Zadaci za vježbu

Zadatak 1.

$$A=\begin{bmatrix}1&2&1\\2&2&3\\-1&-3&0\end{bmatrix}$$

LU faktorizacija oblje

$\det[A(1:k,1:k)]\neq 0$, za $k=1,2,n-1$

$$\begin{matrix}1&2&1\\2&2&3\end{matrix}\quad 1\cdot(-2)\}+\Rightarrow\quad \begin{matrix}-2&-4&-2\\2&2&3\end{matrix}\}+\quad \begin{bmatrix}1&2&1\\0&-2&1\\-1&-3&0\end{bmatrix}\begin{matrix}\\\\\end{matrix}+=\begin{bmatrix}1&2&1\\0&-2&1\\0&-1&1\end{bmatrix}\begin{matrix}\cdot\frac{1}{(-2)}\\\\\cdot(-1)\end{matrix}+$$

pivot 1, multiplikatori $-2, 1 \Rightarrow$ $\begin{bmatrix}1&0&0\\-2&1&0\\1&0&1\end{bmatrix}=M_1$

$$A=\begin{bmatrix}1&2&1\\0&-2&1\\0&0&\frac{1}{2}\end{bmatrix}\qquad M_2=\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}$$

$$M_1\cdot M_2=\begin{bmatrix}1&0&0\\-2&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-\frac{1}{2}&1\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\1&-\frac{1}{2}&1\end{bmatrix}$$

$$L=(M_1\cdot M_2)^{-1}=\begin{bmatrix}1&0&0\\2&1&0\\-1&\frac{1}{2}&1\end{bmatrix}\qquad U=\begin{bmatrix}1&2&1\\0&-2&1\\0&0&\frac{1}{2}\end{bmatrix}$$

$$\det A=\det U=1\cdot(-2)\cdot\frac{1}{2}+2\cdot 0-0(\ )+0(\ )=-1+0=-1$$

Zadatak 2.

$$A=\begin{bmatrix}2&1&1&0\\4&3&3&1\\8&7&9&5\\6&7&9&8\end{bmatrix}$$

a) $LU=?$

$$\begin{matrix}\\-2\\-4\\-3\end{matrix}\quad\begin{bmatrix}2&1&1&0\\0&1&1&1\\0&3&5&5\\0&4&6&8\end{bmatrix}=A\quad\Rightarrow\quad M_1=\begin{bmatrix}1&0&0&0\\-2&1&0&0\\-4&0&1&0\\-3&0&0&1\end{bmatrix}\qquad M_1^{-1}=\begin{bmatrix}1&0&0&0\\2&1&0&0\\4&0&1&0\\3&0&0&1\end{bmatrix}$$

$$\begin{matrix}\\\\-3\\-4\end{matrix}\quad\begin{bmatrix}2&1&1&0\\0&1&1&1\\0&0&2&2\\0&0&2&4\end{bmatrix}=A\quad\Rightarrow\quad M_2=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-3&1&0\\0&-4&0&1\end{bmatrix}\qquad M_2^{-1}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&3&1&0\\0&4&0&1\end{bmatrix}$$

$$\begin{matrix}\\\\\\-1\end{matrix}\quad\begin{bmatrix}2&1&1&0\\0&1&1&1\\0&0&2&2\\0&0&0&2\end{bmatrix}=U\quad\Rightarrow\quad M_3=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-1&1\end{bmatrix}\qquad M_3^{-1}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{bmatrix}$$

$$L=M_1^{-1}\cdot M_2^{-1}\cdot M_3^{-1}=\begin{bmatrix}1&0&0&0\\2&1&0&0\\4&0&1&0\\3&0&0&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&3&1&0\\0&4&0&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\2&1&0&0\\4&3&1&0\\3&4&1&1\end{bmatrix}$$

b) Matrica permutacija P i LU dekompozicija matrice PU tako da je $PA = LU$

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$$

za pivota treba izabrati u 1. stupcu najveći broj i taj redak staviti na mjesto 1. retka

