DZ 2 | 3c

$$\begin{tikzcd} K \arrow[dr] & & C \arrow[dl] \\ & M & \end{tikzcd} \qquad K \perp C$$

$$P(K \mid C, M) \leq P(K \mid M)$$

$M \not\perp C \mid K \Rightarrow$ ne vrijedi !

iz zad. $P(K) = 0.2$

$P(C) = 0.01$

$K \perp C$

K i C su jedini mogući uzroci za M

$$P(K=0, C=0, M=1) = 0$$

K	C	M	P(K, C, M)
0	0	0	0.795
0	0	1	0
0	1	0	0.0025
0	1	1	0.0025
1	0	0	0.0975
1	0	1	0.0975
1	1	0	0.0025
1	1	1	0.0025

0.01

+ 0.2

$= 1$

Ništa osim kiše i cijevi ne može uzrokovati mokro tlo

$$P(K=1 \mid C=1, M=1) \stackrel{?}{\leq} P(K=1 \mid M=1)$$ ← ovo dokazujemo

$$\frac{P(K=1, C=1, M=1)}{P(C=1, M=1)} \stackrel{?}{\leq} \frac{P(K=1, M=1)}{P(M=1)}$$

$$0.5 \leq 0.9756$$

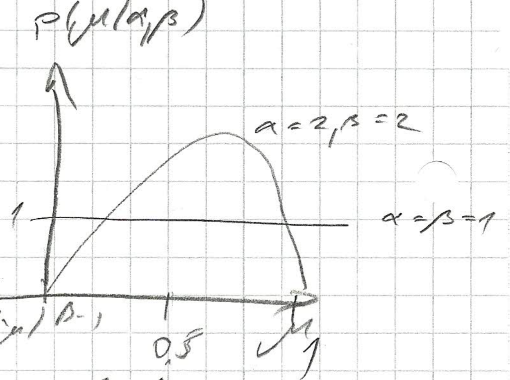
---

uvodimo pretpost. ako je ~~pukla~~ cijev ⇒ sigurno mokro

pada kiša ⇒ sigurno mokro

8.

$\hat{\theta}_{MAP} \arg\max_{\theta} \quad P(D|\theta)\cdot P(\theta)$



$$P(\theta) = B(\mu|\alpha,\beta) \cdot \mu^{\alpha-1}(1-\mu)^{\beta-1}$$

$$\underbrace{P(D|\theta)\cdot P(\theta)}_{\mathcal{L}(\theta|N,m)} = \mu^{m}(1-\mu)^{N-m} \cdot \frac{1}{B(\alpha,\beta)} \cdot \mu^{(\alpha-1)} \cdot (1-\mu)^{\beta-1}$$

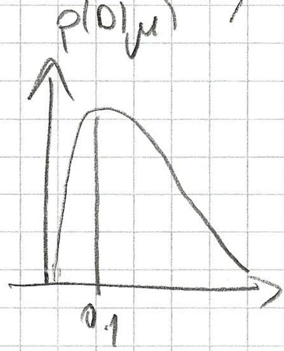
$$= \frac{\mu^{m+\alpha-1} \cdot (1-\mu)^{N-m+\beta-1}}{B(\alpha,\beta)}$$

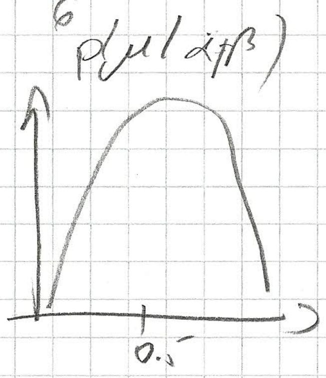
$\Downarrow$

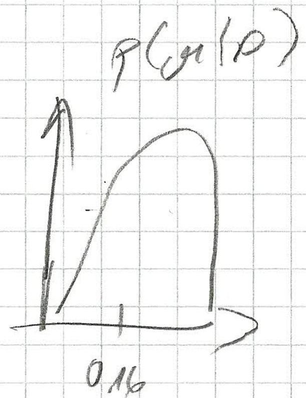
$\boxed{\alpha' = m+\alpha}$ $\qquad$ $\boxed{\beta' = N-m+\beta}$ $\qquad$ BETA DIST. uz α' i β'

c) $\left.\begin{array}{l}\alpha=\beta=2\\ N=10\\ m=1\end{array}\right\}$ $\begin{array}{l}\alpha'=3\\ \beta'=11\end{array}$

$$MOD = \frac{\alpha'-1}{\alpha'+\beta'-2} = \frac{1}{6}$$







$N\uparrow$ $\beta\downarrow$ mod$\downarrow$ $\qquad$ μ_{MAP} se približava μ_{ML}

