1.

$A = \begin{bmatrix} 3 & 5 \\ 6 & 2 \end{bmatrix}$ $b = (3, 3)^T$

$\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$

$A^{-1} = ?$

$Ax = b$

$A^{-1} = \frac{1}{\det A} \begin{bmatrix} 2 & -6 \\ -5 & 3 \end{bmatrix}^T$

$\det A = 6 - 30 = -24 = 4$

$4^{-1} \cdot 4 = 1$

$4 \cdot 2 = 1$

$= \frac{1}{4} \begin{bmatrix} 2 & -5 \\ -6 & 3 \end{bmatrix}$

$= 2 \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 4 & 4 \\ 2 & 6 \end{bmatrix}$ //

$Ax = b \quad / \ A^{-1}$

$x = A^{-1} b = \begin{bmatrix} 4 & 4 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 24 \\ 24 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ //

(2.)

$$A = \begin{bmatrix} a-c & a+2b \\ 3a-b & b+c \end{bmatrix}$$

$a, b, c \in \mathbb{R}$

24 = 4

a) 1) $\alpha A \in X$

$$\alpha \begin{bmatrix} a-c & a+2b \\ 3a-b & b+c \end{bmatrix} = \begin{bmatrix} \alpha a - \alpha c & \alpha a + \alpha 2b \\ \alpha 3a - \alpha b & \alpha b + \alpha c \end{bmatrix} =$$

$$= \left/ \begin{matrix} \alpha a = A \\ \alpha c = C \\ \alpha b = B \end{matrix} \right/ = \begin{bmatrix} A-C & A+2B \\ 3A-B & B+C \end{bmatrix} \in X$$

2) $A_1 + A_2 \in X$

$$A_1 + A_2 = \begin{bmatrix} a_1 - c_1 & a_1 + 2b_1 \\ 3a_1 - b_1 & b_1 + c_1 \end{bmatrix} + \begin{bmatrix} a_2 - c_2 & a_2 + 2b_2 \\ 3a_2 - b_2 & b_2 + c_2 \end{bmatrix} =$$

$$= \begin{bmatrix} (a_1+a_2) - (c_1+c_2) & (a_1+a_2) + 2(b_1+b_2) \\ 3(a_1+a_2) - (b_1+b_2) & (b_1+b_2) + (c_1+c_2) \end{bmatrix} = \left/ \begin{matrix} a_1+a_2 = a \\ b_1+b_2 = b \\ c_1+c_2 = c \end{matrix} \right/$$

$$= \begin{bmatrix} a-c & a+2b \\ 3a-b & b+c \end{bmatrix} \in X$$

b)
$$A = \underbrace{\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}}_{e_1} a + \underbrace{\begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}}_{e_2} b + \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{e_3} c$$

e_1, e_2, e_3 čine bazu jer:

1) razapinju X

2) su linearno nezavisni

$\Rightarrow a e_1 + b e_2 + c e_3 = 0$ $\Rightarrow$ e_1, e_2, e_3 su lin. nezav.

$a - c = 0 \Rightarrow a = c$

$a + 2b = 0$

$3a - b = 0 \qquad 3c - b = 0$

$b + c = 0 \qquad c + b = 0 \qquad \Rightarrow c = 0, \; a = 0 \Rightarrow b = 0$

$\dim X = 3$

(3.) $F : M_{2,2} \to M_{2,2}$

$$F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b & 2b+d \\ a-b-d & a+b+c \end{bmatrix}$$

---

a) $$F\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad F\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \quad F\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

b) $$\det(A) = (-1)\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & -1 & -1 \end{vmatrix} = (-1)\left(\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}\right)$$

$$= (-1)(-2+1+1) = 0$$

$\Downarrow$

Operator nema inverz.

④ $A: \mathbb{R}^2 \to \mathbb{R}^2$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \qquad e_1 = (1, -1)^T \quad f_1 = (1, 2)^T$$
$$e_2 = (2, 1)^T \quad f_2 = (4, -1)^T$$

$$f_1 = -e_1 + e_2 \qquad T = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \qquad \det T = -1 - 2 = -3$$
$$f_2 = 2e_1 + e_2$$

$$A' = T^{-1} A T \qquad T^{-1} = \frac{1}{\det T} \begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}^T = \frac{-1}{3} \begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix}$$

$$= \left(-\frac{1}{3}\right) \begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \left(-\frac{1}{3}\right) \begin{bmatrix} -1 & -6 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} = \left(-\frac{1}{3}\right) \begin{bmatrix} -5 & -8 \\ -4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{3} & \frac{8}{3} \\ \frac{4}{3} & \frac{1}{3} \end{bmatrix}$$