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad P_1 \Rightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 8 & 7 & 9 & 5 \\ 4 & 3 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 6 & 7 & 9 & 8 \end{bmatrix} \quad \begin{matrix} -\frac{1}{2} \\ -\frac{1}{4} \\ -\frac{6}{8}\left(-\frac{1}{4}\right) \end{matrix} \quad \begin{matrix} 8 & 7 & 9 & 5 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} \\ 0 & -\frac{3}{4} & -\frac{5}{4} & -\frac{5}{4} \\ 0 & \frac{3}{4} & \frac{9}{4} & \frac{13}{4} \end{matrix} \Rightarrow M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{4} & 1 & 0 & 0 \\ \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 8 & 7 & 9 & 5 \\ 0 & \frac{3}{4} & \frac{9}{4} & \frac{13}{4} \\ 0 & -\frac{3}{4} & -\frac{5}{4} & -\frac{5}{4} \\ 0 & -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} \end{bmatrix} \quad \begin{matrix} 8 & 7 & 9 & 5 \\ 0 & \frac{3}{4} & \frac{9}{4} & \frac{13}{4} \end{matrix}$$

$$\begin{matrix} 0 & 0 & -\frac{8}{12} & -\frac{12}{12} \\ 0 & 0 & -\frac{6}{4} & -1 \end{matrix} \quad M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & -\frac{3}{4} & 0 & 1 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 8 & 7 & 9 & 5 \\ 0 & \frac{3}{4} & \frac{9}{4} & \frac{13}{4} \\ 0 & 0 & -\frac{6}{4} & -1 \\ 0 & 0 & -\frac{2}{4} & -\frac{1}{2} \end{bmatrix} \Rightarrow$$

$$\begin{matrix} 0 & 0 & -\frac{6}{4} & -1 \\ 0 & 0 & 0 & -\frac{1}{6} \end{matrix} \quad \left(-\frac{2}{6}\left(-\frac{1}{3}\right)\right) \quad \frac{1}{3} - \frac{1}{12} = \frac{4-1}{12} = \frac{1}{4}$$

$$\Rightarrow M_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & +\frac{1}{3} & 1 \end{bmatrix} \Rightarrow M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & -\frac{3}{4} & 0 & 1 \end{bmatrix}$$

$$M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{4} & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{4} & 0 & 0 & 1 \end{bmatrix}$$

$$L = M_1^{-1} M_2^{-1} M_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{2}{3} & 1 & 0 \\ \frac{1}{4} & -\frac{3}{4} & \frac{1}{3} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 8 & 7 & 9 & 5 \\ 0 & \frac{3}{4} & \frac{9}{4} & \frac{13}{4} \\ 0 & 0 & -\frac{6}{4} & -1 \\ 0 & 0 & 0 & -\frac{1}{6} \end{bmatrix}$$

$$P = P_3 P_2 P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Naloga 3.

$Ax = b$ Gaussova eliminacija brez pivotiranja

$L = U = I$ $A = L \cdot U$

a) $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$x_1 + x_2 - x_3 = 1$
$x_1 + 2x_2 - 2x_3 = 0$
$-2x_1 + x_2 + x_3 = 1$

$x_1 + x_2 - x_3 = 1$ | $\cdot(-1)$, $\cdot 2$
$0 + x_2 - x_3 = -1$
$0 + 3x_2 - x_3 = 3$ | $\cdot(-3)$

$x_1 + x_2 - x_3 = 1$
$x_2 - x_3 = -1$
$0 + 2x_3 = 6$

$M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ $M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$

$U = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -10 \end{bmatrix}$ $L = M_1^{-1} \cdot M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix}$