1. a) $P(1,1) = 0.2$
$P(1,2) = 0.05$
$P(1,3) = 0.3$
$P(2,1) = 0.05$
$P(2,2) = 0.3$
$P(2,3) = 0.1$

	1	2	3	
1	0.2	0.05	0.3	0.55
2	0.05	0.3	0.1	0.45
	0.25	0.35	0.4	

marginalne $\leftarrow$ $P(X) = ?$
$P(Y) = ?$
$P(X|Y) = ?$
$P(Y|X) = ?$

marginalne

$P(x=1) = 0.55$ $\quad$ $P(y=1) = 0.25$

$P(x=2) = 0.45$ $\quad$ $P(y=2) = 0.35$

$P(y=3) = 0.4$

$$P(x=1|y=1) = \frac{P(x=1, y=1)}{P(y=1)} = \frac{0.2}{0.25} = 0.8$$

$$P(x=1|y=2) = \frac{0.05}{0.35} = 0.143$$

$$P(x=1|y=3) = \frac{0.3}{0.4} = 0.75$$

$$P(x=2|y=1) = \frac{0.05}{0.25} = 0.2$$

$$P(x=2|y=2) = \frac{0.3}{0.35} = 0.86$$

$$P(x=2|y=3) = \frac{0.1}{0.4} = 0.25$$

$$P(y=1|x=1) = \frac{P(x=1, y=1)}{P(x=1)} = \frac{0.2}{0.55} = 0.364$$

$$P(y=1|x=2) = \frac{0.05}{0.45} = 0.111$$

$$P(y=2|x=1) = \frac{0.05}{0.55} = 0.091$$

$$P(y=2|x=2) = \frac{0.3}{0.45} = 0.667$$

$$P(y=3|x=1) = \frac{0.3}{0.55} = 0.545$$

$$P(y=3|x=2) = \frac{0.1}{0.45} = 0.222$$

Bayes: $P(y|x) = \dfrac{P(x|y) \cdot P(y)}{\sum_y P(x|y) P(y)} =$

$$P(y=1|x=1) = \frac{0.8 \cdot 0.25}{0.55} = 0.364 \ldots$$

b) $E[X] = 1\cdot 0.55 + 2\cdot 0.45 = 1.45$

$E[Y] = 1\cdot 0.25 + 2\cdot 0.35 + 3\cdot 0.4 = 2.15$

$Var(X) = E[X^2] - E[X]^2$

$= 1\cdot 0.55 + 4\cdot 0.45 - 1.45^2 = 0.2475$

$Var(Y) = 1\cdot 0.25 + 4\cdot 0.35 + 9\cdot 0.4 - 2.15^2 = 0.6275$

$Cov(X,Y) = E[(X - E[X])(Y - E(Y))]$

$\Sigma = \begin{bmatrix} 0.2475 & -0.0175 \\ -0.0175 & 0.6275 \end{bmatrix}$

$= (1-1.45)(1-2.15)\cdot 0.2 + (1-1.45)(2-2.15)\cdot 0.05$

$+ (1-1.45)(3-2.15)\cdot 0.3 + (2-1.45)\cdot(1-2.15)\cdot 0.05$

$+ (2-1.45)(2-2.15)\cdot 0.3 + (2-1.45)(3-2.15)\cdot 0.1 =$

$\Sigma = \begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(Y,X) & Var(Y) \end{bmatrix}$

$= 0.1035 + 0.003375 - 0.11475 - 0.031625 - 0.02475$

$+ 0.04675 = \cancel{-0.04475}\ -0.0175$

$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{-0.04475}{\sqrt{0.2475}\sqrt{0.6275}} \quad -0.044$

c) DOKAZATI!

1) $Var(X) = E[X^2] - E[X]^2$

$Var = E[(X - E[X])^2] = E[X^2 - 2X E[X] + E[X]^2]$

$= E[X^2] - 2E[X E[X]] + E[X]^2 =$

$= E[X^2] - 2E[X]^2 + E[X]^2 = E[X^2] - E[X]^2$

2) $Var(aX) = a^2 Var(X)$

$Var(aX) = E[(aX)^2] - E[aX]^2 = a^2 E[X^2] - a^2 E[X]^2 =$

$= a^2 (E[X^2] - E[X]^2) = a^2 Var(X)$

3) $Cov(X,Y) = E[(X - E[X])(Y - E[Y])] =$

$= E[XY - XE[Y] - YE[X] + E[X]E[Y]]$

$= E[XY] - E[X]E[Y] - E[Y]E[X] + E[X]E[Y] =$

$= E[XY] - E[X]E[Y]$

2. a) NEZAVISNOST SLUČ. VARIJABLI

LIN. ZAVISNOST = KORELIRANOST

$$P(X) \perp P(Y)$$

$$P(XY) = P(X)\,P(Y)$$

$$P(X|Y) = P(X) \qquad P(Y|X) = P(Y)$$

b) $\rho_{X,Y} = -0.044$ $\rightarrow$ nisu linearno zav., ali jesu zavisne
(jer za nez. varijable vrijedi $\rho_{X,Y} = 0$)

c) 1) DOB I VEL. CIPELA => (LIN.) ZAVISNE

2) DOB I SATI SPAVANJA => NEZAV.

