabilna na $[\alpha,\beta]$, $\varphi: [\alpha,\beta] \longrightarrow [a,b]$ i neka je φ injekcija. Tada uz supstituciju $\boxed{x=\varphi(t)}$ vrijedi:

$$\int f(x)dx = \int f(\varphi(t))\varphi'(t)\,dt$$

Dokaz:

$$\frac{d}{dt}\left[\int f(x)dx\right] = \frac{d}{dt}\left[F(x)+c\right] = \frac{d}{dt}\left[F(\varphi(t))+c\right] = F'(\varphi(t))\cdot\varphi' t =$$

$$= f(\varphi(t))\cdot\varphi'(t)$$

$\Downarrow$

ONO ŠTO JE „NAJRUŽNIJE"

Pr.) $\int (3x+14)^{69}dx = \left| \begin{array}{l} 3x+14 = t \\ 3dx = dt \\ dx = \frac{dt}{3} \end{array} \right| = \int t^{69}\frac{dt}{3} = \frac{1}{3}\cdot\frac{t^{70}}{70} + C$

derivirati

! OBAVEZNO dx !!!

$= \frac{1}{3}\cdot\frac{(3x+14)^{70}}{70} + C$ //

ZENITO:

$$\int f(ax+b)\,dx = \left| \begin{array}{l} ax+b=t \\ a\,dx=dt \\ dx=\frac{dt}{a} \end{array} \right| = \frac{1}{a}\int f(t)\,dt = \frac{1}{a}F(t)+C = \frac{1}{a}F(ax+b)+C$$

$$\int \cos(\pi x)\,dx = \frac{1}{\pi}\sin(\pi x)+C$$

!!

$$\int e^{-\frac{1}{7}x+5}\,dx = \frac{1}{-\frac{1}{7}}e^{-\frac{1}{7}x+5}+C$$

$$\int x^2 ch(x^3+1)\,dx = \left| \begin{array}{l} x^3+1=t \\ 3x^2\,dx=dt \\ dx=\frac{dt}{3x^2} \end{array} \right| =$$

AKO OSTANE x sa t-om POD INTEGRALOM ONDA JE SUPSTITUC. KRIVA !!

TAKO MORA BITI

NE SMIJE BITI x i t U ISTO VRIJEME !

$$= \int \cancel{x^2}\, ch\,t \cdot \frac{dt}{3\cancel{x^2}} =$$

NE ZABORAVI !

$$= \frac{1}{3}\int ch\,t\,dt = \frac{1}{3}sh(x^3+1)+C$$

$$\int tg\,x\,dx = \text{NIJE TABLIČNO !!}$$

$$= \int \frac{\sin x}{\cos x}dx = \left| \begin{array}{l} \cos x=t \\ -\sin x\,dx=dt \\ dx=\frac{dt}{-\sin x} \end{array} \right| = -\int\frac{dt}{t} = -\ln|t|+C = -\ln|\cos x|+C$$

$$\int \frac{dx}{\sin x} = \int \frac{\sin x}{\sin^2 x}dx = \int \frac{\sin x}{1-\cos^2 x}dx = \left| \begin{array}{l} \cos x=t \\ -\sin x\,dx=dt \\ dx=\frac{dt}{-\sin x} \end{array} \right| =$$

$$= \int \frac{dt}{t^2-1} = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C$$

$$= \frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right|+C$$

TIN IKER !
ŠARIĆ
GAY

M.DZ-4.)
A) $\int \frac{\ln^3 x}{x}\,dx = \frac{\ln^4}{4} + c = \ln + c$!!! TAKO OBAVEZNO NA ISPITU !!

NEMOJ $\ln^3 x = t$ jer je PRESLOŽENA!

$$= \left| \begin{array}{l} \ln x = t \\ \frac{1}{x}dx = dt \\ dx = \frac{dt}{\frac{1}{x}} \end{array} \right| = \int t^3\,dt = \frac{t^4}{4} + c = \frac{\ln^4 x}{4} + c$$

Z1-09-4.)

$$\int \frac{\operatorname{arctg} x + x}{x^2+1}\,dx = \int \frac{\operatorname{arctg} x}{x^2+1}\,dx + \int \frac{x}{x^2+1}\,dx$$

NAJRUŽNIJE

$$\left| \begin{array}{l} \operatorname{arctg} x = t \\ \frac{1}{1+x^2}dx = dt \\ dx = \frac{dt}{\frac{1}{1+x^2}} \end{array} \right| \qquad \left| \begin{array}{l} x^2+1 = k \\ 2x\,dx = dk \\ dx = \frac{dk}{2x} \end{array} \right|$$

$$= \int t\,dt + \frac{1}{2}\int \frac{dk}{k} = \frac{t^2}{2} + \frac{1}{2}\ln|k| + c =$$

$$= \frac{(\operatorname{arctg} x)^2}{2} + \frac{1}{2}\ln|x^2+1| + C$$

M.DZ-4.)
B) $\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx = \left| \begin{array}{l} e^x = t \\ e^x dx = dt \\ dx = \frac{dt}{e^x} \end{array} \right| = \int \frac{t}{\sqrt{t+1}}\,dt = \left| \begin{array}{l} t+1 = u \\ dt = du \end{array} \right| =$

$\frac{e^x}{\sqrt{t+1}} \cdot \frac{dt}{e^x}$ $\quad u-1$

$$= \int \frac{u-1}{\sqrt{u}}\,du = \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c =$$

$$= \frac{2}{3}(e^x+1)^{\frac{3}{2}} - 2\sqrt{e^x+1} + C$$

Pr.) $\int \frac{dx}{x^2+2x+4} = \int \frac{dx}{(x+1)^2+3} = \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x+1}{\sqrt{3}} + C$

$\longrightarrow (x+1)^2 - 1$

LINEARNI POMAK NE UTJEČE !

$\int \frac{1\,dx}{x^2+a^2} = \ldots$

$\longrightarrow$ TABLICA (prijašnji primjer) !

SMETA → MORAMO SUPST.

-3.)

$$\int \frac{2x-1}{\sqrt{x^2-2x+10}}\,dx = \int \frac{2x-1}{\sqrt{(x-1)^2+9}}\,dx = \left| \begin{array}{l} x-1=t \\ dx=dt \\ x=t+1 \end{array} \right| =$$

$$= \int \frac{2t+1}{\sqrt{t^2+9}}\,dt = 2\underbrace{\int \frac{t}{\sqrt{t^2+9}}\,dt}_{\text{NOVA SUPST.}} + \underbrace{\int \frac{dt}{\sqrt{t^2+9}}}_{\text{TABLICA!}} = \underbrace{\int \frac{du}{\sqrt{u}}}_{u^{-\frac{1}{2}}} + \ln\left|t+\sqrt{t^2+9}\right| + C$$

$$\left| \begin{array}{l} t^2+9=u \\ 2t\,dt=du \end{array} \right|$$

$$= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + \ln\left|x-1+\sqrt{(x-1)^2+9}\right| + C =$$

$$= 2\sqrt{(x-1)^2+9} + \ln\left|x-1+\sqrt{(x-1)^2+9}\right| + C$$

-08.-4.)

$$\int \frac{x^9}{x^{10}-2x^5+5}\,dx = \left| \begin{array}{l} x^5=t \\ 5x^4dx=dt \\ x^4dx=\frac{dt}{5} \end{array} \right| = \frac{1}{5}\int \frac{t}{t^2-2t+5}\,dt =$$

$x^9 \to x^5 \cdot x^4$

!!

$$= \frac{1}{5}\int \frac{t}{(t-1)^2+4}\,dt = \left| \begin{array}{l} t-1=u \\ dt=du \end{array} \right| = \frac{1}{5}\int \frac{u+1}{u^2+4}\,du = \frac{1}{5}\underbrace{\int \frac{u}{u^2+4}\,du}_{\left| \begin{array}{l} u^2+4=s \\ 2u\,du=ds \end{array} \right|} + \frac{1}{5}\underbrace{\int \frac{du}{u^2+4}}_{\text{tablica}} =$$

$t \to u+1$

$$= \frac{1}{10}\ln\left|(x^5-1)^2+4\right| + \frac{1}{10}\operatorname{arctg}\frac{x^5-1}{10} + C$$

JASNO KO IZ 3.141569CHKE!!!

-7.)

$$\underbrace{\int \sqrt{a^2-x^2}\,dx}_{} = \left| \begin{array}{l} x=a\sin t \\ dx=a\cos t\,dt \end{array} \right| = \int \underbrace{\sqrt{a^2-a^2\sin^2 t}}_{a^2(1-\sin^2 t)} \cdot a\cos t\,dt =$$

$t=\arcsin\frac{x}{a}$

DOBRO PAMTI !!

$$= \int \sqrt{a^2\cos^2 t} \cdot a\cos t\,dt = a^2\int \cos^2 t\,dt =$$

$|\cos t|$ $\quad$ $\cos^2 t$

$$\boxed{\begin{array}{l} \cos^2 x = \frac{1+\cos 2x}{2} \\ \sin^2 x = \frac{1-\cos 2x}{2} \end{array}}$$

$$= a^2\int \frac{1+\cos 2t}{2}\,dt = \frac{a^2}{2}\left(t+\frac{1}{2}\sin 2t\right) + C =$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{a^2}{4}\sin\Big(2\overset{k}{\arcsin\frac{x}{a}}\Big) + C =$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{a^2}{4}\,2\sin\left(\arcsin\frac{x}{a}\right)\cos\left(\arcsin\frac{x}{a}\right) + C = \longrightarrow$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{a\cancel{x}\cdot x}{2\cdot\cancel{a}}\cdot\sqrt{1-\left(\frac{x}{a}\right)^2} + C =$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{ax}{2}\cdot\sqrt{1-\left(\frac{x}{a}\right)^2} + C$$

11.DZ-11.)

$$\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx = \left| \begin{array}{l} t = \sin x \\ \cos x\,dx = dt \\ dx = \frac{dt}{\cos x} \end{array} \right| =$$

$$= \int \frac{1-t^2}{\underbrace{\sqrt{t}}_{t^{-\frac{1}{2}}}}\,dt = \int \left(t^{-\frac{1}{2}} - t^{\frac{3}{2}}\right) dt = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} - \frac{t^{\frac{5}{2}}}{\frac{5}{2}} + C =$$

$$= 2\sqrt{\sin x} - \frac{2}{5}(\sin x)^{\frac{5}{2}} + C$$

Pr.) $\int x^2 \sin x\,dx = \{\text{NIJEDNA SUPSTITUCIJA}\} = \cancel{\frac{x^3}{3}\cdot(-\cos x) + C}$ NE!

$\longrightarrow$ INTEGRIRANJE NE VRIJEDI ZA UMNOŽAK I KVOCIJENT!

$\int a\cdot b\,dx \neq \int a\,dx \cdot \int b\,dx$!!

TM „PARCIJALNA INTEGRACIJA"

Neka su f i g diferencijabilne funkcije. Ako postoje sljedeći integrali, tada vrijedi:

$$\int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\,g(x)\,dx$$

Dokaz: $\rightarrow$ ČESTO NA ISPITIMA

$$\left[f(x)\,g(x) - \int f'(x)\,g(x)\,dx\right]' = \cancel{f'(x)g(x)} + f(x)g'(x) - \cancel{f'(x)g(x)} =$$

$$= f(x)\,g'(x) \quad \text{II}$$

TRIVIJALNO

→ kraće pišemo:

$$\boxed{\int u \cdot dv = u \cdot v - \int v\,du}$$

DERIVIRAMO

INTEGRIRAMO

POANTA: DOBITI JEDNOSTAVNIJI INTEGRAL

$$\int \underbrace{x}_{u}\,\underbrace{e^x dx}_{dv} = \left| \begin{array}{ll} u = x & dv = e^x dx \\ du = 1\,dx & v = e^x \end{array} \right| = x \cdot e^x - \int e^x dx = x \cdot e^x - e^x + C$$

„SVE" ZNAMO DERIVIRATI!

UVIJEK ODABERI ONO ŠTO ZNAŠ INTEGRIRATI!

→ dakle, ne mora uvijek prvo biti u, a drugo dv!!

2.2.)

$$\int \underbrace{x^2}_{u}\,\underbrace{\ln x\,dx}_{dv} =$$

dv → NEMAMO $\int \ln x\,dx$ NE ZNAMO!!!

zato:

$$\left| \begin{array}{ll} u = \ln x & dv = x^2 dx \\ du = \frac{1}{x} dx & v = \frac{x^3}{3} \end{array} \right|$$

$$= \frac{x^3}{3}\ln x - \int \frac{x^{\cancel{3}\,2}}{3} \cdot \frac{1}{\cancel{x}} dx = \frac{x^3}{3}\ln x - \frac{1}{3} \cdot \frac{x^3}{3} + C$$

$$\int \underbrace{\arcsin x}_{u}\,\underbrace{dx}_{dv}$$

u → JER TO ZNAMO DERIVIRATI!!

$$= \left| \begin{array}{ll} u = \arcsin x & dv = dx \\ du = \frac{1}{\sqrt{1-x^2}} dx & v = x \end{array} \right| =$$

$$= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx = \left| \begin{array}{l} 1 - x^2 = t \\ -2x\,dx = dt \\ -x\,dx = \frac{dt}{2} \end{array} \right| =$$

$$= x \arcsin x + \frac{1}{2} \int \frac{dt}{\sqrt{t}} = x \arcsin x + \frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C =$$

$$= x \cdot \arcsin x + \sqrt{1-x^2} + C$$

→ NAJČEŠĆE NA ISPITU! SVE NAS ISPITAJU!

