7. SLUČAJNI VEKTORI

7.1. SLUČAJNI 2-D VEKTORI (3.2.)

$X \sim \begin{pmatrix} x_1, & \dots, & x_n \\ p_1, & \dots, & p_n \end{pmatrix}$ $\qquad Y \sim \begin{pmatrix} y_1, & \dots, & y_m \\ p_1, & \dots, & p_m \end{pmatrix}$

X\Y	y_1	...	y_m	
x_1	p_{11}	...	p_{1m}	$= p_1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_n	p_{n1}	...	p_{nm}	$= p_n$
	q_1	...	q_m	1

→ razdioba diskr. slučajnog vektora (X, Y)

marginalna razdioba od X

marginalna razdioba od Y

$$\sum_{i=1}^{n} p_i = \sum_{j=1}^{m} q_j = 1$$

$$\sum_j p_{ij} = \sum_j (X = x_i, Y = y_j) = P(X = x_i) = p_i$$

UVJET NEZAVISNOSTI ⇒ X i Y nezavisni ⇐ $p_{ij} = p_i \cdot q_j \quad \forall i,j$

2.DZ. 10. Bacamo 2 kocke, X = min, Y = max okrenutih brojeva.

a) $(X, Y) = ?$

b) odredi marg. razd.

c) ispitaj nezavisnost

d) koreliranost i koef. korel.

X/Y	1	2	3	4	5	6	
1	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{11}{36}$
2	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{9}{36}$
3	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{7}{36}$
4	0	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{5}{36}$
5	0	0	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$
6	0	0	0	0	0	$\frac{1}{36}$	$\frac{1}{36}$
	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$	1

c) Nisu nezavisni:

$\frac{1}{36} \neq \frac{11}{36} \cdot \frac{1}{36} \qquad \frac{2}{36} \neq \frac{11}{36} \cdot \frac{3}{36}$ itd.

kovarijacijski moment

d) $\text{cov} = E(X \cdot Y) - E(X) \cdot E(Y)$

$$E(X \cdot Y) = \sum_{i,j} p_{ij} \cdot x_i \cdot y_j$$

$E(X) = \frac{91}{36} \qquad E(Y) = \frac{161}{36}$

$E(X \cdot Y) = \frac{49}{4}$

$\boxed{\text{cov}(X, Y) = 0{,}94522}$

$$r(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X \cdot \sigma_Y}$$

→ koef. korel.

1. MI 2010. 5

Zadana je razdioba slučajnog vektora (X, Y)

X / Y	0	1	
-1	$1/4$	$1/6$	$\frac{10}{24}$
0	$1/6$	$1/8$	$\frac{7}{24}$
1	$1/8$	$1/6$	$\frac{7}{24}$
	$\frac{13}{24}$	$\frac{11}{24}$	1

a) Jesu li X i Y nezavisne?

b) $P(X \geq 0 \mid Y = 1)$

c) odredi razdiobu (Z, W):

$$Z = X + Y \qquad W = (XY)$$

a) $\frac{13}{24} \cdot \frac{10}{24} = \frac{65}{288} \neq \frac{1}{4}$ $\to$ $\boxed{\text{ZAVISNI}}$

b) $$P(X \geq 0 \mid Y = 1) = \frac{P(X \geq 0, Y = 1)}{P(Y = 1)} = \frac{\frac{1}{8} + \frac{1}{6}}{\frac{11}{24}} = \frac{\frac{7}{\cancel{24}}}{\frac{11}{\cancel{24}}} = \frac{7}{11}$$

$$\boxed{P(X \geq 0 \mid Y = 1) = \frac{7}{11}}$$

c) • fiksirati Z:

✓ $Z = -1$ $\quad (X = -1, Y = 0) \Rightarrow W = 0$

✓ $Z = 0$ $\quad (X = 0, Y = 0) \Rightarrow W = 0$ $\quad (X = -1, Y = 1) \Rightarrow W = -1$

✓ $Z = 1$ $\quad (X = 0, Y = 1) \Rightarrow W = 0$ $\quad (X = 1, Y = 0) \Rightarrow W = 0$

$Z = 2$ $\quad (X = 1, Y = 1) \Rightarrow W = 1$

Z / W	-1	0	1	
-1	0	$\frac{1}{4}$	0	$\frac{1}{4}$
0	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{3}$
1	0	$\frac{1}{8} + \frac{1}{8}$	0	$\frac{1}{4}$
2	0	0	$\frac{1}{6}$	$\frac{1}{6}$
	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	1

94 / pr. 3.5. Bacamo 2 kocke: X = broj na prvoj kocki,

Y = veći od 2 broja

(X, Y) = ?

vjerojatnost da je na 1. kocki pao 2, a 4 je veći (24)

$p = \frac{1}{36}$

X \ Y	1	2	3	4	5	6	
1	p	p	p	p	p	p	6p
2	0	2p	p	p	p	p	6p
3	0	0	3p	p	p	p	6p
4	0	0	0	4p	p	p	6p
5	0	0	0	0	5p	p	6p
6	0	0	0	0	0	6p	6p
	p	3p	5p	7p	9p	11p	1

vjerojatnost da je na 1. kocki pao 3 i da je 3 veći od 2 broja (31 32 33)

96 / pr. 3.7.

$X_1, X_2 \sim \begin{pmatrix} 0 & 1 & 2 \\ 0{,}3 & 0{,}5 & 0{,}2 \end{pmatrix}$

X_1 i X_2 nezavisne

a) odredi $Y = X_1 + X_2$

b) odredi $Z = X_1 \cdot X_2$

a) $Y \sim \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ p_1 & p_2 & p_3 & p_4 & p_5 \end{pmatrix}$

$p_1 = 0{,}3 \cdot 0{,}3$ $p_2 = (0\ 1$ ili $1\ 0) = 0{,}3 \cdot 0{,}5 \cdot 2$ $p_3 = \overset{(02\ 20)}{0{,}3 \cdot 0{,}2 \cdot 2} + \overset{(11)}{0{,}5^2}$

$p_4 = 0{,}5 \cdot 0{,}2 \cdot 2$ $p_5 = 0{,}2 \cdot 0{,}2$

$$\boxed{Y \sim \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0{,}09 & 0{,}3 & 0{,}37 & 0{,}2 & 0{,}04 \end{pmatrix}}$$

b) $Z \sim \begin{pmatrix} 0 & 1 & 2 & 4 \\ q_1 & q_2 & q_3 & q_4 \end{pmatrix}$

$q_1 = 0{,}3 \cdot [0{,}3 + (0{,}5 + 0{,}2) \cdot 2]$ $q_2 = 0{,}5 \cdot 0{,}5$ $q_3 = 0{,}5 \cdot 0{,}2 \cdot 2$

$q_4 = 0{,}2 \cdot 0{,}2$

$$\boxed{Z = \begin{pmatrix} 0 & 1 & 2 & 4 \\ 0{,}51 & 0{,}25 & 0{,}2 & 0{,}04 \end{pmatrix}}$$

AUDITORNE 2.5.2012.

ZAD) Prošle god od 540 studenata koji su slušali i SIS i VIS: 450 prošlo VIS, a 360 SIS, a 330 jedno i drugo. $X=1$ ako je student prošao VIS, inače 0, $Y=1$ ako je -"- -"- SIS, inače 0.

Odredi korelaciju (X,Y).

		SIS		
	X\Y	0	1	
VIS	0	$\frac{60}{540}$	$\frac{360-330}{540}$	$\frac{90}{540}$
	1	$\frac{450-330}{540}$	$\frac{330}{540}$	$\frac{450}{540}$
		$\frac{180}{540}$	$\frac{360}{540}$	1

$E(X)=0{,}83$

$E(Y)=0{,}67$

$D(X)=0{,}141$

$D(Y)=0{,}221$

$\operatorname{cov}(X,Y)=E(XY)-E(X)E(Y)=0{,}0539$

$\hookrightarrow \neq 0 \rightarrow$ korelirane (pozitivno)

$$r(X,Y)=\frac{\operatorname{cov}(X,Y)}{\sqrt{D(X)\cdot D(Y)}}=0{,}305$$

Z1 2008 10) Kocka: X = dvostruko od onog na kocki

$Y=1$ za paran, -1 za neparan

a) $(X,Y)=?$ ✓ b) marginalne ✓

a)

X\Y	-1	1	
2	$\frac{1}{6}$	0	$\frac{1}{6}$
4	0	$\frac{1}{6}$	$\frac{1}{6}$
6	$\frac{1}{6}$	0	$\frac{1}{6}$
8	0	$\frac{1}{6}$	$\frac{1}{6}$
10	$\frac{1}{6}$	0	$\frac{1}{6}$
12	0	$\frac{1}{6}$	$\frac{1}{6}$
	$\frac{3}{6}$	$\frac{3}{6}$	1

c) nezavisnost: Nisu nezavisne jer

npr.: $\frac{1}{6}\cdot\frac{1}{2}=\frac{1}{12}\neq\frac{1}{6}$

d) Odredi disperziju od $Z=X+Y$

$$Z\sim\begin{pmatrix}1 & 3 & 5 & 9 & 13\\ \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}\end{pmatrix}$$

$E(Z)=7$

$D(Z)=E(Z^2)-E(Z)^2=\frac{44}{3}$

(MI. 2011. 5)

$X \backslash Y$	-1	1	
-1	$\cancel{\frac{1}{4}}$	$\cancel{\frac{1}{6}}$	$\frac{10}{24}$
0	$\cancel{\frac{1}{6}}$	$\cancel{\frac{1}{8}}$	$\frac{7}{24}$
1	$\cancel{\frac{1}{8}}$	$\cancel{\frac{1}{6}}$	$\frac{7}{24}$
	$\frac{13}{24}$	$\frac{11}{24}$	1

b) nezavisnost: Nisu nezavisni:

$$\frac{1}{4} \neq \frac{10}{24} \cdot \frac{13}{24}$$

c) $$P(Y=1 \mid X \geq 0) = \frac{P(Y=1, X \geq 0)}{P(X \geq 0)}$$

$$= \frac{\frac{1}{8}+\frac{1}{6}}{\frac{14}{24}} = \frac{\frac{7}{\cancel{24}}}{\frac{14}{\cancel{24}}} = \frac{1}{2}$$

d) $Z = |X-Y|$

$W = \frac{X}{Y}$ odredi (Z,W), ispitaj nezavisnost

$Z \backslash W$	-1	0	1	
0	0	0	$\frac{1}{4}+\frac{1}{6}$	$\frac{10}{24}$
1	0	$\frac{1}{6}+\frac{1}{8}$	0	$\frac{7}{24}$
2	$\frac{1}{6}+\frac{1}{8}$	0	0	$\frac{7}{24}$
	$\frac{7}{24}$	$\frac{7}{24}$	$\frac{10}{24}$	1

$Z=0$: $X=-1, Y=-1$ $W=1$ (uz $Z=0$ jedino preživi $W=1$, ostale
$X=1, Y=1$ $W=1$ vrijednosti uz W su 0)

$Z=1$: $X=0$ $Y=1$ $W=0$
$X=0$ $Y=1$ $W=0$

$Z=2$: $X=-1$ $Y=1$ $W=-1$
$X=1$ $Y=-1$ $W=-1$

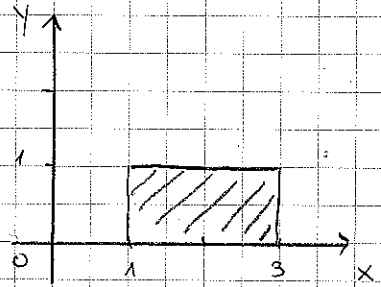
Nisu nezavisne: $\frac{7}{24} \neq \frac{7}{24} \cdot \frac{7}{24}$

DVOSTRUKI INTEGRALI

$\iint\limits_{\Omega} f(x,y)\,dx\,dy$ → Dvostruki integral od $f(x,y)$ po području Ω

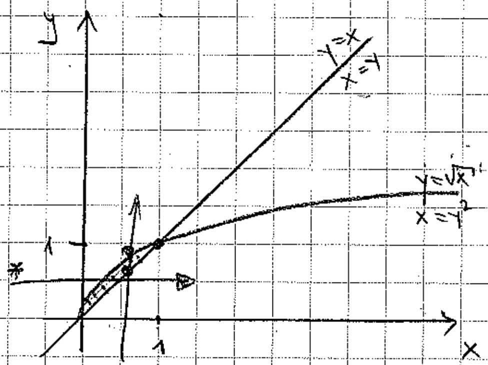
→ predstavlja volumen ispod plohe $f(x,y)$

pr) $\Omega = [1,3] \times [0,1]$



$$\int_1^3 dx \int_0^1 f(x,y)\,dy = \int_0^1 dy \int_1^3 f(x,y)\,dx$$

pr) $\Omega = \{(x,y) : \underset{x=y}{x} \leq y \leq \underset{y=\sqrt{x}}{\sqrt{x}}\}$



$$\int_0^1 dx \int_{x}^{\sqrt{x}} f(x,y)\,dy$$

$\sqrt{x}$ → fja koja ograničava od gore

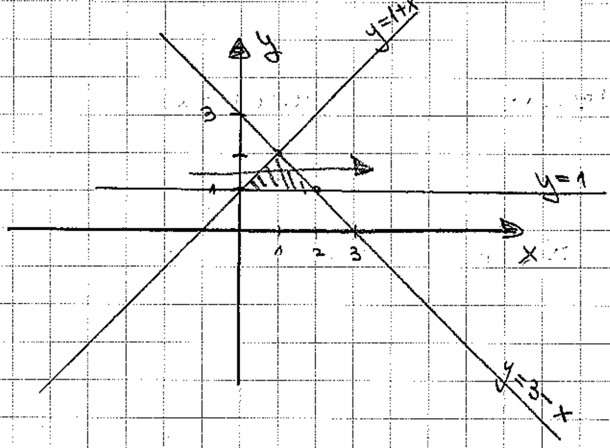
x → fja koja ograničava od dolje

na vanjskim su uvijet konst, a u unutarnjem fje

$$\int_0^1 dx \int_x^{\sqrt{x}} f(x,y)\,dy = \int_0^1 dy \int_{y^2}^{y} f(x,y)\,dy$$

gledamo u smjeru strelice *

pr) $\Omega \ldots \quad y=1+x, \quad y=3-x, \quad y=1$



$$\int_0^1 dx \int_1^{1+x} f(x,y)\,dy + \int_1^2 dx \int_1^{3-x} f(x,y)\,dy =$$

$$= \int_1^2 dy \int_{y-1}^{3-y} f(x,y)\,dx$$

pr) Računanje:

$$\int_0^1 dx \int_x^2 (2x + x^2 y)\,dy = \int_0^1 \left(2xy + x^2 y^2 \cdot \frac{1}{2}\right)\Big|_x^2 dx =$$

$$= \int_0^1 \left(4x + 2\cancel{x^2} - 2\cancel{x^2} - \frac{x^4}{2}\right) dx =$$

$$= \left(2x^2 - \frac{x^5}{10}\right)\Big|_0^1 = 2 - \frac{1}{10} = \frac{19}{10}$$

$$P = \iint_\Omega dx\,dy$$

7.2. Neprekinuti slučajni vektori

def. n-dimenzionalni slučajni vektor je uređena n-torka sl. var.

$$X = (X_1, X_2, \dots, X_n)$$

F-ja razdiobe se definira kao:

$$F(x_1, x_2, \dots, x_n) = P(X_1 < x_1, X_2 < x_2, \dots, X_n < x_n)$$

oznaka $(n=2)$:

$$(X, Y) \Rightarrow \boxed{F(x,y) = P(X < x, Y < y)}$$

def. za sl. vektor (X, Y) kažemo da je kontinuiran ako postoji nenegativna f-ja $f: \mathbb{R}^2 \to [0, \infty\rangle$ takva da $\forall x, y$ vrijedi:

$$\boxed{F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\, dx\, dy}$$

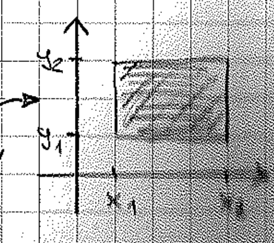
$f(x,y)$ = f-ja gustoće i vrijedi:

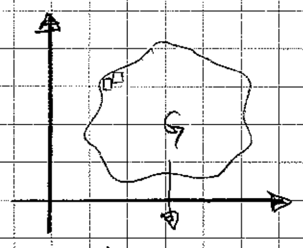
$$\boxed{f(x,y) = \frac{\partial^2 F(x,y)}{\partial x\, \partial y}}$$

izvod - kako računamo vjerojatnost:

$$P(X < x, Y < y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\, dx\, dy$$

$$\rightarrow P(x_1 < X < x_2, y_1 < Y < y_2) = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)\, dx\, dy$$





podijelimo na pravokutnike

zbog aditivnosti:

$$\boxed{P((X,Y) \in G) = \iint_G f(x,y)\, dx\, dy}$$

izvod - Kako dobiti razdiobe od X i Y?

gledamo samo X, pa Y može biti bilo što

$$\int_{-\infty}^{\infty} f(x)dx = F_X(x) = P(X<x) = P(X<x, Y<\infty) = F(x,\infty) =$$

$$= \int_{-\infty}^{x} dx \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-\infty}^{\infty} f(x)\,dx$$

funkcija kojom je y omeđen

$$\boxed{\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy \\ f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx \end{aligned}}$$

→ marginalne gustoće

tm Komponente X i Y sl. vektora (X,Y) su nezavisne ako i samo ako da je

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

dz: ($\Rightarrow$) $F(x,y) = P(X<x, Y<y) = P(X<x)\cdot P(Y<y) = F_X(x)\cdot F_Y(y)$ $/dxdy$

koristimo nezavisnost

$$f(x,y) = f_x(x)\cdot f_y(y)$$

($\Leftarrow$)

koristimo pretpostavku

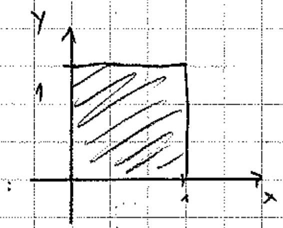
$$P(X\in A, Y\in B) = P((X,Y)\in G) = \iint_G f(x,y)\,dx\,dy =$$

$$= \iint_G f_X(x)\, f_Y(y)\,dx\,dy = \int_A f_X(x)\,dx \cdot \int_B f_Y(y)\,dy =$$

$$= P(X\in A)\cdot P(Y\in B)$$

2. MI 2007. 6) $f(x,y) = \begin{cases} cxy, & x \in [0,1], y \in [0,1] \\ 0, & \text{inače} \end{cases}$

a) $c = ?$



$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1 dx \int_0^1 cxy\,dy = \ldots = \frac{c}{4} = 1$$

$$\boxed{c = 4}$$

$$f(x,y) = \begin{cases} 4xy, & x \in [0,1], y \in [0,1] \\ 0, & \text{inače} \end{cases}$$

b) odredi marginalne gustoće

$$f_X(x) = \int_0^1 4xy\,dy = 2xy^2 \Big|_0^1 = 2x$$

$$\boxed{f_X(x) = 2x} \quad x \in [0,1] \quad !$$

$$f_Y(y) = \int_0^1 4xy\,dx = 2y \quad \boxed{f_Y(y) = 2y} \quad y \in [0,1]$$

c) nezavisnost:

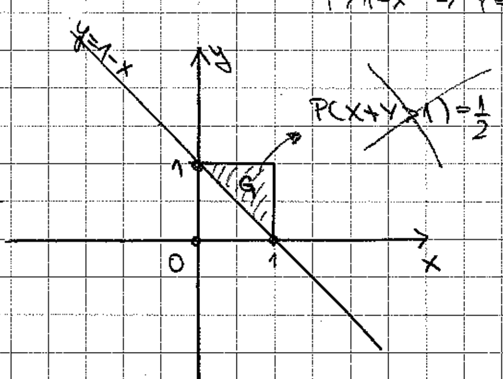
$$f_X(x) \cdot f_Y(y) = 4xy = f(x,y)$$

$$\Downarrow$$

NEZAVISNI

d) $P(X+Y>1)$

$Y > 1-x \Rightarrow Y = 1-x$



!