$x_3 = 3$ $x_2 = -1 + x_3 = -1 + 3 = 2$ $x_1 = 1 - x_2 + x_3 = 1 - 2 + 3 = 2$

$x = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

$Ax = b \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

b) $A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

$4x_1 + 3x_2 + 2x_3 + x_4 = 1$ | $\cdot(-\frac{3}{4})$
$3x_1 + 4x_2 + 3x_3 + 2x_4 = 1$
$2x_1 + 3x_2 + 4x_3 + 3x_4 = -1$ | $\cdot(-\frac{1}{2})$
$x_1 + 2x_2 + 3x_3 + 4x_4 = -1$ | $\cdot(-\frac{1}{4})$

$4x_1 + 3x_2 + 2x_3 + x_4 = 1$ | $\frac{1}{4}$
$0 + \frac{7}{4}x_2 + \frac{6}{4}x_3 + \frac{5}{4}x_4 = \frac{1}{4}$
$0 + \frac{3}{2}x_2 + 3x_3 + \frac{5}{2}x_4 = -\frac{3}{2}$ | $\cdot(-\frac{6}{7})$
$0 + \frac{5}{4}x_2 + \frac{5}{2}x_3 + \frac{15}{4}x_4 = -\frac{5}{4}$ | $\cdot(-\frac{5}{7})$

$\Rightarrow M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{4} & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{4} & 0 & 0 & 1 \end{bmatrix}$

$\frac{7}{4}x_2 + \frac{6}{4}x_3 + \frac{5}{4}x_4 = \frac{1}{4}$
$0 + \frac{12}{7}x_3 + \frac{10}{7}x_4 = -\frac{12}{7}$ | $\cdot(-\frac{10}{12})$
$0 + \frac{10}{7}x_3 + \frac{20}{7}x_4 = -\frac{10}{7}$

$\Rightarrow M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{6}{7} & 1 & 0 \\ 0 & \frac{5}{7} & 0 & 1 \end{bmatrix}$

$\frac{12}{7}x_3 + \frac{10}{7}x_4 = -\frac{12}{7}$
$0 + \frac{5}{3}x_4 = 0$

$M_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{5}{6} & 1 \end{bmatrix}$

$x_4 = 0$

$\frac{12}{7}x_3 = -\frac{12}{7} - \frac{10}{7} \cdot 0$

$x_3 = -\frac{12}{7} / \frac{12}{7} = -1$

$\frac{7}{4}x_2 = \frac{1}{4} + \frac{6}{4} = \frac{7}{4}$ $x_2 = 1$

$4x_1 = 1 - 3 + 2 = 0$ $x_1 = 0$

$x = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

$\frac{7}{4} \cdot \frac{7}{4} = \frac{49}{16}$ $\frac{6}{4} \cdot \frac{7}{4} = \frac{42}{16} = \frac{21}{8}$ $\frac{5}{4} \cdot \frac{7}{4} = \frac{35}{16}$

$\frac{6}{7} \cdot \frac{6}{4} = \frac{36}{28} = \frac{9}{7}$ $\frac{5}{4} \cdot \frac{6}{7} = \frac{30}{28} = \frac{15}{14}$ $\frac{1}{4} \cdot \frac{6}{7} = \frac{6}{28} = \frac{3}{14}$

$\frac{5}{4} \cdot \frac{5}{7} = \frac{25}{28}$

$\frac{12}{7} \cdot \frac{6}{7} = \frac{72}{49}$ $\frac{10}{7} \cdot \frac{6}{7} = \frac{60}{49}$ $\frac{10}{7} \cdot \frac{5}{7} = \frac{50}{49}$

$\frac{12}{7} \cdot \frac{5}{6} = \frac{60}{42} = \frac{10}{7}$ $\frac{10}{7} \cdot \frac{5}{6} = \frac{50}{42} = \frac{25}{21}$

$\frac{20}{7} \cdot \frac{5}{6} = \frac{100}{42} = \frac{50}{21}$

$\frac{7}{4} - \frac{21}{8} = \frac{14 - 21}{8} = -\frac{7}{8}$ $\frac{6}{4} - \frac{9}{7} = \frac{42 - 36}{28} = \frac{6}{28} = \frac{3}{14}$ $\frac{5}{4} - \frac{35}{16} = \frac{20 - 35}{16} = -\frac{15}{16}$