3) RAZINA BUKE I UDALJ. OD IZVOR => SLAV. NEG. LIN ZAVISNOST

4) RAZINA UNOSA MASNOĆA I RIZIK OD BOLESTI SRCA => SAV. POZ. LIN. ZAVISNOST

d) DOKAZATI DA SU NEZAVISNE VAR. LINEARNO NEKOLINEARNE

VRIJEDI : $P(XY) = P(X) \cdot P(Y)$
$P(X|Y) = P(X) \qquad P(Y|X) = P(Y)$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$
$$= E[X]E[Y] - E[X]E[Y] = 0$$

$\Longrightarrow \rho_{X,Y} = 0 \Rightarrow$ nekorelirane su

3. a) $P(X|Y) = P(X)$ $P(Y|X) = P(Y)$

$P(X,Y) = P(X|Y) \cdot P(Y)$ $P(X,Y) = P(Y|X) \cdot P(X)$

$P(X) \cdot P(Y) = P(X|Y) \cdot P(Y)$ $P(\cancel{X}) \cdot P(Y) = P(Y|X) \cdot P(\cancel{X})$

$P(X) = P(X|Y)$ $P(Y) = P(Y|X)$

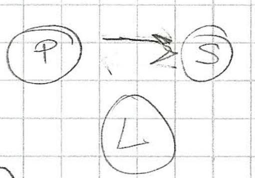
DA LI JE TO TO? NIJE

$P(X|YZ) = P(X|Z)$

$P(XY|Z) = P(X|Z) \cdot P(Y|Z)$

b)

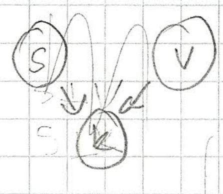
i) P i L => nezavisne $P \perp L$
S i L => nezavisne $S \perp L$



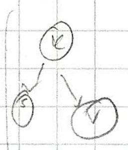
$P \perp L | S$

$P(P|S) = P(P|L,S)$

ii) sve su zavisne



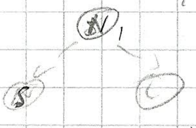
$S \perp V | K$



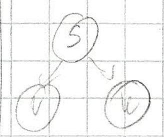
$S \perp V | K$

$P(S|K) = P(S|V,K)$

iii) => zavisne $L \perp N$ $L \perp S | N$

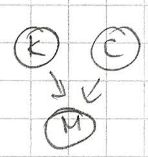


$L \perp S | N$? $P(L|N) = P(L|SN)$
$P(S|N) = P(S|L,N)$



$K \perp V | S$

iv) K, C, M



$K \perp C$ $\cancel{M \perp K | C}$ $\cancel{M \perp C | K}$

$P(K|C) = P(K)$
$P(C|K) = P(C)$

c) $P(K) = 0.2$
$P(C) = 0.01$
$K \perp C$

$P(K|M) = \frac{P(M|K) \cdot P(K)}{P(M)}$

$P(KCM) = ?$

$P(K|CM) \leq P(K|M)$ => pokazati

$P(M|C) \cdot P($

ili

$P(KCM) = P(K|CM) \cdot P(CM) = P(K|CM)\, P(C|M) \cdot P(M)$

$= P(M|KC) \cdot P(KC) = P(K) \cdot P(C) \cdot P(M|KC)$

$P(K|CM) = \frac{P(M|KC) \cdot P(K|C)}{P(K|M)}$

$P(X|YZ) = P(X|Z)$

$P(XY|Z) = P(X|Z) \cdot P(Y|Z)$

$\frac{P(X,Y,Z)}{P(Z)} = \frac{P(Y|Z)\, P(X|YZ)}{P(\cancel{Z})} = P(X|Z)\, P(Y|\cancel{Z})$

4.

