VJEŽBE IZ MEHANIKE I

I. DIO (UVOD, UVJETI RAVNOTEŽE , TRENJE)

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GRAFIČKO I ANALITIČKO ZBRAJANJE SILA

Sile su vektori pa za njih vrijede pravila o zbrajanju i množenju vektora.

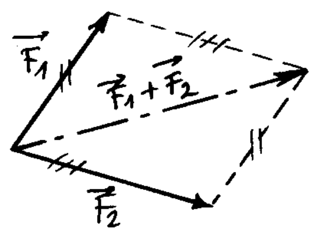
GRAFIČKO ZBRAJANJE

Vektori sila grafički se predočavaju strelicama čija duljina u nekom mjerilu odgovara njihovoj veličini (iznosu ili modulu). Uz grafički prikaz koji nazivamo planom sila treba uvijek definirati mjerilo:

$$1\,\text{cm} \triangleq \ldots\ N$$

Ovo se čita: 1 cm duljine vektora sile odgovara veličini od .... N. Grafički se sile mogu zbrajati s pomoću pravila vektorskog paralelograma i s pomoću pravila vektorskog poligona.

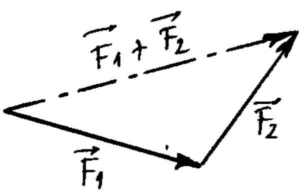
Pravilo vektorskog paralelograma



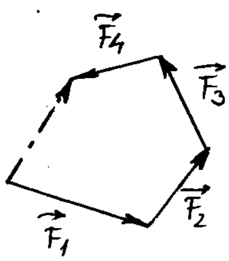
$1\,\text{cm} \triangleq \ldots\ N$

Pravilo vektorskog poligona

$1\,\text{cm} \triangleq \ldots\ N$



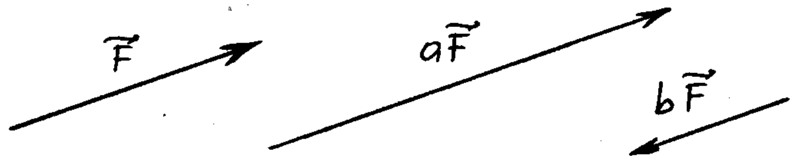
$\vec{F}_R = \sum_{i=1}^{4} \vec{F}_i$



$\vec{F}_R = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = \sum_{i=1}^{4} \vec{F}_i$

MNOŽENJE VEKTORA SKALAROM

Skalar je veličina čiju vrijednost određuje samo njezin iznos. Skalar može imati pozitivan ili negativan predznak (npr. temperatura).

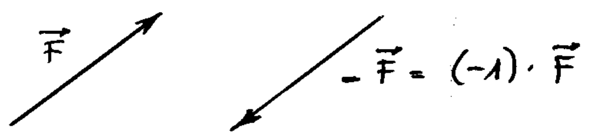


Ako su a i b skalari, pri čemu je

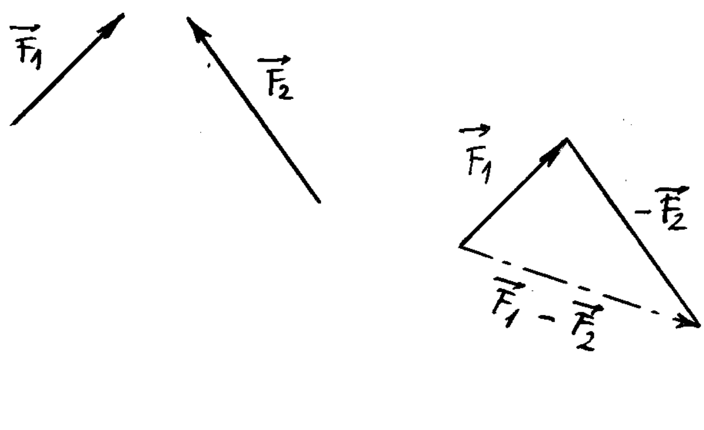
$$a > 0 \;, \quad |a| > 1 \;;$$
$$b < 0 \;, \quad |b| < 1 \;,$$

$a\vec{F}$ je vektor čiji je iznos a puta veći od iznosa $\vec{F}$, leži na istom pravcu i ima isti smisao kao $\vec{F}$, dok je $b\vec{F}$ vektor čiji je iznos $|b|$ puta manji od iznosa $\vec{F}$, leži na istom pravcu kao $\vec{F}$, ali ima suprotan smisao od $\vec{F}$.

ODUZIMANJE VEKTORA



$$\vec{F}_1 - \vec{F}_2 = \vec{F}_1 + (-\vec{F}_2)$$

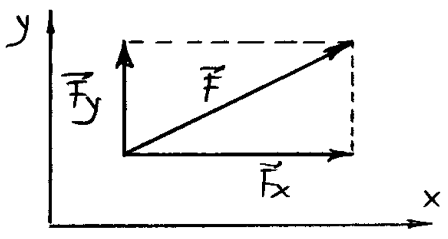


ANALITIČKO ZBRAJANJE VEKTORA SILA

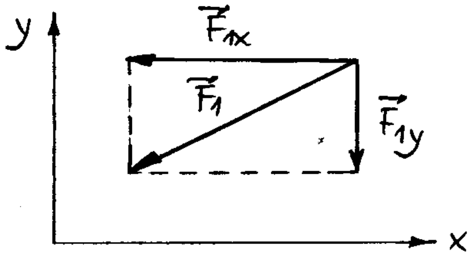
Analitičko računanje s vektorima zahtijeva da se vektori zapišu u takvom obliku koji će s jedne strane potpuno definirati sve tri značajke vektora (veličinu, pravac i smisao), a s druge strane omogućiti analitičko izvođenje računskih operacija (zbrajanje, oduzimanje, skalarno i vektorsko množenje).

ANALITIČKI ZAPIS VEKTORA SILE KOJA LEŽI U RAVNINI xy

Silu $\vec{F}$ možemo prikazati kao zbroj dviju sila $\vec{F}_x$ i $\vec{F}_y$ koje su paralelne osima koordinatnog sustava x i y. Ove sile nazivamo komponentama sile $\vec{F}$.

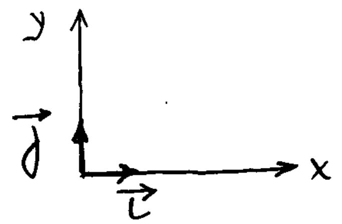


$\vec{F} = \vec{F}_x + \vec{F}_y$



$\vec{F}_1 = \vec{F}_{1x} + \vec{F}_{1y}$

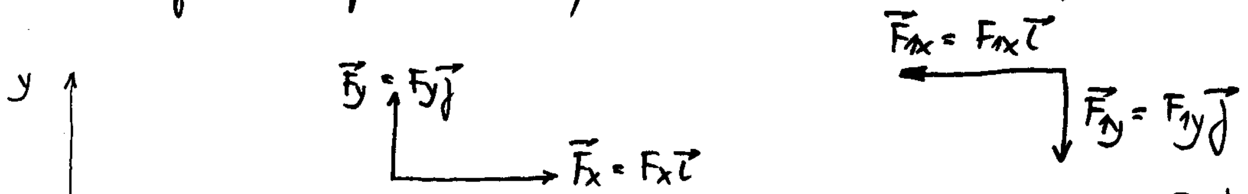
Ako su iznosi komponenata $\vec{F}_{1x}$ i $\vec{F}_{1y}$ sile $\vec{F}_1$ jednaki iznosima komponenata $\vec{F}_x$ i $\vec{F}_y$ sile $\vec{F}$, lako se možemo uvjeriti da su iznosi sila $\vec{F}$ i $\vec{F}_1$ jednaki i da te sile leže na istom pravcu, no očito je da im je smisao suprotan. Ako bi uz jednake iznose komponenata $\vec{F}$ i $\vec{F}_1$ komponente paralelne osi x imale isti smisao, a one paralelne osi y suprotan smisao, sile $\vec{F}$ i $\vec{F}_1$ imale bi samo isti iznos, ali bi ležale na različitim pravcima. Jednako tako bi uz jednaki iznos ležale na različitim pravcima ako bi komponente paralelne osi y imale isti smisao, a one paralelne osi x suprotan smisao. Očito je da smisao komponenata određuje pravac i smisao vektora sile. Zbog toga se smisao komponenata mora jasno definirati. Najprije na osima koordinatnog sustava uvedemo jedinične vektore ili ortove:



$|\vec{i}| = 1$

$|\vec{j}| = 1$

Komponente $\vec{F}_x$ i $\vec{F}_y$ zatim prikazujemo kao umnožak skalara F_x i F_y s odgovarajućim jediničnim vektorima $\vec{i}$ i $\vec{j}$. Pri tome F_x i F_y mogu imati pozitivan ili negativan predznak, a iznos im se mjeri u [N].



$F_x > 0$, $|F_x|$ [N]

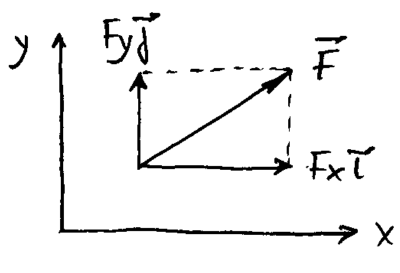
$F_y > 0$, $|F_y|$ [N]

$F_{1x} < 0$, $|F_{1x}|$ [N]

$F_{1y} < 0$, $|F_{1y}|$ [N]

ANALITIČKI ZAPIS VEKTORA SILE $\vec{F}$ je konačno prikaz vektora $\vec{F}$ kao zbroja komponenata $F_x\vec{i}$ i $F_y\vec{j}$:

$$\vec{F} = F_x\vec{i} + F_y\vec{j} \quad [N]$$



Ovim zapisom jednoznačno je određena veličina (iznos) sile $\vec{F}$:

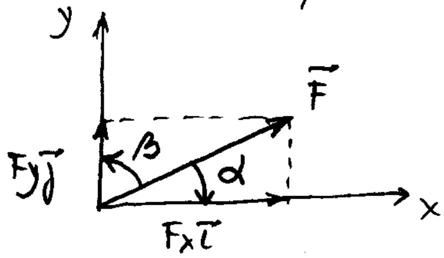
$$|\vec{F}| = \sqrt{F_x^2 + F_y^2}$$

Ovdje smo uvažili da je $(|F_x|)^2 = F_x^2$, $(|F_y|)^2 = F_y^2$.

Također je jednoznačno određen pravac i smisao na tom pravcu vektora $\vec{F}$. Kosinusi kutova α i β što ih vektor $\vec{F}$ zatvara s osima x i y su određeni relacijama:

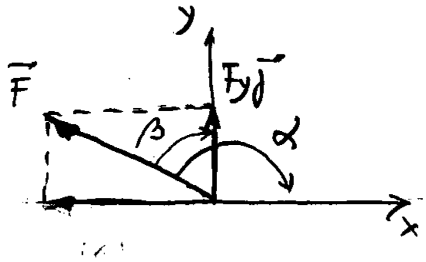
$$\cos\alpha = \cos\sphericalangle(\vec{F}, x) = \frac{F_x}{|\vec{F}|},$$

$$\cos\beta = \cos\sphericalangle(\vec{F}, y) = \frac{F_y}{|\vec{F}|}.$$



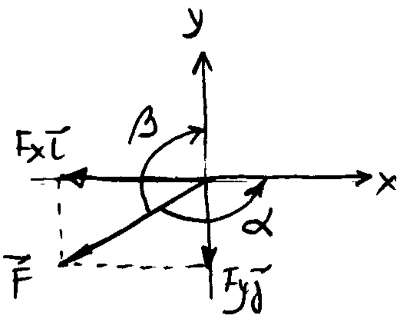
$F_x > 0$, $\cos\alpha = \frac{F_x}{|\vec{F}|} > 0$, $0 < \alpha < 90°$;

$F_y > 0$, $\cos\beta = \frac{F_y}{|\vec{F}|} > 0$, $0 < \beta < 90°$



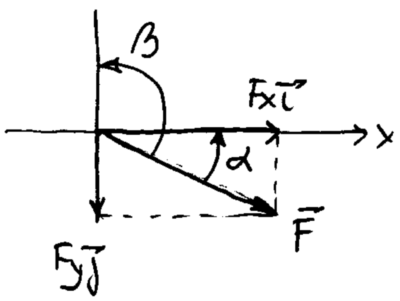
$F_x < 0$, $\cos\alpha = \frac{F_x}{|\vec{F}|} < 0$, $90° < \alpha < 180°$

$F_y > 0$, $\cos\beta = \frac{F_y}{|\vec{F}|} > 0$, $0 < \beta < 90°$



$F_x < 0$, $\cos\alpha = \frac{F_x}{|\vec{F}|} < 0$, $90° < \alpha < 180°$

$F_y < 0$, $\cos\beta = \frac{F_y}{|\vec{F}|} < 0$, $90° < \alpha < 180°$

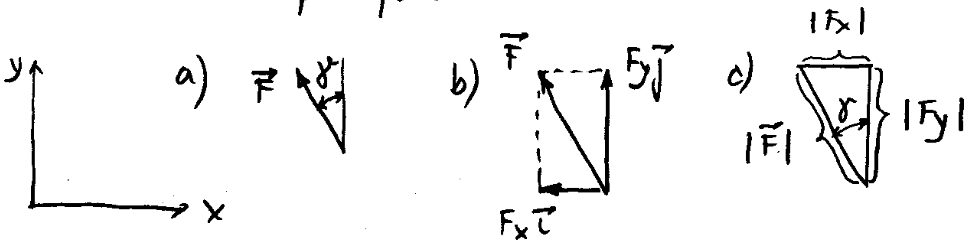


$F_x > 0$, $\cos\alpha = \frac{F_x}{|\vec{F}|} > 0$, $0 < \alpha < 90°$

$F_y < 0$, $\cos\beta = \frac{F_y}{|\vec{F}|} < 0$, $90° < \beta < 180°$

ODREĐIVANJE KOMPONENATA

Praktično određivanje komponenata sile $\vec{F}$ koja djeluje u ravnini xy pokazat ćemo na dva jednostavna primjera.



Zadano: $|\vec{F}|$, γ (sl. a).

Iz sl. b slijedi $F_x < 0$, $F_y > 0$, a iz sl. c slijedi

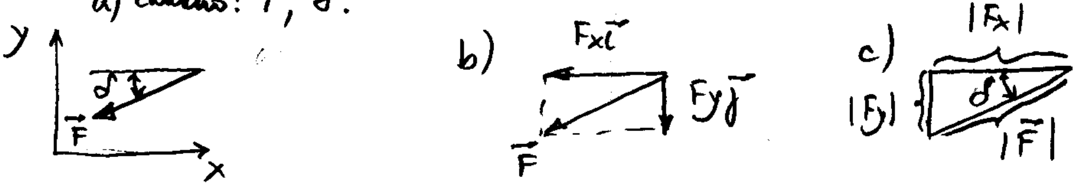
$$\frac{|F_x|}{|\vec{F}|} = \sin\gamma \,, \quad |F_x| = |\vec{F}| \sin\gamma$$

$$\frac{|F_y|}{|\vec{F}|} = \cos\gamma \,, \quad |F_y| = |\vec{F}| \cos\gamma$$

Analitički zapis $\vec{F}$ glasi

$$\vec{F} = -|\vec{F}| \sin\gamma \, \vec{i} + |\vec{F}| \cos\gamma \, \vec{j} \quad [N]$$

a) zadano: $\vec{F}$, δ.



Analogno prvom primjeru dobijemo

$$F_x < 0 \,, \quad F_y < 0 \,, \qquad |F_x| = |\vec{F}| \cos\delta \,, \quad |F_y| = |\vec{F}| \sin\delta$$

$$\vec{F} = -|\vec{F}| \cos\delta \, \vec{i} - |\vec{F}| \sin\delta \, \vec{j}$$

ANALITIČKO ZBRAJANJE $\vec{F}_1$ i $\vec{F}_2$ KOJE LEŽE U RAVNINI xy

Ako su poznati analitički zapisi $\vec{F}_1$ i $\vec{F}_2$ njihov zbroj bit će

$$\vec{F}_R = \vec{F}_1 + \vec{F}_2 = (F_{1x}\vec{i} + F_{1y}\vec{j}) + (F_{2x}\vec{i} + F_{2y}\vec{j}) =$$
$$= F_{1x}\vec{i} + F_{1y}\vec{j} + F_{2x}\vec{i} + F_{2y}\vec{j}$$

Za zbrajanje vektora vrijedi zakon komutacije

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

pa možemo pisati

$$\vec{F}_R = F_{1x}\vec{i} + F_{2x}\vec{i} + F_{1y}\vec{j} + F_{2y}\vec{j}$$

Primijenimo li zakon asocijacije s obzirom na skalare

$$m\vec{a} + n\vec{a} = (m+n)\vec{a}$$

dobit ćemo

$$\vec{F}_R = (F_{1x} + F_{2x})\vec{i} + (F_{1y} + F_{2y})\vec{j}$$

Ako definiramo komponente $\vec{F}_R$ kako slijedi

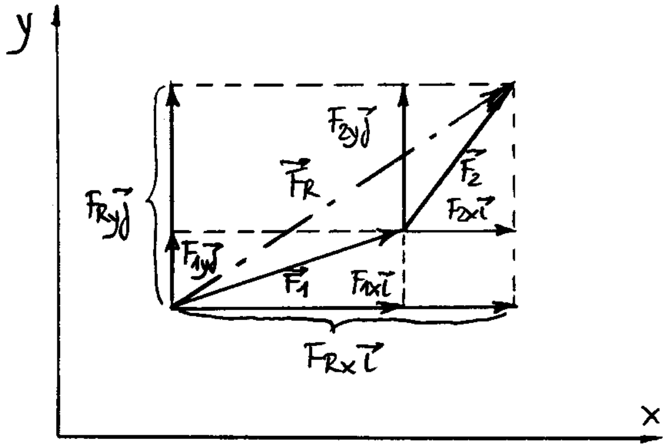
$$\vec{F}_{Rx} = (F_{1x} + F_{2x})\vec{i} = F_{Rx}\vec{i},$$
$$\vec{F}_{Ry} = (F_{1y} + F_{2y})\vec{j} = F_{Ry}\vec{j},$$

dobit ćemo konačno

$$F_{Rx} = F_{1x} + F_{2x}, \quad F_{Ry} = F_{1y} + F_{2y},$$

odnosno riječima: komponenta zbroja $\vec{F}_1$ i $\vec{F}_2$ paralelna osi x jednaka je zbroju x-komponenata tih sila, a komponenta zbroja $\vec{F}_1$ i $\vec{F}_2$ paralelna osi y jednaka je zbroju y-komponenata tih sila.

Ovako se može zbrojiti n sila, pa općenito vrijedi:
$$F_{Rx} = \sum_{i=1}^{n} F_{ix}, \quad F_{Ry} = \sum_{i=1}^{n} F_{iy}.$$



Iz prikaza grafičkog zbrajanja $\vec F_1$ i $\vec F_2$ po pravilu vektorskog poligona uočava se da je

$$F_{Rx}\vec i = F_{1x}\vec i + F_{2x}\vec i ,$$

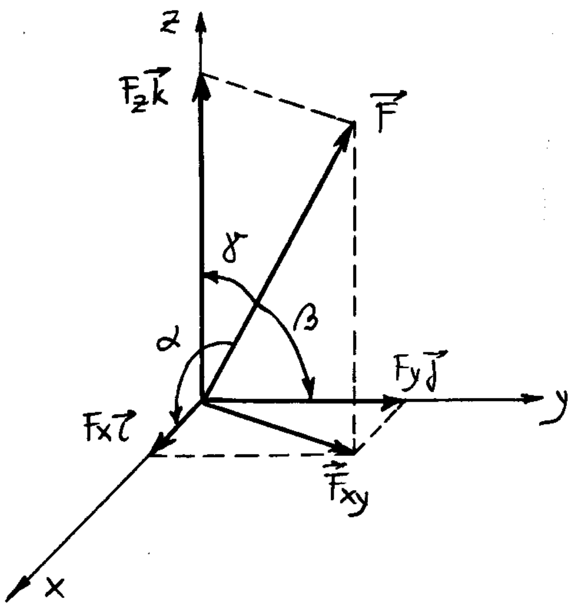
$$F_{Ry}\vec j = F_{1y}\vec j + F_{2y}\vec j ,$$

odnosno da vrijedi

$$F_{Rx} = F_{1x} + F_{2x} \quad , \qquad F_{Ry} = F_{1y} + F_{2y} .$$

Na kraju valja upozoriti da komponente F_{ix} i F_{iy} imaju odgovarajući predznak, što treba uvažiti pri zbrajanju.

ANALITIČKI ZAPIS VEKTORA SILE KOJA DJELUJE NA OPĆEM PRAVCU U PROSTORU



Silu $\vec F$ možemo najprije prikazati kao zbroj komponente $\vec F_{xy}$ koja leži u ravnini xy i komponente $\vec F_z$ paralelne osi z:

$$\vec F = \vec F_{xy} + \vec F_z$$

Zatim $\vec F_{xy}$ prikažemo kao zbroj komponente $\vec F_x$ paralelne osi x i komponente $\vec F_y$ paralelne osi y:

$$\vec F_{xy} = \vec F_x + \vec F_y$$

Konačno dobivamo

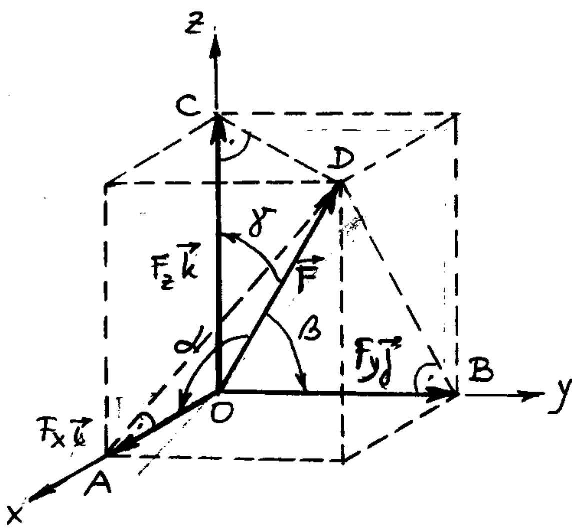
$$\vec{F} = \vec{F}_{xy} + \vec{F}_z = (\vec{F}_x + \vec{F}_y) + \vec{F}_z =$$
$$= \vec{F}_x + \vec{F}_y + \vec{F}_z$$

Ako uvedemo ort $\vec{k}$ na osi z i komponentu F_z analogno uvođenju F_x i F_y, možemo pisati

$$\vec{F}_z = F_z \vec{k} .$$

Slijedi analitički zapis sile $\vec{F}$ na općem pravcu u prostoru

$$\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k} \qquad [N] .$$



Pokazat ćemo da je iz ovog zapisa moguće odrediti iznos sile $\vec{F}$ te pravac i smisao njenog vektora.
Iz slike je uočljivo da vektor $\vec{F}$ leži na dijagonali kvadra čiji su bridovi komponente $\vec{F}_x$, $\vec{F}_y$ i $\vec{F}_z$.

Iznos $\vec{F}$ jednak je dakle duljini dijagonale tog kvadra:

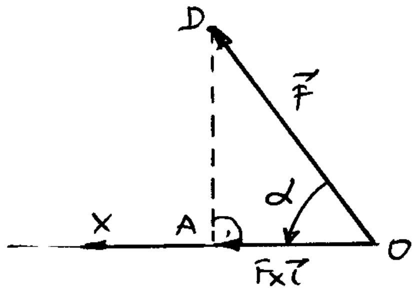
$$|\vec{F}| = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

U ovom izrazu nismo pisali $|F_x|$, $|F_y|$ i $|F_z|$ jer je $F_x^2 = (|F_x|)^2$, $F_y^2 = (|F_y|)^2$, $F_z^2 = (|F_z|)^2$.

Pravac i smisao vektora $\vec{F}$ određujemo kutovima α, β i γ koje vektor $\vec{F}$ zatvara s osima koordinatnog sustava:

$$\alpha = \sphericalangle(\vec{F}, x), \quad \beta = \sphericalangle(\vec{F}, y), \quad \gamma = \sphericalangle(\vec{F}, z).$$

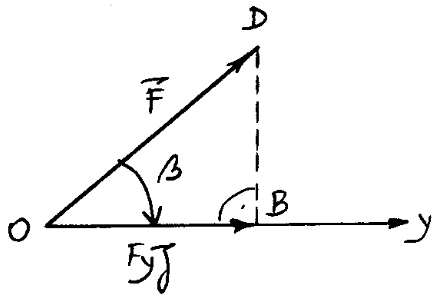
Ove kutove možemo odrediti iz pravokutnih trokuta ΔOAD, ΔOBD i ΔOCD. Valja uočiti da je plošna dijagonala kvadra AD okomita na os x, plošna dijagonala BD okomita na os y, a plošna dijagonala CD okomita na os z.



$$\cos\alpha = \frac{\overline{OA}}{\overline{OD}} = \frac{F_x}{|\vec{F}|},$$

$F_x > 0, \quad 0 < \alpha < 90^\circ$

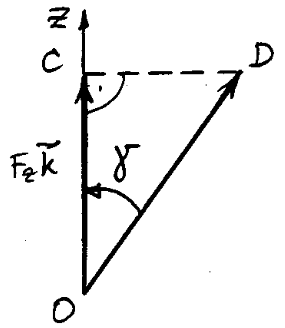
$F_x < 0, \quad 90^\circ < \alpha < 180^\circ$



$$\cos\beta = \frac{\overline{OB}}{\overline{OD}} = \frac{F_y}{|\vec{F}|},$$

$F_y > 0, \quad 0 < \beta < 90^\circ$

$F_y < 0, \quad 90^\circ < \beta < 180^\circ$



$$\cos\gamma = \frac{\overline{OC}}{\overline{OD}} = \frac{F_z}{|\vec{F}|}$$

$F_z > 0, \quad 0 < \gamma < 90^\circ$

$F_z < 0, \quad 90^\circ < \gamma < 180^\circ$

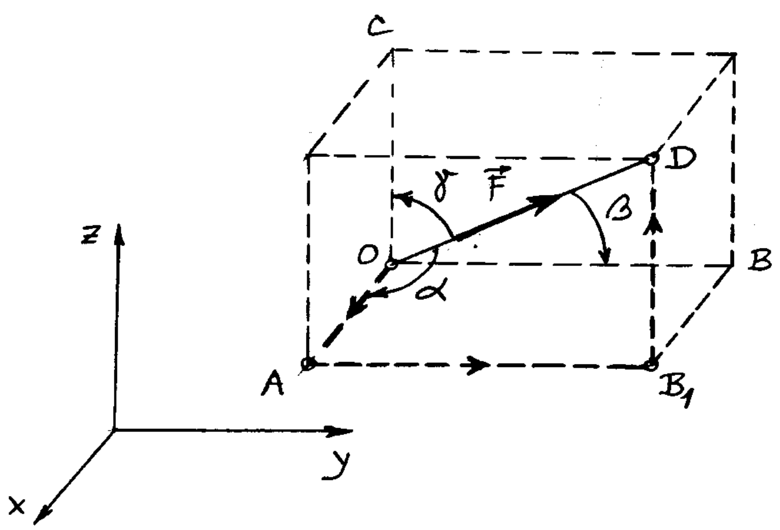
ODREĐIVANJE KOMPONENATA F_x, F_y I F_z

Neka su zadane točke O i D kroz koje prolazi pravac sile $\overline{F}$. Komponente te sile mogu se odrediti ako se izračunaju kosinusi kutova α, β i γ što ih $\overline{F}$ zatvara s osima x, y i z:

$$F_x = |\overline{F}| \cos\alpha, \quad F_y = |\overline{F}| \cos\beta, \quad F_z = |\overline{F}| \cos\gamma .$$

Za izračunavanje kosinusa kutova α, β i γ treba definirati kvadar kojem je prostorna dijagonala dužina $\overline{OD}$, a bridovi su mu paralelni s osima x, y i z. To ćemo učiniti tako da iz točke O u točku D „idemo" putovima $O \rightarrow A$ paralelno osi x, $A \rightarrow B_1$ paralelno osi y i $B_1 \rightarrow D$ paralelno osi z. Ovi putovi definiraju bridove kvadra paralelne osima x, y i z:

$$\overline{OA} = a, \quad \overline{AB_1} = b, \quad \overline{B_1D} = c .$$



Prostorna dijagonala kvadra iznosi

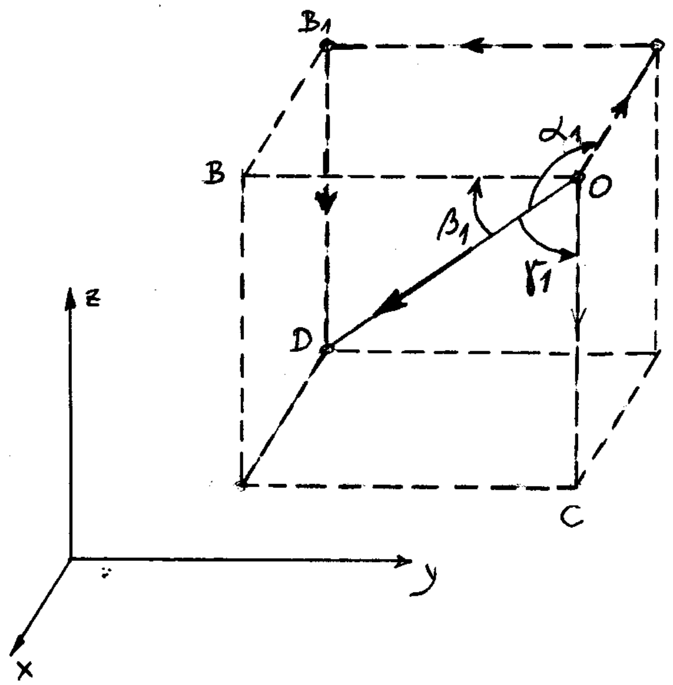
$$\overline{OD} = d = \sqrt{a^2 + b^2 + c^2}$$

Slijede kosinusi kutova α, β i γ:

$$\cos\alpha = \frac{a}{d}, \quad \cos\beta = \frac{b}{d}, \quad \cos\gamma = \frac{c}{d}.$$

Ako prostorna dijagonala zatvara kut manji od 90° s NEGATIVNIM smjerom koordinatne osi, označavamo taj kut indeksom 1:

$$\alpha_1, \ \beta_1, \ \gamma_1 .$$



$\overline{OA} = a$, $\overline{AB_1} = b$, $\overline{B_1D} = c$

$d = \sqrt{a^2 + b^2 + c^2}$

$\cos\alpha_1 = \frac{a}{d}$

$\cos\beta_1 = \frac{b}{d}$

$\cos\gamma_1 = \frac{c}{d}$

Slijede komponente sile $\overline{F}$:

$F_x = |\overline{F}| \cos\alpha$ ili $F_x = -|\overline{F}| \cos\alpha_1$,

$F_y = |\overline{F}| \cos\beta$ ili $F_y = -|\overline{F}| \cos\beta_1$,

$F_z = |\overline{F}| \cos\gamma$ ili $F_z = -|\overline{F}| \cos\gamma_1$.

ZBRAJANJE SILA $\vec{F}_1$ I $\vec{F}_2$ KOJE LEŽE NA OPĆIM PRAVCIMA U PROSTORU

Analogno analitičkom zbrajanju sila koje leže u ravnini xy možemo pisati:

$$\vec{F}_R = \vec{F}_1 + \vec{F}_2 = (F_{1x}\vec{i} + F_{1y}\vec{j} + F_{1z}\vec{k}) + (F_{2x}\vec{i} + F_{2y}\vec{j} + F_{2z}\vec{k}) =$$

$$= (F_{1x}\vec{i} + F_{2x}\vec{i} + F_{1y}\vec{j} + F_{2y}\vec{j} + F_{1z}\vec{k} + F_{2z}\vec{k} =$$

$$= (F_{1x} + F_{2x})\vec{i} + (F_{1y} + F_{2y})\vec{j} + (F_{1z} + F_{2z})\vec{k} =$$

$$= F_{Rx}\vec{i} + F_{Ry}\vec{j} + F_{Rz}\vec{k}$$

$$F_{Rx} = F_{1x} + F_{2x}, \quad F_{Ry} = F_{1y} + F_{2y}, \quad F_{Rz} = F_{1z} + F_{2z}$$

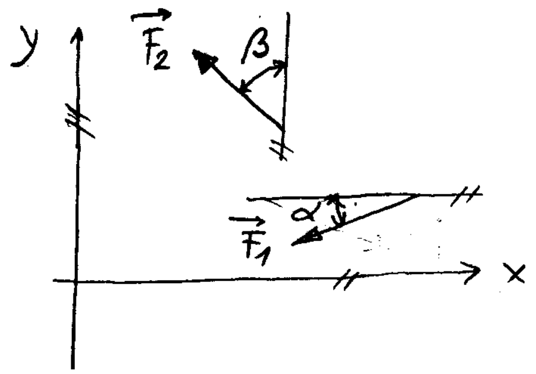
Općenito vrijedi:

$$\vec{F}_R = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_i + \ldots + \vec{F}_n = \sum_{i=1}^{n} \vec{F}_i$$

$$F_{Rx} = \sum_{i=1}^{n} F_{ix}, \quad F_{Ry} = \sum_{i=1}^{n} F_{iy}, \quad F_{Rz} = \sum_{i=1}^{n} F_{iz}$$

$$\vec{F}_R = F_{Rx}\vec{i} + F_{Ry}\vec{j} + F_{Rz}\vec{k}$$

Odrediti grafički i analitički rezultantu sila $\vec{F}_1$ i $\vec{F}_2$. Sile djeluju u ravnini xy, a smjerovi su im zadani slikom. Veličine (iznosi) sila su: $|\vec{F}_1| = 4\,N$ i $|\vec{F}_2| = 5\,N$. Zadano: $\alpha = 30°$, $\beta = 45°$.

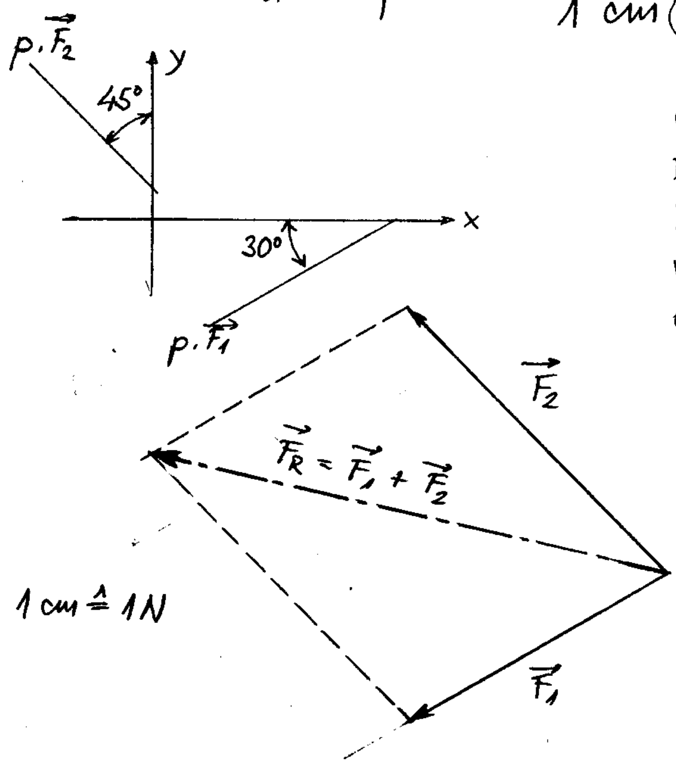


GRAFIČKI

a) ZBRAJANJE POMOĆU PRAVILA VEKTORSKOG PARALELOGRAMA

Vektore sila $\vec{F}_1$ i $\vec{F}_2$ predstavljamo grafički u mjerilu $1\,cm \triangleq 1\,N$

OVAJ ZNAK ČITA SE „ODGOVARA", DAKLE 1 cm DULJINE STRELICE KOJOM PREDSTAVLJAMO GRAFIČKI VEKTOR SILE ODGOVARA IZNOSU OD 1 N.



1) Nacrtamo osi x i y i povučemo pravce djelovanja sila $\vec{F}_1$ i $\vec{F}_2$.

2) Iz zajedničkog početka nacrtamo u mjerilu vektore $\vec{F}_1$ i $\vec{F}_2$ koji tvore dvije stranice paralelograma.

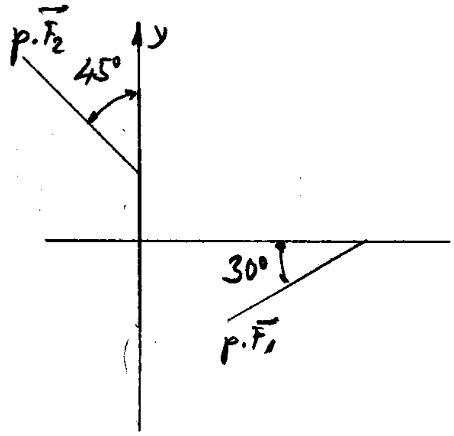
3) Nacrtamo ostale dvije stranice paralelograma.

4) Zbroj $\vec{F}_1$ i $\vec{F}_2$ rezultanta $\vec{F}_R$, leži na dijagonali paralelograma. Vektor $\vec{F}_R$ ima početak u zajedničkom početku vektora $\vec{F}_1$ i $\vec{F}_2$.

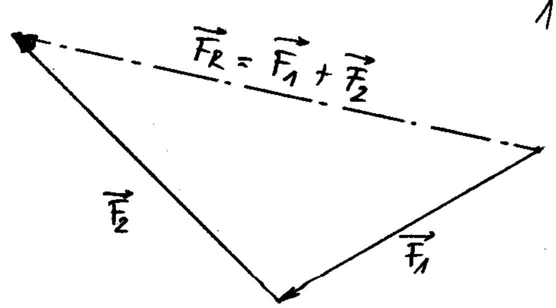
5) Izmjerimo duljinu $\vec{F}_R$ u [cm] i pomnožimo s mjerilom. Dobijemo iznos $\vec{F}_R$ u [N].

$$|\vec{F}_R| = \underset{[cm]}{7{,}2} \cdot \underset{[1\,cm \triangleq 1\,N]}{1} = 7{,}2\ N$$

b) ZBRAJANJE POMOĆU PRAVILA VEKTORSKOG POLIGONA



1 cm ≙ 1 N

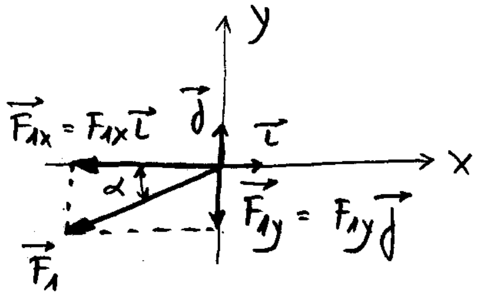


1) Nacrtamo osi x i y, i povučemo pravce djelovanja sile $\vec{F}_1$ i $\vec{F}_2$.
2) Nacrtamo u mjerilu $\vec{F}_1$.
3) Iz vrha vektora $\vec{F}_1$ nacrtamo u mjerilu vektor $\vec{F}_2$.
4) Zbroj $\vec{F}_R = \vec{F}_1 + \vec{F}_2$ jest vektor čiji je početak u početku prvog vektora koji je nacrtan, tj. $\vec{F}_1$, a vrh u vrhu drugog vektora, tj. $\vec{F}_2$.
5) Izmjerimo $\vec{F}_R$ u [cm] i pomnožimo s mjerilom:

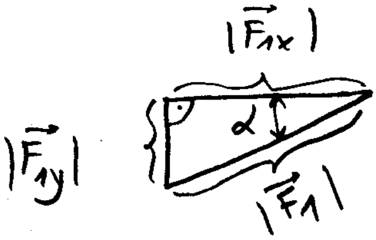
$$|\vec{F}_R| = 7{,}2 \cdot 1 = 7{,}2\ N$$

ANALITIČKI

Određivanje komponenata zadanih sila u pravcima osi koordinatnog sustava



Komponente $\vec{F}_{1x}$ i $\vec{F}_{1y}$ su PROJEKCIJE sile $\vec{F}_1$ na osi koordinatnog sustava x i y.



Uočimo pravokutni trokut čije su katete $|\vec{F}_{1x}|$ i $|\vec{F}_{1y}|$, a hipotenuza $|\vec{F}_1|$.

Vrijedi:

$$\frac{|\vec{F}_{1x}|}{|\vec{F}_1|} = \cos\alpha \implies |\vec{F}_{1x}| = |\vec{F}_1| \cdot \cos\alpha = 4 \cdot \cos 30° = 4 \cdot 0,866 = 3,464 \text{ N}$$

$$\frac{|\vec{F}_{1y}|}{|\vec{F}_1|} = \sin\alpha \implies |\vec{F}_{1y}| = |\vec{F}_1| \cdot \sin\alpha = 4 \cdot \sin 30° = 4 \cdot 0,5 = 2,0 \text{ N}$$

Komponenta $\vec{F}_{1x}$ usmjerena je suprotno od osi x, pa je $\vec{F}_{1x} = F_{1x}\vec{i} = -|\vec{F}_{1x}|\vec{i} = -3,464\vec{i}$ [N],

odnosno skalarna veličina F_{1x} koja množi jedinični vektor $\vec{i}$ (ort) na osi x ima negativan predznak:

$$F_{1x} = -3,464 \text{ N}$$

Komponenta $\vec{F}_{1y}$ usmjerena je suprotno od osi y, pa analogno vrijedi

$$\vec{F}_{1y} = F_{1y}\vec{j} = -|\vec{F}_{1y}|\vec{j} = -2\vec{j} \text{ [N]} \quad ; \quad F_{1y} = -2 \text{ N}.$$

 4

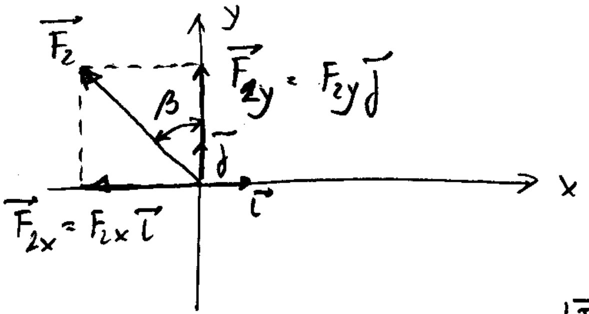
Zbroj komponenata $\vec{F}_{1x}$ i $\vec{F}_{1y}$ po pravilu vektorskog paralelograma daje ANALITIČKI ZAPIS vektora zadane sile $\vec{F}$:

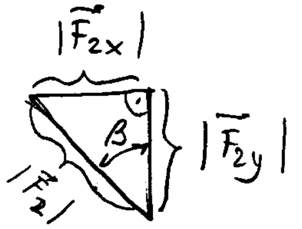
$$\vec{F} = \vec{F}_{1x} + \vec{F}_{1y} = (-3,464)\vec{i} + (-2)\vec{j} =$$
$$= -3,464\vec{i} - 2\vec{j} \quad [N]$$

Opaska: [N] - jedinica N u uglatoj zagradi piše se uz analitički zapis kao oznaka jedinice kojom se mjere komponente

3,464 N - bez uglate zagrade - normalno zapisan iznos u Njutnima

ANALOGNO SE DOBIJE ANALITIČKI ZAPIS $\vec{F}_2$:





$$\frac{|\vec{F}_{2x}|}{|\vec{F}_2|} = \sin\beta \Rightarrow |\vec{F}_{2x}| = |\vec{F}_2| \cdot \sin\beta =$$
$$= 5 \cdot \sin 45° =$$
$$= 5 \cdot 0,707 =$$
$$= 3,5355 \text{ N}$$

$$\frac{|\vec{F}_{2y}|}{|\vec{F}_2|} = \cos\beta \Rightarrow |\vec{F}_{2y}| = |\vec{F}_2| \cos\beta =$$
$$= 5 \cdot \cos 45° =$$
$$= 5 \cdot 0,707 =$$
$$= 3,5355 \text{ N}$$

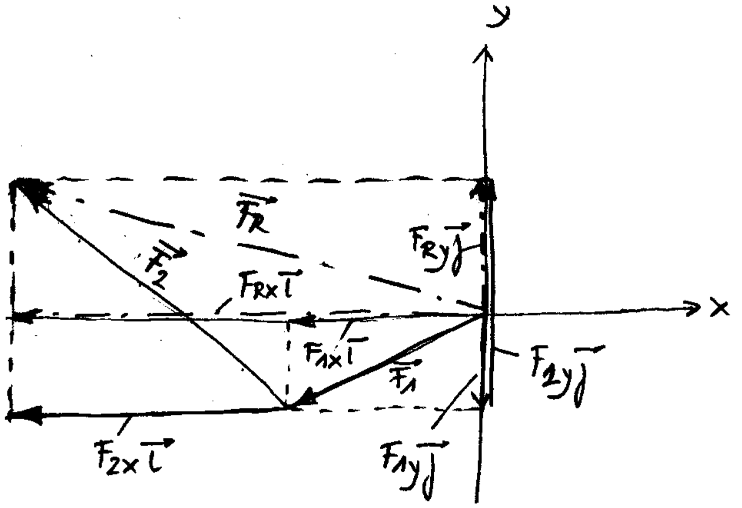
$$\vec{F}_{2x} = F_{2x}\vec{i} = -|\vec{F}_{2x}|\vec{i} = -3{,}5355\vec{i}\ [N]$$

$$F_{2x} = -3{,}5355\,N$$

$$\vec{F}_{2y} = F_{2y}\vec{j} = +|\vec{F}_{2y}|\vec{j} = 3{,}5355\vec{j}\ [N]$$

$$\vec{F}_2 = \vec{F}_{2x} + \vec{F}_{2y} = (-3{,}5355)\vec{i} + (3{,}5355)\vec{j} = $$
$$= -3{,}5355\vec{i} + 3{,}5355\vec{j}\ [N]$$

Analitičko zbrajanje $\vec{F}_1$ i $\vec{F}_2$



NA GRAFIČKOM PRIKAZU SE UOČAVA:

$\vec{F}_{Rx} = \vec{F}_{1x} + \vec{F}_{2x}$

$\vec{F}_{Ry} = \vec{F}_{1y} + \vec{F}_{2y}$

$$\vec{F}_R = \vec{F}_1 + \vec{F}_2 =$$
$$= (F_{1x}\vec{i} + F_{1y}\vec{j}) + (F_{2x}\vec{i} + F_{2y}\vec{j}) =$$
$$= \underbrace{(F_{1x}\vec{i} + F_{2x}\vec{i})}_{} + \underbrace{(F_{1y}\vec{j} + F_{2y}\vec{j})}_{} =$$

OVO JE MOGUĆE PISATI JER ZA ZBRAJANJE VEKTORA VRIJEDI ZAKON KOMUTACIJE

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

$$= \underbrace{(F_{1x} + F_{2x})}_{F_{Rx}}\vec{i} + \underbrace{(F_{1y} + F_{2y})}_{F_{Ry}}\vec{j} =$$

$$= F_{Rx}\vec{i} + F_{Ry}\vec{j}$$

$$\Rightarrow F_{Rx} = F_{1x} + F_{2x} = (-3{,}464) + (-3{,}5355) = -6{,}9995\,N$$

$$F_{Ry} = F_{1y} + F_{2y} = (-2) + 3{,}5355 = 1{,}5355\,N$$

$$\vec{F} = -6{,}9995\vec{i} + 4{,}5355\vec{j}\ [N]$$

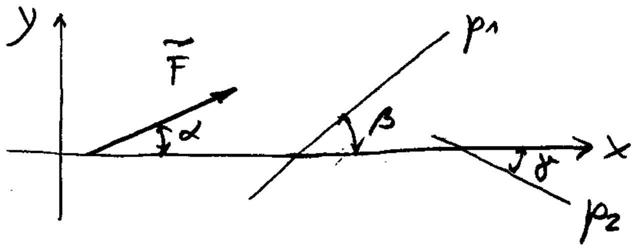
PO PITAGORINOM POUČKU:

$$|\vec{F}_R| = \sqrt{F_{Rx}^2 + F_{Ry}^2} = \sqrt{(-6{,}9995)^2 + 1{,}5355^2}$$

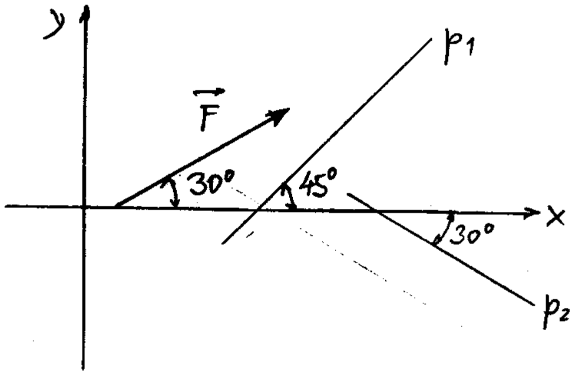
$$\boxed{|\vec{F}_R| = 7{,}1659\,N}$$

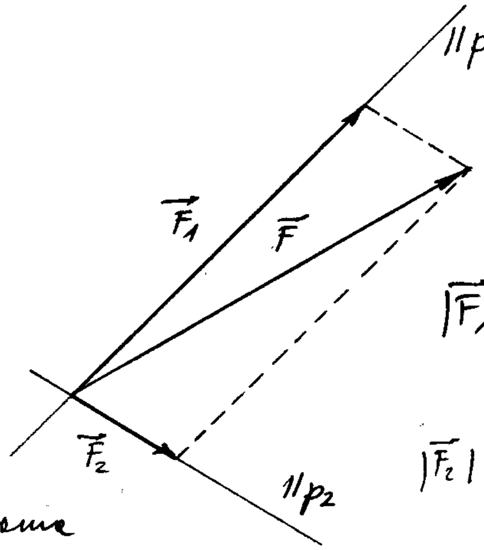
Silu $\vec{F}$ treba rastaviti na komponente $\vec{F}_1$ i $\vec{F}_2$ čiji su pravci djelovanja paralelni sa zadanim pravcima p_1 i p_2.

Zadano: $|\vec{F}| = 6\ N$, $\alpha = 30°$, $\beta = 45°$, $\gamma = 30°$.



GRAFIČKI POMOĆU PRAVILA VEKTORSKOG PARALELOGRAMA





$1\ cm \mathrel{\hat{=}} 1 N$

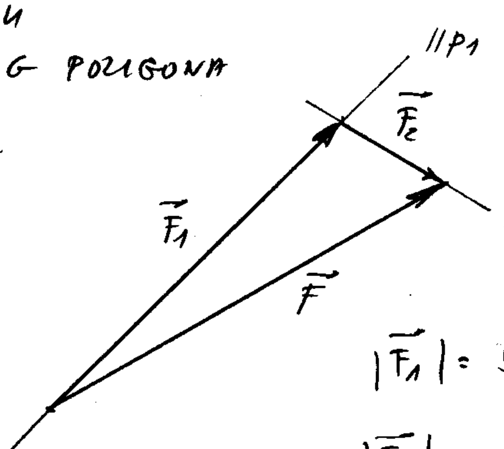
$|\vec{F}_1| = 5{,}4 \cdot 1 = 5{,}4\ N$

$|\vec{F}_2| = 1{,}6 \cdot 1 = 1{,}6\ N$

Kroz početak $\vec{F}$ povuku se paralele s p_1 i p_2 na kojima leže stranice paralelograma čija je dijagonala sila $\vec{F}$. Iz vrha $\vec{F}$ povuku se preostale dvije stranice i tako se definira paralelogram. Izmjere se stranice i tako dobiju iznosi komponenata F_1 i F_2.

GRAFIČKI POMOĆU PRAVILA VEKTORSKOG POLIGONA

Kroz početak $\vec{F}$ povuče se paralela s p_1, a kroz vrh $\vec{F}$ paralela s p_2. Tako se definira trokut sila. Izmjere se stranice i dobiju se iznosi $\vec{F}_1$ i $\vec{F}_2$.



$1\ cm \mathrel{\hat{=}} 1 N$

$|\vec{F}_1| = 5{,}4 \cdot 1 = 5{,}4\ N$

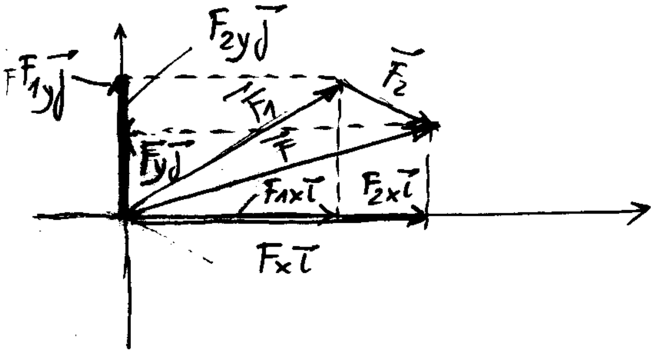
$|\vec{F}_2| = 1{,}6 \cdot 1 = 1{,}6\ N$

U oba slučaja grafički je predstavljena jednadžba

$$\vec{F} = \vec{F_1} + \vec{F_2} .$$

U ovoj je jednadžbi poznat zbroj, tj. $\vec{F}$ te pravci na kojima leže sile koje se zbrajaju. Obrnutim putem primjenjuju se pravila o zbrajanju vektora.

ANALITIČKI



$$F_x\vec{i} = F_{1x}\vec{i} + F_{2x}\vec{i}$$

$$F_{1x} = |\vec{F_1}| \cdot \cos\beta$$

$$F_{2x} = |\vec{F_2}| \cdot \cos\gamma$$

$$F_x = |\vec{F}| \cdot \cos\alpha$$

OZNAČIMO: $|\vec{F_1}| = F_1$, $|\vec{F_2}| = F_2$, $|\vec{F}| = F$

$$\Longrightarrow \quad F\cos\alpha\,\vec{i} = (F_1\cos\beta + F_2\cos\gamma)\vec{i}$$

$$F\cos\alpha = F_1\cos\beta + F_2\cos\gamma \qquad (1)$$

$$F_y\vec{j} = F_{1y}\vec{j} + F_{2y}\vec{j}$$

$$F_{1y} = |\vec{F_1}| \sin\beta$$

$$F_{2y} = -|\vec{F_2}| \sin\gamma$$

$$F_y = |\vec{F}| \sin\alpha$$

$$\Rightarrow \quad F\sin\alpha\,\vec{j} = [F_1\sin\beta + (-F_2)\sin\gamma]\,\vec{j}$$

$$F\sin\alpha = F_1\sin\beta - F_2\sin\gamma \qquad (2)$$

$$F\cos\alpha = F_1\cos\beta + F_2\cos\gamma \quad /\cdot\sin\gamma \quad /\cdot\sin\beta$$
$$F\sin\alpha = F_1\sin\beta - F_2\sin\gamma \quad /\cdot\cos\gamma \quad /\cdot(-\cos\beta)$$

1\. $\oplus$

$$F\cos\alpha\sin\gamma + F\sin\alpha\cos\gamma = F_1\cos\beta\sin\gamma + F_1\sin\beta\cos\gamma$$

$$F\,(\underbrace{\sin\alpha\cos\gamma + \cos\alpha\sin\gamma}_{=\sin(\alpha+\gamma)}) = F_1\,(\underbrace{\sin\beta\cos\gamma + \cos\beta\sin\gamma}_{=\sin(\beta+\gamma)})$$

$$F\sin(\alpha+\gamma) = F_1\sin(\beta+\gamma)$$

$$\underline{\underline{F_1}} = F\,\frac{\sin(\alpha+\gamma)}{\sin(\beta+\gamma)} = 6\cdot\frac{\sin(30^\circ+30^\circ)}{\sin(45^\circ+30^\circ)} =$$

$$= 6\,\frac{\sin 60^\circ}{\sin 75^\circ} = \underline{\underline{5{,}379\ N}}$$

2\. $\oplus$

$$F\cos\alpha\sin\beta - F\sin\alpha\cos\beta = F_2\cos\gamma\sin\beta + F_2\sin\gamma\cos\beta$$

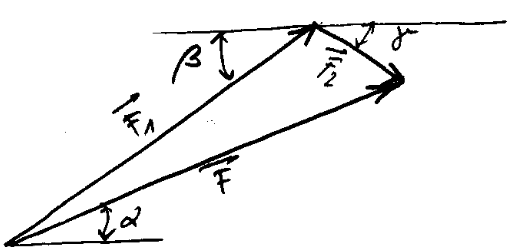
$$F\,(\underbrace{\sin\beta\cos\alpha - \cos\beta\sin\alpha}_{=\sin(\beta-\alpha)}) = F_2\,(\underbrace{\sin\beta\cos\gamma + \cos\beta\sin\gamma}_{=\sin(\beta+\gamma)})$$

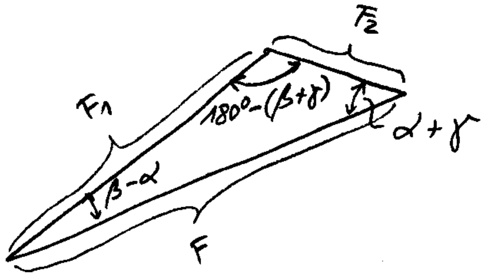
$$F\sin(\beta-\alpha) = F_2\sin(\beta+\gamma)$$

$$\underline{\underline{F_2}} = F\,\frac{\sin(\beta-\alpha)}{\sin(\beta+\gamma)} = 6\,\frac{\sin(45^\circ-30^\circ)}{\sin(45^\circ+30^\circ)} =$$

$$= 6\,\frac{\sin 15^\circ}{\sin 75^\circ} = \underline{\underline{1{,}608\,N}}$$

DIREKTNO IZRAČUNAVANJE IZ SKICE GRAFIČKOG RJEŠENJA PRAVILOM VEKTORSKOG POLIGONA





SINUSOV POUČAK:

$$\frac{F}{\sin[180^\circ-(\beta+\gamma)]} = \frac{F_1}{\sin(\alpha+\gamma)}$$

$\sin(180^\circ - x) = \sin x$

$$F_1 = F\,\frac{\sin(\alpha+\gamma)}{\sin(\beta+\gamma)} = 6\,\frac{\sin 60^\circ}{\sin 75^\circ} =$$

$$= 5{,}379\,N$$

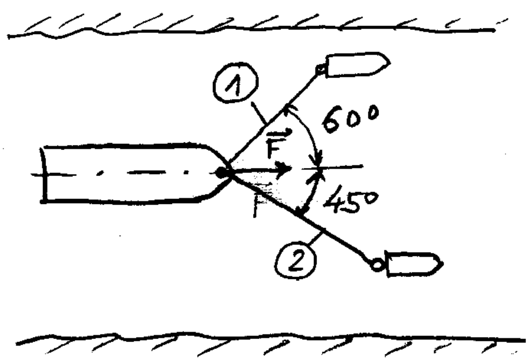
$$\frac{F}{\sin[180^\circ-(\beta+\gamma)]} = \frac{F_2}{\sin(\beta-\alpha)}$$

$$F_2 = F\,\frac{\sin(\beta-\alpha)}{\sin(\beta+\gamma)} = 6\,\frac{\sin 15^\circ}{\sin 75^\circ} =$$

$$= 1{,}608\,N$$

10

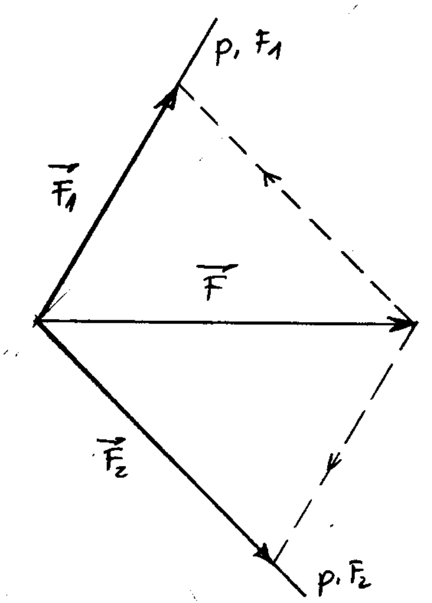
Teglenicu („šlep") vuku dva tegljača prema slici. Ako je za plovidbu teglenice potrebna vučna sila iznosa $F = 100\ kN$, kolike će biti sile u užetima 1 i 2 ?



Tegljač 1 vuče teglenicu silom F_1 preko užeta 1, a tegljač 2 vuče teglenicu silom F_2 preko užeta 2.

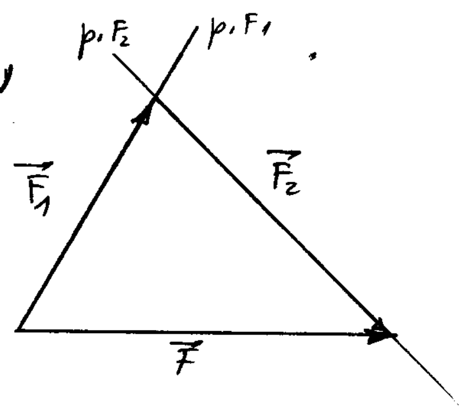
GRAFIČKI POMOĆU VEKTORSKOG PARALELOGRAMA I POMOĆU VEKTORSKOG POLIGONA RASTAVIMO ZADANU SILU $\vec{F}$ NA KOMPONENTE $\vec{F}_1$ I $\vec{F}_2$ PO ZADANIM PRAVCIMA UŽETA 1 i 2.

$1\ cm \triangleq 20\ kN$



$F_1 = 3{,}65 \cdot 20 = 73\ kN$

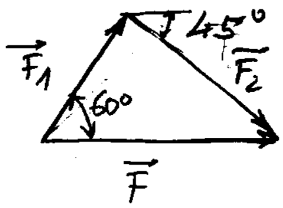
$F_2 = 4{,}5 \cdot 20 = 90\ kN$

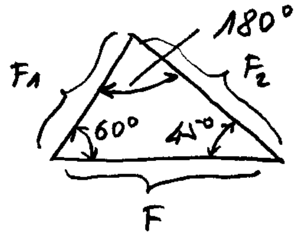


$F_1 = 3{,}65 \cdot 20 = 73\ kN$

$F_2 = 4{,}5 \cdot 20 = 90\ kN$

ANALITIČKI IZ SKICE ~~GRAFIČKOG~~ RJEŠENJA





SINUSOV POUČAK

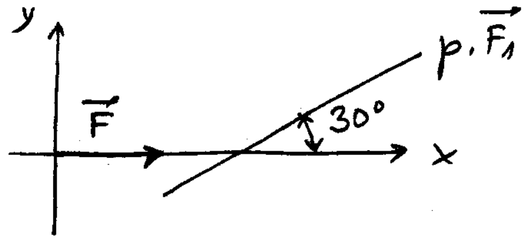
$$\frac{F_1}{\sin 45°} = \frac{F}{\sin 75°}$$

$$F_1 = F \frac{\sin 45°}{\sin 75°} = 100 \cdot \frac{\sin 45°}{\sin 75°} = 73,205\ kN$$

$$\frac{F_2}{\sin 60°} = \frac{F}{\sin 75°}$$

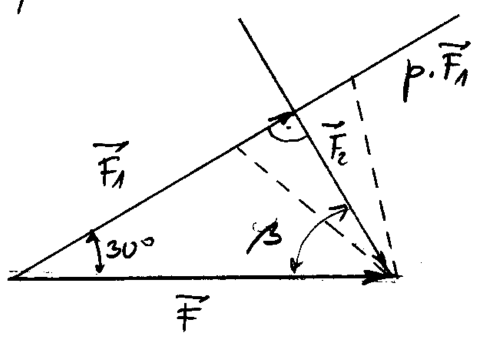
$$F_2 = F \frac{\sin 60°}{\sin 75°} = 100 \cdot \frac{\sin 60°}{\sin 75°} = 89,656\ kN$$

Silu $\vec{F}$ iznosa 5 N treba rastaviti na dvije komponente. Pravac djelovanja komponente $\vec{F_1}$ zadan je, a pravac djelovanja komponente $\vec{F_2}$ treba odrediti tako da joj iznos bude minimalan.



Jednadžbu $\vec{F} = \vec{F_1} + \vec{F_2}$ predočimo grafički primjenom pravila vektorskog poligona.
Iz početka vektora $\vec{F}$ povučemo pravac $\vec{F_1}$ koji je zadan. Iz vrha vektora $\vec{F}$ treba povući pravac komponente $\vec{F_2}$ tako da treća stranica trokuta sila, tj. komponenta $\vec{F_2}$, bude najmanje duljine. Očito će to biti najkraća udaljenost od vrha vektora $\vec{F}$ do pravca sile $\vec{F_1}$, a to je dužina okomita na pravac $\vec{F_1}$!

1 cm ≙ 1 N



CRTKANE LINIJE POKAZUJU DA SVI MOGUĆI OSTALI VEKTORI $\vec{F_2}$ IMAJU VEĆU DULJINU OD ONOG OKOMITOG NA PRAVAC KOMPONENTE $\vec{F_1}$.

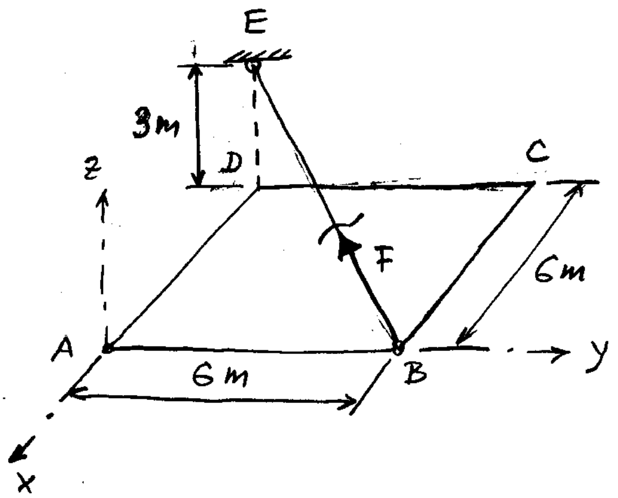
$F_2 = 2{,}5 \cdot 1 = 2{,}5\ N$

$F_1 = 4{,}3 \cdot 1 = 4{,}3\ N$

$\beta = 60°$

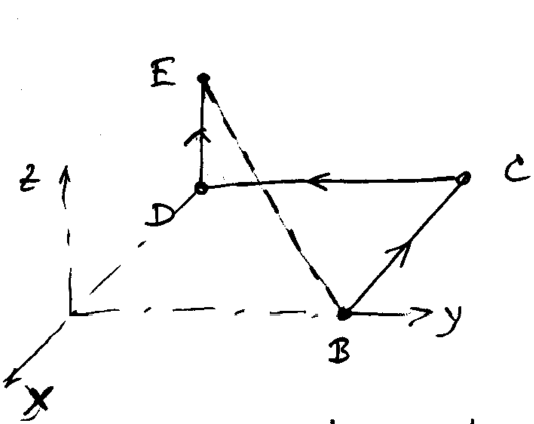
ANALITIČKI: $F_2 = F \sin 30° = 5 \cdot 0{,}5 = 2{,}5\ N$

$F_1 = F \cos 30° = 5 \cdot 0{,}866 = 4{,}33\ N$



Kvadratna ploča ABCD prema slici vezana je, pored ostalih veza, užetom BE. Odrediti komponente F_x, F_y i F_z sile F u užetu ako je zadano F = 10 kN.

Pravac BE je pravac djelovanja sile F. Da bismo našli komponente sile, moramo odrediti kosinuse kutova između pravca BE i paralela s osima koordinatnog sustava Axyz.
Treba konstruirati kvadar čija je prostorna dijagonala dužina $\overline{BE}$. Bridove tog kvadra možemo najlakše odrediti tako da krenemo iz točke B u točku E idući putovima koji su paralelni osima x, y i z. To će biti putovi B → C paralelno osi x, pa C → D paralelno osi y i D → E paralelno osi z:

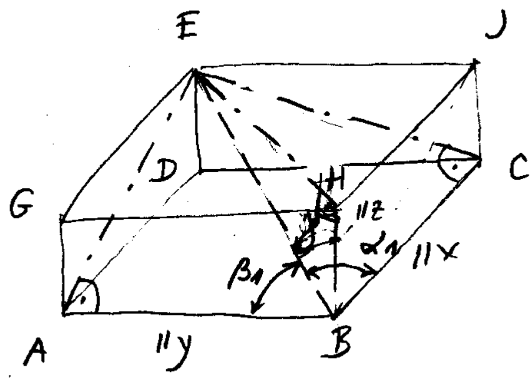


Putovi B → C, C → D i D → E definiraju bridove kvadra kojem je prostorna dijagonala dužina $\overline{BE}$.

Duljina dijagonale $\overline{BE}$ → $D = \sqrt{\overline{BC}^2 + \overline{CD}^2 + \overline{DE}^2} =$

$= \sqrt{6^2 + 6^2 + 3^2} =$

$= \sqrt{81} = 9\,m$



Uočimo pravokutni trokut ΔBCE. Dužina $\overline{BC} \parallel x$ je ⊥ na plohu CDEJ koja je $\parallel$ yz ravnini, pa je, dakle, okomita i na dužinu $\overline{CE}$ koja je dijagonala pravokutnika CDEJ. Dakle kut uz točku C je pravi kut.

Iz pravokutnog trokuta ΔBCE očitamo

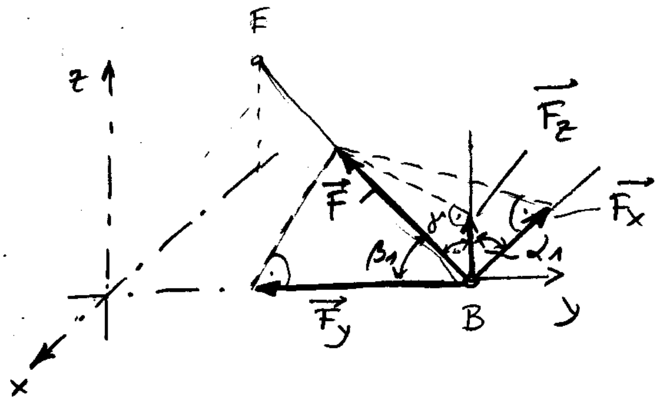
$$\cos \alpha_1 = \frac{\overline{BC}}{\overline{BE}} = \frac{6}{9} = \frac{2}{3}$$

Analogno zaključujemo da su ΔABE i ΔBHE i pravokutni trokuti, pa očitavamo iz njih

$$\cos \beta_1 = \frac{\overline{AB}}{\overline{BE}} = \frac{6}{9} = \frac{2}{3}$$

$$\cos \gamma_1 = \frac{\overline{BH}}{\overline{BE}} = \frac{3}{9} = \frac{1}{3}$$

Kutovi α_1 i β_1 su kutovi između dužine $\overline{BE}$, odnosno vektora sile $\vec{F}$ i negativnih smjerova osi x i y, dok je γ kut između dužine $\overline{BE}$, odnosno vektora $\vec{F}$ i pozitivnog smjera osi z.



Iznosi komponenata sile $\vec{F}$ su

$$|\vec{F_x}| = |\vec{F}| \cos\alpha_1 = 10 \cdot \frac{2}{3} = 6{,}67 \text{ kN}$$

$$|\vec{F_y}| = |\vec{F}| \cos\beta_1 = 10 \cdot \frac{2}{3} = 6{,}67 \text{ kN}$$

$$|\vec{F_z}| = |\vec{F}| \cos\gamma = 10 \cdot \frac{1}{3} = 3{,}33 \text{ kN}$$

$\vec{F_x}$ i $\vec{F_y}$ su NEGATIVNO USMJERENE, A $\vec{F_z}$ POZITIVNO PA SLIJEDI :

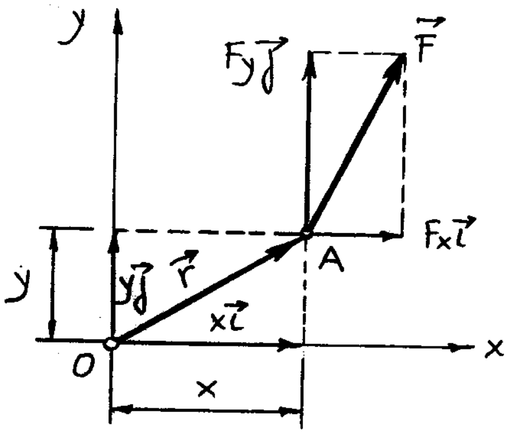
$$F_x = -6{,}67 \text{ kN}$$

$$F_y = -6{,}67 \text{ kN}$$

$$F_z = 3{,}33 \text{ kN}$$

$$\vec{F} = -6{,}67\vec{i} - 6{,}67\vec{j} + 3{,}33\vec{k} \quad [\text{kN}]$$

ODREĐIVANJE MOMENTA SILE KOJA LEŽI U RAVNINI xy S OBZIROM NA NEKU TOČKU TE RAVNINE



Treba odrediti moment sile

$$\vec{F} = F_x\vec{i} + F_y\vec{j}\,,$$

s hvatištem u točki A, s obzirom na točku O.

U točku O postavimo ishodište koordinatnog sustava Oxy i odredimo koordinate x i y hvatišta A.

Vektor položaja hvatišta A u odnosu na točku O jest

$$\vec{r} = x\vec{i} + y\vec{j}$$

Po definiciji momenta sile $\vec{F}$ s obzirom na točku O slijedi

$$\vec{M}_O^F = \vec{r} \times \vec{F} = (x\vec{i} + y\vec{j}) \times (F_x\vec{i} + F_y\vec{j}) =$$

$$= x\vec{i} \times F_x\vec{i} + x\vec{i} \times F_y\vec{j} + y\vec{j} \times F_x\vec{i} + y\vec{j} \times F_y\vec{j} =$$

$$= xF_x(\vec{i}\times\vec{i}) + xF_y(\vec{i}\times\vec{j}) + yF_x(\vec{i}\times\vec{j}) + yF_y(\vec{j}\times\vec{j})$$

U ovom računu primijenjeni su zakon distribucije za množenje vektora

$$(\vec{a}+\vec{b}) \times (\vec{c}+\vec{d}) = \vec{a}\times\vec{c} + \vec{a}\times\vec{d} + \vec{b}\times\vec{c} + \vec{b}\times\vec{d}$$

i zakon asocijacije s obzirom na skalare

$$m\vec{a} \times n\vec{b} = mn(\vec{a}\times\vec{b}) \,, \qquad m, n \ldots \text{skalari} .$$

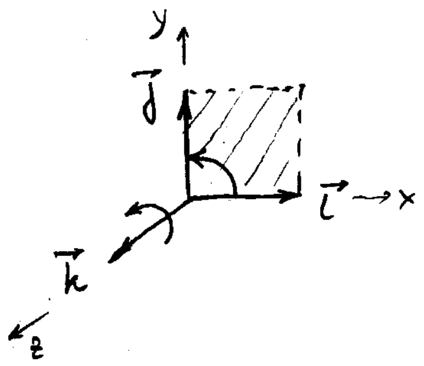
Po definiciji vektorskog produkta iznosi $\vec{i}\times\vec{i}$ i $\vec{j}\times\vec{j}$ jednaki su nuli:

$$|\vec{i}\times\vec{i}| = |\vec{i}|\cdot|\vec{i}|\cdot \sin\sphericalangle(\vec{i},\vec{i}) = 1\cdot 1\cdot \sin 0° = 0$$

$$|\vec{j}\times\vec{j}| = |\vec{j}|\cdot|\vec{j}|\cdot \sin\sphericalangle(\vec{j},\vec{j}) = 1\cdot 1\cdot \sin 0° = 0$$

Stoga vrijedi

$$\vec{i} \times \vec{i} = \vec{0} \quad , \quad \vec{j} \times \vec{j} = \vec{0} .$$

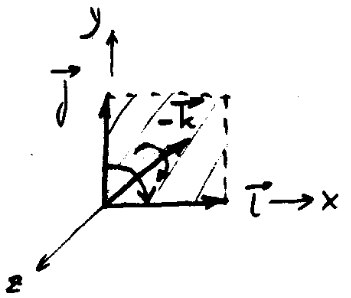


Iznos $\vec{i} \times \vec{j}$ jest

$$|\vec{i} \times \vec{j}| = |\vec{i}| \cdot |\vec{j}| \cdot \sin \sphericalangle (\vec{i}, \vec{j}) = 1 \cdot 1 \cdot \sin 90^\circ = 1 .$$

Pravac vektorskog produkta $\vec{i} \times \vec{j}$ okomit je na ravninu xy u kojoj leže ortovi $\vec{i}$ i $\vec{j}$, a to je pravac osi z. Smisao $\vec{i} \times \vec{j}$ je po pravilu desne ruke pozitivan smjer osi z. Prema tome $\vec{i} \times \vec{j}$ je vektor okomit na ravninu xy usmjeren u pozitivnom smjeru osi z, a iznos mu je 1. To je dakle ort $\vec{k}$:

$$\vec{i} \times \vec{j} = \vec{k}$$



Iznos $\vec{j} \times \vec{i}$ jest

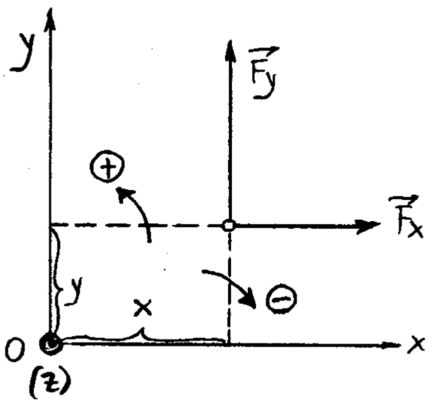
$$|\vec{j} \times \vec{i}| = |\vec{j}| \cdot |\vec{i}| \cdot \sin \sphericalangle (\vec{j}, \vec{i}) = 1 \cdot 1 \cdot \sin 90^\circ = 1 .$$

Pravac vektorskog produkta $\vec{j} \times \vec{i}$ okomit je na ravninu xy, a smisao mu je po pravilu desne ruke suprotan pozitivnom smjeru osi z. Prema tome $\vec{j} \times \vec{i}$ je vektor okomit na ravninu xy usmjeren suprotno od pozitivnog smjera osi z, a iznos mu je 1. To je dakle vektor $-\vec{k}$:

$$\vec{j} \times \vec{i} = -\vec{k}$$

Konačno dobivamo

$$\vec{M}_O^F = x F_y \vec{k} + y F_x (-\vec{k}) = (x F_y - y F_x) \vec{k}$$



U izvodu smo pretpostavili da je $F_x > 0$, $F_y > 0$, $x > 0$, $y > 0$.

Ako $\vec{M}_O^F$ pozitivno usmjeren, tj. ima smisao pozitivnog smjera osi z, mora biti

$$M_O^F = x F_y - y F_x > 0 .$$

Ovo možemo interpretirati na slijedeći način:

Komponenta $\vec{F}_y$ daje pozitivan doprinos momentu $\vec{M}_O^F$ iznosa $x F_y$. Ona nastoji izazvati rotaciju oko osi z u smjeru obrnuto od kazaljke na satu, što je pozitivan smjer.

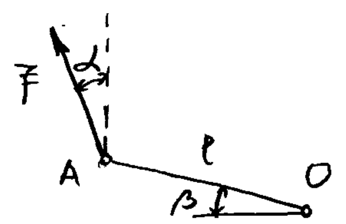
Komponenta $\vec{F}_x$ daje negativan doprinos momentu $\vec{M}_O^F$ iznosa $y F_x$. Ona nastoji izazvati rotaciju oko osi z u smjeru kazaljke na satu, što je negativan smjer.

Ako koordinatu x shvatimo kao krak $\vec{F}_y$, tj. udaljenost od točke O do pravca $\vec{F}_y$, a koordinatu y kao krak $\vec{F}_x$, tj. udaljenost od točke O do pravca $\vec{F}_x$, bit će doprinosi $\vec{F}_y$ i $\vec{F}_x$ momentu $\vec{M}_O^F$ jednaki umnošcima kraka i iznosa sile s predznakom koji ovisi o smjeru rotacije koju sila nastoji izazvati oko točke O (za rotaciju suprotnu kazaljki na satu predznak je pozitivan, a za rotaciju u smjeru kazaljke na satu predznak je negativan):

$$M_O^F = x F_y + (-y F_x) = x F_y - y F_x .$$

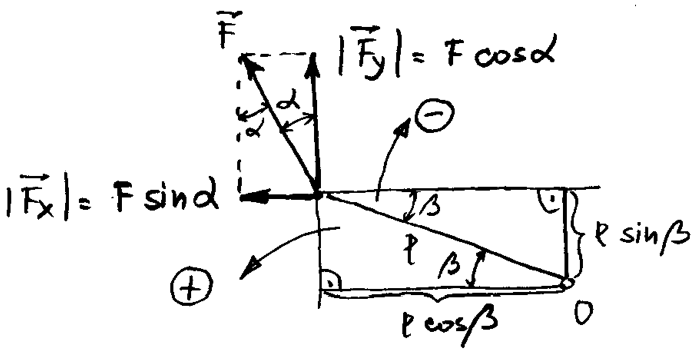
Moment neke sile u ravnini xy oko točke O u toj ravnini možemo dakle odrediti tako da silu u njenom hvatištu rastavimo na komponente paralelne osima x i y, odredimo krakove tih komponenata oko točke O kao udaljenosti od točke O do pravca komponente i doprinose momentu izračunavamo kao umnoške iznosa komponente i iznosa njenog kraka, pri čemu tome umnošku dajemo pozitivan predznak ako komponenta nastoji izazvati rotaciju oko O suprotnu smjeru kazaljke na satu, odnosno negativan predznak ako komponenta nastoji izazvati rotaciju oko O u smjeru kazaljke na satu.

Primjer



Zadano: $|\vec{F}| = F$, $\overline{OA} = \ell$, α, β.

Treba odrediti M_O^F.



Iznosi komponenata su $|\vec{F}_x| = F \sin\alpha$, $|\vec{F}_y| = F\cos\alpha$. Krak $\vec{F}_x$ oko O je udaljenost od O do pravca $\vec{F}_x$. To je $\ell \sin\beta$. Krak $\vec{F}_y$ oko O je udaljenost od O do pravca $\vec{F}_y$. To je $\ell\cos\beta$.