$\frac{6}{7} - \frac{15}{14} = \frac{12 - 15}{14} = -\frac{3}{14}$ $\frac{5}{4} - \frac{25}{28} = \frac{35 - 25}{28} = \frac{10}{28} = \frac{5}{14}$ $\frac{1}{4} - \frac{3}{14} = \frac{7 - 6}{28} = \frac{1}{28}$

$\frac{12}{7} - \frac{10}{7} = \frac{2}{7}$ $\frac{10}{7} - \frac{20}{7} = -\frac{10}{7}$ $\frac{5}{3} - \frac{5}{3} = 0$

$\frac{12}{7} - \frac{50}{21} = \frac{36 - 50}{21} = -\frac{14}{21} = -\frac{2}{3}$ $\frac{10}{7} - \frac{25}{21} = \frac{30 - 25}{21} = \frac{5}{21}$

$\frac{20}{7} - \frac{50}{21} = \frac{60 - 50}{21} = \frac{10}{21}$

$\frac{12}{7} \cdot \frac{5}{6} = \frac{60}{42} = \frac{10}{7}$ $\frac{10}{7} \cdot \frac{5}{6} = \frac{50}{42} = \frac{25}{21}$ $\frac{20}{7} \cdot \frac{5}{6} = \frac{100}{42} = \frac{50}{21}$

$-\frac{12}{7} \cdot (-\frac{5}{6}) = \frac{60}{42} = \frac{10}{7}$ $\frac{10}{7} - \frac{10}{7} = 0$

c)

$$A=\begin{bmatrix}1&-1&1&-1\\-1&3&-3&3\\2&-4&7&-7\\-3&7&-10&14\end{bmatrix}\qquad b=\begin{bmatrix}0\\2\\-2\\8\end{bmatrix}$$

$$\begin{array}{rl}&x_1-x_2+x_3-x_4=0\\(1)&-x_1+3x_2-3x_3+3x_4=2\\-2&2x_1-4x_2+7x_3-7x_4=-2\\3&-3x_1+7x_2-10x_3+14x_4=8\end{array}\qquad \begin{array}{l}x_1-x_2+x_3-x_4=0\\0+2x_2-2x_3+2x_4=2\\0-2x_2+5x_3-5x_4=-2\\0+4x_2-7x_3+11x_4=8\end{array}\qquad M_1^{-1}=\begin{bmatrix}1&0&0&0\\-1&1&0&0\\2&0&1&0\\-3&0&0&1\end{bmatrix}$$

$$\begin{array}{rl}&2x_2-2x_3+2x_4=2\\(1)&0+3x_3-3x_4=0\\(-2)&0+3x_3+7x_4=4\end{array}\qquad \rightarrow M_2^{-1}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\0&2&0&1\end{bmatrix}$$

$$\begin{array}{rl}&3x_3-3x_4=0\\(1)&0+4x_4=4\end{array}\qquad M_3^{-1}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-1&1\end{bmatrix}$$

$x_4=1$

$3x_3=3x_4$

$x_3=1$

$2x_2=2+2x_3-2x_4$

$=2+2-2$

$x_2=1$

$x_1=0+x_2-x_3+x_4$

$=0+1-1+1$

$=1$

$$x=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

Zadatak 4. Gauss eliminacija s pivotiranjem

$Ax=b$ $\quad P,L,U$ $\quad PA=LU$

$$A=\begin{bmatrix}1&2&3&4\\4&2&0&1\\1&-1&0&-1\\2&3&-2&-2\end{bmatrix}\qquad b=\begin{bmatrix}1\\7\\0\\10\end{bmatrix}$$