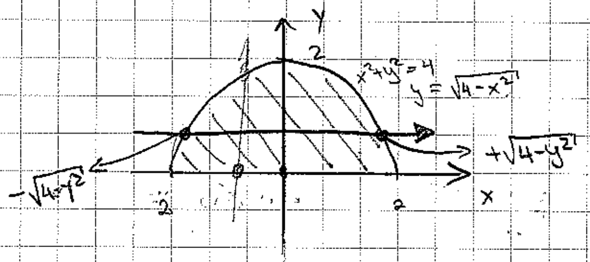
$$P(X+Y>1) = \iint_G f(x,y)\,dx\,dy =$$

$$= \int_0^1 dx \int_{1-x}^1 4xy\,dy = \ldots = \frac{5}{6}$$

$$\boxed{P(X+Y>1) = \frac{5}{6}}$$

2. MI 2008 5.)

$$f(x,y) = C \quad za \quad x^2+y^2 \leq 4, \quad y \geq 0$$



a) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{-2}^{2} dx \int_{0}^{\sqrt{4-x^2}} C\,dy = C \underbrace{\iint dx\,dy}_{\text{površina}} = C \cdot \frac{1}{2} r^2 \pi = C \cdot 2\pi = 1$$

$$\boxed{C = \frac{1}{2\pi}}$$

b) $$f_X(x) = \int_{0}^{\sqrt{4-x^2}} \frac{1}{2\pi}\,dy = \frac{1}{2\pi}\sqrt{4-x^2} \qquad \boxed{f_X(x) = \frac{1}{2\pi}\sqrt{4-x^2}} \quad x \in [-2,2]$$

$$f_Y(y) = \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \frac{1}{2\pi}\,dx = \frac{1}{\pi}\sqrt{4-y^2}$$

$$\boxed{f_Y(y) = \frac{1}{\pi}\sqrt{4-y^2}} \quad y \in [0,2]$$

nezavisnost:

$$f(x,y) \neq f_X(x) \cdot f_Y(y) \rightarrow \text{ZAVISNI}$$

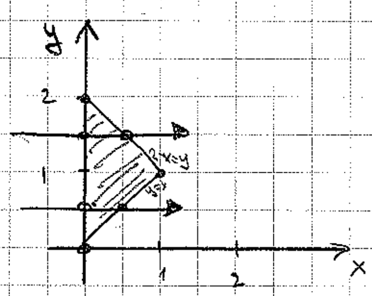
c) $E(Y) = ?$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_{0}^{2} y \frac{1}{\pi}\sqrt{4-y^2}\,dy = \left| 4-y^2 = t \right| = \ldots =$$

$$\boxed{E(Y) = \frac{3\pi}{8}}$$

MI 2010. 5) Vektor ima jednoliku razdiobu na trokutu:

$\triangle$ $(0,0),\ (1,1),\ (0,2)$



$f(x,y) = \frac{1}{P_\Omega}$ → JEDNOLIKA RAZDIOBA (1/mjera)

$$f(x,y) = \frac{1}{\frac{1}{2}\cdot 2\cdot 1} = 1$$

a) marginalne:

$$f_X(x) = \int_x^{2-x} 1\,dy = y\Big|_x^{2-x} = (2-x-x) = \boxed{2-2x} \quad x \in [0,1]$$

$$f_Y(y) = \int_0^{y} 1\,dx = y \quad y = [0,1]$$

$$f_Y(y) = \int_0^{2-y} 1\,dx = 2-y \quad y \in [1,2]$$

ne zbrajati zbog različitog područja

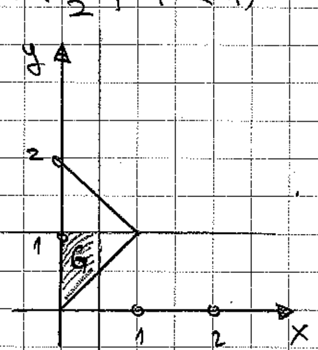
$$\boxed{f_Y(y) = \begin{cases} y, & y \in [0,1] \\ 2-y, & y \in [1,2] \end{cases}}$$

b) očito su zavisni ($f_Y(y)$ uopće ne možemo pomnožiti s $f_X(x)$)

c) $$P\left(X<\tfrac{1}{2} \mid Y<1\right) = \frac{P\left(X<\frac{1}{2}, Y<1\right)}{P(Y<1)} = \frac{\iint_G 1\,dx\,dy}{\iint_B 1\,dx\,dy} =$$

→ površina

→ površina



$$= \frac{\frac{1}{4}+\frac{1}{8}}{\frac{1}{2}\cdot 1\cdot 1} = \frac{3}{4}$$

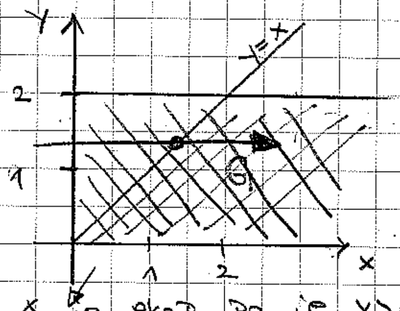
5. DZ. "3" X ima eksp. razdiobu s očekivanjem $\frac{1}{2}$, a Y ima jednoliku na int. $[0,2]$. Ako su X i Y nezavisni, izračunaj vjerojatnost da Y poprimi manju vrijednost od X.

$E = \frac{1}{2} \rightarrow \lambda = 2$

$X \sim E(2)$, $Y \sim U[0,2]$

$f_X(x) = 2e^{-2x}$, $x > 0$ $\qquad f_Y(y) = \frac{1}{2-0}$ $\quad y \in [0,2]$

$$f(x,y) = f_X(x) \cdot f_Y(y) = e^{-2x}$$



X je eksp. pa je $x > 0$

$$P(Y < X) = \iint_G e^{-2x}\, dx\, dy$$

$$P(Y < X) = \int_0^2 dy \int_y^\infty e^{-2x}\, dx = \ldots$$

$$\boxed{P(Y < X) = \frac{1}{4} - \frac{1}{4} e^{-4}}$$

PTI 2007. 3

$$f(x,y)=\frac{c}{x^2+y^2+x^2y^2+1}\quad), x,y\in\mathbb{R}$$

a) $c=?$

$$c\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}\frac{dy}{\underbrace{x^2+y^2+x^2y^2+1}_{\substack{y^2(1+x^2)+1+x^2\\(1+x^2)(y^2+1)}}}=c\int_{-\infty}^{\infty}\frac{dx}{1+x^2}\int_{-\infty}^{\infty}\frac{dy}{y^2+1}=$$

$$=c\,\operatorname{arctg}x\Big|_{-\infty}^{\infty}\operatorname{arctg}y\Big|_{-\infty}^{\infty}=c\,\pi\cdot\pi=c\cdot\pi^2=1$$

$$\boxed{c=\frac{1}{\pi^2}}$$

b)
$$f_x(x)=\int_{-\infty}^{\infty}\frac{dy}{\pi^2(1+x^2)(1+y^2)}=\frac{1}{\cancel{\pi^2}(1+x^2)}\,\operatorname{arctg}\cancel{y}\Big|_{-\infty}^{\infty}$$

$$\boxed{f_x(x)=\frac{1}{\pi(1+x^2)}}\quad x\in\mathbb{R}$$

$$\boxed{f_y(y)=\frac{1}{\pi(1+y^2)}}$$

nezavisnost: $f_x(x)\cdot f_y(y)=f(x,y)$

$\hookrightarrow$ nezavisne

c) $F(x,y)=?$

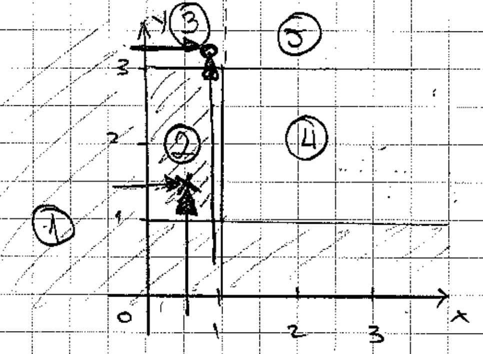
$$F(x,y)=\int_{-\infty}^{x}dx\int_{-\infty}^{y}\frac{dy}{\pi^2(1+x^2)(1+y^2)}=\frac{1}{\pi^2}\operatorname{arctg}x\Big|_{-\infty}^{x}\operatorname{arctg}y\Big|_{-\infty}^{y}$$

$$F(x,y)=\frac{1}{\pi^2}\left(\operatorname{arctg}x+\frac{\pi}{2}\right)\left(\operatorname{arctg}y+\frac{\pi}{2}\right)$$

5. DZ "1" Vektor ima gustoću:

$$f(x,y) = c(5-x-y) \quad ; \quad 0<x<1 \quad ; \quad 1<y<3$$

Odredi f-ju razdiobe!



② ako idemo od $-\infty$ do x sve do granice je $f(x,y)=0$

③ da dođemo iz $-\infty$ do 0 trebamo proći sve $f(x,y)$ (y koord.) - po x samo 0

$$\int_0^1 dx \int_1^3 c(5-x-y)\,dy = \ldots = 5c = 1 \Rightarrow \boxed{c=\frac{1}{5}}$$

$$F(x,y) = \int_{-\infty}^{x} dx \int_{-\infty}^{y} f(x,y)\,dy$$

① $\displaystyle\int_{-\infty}^{x} dx \int_{-\infty}^{y} 0 \cdot dy = 0 \qquad x<0$ ili $y<1$

② $\displaystyle\int_0^x dx \int_1^y \frac{1}{5}(5-x-y)\,dy = \frac{1}{10}x(y-1)(3-x-y)$

$x \in [0,1]$ i $y \in [1,3]$

③ $\displaystyle\int_0^x dx \int_1^3 \frac{1}{5}(5-x-y)\,dy = \frac{1}{5}x(6-x) \qquad x\in[0,1]$; $y>3$

④ $\displaystyle\int_0^1 dx \int_1^y \frac{1}{5}(5-x-y)\,dy = \frac{1}{10}(y-1)(8-y) \qquad y\in[1,3]$ i $x>1$

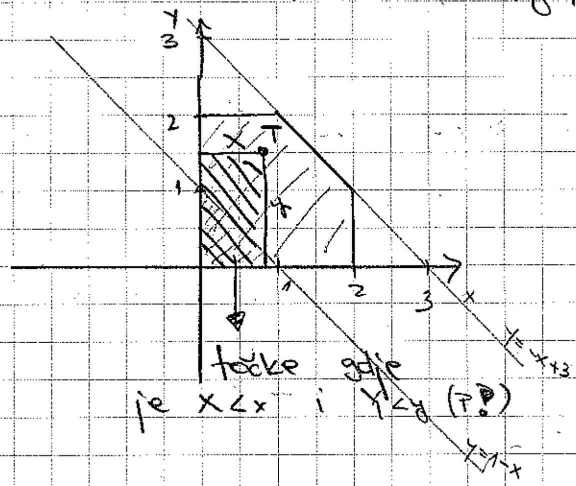
⑤ $\displaystyle\int_0^1 dx \int_1^3 \frac{1}{5}(5-x-y)\,dy = 1 \rightarrow$ (pokupili smo cijelo područje $f(x,y)$)

$x>1$; $y>3$

21.2009. b) Biramo nasreću točku unutar $\Omega = \{x,y \in [0,2],\ x+y<3\}$

Neka je X x-koord., a Y y-koord.

Odredi $F(x,y)$, marginalne ...



$$F(x,y) = P(X<x, Y<y)$$

$$= \frac{x\cdot y}{3+\frac{1}{2}} = \frac{xy}{\frac{7}{2}}$$

$f(x,y) = \frac{\partial^2 F}{\partial x \partial y} = \frac{2}{7}$ → uniformna razdioba (x i y biramo!)
$\frac{1}{P}$

b) $f_X(x),\ f_Y(y) = ?$

$$f_X(x) = \int_0^2 \frac{2}{7}\,dy = \frac{2}{7}y\Big|_0^2 = \frac{4}{7} \qquad x \in [0,1]$$

$$f_X(x) = \int_0^{-x+3} \frac{2}{7}\,dy = \frac{2}{7}y\Big|_0^{-x+3} = \frac{2}{7}(3-x) \qquad x \in [1,2]$$

$$\boxed{f_X(x) = \begin{cases} \frac{4}{7}, & x[0,1] \\ \frac{2}{7}(3-x) & x\in[1,2] \end{cases}}$$

$$f_Y(y) = \int_0^2 \frac{2}{7}\,dx = \frac{2}{7}x\Big|_0^2 = \frac{4}{7} \qquad y \in [0,1]$$

$$f_Y(y) = \int_2^{3-y} \frac{2}{7}\,dx = \frac{2}{7}x\Big|_0^{3-y} = \frac{2}{7}(3-y) \qquad y[1,2]$$

$$\boxed{f_Y(y) = \begin{cases} \frac{4}{7}, & y\in[0,1] \\ \frac{2}{7}(3-y), & y\in[1,2] \end{cases}}$$

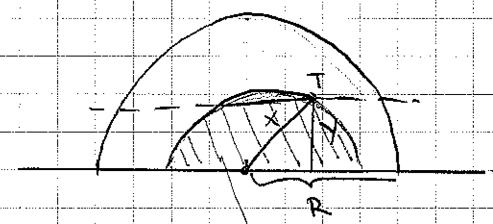
c) $$\underbrace{P(X+Y<1)}_{y<1-x} = \iint_G f(x,y)\,dx\,dy = \underbrace{\int_0^1 dx \int_0^{1-x}}_{\substack{\text{POVRŠINA} \\ \iint dx\,dy}} \frac{2}{7}\,dy = \frac{\cancel{2}}{7}\cdot\frac{1}{\cancel{2}} = \frac{1}{7}$$

$$\boxed{P(X+Y<1) = \frac{1}{7}}$$

5. DZ ② Biramo točku T unutar polukruga polumjera r.

x = udaljenost do središta, y = udaljenost do središta.

$f(x,y) = ?$



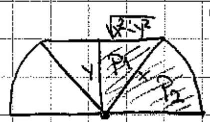
$x \in [0, R]$

$y \in [0, R]$

$x \geq y$

!

$$F(x,y) = P(X < x, Y < y) = \frac{2 \cdot \left[\overbrace{\frac{1}{2} y \sqrt{x^2 - y^2}}^{P_1} + \overbrace{x^2 \cdot \frac{1}{2} \arcsin \frac{y}{x}}^{P_2} \right]}{\frac{1}{2} R^2 \pi}$$



$P_2 = R^2 \pi \cdot \frac{\alpha}{2\pi}$

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y} = \frac{4x}{R^2 \pi \sqrt{x^2 - y^2}} \qquad 0 \leq y \leq x \leq R$$

~~[illegible]~~

• očekivanje sl. vektora: $\boxed{E(X,Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dx\,dy}$

tm (svojstva očekivanja):

a) $E(sX+tY) = sE(X) + tE(Y)$

b) Ako su X i Y NEZAVISNI: $E(XY)=E(X)E(Y)$

dz:

a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (sX+tY) f(x,y)\,dx\,dy = s\iint x f(x,y)\,dx\,dy + t\iint y f(x,y)\,dx\,dy$

$=$

7.3. UVJETNE RAZDIOBE

- motivacija: npr. Biramo $Y \in [0,2]$ i zatim biramo $X \in [y,2]$. Razdioba od X = ?

def. Neka je $f(x,y)$ gustoća vektora (X,Y) te neka je poznata razdioba od Y. Tada se uvjetna gustoća od X uz uvjet $Y=y$ definira s

$$\boxed{f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}}$$

- (češće koristimo vezu: $f(x,y) = f_{X|Y=y}(x) \cdot f_Y(y)$)

Uočimo: marg. gustoća od X je:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y=y}(x) \cdot f_Y(y)\, dy$$

def. Uvjetno očekivanje var. X koja ovisi o realizacijama od Y je:

$$\boxed{E(X|Y=y) = \int_{-\infty}^{\infty} x \cdot f_{X|Y=y}(x)\, dx}$$

te je tada:

$$\boxed{E(X) = \int_{-\infty}^{\infty} E(X|Y=y) f_Y(y)\, dy}$$

def. Vjerojatnost događaja A koji ovisi o realizacijama od X:

$$\boxed{P(A) = \int_{-\infty}^{\infty} P(A|X=x) f_X(x)\, dx}$$

2MI 2007. 7) Biramo $y \in [0,2]$ i zatim $x \in [y,2]$.