$\vec{F}_x$ nastoji izazvati rotaciju suprotno od smjera kazaljke na satu, a $\vec{F}_y$ u smjeru kazaljke na satu. Slijedi

$$M_O^F = + F\sin\alpha \cdot \ell \sin\beta - F\cos\alpha \cdot \ell\cos\beta ,$$

ODREĐIVANJE MOMENTA SILE KOJA LEŽI NA OPĆEM PRAVCU U PROSTORU S OBZIROM NA TOČKU O KOJA NE LEŽI NA TOM PRAVCU

U točku O postavi se koordinatni sustav Oxyz. Sila $\vec{F}$ rastavi se u komponente paralelne osima x, y i z:

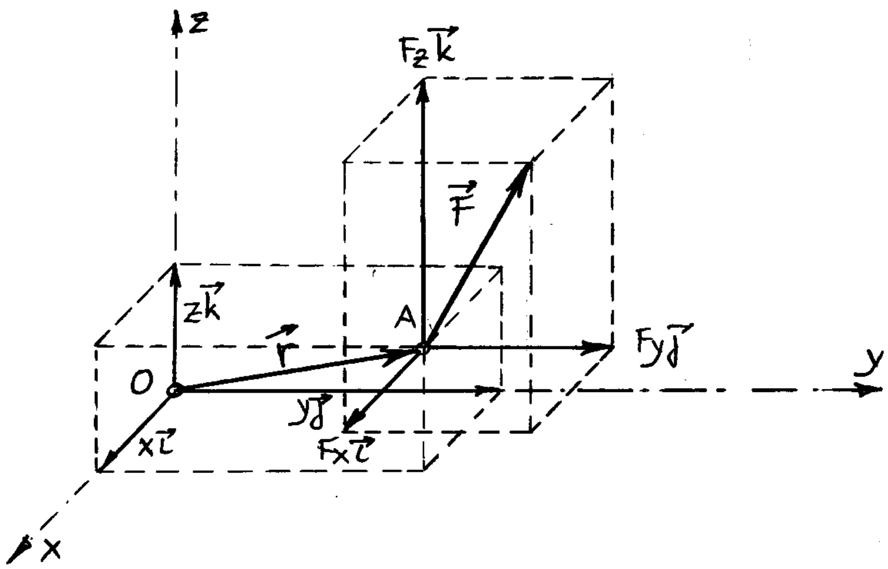
$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

Ako su x, y i z koordinate hvatišta A sile $\vec{F}$, vektor položaja hvatišta s obzirom na točku O jest

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

Uzet ćemo da je

$$x > 0,\ y > 0,\ z > 0 \quad i \quad F_x > 0,\ F_y > 0,\ F_z > 0.$$



Po definiciji je

$$\vec{M}_O^F = \vec{r} \times \vec{F} = (x\vec{i} + y\vec{j} + z\vec{k}) \times (F_x\vec{i} + F_y\vec{j} + F_z\vec{k}) =$$

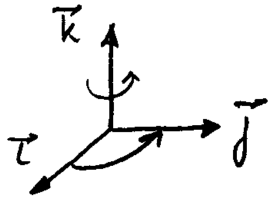
$$= xF_x(\vec{i}\times\vec{i}) + xF_y(\vec{i}\times\vec{j}) + xF_z(\vec{i}\times\vec{k}) +$$

$$+ yF_x(\vec{j}\times\vec{i}) + yF_y(\vec{j}\times\vec{j}) + yF_z(\vec{j}\times\vec{k}) +$$

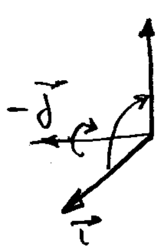
$$+ zF_x(\vec{k}\times\vec{i}) + zF_y(\vec{k}\times\vec{j}) + zF_z(\vec{k}\times\vec{k})$$

Prema definiciji vektorskog produkta slijedi

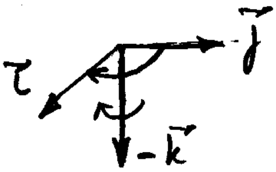
$$\vec{i} \times \vec{i} = \vec{0} \quad , \quad \vec{j} \times \vec{j} = \vec{0} \quad , \quad \vec{k} \times \vec{k} = \vec{0}$$



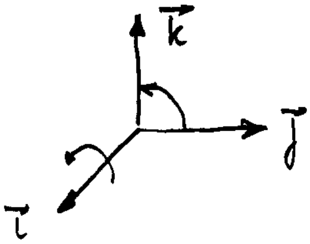
$\vec{i} \times \vec{j} = \vec{k}$



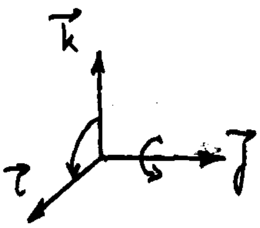
$\vec{i} \times \vec{k} = -\vec{j}$



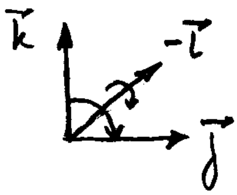
$\vec{j} \times \vec{i} = -\vec{k}$



$\vec{j} \times \vec{k} = \vec{i}$



$\vec{k} \times \vec{i} = \vec{j}$



$\vec{k} \times \vec{j} = -\vec{i}$

Uvrštenjem u izraz za $\vec{M}_0^F$ dobivamo

$$\begin{aligned}
\vec{M}_0^F = {} & x F_y \vec{k} + x F_z (-\vec{j}) + \\
& + y F_x (-\vec{k}) + y F_z \vec{i} + \\
& + z F_x \vec{j} + z F_y (-\vec{i}) = \\
= {} & (y F_z - z F_y)\vec{i} + (z F_x - x F_z)\vec{j} + (x F_y - y F_x)\vec{k} = \\
= {} & M_x \vec{i} + M_y \vec{j} + M_z \vec{k}
\end{aligned}$$

Komponente vektora $\vec{M}_0^F$

$$M_x = y F_z - z F_y \,,$$

$$M_y = z F_x - x F_z \,,$$

$$M_z = x F_y - y F_x$$

nazivamo redom „moment oko osi x", „moment oko osi y" i „moment oko osi z". Ovi nazivi bit će objašnjeni kasnije.

Vektorski produkt $\vec{M}_O^F = \vec{r} \times \vec{F}$ može se izračunati i s pomoću simboličke determinante:

$$\vec{M}_O^F = \vec{r} \times \vec{F} = (x\vec{i} + y\vec{j} + z\vec{k}) \times (F_x\vec{i} + F_y\vec{j} + F_z\vec{k}) =$$

$$= \begin{vmatrix} \overset{1,1}{\vec{i}} & \overset{1,2}{\vec{j}} & \overset{1,3}{\vec{k}} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = (-1)^{1+1}\vec{i}\begin{vmatrix} y & z \\ F_y & F_z \end{vmatrix} +$$

$$+ (-1)^{1+2}\vec{j}\begin{vmatrix} x & z \\ F_x & F_z \end{vmatrix} +$$

$$+ (-1)^{1+3}\vec{k}\begin{vmatrix} x & y \\ F_x & F_y \end{vmatrix} =$$

$$= \vec{i}\,(yF_z - zF_y) - \vec{j}\,(xF_z - zF_x) + \vec{k}\,(xF_y - yF_x)$$

Ovdje su redom uzimani elementi prvog retka $(r=1)$ te prvog stupca $(s=1)$, drugog stupca $(s=2)$ i trećeg stupca $(s=3)$. Križanjem retka i stupca kojem pripada element dobivene su subdeterminante koje množi element pomnožen s $(-1)^{r+s}$.

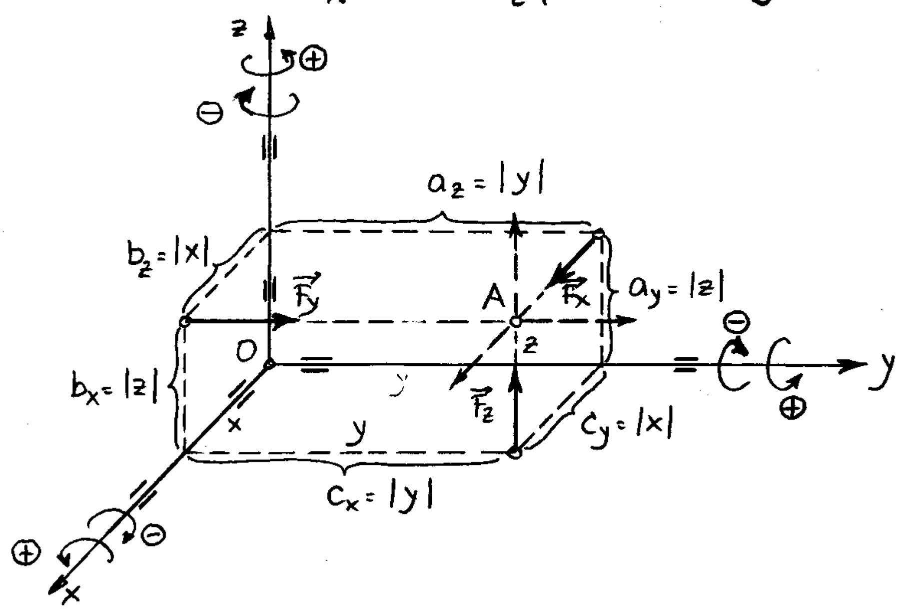
Izraz za $\vec{M}_O^F$ možemo interpretirati na slijedeći način:

1. Zamislimo da se prostor u kojem djeluje sila $\vec{F}$ može rotirati redom oko osi x, pa oko osi y i konačno oko osi z.
2. Silu $\vec{F}$ rastavimo u komponente paralelne osima x, y i z.
3. Komponente pomaknemo po pravcima njihovog djelovanja do probodišta tih pravaca s koordinatnim ravninama na koje su okomite.

4. Neka se prostor u kojem djeluju komponente može okretati oko osi x. Rotaciju oko osi x mogu izazvati samo komponente $\vec{F}_y$ i $\vec{F}_z$ okomite na tu os, dok komponenta $\vec{F}_x$ paralelna osi x ne može izazvati rotaciju oko te osi.

5. Ako se krakovi $\vec{F}_y$ i $\vec{F}_z$ odrede kao udaljenosti njihovih pravaca od osi x, što znači kao udaljenost b_x probodišta $\vec{F}_y$ i ravnine xz od osi x i kao udaljenost c_x probodišta $\vec{F}_z$ i ravnine xy od osi x, $\vec{F}_y$ nastoji rotirati prostor oko osi x momentom $b_x \cdot |\vec{F}_y|$ u smjeru kazaljke na satu gledano s vrha osi x, dok $\vec{F}_z$ nastoji rotirati prostor oko osi x momentom $c_x \cdot |\vec{F}_z|$ u smjeru suprotno od kazaljke na satu gledano s vrha osi x. Ako ove rotacije označimo prvu kao negativnu, a drugu kao pozitivnu i te predznake damo momentima koji ih nastoje izazvati, bit će „moment oko osi x"

$$M_x = c_x \cdot |\vec{F}_z| - b_x \cdot |\vec{F}_y|$$



Analogno se mogu odrediti „moment oko osi y" i „moment oko osi z":

$$M_y = a_y \cdot |\vec{F}_x| - c_y \cdot |\vec{F}_z| ,$$

$$M_z = b_z \cdot |\vec{F}_y| - a_z \cdot |\vec{F}_x| .$$

Uz $F_x > 0$, $F_y > 0$ i $F_z > 0$ bit će $|\vec{F}_x| = F_x$, $|\vec{F}_y| = F_y$, $|\vec{F}_z| = F_z$, a uz $x > 0$, $y > 0$ i $z > 0$ bit će $a_y = |z| = z$, $a_z = |y| = y$, $b_x = |z| = z$, $b_z = |x| = x$, $c_x = |y| = y$, $c_y = |x| = x$, pa slijedi

$$M_x = y F_z - z F_y ,$$

$$M_y = z F_x - x F_z ,$$

$$M_z = x F_y - y F_x .$$

„MOMENTI OKO OSI" SU KOMPONENTE VEKTORA MOMENTA SILE $\vec{F}$ S OBZIROM NA TOČKU O :

$$\vec{M}_O^F = M_x \vec{i} + M_y \vec{j} + M_z \vec{k}$$

Iz ovog razmatranja slijedi uobičajeni postupak određivanja momenta neke sile koja djeluje na općem pravcu u prostoru s obzirom na proizvoljnu točku O :

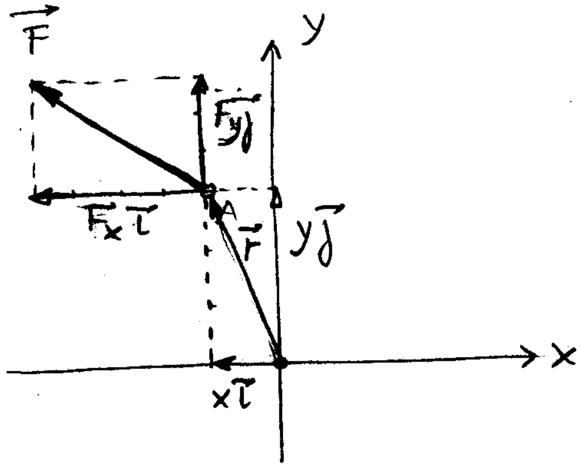
Najprije u točki O postavimo osi koordinatnog sustava Oxyz. Zatim silu $\vec{F}$ rastavimo na komponente paralelne tim osima i odredimo krakove tih komponenata s obzirom na osi x, y i z kao udaljenosti od probodišta komponente s koordinatnom ravninom na koju

je okomita do osi koje određuju tu ravninu.
Konačno moment oko neke osi izračunavamo kao zbroj umnožaka iznosa komponenata okomitih na tu os i njihovih krakova s obzirom na nju, s time da umnošcima (tj. momentima) dajemo pozitivan predznak ako komponenta nastoji izazvati rotaciju oko osi u smjeru suprotnom smjeru kazaljke na satu gledano s vrha osi, odnosno negativan predznak ako komponenta nastoji izazvati rotaciju u smjeru kazaljke na satu gledano s vrha osi.

Odrediti moment sile

$$\vec{F} = -4\vec{i} + 3\vec{j} \quad [N]$$

s hvatištem u točki A $(-1\,m, 3\,m)$ s obzirom na točku O.



Analitički:

$\vec{M}_O^F = \vec{r} \times \vec{F}$ → vektor sile

vektor položaja hvatišta sile s obzirom na točku O

$$\vec{r} = x\vec{i} + y\vec{j}$$

$\left.\begin{matrix} x = -1\,m \\ y = 3\,m \end{matrix}\right\}$ koordinate točke A hvatišta A u koord. sustavu Oxy

$$\vec{r} = -1\vec{i} + 3\vec{j} \quad [m]$$

$$\vec{M}_O^F = (-1\vec{i} + 3\vec{j}) \times (-4\vec{i} + 3\vec{j}) =$$

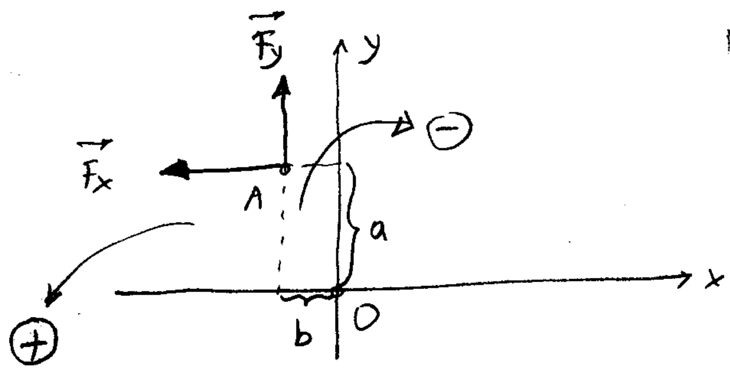
$$= (-1)(-4)\underbrace{(\vec{i} \times \vec{i})}_{=\vec{0}} + (-1)(3)\underbrace{(\vec{i} \times \vec{j})}_{=\vec{k}} +$$

$$+ 3\cdot(-4)\underbrace{(\vec{j} \times \vec{i})}_{=-\vec{k}} + 3\cdot 3\underbrace{(\vec{j} \times \vec{j})}_{=\vec{0}}$$

$$\underline{\underline{\vec{M}_O^F}} = -3\vec{k} + (-12)(-\vec{k}) = (-3 + 12)\vec{k} = \underline{\underline{9\vec{k}\ [Nm]}}$$

$$|M_O^F| = 9\ Nm$$

Računom „SILA PUTA KRAK":



$|\vec{F_x}| = |-4| = 4N$

$|\vec{F_y}| = |3| = 3N$

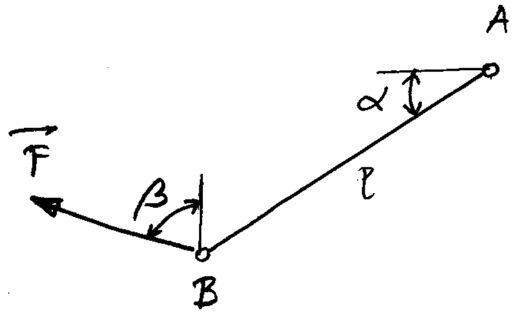
krak $\vec{F_x}$, $a = |y_A| = 3m$

krak $\vec{F_y}$, $b = |x_A| = |-1| = 1m$

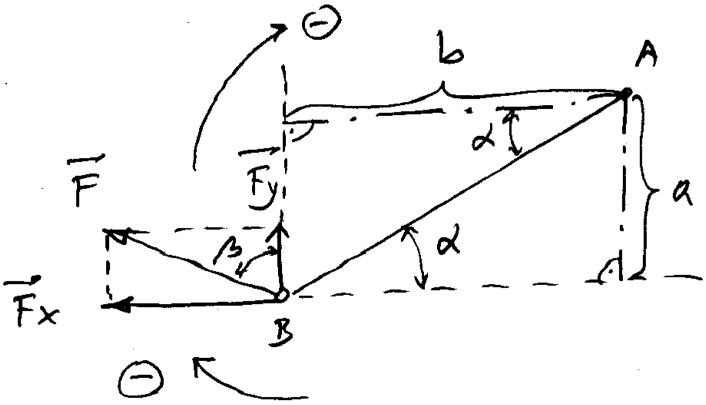
$\vec{F_x}$ bi rotirala xy ravninu oko O u smjeru obrnuto od kazaljke na satu, što daje pozitivan predznak, a $\vec{F_y}$ u smjeru kazaljke na satu, što daje negativan predznak momentu.

$$\Rightarrow \underline{\underline{M_O^F}} = |\vec{F_x}|\cdot a - |\vec{F_y}|\cdot b = 4\cdot 3 - 3\cdot 1 = 12 - 3 = \underline{\underline{9\ Nm}}$$

Odrediti moment sile $\vec{F}$ s obzirom na točku A ako je zadano: $|\vec{F}| = F$, $\overline{AB} = \ell$, α i β.



Silu $\vec{F}$ rastavimo na komponente $\vec{F_x}$ i $\vec{F_y}$ i za te komponente tražimo krakove s obzirom na točku A:



$|\vec{F_x}| = |\vec{F}| \sin\beta = F \sin\beta$

$|\vec{F_y}| = |\vec{F}| \cos\beta = F \cos\beta$

Krak $\vec{F_x}$ oko A: odredimo kao udaljenost od A do pravca djelovanja sile $\vec{F_x}$, tj. spustimo iz A okomicu na taj pravac.

$\Rightarrow a = \overline{AB} \cdot \sin\alpha = \ell \sin\alpha$

Krak $\vec{F_y}$ nađemo tako da iz A spustimo okomicu na pravac $\vec{F_y}$.

$\rightarrow b = \overline{AB} \cdot \cos\alpha = \ell \cos\alpha$

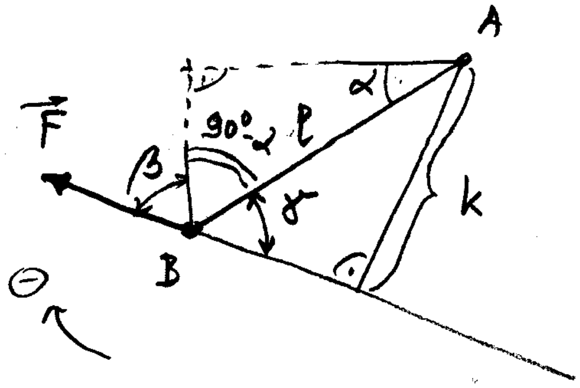
Konačno dobijemo, budući da i $\vec{F_x}$ i $\vec{F_y}$ daju rotacije u smjeru kazaljke na satu

$$M_A^F = -|\vec{F_x}| \cdot a - |\vec{F_y}| \cdot b = -F \sin\beta\, \ell \sin\alpha - F\cos\beta\, \ell \cos\alpha$$

$\longrightarrow$

$$\underline{\underline{M_A^F}} = -F\ell(\sin\beta\sin\alpha + \cos\beta\cos\alpha) =$$
$$= \underline{\underline{-F\ell\cos(\beta-\alpha)}}$$

ovo bismo mogli dobiti i direktno:



$$\gamma = 180° - \beta - (90° - \alpha) =$$
$$= 180° - \beta - 90° + \alpha = 90° - (\beta - \alpha)$$

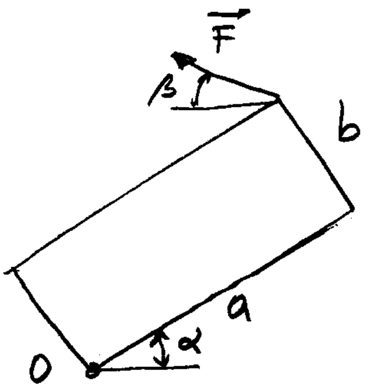
$$k = \ell \sin\gamma = \ell\sin[90° - (\beta-\alpha)] =$$
$$= \ell\cos(\beta-\alpha)$$

$$M_A^F = -|\vec{F}|\cdot k = -F\cdot\ell\cos(\beta-\alpha)$$

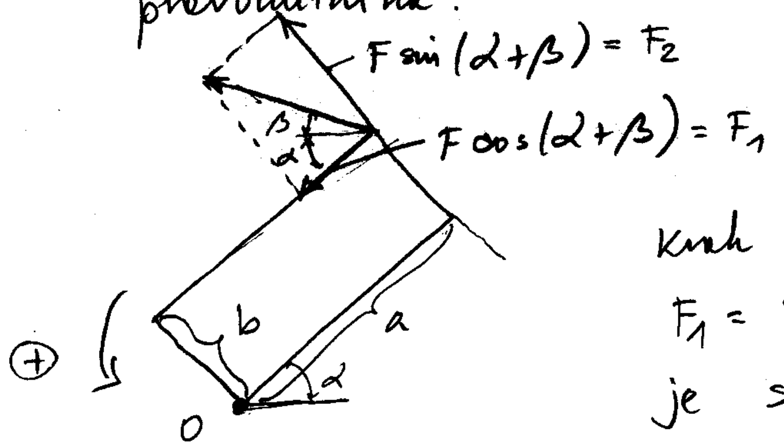
U OVOM JE SLUČAJU NAJLAKŠE IZRAČUNATI M_A^F PRVIM NAČINOM, KOJI SE U PRAVILU UVIJEK KORISTI, A TO JE:

1) RASTAVITI SILU NA x i y KOMPONENTU
2) ODREDITI KRAKOVE TIH KOMPONENATA S OBZIROM NA TOČKU A
3) M_A^F PRIKAZATI KAO ZBROJ MOMENATA KOMPONENATA S OBZIROM NA TOČKU A RAČUNOM „SILA PUTA KRAK".

Odrediti moment sile $\vec{F}$ s obzirom na točku O ako je zadano: $|\vec{F}| = F$, a, b, α i β.



Jednostavniji način je u ovom slučaju rastaviti silu na komponente po pravcima stranice pravokutnika:

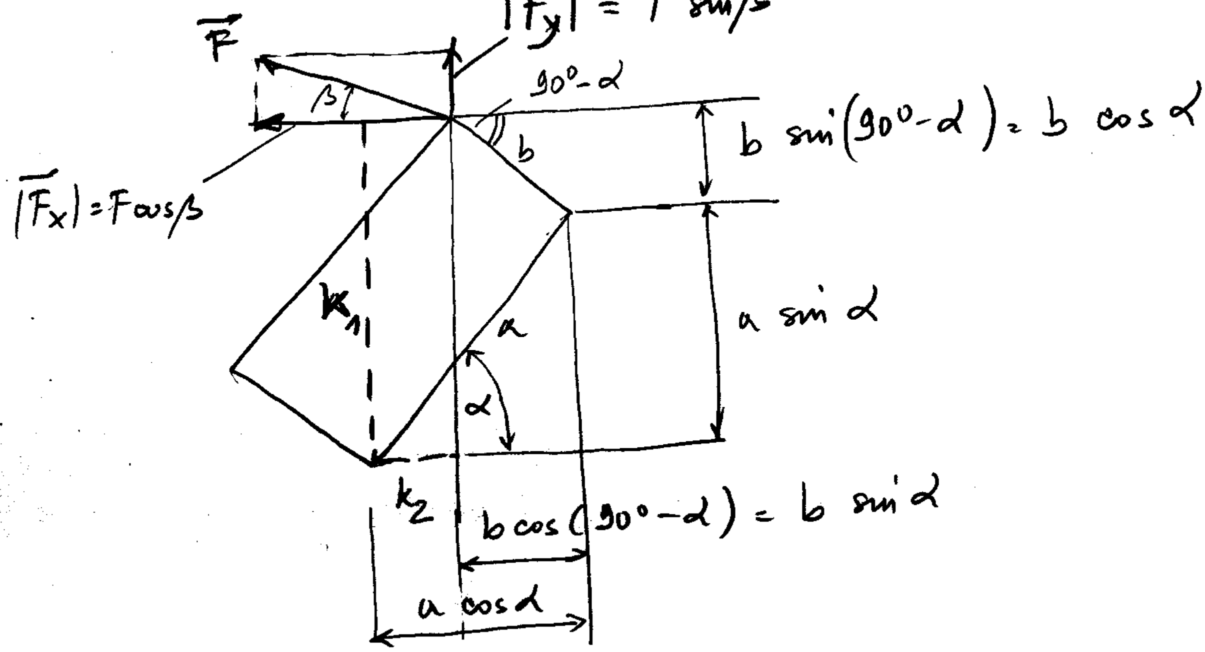


Krak komponente
$F_1 = F\cos(\alpha+\beta)$
je stranica b,
a krak komponente
$F_2 = F\sin(\alpha+\beta)$
je stranica a.

$$\rightarrow \quad M_O^F = F\cos(\alpha+\beta)\cdot b + F\sin(\alpha+\beta)\cdot a = F\left[a\sin(\alpha+\beta) + b\cos(\alpha+\beta)\right]$$

Ako se $\vec{F}$ rastavi na x i y komponente, teško je u ovom slučaju odrediti njihove krakove.

⟶

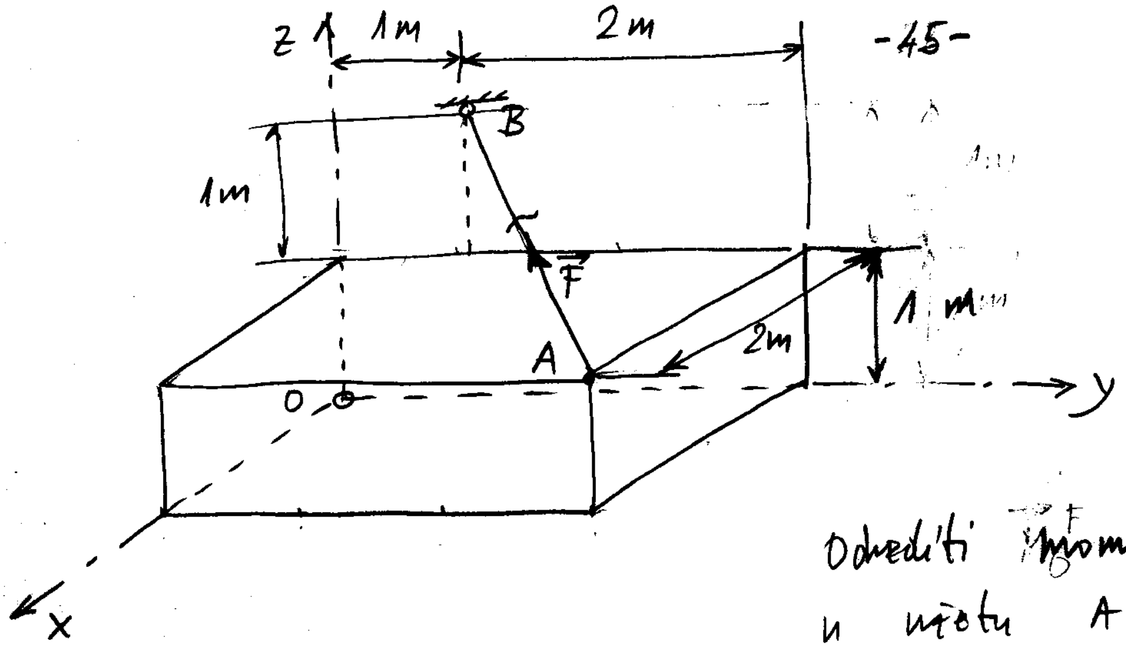


$$k_1 = a \sin\alpha + b\cos\alpha$$

$$k_2 = a\cos\alpha - b\sin\alpha$$

$$M_O^F = F\cos\beta\,(a\sin\alpha + b\cos\alpha) +$$
$$+ F\sin\beta\,(a\cos\alpha - b\sin\alpha) =$$
$$= F\,[\,a\,(\sin\alpha\cos\beta + \cos\alpha\sin\beta) +$$
$$+ b\,(\cos\alpha\cos\beta - \sin\alpha\sin\beta)] =$$
$$= F\,[\,a\sin(\alpha+\beta) + b\cos(\alpha+\beta)]$$

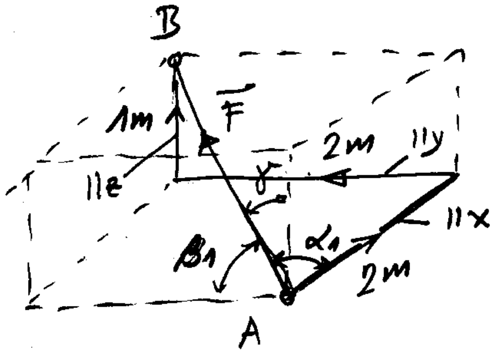
F



Odrediti moment sile $\vec{F}$ u užetu AB s obzirom na točku O.

Zadano: $|\vec{F}| = 120$ N.

Određivanje komponenata sile $\vec{F}$:



$$D = \overline{AB} = \sqrt{2^2 + 2^2 + 1^2} = 3\,m$$

$$\cos\alpha_1 = \frac{2}{3}$$

$$\cos\beta_1 = \frac{2}{3}$$

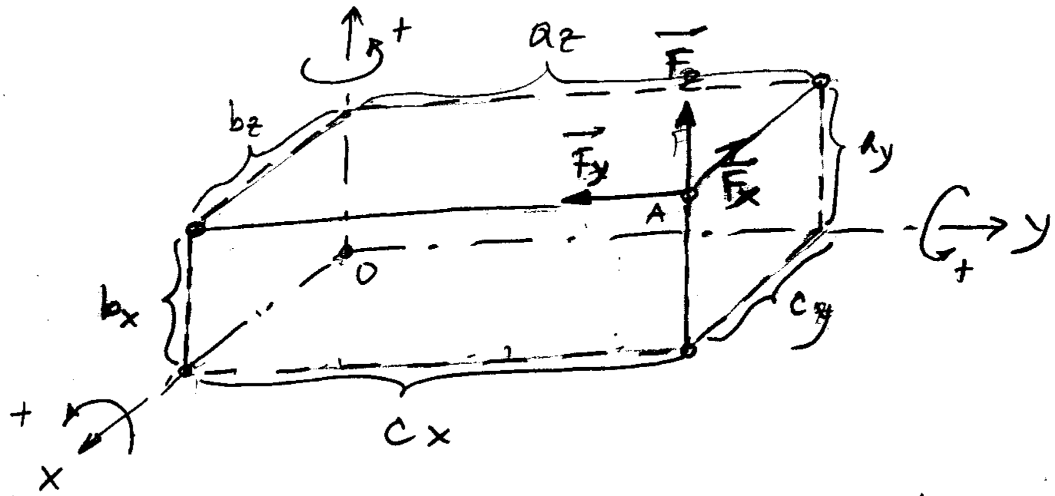
$$\cos\gamma = \frac{1}{3}$$

$$|\vec{F}_x| = |\vec{F}|\cos\alpha_1 = 120 \cdot \frac{2}{3} = 80\ N$$

$$|\vec{F}_y| = |\vec{F}| \cdot \cos\beta_1 = 120 \cdot \frac{2}{3} = 80\ N$$

$$|\vec{F}_z| = |\vec{F}|\cos\gamma = 120 \cdot \frac{1}{3} = 40\ N$$

Ucrtamo komponente $\vec{F}$ u hvatištu A i tražimo krakove tih komponenata kao udaljenosti pravca djelovanja komponente od osi koordinatnog sustava. Te se udaljenosti uzimaju kao udaljenosti od probodišta pravca djelovanja komponente s koordinatnom ravninom u kojoj leži koordinatna os do te osi.



Krak $\vec{F_x}$ oko osi y : $a_y = 1\,m$

" $\vec{F_x}$ —"— z : $a_z = 3\,m$

" $\vec{F_y}$ —"— x : $b_x = 1\,m$

" $\vec{F_y}$ —"— z : $b_z = 2\,m$

" $\vec{F_z}$ —"— x : $c_x = 3\,m$

" $\vec{F_z}$ —"— y : $c_y = 2\,m$

AKO KOMPONENTA SILE NASTOJI ROTIRATI TIJELO NA KOJE DJELUJE OKO NEKE KOORDINATNE OSI TAKO DA JE TO ROTACIJA OBRNUTA OD KAZALJKE NA SATU GLEDANO S VRHA TE OSI, „MOMENT OKO OSI" TE KOMPONENTE JE POZITIVAN, A ZA SUPROTNU ROTACIJU JE NEGATIVAN.

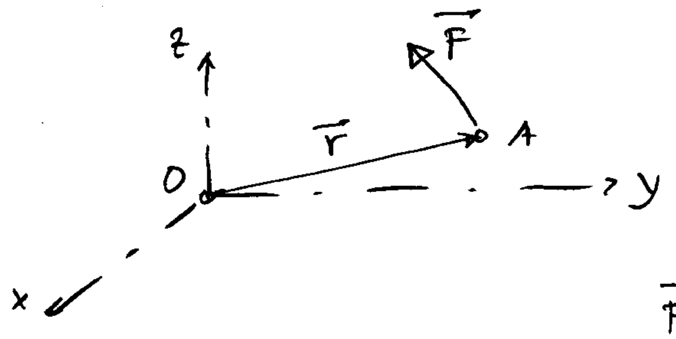
Tako dobivamo komponente momenta sile $\vec{F}$ s obzirom na točku 0 kao „momente komponenata sile $\vec{F}$ oko osi koordinatnog sustava s ishodištem u točki 0":

$$M_x = |\vec{F_y}| \cdot b_x + |\vec{F_z}| \cdot c_x = 80 \cdot 1 + 40 \cdot 3 = 200 \, Nm$$
$$M_y = -|\vec{F_x}| \cdot a_y - |\vec{F_z}| \cdot c_y = -80 \cdot 1 - 40 \cdot 2 = -160 \, Nm$$
$$M_z = |\vec{F_x}| \cdot a_z - |\vec{F_y}| \cdot b_z = 80 \cdot 3 - 80 \cdot 2 = 80 \, Nm$$

$$\vec{M}_0^F = M_x \vec{i} + M_y \vec{j} + M_z \vec{k} =$$
$$= 200 \vec{i} - 160 \vec{j} + 80 \vec{k} \quad [Nm]$$

$$|\vec{M}_0^F| = \sqrt{200^2 + (-160)^2 + 80^2} = 268{,}33 \, Nm$$

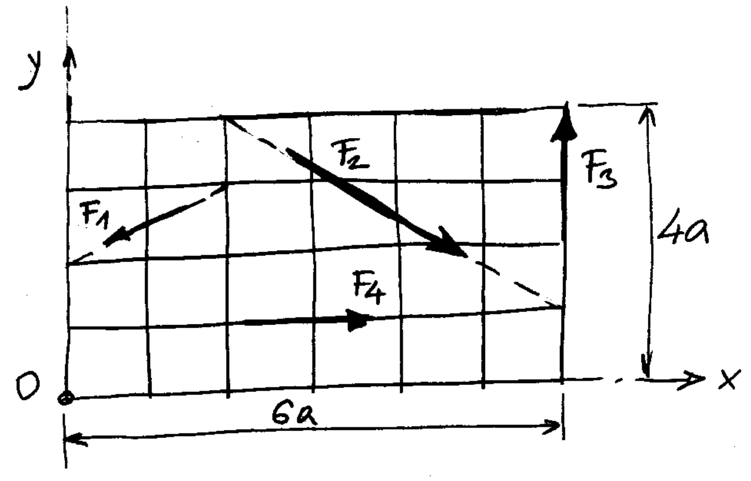
ANALITIČKI



$$A(2, 3, 1) \quad [m]$$
$$\vec{r} = x_A \vec{i} + y_A \vec{j} + z_A \vec{k}$$
$$\vec{r} = 2\vec{i} + 3\vec{j} + \vec{k} \quad [m]$$
$$\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k} =$$
$$= -80\vec{i} - 80\vec{j} + 40\vec{k} \quad [N]$$

$$\vec{M}_0^F = \vec{r} \times \vec{F} = \underbrace{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ -80 & -80 & 40 \end{vmatrix}}_{\text{SIMBOLIČKA DETERMINANTA}} =$$

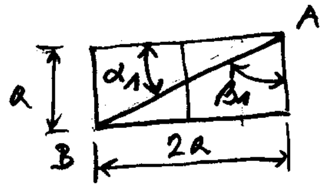
$$= \vec{i}\,[3 \cdot 40 - 1 \cdot (-80)] - \vec{j}\,[2 \cdot 40 - 1(-80)] +$$
$$+ \vec{k}\,[2(-80) - 3(-80)] = 200\vec{i} - 160\vec{j} + 80\vec{k} \quad [Nm]$$



Odrediti rezultantu zadanog sustava sila koje djeluju u xy ravnini te s pomoću momentnog pravila odrediti položaj rezultante s obzirom na točku O.

Zadano: $F_1 = 10\,N$, $F_2 = 20\,N$,
$F_3 = 5\,N$, $F_4 = 5\,N$
$a = 1\,m$.

Određivanje komponenata sila:

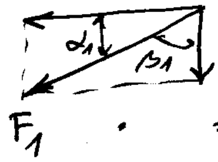


$$\overline{AB} = \sqrt{(2a)^2 + a^2} = \sqrt{5}\,a$$

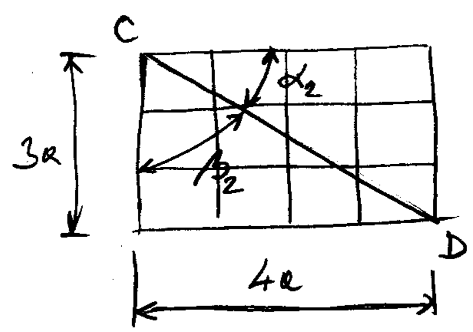
$$\cos\alpha_1 = \frac{2a}{\sqrt{5}\,a} = \frac{2}{\sqrt{5}}$$

$$\cos\beta_1 = \frac{a}{\sqrt{5}\,a} = \frac{1}{\sqrt{5}}$$

$$F_1\cos\alpha_1 = F_1\cdot\frac{2}{\sqrt{5}}$$



$$F_1\cos\beta_1 = F_1\cdot\frac{1}{\sqrt{5}}$$

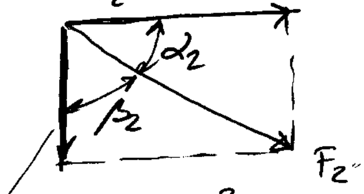


$$\overline{CD} = \sqrt{(4a)^2 + (3a)^2} = 5a$$

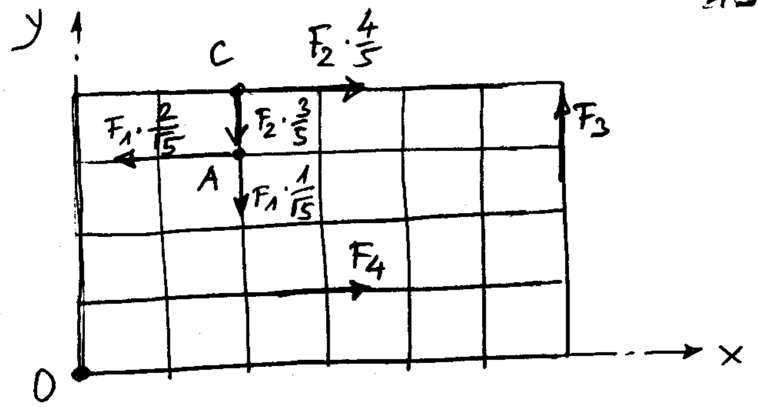
$$\cos\alpha_2 = \frac{4a}{5a} = \frac{4}{5}$$

$$\cos\beta_2 = \frac{3a}{5a} = \frac{3}{5}$$

$$F_2\cos\alpha_2 = F_2\cdot\frac{4}{5}$$



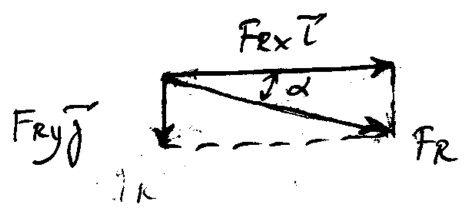
$$F_2\cos\beta_2 = F_2\cdot\frac{3}{5}$$



$$F_{Rx} = \sum F_{xi} = F_{x1} + F_{x2} + F_{x3} + F_{x4} =$$
$$= -F_1 \cdot \frac{2}{\sqrt{5}} + F_2 \cdot \frac{4}{5} + 0 + F_4 =$$
$$= -10 \cdot \frac{2}{\sqrt{5}} + 20 \cdot \frac{4}{5} + 5 = 29{,}944\ N$$

$$F_{Ry} = \sum F_{yi} = F_{y1} + F_{y2} + F_{y3} + F_{y4} =$$
$$= -F_1 \frac{1}{\sqrt{5}} - F_2 \cdot \frac{3}{5} + F_3 + 0 =$$
$$= -10 \cdot \frac{1}{\sqrt{5}} - 20 \cdot \frac{3}{5} + 5 = -11{,}472\ N$$

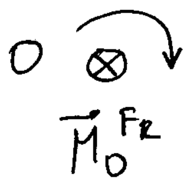
$$\vec{F_R} = 29{,}944\,\vec{i} - 11{,}472\,\vec{j}\ [N]$$
$$|\vec{F_R}| = \sqrt{29{,}944^2 + (-11{,}472)^2} = 32{,}066\ N$$

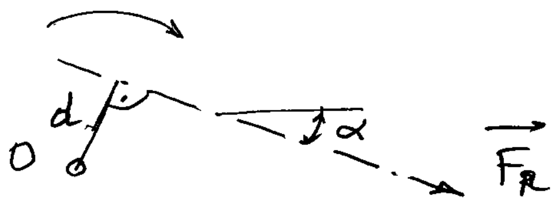


$$\cos\alpha = \frac{F_{Rx}}{F_R} = \frac{29{,}944}{32{,}066} = 0{,}9338$$
$$\alpha = 20{,}96^\circ$$

$$\sum_i M_0^{F_i} = M_0^{F_1} + M_0^{F_2} + M_0^{F_3} + M_0^{F_4} =$$
$$= \left(F_1 \cdot \frac{2}{\sqrt{5}} \cdot 3a - F_1 \cdot \frac{1}{\sqrt{5}} \cdot 2a\right) + \left(-F_2 \cdot \frac{4}{5} \cdot 4a - F_2 \cdot \frac{3}{5} \cdot 2a\right) +$$
$$+ F_3 \cdot 6a - F_4 \cdot a =$$
$$= 10 \cdot \frac{2}{\sqrt{5}} \cdot 3 \cdot 1 - 10 \cdot \frac{1}{\sqrt{5}} \cdot 2 \cdot 1 - 20 \cdot \frac{4}{5} \cdot 4 \cdot 1 - 20 \cdot \frac{3}{5} \cdot 2 \cdot 1 +$$
$$+ 5 \cdot 6 \cdot 1 - 5 \cdot 1 = -45{,}111\ Nm$$

$$\vec{M}_O^{F_R} = \sum_i \vec{M}_O^{F_i} = -45{,}111\,\vec{k} \quad [Nm]$$

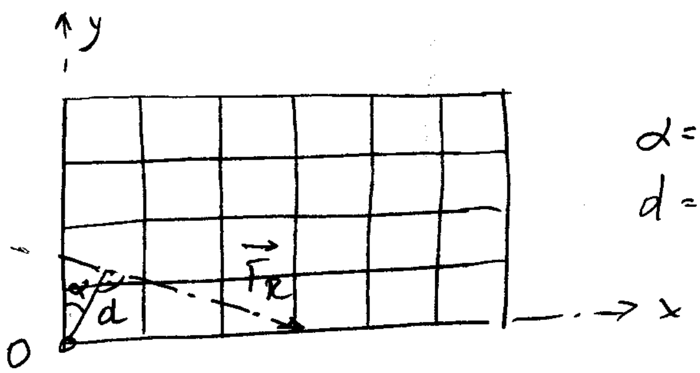




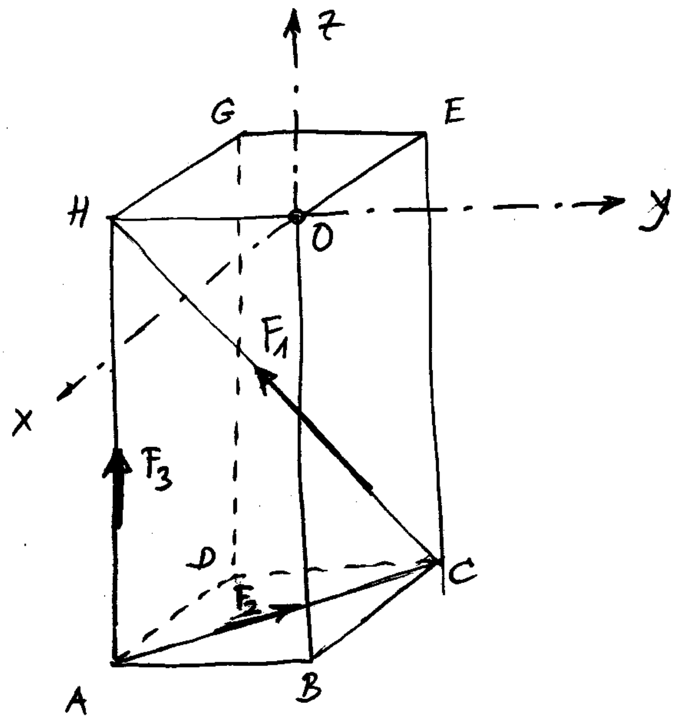
Rezultanta $\vec{F_R}$ mora s obzirom na O imati moment $\vec{M}_O^{F_R} = -45{,}111\,\vec{k}$ [Nm].
Iznos tog momenta mora biti

$$d \cdot |\vec{F_R}| = |\vec{M}_O^{F_R}|$$

$$\Rightarrow \underline{\underline{d}} = \frac{|\vec{M}_O^{F_R}|}{|\vec{F_R}|} = \frac{45{,}111}{32{,}066} = \underline{\underline{1{,}4068\ m}}$$



$\alpha = 20{,}96°$
$d = 1{,}4068\ m$



Reducirati zadani sustav sila na točku O.

Zadano: $F_1 = 130\,N$,

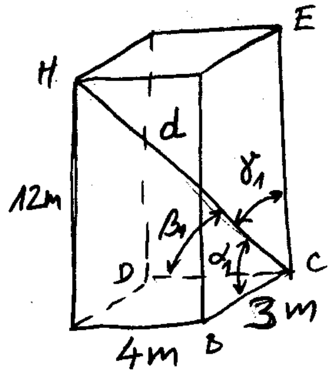
$F_2 = 50\,N$, $F_3 = 80\,N$,

$\overline{AB} = \overline{CD} = \overline{HO} = \overline{GE} = 4\,m$,

$\overline{AD} = \overline{BC} = \overline{HG} = \overline{OE} = 3\,m$,

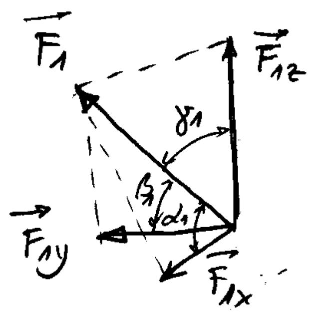
$\overline{AH} = \overline{BO} = \overline{CE} = \overline{DG} = 12\,m$.

Komponente sila;



$$d = \sqrt{3^2 + 4^2 + 12^2} = 13\,m$$

$$\cos\alpha_1 = \frac{3}{13}, \quad \cos\beta_1 = \frac{4}{13}, \quad \cos\gamma_1 = \frac{12}{13}$$

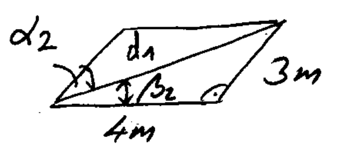


IZNOSI:

$$F_{1x} = F_1 \cos\alpha_1 = 130 \cdot \frac{3}{13} = 30\,N$$

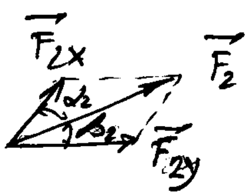
$$F_{1y} = F_1 \cos\beta_1 = 130 \cdot \frac{4}{13} = 40\,N$$

$$F_{1z} = F_1 \cos\gamma_1 = 130 \cdot \frac{12}{13} = 120\,N$$



$$d_1 = \sqrt{3^2 + 4^2} = 5\,m$$

$$\cos\alpha_1 = \frac{3}{5}, \quad \cos\beta_2 = \frac{4}{5}$$



IZNOSI:

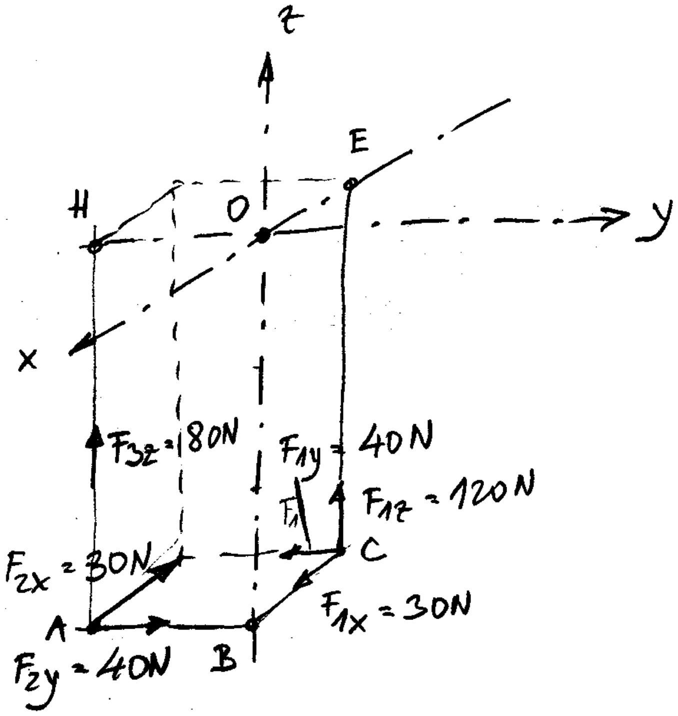
$$F_{2x} = F_2 \cos\alpha_2 = 50 \cdot \frac{3}{5} = 30\,N$$

$$F_{2y} = F_2 \cos\beta_2 = 50 \cdot \frac{4}{5} = 40\,N$$

$$F_{2z} = 0$$

IZNOSI:

$\vec{F_3} = \vec{F_{3z}}$ $\qquad F_{3x} = 0 \,, \quad F_{3y} = 0 \,, \quad F_{3z} = 80\,N$



F_{1x} probada yz ravninu u B

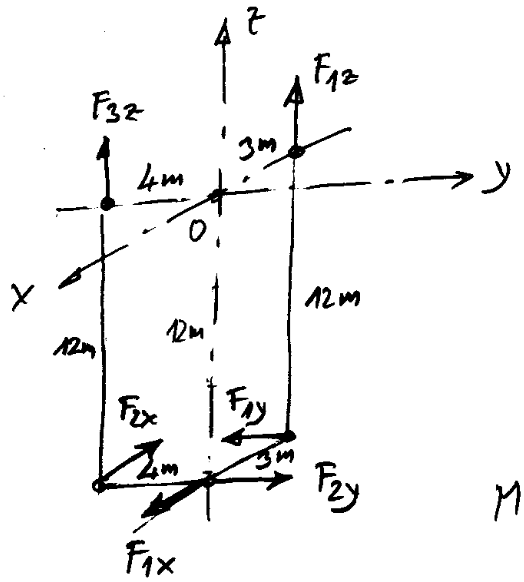
F_{1y} -"- xz -"- u C

F_{1z} -"- xy -"- u E

F_{2x} -"- yz -"- u A

F_{2y} -"- xz -"- u B

F_{3z} -"- xy -"- u H



$$F_{Rx} = \Sigma F_{ix} = F_{1x} - F_{2x} = 30 - 30 = 0$$

$$F_{Ry} = \Sigma F_{iy} = -F_{1y} + F_{2y} = -40 + 40 = 0$$

$$F_{Rz} = \Sigma F_{iz} = F_{1z} + F_{3z} = 120 + 80 = 200\,N$$

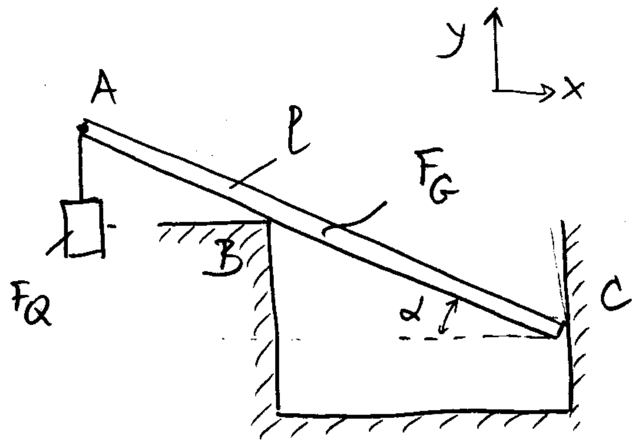
$$\underline{\underline{\vec{F_R} = 200\,\vec{k}\ [N]}} \,, \quad |\vec{F_R}| = 200\,N$$

$$M_{Rx} = \Sigma M_x^{F_i} = -F_{1y} \cdot 12 - F_{3z} \cdot 4 = = -40 \cdot 12 - 80 \cdot 4 = -800\,Nm$$

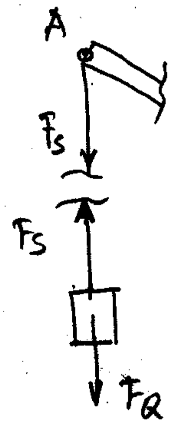
$$M_{Ry} = \Sigma M_y^{F_i} = -F_{1x} \cdot 12 + F_{1z} \cdot 3 + F_{2x} \cdot 12 = = -30 \cdot 12 + 120 \cdot 3 + 30 \cdot 12 = 360\,N$$

$$M_{Rz} = \Sigma M_z^{F_i} = F_{1y} \cdot 3 - F_{2x} \cdot 4 = = 40 \cdot 3 - 30 \cdot 4 = 0$$

$$\vec{M_R} = -800\,\vec{i} + 360\,\vec{j}\ [Nm] \,, \quad |\vec{M_R}| = \sqrt{(-800)^2 + 360^2} = 877{,}268\,Nm$$



U točki A homogene grede AC, duljine l, zanemarivih poprečnih dimenzija i težine F_G, obješen je teret težine F_Q. Greda se oslanja o glatku podlogu u B te pritišće u C na vertikalni glatki zid. Potrebno je gredu osloboditi veza s okolinom.

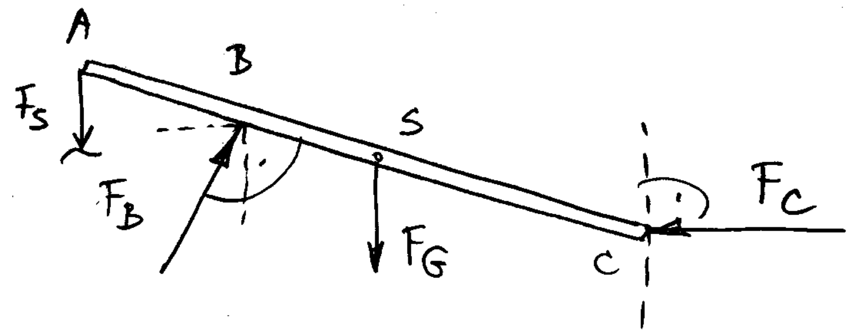


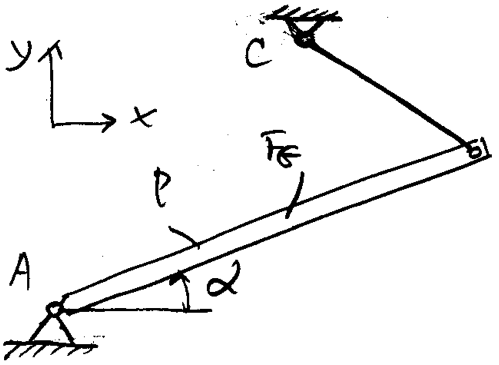
Uže na kojem visi uteg u mislima prerežemo i zamijenimo reakcijom F_s koja djeluje po pravcu užeta, usmjerene od utega, odnosno od grede, u točki A.

Po principu akcije i reakcije, ove dvije suprotno usmjerene sile su jednakog iznosa.

U B greda se oslanja na glatki brid „stepenice". Ako se ona „stepenice" uklone kao veza, zamijenit će je reakcija okomita na zajedničku tangencijalnu ravninu, a te prolazi donjom plohom grede. U dvodimenzijskom prikazu, pravac reakcije u B je OKOMIT na gredu. U C se greda oslanja na glatki vertikalni zid. Ako se taj zid kao veza ukloni, zamijenit će ga reakcija okomita na zajedničku tangencijalnu ravninu,

a ta je podudarna sa ravninom zida.
Aktivne sile, težina F_G, djeluje u geometrijskom težištu grede. Kod homogenih tijela, tj. tijela jednolike gustoće, težište kao hvatište sile težine podudara se s geom. težištem, tj. težištem volumena tijela.

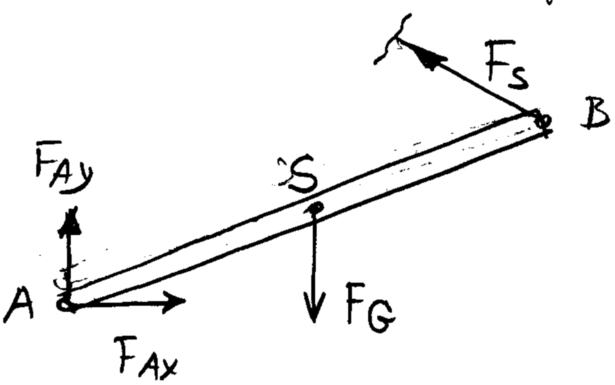


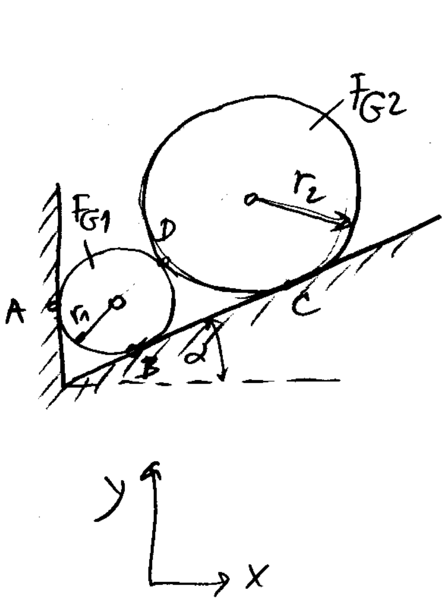


Homogena greda duljine l, zanemarivih poprečnih dimenzija i težine F_G, zglobno je vezana za nepomičnu podlogu u A te učvršćena u točki B užetom BC.

Potrebno je gredu osloboditi veze s okolinom.

Zglobnu vezu u A, koja sprječava pomak točke A u ravnini xy, zamijenit će reakcija čiji pravac i smjer u xy ravnini nije poznat, pa se predstavlja sa svoje dvije nepoznate komponente F_{Ax} i F_{Ay} čiji se smjerovi pretpostavljaju. Uže BC zamijenit će sila F_S koja djeluje po pravcu užeta, usmjerena <u>od</u> grede. Aktivna je sila F_G koja djeluje u geom. težištu homogene grede.

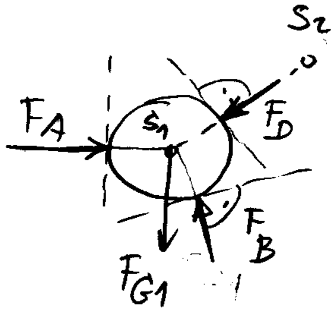




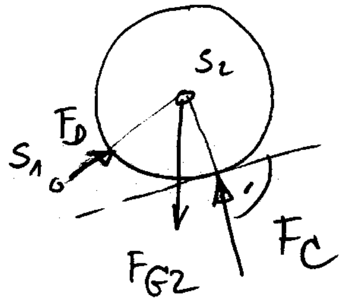
Homogena kugla polumjera r_1 i težine F_{G1} dodiruje glatku podlogu i glatki zid u A, odnosno u B. Na onu kuglu naslanja se u D druga homogena kugla polumjera r_2 i težine F_{G2}, koja se oslanja na glatku podlogu u C.

Potrebno je obje kugle osloboditi veza s okolinom.

Manja kugla oslanja se na glatki zid u A i glatku podlogu u B te na veću kuglu u D. Ako se ~~zid~~, podloga i oslanjajuća kugla uklone, zamijenit će ih reakcije okomite na zajedničke tangencijalne ravnine, usmjerene prema kugli. U A zajednička tang. ravnina podudara se s ravninom vertikalnog zida, u B s ravninom kose podloge, a u D s ravninom okomitom na spojnicu središta kugli. Aktivna sila F_{G1} djeluje u geom. težištu kugle.

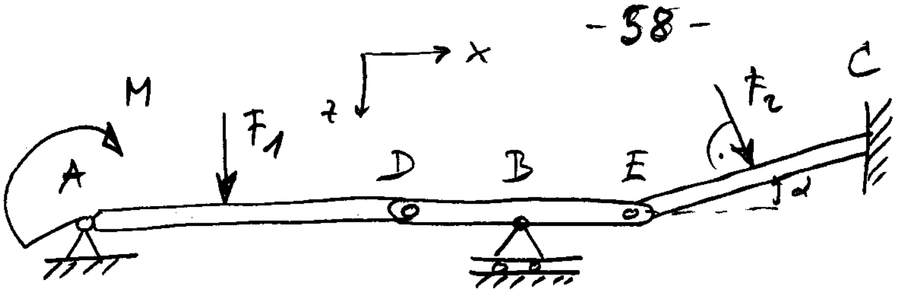


Veća kugla oslanja se u C na kosu podlogu i u D na manju kuglu. Analogno dobijemo:



Reakcije u D, suprotno usmjerene za malu i veliku kuglu, po principu akcije i reakcije jednakog su iznosa.

Reakcije koje djeluju na kugle PROLAZE KROZ SREDIŠTA KUGLI jer okomica na bilo koju tangencijalnu ravninu položenu na kuglu mora prolaziti kroz njezino središte.



Nosač ADE oslonjen je u A na nepomični oslonac i u B na pomični. Nosač se sastoji od dijelova AD i DE koji su zglobno vezani u D.

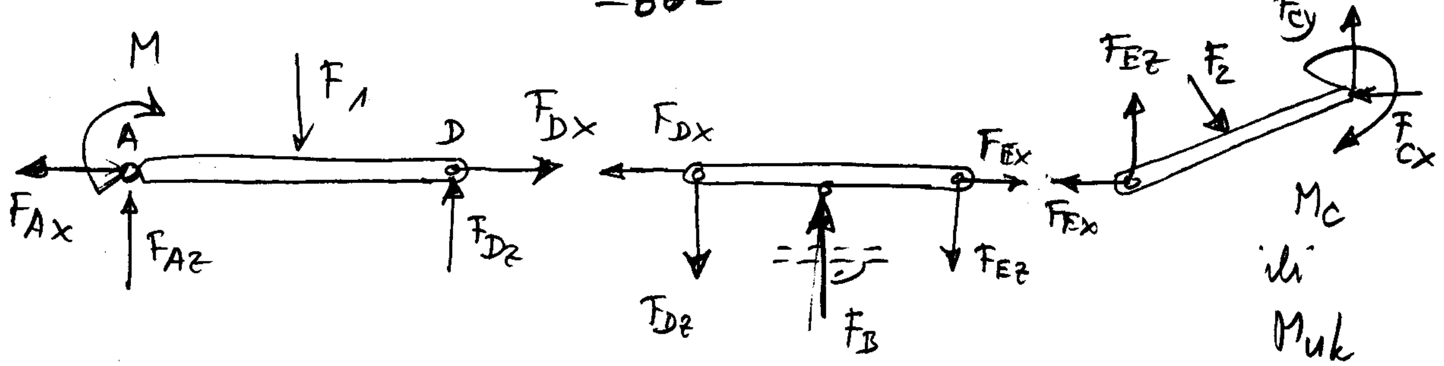
Na desnom kraju, u E, nosač ADE vezan je zglobno s nosačem EC koji je uklješten u nepomičnu okolinu u C. Nosači su opterećeni silama F_1 i F_2 te spregom M. Potrebno je nosače osloboditi veza.

Zglobna veza u A onemogućava pomak točke A u xz ravnini, u odnosu na nepomičnu podlogu, a zglobna veza dijelova AD i DE nosača ADE u točki D sprečava pomak točke D u xz ravnini, tj. točke D kao točke dijela AD u odnosu na dio DE, odnosno točke D kao točke dijela DE u odnosu na dio AD. U točki E nosači ADE i EC povezani su zglobnom vezom koja sprečava pomak točke E kao točke nosača ADE u odnosu na nosač EC, odnosno točke E kao točke nosača EC u odnosu na nosač ADE. U sva tri slučaja uklonjene zglobne veze zamijenit će reakcije čiji pravci i smjerovi u xz ravnini nisu poznati. Predstavljaju se svojim nepoznatim komponentama čiji se smjerovi pretpostavljaju uz poštovanje principa akcije i reakcije u zglobovima D i E.

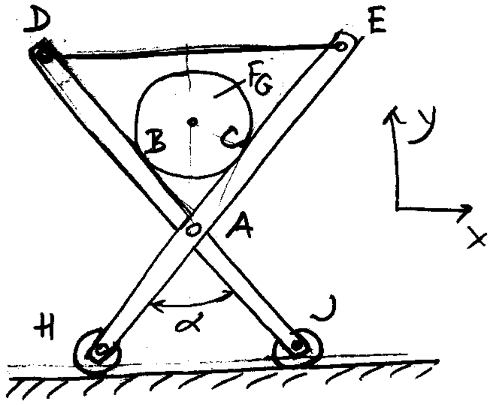
U B je tzv. pomični oslonac koji spriječava pomak točke B u smjeru okomitom na ravninu mogućeg pomaka oslonca, a to je pravac paralelan osi z. Uklonjeni oslonac u B zamijenit će reakcija paralelna osi z, tj. okomita na ravninu mogućeg pomaka oslonca. Smjer reakcije F_B pretpostavljamo. On je moguć i prema nosaču i od nosača.

U C je nosač CE uklješten. Uklještenje onemogućava pomak točke C u xz ravnini i rotaciju nosača CE oko točke C. Uklonjeno uklještenje zamjenjujemo reakcijom nepoznatog pravca i smjera u xz ravnini koju predstavljamo s upravo dvije nepoznate komponente čije smjerove pretpostavljamo i tzv. MOMENT UKLJEŠTENJA (koji ~~zamjenjuje~~ zamjenjuje efekt sprečavanja rotacije uklještenja).

3



Moment uklještenja bolje je označavati M_{uk} nego M_C jer će se s M_C kasnije označavati moment savijanja u točki C.

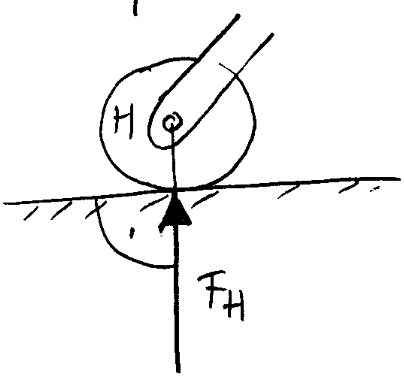


Gladak valjak težine F_G oslonjen je na dvije jednake glatke grede zglobno spojene u A i povezane užetom DE. Grede su preko valjčića u H i J oslonjene na glatku horizontalnu podlogu.

Potrebno je osloboditi veza s okolinom sustav kao cjelinu i svako od tijela zasebno.

VALJČIĆI U H I J

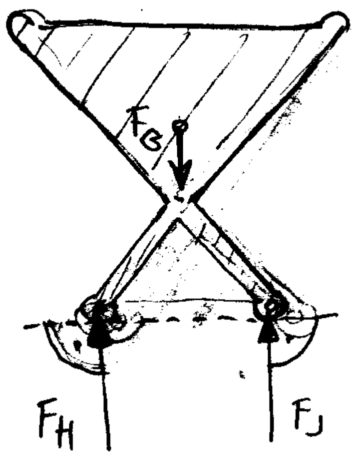
Valjčići se oslanjaju na glatku horizontalnu podlogu. Ako u mislima uklonimo tu podlogu, zamijenit ćemo je reakcijom okomitom na zajedničku tangencijalnu ravninu, a ta je identična horizontalnoj ~~ravnini~~; usmjerenom prema valjčiću. Reakcija je okomita na obod valjčića, pa dakle njen ~~pravac~~ prolazi kroz uzdužnu os valjčića, tj. u dvodimenzijskom prikazu kroz točku H.

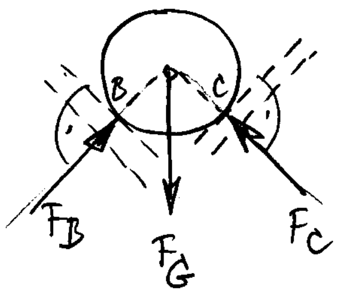


Zato možemo smatrati da F_H djeluje u H (silu smo pomaknuli duž njenog pravca djelovanja).

SUSTAV TRI TIJELA (VALJAK i DVIJE GREDE) OSLOBOĐEN VEZA S OKOLINOM

Sustav tijela promatramo kao KRUTU CJELINU. Ovaj se postupak naziva SOLIDIFIKACIJA. Tu cjelinu oslobađamo veza s okolinom, a to je oslanjanje preko valjčića na glatkoj horizontalnoj ravnini. Jedina aktivna sila je težina valjka F_G u njegovom težištu.

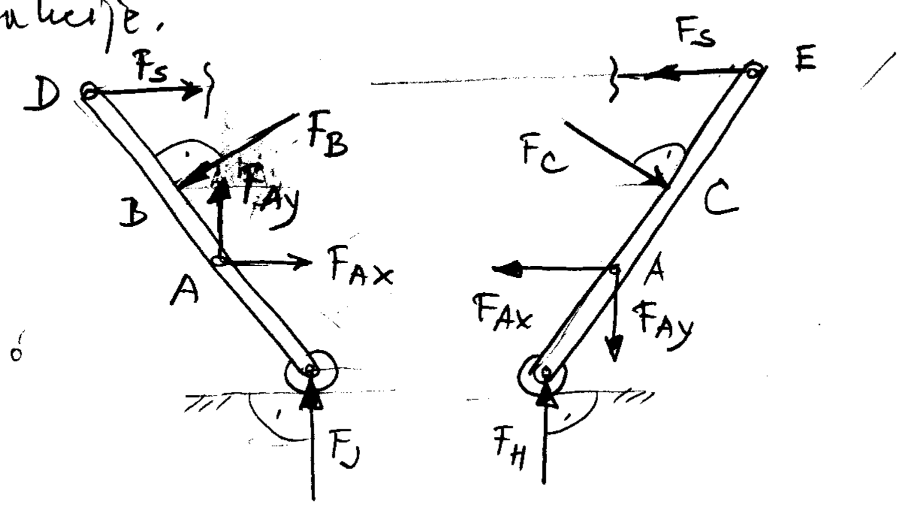


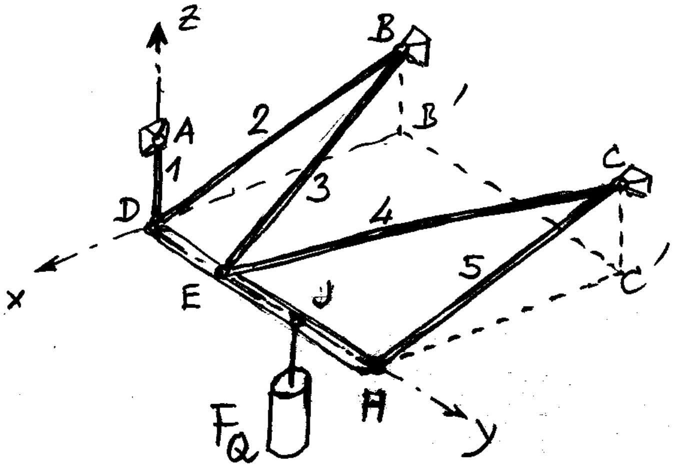


Valjak se oslanja u B i C na grede. Ako se grede u mislima uklone, zamijenit će ih reakcije okomite na zajedničku tangencijalnu ravninu grede i valjka, a ta je određena plohom grede na koju se oslanja valjak, usmjerena prema valjku. Pravci reakcija u B i C, okomiti na tangentu na kružnicu u dvodimenzijskom prikazu, prolaze kroz uzdužnu os valjka, odnosno u ovom prikazu kroz središte kružnice.

Grede su međusobno vezane užetom DE. Ako se ono u mislima prereže, zamijenit će ga reakcije po pravcu užeta, usmjerene od greda u D i E. Po principu akcije i reakcije to su obje jednake i suprotno usmjerene sile.

U A Grede su vezane zglobnom vezom koja sprečava pomak točke A jedne grede u odnosu na drugu U RAVNINI (xy ravnini) u kojoj grede leže. Uklanjanje te zglobne veze zamijenit će se reakcijom kojoj je pravac i smjer u xy ravnini nepoznat, pa se predstavlja sa svoje dvije komponente čiji se smjerovi pretpostavljaju. Smjerovi komponenata su suprotni na gredama, po principu akcije i reakcije. U B i u C na grede djeluju F_B, odnosno F_C u smjeru suprotnom od njihovog smjera djelovanja na valjak – i opet po principu akcije i reakcije.





Homogena greda DH težine F_G opterećena je teretom težine F_Q. Pri tome je u prostoru nose vertikalni štap 1 i kosi štapovi 2, 3, 4 i 5. Štapovi su zglobno vezani za okolinu u A, B i C.

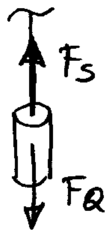
Poznato je:

$\overline{AD} = \overline{DE} = \overline{EJ} = \overline{JH} = \overline{BB'} = \overline{CC'} = a$,

$\overline{DB'} = \overline{B'C'} = \overline{C'H} = 3a$.

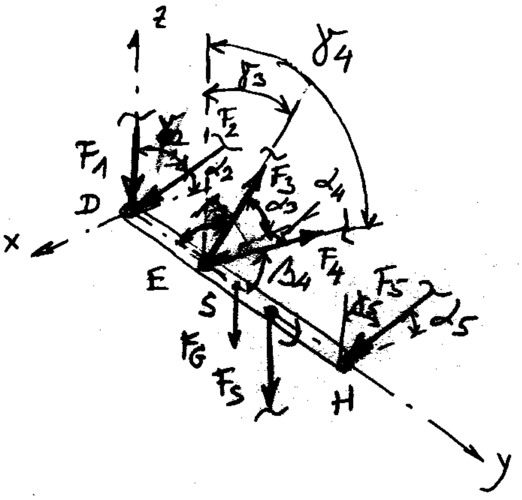
Gredu je potrebno osloboditi veza i odrediti kutove pod kojima su usmjerene sile u štapovima.

Teret težine F_Q vezan je s gredom užetom pričvršćenim u točki J. ~~Teret~~ oslobađamo ~~te~~ veze tako da se uže u mislima presječe te se ucrta reakcija veze F_S na pravcu užeta usmjerena od tereta:



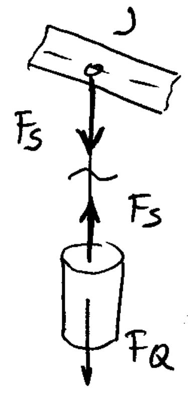
Ucrtamo i aktivnu silu koju djeluje na teret –
– to je težina F_Q.

Greda DH vezana je s okolinom štapovima 1, 2, 3, 4 i 5, a s teretom užetom u J.



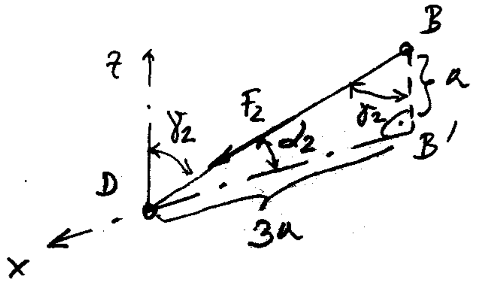
Presječemo u mislima štapove i te uklonjene veze zamijenimo reakcijama s hvatištem na mjestu vezanja štapa na gredu. Reakcije djeluju PO PRAVCIMA štapova, a PRETPOSTAVLJAMO DA F_1, F_2 i F_5 djeluju prema gredi, a F_3 i F_4 da su usmjerene od grede.

Uže u točki J zamijenimo reakcijom F_S koja djeluje po pravcu užeta, usmjerena od grede. Ta je sila, po principu akcije i reakcije jednakog iznosa i suprotnog smjera od reakcije koja zamjenjuje, to isto „privezano" ~~uže~~ kao vezu tereta s gredom.



Aktivna sila koja djeluje na gredu je težina F_G u težištu S na sredini grede.
(Kod homogenih tijela, tj. tijela jednolike gustoće, težište se podudara s geometrijskim težištem.)

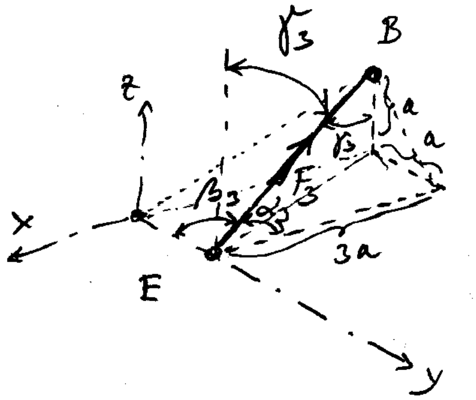
Određivanje kutova koje zatvaraju pravci reakcija, tj. sile u štapovima s koordinatnim osima:



$$\overline{DB} = \sqrt{(3a)^2 + a^2} = \sqrt{10}\,a$$

$$\cos\alpha_2 = \frac{3a}{\sqrt{10}\,a} = \frac{3}{\sqrt{10}}$$

$$\cos\gamma_2 = \frac{a}{\sqrt{10}\,a} = \frac{1}{\sqrt{10}}$$

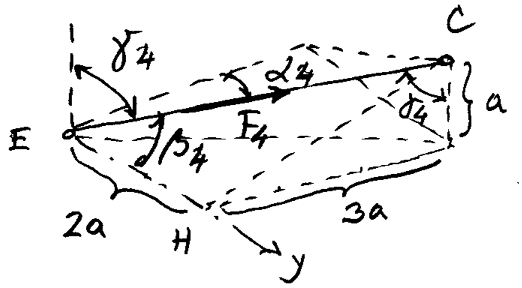


$$\overline{EB} = \sqrt{(3a)^2 + a^2 + a^2} = \sqrt{11}\,a$$

$$\cos\alpha_3 = \frac{3a}{\sqrt{11}\,a} = \frac{3}{\sqrt{11}}$$

$$\cos\beta_3 = \frac{a}{\sqrt{11}\,a} = \frac{1}{\sqrt{11}}$$

$$\cos\gamma_3 = \frac{a}{\sqrt{11}\,a} = \frac{1}{\sqrt{11}}$$

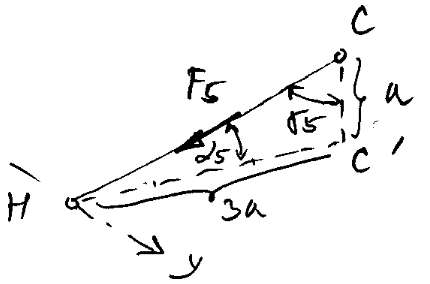


$$\overline{EC} = \sqrt{(2a)^2 + (3a)^2 + a^2} = \sqrt{14}\,a$$

$$\cos\alpha_4 = \frac{3a}{\sqrt{14}\,a} = \frac{3}{\sqrt{14}}$$

$$\cos\beta_4 = \frac{2a}{\sqrt{14}\,a} = \frac{2}{\sqrt{14}}$$

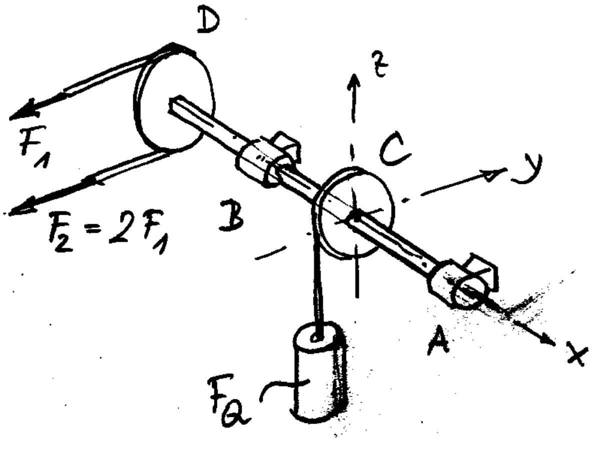
$$\cos\gamma_4 = \frac{a}{\sqrt{14}\,a} = \frac{1}{\sqrt{14}}$$



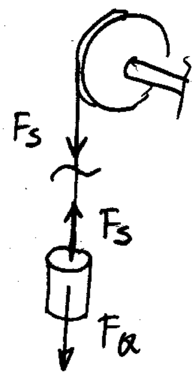
$$\overline{HC} = \sqrt{(3a)^2 + a^2} = \sqrt{10}\,a$$

$$\cos\alpha_5 = \frac{3a}{\sqrt{10}\,a} = \frac{3}{\sqrt{10}}$$

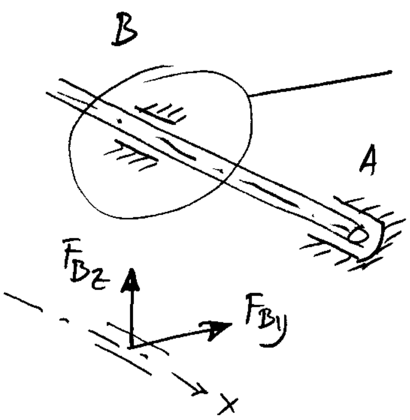
$$\cos\gamma_5 = \frac{a}{\sqrt{10}\,a} = \frac{1}{\sqrt{10}}$$



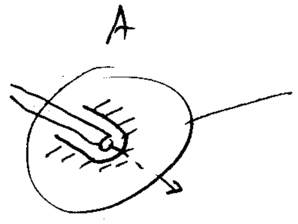
Na vretilu oslonjenom u radijalno-aksijalnom ležaju A i radijalnom ležaju B nalaze se remenice D polumjera r_1 i kolotur C polumjera r_2. Na obod kolotura učvršćen je jedan kraj užeta na čijem je donjem kraju vezan teret težine F_Q. Na remenicu djeluju sile F_1 i $F_2 = 2F_1$ prema slici. Poznato je $\overline{AC} = \overline{CB} = \overline{BD} = a$.
Vretilo je potrebno osloboditi veza.