Zad.)

$$\int \sqrt{x}\,\sin(\sqrt{x})\,dx = \text{PRVO SUPSTITUCIJA S MJESECA} = \left| \begin{array}{l} \sqrt{x} = t \\ \frac{1}{2\sqrt{x}}\,dx = dt \\ dx = 2\sqrt{x}\,dt \end{array} \right| =$$

SMANJUJEMO POTENCIJU I OLAKŠAVAMO SI

$$= 2\int t^2 \sin t\,dt = \text{S MJESECA PARCIJALNA} = \left| \begin{array}{ll} u = t^2 & dv = \sin t \\ du = 2t\,dt & v = -\cos t \end{array} \right| =$$

$$= -2t^2\cos t + 4\underbrace{\int t\cos t\,dt}_{\text{parcijalno opet!}} = \left| \begin{array}{ll} u = t & dv = \cos t\,dt \\ du = dt & v = \sin t \end{array} \right| =$$

$$= -2t^2\cos t + 4\cdot\left(t\sin t - \int \sin t\,dt\right) =$$

$$= -2t^2\cos t + 4t\sin t + 4\cos t + C =$$

$$= -2x\cos(\sqrt{x}) + 4\sqrt{x}\cdot\sin(\sqrt{x}) + 4\cos(\sqrt{x}) + C$$

$\Longrightarrow$ SIGURNO TAKO NEŠTO NA ISPITU

Z1-08-3.)

$$\int \underbrace{e^{-x}}_{u}\,\underbrace{\sin x\,dx}_{dv} = \text{PARCIJALNA S MARSA BUREK ??} = \left| \begin{array}{ll} u = e^{-x} & dv = \sin x\,dx \\ du = -e^{-x}\,dx & v = -\cos x \end{array} \right|$$

$$= -e^{-x}\cos x - \int e^{-x}\cos x\,dx = \left| \begin{array}{ll} u = e^{-x} & dv = \cos x\,dx \\ du = -e^{-x}\,dx & v = \sin x\,dx \end{array} \right| =$$

CIKLIČKI $\int$

↳ ILKO IH JAKO VOLI ILKO <3

$$= -e^{-x}\cos x - \left[e^{-x}\sin x + \underbrace{\int e^{-x}\sin x\,dx}_{\text{POČETNI INTEGRAL}}\right]$$

$$\boxed{I = \int e^{-x}\sin x\,dx}$$

$\rightsquigarrow$ CIKLIČKI INTEGRAL

SIGURNO NA Z1 !

$$I = -e^{-x}\cos x - e^{-x}\sin x - I$$

$$2I = -e^{-x}(\cos x - \sin x) \quad |:2$$

$$\boxed{I = -\frac{1}{2}e^{-x}(\cos x - \sin x) + C}$$

INTEGRIRATI

$$\int \frac{x\cos x}{\sin^2 x}\,dx = \left| \begin{array}{ll} u = x & dv = \frac{\cos}{\sin^2 x}\,dx \\ du = dx & v = -\frac{1}{\sin x} \end{array} \right| =$$

$$\int \frac{\cos x}{\sin^2 x}\,dx = \left| \begin{array}{l} \sin x = t \\ \cos x\,dx = dt \end{array} \right| =$$

UVIJEK DOLINJU ZA SUPST.

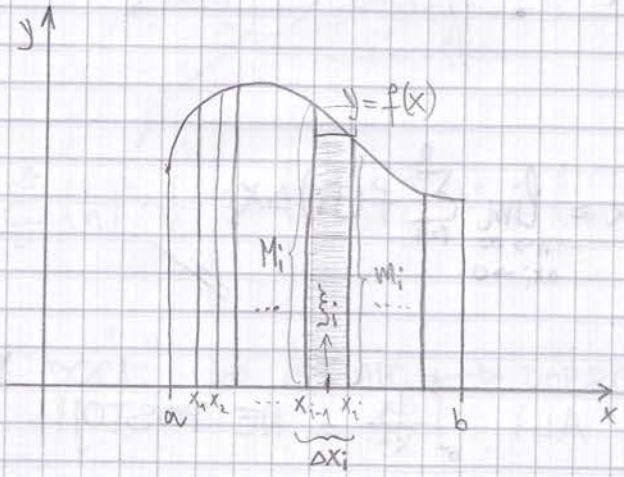
$$= \int \frac{dt}{t^2} = \frac{t^{-1}}{-1} = -\frac{1}{\sin x} + C$$

$$= \frac{-x}{\sin x} + \int \frac{dx}{\sin x} =$$

prošlo predavanje $\frac{\cdot \sin x}{\cdot \sin x}$

$$= -\frac{x}{\sin x} - \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C$$

11.2. ODREĐENI INTEGRAL



$$a = x_0 < x_1 < x_2 < \dots < x_{i-1} < x_i < \dots x_n < b$$

→ SUBDIVIZIJA INTERVALA $[a,b]$

$\Delta x_i = x_i - x_{i-1}$

$m_i = \inf f(x)$ → najmanja vrijednost $f(x)$ na $[x_{i-1}, x_i]$

$M_i = \sup f(x)$ → najveća —||— na $[x_{i-1}, x_i]$

$s_\Delta = \sum_{i=1}^{n} m_i \cdot \Delta x_i$ → DONJA INTEGRALNA SUMA

$S_\Delta = \sum_{i=1}^{n} M_i \Delta x_i$ → GORNJA INT. SUMA

$\sigma_\Delta = \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i$

$m(b-a) \leq s_\Delta \leq \sigma_\Delta \leq S_\Delta \leq M(b-a)$ (*) (očito sa slike)

$m = \inf(f(x))$ na $[a,b]$

$M = \sup(f(x))$ na $[a,b]$

$$\left.\begin{array}{ll} I_* = \sup\limits_{\Delta} s_\Delta & \text{donji Riemannov integral} \\ I^* = \inf\limits_{\Delta} S_\Delta & \text{gornji Riemanov integral} \end{array}\right\} I_* \leq I^* \quad \text{(očito)}$$

def. Kažemo da je funkcija $f(x)$ integrabilna (u Riemannovom smislu) na $[a,b]$ ako je $I_* = I^* = I$ i tada taj I nazivamo ODREĐENI INTEGRAL funkcije $f(x)$ na $[a,b]$ i označavamo ga sa

$$\int_a^b f(x)\,dx \quad .$$

analogna def:

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

Pr.) $\int \frac{dx}{x^2} = -\frac{1}{x} + C$ ALI $\int_0^1 \frac{dx}{x^2} =$ NE POSTOJI !!

TM

Omeđena funkcija $f(x)$ na $[a,b]$ je integrabilna ako i samo ako za svaki $\varepsilon > 0$ postoji razdioba Δ tako da je $S_\Delta - s_\Delta \leq \varepsilon$.

Dokaz: PMF !!

TM

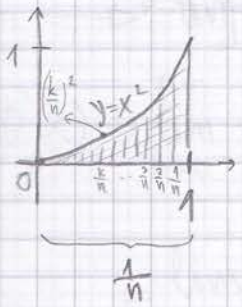
Ako je $f(x)$ neprekinuta na $[a,b]$ onda je integrabilna na $[a,b]$

VAŽNO^2

Ako je $f(x)$ omeđena na $[a,b]$ i ima samo konačno mnogo točki prekida, tada je integrabilna na $[a,b]$.

Ako je $f(x)$ omeđena i monotona na $[a,b]$ tada je integrabilna na $[a,b]$. → ANALOGIJA SA KONVERGENCIJOM

$$\int_0^1 x^2\,dx = ?$$



po definiciji

$$\int_0^1 x^2dx = \lim_{x\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2\cdot\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}k^2 = \lim_{n\to\infty}\frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\begin{matrix}\sim 2n^3\\ \sim 6n^3\end{matrix} = \frac{1}{3}$$

„SMORI SE DA ĆE DOĆI" po definic. → DOBRO PROUČITI !!!

SVOJSTVA.

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

$$\int_a^b c\cdot f(x)dx = c\cdot\int_a^b f(x)dx$$

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

DOKAZE NE TREBAMO, IAKO SU NARAVNO TRIVIC.

$\boxed{TM}$ Ako je $f(x) \geq g(x)$ na $[a,b]$ tada je $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

$\boxed{TM}$ „TEOREM SREDNJE VRIJEDNOSTI INTEGRALNOG RAČUNA"

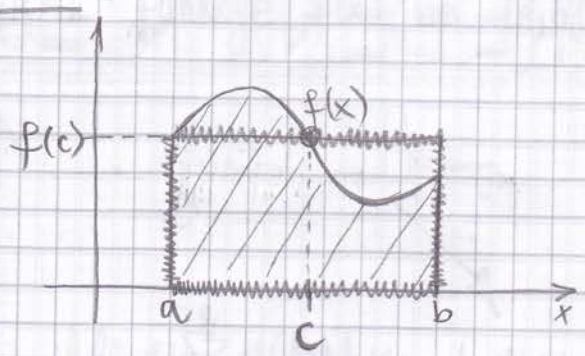
Neka je f neprekinuta funkcija na $[a,b]$. Tada postoji $c \in (a,b)$ tako da

$$\boxed{\int_a^b f(x)dx = f(c)\cdot(b-a)}.$$

TAKODER!

→ geom. interpretacija:

ISPIT



→ površina ispod krivulje $f(x)$ od a do b jednaka je površini pravokutnika

→ NIJE BILO NITI JEDNOG ZI DO SADA BEZ OVOG $\boxed{TM}$ (IL)I ZADATKA S NJIME

Dokaz: (*)

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a) \quad |:(b-a)$$

$$m \leq \frac{\int_a^b f(x)dx}{b-a} \leq M \Rightarrow f(c) = \frac{\int_a^b f(x)dx}{b-a} \quad \square$$

→ potrebno je lijepo objasniti da budući da je funkcija f neprekinuta, onda poprima sve vrijednosti između m i M, pa poprima i ovu vrijednost (u sredini nejednakosti), stoga postoji c iz intervala takav da je $f(c)$ = bla bla ...

„OSNOVNI TEOREM INTEGRALNOG RAČUNA"

Neka je f neprekinuta funkcija na $[a,b]$ i neka je $x \in [a,b]$. Tada je $\phi(x) = \int_a^x f(x)\,dx$ diferencijalna i vrijedi:

$$\phi'(x) = \frac{d}{dx}\int_a^x f(x)\,dx = f(x).$$

!

ILKO MJENJA OZNAKE! Ö

Dokaz: GOTOVO → UVIJEK DOĐE DOKAZ OVOG TEOREMA NA ISPIT!!

$$\phi'(x) = \lim_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h} = \lim_{h\to 0}\frac{\int_a^{x+h} f(x)\,dx - \int_a^x f(x)\,dx}{h} =$$

$$= \lim_{h\to 0}\frac{\int_x^{x+h} f(x)\,dx}{h} \overset{\text{TEOR. SREDNJE VRIJEDN. I.R.}}{=} \lim_{h\to 0}\frac{f(c_h)\cdot \cancel{h}}{\cancel{h}} = \quad f \text{ NEPREKINUTA}$$

(postoji $c_h \in [x, x+h]$) → prethodni TM

SMANJUJE SE

$$= f(x) \quad \Pi$$

„NEWTON-LEIBNIZOVA FORMULA"

Neka je $f(x)$ neprekinuta funkcija na $[a,b]$, te neka je $F(x)$ bilokoja primitivna funkcija od $f(x)$. Tada vrijedi

$$\boxed{\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)}$$

Dokaz: (VOLE GA STAVITI KAO I ONE PRETHODNE)

ZADNJI DOKAZ U MAT 1

$\phi(x)$ iz prethodnog TM je primitivna fja od $f(x)$ pa je bilo koja druga primitivna fja $F(x) = \phi(x) + C$

$$F(b) - F(a) = \phi(b) - \phi(a) = \int_a^b f(x)\,dx - \underbrace{\int_a^a f(x)\,dx}_{0} = \int_a^b f(x)\,dx \quad \Pi$$

Pr.) $\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$ //

GRRR

$\boxed{\text{TM}}$ „METODA SUPSTITUCIJE"

$$\int_a^b f(x)\,dx = \left| \begin{array}{l} x=\varphi(t) \\ \varphi(\alpha)=a \\ \varphi(\beta)=b \end{array} \right| = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

$\boxed{\text{TM}}$ „PARCIJALNA INTEGRACIJA"

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

11. DZ. – 7.)

NAJRUŽNIJI $\quad x^2=t-1$

$$\int_0^{\sqrt{3}} x^5\sqrt{x^2+1}\,dx = \left| \begin{array}{l} x^2+1=t \\ 2x\,dx=dt \\ dx=\frac{dt}{2x} \\ x=0\to\alpha=1 \\ x=1\to\beta=4 \end{array} \right| = \int_1^4 x^{\cancel{5}\,4}\sqrt{t}\,\frac{dt}{2\cancel{x}} = \frac{1}{2}\int_1^4 \underbrace{(t-1)^2}_{(t^2-2t+1)}\sqrt{t}\,dt =$$

PAZI!

$$= \frac{1}{2}\int_1^4 \left(t^{\frac{5}{4}} - 2t^{\frac{3}{2}} + t^{\frac{1}{2}}\right)dt = \frac{1}{2}\left[\frac{t^{\frac{7}{2}}}{\frac{7}{2}} - 2\cdot\frac{t^{\frac{5}{2}}}{\frac{5}{2}} + \frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]\Bigg|_1^4 = \dots = \frac{848}{105}$$ //

!! PAZI → SADA SE NE VRAĆA x u t !!

GRANICE PROMJENJENE $\Big|_1^4$!!!

$1-\sin^2 t = 1-t$

-4.) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 x \cos^3 x \, dx = \left| \begin{array}{l} \sin x = t \\ \cos x \, dx = dt \\ x=\frac{\pi}{6} \to \alpha = \frac{1}{2} \\ x = \frac{\pi}{2} \to \beta = 1 \end{array} \right| = \int_{\frac{1}{2}}^{1} t^2 \cos^2 x \, dt = \int_{\frac{1}{2}}^{1} t^2 (1-t^2) \, dt =$$

$$= \int_{\frac{1}{2}}^{1} (t^2 - t^4) \, dt = \left(\frac{t^3}{3} - \frac{t^5}{5} \right) \Big|_{\frac{1}{2}}^{1} = \left(\frac{1^3}{3} - \frac{1^5}{5} \right) - \left(\frac{\frac{1}{8}}{3} - \frac{\frac{1}{3^2}}{5} \right)$$ OK.

$$= \frac{47}{480}$$

-16.) NAJROŽNIJE

$$\int_0^1 e^{-\sqrt{x}} \, dx = \left| \begin{array}{l} -\sqrt{x} = t \\ -\frac{1}{2\sqrt{x}} dx = dt \\ dx = -2\sqrt{x} \, dt \\ dx = -2t \, dt \\ x=0 \to \alpha = 0 \\ x = 1 \to \beta = -1 \end{array} \right| = \int_0^{-1} \underbrace{2t}_{u} \cdot \underbrace{e^t \, dt}_{dv} = \{ \text{PARCIJALNA} \} =$$

$$= \left| \begin{array}{ll} u = 2t & dv = e^t dt \\ du = 2 \, dt & v = e^t \end{array} \right| = 2te^t \Big|_0^{-1} - \int_0^{-1} 2e^t \, dt =$$

$$= -2e^{-1} - 0 - 2e^t \Big|_0^{-1} = -2e^{-1} - (2e^{-1} - 2) = -4e^{-1} + 2$$

-4.) Odredi parametar a tako da vrijedi:

$$\int_1^a \frac{\ln^3 x}{x} \, dx = 4 \quad , \quad a = ?$$

najtužniji

$$\int_1^a \frac{\ln^3 x}{x} \, dx = \left| \begin{array}{l} \ln x = t \\ \frac{1}{x} dx = dt \\ x = 1 \to \alpha = 0 \\ x = a \to \beta = \ln a \end{array} \right| = \int_0^{\ln a} t^3 \, dt = \frac{t^4}{4} \Big|_0^{\ln a} = \frac{(\ln a)^4}{4} = 4$$

$$(\ln a)^4 = 16 \quad | \sqrt[4]{\ }$$

$$\boxed{\ln a = \pm 2}$$

$$\boxed{a = e^{\pm 2}}$$

PARNI $\sqrt{\ }$
UVIJEK $\pm$!

→ ILKO IH JAKO VOLI !

!!