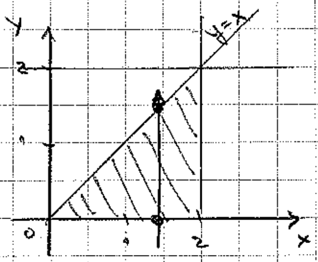
Odredi gustoću i očekivanje od X.

$f_Y(y) = \frac{1}{2}$ (uniformna) $y \in [0,2]$

$f_{X|Y=y} = \frac{1}{2-y}$

$f(x,y) = f_Y(y) \cdot f_{X|Y=y}$

$f(x,y) = \frac{1}{2(2-y)}$ $0 \leq y \leq x \leq 2$



$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^x \frac{1}{2(2-y)}\,dy =$

$f_X(x) = \frac{1}{2}[\ln 2 - \ln(2-x)]$ $x \in [0,2]$

$E(X) = ?$

i) $E(X) = \int x \cdot f_X(x)\,dx = \frac{1}{2}\int_0^2 x(\ln 2 - \ln(2-x))\,dx$

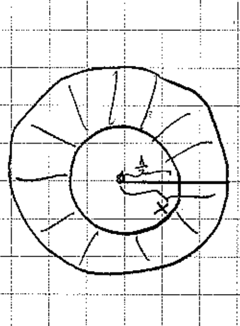
ii) $E(X|Y=y) = \int_{-\infty}^{\infty} x \cdot f_{X|Y=y}(x)\,dx = \int_y^2 x \cdot \frac{1}{2-y}\,dx =$

$= \frac{1}{2}x^2 \frac{1}{2-y}\Big|_y^2 = \frac{1}{2}(y+2)$

$E(X) = \int_{-\infty}^{\infty} E(X|Y=y) \cdot f_Y(y)\,dy = \int_0^2 \frac{1}{2}(y+2) \cdot \frac{1}{2}\,dy =$

$= \frac{1}{4}\left(\frac{y^2}{2} + 2y\right)\Big|_0^2 = \frac{3}{2}$

2. MI 2011 6) Radijus kruga je jednoliko distr. var. na [1,2]. Biramo točku T unutar tog kruga. Izračunaj vjerojatnost da je udaljenost $|TS| > \frac{1}{2}$.

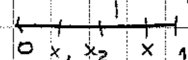


$$f_X(x) = \frac{1}{2-1} = 1$$

$$P(A \mid X=x) = \frac{x^2\pi - \frac{1}{4}\pi}{x^2\pi} = 1 - \frac{1}{4x^2}$$

$$P(A) = \int_{-\infty}^{\infty} P(A \mid X=x)\, f_X(x)\, dx = \int_1^2 \left(1 - \frac{1}{4x^2}\right) dx = \frac{7}{8}$$

2. MI 2008 6) Biramo x_1 i x_2 iz [0, 1]. Neka je sl. var. $X = \max\{x_1, x_2\}$. Zatim biramo Y iz $[0, X]$. Gustoća, očekivanje od Y=?



ako je $x_2 < x$ onda je $x_1 < x$

$f_{x_1}(x) = 1 \qquad f_{x_2}(x) = 1$

$$F_X(x) = P(X < x) = P(\max\{x_1, x_2\} < x) =$$

$$= P(x_1 < x,\ x_2 < x) \underset{\text{nezavisne su}}{=} P(x_1 < x) \cdot P(x_2 < x) =$$

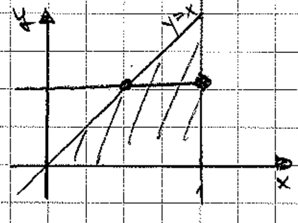
$$= F_{x_1}(x) \cdot F_{x_2}(x) = \int_{-\infty}^{\infty} f_{x_1}(x)\,dx \cdot \int_{-\infty}^{\infty} f_{x_2}(x)\,dx =$$

$$= \int_0^x dx \cdot \int_0^x dx = x^2 \qquad \boxed{F_X(x) = x^2}$$

$$f_X(x) = F_X'(x) = 2x \qquad \boxed{f_X(x) = 2x}$$

$$\boxed{f_{Y|X=x}(y) = \frac{1}{x}} \ \text{uniformna} \qquad y \in [0, x] \Rightarrow f(x,y) = f_X(x) \cdot f_{Y|X=x}(y)$$

$$= \cancel{2x} \cdot \frac{1}{\cancel{x}} = 2$$



$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_y^1 2\,dx = 2 - 2y$$

$$\boxed{f_Y(y) = 2 - 2y} \quad y \in [0,1]$$

$$E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy \qquad \boxed{E(Y) = \frac{1}{3}}$$

8. FUNKCIJE SLUČAJNIH VEKTORA

- pitanje: Ako znamo razdiobu od X i Y, kolika je razdioba od $Z = \varphi(X, Y)$?

Zad) X i Y su nezavisne s eksp. razdiobom s parametrom 2. Nađi razdiobu od $Z = X + Y$.

$$f_X(x) = \lambda e^{-\lambda x} = 2e^{-2x}, \quad x > 0$$

$$f_Y(y) = 2e^{-2y}, \quad y > 0$$

$$f(x,y) \underset{\text{nezavisne X i Y}}{=} f_X(x) \cdot f_Y(y) = \boxed{4e^{-2x-2y} = f(x,y)}$$

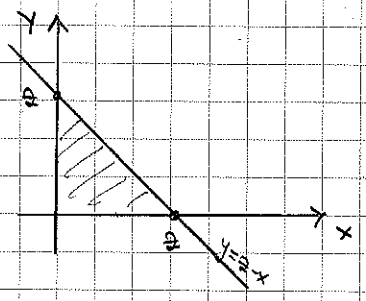
$$z \in \langle 0, \infty \rangle$$

$G(z) = ?$

gledamo kao konst.

$$G(z) = P(Z < z) = P(\underbrace{X + Y < z}_{G}) = \iint_G f(x,y)\,dx\,dy$$

$y < z - x$



$$G(z) = \int_0^z dx \int_0^{z-x} 4e^{-2x-2y}\,dy = \dots$$

$$\boxed{G(z) = 1 - (1 + 2z)e^{-2z}} \quad z > 0$$

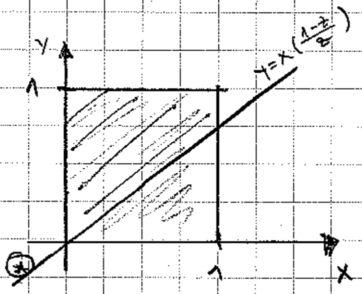
$$g(z) = G'(z)$$

$$\boxed{g(z) = 4ze^{-2z}} \quad z > 0$$

(službeno rj. krivo)

5. DZ (5) Sl. vektor (X,Y) ima jedn. razd. na $[0,1] \times [0,1]$.

Odredi gustoću od var. $Z = \frac{X}{X+Y}$.



$f(x,y) = \frac{1}{P}$ $f(x,y) = 1$

$z \in [0,1]$

$x < zx + zy$

$zy > x - zx$

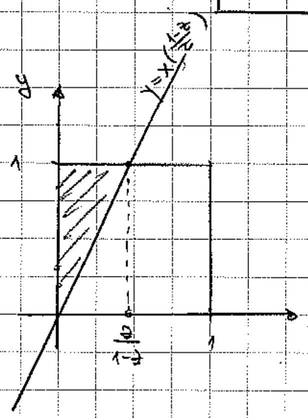
$y > \frac{x - zx}{z} = x\left(\frac{1}{z} - 1\right)$ konst. — uvjet > 0

$G(z) = P(Z < z) = P\left(\frac{X}{X+Y} < z\right) =$

$= P\left(y > x\left(\frac{1-z}{z}\right)\right) = \int_0^1 dx \int_{x\left(\frac{1-z}{z}\right)}^1 1 \, dy =$ (može kao površina)

$\boxed{G(z) = 1 + \frac{1}{2} \frac{z-1}{z}}$ $z = (\frac{1}{2}, 1)$

(ako je $z = \frac{1}{2}$ pravac * je $y = x$, za $z > \frac{1}{2}$ je površina drukčija)



$G(z) = P\left(Y > X\frac{1-z}{z}\right) = \int_0^{\frac{z}{1-z}} dx \int_{x\frac{1-z}{z}}^1 dy = \dots$

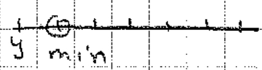
$\boxed{G(z) = \frac{1}{2} \frac{z}{1-z}}$ $z \in (0, \frac{1}{2})$

$g(z) = G'(z)$

$$g(z) = \begin{cases} \frac{1}{2} \cdot \frac{1}{z^2}, & z \in (0, \frac{1}{2}) \\ \frac{1}{2} \cdot \frac{1}{(1-z)^2}, & z \in (0, \frac{1}{2}) \end{cases}$$

8.7.) Neka su $X_1, \dots, X_n$ nezavisne s eksp. razdiobom s parametrima $\lambda_1, \lambda_2, \dots, \lambda_n$

ako $\min \geq y \Rightarrow$ svi su $\geq y$



$$Y = \min\{X_1, \dots, X_n\}$$

$$G(y) = P(Y < y) = P(\min\{X_1, \dots, X_n\} < y) =$$

$$= 1 - P(\min\{X_1, \dots, X_n\} \geq y) = 1 - P(X_1 \geq y, X_2 \geq y, \dots, X_n \geq y) =$$

$$= 1 - P(X_1 \geq y) \cdot \ldots \cdot P(X_n \geq y) = 1 - e^{-(\lambda_1 + \lambda_2 + \dots + \lambda_n) y}$$

$\sim E(\lambda_1)$
$\geq y) = 1 - F_1(y)$
$= (1 - e^{-\lambda_1 y})$
$e^{-\lambda_1 y}$

$$Y \sim E(\lambda_1 + \dots + \lambda_n)$$

$$Y = \Psi(X) \rightarrow g(y) = f(x) \left| \frac{dx}{dy} \right|$$

$x = \Psi^{-1}(x)$

vrijedi za 1 var, zanima nas za n

izvod $(X_1, \dots, X_n) \Rightarrow f(x_1, \dots, x_n)$ $\quad Y_1 = \Psi_1(X_1)$, $\vdots$, $Y_n = \Psi_n(X_n)$ $\quad g(y_1, \dots, y_n) = ?$

$$P((X_1, \dots, X_n) \in G) = P((y_1, \dots, y_n) \in G)$$

Ψ_1^{-1}

$$\int_G \cdots \int f(x_1, \dots, x_n)\, dx_1 \dots dx_n = \left| \begin{array}{l} x_1 = \chi_1(y_1) \\ \vdots \\ x_n = \chi_n(y_n) \end{array} \quad J = \frac{\partial(x_1, \dots, x_n)}{\partial(y_1, \dots, y_n)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial x_n}{\partial y_n} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix} \right| =$$

$$= \int_{G'} \cdots \int \underbrace{f(x_1, \dots, x_n)\, |J|}_{g(y_1, \dots, y_n)}\, dy_1 \dots dy_n \Rightarrow$$

$$\Rightarrow g(y_1, \dots, y_n) = f(x_1, \dots, x_n) \left| \frac{\partial(x_1, \dots, x_n)}{\partial(y_1, \dots, y_n)} \right|$$

$x_1 = x_1(y_1)$
$x_n = x_n(y_n)$

//

- mi gledamo samo $n=2$:

$$z = \varphi(x,y) \Rightarrow \left.\begin{matrix} x = x \\ y = \chi(x,z) \end{matrix}\right\} \; J = \frac{\partial(x,y)}{\partial(x,z)} =$$

$$= \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial z} \end{vmatrix} = \frac{\partial y}{\partial z}$$

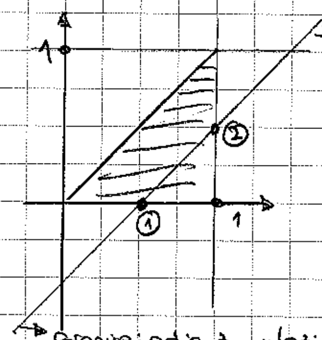
$$g(x,z) = f(x,y) \left| \frac{\partial y}{\partial z} \right|$$

$$\boxed{g(z) = \int_{-\infty}^{\infty} f(x,y) \left| \frac{\partial y}{\partial z} \right| dx}$$

uvrstiti Y !

Buričev princip:

2-MI 2011. 7) slučajni vektor (X,Y) ima gustoću $f(x,y) = cx$, $0 \le y \le x \le 1$. Odredi $g(z)$ za $z = X - Y$.



$$\int_0^1 dx \int_0^x cx\, dy = \frac{1}{3} c = 1 \rightarrow \underline{c = 3}$$

$$z = X - Y \rightarrow \underline{z \in (0,1)}$$

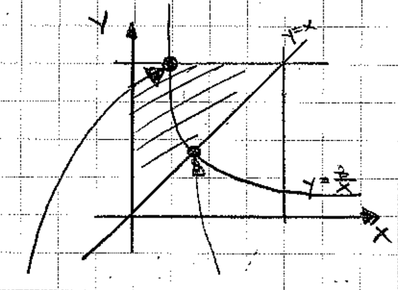
$\hookrightarrow y = x - z \qquad \left|\frac{\partial y}{\partial z}\right| = |-1| = \underline{1}$

→ granice: gdje z ulazi i izlazi iz područja

$$g(z) = \int_{z\,①}^{1\,②} 3x \cdot 1\, dx = \frac{3x^2}{2}\Big|_z^1 \qquad \boxed{g(z) = \frac{3}{2}(1 - z^2)} \quad z \in (0,1)$$

5. DZ 19) $f(x,y) = c(X+Y) \quad 0 \le x < y \le 1$

$z = X \cdot Y \quad , \quad g(z) = ?$



$$\int_0^1 dx \int_x^1 c(x+y)\,dy = \frac{1}{2} c = 1 \qquad c = 2$$

$y = 1 = \frac{z}{x} \qquad y = x = \frac{z}{x}$

$x = z \qquad x = \sqrt{z}$

$$z = X \cdot Y \rightarrow z \in (0,1)$$

$$Y = \frac{z}{x} \qquad \left|\frac{\partial Y}{\partial z}\right| = \frac{1}{x}$$

$$g(z) = \int_z^{\sqrt{z}} 2\left(x + \frac{z}{x}\right) \cdot \frac{1}{x}\, dx$$

$$\boxed{g(z) = 2 - 2z} \qquad z \in (0,1)$$

2. VII 2008 7) X i Y su nezavisne sl.var. $Y \sim E(1)$, $X \sim U[1,3]$.

Odredi razdiobu od $Z = \frac{Y}{X}$ i izračunaj $P\left(\frac{Y}{X} < 1\right)$?