a) FUNKCIJA IZGLEDNOSTI $\mathcal{L} : \vec{\theta} \to p(D|\vec{\theta})$

$D \Rightarrow$ iid - nezavisno i identično distribuirani

$x^{(i)} \sim p(x|\theta)$ - pokoravaju se pdf-u

$$\mathcal{L}(\vec{\theta}|D) = p(D|\vec{\theta}) = p(\vec{x}^{(1)} \vec{x}^{(2)} \dots \vec{x}^{(n)} | \vec{\theta})$$

$$\underline{\underline{\text{iid}}} \to = \prod_{i=1}^{n} p(\vec{x}^{(i)} | \vec{\theta}) \Rightarrow \ln \dots$$

b) $D = \{x^{(i)}\}_i = \{-2, -1, 1, 3, 5, 7\}$

pret. Gaussova distribucija $x^{(i)} \sim \mathcal{N}(\mu, \sigma^2)$

$$\mathcal{L}(\mu, \sigma^2 | D) = ?$$

$$\mathcal{L}(\mu, \sigma^2|D) = p(D|\mu, \sigma^2) = \prod_{i=1}^{n} p(x^{(i)} | \mu, \sigma^2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x^{(i)} - \mu)^2}{2\sigma^2}\right)$$

$\mu = 0$, $\sigma^2 = 1$

$$\mathcal{L}(0,1|D) = p(D|0,1) = \prod_{i=1}^{6} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^{(i)2}}{2}\right)$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^6 \cdot e^{\left(-\frac{4}{2} - \frac{1}{2} - \frac{1}{2} - \frac{9}{2} - \frac{25}{2} - \frac{49}{2}\right)} = 1.9 \cdot 10^{-22}$$

$$\cancel{p(D|0,1) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}}}$$

$p(D|0,1) =$ to je isto to $= 1.9 \cdot 10^{-22}$

$\to$ to je vj. uzorka uz te parametre

c) Novčić bačen N puta

$$D = \{ \underbrace{G_1, \dots, G}_{m}, \underbrace{P_1, \dots, P}_{N-m} \}$$

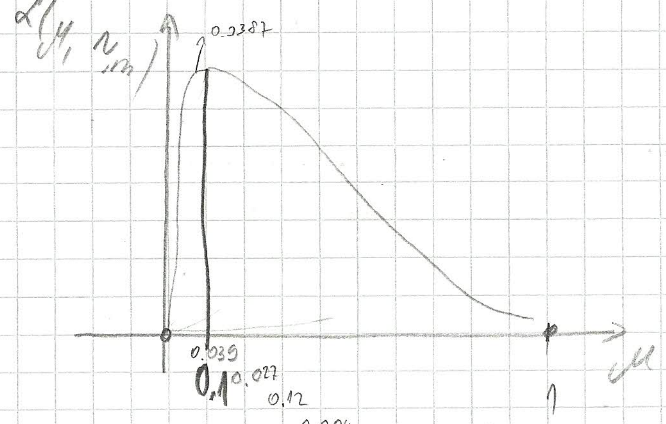
$$\mathcal{L}(\mu | D) = \mathcal{L}(\mu | N, m) = ?$$

$$\mathcal{L}(\mu | N, m) = P(N, m | \mu) =$$

$$= \prod_{i=1}^{N} \mu^{m} (1-\mu)^{N-m}$$

d) $N = 10$, $m = 1$

$$\mathcal{L}(\mu | 10, 1) = \mu (1-\mu)^{9}$$



0.027
0.12
0.004
0.0005
0.0002
0.00001
0.0000004
$9 \cdot 10^{-10}$

najizglednija = ? ? → 0.1

$$\hat{\mu}_{ML} = \frac{1}{10} \cdot (1 + \underbrace{0 + \dots + 0}_{9}) = \frac{1}{10}$$

5\. a) ML - procjenitelj = ?

$$\hat{\theta}_{ML} = \underset{\vec{\theta}}{\operatorname{argmax}}\, \mathcal{L}(\vec{\theta}|D) = \underset{\vec{\theta}}{\operatorname{argmax}}\, P(D|\vec{\theta})$$

b) μ_{ML} za Bernoulli $p(x, \mu)$

$$\ln \mathcal{L}(\mu|D) = \ln \prod_{i=1}^{N} p(x^{(i)}|\mu)$$

$$= \sum_{i=1}^{N} \ln \mu^{x^{(i)}} \cdot (1-\mu)^{1-x^{(i)}} =$$

$$= \sum_{i=1}^{N} x^{(i)} \cdot \ln \mu + \left(N - \sum_{i=1}^{N} x^{(i)}\right)(1-\mu) \quad \Big/ \frac{d\mathcal{L}}{d\mu}$$

$$0 = \frac{1}{\mu}\sum_{i=1}^{N} x^{(i)} - \frac{1}{1-\mu}\left(N - \sum_{i=1}^{N} x^{(i)}\right) \quad \Big/ \cdot \mu(1-\mu)$$