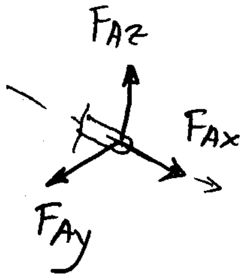
Uže zamijenimo reakcijom F_S. Sila F_S djeluje po pravcu užeta usmjerena <u>od</u> tereta, odnosno <u>od</u> kolotura.



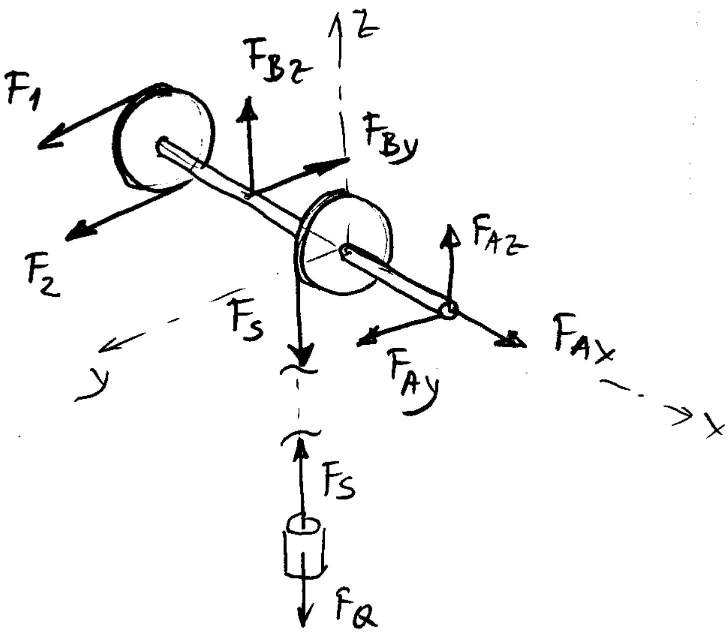
RADIJALNI LEŽAJ SPRJEČAVA POMAK VRATILA U SMJERU OKOMITOM NA SVOJU UZDUŽNU OS, PODUDARNU S UZDUŽNOM OSI VRATILA. AKO SE RADIJALNI LEŽAJ KAO VEZA UKLONI, ZAMIJENIT ĆE GA REAKCIJA KOJA JE OKOMITA NA UZDUŽNU OS VRATILA. PRAVAC I SMJER OVE REAKCIJE U ODNOSU NA OSI OKOMITE NA UZDUŽNU OS <u>NIJE</u> POZNAT, PA SE REAKCIJA PREDSTAVLJA SA SVOJE DVIJE NEPOZNATE KOMPONENTE OKOMITE NA UZDUŽNU OS ČIJI SE SMJEROVI PRETPOSTAVE.

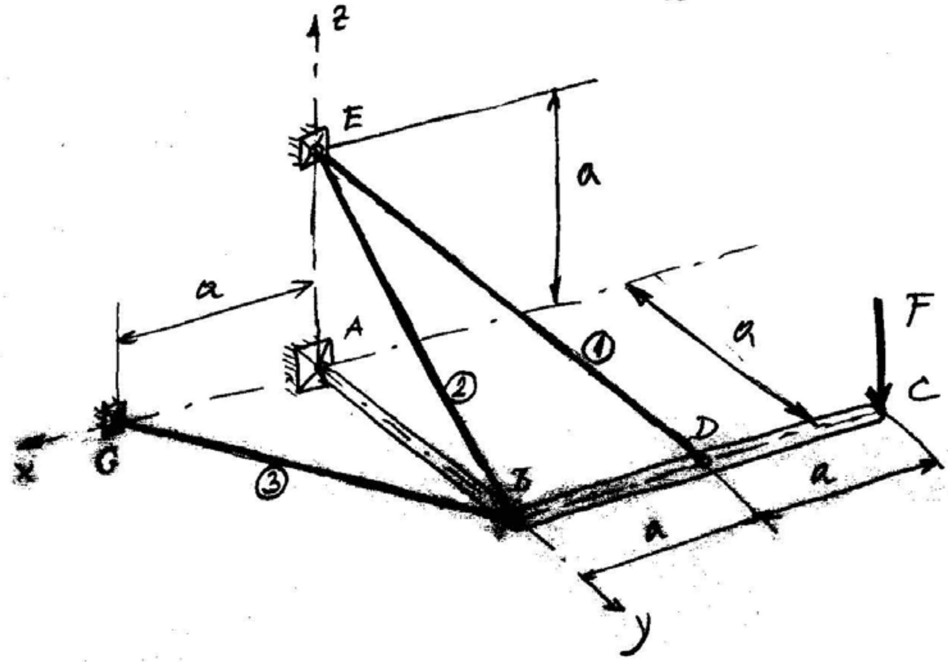


RADIJALNO AKSIJALNI LEŽAJ SPRJEČAVA POMAK VRATILA I U UZDUŽNOM SMJERU I U SMJERU OKOMITOM NA UZDUŽNU OS. AKO SE OVAKAV LEŽAJ KAO VEZA UKLONI, ZAMIJENIT ĆE GA REAKCIJA, KOJA LEŽI NA OPĆEM PRAVCU U PROSTORU KOJI NIJE POZNAT I DJELUJE U SMJERU KOJI NIJE POZNAT, PREDSTAVLJA SE SA SVOJE TRI NEPOZNATE KOMPONENTE ČIJI SE SMJEROVI PRETPOSTAVLJAJU.

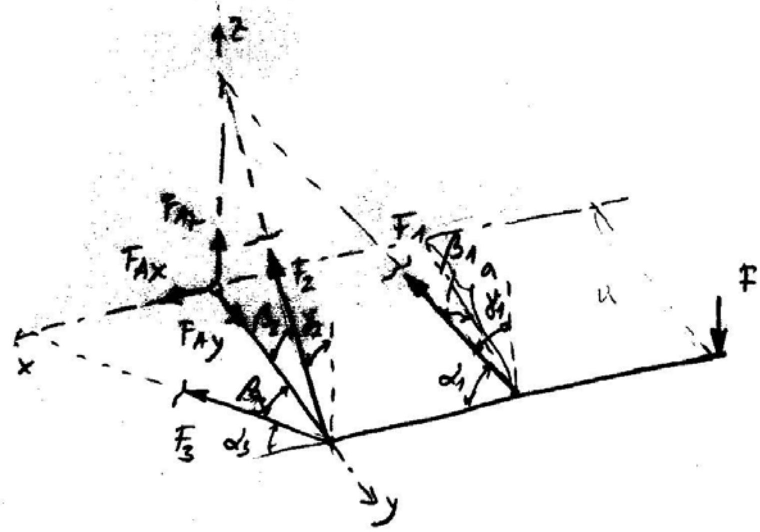


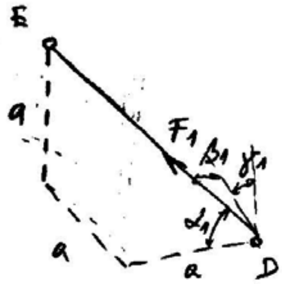
(y i z komponente „sprječavaju" pomak okomit na uzdužnu os, a x komponenta „sprječava" uzdužni pomak.)





Na zakrivljeni štap ABC koji je vezan za podlogu sfernim zglobom u A i zglobno vezanim štapovima 1, 2 i 3 djeluje sila F prema slici. Potrebno je odrediti sile u zglobu A i u štapovima 1, 2 i 3. Zadano: $F = 2$ kN.





$$\overline{DE} = d_1 = \sqrt{a^2+a^2+a^2} = \sqrt{3}\,a$$

$$\cos\alpha_1 = \frac{a}{d_1} = \frac{1}{\sqrt{3}}$$

$$\cos\beta_1 = \frac{a}{d_1} = \frac{1}{\sqrt{3}}$$

$$\cos\gamma_1 = \frac{a}{d_1} = \frac{1}{\sqrt{3}}$$

$$\overline{BE} = d_2 = \sqrt{a^2+a^2} = \sqrt{2}\,a$$

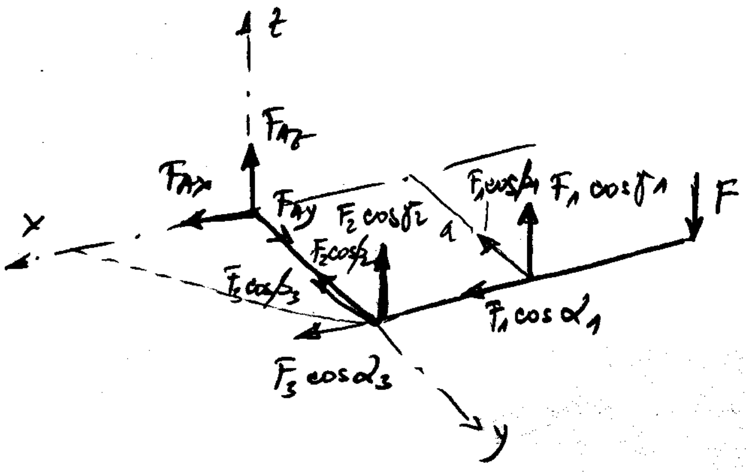
$$\cos\beta_2 = \frac{a}{d_1} = \frac{1}{\sqrt{2}}$$

$$\cos\gamma_2 = \frac{a}{d_1} = \frac{1}{\sqrt{2}}$$

$$\overline{BG} = d_3 = \sqrt{a^2+a^2} = \sqrt{2}\,a$$

$$\cos\alpha_3 = \frac{a}{d_3} = \frac{1}{\sqrt{2}}$$

$$\cos\beta_3 = \frac{a}{d_3} = \frac{1}{\sqrt{2}}$$



$\Sigma F_x = 0 \qquad F_{Ax} + F_1 \cos\alpha_1 + F_3 \cos\alpha_3 = 0 \qquad (1)$

$\Sigma F_y = 0 \qquad F_{Ay} - F_1 \cos\beta_1 - F_2 \cos\beta_2 - F_3 \cos\beta_3 = 0 \qquad (2)$

$\Sigma F_z = 0 \qquad F_{Az} + F_1 \cos\gamma_1 + F_2 \cos\gamma_2 - F = 0 \qquad (3)$

$\Sigma M_x = 0 \qquad F_1 \cos\gamma_1 \cdot a + F_2 \cos\gamma_2 \cdot a - F \cdot a = 0 \qquad (4)$

$\Sigma M_y = 0 \qquad F_1 \cos\gamma_1 \cdot a - F \cdot 2a = 0 \qquad (5)$

$\Sigma M_z = 0 \qquad -F_3 \cos\alpha_3 \cdot a = 0 \qquad (6)$

(sile F_1 i F_2 sijeku os z !)

$(6) \rightarrow \underline{F_3 = 0}$

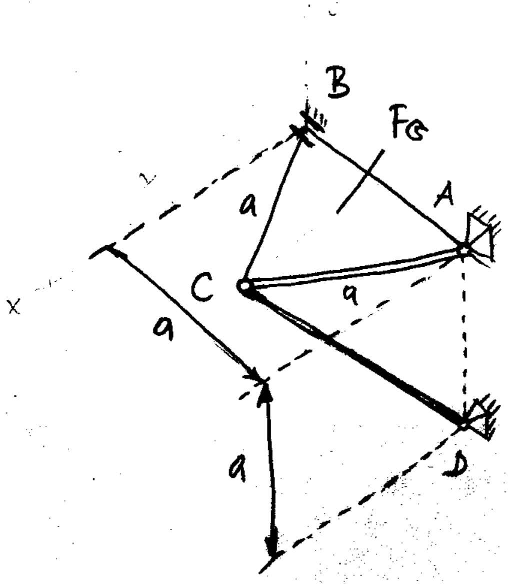
$(5) \rightarrow \underline{F_1} = \frac{2}{\cos\gamma_1} F = \frac{2}{\frac{1}{\sqrt{3}}} \cdot 2 = 4\sqrt{3} = \underline{6{,}928 \text{ kN}}$

$(4) \Rightarrow \underline{F_2} = \frac{F - F_1 \cos\gamma_1}{\cos\gamma_2} = \frac{2 - 4\sqrt{3} \cdot \frac{1}{\sqrt{3}}}{\frac{1}{\sqrt{2}}} = -2\sqrt{2} = \underline{-2{,}828 \text{ kN}}$

$(1) \Rightarrow \underline{F_{Ax}} = -F_1 \cos\alpha_1 - F_3 \cos\alpha_3 = -4\sqrt{3} \cdot \frac{1}{\sqrt{3}} - 0 = \underline{-4 \text{ kN}}$

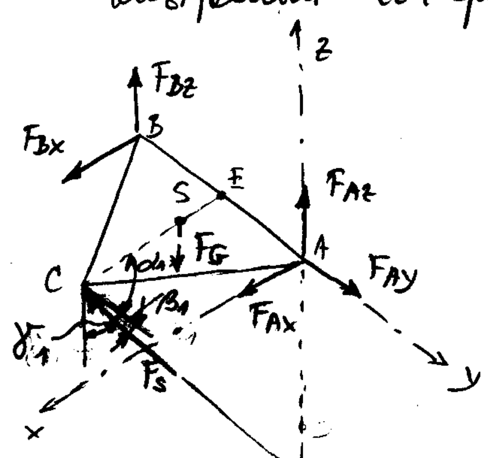
$(2) \Rightarrow \underline{F_{Ay}} = F_1 \cos\beta_1 + F_2 \cos\beta_2 + F_3 \cos\beta_3 = 4\sqrt{3} \cdot \frac{1}{\sqrt{3}} + (-2\sqrt{2}) \cdot \frac{1}{\sqrt{2}} + 0 = \underline{2 \text{ kN}}$

$(3) \Rightarrow \underline{F_{Az}} = -F_1 \cos\gamma_1 - F_2 \cos\gamma_2 + F = -4\sqrt{3} \cdot \frac{1}{\sqrt{3}} - (-2\sqrt{2}) \cdot \frac{1}{\sqrt{2}} + 2 = \underline{0}$

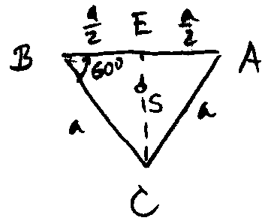


Homogena ploča ABC u obliku istostraničnog trokuta stranice a i težine $F_G = 3$ kN nalazi se u horizontalnoj ravnini, a vezana je na vertikalni zid radijalno-aksijalnim ležajem u A (ili zglobom A) i radijalnim ležajem u B te štapom CD prema slici. Potrebno je odrediti reakcije u ležajevima i silu u štapu.

U zadacima s prostornim sustavom sila dobro je ishodište koordinatnog sustava odabrati u radijalno-aksijalnom ležaju ili prostornom zglobu (odnosno nepomičnom osloncu) jer u tom se slučaju u momentnim jednadžbama ravnoteže neće javljati 3 nepoznanice - komponente reakcije u radijalno-aksijalnom ležaju.

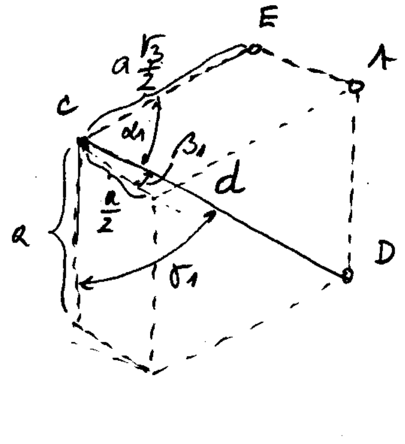


Kosinusi kutova što ih zatvara pravac sile u štapu s paralelama s koordinatnim osima



$\overline{CE} = v = a \sin 60° = a \cdot \frac{\sqrt{3}}{2}$

$\overline{ES} = \frac{1}{3} v = \frac{1}{3} a \frac{\sqrt{3}}{2}$



$$\underline{\underline{d}} = \sqrt{\left(a\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{a}{2}\right)^2 + a^2} = \sqrt{2a^2} = \underline{\underline{\sqrt{2}a}}$$

$$\cos\alpha_1 = \frac{a\frac{\sqrt{3}}{2}}{\sqrt{2}a} = \frac{\sqrt{3}}{2\sqrt{2}}$$

$$\cos\beta_1 = \frac{\frac{a}{2}}{\sqrt{2}a} = \frac{1}{2\sqrt{2}}$$

$$\cos\gamma_1 = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$$

$\Sigma F_x = 0$

$$F_{Ax} + F_{Bx} + F_S \cos\alpha_1 = 0$$

$$F_{Ax} + F_{Bx} + F_S \cdot \frac{\sqrt{3}}{2\sqrt{2}} = 0 \qquad (1)$$

$\Sigma F_y = 0$

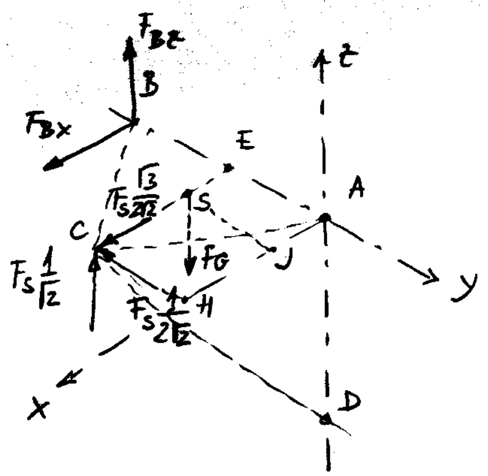
$$F_{Ay} - F_S \cos\beta_1 = 0$$

$$F_{Ay} - F_S \frac{1}{2\sqrt{2}} = 0 \qquad (2)$$

$\Sigma F_z = 0$

$$F_{Az} + F_{Bz} + F_S \cos\gamma_1 - F_G = 0$$

$$F_{Az} + F_{Bz} + F_S \frac{1}{\sqrt{2}} - F_G = 0 \qquad (3)$$



$\Sigma M_x = 0$

$$-F_{Bz} \cdot \overline{BA} - F_S \frac{1}{\sqrt{2}} \cdot \overline{CH} + F_G \, \overline{SJ} = 0$$

$$-F_{Bz} \cdot a - F_S \frac{1}{\sqrt{2}} \cdot \frac{a}{2} + F_G \cdot \frac{a}{2} = 0 \qquad (4)$$

(y - komponenta F_S siječe os x pa joj je krak = 0 !)

$\Sigma M_y = 0$

$$-F_S \cdot \frac{1}{\sqrt{2}} \cdot \overline{CE} + F_G \cdot \overline{SE} = 0$$

$$-F_S \cdot \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} a + F_G \cdot \frac{1}{3} \cdot \frac{\sqrt{3}}{2} a = 0 \qquad (5)$$

(x - komponenta F_S siječe os y pa joj je krak = 0 !)

$\Sigma M_z = 0$

$$F_{Bx} \cdot \overline{BA} = 0$$

$$F_{Bx} \cdot a = 0 \qquad (6)$$

(sila F_S leži na pravcu CD koji siječe os z, pa je sila nema moment oko te osi !)

$$(6) \Rightarrow \quad \underline{\underline{F_{Bx} = 0}}$$

$$(5) \Rightarrow \quad F_S \cdot \frac{1}{\sqrt{2}} = F_G \cdot \frac{1}{3}$$

$$\underline{\underline{F_S}} = \frac{\sqrt{2}}{3} F_G = \frac{\sqrt{2}}{3} \cdot 3 = \underline{\underline{1{,}414\ kN}}$$

$$\rightarrow (4) \Rightarrow \quad -F_{Bz} \cdot \cancel{a} - \frac{\sqrt{2}}{3} F_G \cdot \frac{1}{\cancel{\sqrt{2}}} \cdot \frac{\cancel{a}}{2} + F_G \cdot \frac{\cancel{a}}{2} = 0$$

$$\underline{\underline{F_{Bz}}} = F_G \left(\frac{1}{2} - \frac{1}{6} \right) = \frac{2}{6} F_G = \frac{1}{3} F_G = \frac{1}{3} \cdot 3 = \underline{\underline{1\ kN}}$$

$$\rightarrow (1) \quad F_{Ax} + 0 + \frac{\cancel{\sqrt{2}}}{3} F_G \cdot \frac{\sqrt{3}}{2\cancel{\sqrt{2}}} = 0$$

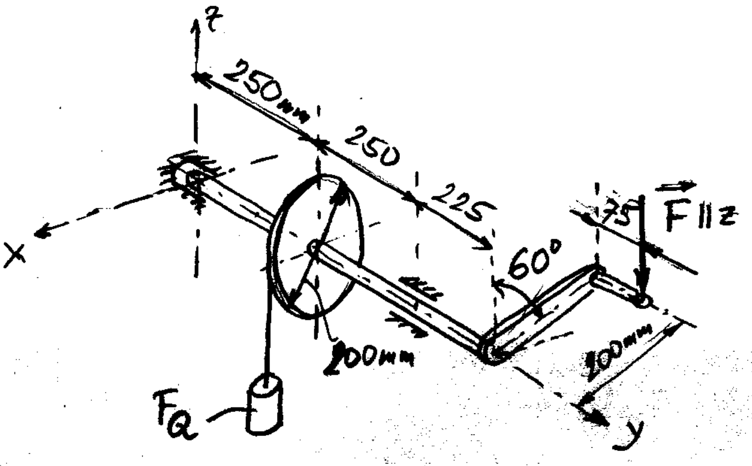
$$\underline{\underline{F_{Ax}}} = -\frac{\sqrt{3}}{6} \cdot 3 = -\frac{\sqrt{3}}{2} = \underline{\underline{-0{,}866\ kN}}$$

$$\rightarrow (2) \quad F_{Ay} - \frac{\cancel{\sqrt{2}}}{3} F_G \cdot \frac{1}{2\cancel{\sqrt{2}}} = 0$$

$$\underline{\underline{F_{Ay}}} = \frac{1}{6} \cdot 3 = \underline{\underline{0{,}5\ kN}}$$

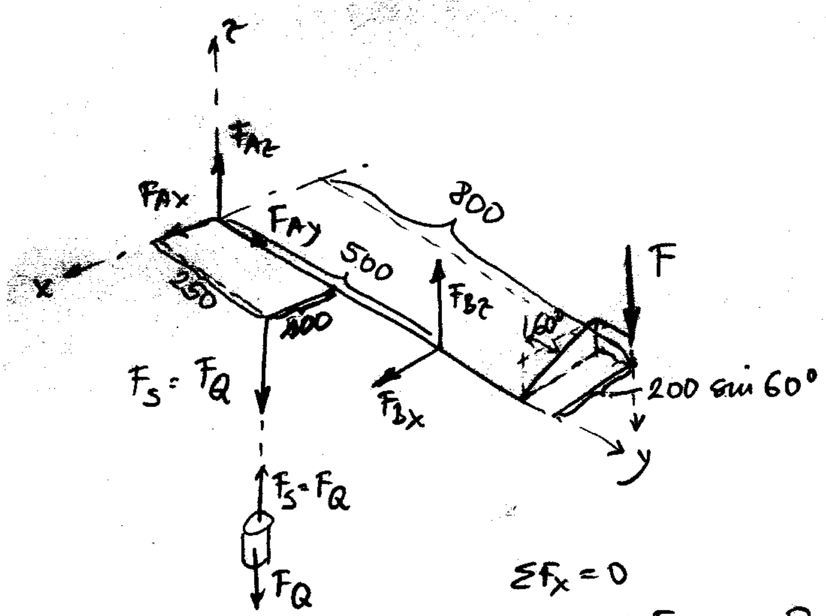
$$\rightarrow (3) \quad F_{Az} + F_{Bz} + \frac{\cancel{\sqrt{2}}}{3} F_G \cdot \frac{1}{\cancel{\sqrt{2}}} - F_G = 0$$

$$\underline{\underline{F_{Az}}} = -1 - \frac{1}{3} \cdot 3 + 3 = \underline{\underline{1\ kN}}$$



Uređaj za podizanje tereta pridržava se u stanju ravnoteže vertikalnom silom F koja djeluje na kraju ručice E, prema slici. Potrebno je odrediti iznos sile F i reakcije u radijalno-aksijalnom ležaju A i radijalnom ležaju B.

Zadano: $F_Q = 800$ N.



$\Sigma F_x = 0$

$$F_{Ax} + F_{Bx} = 0 \qquad (1)$$

$\Sigma F_y = 0$

$$\underline{F_{Ay} = 0} \qquad (2)$$

$\Sigma F_z = 0$

$$F_{Az} + F_{Bz} - F - F_Q = 0 \qquad (3)$$

$\Sigma M_x = 0$

$$F_{Bz} \cdot 500 - F \cdot 800 - F_Q \cdot 250 = 0 \qquad (4)$$

$\Sigma M_y = 0$

$$-F \cdot 200 \sin 60° + F_Q \cdot 100 = 0 \qquad (5)$$

$\Sigma M_z = 0$

$$-F_{Bx} \cdot 500 = 0 \qquad (6)$$

$$\Rightarrow \underline{F_{Bx} = 0}$$

$$(5) \Rightarrow \underline{F} = F_Q \cdot \frac{100}{200 \sin 60^\circ} = 800 \cdot \frac{100}{200 \cdot 0{,}866} = \underline{461{,}88\ N}$$

$$(6) \Rightarrow \underline{F_{Bz}} = \frac{1}{500}\left(461{,}88 \cdot 800 + 800 \cdot 250\right) = \underline{1139{,}01\ N}$$

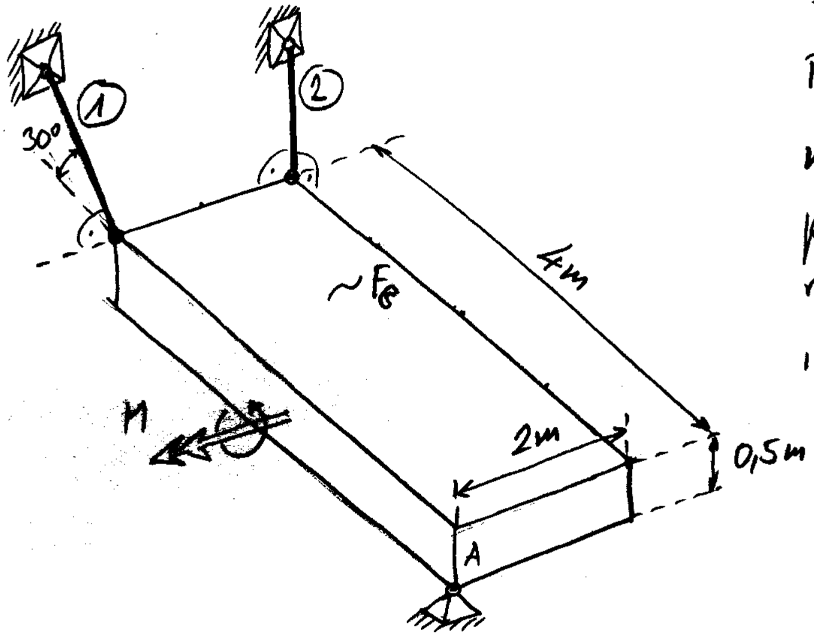
$$(1) \Rightarrow \underline{F_{Ax}} = -F_{Bx} = \underline{\emptyset}$$

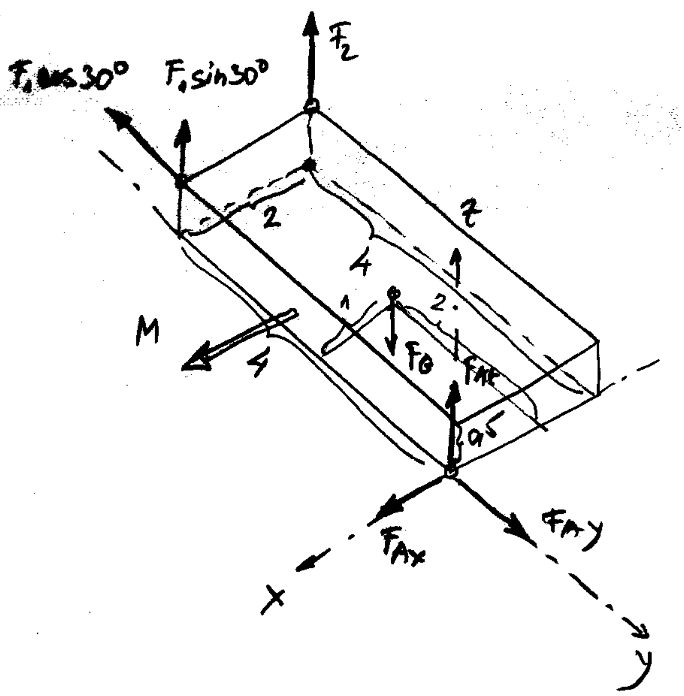
$$(3) \Rightarrow \underline{F_{Az}} = -F_{Bz} + F + F_Q = \underline{122{,}87\ N}$$

 1

Zad. 4.25

Homogeni blok težine $F_G = 3\ kN$ opterećen je momentom $M = 5\ kNm$ prema slici. Odrediti reakcije u osloncu A i sile u štapovima 1 i 2.





$\Sigma F_x = 0$

$F_{Ax} = 0$ (1)

$\Sigma F_y = 0$

$F_{Ay} - F_1 \cos 30° = 0$ (2)

$\Sigma F_z = 0$

$F_{Az} + F_1 \sin 30° + F_2 - F_G = 0$ (3)

$\Sigma M_x = 0$

$F_1 \cos 30° \cdot 0{,}5 - F_1 \sin 30° \cdot 4 - F_2 \cdot 4 +$

$+ M + F_G \cdot 2 = 0$ (4)

$\Sigma M_y = 0$

$F_2 \cdot 2 - F_G \cdot 1 = 0$ (5)

$\Sigma M_z = 0$

$0 = 0$

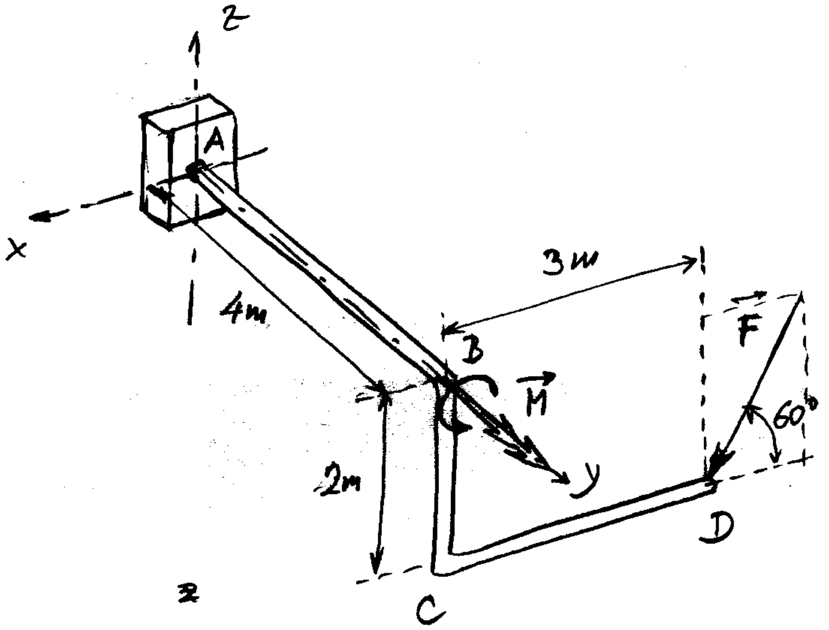
(5) $\Rightarrow$ $F_2 = \frac{1}{2} F_G = 1{,}5\ kN$

(4) $\Rightarrow$ $F_1 = \dfrac{1{,}5 \cdot 4 - 5 - 3 \cdot 2}{0{,}866 \cdot 0{,}5 - 0{,}5 \cdot 4} = 3{,}1908\ kN$

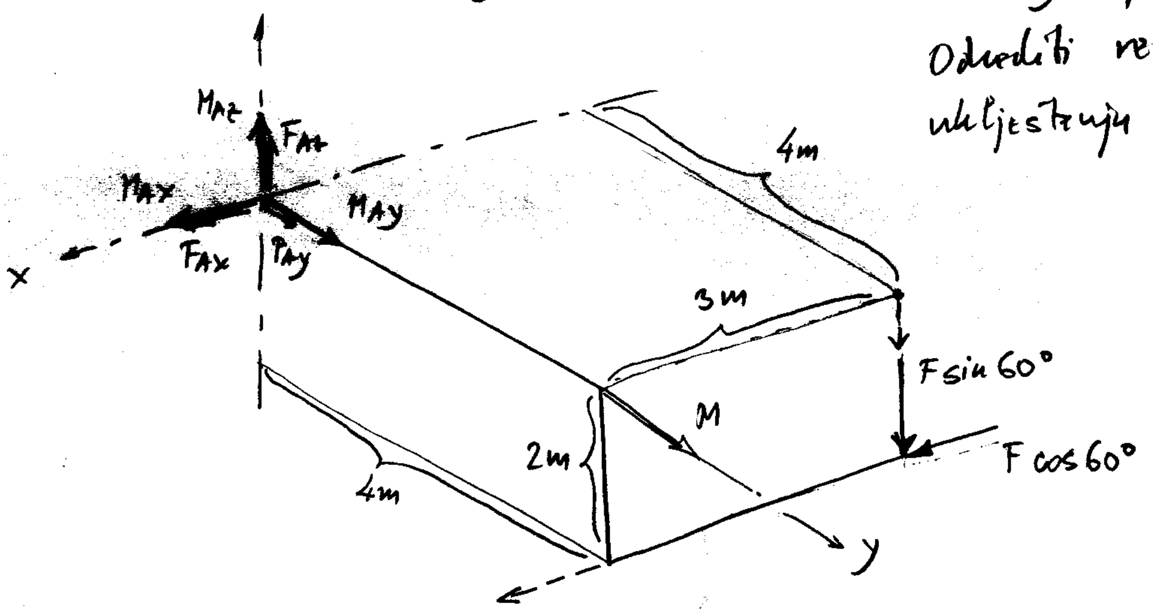
(2) $\Rightarrow$ $F_{Ay} = 3{,}1908 \cdot 0{,}866 = 2{,}7633\ kN$

(3) $\Rightarrow$ $F_{Az} = -3{,}1908 \cdot 0{,}5 - 1{,}5 + 3 = 0{,}0954\ kN$

Zad. 4.21



Zakrivljeni štap ABCD ukliješten je u A i opterećen u D silom F = 2 kN koja leži u ravnini paralelnoj s xz ravninom, i u B momentom M = 3 kNm čiji vektor leži na osi y prema slici. Odrediti reakcije u uklještenju A.



$\Sigma F_x = 0 \qquad F_{Ax} + F\cos 60° = 0 \qquad (1) \quad \rightarrow \quad \underline{F_{Ax}} = -2 \cdot 0{,}5 = \underline{-1\ kN}$

$\Sigma F_y = 0 \qquad \underline{F_{Ay} = 0} \qquad (2)$

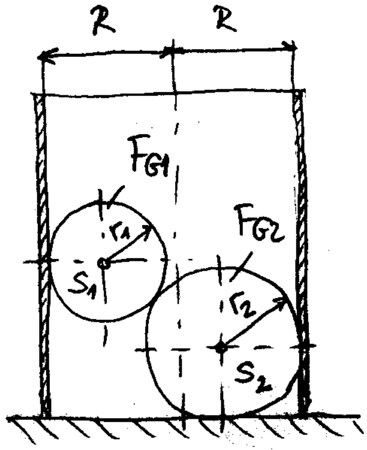
$\Sigma F_z = 0 \qquad F_{Az} - F\sin 60° = 0 \qquad (3) \quad \rightarrow \quad \underline{F_{Az}} = 2 \cdot 0{,}866 = \underline{1{,}732\ kN}$

$\Sigma M_x = 0 \qquad M_{Ax} - F\sin 60° \cdot 4 = 0 \quad \Rightarrow \quad \underline{M_{Ax}} = 2 \cdot 0{,}866 \cdot 4 = \underline{6{,}928\ kNm}$

$\Sigma M_y = 0 \qquad M_{Ay} + M - F\sin 60° \cdot 3 - F\cos 60° \cdot 2 = 0$

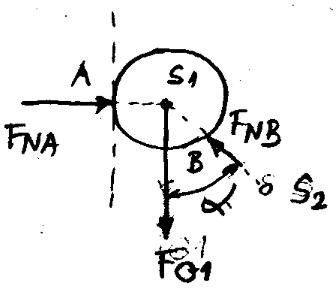
$\Rightarrow \quad \underline{M_{Ay}} = -3 + 2 \cdot 0{,}866 \cdot 3 + 2 \cdot 0{,}5 \cdot 2 = \underline{4{,}196\ kNm}$

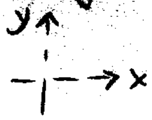
$\Sigma M_z = 0 \qquad M_{Az} - F\cos 60° \cdot 4 = 0 \quad \Rightarrow \quad \underline{M_{Az}} = 2 \cdot 0{,}5 \cdot 4 = \underline{4\ kNm}$



Kružni homogeni tankostjeni cilindar polumjera R otvoren je na oba kraja. Leži na glatkoj horizontalnoj površini, prema slici. U cilindru se nalaze dvije kugle polumjera r_1 i r_2 i težina F_{G1} i F_{G2}. Zanemarujući trenje, treba naći minimalnu težinu F_Q cilindra pri kojoj se on još neće prevrnuti.

Sustav se sastoji od 3 tijela, razmotrimo ravnotežu svakog od njih:





$\Sigma F_x = 0$

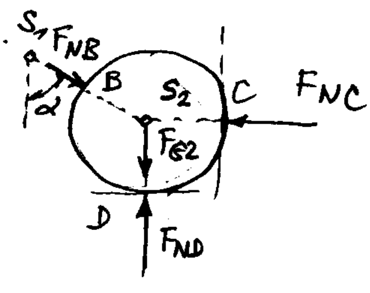
$$F_{NA} - F_{NB} \sin\alpha = 0 \qquad (1)$$

$\Sigma F_y = 0$

$$F_{NB} \cos\alpha - F_{G1} = 0 \qquad (2)$$

$$(2) \Rightarrow F_{NB} = \frac{F_{G1}}{\cos\alpha} \rightarrow (1) \rightarrow F_{NA} = F_{G1} \frac{\sin\alpha}{\cos\alpha}$$

$$\underline{F_{NA} = F_{G1} \cdot \operatorname{tg}\alpha}$$



$\Sigma F_x = 0$

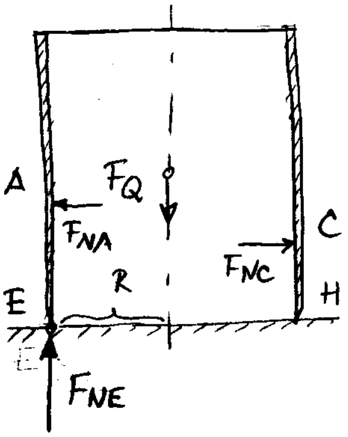
$$F_{NB} \sin\alpha - F_{NC} = 0 \qquad (3)$$

$\Sigma F_y = 0$

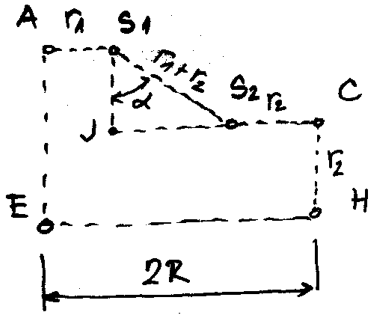
$$-F_{NB} \cos\alpha + F_{ND} - F_{G2} = 0 \qquad (4)$$

$$(3) \Rightarrow F_{NC} = F_{NB} \sin\alpha \;; \qquad \text{iz } (2)\ F_{NB} = \frac{F_{G1}}{\cos\alpha}$$

$$\Rightarrow \underline{F_{NC}} = F_{G1} \cdot \frac{\sin\alpha}{\cos\alpha} = \underline{F_{G1} \cdot \operatorname{tg}\alpha = F_{NA}}$$



U trenutku neposredno prije prevrtanja cilindar će još biti PRISLONJEN svojim donjim rubom uz horizontalnu podlogu, ali samo u točki E postojat će međupritisak između cilindra i podloge.



$$\overline{JS_2} = 2R - r_1 - r_2$$

$$\overline{AE} - \overline{CH} = \overline{S_1J} = (r_1+r_2)\cos\alpha$$

$$\overline{JS_2} = (r_1+r_2)\sin\alpha = 2R - r_1 - r_2$$

$\Sigma F_x = 0$

$$F_{NC} - F_{NA} = 0$$

$$F_{G1}\,tg\alpha - F_{G1}\,tg\alpha = 0$$

$\Sigma F_y = 0$

$$F_{NE} - F_Q = 0$$

$\Sigma M_E = 0$

$$F_{NA}\cdot\overline{AE} - F_{NC}\cdot\overline{CH} - F_Q\cdot R = 0$$

$$F_{G1}\,tg\alpha\cdot\overline{AE} - F_{G1}\,tg\alpha\cdot\overline{CH} = F_Q\cdot R$$

$$F_{G1}\,tg\alpha\,(\overline{AE} - \overline{CH}) = F_Q\cdot R$$

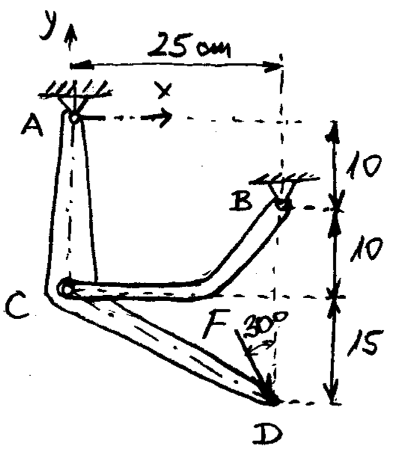
$$F_{G1}\,tg\alpha\cdot(r_1+r_2)\cdot\cos\alpha = F_Q\cdot R$$

$$F_Q\cdot R = F_{G1}\,(r_1+r_2)\,\frac{\sin\alpha}{\cancel{\cos\alpha}}\cdot\cancel{\cos\alpha} =$$

$$F_Q\cdot R = F_{G1}\cdot\overline{JS_2}$$

$$F_Q\cdot R = F_{G1}\cdot(2R - r_1 - r_2) \Longrightarrow \boxed{F_Q = F_{G1}\left(2 - \frac{r_1+r_2}{R}\right)}$$

Zad. 4.38

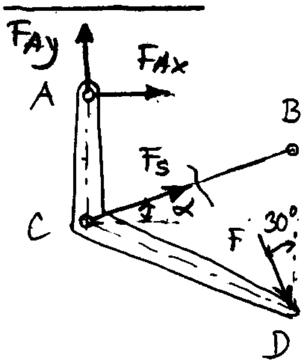


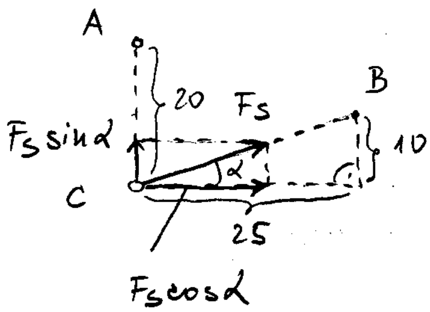
Odrediti reakcije u osloncima A i B i u zglobu C krute grede ACD analitički i grafički. Greda BC je zglobno vezana u C s gredom ACD.

Zadano: $F = 150\,N$.

GREDA CB VEZANA JE ZGLOBOVIMA NA SVOJIM ~~KRAJEVIMA~~. IZMEĐU TIH ZGLOBNO VEZANIH KRAJEVA NA GREDU NE DJELUJU SILE, ZBOG TOGA SE GREDA CB MOŽE SMATRATI ŠTAPOM CB KOJI VEŽE GREDU ACD S NEPOMIČNOM OKOLINOM.

Analitički

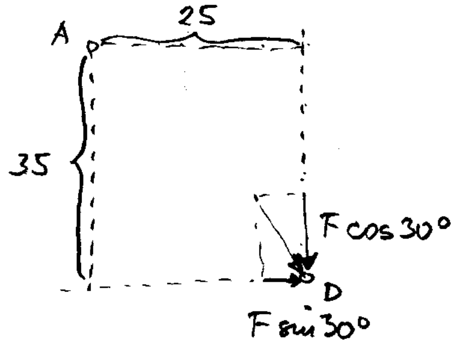




$\overline{CB} = \sqrt{25^2+10^2} = 26{,}926\,cm$

$\cos\alpha = \dfrac{25}{26{,}926} = 0{,}9285$

$\sin\alpha = \dfrac{10}{26{,}926} = 0{,}3714$



$\Sigma F_x = 0$

$F_{Ax} + F_S \cos\alpha + F \sin 30° = 0 \qquad (1)$

$\Sigma F_y = 0$

$F_{Ay} + F_S \sin\alpha - F \cos 30° = 0 \qquad (2)$

$\Sigma M_A = 0$

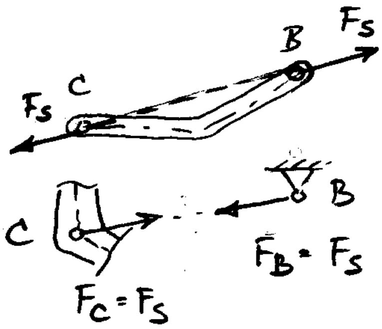
$F_S \cos\alpha \cdot 20 + F_S \sin\alpha \cdot 0 + F \sin 30° \cdot 35 - F \cos 30° \cdot 25 = 0 \qquad (3)$

$$(3) \Rightarrow \quad \underline{F_S} = \frac{150 \cdot \cos 30^\circ \cdot 25 - 150 \cdot \sin 30^\circ \cdot 35}{0{,}9285 \cdot 20} = \underline{33{,}528\ N}$$

$$(1) \Rightarrow \quad F_{Ax} = -F_S \cos\alpha - F \sin 30^\circ = -106{,}13\ N$$

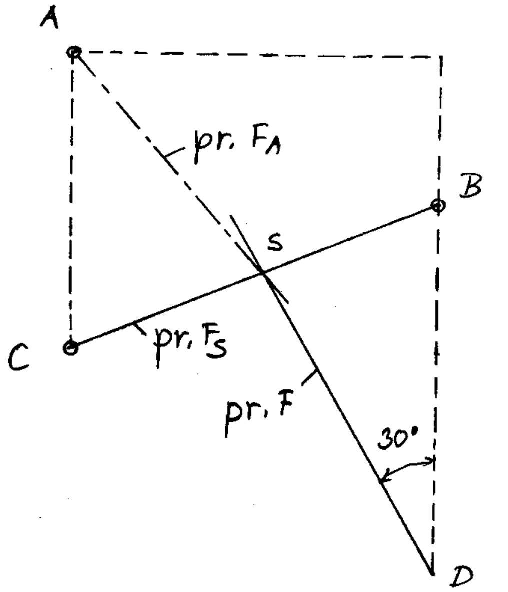
$$(2) \Rightarrow \quad F_{Ay} = -F_S \sin\alpha + F \cos 30^\circ = 117{,}45\ N$$

$$\underline{F_A} = \sqrt{F_{Ax}^2 + F_{Ay}^2} = \underline{158{,}3\ N}$$



Grafički

PLAN POLOŽAJA $1\,cm \hat{=} 2{,}5\,cm$



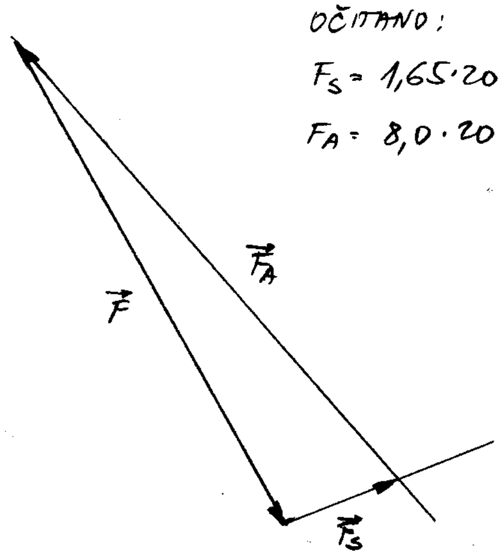
PLAN SILA

$1\,cm \hat{=} 20\,N$

OČITANO:

$F_S = 1{,}65 \cdot 20 = 33\ N$

$F_A = 8{,}0 \cdot 20 = 160\ N$

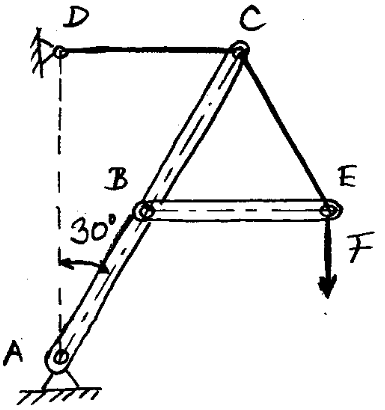


Komentar grafičkog rješenja

Na gredu ACD djeluje zadana sila F, u D, reakcija F_S u zglobu C čiji je pravac poznat (pravac CB) i reakcija F_A u zglobu A. čiji pravac nije poznat.

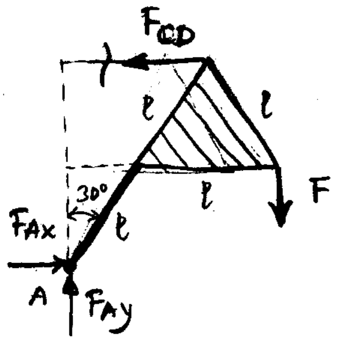
U planu položaja pravci F, F_S i F_A moraju se sjeći u jednoj točki. Odredimo u mjerilu položaje točaka A, C, B i D i ucrtamo u D pravac sile F. Povučemo pravac kroz C i B, tj. pravac sile F_S i nađemo njegovo sjecište S s pravcem sile F, kroz S i točku A, kao hvatište sile F_A, prolazi pravac sile F_A.

U planu sila F_A, F_S i F_A moraju tvoriti zatvoreni poligon sila – „trokut sila". Nacrtamo najprije u mjerilu silu F, u zatim kroz kraj vektora $\vec{F}$ ucrtamo pravac sile F_S (paralelno s tim pravcem u planu položaja), a kroz početak vektora $\vec{F}$ ucrtamo pravac sile F_A (paralelno s tim pravcem u planu položaja). Tako se formira trokut čije stranice definiraju sile $\vec{F}_S$ i $\vec{F}_A$.



Konstrukcija prema slici opterećena je silom F prema slici. Odrediti sile u štapovima CE i CD te reakciju u A.
Zadano: $\overline{AB} = \overline{BC} = \overline{CE} = \overline{BE} = l$, $F = 10$ kN.

Primijenit ćemo postupak solidifikacije i najprije razmotriti ravnotežu cjeline koju čine grede AC i BE te štap CE kao kruto tijelo.



$\Sigma F_x = 0$
$$F_{Ax} - F_{CD} = 0 \qquad (1)$$
$\Sigma F_y = 0$
$$F_{Ay} - F = 0 \qquad (2)$$
$\Sigma M_A = 0$
$$F_{CD} \cdot 2l\cos 30° - F(l\sin 30° + l) = 0 \qquad (3)$$

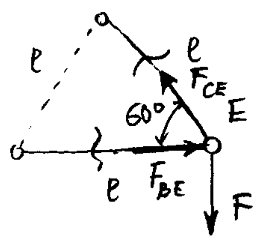
$$(3) \Rightarrow F_{CD} = F\,\frac{1+\sin 30°}{2\cos 30°} = F\,\frac{1+\frac{1}{2}}{2\cdot\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{2}F = \underline{8{,}66\text{ kN}}$$

$$\rightarrow (1) \Rightarrow F_{Ax} = F_{CD} = \underline{\frac{\sqrt{3}}{2}F} = 8{,}66\text{ kN}$$

$$\rightarrow (2) \Rightarrow F_{Ay} = F = 10\text{ kN}$$

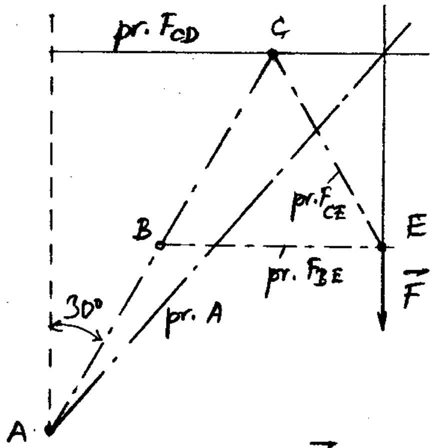
$$F_A = \sqrt{F_{Ax}^2 + F_{Ay}^2} = \underline{13{,}23\text{ kN}}$$

Greda BE nije opterećena između svojih zglobno vezanih krajeva, pa je možemo tretirati KAO ŠTAPNU VEZU. Razmatramo RAVNOTEŽU TOČKE E na koju djeluje skupina KONKURENTNIH SILA.



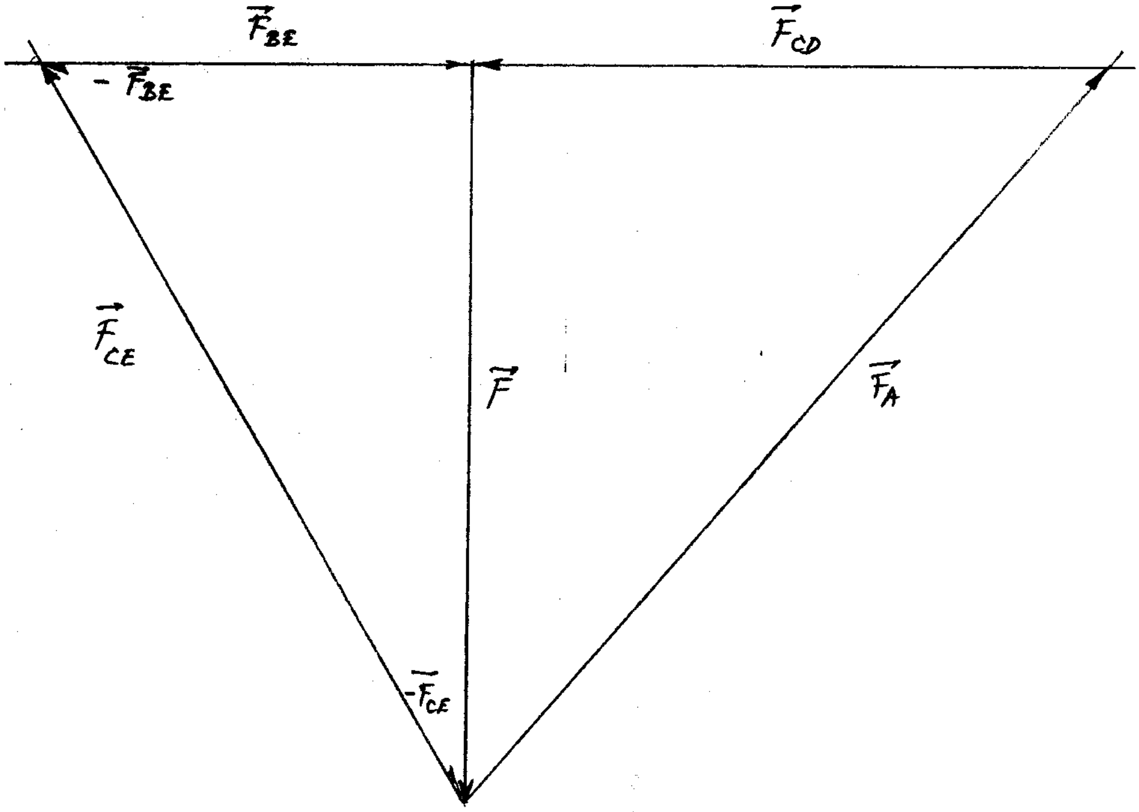
$\Sigma F_x = 0$
$$F_{BE} - F_{CE}\cos 60° = 0 \qquad (4)$$
$\Sigma F_y = 0$
$$F_{CE}\sin 60° - F = 0 \qquad (5)$$

$$(5) \Rightarrow F_{CE} = \frac{F}{\sin 60°} = \frac{F}{\frac{\sqrt{3}}{2}} = \underline{\frac{2}{\sqrt{3}}}F = \underline{11{,}55\text{ kN}}$$



$1\,cm \mathrel{\hat{=}} \frac{1}{3}\,\ell$

U ovom planu položaja mjerilo dužina nije bitno, moraju biti prikazani kutovi, a to će biti ako se ucrta kut od 30° postigne $\overline{AB} = \overline{BC} = \overline{CE} = \overline{BE}$ i pr. $F_{CD} \parallel BE$.



1. TROKUT SILA ČINE SILE KOJE DJELUJU NA CJELINU KONSTRUKCIJE PROMATRANU KAO KRUTO TIJELO:

$$\vec{F} + \vec{F}_A + \vec{F}_{CD} = \vec{O}$$

PRAVAC F_A PROLAZI U PLANU POLOŽAJA KROZ SJECIŠTE PRAVACA F I F_{CD}.

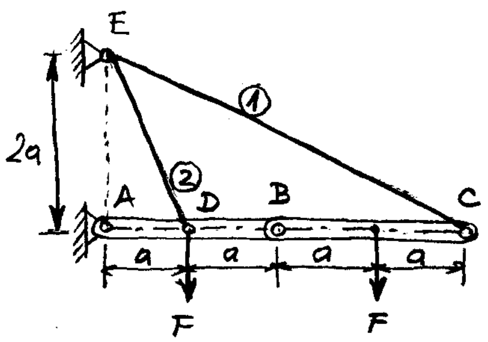
2. TROKUT SILA ČINE SILE KOJE DJELUJU NA TOČKU E:

$$\vec{F} + \vec{F}_{CE} + \vec{F}_{BE} = \vec{O}$$

MOŽE SE UOČITI I ZATVORENI POLIGON SILA KOJE DJELUJU NA GREDU ABC:

$$\vec{F}_A + \vec{F}_{CD} + (-\vec{F}_{BE}) + (-\vec{F}_{CE}) = \vec{O}$$

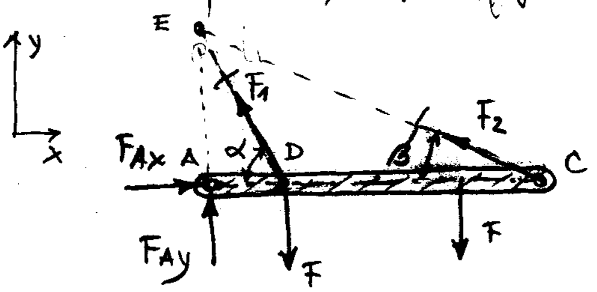
Zad. 4.47

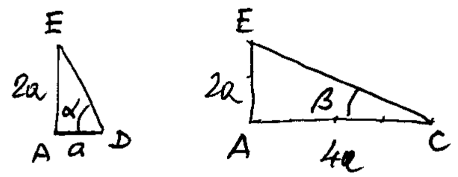


Odrediti sile u užetima 1 i 2 te reakciju u zglobu A konstrukcije zadane i opterećene prema slici. Zadatak riješiti analitički i grafički, ako je zadano:
$F = 4\ kN$, $a = 2m$.

Konstrukcija se sastoji od greda AB i BC vezanih zglobom u B.

Najprije ćemo razmotriti ravnotežu konstrukcije kao cjeline, tj. zglobno vezanih greda AB i BC:





$\overline{DE} = \sqrt{a^2+4a^2} = \sqrt{5}\,a$

$\sin\alpha = \frac{2a}{\sqrt{5}a} = \frac{2}{\sqrt{5}}$

$\cos\alpha = \frac{a}{\sqrt{5}a} = \frac{1}{\sqrt{5}}$

$\overline{CE} = \sqrt{16a^2+4a^2} = \sqrt{20a} = 2\sqrt{5}a$

$\sin\beta = \frac{2a}{2\sqrt{5}a} = \frac{1}{\sqrt{5}}$

$\cos\beta = \frac{4a}{2\sqrt{5}a} = \frac{2}{\sqrt{5}}$

$\Sigma F_x = 0$

$$F_{Ax} - F_1\cos\alpha - F_2\cos\beta = 0 \qquad (1)$$

$\Sigma F_y = 0$

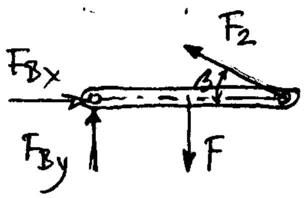
$$F_{Ay} + F_1\sin\alpha + F_2\sin\beta - F - F = 0 \qquad (2)$$

$\Sigma M_A = 0$

$$F_1\sin\alpha\cdot a + F_2\sin\beta\cdot 4a - F\cdot a - F\cdot 3a = 0 \qquad (3)$$

U ove 3 jednadžbe javljaju se 4 nepoznanice. Zbog toga potrebno je još razmotriti ravnotežu ili grede AB ili grede BC što će povećati broj nepoznanica za 2, tj. javit će se nove nepoznanice komponente reakcije u zglobu B F_{Bx} i F_{By}, ali će se postaviti i 3 nove jednadžbe ravnoteže. Tako ćemo imati ukupno 6 jednadžbi sa 6 nepoznanica.

Što je povoljnije, razmatrati gredu AB ili BC?
U jednadžbama sila javit će se nove nepoznanice F_{Bx} i F_{By} za obje grede, ali u momentnim jednadžbama, ako se kao točka s obzirom na koju se postavljaju jednadžbe odabere točka B, ove se nove nepoznanice neće javiti. U jednadžbi $\Sigma M_B = 0$ za gredu AB bit će nepoznanice F_{Ay} i F_1, a u jednadžbi $\Sigma M_B = 0$ za gredu BC bit će nepoznanica samo F_2. Tako će ova jednadžba ($\Sigma M_B = 0$ za gredu BC) i jednadžba (3) za cjelinu ABC činiti podsustav od 2 jednadžbe s 2 nepoznanice. Zbog toga je povoljnije razmatrati ravnotežu grede BC.



$\Sigma F_x = 0$

$$F_{Bx} - F_2 \cos\beta = 0 \qquad (4)$$

$\Sigma F_y = 0$

$$F_{By} + F_2 \sin\beta - F = 0 \qquad (5)$$

$\Sigma M_B = 0$

$$F_2 \sin\beta \cdot 2a - F \cdot a = 0 \qquad (6)$$

$$(6) \Rightarrow \quad F_2 = \frac{1}{2\sin\beta} F = \frac{1}{2 \cdot \frac{1}{\sqrt{5}}} F = \underline{\frac{\sqrt{5}}{2} F} = \underline{4{,}47\ kN}$$

$$\to (4) \Rightarrow \quad F_{Bx} = F_2 \cos\beta = \frac{\sqrt{5}}{2} F \cdot \frac{2}{\sqrt{5}} = \underline{F} = \underline{4\ kN}$$

$$\to (5) \Rightarrow \quad F_{By} = F - F_2 \sin\beta = F - \frac{\sqrt{5}}{2} F \cdot \frac{1}{\sqrt{5}} = \underline{\frac{1}{2} F} = \underline{2\ kN}$$

$$F_B = \sqrt{F_{Bx}^2 + F_{By}^2} = 4{,}47\ kN$$

$$\to (3) \Rightarrow \quad F_1 = \frac{4}{\sin\alpha}\left(F - F_2 \sin\beta\right) = \frac{4}{\frac{2}{\sqrt{5}}}\left(F - \frac{\sqrt{5}}{2} F \cdot \frac{1}{\sqrt{5}}\right) = \underline{\sqrt{5} F} = \underline{8{,}94\ kN}$$

$$\to (1) \Rightarrow F_{Ax} = F_1 \cos\alpha + F_2 \cos\beta = \sqrt{5} F \cdot \frac{1}{\sqrt{5}} + \frac{\sqrt{5}}{2} F \cdot \frac{2}{\sqrt{5}} = \underline{2F} = \underline{8\ kN}$$

$$\to (2) \Rightarrow F_{Ay} = 2F - F_1 \sin\alpha - F_1 \sin\beta = 2F - \sqrt{5} F \cdot \frac{2}{\sqrt{5}} - \frac{\sqrt{5}}{2} F \cdot \frac{1}{\sqrt{5}} = \underline{-\frac{1}{2} F} = -2\ kN$$

$$F_A = \sqrt{F_{Ax}^2 + F_{Ay}^2} = \underline{8{,}25\ kN}$$

Grafičko rješenje moguće je ako se najprije razmotri ravnoteža grede BC i odrede $\vec{F}_1$ i $\vec{F}_B$, iz trokuta sila

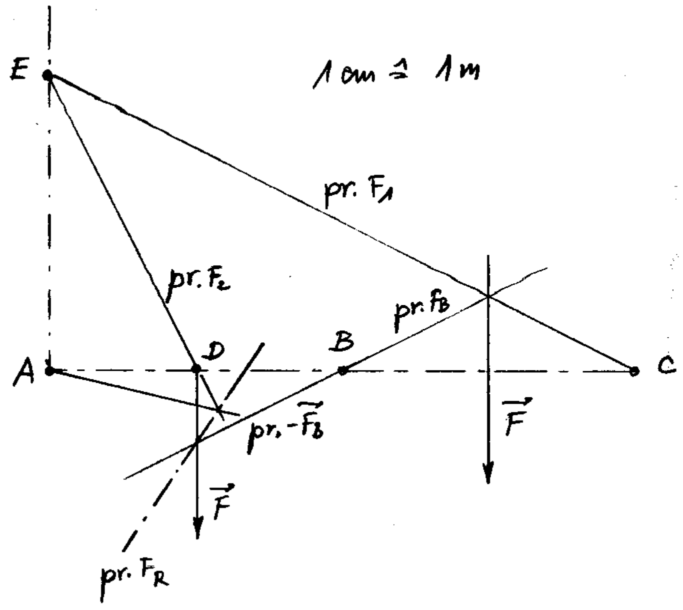
$$\vec{F} + \vec{F}_B + \vec{F}_1 = \vec{O}$$

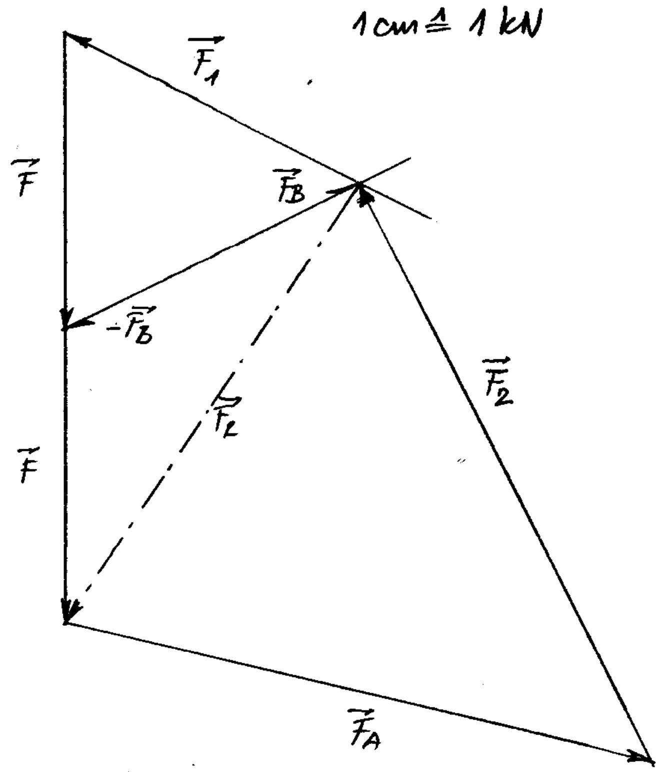
Zatim se razmatra ravnoteža grede AB opterećene rezultantom sila $-\vec{F}_B$ i $\vec{F}$. Rezultanta

$$\vec{F}_R = -\vec{F}_B + \vec{F}$$

odredi se u planu sila. U planu položaja rezultanta prolazi kroz sjecište pravca $-\vec{F}_B$ kroz B i pravca sile $\vec{F}$ u D. Pravac $\vec{F}_R$ povuče se paralelno sa $\vec{F}_R$ u planu sila kroz to sjecište u planu položaja.
Zatim se $\vec{F}_A$ i $\vec{F}_2$ odrede iz trokuta sila

$$\vec{F}_R + \vec{F}_A + \vec{F}_2 = \vec{O}$$

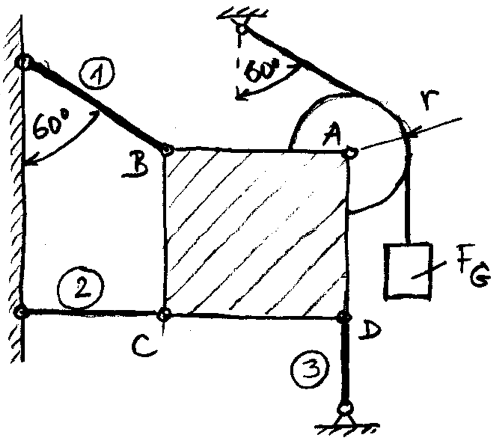




1) pr. F_B kroz sj. pr. F i pr. F_1
2) pr. F_R kroz sj. pr. $-\vec{F}_B$ i pr. F u D
3) pr. F_A kroz sj. pr. F_R i pr. F_2

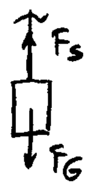
OČITANO:
$F_B = 4,5$ kN
$F_A = 8,3$ kN
$F_1 = 4,5$ kN
$F_2 = 8,9$ kN

Zad. 4.56



Sustav prema slici, koji se nalazi u vertikalnoj ravnini, sastoji se od kolotura zanemarive težine zglobno vezanog u A i kvadratne ploče ABCD stranice a također zanemarive težine. Uteg težine $F_G = 6$ kN slobodno je obješen o uže prebačeno preko kolotura. Potrebno je odrediti sile u štapovima 1, 2 i 3. Zadatak riješiti analitički i grafički. Zadano: a, $r = 0{,}3a$.

Ravnoteža utega:

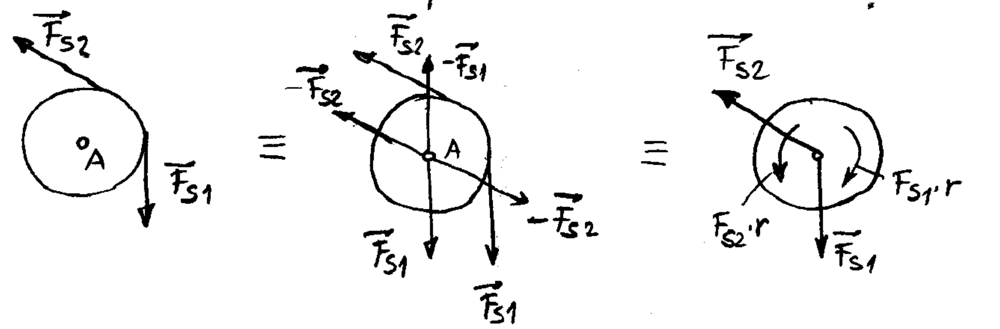


$$\Sigma F_y = 0$$

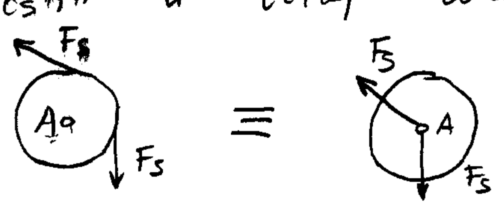
$$F_S - F_G = 0$$

$$\underline{F_S = F_G = 6\ \text{kN}}$$

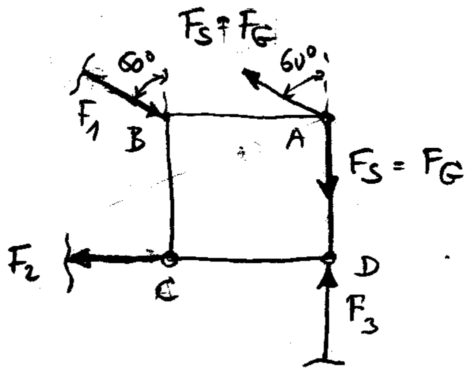
Sile u užetu s lijeve i desne strane kolotura reducirat ćemo na ležaj kolotura A:



Kako je $|\vec{F}_{S1}| = |\vec{F}_{S2}| = F_S$, spregovi $F_{S2} \cdot r$ i $F_{S1} \cdot r$ imaju isti jednaki iznos. Kako su suprotnog smjera, oni se spregovi ponište i ostaju u ležaju A sile $\vec{F}_{S1}$ i $\vec{F}_{S2}$. Dakle, sile u užetu sa lijeve i desne strane kolotura mogu se premjestiti u ležaj kolotura:



Ravnoteža ploče ABCD



$\Sigma F_x = 0$

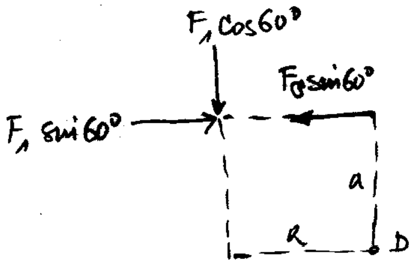
$$-F_1 \sin 60° + F_2 - F_G \sin 60° = 0 \quad (1)$$

$\Sigma F_y = 0$

$$-F_1 \cos 60° + F_3 + F_G \cos 60° - F_G = 0 \quad (2)$$

$\Sigma M_D = 0$

$$F_1 \cos 60° \cdot a - F_1 \sin 60° \cdot a + F_G \sin 60° \cdot a = 0 \quad (3)$$



$$(3) \Rightarrow F_1 = \frac{F_G \sin 60°}{\sin 60° - \cos 60°} = \frac{F_G \frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2} - \frac{1}{2}} = \underline{\frac{\sqrt{3}}{\sqrt{3}-1} F_G} =$$

$$= \underline{14,2 \text{ kN}}$$

$$\rightarrow (1) \Rightarrow F_2 = F_1 \sin 60° - F_G \sin 60° = \frac{\sqrt{3}}{\sqrt{3}-1} F_G \frac{\sqrt{3}}{2} - F_G \cdot \frac{\sqrt{3}}{2} =$$

$$= \frac{\sqrt{3}}{2} \left(\frac{\sqrt{3} - \sqrt{3} + 1}{\sqrt{3} - 1} \right) F_G = \underline{\frac{\sqrt{3}}{2(\sqrt{3}-1)} F_G} =$$

$$= \underline{7,1 \text{ kN}}$$

$$\rightarrow (2) \Rightarrow F_3 = F_1 \cos 60° - F_G \cos 60° + F_G =$$

$$= \frac{\sqrt{3}}{\sqrt{3}-1} F_G \cdot \frac{1}{2} - F_G \cdot \frac{1}{2} + F_G =$$

$$= F_G \cdot \frac{1}{2} \left(\frac{\sqrt{3} + \sqrt{3} + 1}{\sqrt{3} - 1} \right) = \underline{\frac{2\sqrt{3} - 1}{2(\sqrt{3}-1)} F_G} =$$

$$= \underline{10,1 \text{ kN}}$$

Grafički se zadatak riješava Culmannovom metodom. Najprije se u planu sila zbroje sile $\vec{F}_{S1}$ i $\vec{F}_{S2}$ iznosa F_R:

$$\vec{F}_{S1} + \vec{F}_{S2} = \vec{F}_R$$

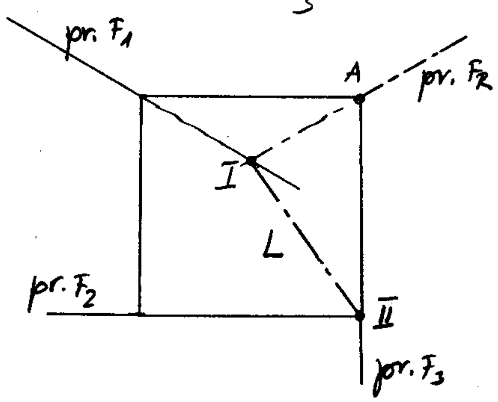
$\vec{F}_R$ prolazi kroz A. Odredi se sjecište $\vec{F}_R$ i pravca $\vec{F}_1$, točka $\underline{I}$ te sjecište pravca $\vec{F}_2$ i $\vec{F}_3$, točka $\underline{II} \equiv D$. Kroz $\underline{I}$ i $\underline{II}$ povuče se Culmannova linija L. U planu sila najprije se nacrta trokut sile koje se sijeku u točki $\underline{I}$:

$$\vec{F}_R + \vec{F}_1 + \vec{F}_L = \vec{0}$$

a zatim trokut sile koje se sijeku u točki $\underline{II}$:

$$-\vec{F}_L + \vec{F}_3 + \vec{F}_2 = \vec{0}$$

$1\,cm \mathrel{\hat{=}} \frac{1}{3}a$



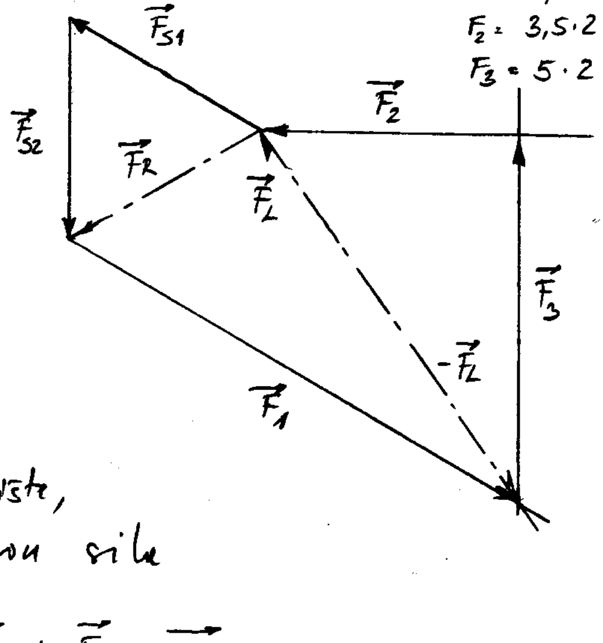
$1\,cm \mathrel{\hat{=}} 2\,kN$

OČITANO:

$F_1 = 7{,}1 \cdot 2 = 14{,}2\,kN$

$F_2 = 3{,}5 \cdot 2 = 7\,kN$

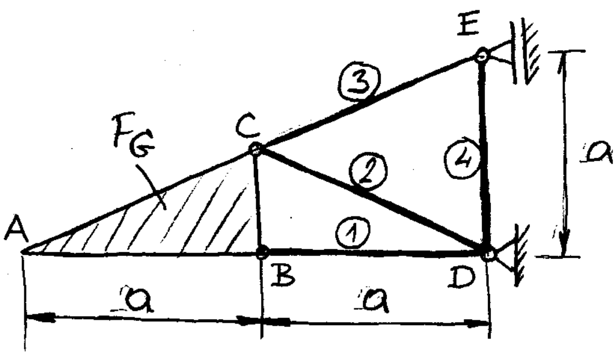
$F_3 = 5 \cdot 2 = 10\,kN$



Kako se $\vec{F}_L$ i $-\vec{F}_L$ ponište, ostaje zatvoreni poligon sile

$$\vec{F}_{S1} + \vec{F}_{S2} + \vec{F}_1 + \vec{F}_3 + \vec{F}_2 = \vec{0}$$

Zad. 4.71



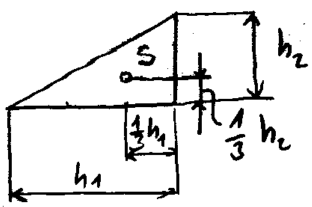
Homogena trokutasta ploča ABC težine F_G nalazi se u vertikalnoj ravnini vezana sustavom štapova za podlogu. Odredite

a) sile u štapovima 1, 2 i 3,

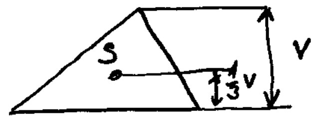
b) reakcije u D i E.

Zadano: $F_G = 6$ kN.