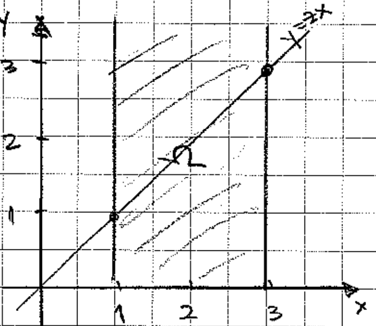
$f_Y(y) = e^{-y}, \; y > 0 \qquad f_X(x) = \frac{1}{2} \quad x \in [1,3]$

$$f(x,y) = \frac{1}{2} e^{-y}$$

1. način) formula:

$y = \chi(x,z)$

$$g(z) = \int_{-\infty}^{\infty} f(x,y) \left|\frac{\partial y}{\partial z}\right| dx$$



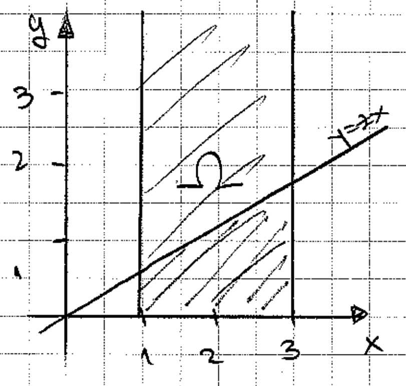
$$z = \frac{y}{x} \rightarrow y = z \cdot x \qquad \left|\frac{\partial Y}{\partial z}\right| = x$$

$$g(z) = \int_1^3 \frac{1}{2} e^{-zx} \cdot x\, dx$$

$$\boxed{g(z) = \frac{1}{2z^2}\left(e^{-z} - e^{-3z}\right) + \frac{1}{2z}\left(e^{-z} - 3e^{-3z}\right)}$$

$$z \in \langle 0, +\infty \rangle$$

2. način) po definiciji



$$G(z) = P(Z < z) = P\left(\frac{Y}{X} < z\right) = P(Y < z \cdot x)$$

$$= \iint_\Omega f(x,y)\,dx\,dy = \int_1^3 dx \int_0^{zx} \frac{1}{2} e^{-y}\,dy$$

$$\boxed{G(z) = 1 + \frac{1}{2z}\left(e^{-3z} - e^{-z}\right)} \quad z \in \langle 0, +\infty\rangle$$

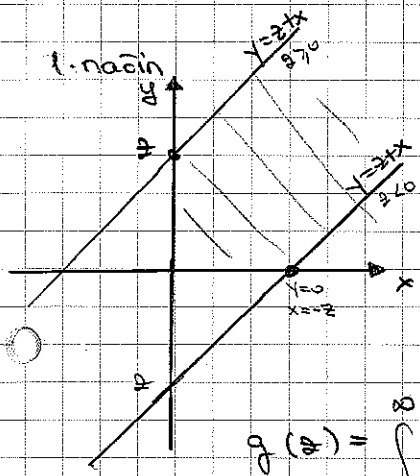
$$g(z) = G'(z) = \dots \qquad P(Z<1) = G(1) \qquad \boxed{P(Z<1) = 1 + \frac{1}{2}\left(e^{-3} - e^{-1}\right)}$$

2. MI 2010. 6) X i Y su nezavisne $\sim E(\frac{1}{2})$. Odredi razdiobu

$Z = Y - X$. $g(z) = ?$

$$f_X(x) = \frac{1}{2} e^{-\frac{1}{2}x} \quad x>0$$
$$f_Y(y) = \frac{1}{2} e^{-\frac{1}{2}y} \quad y>0$$
$$\Big\} \; f(x,y) = \frac{1}{4} e^{-\frac{1}{2}(x+y)}$$

1. način



$Z = Y - X \rightarrow z \in \langle -\infty, \infty\rangle$

$\hookrightarrow Y = z + x \qquad \left|\frac{\partial Y}{\partial z}\right| = 1$

$$g(z) = \int_{-\infty}^{\infty} f(x,y) \left|\frac{\partial y}{\partial z}\right| dx$$

$$g(z) = \int_0^{\infty} \frac{1}{4} e^{-\frac{1}{2}(x+z+x)} \cdot 1 \; dx \qquad \text{I) } z \geq 0$$

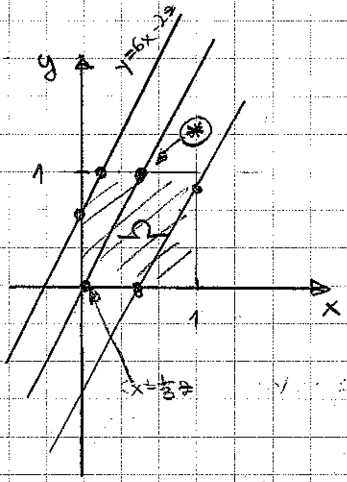
$$\boxed{g(z) = \frac{1}{4} e^{-\frac{1}{2}z}, \quad z \geq 0}$$

$z<0$: II $\quad g(z) = \int_{-z}^{\infty} \frac{1}{4} e^{-\frac{1}{2}(x+z+x)} \cdot 1 \; dx$

$$\boxed{g(z) = \frac{1}{4} e^{\frac{1}{2}z}, \quad z \leq 0}$$

2. PTI 2007. 7) $f(x,y) = x^2 + Cy \qquad 0 \le x, y \le 1$

Izračunaj fju gustoće od: $z = 3x - \frac{1}{2}Y$.



$$\int_0^1 dx \int_0^1 (x^2 + Cy)\,dy = \dots = 1$$

$$\boxed{C = \frac{4}{3}}$$

$z \in [-\frac{1}{2}, 3]$ (uvrstimo min x, max Y / max x, min Y)

$y = 6x - 2z \qquad \left|\frac{\partial y}{\partial z}\right| = 2$

y=1
=6x-2z
$x = \frac{1+2z}{6}$

I $\quad g(z) = \int_0^{\frac{1+2z}{6}} \left[x^2 + \frac{4}{3}(6x - 2z)\right] \cdot 2\,dx$

$$\boxed{g(z) = \frac{73}{324} + \frac{z}{54} - \frac{23}{27}z^2 + \frac{2z^3}{81}}$$ $z \in [-\frac{1}{2}, 0]$

II $\quad g(z) = \int_{\frac{1}{3}z}^{\frac{1+2z}{6}} \left[x^2 + \frac{4}{3}(6x - 2z)\right] \cdot 2\,dx$

$z \in [0, \frac{5}{2}]$

$$\boxed{g(z) = \frac{73}{324} + \frac{z}{54} + \frac{z^2}{27}}$$

III $\quad g(z) = \int_{\frac{1}{3}z}^{1} \left[x^2 + \frac{4}{3}(6x - 2z)\right] \cdot 2\,dx$

$z \in [\frac{5}{2}, 3]$

$$\boxed{g(z) = \frac{26}{3} - \frac{16}{3}z + \frac{8}{9}z^2 - \frac{2}{21}z^3}$$

9. TEORIJA VJEROJATNOSTI

- motivacija:

  - indikatorska varijabla (bilježi da li se neki događaj dogodio ili ne : 0, 1)

$$I_k \sim \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix} \sim B(1,p)$$

↳ n puta ponavljamo: $X_n = I_1 + I_2 + \dots + I_n$

↳ br. koliko puta se dogodio neki događaj

$$X_n \sim B(n,p)$$

- intuitivno:

$$\frac{X_n}{n} \xrightarrow{?} p$$ kakav je to limes slučajnih varijabli?

9.1. ZAKON VELIKIH BROJEVA

def. Niz sl. var. (X_n) <u>konvergira po vjerojatnosti</u> prema sl. var. X ako za $\forall \varepsilon > 0$:

$$\lim_{n \to \infty} P(|X_n - X| > \varepsilon) = 0$$

Pišemo: $X_n \xrightarrow{P} X$

tm a) Nejednakost Markova: Ako X poprima nenegativne vrijednosti onda $\forall \varepsilon > 0$ vrijedi:

$$\boxed{P(X \geq \varepsilon) \leq \frac{E(X)}{\varepsilon}}$$

b) L^p nejednakost: Za svaku sl. var. X s konačnim očekivanjem vrijedi:

$$P(|X - E(X)| \geq \varepsilon) \leq \frac{E(|X - E(X)|^p)}{\varepsilon^p}$$

c) Čebiševljeva nejedn.: specijalno za $p=2$ imamo:

$$\boxed{P(|X-E(X)|\geq \varepsilon)\leq \frac{D(X)}{\varepsilon^2}}$$

dokaz:

a) $P(X\geq \varepsilon)=\int_{\varepsilon}^{+\infty} f(x)\,dx \leq \int_{\varepsilon}^{+\infty} \underbrace{\frac{x}{\varepsilon}}_{\geq 1} f(x)\,dx \leq \frac{1}{\varepsilon}\int_0^{+\infty} x f(x)\,dx = \frac{E}{\varepsilon}$ //

b) $P(|X-E(X)|\geq \varepsilon)=P(|X-E(X)|^p\geq \varepsilon^p) \underset{\text{prema a)}}{\leq} \frac{E(|X-E(X)|^p)}{\varepsilon^p}$ //

c) uvrstimo 2

DZ. 1) Broj sunčanih dana je sl. var. s $E(X)=75$. Dokažite da je vjerojatnost da u toku 1 godine ne bude više od 200 sunčanih dana veća od $\frac{5}{8}$.

$E(X)=75$ $X=$ broj sunčanih dana

$P(X\leq 200)>\frac{5}{8}$ → dokaži

$P(X<200)=1-\underbrace{P(X\geq 200)}_{\leq \frac{E(X)}{\varepsilon}} > 1-\frac{75}{200}=\frac{5}{8}$ //

jer je − ispred $\leq \frac{E(X)}{\varepsilon}$

6. DZ. 2.) $E(X)=25$ km/h $\sigma(X)=4.5$ km/h. Koliku brzinu vjetra možemo očekivati s vjerojatnosti ne manjom od 0,9?

$P(|X-E(X)|\geq\varepsilon)\leq\frac{D(X)}{\varepsilon^2}$ → $P(|X-E(X)|<\varepsilon)>1-\frac{D(X)}{\varepsilon^2}>0{,}9$

$1-\frac{\sigma^2}{\varepsilon^2}>0{,}9$

$|X-E(X)|<\varepsilon$

$|X-25|<14{,}23$

$\varepsilon=14{,}23$

$-14{,}23<X-25<14{,}23$

$$\boxed{X=\langle 10{,}77,\ 39{,}23\rangle}$$

·(ZAKON VELIKIH BROJEVA nastavak)
- zanima nas: · aritmetička sr. rezultata $(X_n)_{n \geq N}$

$$\frac{X_1 + \dots + X_n}{n} \quad (*)$$

· da li aritm. sr. konvergira

→ ZVB objašnjavaju konvergencije ovog tipa

- ovisi o tipu konvergencije:

1) konv. po vjerojatnosti (slabi)

2) konv. gotovo sigurno (jaki)

- cilj: naći uvjete na sl. var. (X_n) tako da imamo konv. $(*)$ po vjerojatnosti ili g.s.

SLABI ZAKON VELIKIH BROJEVA (SZVB)

def.
$X_n \xrightarrow{P} Y$ ako $\forall \varepsilon > 0$ $\lim_{n\to\infty} P(|X_n - Y| \geq \varepsilon) = 0$ (konv. po vjerojat.)

def. Kažemo da niz $(X_n)_{n\in\mathbb{N}}$ zadovoljava SZVB ako

$$\frac{1}{n} \sum_{k=1}^{n} [X_k - E(X_k)] \xrightarrow{P} 0$$

tm. DOVOLJNI UVJETI ZA SZVB: Ako var. $(X_n)_{n\in\mathbb{N}}$ zadovoljavaju uvjet:

$$\lim_{n\to\infty} \frac{1}{n^2} \cdot D\left(\sum_{k=1}^{n} X_k\right) = 0$$

tada $(X_n)_{n\in\mathbb{N}}$ zadovoljava SZVB.

→ (tražimo da $D(X_n)$ postoji)

→ (Čebiševljeva nejednakost za dokaz)

tražimo da $P(|\sum_{k=1}^{n} (X_k - E(X_k))| \geq \varepsilon) \to 0$

- specijalni slučajevi kada SZVB vrijedi:
  (1) (X_n) nekoreliranih s ograničenom varijancom
  (2) (X_n) nezavisnih s istom varijancom $D(X_n)=\sigma^2$
  (3) (X_n) nezavisnih i jednako distribuiranih s konačnom varijancom

dokaz: ako su nezavisne (X_n) tada je disperzija sume = suma disperzija, a budući da su jednako distribuirane, sve imaju jednaku $D(X_n)=\sigma^2$

$$\lim_{n\to\infty} \frac{1}{n^2} \cdot D\left(\sum_{k=1}^{n} X_n\right) = \lim_{n\to\infty} \frac{1}{n^2} \cdot \overbrace{\sum_{k=1}^{n} D(X_n)}^{n \text{ disperzija}} =$$

$$= \lim_{n\to\infty} \frac{1}{n^{\cancel{2}}} \cdot \cancel{n} \cdot \sigma^2 = \lim_{n\to\infty} \frac{\sigma^2}{n} = 0$$

konst

prim. Promotrimo jednostavni slučaj

$$X_n = I_n \sim \begin{pmatrix} 0 & 1 \\ Q & P \end{pmatrix} \rightarrow \text{niz nezavisnih indik. sl. var.}$$

$$S_n = \sum_{k=1}^{n} I_k = \text{broj uspjeha u } n \text{ pokusa}$$

$$\frac{1}{n} \cdot S_n = \frac{1}{n}\sum_{k=1}^{n} I_k = \text{prosječan broj uspjeha}$$

za veliki n:
→ $(I_n)_{n\in N}$ zadovoljava specijalni slučaj (3)

$$m(I_n) = P \qquad D(I_n) = P\cdot Q$$

→ vrijedi SZVB:

$$\frac{1}{n} \cdot \sum_{k=1}^{n} [I_k - \underbrace{E(I_k)}_{P}] \xrightarrow{P} 0$$

$$\frac{1}{n} \cdot \sum_{k=1}^{n} I_k - P \xrightarrow{P} 0$$

vjerojatnost da je odstupanje od teoretske vjerojatnosti 0

$$\boxed{\frac{S_n}{n} \xrightarrow{P} P} \qquad \lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - P\right| \geq \varepsilon\right) = 0$$

→ možemo dobiti ocjene na odstupanja prosječnog broja uspjeha od vjerojatnosti uspjeha

→ Koristimo Čebiševljevu nejednakost

$$P\left(\left|\frac{S_n}{n} - p\right| \geq \varepsilon\right) \leq \frac{D\left(\frac{1}{n} \cdot S_n\right)}{\varepsilon^2}$$

$$P(|X - m| \geq \varepsilon) \leq \frac{D(X)}{\varepsilon^2}$$

zbog nezavisnosti $D(\Sigma) = \Sigma D$

$$D\left(\frac{1}{n} \cdot S_n\right) = \frac{1}{n^2} D(S_n) = \frac{1}{n^2} \cdot \sum_{k=1}^{n} D(I_k) = \frac{1}{n^{\cancel{2}}} \cdot \cancel{n} p q$$

$$= \frac{pq}{n}$$

$$\hookrightarrow P\left(\left|\frac{S_n}{n} - p\right| \geq \varepsilon\right) \leq \frac{pq}{n \cdot \varepsilon^2} \leq \frac{1}{4 \cdot n \cdot \varepsilon^2}$$

$pq = p(1-p) \leq \frac{1}{4}$ za $p \in \langle 0, 1 \rangle$ (max je za $p = \frac{1}{2}$ pa je $p \cdot (1-p) = \frac{1}{4}$ max)

primjer: Obavili smo pokus 1000 puta i tražimo poklapanje do na 1. decimalu.

$n = 1000 \qquad \varepsilon = 0{,}1$

$$P\left(\left|\frac{S_n}{n} - p\right| \geq 0{,}1\right) \leq \frac{1}{4 \cdot 1000 \cdot 0{,}1^2} = 0{,}025$$

p – nepoznato

komplement: $P\left(\left|\frac{S_n}{n} - p\right| < 0{,}1\right) \geq 1 - 0{,}025 = 0{,}975$

↳ u barem 97,5 % slučajeva će relativni uspjeh odstupati od vjerojatnosti uspjeha za manje od 0.1

SZVB → vj. odstupanja → 0

želimo jači rezultat: $\frac{S_n}{n} \to p$: JZVB

def. Kažemo da $(X_n)_{n \in \mathbb{Z}}$ konvergira gotovo sigurno sl. var. Y ako:

$$P\left(\lim_{n\to\infty} X_n = Y\right) = 1$$

označavamo: $X_n \xrightarrow{(g.s.)} Y$

napomena: $K = \{\lim_{n\to\infty} X_n = Y\} = \{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = Y(\omega)\}$

$$K \in \mathcal{F} \quad i \quad P(K) = 1$$

JAKI ZAKON VELIKIH BROJEVA (JZVB)

def. Kažemo da $(X_n)_{n\in\mathbb{N}}$ zadovoljava JZVB ako

$$\frac{1}{n}\sum_{k=1}^{n}\left[X_k - E(X_k)\right] \xrightarrow{g.s.} 0$$

tm. Odnos konv. po vjerojatnosti i g.s.

$$X_n \xrightarrow{(g.s.)} Y \Rightarrow X_n \xrightarrow{P} Y$$

posljedica: JZVB povlači SZVB

tm. Neka je $(X_n)_{n\in\mathbb{Z}}$ niz nezavisnih jednako distribuiranih sl. var. s očekivanjem m: tada vrijedi JZVB, tj.

$$\frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{g.s.} m$$

primjer) $(I_n)_{n\in\mathbb{N}}$ nezavisne indikatorske, jednako distrib.

$$I_n \sim \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix} \qquad E(I_n) = p = m$$

$\longrightarrow$ JZVB vrijedi:

$$\underbrace{\frac{1}{n}\sum_{k=1}^{n} I_k}_{\frac{S_n}{n} = \text{prosj. br. uspjeha}} \xrightarrow{g.s.} p$$

$$P(\{\omega \in \Omega : \frac{S_n(\omega)}{n} \longrightarrow p\}) = 1$$ (skoro svugdje ima konv. $\frac{S_n}{n}$)

- zanimaju nas vjerojatnosti da svaki od el. X_n upada u A:

$$\lim_{n\to\infty} P(X_n \in A)$$ (limes nekih distribucija)

$$A = \langle -\infty, x\rangle$$

$$\downarrow$$

$$P(X_n \in \langle -\infty, x\rangle) = P(X_n < x) = F_{X_n}(x)$$

KONVERGENCIJA PO DISTRIBUCIJI I KARAKT. F-je

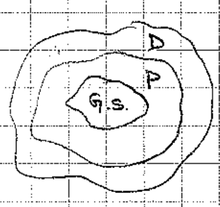
def. Kažemo da niz $(X_n)_{n\in\mathbb{N}}$ sl. var. <u>konvergira po distribuciji</u> prema sl. var. Y ako za pripadni niz f-ja distribucije vrijedi

$$\lim_{n\to\infty} F_{X_n}(x) = F_Y(x) \quad \text{za svaki } x \text{ u kojem je } F_Y \text{ neprekinuta}$$

Označavamo: $X_n \xrightarrow{D} Y$

– odnos konv. po distribuciji s ostalim konv.:

tm. $X_n \xrightarrow{P} Y \Longrightarrow X_n \xrightarrow{D} Y$



– ideja: povezati konv. po distribuciji s karak. funkcijama

tm. $X_n \xrightarrow{D} X \iff \vartheta_{X_n}(t) \to \vartheta_X(t) \quad \forall t$

$\vartheta_X(t) = E[e^{itX}]$ → karakteristična f-ja primjer) Aproksimacija binomne sl. var. Poissonovom

$$X_{n,p_n} \sim B(n, p_n) \qquad p_n \xrightarrow{n\to\infty} 0 \qquad n \cdot p_n \xrightarrow{n\to\infty} \lambda > 0$$

$\Longrightarrow X_{n,p_n} \xrightarrow{D} P(\lambda)$ Pokažimo da ovo vrijedi:

$$Y \sim P(\lambda) \qquad X_{n,p_n} \xrightarrow{D} Y \Longleftrightarrow \vartheta_{X_{n,p_n}}(t) \xrightarrow{n\to\infty} \vartheta_Y(t)$$

$$\vartheta_{X_{n,p_n}}(t) = (q_n + p_n e^{it})^n = (1 - p_n + p_n e^{it})^n =$$

$$= [1 + p_n(e^{it} - 1)]^n$$

$$= \left\{[1 + p_n(e^{it} - 1)]^{\frac{1}{p_n}}\right\}^{\overset{\to\lambda}{n p_n}} \to \left(e^{e^{it}-1}\right)^{\lambda} =$$

$$= e^{\lambda(e^{it}-1)} = \vartheta_Y(t) \qquad \forall t$$

CENTRALNI GRANIČNI TEOREM (CGT)

- tm. $(X_n)_{n\in\mathbb{N}}$ niz nezavisnih jednako distribuiranih s očekivanjem m i disperzijom σ^2. Tada za normirani zbroj

$$\frac{\sum_{i=1}^{n} X_i - nm}{\sigma\sqrt{n}} \xrightarrow{D} N(0,1)$$

- specijalni slučaj: Moivre Laplaceov teorem:

$$\frac{B(n,p) - np}{\sqrt{npq}} \xrightarrow{D} N(0,1)$$

dokaz: uzmemo $X_i = I_i$ indikatorske s param. p i nezav.

$$\sum_{i=1}^{n} I_i \sim B(n,p) \qquad \mathbb{E} I_i = p = m$$

$$D(I_i) = pq = \sigma^2$$

$$\underset{\substack{\downarrow D \\ N(0,1)}}{\frac{\sum_{i=1}^{n} X_i - nm}{\sigma\sqrt{n}}} \stackrel{D}{=} \underset{\substack{\downarrow D \\ N(0,1)}}{\frac{B(n,p) - np}{\sqrt{npq}}}$$

(6. DZ. 5) $X_1, \dots, X_{10}$ nezav. sl. var. s jednolikom razdiobom na $[0,1]$.