$Ax=b$ / P slijeva $\quad PAx=Pb \Rightarrow LUx=Pb$

$Ly=Pb$ $\qquad Ux=y$

$$A_1=\begin{bmatrix}4&2&0&1\\2&3&-2&-2\\1&2&3&4\\1&-1&0&-1\end{bmatrix}\qquad P_1=\begin{bmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{bmatrix}$$

$$\begin{array}{rl|cccc}&&1&4&2&0&1\\-2&&0&2&-2&-\frac{3}{2}\\-\frac{1}{4}&&0&\frac{3}{2}&-3&\frac{15}{4}\\-\frac{1}{4}&&0&-\frac{3}{2}&0&-\frac{5}{4}\end{array}\qquad M_1^{-1}\begin{bmatrix}1&0&0&0\\\frac{1}{2}&1&0&0\\\frac{1}{4}&0&1&0\\\frac{1}{4}&0&0&1\end{bmatrix}\qquad \begin{array}{cccc}2&-2&-\frac{3}{2}\\-\frac{3}{4}&0&\frac{9}{2}&\frac{45}{8}\\\frac{3}{4}&0&-\frac{3}{2}&-\frac{25}{8}\end{array}$$

$$-\frac{3}{2}=-\frac{7}{4}$$

$$\frac{3}{2}=\frac{3}{4}$$

$$P_2=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$M_2^{-1}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&\frac{3}{4}&1&0\\0&-\frac{3}{4}&0&1\end{bmatrix}$$

$$\begin{array}{|c|c|} \hline \frac{9}{2} & \frac{45}{8} \\ \hline 0 & -\frac{10}{8} \\ \hline \end{array} \quad +\frac{1}{3}$$

$$M_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} & 1 \end{bmatrix} \qquad P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$L = M_1^{-1} \cdot M_2^{-1} \cdot M_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 1 & 0 \\ \frac{1}{4} & -\frac{3}{4} & -\frac{1}{3} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 4 & 2 & 0 & 1 \\ 0 & 2 & -2 & -\frac{5}{2} \\ 0 & 0 & \frac{9}{2} & \frac{45}{8} \\ 0 & 0 & 0 & -\frac{5}{4} \end{bmatrix}$$

$$P = P_1 \cdot P_2 \cdot P_3 = P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$b = P \cdot b = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 1 \\ 0 \end{bmatrix}$$

$Ly = Pb$ $\qquad Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 1 & 0 \\ \frac{1}{4} & -\frac{3}{4} & -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 1 \\ 0 \end{bmatrix}$$

$y_1 = 7$

$y_2 = 10 - \frac{7}{2} = \frac{13}{2}$

$y_3 = 1 - \frac{1}{4} \cdot 7 - \frac{3}{4} \cdot \frac{13}{2}$

$= 1 - \frac{7}{4} - \frac{39}{8} = \frac{8 - 14 - 39}{8}$

$= \frac{-45}{8}$

$y_4 = 0 - \frac{1}{4} \cdot 7 + \frac{3}{4} \cdot \frac{13}{2} + \frac{1}{3} \left(-\frac{45}{8}\right)$

$= 0 - \frac{7}{4} + \frac{39}{8} - \frac{15}{8}$

$= \frac{0 - 14 + 39 - 15}{8} = \frac{16}{8} = \frac{5}{4}$

$$y = \begin{bmatrix} 7 \\ \frac{13}{2} \\ -\frac{45}{8} \\ \frac{5}{4} \end{bmatrix}^T$$

$Ux = y$

$$\begin{bmatrix} 4 & 2 & 0 & 1 \\ & 2 & -2 & -\frac{5}{2} \\ & & \frac{9}{2} & \frac{45}{8} \\ & & & -\frac{5}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ \frac{13}{2} \\ -\frac{45}{8} \\ \frac{5}{4} \end{bmatrix}$$

$-\frac{5}{4} x_4 = \frac{5}{4}$ $\qquad \boxed{x_4 = -1}$

$\frac{9}{2} x_3 - \frac{45}{8} = -\frac{45}{8}$ $\qquad \frac{9}{2} x_3 = 0$ $\qquad \boxed{x_3 = 0}$

$2x_2 - 2x_3 - \frac{5}{2} x_4 = \frac{13}{2}$ $\qquad 2x_2 = \frac{13}{2} + 0 + \frac{5}{2} \cdot (-1) = 2x_2 = 4$ $\qquad \boxed{x_2 = 2}$