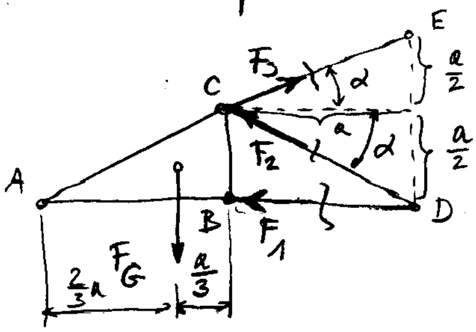
Težište homogene ploče ABC podudara se s težištem trokuta ABC:



općenito:



Ravnoteža ploče



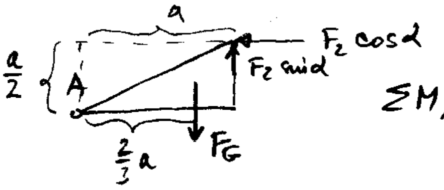
$$\overline{CE} = \sqrt{\left(\frac{a}{2}\right)^2 + a^2} = \frac{\sqrt{5}}{2}a$$

$$\sin\alpha = \frac{\frac{a}{2}}{\overline{CE}} = \frac{\frac{a}{2}}{\frac{\sqrt{5}}{2}a} = \frac{1}{\sqrt{5}}$$

$$\cos\alpha = \frac{a}{\overline{CE}} = \frac{a}{\frac{\sqrt{5}}{2}a} = \frac{2}{\sqrt{5}}$$

Postavimo 3 momentne jednadžbe s obzirom na točke C, A i D koje ne leže na istom pravcu:

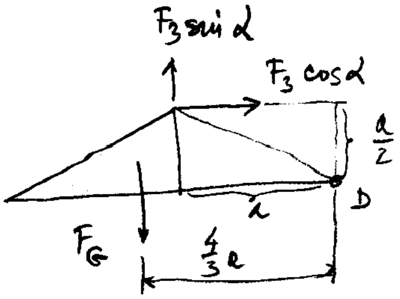
$$\Sigma M_A = 0 \quad -F_1 \cdot \frac{a}{2} + F_G \cdot \frac{a}{3} = 0 \quad \Rightarrow \quad F_1 = \underline{\frac{2}{3}F_G} = \underline{4\ \text{kN}}$$



$$\Sigma M_A = 0 \quad F_2 \sin\alpha \cdot a + F_2 \cos\alpha \cdot \frac{a}{2} - F_G \cdot \frac{2}{3}a = 0$$

$$F_2 = \frac{\frac{2}{3}}{\sin\alpha + \frac{1}{2}\cos\alpha} F_G = \frac{\frac{2}{3}}{\frac{1}{\sqrt{5}} + \frac{1}{2}\cdot\frac{2}{\sqrt{5}}} F_G =$$

$$= \underline{\frac{\sqrt{5}}{3}F_G} = \underline{4{,}47\ \text{kN}}$$



$\Sigma M_D = 0$

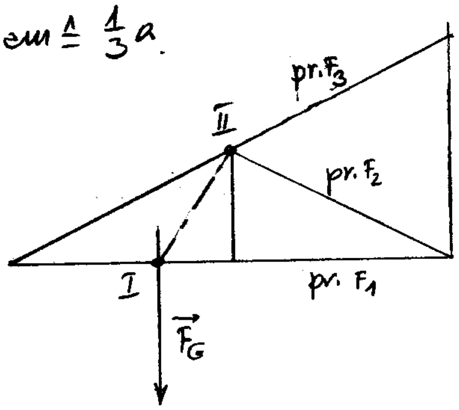
$$-F_3 \sin\alpha \cdot a - F_3 \cos\alpha \cdot \frac{a}{2} + F_G \cdot \frac{4}{3} a = 0$$

$$F_3 = \frac{\frac{4}{3}}{\sin\alpha + \frac{1}{2}\cos\alpha} F_G = \frac{\frac{4}{3}}{\frac{1}{\sqrt{5}} + \frac{1}{2} \cdot \frac{2}{\sqrt{5}}} \cdot F_G =$$

$$= \underline{\frac{2\sqrt{5}}{3} F_G} = \underline{8{,}94\ kN}$$

Grafički Culmannovom metodom

1cm $\hat{=}$ $\frac{1}{3}a$



1) I u sj. $\vec{F}_G$ i pr. F_1

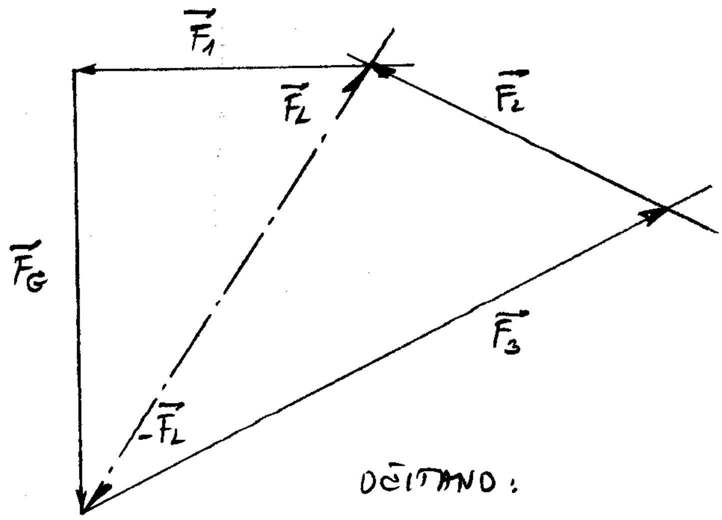
2) II u sj. pr. F_2 i pr. F_3

1cm $\hat{=}$ 1kN

I $\quad \vec{F}_G + \vec{F}_L + \vec{F}_1 = \vec{0}$

II $\quad -\vec{F}_L + \vec{F}_3 + \vec{F}_2 = \vec{0}$

$\Rightarrow \vec{F}_G + \vec{F}_3 + \vec{F}_2 + \vec{F}_1 = \vec{0}$



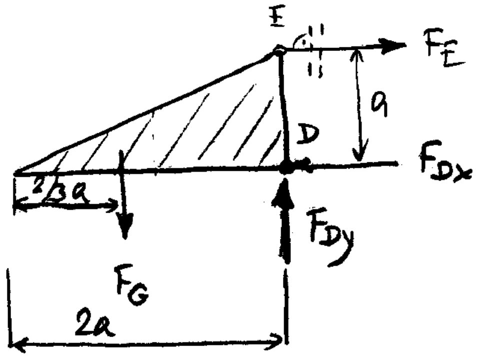
OČITANO:

$F_1 = 4\ kN$

$F_2 = 4{,}5\ kN$

$F_3 = 9\ kN$

Reakcije u nepomičnom osloncu D i pomičnom osloncu E određujemo iz razmatranja ravnoteže cjeline sastavljene od ploče i štapova 1, 2, 3 i 4 koju smatramo krutim tijelom:



$\Sigma F_x = 0$

$-F_{Dx} + F_E = 0 \qquad (1)$

$\Sigma F_y = 0$

$F_{Dy} - F_G = 0 \qquad (2)$

$\Sigma M_D = 0$

$-F_E \cdot a + F_G\left(2a - \frac{2}{3}a\right) = 0 \qquad (3)$

$(3) \Rightarrow \quad F_E = \frac{4}{3} F_G = \underline{8\ kN}$

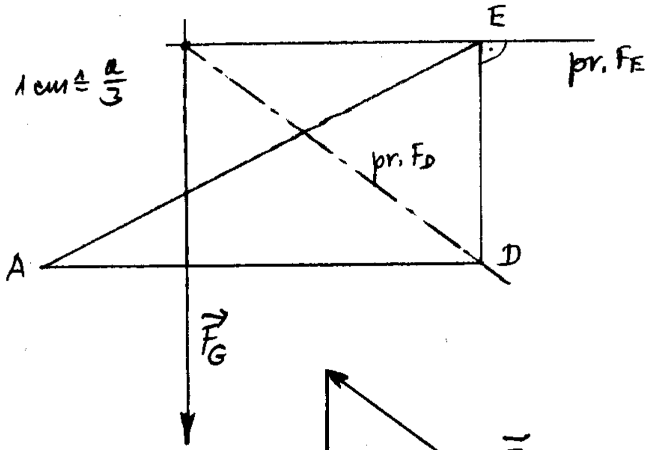
$\rightarrow (1) \Rightarrow \quad F_{Dx} = F_E = \frac{4}{3} F_G = \underline{8\ kN}$

$(2) \Rightarrow \quad F_{Dy} = F_G = \underline{6\ kN}$

$\left.\right\} \; F_D = \sqrt{F_{Dx}^2 + F_{Dy}^2} = \underline{10\ kN}$

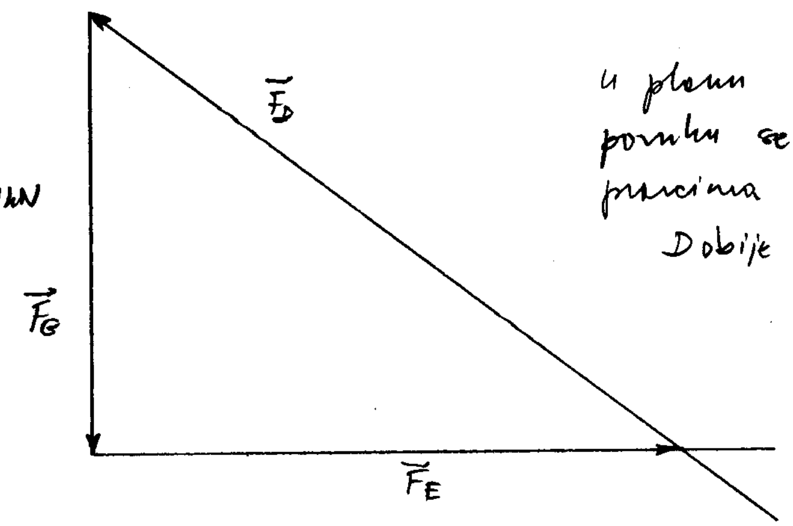
<u>Grafički</u> – trokut sila

$1\,cm \mathrel{\hat=} \frac{a}{3}$



U planu položaja dobije se pravac $\vec{F}_D$ koji prolazi sjecištem pravaca $\vec{F}_G$ i $\vec{F}_E$ te točkom D.

$1\,cm \mathrel{\hat=} 1\,kN$



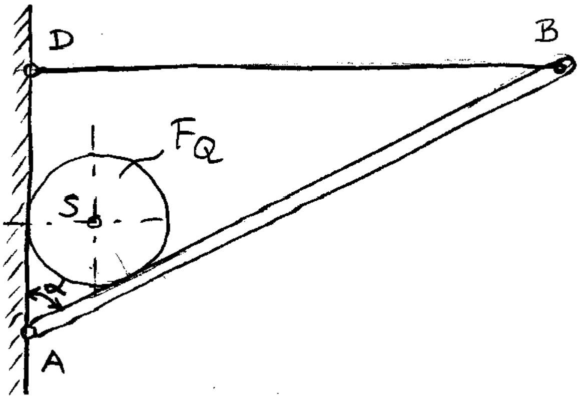
U planu sila uz $\vec{F}_G$ povuku se paralele s pravcima $\vec{F}_E$ i $\vec{F}_D$. Dobije se trokut sila

$\vec{F}_G + \vec{F}_E + \vec{F}_D = \vec{0}$

OČITANO:

$F_D = 10\ kN$

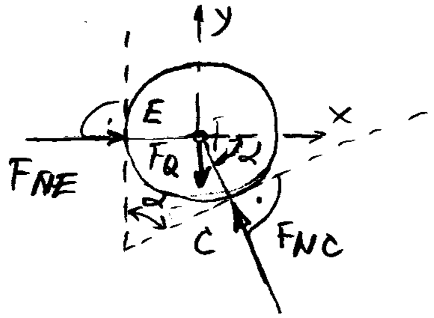
$F_E = 8\ kN$



Kružna ploča polumjera r i težine F_Q uglavljena je između vertikalnog zida i kosog štapa AB, prema slici. Štap je vezan donjim krajem na zid preko zgloba A, dok je gornji kraj vezan preko horizontalnog užeta BD. Težina štapa se zanemaruje. Treba naći iznos kuta α koji štap treba zatvarati sa zidom da bi u užetu BD sila zatezanja bila minimalna. Trenje se zanemaruje.

Razmotrit ćemo najprije ravnotežu kružne ploče, a zatim štapa AB.

Ploča se oslanja u C na štap, a u E na vertikalni zid.



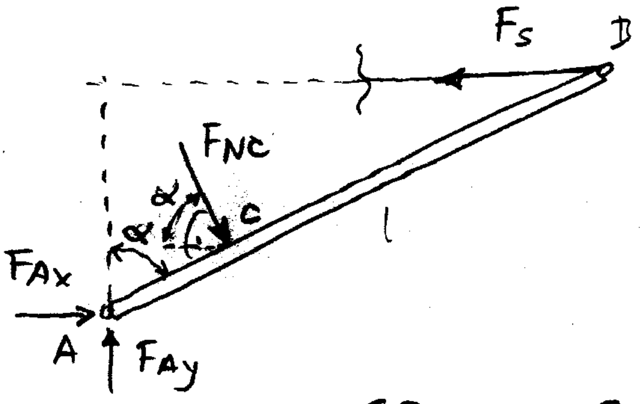
F_Q, F_{NE} i F_{NC} tvore konkurentni sustav sila. Jednadžbe ravnoteže su $\Sigma F_x = 0$ i $\Sigma F_y = 0$.
Kut između pravca F_{NC} i osi x jednak je kutu α između štapa AB i zida jer su mu kraci okomiti na krakove kuta α.

$\Sigma F_x = 0 \qquad F_{NE} - F_{NC}\cos\alpha = 0 \qquad (1)$

$\Sigma F_y = 0 \qquad F_{NC}\sin\alpha - F_Q = 0 \qquad (2)$

Iz (2) slijedi

$$F_{NC} = \frac{F_Q}{\sin\alpha}$$

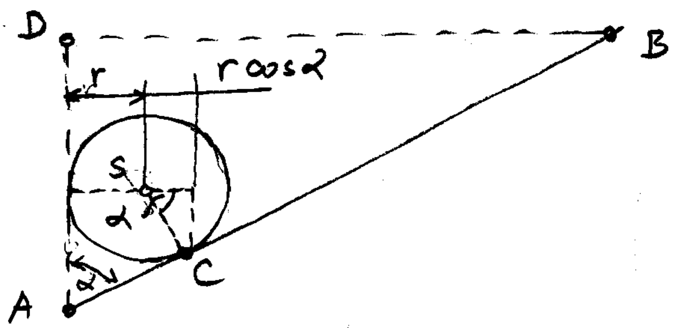


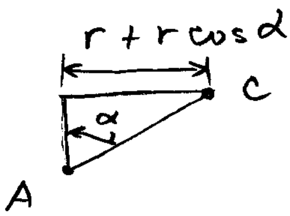
Na štap AB djeluje F_{NC}, sila u užetu F_S i reakcija u zglobu A s komponentama F_{Ax} i F_{Ay}

$\Sigma F_x = 0 \qquad F_{Ax} + F_{NC}\cos\alpha - F_S = 0 \qquad (3)$

$\Sigma F_y = 0 \qquad F_{Ay} - F_{NC}\sin\alpha = 0 \qquad (4)$

$\Sigma M_A = 0 \qquad F_S \cdot \overline{AB}\cos\alpha - F_{NC}\cdot\overline{AC} = 0 \quad (5)$





$$\frac{r + r\cos\alpha}{\overline{AC}} = \sin\alpha$$

$$\overline{AC} = \frac{r(1+\cos\alpha)}{\sin\alpha}$$

$$\overline{AB} = l$$

$\rightarrow (5) \quad \Rightarrow \qquad F_S\, l\cos\alpha - \frac{F_Q}{\sin\alpha}\cdot r\,\frac{(1+\cos\alpha)}{\sin\alpha} = 0$

$$\Rightarrow \quad F_S = F_Q \cdot \frac{r}{e} \quad \frac{1+\cos\alpha}{\cos\alpha \cdot \sin^2\alpha}$$

$$\sin^2\alpha = 1-\cos^2\alpha = (1+\cos\alpha)(1-\cos\alpha)$$

$$\Rightarrow \quad \frac{1+\cos\alpha}{\cos\alpha \cdot \sin^2\alpha} = \frac{\cancel{1+\cos\alpha}}{\cos\alpha\,(\cancel{1+\cos\alpha})(1-\cos\alpha)} =$$

$$= \frac{1}{\cos\alpha\,(1-\cos\alpha)}$$

$$F_S = F_Q \cdot \frac{r}{e} \cdot \frac{1}{\cos\alpha\,(1-\cos\alpha)}$$

UVJET EKSTREMNE VRIJEDNOSTI $\quad \dfrac{dF_S}{d\alpha} = 0$

$$F_Q \cdot \frac{r}{e} \cdot \left\{ -1 \left[\cos\alpha\,(1-\cos\alpha)\right]^{-2} \cdot \left[-\sin\alpha\,(1-\cos\alpha) + \cos\alpha \cdot \sin\alpha\right] \right\} = 0$$

$$\Rightarrow \quad \sin\alpha - \sin\alpha\cos\alpha - \sin\alpha\cos\alpha = 0$$

$$\sin\alpha\,(1-2\cos\alpha) = 0$$

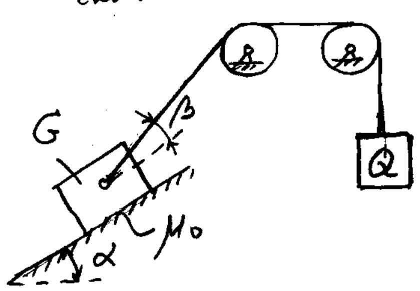
$\sin\alpha = 0 \quad \Rightarrow \quad \alpha_1 = 0$ MAX. IZ... F_S (TL... SKI)

$$1 - 2\cos\alpha = 0$$

$\cos\alpha = \frac{1}{2} \quad \Rightarrow \quad \alpha_2 = 60°$ M... IZ... F_S

$\Rightarrow$ za $\alpha = 60°$, F_S ima najmanji izn

Zad. 5.1

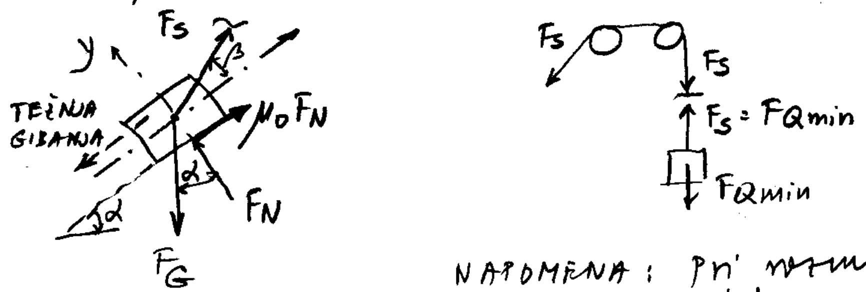


Odrediti minimalni i maksimalni iznos težine tereta Q za koji će teret G još biti u ravnotežnom položaju.

Zadano: $F_G = 3\ kN$, $\alpha = 45°$, $\beta = 15°$, $\mu_0 = 0{,}18$.

Za minimalni iznos težine tereta Q sila u užetu bit će taman dostatna da uz silu trenja graničnog iznosa $F_{Tg} = \mu_0 F_N$ spriječi gibanje tereta G niz kosinu. Za maksimalni iznos težine tereta Q sila u užetu neće još biti dovoljna da bi svladala otpor sile trenja graničnog iznosa i komponentu težine tereta G paralelnu kosini, te izazvala gibanje tereta G uz kosinu:

a) TEŽNJA GIBANJA NIZ KOSINU



NAPOMENA: Pri razmatranju ravnoteže bloka na kosini dobro je postaviti os x paralelnu kosini i os y okomito na nju. Time se dobiju jednostavnije građene jednadžbe ravnoteže.

$\Sigma F_x = 0$

$F_S \cos\beta + \mu_0 F_N - F_G \sin\alpha = 0 \quad (1)$

$\Sigma F_y = 0$

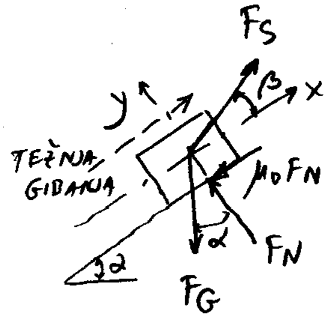
$F_N + F_S \sin\beta - F_G \cos\alpha = 0 \quad (2)$

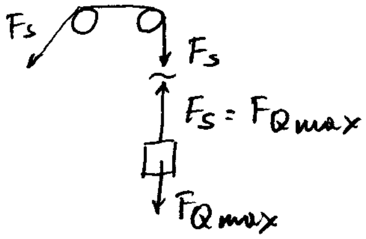
$(2) \Rightarrow F_N = F_G \cos\alpha - F_S \sin\beta$

$\rightarrow (1) \quad F_S \cos\beta - \mu_0 F_S \sin\beta + \mu_0 F_G \cos\alpha - F_G \sin\alpha = 0$

$$F_S = F_{Q\,min} = F_G \frac{\sin\alpha - \mu_0 \cos\alpha}{\cos\beta - \mu_0 \sin\beta} = 1{,}892\ kN$$

b) TEŽNJA GIBANJA UZ KOSINU





$\Sigma F_x = 0$

$$F_S \cos\beta - \mu_0 F_N - F_G \sin\alpha = 0 \qquad (3)$$

$\Sigma F_y = 0$

$$F_N + F_S \sin\beta - F_G \cos\alpha = 0 \qquad (4)$$

$$(4) \Rightarrow \quad F_N = F_G \cos\alpha - F_S \sin\beta$$

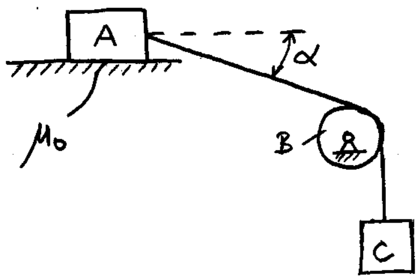
$$\rightarrow (3) \Rightarrow \quad F_S \cos\beta + \mu_0 F_S \sin\beta - \mu_0 F_G \cos\alpha - F_G \sin\alpha = 0$$

$$F_S = F_{Q\,max} = F_G \, \frac{\sin\alpha + \mu_0 \cos\alpha}{\cos\beta + \mu_0 \sin\beta} = 2{,}472 \text{ kN}$$

UVJET RAVNOTEŽE TERETA G

$$1{,}892 \text{ kN} \leq F_Q \leq 2{,}472 \text{ kN}$$

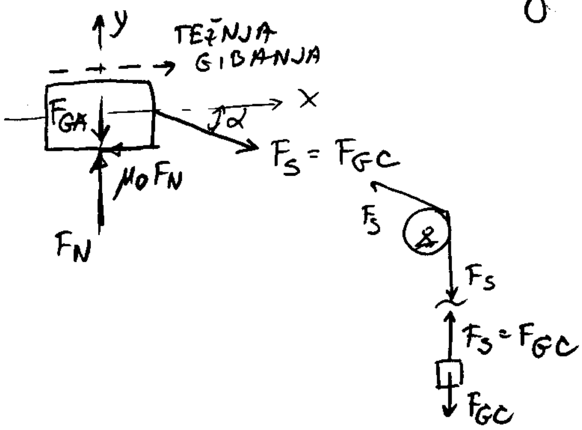
Zad. 5.10



Blokovi A i C vezani su užetom koje je prebačeno preko kolotura B, prema slici. Ako je poznat koeficijent (faktor) trenja μ_0 između podloge i bloka A, odrediti kut α za koji se još zadani sustav nalazi u mirovanju.

Zadano: $F_{GA} = 100\,N$, $F_{GC} = 80\,N$, $\mu_0 = 0{,}5$.

Sila u užetu, jednaka težini bloka C, nastoji pomaknuti blok A u desno. Tome se suprostavlja sila trenja između podloge i bloka A svojim graničnim iznosom $F_{Tg} = \mu_0 F_N$



RAVNOTEŽA BLOKA A:

$\Sigma F_x = 0$

$$F_{GC}\cos\alpha - \mu_0 F_N = 0 \qquad (1)$$

$\Sigma F_y = 0$

$$F_N - F_{GC}\sin\alpha - F_{GA} = 0 \qquad (2)$$

$$(2) \Rightarrow F_N = F_{GC}\sin\alpha + F_{GA}$$

$$\rightarrow (1) \Rightarrow F_{GC}\cos\alpha - \mu_0 F_{GC}\sin\alpha - \mu_0 F_{GA} = 0 \quad /: F_{GC}$$

$$\cos\alpha - \mu_0 \sin\alpha = \mu_0 \frac{F_{GA}}{F_{GC}} \qquad (A)$$

$$\mu_0 = \operatorname{tg}\varphi_0 = \frac{\sin\varphi_0}{\cos\varphi_0} \longrightarrow (A) \qquad \boxed{\varphi_0 = \operatorname{arctg}\mu_0 = \operatorname{arctg} 0{,}5 = 26{,}565^\circ}$$

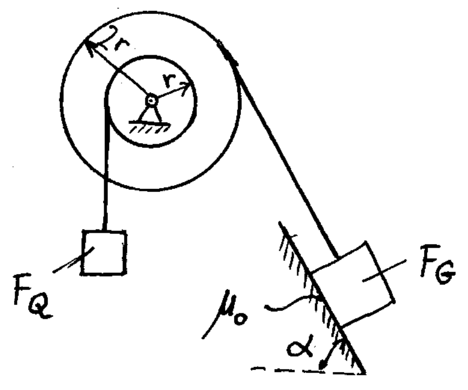
$$\cos\alpha - \frac{\sin\varphi_0}{\cos\varphi_0}\sin\alpha = \frac{\sin\varphi_0}{\cos\varphi_0}\cdot\frac{F_{GA}}{F_{GC}} \quad /\cdot\cos\varphi_0$$

$$\underbrace{\cos\alpha\cos\varphi_0 - \sin\alpha\sin\varphi_0}_{\cos(\alpha+\varphi_0)} = \sin\varphi_0 \cdot \frac{F_{GA}}{F_{GC}}$$

$$\alpha + \varphi_0 = \arccos\left(\frac{F_{GA}}{F_{GC}}\sin\varphi_0\right) \Rightarrow \boxed{\alpha = \arccos\left(\frac{F_{GA}}{F_{GC}}\sin\varphi_0\right) - \varphi_0}$$

$$\underline{\underline{\alpha = 29{,}447^\circ}}$$

Zad. 5.26



Na kolotur zanemarive težine, sastavljen od dva kruto spojena diska polumjera r i 2r obješeni su utezi težina F_G i F_Q. Potrebno je odrediti minimalni iznos težine utega F_Q da bi sustav ostao u stanju mirovanja.

Zadano: $F_G = 8$ kN, $\mu_o = 0{,}2$, $\alpha = 60°$.

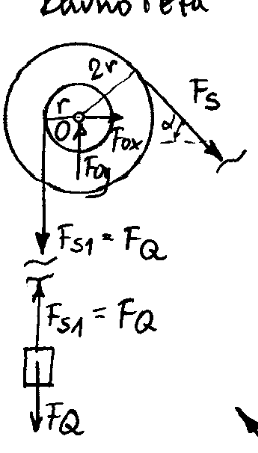
Minimalni iznos težine utega F_Q jest onaj iznos koji će biti taman dostatan da uz djelovanje sile trenja s njezinim graničnim iznosom

$$F_{Tg} = \mu_o F_N$$

spriječi gibanje utega težine F_G niz kosinu.

(Napomena: Užad na čijim krajevima su vezani utezi drugim krajem su vezana za obod diskova i omotana oko njih.)

Ravnoteža kolotura

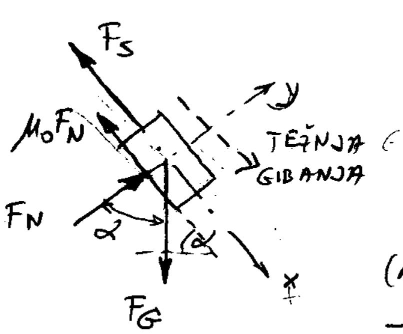


$$\Sigma F_x = 0 \quad F_{ox} + F_S \cos\alpha = 0 \qquad (1)$$

$$\Sigma F_y = 0 \quad F_{oy} - F_S \sin\alpha - F_Q = 0 \qquad (2)$$

$$\Sigma M_O = 0 \quad F_Q \cdot r - F_S \cdot 2r = 0 \qquad (3)$$

$$(3) \Rightarrow \quad F_S = \frac{1}{2} F_Q \qquad (A)$$



Ravnoteža utega F_G

$$\Sigma F_x = 0 \quad -F_S - \mu_o F_N + F_G \sin\alpha = 0 \qquad (4)$$

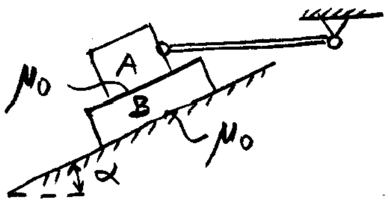
$$\Sigma F_y = 0 \quad F_N - F_G \cos\alpha = 0 \qquad (5)$$

$$(5) \Rightarrow \quad F_N = F_G \cos\alpha$$

$$(A) \text{ i } (5) \rightarrow (4):$$

$$-\frac{1}{2} F_Q - \mu_o F_G \cos\alpha + F_G \sin\alpha = 0$$

$$\Rightarrow \underline{F_Q} = 2 F_G (\sin\alpha - \mu_o \cos\alpha) = 2 \cdot 8 (\sin 60° - 0{,}2 \cos 60°) = \underline{12{,}256 \text{ kN}}$$



Potrebno je odrediti najmanji iznos težine bloka A da bi blok B ostao u stanju mirovanja na kosini nagiba $\alpha = 30°$.

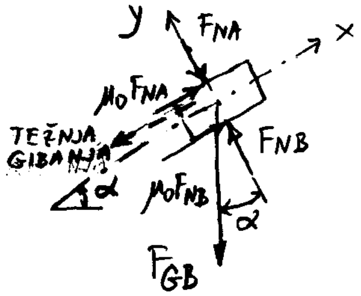
Zadano: $\mu_0 = 0{,}25$, $F_{GB} = 160$ N.

Najmanji iznos težine bloka A jest ona težina tog bloka koja će biti taman dostatna da uz djelovanje sila trenja između bloka A i bloka B i između kosine i bloka B s njihovim graničnim iznosima

$$F_{TA} = \mu_0 F_{NA} \quad \text{i} \quad F_{TB} = \mu_0 F_{NB}$$

spriječi gibanje bloka B niz kosinu.

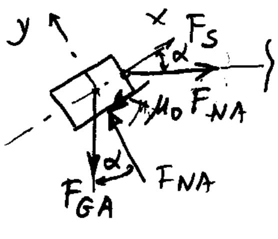
Ravnoteža bloka B:



$$\Sigma F_x = 0 \quad \mu_0 F_{NA} + \mu_0 F_{NB} - F_{GB} \sin\alpha = 0 \quad (1)$$

$$\Sigma F_y = 0 \quad F_{NB} - F_{NA} - F_{GB} \cos\alpha = 0 \quad (2)$$

Ravnoteža bloka A:



$$\Sigma F_x = 0 \quad F_S \cos\alpha - \mu_0 F_{NA} - F_{GA} \sin\alpha = 0 \quad (3)$$

$$\Sigma F_y = 0 \quad F_{NA} - F_S \sin\alpha - F_{GA} \cos\alpha = 0 \quad (4)$$

$$(3)/\cdot\sin\alpha + (4)\cdot/\cos\alpha \Rightarrow$$

$$-\mu_0 F_{NA} \sin\alpha + F_{NA} \cos\alpha - F_{GA} \sin^2\alpha - F_{GA} \cos^2\alpha = 0$$

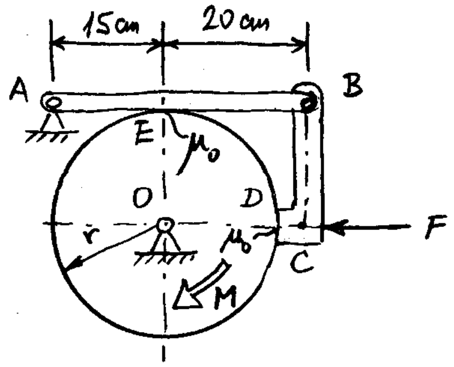
$$F_{NA}(\cos\alpha + \mu_0 \sin\alpha) = F_{GA}(\underbrace{\sin^2\alpha + \cos^2\alpha}_{=1})$$

$$F_{GA} = F_{NA}(\cos\alpha - \mu_0 \sin\alpha) \quad (A)$$

$$(1) + (2)/\cdot(-\mu_0) \Rightarrow$$

$$\mu_0 F_{NA} + \mu_0 F_{NA} - F_{GB} \sin\alpha + \mu_0 F_{GB} \cos\alpha = 0 \Rightarrow F_{NA} = F_{GB} \frac{\sin\alpha - \mu_0 \cos\alpha}{2\mu_0} \quad (B)$$

$$(B) \to (A) \Rightarrow \underline{F_{GA}} = F_{GB} \frac{(\sin\alpha - \mu_0 \cos\alpha)(\cos\alpha - \mu_0 \sin\alpha)}{2\mu_0} = \underline{67{,}22\ N}$$

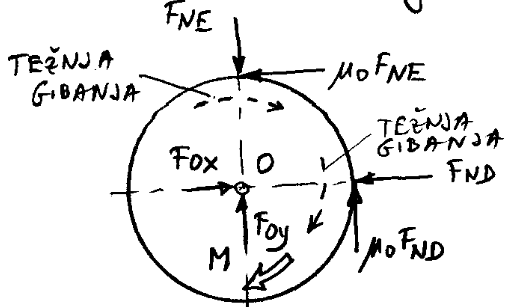


Hrapav valjak polumjera r koči se preko poluge AB i ručice BCD, prema slici. Ako na ručicu djeluje sila F, odrediti najveći iznos momenta M kod kojeg će valjak još mirovati. Faktor trenja μ_0 na svim dodirnim površinama je jednak.

Zadano: $F = 150$ N, $r = 15$ cm, $\mu_0 = 0{,}2$.

Sile trenja koje djeluju na obod valjka u E i u D sprečavaju rotaciju valjka oko O uslijed djelovanja momenta M. Najveći iznos momenta M kojem se te sile suprotstavljaju određen je najvećim iznosom koji sile trenja mogu imati, a to su granične vrijednosti

$$F_{TEg} = \mu_0 F_{NE} \quad i \quad F_{TDg} = \mu_0 F_{ND}$$



Ravnoteža valjka

$\Sigma F_x = 0 \quad F_{Ox} - F_{ND} - \mu_0 F_{NE} = 0 \qquad (1)$

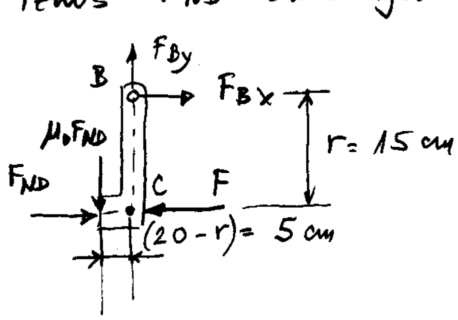
$\Sigma F_y = 0 \quad F_{Oy} - F_{NE} + \mu_0 F_{ND} = 0 \qquad (2)$

$\Sigma M_O = 0$

$\mu_0 F_{NE} \cdot r + \mu_0 F_{ND} \cdot r - M = 0 \qquad (3)$

$$(3) \quad \Rightarrow \quad M = \mu_0 (F_{NE} + F_{ND}) \cdot r \qquad (A)$$

Iznos F_{ND} određujemo iz ravnoteže ručice BCD:



$\Sigma F_x = 0 \quad F_{Bx} + F_{ND} - F = 0 \qquad (4)$

$\Sigma F_y = 0 \quad F_{By} - \mu_0 F_{ND} = 0 \qquad (5)$

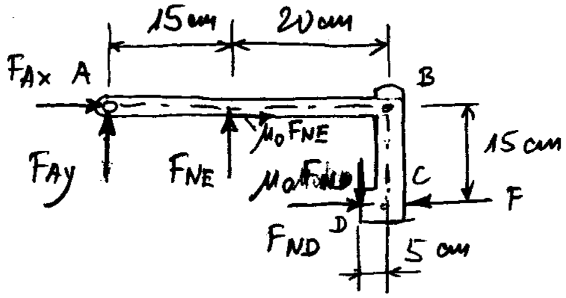
$\Sigma M_B = 0$

$F_{ND} \cdot 15 + \mu_0 F_{ND} \cdot 5 - F \cdot 15 = 0 \qquad (6)$

$$(6) \Rightarrow \quad F_{ND} = \frac{F}{1 + \frac{1}{3}\mu_0} = \frac{150}{1 + \frac{1}{3}\cdot 0{,}2} =$$

$$F_{ND} = 140{,}625 \text{ N}$$

Iznos F_{NE} može se sada odrediti iz razmatranja ravnoteže poluge AB i ručice BCD kao krute cjeline (primjena principa solidifikacije):



$\Sigma F_x = 0$

$$F_{Ax} + \mu_0 F_{NE} + F_{ND} - F = 0 \quad (7)$$

$\Sigma F_y = 0$

$$F_{Ay} + F_{NE} - \mu_0 F_{ND} = 0 \quad (8)$$

$\Sigma M_A = 0$

$$F_{NE}\cdot 15 + \mu_0 F_{NE}\cdot 0 + F_{ND}\cdot 15 -$$
$$- \mu_0 F_{ND}\,(15 + 20 - 5) - F\cdot 15 = 0 \quad (9)$$

Krak sile trenja $\mu_0 F_{NE}$ s obzirom na A uzima se približno jednak nuli jer debljina poluge AB nije definirana.

$$(9) \Rightarrow \quad F_{NE} = F - F_{ND}\,(1 - 2\mu_0) = 150 - 140{,}625\,(1 - 2\cdot 0{,}2)$$

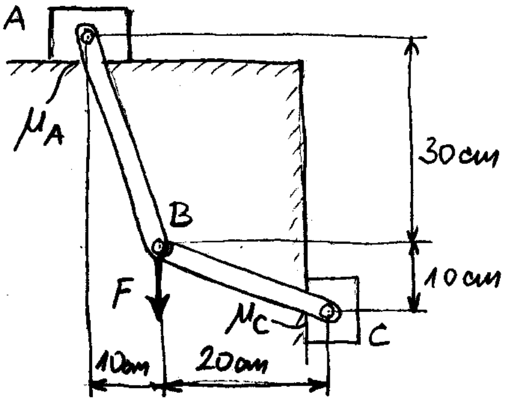
$$F_{NE} = 65{,}625 \text{ N}$$

Uvrstimo iznose F_{ND} i F_{NE} u (A):

$$M = 0{,}2\,(65{,}625 + 140{,}625)\cdot 0{,}15$$

$r = 15 \text{ cm} = 0{,}15 \text{ m}$

$$\underline{\underline{M = 6{,}1875 \text{ Nm}}}$$

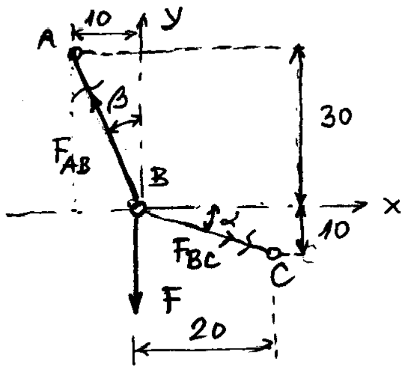


Blokovi A i C vezani su štapovima AB i BC čija se težina zanemaruje. Odrediti analitički i grafički minimalni iznos sile F koja mora djelovati u zglobu B da bi sustav tereta bio u ravnoteži u položaju danom na slici. Koliki je u tom primjeru koeficijent (faktor) statičkog trenja μ_A između bloka A i podloge?

Zadano: $F_{GA} = 300\ N$, $F_{GC} = 150\ N$, $\mu_C = 0{,}45$.

Analitički

Ravnoteža točke B vezane štapovima AB i BC:



$$\overline{AB} = \sqrt{10^2+30^2} = 10\sqrt{10}\ cm$$

$$\sin\beta = \frac{10}{10\sqrt{10}} = \frac{1}{\sqrt{10}}$$

$$\cos\beta = \frac{30}{10\sqrt{10}} = \frac{3}{\sqrt{10}}$$

$$\overline{BC} = \sqrt{20^2+10^2} = 10\sqrt{5}$$

$$\sin\alpha = \frac{10}{10\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$\cos\alpha = \frac{20}{10\sqrt{5}} = \frac{2}{\sqrt{5}}$$

$$\Sigma F_x = 0 \qquad F_{BC}\cos\alpha - F_{AB}\sin\beta = 0 \qquad (1)$$

$$\Sigma F_y = 0 \qquad F_{AB}\cos\beta - F_{BC}\sin\alpha - F = 0 \qquad (2)$$

$(1)\cdot\sin\alpha + (2)\cdot\cos\alpha \rightarrow$

$$\Rightarrow -F_{AB}\sin\beta\sin\alpha + F_{AB}\cos\beta\cos\alpha - F\cos\alpha = 0$$

$$F_{AB} = F\,\frac{\cos\alpha}{\cos\beta\cos\alpha - \sin\beta\sin\alpha} = F\cdot\frac{\frac{2}{\sqrt{5}}}{\frac{3}{\sqrt{10}}\cdot\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{10}}\cdot\frac{1}{\sqrt{5}}}$$

$$F_{AB} = \frac{2}{5}\sqrt{10}\,F \qquad (A)$$

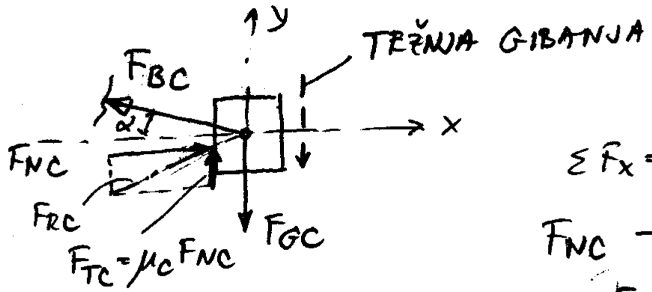
$$\rightarrow (1) \Rightarrow \quad F_{BC} = F_{AB}\,\frac{\sin\beta}{\cos\alpha} =$$

$$= \frac{\cancel{2}}{5}\,\cancel{\sqrt{10}}\,F \cdot \frac{\frac{1}{\cancel{\sqrt{10}}}}{\frac{\cancel{2}}{\sqrt{5}}}$$

$$F_{BC} = \frac{\sqrt{5}}{5}\,F \qquad (3)$$

UZ MINIMALNI IZNOS SILE F SILA F_{BC} U ŠTAPU BC BIT ĆE TAMAN DOVOLJNA DA UZ GRANIČNU SILU TRENJA $F_{TC} = \mu_C F_{NC}$ SPRIJEČI GIBANJE BLOKA C PREMA DOLJE.

Ravnoteža bloka C



$$\Sigma F_x = 0$$

$$F_{NC} - F_{BC}\cos\alpha = 0 \qquad (3)$$

$$\Sigma F_y = 0$$

$$\mu_C F_{NC} + F_{BC}\sin\alpha - F_{GC} = 0 \qquad (4)$$

$$(3) \Rightarrow \quad F_{NC} = F_{BC}\cos\alpha$$

$$\rightarrow (4) \Rightarrow \quad \mu_C F_{BC}\cos\alpha + F_{BC}\sin\alpha - F_{GC} = 0$$

$$F_{BC} = \frac{F_{GC}}{\mu_C\cos\alpha + \sin\alpha} = \frac{F_{GC}}{\mu_C\frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}}} =$$

$$= \frac{\sqrt{5}}{(2\mu_C + 1)}\,F_{GC}$$

Uvrsti se F_{BC} prema (3)

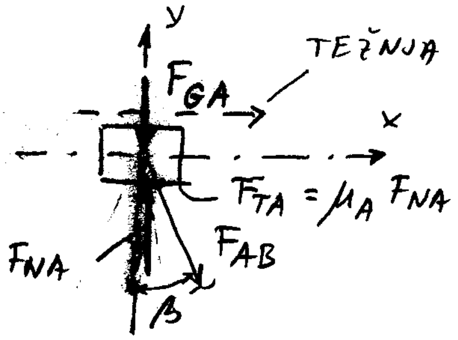
$$\frac{\cancel{\sqrt{5}}}{5}\,F = \frac{\cancel{\sqrt{5}}}{(2\mu_C + 1)}\,F_{GC}$$

$$\Rightarrow \quad \underline{\underline{F}} = \frac{5}{2\mu_C + 1}\,F_{GC} = \frac{5}{2\cdot 0{,}45 + 1}\cdot 150 = \underline{\underline{394{,}7\ \text{N}}}$$

$\rightarrow (B) \Rightarrow \quad F_{BC} = \frac{\sqrt{5}}{5} \cdot 394{,}7 = 176{,}5\ N$

$\rightarrow (A) \Rightarrow \quad F_{AB} = \frac{2}{5}\sqrt{10} \cdot 394{,}7 = 499{,}3\ N$

ravnoteža bloka A



uslijed djelovanja sile F_{AB} u štapu AB blok se teži pomaknuti u desno. Tome se suprostavlje sila trenja F_{TA} čiji je granični iznos $\mu_A F_{NA}$.

$\Sigma F_x = 0$

$$-\mu_A F_{NA} + F_{AB} \sin\beta = 0 \qquad (5)$$

$\Sigma F_y = 0$

$$F_{NA} - F_{AB}\cos\beta - F_{GA} = 0 \qquad (6)$$

$(6) \Rightarrow \quad F_{NA} = F_{AB}\cos\beta + F_{GA}$

$\rightarrow (5) \Rightarrow \quad \mu_A = \dfrac{F_{AB}\sin\beta}{F_{AB}\cos\beta + F_{GA}}$

uvrsti se (A):

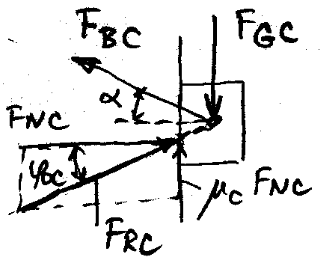
$$\mu_A = \frac{\frac{2}{5}\sqrt{10}\,F \cdot \frac{1}{\sqrt{10}}}{\frac{2}{5}\sqrt{10}\,F \cdot \frac{3}{\sqrt{10}} + F_{GA}} = \frac{\frac{2}{5}F}{\frac{6}{5}F + F_{GA}}$$

$$\underline{\underline{\mu_A}} = \frac{2 \cdot 394{,}7}{6 \cdot 394{,}7 + 5 \cdot 300} = \underline{\underline{0{,}204}}$$

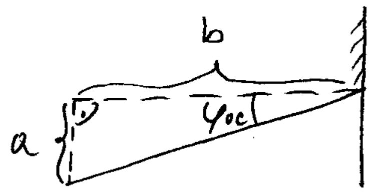
Grafički

Najprije se razmatra ravnoteža bloka C uz pretpostavku težnje gibanja prema dolje i graničnog iznosa sile trenja $F_{TC} = \mu_C F_{NC}$ što određuje pravac totalne reakcije F_{RC} koja je od normale otklonjena za kut

$$\varphi_{0C} = \mathrm{arctg}\, \mu_C$$



Kut φ_{0C} konstruiramo pomoću njegovog tangensa



$$\frac{a}{b} = \mathrm{tg}\, \varphi_{0C} = \mu_C$$

$$a = b \cdot \mathrm{tg}\, \varphi_{0C} = b \cdot \mu_C$$

$$a = b \cdot 0{,}45$$

Ako odaberemo u planu položaja

$b = 5$ cm,

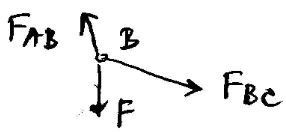
bit će

$a = 5 \cdot 0{,}45 = 2{,}25$ cm

Prvi trokut sila u planu sila je dakle

$$\vec{F}_{GC} + \vec{F}_{BC} + \vec{F}_{RC} = \vec{0}$$

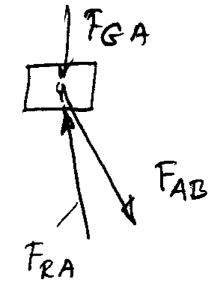
Zatim razmatramo ravnotežu točke B:



U planu sila promijenimo smjer vektora $\vec{F}_{BC}$ i nacrtamo trokut sila

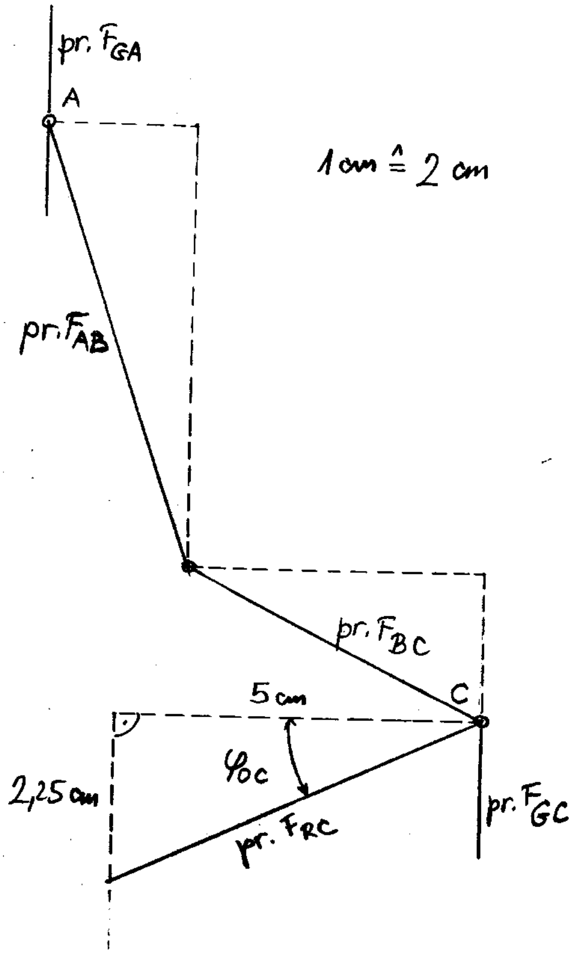
$$-\vec{F}_{BC} + \vec{F} + \vec{F}_{AB} = \vec{0}$$

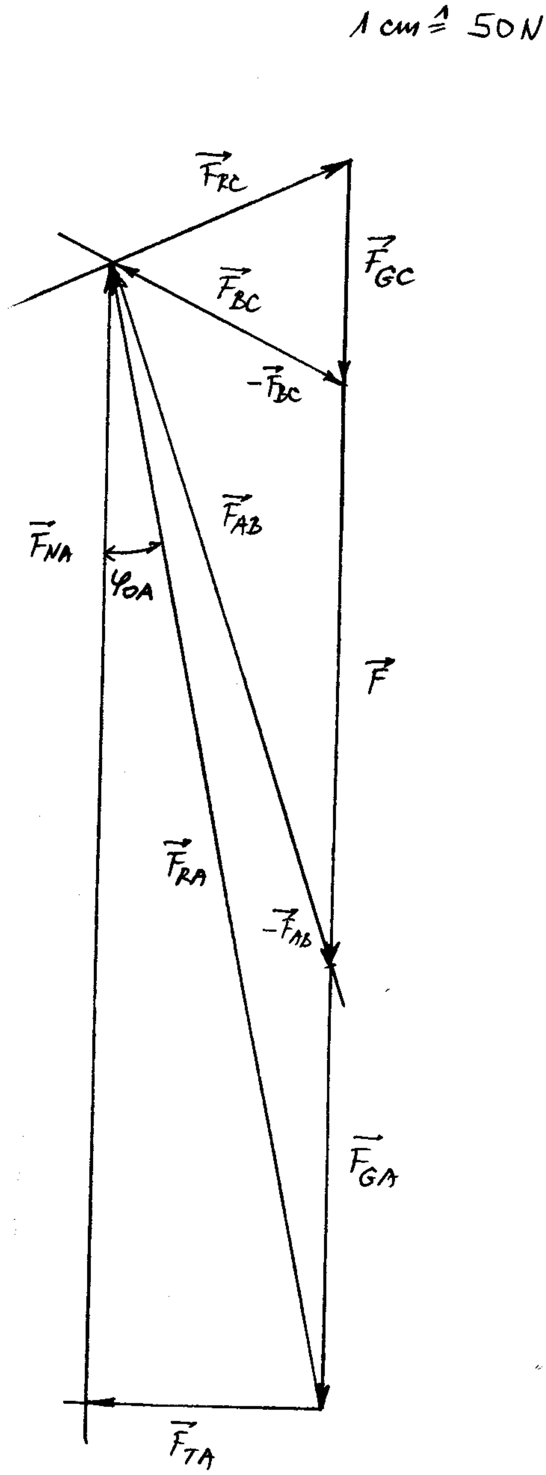
Konačno razmotrimo ravnotežu bloka A:



U planu sila promijenimo smjer vektora $\vec{F}_{AB}$, pa iz vrha $-\vec{F}_{AB}$ nacrtamo vektor $\vec{F}_{GA}$, čiji je iznos zadan ($F_{GA} = 300 N$) i dovršimo trokut sila tako da nacrtamo iz vrha $\vec{F}_{GA}$ do početka $-\vec{F}_{BA}$ vektor $\vec{F}_{RA}$ (totalna reakcija u A).
Iscrtamo komponente sile $\vec{F}_{RA}$: normalnu reakciju $\vec{F}_{NA}$ i silu trenja $\vec{F}_{TA}$. Odnos veličina ovih sila daje trenja koeficijent μ_A:

$$\mu_A = \frac{|\vec{F}_{TA}|}{|\vec{F}_{NA}|}$$





OČITANO:

$F_{RC} = 3{,}5 \cdot 50 = 175\,N$

$F_{BC} = 3{,}5 \cdot 50 = 175\,N$

$F = 7{,}9 \cdot 50 = 395\,N$

$F_{AB} = 10 \cdot 50 = 500\,N$

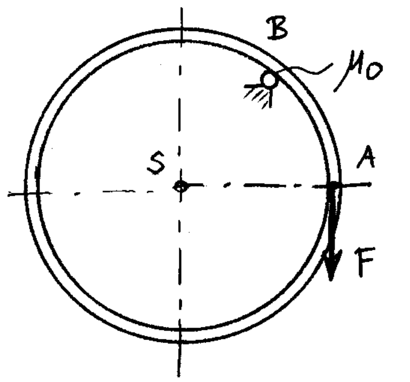
$F_{RA} = 15{,}8 \cdot 50 = 790\,N$

$F_{TA} = 3{,}2 \cdot 50 = 160\,N$

$F_{NA} = 15{,}6 \cdot 50 = 780\,N$

$$\mu_A = tg\,\varphi_{0A} = \frac{F_{TA}}{F_{NA}} = 0{,}205$$

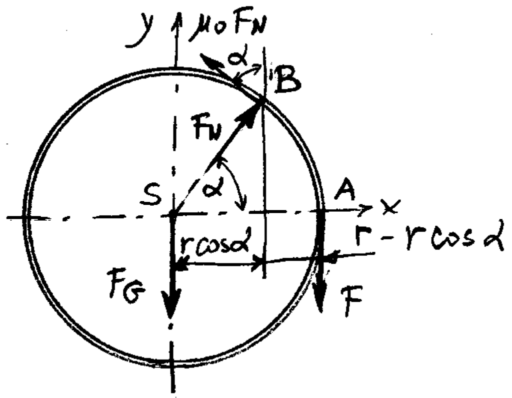
Zad. 5.35



Homogen prsten oslanja se na mali kružni nepomični oslonac B. Prsten ima težinu F_G. U točki A na prsten djeluje sila $F = 0,4\, F_G$. Koji je koeficijent trenja između oslonca i prstena potreban da bi prsten bio u ravnoteži u položaju kao na slici?

Uslijed djelovanja sile F prsten teži kliziti preko oslonca B u desno. Sila trenja graničnog iznosa $F_{Tg} = \mu_0 F_N$ sprječava to gibanje.

Položaj oslonca B nije zadan u odnosu na točku S. Definirat ćemo je kutom α.



Ravnoteža prstena:

$$\Sigma F_x = 0 \qquad \mu_0 F_N \sin\alpha - F_N \cos\alpha = 0 \quad (1) \quad / : F_N$$

$$\mu_0 \sin\alpha = \cos\alpha$$

$$\mu_0 = \frac{\cos\alpha}{\sin\alpha} = \operatorname{ctg}\alpha \qquad (A)$$

$$\Sigma F_y = 0 \qquad F_N \sin\alpha + \mu_0 F_N \cos\alpha - F_G - F = 0 \quad (2)$$

$$\Sigma M_S = 0 \qquad \mu_0 F_N \cdot r - F \cdot r = 0 \quad (3)$$

$$(3) \Rightarrow F_N = \frac{1}{\mu_0} F$$

$$\rightarrow (2) \Rightarrow \frac{1}{\mu_0} F \sin\alpha + \cancel{\mu_0} \cdot \frac{1}{\cancel{\mu_0}} F \cos\alpha - F_G - F = 0 \quad / \cdot \mu_0$$

$$F \sin\alpha + \mu_0 F \cos\alpha - \mu_0 F_G - \mu_0 F = 0, \quad F = 0{,}4 F_G$$

$$0{,}4 F_G (\sin\alpha + \mu_0 \cos\alpha) - \mu_0 F_G - \mu_0 \cdot 0{,}4 F_G = 0 \quad / : F_G$$

$$0{,}4 (\sin\alpha + \mu_0 \cos\alpha) = 1{,}4 \mu_0$$

uvrsti se (A) $\mu_0 = \dfrac{\cos\alpha}{\sin\alpha}$

$$0{,}4 \left(\sin\alpha + \frac{\cos\alpha}{\sin\alpha} \cdot \cos\alpha \right) = 1{,}4 \frac{\cos\alpha}{\sin\alpha} \quad / : 0{,}4 \quad / \cdot \sin\alpha$$

$$\sin^2\alpha + \cos^2\alpha = \frac{1{,}4}{0{,}4} \cdot \cos\alpha$$

$$\Rightarrow \cos\alpha = \frac{0{,}4}{1{,}4} = \frac{2}{7}$$

$$\alpha = \arccos \frac{2}{7} = 73{,}4^\circ$$

$$\Rightarrow \underline{\underline{\mu_0}} = \mathrm{ctg}\,\alpha = \frac{1}{\mathrm{tg}\, 73{,}4^\circ} = \underline{\underline{0{,}298}}$$

Ako se postavi momentna jednadžba $\Sigma M_B = 0$, može se dobiti isti rezultat jednostavnije:

$$\Sigma M_B = 0 \qquad F_G \cdot r \cos\alpha - F \cdot (r - r\cos\alpha) = 0 \quad / : r$$

$$\text{uz } F = 0{,}4 F_G \Rightarrow F_G \cos\alpha - 0{,}4 F_G + 0{,}4 F_G \cos\alpha = 0 \quad / : F_G$$

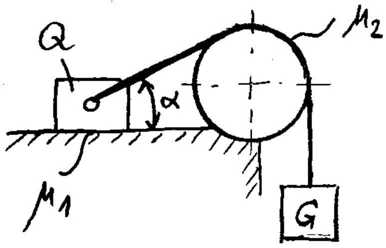
$$\cos\alpha + 0{,}4 \cos\alpha = 0{,}4$$

$$\cos\alpha = \frac{0{,}4}{1{,}4} = \frac{2}{7}$$

$$\rightarrow \alpha = \arccos \frac{2}{7} = 73{,}4^\circ$$

$$\rightarrow (A) \Rightarrow \mu_0 = \mathrm{ctg}\,\alpha = \frac{1}{\mathrm{tg}\, 73{,}4^\circ} = 0{,}298$$

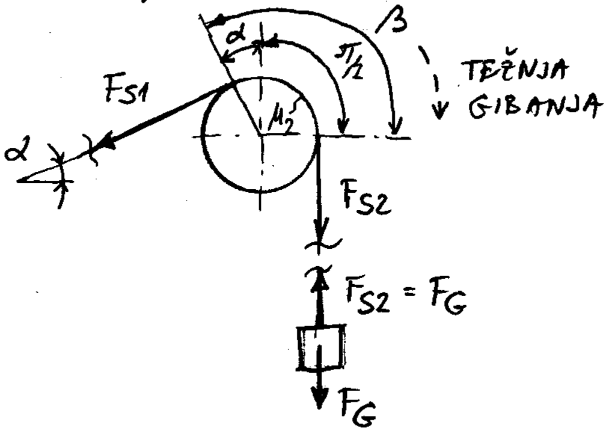
Zad. 5.2



Dva bloka povezana su nerastegljivim užetom prebačenim preko nepomičnog hrapavog valjka. Odrediti minimalnu težinu bloka Q da bi sustav bio u ravnoteži, ako je zadano:

$F_G = 100\ N$, $\mu_1 = 0{,}2$, $\mu_2 = \frac{1}{\pi}$, $\alpha = 30°$.

Minimalni iznos težine bloka Q bit će taman dostatan da uz granični iznos sile trenja između bloka Q i podloge i trenja užeta spriječi gibanje sustava uslijed djelovanja težine bloka G.



$$\beta = \alpha + \frac{\pi}{2} = \frac{\pi}{6} + \frac{\pi}{2} = \frac{\pi + 3\pi}{6} = \frac{2}{3}\pi$$

$$\alpha^0 = 30° \longrightarrow \alpha = \frac{30}{180}\pi = \frac{\pi}{6}$$

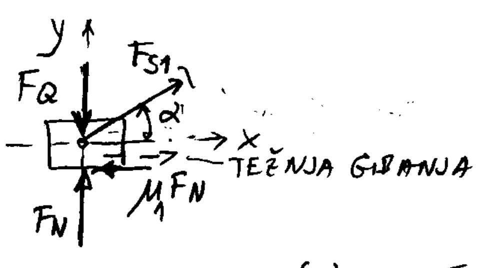
$$F_{S2} > F_{S1}$$

$$F_{S2} = F_{S1} \cdot e^{\mu_2 \beta}$$

$$F_G = F_{S1} \cdot e^{\frac{1}{\pi} \cdot \frac{2}{3}\pi}$$

$$\Rightarrow F_{S1} = \frac{F_G}{e^{\frac{2}{3}}} \qquad (A)$$

Ravnoteža bloka Q



$$\Sigma F_x = 0 \qquad F_{S1}\cos\alpha - \mu_1 F_N = 0 \qquad (1)$$

$$\Sigma F_y = 0 \qquad F_N + F_{S1}\sin\alpha - F_Q = 0 \qquad (2)$$

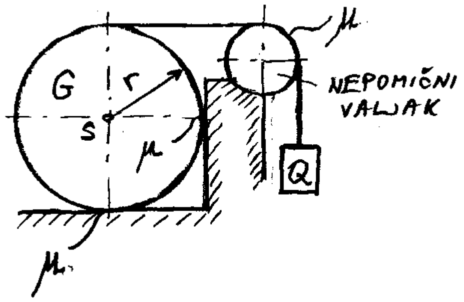
$$(2) \Rightarrow F_N = F_Q - F_{S1}\sin\alpha$$

$$\rightarrow (1) \Rightarrow F_{S1}\cos\alpha + \mu_1 F_{S1}\sin\alpha - \mu_1 F_Q = 0$$

$$F_Q = \frac{\cos\alpha + \mu_1 \sin\alpha}{\mu_1} F_{S1}$$

Uvrsti se (A) $\rightarrow$ $\underline{F_Q} = \dfrac{\cos\alpha + \mu_1 \sin\alpha}{\mu_1 e^{2/3}} \cdot F_G = \underline{247{,}987\ N}$

Zad. 5.3

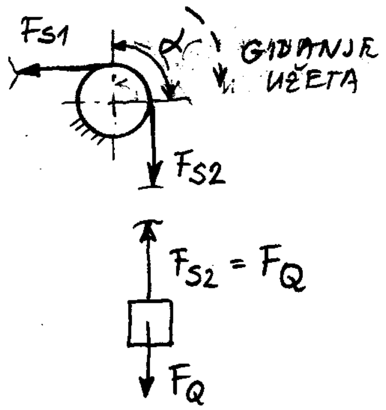


Kolika treba biti težina valjka G za jednoliko spuštanje tereta Q, ako je koeficijent trenja na svim dodirnim površinama jednak ?

Zadano: $F_Q = 150\ N$, $\mu_0 = 0{,}3$.

Za jednoliko pravocrtno gibanje tereta Q i jednoliku rotaciju valjka G oko uzdužne osi kroz S, sustav je u ravnoteži.

TRENJE UŽETA



$$\alpha = \frac{\pi}{2}$$

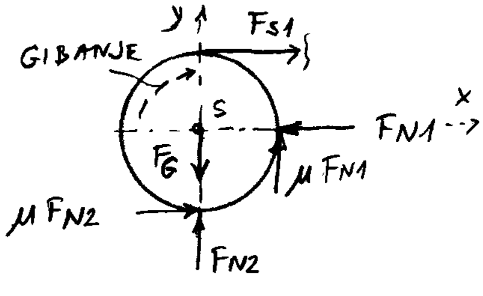
$$F_{S2} > F_{S1}$$

$$F_{S2} = F_{S1}\, e^{\mu \alpha}$$

$$F_Q = F_{S1}\, e^{\mu \frac{\pi}{2}}$$

$$\rightarrow F_{S1} = \frac{F_Q}{e^{\mu \frac{\pi}{2}}} \qquad (A)$$

RAVNOTEŽA VALJKA



SILE TRENJA SUPROTSTAVLJAJU SE ROTACIJI VALJKA

$$\Sigma F_x = 0$$

$$F_{S1} - F_{N1} + \mu F_{N2} = 0 \qquad (1)$$

$$\Sigma F_y = 0$$

$$F_{N2} + \mu F_{N1} - F_G = 0 \qquad (2)$$

$$\Sigma M_S = 0$$

$$\mu F_{N1} \cdot r + \mu F_{N2} \cdot r - F_{S1} \cdot r = 0 \qquad (3)$$

$(1) + (2) / \cdot (-\mu) \Rightarrow$

$$F_{S1} - F_{N1} - \mu^2 F_{N1} + \mu F_G = 0$$

$$F_{N1} = \frac{F_{S1} + \mu F_G}{1+\mu^2} \qquad (B)$$

$(1) / \cdot \mu + (2) \Rightarrow$

$$\mu F_{S1} + \mu^2 F_{N2} + F_{N2} - F_G = 0$$

$$F_{N2} = \frac{F_G - \mu F_{S1}}{1+\mu^2} \qquad (C)$$

$(B) \text{ i } (C) \rightarrow (3) \Rightarrow$

$$\frac{\mu}{1+\mu^2}\left(F_{S1} + \mu F_G\right) + \frac{\mu}{1+\mu^2}\left(F_G - \mu F_{S1}\right) - F_{S1} = 0$$

$$\mu F_{S1} + \mu^2 F_G + \mu F_G - \mu^2 F_{S1} - F_{S1} - \mu^2 F_{S1} = 0$$

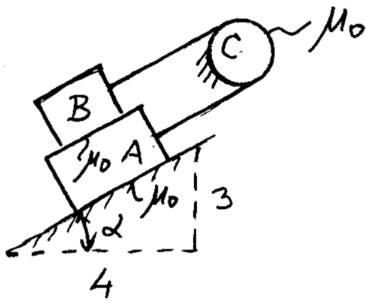
$$F_G \, \mu (1+\mu) = F_{S1}\left(1 - \mu + 2\mu^2\right)$$

Uvrsti se (A)

$$F_G = \frac{F_Q}{e^{\mu \frac{\pi}{2}}} \; \frac{1-\mu+2\mu^2}{\mu(1+\mu)}$$

$$\underline{\underline{F_G}} = \frac{150}{e^{0{,}3 \cdot \frac{\pi}{2}}} \; \frac{1 - 0{,}3 + 2 \cdot 0{,}3^2}{0{,}3(1+0{,}3)} = \underline{\underline{211{,}277 \text{ N}}}$$

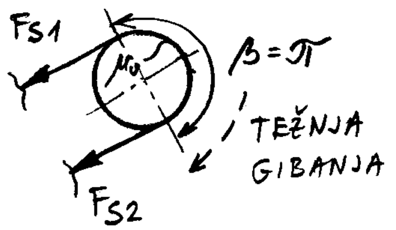
Zad. 5.18



Prizmatični blokovi A težine F_G i B težine F_Q nalaze se na kosini nagiba α, a povezani su užetom koje je paralelno s kosinom i prebačeno preko hrapavog nepomičnog valjka C, prema slici. Potrebno je odrediti najmanji iznos težine bloka B da blok A ne započne klizanje niz kosinu, ako je zadano: $\mu_0 = 0,2$ (koeficijent statičkog trenja na svim dodirnim površinama), $F_G = 1000$ N.

Minimalna težina F_Q bloka B taman je dostatna da uz djelovanje sila trenja s njihovim graničnim iznosom te uz djelovanje trenja užeta spriječi klizanje bloka A niz kosinu.

Trenje užeta

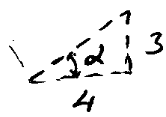


$$F_{S2} > F_{S1}$$

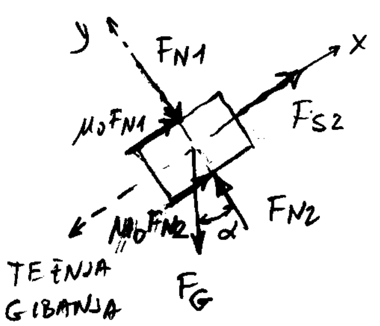
$$F_{S2} = F_{S1} \cdot e^{\mu_0 \beta}$$

$$F_{S2} = F_{S1} \cdot e^{\mu_0 \pi} \qquad (A)$$

Ravnoteža bloka A



$$\operatorname{tg} \alpha = \frac{3}{4}$$



$$\Sigma F_x = 0$$

$$F_{S2} + \mu_0 F_{N1} + \mu_0 F_{N2} - F_G \sin\alpha = 0 \qquad (1)$$

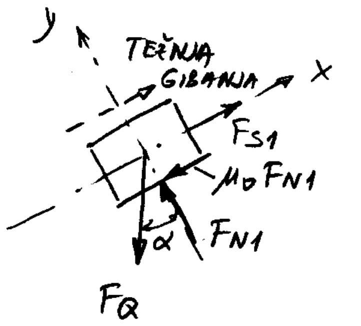
$$\Sigma F_y = 0$$

$$F_{N2} - F_{N1} - F_G \cos\alpha = 0 \qquad (2)$$

$$(1) + (2) / \cdot (-\mu_0) \Longrightarrow F_{S2} + \mu_0 F_{N1} + \mu_0 F_{N1} - F_G \sin\alpha + \mu_0 F_G \cos\alpha = 0$$

$$\Rightarrow \quad F_{N1} = \frac{1}{2\mu_0}\left[F_G\left(\sin\alpha - \mu_0\cos\alpha\right) - F_{S2}\right] \qquad (B)$$

Ravnoteža bloka B



$$\Sigma F_x = 0$$

$$F_{S1} - \mu_0 F_{N1} - F_Q \sin\alpha = 0 \qquad (3)$$

$$\Sigma F_y = 0$$

$$F_{N1} - F_Q\cos\alpha = 0 \qquad (4)$$

$$(4) \Rightarrow \quad F_{N1} = F_Q\cos\alpha \qquad (C)$$

$$(C) \rightarrow (B) \quad \Rightarrow \quad F_{S2} = F_G(\sin\alpha - \mu_0\cos\alpha) - 2F_Q\mu_0\cos\alpha \quad (D)$$

$$(C) \rightarrow (3) \quad \Rightarrow \quad F_{S1} = F_Q(\sin\alpha + \mu_0\cos\alpha) \quad (E)$$

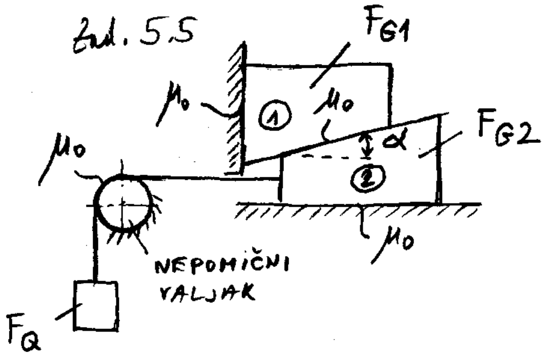
$$(D) \text{ i } (E) \rightarrow (A) \Rightarrow$$

$$F_G(\sin\alpha - \mu_0\cos\alpha) - 2F_Q\mu_0\cos\alpha = F_Q(\sin\alpha + \mu_0\cos\alpha)\cdot e^{\mu_0\pi} \quad /:\cos\alpha$$

$$F_G(\operatorname{tg}\alpha - \mu_0) = F_Q\left[2\mu_0 + (\operatorname{tg}\alpha + \mu_0)e^{\mu_0\pi}\right]$$

$$\underline{F_Q} = F_G\frac{\operatorname{tg}\alpha - \mu_0}{(\operatorname{tg}\alpha + \mu_0)e^{\mu_0\pi} + 2\mu_0} = \underline{252{,}2 \text{ N}}$$

Zad. 5.5

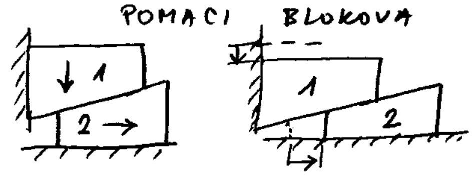


Odrediti granične iznose težine tereta F_Q za koje će sustav zadan slikom još biti u ravnoteži.

Zadano: $F_{G1} = 300\ N$, $F_{G2} = 200\ N$, $\alpha = 30°$, $\mu_0 = 0{,}15$.

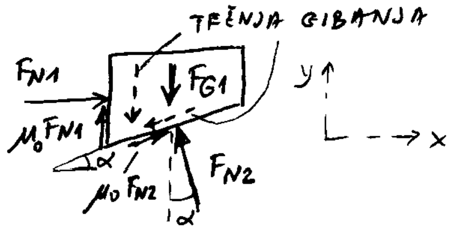
Uz djelovanje sila trenja njihovim graničnim vrijednostima na svim dodirnim površinama, $F_{Q\min}$ će taman spriječiti pomak bloka 1 prema dolje i bloka 2 u desno, a $F_{Q\max}$ još neće izazvati pomak bloka 2 u lijevo i bloka 1 prema gore.

POMACI BLOKOVA



1) određivanje $F_{Q\min}$

Ravnoteža bloka 1



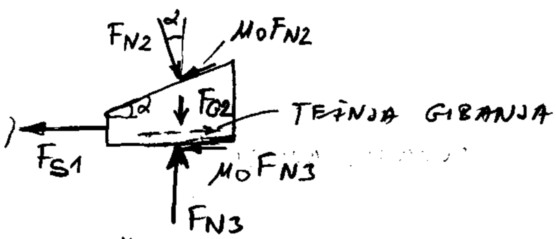
$\Sigma F_x = 0$

$$F_{N1} + \mu_0 F_{N2} \cos\alpha - F_{N2} \sin\alpha = 0 \quad (1)$$

$\Sigma F_y = 0$

$$\mu_0 F_{N1} + \mu_0 F_{N2} \sin\alpha + F_{N2} \cos\alpha - F_{G1} = 0 \quad (2)$$

Ravnoteža bloka 2

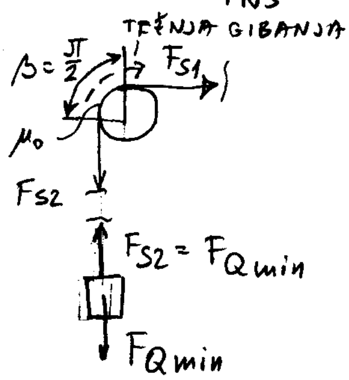


$\Sigma F_x = 0$

$$F_{N2} \sin\alpha - \mu_0 F_{N2} \cos\alpha - \mu_0 F_{N3} - F_{S1} = 0 \quad (3)$$

$\Sigma F_y = 0$

$$-F_{N2} \cos\alpha - \mu_0 F_{N2} \sin\alpha + F_{N3} - F_{G2} = 0 \quad (4)$$



TRENJE UŽETA

$$F_{S1} > F_{S2}$$

$$F_{S1} = F_{S2}\, e^{\mu_0 \beta}$$

$$F_{S1} = F_{Q\min}\, e^{\mu_0 \frac{\pi}{2}} \quad (5)$$

$(1)/\cdot\mu_0 + (2)/\cdot(-1) \Rightarrow$

$$\mu_0^2 F_{N2}\cos\alpha - \mu_0 F_{N2}\sin\alpha - \mu_0 F_{N2}\sin\alpha - F_{N2}\cos\alpha + F_{G1} = 0$$

$$F_{N2}\left(\cos\alpha + 2\mu_0\sin\alpha - \mu_0^2\cos\alpha\right) = F_{G1}$$

$$F_{N2} = \frac{F_{G1}}{(1-\mu_0^2)\cos\alpha + 2\mu_0\sin\alpha} \qquad (A)$$

$(3) + (4)/\cdot\mu_0 \Rightarrow$

$$F_{N2}\sin\alpha - \mu_0 F_{N2}\cos\alpha - \mu_0 F_{N2}\cos\alpha - \mu_0^2 F_{N2}\sin\alpha - F_{S1} - \mu_0 F_{G2} = 0$$

$$F_{N2}\left(\sin\alpha - 2\mu_0\cos\alpha - \mu_0^2\sin\alpha\right) = F_{S1} + \mu_0 F_{G2}$$

$$F_{N2} = \frac{F_{S1} + \mu_0 F_{G2}}{(1-\mu_0^2)\sin\alpha - 2\mu_0\cos\alpha} \qquad (B)$$

Izjednače se desne strane izraza (A) i (B):

$$\Rightarrow \quad F_{S1} + \mu_0 F_{G2} = \frac{(1-\mu_0^2)\sin\alpha - 2\mu_0\cos\alpha}{(1-\mu_0^2)\cos\alpha + 2\mu_0\sin\alpha} F_{G1}$$

$$F_{S1} = \frac{(1-\mu_0^2)\sin\alpha - 2\mu_0\cos\alpha}{(1-\mu_0^2)\cos\alpha + 2\mu_0\sin\alpha} F_{G1} - \mu_0 F_{G2}$$

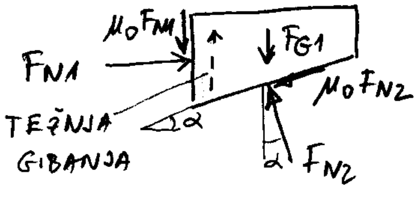
$$\rightarrow (5) \Rightarrow \quad F_{Qmin} = \frac{1}{e^{\mu_0\frac{\pi}{2}}}\left[\frac{(1-\mu_0^2)\sin\alpha - 2\mu_0\cos\alpha}{(1-\mu_0^2)\cos\alpha + 2\mu_0\sin\alpha} F_{G1} - \mu_0 F_{G2}\right]$$

$$F_{Qmin} = 30{,}75 \text{ N}$$

2) Određivanje F_{Qmax}

POMACI BLOKOVA

Ravnoteža bloka 1



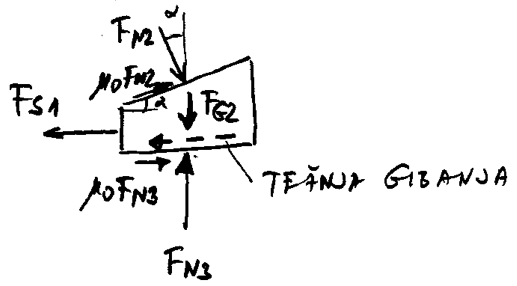
$\Sigma F_x = 0$

$$F_{N1} - F_{N2}\sin\alpha - \mu_0 F_{N2}\cos\alpha = 0 \qquad (1)$$

$\Sigma F_y = 0$

$$-\mu_0 F_{N1} + F_{N2}\cos\alpha - \mu_0 F_{N2}\sin\alpha - F_{G1} = 0 \qquad (2)$$

Ravnoteža bloka 2

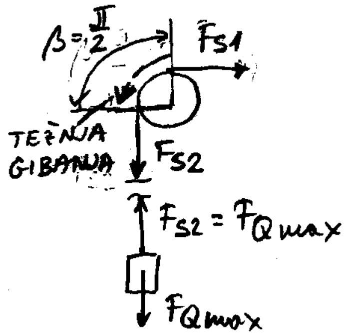


$\Sigma F_x = 0$

$$F_{N2} \sin\alpha + \mu_0 F_{N2} \cos\alpha + \mu_0 F_{N3} - F_{S1} = 0 \quad (3)$$

$\Sigma F_y = 0$

$$-F_{N2} \cos\alpha + \mu_0 F_{N2} \sin\alpha + F_{N3} - F_{G2} = 0 \quad (4)$$



TRENJE UŽETA

$$F_{S2} > F_{S1}$$

$$F_{S2} = F_{S1} \cdot e^{\mu_0 \beta}$$

$$F_{Q\,max} = F_{S1}\, e^{\mu_0 \frac{\pi}{2}} \qquad (5)$$

$(1)/\cdot\mu_0 + (2) \rightarrow$

$$-\mu_0 F_{N2} \sin\alpha - \mu_0^2 F_{N2} \cos\alpha + F_{N2} \cos\alpha - \mu_0 F_{N2} \sin\alpha - F_{G1} = 0$$

$$F_{N2} = \frac{F_{G1}}{(1-\mu_0^2)\cos\alpha - 2\mu_0 \sin\alpha} \qquad (A)$$

$(3) + (4)/\cdot(-\mu_0) \Rightarrow$

$$F_{N2} \sin\alpha + \mu_0 F_{N2} \cos\alpha + \mu_0 F_{N2} \cos\alpha - \mu_0^2 F_{N2} \sin\alpha - F_{S1} + \mu_0 F_{G2} = 0$$

$$F_{N2} = \frac{F_{S1} - \mu_0 F_{G2}}{(1-\mu_0^2)\sin\alpha + 2\mu_0 \cos\alpha} \qquad (B)$$

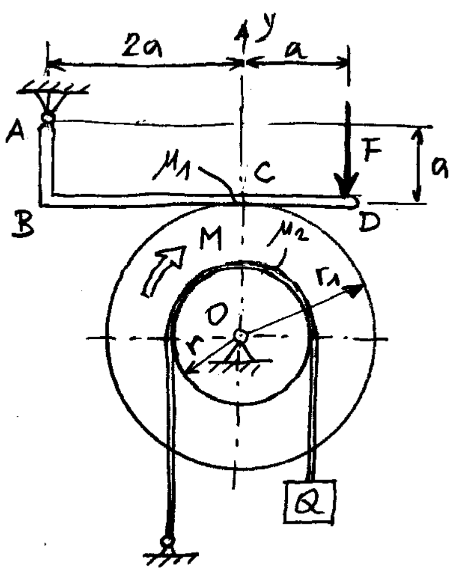
Izjednače se desne strane izraza (A) i (B)

$$F_{S1} - \mu_0 F_{G2} = \frac{(1-\mu_0^2)\sin\alpha + 2\mu_0\cos\alpha}{(1-\mu_0^2)\cos\alpha - 2\mu_0 \sin\alpha} F_{G1}$$

$$F_{S1} = \frac{(1-\mu_0^2)\sin\alpha + 2\mu_0\cos\alpha}{(1-\mu_0^2)\cos\alpha - 2\mu_0 \sin\alpha} F_{G1} + \mu_0 F_{G2}$$

$$\rightarrow (5) \Rightarrow \quad F_{Q\,max} = e^{\mu_0 \frac{\pi}{2}} \left[\frac{(1-\mu_0^2)\sin\alpha + 2\mu_0\cos\alpha}{(1-\mu_0^2)\cos\alpha - 2\mu_0 \sin\alpha} F_{G1} + \mu_0 F_{G2} \right]$$

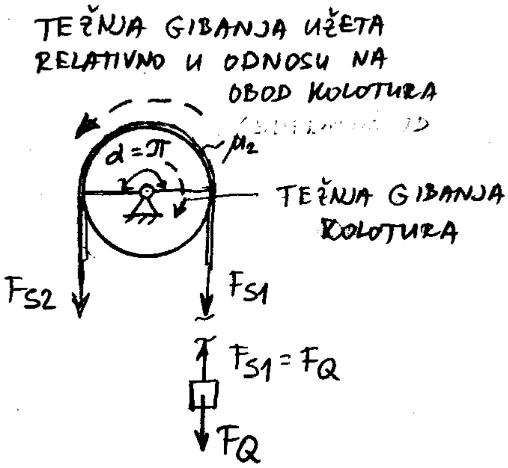
$$F_{Q\,max} = 446{,}04 \text{ N}$$



Zad. 5.22

Teret Q vezan je užetom koje je prebačeno preko hrapavog koloture polumjera r. Ovaj je kolotur kruto spojen s koloturom polumjera r_1. Ako na ovaj dvostruki kolotur djeluje moment M, te ako se na obod većeg koloture djeluje kočenjem s pomoću poluge ABCD, odrediti iznos težine tereta Q za koji će kolotur još mirovati.

Zadano: $F = 100\ N$, $r = 0{,}5\ m$, $r_1 = 1\ m$, $a = 0{,}6\ m$, $\mu_1 = 0{,}4$, $\mu_2 = 0{,}3$, $M = 300\ Nm$.