Z1-08'-7.)

$$\int_0^{\frac{3}{2}} \sqrt{9-x^2}\,dx = \left| \begin{array}{l} x = 3\sin t \\ dx = 3\cos t\,dt \\ x=0 \to \alpha = 0 \\ x=\frac{3}{2} \to \frac{3}{2} = 3\sin t \\ \quad \sin t = \frac{1}{2} \\ \quad \to \beta = \frac{\pi}{6} \end{array} \right| = \int_0^{\frac{\pi}{6}} 3\cos t \cdot 3\cos t\,dt =$$

$9 - 9\sin^2 t$

$9(1-\sin^2 t) =$

$9\cos^2 t$

$$= 9\int_0^{\frac{\pi}{6}} \cos^2 t\,dt = \frac{9}{2}\int_0^{\frac{\pi}{6}} (1+\cos 2t)\,dt = \frac{9}{2}\left(t + \frac{1}{2}\sin 2t\right)\Big|_0^{\frac{\pi}{6}} =$$

$$= \frac{9}{2}\left(\frac{\pi}{6} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) - \frac{9}{2}\left(0 + \frac{1}{2}\cdot 0\right) = \frac{3\pi}{4} + \frac{9\sqrt{3}}{8}$$

11.ZZV-3.)

$$\int_0^1 x\ln(x^2+1)\,dx = \left| \begin{array}{l} x^2+1 = t \\ 2x\,dx = dt \\ x\,dx = \frac{dt}{2} \\ x=0 \to \alpha = 1 \\ x=1 \to \beta = 2 \end{array} \right| = \int_1^2 \ln t\,\frac{dt}{2} = \frac{1}{2}\int_1^2 \ln t\,dt =$$

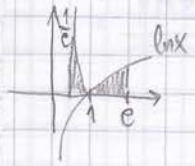
$$= \left| \begin{array}{ll} u = \ln t & dv = dt \\ du = \frac{1}{t}\,dt & v = t \end{array} \right| = \frac{1}{2}\,t\cdot\ln t\Big|_1^2 - \frac{1}{2}\int_1^2 \frac{1}{\cancel{t}}\cancel{t}\,dt = \left(\frac{1}{2}t\ln t - \frac{1}{2}t\right)\Big|_1^2 =$$

$$= (\ln 2 - 1) - \left(0 - \frac{1}{2}\right) = \ln 2 - \frac{1}{2}$$

prošli primjeri napr

11.DZ-20.)

$$\int_{\frac{1}{e}}^{e} |\ln x|\,dx = \int_{\frac{1}{e}}^{1} -\ln x\,dx + \int_1^e \ln x\,dx = -(x\ln x - x)\Big|_{\frac{1}{e}}^{1} + (x\ln x - x)\Big|_1^e =$$



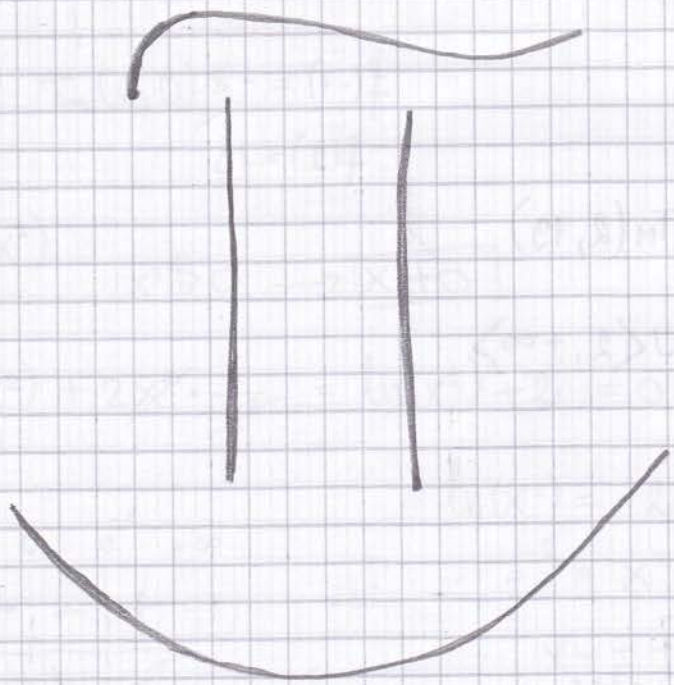
$$= 1 + \left(\frac{1}{e}\ln\frac{1}{e} - \frac{1}{e}\right) + (e\ln e - e) - (0 - 1) =$$

$$= 2 - \frac{2}{e}$$

BOŽIĆNI POKLONI ZA POD BOR:

1) $f(x) = \frac{\sqrt{x}}{\ln^2 x}$ ⟶ Odrediti sve i nacrtaj kvalitativni graf!

2) $\int_1^{\sqrt{3}} \frac{\operatorname{arctg}\frac{1}{x}}{x^2}\, dx = ?$



PETAK → INTEGRALI, neki ekstremi ... graf neće – previšo!

12.1. INTEGRIRANJE RACIONALNIH FUNKCIJA

PZI-06-4)

$$\int \frac{x^2}{x^2+x+1}dx = \int \frac{x^2+x+1-x-1}{x^2+x+1}dx = \int dx + \int \frac{-x-1}{x^2+x+1}dx = x + \int \frac{-x-1}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$= \left| \begin{array}{l} x+\frac{1}{2}=t \\ dx=dt \\ \Rightarrow x=t-\frac{1}{2} \end{array} \right| = x + \int \frac{-t-\frac{1}{2}}{t^2+\frac{3}{4}}dt = x + \int \frac{-t}{\underbrace{t^2+\frac{3}{4}}_{u}}dt - \frac{1}{2}\int \frac{dt}{t^2+\frac{3}{4}} =$$

$$= x - \frac{1}{2}\ln|u| - \frac{1}{2}\frac{1}{\frac{\sqrt{3}}{2}}\operatorname{arctg}\frac{t}{\frac{\sqrt{3}}{2}} + C =$$

$$= x - \frac{1}{2}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C$$

12.DZ-2.)

$$\int \frac{x^3dx}{x^2+2x+4}$$ → UVIJEK KADA JE STUPANJ BROJNIKA VEĆI OD STUPNJA NAZIVNIKA

PODJELITI POLINOME !!

$$x^3 : (x^2+2x+4) = x-2$$
$$\underline{-x^3 \mp 2x^2 \pm 4x}$$
$$\underline{-\pm 2x^2 \mp 4x \mp 8}$$
$$8$$

OSTATAK !

$$= \int \left(x-2+\frac{8}{\underbrace{x^2+2x+4}_{(x+1)^2+3}}\right)dx = \frac{x^2}{2} - 2x + 8\cdot\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x+1}{\sqrt{3}} + C$$

$$\int \frac{P_n(x)}{Q_2(x)}\,dx = \begin{cases} \text{svedemo na potpuni kvadrat + supstitucija, ako je } n<2 \\ \text{dodamo i oduzmemo da svedemo na 1. slučaj, ako je } n=2 \\ \text{dijelimo polinome i opet vidi 1. slučaj, ako je } n>2 \end{cases}$$

→ ako je u nazivniku $Q_m(x)$, za $m>2$:

1. FAKTORIZACIJA NAZIVNIKA (prikazati u obliku umnoška)

2. RASTAV NA PARCIJALNE RAZLOMKE

→ poanta: dobiti ove „jednostavnije" integrale koje znamo

$$\frac{3}{(x^2-x-2)(x^2+x+2)} =$$

$x_{1,2} = \begin{cases} 1 \\ -2 \end{cases}$ $\quad x_{1,2} = \in \mathbb{C}$ → NEMA REALNE FAKTORIZACIJE

!! UVIJEK POLINOM 1. STUPNJA

$$= \frac{3}{(x-1)(x+2)(x^2+x+2)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{Cx+D}{x^2+x+2}$$

$$\frac{1}{(x^2-4)(x^3+8)} = \frac{1}{(x-2)(x+2)^2(x^2-2x+4)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{Dx+E}{x^2-2x+4}$$

$(x-2)(x+2)(x+2)(x^2-2x+4)$

→ SA SVAKOM POTENCIJOM SE POJAVLJUJE

VJEROJATNO U ISPITU

5.) $$\int \frac{dx}{x^3+x^2} = \int \frac{-1}{x}\,dx + \int \frac{1}{x^2}\,dx + \int \frac{1}{x+1}\,dx = -\ln|x| - \frac{1}{x} + \ln|x+1| + C$$

$$\frac{1}{(x+1)x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} = \frac{Ax(x+1) + B(x+1) + Cx^2}{x^2(x+1)}$$

$$Ax(x+1) + B(x+1) + Cx^2 = 1$$

→ UVRSTITI NULTOČKE: $x=0 \to \boxed{B=1}$

$x=-1 \to \boxed{C=1}$

neki prikladan x, npr. $x=1 \to 2A+2\times+1=1$

$\boxed{A=-1}$

21-10-4.) $\displaystyle\int \frac{7x+1}{\underbrace{2x^3-x^2+4x-2}_{x^2(2x-1)+2(2x-1)}}\,dx =$

$\hookrightarrow$ DA NIJE BILO NAŽJEŠTENO POGODILI BI JEDNU NULTOČKU $\rightarrow$ DJELITELJ SLOBODNOG ČLANA ... (5. TJEDAN)

$$\frac{7x+1}{(2x-1)(x^2+2)} = \frac{A}{2x-1} + \frac{Bx+C}{x^2+2}$$

$$A(x^2+2)+(Bx+C)(2x-1) = 7x+1$$

$x=\frac{1}{2}$: $\frac{9}{4}A = \frac{9}{2}$

$$\boxed{A=2}$$

neki proizvoljan, najljepše : $x=0 \longrightarrow 4-C=1$

$$\boxed{C=3}$$

$x=1 \longrightarrow 6+B+3=8$

$$\boxed{B=-1}$$

$$= \int \frac{2}{2x-1}\,dx + \int \frac{-x+3}{x^2+2}\,dx = \dots = 2\cdot\frac{1}{2}\ln|2x-1| - \frac{1}{2}\ln|x^2+2| + \frac{3}{\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}} + C$$

12.DZ-6.) $\displaystyle\int_0^1 \frac{dx}{(x^2+1)^3} =$

$$I_n = \int \frac{dx}{(x^2+1)^n} = \int \frac{x^2+1-x^2}{(x^2+1)^n}\,dx = \underbrace{\int \frac{dx}{(x^2+1)^{n-1}}}_{I_{n-1}} - \int \frac{\overset{u}{x}\cdot\overset{dv}{x}}{(x^2+1)^n}\,dx = \left| \begin{array}{ll} u=x & dv=\frac{x}{(x^2+1)^n} \\ du=dx & v=\frac{1}{2}\frac{\ldots}{\ldots} \end{array} \right.$$

$$= I_{n-1} + \frac{x}{2(n-1)(x^2+1)^{n-1}} - \frac{1}{2(n-1)}\underbrace{\int \frac{dx}{(x^2+1)^{n-1}}}_{I_{n-1}}$$

$$\boxed{I_n = \frac{2n-3}{2n-2} I_{n-1} + \frac{x}{2(n-1)(x^2+1)^{n-1}}} \quad \boxed{\text{za } n \geq 2} \;!!$$

$\underline{n=3}$

$$I_3 = \frac{3}{4} I_2 + \frac{x}{4(x^2+1)^2}\Big|_0^1 = \frac{3}{4}\overbrace{\left(\frac{1}{2} I_1 + \frac{x}{2(x^2+1)}\Big|_0^1\right)}^{I_2} + \frac{1}{16} =$$

$$I_1 = \int \frac{dx}{(x^2+1)^1} = \operatorname{arctg} x$$

$$\frac{3}{8}\operatorname{arctg}x\Big|_0^1 + \frac{3}{16} + \frac{1}{16} = \frac{3\pi}{32} + \frac{1}{4}$$

→ DOBRO NAUČITI OVE SA IZVODOM REKURZIVNE FORMULE

by Burek

12-5.)

$$\int \frac{dx}{x\ln^3 x + 4x\ln x} = \left| \begin{array}{l} \ln x = t \\ \frac{1}{x}dx = dt \\ dx = x\,dt \end{array} \right| = \int \frac{\cancel{x}\,dt}{\cancel{x}(t^3+4t)} =$$

NAJRUŽNIJE

$$\frac{1}{t(t^2+4)} = \frac{A}{t} + \frac{Bt+C}{t^2+4}$$

$$A(t^2+4) + (Bt+C)t = 1$$

$t=0$: $4A = 1 \longrightarrow \boxed{A = \frac{1}{4}}$

$$\left.\begin{array}{l} t=1: \ \frac{5}{4} + B + \cancel{C} = 1 \\ t=-1: \ \frac{5}{4} + B - \cancel{C} = 1 \end{array}\right\} +$$

$$\boxed{B = \frac{-1}{4}} \quad \boxed{C = 0}$$

$$\int \frac{\frac{1}{4}}{t}\,dt + \int \frac{-\frac{1}{4}t}{\underbrace{t^2+4}_{u}}\,dt = \frac{1}{4}\ln|t| - \frac{1}{8}\ln|t^2+4| + C =$$

$$= \frac{1}{4}\ln|\ln x| - \frac{1}{8}\ln|\ln^2 x + 4| + C$$

→ do srijede prvih 7 zad. od 12.

NAPOMENA:

12. DZ – 7.)

$$\int_0^1 \frac{dx}{(x+1)(x^2+x+1)^2} =$$

$$\frac{A}{x+1} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2} \cdots$$

rekurzivnom

12.2. INTEGRIRANJE TRIGONOMETRIJSKIH FUNKCIJA

PRIKLADNE SUPSTITUCIJE

npr. $\sin x = t$, $\cos x = t$ $\longrightarrow$ pogledaj 11. dz $\int \frac{\sin x}{\sqrt{\cos x}}dx = \left| \begin{array}{l} \cos x = t \\ -\sin x\, dx = dt \\ \dots \end{array} \right|$

$\int R(\sin x, \cos x)dx$ = UNIVERZALNA SUPSTITUCIJA

(UVIJEK USPJEVA)

$$\boxed{t = \operatorname{tg}\frac{x}{2}}$$

$\hookrightarrow dx = \frac{2dt}{1+t^2}$ $\Longrightarrow$ $\boxed{\sin x = \frac{2t}{1+t^2}}$

$$\boxed{\cos x = \frac{1-t^2}{1+t^2}}$$

$$\int R(\sin^2 x, \cos^2 x)\, dx = \left| \begin{array}{l} \boxed{t = \operatorname{tg} x} \\ \hookrightarrow dx = \frac{dt}{1+t^2} \end{array} \longrightarrow \begin{array}{l} \boxed{\sin x = \frac{t}{\sqrt{1+t^2}}} \\ \boxed{\cos x = \frac{1}{\sqrt{1+t^2}}} \end{array} \right|$$

$\rightarrow$ knjižica, str. 19 $\longrightarrow$ ZNATI, sve isto sa hiperbolnim!