$4x_1 = 7 - 2x_2 - 0x_3 - x_4$

$= 7 - 4 - 0 + 1 = 4$

$\boxed{x_1 = 1}$

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

Zadatak 5 Gauss-porajdelo

$$A=\begin{bmatrix}1&2&3&1\\2&-1&9&-7\\-3&4&-3&19\\4&-2&6&-21\end{bmatrix}\quad b=\begin{bmatrix}8\\30\\-16\\24\end{bmatrix}$$

$$P_1=\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix}$$

$$P=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\quad A_1=\begin{bmatrix}4&-2&6&-21\\2&-1&9&-7\\-3&4&-3&19\\1&2&3&1\end{bmatrix}$$

	4	-2	6	-21
$-\frac{1}{4}$	0	0	6	$\frac{7}{2}$
$+\frac{3}{4}$	0	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{13}{4}$
$-\frac{1}{4}$	0	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{11}{2}$

$$M_1^{-1}=\begin{bmatrix}1&0&0&0\\\frac{3}{4}&1&0&0\\\frac{1}{2}&0&1&0\\\frac{1}{4}&0&0&1\end{bmatrix}$$

$$P_2=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

$-\frac{1}{4}$

$\frac{5}{2}$	$\frac{3}{2}$	$\frac{13}{4}$	
0	6	$\frac{7}{2}$	
$\frac{5}{2}$	$\frac{3}{2}$	$\frac{11}{2}$	

$\frac{5}{2}$	$\frac{3}{2}$	$\frac{13}{4}$	
0	0	6	$\frac{7}{2}$
0	-1	0	2

$$M_2^{-1}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&1&0&1\end{bmatrix}$$

$$P=P_2\cdot P_1=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix}=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}$$ ✓✓

$$U=\begin{bmatrix}4&-2&6&-21\\0&\frac{5}{2}&\frac{3}{2}&\frac{13}{4}\\0&0&6&\frac{7}{2}\\0&0&0&2\end{bmatrix}\quad L=\begin{bmatrix}1&0&0&0\\-\frac{3}{4}&1&0&0\\\frac{1}{2}&0&1&0\\\frac{1}{4}&1&0&1\end{bmatrix}$$ ✓✓

$Ax=b \mid \cdot P$ $\quad PAx=Pb$

$LUx=Pb \quad Ly=\tilde{b} \quad Ux=y$

$$\tilde{b}=Pb \Rightarrow \begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}8\\30\\-16\\24\end{bmatrix}=\begin{bmatrix}24\\-16\\30\\8\end{bmatrix}$$

$$Ly=\tilde{b}\quad \begin{bmatrix}1&0&0&0\\-\frac{3}{4}&1&0&0\\\frac{1}{2}&0&1&0\\\frac{1}{4}&1&0&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=\begin{bmatrix}24\\-16\\30\\8\end{bmatrix}$$

$y_1=24$
$y_2=-16+18=2$
$y_3=30-12=18$
$y_4=8-6-2=0$

$$Ux=y\rightarrow\quad \begin{bmatrix}4&-2&6&-21\\0&\frac{5}{2}&\frac{3}{2}&\frac{13}{4}\\0&0&6&\frac{7}{2}\\0&0&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}24\\2\\18\\0\end{bmatrix}$$

$4x_1=24+2x_2-6x_3+21x_4=24+2-18+0=4$

$\frac{5}{2}x_2=2-\frac{3}{2}x_3-\frac{13}{4}x_4=2-\frac{9}{2}-0=\frac{-5}{2}=\frac{-5}{2}$

$6x_3=18-\frac{7}{2}\cdot 0 \Rightarrow x_3=3$ $\quad x_2=-\frac{5}{2}/\frac{5}{2}=$

$\rightarrow x_4=0$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}1\\-1\\3\\0\end{bmatrix}$$ ✓✓