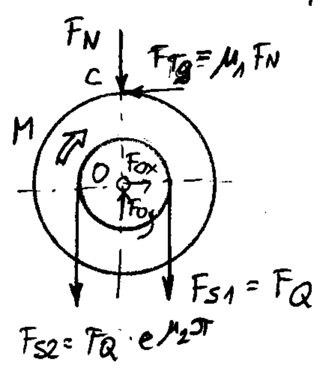


Kolotur u odnosu na mirujuće uže teži rotirati u smjeru kazaljke na satu. Uže relativno u odnosu na obod koloture teži se gibati u suprotnom smjeru. Taj smjer određuje da je

$$F_{S2} > F_{S1}$$

$$F_{S2} = F_{S1} \cdot e^{\mu_2 \alpha}$$

$$F_{S2} = F_Q \cdot e^{\mu_2 \pi} \qquad (1)$$



Ravnoteža dvostrukog koloture:

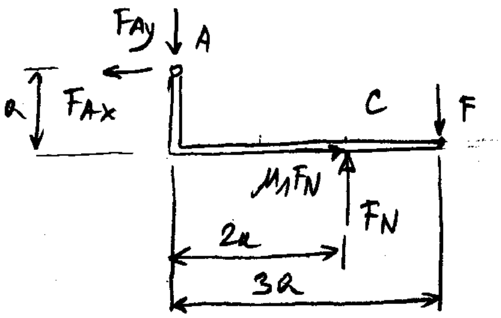
$$\Sigma M_O = 0$$

$$\mu_1 F_N r_1 + F_{S2} r - F_{S1} r - M = 0 \qquad (2)$$

U zadatku se traži minimalna težina F_Q za koju će još kolotur mirovati. U tom slučaju u C na obod većeg kolotura djeluje granična sila trenja $\mu_1 F_N$.

$$(1) \rightarrow (2) \Rightarrow \quad F_Q\left(e^{\mu_2 \pi}-1\right) r=M-\mu_1 F_N r_1 \quad (A)$$

Ravnoteža poluge ABCD:



$$\Sigma M_A=0$$

$$F_N \cdot 2a+\mu_1 F_N \cdot a-F \cdot 3a=0 \quad / : a$$

$$\Rightarrow F_N=\frac{3}{2+\mu_1} F \quad (B)$$

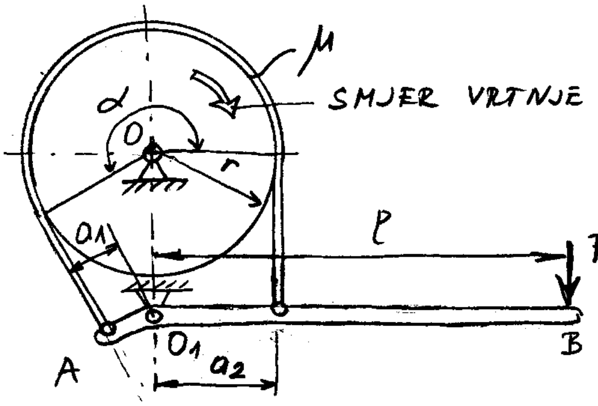
$$(B) \rightarrow (A) \Rightarrow$$

$$F_Q\left(e^{\mu_2 \pi}-1\right) r=M-\mu_1 \frac{3}{2+\mu_1} F r_1$$

$$F_Q=\frac{1}{\left(e^{\mu_2 \pi}-1\right) r}\left(M-\frac{3 \mu_1}{2+\mu_1} F r_1\right)$$

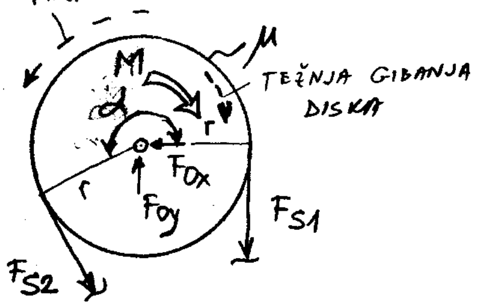
$$F_Q=319,22 \text{ N}$$

Zad. 5.9



Kolikom se silom F mora djelovati na polugu AO_1B diferencijalne pojasne kočnice prema slici, da bi se na disku kočnice proizveo moment otpora trenja M, ako je zadano: $M, r, a_1, a_2, l, \mu, \alpha$?

TEŽNJA RELATIVNOG GIBANJA POJASA U ODNOSU NA OBOD DISKA



Momentu koji nastoji zarotirati disk mora se suprostaviti moment otpora trenja jednakog iznosa ali suprotnog smjera. Taj moment proizvode sile F_{S1} i F_{S2} koje zatežu pojas.

Ravnoteža diska

$$\Sigma M_O = 0$$

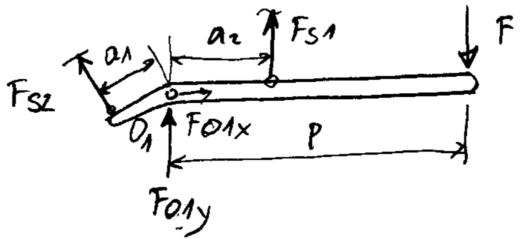
$$F_{S2} \cdot r - F_{S1} \cdot r - M = 0$$

$$\Rightarrow M = (F_{S2} - F_{S1}) \cdot r \qquad (1)$$

Trenje pojasa:

$$F_{S2} > F_{S1} \quad \Longrightarrow \quad F_{S2} = F_{S1} \cdot e^{\mu\alpha} \qquad (2)$$

Ravnoteža poluge AO_1C :



$$\Sigma M_{O1} = 0$$

$$F_{S1}\, a_2 - F_{S2}\, a_1 - F\, l = 0 \qquad (3)$$

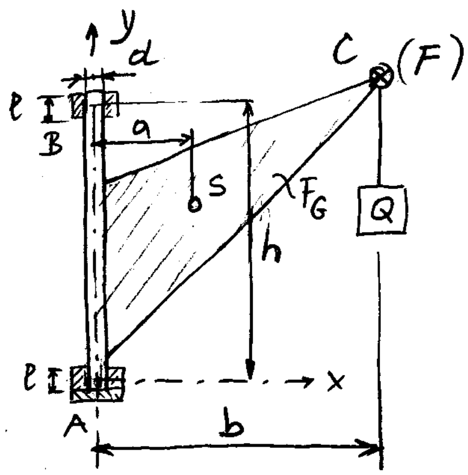
$$(2) \rightarrow (1) \rightarrow \quad F_{S1} = \frac{M}{r} \cdot \frac{1}{e^{\mu\alpha} - 1} \qquad (a)$$

$$\rightarrow (2) \rightarrow \quad F_{S2} = \frac{M}{r} \frac{e^{\mu\alpha}}{e^{\mu\alpha} - 1} \qquad (b)$$

(a) i (b) $\rightarrow$ (3) $\Rightarrow$

$$\underline{F = \frac{M}{r} \; \frac{a_2 - a_1 \cdot e^{\mu\alpha}}{l\,(e^{\mu\alpha} - 1)}}$$

Zad. 5.23



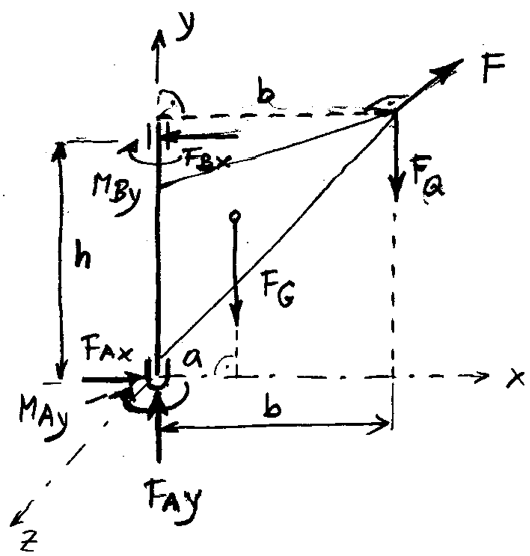
Konzolna dizalica, zadana prema slici, opterećena je na kuki težinom tereta Q. Potrebno je odrediti vodoravnu silu F koja djeluje u točki C za okretanje dizalice. Kolike su komponente reakcije u radijalnom B i radijalno-aksijalnom A ležaju? Odrediti specifične tlakove u ležajima. Koeficijenti trenja u oba ležaja su jednaki i iznose $\mu = 0{,}1$.

Zadano: $F_Q = 2\ kN$, $F_G = 16\ kN$,

$a = 0{,}75\ m$, $b = 3\ m$, $h = 3\ m$,

$d = 70\ mm$ (promjer rukavca),

$l = 70\ mm$ (duljina rukavca ležaja).



$$\Sigma F_x = 0 \qquad F_{Ax} - F_{Bx} = 0 \qquad (1)$$

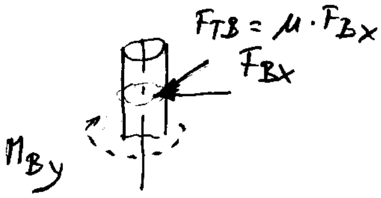
$$\Sigma F_y = 0 \qquad F_{Ay} - F_G - F_Q = 0 \qquad (2)$$

$$\Sigma M_z = 0 \qquad F_{Bx} \cdot h - F_G \cdot a - F_Q \cdot b = 0 \quad (3)$$

$$(3) \Rightarrow \quad F_{Bx} = \frac{F_G \cdot a + F_Q \cdot b}{h} = \frac{16 \cdot 0{,}75 + 2 \cdot 3}{3} = 6\ kN$$

$$\rightarrow (1) \Rightarrow \quad F_{Ax} = F_{Bx} = 6\ kN$$

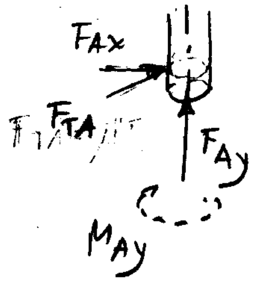
$$(2) \Rightarrow \quad F_{Ay} = F_G + F_Q = 16 + 2 = 18\ kN$$



MOMENT OTPORA TRENJA U LEŽAJU B:

$$M_{By} = F_{TB} \cdot \frac{d}{2} = \mu F_{Bx} \cdot \frac{d}{2} =$$

$$= 0{,}1 \cdot 6 \cdot 10^3 \cdot \frac{1}{2} \cdot 0{,}07 = 21\ \text{Nm}$$



MOMENT OTPORA TRENJA U LEŽAJU A:

– od radijalne komponente ležaja

$$M_{Ayr} = F_{TA} \cdot \frac{d}{2} = \mu F_{Ax} \cdot \frac{d}{2} =$$

$$= 0{,}1 \cdot 6 \cdot 10^3 \cdot \frac{1}{2} \cdot 0{,}07 = 21\ \text{Nm}$$

– od aksijalne komponente ležaja

$$M_{Aya} = \frac{2}{3} \mu F_{Ay} \cdot \frac{d}{2} =$$

$$= \frac{2}{3} \cdot 0{,}1 \cdot 18 \cdot 10^3 \cdot 0{,}07 = 84\ \text{Nm}$$

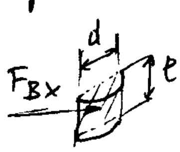
$$\Rightarrow M_{Ay} = M_{Ayr} + M_{Aya} = 21 + 84 = 105\ \text{Nm}$$

$$\Sigma M_y = 0$$

$$F \cdot b - M_{Ay} - M_{By} = 0$$

$$\underline{\underline{F}} = \frac{M_{Ay} + M_{By}}{b} = \frac{105 + 21}{3} = \frac{126}{3} = \underline{\underline{42\ N}}$$

Specifični tlakovi



$$p_{Bx} = \frac{F_{Bx}}{d \cdot e} = \frac{6 \cdot 10^3}{7 \cdot 10^{-2} \cdot 7 \cdot 10^{-2}} = 0{,}12245 \cdot 10^7 \frac{N}{m^2} =$$

$$= 1{,}2245 \cdot 10^6\ \text{Pa} =$$

$$= 1{,}2245\ \text{MPa}$$

$$p_{Ax} = \frac{F_{Ax}}{d \cdot e} = \frac{6 \cdot 10^3}{7 \cdot 10^{-2} \cdot 7 \cdot 10^{-2}} =$$

$$= 1{,}2245\ \text{MPa}$$

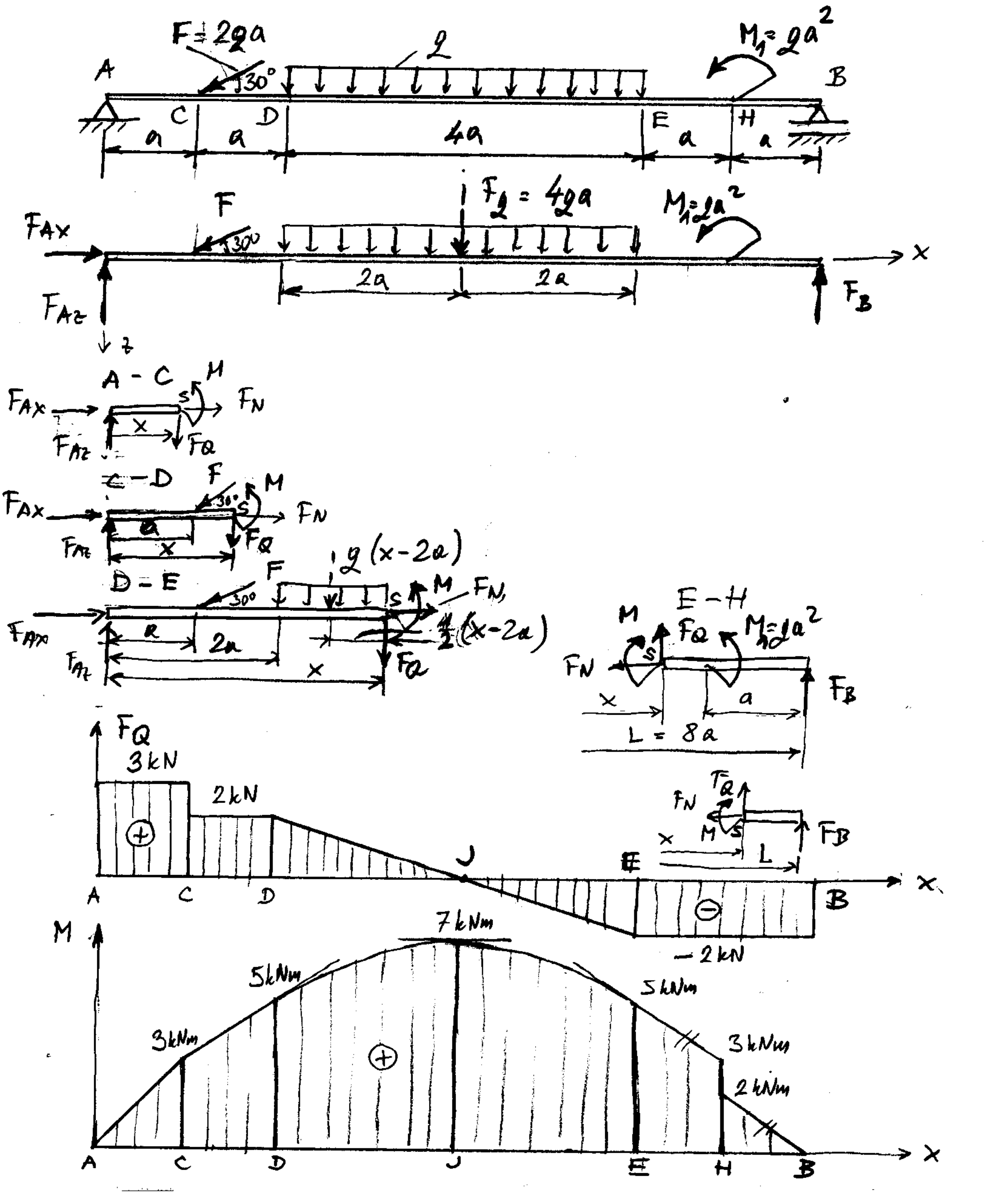
$$p_{Ay} = \frac{F_{Ay}}{\frac{d^2 \pi}{4}} = 4{,}6772\ \text{MPa}$$

VJEŽBE IZ MEHANIKE I

II. DIO (RAVNI NOSAČI, OKVIRNI NOSAČI, REŠETKASTI NOSAČI)

SASTAVIO prof. dr. sc. J. Saucha

Za jednostavnu gredu zadanu i opterećenu prema slici potrebno je odrediti zakonitosti promjene poprečne sile $F_Q(x)$ i momenta savijanja $M(x)$ te nacrtati dijagrame poprečnih sila i momenata savijanja kao grafove funkcija $F_Q(x)$ i $M(x)$. Zatim treba skicirati F_Q-dijagram i M-dijagram primjenom postupka pojednostavljene konstrukcije tih dijagrama. Zadano: $q = 1$ kN/m, $a = 1$ m.



Jednadžbe ravnoteže nosača:

$\Sigma F_x = 0 \qquad F_{Ax} - F\cos 30^\circ = 0$

$\Rightarrow \underline{F_{Ax}} = F\cos 30^\circ = 2qa \cdot \frac{\sqrt{3}}{2} = \sqrt{3}qa = 1{,}73 \cdot 1 \cdot 1 = \underline{1{,}73\,kN}$

$\Sigma F_z = 0 \qquad -F_{Az} + F\sin 30^\circ + F_q - F_B = 0 \quad ; \quad F_q = q \cdot 4a = 4qa$

$\Rightarrow \underline{F_{Az} + F_B} = 2qa \cdot \frac{1}{2} + 4qa = \underline{5qa} \qquad (1)$

$\Sigma M_A = 0 \qquad F_B \cdot 8a + M - F_q \cdot 4a - F\sin 30^\circ \cdot a = 0$

$\Rightarrow F_B = \frac{1}{8a}\left(-qa^2 + 4qa \cdot 4a + 2qa \cdot \frac{1}{2} \cdot a\right) = \frac{16qa^2}{8a} \qquad (2)$

$\underline{F_B} = 2qa = 2 \cdot 1 \cdot 1 = \underline{2\,kN}$

$\to (1) \Rightarrow \underline{F_{Az}} = 5qa - 2qa = 3qa = 3 \cdot 1 \cdot 1 = \underline{3\,kN}$

Granice područja su presjeci A i B (krajevi nosača, odnosno mjesta na kojima djeluju vanjske sile F_{Az} i F_{Ax} te F_B), C i H (mjesta na kojima djeluju vanjska sila F, odnosno vanjski moment M) te D i E (na kojima počinje, odnosno prestaje djelovati kontinuirano opterećenje q).

<u>Područje A - C</u> $\Rightarrow 0 \leq x \leq a$

Uvjeti ravnoteže lijevog dijela nosača:

$\Sigma F_x = 0 \qquad F_N + F_{Ax} = 0 \quad \Rightarrow \quad F_N = -F_{Ax} = -1{,}73\,kN$

$\Sigma F_z = 0 \qquad F_Q - F_{Az} = 0 \quad \Rightarrow \quad F_Q = F_{Az} = 3\,kN$

$\Sigma M_S = 0 \qquad M - F_{Az} \cdot x = 0 \quad \Rightarrow \quad M = F_{Az} \cdot x = 3x \; [kNm]$ - PRAVAC!

za $x = 0$, $\underline{M_A = 0}$; za $x = a = 1\,m$, $\underline{M_C} = 3 \cdot 1 = \underline{3\,kNm}$

Područje C - D $\Rightarrow a \le x \le 2a$

Uvjeti ravnoteže lijevog dijela nosača:

$\Sigma F_x = 0 \qquad F_N + F_{Ax} - F\cos 30° = 0 \Rightarrow \underline{F_N} = -F_{Ax} + 2qa \cdot \frac{\sqrt{3}}{2} =$

$= -1{,}73 + 2 \cdot 1 \cdot 1 \cdot \frac{\sqrt{3}}{2} = \underline{0}$

$\Sigma F_z = 0 \qquad F_Q - F_{Az} + F\sin 30° = 0 \Rightarrow \underline{F_Q} = F_{Az} - 2qa \cdot \frac{1}{2} =$

$= 3 - 2 \cdot 1 \cdot 1 \cdot \frac{1}{2} = \underline{2\,kN}$

$\Sigma M_S = 0 \qquad M - F_{Az} \cdot x + F\sin 30° \cdot (x - a) = 0$

$\Rightarrow M = F_{Az} x - 2qa \cdot \frac{1}{2}(x - a) = 3x - 2 \cdot 1 \cdot 1 \cdot \frac{1}{2}(x - 1)$

$M = \underbrace{2x + 1}_{\text{PRAVAC!}} \quad [kNm]$

za $x = a = 1\,m$, $\underline{M_C} = 2 \cdot 1 + 1 = \underline{3\,kNm}$

za $x = 2a = 2\,m$, $\underline{M_D} = 2 \cdot 2 + 1 = \underline{5\,kNm}$

Područje D - E $\Rightarrow 2a \le x \le 6a$

Uvjeti ravnoteže lijevog dijela nosača

$\Sigma F_x = 0 \qquad F_N + F_{Ax} - F\cos 30° = 0 \Rightarrow \underline{F_N} = -F_{Ax} + F\cos 30° = \underline{0}$

$\Sigma F_z = 0 \qquad F_Q - F_{Az} + F\sin 30° + q(x - 2a) = 0$

$\rightarrow F_Q = F_{Az} - 2qa \cdot \frac{1}{2} - q(x - 2a) = 3 - 2 \cdot 1 \cdot 1 \cdot \frac{1}{2} - 1 \cdot (x - 2 \cdot 1)$

$F_Q = \underbrace{4 - x}_{\text{PRAVAC!}} \quad [kN], \quad x\,[m]$

za $x = 2a = 2\,m$, $\underline{F_{QD}} = 4 - 2 = \underline{2\,kN}$

za $x = 6a = 6\,m$, $\underline{F_{QE}} = 4 - 6 = \underline{-2\,kN}$

Nultočka u F_Q dijagramu:

$\underline{F_{QJ} = 0} \Rightarrow 4 - x_J = 0 \rightarrow \underline{x_J = 4\,m}$

$\Sigma M_S = 0 \qquad M - F_{Az} \cdot x + F\sin 30° (x - a) + q(x - 2a) \cdot \frac{1}{2}(x - 2a) = 0$

$\rightarrow M = F_{Az} \cdot x - 2qa \cdot \frac{1}{2}(x - a) - \frac{1}{2}(x - 2a)^2 =$

$= 3 \cdot x - 2 \cdot 1 \cdot 1 \cdot \frac{1}{2}(x - 1) - \frac{1}{2}(x - 2 \cdot 1)^2 = 3x - x + 1 - \frac{1}{2}(x^2 - 4x + 4)$

$$\Rightarrow M = \underbrace{-\frac{1}{2}x^2 + 4x - 1}_{\text{PARABOLA 2o REDA!}} \quad [kNm], \quad x[m]$$

za $x = 2a = 2m$, $\underline{M_D} = -\frac{1}{2}\cdot 2^2 + 4\cdot 2 - 1 = \underline{5\,kNm}$

za $x = 6a = 6m$, $\underline{M_E} = -\frac{1}{2}\cdot 6^2 + 4\cdot 6 - 1 = \underline{5\,kNm}$

Na mjestu nultočke u F_Q dijagramu, za $x_J = 4m$, u M-dijagramu bit će ekstremne vrijednost $M_J = M_{max}$, odnosno tjeme parabole 2o reda:

za $x = x_J = 4m$, $\underline{M_J} = -\frac{1}{2}\cdot 4^2 + 4\cdot 4 - 1 = \underline{7\,kNm} = \underline{M_{max}}$

<u>Područje E – H</u> $\Rightarrow$ $6a \leq x \leq 7a$

Uvjeti ravnoteže desnog dijela nosača:

$\Sigma F_x = 0 \quad -F_N = 0 \quad \Rightarrow \quad F_N = 0$

$\Sigma F_z = 0 \quad -F_Q - F_B = 0 \quad \Rightarrow \quad \underline{F_Q} = -F_B = \underline{-2\,kN}$

$\Sigma M_S = 0 \quad -M + M_1 + F_B\cdot(L - x) = 0$

$\Rightarrow M = qa^2 + F_B(8a - x) = 1\cdot 1^2 + 2(8\cdot 1 - x)$

$$M = \underbrace{17 - 2x}_{\text{pravac!}} \quad [kNm], \quad x[m]$$

za $x = 6a = 6m$, $\underline{M_E} = 17 - 2\cdot 6 = \underline{5\,kNm}$

za $x = 7a = 7m$, $\underline{M_H^L} = 17 - 2\cdot 7 = \underline{3\,kNm}$

<u>Područje H – B</u> $\Rightarrow$ $7a \leq x \leq 8a$

Uvjeti ravnoteže desnog dijela nosača:

$\Sigma F_x = 0 \quad -F_N = 0 \quad \Rightarrow \quad \underline{F_N = 0}$

$\Sigma F_z = 0 \quad -F_Q - F_B = 0 \quad \Rightarrow \quad \underline{F_Q} = -F_B = \underline{-2\,kN}$

$\Sigma M_S = 0 \quad -M + F_B\cdot(L - x) = 0 \quad \Rightarrow \quad M = F_B(8a - x) = 2(8\cdot 1 - x)$

$M = 16 - 2x \;[kNm], \quad x\,[m]$

za $x = 7a = 7m$, $\underline{M_H^D} = 16 - 2\cdot 7 = \underline{2 \text{ kNm}}$

za $x = 8a = 8m$, $\underline{M_B} = 16 - 2\cdot 8 = \underline{0}$

Prema izvedenim izrazima za $F_Q(x)$ i $M(x)$ nacrtani su F_Q-dijagram i M-dijagram.
U F_Q dijagramu u područjima A–C, C–D, E–H i H–B konstantne su vrijednosti F_Q, a u području D–E na kojem djeluje kontinuirano opterećenje q, $F_Q(x)$ je pravac s negativnim koeficijentom smjera.
U M-dijagramu u područjima A–C i C–D su pravci s pozitivnim koeficijentima smjera (u skladu s pozitivnim konstantnim veličinama F_Q u tim područjima), a u područjima E–H i H–B su paralelni pravci s negativnim koeficijentom smjera (u skladu s jednakim negativnim veličinama F_Q u tim područjima).
U području D–E je parabola 2. reda, u skladu s pravcem u F_Q dijagramu. Od D do J tangente na parabolu imaju pozitivne nagibe koji postaju sve manji i padaju na nulu na mjestu J gdje je $F_Q = 0$. Od J do E tangente na parabolu imaju negativne nagibe koji postaju sve veći po iznosu. Sve je ovo u skladu s promjenom veličine F_Q (od D do J linearno F_Q opada s 2kN na 0, a od J do E s 0 na −2 kN).

U D, parabola se TANGENCIJALNO nastavlja iz pravca, budući da su nagibi tangenti prilijeva (tangenta na pravac je sam pravac!) i s desna jednaki, jer je $F_{QD}^{L} = F_{QD}^{D}$.

U E, pravac se TANGENCIJALNO nastavlja iz parabole, zbog istog razloga, tj. zbog toga što je $F_{QE}^{L} = F_{QE}^{D}$.

Pojednostavljena konstrukcija F_Q i M dijagrama

Konstrukciju F_Q dijagrama započinjemo na lijevom kraju grede, na kraju A. Tu djeluje negativno usmjerena u odnosu na os z vanjska sila reakcija F_{Az}. To će izazvati skokovit pozitivan prirast F_Q za iznos F_{Az}, tj. za $+3qa$. Od A do C F_Q zadržava konstantan iznos $+3qa$. U C komponenta sile F u z smjeru, $F\sin 30°$, usmjerena je pozitivno, pa izaziva negativan prirast poprečne sile za svoj iznos, tj. za $-F\sin 30° = -qa$, pa će F_Q poprimiti vrijednost $3qa - qa = 2qa$. Od C do D taj iznos ostaje nepromijenjen. U D počinje djelovati kontinuirano opterećenje q koje će od D do E izazvati LINEARAN NEGATIVAN PRIRAST F_Q za $-q \cdot 4a$, pa F_Q sa vrijednosti $2qa$ u D linearno pada na vrijednost $-2qa$ u E.

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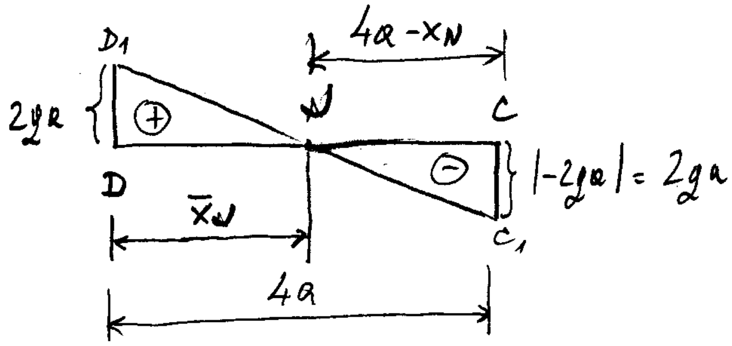
Od E do B ne djeluje niti jedna vanjska sila u z smjeru, pa vrijednost F_Q ostaje konstantna, tj. $-2qa$. U B djeluje u negativnom smjeru reakcija F_B što izaziva pozitivan skokovit prirast F_Q za iznos F_B, tj. za $+2qa$. Tako na desnom kraju grede F_Q sa $-2qa$ prirastom $+2qa$ pada na 0.

Položaj nultočke u F_Q dijagramu možemo odrediti iz sličnosti trokuta:



$$\frac{\bar{x}_N}{\overline{DD_1}} = \frac{4a - \bar{x}_N}{\overline{CC_1}}$$

$$\bar{x}_N = \frac{\overline{DD_1}}{\overline{CC_1}}(4a - \bar{x}_N)$$

$$\bar{x}_N = \frac{2qa}{2qa}(4a - \bar{x}_N) \Longrightarrow \bar{x}_N = 4a - \bar{x}_N$$

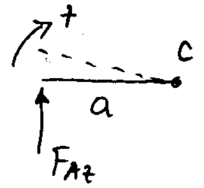
$$2\bar{x}_N = 4a \Rightarrow \underline{\underline{\bar{x}_N = 2a}}$$

Za konstrukciju M dijagrama potrebno je izračunati iznose M na granicama područja i na mjestu nultočke u F_Q-dijagramu.

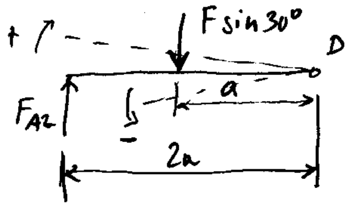
Na krajevima nosača ne djeluju vanjski momenti pa su momenti savijanja tu jednaki nuli, tj

$$M_A = 0 \quad i \quad M_B = 0.$$

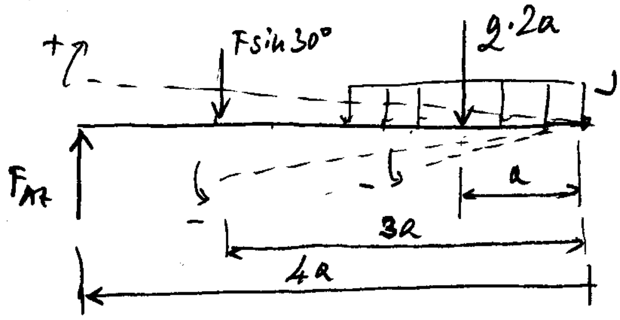
Određivanje M_C, M_D, M_J, M_E, M_H^L i M_H^D:



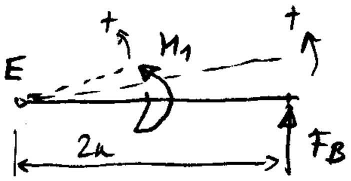
$$\underline{M_C} = +F_{Az}\cdot a = +3qa\cdot a = \underline{+3qa^2}$$



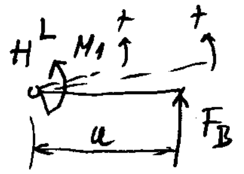
$$\underline{M_D} = +F_{Az}\cdot 2a - F\sin 30^\circ\cdot a =$$
$$= +3qa\cdot 2a - 2qa\cdot\tfrac{1}{2}\cdot a = \underline{+5qa^2}$$



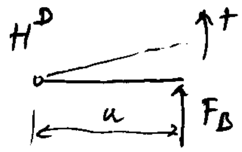
$$\underline{M_J} = +F_{Az}\cdot 4a - F\sin 30^\circ\cdot 3a -$$
$$-(q\cdot 2a)\cdot a =$$
$$= +3qa\cdot 4a - 2qa\cdot\tfrac{1}{2}\cdot 3a -$$
$$-2qa^2 = \underline{+7qa^2}$$



$$\underline{M_E} = +F_B\cdot 2a + M_1 = +2qa\cdot 2a + qa^2 =$$
$$= \underline{5qa^2}$$



$$\underline{M_H^L} = +F_B\cdot a + M_1 = +2qa\cdot a + qa^2 =$$
$$= \underline{3qa^2}$$



$$M_H^D = +F_B\cdot a = +2qa\cdot a = +2qa^2$$

U presjeku H djeluje vanjski moment M koji izaziva skok u M-dijagramu. Ako promatramo desni dio nosača, presjek neposredno lijevo od H, tj. H^L, obuhvaća i vanjski moment M_1, a presjek neposredno desno od H, tj. H^D, <u>ne</u> obuhvaća M_1, kako je pokazano. Vrijedi

$$M_H^D = M_H^L - M_1$$
$$2qa^2 = 3qa^2 - qa^2$$

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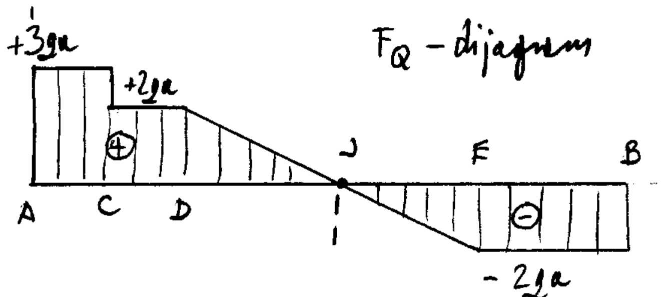
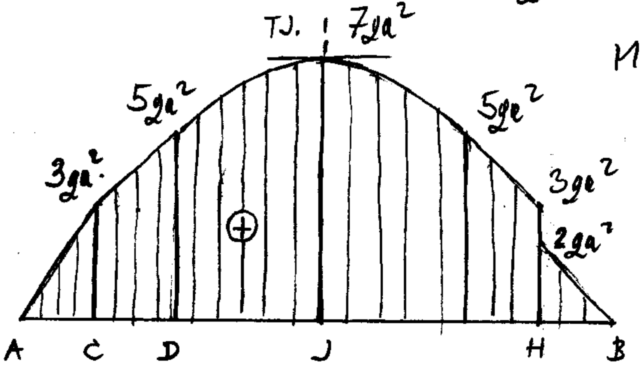
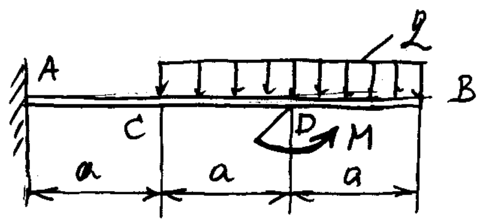
U M-dijagram nanose se ordinate koje prikazuju izračunate momente savijanja na odgovarajućim mjestima, tj. za odgovarajuće apscise x. Dijagramske točke povezuju se pravcima u područjima A-C, C-D, E-H i H-B u skladu s konstantnim iznosima F_Q u tim područjima. U području D-E dijagramska linija je parabola 2o reda koja ima tjeme u mjestu J, u skladu s F_Q-dijagramom.



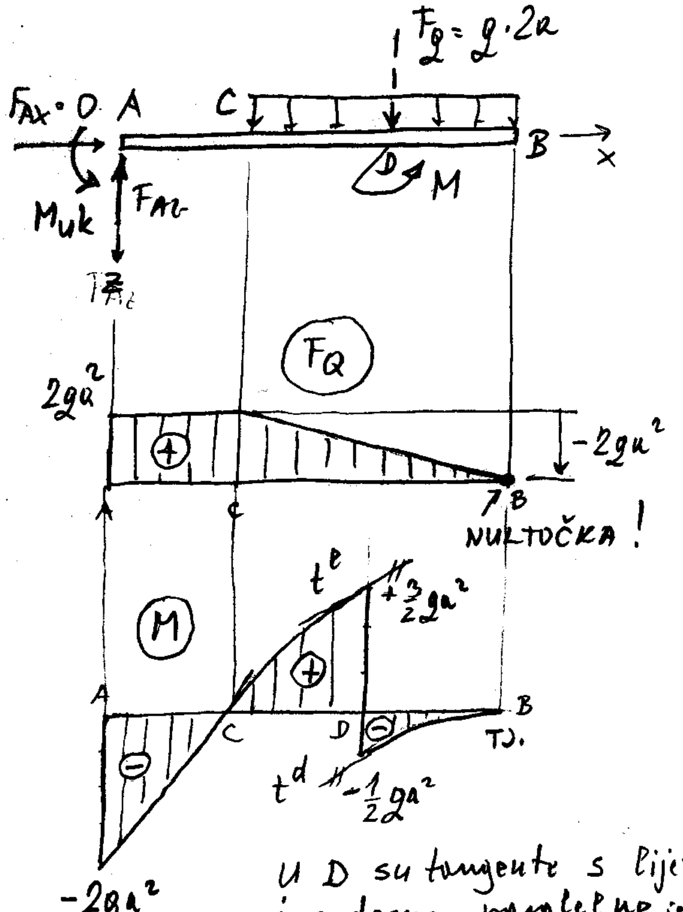
F_Q - dijagram



M - dijagram



Za konzolu zadanu i opterećenu prema slici potrebno je skicirati i kotirati F_Q i M dijagrame.
Zadano: q, a, $M = 2qa^2$.



U D su tangente s lijeva i s desna paralelne jer je $F_{QD}^L = F_{QD}^D$!

Određivanje reakcije:

$\Sigma F_x = 0 \rightarrow F_{Ax} = 0$

$\Sigma F_z = 0 \quad -F_{Az} + q \cdot 2a = 0$

$\underline{F_{Az} = 2qa}$

$\Sigma M_A = 0 \quad M_{uk} - (q \cdot 2a) \cdot 2a + M = 0$

$\underline{M_{uk}} = 4qa^2 - 2qa^2 = \underline{2qa^2}$

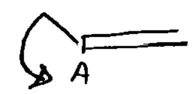
F_Q dijagram:

U A skok s 0 za $+F_{Az}$ na $+2qa$, od A→C F_Q = konst., od C→B linearni negativni prirast za ukupno $-2qa$ i u B → 0 i nultočka.

M-dijagram:

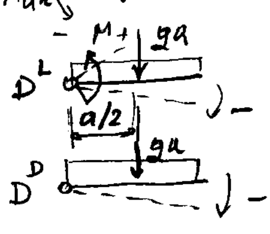
Granice područja: A, C, D i B.

$\underline{M_B = 0}$ (kraj nosača bez vanjskog momenta)



$\underline{M_A} = -M_{uk} = \underline{-2qa^2}$

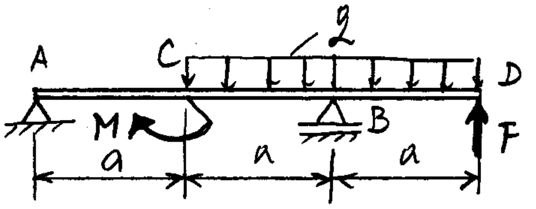
$\underline{M_C} = +F_{Az} \cdot a - M_{uk} = +2qa \cdot a - 2qa^2 = \underline{0}$

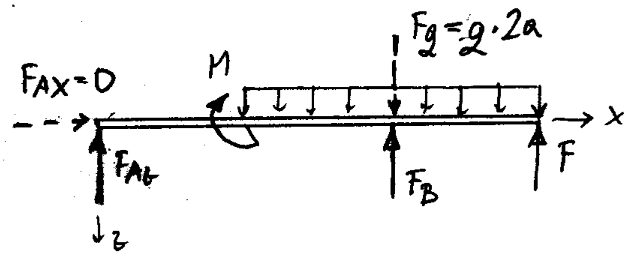


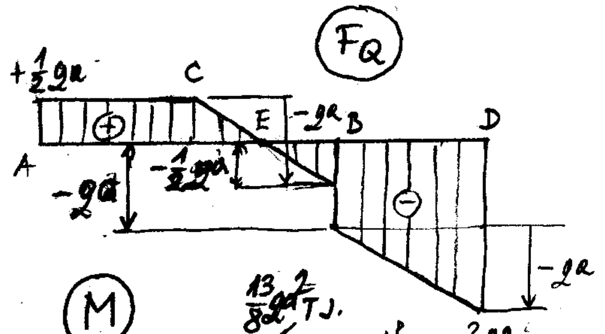
$M_D^L = +M - qa \cdot \frac{1}{2}a = +2qa^2 - \frac{1}{2}qa^2 = +\frac{3}{2}qa^2$

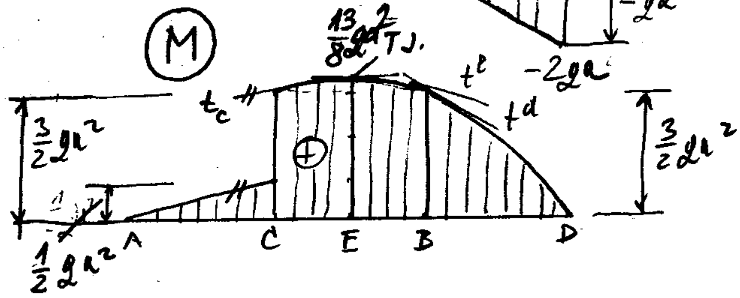
$M_D^D = -qa \cdot \frac{1}{2}a = -\frac{1}{2}qa^2$

Zad. 6.26

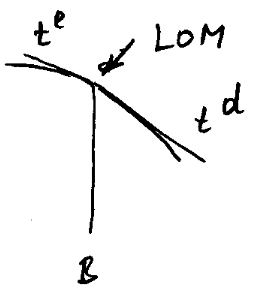








Pravac A − C i t_C u C su paralelni jer je $F_{QC}^L = F_{QC}^D$.



U B t^l je blaže nagnuta od t^d, jer je $|F_{QB}^L| < |F_{QB}^D|$. Linija ima lom, odnosno radi se o 2 različite parabole 2. reda, jedna u području C − B, a druga u području B − D.

Za gredu s prepustom zadanu i opterećenu prema slici potrebno je skicirati i kotirati F_Q i M dijagrame. Zadano: q, a, $F = 2qa$, $M = qa^2$.

Reakcije:

$\Sigma F_z = 0 \quad -F_{Az} - F_B + q \cdot 2a - F = 0$

$F_{Az} + F_B - 2qa + 2qa = 0$

$F_{Az} + F_B = 0 \quad (1)$

$\Sigma M_A = 0$

$F_B \cdot 2a + F \cdot 3a - (q \cdot 2a) \cdot 2a - M = 0$

$F_B = \frac{1}{2a}(-2qa \cdot 3a + 4qa^2 + qa^2)$

$F_B = -\frac{1}{2}qa \quad (2)$

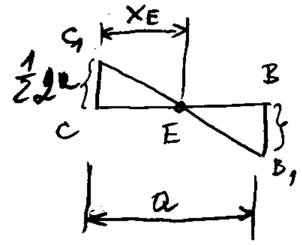
↳ F_B je usmjerena suprotno od pretpostavljenog smjera: $\downarrow F_B$)

$\rightarrow (1) \Rightarrow F_{Az} = -F_B = -(-\frac{1}{2}qa) = \frac{1}{2}qa$

F_Q - dijagram

U A skok s 0 za $+F_{Az}$ na $+\frac{1}{2}qa^2$, do C $F_Q =$ konst., Od C do B negativni linearni prirast za $-qa$, pa s $+\frac{1}{2}qa^2$ u C F_Q pada na $-\frac{1}{2}qa$ u B^L. U B skokovit prirast za $-F_B$, tj. za $-\frac{1}{2}qa$, pa $\rightarrow F_Q^D = -qa$. Od B^D do D negativni linearni prirast za $-qa$, pa su $-qa$ F_Q pada na $-2qa$ u D. U D skokovit prirast $+F = +2qa$ daje $F_Q = 0$.

Nultočka u F_Q-dijagramu:



$$\frac{\bar{x}_E}{\overline{CC_1}} = \frac{(a-\bar{x}_E)}{\overline{BB_1}}$$

$$\bar{x}_E = \frac{\overline{CC_1}}{\overline{BB_1}}(a-\bar{x}_E)$$

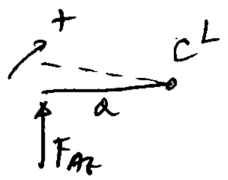
$$\bar{x}_E = \frac{\frac{1}{2}qa}{\frac{1}{2}qa}(a-\bar{x}_E)$$

$$\bar{x}_E = a-\bar{x}_E \Rightarrow \underline{\bar{x}_E = \frac{1}{2}a}$$

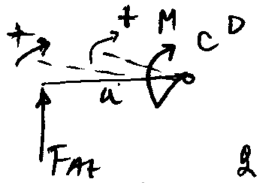
M-dijagram

Granice područja: A, C, B i D, nultočka E.

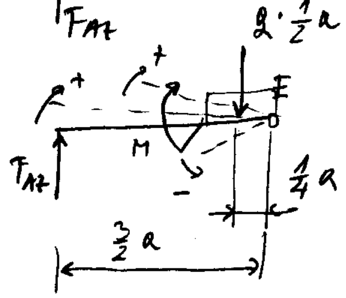
$M_A = 0$ i $M_D = 0$ (krajevi nosača bez vanjskih momenata)



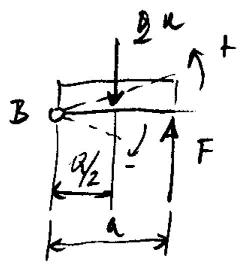
$$\underline{M_C^L} = +F_{Az}\cdot a = +\frac{1}{2}qa\cdot a = \underline{+\frac{1}{2}qa^2} \rightarrow \Rightarrow \underline{M_C^D = M_C^L + M}$$



$$\underline{M_C^D} = +F_{Az}\cdot a + M = +\frac{1}{2}qa\cdot a + qa^2 = \underline{+\frac{3}{2}qa^2}$$



$$\underline{M_E} = +F_{Az}\cdot\frac{3}{2}a + M - \left(q\cdot\frac{1}{2}a\right)\cdot\frac{1}{4}a =$$
$$= \frac{1}{2}qa\cdot\frac{3}{2}a + qa^2 - \frac{1}{8}qa^2 =$$
$$= qa^2\,\frac{6+8-1}{8} = \underline{\frac{13}{8}qa^2}$$

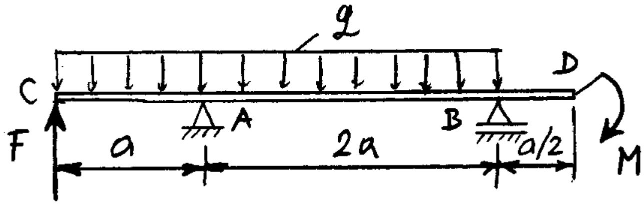


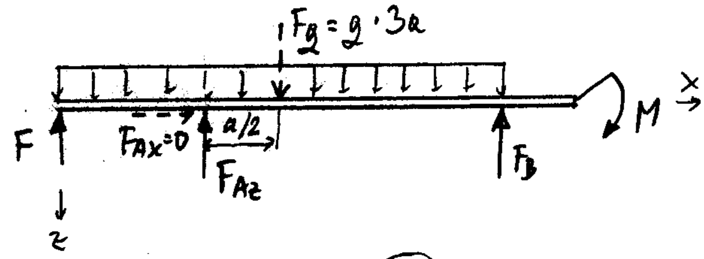
$$\underline{M_B} = +F\cdot a - (qa)\cdot\frac{a}{2} = +2qa\cdot a - \frac{1}{2}qa^2 =$$
$$= \underline{+\frac{3}{2}qa^2}$$

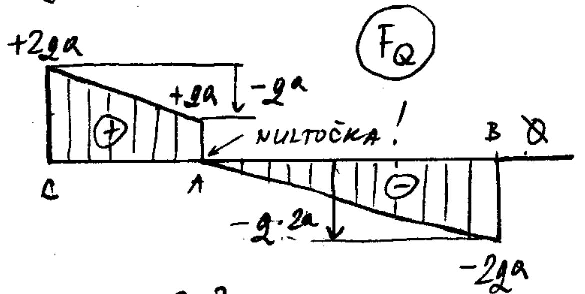
Zad. 6.28

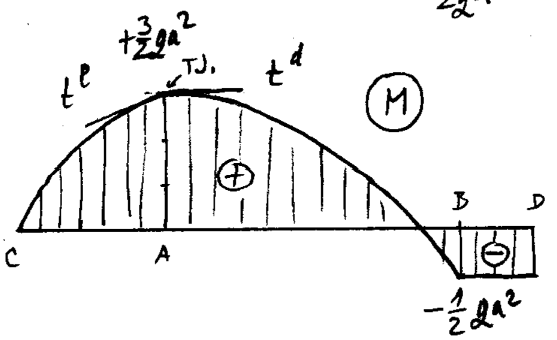
Za gredu s dva prepusta zadanu i opterećenu prema slici potrebno je skicirati i kotirati F_Q i M dijagrame.

Zadano: q, a, $F = 2qa$, $M = \frac{1}{2}qa^2$.









Reakcije:

$\Sigma F_z = 0$

$-F - F_{Az} + q \cdot 3a - F_B = 0$

$F_{Az} + F_B = -F + 3qa = -2qa + 3qa$

$F_{Az} + F_B = qa \qquad (1)$

$\Sigma M_A = 0$

$F_B \cdot 2a - M - (q \cdot 3a) \cdot \frac{a}{2} - F \cdot a = 0$

$F_B = \frac{1}{2a}\left(M + \frac{3}{2}qa^2 + Fa\right) =$

$= \frac{1}{2a}\left(\frac{1}{2}qa^2 + \frac{3}{2}qa^2 + 2qa \cdot a\right)$

$\underline{F_B = 2qa} \qquad (2)$

$\rightarrow (1) \Rightarrow F_{Az} = qa - F_B =$

$= qa - 2qa$

$\underline{F_{Az} = -qa}$ (F_{Az} je usmjerena suprotno od pretpostavljenog, tj. $\downarrow F_{Az}$!)

U A: $F_{QA}^L = +qa \Rightarrow t^l$ je pozitivno nagnuta,

$F_{QA}^D = 0 \Rightarrow$ NULTOČKA $\Rightarrow$

$\Rightarrow t^d \parallel x$, dijagramska linija, parabola 2. reda ima TJEME!

LOM DIJAGRAMSKE LINIJE – TU SE SPAJAJU 2 RAZLIČITE PARABOLE 2. REDA!

F_Q- dijagram:

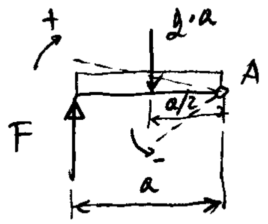
U C skokovit prirast za $+F$ uzrokuje skok F_Q s 0 na $+2qa$. Od C do A negativni linearni prirast $-qa$ smanjuje F_Q na $F_{QA}^L = +qa$. U A skokovit prirast za $-F_{Az} \Rightarrow F_{QA}^D = 0$. Od A do B negativni linearni prirast $-q \cdot 2a$ dovodi F_Q do iznosa $-2qa$. U B skokovit prirast za $+F_B = +2qa$ daje $F_{QB}^D = 0$. Od B do D NEMA SILA U z SMJERU $\Rightarrow$ $F_Q = F_{QB}^D = 0 = \underline{\text{konst.}}$

M - dijagram:

Granice područja: C, A, B, D.

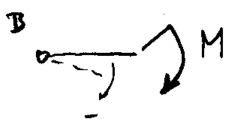
$\underline{M_C = 0}$ (kraj nosača bez vanjskog momenta)

$\underline{M_D} = -M = \underline{-\frac{1}{2} q a^2}$



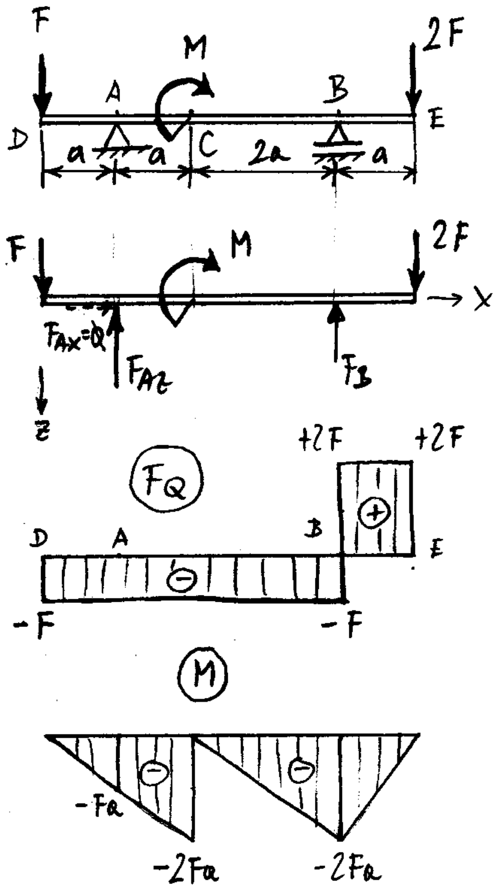
$\underline{M_A} = +F \cdot a - (qa) \cdot \frac{a}{2} = +2qa \cdot a - \frac{1}{2} q a^2 =$

$= \underline{+\frac{3}{2} q a^2}$



$\underline{M_B} = -M = \underline{-\frac{1}{2} q a^2}$

Zad. 6.24



Za nosač zadan i opterećen prema slici treba skicirati i kotirati F_Q i M dijagrame.

Zadano: a, F, $M = 2Fa$.

Reakcije:

$\Sigma F_z = 0$

$-F_{Az} - F_B + F + 2F = 0$

$F_{Az} + F_B = 3F \quad (1)$

$\Sigma M_A = 0$

$F_B \cdot 3a - 2F \cdot 4a - M + F \cdot a = 0$

$F_B = \frac{1}{3a}(8Fa + 2Fa - Fa)$

$\underline{F_B = 3F}$

$\rightarrow (1) \Rightarrow F_{Az} = 3F - F_B = 3F - 3F$

$\underline{F_{Az} = 0}$

F_Q dijagram:

U D skok s 0 na $-F$ uzrokuje sila F. Od D do B ostaje F_Q konstantna jer nema sile u z smjeru ($F_{Az} = 0$!). U B sila F_B uzrokuje skok s $-F$ na $+2F$. Od B do E F_Q = konst. U E sila $2F$ uzrokuje skok s $+2F$ na 0.

M dijagram:

Granice područja: D, A, C, B, E.

$\left.\begin{array}{l} M_D = 0 \\ M_E = 0 \end{array}\right\}$ Krajevi nosača bez vanjskih momenata!

$\underline{M_A} = -F \cdot a = \underline{-Fa}$

$M_C^L = -F \cdot 2a = \underline{-2Fa}$

$\underline{M_C^D} = -F \cdot 2a + M = -2Fa + 2Fa = \underline{0}$

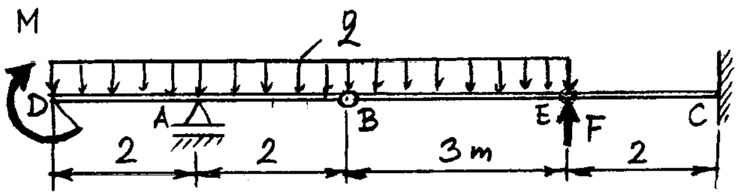
$\Rightarrow M_C^D = M_C^L + M$

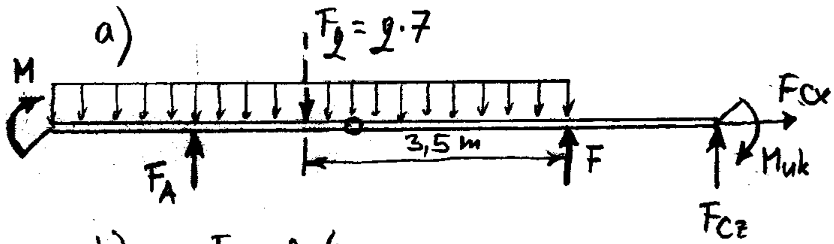
$\underline{M_B} = -2F \cdot a = \underline{-2Fa}$

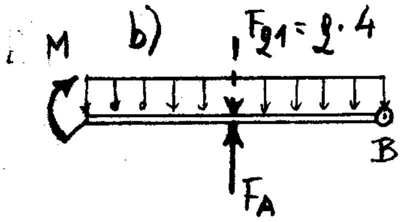
Zad. 6.34

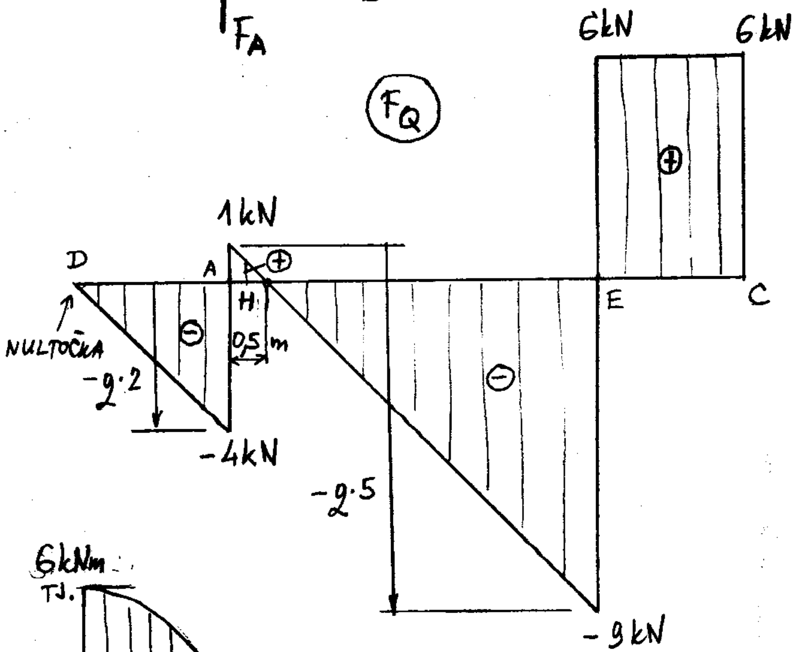
Za Gerberov nosač zadan i opterećen prema slici treba odrediti reakcije u osloncima te skicirati i kotirati F_Q i M dijagrame.

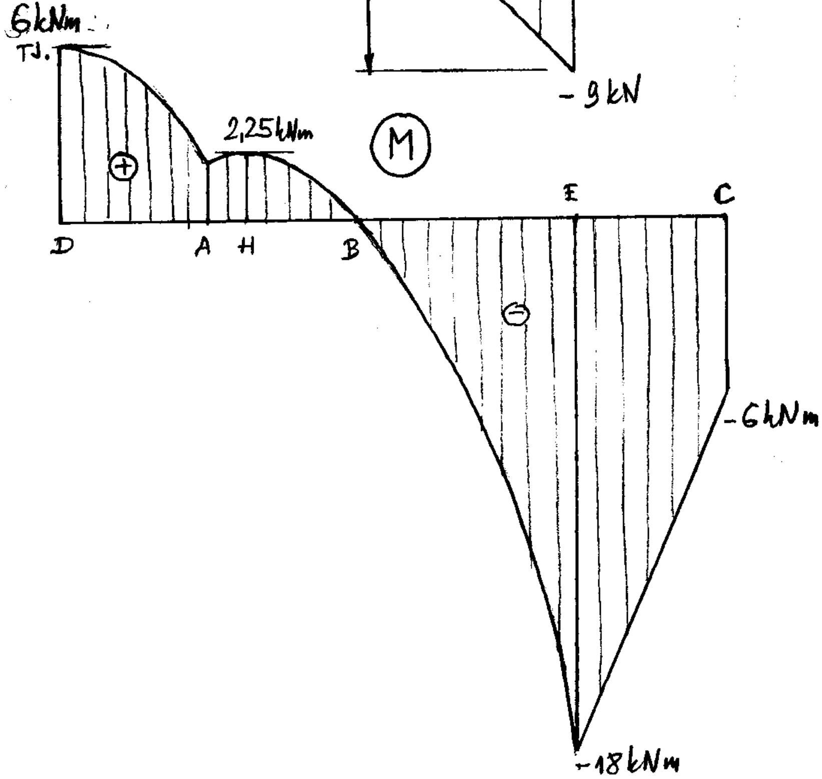
Zadano: $q = 2\ kN/m$, $M = 6\ kNm$, $F = 15\ kN$.











Ravnoteža nosača kao cjeline (sl. a):

$\Sigma F_x = 0 \quad \underline{F_{Cx} = 0}$

$\Sigma F_z = 0$

$-F_A - F_{Cz} + q\cdot 7 - F = 0$

$F_A + F_{Cz} = 2\cdot 7 - 15$

$F_A + F_{Cz} = -1 \ [kN] \quad (1)$

$\Sigma M_C = 0$

$-F_A\cdot 7 - M_{uk} - M + q\cdot 7\cdot 5{,}5 - F\cdot 2 = 0$

$F_A\cdot 7 + M_{uk} = 2\cdot 7\cdot 5{,}5 - 6 - 15\cdot 2$

$F_A\cdot 7 + M_{uk} = 41 \ [kNm] \quad (2)$

Dopunska jednadžba (sl. b):

$\Sigma M_B^L = 0$

$-F_A\cdot 2 + q\cdot 4\cdot 2 - M = 0$

$F_A\cdot 2 = 2\cdot 4\cdot 2 - 6$

$F_A\cdot 2 = 16 - 6$

$\underline{F_A = 5\ kN} \quad (3)$

$(3)\rightarrow(2) \Rightarrow 5\cdot 7 + M_{uk} = 41$

$M_{uk} = 41 - 35$

$\underline{M_{uk} = 6\ kNm}$

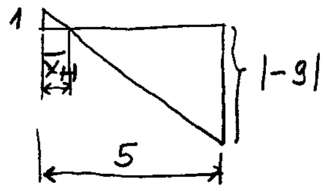
$(3)\rightarrow(1) \Rightarrow 5 + F_{Cz} = -1$

$\underline{F_{Cz} = -6\ kN} \Rightarrow \downarrow F_{Cz} = 6\ kN$

F_Q- dijagram :

Od D do A negativni linearni prirast za $-q\cdot 2 = -4\,kN$. U A skokovit pozitivni prirast za $+F_A = +5\,kN$. Od A do E negativni linearni prirast za $-q\cdot 5 = -10\,kN$. U E skokovit pozitivni prirast za $+F = +15\,kN$. Od E do C F_Q = konst. U C negativni skokovit prirast za $-F_C = -6\,kN$.

Nultočka H :



$$\frac{\bar{x}_H}{1} = \frac{5-\bar{x}_H}{9} \quad /\cdot 9$$

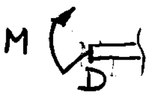
$$9\bar{x}_H = 5 - \bar{x}_H$$

$$10\bar{x}_H = 5 \quad \rightarrow \quad \underline{\bar{x}_H = 0{,}5\,m}$$

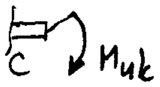
M- dijagram :

Granice područja: D, A, E, C. Nultočke u F_Q- dijagramu : H, D.
Zglob : B.

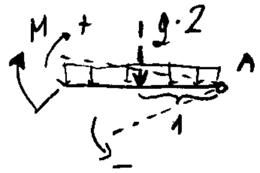
$\underline{M_B = 0}$ (ZGLOB!)



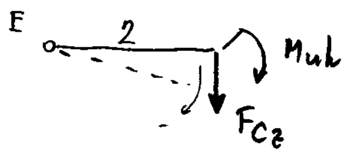
$\underline{M_D} = +M = \underline{6\,kNm}$



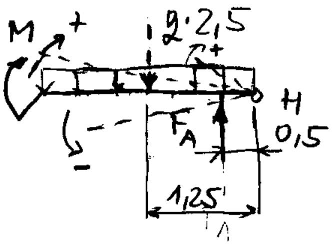
$\underline{M_C} = -M_{uk} = \underline{-6\,kNm}$



$\underline{M_A} = +M - (q\cdot 2)\cdot 1 = 6 - 2\cdot 2\cdot 1 = \underline{2\,kNm}$

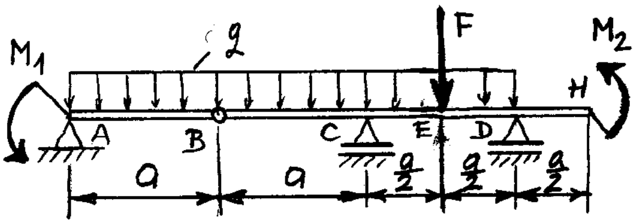


$\underline{M_E} = -M_{uk} - F_{Cz}\cdot 2 = -6 - 6\cdot 2 = \underline{-18\,kNm}$



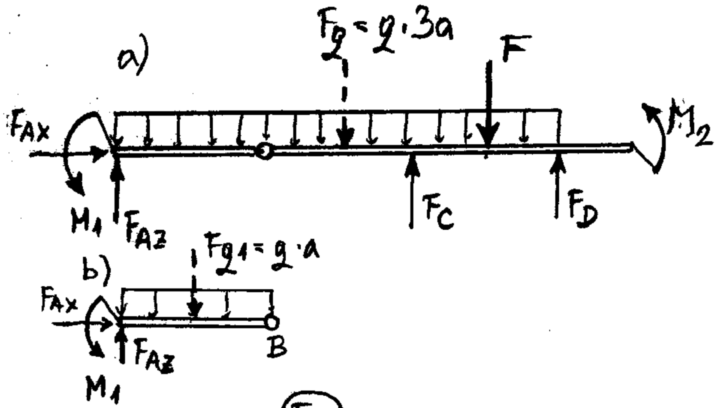
$\underline{M_H} = +M - (q\cdot 2{,}5)\cdot 1{,}25 + F_A\cdot 0{,}5 =$
$= 6 - (2\cdot 2{,}5)\cdot 1{,}25 + 5\cdot 0{,}5 = 6 - 6{,}25 + 2{,}5 =$
$= \underline{2{,}25\,kNm}$

Zad. 6.38



Za Gerberov nosač zadan i opterećen prema slici treba odrediti reakcije u osloncima te skicirati i kotirati F_Q i M dijagrame.

Zadano: q, a, $F = qa$, $M_1 = qa^2$, $M_2 = \frac{1}{2}qa^2$.



Ravnoteža nosača kao cjeline (sl. a):

$\Sigma F_x = 0 \qquad \underline{F_{Ax} = 0}$

$\Sigma F_z = 0$

$-F_{Az} - F_C - F_D + q \cdot 3a + F = 0$

$F_{Az} + F_C + F_D = 3qa + qa$

$F_{Az} + F_C + F_D = 4qa \qquad (1)$

$\Sigma M_D = 0$

$-F_{Az} \cdot 3a - F_C \cdot a + q \cdot 3a \cdot \frac{3a}{2} +$

$+ F \cdot \frac{a}{2} + M_1 + M_2 = 0$

$F_{Az} \cdot 3a + F_C \cdot a = \frac{9}{2}qa^2 + \frac{qa^2}{2} + qa^2 + \frac{qa^2}{2}$

$3F_{Az} + F_C = \frac{13}{2}qa \qquad (2)$

Dopunska jednadžba (sl. b):

$\Sigma M_B^L = 0$

$-F_{Az} \cdot a + M_1 + (q \cdot a) \cdot \frac{a}{2} = 0$

$F_{Az} \cdot a = qa^2 + \frac{1}{2}qa^2$

$\underline{F_{Az} = \frac{3}{2}qa} \qquad (3)$

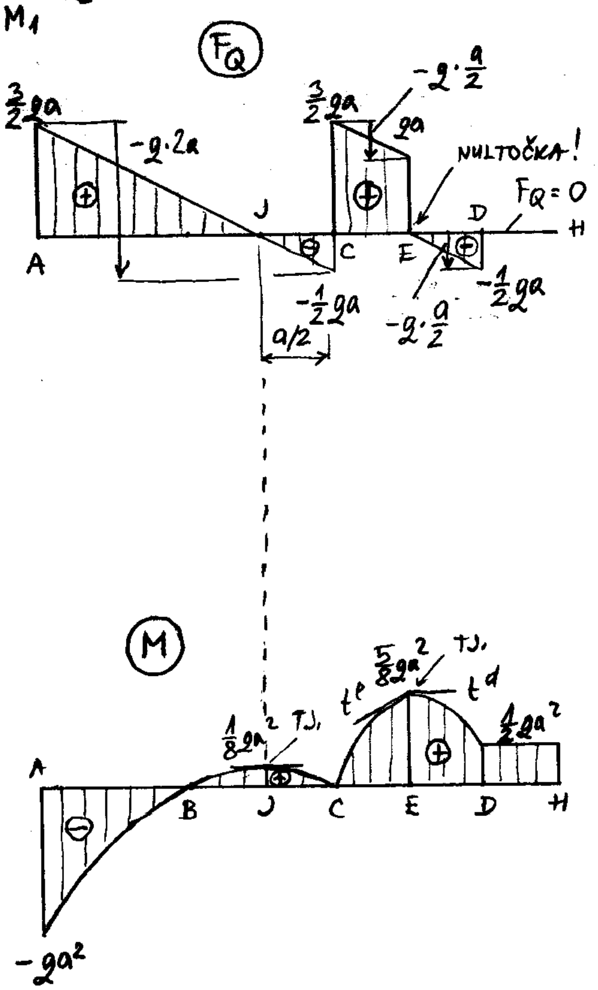
$(3) \rightarrow (2) \Rightarrow F_C = \frac{13}{2}qa - 3 \cdot \frac{3}{2}qa$

$\underline{F_C = 2qa}$

$(1) \Rightarrow F_D = 4qa - F_{Az} - F_C =$

$= 4qa - \frac{3}{2}qa - 2qa$

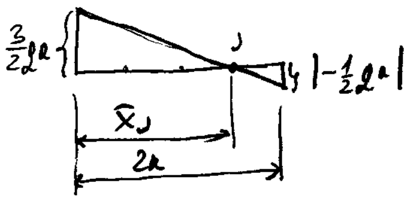
$\underline{F_D = \frac{1}{2}qa}$



F_Q-dijagram:

U A skokovit pozitivan prirast za $+F_{Az} = +\frac{3}{2}qa$. Od A do C negativni linearni prirast za $-q\cdot 2a$. U C skokovit pozitivan prirast za $+F_C = +2qa$. Od C do E negativni linearni prirast za $-q\cdot\frac{a}{2}$. U E skokovit negativni prirast za $-F=-qa$. Od E do D negativni linearni prirast za $-q\cdot\frac{a}{2}$. U D skokovit pozitivan prirast za $+F_D = +\frac{1}{2}qa$. Od D do H $F_Q = 0 =$ konst.

Nultočka J:



$$\frac{\bar{x}_J}{\frac{3}{2}qa} = \frac{2a-\bar{x}_J}{\frac{1}{2}qa}$$

$$\bar{x}_J = 3(2a-\bar{x}_J)$$

$$4\bar{x}_J = 6a \Rightarrow \underline{\bar{x}_J = \frac{3}{2}a}$$

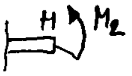
M-dijagram:

Granice područja: A, C, E, D, H.

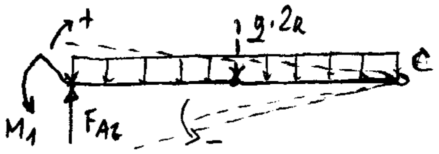
Nultočke u F_Q-dijagramu: J i E. Zglob: B.

$\underline{M_B = 0}$ (zglob!)

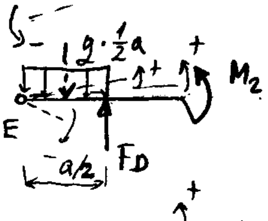
$\underline{M_A} = -M_1 = \underline{-qa^2}$



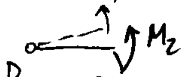
$\underline{M_H} = M_2 = \underline{\frac{1}{2}qa^2}$



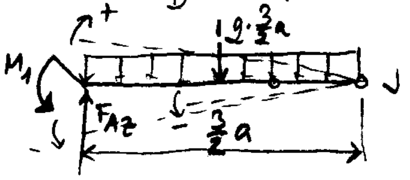
$\underline{M_C} = +F_{Az}\cdot 2a - M_1 - (q\cdot 2a)\cdot a =$
$= \frac{3}{2}qa\cdot 2a - qa^2 - 2qa^2 = \underline{0}$



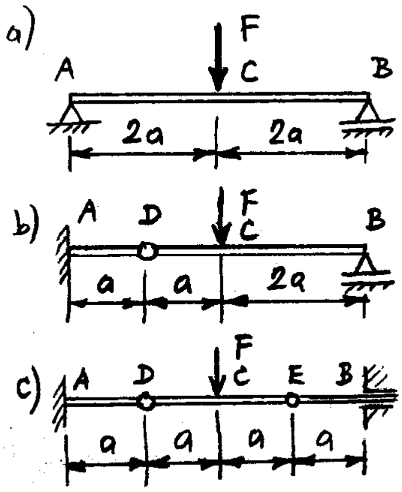
$\underline{M_E} = M_2 + F_D\cdot\frac{a}{2} - (q\cdot\frac{1}{2}a)\cdot\frac{1}{4}a = \frac{1}{2}qa^2 + \frac{1}{2}qa\cdot\frac{a}{2} - \frac{1}{8}qa^2 =$
$= \underline{\frac{5}{8}qa^2}$



$\underline{M_D} = M_2 = \underline{\frac{1}{2}qa^2}$



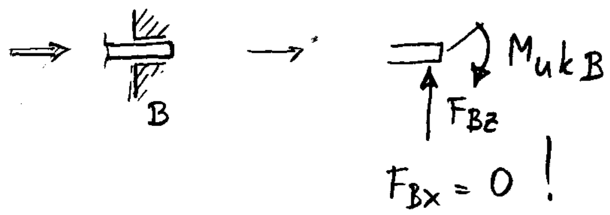
$\underline{M_J} = F_{Az}\cdot\frac{3}{2}a - M_1 - (q\cdot\frac{3}{2}a)\cdot\frac{3}{4}a = \frac{3}{2}qa\cdot\frac{3}{2}a - qa^2 - \frac{9}{8}qa^2 =$
$= \underline{\frac{1}{8}qa^2}$

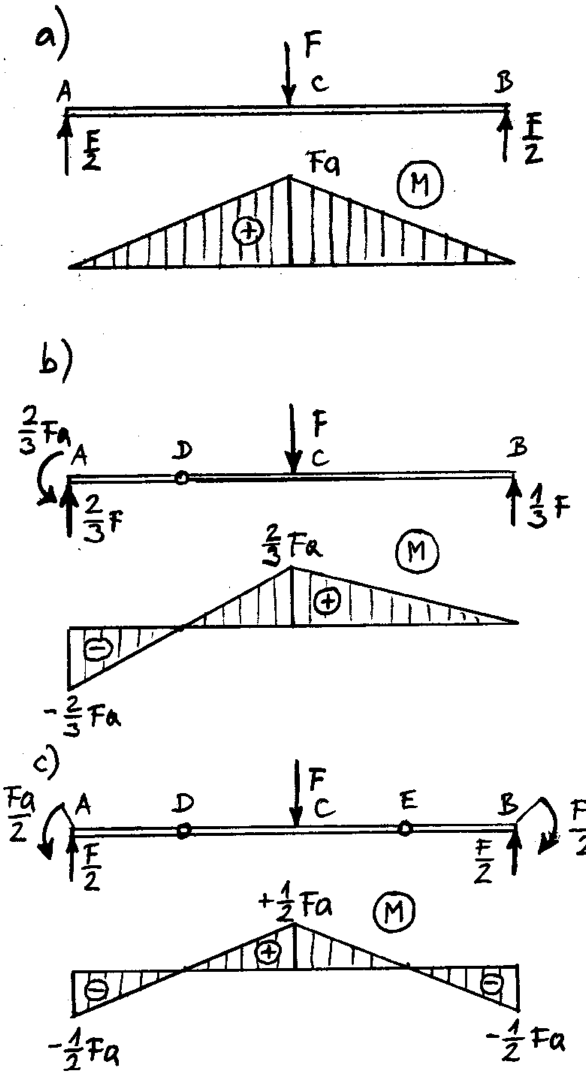


Za tri nosača jednakih raspona 4a (jednostavna greda na sl. a, Gerberov nosač s jednim zglobom na sl. b i Gerberov nosač s dva zgloba na sl. c) treba skicirati i kotirati M-dijagrame te usporediti maksimalne iznose momenata savijanja.

Zadano: F, a.

NAPOMENA: Oslonac u B nosača na sl. c sprječava pomak kraja B u smjeru z osi i rotaciju oko osi okomite na uzdužnu os x, dok je pomak kraja B u smjeru osi x moguć.





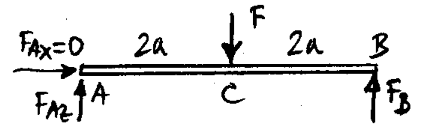
$$M_{max}\ b) = \frac{2}{3} M_{max}\ a)$$

$$M_{max}\ c) = \frac{1}{2} M_{max}\ a)$$

MOŽE SE USTANOVITI DA SE UGRADNJOM ZGLOBOVA MOŽE UTJECATI NA IZNOS MAKSIMALNOG MOMENTA SAVIJANJA. U ovom slučaju, ugradnja 1 zgloba (u D) smanjila je M_{max} za 33%, a ugradnja 2 zgloba (u D i E) smanjila je M_{max} za 50%.

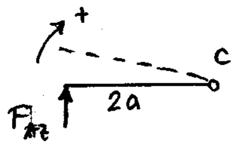
21

a)



$\Sigma F_z = 0 \quad -F_{Az} + F_B + F = 0$

$\Sigma M_A = 0 \quad F_B \cdot 4a - F \cdot 2a = 0 \Rightarrow \underline{F_B = \frac{1}{2}F}$

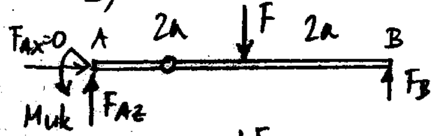


$\underline{M_A} = \underline{M_B} = \underline{0}$ (Krajevi nosača!)

$\underline{F_{Az}} = F - F_B = \underline{\frac{1}{2}F}$

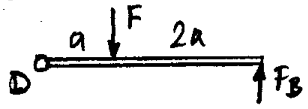
$\underline{M_C} = F_{Az} \cdot 2a = \frac{1}{2}F \cdot 2a = \underline{Fa}$

b)



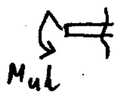
$\Sigma F_z = 0 \quad -F_{Az} - F_B + F = 0 \quad (1)$

$\Sigma M_A = 0 \quad F_B \cdot 4a - F \cdot 2a + M_{uk} = 0 \quad (2)$



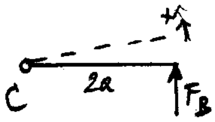
Dopunska jednadžba: $\Sigma M_D^D = 0 \quad F_B \cdot 3a - F \cdot a = 0 \quad (3)$

$\rightarrow \underline{F_B = \frac{1}{3}F}$



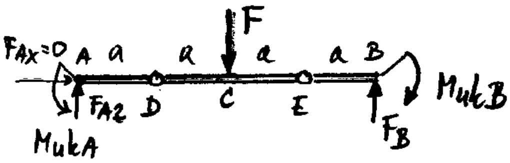
$\underline{M_A} = -M_{uk} = \underline{-\frac{2}{3}Fa}$ $\quad \underline{M_D = 0}$ (zglob!)

$\rightarrow (2) \Rightarrow \underline{M_{uk}} = 2Fa - \frac{1}{3}F \cdot 4a = \underline{\frac{2}{3}Fa}$



$\underline{M_C} = F_B \cdot 2a = \frac{1}{3}F \cdot 2a = \underline{\frac{2}{3}Fa}$

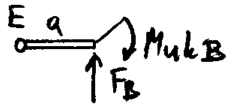
c)



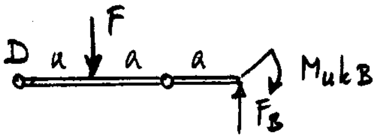
$\Sigma F_z = 0 \quad -F_{Az} - F_B + F = 0 \quad (1)$

$\Sigma M_A = 0 \quad F_B \cdot 4a - M_{ukB} - F \cdot 2a + M_{ukA} = 0 \quad (2)$

Dopunske jednadžbe:



$\Sigma M_E^D = 0 \quad F_B \cdot a - M_{ukB} = 0 \quad (3)$

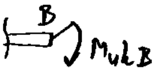


$\Sigma M_D^D = 0 \quad F_B \cdot 3a - M_{ukB} - F \cdot a = 0 \quad (4)$

$(4) - (3) \Rightarrow 2F_B a - Fa = 0$

$\Rightarrow \underline{F_B = \frac{1}{2}F}$

$\underline{M_A} = -M_{ukA} = \underline{-\frac{1}{2}Fa}$

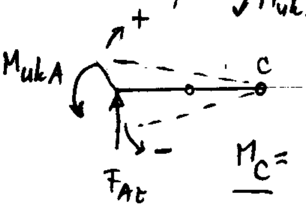


$\underline{M_B} = -M_{ukB} = \underline{-\frac{1}{2}Fa}$

$\rightarrow (1) \Rightarrow \underline{F_{Az}} = F - \frac{1}{2}F = \underline{\frac{1}{2}F}$

$\rightarrow (3) \Rightarrow \underline{M_{ukB}} = \frac{1}{2}F \cdot a = \underline{\frac{1}{2}Fa}$

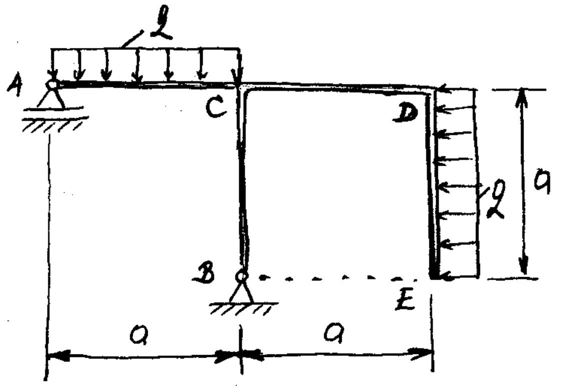
$\rightarrow (2) \Rightarrow \underline{M_{ukA}} = 2Fa + \frac{1}{2}Fa - \frac{1}{2}F \cdot 4a = \underline{\frac{1}{2}Fa}$



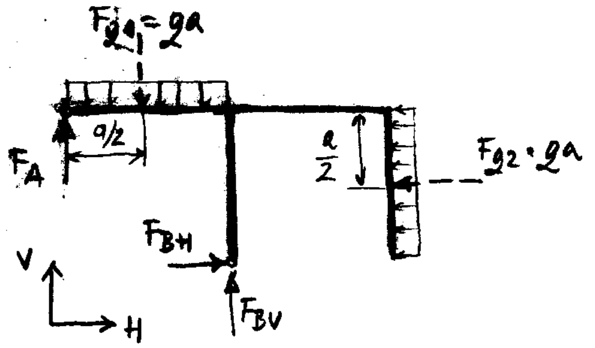
$\underline{M_C} = F_{Az} \cdot 2a - M_{ukA} = \frac{1}{2}F \cdot 2a - \frac{1}{2}Fa = \underline{\frac{1}{2}Fa}$

$\underline{M_D = M_E = 0}$ (zglobovi!)