12. DZ – 12.)

$$\int \frac{dx}{\cos x + 2\sin x + 3} = \left\{ \text{UNIVERZALNA SUPS.} \right\} =$$

$$= \int \frac{\frac{2dt}{1+t^2}}{\frac{1-t^2}{1+t^2} + 2\cdot\frac{2t}{1+t^2} + 3} = \int \frac{\frac{2dt}{\cancel{1+t^2}}}{\frac{1-t^2+4t+3+3t^2}{\cancel{1+t^2}}} = \int \frac{\cancel{2}dt}{\cancel{2}(t^2+2t+2)} = \int \frac{dt}{t^2+2t+2} =$$

$$= \int \frac{dt}{(t+1)^2+1} = \operatorname{arctg}(t+1) + C = \operatorname{arctg}\left(\operatorname{tg}\frac{x}{2} + 1\right) + C$$

NE ZABORAVI GRANICE!!

PZI – 09 – 4.)

$$\int_0^{\frac{\pi}{3}} \frac{dx}{1+8\cos^2 x} \overset{t=\operatorname{tg}x}{=} \int_0^{\sqrt{3}} \frac{\frac{dt}{1+t^2}}{1+8\frac{1}{1+t^2}} = \int_0^{\sqrt{3}} \frac{\frac{dt}{\cancel{1+t^2}}}{\frac{1+8+t^2}{\cancel{1+t^2}}} = \int_0^{\sqrt{3}} \frac{dt}{t^2+9} = \frac{1}{3}\operatorname{arctg}\left(\frac{t}{3}\right)$$

$$= \frac{\pi}{18}$$

!!!

HIPERBOLNI

12. DZ – 17.)

$$\int \frac{dx}{\operatorname{ch}^4 x} \overset{t=\operatorname{th}x}{=} \int \frac{\frac{dt}{1-t^2}}{\left(\frac{1}{1-t^2}\right)^2} = \int (1-t^2)\,dt = t - \frac{t^3}{3} + C = \operatorname{th}x - \frac{(\operatorname{th}x)^3}{3} + C$$

!!! OBIČNA SUPST.

12. DZ – 13.)

$$\int \frac{dx}{\sin^2 x \cos x} = \left| \begin{array}{l} \sin x = t \\ \cos x\, dx = dt \\ dx = \frac{dt}{\cos x} \end{array} \right| = \int \frac{dt}{t^2 \underbrace{\cos^2 x}_{1-\sin^2 x}} = \int \frac{dt}{t^2 - t^4} = \dots$$

DZ

$$\frac{1}{t^2(1-t^2)} = \frac{1}{t^2(1-t)(1+t)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{1+t} + \frac{D}{1-t} \dots$$

c-7) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos x\,dx}{\sin^3 x - 2\sin^2 x + \sin x} = \left| \begin{array}{l} \sin x = t \\ \cos x\,dx = dt \\ x = \frac{\pi}{6} \to \alpha = \frac{1}{2} \\ x = \frac{\pi}{4} \to \beta = \frac{\sqrt{2}}{2} \end{array} \right| =$$

$$= \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \frac{dt}{\underbrace{t^3 - 2t + t}_{t(t^2-2t+1)}} =$$

$$\frac{1}{t(t-1)^2} = \frac{A}{t} + \frac{B}{t-1} + \frac{C}{(t-1)^2}$$

$$A(t-1)^2 + Bt(t-1) + Ct = 1$$

$t=0$: $\boxed{A=1}$

$t=1$: $\boxed{C=1}$

$t=7$: $36 + 42B + 7 = 1$

$\boxed{B=-1}$

$$= \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \left(\frac{1}{t} + \frac{-1}{t-1} + \frac{1}{(t-1)^2} \right) dt = \left[\ln|t| - \ln|t-1| + \frac{(t-1)^{-1}}{-1} \right] \Bigg|_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} = \dots =$$

$$= \dots = \frac{2-2\sqrt{2}}{\sqrt{2}-2} + \ln \frac{\sqrt{2}}{2-\sqrt{2}}$$

UMNOŠCI SINUSA I KOSINUSA

1) općenito

$$\int \sin x \underbrace{\cos 2x \sin 2x}_{\frac{1}{2}\sin 4x}\, dx = \frac{1}{2} \int \underbrace{\sin x \sin 4x}_{\text{FORMULA PRETVORBE}}\, dx = \frac{1}{2} \int \frac{1}{2} \left[\cos(-3x) - \cos(5x) \right] dx =$$

$$= \frac{1}{4} \left(\frac{1}{3} \sin 3x - \frac{1}{5} \sin 5x \right) + C$$

ii) neparne potencije $\longrightarrow$ $\sin x = t$, $\cos x = t$ (onaj koji nije u

12.ZZV-2.)

jer će sinus svesti na parnu!

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^6 x \, dx = \left| \begin{array}{l} \cos x = t \\ -\sin x \, dx = dt \\ x=0 \to t=1 \\ x=\frac{\pi}{2} \to t=0 \end{array} \right| =$$

$$= -\int_1^0 \underbrace{\sin^2 x}_{1-t^2} t^6 dt = -\int_1^0 (t^6 - t^8) dt = \left(-\frac{t^7}{7} + \frac{t^9}{9}\right)\Big|_1^0 = \frac{2}{63}$$

iii) samo parne potencije : $\sin^2 x = \dfrac{1-\cos 2x}{2}$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

ZZV-1LKO-40.)

$$\int \sin^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) \, dx =$$

$$= \frac{1}{4}x - \frac{2}{4}\cdot\frac{1}{2}\sin 2x + \frac{1}{4}\int \frac{1+\cos 4x}{2} dx =$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C$$

12.DZ-15.)

$$\int_0^{\frac{\pi}{2}} \sin^8 x \, dx = \cancel{\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2x}{2}\right)^4 dx} = \text{NE} \; !!!$$

<u>REKURZIVNA</u> :

$$I_n = \int \sin^n x \, dx = \int \underbrace{\sin^{n-1} x}_{u} \cdot \underbrace{\sin x \, dx}_{dv} = \left| \begin{array}{ll} u = \sin^{n-1} x & dv = \sin x \, dx \\ du = (n-1)\sin^{n-2} x \cdot \cos x \, dx & v = -\cos x \end{array} \right|$$

$$= -\cos x \sin^{n-1} x + \int (n-1)\sin^{n-2} x \underbrace{\cos^2 x}_{1-\sin^2 x} dx =$$

$$I_n = -\cos x \sin^{n-1} x + (n-1)\underbrace{\int \sin^{n-2} x \, dx}_{I_{n-2}} - (n-1)\underbrace{\int \sin^n x \, dx}_{I_n}$$

$$I_n(\cancel{1}+n-\cancel{1}) = -\cos x \sin^{n-1} x + (n-1) I_{n-2} \quad | : n$$

$$\boxed{I_n = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\cdot I_{n-2}}$$

$$= -\frac{1}{8}\cos x \sin^7 x\Big|_0^{\frac{\pi}{2}} + \frac{7}{8} I_6 = \frac{7}{8}\left[\left(-\frac{1}{6}\cos x \sin^5 x\right)\Big|_0^{\frac{\pi}{2}} + \frac{5}{6} I_4\right] =$$

$$= \frac{35}{48}\left[\left(-\frac{1}{4}\cos x \sin^3 x\right)\Big|_0^{\frac{\pi}{2}} + \frac{3}{4} I_2\right] = \frac{35}{64}\left[\left(-\frac{1}{2}\cos x \sin x\right)\Big|_0^{\frac{\pi}{2}} + \frac{1}{2} I_0\right] =$$

$$= \frac{35}{128} x \Big|_0^{\frac{\pi}{2}} = \frac{35\pi}{256}$$

DZ-18.) $\int sh^2 x \, ch^4 x \, dx =$

HIPERBOLNE FORMULE

I. način: $\int \left(\frac{ch 2x - 1}{2}\right)\left(\frac{ch 2x + 1}{2}\right)^2 dx = \dots = DZ$

II. način: $\int \left(\frac{e^x - e^{-x}}{2}\right)^2 \left(\frac{e^x + e^{-x}}{2}\right)^4 dx = \dots = DZ$

III. način: $\{ch^2 x - sh^2 x = 1\}$

$$\int (ch^2 x - 1)\, ch^4 x \, dx = \int ch^6 x \, dx - \int ch^4 x \, dx =$$

$\dots$

$$I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n} sh x \, ch^{n-1} x \qquad I_n = \int ch^n x \, dx$$

$$I_6 - I_4 = \dots =$$

(x)

VJEŽBA ZA ŠKOLSKU by BUREK

1.) $$\int_0^{16} \frac{dx}{(1+\sqrt[4]{x})^2} = \left| \begin{array}{l} 1+\sqrt[4]{x} = t \\ \frac{1}{4}x^{-\frac{3}{4}}dx = dt \\ dx = 4x^{\frac{3}{4}}dt \\ x=0 \to \alpha = 1 \\ x=3 \to \beta = 3 \\ \sqrt[4]{x} = t-1 \end{array} \right| = \int_1^3 \frac{4x^{\frac{3}{4}}\,dt}{t^2} = \int_1^3 \frac{4(t-1)^3\,dt}{t^2} =$$

$(\sqrt[4]{x})^3$

$$= 4\int_1^3 \frac{t^3-3t^2+3t-1}{t^2}\,dt = 4\int_1^3 \left(t-3+\frac{3}{t}-\frac{1}{t^2}\right)dt = 4\left(\frac{t^2}{2}-3t+3\ln t-\frac{t^{-1}}{-1}\right)$$

$$= 12\ln 3 - \frac{32}{3}$$

2.) $$\int \frac{x}{\sqrt{4x-x^2}}\,dx =$$

$$4x-x^2 = -(x^2-4x)$$
$$= -(\underbrace{x^2-4x+4}_{(x-2)^2}-4)$$
$$= 4-(x-2)^2$$

$$= \int \frac{x}{\sqrt{4-(x-2)^2}}\,dx = \left| \begin{array}{l} x-2 = t \\ dx = dt \end{array} \right| = \int \frac{t+2}{\sqrt{4-t^2}}\,dt = \int \frac{t}{\sqrt{4-t^2}}\,dt + \int \frac{2}{\sqrt{4-t^2}}\,dt =$$

$x = t+2$

$$\left| \begin{array}{l} 4-t^2 = u \\ -2t\,dt = du \end{array} \right|$$

$$= -\frac{1}{2}\int \frac{du}{\sqrt{u}} + 2\arcsin\frac{t}{2} + C =$$

$u^{-\frac{1}{2}}$

$$= \cancel{\frac{1}{2}}\,\frac{u^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} + 2\arcsin\frac{x-2}{2} + C$$

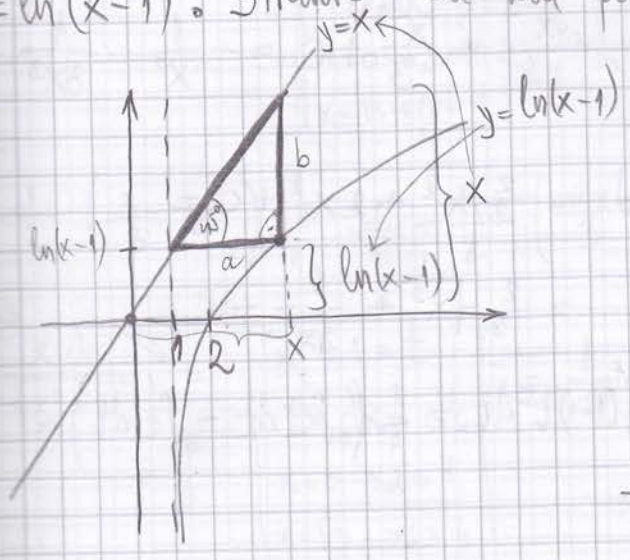
3.) $$\int_1^{\sqrt{3}} \frac{\operatorname{arctg}\frac{1}{x}}{x^2}\,dx = \left| \begin{array}{l} \frac{1}{x} = t \\ -\frac{1}{x^2}dx = dt \end{array} \right| = -\int_1^{\frac{1}{\sqrt{3}}} \underbrace{\operatorname{arctg} t}_{u}\,\underbrace{dt}_{dv} = \left| \begin{array}{ll} u = \operatorname{arctg} t & dv = \\ du = \frac{1}{1+t^2}dt & v = \end{array} \right|$$

$$= -t\operatorname{arctg} t \Big|_1^{\frac{1}{\sqrt{3}}} + \int_1^{\frac{1}{\sqrt{3}}} \frac{t}{1+t^2}\,dt =$$

u

$$= -\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{1}{\sqrt{3}} - (-\operatorname{arctg} 1) + \frac{1}{2}\ln|1+t^2|\Big|_1^{\frac{1}{\sqrt{3}}} = \dots$$

Odredi pravokutni trokut minimalne površine čija hipotenuza leži na pravcu $y=x$, a vrh nasuprot hipotenuze leži na krivulji $y=\ln(x-1)$. Stranice su mu paralelne sa koordinatnim osima.



$$P=\frac{a\cdot b}{2}$$

$$b=x-\ln(x-1)$$

$y=x$ ide pod $45°$ $\Rightarrow$ jednakokračan $\triangle$

$\underline{a=b}$

$$\hookrightarrow P=\frac{1}{2}\left(x-\ln(x-1)\right)^2$$

$$P'=\frac{1}{2}\cdot 2\left(x-\ln(x-1)\right)\cdot\left(1-\frac{1}{x-1}\right)$$

$$P'=\frac{(x-\ln(x-1))(x-2)}{x-1}=0$$

$x-2=0$ $\quad$ $x=\ln(x-1)$

$\boxed{x=2}$ $\quad$ NIKADA $\quad$ GRAF NEMAJU PRESJEK!

OBAVEZNO DOKAZATI!

TABLICA!

	1		2	
P'		$-$		$+$
P		$\searrow$		$\nearrow$

MINIMUM WOOHOO!

$$P_{min}=P(2)=\frac{1}{2}(2-\ln 1)^2=2$$

12.3. INTEGRIRANJE IRACIONALNIH FUNKCIJA

→ cilj: ODABRATI SUPSTITUCIJU DA SE RIJEŠITO KORJENA

1. supstitucija t^n, gdje je n najmanji zajednički višekratnik svih korjena!

$\sqrt[2]{\ }, \sqrt[3]{\ } \to 6$!!