$$0 = (1-\mu)\sum_{i=1}^{N} x^{(i)} - \mu\left(N - \sum_{i=1}^{N} x^{(i)}\right)$$

$$0 = \cancel{-\mu \sum_{i=1}^{N} x^{(i)}} + \sum_{i=1}^{N} x^{(i)} - \mu N + \cancel{\mu \sum_{i=1}^{N} x^{(i)}}$$

$$\mu N = \sum_{i=1}^{N} x^{(i)}$$

$$\boxed{\mu = \frac{1}{N}\sum_{i=1}^{N} x^{(i)}}$$

c) Gaussova razdioba $p(\vec{x}|\mu, \sigma^2)$

$$p(\vec{x}|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(\vec{x}^{(i)} - \mu)^2}{2\sigma^2}\right)$$

$$\mathcal{L}(\mu, \sigma^2|\vec{x}) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x^{(i)} - \mu)^2}{2\sigma^2}\right)$$

$$\ln \mathcal{L}(\mu, \sigma^2|\vec{x}) = \sum_{i=1}^{N} \ln \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x^{(i)} - \mu)^2}{2\sigma^2}\right)$$

$$= \sum_{i=1}^{N}\left(\ln \frac{1}{\sqrt{2\pi}} + \ln \frac{1}{\sigma} + \ln \cancel{\exp}\left(-\frac{(x^{(i)} - \mu)^2}{2\sigma^2}\right)\right)$$

$$= \sum_{i=1}^{N} -\frac{1}{2}\ln 2\pi - \ln\sigma - \frac{(x^{(i)}-\mu)^2}{2\sigma^2}$$

$$= -\frac{N}{2}\ln 2\pi - N\ln\sigma - \sum_{i=1} \frac{(x^{(i)}-\mu)^2}{2\sigma^2}$$

$$\frac{\partial \ln \mathcal{L}}{\mu} \Rightarrow 0 = \frac{\sum_{i=1}^{N} \cancel{2}(x^{(i)}-\mu)}{\cancel{2}\sigma^2} \quad / \cdot \sigma^2$$

$$\sum_{i=1}^{N} x^{(i)} - N\mu = 0$$

$$\boxed{\hat{\mu}_{ML} = \frac{1}{N} \cdot \sum_{i=1}^{N} x^{(i)}}$$

$$\frac{\partial \ln \mathcal{L}}{\partial\sigma} \Rightarrow 0 = -\frac{N}{\sigma} - \left(-\cancel{2} \cdot \frac{\sum_{i=1}^{N}(x^{(i)}-\mu)^2}{\cancel{2}\sigma^3}\right) \quad / \cdot \sigma^3$$

$$0 = -N\sigma^2 + \sum_{i=1}^{N} (x^{(i)}-\mu)^2$$

$$\boxed{\hat{\sigma}^2_{ML} = \frac{\sum_{i=1}^{N} (x^{(i)}-\hat{\mu}_{ML})^2}{N}}$$

(6) a) DOKAZ $\hat{\mu}_{ML}$ je nepristran

NEPRISTRANI PROCJENITELJ: $E[\Theta] = \theta$

$$E[\hat{\mu}_{ML}] = E\left[\frac{1}{N}\sum_{i=1}^{N} x^{(i)}\right] =$$

$$= \frac{1}{N}\sum_{i=1}^{N} E[x^{(i)}] = \frac{1}{N}\cdot N \cdot \mu = \mu \quad \Rightarrow \text{NEPRISTRAN}$$

$$E[\hat{\sigma}^2_{ML}] =$$

$$\hat{\sigma}^2_{ML} = \frac{1}{N}\sum_{i=1}^{N} (x^{(i)}-\hat{\mu}_{ML})^2 = \frac{1}{N}\sum_{i=1}^{N} \left(x^{(i)2} - 2x^{(i)}\hat{\mu}_{ML} + \hat{\mu}_{ML}^2\right)$$

$$= \frac{1}{N}\left(\sum_{i=1}^{N} (x^{(i)})^2 - N\hat{\mu}_{ML}^2\right)$$

$$E[\hat{\sigma}_{ML}^2] = E\left[\frac{1}{N}\sum_{i=1}^{N}(x^{(i)} - \hat{\mu}_{ML})^2\right]$$

$$= \frac{1}{N} E\left[\sum_{i=1}^{N} x^{(i)2} - N\hat{\mu}_{ML}^2\right]$$

$$Var(x) = \sigma^2 = E[x^2] - E[x]^2$$

$$E[x^2] = \sigma^2 + E[x]^2 = \sigma^2 + \mu^2$$

$$Var(\mu) = E[\hat{\mu}^2] - E[\mu]^2 = \frac{\sigma^2}{N} \qquad E[\hat{\mu}^2] = \frac{\sigma^2}{N} + \mu^2$$

$$= \frac{1}{N} N E\left[\vec{x}^2 - N\hat{\mu}_{ML}^2\right]$$

$$= E[x^2] - E[\hat{\mu}_{ML}^2]$$

$$= \sigma^2 + \cancel{\mu^2} - \frac{\sigma^2}{N} - \cancel{\mu^2} = \frac{N-1}{N}\sigma^2$$