Zd. 6.48



Za okvirni nosač zadan i opterećen prema slici odrediti reakcije u osloncima te skicirati i kotirati F_Q-, F_N- i M- dijagrame duž konture nosača.
Zadano: q, a.



$\Sigma F_H = 0 \qquad F_{BH} - qa = 0 \qquad (1)$

$\underline{F_{BH} = qa}$

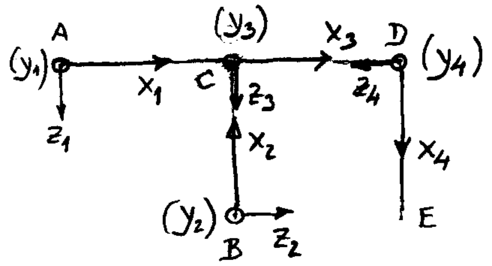
$\Sigma F_V = 0 \qquad F_A + F_{BV} - qa = 0 \qquad (2)$

$\Sigma M_B = 0 \qquad -F_A \cdot a + qa \cdot \frac{a}{2} + qa \cdot \frac{a}{2} = 0 \quad (3)$

$\underline{F_A = qa}$

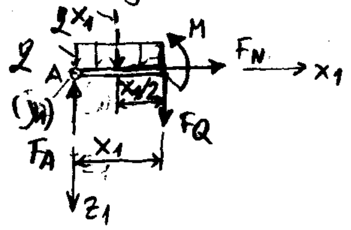
$\rightarrow (2) \Rightarrow \underline{F_{BV}} = qa - F_A = \underline{0}$

LOKALNI KOORDINATNI SUSTAVI



METODA PRESJEKA

Područje A - C $\qquad 0 \le x_1 \le a$



$\Sigma F_x = 0 \qquad \underline{F_N = 0}$

$\Sigma F_z = 0 \qquad F_Q - F_A + qx_1 = 0 \Rightarrow \underline{F_Q = qa - qx_1}$

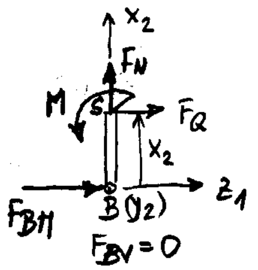
$F_{QA} = F_Q(x_1 = 0) = qa \quad , \quad \underrightarrow{F_{QC}} = F_Q(x_1 = a) = 0$

$\Sigma M_S = 0 \qquad \underline{M} - F_A \cdot x_1 + qx_1 \cdot \frac{x_1}{2} = 0$

$\underline{M = qax_1 - \frac{1}{2} q x_1^2}$ (parabola 2. reda)

$M_A = M(x_1 = 0) = 0 \quad , \quad \underrightarrow{M_C} = M(x_1 = a) = \frac{1}{2} q a^2$

Područje B – C $\quad 0 \le x_2 \le a$



$\Sigma F_x = 0 \qquad \underline{F_N = 0}$

$\Sigma F_z = 0 \qquad F_Q + F_{BH} = 0 \Rightarrow \underline{F_Q} = -F_{BH} = \underline{-qa}$

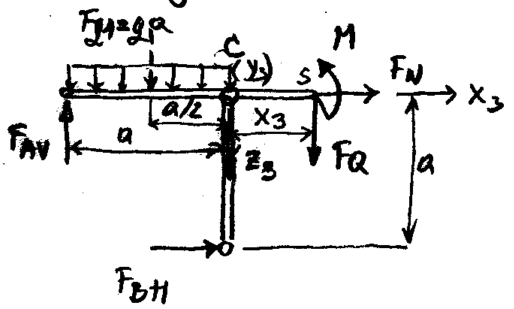
$\Sigma M_S = 0 \qquad M + F_{BH}\cdot x_2 = 0$

$\underline{M} = -F_{BH}\cdot x_2 = \underline{-qax_2}$ (pravac)

$M_B = M(x_2=0) = 0$

$M_C = M(x_2 = a) = -qa\cdot a = -qa^2$

Područje C – D $\quad 0 \le x_3 \le a$



$\Sigma F_x = 0 \qquad F_N + F_{BH} = 0 \Rightarrow \underline{F_N = -qa}$

$\Sigma F_z = 0 \qquad F_Q - F_{AV} + qa = 0$

$\underline{F_Q} = qa - qa = \underline{0}$

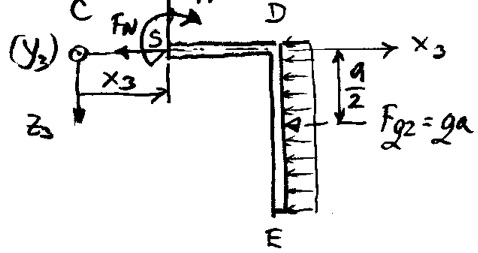
$\Sigma M_S = 0 \qquad M - F_{AV}(a+x_3) + qa\left(\frac{a}{2}+x_3\right) + F_{BH}\cdot a = 0$

$\underline{M} = qa(a+x_3) + qa\left(\frac{a}{2}+x_3\right) - qa\cdot a =$

$= qa^2 + qax_3 - \frac{1}{2}qa^2 - qax_3 - qa^2 =$

$= \underline{-\frac{1}{2}qa^2}$ (konst.)

Jednostavnije je razmatrati ravnotežu desnog dijela nosača omeđenog negativnim presjekom:



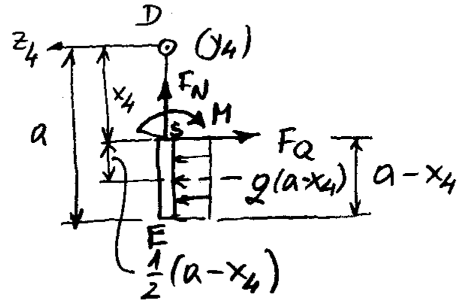
$\Sigma F_x = 0 \qquad -F_N - F_{q2} = 0 \Rightarrow \underline{F_N = -qa}$

$\Sigma F_z = 0 \qquad -F_Q = 0 \Rightarrow \underline{F_Q = 0}$

$\Sigma M_S = 0 \qquad -M - qa\cdot\frac{a}{2} = 0$

$\underline{M = -\frac{1}{2}qa^2}$ (konst.)

Područje D – E $\qquad 0 \le x_4 \le a$



$\Sigma F_x = 0 \qquad -F_N = 0 \qquad \underline{F_N = 0}$

$\Sigma F_z = 0 \qquad -F_Q + q(a-x_4) = 0$

$$\underline{F_Q = qa - qx_4}$$

$\uparrow F_{QD} = F_Q(x_4 = 0) = qa ,$

$F_{QE} = F_Q(x_4 = a) = qa - qa = 0$

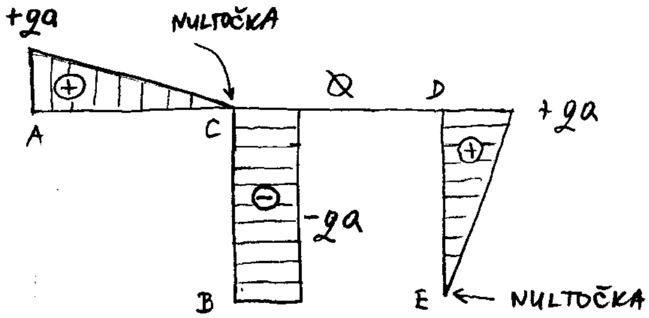
$\Sigma M_S = 0 \qquad -M - q(a-x_4)\cdot\frac{a-x_4}{2} = 0$

$$\underline{M = -\frac{1}{2}q(a-x_4)^2} \quad \text{(parabola 2. reda)}$$

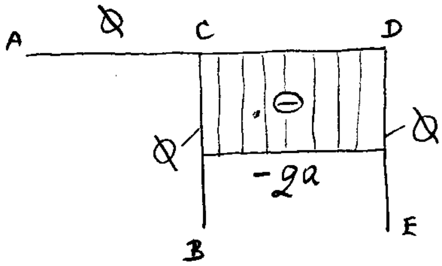
$\uparrow M_D = M(x_4 = 0) = -\frac{1}{2}qa^2$

$M_E = M(x_4 = a) = 0$

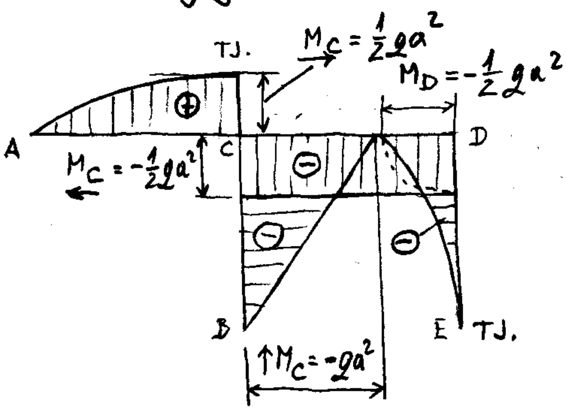
F_Q – dijagram



F_N – dijagram



M – dijagram



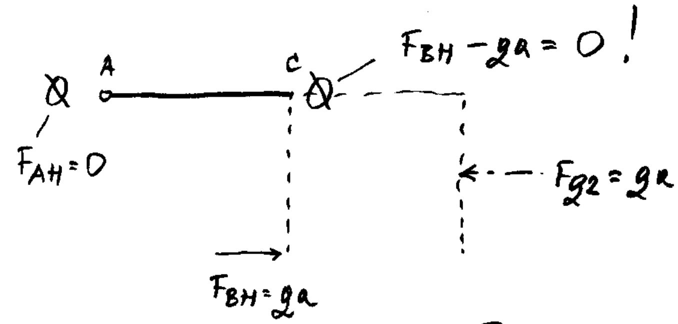
$\underset{\rightarrow}{M_C} + \uparrow M_C = \underset{\leftarrow}{M_C}$

$\frac{1}{2}qa^2 + (-qa^2) = -\frac{1}{2}qa^2$

Pojednostavljena konstrukcija dijagrama

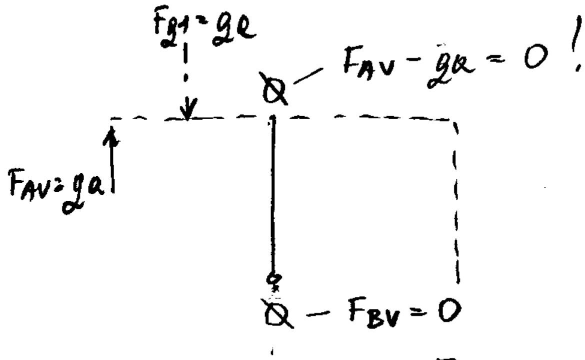
F_N - dijagram

Dio nosača A – C



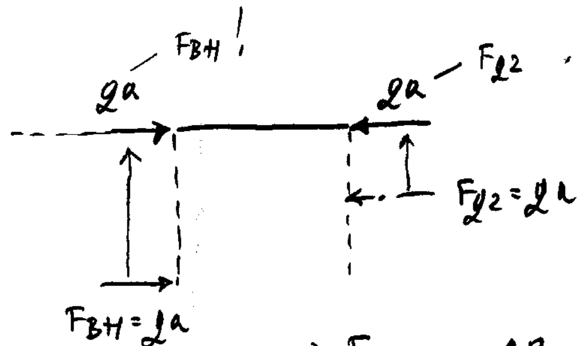
$\Rightarrow$ $\underline{F_N = 0}$

Dio nosača B – C



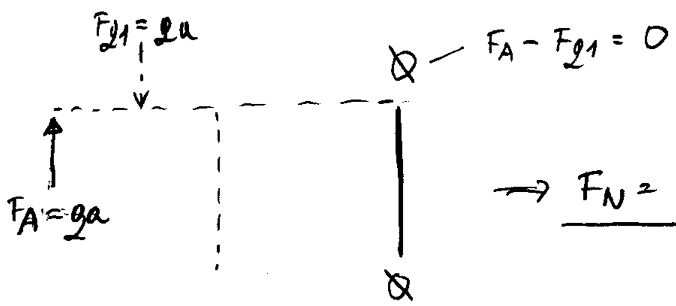
$\Rightarrow$ $\underline{F_N = 0}$

Dio nosača C – D



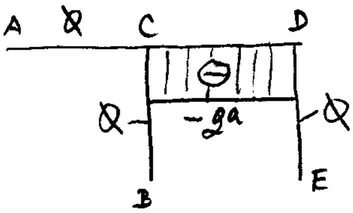
$\Rightarrow$ $\underline{F_N = -qa}$

Dio nosača D – E



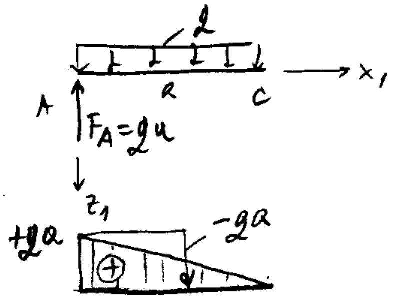
$\Rightarrow$ $\underline{F_N = 0}$

F_N - dijagram



F_Q - dijagram

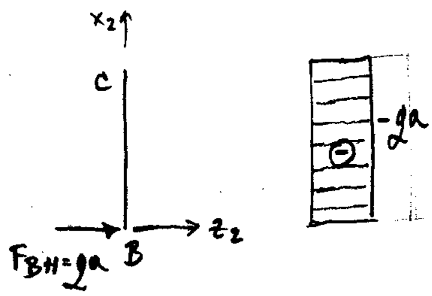
Dio nosača A – C



Za $x_1 = 0$ F_Q ima skokovit prirast za $+F_A = +qa$, pa sa 0 skače na $F_Q = +qa$.

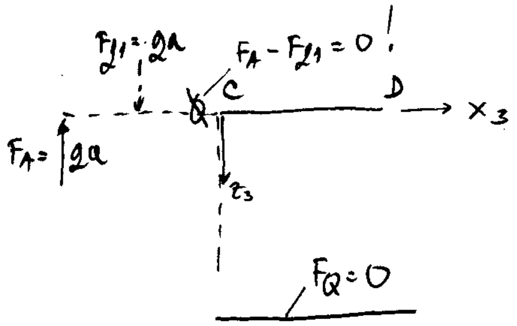
Od $x_1 = 0$ do $x_1 = a$ F_Q ima negativni linearni prirast za ukupno $-qa$, pa je u $\underrightarrow{C}$ $\underrightarrow{F_{QC}} = qa - qa = 0$.

Dio nosača B – C

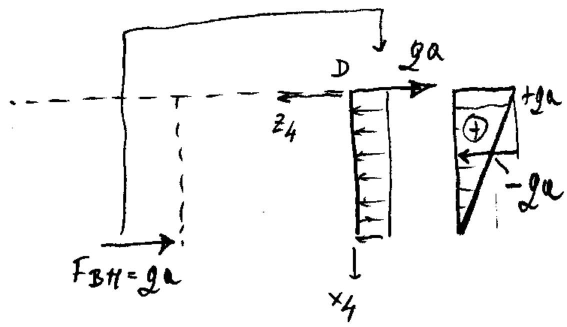


Za $x_2 = 0$ F_Q ima skokovit prirast za $-F_{BH} = -qa$, pa sa 0 skače na $F_Q = -qa$. Od A do ↑C ovaj iznos ostaje konstantan.

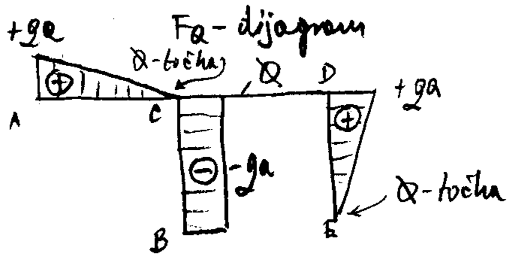
Dio nosača C – D



Za $x_3 = 0$ rezultanta sila lijevo od C u smjeru z_3 iznosi $F_A - F_{q1} = 0$. Zbog toga nema skokovitog prirasta koji bi izravno skok iznosa F_Q sa nule! Od C do D ostaje konstantan iznos $F_Q = 0$!



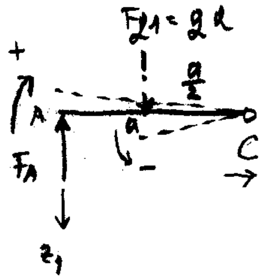
Za $x_4 = 0$ rezultanta sila lijevo od D paralelna z_4 iznosi $F_{BH} = qa$ usmjerena u negativnom smjeru z_4. F_Q ima skokovit prirast za $+qa$ pa sa 0 skače na $F_Q = +qa$.



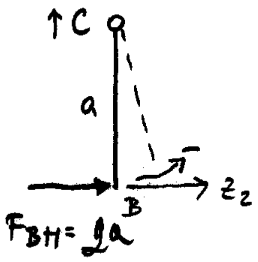
Od $x_4 = 0$ do $x_4 = a$ F_Q ima negativni linearni prirast za ukupno $-qa$, pa je u E $F_{QE} = +qa - qa = 0$.

M - dijagram

$$\left.\begin{array}{l} M_A = 0 \\ M_B = 0 \\ M_E = 0 \end{array}\right\} \text{ KRAJEVI NOSAČA BEZ VANJSKIH MOMENATA!}$$

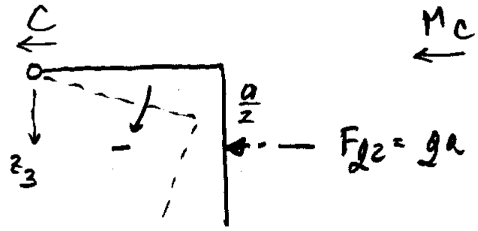


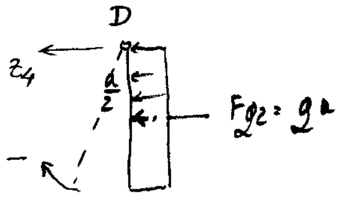
$$\underset{\rightarrow}{M_C} = +F_A\cdot a - F_{q1}\cdot\frac{a}{2} = +qa\cdot a - qa\cdot\frac{a}{2} = +\frac{1}{2}qa^2$$



$$\uparrow M_C = -F_{BH}\cdot a = -qa\cdot a = -qa^2$$

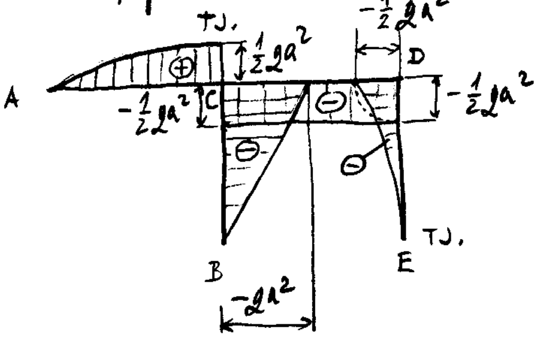
$$\underset{\leftarrow}{M_C} = -F_{q2}\cdot\frac{a}{2} = -qa\cdot\frac{a}{2} = -\frac{1}{2}qa^2$$





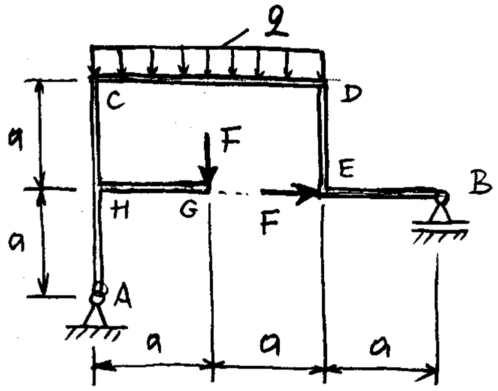
$$M_D = -F_{q2}\cdot\frac{a}{2} = -\frac{1}{2}qa^2$$

M-dijagram



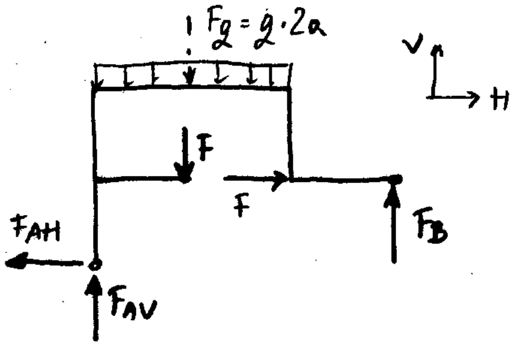
U dijagram se ucrtaju izračunati iznosi u $\underset{\rightarrow}{C}$, ↑C, $\underset{\leftarrow}{C}$ i D, a u A, B i E iznosi su 0. Ucrtane ordinate povežu se linijama koje slijede F_Q dijagram: od A do $\underset{\rightarrow}{C}$ parabola 2. reda s tjemenom u C, od B do ↑C pravac, od $\underset{\leftarrow}{C}$ do D konstantni iznos i od D do E parabola 2. reda s tjemenom u E.

Zad. 6.42



Za okvirni nosač zadan i opterećen prema slici odrediti reakcije u osloncima te skicirati i kotirati F_N-, F_Q- i M-dijagrame.

Zadano: q, a, $F = \frac{1}{2} q a$.



$\Sigma F_H = 0 \quad -F_{AH} + F = 0$ (1)

$\underline{F_{AH}} = F = \underline{\frac{1}{2} q a}$

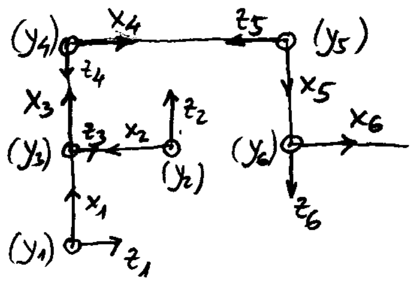
$\Sigma F_V = 0 \quad F_{AV} + F_B - q 2a - F = 0$ (2)

$\Sigma M_A = 0 \quad F_B \cdot 3a - (q \cdot 2a) \cdot a - F \cdot a - F \cdot a = 0$ (3)

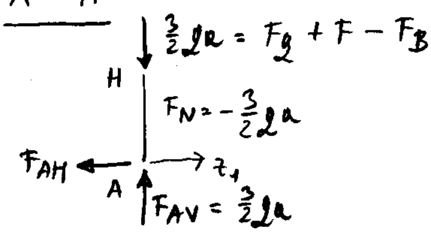
$\underline{F_B} = \frac{1}{3}(2qa + F + F) = \underline{qa}$

$\rightarrow (2) \Rightarrow \underline{F_{AV}} = q \cdot 2a + F - F_B = \underline{\frac{3}{2} q a}$

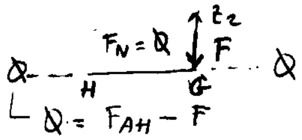
LOKALNI KOORDINATNI SUSTAVI



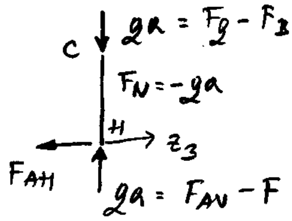
A − H



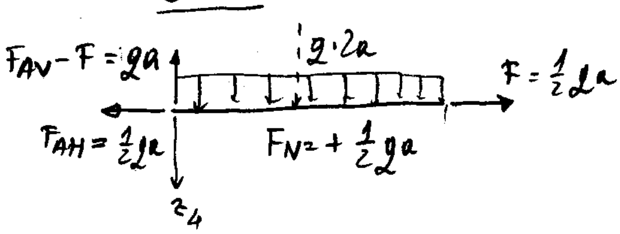
G − H

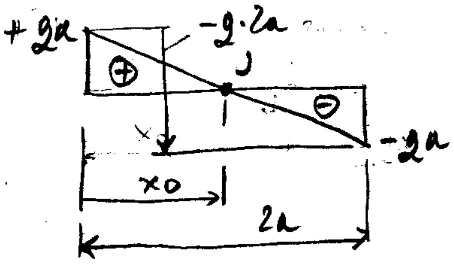


H − C



C − D





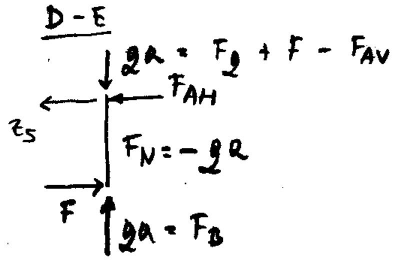
NULTOČKA:

$$\frac{x_0}{qa} = \frac{2a - x_0}{|-qa|}$$

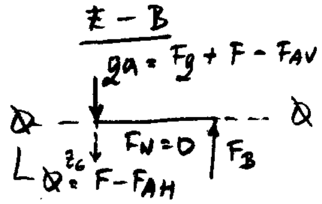
$$\Rightarrow x_0 = 2a - x_0$$

$$2x_0 = 2a \Longrightarrow \underline{\underline{x_0 = a}}$$

D − E

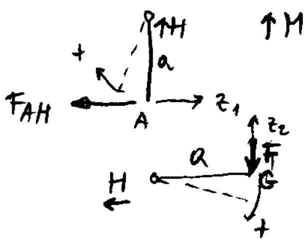


E − B



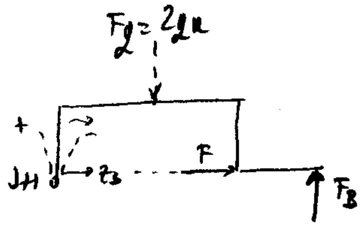
Momenti savijanja

$$\left.\begin{array}{l} M_A = 0 \\ M_B = 0 \\ M_G = 0 \end{array}\right\} \text{KRAJEVI NOSAČA BEZ VANJSKIH MOMENATA!}$$



$$\uparrow M_H = + F_{AH} \cdot a = + \frac{1}{2} qa \cdot a = + \frac{1}{2} qa^2$$

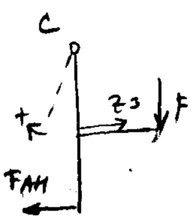
$$\overleftarrow{M_H} = + F \cdot a = + \frac{1}{2} qa \cdot a = + \frac{1}{2} qa^2$$



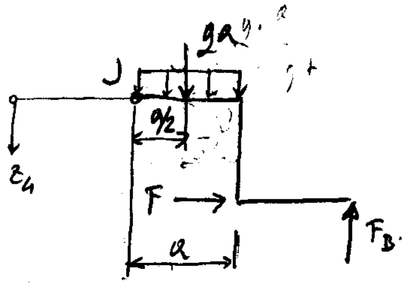
$$\downarrow M_H = - F_q \cdot a + F_B \cdot 3a =$$

$$= - 2qa \cdot a + qa \cdot 3a = + qa^2$$

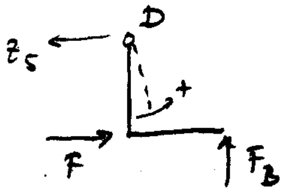
$$\rightarrow \underline{\underline{\downarrow M_H = \uparrow M_H + \overleftarrow{M_H}}}$$



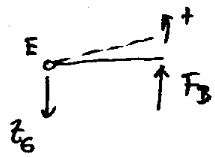
$$M_C = + F_{AH} \cdot 2a + F \cdot a = + \frac{1}{2} qa \cdot 2a + \frac{1}{2} qa \cdot a = + \frac{3}{2} qa^2$$



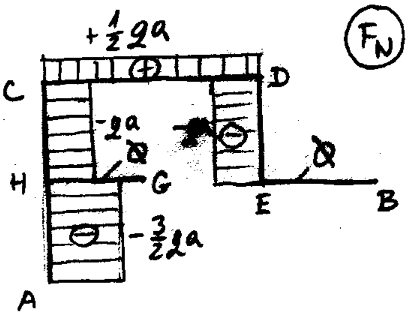
$$M_J = +F\cdot a + F_B\cdot(a+a) - (qa)\cdot\frac{a}{2} =$$
$$= +\frac{1}{2}qa\cdot a + qa\cdot 2a - \frac{1}{2}qa^2 =$$
$$= 2qa^2$$

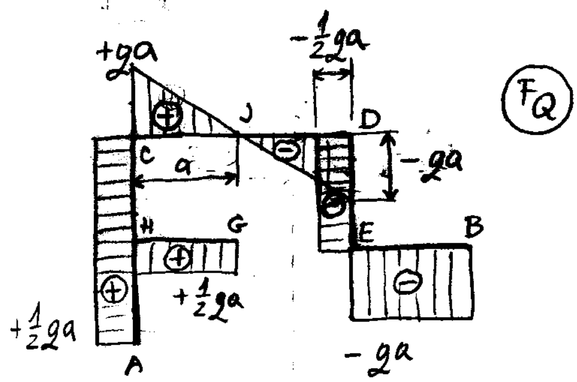


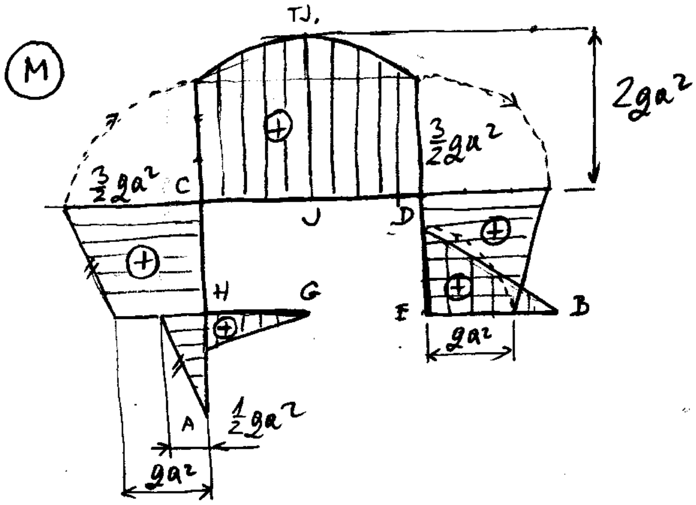
$$M_D = +F\cdot a + F_B\cdot a = +\frac{1}{2}qa\cdot a + qa\cdot a = +\frac{3}{2}qa^2$$



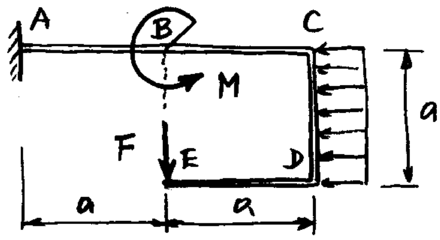
$$M_E = +F_B\cdot a = +qa\cdot a = +qa^2$$



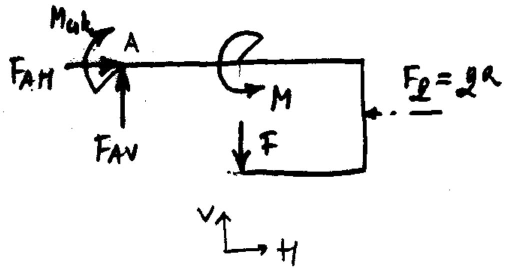




Zad. 6.50



Za konzolni okvirni nosač odrediti reakcije u uklještenju te skicirati i kotirati F_N-, F_Q- i M-dijagrame.
Zadano: q, a, $F = qa$, $M = 2qa^2$.



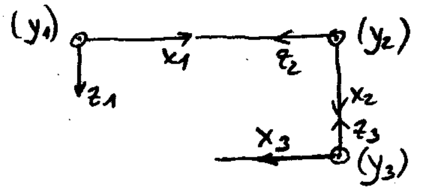
$\Sigma F_H = 0 \quad F_{AH} - F_q = 0 \rightarrow \underline{F_{AH} = qa}$

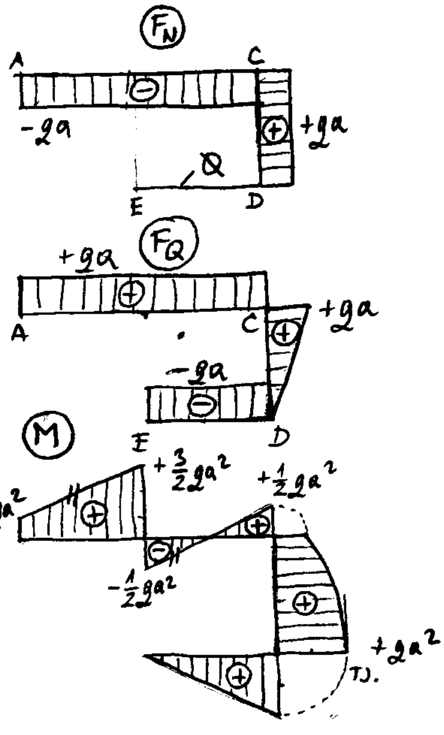
$\Sigma F_V = 0 \quad F_{AV} - F = 0 \rightarrow \underline{F_{AV} = qa}$

$\Sigma M_A = 0 \quad -M_{uk} + M - F_q \cdot \frac{a}{2} - F \cdot a = 0$

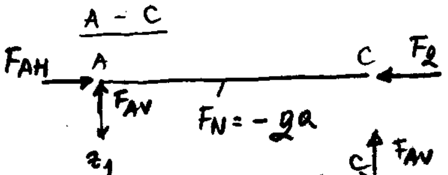
$\underline{M_{uk}} = M - qa \cdot \frac{a}{2} - F \cdot a = \underline{\frac{1}{2} qa^2}$

LOKALNI KOORDINATNI SUSTAVI



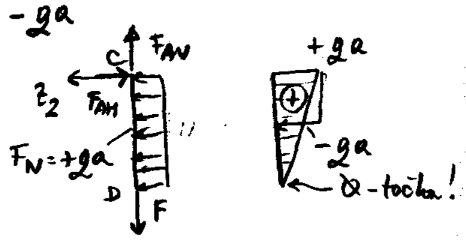


A – C



$F_N = -qa$

C – D



$F_N = +qa$

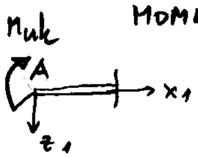
Q-točka!

D – E

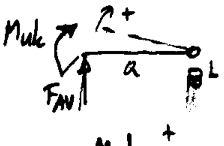
$F_N = 0$

$Q = F_{AH} - qa$

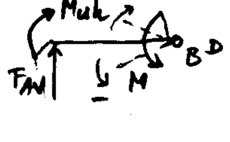
MOMENTI SAVIJANJA



$M_A = +M_{uk} = +\frac{1}{2} qa^2$

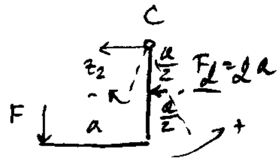


$M_B^L = +M_{uk} + F_{AV} \cdot a = +\frac{1}{2}qa^2 + qa \cdot a = +\frac{3}{2} qa^2$

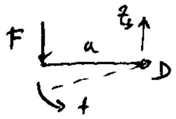


$M_B^D = +M_{uk} + F_{AV} \cdot a - M = +\frac{1}{2}qa^2 + qa \cdot a - 2qa^2 = -\frac{1}{2} qa^2$

$M_E = 0$ SLOBODNI KRAJ BEZ VANJSKOG MOMENTA!

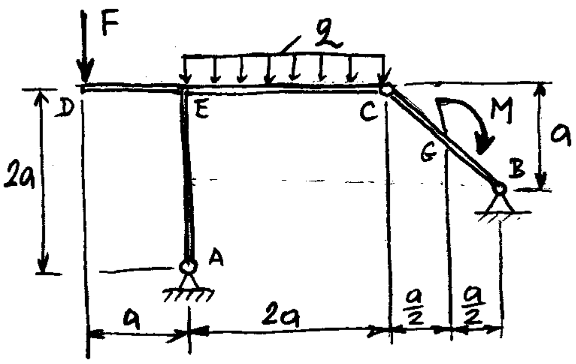


$M_C = -F_q \cdot \frac{a}{2} + F \cdot a = -qa \cdot \frac{a}{2} + qa \cdot a = +\frac{1}{2} qa^2$



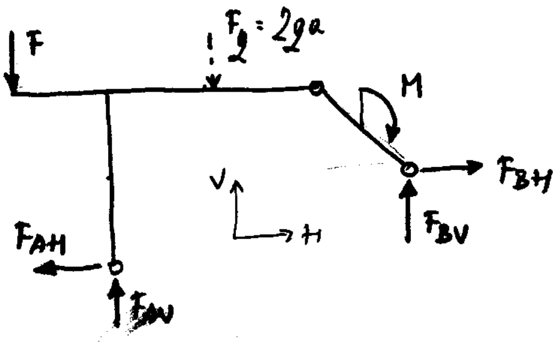
$M_D = +F \cdot a = +qa \cdot a = +qa^2$

Zad. 6.53 modificiran



Za trozglobni okvirni nosač zadan i opterećen prema slici odrediti reakcije u osloncima te skicirati i kotirati F_N-, F_Q- i M-dijagrame.

Zadano: q, a, $F=2qa$, $M=2qa^2$.



Jednadžbe ravnoteže:

$\sum F_H = 0 \qquad -F_{AH} + F_{BH} = 0 \qquad (1)$

$\sum F_V = 0 \qquad F_{AV} + F_{BV} + F - 2qa = 0 \quad (2)$

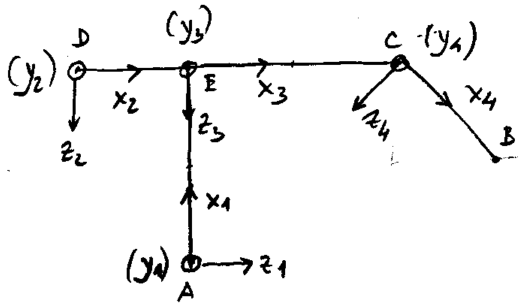
$\sum M_A = 0$

$$-F_{BH}\cdot a + F_{BV}\cdot 3a - M - (2qa)\cdot a + F\cdot a = 0 \qquad (3)$$

$$-F_{BH} + 3F_{BV} = \frac{M}{a} + 2qa - F$$

$$\boxed{-F_{BH} + 3F_{BV} = 2qa} \quad (A)$$

LOKALNI KOORDINATNI SUSTAVI



Dopunska jednadžba:

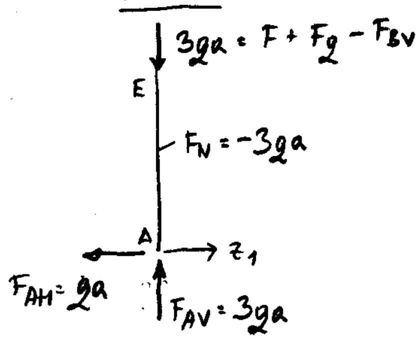
$M_C^D = 0$

$F_{BH}\cdot a + F_{BV}\cdot a - M = 0 \qquad (4)$

$$F_{BH} + F_{BV} = \frac{M}{a}$$

$$\boxed{F_{BH} + F_{BV} = 2qa} \quad (B)$$

A − E



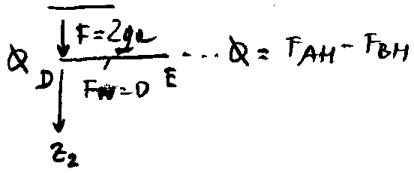
$(A)+(B) \Rightarrow 4F_{BV} = 4qa \Rightarrow \underline{F_{BV} = qa}$

$\rightarrow (B) \Rightarrow \underline{F_{BH}} = 2qa - F_{BV} = \underline{qa}$

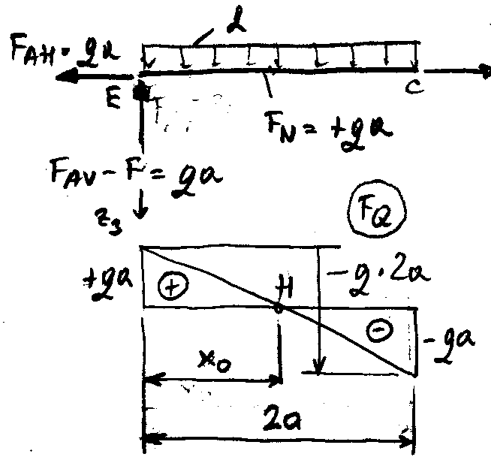
$\rightarrow (1) \Rightarrow \underline{F_{AH}} = F_{BH} = \underline{qa}$

$\rightarrow (2) \Rightarrow \underline{F_{AV}} = 2qa + F - F_{BV} = \underline{3qa}$

D − E



E - C

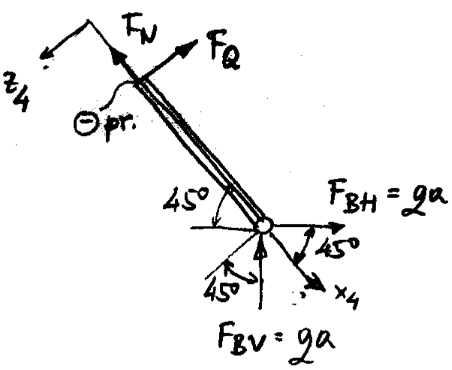


NULTOČKA :

$$\frac{x_0}{qa} = \frac{2a - x_0}{|-qa|} \longrightarrow x_0 = 2a - x_0$$

$$2x_0 = 2a \Rightarrow \underline{x_0 = a}$$

F_N i F_Q na dijelu C - B odredit ćemo metodom presjeka:



$$\Sigma F_x = 0 \quad -F_N + F_{BH}\cos 45° - F_{BV}\sin 45° = 0$$

$$\underline{F_N} = qa \cdot \frac{\sqrt{2}}{2} - qa \cdot \frac{\sqrt{2}}{2} = \underline{0}$$

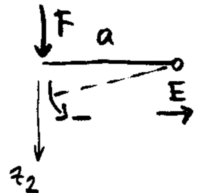
$$\Sigma F_z = 0 \quad -F_Q - F_{BH}\sin 45° - F_{BV}\cos 45° = 0$$

$$\underline{F_Q} = -qa\frac{\sqrt{2}}{2} - qa\frac{\sqrt{2}}{2} = \underline{-\sqrt{2}qa}$$

MOMENTI SAVIJANJA

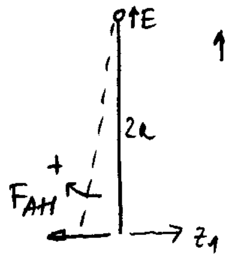
$$\left.\begin{array}{l} M_A = 0 \\ M_B = 0 \\ M_D = 0 \end{array}\right\} \text{KRAJEVI NOSAČA BEZ VANJSKIH MOMENATA!}$$

$M_C = 0$ ZGLOB!

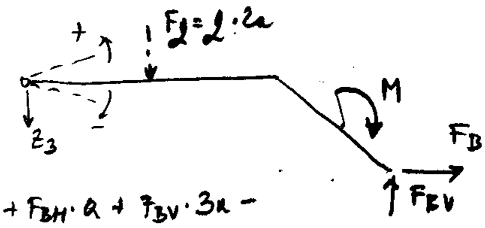


$$\underset{\rightarrow}{M_E} = -F \cdot a = -2qa \cdot a = -2qa^2$$

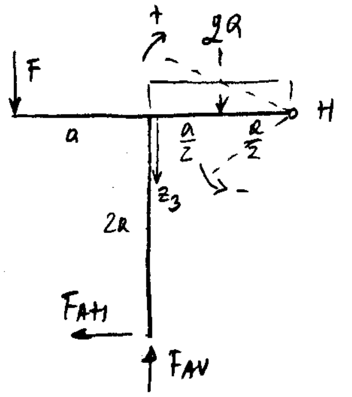
$$\underset{\leftarrow}{M_E} = \underset{\rightarrow}{M_E} + \uparrow M_E = -2qa^2 + 2qa^2 = 0$$



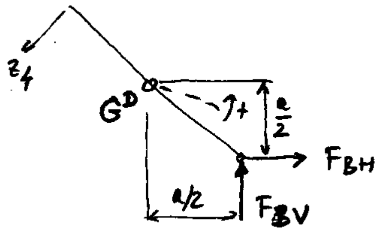
$$\uparrow M_E = +F_{AH} \cdot 2a = +qa \cdot 2a = +2qa^2$$



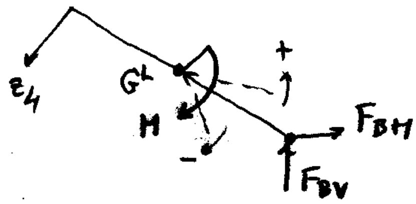
$$\underset{\leftarrow}{M_E} = +F_{BH} \cdot a + F_{BV} \cdot 3a - (q \cdot 2a) \cdot a - M = qa \cdot a + qa \cdot 3a - 2qa^2 - 2qa^2 = 0$$



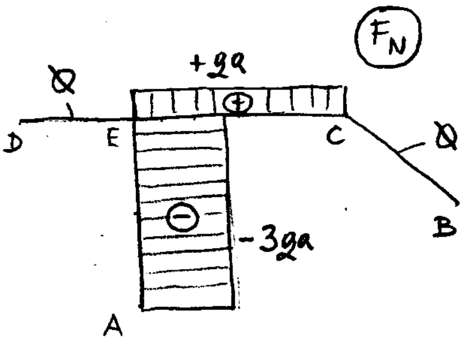
$$M_H = +F_{AH}\cdot 2a + F_{AV}\cdot a - F\cdot 2a - (qa)\cdot\frac{a}{2} =$$
$$= +qa\cdot 2a + 3qa\cdot a - 2qa\cdot 2a - \frac{1}{2}qa^2 = \frac{1}{2}qa^2$$

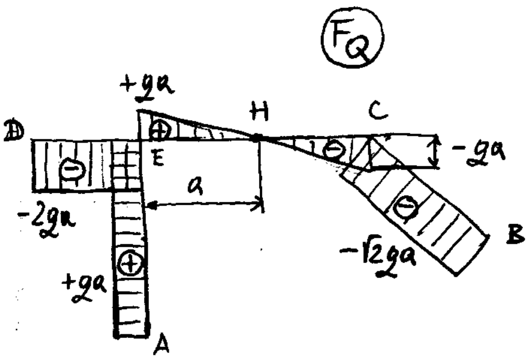


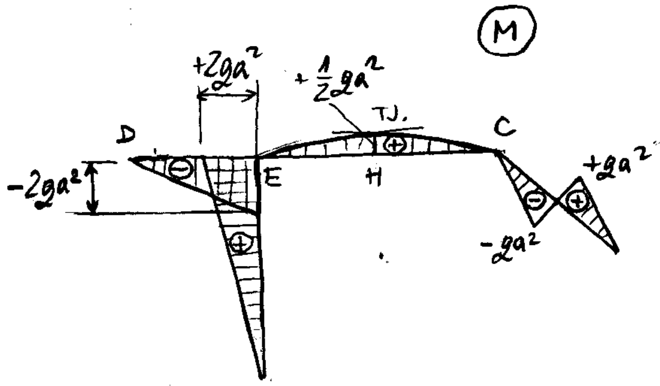
$$M_G^D = +F_{BH}\cdot\frac{a}{2} + F_{BV}\cdot\frac{a}{2} = +qa\cdot\frac{a}{2} + qa\cdot\frac{a}{2} = +qa^2$$



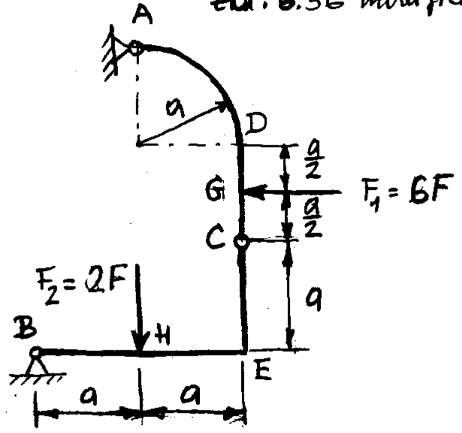
$$M_G^L = +F_{BH}\cdot\frac{a}{2} + F_{BV}\cdot\frac{a}{2} - M =$$
$$= +qa\cdot\frac{a}{2} + qa\cdot\frac{a}{2} - 2qa^2 = -qa^2$$



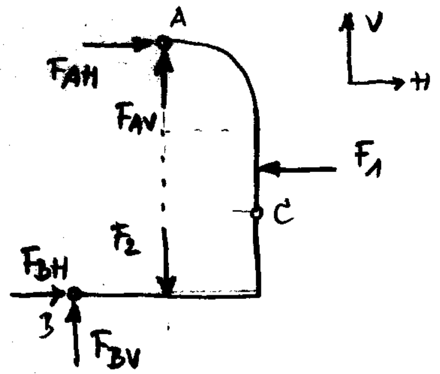




Zad. 6.56 modificiran



Za trozglobni nosač zadan i opterećen prema slici odrediti reakcije u osloncima i skicirati i kotirati F_N-, F_Q- i M-dijagrame. Zadano: F, a.



Jednadžbe ravnoteže:

$$\Sigma F_H = 0 \qquad F_{AH} + F_{BH} - F_1 = 0 \qquad (1)$$

$$\Sigma F_V = 0 \qquad F_{AV} + F_{BV} - F_2 = 0 \qquad (2)$$

$$\Sigma M_A = 0 \qquad F_{BH} \cdot 3a - F_{BV} \cdot a - F_1 \cdot \tfrac{3}{2}a = 0 \qquad (3)$$

Dopunska jednadžba:

$$M_C^D = 0 \qquad F_{BH} \cdot a - F_{BV} \cdot 2a + F_2 \cdot a = 0 \qquad (4)$$

$$(4) \Rightarrow \quad F_{BH} = 2F_{BV} - F_2$$

$$\rightarrow (3) \Rightarrow \quad 6F_{BV} - 3F_2 - F_{BV} - \tfrac{3}{2}F_1 = 0$$

$$5F_{BV} = \tfrac{3}{2}F_1 + 3F_2 = \tfrac{3}{2} \cdot 6F + 3 \cdot 2F$$

$$5F_{BV} = 9F + 6F \Longrightarrow \underline{F_{BV} = 3F}$$

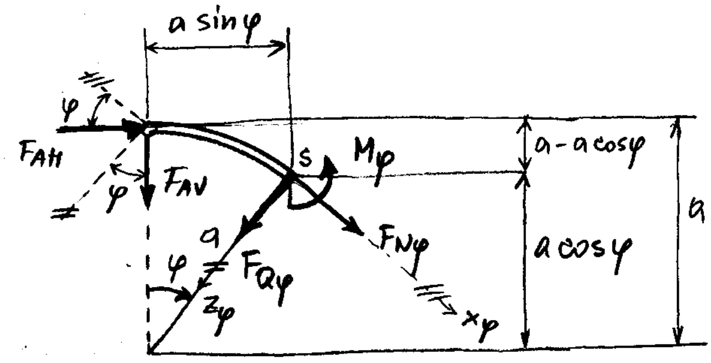
$$\underline{F_{BH}} = 2 \cdot 3F - 2F = \underline{4F}$$

$$\rightarrow (1) \rightarrow \quad \underline{F_{AH}} = F_1 - F_{BH} = 6F - 4F = \underline{2F}$$

$$\rightarrow (2) \Rightarrow \quad \underline{F_{AV}} = F_2 - F_{BV} = 2F - 3F = \underline{-F}$$

$$\Longrightarrow \quad \downarrow F_{AV} = F$$

Na zakrivljenom dijelu nosača od A do D odredit ćemo unutarnje sile metodom presjeka:



$$\Sigma F_{x\varphi} = 0$$

$$F_{N\varphi} + F_{AH}\cos\varphi + F_{AV}\sin\varphi = 0$$

$$\underline{F_{N\varphi} = -2F\cos\varphi - F\sin\varphi}$$

$$\Sigma F_{z\varphi} = 0$$

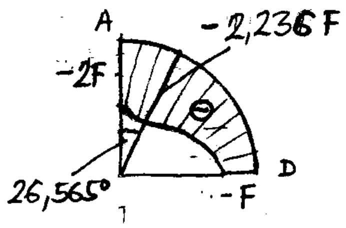
$$F_{Q\varphi} - F_{AH}\sin\varphi + F_{AV}\cos\varphi = 0$$

$$\underline{F_{Q\varphi} = 2F\sin\varphi - F\cos\varphi}$$

$$\Sigma M_S = 0 \qquad M_\varphi - F_{AH}(a - a\cos\varphi) + F_{AV}\,a\sin\varphi = 0$$

$$\underline{M_\varphi = 2Fa(1-\cos\varphi) + Fa\sin\varphi}$$

$\underline{F_{N\varphi}\text{- dijagram}}$



za $\varphi = 0$ $\quad F_{N\varphi=0} = -2F\cos 0 - F\sin 0 = -2F$

za $\varphi = 90^\circ$ $\quad F_{N\varphi=90^\circ} = -2F\cos 90^\circ - F\sin 90^\circ = -F$

$$\frac{dF_N}{d\varphi} = 0 \Rightarrow -2F(-\sin\varphi) - F\cos\varphi = 0$$

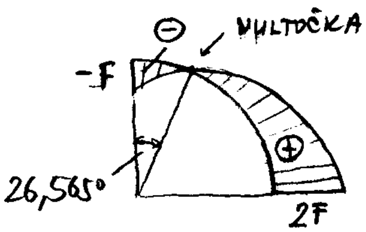
$$2\sin\varphi = \cos\varphi \quad /:2\cos\varphi$$

$$\operatorname{tg}\varphi = \frac{1}{2} \Rightarrow \varphi = 26{,}565^\circ$$

$$F_{N\varphi=26{,}565^\circ} = -2F\cos 26{,}565^\circ - F\sin 26{,}565^\circ =$$

$$= -2{,}236\,F = F_{N\min}$$

$\underline{F_{Q\varphi}\text{- dijagram}}$



za $\varphi = 0$ $\quad F_{Q\varphi=0} = 2F\sin 0 - F\cos 0 = -F$

za $\varphi = 90^\circ$ $\quad F_{Q\varphi=90^\circ} = 2F\sin 90^\circ - F\cos 90^\circ = 2F$

$$F_{Q\varphi} = 0 \Rightarrow 2F\sin\varphi - F\cos\varphi = 0$$

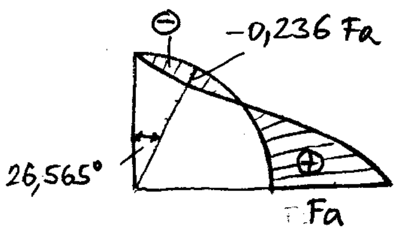
$$2\sin\varphi = \cos\varphi \quad /:2\cos\varphi$$

$$\operatorname{tg}\varphi = \frac{1}{2} \Rightarrow \varphi = 26{,}565^\circ$$

M_φ - dijagram

za $\varphi = 0$ $\quad M_{\varphi=0} = 2Fa(1-\cos 0) - Fa \sin 0 = 0$

Ovaj smo rezultat mogli dobiti jer je za $\varphi=0$ u A kraj nosača bez vanjskog momenta!

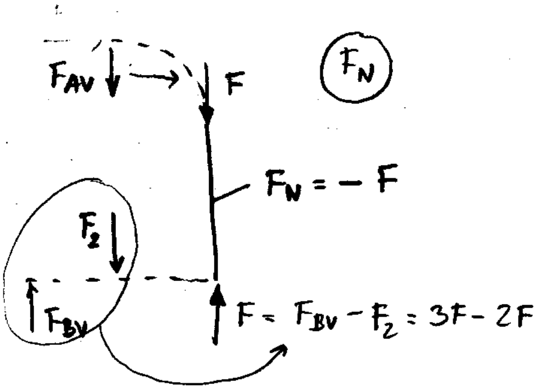


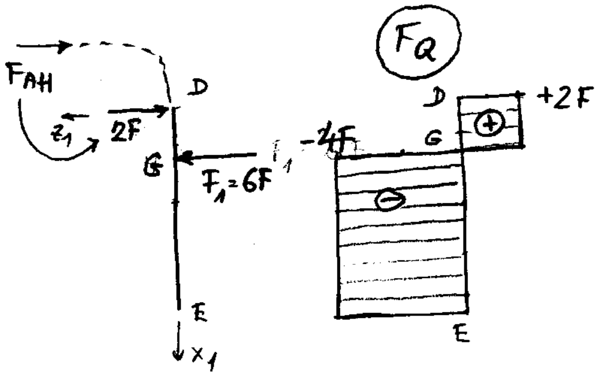
za $\varphi = 90°$ $\quad M_{\varphi=90°} = 2Fa(1-\cos 90°) + Fa \sin 90° = Fa$

u nultočki $F_{Q\varphi}$ dijagrama:

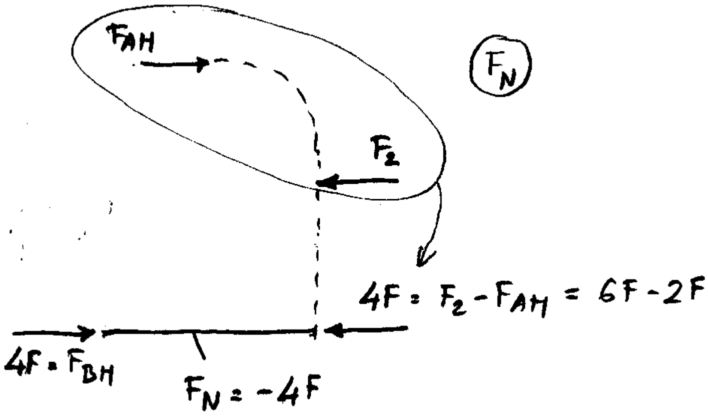
$$M_{\varphi\,min} = 2Fa(1-\cos 26{,}565°) - Fa \sin 26{,}565° = -0{,}236\,Fa$$

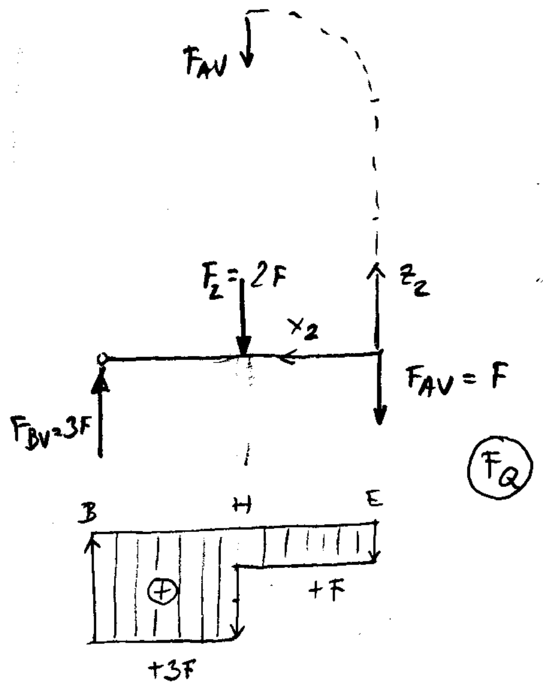
Dio nosača D - E



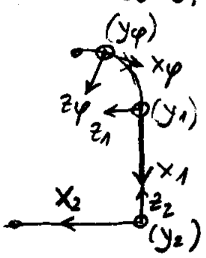


Dio nosača E - B

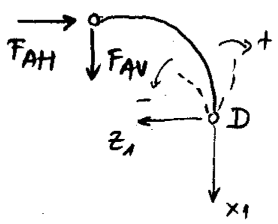




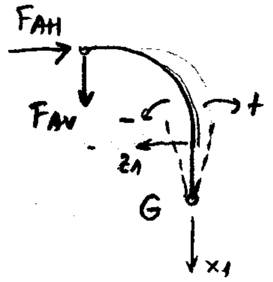
LOKALNI KOORD. SUSTAVI



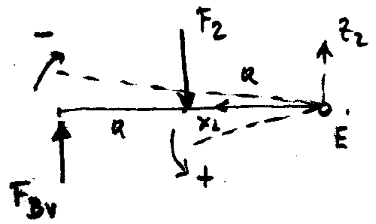
Momenti savijanja



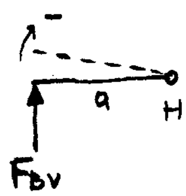
$$M_D = +F_{AH}\cdot a - F_{AV}\cdot a = 2Fa - Fa = Fa$$



$$M_G = +F_{AH}\cdot \frac{3}{2}a - F_{AV}\cdot a = 2Fa\cdot\frac{3}{2}a - Fa = 2Fa$$



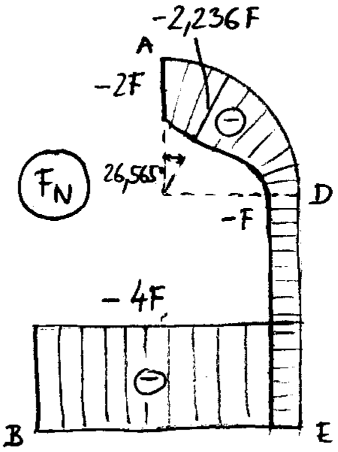
$$M_E = -F_{BV}\cdot 2a + F_2\cdot a = -3F\cdot 2a + 2F\cdot a = -4Fa$$

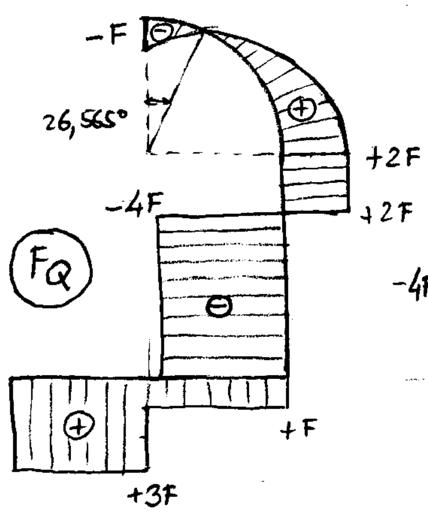


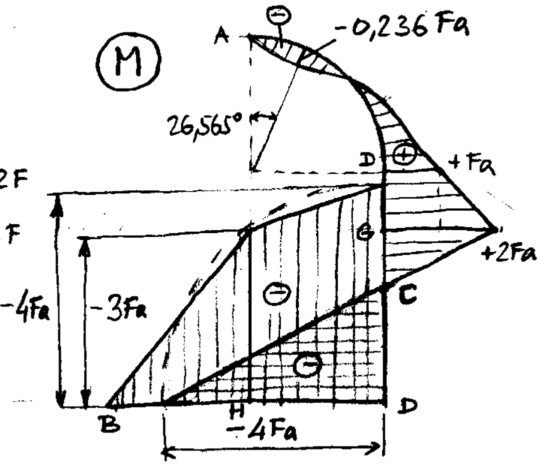
$$M_H = -F_{BV}\cdot a = -3F\cdot a = -3Fa$$

$$\left.\begin{matrix} M_B = 0 \\ M_A = 0 \end{matrix}\right\}$$ KRAJEVI NOSAČA BEZ VANJSKOG MOMENTA!

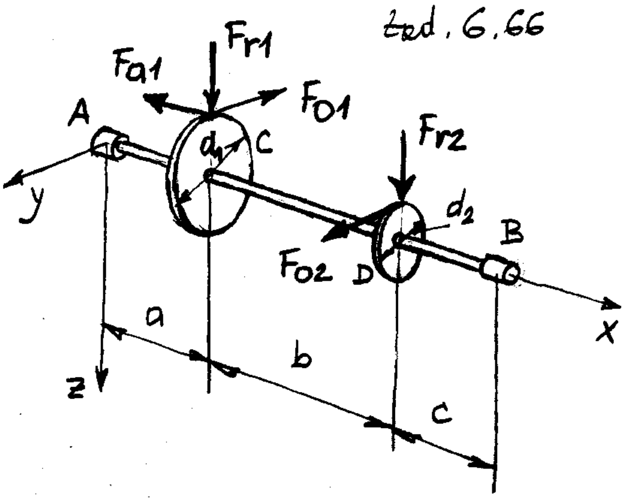
$M_C = 0$ ZGLOB!







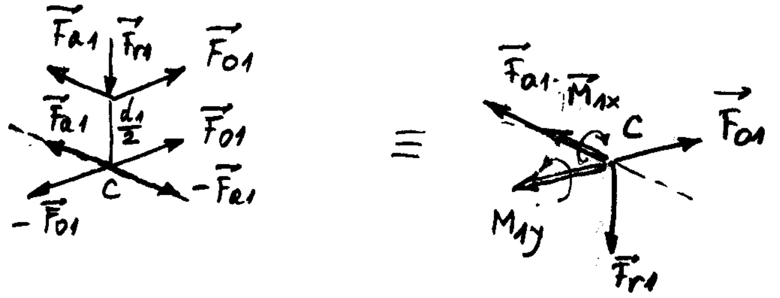
Zad. 6.66



Za vratilo ACDB opterećeno prema slici treba odrediti reakcije u radijalnoaksijalnom ležaju A i radijalnom ležaju B, te skicirati i kotirati dijagrame unutrašnjih sila F_N, F_{Qy} i F_{Qz} i momenata M_x, M_y i M_z.

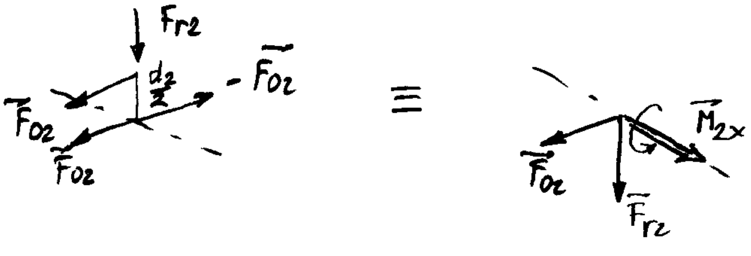
Zadano: $d_1 = 40$ cm, $d_2 = 25$ cm, $F_{o1} = 0{,}8$ kN, $F_{a1} = 0{,}185$ kN, $F_{r1} = 0{,}3$ kN, $F_{r2} = 1{,}17$ kN, $a = 14$ cm, $b = 27$ cm, $c = 15$ cm.

Sile s oboda diskova C i D reduciramo u težišta presjeka vratila u C i D:



$|\vec{M}_{1x}| = F_{o1} \cdot \frac{d_1}{2}$

$|\vec{M}_{1y}| = F_{a1} \cdot \frac{d_1}{2}$



$|\vec{M}_{2x}| = F_{o2} \cdot \frac{d_2}{2}$

Iz jednadžbe ravnoteže vratila

$$\Sigma M_x = 0 \qquad -M_{1x} + M_{2x} = 0$$

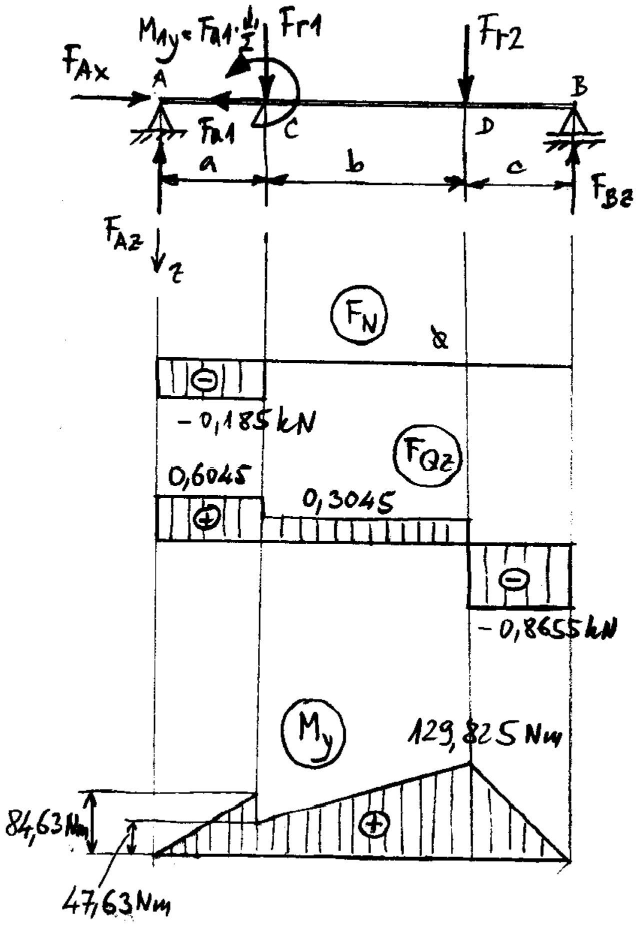
$$-F_{o1} \cdot \frac{d_1}{2} + F_{o2} \cdot \frac{d_2}{2} = 0$$

dobije se

$$F_{o2} = F_{o1} \cdot \frac{d_1}{d_2} = 0{,}8 \cdot \frac{40}{25} = 1{,}28 \text{ kN}$$

Unutarnje sile u vratilu možemo odrediti primjenom METODE SUPERPOZICIJE. Najprije razmatramo vratilo opterećeno silama u xz ravnini (F_{r1}, F_{r2}, F_{a1} i spreg $\bar{F}_{a1}, -\bar{F}_{a1}$ koji smo zamijenili njegovim momentom M_{1y} iznosa $F_{a1} \cdot \frac{d}{2}$). Iz ovog razmatranja odredit ćemo F_N, F_{Qz} i M_y. Zatim razmatramo vratilo opterećeno silama u xy ravnini (F_{o1} i F_{o2}) što će nam omogućiti određivanje F_{Qy} i M_z. Konačno razmatramo vratilo opterećeno spregovima čiji su momenti M_{1x} i M_{2x} što će nam omogućiti određivanje M_x.

U prvom slučaju radi se o jednostavnoj gredi opterećenoj silama F_{r1}, F_{r2} i F_{a1} te momentom M_{1y}:



$$\Sigma F_x = 0 \qquad F_{Ax} - F_{a1} = 0 \quad (1)$$

$$\Sigma F_z = 0 \qquad -F_{Az} - F_{Bz} + F_{r1} + F_{r2} = 0 \quad (2)$$

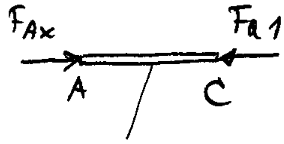
$$\Sigma M_A = 0 \qquad F_{Bz}(a+b+c) + F_{a1} \cdot \frac{d_1}{2} - F_{r1} \cdot a - F_{r2}(a+b) = 0 \quad (3)$$

$$(3) \rightarrow F_{Bz} = \frac{F_{r1} a + F_{r2}(a+b) - F_{a1} \cdot \frac{d_1}{2}}{a+b+c} = \frac{0{,}3 \cdot 14 + 1{,}17(14+27) - 0{,}185 \cdot \frac{40}{2}}{14+27+15} = 0{,}8655 \text{ kN}$$

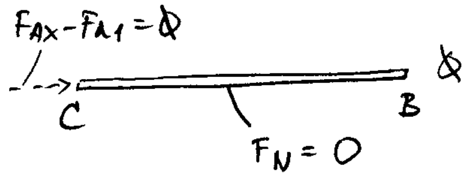
$$(2) \rightarrow F_{Az} = F_{r1} + F_{r2} - F_{Bz} = 0{,}3 + 1{,}17 - 0{,}8655 = 0{,}6045 \text{ kN}$$

$$(1) \rightarrow F_{Ax} = F_{a1} = 0{,}185 \text{ kN}$$

F_N - dijagram :



$F_N = -F_{Ax} = -0{,}185\,kN$



F_{Qz} - dijagram :

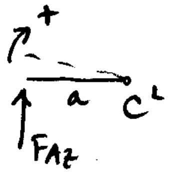
A - C $\quad F_{Qz} = F_{Az} = 0{,}6045\,kN$

C - D $\quad F_{Qz} = F_{Az} - F_{r1} = 0{,}6045 - 0{,}3 = 0{,}3045\,kN$

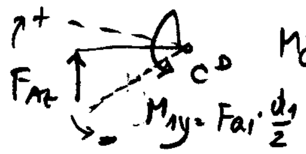
D - B $\quad F_{Qz} = F_{Az} - F_{r1} - F_{r2} = 0{,}6045 - 0{,}3 - 1{,}17 =$

$= -0{,}8655\,kN$

M_y - dijagram : $\quad M_{Ay} = 0, \quad M_{By} = 0 \quad$ (krajevi nosača !)

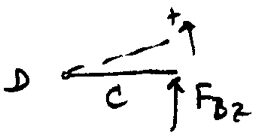


$M_{Cy}^{L} = +F_{Az} \cdot a = 604{,}5 \cdot 0{,}14 = 84{,}63\,Nm$



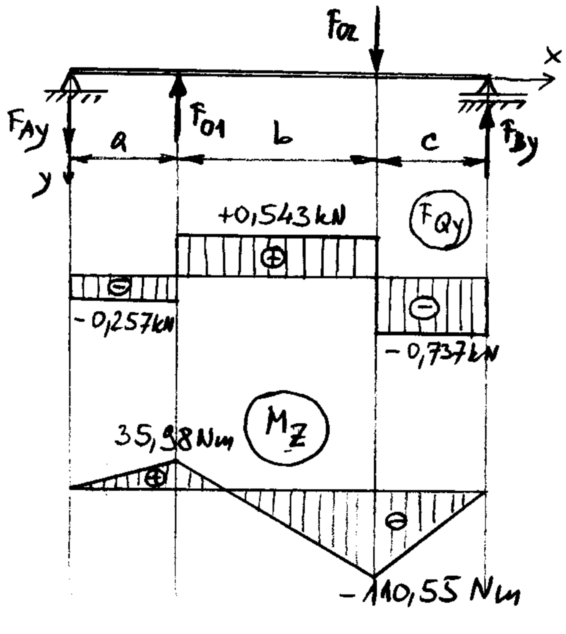
$M_{Cy}^{D} = +F_{Az} \cdot a - M_{1y} = 604{,}5 \cdot 0{,}14 -$

$- 185 \cdot 0{,}2 = 47{,}63\,Nm$



$M_{Dy} = +F_{Bz} \cdot c = 865{,}5 \cdot 0{,}15 = 129{,}825\,Nm$

U drugom slučaju radi se o jednostavnoj gredi opterećenoj silama F_{01} i F_{02}



$\Sigma F_y = 0 \quad F_{Ay} - F_{01} + F_{02} - F_{By} = 0 \quad (1)$

$\Sigma M_A = 0 \quad -F_{By} \cdot (a+b+c) + F_{02} \cdot (a+b) -$

$- F_{01} \cdot a = 0 \quad (2)$

$(2) \Rightarrow F_{By} = \dfrac{F_{02}(a+b) - F_{01} \cdot a}{a+b+c} =$

$= \dfrac{1{,}28\,(14+27) - 0{,}8 \cdot 14}{14+27+15} = 0{,}737\,kN$

$(1) \Rightarrow F_{Ay} = F_{01} - F_{02} + F_{By} = 0{,}8 - 1{,}28 + 0{,}737 =$

$= 0{,}257\,kN$

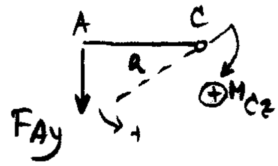
F_{Qy} - dijagram:

A – C $\quad F_{Qy} = -F_{Ay} = -0{,}257\ kN$

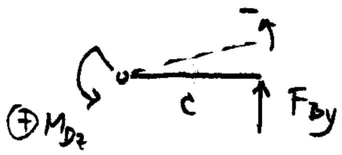
C – D $\quad F_{Qy} = -F_{Ay} + F_{O1} = -0{,}257 + 0{,}8 = 0{,}543\,kN$

D – B $\quad F_{Qy} = -F_{Ay} + F_{O1} - F_{O2} = -0{,}257 + 0{,}8 - 1{,}28 =$
$\quad = -0{,}737\ kN$

M_z - dijagram: $\quad M_{Az} = 0, \quad M_{Bz} = 0 \quad$ (krajevi nosača!)

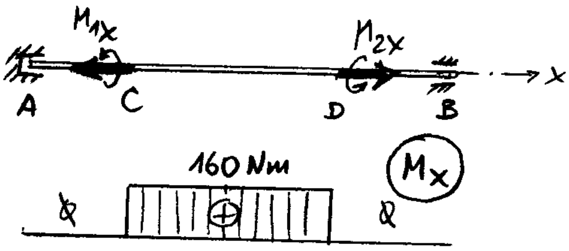


$M_{Cz} = +F_{Ay} \cdot a = +257 \cdot 0{,}14 =$
$= +35{,}98\ Nm$



$M_{Dz} = -F_{By} \cdot c = -737 \cdot 0{,}15 =$
$= -110{,}55\ Nm$

Konačno, promatramo vratilo opterećeno samo momentima M_{1x} i M_{2x}:



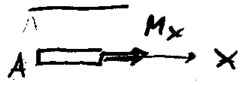
$\Sigma M_x = 0 \quad -M_{1x} + M_{2x} = 0$

$-F_{O1} \cdot \frac{d_1}{2} + F_{O2} \cdot \frac{d_2}{2} = 0$

$-0{,}8 \cdot \frac{0{,}4}{2} + 1{,}28 \cdot \frac{0{,}25}{2} = 0$

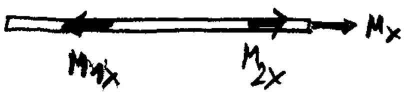
$-0{,}16 + 0{,}16 = 0$ ✓

A – C



$\Sigma M_x = 0$
$\Rightarrow M_x = 0$

D – B

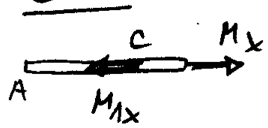


$\Sigma M_x = 0$

$M_x - M_{1x} + M_{2x} = 0$

$M_x = 0{,}16 - 0{,}16 = 0$

C – D

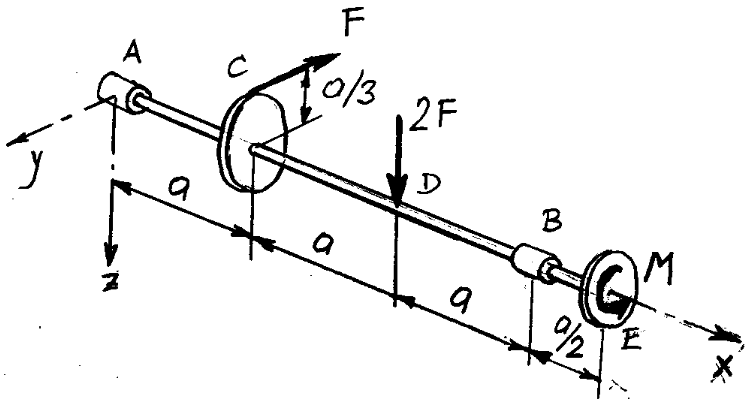


$\Sigma M_x = 0$

$M_x - M_{1x} = 0$

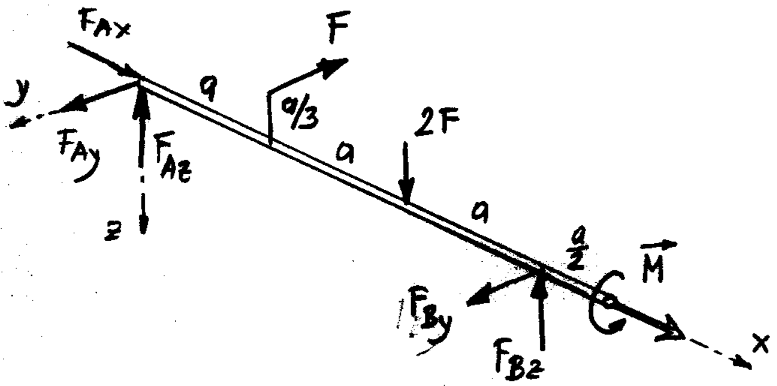
$M_x = M_{1x} = F_{O1} \cdot \frac{d_1}{2} =$
$= 800 \cdot \frac{0{,}4}{2} = 160\,Nm$

Zad. 6.64



Za vratilo ACDBE opterećeno prema slici treba odrediti reakcije u radijalnoaksijalnom ležaju A i radijalnom ležaju B te skicirati i kotirati dijagrame unutrašnjih sila F_N, F_{Qz} i F_{Qy} i momenata M_x, M_y i M_z.

Zadano: $M = 1200$ Nm,
$a = 30$ cm.