5. $A, B : \mathbb{R}^2 \to \mathbb{R}^3$

$A(a, b) = (a+b,\ a-2b,\ a+2b)$

$B(a, b) = A(b, 2a)$

$A(1, 0) = (1, 1, 1)$ $\qquad$ $B(1, 0) = (2, -4, 4)$

$A(0, 1) = (1, -2, 2)$ $\qquad$ $B(0, 1) = (1, 1, 1)$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ -4 & 1 \\ 4 & 1 \end{bmatrix}$$

$$\| B - A \|_1 = \left\| \begin{bmatrix} 1 & 0 \\ -5 & 3 \\ 3 & -1 \end{bmatrix} \right\|_1 = \max\{9, 4\} = 9$$

7) a) Neumannov red konvergira ako i samo ako $r(A) < 1$.

b)

$$A = \begin{bmatrix} \lambda-2 & 1 \\ 0 & 3-\lambda \end{bmatrix}$$

$$|\mu I - A| = \begin{vmatrix} \mu-\lambda+2 & -1 \\ 0 & \mu-3+\lambda \end{vmatrix} = (\mu-\lambda+2)(\mu-3+\lambda) = 0$$

$\mu_1 = \lambda - 2$

$\mu_2 = -\lambda + 3$

$r(A) < 1$

$r(A) = \max\{|\mu_1|, |\mu_2|\}$

1° $\mu_1 \geq \mu_2 \rightarrow \lambda - 2 \geq -\lambda + 3$

$\lambda \geq \frac{5}{2}$

$-1 < \mu_1 < 1$

$-1 < \lambda - 2 < 1$

$\lambda > 1$ i $\lambda < 3$

$\lambda \in [\frac{5}{2}, 3\rangle$

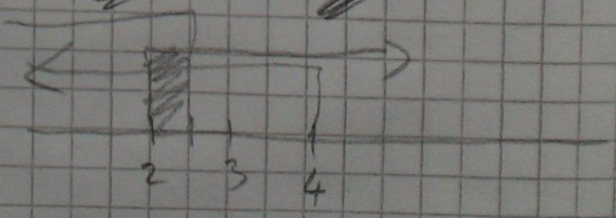
2° $\mu_2 > \mu_1 \rightarrow -\lambda + 3 > \lambda - 2$

$\lambda < \frac{5}{2}$

$-1 < \mu_2 < 1$

$-1 < -\lambda + 3 < 1$

$\lambda < 4$ i $\lambda > 2$



$\lambda \in \langle 2, \frac{5}{2} \rangle$

$\rightarrow$ unija

$\lambda \in \langle 2, 3 \rangle$

8. a) T: $(Ax|y) = (x|A^T y)$

$$(Ax)^T \cdot y = x^T A^T y = (x^T) \cdot (A^T y) = (x|A^T y)$$

b)

knjiga - 135. str.

c) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\|A\|_2 = ?$

$$\|A\|_2 = \sqrt{r(A^T A)}$$

$$A^T A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$|\lambda I - A^T A| = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda-2)(\lambda-1) - 1 = 0$$