$\Rightarrow$ $\hat{\sigma}_{ML}^2$ je PRISTRAN

$$b(\hat{\sigma}_{ML}^2) = E[\hat{\sigma}_{ML}^2] - \sigma^2 = \frac{N-1}{N}\sigma^2 - \sigma^2 = -\frac{\sigma^2}{N}$$

b) Ova razlika dolazi do izražaja kod malenih uzoraka, pa stoga nije veliki problem. :)

$$N\uparrow_{N\to\infty} \Rightarrow b(\hat{\sigma}_{ML}^2)\downarrow$$

7. $D = \{x^{(i)}\}_{i=1}^{6}$

$x^{(1)} = (9.59, -0.75, 2.88)$ $\qquad$ $x^{(4)} = (2.24, 0.02, 0.67)$

$x^{(2)} = (2.30, 0.37, 0.69)$ $\qquad$ $x^{(5)} = (6.59, -0.20, 1.98)$

$x^{(3)} = (8.87, -0.84, 2.66)$ $\qquad$ $x^{(6)} = (3.69, 0.35, 1.11)$

a) $\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x^{(i)} = \frac{1}{6}\sum_{i=1}^{6} x^{(i)} =$

$= \frac{1}{6} \cdot (33.28 \quad -1.05 \quad 9.99) = (5.55, -0.175, 1.665)$

$$\hat{\Sigma} = \frac{1}{6}\sum_{i=1}^{6} (\vec{x}^{(i)} - \vec{\hat{\mu}})(\vec{x}^{(i)} - \vec{\hat{\mu}})^T$$

$$= \frac{1}{6}\Bigg(\begin{bmatrix} 4.04 \\ -0.575 \\ 1.215 \end{bmatrix} [4.04, -0.575 \quad 1.215] + \begin{bmatrix} -3.25 \\ 0.545 \\ -0.975 \end{bmatrix} [-3.25 \quad 0.545 \quad -0.975]$$

$$+ \begin{bmatrix} \overset{3.32}{\cancel{4.83}} \\ -0.665 \\ \underset{0.995}{\cancel{-0.975}} \end{bmatrix} [\overset{3.32}{\cancel{4.83}} \quad -0.665 \quad \overset{0.995}{\cancel{-0.975}}] + \begin{bmatrix} -3.31 \\ 0.195 \\ -0.995 \end{bmatrix} [-3.31 \quad 0.195 \quad -0.995]$$

$$\overset{-0.025}{+} \begin{bmatrix} 1.04 \\ \cancel{0.025} \\ 0.315 \end{bmatrix} [1.04 \quad \overset{-0.025}{\cancel{0.025}} \quad 0.315] + \begin{bmatrix} -1.86 \\ 0.525 \\ -0.555 \end{bmatrix} [-1.86 \quad 0.525 \quad -0.555] \Bigg)$$

$$= \frac{1}{6}\Bigg(\begin{bmatrix} 16.3216 & -2.323 & 4.9086 \\ -2.323 & 0.330625 & \overset{-0.698625}{\cancel{-0.71875}} \\ 4.9086 & \underset{-0.698625}{\cancel{-0.71875}} & \underset{1.47623}{\cancel{1.25796225}} \end{bmatrix} + \begin{bmatrix} 10.5625 & -1.77125 & 3.16875 \\ -1.77125 & 0.297025 & -0.531375 \\ 3.16875 & -0.531375 & 0.950625 \end{bmatrix}$$

$$+ \begin{bmatrix} \overset{11.0224}{\cancel{23.3289}} & \overset{-2.2078}{\cancel{-3.21195}} & \overset{3.3034}{\cancel{4.70925}} \\ \overset{-2.2078}{\cancel{-3.21195}} & 0.442225 & \overset{-0.661675}{\cancel{0.648375}} \\ \underset{3.3034}{\cancel{-4.70925}} & \underset{-0.661675}{\cancel{0.648375}} & \underset{0.990025}{\cancel{0.950625}} \end{bmatrix} + \begin{bmatrix} 10.9561 & -0.64545 & 3.29345 \\ -0.64545 & 0.038025 & -0.194025 \\ 3.29345 & -0.194025 & 0.990025 \end{bmatrix}$$