$\Sigma F_x = 0 \qquad F_{Ax} = 0 \qquad (1)$

$\Sigma F_y = 0 \qquad -F_{Ay} + F - F_{By} = 0 \qquad (2)$

$\Sigma F_z = 0 \qquad -F_{Az} + 2F - F_{Bz} = 0 \qquad (3)$

$\Sigma M_x = 0 \qquad -F \cdot \frac{a}{3} + M = 0 \qquad (4)$

$\Sigma M_y = 0 \qquad F_{Bz} \cdot 3a - 2F \cdot 2a = 0 \qquad (5)$

$\Sigma M_z = 0 \qquad -F_{By} \cdot 3a + F \cdot a = 0 \qquad (6)$

$(4) \Rightarrow \quad F = \frac{3M}{a} = \frac{3 \cdot 1200}{0{,}3} = 12000\ \text{N} = 12\ \text{kN}$

$(5) \Rightarrow \quad F_{Bz} = \frac{4}{3} F = \frac{4}{3} \cdot 12 = 16\ \text{kN}$

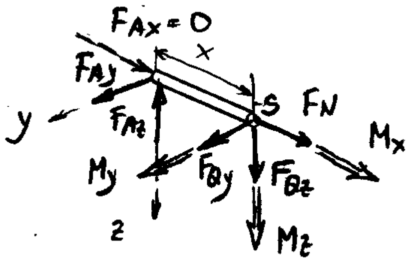
$(6) \Rightarrow \quad F_{By} = \frac{1}{3} F = \frac{1}{3} \cdot 12 = 4\ \text{kN}$

$(2) \Rightarrow \quad F_{Ay} = F - F_{By} = 12 - 4 = 8\ \text{kN}$

$(3) \Rightarrow \quad F_{Az} = 2F - F_{Bz} = 2 \cdot 12 - 16 = 8\ \text{kN}$

Određivanje unutrašnjih sila metodom presjeka

A – C



$\sum F_x = 0 \qquad F_N = 0$

$\sum F_y = 0 \qquad F_{Qy} + F_{Ay} = 0$

$F_{Qy} = -F_{Ay} = -8\ kN$

$\sum F_z = 0 \qquad F_{Qz} - F_{Az} = 0$

$F_{Qz} = F_{Az} = 8\ kN$

$\sum M_{Sx} = 0 \qquad M_x = 0$

$\sum M_{Sy} = 0 \qquad M_y - F_{Az} \cdot x = 0$

$M_y = F_{Az} \cdot x$

$M_{yA} = M_y(x=0) = 0$

$M_{yC} = M_y(x=a) = F_{Az} \cdot a = 8 \cdot 0{,}3 = 2{,}4\ kNm$

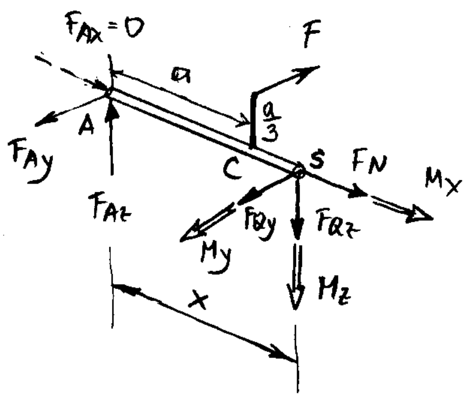
$\sum M_{Sz} = 0 \qquad M_z - F_{Ay} \cdot x = 0$

$M_z = F_{Ay} \cdot x$

$M_{zA} = M_z(x=0) = 0$

$M_{zC} = M_z(x=a) = F_{Ay} \cdot a = 8 \cdot 0{,}3 = 2{,}4\ kNm$

C – D



$\sum F_x = 0 \qquad F_N = 0$

$\sum F_y = 0 \qquad F_{Qy} + F_{Ay} - F = 0$

$F_{Qy} = -F_{Ay} + F = -8 + 12 = 4\ kN$

$\sum F_z = 0 \qquad F_{Qz} - F_{Az} = 0$

$F_{Qz} = F_{Az} = 8\ kN$

$\sum M_x = 0 \qquad M_x + F \cdot \frac{a}{3} = 0$

$M_x = F \cdot \frac{a}{3} = 12 \cdot \frac{0{,}3}{3} = 1{,}2\ kNm$

$\sum M_{Sy} = 0 \qquad M_y - F_{Az} \cdot x = 0$

$M_y = F_{Az} \cdot x \quad , \qquad M_{yC} = M_y(x=a) = F_{Az} \cdot a = 8 \cdot 0{,}3 = 2{,}4\ kNm$

$M_{yD} = M_y(x=2a) = F_{Az} \cdot 2a = 8 \cdot 2 \cdot 0{,}3 = 4{,}8\ kNm$

$$\Sigma M_{Sz} = 0 \qquad M_z - F_{Ay} \cdot x + F(x-a) = 0$$

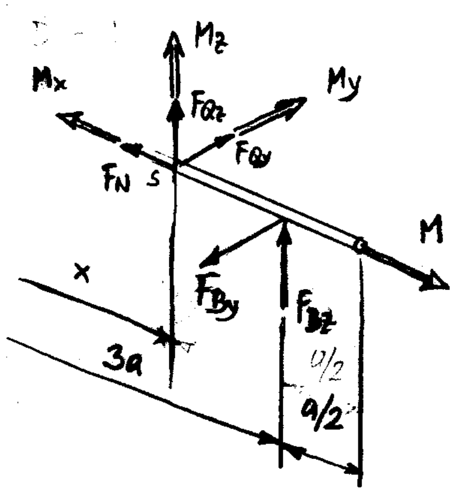
$$M_z = F_{Ay} \cdot x - F(x-a)$$

$$M_{zC} = M_z(x=a) = F_{Ay} \cdot a - F(a-a) = 8 \cdot 0{,}3 - 0 = 2{,}4 \text{ kNm}$$

$$M_{zD} = M_z(x=2a) = F_{Ay} \cdot 2a - F(2a-a) =$$

$$= 8 \cdot 2 \cdot 0{,}3 - 12 \cdot 0{,}3 = 1{,}2 \text{ kNm}$$

D-B



$$\Sigma F_x = 0 \qquad -F_N = 0 \quad \Rightarrow F_N = 0$$

$$\Sigma F_y = 0 \qquad -F_{Qy} + F_{By} = 0$$

$$F_{Qy} = F_{By} = 4 \text{ kN}$$

$$\Sigma F_z = 0 \qquad -F_{Qz} - F_{Bz} = 0$$

$$F_{Qz} = -F_{Bz} = -16 \text{ kN}$$

$$\Sigma M_{Sx} = 0 \qquad -M_x + M = 0$$

$$M_x = M = 1{,}2 \text{ kNm}$$

$$\Sigma M_{Sy} = 0 \qquad -M_y + F_{Bz} \cdot (3a - x) = 0$$

$$M_y = F_{Bz}(3a - x)$$

$$M_{yD} = M_y(x=2a) = F_{Bz}(3a-2a) = 16 \cdot 0{,}3 = 4{,}8 \text{ kNm}$$

$$M_{yB} = M_y(x=3a) = F_{Bz}(3a-3a) = 0$$

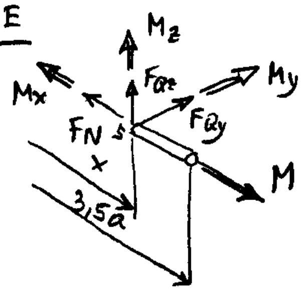
$$\Sigma M_{Sz} = 0 \qquad -M_z + F_{By} \cdot (3a - x) = 0$$

$$M_z = F_{By}(3a - x)$$

$$M_{zD} = M_z(x=2a) = F_{By}(3a-2a) = 4 \cdot 0{,}3 = 1{,}2 \text{ kNm}$$

$$M_{zB} = M_z(x=3a) = F_{By}(3a-3a) = 0$$

B - E



$\Sigma F_x = 0 \quad -F_N = 0 \rightarrow F_N = 0$

$\Sigma F_y = 0 \quad -F_{Qy} = 0 \rightarrow F_{Qy} = 0$

$\Sigma F_z = 0 \quad -F_{Qz} = 0 \rightarrow F_{Qz} = 0$

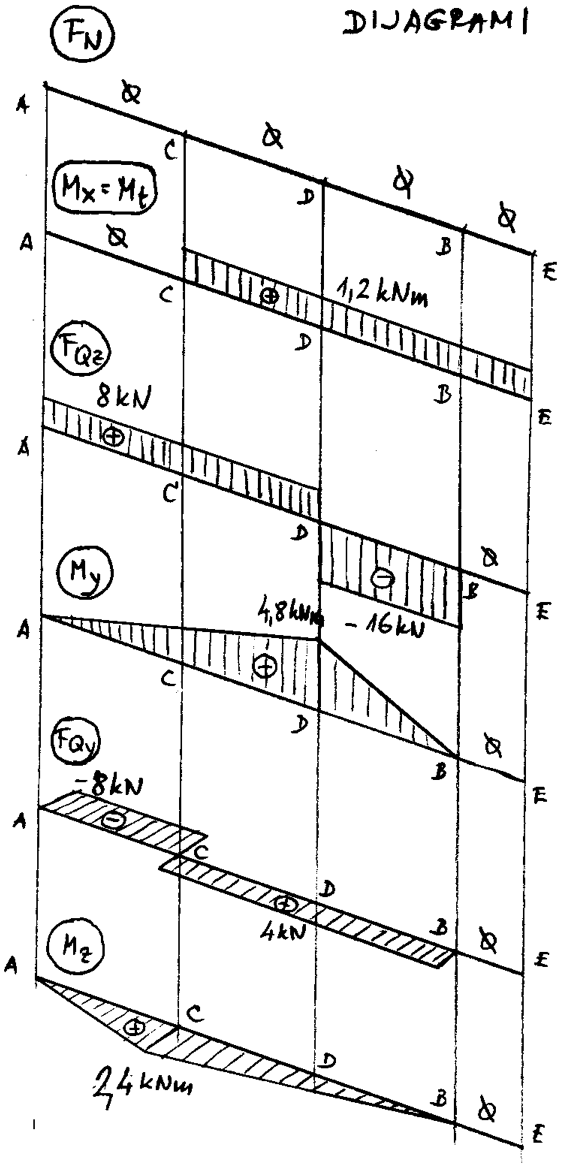
$\Sigma M_{sx} = 0 \quad -M_x + M = 0$

$M_x = M = 1{,}2\ kNm$

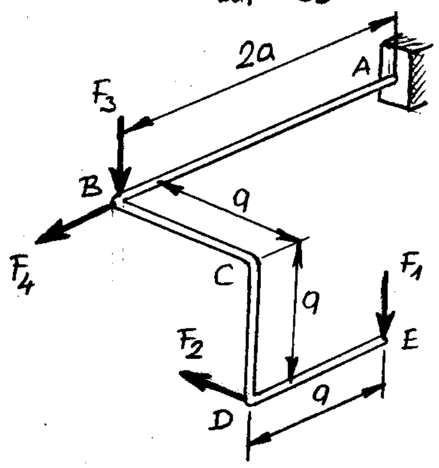
$\Sigma M_{sy} = 0 \quad -M_y = 0 \rightarrow M_y = 0$

$\Sigma M_{sz} = 0 \quad -M_z = 0 \rightarrow M_z = 0$

DIJAGRAMI

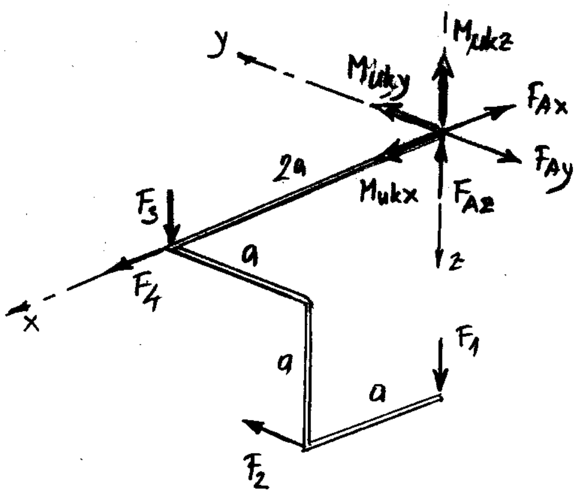


Zad. 6.59



Za prostorni konzolni nosač treba odrediti komponente reakcije na mjestu uklještenja A te skicirati i kotirati dijagrame unutrašnjih sila F_N, F_{Qy} i F_{Qz} i momenta M_x, M_y i M_z.

Zadano: a, $F_1 = F_2 = F_3 = F$, $F_4 = 2F$.



$\Sigma F_x = 0 \quad -F_{Ax} + F_4 = 0$

$\underline{F_{Ax}} = F_4 = \underline{2F}$

$\Sigma F_y = 0 \quad -F_{Ay} + F_2 = 0$

$\underline{F_{Ay}} = F_2 = \underline{F}$

$\Sigma F_z = 0 \quad -F_{Az} + F_1 + F_3 = 0$

$\underline{F_{Az}} = F_1 + F_3 = F + F = \underline{2F}$

$\Sigma M_x = 0 \quad M_{ukx} - F_1 \cdot a - F_2 \cdot a = 0$

$\underline{M_{ukx}} = F_1 a + F_2 a = Fa + Fa = \underline{2Fa}$

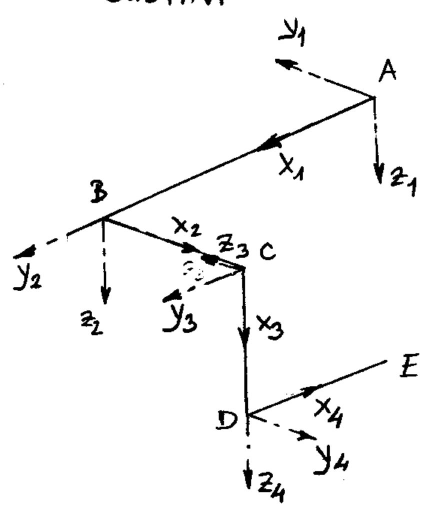
$\Sigma M_y = 0 \quad M_{uky} - F_1 \cdot a - F_3 \cdot 2a = 0$

$\underline{M_{uky}} = F_1 a + 2F_3 a = Fa + 2Fa = \underline{3Fa}$

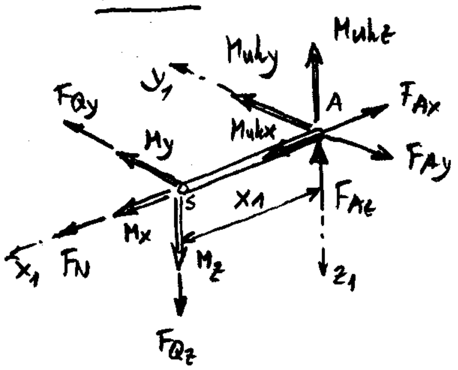
$\Sigma M_z = 0 \quad M_{ukz} - F_2 \cdot 2a = 0$

$\underline{M_{ukz}} = 2F_2 a = \underline{2Fa}$

LOKALNI KOORDINATNI SUSTAVI



A - B



⊕ presjek!

$$\Sigma F_x = 0 \qquad F_N - F_{Ax} = 0$$
$$F_N = F_{Ax} = 2F$$

$$\Sigma F_y = 0 \qquad F_{Qy} - F_{Ay} = 0$$
$$F_{Qy} = F_{Ay} = F$$

$$\Sigma F_z = 0 \qquad F_{Qz} - F_{Az} = 0$$
$$F_{Qz} = F_{Az} = 2F$$

$$\Sigma M_{Sx} = 0 \qquad M_x + M_{ukx} = 0$$
$$M_x = -M_{ukx} = -2Fa$$

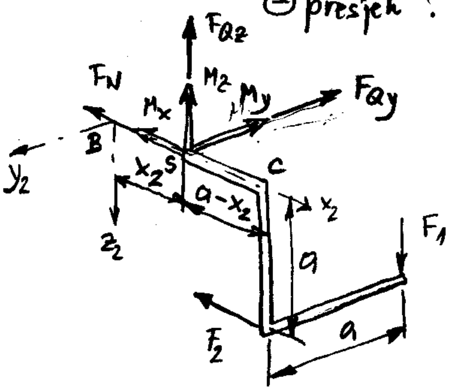
$$\Sigma M_{Sy} = 0 \qquad M_y + M_{uky} + F_{Az} \cdot x_1 = 0$$
$$M_y = -M_{uky} + F_{Az} \cdot x_1 =$$
$$= -3Fa + 2Fx_1$$

$$M_{yA} = M_y(x_1 = 0) = -3Fa$$
$$M_{yB} = M_y(x_1 = 2a) = -3Fa + 2F \cdot 2a = Fa$$

$$\Sigma M_{Sz} = 0 \qquad M_z - M_{ukz} + F_{Ay} x_1 = 0$$
$$M_z = M_{ukz} - F_{Ay} \cdot x_1 =$$
$$= 2Fa - Fx_1$$
$$M_{zA} = M_z(x_1 = 0) = 2Fa$$
$$M_{zB} = M_z(x_1 = 2a) = 2Fa - F \cdot 2a = 0$$

B – C

⊖ presjek !



$$\Sigma F_x = 0 \quad -F_N - F_2 = 0$$

$$F_N = -F_2 = -F$$

$$\Sigma F_y = 0 \quad -F_{Qy} = 0 \Longrightarrow F_{Qy} = 0$$

$$\Sigma F_z = 0 \quad -F_{Qz} + F_1 = 0$$

$$F_{Qz} = F_1 = F$$

$$\Sigma M_{Sx} = 0 \quad -M_x - F_1 \cdot a = 0$$

$$M_x = -F_1 a = -Fa$$

$$\Sigma M_{Sy} = 0 \quad -M_y - F_1 \cdot (a - x_2) - F_2 \cdot a = 0$$

$$M_y = -F_1(a - x_2) - F_2 a =$$

$$= -2Fa + Fx_2$$

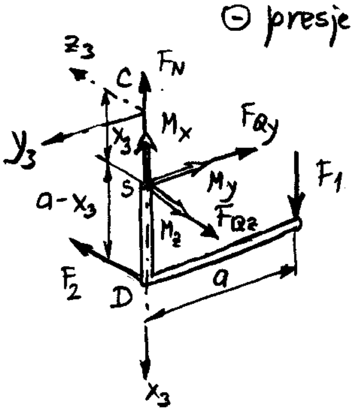
$$M_{yB} = M_y(x_2 = 0) = -2Fa$$

$$M_{yC} = M_y(x_2 = a) = -2Fa + Fa = -Fa$$

$$\Sigma M_{Sz} = 0 \quad -M_z = 0 \qquad M_z = 0$$

C – D

⊖ presjek !



$$\Sigma F_x = 0 \quad -F_N + F_1 = 0.$$

$$F_N = F_1 = F$$

$$\Sigma F_y = 0 \quad -F_{Qy} = 0 \rightarrow F_{Qy} = 0$$

$$\Sigma F_z = 0 \quad -F_{Qz} + F_2 = 0$$

$$F_{Qz} = F_2 = F$$

$$\Sigma M_{Sx} = 0 \quad -M_x = 0 \rightarrow M_x = 0$$

$$\Sigma M_{Sy} = 0 \quad -M_y - F_2 \cdot (a - x_3) = 0$$

$$M_y = -F_2(a - x_3) = -Fa + Fx_3$$

$$M_{yC} = M_y(x_3 = 0) = -Fa$$

$$M_{yD} = M_y(x_3 = a) = -Fa + Fa = 0$$

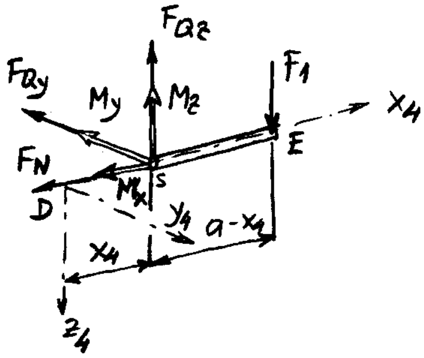
$$\Sigma M_{Sz} = 0 \quad -M_z + F_1 a = 0$$

$$M_z = F_1 a = Fa$$

4

D - E

⊖ presjek !



$\Sigma F_x = 0 \quad -F_N = 0 \Rightarrow F_N = 0$

$\Sigma F_y = 0 \quad -F_{Qy} = 0 \Rightarrow F_{Qy} = 0$

$\Sigma F_z = 0 \quad -F_{Qz} + F_1 = 0$

$F_{Qz} = F_1 = F$

$\Sigma M_{Sx} = 0 \quad -M_x = 0 \rightarrow M_x = 0$

$\Sigma M_{Sy} = 0 \quad -M_y - F_1(a - x_4) = 0$

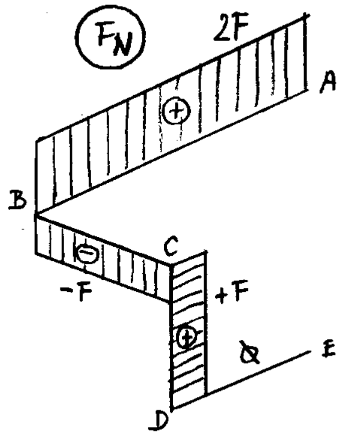
$M_y = -F_1(a - x_4) = -Fa + Fx_4$

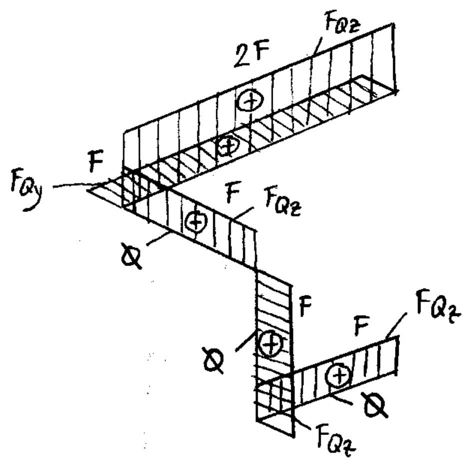
$M_{yD} = M_y(x_4 = 0) = -Fa$

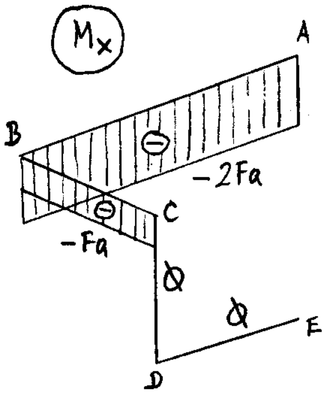
$M_{yE} = M_y(x_4 = a) = -Fa + Fa = 0$

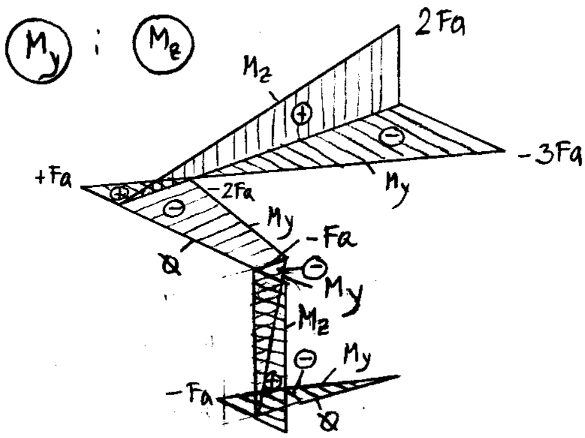
$\Sigma M_{Sz} = 0 \quad -M_z = 0 \Rightarrow M_z = 0$

DIJAGRAMI

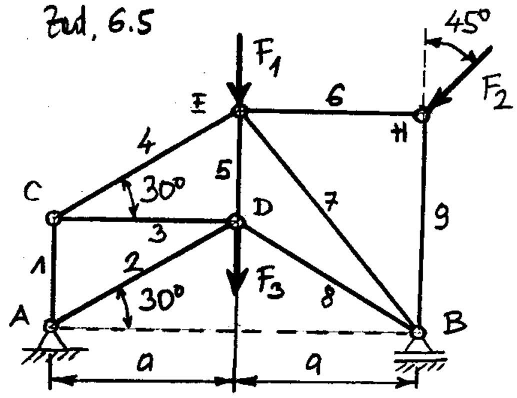






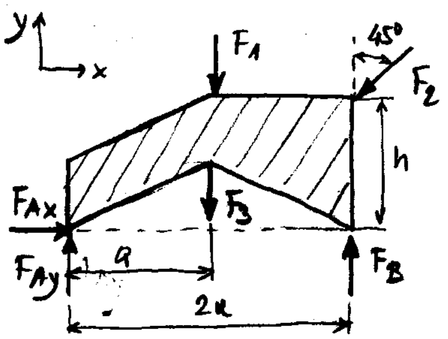


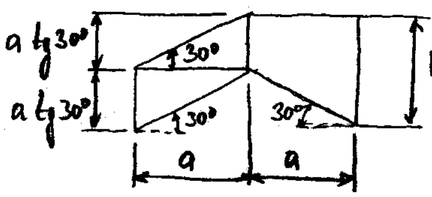
Zad. 6.5



Za rešetkasti nosač zadan i opterećen prema slici treba odrediti reakcije u osloncima i sile u štapovima.

Zadano: $F_1 = 3\ kN$, $F_2 = 2\ kN$, $F_3 = 3\ kN$, $a = 3\ m$.





$$h = 2a\,tg\,30° = 2a\cdot\frac{\sqrt{3}}{3}$$

(1) $\quad \Sigma F_x = 0 \qquad F_{Ax} - F_2 \sin 45° = 0 \quad \Longrightarrow \quad \underline{F_{Ax}} = F_2 \sin 45° = 2\cdot\frac{\sqrt{2}}{2} = \underline{1{,}414\ kN}$

(2) $\quad \Sigma F_y = 0 \qquad F_{Ay} + F_B - F_1 - F_2 \cos 45° - F_3 = 0$

(3) $\quad \Sigma M_A = 0 \qquad F_B\cdot 2a - F_1\cdot a - F_3\cdot a - F_2 \cos 45°\cdot 2a + F_2 \sin 45°\cdot h = 0$

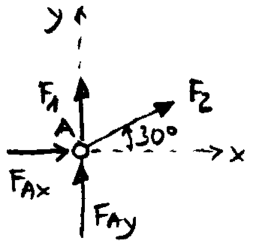
$$F_B = \frac{1}{2a}\left(F_1 a + F_3 a + F_2\frac{\sqrt{2}}{2}\cdot 2a - F_2\frac{\sqrt{2}}{2}\cdot 2a\frac{\sqrt{3}}{3}\right) =$$

$$= \frac{1}{2}F_1 + \frac{1}{2}F_3 + \frac{\sqrt{2}}{2}F_2\left(1 - \frac{\sqrt{3}}{3}\right) = \frac{3}{2} + \frac{3}{2} + \frac{\sqrt{2}}{2}\cdot 2\left(1 - \frac{\sqrt{3}}{3}\right)$$

$$\underline{F_B = 3{,}598\ kN}$$

$$F_{Ay} = F_1 + F_2\frac{\sqrt{2}}{2} + F_3 - F_B = 3 + 2\frac{\sqrt{2}}{2} + 3 - 3{,}598$$

$$\underline{F_{Ay} = 3{,}816\ kN}$$



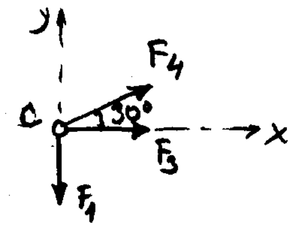
$\Sigma F_x = 0 \qquad F_2 \cos 30° + F_{Ax} = 0$

$$F_2 = -\frac{F_{Ax}}{\cos 30°} = -\frac{1{,}414}{\frac{\sqrt{3}}{2}} = -1{,}633\ kN$$

$\Sigma F_y = 0 \qquad F_1 + F_2 \sin 30° + F_{Ay} = 0$

$$F_1 = -F_{Ay} - F_2 \sin 30° = -3{,}816 - (-1{,}633)\cdot\frac{1}{2}$$

$$F_1 = -2{,}9996 \approx -3{,}0\ kN$$

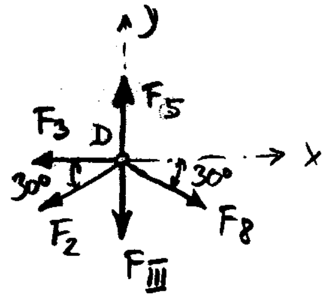


$\Sigma F_y = 0 \qquad F_4 \sin 30^\circ - F_1 = 0$

$$F_4 = \frac{F_1}{\sin 30^\circ} = \frac{-3{,}0}{\frac{1}{2}} = -6{,}0 \text{ kN}$$

$\Sigma F_x = 0 \qquad F_3 + F_4 \cos 30^\circ = 0$

$$F_3 = -F_4 \cos 30^\circ = -(-6) \cdot \frac{\sqrt{3}}{2} = 5{,}196 \text{ kN}$$



$F_{\underline{III}}$ ... OZNAKA ZA SILU OPTEREĆENJA

$$F_{\underline{III}} = F_3 = 3 \text{ kN}$$

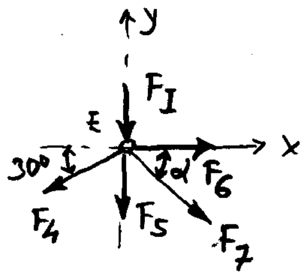
$\Sigma F_x = 0 \qquad F_8 \cos 30^\circ - F_2 \cos 30^\circ - F_3 = 0$

$$F_8 = F_2 + \frac{F_3}{\cos 30^\circ} = -1{,}633 + \frac{5{,}196}{\frac{\sqrt{3}}{2}} = 4{,}367 \text{ kN}$$

$\Sigma F_y = 0 \qquad F_5 - F_2 \sin 30^\circ - F_8 \sin 30^\circ - F_{\underline{III}} = 0$

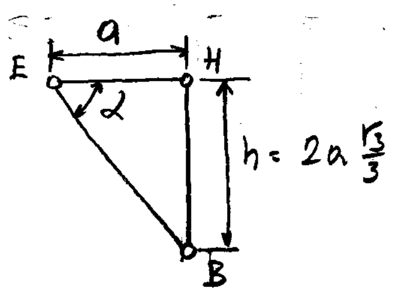
$$F_5 = F_2 \sin 30^\circ + F_8 \sin 30^\circ + F_{\underline{III}} =$$

$$= -1{,}633 \cdot \frac{1}{2} + 4{,}367 \cdot \frac{1}{2} + 3 = 4{,}367 \text{ kN}$$



$F_{\underline{I}}$ ... OZNAKA ZA SILU OPTEREĆENJA

$$F_{\underline{I}} = F_1 = 3 \text{ kN}$$



$$\overline{EB} = \sqrt{a^2 + \left(2a\frac{\sqrt{3}}{3}\right)^2} = a\sqrt{\frac{7}{3}}$$

$$\sin\alpha = \frac{2a\frac{\sqrt{3}}{3}}{a\sqrt{\frac{7}{3}}} = \frac{2}{\sqrt{7}} = 0{,}7559$$

$$\cos\alpha = \frac{a}{a\sqrt{\frac{7}{3}}} = \sqrt{\frac{3}{7}} = 0{,}65465$$

$\Sigma F_y = 0 \qquad -F_7 \sin\alpha - F_5 - F_4 \sin 30^\circ - F_I = 0$

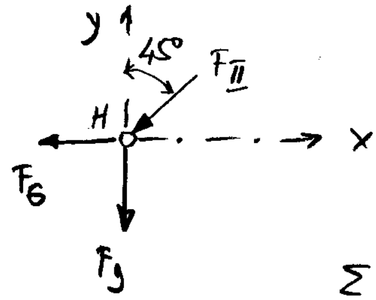
$$F_7 = -\frac{1}{\sin\alpha}\left(F_5 + F_4 \sin 30^\circ + F_I\right) =$$

$$= -\frac{1}{0{,}7559}\left[4{,}367 + (-6) \cdot 0{,}5 + 3\right] = -5{,}777 \text{ kN}$$

$\Sigma F_x = 0 \qquad F_6 + F_7 \cos\alpha - F_4 \cos 30^\circ = 0$

$$F_6 = -F_7 \cos\alpha + F_4 \cos 30^\circ = -(-5{,}777) \cdot 0{,}65465 + (-6) \cdot \frac{\sqrt{3}}{2} =$$

$$= -1{,}414 \text{ kN}$$



$F_{\overline{II}}$ ... OZNAKA ZA SILU OPTEREĆENJA

$F_{\overline{II}} = F_2 = 2\,kN$

$\Sigma F_x = 0 \quad -F_6 - F_{\overline{II}} \sin 45^\circ = 0$

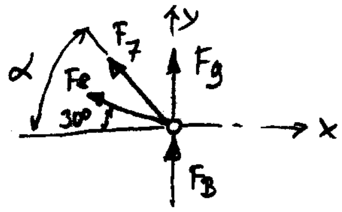
$F_6 = -F_{\overline{II}} \sin 45^\circ = -2 \cdot \frac{\sqrt{2}}{2} = -1{,}414\,kN$

DOBILI SMO ISTI REZULTAT!

$\Sigma F_y = 0 \quad -F_9 - F_{\overline{II}} \cos 45^\circ = 0$

$F_9 = -F_{\overline{II}} \cos 45^\circ = -2 \cdot \frac{\sqrt{2}}{2} = -1{,}414\,kN$

KONTROLA – RAVNOTEŽA ČVORA B :



$\Sigma F_x = 0$

$-F_7 \cos\alpha - F_8 \cos 30^\circ = 0$

$-(-5{,}777) \cdot 0{,}65465 - 4{,}367 \cdot \frac{\sqrt{3}}{2} = 0$

$3{,}7819 - 3{,}7819 = 0$ ✓

$\Sigma F_y = 0$

$F_7 \sin\alpha + F_8 \sin 30^\circ + F_9 + F_B = 0$

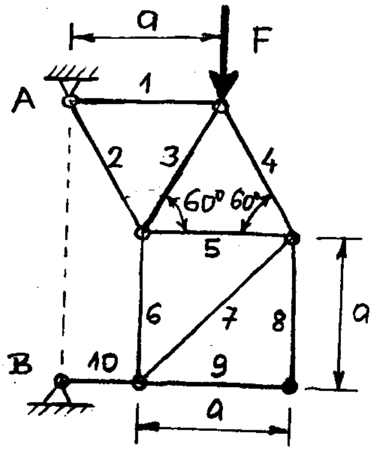
$-5{,}777 \cdot 0{,}7559 + 4{,}367 \cdot 0{,}5 - 1{,}414 + 3{,}598 = 0$

$-5{,}781 + 5{,}781 = 0$ ✓

ŠTAP	1	2	3	4	5	6	7	8	9
VLAK [kN]			5,196		4,367			4,367	
TLAK [kN]	3,0	1,633		6,0		1,414	5,777		1,414

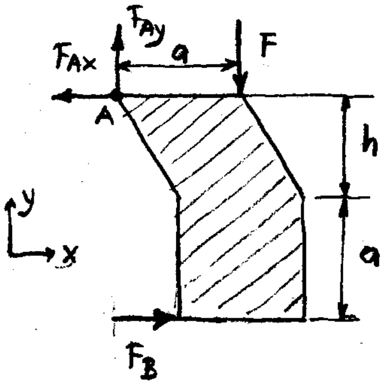
 1

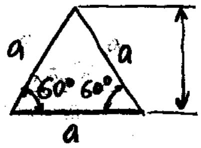
Zad. 6.19



Za rešetkasti nosač zadan i opterećen prema slici odrediti reakcije u osloncima i sile u štapovima. Sile u štapovima 4, 5 i 6 treba odrediti i grafički.

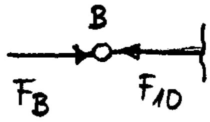
Zadano: $F = 10\ kN$.





$h = a \cdot \sin 60° = a \frac{\sqrt{3}}{2}$

REAKCIJA U B:



SILA U ŠTAPU 10, F_{10}, DJELUJE PO UZDUŽNOJ OSI ŠTAPA. IZ RAVNOTEŽE TOČKE B SLIJEDI DA REAKCIJA F_B MORA DJELOVATI PO ISTOM PRAVCU.

(1) $\Sigma F_x = 0 \qquad -F_{Ax} + F_B = 0$

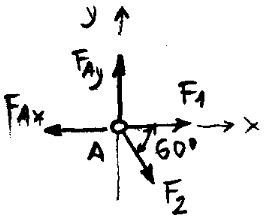
(2) $\Sigma F_y = 0 \qquad F_{Ay} - F = 0$

$$\underline{F_{Ay} = F = 10\ kN}$$

(3) $\Sigma M_A = 0 \qquad F_B \cdot (a+h) - F \cdot a = 0$

$$\underline{F_B} = F \frac{a}{a+h} = F \frac{\cancel{a}}{\cancel{a} + \cancel{a}\frac{\sqrt{3}}{2}} = 10 \frac{1}{1+\frac{\sqrt{3}}{2}} = \underline{5,359\ kN}$$

$(1) \Rightarrow \quad \underline{F_{Ax}} = F_B = \underline{5,359\ kN}$

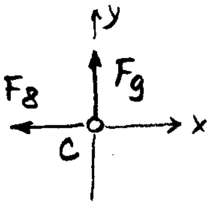


$\Sigma F_y = 0 \qquad -F_2 \sin 60° + F_{Ay} = 0$

$$F_2 = \frac{F_{Ay}}{\sin 60°} = \frac{10}{\frac{\sqrt{3}}{2}} = 11,547\ kN$$

$\Sigma F_x = 0 \qquad F_1 + F_2 \cos 60° - F_{Ax} = 0$

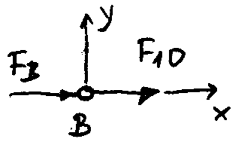
$$F_1 = F_{Ax} - F_2 \cos 60° = 5,359 - 11,547 \cdot 0,5 = -0,4145\ kN$$



$\Sigma F_x = 0 \qquad -F_8 = 0 \quad \Rightarrow \quad F_8 = 0$

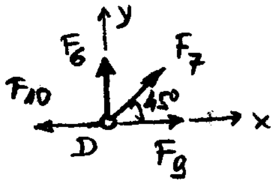
$\Sigma F_y = 0 \qquad F_9 = 0$

RAZMATRANJE RAVNOTEŽE ZGLOBA C NAKON ZGLOBA A POJEDNOSTAVLJUJE RAČUN !



$\Sigma F_x = 0 \qquad F_{10} + F_8 = 0$

$F_{10} = -F_8 = -5{,}359\ kN$



$\Sigma F_x = 0 \qquad F_7 \cos 45^\circ - F_{10} + F_9 = 0$

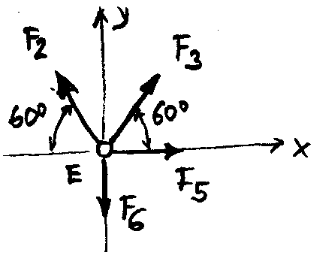
$F_7 = \frac{1}{\cos 45^\circ}(F_{10} - F_9) =$

$= \frac{1}{\frac{\sqrt{2}}{2}}(-5{,}359 - 0) = -7{,}579\ kN$

$\Sigma F_y = 0 \qquad F_6 + F_7 \sin 45^\circ = 0$

$F_6 = -F_7 \sin 45^\circ = -(-7{,}579)\cdot \frac{\sqrt{2}}{2} =$

$= 5{,}359\ kN$



$\Sigma F_y = 0 \qquad F_3 \sin 60^\circ + F_2 \sin 60^\circ - F_6 = 0$

$F_3 = -F_2 + \frac{F_6}{\sin 60^\circ} = -11{,}547 + \frac{5{,}359}{\frac{\sqrt{3}}{2}} =$

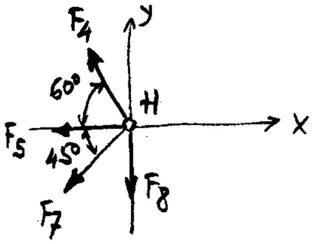
$= -5{,}359\ kN$

$\Sigma F_x = 0 \qquad F_5 + F_3 \cos 60^\circ - F_2 \cos 60^\circ = 0$

$F_5 = (F_2 - F_3)\cos 60^\circ =$

$= [11{,}547 - (-5{,}359)]\cdot 0{,}5 =$

$= 8{,}453\ kN$



$$\Sigma F_x = 0 \qquad -F_4 \cos 60° - F_5 - F_7 \cos 45° = 0$$

$$F_4 = -\frac{1}{\cos 60°}\left(F_5 + F_7 \cos 45°\right) =$$

$$= -\frac{1}{0{,}5}\left[8{,}453 + (-7{,}579)\cdot\frac{\sqrt{2}}{2}\right] =$$

$$= -6{,}188 \text{ kN}$$

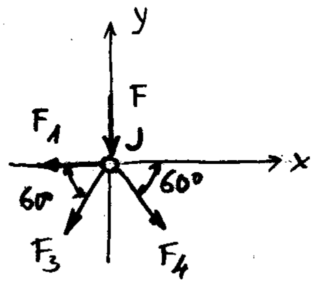
KONTROLA:

$$\Sigma F_y = 0 \qquad F_4 \sin 60° - F_7 \sin 45° - F_8 = 0$$

$$F_4 = \frac{1}{\sin 60°}\left(F_7 \sin 45° + F_8\right) =$$

$$= \frac{1}{\frac{\sqrt{3}}{2}}\left[(-7{,}579)\cdot\frac{\sqrt{2}}{2} + 0\right] =$$

$$= -6{,}188 \text{ kN} \checkmark$$



KONTROLA:

$$\Sigma F_x = 0 \qquad F_4 \cos 60° - F_1 - F_3 \cos 60° = 0$$

$$-6{,}188 \cdot 0{,}5 - (-0{,}4145) - (-5{,}359)\cdot 0{,}5 = 0$$

$$-3{,}094 + 3{,}094 = 0 \checkmark$$

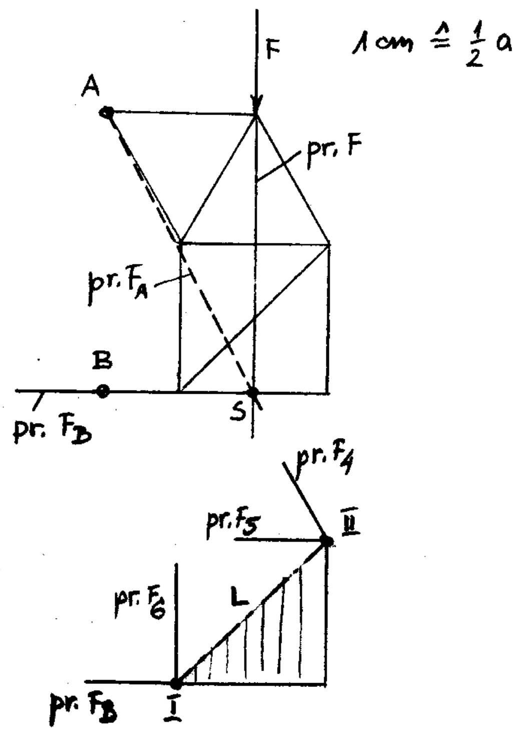
$$\Sigma F_y = 0 \qquad -F_3 \sin 60° - F_4 \sin 60° - F = 0$$

$$-(-5{,}359)\cdot\frac{\sqrt{3}}{2} - (-6{,}188)\cdot\frac{\sqrt{3}}{2} - 10 =$$

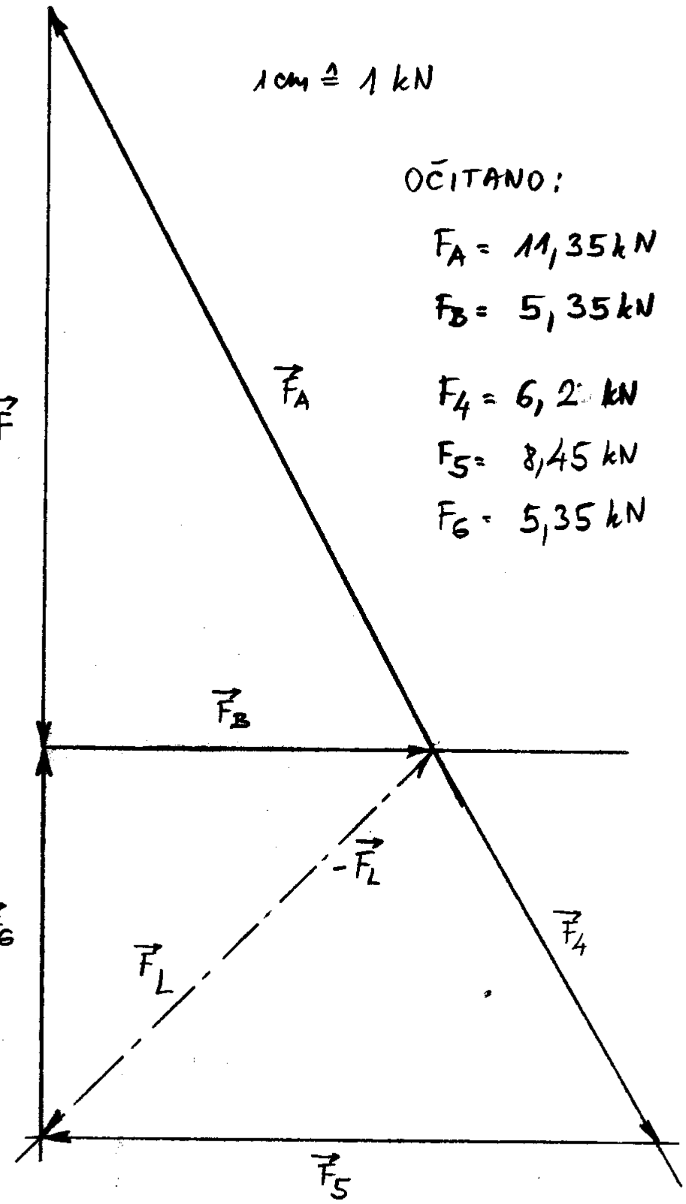
$$10 - 10 = 0 \checkmark$$

ŠTAP	1	2	3	4	5	6	7	8	9	10
VLAK [kN]		11,547			8,453	5,359		Ø	Ø	
TLAK [kN]	0,4145		5,359	6,188			7,579			5,359

PLAN POLOŽAJA:



PLAN SILA



OČITANO:

F_A = 11,35 kN
F_B = 5,35 kN

F_4 = 6,2 kN
F_5 = 8,45 kN
F_6 = 5,35 kN

Najprije se grafički odrede reakcije F_A i F_B pomoću metode trokuta sila. U planu položaja odredi se sjecište S pravca sile F i reakcije F_B koji su poznati, a zatim se kroz A i S ucrta pravac reakcije F_A. U planu sila nacrta se trokut koji prikazuje jednadžbu $\vec{F} + \vec{F}_B + \vec{F}_A = \vec{0}$.

Zatim se Culmannovom metodom odrede sile u štapovima 4, 5 i 6. U planu položaja (nacrtan je posebno, radi jasnoće) odrede se sjecišta I pravca F_B i F_6 i II pravca F_4 i F_5. Kroz I i II povuče se Culmannova linija L. U planu sila prikaže se najprije jednadžba $\vec{F}_B + \vec{F}_L + \vec{F}_6 = \vec{0}$, a zatim jednadžba $-\vec{F}_L + \vec{F}_4 + \vec{F}_5 = \vec{0}$.

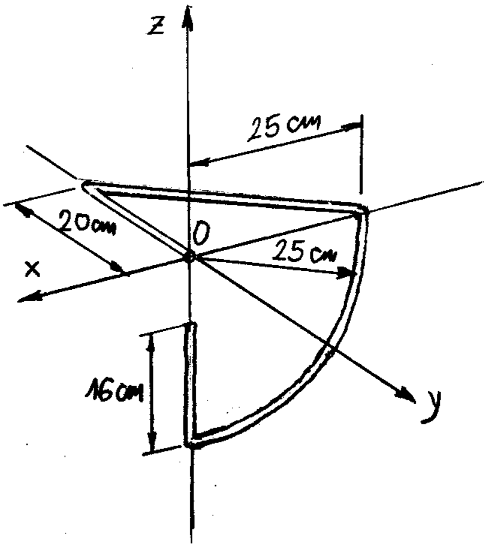
VJEŽBE IZ MEHANIKE I

III. DIO (TEŽIŠTA, MOMENTI INERCIJE)

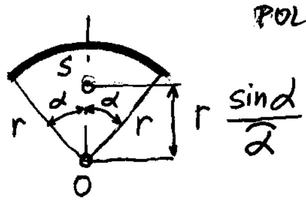
SASTAVIO prof. dr. sc. J. Saucha

Zad. 7.11

Za homogenu materijalnu liniju zadanu slikom treba odrediti koordinate težišta.

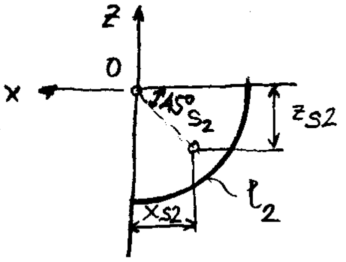


POLOŽAJ TEŽIŠTA KRUŽNOG LUKA



U zadatku $\alpha = 45^\circ$, $\hat{\alpha} = \frac{\pi}{4}$, $r = 25$ cm

$$\overline{OS_2} = r\,\frac{\sin 45^\circ}{\frac{\pi}{4}} = r\,\frac{\frac{\sqrt{2}}{2}}{\frac{\pi}{4}} = \frac{2\sqrt{2}}{\pi}\cdot 25 \text{ cm}$$

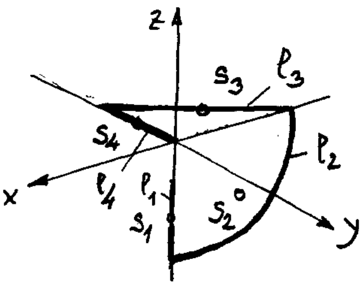


$$x_{S2} = -\overline{OS_2}\cdot\cos 45^\circ = -\frac{2\sqrt{2}}{\pi}\cdot 25\cdot\frac{\sqrt{2}}{2} = -\frac{50}{\pi} \text{ cm}$$

$$z_{S2} = -\overline{OS_2}\cdot\sin 45^\circ = -\frac{2\sqrt{2}}{\pi}\cdot 25\cdot\frac{\sqrt{2}}{2} = -\frac{50}{\pi} \text{ cm}$$

$$y_{S2} = 0$$

DULJINA LUKA $l_2 = r\cdot 2\hat{\alpha} = r\cdot 2\cdot\frac{\pi}{4} = 25\cdot\frac{\pi}{2} = \frac{25}{2}\pi$ cm



$l_1 = 16$ cm, $x_{S1} = 0$, $y_{S1} = 0$, $z_{S1} = -\left(25 - \frac{1}{2}\cdot 16\right) = -17$ cm

$l_3 = \sqrt{25^2 + 20^2} = 32{,}0156$ cm, $x_{S3} = -\frac{1}{2}\cdot 25 = -12{,}5$ cm, $y_{S3} = -\frac{1}{2}\cdot 20 = -10$ cm, $z_{S3} = 0$

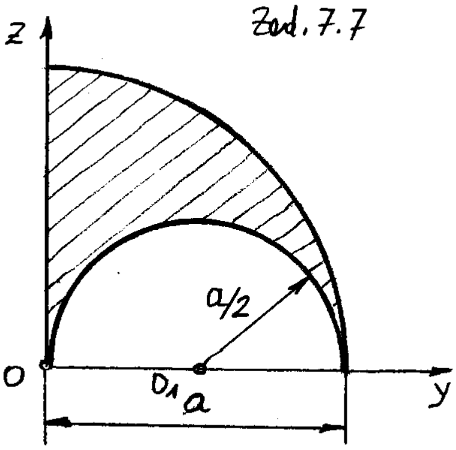
$l_4 = 20$ cm, $x_{S4} = 0$, $y_{S4} = -\frac{1}{2}\cdot 20 = -10$ cm, $z_{S4} = 0$

$$x_S = \frac{\sum_i x_i l_i}{\sum_i l_i} = \frac{0\cdot 16 + \left(-\frac{50}{\pi}\right)\cdot\frac{25}{2}\pi + (-12{,}5)\cdot 32{,}0156 + 0\cdot 20}{16 + \frac{25}{2}\pi + 32{,}0156 + 20} = -9{,}5558 \text{ cm}$$

$$y_S = \frac{\sum_i y_i l_i}{\sum_i l_i} = \frac{0\cdot 16 + 0\cdot\frac{25}{2}\pi + (-10)\cdot 32{,}0156 + (-10)\cdot 20}{16 + \frac{25}{2}\pi + 32{,}0156 + 20} = -4{,}8483 \text{ cm}$$

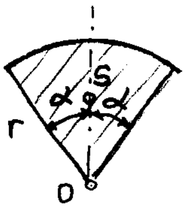
$$z_S = \frac{\sum_i z_i l_i}{\sum_i l_i} = \frac{(-17)\cdot 16 + \left(-\frac{50}{\pi}\right)\cdot\frac{25}{2}\pi + 0\cdot 32{,}0156 + 0\cdot 20}{16 + \frac{25}{2}\pi + 32{,}0156 + 20} = -8{,}3609 \text{ cm}$$

Zad. 7.7



Iz četvrtine kruga polumjera a izrezan je polukrug polumjera a/2 prema slici. Odrediti koordinate težišta preostale površine, ako je zadano:

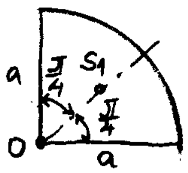
$a = 12$ cm.



KRUŽNI ISJEČAK

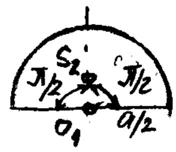
$$\overline{OS} = \frac{2}{3} r \, \frac{\sin\alpha}{\alpha}$$

Polukružni otvor u zadanoj površini smatramo negativnom površinom A_2:



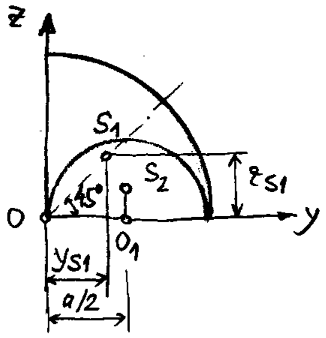
$$A_1 = \frac{1}{4} a^2 \pi$$

$$\overline{OS_1} = \frac{2}{3} a \cdot \frac{\sin 45^\circ}{\frac{\pi}{4}} = \frac{2}{3} a \cdot \frac{4}{\pi} \cdot \frac{\sqrt{2}}{2} = \frac{4\sqrt{2}}{3\pi} a$$



$$A_2 = -\frac{1}{2}\left(\frac{a}{2}\right)^2 \pi = -\frac{1}{8} a^2 \pi$$

$$\overline{O_1 S_2} = \frac{2}{3} \cdot \frac{a}{2} \cdot \frac{\sin 90^\circ}{\frac{\pi}{2}} = \frac{2}{3} \cdot \frac{a}{2} \cdot \frac{2}{\pi} \cdot 1 = \frac{2}{3\pi} a$$



$$A_1 = \frac{1}{4} a^2 \pi$$

$$y_{S1} = \overline{OS_1} \cos 45^\circ = \frac{4\sqrt{2}}{3\pi} a \cdot \frac{\sqrt{2}}{2} = \frac{4}{3\pi} a$$

$$z_{S1} = \overline{OS_1} \sin 45^\circ = \frac{4\sqrt{2}}{3\pi} a \cdot \frac{\sqrt{2}}{2} = \frac{4}{3\pi} a$$

$$A_2 = -\frac{1}{8} a^2 \pi$$

$$y_{S2} = \frac{a}{2}$$

$$z_{S2} = \overline{O_1 S_2} = \frac{2}{3\pi} a$$

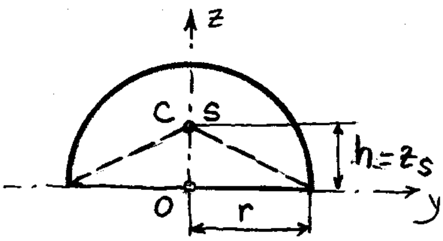
$$y_S = \frac{y_{S1} A_1 + y_{S2} A_2}{A_1 + A_2} = \frac{\frac{4}{3\pi} a \cdot \frac{1}{4} a^2 \pi + \frac{a}{2} \cdot \left(-\frac{1}{8} a^2 \pi\right)}{\frac{1}{4} a^2 \pi + \left(-\frac{1}{8} a^2 \pi\right)} = \frac{\frac{1}{3\pi} - \frac{1}{16}}{\frac{1}{8}} a =$$

$$= \left(\frac{8}{3\pi} - \frac{1}{2}\right) a = 0{,}3488\, a = 4{,}186 \text{ cm}$$

$$z_S = \frac{z_{S1} A_1 + z_{S2} A_2}{A_1 + A_2} = \frac{\frac{4}{3\pi} a \cdot \frac{1}{4} a^2 \pi + \frac{2}{3\pi} a \left(-\frac{1}{8} a^2 \pi\right)}{\frac{1}{4} a^2 \pi + \left(-\frac{1}{8} a^2 \pi\right)} = \frac{\frac{1}{3\pi} - \frac{1}{12\pi}}{\frac{1}{8}} a =$$

$$= \frac{2}{\pi} a = 0{,}6366\, a = 7{,}639 \text{ cm}$$

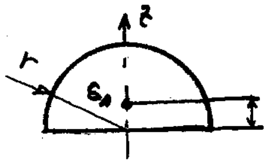
Zad. 7.5



Iz polukugle polumjera r treba izrezati pravilni stožac, čija je osnovica zajednička s osnovicom polukugle, kako je prikazano na slici. Kolika mora biti visina h stošca, ako njegov vrh C mora biti istovremeno i težište preostalog tijela?

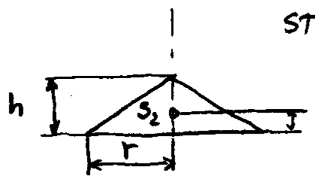
Zadano tijelo je polukugla sa šupljinom u obliku stošca. Smatrat ćemo ga složenim tijelom koje se sastoji od polukugle i negativnog volumena u obliku stošca.

POLUKUGLA



$$z_{S1} = \frac{3}{8} r \qquad V_1 = \frac{1}{2} \cdot \underbrace{\frac{4}{3} r^3 \pi}_{\text{VOLUMEN KUGLE}} = \frac{2}{3} r^3 \pi$$

STOŽAC



$$z_{S2} = \frac{h}{4} \qquad V_2 = -\frac{1}{3} B \cdot h = -\frac{1}{3} r^2 \pi h$$

Za polukuglu sa stožastom šupljinom traži se da bude

$$z_S = h$$

$$z_S = \frac{z_{S1} V_1 + z_{S2} V_2}{V_1 + V_2}$$

$$h = \frac{\frac{3}{8} r \cdot \frac{2}{3} r^3 \pi + \frac{h}{4}\left(-\frac{1}{3} r^2 \pi h\right)}{\frac{2}{3} r^3 \pi + \left(-\frac{1}{3} r^2 \pi h\right)} = \frac{\frac{3}{4} r^2 - \frac{1}{4} h^2}{2r - h}$$

$$h = \frac{3r^2 - h^2}{4(2r - h)} \quad / \cdot 4(2r - h)$$

$$4h(2r-h) = 3r^2 - h^2$$

$$-4h^2 + 8rh + h^2 - 3r^2 = 0 \qquad / \cdot (-1)$$

$$3h^2 - 8rh + 3r^2 = 0$$

$$h_{1,2} = \frac{8r \pm \sqrt{64r^2 - 36r^2}}{6} = \frac{8 \pm \sqrt{28}}{6} r = \frac{\overset{4}{\cancel{8}} \pm \overset{1}{\cancel{2}}\sqrt{7}}{\underset{3}{\cancel{6}}} r$$

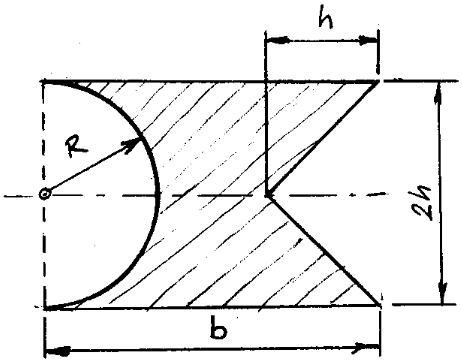
$$h_{1,2} = (4 \pm \sqrt{7}) \frac{r}{3}$$

$$h_1 = \frac{4-\sqrt{7}}{3} r = 0,4514\, r$$

$$h_2 = \frac{4+\sqrt{7}}{3} r = 2,2153\, r$$

Drugi korijen nema fizikalnog smisla pa je rješenje zadatka korijen h_1 :

$$h = \frac{4-\sqrt{7}}{3} r = 0,4514\, r$$

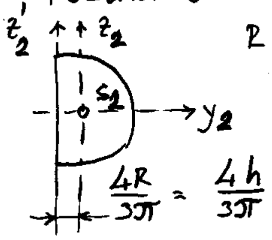


Za presjek zadan slikom treba odrediti položaj težišta i glavne težišne momente inercije.

Zadano: $b = 8$ cm, $h = 3$ cm.

Zadani presjek ima jednu os simetrije. Težište leži na toj osi, a to je ujedno i jedna od glavnih težišnih osi inercije. Početni koordinatni sustav postavljamo tako da os y' leži na simetrali, a os z' poleži lijevim rubom presjeka. Zadani presjek promatramo kao pravokutnik s polukružnim il trokutnim otvorom. Pri određivanju položaja težišta površinama otvora dajemo NEGATIVAN predznak, a pri određivanju momenata inercije NEGATIVAN predznak dajemo momentima inercije otvora.

POLUKRUG



$R = h$

$$A_2 = \frac{R^2\pi}{2} = \frac{h^2\pi}{2}$$

$$\underline{I_{y2}} = \frac{1}{2} I_{y} = \frac{1}{2}\frac{D^4\pi}{64} = \frac{1}{2}\cdot\frac{(2h)^4\pi}{64} = \underline{\frac{h^4\pi}{8}}$$

$$I'_{z2} = \frac{1}{2} I_{z} = \frac{1}{2}\frac{D^4\pi}{64} = \frac{h^4\pi}{8}$$

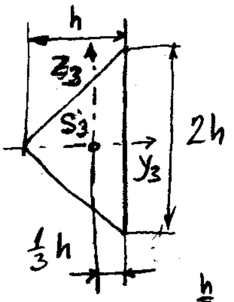
STEINEROVO PRAVILO: $I'_{z2} = I_{z2} + \left(\frac{4R}{3\pi}\right)^2 \cdot |A_1|$

$$\Rightarrow \underline{I_{z2}} = I'_{z2} - \left(\frac{4R}{3\pi}\right)^2 \cdot |A_1| = \frac{h^4\pi}{8} - \left(\frac{4h}{3\pi}\right)^2 \cdot \frac{h^2\pi}{2} =$$

$$= \frac{h^4\pi}{8} - \frac{8h^4}{9\pi} = \frac{h^4\pi}{8}\left[1 - \left(\frac{8}{3\pi}\right)^2\right] =$$

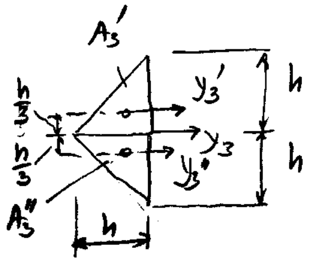
$$= \underline{\frac{h^4\pi}{8} \cdot 0{,}2795}$$

TROKUT



$$\underline{A_3} = \frac{1}{2}\cdot 2h\cdot h = \underline{h^2}$$

$$\underline{I_{z3}} = \frac{(2h)\cdot h^3}{36} = \underline{\frac{h^4}{18}}$$



STEINEROVO PRAVILO:

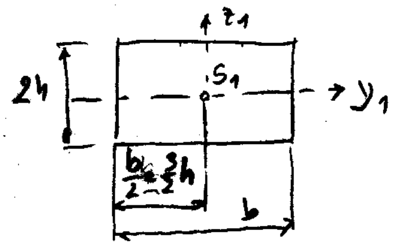
$$\underline{I_{y3}} = \left[I_{y3}' + \left(\frac{h}{3}\right)^2 A_3'\right] + \left[I_{y3}'' + \left(\frac{h}{3}\right)^2 A_3''\right] =$$

$$= \left[\frac{h\cdot h^3}{36} + \frac{h^2}{9}\cdot\frac{h\cdot h}{2}\right] + \left[\frac{h\cdot h^3}{36} + \frac{h^2}{9}\cdot\frac{h\cdot h}{2}\right] =$$

$$= \frac{3}{36}h^4 + \frac{3}{36}h^4 = \frac{1}{12}h^4 + \frac{1}{12}h^4 =$$

$$= \underline{\frac{h^4}{6}}$$

PRAVOKUTNIK

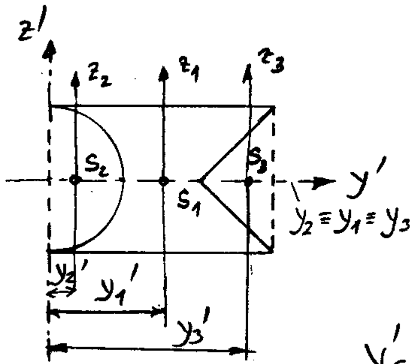


$$b = 3h$$

$$\underline{A_1} = b\cdot 2h = 3h\cdot 2h = \underline{6h^2}$$

$$\underline{I_{y1}} = \frac{b(2h)^3}{12} = \frac{3h\cdot 8h^3}{12} = \underline{2h^4}$$

$$\underline{I_{z1}} = \frac{2h\cdot b^3}{12} = \frac{2h\,(3h)^3}{12} = \frac{2h\cdot 27h^3}{12} = \underline{\frac{9}{2}h^4}$$



$$y_1' = \frac{3}{2}h, \quad z_1' = 0, \quad A_1 = 6h^2$$

$$y_2' = \frac{4h}{3\pi}, \quad z_2' = 0, \quad A_2 = -\frac{\pi}{2}h^2$$

$$y_3' = \left(b - \frac{1}{3}h\right) = 3h - \frac{1}{3}h = \frac{8}{3}h, \quad z_3' = 0, \quad A_3 = -h^2$$

$$\underline{y_s'} = \frac{y_1' A_1 + y_2' A_2 + y_3' A_3}{A_1 + A_2 + A_3} =$$

$$= \frac{\frac{3}{2}h\cdot 6h^2 + \frac{4}{3\pi}h\left(-\frac{\pi}{2}h^2\right) + \frac{8}{3}h\left(-h^2\right)}{6h^2 + \left(-\frac{\pi}{2}h^2\right) + \left(-h^2\right)} =$$

$$= \frac{9 - \frac{2}{3} - \frac{8}{3}}{6 - \frac{\pi}{2} - 1}\,h = \frac{\frac{17}{3}}{\frac{10-\pi}{2}}\,h = \frac{34}{3(10-\pi)}\,h =$$

$$= 1{,}65247\,h = 4{,}9574\ \text{cm} \approx \underline{4{,}96\ \text{cm}}$$

$$\underline{z_s'} = \underline{0}$$

$$I_y' = (I_{y1} + 0^2 \cdot A_1) - (I_{y2} + 0^2 \cdot A_2) - (I_{y3} + 0^2 \cdot A_3) =$$
$$= I_{y1} - I_{y2} - I_{y3} = 2h^4 - \frac{h^4\pi}{8} - \frac{h^4}{6} = 1{,}440634\, h^4 =$$
$$= 116{,}691\ \text{cm}^4$$

Os y' identična je osi y kroz težište zadanog presjeka, pa slijedi

$$\underline{I_y} = I_y' = \underline{116{,}691\ \text{cm}^4}$$

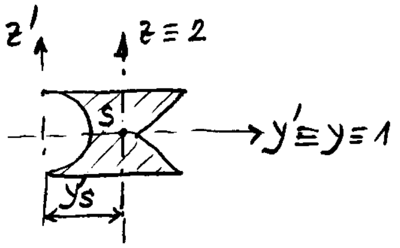
$$I_z' = (I_{z1} + y_1'^2 |A_1|) - (I_{z2} + y_2'^2 |A_2|) - (I_{z3} + y_3'^2 |A_3|) =$$
$$= \left[\frac{9}{2}h^4 + \left(\frac{3}{2}h\right)^2 \cdot 3h^2\right] - \left[\frac{h^4\pi}{8} \cdot 0{,}2795 + \left(\frac{4}{3\pi}h\right)^2 \cdot \frac{\pi}{2}h^2\right] -$$
$$- \left[\frac{h^4}{18} + \left(\frac{8}{3}h\right) \cdot h^2\right]$$

Izraz u drugoj uglatoj zagradi dat će rezultat $I_{z2}' = \frac{h^4\pi}{8}$ za polukrug:

$$\frac{h^4\pi}{8} \cdot 0{,}2795 = \frac{h^4\pi}{8}\left[1 - \left(\frac{8}{3\pi}\right)^2\right] = \frac{h^4\pi}{8} - \frac{8h^4}{9\pi} =$$
$$= \frac{h^4\pi}{8} - \left(\frac{4h}{3\pi}\right)^2 \cdot \frac{\pi}{2}h^2$$

$$\Rightarrow \frac{h^4\pi}{8} \cdot 0{,}2795 + \left(\frac{4}{3\pi}h\right)^2 \frac{\pi}{2}h^2 = \frac{h^4\pi}{8} - \left(\frac{4h}{3\pi}\right)^2 \frac{\pi}{2}h^2 + \left(\frac{4h}{3\pi}\right)^2 \frac{\pi}{2}h^2 =$$
$$= \frac{h^4\pi}{8}$$

$$I_z' = \left(\frac{9}{2} + \frac{27}{2}\right)h^4 - \frac{\pi}{8}h^4 - \left(\frac{1}{18} + \frac{64}{9}\right)h^4 =$$
$$= \left(18 - \frac{\pi}{8} - \frac{129}{18}\right)h^4 = 10{,}44063\, h^4$$



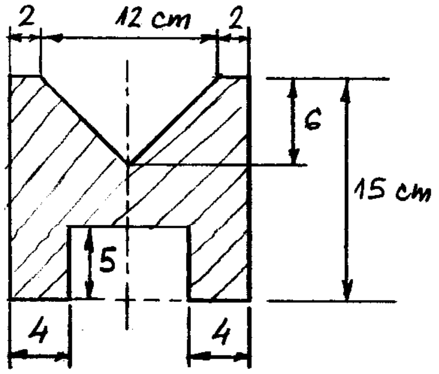
STEINEROVO PRAVILO:

$$I_z' = I_z + y_S'^2 A, \quad A = A_1 - A_2 - A_3$$

$$\Rightarrow \underline{I_z} = I_z' - y_S'^2 (A_1 - A_2 - A_3) = 10{,}44063\, h^4 -$$
$$- (1{,}65247\, h)^2 \left(6h^2 - \frac{\pi}{2}h^2 - h^2\right) =$$
$$= 1{,}0766506\, h^4 = \underline{87{,}209\ \text{cm}^4}$$

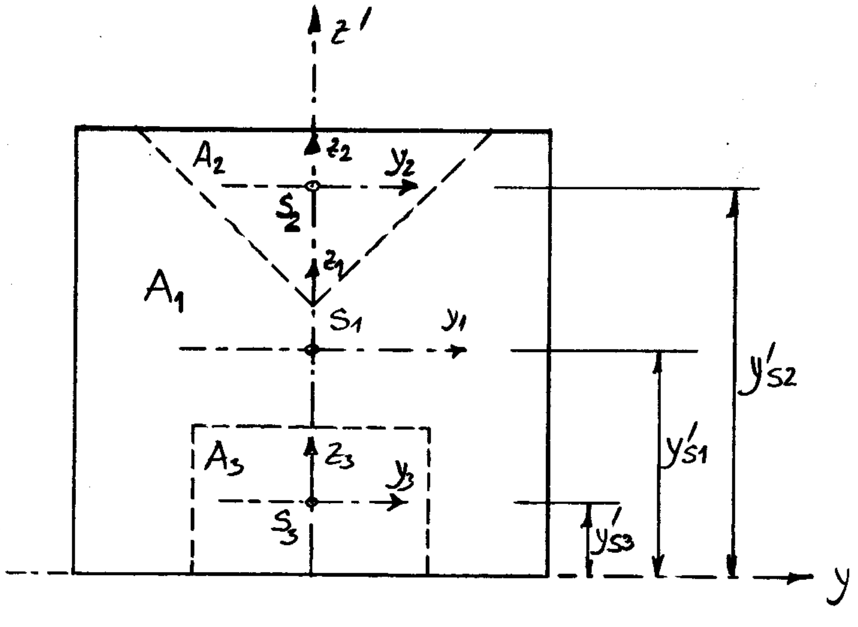
Osi y i z su GLAVNE TEŽIŠNE OSI, tj. $y \equiv 1$, $z \equiv 2$, a I_y i I_z su GLAVNI TEŽIŠNI MOMENTI INERCIJE: $I_y = I_1$, $I_z = I_2$

Zad. 7.19



Za presjek zadan slikom treba odrediti položaj težišta i glavne težišne momente inercije.

Zadani presjek ima jednu os simetrije. Težište S presjeka leži na toj simetrali. Os simetrije je jedna od glavnih težišnih osi inercije. Početni koordinatni sustav $y'z'$ postavljamo tako da os z' leži na simetrali, a os y' prolazi donjim rubom presjeka. Zadani presjek promatramo kao pravokutnik s trokutnim i pravokutnim otvorom. Pri određivanju položaja težišta površinama otvora dajemo NEGATIVNI PREDZNAK, a pri određivanju momenata inercije NEGATIVAN PREDZNAK dajemo momentima inercije otvora.



$A_1 = 16 \cdot 15 = 240 \text{ cm}^2$

$y'_{S1} = 0, \quad z'_{S1} = \frac{15}{2} = 7{,}5 \text{ cm}$

$A_2 = -\left(\frac{12 \cdot 6}{2}\right) = -36 \text{ cm}^2$

$y'_{S2} = 0, \quad z'_{S2} = 15 - \frac{1}{3} 6 = 13 \text{ cm}$

$A_3 = -(8 \cdot 5) = -40 \text{ cm}^2$

$y'_{S3} = 0, \quad z'_{S3} = \frac{5}{2} = 2{,}5 \text{ cm}$

$$y'_S = 0$$

$$z'_S = \frac{z'_{S1} A_1 + z'_{S2} A_2 + z'_{S3} A_3}{A_1 + A_2 + A_3} = \frac{7{,}5 \cdot 240 + 13 \cdot (-36) + 2{,}5 \cdot (-40)}{240 + (-36) + (-40)}$$

$$z'_S = 7{,}5122 \text{ cm}$$

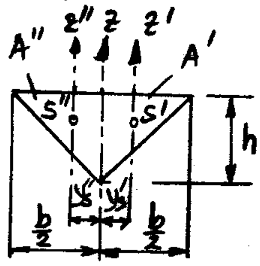
Puni pravokutni presjek i oba otvora simetrični su s obzirom na osi z_i koje prolaze njihovim težištima pa su centrifugalni momenti inercije

$$I_{y_i' z_i'} = 0$$

Za pravokutnike su

$$I_{y_i} = \frac{b_i h_i^3}{12}, \quad I_{z_i} = \frac{h_i b_i^3}{12}$$

Za trokut je $I_y = \dfrac{bh^3}{36}$, a I_z određujemo primjenom Steinerovog pravila kako slijedi:



$$I_z = \left(I_z' + y_s'^2 A'\right) + \left(I_z'' + y_s''^2 \cdot A''\right) =$$
$$= \left[\frac{h\cdot\left(\frac{b}{2}\right)^3}{36} + \left(\frac{1}{3}\cdot\frac{b}{2}\right)^2 \cdot \frac{h\frac{b}{2}}{2}\right] +$$
$$+ \left[\frac{h\cdot\left(\frac{b}{2}\right)^3}{36} + \left(-\frac{1}{3}\cdot\frac{b}{2}\right)\cdot\frac{h\cdot\frac{b}{2}}{2}\right] =$$
$$= 2\,hb^3\left(\frac{1}{288} + \frac{1}{144}\right) = \frac{3}{144}hb^3 = \frac{hb^3}{48}$$

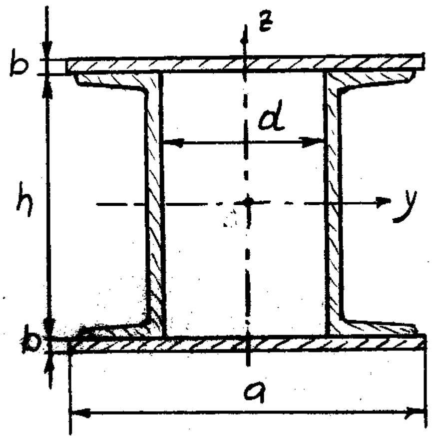
$$I_y' = \left(I_{y1} + z_{s1}'^2 A_1\right) - \left(I_{y2} + z_{s2}'^2 |A_2|\right) - \left(I_{y3} + z_s'^2 |A_3|\right) =$$
$$= \left(\frac{16\cdot 15^3}{12} + 7{,}5^2\cdot 240\right) - \left(\frac{12\cdot 6^3}{36} + 13^2\cdot|-36|\right) - \left(\frac{8\cdot 5^3}{12} + 2{,}5\cdot|-40|\right) =$$
$$= 11\,510{,}67\ \text{cm}^4$$

$$I_y = I_y' - z_s^2\left(A_1 + A_2 + A_3\right) = 11\,510{,}67 - 7{,}5122^2\left[240 + (-36) + (-40)\right] =$$
$$= 2255{,}63\ \text{cm}^4$$

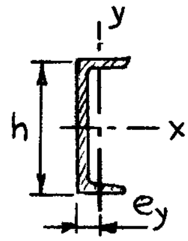
$$I_z = I_z' = \left(I_{z1} + \emptyset\right) - \left(I_{z2} + \emptyset\right) - \left(I_{z3} + \emptyset\right) =$$
$$= \frac{15\cdot 16^3}{12} - \frac{6\cdot 12^3}{48} - \frac{5\cdot 8^3}{12} = 4690{,}63\ \text{cm}^4$$

$$\longrightarrow \quad I_z = I_1 = I_{max}, \quad I_y = I_2 = I_{min}$$

Zad. 7.27



Dva profila [NP 180 spojena su s dvije lamelne ploče širine $a = 24$ cm i debljine $b = 1$ cm prema slici. Treba izračunati razmak između profila ($d = ?$), ako se traži da sastavljeni presjek zadovoljava uvjet $I_y = I_z$.



Iz tablica:

[NP 180

$A = 28\ cm^2$

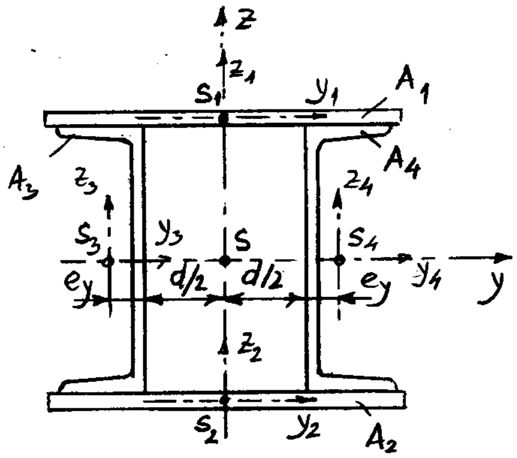
$h = 180\ mm$

$e_y = 1{,}92\ cm$

$I_x = 1350\ cm^4$

$I_y = 114\ cm^4$

Osi y i z su osi simetrije presjeka. Težište presjeka je u njihovom sjecištu. I_y i I_z određujemo Steinerovim pravilom.



$A_1 = A_2 = a \cdot b = 24 \cdot 1 = 24\ cm^2$

$y_{S1} = 0, \quad z_{S1} = \frac{h}{2} + \frac{b}{2} = \frac{18}{2} + \frac{1}{2} = 9{,}5\ cm$

$y_{S2} = 0, \quad z_{S2} = -\left(\frac{h}{2} + \frac{b}{2}\right) = -9{,}5\ cm$

$A_3 = A_4 = 28\ cm^2$

$y_{S3} = -\left(e_y + \frac{d}{2}\right) = -\left(1{,}92 + \frac{d}{2}\right), \quad z_{S3} = 0$

$y_{S4} = e_y + \frac{d}{2} = 1{,}92 + \frac{d}{2}, \quad z_{S4} = 0$

$$I_y = \left(I_{y1} + z_{S1}^2 A_1\right) + \left(I_{y2} + z_{S2}^2 A_2\right) + \left(I_{y3} + z_{S3}^2 A_3\right) + \left(I_{y4} + z_{S4}^2 A_4\right) =$$

$$= \left(\frac{24 \cdot 1^3}{12} + 9{,}5 \cdot 24\right) + \left[\frac{24 \cdot 1^3}{12} + (-9{,}5)^2 \cdot 24\right] +$$

$$+ \left(1350 + 0\right) + \left(1350 + 0\right) = 7036\ cm^4$$

$$I_z = (I_{z1} + y_{s1}^2 A_1) + (I_{z2} + y_{s2}^2 A_2) + (I_{z3} + y_{s3}^2 A_3) + (I_{z4} + y_{s4}^2 A_4) =$$

$$= \left(\frac{1 \cdot 24^3}{12} + 0\right) + \left(\frac{1 \cdot 24^3}{12} + 0\right) +$$

$$+ \left\{114 + \left[-\left(1{,}92 + \frac{d}{2}\right)\right]^2 \cdot 28\right\} + \left[114 + \left(1{,}92 + \frac{d}{2}\right)^2 \cdot 28\right] =$$

$$= 2532 + \left(1{,}92 + \frac{d}{2}\right)^2 \cdot 56$$

U zadatku se traži da bude $I_z = I_y$:

$$\Rightarrow 2532 + \left(1{,}92 + \frac{d}{2}\right)^2 \cdot 56 = 7036$$

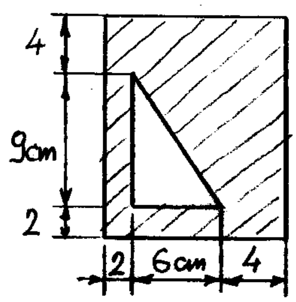
$$\left(1{,}92 + \frac{d}{2}\right)^2 = \frac{7036 - 2532}{56} = 80{,}42857$$

$$\frac{d}{2} = \sqrt{80{,}42857} - 1{,}92$$

$$\frac{d}{2} = 7{,}0482$$

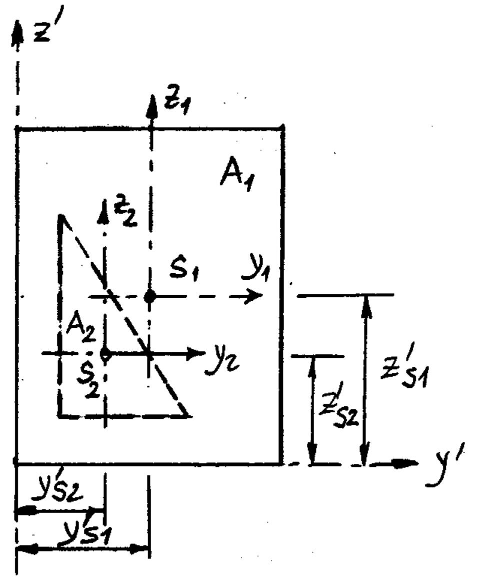
$$d = 14{,}0964 \text{ cm} \Rightarrow \underline{\underline{d = 141 \text{ mm}}}$$

Zad. 7.21



Za zadani presjek odrediti položaj težišta, glavne težišne momente inercije i glavne težišne osi inercije.

Osi y' i z' početnog koordinatnog sustava položit ćemo po donjem, odnosno po lijevom rubu presjeka. Zadani presjek promatramo kao pravokutnik s trokutnim otvorom.