Pomoću CGT izračunati: $P\left(\sum_{i=1}^{10} X_i > 6\right) = ?$

$$E(X_1) = \frac{b-a}{2} = \frac{1-0}{2} = \frac{1}{2}$$

$$D(X_1) = \frac{(b-a)^2}{12} = \frac{1}{12} \qquad \sigma = \frac{\sqrt{12}}{12}$$

$$\frac{\sum_{i=1}^{10} X_i - 10\cdot\frac{1}{2}}{\frac{\sqrt{12}}{12}\cdot\sqrt{10}} \xrightarrow{D} N(0,1)$$

$$P\left(\sum_{i=1}^{10} X_i > 6\right) = 1 - P\left(\sum_{i=1}^{10} X_i < 6\right) =$$

$$= 1 - P\left(\frac{\sum_{i=1}^{10} X_i - 5}{\frac{\sqrt{10}}{\sqrt{12}}} < \frac{6-5}{\frac{\sqrt{10}}{\sqrt{12}}}\right) = 1 - \Phi\left(\frac{6-5}{\frac{\sqrt{10}}{\sqrt{12}}}\right)$$

$$= 1 - \frac{1}{2}\left[1 + \Phi^*(\sqrt{12})\right]$$

$$= 0{,}1349$$

treba naštimat D i L na CGT

ZI 2011. 1b) $\left.\begin{matrix} y_1, \dots, y_{100} \\ [y_1], \dots, [y_{100}] \end{matrix}\right\}$ $[y_i] - y_i \sim U(-0{,}5, 0{,}5)$

nezavisne

Pogreška zbroja: $\sum_{i=1}^{100} [y_i] - \sum_{i=1}^{100} y_i = \sum_{i=1}^{100} (\underbrace{[y_i] - y_i}_{x_i \text{ greške}})$

$$P\left(\sum_{i=1}^{100} x_i \in \langle -t, t \rangle\right) \geq 0{,}95$$

$$\shortparallel$$

$$P\left(\left|\sum_{i=1}^{100} x_i\right| < t\right)$$

(x_i) nezavisne, jednako distr. $\sim U(-0{,}5, 0{,}5)$

$m = 0 \qquad \sigma^2 = \frac{1}{12} \quad \Rightarrow$ CGT

$$\frac{\sum_{i=1}^{100} x_i - 100 \cdot 0}{\sqrt{\frac{1}{12}} \sqrt{100}} \approx N(0,1)$$

$$P\left(\left|\sum_{i=1}^{100} x_i\right| < t\right) = P\left(\left|\frac{\sum_{i=1}^{100} x_i}{\frac{10}{\sqrt{12}}}\right| < \frac{t}{\frac{10}{\sqrt{12}}}\right) \approx \Phi^*\left(\frac{t}{\frac{10}{\sqrt{12}}}\right) \geq 0{,}95$$

$$\Phi^*(1{,}96) = 0{,}95$$

$$\frac{t}{\frac{10}{\sqrt{12}}} = 1{,}96$$

$$\boxed{t = 5{,}658}$$

10. MATEMATIČKA STATISTIKA

TOČKASTE PROCJENE PARAMETARA

Ω populacija $\longrightarrow$ obilježje X

$\longrightarrow$ distribucija $F \rightarrow$ nepoznata

uzeti uzorak $\longrightarrow$ $m, \sigma^2 \longrightarrow$ nepoznato

X sl. var. , populacija

$\hookrightarrow F, f, m, \sigma^2$

$\hookrightarrow F$ može ovisiti o nekim parametrima

$\vartheta_1, \dots, \vartheta_k$

def. Za $X_1, \dots, X_n$ sl. var. kažemo da su nezavisne kopije sl. var. X ako vrijedi:

(1) $X_1, \dots, X_n$ nezavisne

(2) $X_i \sim F$

$(X_1, \dots, X_n)$ UZORAK

$(x_1, \dots, x_n)$ REALIZACIJA

Pretpostavimo da razdioba F ovisi samo o ϑ:

$$F(x) = F_\vartheta(x)$$

$$f(x) = f_\vartheta(x)$$

ϑ procijeniti pomoću $X_1, \dots, X_n$

$\hat{u} = g(x_1, \dots, x_n)$ procjena za ϑ

$\Theta = g(X_1, \dots, X_n)$ STATISTIKA

$g(X_1, \ldots, X_n) \rightarrow$ statistika

$\Theta \rightarrow$ procjenitelj parametra ϑ

$\hat{u} \rightarrow$ procjena parametra ϑ

Pitanje: 1) Koje statistike su dobre za m i σ^2?

2) Što su to dobre statistike?

- statistika za procjenu očekivanja $a = E(X)$:

$$\overline{X} = \frac{1}{n} \cdot (X_1 + \ldots + X_n) \qquad \text{SREDINA UZORKA}$$

$\hookrightarrow$ sl. var., statistika

$$E[\overline{X}] = \frac{1}{n}[E(X_1) + \ldots + E(X_n)] = \frac{1}{n} \cdot n \cdot a = a$$

$$D(\overline{X}) = \frac{1}{n^2}[D(X_1) + \ldots + D(X_n)] = \frac{1}{n^{\cancel{2}}} \cdot \cancel{n} \cdot \sigma^2 = \frac{\sigma^2}{n}$$

- dobre statistike:

1) Θ = nepristrana statistika (procjenitelj) za ϑ ako je

$$E[\Theta] = \vartheta$$

$\hookrightarrow \overline{X}$ nepr. procjenitelj za očekivanje

2) Ako niz statistika $\Theta_n = \Theta_n(X_1, \ldots, X_n)$ konvergira po vjerojatnosti prema param ϑ tada Θ_n zovemo valjanom statistikom

$$\forall \varepsilon > 0 \quad P(|\Theta_n - \vartheta| \geq \varepsilon) \rightarrow 0 \quad n \to \infty$$

3) Θ_1 i Θ_2 dvije nepristr. stat. za ϑ. Θ_1 bolja od Θ_2 ako $D(\Theta_1) < D(\Theta_2)$

tm. Nepristrana statistika je valjana ako joj disperzija teži u 0.

dokaz:

$$E[\Theta_n] = \vartheta \qquad D(\Theta_n) \to 0 \quad {}_{n\to\infty} \qquad \text{želimo: } \Theta_n \xrightarrow{P} \vartheta$$

$$0 \leq P(|\Theta_n - \vartheta| \geq \varepsilon) \underset{\text{čebiš.}}{\leq} \frac{D(\Theta_n)}{\varepsilon^2} \to 0 \Rightarrow$$

→ $\bar{X}$ valjana za a zbog:

$$D(\bar{X}) = \frac{\sigma^2}{n} \to 0 \quad \checkmark$$

- statistika za procjenu disperzije:
  - a poznat $\qquad D^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - a)^2$
  - a nepoznat $\qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

→ D^2 i S^2 su nepristrani i valjani procj. za σ^2

$$E(D^2) = \sigma^2$$

$$D(D^2) = \frac{1}{n}(\mu_4 - \sigma^4) \to 0 \quad \checkmark$$

$$\Theta = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \Rightarrow E(\Theta) = \frac{n-1}{n}\sigma^2$$

$$E\left[\frac{n}{n-1}\Theta\right] = \sigma^2$$

$$E(S^2) = \sigma^2$$

pr. 10.3. 6 rezultata: 3540, 3582, 3555, 3572, 3564, 3548 m

a) $a = 3560$ b) a nepoznata

a) $d^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - a)^2 = \frac{1}{6}\sum_{i=1}^{6}(x_i - 3560)^2 = 232,17\ m^2$

b) $s^2 = \frac{1}{6-1}\sum_{i=1}^{6}(x_i - \bar{x})^2 = \frac{1}{5}\left[(3540 - 3561.17)^2 + \dots + ()^2\right] =$

$$\boxed{s^2 = 276,97\ m^2}$$

→ podaci mogu biti podijeljeni unaprijed u razrede i dani u tablici:

vrijednosti	x_1	x_2	...	x_r
frekvencije	n_1	n_2	...	n_r

$n = n_1 + \dots + n_r$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{r} x_i \cdot n_i$$

$$s^2 = \frac{1}{n-1}\left(\sum_{i=1}^{r} n_i \cdot x_i^2 - n\cdot\bar{x}^2\right)$$

pr. 10.4. $\bar{x}, s^2 = ?$

x_i	2560	2600	2620	2650	2700
n_i	2	3	10	4	1

* $E(x-c) = E(x) - c$
$D(x-c) = D(x)$

$c = 2620$

$\bar{x} = \frac{1}{20}\sum_{i=1}^{5}(x_i - 2620)n_i + 2620$ $\boxed{\bar{x} = 2621}$

$s^2 = \frac{1}{19}\left(\sum_{i=1}^{5} n_i\cdot(x_i - 2620)^2 - n\bar{x}^2\right)$

$$\boxed{s^2 = 967,4}$$

KRITERIJ NAJVEĆE IZGLEDNOSTI

- zakon razdiobe $f(x, \vartheta)$

X diskretna $f(x, \vartheta) = P_\vartheta(X = x)$

X neprekidna $f(x, \vartheta) = f_\vartheta(x)$

$x_1, \dots, x_n$ iz populacije X sa zakonom razdiobe $f(x, \vartheta)$

def. F-ja IZGLEDNOSTI

$$F(\vartheta, x_1, \dots, x_n) = f(\vartheta, x_1) \cdot f(\vartheta, x_2) \cdot \dots \cdot f(\vartheta, x_n)$$

$\hat{\vartheta}$ procjenitelj za ϑ za koji F postiže globalni maximum.

→ kriterij za odabir $\hat{\vartheta}$

pr. 10.6. Procjena parametra eksp. razdiobe.

$$X \sim Exp(\lambda) \qquad f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

$x_1, \dots, x_n$ realizacija ⇒ $\hat{\lambda}$?

$f(x, \lambda) = \lambda e^{-\lambda x}$

$$F(\lambda, x_1, \dots, x_n) = \lambda e^{-\lambda x_1} \cdot \lambda e^{-\lambda x_2} \cdot \dots \cdot \lambda e^{-\lambda x_n} =$$
$$= \lambda^n e^{-\lambda(x_1 + \dots + x_n)} \rightarrow \text{maksimizirati}$$

$$\frac{dL}{d\lambda} = 0$$

$$\frac{\partial L}{\partial \lambda} = n\lambda^{n-1} e^{-\lambda(x_1 + \dots + x_n)} - n\lambda^{n-1}(x_1 + \dots + x_n) e^{-\lambda(x_1 + \dots + x_n)} = 0$$

$$\hat{\lambda} = \frac{n}{x_1 + \dots + x_n} = \frac{1}{\overline{x}}$$

$$E(x) = \frac{1}{\lambda} \qquad \overline{x} = \frac{1}{\hat{\lambda}} \quad \Rightarrow \quad \hat{\lambda} = \frac{1}{\overline{x}}$$

VJEŽBA ZA ŠKOLSKU

2. DZ. 7.

X\Y	0	1
0	1/4	5/12
1	5/24	1/8

$E(X+Y) = ?$

$D(X+Y) = ?$

$Z = X+Y$

$$Z \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{4} & \frac{5}{24}+\frac{5}{12} & \frac{1}{8} \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \end{pmatrix}$$

$E(X+Y) = \frac{5}{8} + \frac{1}{4}$ $\boxed{E(X+Y) = \frac{7}{8}}$

$D(X) = E(X^2) - E(X)^2$

$D(X+Y) = \frac{5}{8} + \frac{1}{2} - \frac{49}{64}$ $\boxed{D(X+Y) = \frac{23}{64}}$

2. DZ. 8.

X\Y	0	1	2	
0	$\frac{1}{6}$	0	0	$\frac{1}{6}$
1	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$
2	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{3}{6}$
	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	1

$E(X) = 0\cdot\frac{1}{6} + 1\cdot\frac{2}{6} + 2\cdot\frac{3}{6}$

$\boxed{E(X) = \frac{4}{3}}$

$E(Y) = 0\cdot\frac{1}{6} + 1\cdot\frac{2}{6} + 2\cdot\frac{3}{6}$

$\boxed{E(Y) = \frac{4}{3}}$

$D(X) = 1\cdot\frac{2}{6} + 4\cdot\frac{3}{6} - \frac{16}{9}$ $\boxed{D(X) = \frac{5}{9}}$ $\boxed{D(Y) = \frac{5}{9}}$

$r(X,Y) = \frac{cov(X,Y)}{\sigma_X \cdot \sigma_Y}$ $cov(X,Y) = E(X\cdot Y) - E(X)\cdot E(Y)$

$E(X\cdot Y) = p_{ij}\cdot x_i \cdot y_j = \frac{1}{6}\cdot 0\cdot 0 + \frac{1}{6}\cdot 1\cdot 1 + \frac{1}{6}\cdot 1\cdot 2 + \frac{1}{6}\cdot 2\cdot 1 + \frac{1}{3}\cdot 2\cdot 2$

$E(X\cdot Y) = \frac{13}{6}$

$cov(X,Y) = \frac{7}{18}$ $\sigma_X\cdot\sigma_Y = \sqrt{D(X)D(Y)} = \frac{5}{9}$

$\boxed{r(X,Y) = \frac{7}{10}}$

2.DZ.9 Baca se kocka. $X = (\text{broj okrenut})^2$, $Y = -1$ za broj ≤ 4, $Y = 1$ za broj > 4. $D(Z = X+Y) = ?$

X|Y	−1	1
1	$\frac{1}{6}$	0
4	$\frac{1}{6}$	0
9	$\frac{1}{6}$	0
16	$\frac{1}{6}$	0
25	0	$\frac{1}{6}$
36	0	$\frac{1}{6}$

$$Z \sim \begin{pmatrix} 0 & 3 & 8 & 15 & 26 & 37 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

$E(Z) = \frac{89}{6}$ $\quad E(Z^2) = \frac{781}{2}$

$D(Z) = E(Z^2) - E(Z)^2$

$$\boxed{D(Z) = 170.47}$$

2.DZ.10. Bacamo 2 kocke: $X = \min$, $Y = \max$. Jesu li X i Y korelirane? $r(X,Y) = ?$

X/Y	1	2	3	4	5	6	
1	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{11}{36}$
2	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{9}{36}$
3	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{7}{36}$
4	0	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{5}{36}$
5	0	0	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$
6	0	0	0	0	0	$\frac{1}{36}$	$\frac{1}{36}$
	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$	1

$r(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$

$\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y)$

$E(XY) = \frac{49}{4}$

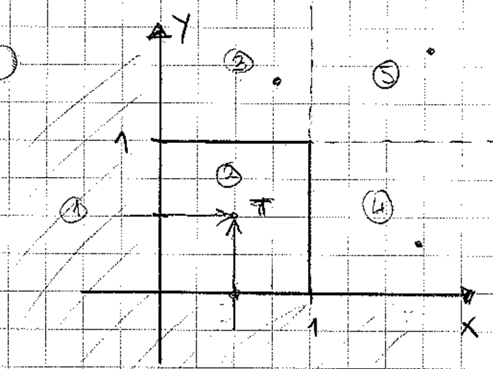
$E(X) = \frac{91}{36}$ $\quad E(Y) = \frac{161}{36}$

$D(X) = 1{,}971$ $\quad D(Y) = \frac{35}{2} = 1{,}971$

$$r(X,Y) = \frac{E(XY) - E(X)E(Y)}{\sqrt{D(X)D(Y)}} = \frac{0{,}945}{1{,}971}$$

$$\boxed{r(X,Y) = 0{,}479}$$

3. DZ. 1. $(X,Y) \sim U([0,1] \times [0,1])$



$$f(x,y) = \frac{1}{1 \cdot 1} = 1$$

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\, dx\, dy$$

① $x < 0 \cup y < 0$ $\quad F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} 0 \, dx\, dy = 0$

② $0 < x, y < 1$
$$F(x,y) = \int_0^x \int_0^y dx\, dy = \int_0^x dx \cdot y = xy$$