$$+ \begin{bmatrix} 1.0816 & -0.026 & 0.3276 \\ -0.026 & 6.25 \cdot 10^{-4} & 7.875 \cdot 10^{-3} \\ 0.3276 & 7.875 \cdot 10^{-3} & 0.099225 \end{bmatrix} + \begin{bmatrix} 3.4596 & -0.9765 & 1.0323 \\ -0.9765 & 0.275625 & -0.291375 \\ 1.0323 & -0.291375 & 0.308025 \end{bmatrix} \Bigg)$$

$$= \begin{bmatrix} \overset{8.8973}{\cancel{10.9517}1667} & \overset{-1.325}{-1.\cancel{4839}03333} & \overset{2.67268}{1.33\cancel{6908}333} \\ \overset{-1.325}{-1.4\cancel{839}03333} & 0.2306916667 & \overset{-0.394867}{-0.1\cancel{7987}91667} \\ \underset{2.67235}{1.33\cancel{6908}333} & \underset{-0.394867}{-0.\cancel{17987}91667} & \underset{0.80236}{0.59\cancel{27145}417} \end{bmatrix}$$

b) $p(X \mid \mu, \Sigma) = ?$ MULTIVARIJANTNA GAUSSOVA RAZ.

$$p(\vec{x} \mid \mu, \Sigma) = \frac{1}{\sqrt[n]{2\pi}\, |\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} (\vec{x}-\mu)^T \Sigma^{-1} (\vec{x}-\mu)\right)$$

Da bi razdioba bila dobro definirana, kovarijacijska matrica Σ mora biti pozitivno definitna

$\Downarrow$

$\vec{x}^T A \vec{x} > 0$ za svaki ne-nul vektor

$\uparrow$

kvadratna forma

// poz. determinatna i nesingularna je (ima inverz)

$\Delta^2 = (x-\mu)^T \Sigma^{-1} (x-\mu)$ <- f-ija gust..

$$\begin{vmatrix} 8.8973 & -1.325 & 2.67268 \\ -1.325 & 0.230692 & -0.394867 \\ 2.67268 & -0.394867 & 0.80236 \end{vmatrix} =$$

* 0 na dij.
ili 1 na x_i, x_j

det je 0

$= 8.8973 \cdot (-1)^2 \cdot (0.230692 \cdot 0.80236 - (-0.394867)^2$

$+ 2 \cdot (-1)^3 (-1.325 \cdot 0.80236 - 2.67268 \cdot (-0.394867))$

$+ 2 \cdot (-1)^4 (-1.325 \cdot (-0.394867) - 2.67268 \cdot 0.23069) = -0.0000234..$

c) $\rho_{x_1,x_2} = \frac{Cov(x_1,x_2)}{\sigma_{x_1} \sigma_{x_2}} = \frac{-1.325}{\sqrt{8.8973 \cdot 0.230692}} = -0.9248$

$\hat{\rho}_{x_1,x_3} = \frac{Cov(x_1,x_3)}{\sigma_{x_1} \sigma_{x_2}} = \frac{2.67268}{\sqrt{8.8973 \cdot 0.80236}} = \overset{0.99}{1.000}$ <-

$\hat{\rho}_{x_2,x_3} = \frac{Cov(x_2,x_3)}{\sigma_{x_2} \sigma_{x_3}} = \frac{-0.394867}{\sqrt{0.230692 \cdot 0.80236}} = -0.917$

$\underline{X_1}$ -> van

$$\begin{vmatrix} 0.230692 & -0.394867 \\ -0.394867 & 0.80236 \end{vmatrix} = 0.03 \Rightarrow \Sigma^{-1} = \frac{1}{0.03} \begin{vmatrix} 0.80236 & +0.394867 \\ +0.394867 & 0.230692 \end{vmatrix}$$

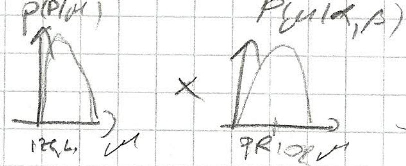
$$= \begin{vmatrix} 26.745 & 13.16 \\ +13.16 & 7.69 \end{vmatrix}$$

x_3 -> van

0.89

$$\Delta^2 = (x-\mu)^T \begin{vmatrix} 26.745 & 13.16 \\ 13.16 & 7.65 \end{vmatrix} (x-\mu)$$