12.DZ-8) $$\int_0^1 \frac{1+\sqrt{x}}{1+\sqrt[3]{x}}\,dx = \left| \begin{array}{l} x = t^{6} \\ dx = 6t^5\,dt \end{array} \right| =$$

$$= \int_0^1 \frac{1+t^3}{1+t^2}\cdot 6t^5\,dt = 6\int_0^1 \frac{t^8+t^5}{t^2+1}\,dt$$ → U BROJNIKU VEĆI STUPANJ: PODIJELITI POLINOME.

$$(t^8+t^5):(t^2+1) = t^6-t^4+t^3+t^2-t-1$$

$$\begin{array}{l}
-t^8 \pm t^6 \\
\hline
-t^6+t^5 \\
+t^6+t^4 \\
\hline
t^5+t^4 \\
-t^5 \pm t^4 \\
\hline
t^4-t^3 \\
-t^4 \pm t^2 \\
\hline
-t^3-t^2 \\
+t^3+t \\
\hline
-t^2+t \\
+t^2+1 \\
\hline
\underline{\underline{t+1}}
\end{array}$$

→ ostatak

$$= 6\int_0^1 \left(t^6-t^4+t^3+t^2-t-1+\frac{t+1}{t^2+1}\right)dt =$$

$$= 6\left(\frac{t^7}{7}-\frac{t^5}{5}+\frac{t^4}{4}+\frac{t^3}{3}-\frac{t^2}{2}-t+\frac{1}{2}\ln|t^2+1|+\operatorname{arctg} t\right)\Big|_0^1 =$$

$$= 6\cdot\left(-\frac{409}{420}+\frac{1}{2}\ln 2+\frac{\pi}{4}\right)$$

9.) $\displaystyle\int_2^3 \sqrt{\frac{x+1}{x-1}}\,dx =$

ŽELIMO SE RIJEŠITI KORJENA !!

$$\frac{x+1}{x-1} = t^2$$

PRVO SE IZRAZI x preko t-a:

$$x+1 = t^2x - t^2$$

$$x(1-t^2) = -1-t^2$$

$$x = \frac{-1-t^2}{1-t^2} = \frac{t^2+1}{t^2-1}$$

$$dx = \frac{2t(t^2-1)-(t^2+1)\cdot 2t}{(t^2-1)^2}$$

PAZI !

$$dx = \frac{-4t}{(t^2-1)^2}\,dt$$

$$x=2 \longrightarrow \alpha = \sqrt{\frac{2+1}{2-1}} = \sqrt{3}$$

$$x=3 \longrightarrow \beta = \sqrt{\frac{3+1}{3-1}} = \sqrt{2}$$

$$= \int_{\sqrt{3}}^{\sqrt{2}} t\cdot\frac{-4t}{(t^2-1)}\,dt =$$

$$\frac{-4t^2}{(t-1)^2(t+1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2}$$

$\vdots$

$$\boxed{A=B=D=-1}$$

$$\boxed{C=1}$$

$$= \int_{\sqrt{3}}^{\sqrt{2}} \left(\frac{-1}{t-1} - \frac{1}{(t-1)^2} + \frac{1}{t+1} - \frac{1}{(t+1)^2}\right)dt = \left(-\ln|t-1| - \frac{(t-1)^{-1}}{-1} + \ln|t+1| - \frac{(t+1)^{-1}}{-1}\right)\Bigg|_{\sqrt{3}}^{\sqrt{2}} =$$

$$= 2\sqrt{2} - \sqrt{3} + \ln\frac{3+2\sqrt{2}}{2+\sqrt{3}} \quad //$$

12.DZ-13.)

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx = \left| \begin{array}{c} 1+\sqrt[4]{x} = t^3 \rightarrow t=\sqrt[3]{1+\sqrt[4]{x}} \\ \text{PRVO IZRAZI } x \text{ !!!} \\ x=(t^3-1)^4 \\ dx = 4(t^3-1)\cdot 3t^2\,dt \end{array} \right| =$$

!!! PAZI

$t^{3/2}$

$$= \int \frac{t}{\cancel{(t^3-1)^2}} 4(t^3-1)^{\cancel{3}} 3t^2\,dt = 12\int (t^6-t^3)\,dt =$$

$$= 12\frac{t^7}{7} - 12\frac{t^4}{4} + C =$$

$$= \frac{12}{7}\left(\sqrt[3]{1+\sqrt[4]{x}}\right)^7 - 3\left(\sqrt[3]{1+\sqrt[4]{x}}\right)^4 + C$$

12.DZ-10.)

$$\int \frac{dx}{x^4\sqrt{1+x^2}} = \left| \begin{array}{c} 1+x^2=t^2 \\ x=\sqrt{t^2-1} \\ dx = \frac{1}{2\sqrt{t^2-1}}\cdot 2t\,dt \\ dx = \frac{t}{\sqrt{t^2-1}}\,dt \end{array} \right| =$$

$$= \int \frac{dt}{(t^2-1)^2\sqrt{t^2-1}}$$ FAIL!!

TRIK:

$$\int \frac{dx}{x^4\sqrt{1+x^2}} = \int \frac{dx}{x^5\cdot\sqrt{\frac{1}{x^2}-1}} = \left| \begin{array}{c} \frac{1}{x^2}+1 = t^2 \rightarrow t=\sqrt{\frac{1}{x^2}+1} \\ x=\frac{1}{\sqrt{t^2-1}} = (t^2-1)^{-\frac{1}{2}} \\ dx = -\frac{1}{2}(t^2-1)^{-\frac{3}{2}}\cdot 2t\,dt \end{array} \right| = \int \frac{-\frac{1}{\cancel{2}}(t^2-1)^{-\frac{3}{2}}\cdot \cancel{2}\cancel{t}\,dt}{(t^2-1)^{-\frac{5}{2}}\cdot \cancel{t}} =$$

$$= -\int (t^2-1)\,dt = -\frac{t^3}{3} + t + C = -\frac{1}{3}\left(\sqrt{\frac{1}{x^2}+1}\right)^3 + \sqrt{\frac{1}{x^2}+1} + C$$

12.2.1, 12.2.2, 12.2.3 $\longrightarrow$ ZA NADOBUDNE !

TRIGONOMETRIJSKE SUPSTITUCIJE

$$\int \sqrt{a^2-x^2}\,dx = \left| x = a\sin t \right|$$

1) $\int R(x, \sqrt{a^2-x^2})\,dx \qquad \left| x = a\sin t \right|$

2) $\int R(x, \sqrt{a^2+x^2})\,dx \qquad \left| x = a\,\mathrm{sh}\,t \right|$

3) $\int R(x, \sqrt{x^2-a^2})\,dx \qquad \left| x = a\,\mathrm{ch}\,t \right|$

PAZI! a^2

14.)

$$\int \sqrt{1-3x^2}\,dx = \int \left(\sqrt{3}\cdot\sqrt{\tfrac{1}{3}-x^2}\right)dx = \sqrt{3}\int\sqrt{\tfrac{1}{3}-x^2}\,dx = \left| \begin{array}{l} x = \frac{1}{\sqrt{3}}\sin t \\ dx = \frac{1}{\sqrt{3}}\cos t\,dt \end{array} \right|$$

$$= \cancel{\sqrt{3}}\int\sqrt{\tfrac{1}{3}\cos^2 t}\cdot\frac{1}{\cancel{\sqrt{3}}}\cos t\,dt = \frac{1}{\sqrt{3}}\int\cos^2 t\,dt = \frac{1}{\sqrt{3}}\int\frac{1+\cos 2t}{2}\,dt =$$

$$= \frac{1}{2\sqrt{3}}t + \frac{1}{4\sqrt{3}}\sin 2t + C = \frac{1}{2\sqrt{3}}\arcsin(\sqrt{3}x) + \frac{1}{4\sqrt{3}}\sin\left(2\arcsin(\sqrt{3}x)\right) + C$$

DZ-14.)

RIJEŠITI SE SINUSOM

$$\int_0^{\frac{\pi}{2}} \cos x\sqrt{3\sin^2 x + 2\cos^2 x}\,dx = \left| \begin{array}{l} \sin x = t \\ \cos x\,dx = dt \\ x=0 \to \alpha = 0 \\ x = \frac{\pi}{2} \to \beta = 1 \end{array} \right| =$$

zbog granica je pozitivno, pa možemo maknuti !!

$$= \int_0^1 \sqrt{2-t^2}\,dt = \left\{ \text{supstitucija (B)} \right\} = \left| \begin{array}{l} t = \sqrt{2}\,\mathrm{sh}\,u \\ dt = \sqrt{2}\,\mathrm{ch}\,u\,du \\ t=0 \to \alpha = 0 \\ t=1 \to \beta = \mathrm{arsh}\frac{1}{\sqrt{2}} \end{array} \right| = \int_0^{\mathrm{arsh}\frac{1}{\sqrt{2}}} \underbrace{\sqrt{2+2\,\mathrm{sh}^2 u}}_{\substack{2(1+\mathrm{sh}^2 u) \\ 2\,\mathrm{ch}^2 u}}\cdot\sqrt{2}\,\mathrm{ch}\,u\,du = \int_0^{\mathrm{arsh}\frac{1}{\sqrt{2}}} \sqrt{2}\,|\mathrm{ch}\,u|\sqrt{2}\,du\,du$$

zbog granica

$$= 2\int_0^{\operatorname{arsh}\frac{1}{\sqrt{2}}} \operatorname{ch}^2 u\,du = 2\int_0^{\operatorname{arsh}\frac{1}{\sqrt{2}}} \frac{\operatorname{ch}2u+1}{2}\,du = \left(\frac{1}{2}\operatorname{sh}2u + u\right)\Big|_0^{\operatorname{arsh}\frac{1}{\sqrt{2}}} =$$

$$= \underbrace{\frac{1}{2}\operatorname{sh}\left(2\operatorname{arsh}\frac{1}{\sqrt{2}}\right)}_{\text{WOLFRAM: } \frac{\sqrt{3}}{2}} + \operatorname{arsh}\frac{1}{\sqrt{2}}$$

12. DZ-20.)

$$\int_0^1 x^2\sqrt{x-x^2}\,dx =$$

$$-(x^2-x) = -\left(\left(x-\frac{1}{2}\right)^2 - \frac{1}{4}\right)$$

$\sqrt{a^2-x^2}$

$x = a\sin t$

$$= \int_0^1 x^2\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}\,dx = \left|\begin{array}{l} x-\frac{1}{2} = \frac{1}{2}\sin t \\ dx = \frac{1}{2}\cos t\,dt \\ x = \frac{1}{2}\sin t + \frac{1}{2} \\ x=0 \to \alpha = -\frac{\pi}{2} \\ x=1 \to \beta = \frac{\pi}{2} \end{array}\right| =$$

$\sin t = \frac{x-\frac{1}{2}}{\frac{1}{2}}$

$t = \arcsin(2x-1)$

NE SMIJE BITI $\frac{3\pi}{2}$!! U SMJERU PORASTA GLEDAMO.

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2}\sin t + \frac{1}{2}\right)^2 \sqrt{\frac{1}{4}\cos^2 t}\cdot\frac{1}{2}\cos t\,dt =$$

$$= \frac{1}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^2 t + 2\sin t + 1)\cos^2 t\,dt =$$

$$= \frac{1}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t\cos^2 t\,dt + \frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin t\cos^2 t\,dt + \frac{1}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\,dt =$$

$(\sin t\cdot\cos t)^2$

$\left(\frac{1}{2}\sin 2t\right)^2$

$$\left|\begin{array}{l} \cos t = u \\ -\sin t\,dt = du \\ x = -\frac{\pi}{2} \to 0 = \alpha \\ x = \frac{\pi}{2} \to \beta = 0 \end{array}\right| \quad \frac{1}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\,dt = \frac{\pi}{32}$$

$$\frac{1}{64}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\cos 4t}{2}\,dt =$$

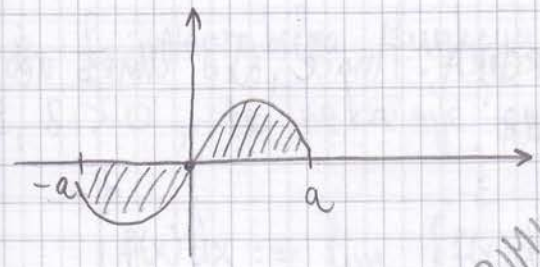
$$-\int_0^0 u^2\,du = 0$$

$$= \frac{\pi}{128}$$

$$= \frac{5\pi}{128}$$

DZ-M.) $\int_{-1}^{1} x\sqrt{\frac{4-x^2+x^4}{4+x^2-x^4}}\,dx = 0$

→ NEPARNE FUNKCIJE NA SIMETRIČNOM INTERVALU IMAJU VRIJEDNOST INTEGRALA 0 !



$\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$ ako je f neparna!

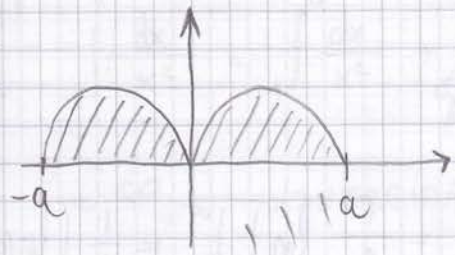
PRIMJETI

→ VRIJEDNOST TOGA INTEGR.!

ALI TO NE PREDSTAVLJA POVRŠINU!

POVRŠINA POSTOJI, RAČUNA SE KAO $2\cdot\int_{0}^{a} f(x)\,dx$, SAMO ŠTO JE JEDAN DIO ISPOD X-OSI ...

→ PARNE FUNKCIJE:



$\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\cdot\int_{0}^{a} f(x)\,dx$, ako je f parna!

BOOBIES!!!

PROVJERA (NE)PARNOSTI:

neparne: $f(-x) = -f(x)$

parne: $f(x) = f(-x)$

13.1. NEPRAVI INTEGRALI

- def. „NEPRAVI INTEGRALI PRVE VRSTE"

Neka je $a \in \mathbb{R}$ i neka je f integrabilna na $[a,b]$, za $\forall b < \infty$. Tada definiramo

$$\int_a^\infty f(x)dx := \lim_{b\to\infty} \int_a^b f(x)\,dx$$

ako taj limes postoji i konačan.

$\longrightarrow$ Tada kažemo da integral KONVERGIRA. Inače, ako limes ne postoji ili je ∞, integral DIVERGIRA.

Analogno:

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$$

$$\int_{-\infty}^\infty f(x)dx = \lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b f(x)dx$$

13. DZ - 1.)

$$\int_0^\infty \frac{x\,dx}{x^2+1} = \left| \begin{array}{l} x^2+1=t \\ 2x\,dx = dt \\ x=0 \to \alpha=1 \\ x=\infty \to \beta=\infty \end{array} \right| = \frac{1}{2}\int_1^\infty \frac{dt}{t} = \frac{1}{2}\ln t\Big|_1^\infty = \frac{1}{2}(\ln(\infty) - \ln 1)$$

$$= \frac{1}{2}(\infty - 0) = \infty \longrightarrow \text{DIVERGIRA}$$

DEK. - 11. - 4.)

$$\int_0^\infty \frac{x\,dx}{x^4+1} = \left| \begin{array}{l} x^2=t \\ 2x\,dx = dt \\ x=0 \to \alpha=0 \\ x=\infty \to \beta=\infty \end{array} \right| = \frac{1}{2}\int_0^\infty \frac{dt}{t^2+1} = \frac{1}{2}\,\mathrm{arctg}\,t\Big|_0^\infty = \frac{\pi}{4}$$

konvergira

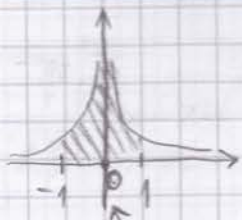
* Ispravnije je pisati:

$$\lim_{b\to\infty} \mathrm{arctg}\,t\Big|_0^b = \lim_{b\to\infty} \mathrm{arctg}\,b - 0$$

U POZADINI PRIČE SU LIMESI!!