③ $0 < x < 1$, $y > 0$ $\quad F(x,y) = \int_0^x dx \int_0^1 dy = \int_0^x dx = x$ (po y prođemo cijelo područje do •)

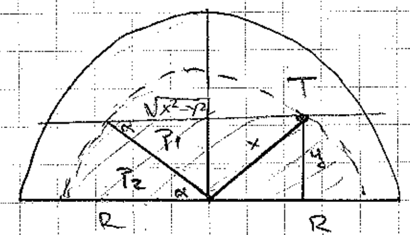
④ $x > 0$, $0 < y < 1$ $\quad F(x,y) = \int_0^1 dx \int_0^y dy = \int_0^1 y\, dx = y$ (po x prođemo cijelo podr. do •)

⑤ $x > 1$, $y > 1$ $\quad F(x,y) = \int_0^1 dx \int_0^1 dy = 1$ (i po x i po y prođemo cijelo podr. do •)

$$F(x,y) = \begin{cases} x \leq 0 \text{ ili } y \leq 0 & \rightarrow 0 \\ 0 < x < 1 \text{ i } 0 < y < 1 & \rightarrow xy \\ 0 < x < 1 \text{ i } y > 0 & \rightarrow x \\ 0 < y < 1 \text{ i } x > 0 & \rightarrow y \\ x > 1 \text{ i } y > 1 & \rightarrow 1 \end{cases}$$

5. DZ. 2. $X = |TS|$ $f(x, y) = ?$

$Y = |Tr|$



$\alpha = \arcsin \frac{y}{x}$

$P_1 = y\sqrt{x^2-y^2} \cdot \frac{1}{2}$ $P_2 = x^2 \pi \cdot \frac{\alpha}{2\pi}$

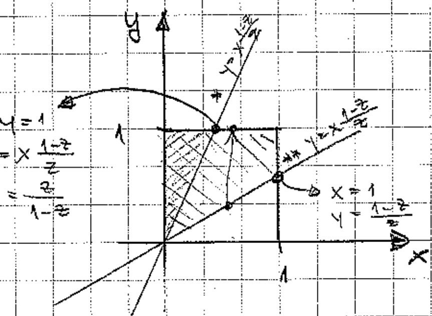
$$F(x, y) = P(X < x, y < Y) = \frac{2P_1 + 2P_2}{R^2 \pi} = \frac{y\sqrt{x^2-y^2} + x^2 \arcsin \frac{y}{x}}{R^2 \pi}$$

$$f(x, y) = \frac{\partial^2 F}{\partial x \partial y} = \dots$$

5. DZ. 15. X i Y nezavisne $\sim U([0,1])$

$f_X(x) = 1$ $f_Y(y) = 1$ $f(x, y) = 1$

$z = x/(x+y)$ $f(z) = ?$ $z \in [0, 1]$



$$F(z) = P(Z < z) = P\left(\frac{X}{X+Y} < z\right)$$

$$\frac{X}{X+Y} < z \qquad Y > \frac{X}{z} - X$$

$$X < z(X+Y)$$

$$F(z) = P\left(Y > X\frac{1-z}{z}\right)$$

granica: $\frac{1-z}{z} = 1$
$1 - z = z$
$z = \frac{1}{2}$

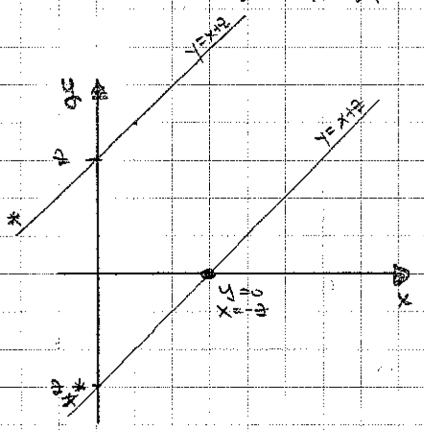
* $z \in [0, \frac{1}{2}]$

$$F(z) = \int_0^{\frac{z}{1-z}} dx \int_{x\frac{1-z}{z}}^1 dy = \iint_{0 \; x\frac{1-z}{z}}^{\frac{z}{1-z} \; 1} dx\,dy = P = \frac{1}{2}\frac{z}{1-z}$$

** $z [\frac{1}{2}, 1]$ $$F(z) = \int_0^1 dx \int_{x\frac{1-z}{z}}^1 dy = \int_0^1 \left(1 - x\frac{1-z}{z}\right)dx = 1 - \frac{1}{2}\frac{1-z}{z}$$ //

5.DZ.20. $f(x,y) = 9e^{-3x-3y}$ $\quad x>0,\ y>0$

$z = Y - X \qquad F(z)=?$



$z \in (-\infty, \infty)$

$y = x + z$

$\left|\frac{\partial y}{\partial z}\right| = 1$

$*\quad z>0 \qquad g(z) = \int_{-\infty}^{\infty} f(x,y)\left|\frac{\partial y}{\partial z}\right| dx = \int_{0}^{\infty} 9e^{-3x-3x-3z}\, dx =$

$= 9e^{-3z} \cdot \frac{1}{-6} e^{-6x} \Big|_{0}^{\infty} = \frac{3}{2} e^{-3z} \qquad z>0$

$**\quad z<0 \qquad g(z) = \int_{-\infty}^{\infty} f(x,y)\left|\frac{\partial y}{\partial z}\right| dx = \int_{-z}^{\infty} 9e^{-3x-3x-3z}\, dx =$

$= 9e^{-3z} \cdot \frac{1}{-6} e^{-6x} \Big|_{-z}^{\infty} = \frac{3}{2} e^{3z} \qquad z<0$

$$g(z) = \begin{cases} \frac{3}{2} e^{3z}, & z<0 \\ \frac{3}{2} e^{-3z}, & z \geq 0 \end{cases}$$

$z<0 \qquad G(z) = \int_{-\infty}^{z} \frac{3}{2} e^{3z}\, dz = \frac{1}{2} e^{3z} \Big|_{-\infty}^{z} = \frac{1}{2} e^{3z}$

$z>0 \qquad G(z) = \int_{0}^{z} \frac{3}{2} e^{-3z}\, dz = -\frac{1}{2} e^{-3z} \Big|_{0}^{z} = -\frac{1}{2} e^{-3z} + \frac{1}{2}$

6\. DZ. 1. X = br. sunčanih dana u 1 god

$E(X) = 75$

$P(X \leq 200) > \frac{5}{8}$

$P(X > \varepsilon) < \frac{E(X)}{\varepsilon}$

$P(X \leq 200) = 1 - P(X > 200) > 1 - \frac{75}{200} = \frac{5}{8}$ //

6\. DZ 2. $E(X) = 25$ km/h $\quad X$ = brzina vjetra

$\sigma(X) = 4{,}5$ km/h $\quad P \geq 0{,}9$

$P(|X - E(X)| > \varepsilon) < \frac{D(X)}{\varepsilon^2}$

$P(|X - E(X)| < \varepsilon) > 1 - \frac{D(X)}{\varepsilon^2} \geq 0{,}9$

$1 - \frac{D(X)}{\varepsilon^2} \geq 0{,}9$

$\frac{D(X)}{\varepsilon^2} \leq 0{,}1$

$\varepsilon^2 = \frac{D(X)}{0{,}1}$

$\varepsilon = 14{,}23$

$|X - E(X)| < \varepsilon$

$X - E(X) < \varepsilon \qquad -X + E(X) < \varepsilon$

$X < 39{,}23 \qquad X > 10{,}77$

$\boxed{X \in \langle 10.77, 39.23 \rangle}$

6\. DZ 3. $E(X) = 1$

$\sigma(X) = 0{,}4$

$A = \{X < 3\} \quad P(A) > 0{,}95$

$P(|X - E(X)| > \varepsilon) < \frac{D(X)}{\varepsilon^2}$

$P(|X - E(X)| < \varepsilon) > 1 - \frac{D(X)}{\varepsilon^2} = 1 - \frac{0{,}4^2}{4} = 0{,}96$ //

$X < \varepsilon + E(X)$

$\varepsilon + E(X) = 3$

$\varepsilon = 2$

6.DZ 4. Bacimo 10 kocaka, pomoću CGT

$$P\left(30 \le \sum_{i=1}^{10} X_i \le 40\right)$$

$E(x) = \frac{7}{2}$ $D(x) = \frac{35}{12}$

$n = 10$

$\sigma(x) = \sqrt{\frac{35}{12}}$

$$\frac{\sum_{i=1}^{10} X_i - n \cdot E(x)}{\sigma \cdot \sqrt{n}} = \frac{\sum_{i=1}^{10} X_i - 35}{\frac{5\sqrt{42}}{6}}$$

$$P\left(\frac{30-35}{\frac{5\sqrt{42}}{6}} \le \frac{\sum_{i=1}^{10} X_i - 35}{\frac{5\sqrt{42}}{6}} \le \frac{40-35}{\frac{5\sqrt{42}}{6}}\right) =$$

$$= \frac{1}{2}\left[\Phi^*(0{,}9258) - \Phi^*(-0{,}9258)\right] = \boxed{\Phi^*(0{,}9258)}$$

6.DZ 5. $X_i = 1, 2, \ldots, 10$ $\sim U([0,1])$ $n = 10$

pomoću CGT: $P\left\{\sum_{i=1}^{10} X_i > 6\right\}$

$$f(x) = 1 \qquad E(x) = \int_0^1 x \, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$D(x) = \int_0^1 x^2 f(x) dx - E(x)^2 = \frac{x^3}{3}\Big|_0^1 - \frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

CGT: $$\frac{\sum_{i=1}^{n} X_i - E(x) \cdot n}{\sqrt{n} \cdot \sqrt{D(x)}} \approx N(0,1) = \frac{\sum_{i=1}^{10} X_i - 5}{\frac{\sqrt{30}}{6}}$$

$$P\left(\sum_{i=1}^{10} X_i > 6\right) = P\left(\frac{\sum_{i=1}^{10} X_i - 5}{\frac{\sqrt{30}}{6}} > \frac{6-5}{\frac{\sqrt{30}}{6}}\right) = \frac{1}{2}\left[1 - \Phi^*(1{,}095)\right]$$

$$\boxed{P\left(\sum_{i=1}^{10} X_i > 6\right) = 0{,}138}$$

$\exists$ 2011. 1.b) $n = 100$ $X_i \sim U[-0.5, 0.5]$ $D(x) = \frac{1}{12}$

$[-t, t] = ?$

$P(-t < x < t) = 0{,}95$ $E(x) = 0$

$$\frac{\sum_{i=1}^{100} X_i - 0 \cdot 100}{10 \cdot \sqrt{\frac{1}{12}}}$$

$$P\left(\frac{-t}{\frac{10}{\sqrt{12}}} < \frac{\sum_{i=1}^{100} X_i}{\frac{10}{\sqrt{12}}} < \frac{t}{\frac{10}{\sqrt{12}}}\right) \approx N(0,1) = 0{,}95$$

$$0{,}95 = \Phi^*\left(\frac{t}{\frac{10}{\sqrt{12}}}\right) \qquad \frac{t}{\frac{10}{\sqrt{12}}} = 1{,}96 \qquad \boxed{t = 5{,}658}$$

*nastavak statistika

Pr 10.10. $X \sim P(\lambda)$ $\lambda > 0$ $\hat{\lambda} = ?$

$$f(x, \lambda) = P_\lambda(X = x) = \frac{\lambda^x}{x!} e^{-\lambda}, \quad x \in \mathbb{N}_0$$

$x_1, \dots, x_n$ realizacija

$$L(\lambda, x_1, \dots, x_n) = \prod_{i=1}^{n} f(x_i, \lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i}}{x_i!} e^{-\lambda} = \frac{\lambda^{x_1 + \dots + x_n}}{x_1! \dots x_2!} e^{-\lambda}$$

$L \to \max$

$$\ell(\lambda, x_1, \dots, x_n) = \ln L \to \max \qquad \ell(\lambda, x_1, \dots, x_n) = (x_1, \dots, x_n) \ln \lambda + (-n\lambda) - \ln(x_1! \dots x_2!)$$

$$\frac{\partial \ell}{\partial \lambda} = 0 \qquad \frac{\partial \ell}{\partial \lambda} = (x_1 + \dots + x_n) \cdot \frac{1}{\lambda} - n - 0 = 0 \Rightarrow \overline{\lambda} = \frac{x_1 + \dots + x_n}{n} = \overline{x}$$

ZI 2008. ① $X_1, \dots, X_n$ iz Rayleighove razdiobe

$$f(x) = 2\lambda^2 x e^{-\lambda^2 x^2}, \quad x>0, \quad \lambda>0, \quad \lambda$$

$$f(x,\lambda) = f(x) = 2\lambda^2 x e^{-\lambda^2 x^2}$$

$$L(\lambda_1, x_1, \dots, x_n) = \prod_{i=1}^{n} 2\lambda^2 x_i e^{-\lambda^2 x_i^2} = 2^n \lambda^{2n} x_1 \cdot \ldots \cdot x_n e^{-\lambda^2(x_1^2 + \dots + x_n^2)}$$

$$\ell(\lambda_1, x_1, \dots, x_n) = \ln L = \ln 2^n + 2n \ln\lambda + \ln(x_1 \cdot \ldots \cdot x_n) - \lambda^2(x_1^2 + \dots + x_n^2)$$

$$\frac{\partial \ell}{\partial \lambda} = 2n \cdot \frac{1}{\lambda} - 2\lambda(x_1^2 + \dots + x_n^2) = 0 \quad \Rightarrow$$

$$\boxed{\hat{\lambda} = \sqrt{\frac{n}{\sum_{i=1}^{n} x_i^2}}}$$

ZI 2010. ① p = vjerojatnost pisma na novčiću, nepoznata

Novčić je bačen 5 puta, pismo se pojavilo 2 puta.

Nakon toga je bačen 8 puta i pojavio se 3 puta.

$\hat{p} = ?$

$$\underbrace{X_1, X_2, X_3, X_4, X_5}_{2 \text{ puta}}, \underbrace{\dots, X_{13}}_{3 \text{ puta}}$$

$$X \sim \overset{\text{Bernulli}}{B}(p)$$

$$X \sim \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$$

$$f(x,p) = P(X=x) = \begin{cases} p, & x=1 \\ 1-p, & x=0 \end{cases}$$

mjesta 1 u prvih 5 bacanja

$$L(p, x_1, \dots, x_{13}) = \prod_{i=1}^{13} f(x_i, p) = f(1,p)^5 f(0,p)^8 \cdot \binom{5}{2}\binom{8}{3} =$$

$$= \binom{5}{2}\binom{8}{3} p^5 (1-p)^8$$

$$\ell(p, x_1, \dots x_{13}) = \ln L = \ln c + 5\ln p + 8\ln(1-p)$$

$$\frac{\partial \ell}{\partial p} = 0 + \frac{5}{p} - \frac{8}{1-p} = 0 \quad / p(1-p) \qquad 5(1-p) - 8p \qquad p = \frac{5}{13}$$

INTERVALNE PROCJENE

INTERVALI POVJERENJA

$P(\vartheta \in \langle c_1, c_2 \rangle) = p \in \langle 0, 1 \rangle$

$X, F \qquad P(X \leq x_p) = p$ → zadamu

→ možemo li naći x_p takav da to vrijedi?

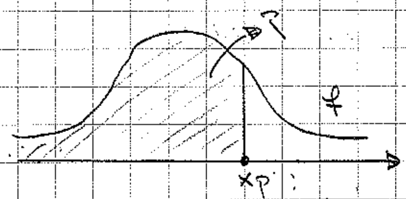
$F(x_p) = p$

↳ Ako takav x_p postoji, zovemo ga kvantil reda p za distribuciju F

(promatramo samo slučajeve za F neprekidna i strogo rast.)