c) $p(\mu|\alpha=2,\beta=2)$

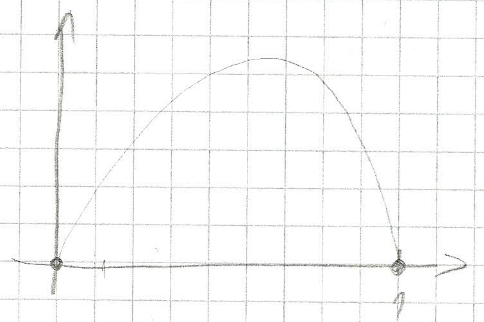


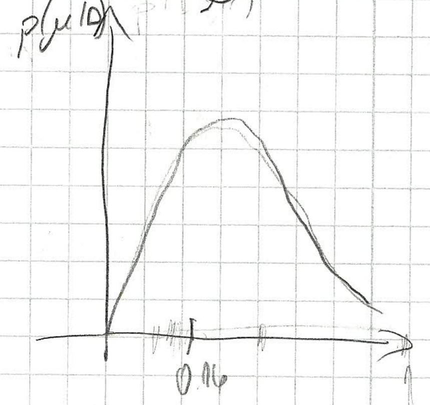
$N=10$ $\quad m=1$

$\alpha' = m+\alpha = 3$

$\beta' = N-m+\beta = 11$

$p(\mu|\alpha=2,\beta=2) = 6\cdot\mu^1(1-\mu)$





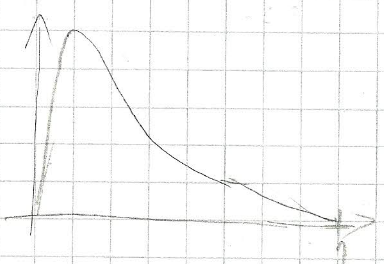
$\mathcal{L}(\mu|N=10,m=1) = \mu^1(1-\mu)^9$

MAX. BETA DIS

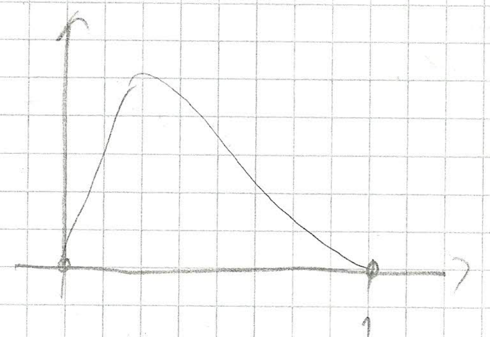
$\Rightarrow$ MOD (najv. vrij)

$$\frac{\alpha-1}{\alpha+\beta-2}$$

$$\frac{3-1}{3+11-2} = \frac{2}{12} = \frac{1}{6}$$



$p(\mu|\alpha=2,\beta=2)\cdot\mathcal{L}(\mu|N=10,m=1) = -6\mu^2(1-\mu)^{10}$



d) $\mu_{ML} = \frac{1}{10}\cdot 1 = 0.1$

$\mu_{MAP} \Rightarrow \ln\left(-6\mu^2(1-\mu)^{10}\right)$

$\ln 6 + 2\ln\mu + 10\ln(1-\mu) = 0$

$\frac{2}{\mu} + \frac{-10}{1-\mu} = 0$

$2(1-\mu) - 10\mu = 0$

$1-\mu-5\mu = 0$

$\mu = 0.167$

razlika bi se smanjila

(4) MAP

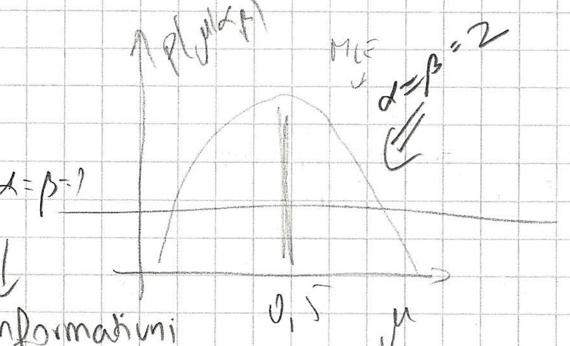
e) $\alpha' = m + \alpha$

$\beta' = N - m + \beta$

ako je $\alpha = 1$, $\beta = 1$ onda se MAP degradira na MLE

(5.) ZA BAYES-a

c) $\alpha = \beta = 1$



neinformativni prior

$\alpha' = m + 1$

$\beta' = N - m + 1$

$$\frac{\alpha'}{\alpha' + \beta'} = \frac{m+1}{m+1+N-m+1} = \frac{m+1}{N+2}$$

kod Bay. za $\alpha = \beta = 1$ dobio se Laplaceov proc.

d)

	$m=0$	$m=1$	$m=9$	$m=10$
Bayes	1/12	2/12	9/12	10/12
MLE	0/10	1/10	9/10	10/10

pogrešno nije 1 za Bayesa i kada sve daju glave