$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^{1} = -1-1 = -2 \quad ?$$

TVZ



NIJE U DOMENI!

NE!! → IMAMO PREKID!!

NEPRAVI INTEGRALI „DRUGE VRSTE"

Neka je f integrabilna funkcija na $[a, c-\varepsilon]$ i $[c+\delta, b]$, za $\varepsilon, \delta > 0$, te neka nije omeđena u $c \in [a,b]$. Tada definiramo:

$$\int_a^b f(x)\,dx := \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\delta \to 0} \int_{c+\delta}^{b} f(x)\,dx$$

Ako taj limes postoji i ako je konačan, kažemo da integral konvergira.

→ Pr.)

$$\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0^-} \frac{dx}{x^2} + \int_{0^+}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^{0^-} + \left(-\frac{1}{x}\right)\Big|_{0^+}^{1} = +\infty - 1 + (-1) + \infty$$

$= \infty$ // → DIVERGIRA!

PAZI!! LIMES U POZADINI PRIČE:

$$\lim_{x \to 0^-} \left(-\frac{1}{x}\right) = +\infty \,//$$

Zad-6.)

$$\int_0^{\pi} tg^2(2x)\,dx \overset{TVZ}{=} \left(\frac{1}{2} tg\,2x - x\right)\Big|_0^{\pi} = -\pi \,//$$

NE!

$$\int \frac{\sin^2 2x}{\cos^2 2x}\,dx = \int \frac{1-\cos^2 2x}{\cos^2 2x}\,dx = \frac{1}{2} tg\,2x - x + C$$

UVIJEK PROVJERITI IMA LI FUNKCIJA PREKID!!!

$D(f)$: $2x \neq \frac{\pi}{2} + k\pi \quad |:2$

$x \neq \frac{\pi}{4} + \frac{k\pi}{2}$ → $\frac{\pi}{4}, \frac{3\pi}{4}$

→ MUNJA!

$$= \left(\frac{1}{2}\operatorname{tg}2x - x\right)\Big|_0^{\frac{\pi}{4}^-} + \left(\frac{1}{2}\operatorname{tg}2x - x\right)\Big|_{\frac{\pi}{4}^+}^{\frac{3\pi}{4}^-} + \left(\frac{1}{2}\operatorname{tg}2x - x\right)\Big|_{\frac{3\pi}{4}^+}^{\pi} =$$

$$= +\infty - \frac{\pi}{4} + \left(+\infty - \frac{3\pi}{4}\right) - \left(-\infty - \frac{\pi}{4}\right) + (-\pi) - \left(-\infty - \frac{3\pi}{4}\right) =$$

$$\lim_{x\to\frac{\pi}{4}^-} \operatorname{tg}2x = \operatorname{tg}\frac{\pi}{2}^- = +\infty$$

$$= \infty - \pi = \infty$$

$\hookrightarrow$ DIVERGIRA!

$\boxed{\text{TM}}$ „KRITERIJ USPOREDBE"

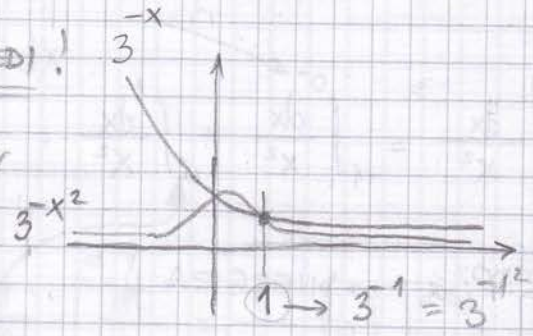
Neka je $|f(x)| \leq g(x)$. Ako konvergira nepravi integral od $g(x)$, tada integral od $f(x)$ koji je također nepravi također konvergira. (za obje vrste)

21-09-07.) Ispitaj konvergenciju:

$$\int_1^\infty \cos x \cdot 3^{-x^2}\,dx =$$

VRIJEDI!

$$\left|\underbrace{\cos x}_{|\cos x| \leq 1} 3^{-x^2}\right| \leq 3^{-x^2} \leq 3^{-x}$$



$$\int_1^\infty 3^{-x}\,dx = \frac{-3^{-x}}{\ln 3}\Big|_1^\infty = \frac{1}{3\ln 3}$$ konvergira!

Dakle, po kriteriju usporedbe konvergira i $\int_1^\infty \cos x\, 3^{-x^2}\,dx$.

$\boxed{\text{TM}}$

Neka je $f(x)$ ekvivalentna s $g(x)$ (kada x teži u "kritičnu tocku" $\to c \to 0$). Tada je $\int f(x)dx \sim \int g(x)dx$, tj. ili oba konvergiraju ili oba divergiraju.

$$\int_0^1 \frac{\ln(1+x^2)}{\sqrt[3]{x^{10}}}\,dx$$ → veća funkcija od ove koju znamo integrirati ???

ne znamo

→ ČIM VIDIMO ln ODMAH SE SJETITI NEIZMJERNIH VEL.!

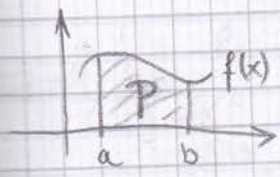
$$\underset{\sim}{x\to 0} \int_0^1 \frac{x^2}{x^{\frac{10}{3}}}\,dx = \int_0^1 x^{-\frac{4}{3}}\,dx = \frac{x^{-\frac{1}{3}}}{-\frac{1}{3}\sqrt[3]{x}}\Big|_{0^+}^{1} = -3-(-\infty) = +\infty$$

↓ divergira

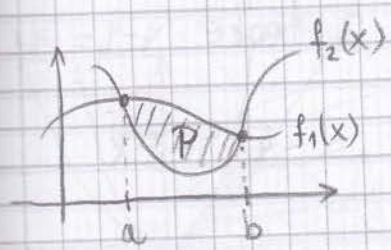
→ pa stoga divergira i početni ! (ili oba ...) ITM

13.2. PRIMJENA INTEGRALA

- POVRŠINA LIKA



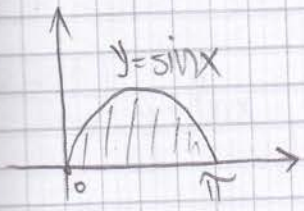
$$\boxed{P = \int_a^b f(x)\,dx}$$



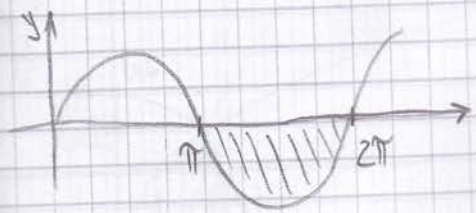
$$\boxed{P = \int_a^b (f_1(x) - f_2(x))\,dx}$$

↳ „GORNJA FUNKCIJA − DONJA FUNKCIJA"

1)



$$P = \int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} = 2$$



TVZ

$$P = \int_{\pi}^{2\pi} -\sin x\,dx \overset{TVZ}{=} -2$$

→ POVRŠINA NE SMIJE BITI NEGATIVNA!

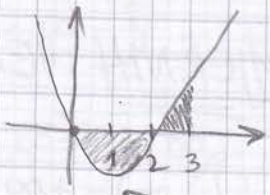
OBAVEZNO NACRTATI!

KADA JE ISPOD X-OSI U integ. STAVLJAMO − !!

Zad.) Izračunaj površinu između parabole $y = x^2 - 2x$ i osi x na intervalu $[0,3]$.

TVZ-ovac:

$$\int_0^3 (x^2-2x)\,dx = \text{pozitivan}$$

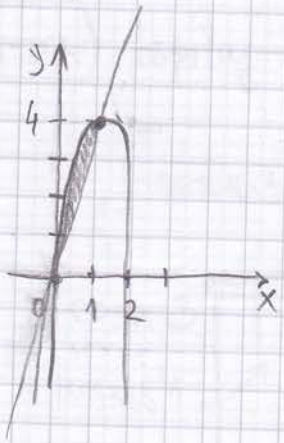


PAZI! ILKO OBOŽAVA OVAKVE STVARI

$$-\int_0^2 (x^2-2x)\,dx + \int_2^3 (x^2-2x)\,dx = \ldots = DZ$$

ispod x-osi

z1-09-14.) (2 boda) Izračunaj površinu lika sa slike:



$$P = \int_0^1 (\text{parabola} - \text{pravac})\,dx$$

p... $y = 4x$ → pravac kroz 2 točke $y - y_1 =$

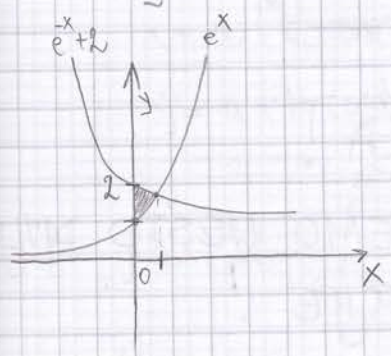
parabola ... $y = a(x-x_1)(x-x_2) = -4x^2 + 8x$

$T(1,4)$

$4 = a(-1)$

$$P = \int_0^1 (-4x^2 + 8x - 4x)\,dx = \frac{2}{3}$$

-8.) Izračunaj površinu lika omeđenog krivuljama $y=e^x$, $y=e^{-x}+2$ i osi y.



$$P=\int_0 (e^{-x}+2-e^x)\,dx$$

$$e^x = e^{-x}+2$$

$$e^x - e^{-x} + 2 = 0 \quad /\cdot e^x$$

$$e^{2x} - 1 - 2e^x = 0$$

$$\underline{e^x = t}$$

$$t^2 - 2t - 1 = 0$$

$$t_{1,2} = \frac{2 \pm \sqrt{4}}{2}$$

$$t_{1,2} = 1 \pm \sqrt{2}$$

e^x ne smije biti negativan!

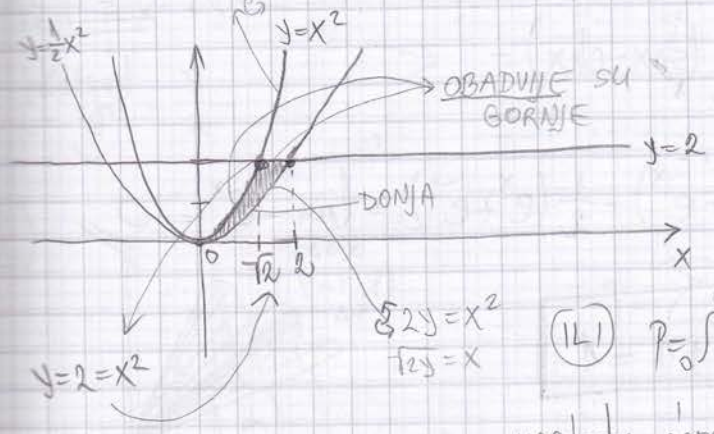
$$\boxed{x = \ln(1+\sqrt{2})}$$

$$P=\int_0^{\ln(1+\sqrt{2})} (e^{-x}+2-e^x)\,dx =$$

$$= \left(-e^{-x}+2x-e^x\right)\Big|_0^{\ln(1+\sqrt{2})} =$$

$$P = 2\ln(1+\sqrt{2}) + 2 - 2\sqrt{2}$$

-07-9) $y=x^2$, $y=\frac{1}{2}x^2$, $y=2$ (u 1. kvadrantu)



$$P=\int_0^{\sqrt{2}} \left(x^2-\frac{1}{2}x^2\right)dx + \int_{\sqrt{2}}^{2} \left(2-\frac{1}{2}x^2\right)dx$$

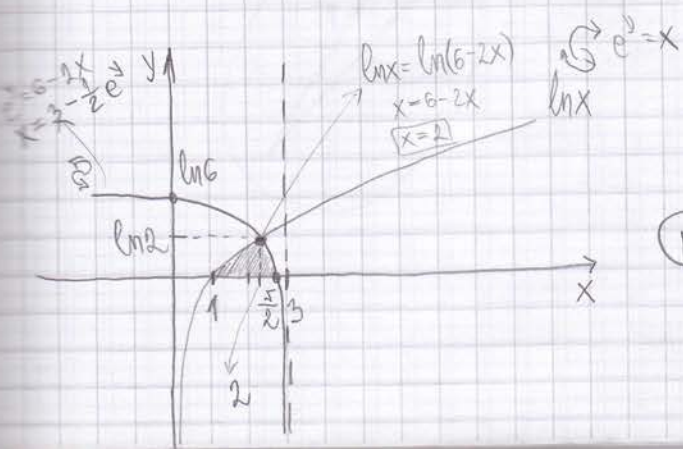
$$\boxed{P=\frac{8}{3}-\frac{4}{3}\sqrt{2}}$$

(ILI) $P=\int_0^2 (\sqrt{2y}-\sqrt{y})\,dy = (\sqrt{2}-1)\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^2 = \dots$

→ metoda čorave kokoši?? ne....

-DZ-12.) $y=\ln x$, $y=\ln(6-2x)$ i os x

$$y=\ln(-2(3-2x))$$



$$P=\int_1^2 \ln x\,dx + \int_2^{\frac{5}{2}} \underbrace{\ln(6-2x)}_{\text{parcijala!}}\,dx = 3\ln 2 - \frac{3}{2}$$

(ILI) OKRENUTI PAPIR!

ROTACIJA OKO y-OSI

$$P=\int_0^{\ln 2} \left(3-\frac{1}{2}e^y-e^y\right)dy = 3\ln 2 - \frac{3}{2}$$

! OKRENUTI CIJELU SLIKU, ROTIRATI I NAĆI INVERZE FUNKCIJA !

→ NE TREBA RASTAVLJATI NA DVA DJELA!

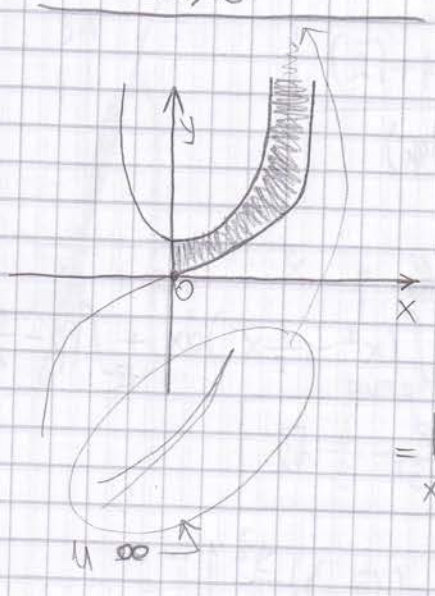
→ UVIJEK AKO NORMALNIM PUTEM DOBIJEMO NEŠTO DIVLJE ONDA OKRENUTI PODRUČJE INTEGRACIJE!

13.DZ-11.) Dokaži da je površina lika omeđena zadanim jednadžbama konačna.