→ specijalno promatramo:

$p = 0.001, 0.01, 0.05, 0.025, 0.9, 0.99, 0.95 \ldots$

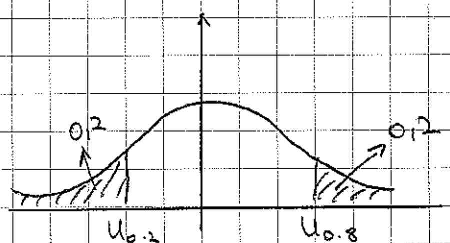


$\int_{-\infty}^{x_p} f(t)\,dt = p$ površina ispod grafa f-je gustoće do x_p jednaka p

Zl 2011. 3) a) Odrediti x_p za stand. normalnu u_p za $p = 0.2$ iz tablice 4 (normalna)

b) u_p $p = 0{,}813$ iz tablice 1

a) $u_{0.2}$ u tablici $\geq 0{,}5$



$u_{0.2} = -u_{0.8} = -0{,}84162$

$\boxed{u_{0.2} = -0{,}84162}$

b) $P(X < u_p) = p \rightarrow \Phi(u_p) = p$

$$\frac{1}{2}\left[1 + \Phi^*(u_p)\right] = p$$

$$\Phi^*(u_p) = 2p - 1$$

$$\Phi^*(u_p) = 0{,}626 \qquad \boxed{u_p = 0{,}889}$$

→ $[c_1, c_2]$ takav da $P(X \in [c_1, c_2]) = p$ zovemo interval povjerenja sl. var. X reda p

$$P(\vartheta \in [c_1, c_2]) = p$$

$[c_1, c_2]$ ima više, mi tražimo najkraći

INTERVALNE PROCJENE ZA PARAMETRE NORMALNE RAZDIOBE a, σ^2

$X \sim N(a, \sigma^2)$

① promatramo intervalnu procjenu za a, ako je σ^2 poznata

$\overline{X} = \frac{X_1 + \ldots + X_n}{n}$ točkovna procjena za očekivanje

$$\overline{X} \sim N\left(a, \frac{\sigma^2}{n}\right) \qquad \frac{\overline{X} - a}{\sqrt{\frac{\sigma^2}{n}}} \sim N(0,1) \qquad \boxed{\frac{\overline{X} - a}{\sigma^2}\sqrt{n} \sim N(0,1)}$$

$$\hat{a} = \overline{X}$$

$$U = \frac{\overline{X} - a}{\sigma}\sqrt{n} \sim N(0,1)$$

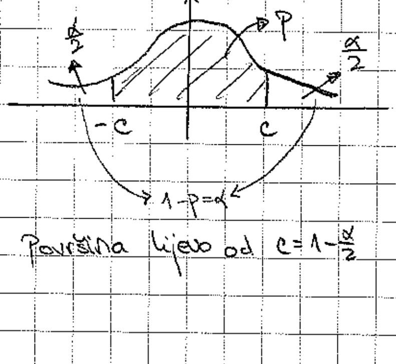
p zadano

p = pouzdanost
$\alpha = 1 - p$ = značajnost

$$P(u \in [-c, c]) = P(|u| \le c) = p$$

$$\Phi^*(c)$$

$\rightarrow c = u_{1-\frac{\alpha}{2}}$



Površina lijevo od $c = 1 - \frac{\alpha}{2}$

$$P(|U| \leq u_{1-\frac{\alpha}{2}}) = 1-\alpha$$

$$P(-u_{1-\frac{\alpha}{2}} \leq U \leq u_{1-\frac{\alpha}{2}}) = 1-\alpha$$

$$P\left(-u_{1-\frac{\alpha}{2}} \leq \frac{\overline{X}-a}{\sigma}\sqrt{n} \leq u_{1-\frac{\alpha}{2}}\right) = 1-\alpha$$

$$P\left(\overline{X} + u_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} \geq a \geq \overline{X} - u_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

P- interval povjerenja za a uz σ^2 poznat:

$$\left[\overline{X} - u_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}, \overline{X} + u_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right] *$$

- POSTUPAK:

1) P zadan $\Rightarrow$ odredimo $\alpha = 1-P = \ldots$
2) iz tablice odredimo kvantil reda $1-\frac{\alpha}{2}$ $u_{1-\frac{\alpha}{2}} = \ldots$
3) $X_1, \ldots, X_n$ uzorak odredimo

$$\overline{x} = \frac{1}{n}(x_1 + \ldots + x_n)$$

$\hookrightarrow$ uvrstimo u formulu *

pr. 11.1.

$$X \sim N(a, 4)$$

x_i	0	1	2	3	4
n_i	1	4	6	12	2

90%-tni int. povj. za a

1) $P = 0.9 \Rightarrow \alpha = 0.1$

2) $u_{1-\frac{\alpha}{2}} = u_{1-0,05} = u_{0,95} \overset{(u)}{=} 1,96$

3) $\overline{x} = \frac{1}{25}\sum_{i=1}^{5} x_i n_i = 2.4 \qquad \sigma^2 = 4 \qquad \sigma = 2$

$$\left[2,4 - 1,96 \cdot \frac{2}{\sqrt{25}}, 2.4 + 1,96 \cdot \frac{2}{\sqrt{25}}\right]$$

$$= \boxed{[1.742, 3.58]}$$

② interval povjerenja za σ^2, ako je a poznat

$\sigma^2 \rightarrow D^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - a)^2 \rightarrow$ točka proc. za σ^2 ako a znamo

$\sigma^2 \geq 0 \longrightarrow$ jednostrani interval $[0, C]$

$\searrow$ dvostrani interval $[c_1, c_2]$

$$P\left(0 \leq \sigma^2 \leq \frac{nd^2}{\chi^2_{n,\alpha}}\right) = 1-\alpha \qquad \frac{nD^2}{\sigma^2} \sim \chi^2(n)$$

$$P\left(\frac{nd^2}{\chi^2_{n,\frac{\alpha}{2}}} \leq \sigma^2 \leq \frac{nd^2}{\chi^2_{n,1-\frac{\alpha}{2}}}\right) = 1-\alpha$$

③ interval povjerenja za a, ako je σ^2 nepoznat

$$\frac{\overline{X}-a}{\sigma}\sqrt{n} \sim N(0,1)$$

zamjenimo s procjeniteljem s^2 jer a nije poznat

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$$

$$\Rightarrow \frac{\frac{\overline{X}-a}{\sigma}\sqrt{n}}{\sqrt{\frac{\frac{(n-1)s^2}{\sigma^2}}{n}}} \sim t(n-1) \qquad u = \frac{\overline{X}-a}{s}\sqrt{n} \sim t(n-1)$$

$$P(|u| \leq c) = p$$

$$c = t_{1-\frac{\alpha}{2}}$$

p-interval povjerenja za a uz σ^2 nepoznat

$$\left[\overline{X} - t_{1-\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}},\ \overline{X} + t_{1-\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}\right]$$

④ interval povjerenja za σ^2 ako je a nepoznato

$$\frac{n s^2}{\sigma^2} \sim \chi^2 (n-1)$$

$$P\left(\chi^2_{\frac{\alpha}{2}} \leq \frac{(n-1)s^2}{\sigma^2} \leq \chi^2_{1-\frac{\alpha}{2}} \right) = 1-\alpha$$

$$\left[\frac{(n-1)s^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \frac{(n-1)s^2}{\chi^2_{n-1,\frac{\alpha}{2}}} \right] \qquad \left[0, \frac{(n-1)s^2}{\chi^2_{n-1,\alpha}} \right]$$

ZI 2009. ②

x_i	20	22	24	26	28	30	31
n_i	3	3	8	7	3	2	1

$n=27$

a) točk. procj. za a i σ^2

$$\hat{a} = \bar{x} = \frac{1}{n} \sum_{i=1}^{7} x_i n_i \qquad \boxed{\hat{a} = 25}$$

$$\hat{\sigma}^2 = S_2 = \frac{1}{n-1} \left(\sum_{i=1}^{7} n_i x_i^2 - n \cdot \bar{x}^2 \right) \qquad \boxed{\hat{\sigma}^2 = 8.846}$$

b) 95% - int. povj. za a, σ^2

1) $P = 0{,}95 \qquad \alpha = 0{,}05$

2) $t_{n-1,1-\frac{\alpha}{2}} = t_{26,1-\frac{0{,}05}{2}} = 2{,}056$

3) $\bar{x} = 25 \qquad S = \sqrt{S^2} = \sqrt{8{,}846}$

$$\left[25 - 2{,}056 \cdot \frac{\sqrt{8{,}846}}{\sqrt{27}}, 25 + 2{,}056 \cdot \frac{\sqrt{8{,}846}}{\sqrt{27}} \right] =$$

$$\boxed{[23.823, 26.177]}$$

$$\chi^2_{n-1,1-\frac{\alpha}{2}} = \chi^2_{26,0.975} = 41{,}923$$

$$\chi^2_{n-1,\frac{\alpha}{2}} = \chi^2_{26,0.025} = 13{,}844$$

$$\boxed{[5.486, 16.613]}$$

PZI. '08. (3)

Iz neke populacije dobili smo sljedeći uzorak:

x_i	20	22	24	26	28	30
n_i	3	2	5	7	3	2

$n = 3+2+5+7+3+2 = 22$ ✓

3 proizvoda su imala veličinu 20

a) ✓ Odredi točkastu procjenu za oček. i disperziju

$\overline{x} = \frac{\sum x_i n_i}{n} = 25$ ✓ $\hat{s}^2 = \frac{1}{n-1}\left(\sum n_i x_i^2 - n \cdot \overline{x}^2\right) = 8{,}667$ ✓

- ako je u zad. poznato da je prosječna vrijednost 25 $\hat{s}^2$ dijelim sa $\frac{1}{n}$?

b) ✓ Odredi 80%-tni interval za oč. i 80%-tni jednostrani interval za disperziju → gledam da je disp nepoznata ili ?

očekivanje norm. razdiobe

$$P\Big(\underset{25}{\overline{x}} - \underbrace{t_{n-1,1-\frac{\alpha}{2}}}_{1{,}323} \cdot \frac{\overset{\sqrt{8{,}667}}{\hat{s}}}{\underset{\sqrt{22}}{\sqrt{n}}} < a < \underset{25}{\overline{x}} + t_{n-1,1-\frac{\alpha}{2}} \cdot \frac{\overset{\sqrt{8{,}667}}{\hat{s}}}{\underset{\sqrt{22}}{\sqrt{n}}}\Big) = p$$ ✓

$p = 0{,}8$

$p = 1-\alpha$

$\alpha = 0{,}2$ → postotak za grešku

studentova razdioba

$t_{n-1,\,1-\frac{\alpha}{2}}$ $\quad n-1 = 21,\ \alpha = 0{,}2$ iz tablice 1,323 → napravljena za $1-\frac{\alpha}{2}$

$$P(24{,}17 \leq a \leq 25{,}83) = 0{,}8$$

formula: disperzija, nepoznato oč., jednostrani interval

$$P\left(0 \leq \sigma^2 \leq \frac{(n-1)\hat{s}^2}{\chi^2_{\underset{21}{n-1},\,\underset{0{,}2}{\alpha}}}\right) = p = 0{,}8$$

↓ iz tablice 15,445

$$P(0 \leq \sigma^2 \leq 11{,}784) = 0{,}8$$

✓

11.5 Intervalna procjena za vjerojatnosti događaja

$$\hat{p} = \frac{X_1 + \dots + X_n}{n} \sim \frac{1}{n} B(n, p) \approx \frac{1}{n} N(np, npq) = N\left(p, \frac{pq}{n}\right)$$

$$\frac{1}{n} \qquad \frac{1}{n^2}$$

$$\vartheta = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}} \Rightarrow P\left(-u_{1-\frac{\alpha}{2}} \leq \vartheta \leq u_{1-\frac{\alpha}{2}}\right) = 1 - \alpha$$

$$\frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}} \leq u_{1-\frac{\alpha}{2}}$$

$p_{1,2} =$ kupus (vrh 2. str.)

$n \geq 100$

$$p_{1,2} = \hat{p} \mp u_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \quad \text{tj.} \quad P(p_1 < p_2) = 1 - \alpha$$

21.07. (4) Od 400 na sreću odabranih maturanata 150 je prošlo prag. Koliki je postotak svi maturanata koji su s pouzdanošću 90% prošli prag?

$$\hat{p} = \frac{150}{400} = 0,374 \qquad n = 400$$

vj. koju smo dobili iz uzorka

U – normalna razdioba

$u_{1-\frac{\alpha}{2}}$ $\qquad \alpha = 1 - p = 1 - 0,9 = 0,1$

– izračunat $1 - \frac{\alpha}{2} = 0,95$ i to je p

$$u_{1-\frac{\alpha}{2}} = 1,64485 \Rightarrow p_{1,2} = 0,375 \mp 0,04$$

$$P(0,335 \leq p \leq 0,415) = 0,9$$

ZI 08. (3) Na izl. anketi od 200 glasača 112 je glasalo za kandidata X.

a) odredi 95% postotni interval povjerenja

$\hat{p} = \frac{112}{200} = 0,56$ $n = 200$

$U_{1-\frac{\alpha}{2}} = U_{0,975} = 1,95996$ $p_{1,2} = 0,56 \mp 0,069$

kvantil norm. razdiobe

$P(0,491 \leq p \leq 0,629) = 0,95$ ✓

b) S kojom vjerojatnosti X može tvrditi da će biti izabran

donja granica veća od 0,5

$\boxed{p_1 > 0,5}$

$\hat{p} - U_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} > 0,5$

?

$U_{1-\frac{\alpha}{2}} < 1,7094$

tražim prvi manji od 1,7

$\underbrace{U_{1-\frac{\alpha}{2}}}_{0,95} = 1,64485$

$1 - \frac{\alpha}{2} = 0,95$

$\alpha = 0,1$ → nivo značajnosti

$p = 1 - \alpha = 0,9$ ✓

90%

?

c) koliko veliki broj ljudi da bi bio siguran 95%, tj. da je nivo značajnosti 5%

$\alpha = 5\% \rightarrow 95\%$

$U_{1-\frac{0,5}{2}} = U_{0,975} = 1,95996$

$\hat{p} - U_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} > 0,5$

0,56

~~13~~ i ~~14~~

$n > 262,93$

$n = 263$

12. Testiranje hipoteza

12.1. Hipoteze i pogreške odlučivanja

pr. Specif. printera kaže da će biti 50 str./min. Na 30 ispisa prosječno je ispisivao 47 stranica po minuti.

nulta hipoteza H_0 ... ono što se testira 50 str./min (specifikacija je istinita)

alternativna hipoteza H_1 ... < 50 str./min (proizvođač laže)

def.

Snaga testa se definira s $S(\vartheta) = P(\text{prihvatili smo } H_1)$.

idealno za $\vartheta \in H_0 : S(\vartheta) = 0$

$\vartheta \in H_1 : S(\vartheta) = 1$

def. Pogreška prve vrste $\alpha = \sup\limits_{\vartheta \in H_0} S(\vartheta)$

→ vjerojatnost da prihvatimo H_1, a istinita je H_0

Pogreška druge vrste $\beta = \sup\limits_{\vartheta \in H_1} (1 - S(\vartheta))$

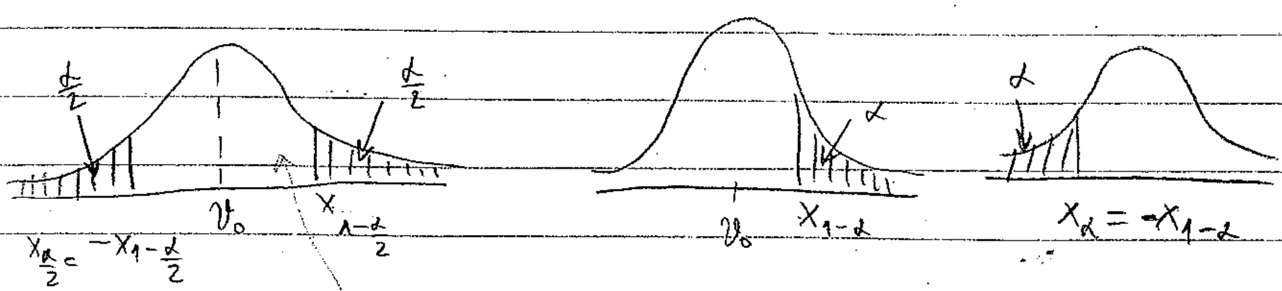
→ vjerojatnost da prihvatimo H_0, a istinita je H_1

– pogreške prve vrste su puno gore i nju uvijek pokušavamo smanjiti (α što manji)

12.2. Testiranje parametarskih hipoteza

H_0 ... parametar koji testiramo $\vartheta = \vartheta_0 \rightarrow \hat{X}$ mjerim uspješnost

dvostrani test	desni test	lijevi test
H_1 ... $\vartheta \neq \vartheta_0$	H_1 ... $\vartheta > \vartheta_0$	H_1 ... $\vartheta < \vartheta_0$



Ako $\hat{X}$ pada u prihvaćamo H_0, inače prihvaćamo H_1

odbacujemo H_0
prihvaćamo (H_1): $|\hat{X}| > X_{1-\frac{\alpha}{2}}$ $\quad$ $\hat{X} > X_{1-\alpha}$ $\quad$ $\hat{X} < -X_{1-\alpha}$

• U-test: testiranje uzorka uz poznatu disp. $\hat{u} = \frac{\overline{X} - a_0}{\sigma / \sqrt{n}}$

• T-test: testiranje uzorka uz nepoznatu disp $\hat{t} = \frac{\overline{X} - a_0}{\hat{S} / \sqrt{n}} \sim t_{n-1}$

ZI 2009. 4) 99.1, 96.9, ..., 97.1, 97.6 $\rightarrow n = 12$

• nije poznata disp. $\Rightarrow$ t-test

$H_0 \ldots \; a = 98 = a_0$

$H_1 \ldots \; a \neq 98$

$\overline{X} = 97{,}6583 \qquad \hat{S}^2 = 0{,}6079$

$\hat{t} = \frac{97{,}6583 - 98}{\sqrt{0{,}6079} / \sqrt{12}} = -1{,}518$

$\alpha = 5\%$

$t_{11, 1-\frac{\alpha}{2}} = 2{,}201$ ($11 = n-1$)

$\Downarrow$

$|\hat{t}| < t_{11, 1-\frac{\alpha}{2}} \rightarrow$ PRIHVAĆAMO H_0,

rafinerija je u pravu

7. DZ. 3) 165, 170, 182, 185, 193, 200, 203, 210 $\rightarrow n = 8$

$\alpha = 5\%$ nivo značajnosti $\qquad a = 200$

$\overline{X} = 188.5$

$H_0 \ldots \; a = 200 = a_0$

$\hat{S}^2 = 253{,}43$

$H_1 \ldots \; a < 200$

$\hat{t} = \frac{188{,}5 - 200}{\sqrt{253{,}43} / \sqrt{8}} = -2{,}04$

$-t_{7, 1-\alpha} = -t_{7, 1-\frac{0{,}1}{2}} = -1{,}895$

$\hat{t} < -t_{n-1, 1-\alpha} \rightarrow -2{,}04 < -1{,}895$

$\Downarrow$

ODBACUJEMO H_0

$\Downarrow$

proizvođač laže

• Hipoteza o proporciji:

$$\hat{u} = \left(\frac{m}{n} - p_0\right)\sqrt{\frac{n}{p_0 q_0}} \sim N(0,1)$$

7.DZ 6.) $n = 500$ $m = 16$

$\alpha = 5\%$ $p_0 = 2\%$

$H_0 \ldots p = 0{,}02 = p_0$ $H_1 \ldots p > 0{,}02$

$$\hat{u} = \left(\frac{m}{n} - p_0\right)\cdot\sqrt{\frac{n}{p_0 q_0}} = 1{,}91663$$

kriterij: $H_1 \ldots a > a_1$ $\hat{u} > u_{1-\alpha}$

$u_{1-\alpha} = u_{0{,}95} = 1{,}64485$ $1{,}91663 > 1{,}64485$

$\Downarrow$

ODBACUJEMO H_0

↳ proizv. LAŽE

ZI 2011-5) $n = 97$

$m = 69$

Uz koji α možemo tvrditi da je bolonjska prolaznost 75%?