$$\lambda^2 - \lambda - 2\lambda + 2 - 1 = 0$$

$$\lambda^2 - 3\lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

$$\lambda_1 = \frac{3+\sqrt{5}}{2}$$

$$\lambda_2 = \frac{3-\sqrt{5}}{2}$$

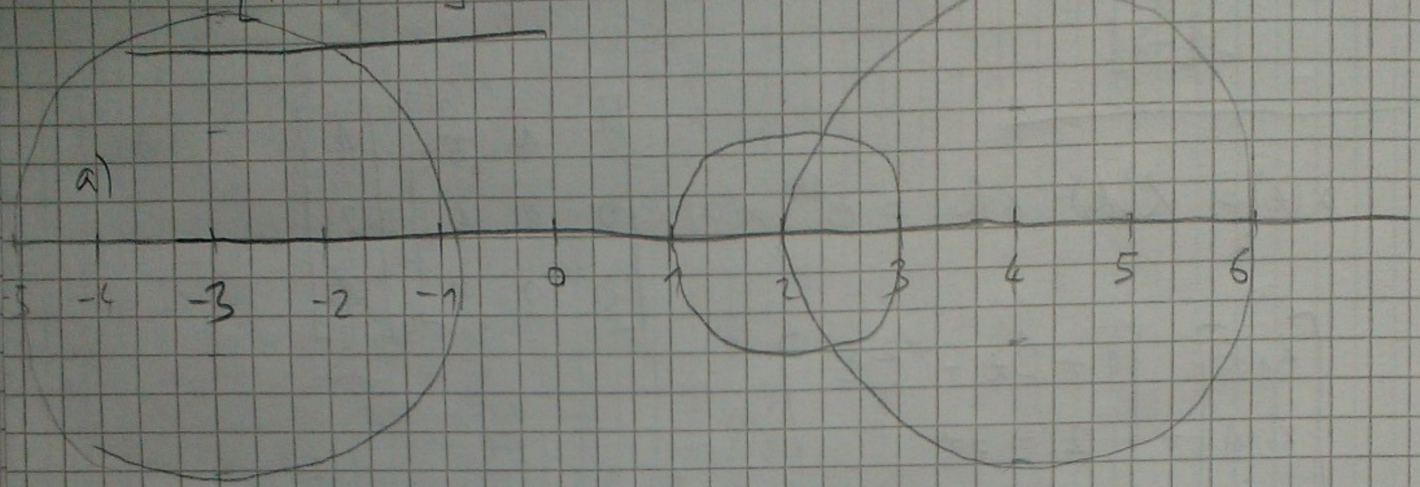
$$r(A^T A) = \frac{3+\sqrt{5}}{2}$$

$$\|A\|_2 = \sqrt{\frac{3+\sqrt{5}}{2}}$$

9.

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 2 & 1 \\ 1 & 1 & -3 \end{bmatrix}$$

a)



Matrica je regularna ako joj je determinanta različita od 0. Budući da niti jedna vlastita vrijednost matrice A ne može biti 0, determinanta matrice različita je od 0. Zbog toga je A regularna matrica.

b) $T = -D^{-1}(L+U)$ $\quad A = L + D + U$

↳ matrica s dijagonalnim elem. matrice A (ostalo 0)

↳ donji trokut matrica A bez dijagonale (—||—)

↳ gornji —||—

$b' = D^{-1} b$

$x^{(0)} = [\dots]$

$x^{(1)} = b'$

$\vdots$

$x^{(k)} = (T^{(k-1)} + \dots + T + I)\, b'$

teorem: metoda konvergira ako i samo ako je $\rho(T) < 1$

c) $x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$T = -D^{-1}(L+U) \qquad D^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$$

$$= -\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & -\frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

1-norma

$\|D^{-1}(L+U)\|_1 =$

$= \max\left\{\frac{1}{3}, \frac{7}{12}, \frac{3}{4}\right\}$

$= \frac{3}{4}$

$r(A) \leq \|A\|_1$

$\Downarrow$

$r(D^{-1}(L+U)) < \frac{3}{4} < 1$

$\Downarrow$

konvergira

$$b' = D^{-1}b = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ -\frac{1}{3} \end{bmatrix}$$

$$x^{(1)} = I b' = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ -\frac{1}{3} \end{bmatrix}$$

$$x^{(2)} = (T+I)b' = \begin{bmatrix} 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & -\frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ -\frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{24} \\ \frac{2}{3} \\ -\frac{1}{12} \end{bmatrix}$$

(10.)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$U \to A \cdot A^T$

$V \to A^T \cdot A$

---

a) $A = \sigma_1 u^1 v^{1T} + \dots$

$$A \cdot A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$\lambda_1 = 4$ $\quad \sigma_1 = 2$

$\lambda_{2,3} = 2$ $\quad \sigma_2 = \sqrt{2}$

$\sigma_3 = \sqrt{2}$

1) $(\lambda_1 I - A \cdot A^T) v_1 = 0$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2) $(\lambda_2 I - A \cdot A^T) v = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow x_3 = 0$$

$$v = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A = U \cdot D \cdot V^T \quad / V^*$$

$$AV = U \cdot D$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1^1 & v_1^2 & v_1^3 \\ v_2^1 & v_2^2 & v_2^3 \\ v_3^1 & v_3^2 & v_3^3 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 2 & 0 & 0 \end{bmatrix}$$

1) $v_1^1 + v_2^1 = 0$
$v_1^1 - v_2^1 = 0$
$v_3^1 = 2$
$\Rightarrow v^1 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \quad \hat{v}^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

2) $v_1^2 + v_2^2 = \sqrt{2}$
$v_1^2 - v_2^2 = 0$
$2v_3^2 = 0$
$\Rightarrow v^2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \quad \hat{v}^2 = \frac{v^2}{\|v^2\|_2} = \frac{v^2}{1} = v^2$

3) $v_1^3 + v_2^3 = 0$
$v_1^3 - v_2^3 = \sqrt{2}$
$v_3^3 = 0$
$v^3 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = \hat{v}^3$

$$A = 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} + \sqrt{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} + \sqrt{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

b)

$$A^+ = V D^+ U^T = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^T$$ 61.

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

c) $\|A\|_F = \sqrt{1^2 + 1^2 + 1^2 + (-1)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

11. a) $e^A = I + A + \frac{A^2}{2!} + \dots + \frac{A^k}{k!}$

Red konvergira za svaku kvadratnu matricu A.

b)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A^3 = 0$

$$e^{At} = I + At + \frac{A^2 t^2}{2}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

c) $\dot{x}(t) = A\,x(t)$ 63.

$$x(0) = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$x(t) = X(t) \cdot x(0) = e^{At}\,x(0) = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 1 + 2t - \frac{3}{2}t^2 \\ 2 - 3t \\ -3 \end{bmatrix}$$