$y \geq \operatorname{sh} x$

$y \leq \operatorname{ch} x$

$x \geq 0$



$$P = \int_0^\infty (\operatorname{ch} x - \operatorname{sh} x)\,dx =$$

$$= (\operatorname{sh} x - \operatorname{ch} x)\Big|_0^\infty = \infty - \infty + 1$$

NEODREĐENI OBLIK

NLO

limes

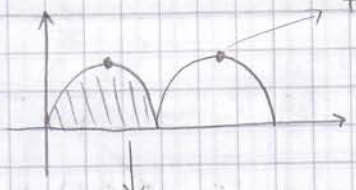
$$\lim_{x\to\infty} (\operatorname{sh} x - \operatorname{ch} x) =$$

$$= \lim_{x\to\infty} \frac{1}{2}(e^x - e^{-x}) - \frac{1}{2}(e^x + e^{-x}) = \lim_{x\to\infty} -e^{-x} = 0$$

$$\boxed{P = 0 + 1 = 1}$$

ZZ-14.) Izračunaj površinu prvog svoda cikloide ö

$$x = a(t - \sin t)$$
$$y = a(1 - \cos t) \qquad t \in [0, 2\pi]$$



točke maksimuma he he

SISOIDA !!!

$$P = \int_a^b y(x)\,dx = \int_0^{2\pi} \underbrace{a(1-\cos t)}_{y}\,\underbrace{a(1-\cos t)\,dt}_{x'}$$

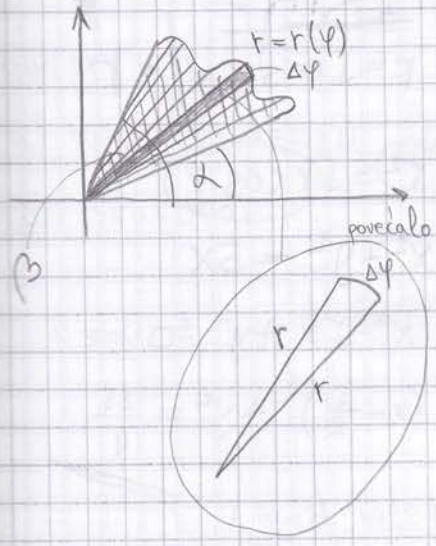
$$P = a^2 \int_0^{2\pi} (1 - \cos 2t + \underbrace{\cos^2 t}_{\frac{1}{2}(1+\cos 2t)})\,dt = 3\pi a^2$$

POVRŠINA U POLARNIM KOORDINATATA

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

$$r^2 = x^2 + y^2$$

$$\operatorname{tg}\varphi = \frac{y}{x}$$



$$P = \sum P_{k,i} = \sum \frac{1}{2} r^2 \Delta\varphi \quad \Big| \lim_{x \to \infty}$$

$$P_{k,i} = \frac{1}{2} r^2 \Delta\varphi$$

kružni isječak

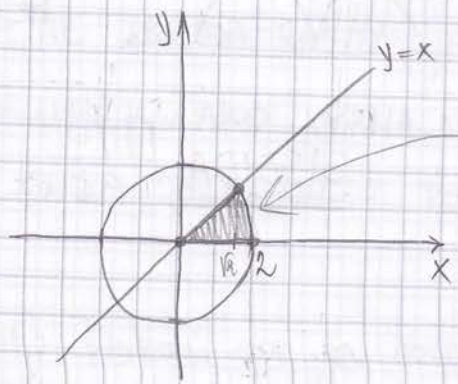
$$\boxed{P = \frac{1}{2} \int_\alpha^\beta r^2 \Delta\varphi}$$

Pr.) Izračunaj površinu lika između krivulja:

$y=\sqrt{4-x^2}$, $y=x$, $y=0$

$y^2=4-x^2$

$\to x^2+y^2=4 \to$ kružnica, $r=$



$y=\sqrt{4-x}$

$\boxed{y>0}$

NE I DOLJE! ZBOG KORJENA

I. NAČIN:

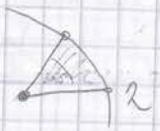
$\longrightarrow$ gornje su: $y=x$ i $y=\sqrt{4-x^2}$

TVZ $\boxed{x=\sqrt{2}}$

$$P=\int_0^{\sqrt{2}} x\,dx+\int_{\sqrt{2}}^{2}\sqrt{4-x^2}\,dx = \left| \begin{array}{c} x=2\sin t \\ \vdots \end{array} \right| \text{ DZ } \frac{\pi}{2}$$

II. NAČIN:

$\frac{\pi}{4} \to 45^\circ$



$$P=\frac{1}{2}\int_0^{\frac{\pi}{4}} 4\,d\varphi=\frac{\pi}{2}$$

13. ZZV – 24.) I_{plok} unutar $r=5\cos\varphi$ (kružnica) i izvan $r=2+\cos\varphi$ (kardioida). Slika!

TRIK:

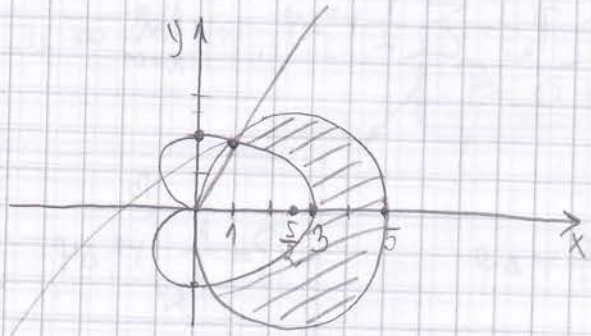
$r=5\cos\varphi \quad /\cdot r$

$x^2+y^2=r^2=5\underbrace{r\cos\varphi}_{=x}$

$x^2+y^2=5x$

$x^2-5x+y^2=0$

$\left(x-\frac{5}{2}\right)^2+y^2=\frac{25}{4}$



$5\cos\varphi=2+\cos\varphi$

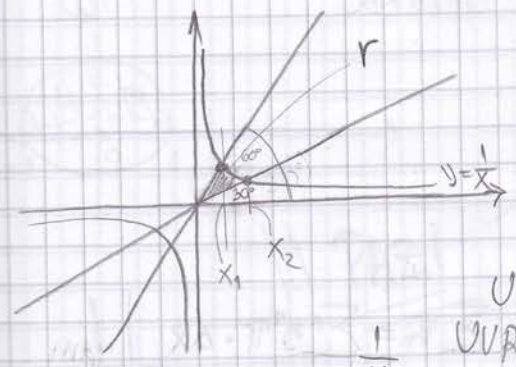
$\cos\varphi=\frac{1}{2}$

$\varphi=\frac{\pi}{3}$

$$P=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \underbrace{(5\cos\varphi)^2}_{\text{DALJNJI } r}-\underbrace{(2+\cos\varphi)^2}_{\text{BLIŽI } r}\,d\varphi$$

$$=\frac{1}{2}\cdot\underbrace{2}_{\text{simetrija}}\cdot\int_0^{\frac{\pi}{3}}\left(25\cos^2\varphi-4-4\cos\varphi-\cos^2\varphi\right)d\varphi=\dots=\frac{8\pi}{3}+\sqrt{3}$$

C-13.) Pravci koji sa osi x zatvaraju 30° i 60° sijeku hiperbolu $\frac{1}{x}$ u prvom kvadrantu. Površina?



$$P=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{d\varphi}{\sin\varphi\cos\varphi}=\left|\begin{array}{l} t=\operatorname{tg}\varphi \\ d\varphi=\frac{dt}{1+t^2} \\ \operatorname{tg}\frac{\pi}{6}=\frac{\sqrt{3}}{3}\ldots \end{array}\right|=$$

$$=\frac{1}{2}\int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}}\frac{\frac{dt}{1+t^2}}{\frac{t}{\sqrt{1+t^2}}\cdot\frac{1}{\sqrt{1+t^2}}}=\frac{1}{2}\int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}}\frac{dt}{t}=$$

$$=\frac{1}{2}\ln t\Big|_{\frac{\sqrt{3}}{3}}^{\sqrt{3}}=\frac{1}{2}\ln 3$$

UVIJEK UVRSTI POLARNE

$$y=r\sin\varphi=\frac{1}{r\cos\varphi}$$

$$\frac{1}{x}$$

$$r^2=\frac{1}{\sin\varphi\cos\varphi}$$

L: {KARTEZIJEV}

$k=\operatorname{tg}\varphi$

$$P=\int_0^{x_1}\left(\sqrt{3}x-\frac{\sqrt{3}}{3}x\right)dx+\int_{x_1}^{x_2}\left(\frac{1}{x}-\frac{\sqrt{3}}{3}x\right)dx$$

→ neće biti uopće gotovih integrala, sve u primjeni

eventualno kombinirano sa teorijom: a) def. odrednog int.

b) izračunaj $\int$

običan integral:

a) izvedi rekurzivnu formulu

b) izračunaj...

8 zad → 4 zad derivacije → 1. graf, 2. Taylor, 3. ekstremni zadatak s riječima, 4. teorija → ili teorija u komb. sa tangentama

4 zad integrala:

1. teorija (parcij.) Newton-Leibniz

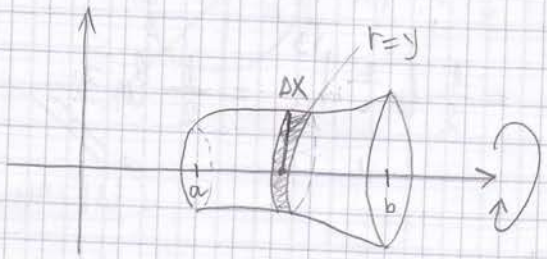
2, 3, 4 → primjena ⇒ površina sigurno, duljina luka...

pitanja <u>općenito</u> → 1.) lin. nezavisnost, rang o parametru → sigurno
→ 2.) funkcije – pomaci, crtanje.

→ 1. ciklus → OSNOVNE DEF. neće biti dokaza

B) VOLUMENI ROTACIONIH TIJELA

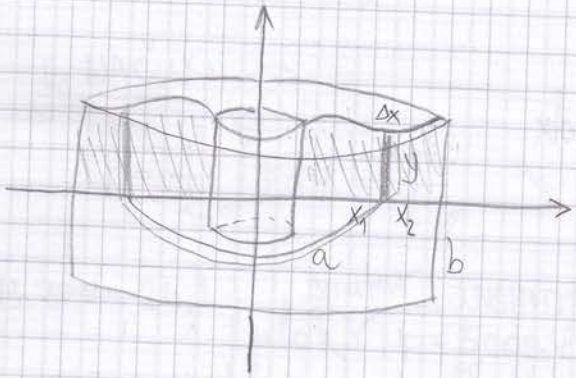
(I) ROTIRANJE OKO OSI X



$$V = \sum V_i = \sum y^2 \pi \cdot \Delta x \quad \Big| \lim_{n\to\infty}$$

$$\boxed{V = \pi \int_a^b y^2 dx}$$

(II) ROTIRANJE OKO OSI Y

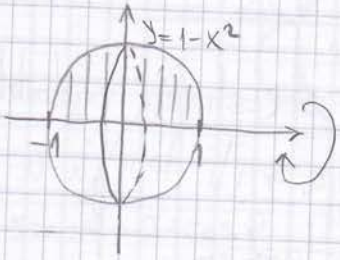


$$V = \sum V_i = 2 \sum y \cdot \pi (x_2^2 - x_1^2)^2 \quad \Big| \lim_{n\to\infty}$$

$$\boxed{V = 2\pi \int_a^b xy\, dx}$$

13. DZ – 16.) $y = 1 - x^2$ i osi X

a) rotiramo oko osi X, $V = ?$

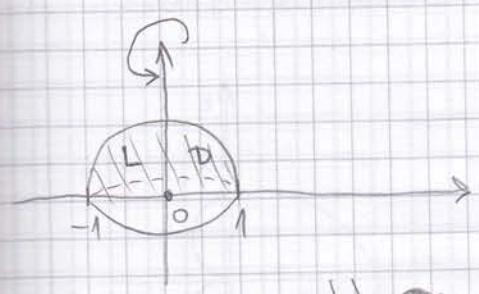


→ POANTA: UVRSTITI U FORMU

$$V = \pi \int_{-1}^{1} (1-x^2)^2 dx = \pi \int_{-1}^{1} (1 - 2x^2 + x^4)\, dx = \dots$$

$$\boxed{V = \frac{16\pi}{15}}$$

rotiramo oko osi y



$$V = 2\pi \int_0^1 x(1-x^2)\,dx$$

!! PAZI

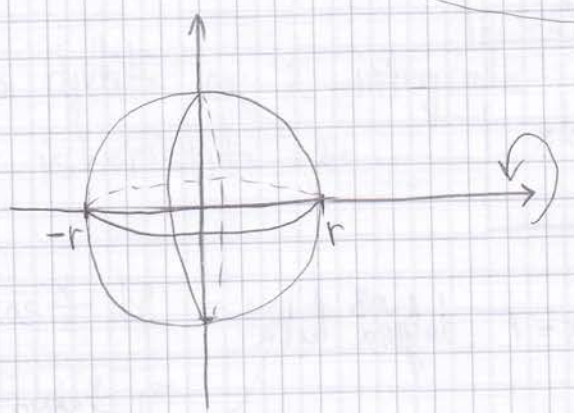
→ kada rotira L ide u D i obratno pa od -1 do 1 je zapravo dupli volumen !!

2008.-7.) Izvedi formulu za volumen kugle polumjera r.

→ mora rotira kružnica polumjera r

$y^2 = r^2 - x^2$

$x^2 + y^2 = r^2$



$$V = \pi \int_{-r}^{r} (r^2 - x^2)\,dx$$

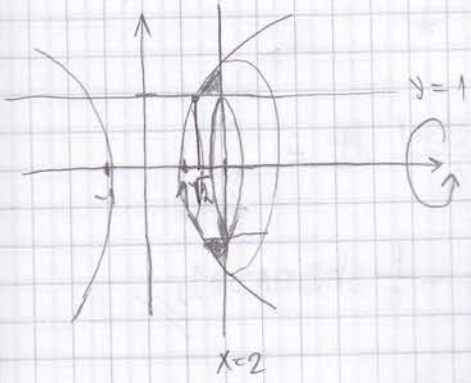
$$V = \pi \left(r^2 x - \frac{x^3}{3}\right)\Big|_{-r}^{r}$$

$$\boxed{V = \frac{4}{3} r^3 \pi}$$

07-9.) Dio krivulje omeđen I. kvadrant $x^2 - y^2 = 1$, $y = 1$, $x = 2$ rotira oko osi x. Volumen?

$x^2 - y^2 = 1$ → HIPERBOLA !