$H_0 \ldots p = 0{,}75 = p_0$

$H_1 \ldots p_1 < 0{,}75$

$$\hat{u} = \left(\frac{m}{n} - p_0\right)\cdot\sqrt{\frac{n}{p_0 q_0}} = -0{,}879$$

kriterij: $H_1 \ldots a < a_0$: $\hat{u} < -u_{1-\alpha}$

↳ moramo prihvatiti pa mora vrijediti:

$\hat{u} \geq -u_{1-\alpha}$

$-0{,}879 \geq -u_{1-\alpha} \quad / \cdot (-1)$

$u_{1-\alpha} \geq 0{,}879$

$u_{0{,}82} = 0{,}91537$

↳ $1 - \alpha = 0{,}82$

$\alpha = 1 - 0{,}82$

$\boxed{\alpha = 18\%}$

12.3. USPOREDBA DVIJU POPULACIJA

Z1 2011. 6)

A	12	9	10	8	7	13	11
B	10	8	9	7	12	10	7

Možemo li zaklj. uz $\alpha = 0{,}1$ da se prosječna prodaja u ove 2 trgovine ne razlikuje?

$H_0 \ldots \mu = \nu$

$H_1 \ldots \mu \neq \nu$

A: $n = 7 \quad \bar{X} = 10 \quad \hat{S}^2_X = \frac{28}{6}$

B: $m = 7 \quad \bar{Y} = 9 \quad \hat{S}^2_Y = \frac{20}{6}$

$$S_Z^2 = \frac{1}{n+m-2}\left[(n-1)\hat{S}^2_X + (m-1)\hat{S}^2_Y\right] = 4$$

$$\hat{t} = \frac{\bar{X}-\bar{Y}}{S_Z}\sqrt{\frac{nm}{n+m}} = 0{,}935$$

$$t_{n+m-2,\,1-\frac{\alpha}{2}} = t_{12,\,0.1} = 1{,}782$$

$\hat{t} < 1{,}782$

$\to$ prihvaćamo H_0, tj.: prosječna prodaja čokolade Π je ista u A i B

"ZI 2008 5)" E grupe su na 1. MI imale prosjek 20.625 $(n=254)$ uz $\sigma=8.07$, a R grupe su imale prosjek 19.697 $(m=299)$ uz $\sigma=8{,}36$. Uz koji α možemo tvrditi da ne postoji razlika između E i R?

$H_0 \dots \mu=\nu$ $\qquad \bar{X}=20{,}625 \quad n=254 \quad \sigma_X=8.07$

$H_1 \dots \mu\neq\nu$ $\qquad \bar{Y}=19{,}697 \quad m=299 \quad \sigma_Y=8{,}36$

$$\sigma_Z^2=\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m}=0{,}49$$

$$\hat{u}=\frac{\bar{X}-\bar{Y}}{\sigma_Z}=1{,}32$$

(jer moramo prihvatiti)

kriterij: $|\hat{u}|>u_{1-\frac{\alpha}{2}} \longrightarrow 1{,}32\leq u_{1-\frac{\alpha}{2}}$

$u_{1-\frac{\alpha}{2}}\geq 1{,}32$

$u_{0{,}91}=1{,}34076$

$\hookrightarrow 1-\frac{\alpha}{2}=0{,}91$

$\frac{\alpha}{2}=0{,}09 \qquad \boxed{\alpha=18\%}$

ZI 2009. 5) 90 od 200 ferovaca $\quad m_1=90 \quad n_1=200$

69 od 150 ferovki $\quad m_2=69 \quad n_2=150$

$\alpha=5\%$

$\hat{p}_1=\frac{m_1}{n_1}=45\% \qquad \hat{p}_2=\frac{m_2}{n_2}=46\%$

$H_0 \dots p_1=p_2$

$H_1 \dots p_1\neq p_2$

$p=\frac{m_1+m_2}{n_1+n_2}=0{,}454 \qquad \sigma^2=p(1-p)=0{,}248$

$\hat{u}=\frac{\hat{p}_1-\hat{p}_2}{\sqrt{\frac{\sigma^2}{n_1}+\frac{\sigma^2}{n_2}}}=-0{,}186 \qquad H_1 \dots p_1\neq p_2 \qquad |\hat{u}|>u_{1-\frac{\alpha}{2}}$

$u_{1-\frac{\alpha}{2}}=u_{1-0{,}025}=u_{0{,}975}=1{,}96 \qquad 0{,}186>1{,}96$ ✗

$\downarrow$

prihvaćamo H_0

TEST PRILAGODBE RAZDIOBAMA (χ^2-test)

ŠI 2010. 6) Bacamo 3 kocke 100 puta i bilježimo broj šestica. Uz nivo značajnosti 5% provjeri hipotezu o ispravnosti kocaka.

0	1	2	3	→ broj šestica
61	29	8	2	→ broj bacanja

x_j	n_j	p_j	np_j	$(n_j-np_j)^2/np_j$
0	61	$\left(\frac{5}{6}\right)^3$	57,87	0,169
1	29	$\binom{3}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$	34,72	0,942
2	8 } 10	$\binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)$	6,94 } 7,4	0,913
3	2	$\left(\frac{1}{6}\right)^3$	0,46	
Σ	100	1	100	

↳ teorijski br. ishoda

→ spajamo razrede međusobno da imaju više od 5 pokušaja
⇓
poboljšamo test

$s.s. = m - r - 1$

stupanj slobode

m → broj razreda nakon spajanja ($m=3$)

r → br. param. koje smo trebali izračunati iz uzorka
$r=0$ → gledamo $B(3,\frac{1}{6})$

$s.s. = 3 - 0 - 1 = 2$

$$\chi^2_{2,1-\alpha} = \chi^2_{2,0.95} = 5.991$$

$\chi^2_q < 5.991$ → prihvaćamo hipotezu (kocke su ispravne)

Z1 2007. c) Tijekom 1 god. bilježili smo br. krivo spojenih poziva i dobili smo sljedeće podatke. Provjerite da li se dobiveni podaci ravnaju po Pois. razdiobi uz $\alpha = 0{,}02$

x_j	n_j	p_j	np_j	$(n_j - np_j)^2 / np_j$
0	93	0,2556	93,31	0,031
1	140	0,3486	127,75	1,277
2	73	0,2377	86,777	2,187
3	31	0,108	39,45	1,81
4	16	0,037	13,45	0,483
5	7	0,01	3,67	6,008
6	2	0,0023	0,83	
7	0	0,000445	0,163	
8	1	0,000076	0,028	
Σ	365	≈ 1	≈ 365	$\chi_q^2 = 11{,}796$

$\downarrow$ br. poziva $\quad \downarrow$ br. dana $= n$

(uz np_j za 5–8: 4,691)

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$E(x) = \lambda = \bar{x}$$

$$\bar{x} = \frac{\sum x_j n_j}{n} = 1{,}364$$

→ spajamo razrede (krećemo od najmanjeg i spajamo sa susjedom)

$$ss = 6 - 1 - 1 = 4$$

$\chi^2_{4,\,0.98} = 11{,}668 < \chi^2_q$ → odbacujemo H0

ne ravnaju se po Pois.

Z1 2011. b) Ravnaju li se po eksp. $\alpha = 5\%$

$\bar{x}_j$	$[a,b]$	n_j	p_j	np_j	$(n_j - np_j)^2 / np_j$
12.5	0 – 25	87	0,613	91,99	0,271
37.5	25 – 50	46	0,237	35,576	3,054
62.5	50 – 75	14 (17)	0,092	13,76 (19,08)	0,227
87.5	75 – 100	3	0,035	5,32	
	Σ	150	1		$\chi^2_q = 3{,}552$

$$P(a<x<b) = F(b) - F(a) = 1 - e^{-\lambda b} - 1 + e^{-\lambda a} = e^{-\lambda a} - e^{-\lambda b}$$

$$p_j = e^{-\lambda a} - e^{-\lambda b}$$

$$E(x) = \frac{1}{\lambda} \rightarrow \lambda = \frac{1}{E(x)} = \frac{1}{\bar{x}} = \frac{1}{26{,}33} = 0{,}038$$

→ zadnji int. sami proširimo i izračunamo sredinu

$$s.s. = 3 - 1 - 1 = 1$$

$\chi_{1,\,0.95} = 3{,}841 > \chi^2_q$ → prihvaćamo H0, tj. ravna se po eksp. razdiobi

// PONAVLJANJE

KPA 3) Igrač starta igru s 0 žetona. Dobiva 2 žetona s $p=\frac{1}{6}$ i 1 žeton s $p=\frac{2}{3}$. Kolika je vjerojatnost da nakon 100 partija ima više od 90 žetona?

→ Ponavljanje pokusa - normalna razdioba

$$X_i \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{pmatrix} \rightarrow \text{razdioba u 1 igri}$$

CGT $$\frac{\sum_{i=1}^{n} X_i - nE(x)}{\sigma(x_i)\cdot\sqrt{n}} \sim N(0,1)$$

$$E(x_i) = 0\cdot\frac{1}{6} + \frac{2}{3} + \frac{2}{6} = 1$$

$$D(x_i) = \frac{2}{3} + 4\cdot\frac{1}{6} - 1 = \frac{1}{3} \rightarrow \sigma(x_i) = \frac{1}{\sqrt{3}}$$

$$P\left(\sum_{i=1}^{100} X_i > 90\right) = P\left(\frac{\sum X_i - 100\cdot 1}{\frac{1}{\sqrt{3}}\cdot\sqrt{100}} > \frac{90-100}{\frac{10}{\sqrt{3}}}\right) = P(x^* > -\sqrt{3}) =$$

$$= \frac{1}{2} - \frac{1}{2}\Phi^*(-\sqrt{3}) = \frac{1}{2} + \frac{1}{2}\Phi^*(\sqrt{3}) = 0,958$$

ZI 2007. 2) Dokažite pravilo 3σ za sl. var. X

$$P(a-3\sigma < x < a+3\sigma) > \frac{8}{9}$$

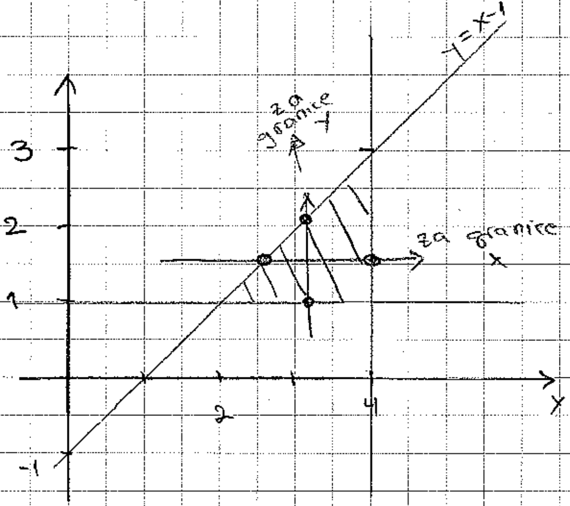
Č.n. $$P(|X-E(x)| > \varepsilon) < \frac{D(x)}{\varepsilon^2}$$

$$P(-3\sigma < X-a < 3\sigma) = P(|x-a| < 3\sigma) > 1 - \frac{D(x)}{\varepsilon^2}$$

$$P(|x-a| < 3\sigma) > 1 - \frac{\sigma^2}{9\sigma^2} = 1 - \frac{1}{9} = \frac{8}{9} \checkmark$$

Zad) Sl. vektor ima $f(x,y) = C(6-x-y)$ na $\{y>1, x<4, x-y-1>0\}$ $y<x-1$

Odredi sve



a) $c=?$

$$\int_2^4 dx \int_1^{x-1} c(6-x-y)\,dy = \dots = 2c = 1 \rightarrow \boxed{c=\frac{1}{2}}$$

b) marginalne

$$f_X(x) = \int_1^{x-1} f(x,y)\,dy = \dots \qquad x \in [2,4]$$

granice su funkcije

$$f_Y(y) = \int_{y+1}^{4} f(x,y)\,dx = \dots \qquad y \in [1,3]$$

granice konst $\quad E(X) = \int_2^4 x \cdot f_X(x)\,dx = \dots$

X i Y nisu nezavisne jer $f(x,y) \neq f_X(x) \cdot f_Y(y)$

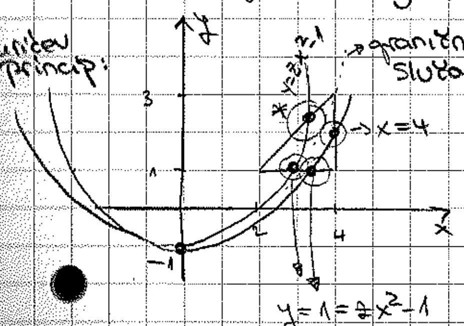
c) $Z = \frac{Y+1}{X^2}$ $\qquad z \in \left[\frac{1}{8}, \frac{1}{2}\right]$ → max y, a min x ! T(2,3) ne može jer to nije u području

za min Y, a max X koji su u području !

$y = zx^2 - 1$ parabola $\qquad \left|\frac{\partial y}{\partial z}\right| = x^2$

$$g(z) = \int \frac{1}{2}\left[6 - x - (zx^2 - 1)\right] \cdot x^2\,dx$$

Burikev princip:



graniční slučaj $T(4,3) \rightarrow 3 = 16z - 1$

$z = \frac{1}{4}$

i) $z \in \left[\frac{1}{8}, \frac{1}{4}\right]$: $g(z) = \int_{\sqrt{\frac{2}{z}}}^{4} \frac{1}{2}\left[6-x-(zx^2-1)\right] x^2\,dx$

ii) $z \in \left[\frac{1}{4}, \frac{1}{2}\right] = \int_{\sqrt{\frac{2}{z}}}^{\frac{1}{z}} \frac{1}{2}\left[6-x-(zx^2-1)\right] x^2\,dx$

$y = 1 = zx^2 - 1$

$x = \sqrt{\frac{2}{z}}$

* $x - 1 = zx^2 - 1$

$x = \frac{1}{z}$

PZ 2007. 3)

$$f(x) = \begin{cases} \frac{x}{a^2} e^{-\frac{x}{a}} & , \quad x > 0 \\ 0 & , \quad x \leq 0 \end{cases}$$

Korištenjem najveće izglednosti odredi a i odredi njenu nepristranost

$$L(x_1, \dots, x_n; a) = f(x_1) \cdot \dots \cdot f(x_n) = \frac{x_1 \cdot \dots \cdot x_n}{a^{2n}} e^{-\frac{1}{a}(x_1 + \dots + x_n)}$$

$$\ln L = \ln(x_1 \cdot \dots \cdot x_n) - 2n \ln a - \frac{1}{a}(x_1 + \dots + x_n) \quad \Big/ \frac{\partial}{\partial a}$$

$$-\frac{2n}{a} + \frac{1}{a^2}(x_1 + \dots + x_n) = 0 \quad / \cdot a$$

$$\boxed{a = \frac{x_1 + \dots + x_n}{2n}} \qquad \boxed{a = \frac{\overline{X}}{2}}$$

nepristr.: $E\left(\frac{\overline{X}}{2}\right) = a$ (uvjet)

$$E\left(\frac{\overline{X}}{2}\right) = E\left(\frac{x_1 + \dots + x_n}{2n}\right) = \frac{1}{2\cancel{n}} \cdot \cancel{n} E(x) = \frac{1}{2} \cdot \int_0^\infty x \cdot f(x)\, dx =$$

$$= \frac{1}{2} \int_0^\infty x \frac{x}{a^2} e^{-\frac{x}{a}} dx = \dots = \frac{1}{2} \cdot 2a = a \quad \checkmark$$

↳ procjena je nepristrana

Zad) Blagonakloni prof. je obećao da prosječna ocjena neće biti < 3.5. Studenti su dobili: 2, 4, 4, 3, 5, 1, 5, 2, 3, 5. Uz nivo značajnosti 10 % odredi je li profesor lagao.

$\alpha = 10\% = 0{,}1$ $\quad n = 10$

uvijek mora biti =

$H_0 \ldots \; a = 3{,}5$

ako je prof lagao $H_1 \ldots \; a < 3{,}5$

(nepoznata disperzija)

$\bar{x} = 3.4$

$s^2 = 2.04$

$\hat{t} = \dfrac{\bar{x} - a_0}{\frac{s}{\sqrt{n}}} = \dfrac{3.4 - 3.5}{\frac{\sqrt{2.04}}{10}}$

$\hat{t} = -0{,}221$

$-t_{n-1, 1-\alpha} = -t_{9, 1-0.1} = -t_{1-\frac{0.2}{2}} = -1{,}383$

stupnjevi slobode

jer je u tablici $t_{1-\frac{\alpha}{2}}$, gledamo za $\alpha = 0.2$

$-1{,}383 < -0{,}221$

$\times$

→ prihvaćamo H_0 → profesor nije prekršio obećanje

12. zad. 26) U prvih 800 decimala broja π

x_j	n_j	p_j	np_j	$(n_j - np_j)^2/np_j$
0	74	1/10	80	0.45
1	92	1/10	80	1.8
2	83	1/10	.	0.1125
3	79	.	.	.
4	80	.	.	.
5	73	.	.	.
6	77	.	.	.
7	75	.	.	.
8	76	.	.	.
9	91	1/10	80	1,5152
Σ	800	1	800	$\chi^2_q = 5,152$

Provjeri da li je pojava svake znam. jednako vjerojatna (da li se ravnaju po jednolikoj razdiobi) uz $\alpha = 0.1$.

stupnji slobode: $s.s. = m - r - 1 = 10 - 0 - 1 = 9$

$$\chi^2_{9, 0.9} = 14,684$$

$\chi^2_q < 14,684 \rightarrow$ prihvaćamo H_0

$\downarrow$

ravna se po jednolikoj