BITNO JE ZNATI FORMUL



PAZI !

$$V = \pi \int_{\sqrt{2}}^{2} \left((x^2 - 1) - 1\right) dx$$

$\boxed{y^2_{GORNJA} - y^2_{DONJA}}$

$y = 1$, $y^2 = x^2 - 1$

$1 = x^2 - 1$

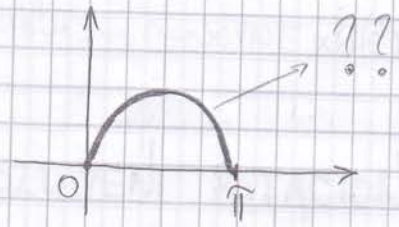
$x = \sqrt{2}$

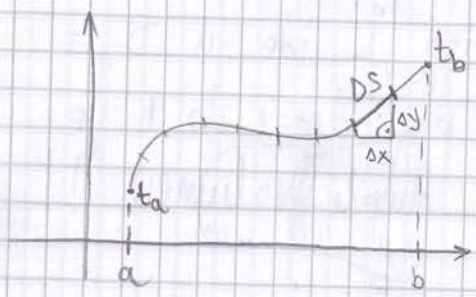
$$V = \pi \left(\frac{x^3}{3} - 2x\right)\Big|_{\sqrt{2}}^{2}$$

$$\boxed{V = \frac{4\pi}{3}(\sqrt{2} - 1)}$$

!

c) DULJINA LUKA





$$s = \sum \Delta s = \sum \sqrt{(\Delta x)^2 + (\Delta y)^2} \cdot \frac{\Delta t}{\Delta t} =$$

$$= \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \cdot \Delta t \quad \Big/ \lim_{n \to \infty}$$

parametarski zadane $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$

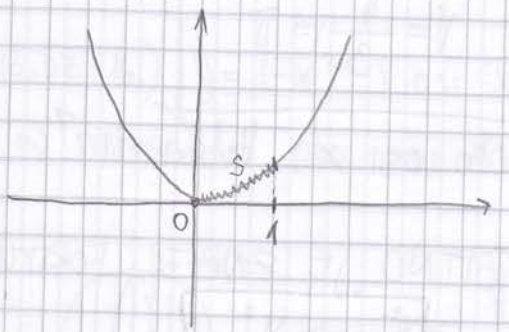
$$\boxed{s = \int_{t_a}^{t_b} \sqrt{(\dot{x}^2) + (\dot{y})^2}\, dt}$$

za $y = y(x)$:

$$\boxed{s = \int_a^b \sqrt{1 + (y'(x))^2}\, dx}$$

13. DZ-18.)

$y = x^2$ od $x = 0$ do $x = 1$, duljina luka



SAMO UVRŠTAVANJE U FORMULU !!

$$s = \int_0^1 \sqrt{1 + 4x^2}\, dx = \left| \begin{array}{l} x = \frac{1}{2} \operatorname{sh} t \\ dx = \frac{1}{2} \operatorname{ch} t\, dt \\ x = 0 \to \alpha = 0 \\ x = 1 \to \beta = \operatorname{arsh} 2 \end{array} \right|$$

$$s = \frac{1}{2} \int_0^{\operatorname{arsh} 2} \operatorname{ch}^2 t\, dt =$$

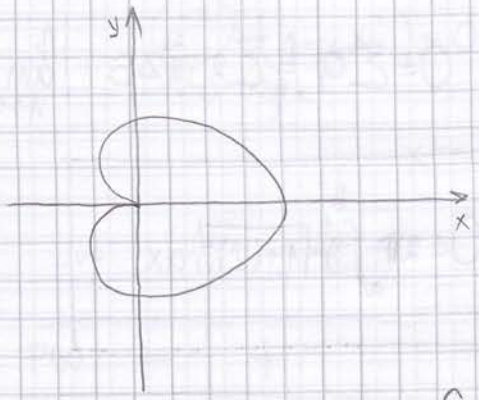
$$= \frac{1}{2} \int_0^{\operatorname{arsh} 2} \frac{\operatorname{ch} 2t + 1}{2}\, dt =$$

$$= \frac{1}{4} \operatorname{arsh} 2 + \frac{1}{8} \operatorname{sh} 2 \operatorname{arsh} 2$$

PONOVITI sh, ch PREKO ln-ova !

↳ u službenim rj. preko ln-ova

2) Izračunaj opseg kardioide $r = a(1+\cos\varphi)$. (Ne treba napamet znati jednadžbe kardioide, astroide... pisat će nam!)



$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}\right\} \quad s = \int \sqrt{(r'\cos\varphi - r\sin\varphi)^2 + (r'\sin\varphi + r\cos\varphi)^2}$$

$$\Rightarrow \boxed{s = \int_\alpha^\beta \sqrt{r^2 + (r')^2}\, d\varphi}$$

$$S = \int_0^{2\pi} \sqrt{a^2(1 + 2\cos\varphi + \cos^2\varphi) + a^2\sin^2\varphi}\, d\varphi =$$

$$= a\int_0^{2\pi} \sqrt{2 + 2\cos\varphi}\, d\varphi =$$

PAZI!!!

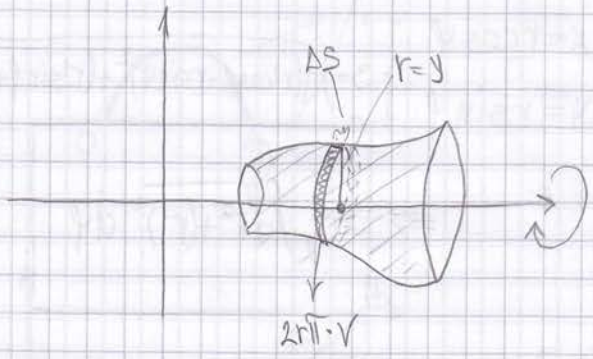
$$= a\sqrt{2}\int_0^{2\pi} \sqrt{1+\cos\varphi}\, d\varphi = 2a\int_0^{2\pi} \left|\cos\frac{\varphi}{2}\right| d\varphi =$$

$$\sqrt{2\cos^2\frac{\varphi}{2}}$$

→ trebali bi dijeliti na 2 integrala, ali je SIMETRIČNO pa možemo iskoristiti 2...

$$= 2a \cdot 2 \underbrace{\int_0^{\pi} \cos\frac{\varphi}{2}\, d\varphi}_{\left.\frac{\sin\frac{\varphi}{2}}{\frac{1}{2}}\right|_0^{\pi}} = \dots = 8a$$

D) OPLOŠJE ROTACIJSKIH TJELA

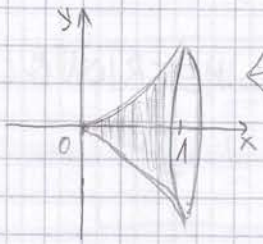


$$O = \sum O_i = \sum 2y\pi\Delta S \quad \Big| \lim_{n\to\infty}$$

$$\boxed{O = 2\pi \int_a^b y\sqrt{1+(y')^2}\,dx}$$

13.ZZV-33.) Izračunaj oplošje tijela koje rotira $y = \frac{x^3}{3}$ za $x \in [0,1]$.

$$O = 2\pi \int_0^1 \frac{x^3}{3}\sqrt{1+x^4}\,dx = \left| \begin{matrix} 1+x^n = t \\ 4x^3dx = dt \end{matrix} \right| = \frac{\pi}{6}\int_0^1 \sqrt{t}\,dt = \dots = \frac{\pi}{9}(2\sqrt{2}-1)$$



BUREK TORNADO !!!

ZADACI KOJE VRLO VJEROJATNO MOŽEMO OČEKIVATI:

PZI-08-14.) Nacrtaj kvalitativni graf funkcije i izračunaj površinu koju zatvara sa svojim asimptotom: $f(x) = xe^{-x^2}$.

→ ILKO ŽIVI ZA OVO, NA ROKU SIGURNO OVAKAV NEKI ZADATAK KOJI SUMIRA SVE!

1.) $D_f = \mathbb{R}$

2.) asimptote: VA → nema

$$HA \to \lim_{x\to+\infty}(xe^{-x^2}) = (\infty\cdot 0) \overset{NLO}{=} \lim_{x\to+\infty}\frac{x}{e^{x^2}} = \left(\frac{\infty}{\infty}\right) \overset{L'H}{=}$$

$$= \lim_{x\to+\infty}\frac{1}{e^{x^2}\cdot 2x} = \frac{1}{\infty} = 0 \quad \boxed{y=0 \text{ je HA}}$$

ISTO

s obje strane

Ekstremi

$$f'(x) = e^{-x^2} + x \cdot e^{-x^2} \cdot (-2x) = 0$$

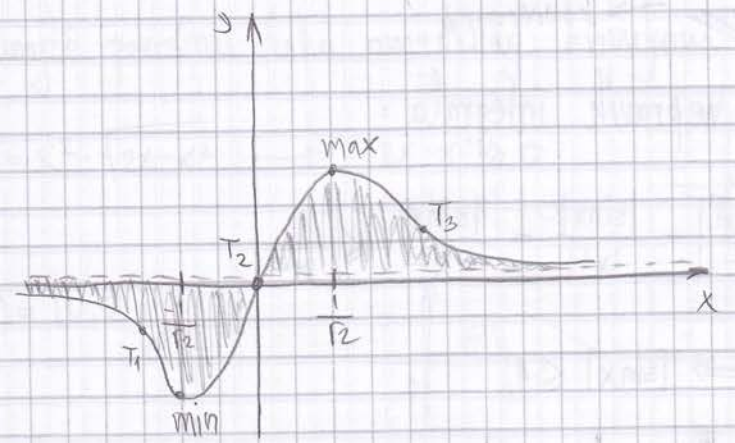
$$e^{-x^2}(1-2x^2) = 0$$

$\hookrightarrow > 0$ uvijek!

PAZI!

$$x^2 = \frac{1}{2} \longrightarrow x = \pm\frac{1}{\sqrt{2}}$$

	$-\infty$	$-\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$	
$f'(x)$	$-$		$+$		$-$
$f(x)$	$\searrow$	min	$\nearrow$	max	$\searrow$



$$P = 2\int_0^\infty x e^{-x^2} dx = \left| \begin{array}{c} x^2 = t \\ 2x\,dx = dt \\ \text{granice iste} \end{array} \right| = \int_0^\infty e^{-t} dt = -e^{-t}\Big|_0^\infty$$

$$\boxed{P = 1}$$

$\Downarrow$

konzultacije $\rightarrow$ četvrtak!

AUDITORNE VJEŽBE

Zad. Izračunajte ili ustvrdite konvergenciju sljedećeg integrala:

$$\int_a^{\infty} \frac{dx}{x \ln^2 x}, \quad a > 1.$$

$$\int_a^{+\infty} \frac{dx}{x \ln^2 x} = \lim_{B \to +\infty} \int_a^B \frac{dx}{x \ln^2 x} =$$

$$\int_a^B \frac{dx}{x \ln^2 x} = \left| \begin{array}{l} t = \ln x \\ dt = \frac{1}{x} dx \\ a \to \ln a \\ B \to \ln B \end{array} \right| = \int_{\ln a}^{\ln B} \frac{dt}{t^2} = -\frac{1}{t} \Big|_{\ln a}^{\ln B}$$

$$= -\frac{1}{\ln B} + \frac{1}{\ln a}$$

$$= \lim_{B \to +\infty} \left(\frac{-1}{\ln B} + \frac{1}{\ln a} \right) = \frac{1}{\ln a} \longrightarrow \text{konvergira!}$$

Zad. Ispitajte konvergenciju nepravih integrala:

a) $$\int_{\frac{\pi}{2}}^{+\infty} \frac{\sin x}{x^2} dx \qquad \left| \frac{\sin x}{x^2} \right| = \frac{|\sin x|}{|x^2|} = \frac{|\sin x|}{x^2} \leq \frac{1}{x^2}$$

$$-1 \leq \sin x \leq 1 \implies |\sin x| \leq 1$$

$$\int_{\frac{\pi}{2}}^{+\infty} \frac{1}{x^2} dx = -\frac{1}{x} \Big|_{\frac{\pi}{2}}^{+\infty} = -\frac{1}{+\infty} + \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

$\longrightarrow$ konvergira!

b) $$\int_1^2 \frac{1}{\ln x} dx = \int_1^2 \frac{dx}{\ln(1+x-1)} = \left| \begin{array}{l} x - 1 = t \\ dx = dt \\ 1 \to 0 \\ 2 \to 1 \end{array} \right| = \int_0^1 \frac{dt}{\ln(1+t)} \sim \int_0^1 \frac{dt}{t} = \ln t \Big|_0^1 =$$

$$= \ln(1+t) \sim t \longrightarrow \text{divergira}$$

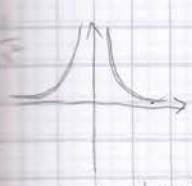
$\ln 0 \to$ NE postoji!

Izračunajte površinu lika omeđenog krivuljom $y=\frac{1}{x^2+2x+3}$ i njezinom asimptotom.

$\longrightarrow$ računamo HA:

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\frac{1}{x^2+2x+3} = \dots = 0$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty}\frac{1}{(x+1)^2+2}\,dx = \left|\begin{array}{l} x+1=t \\ dx=dt \\ -\infty\to-\infty \\ \infty\to\infty \end{array}\right| = \int_{-\infty}^{+\infty}\frac{dt}{t^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\cdot\operatorname{arctg}\frac{t}{\sqrt{2}}\Big|_{-\infty}^{+\infty} =$$



$$= \frac{1}{\sqrt{2}}\left(\operatorname{arctg}(+\infty) - \operatorname{arctg}(-\infty)\right) =$$

$x^2+2x+3=(x+1)^2+2 \longrightarrow$

$$= \frac{1}{\sqrt{2}}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{\pi}{\sqrt{2}}$$

Izračunajte površinu lika omeđenog krivuljom $y=x^2\cdot\sqrt{2x-x^2}$ i osi X.

$f(x)=x^2\cdot\sqrt{2x-x^2} \longrightarrow 2x-x^2\geq 0$

$x(2-x)\geq 0$

$D(f)=[0,2]$

$\longrightarrow$ naša funkcija („nebitno" kako izgleda)

$$P=\int_0^2 f(x)\,dx = \int_0^2\left(x^2\cdot\sqrt{2x-x^2}\right)dx = \dots = \int_0^2 x^2\cdot\sqrt{1-(x-1)^2}\,dx = \left|\begin{array}{l} x-1=\sin t \\ dx=\cos t\,dt \\ 0\to\frac{-\pi}{2} \\ 2\to\frac{\pi}{2} \end{array}\right| =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin t+1)^2\cdot\cos t\cdot\cos t\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\underbrace{(\sin t+1)^2}_{\sin^2 t+2\sin t+1}\cdot\cos^2 t\,dt =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\sin^2 t\cos^2 t + 2\sin t\cos^2 t + \cos^2 t\right)dt =$$

$\longrightarrow$ neparna na simetričnoj domeni

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2 t\cos^2 t\,dt + \underbrace{2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin t\cos^2 t\,dt}_{=0} + \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt}_{=DZ=\frac{\pi}{2}} = \frac{\pi}{8}+\frac{\pi}{2} = \frac{5\pi}{8}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2 t\cos^2 t\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos t\sin t)^2\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1}{2}\sin 2t\right)^2 dt = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2 2t\,dt = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1-\cos 4t}{2}\,dt =$$

$$= \frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\cos 4t)\,dt = \frac{1}{8}\left[t-\frac{1}{4}\sin 4t\right]\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{8}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{\pi}{8}$$

Zad.

$$r = a(1+\cos\varphi)$$

$$P = \frac{1}{2}\int_{\alpha}^{\beta} r^2(\varphi)\,d\varphi = \dots = \frac{3a^2}{2}\pi$$

$$\int_{-\pi}^{\pi} \text{ ili od } \int_{0}^{2\pi}$$