Pravokutnik

$b_1 = 2+6+4 = 12$ cm

$h_1 = 2+9+4 = 15$ cm

$A_1 = b_1 h_1 = 180 \text{ cm}^2$

$y'_{S1} = \frac{1}{2} b_1 = 6$ cm , $z'_{S1} = \frac{1}{2} h_1 = 7{,}5$ cm

Trokutni otvor

$b_2 = 6$ cm , $h_2 = 9$ cm

$A_2 = -\frac{1}{2} b_2 h_2 = -\frac{1}{2} \cdot 6 \cdot 9 = -27 \text{ cm}^2$

$y'_{S2} = 2 + \frac{1}{3} \cdot b_2 = 2 + \frac{1}{3} \cdot 6 = 4$ cm

$z'_{S2} = 2 + \frac{1}{3} \cdot h_2 = 2 + \frac{1}{3} \cdot 9 = 5$ cm

$$y'_S = \frac{y'_{S1} A_1 + y'_{S2} A_2}{A_1 + A_2} = \frac{6 \cdot 180 + 4 \cdot (-27)}{180 + (-27)} = 6{,}353 \text{ cm}$$

$$z'_S = \frac{z'_{S1} A_1 + z'_{S2} A_2}{A_1 + A_2} = \frac{7{,}5 \cdot 180 + 5 \cdot (-27)}{180 + (-27)} = 7{,}941 \text{ cm}$$

$$I_y' = \left(\frac{b_1h_1^3}{12} + z_{s1}'^2 A_1\right) - \left(\frac{b_2h_2^3}{36} + z_{s2}'^2 |A_2|\right) =$$

$$= \left(\frac{12\cdot15^3}{12} + 7{,}5^2\cdot180\right) - \left(\frac{6\cdot9^3}{36} + 5^2\cdot|-27|\right) =$$

$$= 12703{,}5 \text{ cm}^4$$

$$I_z' = \left(\frac{h_1b_1^3}{12} + y_{s1}'^2 A_1\right) - \left(\frac{h_2b_2^3}{36} + y_{s2}'^2 |A_2|\right) =$$

$$= \left(\frac{15\cdot12^3}{12} + 6^2\cdot180\right) - \left(\frac{9\cdot6^3}{36} + 4^2|-27|\right) =$$

$$= 8154 \text{ cm}^4$$

$$I_{yz}' = \left(0 + y_{s1}' z_{s1}' A_1\right) - \left(-\frac{b_2^2h_2^2}{72} + y_{s2}' z_{s2}' |A_2|\right) =$$

$$= \left(0 + 6\cdot7{,}5\cdot180\right) - \left(-\frac{6^2 9^2}{72} + 4\cdot5\cdot|-27|\right) =$$

$$= 7600{,}5 \text{ cm}^4$$

$$I_y = I_y' - z_s'^2(A_1+A_2) = 12703{,}5 - 7{,}941^2\,[180+(-27)] = 3055{,}4 \text{ cm}^4$$

$$I_z = I_z' - y_s'^2(A_1+A_2) = 8154 - 6{,}353^2\,[180+(-27)] = 1978{,}8 \text{ cm}^4$$

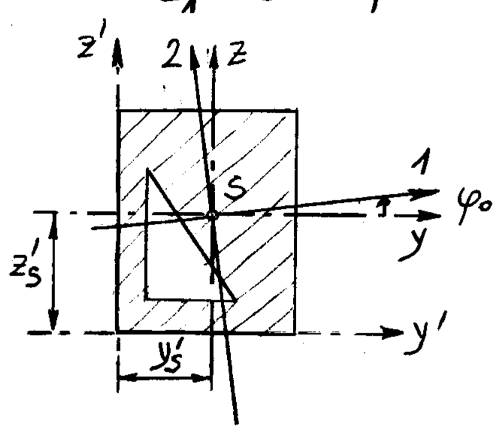
$$I_{yz} = I_{yz}' - y_s' z_s'(A_1+A_2) = 7600{,}5 - 6{,}353\cdot7{,}941\,[180+(-27)] =$$

$$= -118{,}2 \text{ cm}^4$$

$$I_{1,2} = \frac{I_y+I_z}{2} \pm \sqrt{\left(\frac{I_y-I_z}{2}\right)^2 + I_{yz}^2} =$$

$$= \frac{3055{,}4+1978{,}8}{2} \pm \sqrt{\left(\frac{3055{,}4-1978{,}8}{2}\right)^2 + (-118{,}2)^2}$$

$$I_1 = 3068{,}2 \text{ cm}^4 \quad , \quad I_2 = 1966 \text{ cm}^4$$



$$\operatorname{tg} 2\varphi_0 = -\frac{2I_{yz}}{I_y - I_z} =$$

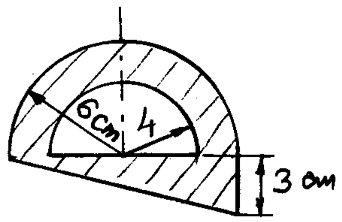
$$= -\frac{2(-118{,}2)}{3055{,}4-1978{,}8} =$$

$$= 0{,}21962$$

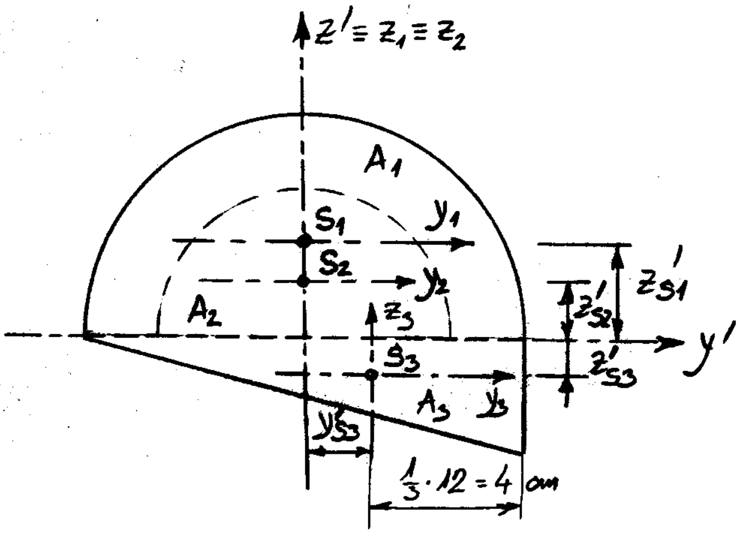
$$2\varphi_0 = 12{,}3868^\circ$$

$$\varphi_0 = 6{,}193^\circ$$

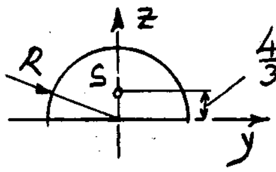
Zad. 7.25



Za zadani presjek odrediti položaj težišta, glavne težišne momente inercije i položaj glavnih težišnih osi inercije.



Os z' početnog koordinatnog sustava postavljamo po simetrali polukrugova, a os y' po njihovoj bazi. Zadani presjek promatramo kao presjek sastavljen od polukruga i trokuta s polukružnim otvorom.



$$I_y = \frac{1}{2}\,\frac{R^4\pi}{4}$$
$$I_z = \frac{1}{2}\,\frac{R^4\pi}{4}$$
$$A = \frac{1}{2}R^2\pi$$

$$A_1 = \frac{1}{2}\cdot 6^2\pi = 18\pi\ \text{cm}^2$$
$$y'_{s1} = 0, \quad z'_{s1} = \frac{4\cdot 6}{3\pi} = \frac{8}{\pi}\ \text{cm}$$
$$A_2 = -\frac{1}{2}\,4^2\pi = -8\pi\ \text{cm}^2$$
$$y'_{s2} = 0, \quad z'_{s2} = \frac{4\cdot 4}{3\pi} = \frac{16}{3\pi}\ \text{cm}$$

$$A_3 = \frac{1}{2}\,b_3 h_3 = \frac{1}{2}\cdot 12\cdot 3 = 18\ \text{cm}^2$$
$$y'_{s3} = 6 - \frac{1}{3}\cdot 12 = 6 - 4 = 2\ \text{cm}$$
$$z'_{s3} = -\frac{1}{3}\cdot 3 = -1\ \text{cm}$$

$$y'_s = \frac{y'_{s1}A_1 + y'_{s2}A_2 + y'_{s3}A_3}{A_1 + A_2 + A_3} = \frac{0 + 0 + 2\cdot 18}{18\pi + (-8\pi) + 18} = 0{,}7285\ \text{cm}$$

$$z'_s = \frac{z'_{s1}A_1 + z'_{s2}A_2 + z'_{s3}A_3}{A_1 + A_2 + A_3} = \frac{\frac{8}{\pi}\cdot 18\pi + \frac{16}{3\pi}\cdot(-8\pi) + (-1)\cdot 18}{18\pi + (-8\pi) + 18} =$$
$$= 1{,}6864\ \text{cm}$$

Momente inercije polukrugova s obzirom na os y' nećemo računati primjenom Steinerovog pravila, već direktno primjenom izraza za aksijalni moment inercije polukruga s obzirom na os koja prolazi njegovom bazom, a to je jednako polovici momenta inercije punog kruga s obzirom na os kroz središte.

$$I_y' = I_{y1}' - I_{y2}' + \left(I_{y3} + z_{S3}'^{2} \cdot A_3 \right) =$$

$$= \frac{1}{2}\frac{6^4\pi}{4} - \frac{1}{2}\frac{4^4\pi}{4} + \left[\frac{12 \cdot 3^3}{36} + (-1)^2 \cdot 18 \right] = 435{,}407 \text{ cm}^4$$

$$I_z' = (I_{z1} + 0) + (I_{z2} + 0) + (I_{z3} + y_{S3}'^{2} A_3) =$$

$$= \frac{1}{2}\frac{6^4\pi}{4} - \frac{1}{2}\frac{4^4\pi}{4} + \left(\frac{3 \cdot 12^3}{36} + 2^2 \cdot 18 \right) = 624{,}407 \text{ cm}^4$$

$$I_{yz}' = (\underbrace{I_{y1z1}}_{\substack{=0 \\ z_1 \text{ os sim.!}}} + 0) - (\underbrace{I_{y2z2}}_{\substack{=0 \\ z_2 \text{ os sim.!}}} + 0) + (I_{y3z3} + y_{S3}' z_{S3}' A_3) =$$

$$= -\frac{12^2 \cdot 3^2}{72} + 2 \cdot (-1) \cdot 18 = -54 \text{ cm}^4$$

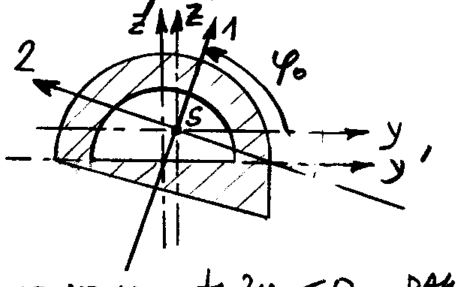
$$I_y = I_y' - z_S'^2 (A_1 + A_2 + A_3) = 435{,}407 - 1{,}6864^2 \cdot 49{,}41593 = 294{,}87 \text{ cm}^4$$

$$I_z = I_z' - y_S'^2 (A_1 + A_2 + A_3) = 624{,}407 - 0{,}7285^2 \cdot 49{,}41593 = 598{,}18 \text{ cm}^4$$

$$I_{yz} = I_{yz}' - y_S' z_S' (A_1 + A_2 + A_3) = -54 - 0{,}7285 \cdot 1{,}6864 \cdot 49{,}41593 =$$

$$= -114{,}71 \text{ cm}^4$$

$$I_{1,2} = \frac{I_y + I_z}{2} \pm \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + I_{yz}^2} = \frac{294{,}87 + 598{,}18}{2} \pm \sqrt{\left(\frac{294{,}87 - 598{,}18}{2}\right)^2 + (-114{,}71)^2}$$



$$\Rightarrow I_1 = 636{,}677 \text{ cm}^4, \quad I_2 = 256{,}373 \text{ cm}^4$$

$$\operatorname{tg} 2\varphi_0 = -\frac{2 I_{yz}}{I_y - I_z} = -\frac{2(-114{,}71)}{294{,}87 - 598{,}18} =$$

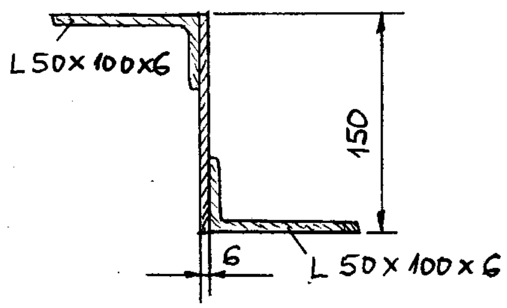
$$= -0{,}75639 \Rightarrow \overline{2\varphi_0} = -37{,}1035^\circ$$

$$2\varphi_0 = 180^\circ - |\overline{2\varphi_0}| = 142{,}8965^\circ$$

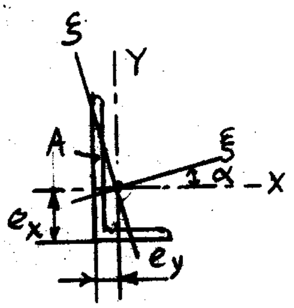
$$\varphi_0 = 71{,}4483^\circ$$

NAPOMENA: $\operatorname{tg} 2\varphi_0 < 0$, DAKLE $2\varphi_0$ JE KUT IZ 2. KVADRANTA. KUT $\overline{2\varphi_0}$ JE KUT IZ 4. KVADRANTA S ISTIM TANGENSOM!

Zad. 7.26



Presjek zadan slikom sastavljen je od pravokutnog profila 150 x 6 mm i dva kutna profila L 50x100x 6 mm. Odrediti položaj težišta, glavne težišne momente inercije i položaj glavnih težišnih osi inercije.



Za profil L 100x50x6 u tablicama valjanih profila nalazimo slijedeće podatke:

$A = 8{,}73\ cm^2$, $e_x = 3{,}49\ cm$, $e_y = 1{,}04\ cm$ (e_x i e_y određuju položaj težišta),

$I_X = 89{,}7\ cm^4$, $I_Y = 15{,}3\ cm^4$, $tg\,\alpha = 0{,}263$.

Kut α određuje položaj osi ξ koja je os 1, tj. glavna težišna os inercije;

Dakle, mora biti

$$tg\,2\alpha = -\frac{2 I_{XY}}{I_X - I_Y}.$$

i odavde

$$I_{XY} = -\frac{1}{2}(I_X - I_Y)\,tg\,2\alpha$$

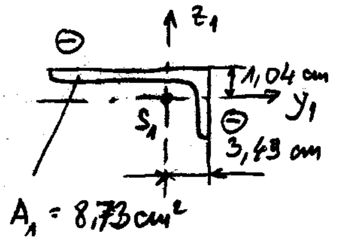
$$tg\,2\alpha = \frac{2\,tg\,\alpha}{1 - tg^2\alpha} \quad \text{(TANGENS DVOSTRUKOG KUTA!)}$$

$$\Rightarrow tg\,2\alpha = \frac{2 \cdot 0{,}263}{1 - 0{,}263^2} = 0{,}56509$$

$$I_{XY} = -\frac{1}{2}(89{,}7 - 15{,}3) \cdot 0{,}56509 = -21{,}0212\ cm^4 \approx -21\ cm^4$$

Na ovaj se način iz podataka koji su dani u tablicama raznostraničnih L profila određuje centrifugalni moment inercije za težišne osi paralelne stranicama profila, tj. za osi X i Y.

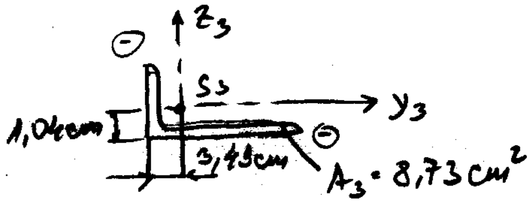
Izračunati I_{XY} je negativan jer veći dio presjeka leži u 2. i 4. kvadrantu koordinatnog sustava XY. Kad bi se profil postavio tako da mu veći dio presjeka bude u 1. i 3. kvadrantu koord. sustava XY, trebalo bi uzeti $I_{XY} = +21\ cm^4$.



Za profile, postavljene u složenom presjeku kako je zadano, vrijedi:

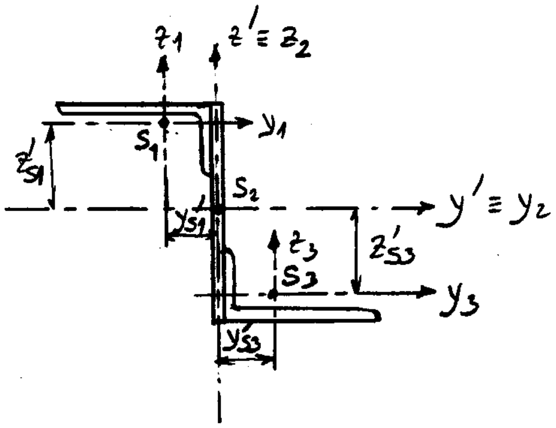
$I_{y1} = I_Y = 15,3\ cm^4$, $I_{z1} = I_X = 83,7\ cm^4$,

$I_{y1z1} = -21\ cm^4$



$I_{y3} = I_Y = 15,3\ cm^4$, $I_{z3} = I_X = 83,7\ cm^4$,

$I_{y1z1} = -21\ cm^4$



Početni koordinatni sustav $y'z'$ postavljamo po simetralama pravokutnog profila.

$A_1 = 8,73\ cm^2$

$y'_{S1} = -\left(\frac{0,6}{2} + 3,49\right) = -3,79\ cm$

$z'_{S1} = \frac{15}{2} - 1,04 = 6,46\ cm$

$A_2 = 0,6 \cdot 15 = 9\ cm^2$

$y'_{S2} = 0$, $z'_{S2} = 0$

$A_3 = 8,73\ cm^2$

$y'_{S3} = \frac{0,6}{2} + 3,49 = 3,79\ cm$

$z'_{S3} = -\left(\frac{15}{2} - 1,04\right) = -6,46\ cm$

$A_1 + A_2 + A_3 = 26,46\ cm^2$

$$y_s' = \frac{y_{s1}' A_1 + y_{s2}' A_2 + y_{s3}' A_3}{A_1 + A_2 + A_3} = \frac{(-3{,}79)\cdot 8{,}73 + 0 + 3{,}79 \cdot 8{,}73}{26{,}46} = \phi$$

$$z_s' = \frac{z_{s1}' A_1 + z_{s2}' A_2 + z_{s3}' A_3}{A_1 + A_2 + A_3} = \frac{6{,}46 \cdot 8{,}73 + 0 + (-6{,}46)\cdot 8{,}73}{26{,}46} = \phi$$

Dakle težišne osi y i z podudaraju se s osima y' i z'.

$$I_y = (I_{y1} + z_{s1}'^2 A_1) + (I_{y2} + z_{s2}'^2 A_2) + (I_{y3} + z_{s3}' A_3) =$$
$$= (15{,}3 + 6{,}46^2 \cdot 8{,}73) + \left(\frac{0{,}6 \cdot 15^3}{12} + \phi\right) + [15{,}3 + (-6{,}46)^2 \cdot 8{,}73] =$$
$$= 927{,}98 \text{ cm}^4$$

$$I_z = (I_{z1} + y_{s1}'^2 A_1) + (I_{z2} + y_{s2}'^2 A_2) + (I_{z3} + y_{s3}'^2 A_3) =$$
$$= [89{,}7 + (-3{,}79)^2 \cdot 8{,}73] + \left(\frac{15 \cdot 0{,}6^3}{12} + \phi\right) + (89{,}7 + 3{,}79^2 \cdot 8{,}73) =$$
$$= 430{,}47 \text{ cm}^4$$

$$I_{yz} = (I_{y1z1} + y_{s1}' z_{s1}' A_1) + (I_{y2z2} + y_{s2}' z_{s2}' A_2) + (I_{y3z3} + y_{s3}' z_{s3}' A_3) =$$
$$= [-21 + (-3{,}79)\cdot 6{,}46 \cdot 8{,}73] + (\phi + \phi) + [-21 + 3{,}79(-6{,}46)\cdot 8{,}73] =$$
$$= -470{,}46 \text{ cm}^4$$

$$I_{1,2} = \frac{I_y + I_z}{2} \pm \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + I_{yz}^2} =$$
$$= \frac{927{,}98 + 430{,}47}{2} \pm \sqrt{\left(\frac{927{,}98 - 430{,}47}{2}\right)^2 + (-470{,}46)^2}$$

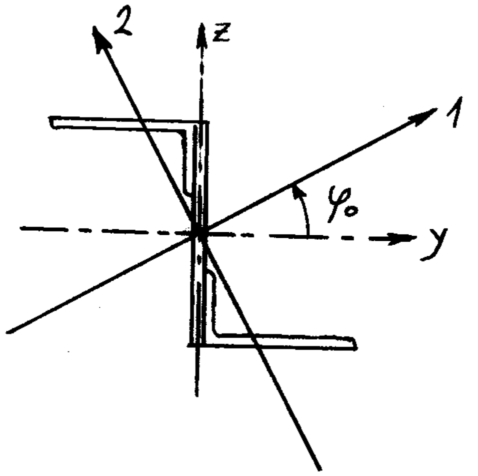
$$I_1 = 1211{,}40 \text{ cm}^4$$

$$I_2 = 147{,}05 \text{ cm}^4$$

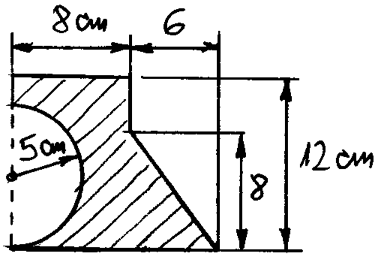
$$\operatorname{tg} 2\varphi_0 = -\frac{2\, I_{yz}}{I_y - I_z} = -\frac{2\,(-470{,}46)}{927{,}98 - 430{,}47} = 1{,}89126$$

$$\Rightarrow \quad 2\varphi_0 = 62{,}1324°$$

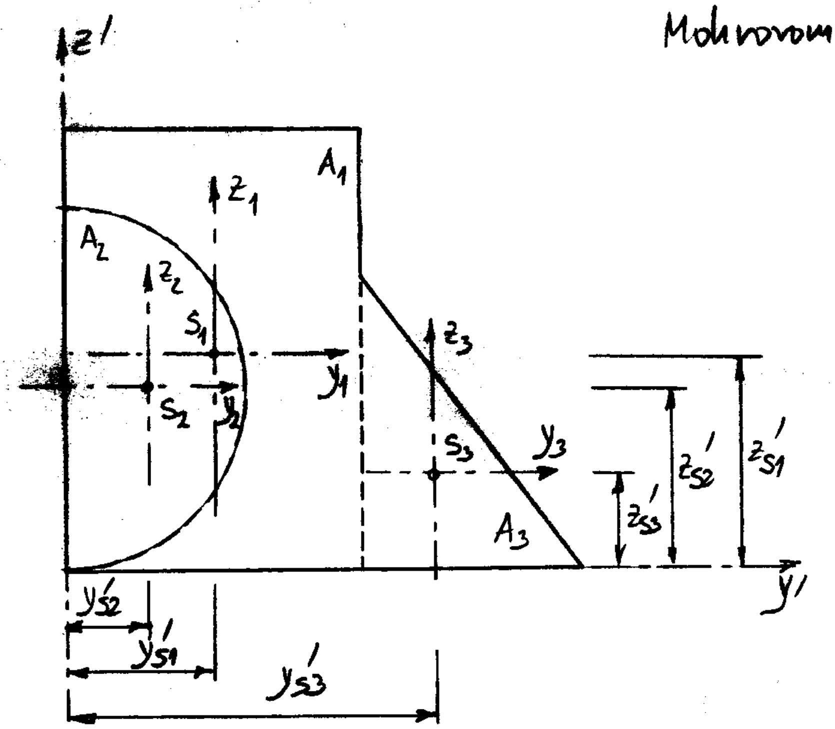
$$\varphi_0 = 31{,}066°$$



Zad. tip J, str. 254 modificiran



Za presjek zadan prema slici odrediti položaj težišta i glavne težišne momente inercije te položaj glavnih težišnih osi. Rezultate kontrolirati Mohrovom kružnicom inercije.



Zadani presjek promatramo kao presjek sastavljen od pravokutnika i trokuta s polukružnim otvorom. Osi y' i z' početnog koordinatnog sustava polažemo po lijevom i donjem rubu presjeka.

$A_1 = b_1 h_1 = 8 \cdot 12 = 96\ cm^2, \quad y'_{s1} = \frac{1}{2} b_1 = 4\ cm, \quad z'_{s1} = \frac{1}{2} h_1 = 6\ cm$

$A_2 = -\frac{1}{2} R^2 \pi = -39{,}27\ cm^2, \quad y'_{s2} = \frac{4R}{3\pi} = 2{,}122\ cm, \quad z'_{s1} = R = 5\ cm$

$A_3 = \frac{1}{2} b_2 h_2 = \frac{1}{2} \cdot 6 \cdot 8 = 24\ cm^2, \quad y'_{s3} = 8 + \frac{1}{3} b_2 = 8 + \frac{1}{3} \cdot 6 = 10\ cm,$

$z'_{s3} = \frac{1}{3} h_2 = \frac{1}{3} \cdot 8 = 2{,}667\ cm$

$$y'_s = \frac{y'_{s1} A_1 + y'_{s2} A_2 + y'_{s3} A_3}{A_1 + A_2 + A_3} = \frac{4 \cdot 96 + 2{,}122(-39{,}27) + 10 \cdot 24}{96 + (-39{,}27) + 24} =$$

$$= \frac{540{,}6667}{80{,}73} = 6{,}6972\ cm$$

$$z'_s = \frac{z'_{s1} A_1 + z'_{s2} A_2 + z'_{s3} A_3}{A_1 + A_2 + A_3} = \frac{6 \cdot 96 + 5(-39{,}27) + 2{,}667 \cdot 24}{96 + (-39{,}27) + 24} =$$

$$= \frac{443,6505}{80,73} = 5,4955 \text{ cm}$$

$$I_y' = \left(I_{y1} + z_{s1}'^2 A_1\right) - \left(I_{y2} + z_{s2}'^2 |A_2|\right) + \left(I_{y3} + z_{s3}'^2 A_3\right) =$$

$$= \left(\frac{8 \cdot 12^3}{12} + 6^2 \cdot 96\right) - \left(\frac{1}{2} \cdot \frac{5^4 \pi}{4} + 5^2 \cdot |-39,271|\right) +$$

$$+ \left(\frac{6 \cdot 8^3}{36} + 2,667^2 \cdot 24\right) = 3636,8154 \text{ cm}^4$$

$$I_z' = \left(I_{z1} + y_{s1}'^2 A_1\right) - \underbrace{\left(I_{z2} + y_{s2}'^2 |A_2|\right)}_{=(I_z')_2} + \left(I_{z3} + y_{s3}'^2 A_3\right) =$$

$$I_z' = \frac{1}{2} \frac{R^4 \pi}{4}$$

$$= \left(\frac{12 \cdot 8^3}{12} + 4^2 \cdot 96\right) - \left(\frac{1}{2} \cdot \frac{5^4 \pi}{4}\right) + \left(\frac{8 \cdot 6^3}{36} + 10^2 \cdot 24\right) =$$

$$= 4250,5631 \text{ cm}^4$$

$$I_{yz}' = \left(I_{y1z1} + y_{s1}' z_{s1}' A_1\right) - \left(I_{y2z2}' + y_{s2}' z_{s2}' |A_2|\right) +$$

$$+ \left(I_{y3z3} + y_{s3}' z_{s3}' A_3\right) =$$

$$= \left(0 + 4 \cdot 6 \cdot 96\right) - \left(0 + 2,122 \cdot 5 \cdot |-39,271|\right) +$$

$$+ \left(-\frac{6^2 8^2}{72} + 10 \cdot 2,667 \cdot 24\right) = 2495,3333 \text{ cm}^4$$

$$I_y = I_y' - z_s'^2 (A_1 + A_2 + A_3) = 3636,8154 - 5,4955^2 \cdot 80,73 =$$
$$= 1198,7266 \text{ cm}^4$$

$$I_z = I_z' - y_s'^2 (A_1 + A_2 + A_3) = 4250,5631^2 - 6,6972^2 \cdot 80,73 =$$
$$= 629,6176 \text{ cm}^4$$

$$I_{yz} = I_{yz}' - y_s' z_s' (A_1 + A_2 + A_3) = 2495,3333 - 6,6972 \cdot 5,4955 \cdot 80,73 =$$
$$= -475,8943 \text{ cm}^4$$

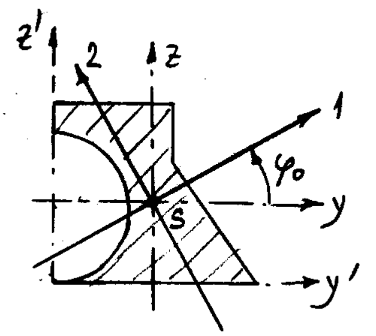
$$I_{1,2} = \frac{I_y + I_z}{2} \pm \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + I_{yz}^2} =$$

$$= \frac{1198,7246 + 629,6176}{2} \pm \sqrt{\left(\frac{1198,7246 - 629,6176}{2}\right)^2 + (-475,8943)^2}$$

$$I_1 = 1468,6493\ cm^4, \qquad I_2 = 359,6929\ cm^4$$

$$tg\,2\varphi_0 = -\frac{2\,I_{yz}}{I_y - I_z} = -\frac{2\cdot(-475,8943)}{1198,7246 - 629,6176} = 1,672425$$

$$\Rightarrow 2\varphi_0 = 59,1234^\circ \rightarrow \varphi_0 = 29,5617^\circ$$



KONSTRUKCIJA MOHROVE KRUŽNICE INERCIJE

1. Odaberemo prikladno mjerilo $1\,cm \hat{=} 100\,cm^4$.
2. U koordinatni sustav I_y, I_{yz} ucrtamo točke $A(I_y, I_{yz})$, tj. $A(1198,7246;\ -475,8943)$ i $B(I_z, -I_{yz})$, tj. $B[629,6176;\ -(-475,8943)]$.
3. Točke A i B spojimo dužinom koja sijeće os apscise, tj. os I_y u središtu S Mohrove kružnice.
4. Ucrtamo kružnicu sa središtem u S, koja prolazi kroz A i B.
5. Kružnica siječe os I_y u točkama 1 i 2 čije apscise su I_1, odnosno I_2.

6. Kroz točku A povučemo pravac paralelan osi apscisa, tj. osi I_y. Ta os siječe Mohrovu kružnicu u točki P, polu Mohrove kružnice.

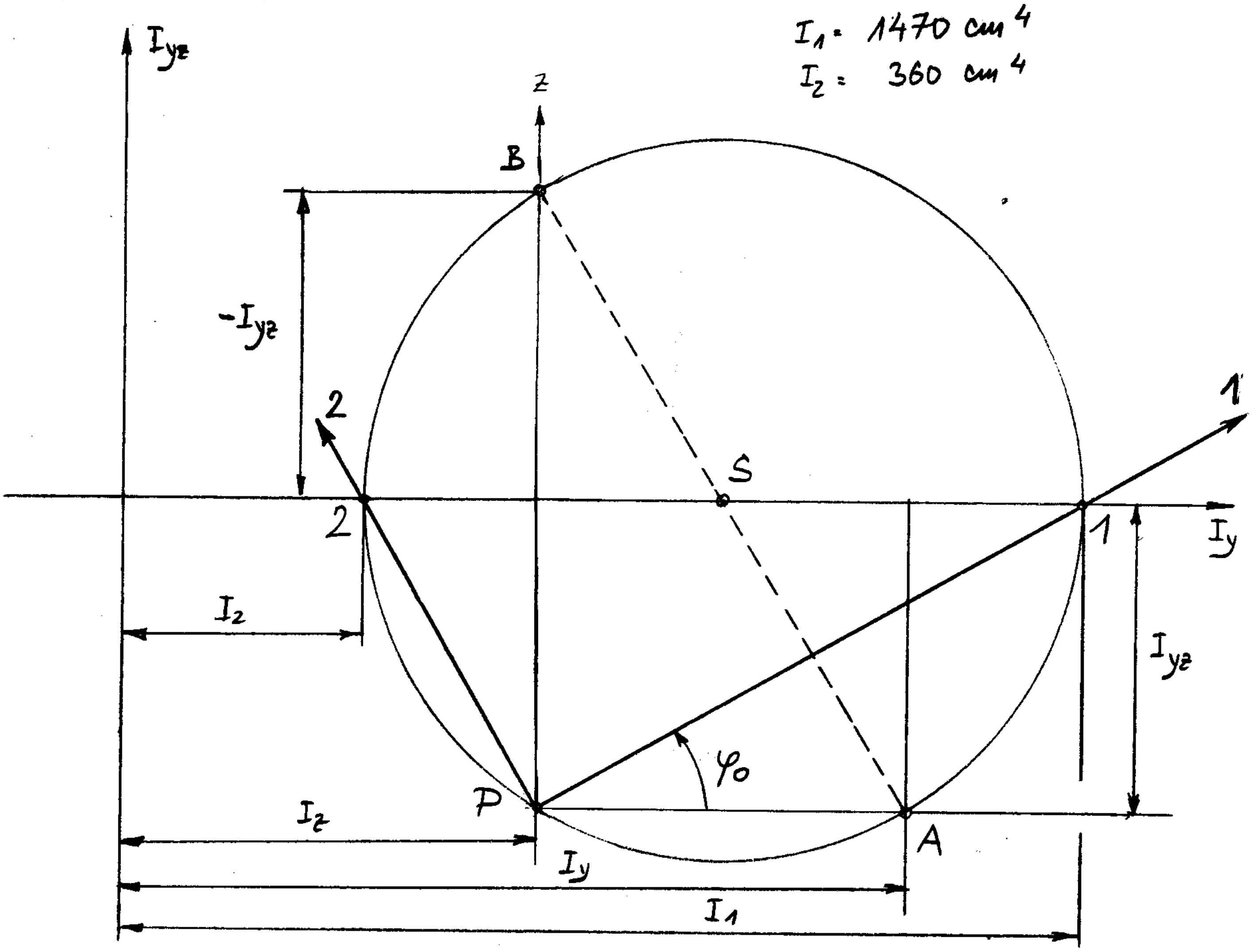
7. Iz P kroz B povlačimo os z koja je okomita na os y.

8. Iz P kroz točku 1 povlačimo glavnu težišnu os 1, a kroz točku 2 glavnu težišnu os 2.

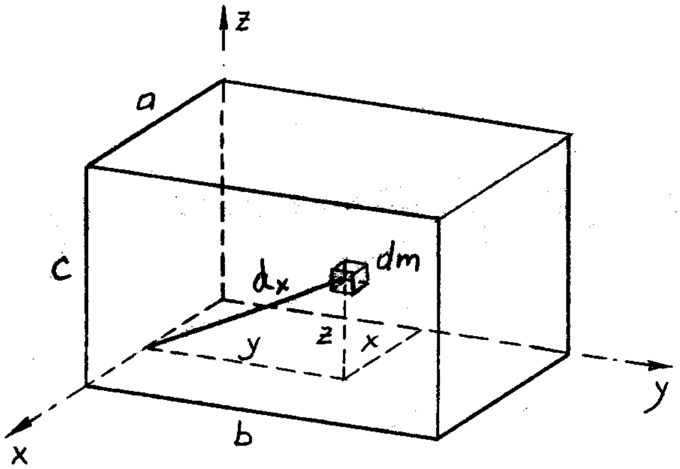
9. Kut između osi y i osi 1 je kut φ_0.

$1\,cm \hat{=} 100\,cm^4$

OČITANO:
$I_1 = 1470\ cm^4$
$I_2 = 360\ cm^4$



Odrediti momente inercije homogenog kvadra mase m i bridova a, b i c s obzirom na osi x, y i z te s obzirom na težišne osi x_s, y_s i z_s paralelne osima x, y i z (sl. a). S pomoću dobivenih izraza ~~odrediti~~ J_{xs}, J_{ys} i J_{zs} za homogenu ploču širine b i visine c te debljine $\delta = a$ i za homogeni štap duljine b (dimenzije poprečnog presjeka su a i c).



sl. a

Po definiciji je

$$J_x = \int_m d_x^2 \, dm \qquad (1)$$

Prema sl. a je

$$d_x^2 = y^2 + z^2 \qquad (2)$$

Element mase dm sadržan je u elementu volumena dV (kvadar bridova dx, dy i dz).

Slijedi

$$dm = \rho \, dV \, , \qquad dV = dx \cdot dy \cdot dz \qquad (3)$$

Gustoća ρ homogenog tijela je konstantna po volumenu tijela i jednaka je omjeru mase i volumena tijela

$$\rho = \frac{m}{V}$$

kako je uvršteno u izrazu (3). Ako se u izraz (1) uvrsti $dm = \frac{m}{V} dV$, J_x se dobije integriranjem po volumenu tijela:

$$J_x = \frac{m}{V} \int_V d_x^2 \, dV$$

Ako se uvrsti dV prema (3) i d_x^2 prema (2) dobije se

$$J_x = \frac{m}{V}\int_V (y^2+z^2)\,dx\,dy\,dz =$$

$$= \frac{m}{V}\int_{x=0}^{x=a} dx \int_{y=0}^{y=b} dy \int_{z=0}^{z=c} (y^2+z^2)\,dz =$$

$$= \frac{m}{V}\int_{x=0}^{x=a} dx \int_{y=0}^{y=b} dy \left| y^2 z + \frac{z^3}{3} \right|_0^c =$$

$$= \frac{m}{V}\int_{x=0}^{x=a} dx \int_{y=0}^{y=b} \left(y^2 c + \frac{c^3}{3} \right) dy =$$

$$= \frac{m}{V}\int_{x=0}^{x=a} dx \left| \frac{y^3}{3} c + \frac{c^3}{3} \cdot y \right|_0^b =$$

$$= \frac{m}{V}\int_{x=0}^{x=a} dx \left(\frac{b^3}{3} c + \frac{c^3}{3} \cdot b \right) =$$

$$= \frac{m}{V}\,\frac{bc}{3}(b^2+c^2)\int_0^a dx = \frac{m}{V}\,\frac{bc}{3}(b^2+c^2)\left| x \right|_0^a =$$

$$= \frac{m}{V}\,\frac{bc}{3}(b^2+c^2)\cdot a = \frac{m}{V}\, abc \cdot \frac{b^2+c^2}{3}$$

Ako se u ovaj rezultat uvrsti $V = abc$, dobije se

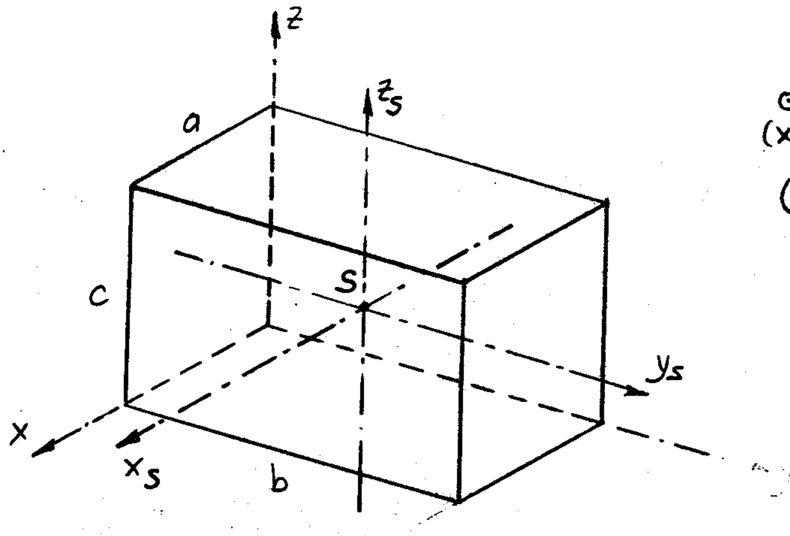
$$J_x = \frac{m}{abc}\cdot abc \cdot \frac{b^2+c^2}{3} = \frac{m(b^2+c^2)}{3} \qquad (4)$$

Analognim računom za J_y i J_z dobiva se

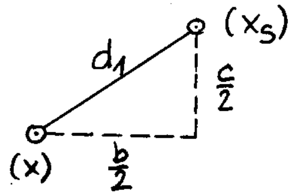
$$J_y = \frac{m(a^2+c^2)}{3} \qquad (5)$$

$$J_z = \frac{m(a^2+b^2)}{3} \qquad (6)$$

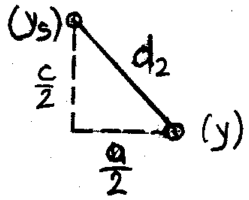
Momenti inercije s obzirom na osi x_s, y_s i z_s izračunat ćemo primjenom Steinerovog pravila.



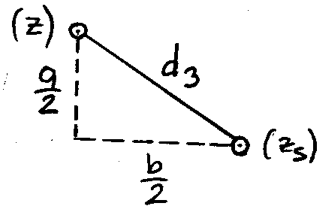
Sl. b



sl. c



sl. d



sl. e

Prema Steinerovom pravilu za osi x, y i z, paralelne težišnim osima x_s, y_s i z_s vrijedi

$$J_x = J_{xs} + d_1^2 \cdot m$$
$$J_y = J_{ys} + d_2^2 \cdot m \qquad (7)$$
$$J_z = J_{zs} + d_3^2 \cdot m$$

Udaljenosti d_1 osi x i x_s, d_2 osi y i y_s i d_3 osi z i z_s prema slikama c, d i e su

$$d_1 = \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2}$$
$$d_2 = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{c}{2}\right)^2} \qquad (8)$$
$$d_3 = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$$

Uvrštenjem izraza (8) u izraze (7) dobivamo

$$J_x = J_{xs} + \left(\frac{b^2}{4} + \frac{c^2}{4}\right) \cdot m$$

$$J_y = J_{ys} + \left(\frac{a^2}{4} + \frac{c^2}{4}\right) \cdot m$$

$$J_z = J_{zs} + \left(\frac{a^2}{4} + \frac{b^2}{4}\right) \cdot m$$

Odavde, uz uvrštenje J_x, J_y i J_z prema (4), (5) i (6), dobivamo

$$J_{xs} = J_x - \frac{b^2+c^2}{4} \cdot m = \frac{m(b^2+c^2)}{3} - \frac{m(b^2+c^2)}{4} =$$

$$= m(b^2+c^2)\left(\frac{1}{3} - \frac{1}{4}\right) = m(b^2+c^2) \cdot \frac{4-3}{12} =$$

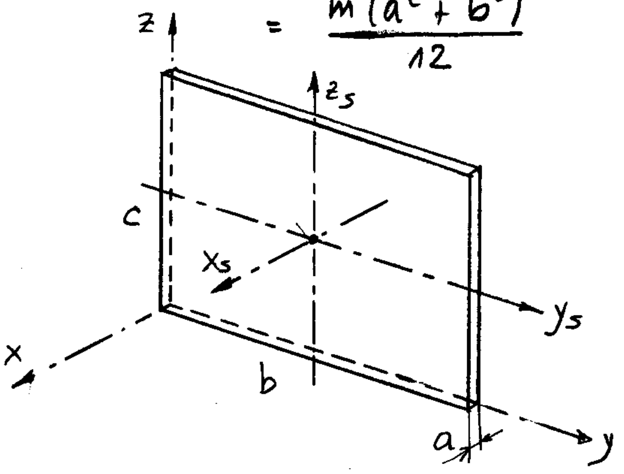
$$= \frac{m(b^2+c^2)}{12} \quad (9)$$

$$J_{ys} = J_y - \frac{a^2+c^2}{4} \cdot m = \frac{m(a^2+c^2)}{3} - \frac{m(a^2+c^2)}{4} =$$

$$= \frac{m(a^2+c^2)}{12} \quad (10)$$

$$J_{zs} = J_z - \frac{a^2+b^2}{4} \cdot m = \frac{m(a^2+b^2)}{3} - \frac{m(a^2+b^2)}{4} =$$

$$= \frac{m(a^2+b^2)}{12} \quad (11)$$



sl. f

Homogenu pravokutnu ploču prema sl. f možemo smatrati kvadrom kojem je brid a (debljina ploče) znatno manji od bridova b i c.

Zbog toga možemo veličinu a^2 zanemariti u odnosu na veličine b^2 i c^2, odnosno možemo smatrati da

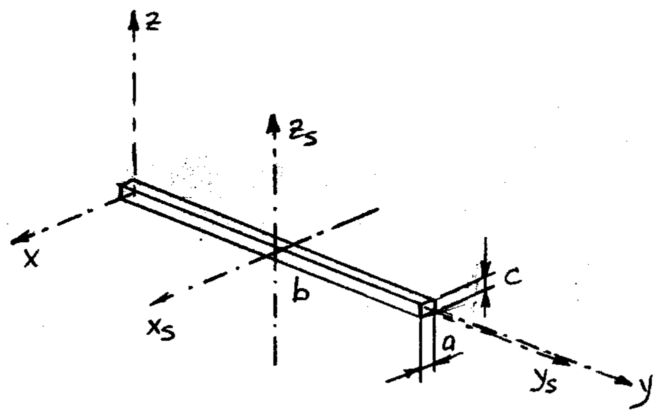
$$a^2 \rightarrow 0 .$$

J_{xs}, J_{ys} i J_{zs} ploče određujemo s pomoću izraza (9), (10) i (11) za odgovarajuće momente inercije kvadra, uvrštavajući $a^2 \to 0$, odnosno $a^2 \approx 0$:

$$J_{xs} = \frac{m(b^2+c^2)}{12} \qquad (12)$$

$$J_{ys} = \frac{mc^2}{12} \qquad (13)$$

$$J_{zs} = \frac{mb^2}{12} \qquad (14)$$



Sl. 9

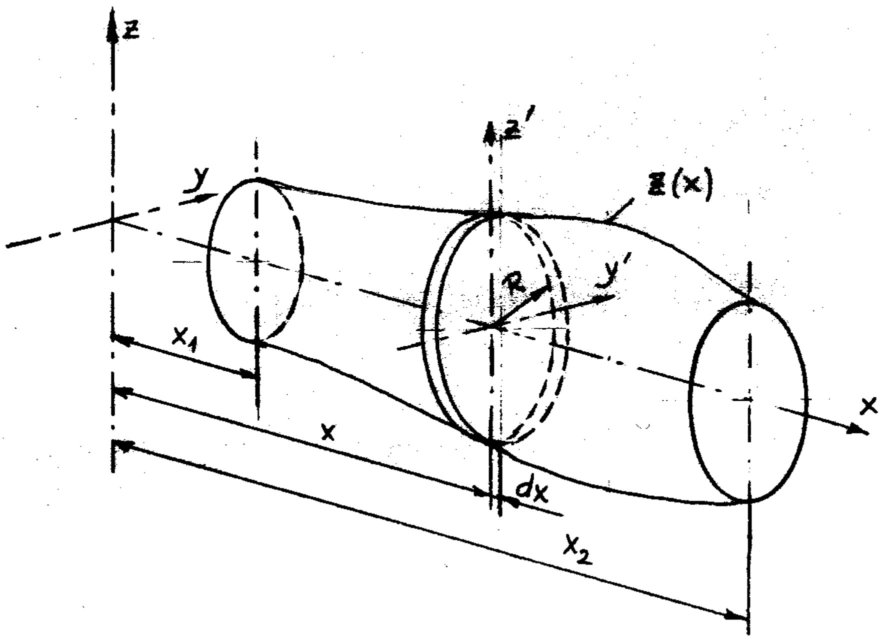
Homogeni štap prema sl. 9 možemo smatrati kvadrom kojem su bridovi a i c (dimenzije poprečnog presjeka) znatno manji od brida b (duljina štapa). Zbog toga možemo veličine a^2 i c^2 zanemariti u odnosu na veličinu b^2, odnosno možemo smatrati da $a^2 \to 0$, $c^2 \to 0$ ili $a^2 \approx 0$, $c^2 \approx 0$.

NAPOMENA:
Na osnovi rezultata (15) može se zaključiti da je, neovisno o obliku poprečnog presjeka, moment inercije homogenog štapa duljine l zanemariv s obzirom na uzdužnu os, a moment inercije s obzirom na bilo koju težišnu os okomitu na uzdužnu os štapa jednak $J_S = \frac{ml^2}{12}$.

Uvrštenjem $a^2 \approx 0$ i $c^2 \approx 0$ u izraze (9), (10) i (11) dobivamo

$$J_{xs} = \frac{mb^2}{12}, \quad J_{ys} = 0, \quad J_{zs} = \frac{mb^2}{12}. \qquad (15)$$

Za osnosimetrično homogeno tijelo mase m i proizvoljne konture z(x) treba odrediti dinamičke momente inercije za os simetrije x i za os z. S pomoću izvedenih izraza odrediti J_x i J_z za homogeni uspravni stožac, kuglu i valjak.

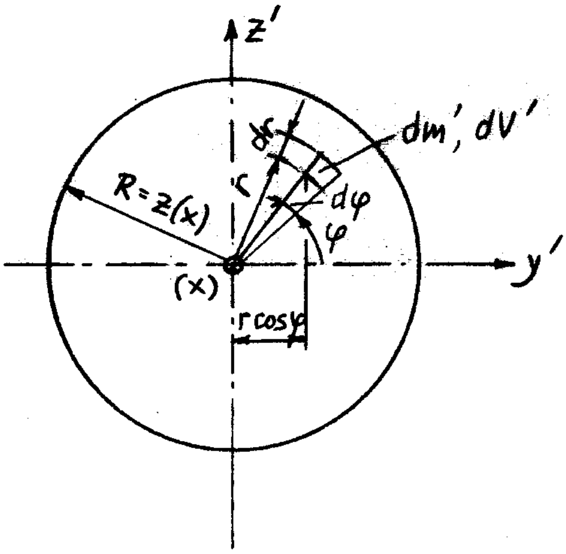


sl. a

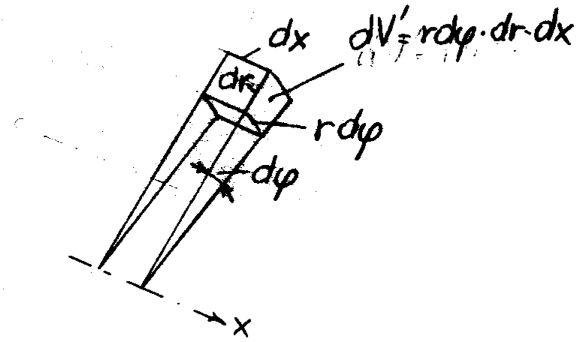
Plašt osnosimetričnog tijela nastaje rotacijom oko osi simetrije x linije koja leži u ravnini xz definirana funkcijom z(x) od $x = x_1$ do $x = x_2$. Baze tog tijela su krugovi polumjera $z(x_1)$ i $z(x_2)$.

Osnosimetrično tijelo promatrat ćemo kao tijelo sastavljeno od elementarnih diskova (valjaka male visine). Polumjeri baza tih diskova su $R = z(x)$ a visine su im $h = dx$. Jedan je takav elementarni disk prikazan na sl. a. Možemo zamisliti da elementarni disk nastaje kao dio osnosimetričnog tijela između dvaju presjeka tijela ravninama okomitim na os simetrije, međusobno udaljenim za dx.

Odredit ćemo momente inercije elementarnog diska s obzirom na os x i s obzirom na os z' paralelnu osi z (sl. a). Masa diska je dm , polumjer baze je $R = z(x)$ a debljina mu je dx. Element mase diska dm' sadržan je u elementu volumena $dV' = r\,d\varphi \cdot dr \cdot dx$ (sl. c). Kako je disk homogen, bit će $dm' = \rho\, dV'$. ~~Za homogeno tijelo gustoća~~ $\rho = \frac{m}{V}$.



sl. b



sl. c

Moment inercije dJ_x diska s obzirom na os x po definiciji je

$$dJ_x = \int_{dm} d_x^2\, dm' .$$

Udaljenost d_x elementa mase dm' od osi x iznosi (sl. b)

$$d_x = r .$$

Ako se uvrsti

$$dm' = \rho\, dV'$$

integriranje po masi dm prelazi u integriranje po volumenu dV elementarnog diska :

$$dJ_x = \int_{dV} r^2 \rho\, dV'$$

Dalje je, uz $dV' = r d\varphi \cdot dr \cdot dx$

$$dJ_x = \int_{dV} r^2 \rho \, r d\varphi dr dx$$

Ovdje je gustoća ρ konstantna, kao i debljina diska dx, a pri integriranju φ se mijenja od 0 do 2π, a r od 0 do $R = z(x)$, pa slijedi

$$dJ_x = \rho dx \int_{\varphi=0}^{\varphi=2\pi} d\varphi \int_{r=0}^{r=R=z(x)} r^3 dr = \rho dx \cdot \left| \varphi \right|_0^{2\pi} \cdot \left| \frac{r^4}{4} \right|_0^{z(x)} =$$

$$= \rho dx \cdot 2\pi \cdot \frac{[z(x)]^4}{4}$$

Ovo možemo pisati u slijedećem obliku

$$dJ_x = \rho \cdot [z(x)]^2 \pi dx \cdot \frac{[z(x)]^2}{2}.$$

Kako je volumen elementarnog diska

$$dV = R^2 \pi \cdot dx = [z(x)]^2 \pi dx$$

bit će

$$\rho \cdot [z(x)]^2 \pi dx = \rho \cdot dV = dm$$

pa konačno dobivamo

$$dJ_x = \frac{dm \cdot [z(x)]^2}{2}$$

Ovo je poznati izraz za moment inercije valjka mase dm i polumjera baze $R = z(x)$ s obzirom na uzdužnu os.

Moment inercije diska s obzirom na os z' po definiciji je

$$dJ_z' = \int_{dm} d_z'^2 dm'$$

Udaljenost d_z' elementa mase dm' od osi z' iznosi (sl. b)

$$d_z' = r \cos\varphi$$

Analogno kao u računu dJ_x dobivamo

$$dJ_z' = \int_{dV} (r\cos\varphi)^2 \rho\, dV' = \int_{dV} r^2\cos^2\varphi \cdot \rho \cdot r\,d\varphi\, dr\, dx =$$

$$= \rho\, dx \int_{\varphi=0}^{\varphi=2\pi} \cos^2\varphi\, d\varphi \int_{r=0}^{r=R=z(x)} r^3\, dr =$$

$$= \rho\, dx \cdot \left| \frac{\varphi}{2} + \frac{\sin 2\varphi}{4} \right|_0^{2\pi} \cdot \left| \frac{r^4}{4} \right|_0^{z(x)} =$$

$$= \rho\, dx \cdot \left(\frac{2\pi}{2} + \frac{\sin 4\pi}{4} - 0 - 0 \right) \cdot \frac{[z(x)]^4}{4} =$$

$$= \rho\, dx\, \pi \frac{[z(x)]^4}{4} = \rho$$

$$= dm \cdot \frac{[z(x)]^2}{4} = \frac{dm \cdot [z(x)]^2}{4}$$

Odredit ćemo moment inercije diska s obzirom na os z koja je za x udaljena od osi z' (sl. a). Kako je os z' težišna os diska, primijenit ćemo Steinerovo pravilo:

$$dJ_z = dJ_z' + x^2 \cdot dm = \frac{dm\,[z(x)]^2}{4} + dm\,x^2 =$$

$$= dm \left\{ \frac{[z(x)]^2}{4} + x^2 \right\}$$

Momente inercije J_x i J_z za cijelo tijelo dobit ćemo kao zbroj odgovarajućih momenata inercije svih elementarnih diskova sadržanih u tijelu, što znači da moramo integrirati dJ_x, odnosno dJ_z po masi tijela:

$$J_x = \int_m dJ_x = \int_m \frac{dm\,[z(x)]^2}{2}$$

$$J_z = \int_m dJ_z = \int_m dm \left\{ \frac{[z(x)]^2}{4} + x^2 \right\}$$

Ako u ove izraze uvrstimo

$$dm = \rho\, dV$$

gdje je dV volumen elementarnog diska

$$dV = R^2\pi \cdot dx = [z(x)]^2 \pi\, dx$$

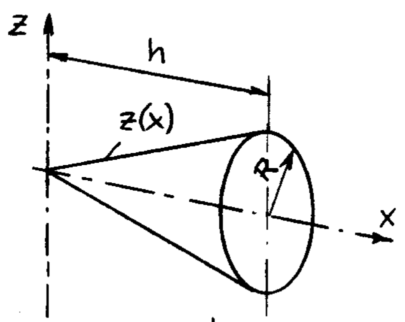
dobit ćemo

$$J_x = \int_{x=x_1}^{x=x_2} \frac{\rho\,[z(x)]^2\pi\, dx \cdot [z(x)]^2}{2} = \rho \frac{\pi}{2} \int_{x=x_1}^{x=x_2} [z(x)]^4 dx \qquad (1)$$

$$J_z = \int_{x=x_1}^{x=x_2} \rho\,[z(x)]^2\pi\, dx \left\{ \frac{[z(x)]^2}{4} + x^2 \right\} =$$

$$= \rho\pi \int_{x=x_1}^{x=x_2} \left\{ \frac{[z(x)]^4}{4} + x^2 [z(x)]^2 \right\} dx \qquad (2)$$

S pomoću izraza (1) i (2) odredit ćemo J_x i J_z za stožac prema sl. d.



sl. d

Ovdje je $z(x) = \frac{R}{h} \cdot x$ (jednadžba pravca na kojem leži izvodnica plašta stošca), $x_1 = 0$ i $x_2 = h$.

Prema izrazu (1) je

$$J_x = \rho \frac{\pi}{2} \int_{x_1=0}^{x_2=h} \left(\frac{R}{h} x\right)^4 dx =$$

$$= \rho \frac{\pi}{2} \cdot \frac{R^4}{h^4} \int_0^h x^4 dx = \rho \frac{\pi}{2} \frac{R^4}{h^4} \cdot \left| \frac{x^5}{5} \right|_0^h =$$

$$= \rho \frac{\pi}{2} \frac{R^4}{h^4} \cdot \frac{h^5}{5} = \rho \cdot \frac{\pi}{10} \cdot R^4 h$$

Volumen stošca je

$$V = \frac{R^2\pi \cdot h}{3} ,$$

a masa mu je

$$m = \rho V = \rho \frac{R^2 \pi h}{3},$$

pa se J_x može transformirati kako slijedi

$$J_x = \rho \frac{\pi}{10} R^4 h \cdot \frac{3}{3} = \rho \frac{R^2 \pi h}{3} \cdot \frac{3}{10} R^2 = \frac{3}{10} m R^2.$$

Prema izrazu (2) je

$$J_z = \rho \pi \int_{x_1=0}^{x_2=h} \left[\frac{1}{4} \left(\frac{R}{h} x\right)^4 + x^2 \left(\frac{R}{h} x\right)^2 \right] dx =$$

$$= \rho \pi \left[\frac{1}{4} \frac{R^4}{h^4} \int_0^h x^4 dx + \frac{R^2}{h^2} \int_0^h x^4 dx \right] =$$

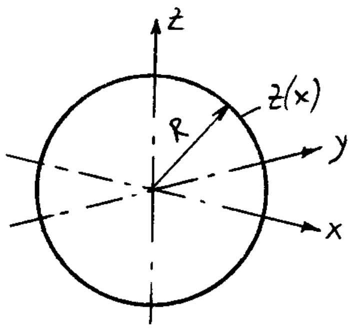
$$= \rho \pi \left[\frac{1}{4} \frac{R^4}{h^4} \left| \frac{x^5}{5} \right|_0^h + \frac{R^2}{h^2} \left| \frac{x^5}{5} \right|_0^h \right] =$$

$$= \rho \pi \left(\frac{1}{4} \frac{R^4}{h^4} \cdot \frac{h^5}{5} + \frac{R^2}{h^2} \cdot \frac{h^5}{5} \right) = \rho \pi \frac{R^2 h}{5} \left(\frac{R^2}{4} + h^2 \right)$$

Dobiveni izraz možemo transformirati kako slijedi

$$J_z = \rho \pi \frac{R^2 h}{5} \left(\frac{R^2}{4} + h^2 \right) \cdot \frac{3}{3} = \rho \frac{\pi R^2 h}{3} \cdot \frac{3}{5} \left(\frac{R^2}{4} + h^2 \right) =$$

$$= \frac{3}{5} m \left(\frac{R^2}{4} + h^2 \right)$$



Sl. e

Za kuglu prema sl. e je $z(x) = \sqrt{R^2 - x^2}$ (jednadžba kružnice kao presječnice plašta kugle ravninom xz), $x_1 = -R$ i $x_2 = R$.

Prema izrazu (1) slijedi

$$J_x = \rho \frac{\pi}{2} \int_{x_1=-R}^{x_2=R} \left(\sqrt{R^2 - x^2} \right)^4 dx =$$

$$= \rho \frac{\pi}{2} \int_{-R}^{R} (R^2 - x^2)^2 dx =$$

$$J_x = \rho\frac{\pi}{2}\int_{-R}^{R}\left(R^4 - 2R^2x^2 + x^4\right)dx = \rho\frac{\pi}{2}\left| R^4x - 2R^2\frac{x^3}{3} + \frac{x^5}{5}\right|_{-R}^{R} =$$

$$= \rho\frac{\pi}{2}\left[R^4R - \frac{2}{3}R^2\cdot R^3 + \frac{1}{5}R^5 - R^4(-R) + 2R^2\frac{(-R)^3}{3} - \frac{(-R)^5}{5}\right] =$$

$$= \rho\frac{\pi}{2}R^5\left(1 - \frac{2}{3} + \frac{1}{5} + 1 - \frac{2}{3} + \frac{1}{5}\right) = \rho\frac{\pi}{2}R^5\frac{30-20+6}{15} =$$

$$= \rho\frac{\pi}{2}\cdot R^5\cdot\frac{16}{15} = \frac{8}{15}\rho\pi R^5$$

Masa kugle je

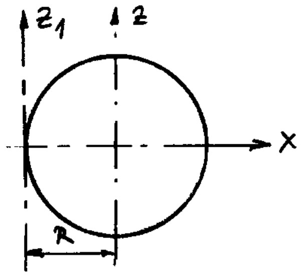
$$m = \rho\cdot V = \rho\cdot\frac{4}{3}R^3\pi$$

pa se izraz za J_x može transformirati kako slijedi

$$J_x = \frac{8}{15}\rho\pi R^5 = \frac{4\cdot 2}{3\cdot 5}\rho\pi R^5 = \rho\cdot\frac{4}{3}R^3\pi\cdot\frac{2}{5}R^2 = \frac{2}{5}mR^2$$

očito je da bismo za račun J_z mogli primijeniti izraz (1) uz zamjenu varijabli x i z. Rezultat bi bio jednak, tj.

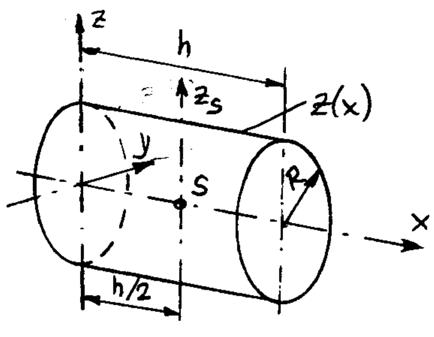
$$J_z = \frac{2}{5}mR^2.$$



Sl. f

Moment inercije kugle za os z_1 koja tangira njen plašt možemo odrediti primjenom Steinerovog pravila. Budući da je os z težišna os, vrijedi

$$J_{z1} = J_z + R^2\cdot m = \frac{2}{5}mR^2 + R^2m = \frac{7}{5}mR^2.$$



Sl. g

Za valjak prema sl. g je $z(x) = R$ (jednadžba pravca paralelnog osi x na kojem leži izvodnica plašta stošca), $x_1 = 0$ i $x_2 = h$.

Prema izrazu (1) je

$$J_x = \rho \frac{\pi}{2} \int_{x_1=0}^{x_2=h} R^4 dx = \rho \frac{\pi}{2} R^4 \left| x \right|_0^h = \rho \frac{\pi}{2} R^4 h$$

Kako je masa valjka

$$m = \rho V = \rho \cdot R^2 \pi h$$

izraz za J_x može se transformirati kako slijedi:

$$J_x = \rho \frac{\pi}{2} R^4 h = \rho R^2 \pi h \cdot \frac{R^2}{2} = \frac{m R^2}{2}$$

Prema izrazu (2) je

$$J_z = \rho \pi \int_{x_1=0}^{x_2=h} \left(\frac{R^4}{4} + x^2 R^2 \right) dx =$$

$$= \rho \pi \left[\int_0^h \frac{R^4}{4} dx + \int_0^h R^2 x^2 dx \right] =$$

$$= \rho \pi \left[\frac{R^4}{4} \left| x \right|_0^h + R^2 \left| \frac{x^3}{3} \right|_0^h \right] =$$

$$= \rho \pi \left(\frac{R^4}{4} \cdot h + R^2 \frac{h^3}{3} \right) = \rho \pi R^2 h \left(\frac{R^2}{4} + \frac{h^2}{3} \right)$$

Uz $m = \rho R^2 \pi h$ slijedi

$$J_z = m \left(\frac{R^2}{4} + \frac{h^2}{3} \right)$$

Moment inercije J_{zS} za težišnu os z_S možemo odrediti primjenom Steinerovog pravila:

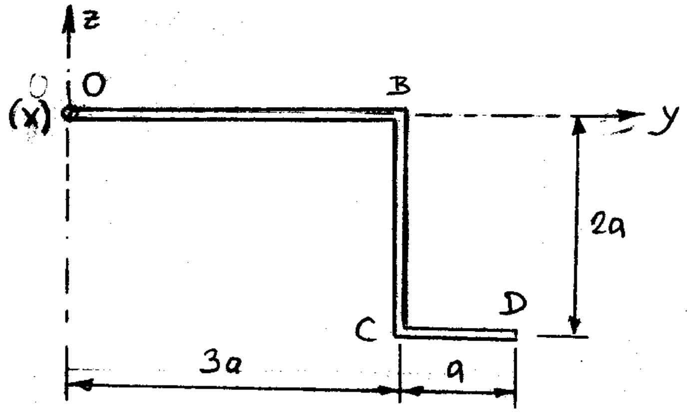
$$J_z = J_{zS} + \left(\frac{h}{2} \right)^2 \cdot m \ .$$

Naime, J_z je moment inercije s obzirom na os z paralelnu težišnoj osi z_S, udaljenu za $\frac{h}{2}$. Slijedi

$$J_{zS} = J_z - \left(\frac{h}{2} \right)^2 \cdot m = m \left(\frac{R^2}{4} + \frac{h^2}{3} \right) - m \frac{h^2}{4} =$$

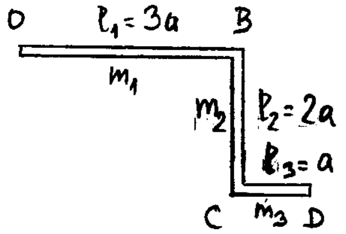
$$= m \left(\frac{R^2}{4} + \frac{4h^2 - 3h^2}{12} \right) = m \left(\frac{R^2}{4} + \frac{h^2}{12} \right)$$

Odrediti moment inercije homogenog sastavljenog štapa prema sl. a s obzirom na os x koja prolazi kroz točku O. Poprečni presjeci dijelova štapa su jednaki, a dijelovi su izrađeni od istog materijala. Zadano: m (ukupna masa), a.



Sl. a

Sastavljeni štap možemo promatrati kao tri kruto spojena homogena štapa. Uz istu gustoću ρ i poprečni presjek A, mase tih štapova proporcionalne su njihovim duljinama. Mase m_1, m_2 i m_3 (sl. b) izrazimo kao dijelove ukupne mase m:



Sl. b

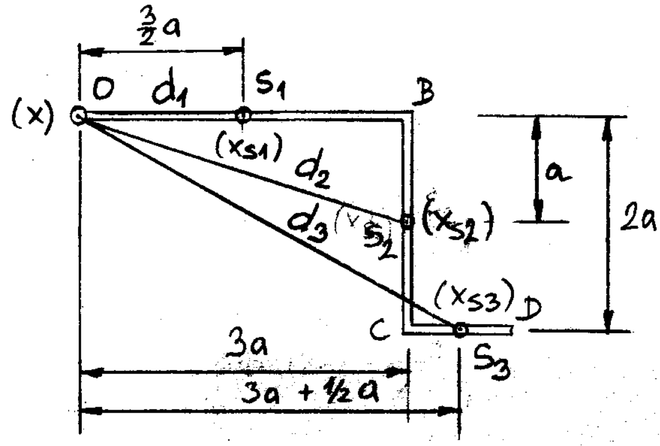
$$\frac{m_1}{m} = \frac{\rho V_1}{\rho V} = \frac{\rho A l_1}{\rho A (l_1 + l_2 + l_3)} =$$

$$= \frac{l_1}{l_1 + l_2 + l_3} = \frac{3a}{3a + 2a + a} = \frac{3a}{6a} = \frac{1}{2}$$

$$m_1 = \frac{1}{2} m$$

$$\frac{m_2}{m} = \frac{\rho V_2}{\rho V} = \frac{\rho A l_2}{\rho A (l_1 + l_2 + l_3)} = \frac{2a}{6a} = \frac{1}{3} \quad , \qquad m_2 = \frac{1}{3} m$$

$$\frac{m_3}{m} = \frac{\rho V_3}{\rho V} = \frac{\rho A l_3}{\rho A (l_1 + l_2 + l_3)} = \frac{a}{6a} = \frac{1}{6} \quad , \qquad m_3 = \frac{1}{6} m$$



Sl. c

Traženi J_x odredit ćemo s pomoću Steinerovog pravila. Za štap OB, štap BC i štap CD momenti inercije s obzirom na težišne osi x_{S1}, x_{S2} i x_{S3}, okomite na njihove uzdužne osi i paralelne sa zadanom osi x, iznose:

$$J_{xS1} = \frac{m_1 l_1^2}{12} = \frac{\frac{1}{2}m(3a)^2}{12} = \frac{9}{24}ma^2 = \frac{3}{8}ma^2,$$

$$J_{xS2} = \frac{m_2 l_2^2}{12} = \frac{\frac{1}{3}m(2a)^2}{12} = \frac{4}{36}ma^2 = \frac{1}{9}ma^2,$$

$$J_{xS3} = \frac{m_3 l_3^2}{12} = \frac{\frac{1}{6}ma^2}{12} = \frac{1}{72}ma^2$$

Udaljenosti d_1, d_2 i d_3 osi x_{S1}, x_{S2} i x_{S3} od osi x iznose (sl. c)

$$d_1 = \frac{3}{2}a$$

$$d_2 = \sqrt{(3a)^2 + a^2} = \sqrt{10}\,a$$

$$d_3 = \sqrt{(\frac{7}{2}a)^2 + (2a)^2} = \frac{\sqrt{65}}{2}a$$

Konačno primjenom Steinerovog pravila dobivamo

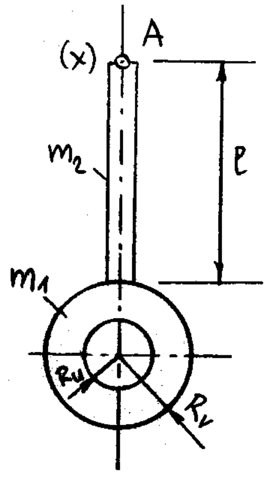
$$J_x = (J_{xS1} + d_1^2 \cdot m_1) + (J_{xS2} + d_2^2 \cdot m_2) + (J_{xS3} + d_3^2 \cdot m_3) =$$

$$= \left[\frac{3}{8}ma^2 + (\frac{3}{2}a)^2 \cdot \frac{1}{2}m\right] + \left[\frac{1}{9}ma^2 + (\sqrt{10}a)^2 \cdot \frac{1}{3}m\right] + \left[\frac{1}{72}ma^2 + (\frac{\sqrt{65}}{2}a)^2 \cdot \frac{1}{6}m\right] =$$

$$= ma^2\left(\frac{3}{8} + \frac{9}{4}\cdot\frac{1}{2}\right) + ma^2\left(\frac{1}{9} + \frac{10}{3}\right) + ma^2\left(\frac{1}{72} + \frac{65}{4}\cdot\frac{1}{6}\right) =$$

$$= ma^2 \frac{27 + 81 + 8 + 240 + 1 + 195}{72} = \frac{552}{72}ma^2 = \frac{69}{9}ma^2 =$$

$$= 7{,}667\,ma^2$$



Sl. a

Složeno tijelo sastoji se od šupljeg homogenog valjka mase m_1 i ~~homogenog~~ štapa mase m_2 koji su kruto spojeni. Treba odrediti moment inercije ovog složenog tijela s obzirom na os x koja prolazi krajem A štapa paralelno s uzdužnom osi valjka i okomito na uzdužnu os štapa.

Zadano: $m_1 = 4m$, $m_2 = m$, $R_u = a$, $R_v = 2a$, $l = 6a$.

Moment inercije šupljeg valjka s obzirom na uzdužnu težišnu os odredit ćemo kao zbroj momenta inercije punog valjka polumjera R_v i momenta inercije šupljine, tj. momenta inercije punog valjka polumjera R_u s NEGATIVNIM predznakom:

$$J_{xs1} = J_{xsv} - J_{xsu} = \frac{m_v R_v^2}{2} - \frac{m_u R_u^2}{2}$$

Odredimo mase punog valjka m_v i odgovarajuću masu šupljine m_u. Ove su mase proporcionalne volumenima:

$$\frac{m_u}{V_u} = \frac{m_v}{V_v}$$

Ovdje je $V_u = R_u^2 \pi \cdot H$, $V_v = R_v^2 \pi \cdot H$, gdje je H visina valjka. Slijedi

$$\frac{m_u}{m_v} = \frac{V_u}{V_v} = \frac{R_u^2 \pi H}{R_v^2 \pi H} = \left(\frac{R_u}{R_v}\right)^2$$

$$m_u = \left(\frac{R_u}{R_v}\right)^2 m_v \qquad (1)$$

Razlika mase punog valjka m_v i mase šupljine m_u more biti jednaka zadanoj masi šupljeg valjka m_1 :

$$m_v - m_u = m_1 \qquad (2)$$

Uvrstimo ... (2) m_u prema (1), dobit ćemo

$$m_v - \left(\frac{R_u}{R_v}\right)^2 m_v = m_1 ,$$

$$m_v = \frac{m_1}{1 - \left(\frac{R_u}{R_v}\right)^2} ,$$

i dalje prema (1)

$$m_u = \left(\frac{R_u}{R_v}\right)^2 m_v = \frac{\left(\frac{R_u}{R_v}\right)^2}{1 - \left(\frac{R_u}{R_v}\right)^2} m_1 .$$

Konačno dobivamo

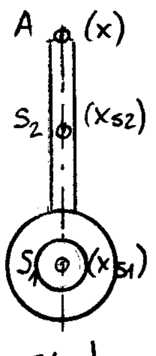
$$J_{xs1} = \frac{m_1}{1-\left(\frac{R_u}{R_v}\right)^2} \cdot \frac{R_v^2}{2} - \frac{\left(\frac{R_u}{R_v}\right)^2}{1-\left(\frac{R_u}{R_v}\right)^2} m_1 \cdot \frac{R_u^2}{2} =$$

$$= \frac{1}{2} \frac{m_1}{1-\left(\frac{R_u}{R_v}\right)^2} \left[R_v^2 - \left(\frac{R_u}{R_v}\right)^2 \cdot R_u^2 \right] =$$

$$= \frac{m_1 R_v^2}{2} \cdot \frac{1}{1-\left(\frac{R_u}{R_v}\right)^2} \cdot \left[1 - \left(\frac{R_u}{R_v}\right)^2 \cdot \frac{R_u^2}{R_v^2} \right] =$$

$$= \frac{m_1 R_v^2}{2} \cdot \frac{1-\left(\frac{R_u}{R_v}\right)^4}{1-\left(\frac{R_u}{R_v}\right)^2} = \frac{m_1 R_v^2}{2} \cdot \frac{\left[1-\left(\frac{R_u}{R_v}\right)^2\right]\cdot\left[1+\left(\frac{R_u}{R_v}\right)^2\right]}{1-\left(\frac{R_u}{R_v}\right)^2} =$$

$$= \frac{m_1 R_v^2}{2} \cdot \left[1 + \left(\frac{R_u}{R_v}\right)^2 \right]$$



Sl. b

Moment inercije štapa s obzirom na njegovu težišnu os je

$$J_{xs2} = \frac{m_2 l^2}{12}$$

Primjenom Steinerovog pravila konačno dobijemo

$$J_x = \left(J_{xs1} + d_1^2 \cdot m_1\right) + \left(J_{xs2} + d_2^2 \cdot m_2\right)$$

Prema slici b udaljenost osi x_{s1} i x je

$$d_1 = \overline{AS_1} = l + R_v = 6a + 2a = 8a$$

a udaljenost osi x_{s2} i x je

$$d_2 = \overline{AS_2} = \frac{1}{2} l = \frac{1}{2} \cdot 6a = 3a$$

Izračunajmo J_{xs1} i J_{xs2} :

$$J_{xs1} = \frac{m_1 R_v^2}{2} \left[1 + \left(\frac{R_u}{R_v} \right)^2 \right] = \frac{4m \cdot (2a)^2}{2} \cdot \left[1 + \left(\frac{a}{2a} \right)^2 \right] =$$

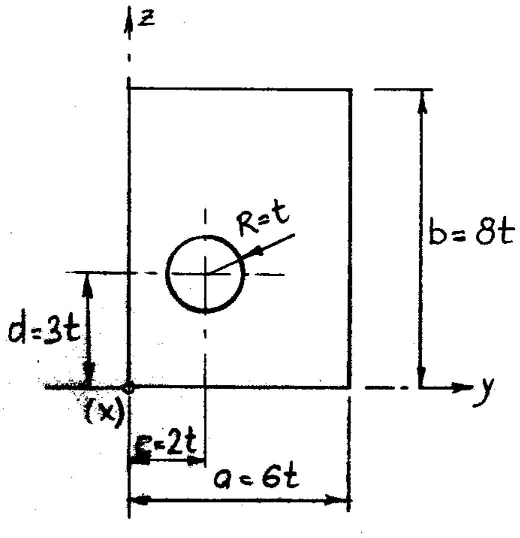
$$= 8ma^2 \left(1 + \frac{1}{4} \right) = 8ma^2 \cdot \frac{5}{4} = 10 ma^2$$

$$J_{xs2} = \frac{m_2 l^2}{12} = \frac{m (6a)^2}{12} = \frac{m \cdot 36a^2}{12} = 3 ma^2$$

Konačno slijedi

$$J_x = \left[10ma^2 + (8a)^2 \cdot 4m \right] + \left[3ma^2 + (3a)^2 \cdot m \right] =$$

$$= 10 ma^2 + 256 ma^2 + 3 ma^2 + 9 ma^2 = 278 ma^2$$



Sl. a

Za homogeni kvadar s provrtom polumjera R, mase m, treba odrediti dinamički moment inercije s obzirom na težišnu os x_S.
Zadano: m, t.

Položaj težišta zadanog tijela određujemo izrazima

$$y_S = \frac{m_1 y_1 - m_2 y_2}{m_1 - m_2},$$

$$z_S = \frac{m_1 z_1 - m_2 z_2}{m_1 - m_2}.$$

Ovdje je m_1 masa kvadra bez provrta, y_1 i z_1 koordinate težišta tog kvadra, a m_2 masa koja bi popunjavala šupljinu provrta, y_2 i z_2 koordinate težišta provrta. Šupljinu dakle promatramo kao valjak NEGATIVNE mase m_2.

Mase m_1 i m_2 odredit ćemo na temelju zadane homogenosti tijela. Naime za homogeno tijelo omjer mase i volumena u kojem je ona sadržana konstantan je i jednak je gustoći ρ. Prema tome, omjer mase m_1 kvadra bez provrta i njegovog volumena V_1 mora biti jednak omjeru mase m_2 i volumena V_2 cilindrične šupljine:

$$\frac{m_1}{V_1} = \frac{m_2}{V_2}$$

Odatle slijedi omjer

$$\frac{m_2}{m_1} = \frac{V_2}{V_1} = \frac{R^2\pi h}{abh} = \frac{R^2\pi}{ab} = \frac{t^2\pi}{6t\cdot 8t} = \frac{\pi t^2}{48t^2} = \frac{\pi}{48}.$$

Ovdje je h debljina kvadra, odnosno njegov brid paralelan osi x. Izrazimo m_2 masom m_1:

$$m_2 = \frac{\pi}{48} m_1 = 0{,}06545\, m_1.$$

Zbroj mase m_1 i negativne mase m_2 mora biti jednak zadanoj masi m kvadra s provrtom;

$$m_1 - m_2 = m$$

$$m_1 - 0{,}06545\, m_1 = m$$

$$0{,}93455\, m_1 = m$$

slijedi

$$m_1 = 1{,}07\, m$$

$$m_2 = 0{,}06545\, m_1 = 0{,}07\, m$$

Koordinate težišta kvadra bez provrta i cilindrične šupljine su (sl. a)

$$y_1 = \frac{a}{2} = 3t, \quad z_1 = \frac{b}{2} = 4t;$$

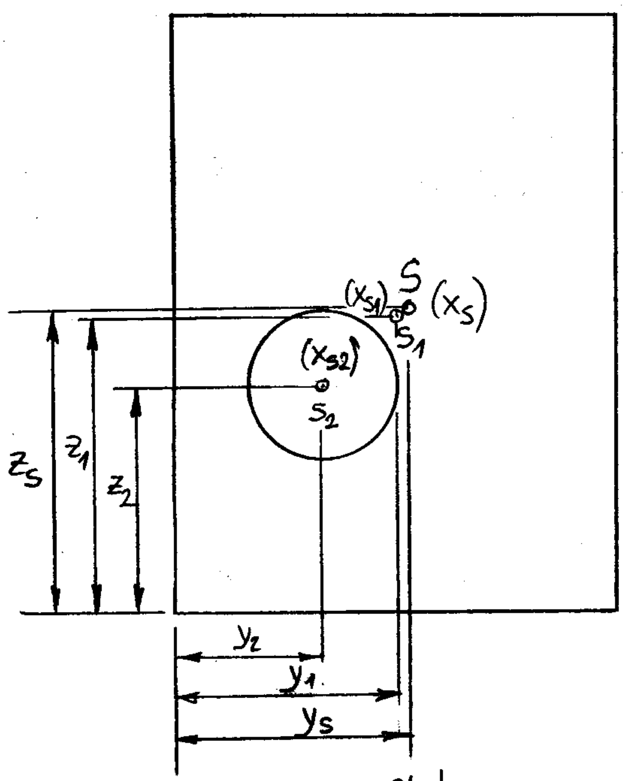
$$y_2 = c = 2t, \quad z_2 = d = 3t.$$

Konačno dobivamo

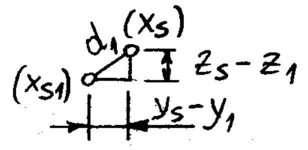
$$y_S = \frac{1{,}07m\cdot 3t - 0{,}07\cdot 2t}{1{,}07m - 0{,}07m} = 3{,}07t$$

$$z_S = \frac{1{,}07m\cdot 4t - 0{,}07\cdot 3t}{1{,}07m - 0{,}07m} = 4{,}07t$$

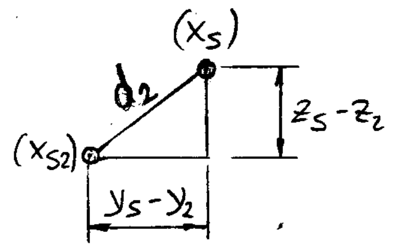
Dinamički moment inercije kvadra s provrtom s obzirom na os x_s dobit ćemo kao zbroj momenta inercije kvadra bez provrta s obzirom na os x_s i momenta inercije šupljine s obzirom na tu os. Moment inercije šupljine određujemo kao moment inercije veličine mase m_2 s NEGATIVNIM predznakom.



Sl. b



Sl. c



Sl. d

Moment inercije kvadra bez provrta s obzirom na os x_s paralelnu njegovoj težišnoj osi x_{s1} određujemo Steinerovim pravilom:

$$J'_{xs} = J_{xs1} + d_1^2 \cdot m_1$$

Prema sl. c ~~kvadrat~~ udaljenosti osi x_{s1} i x_s iznosi

$$d_1^2 = (y_s - y_1)^2 + (z_s - z_1)^2 = (3{,}07t - 3t)^2 + (4{,}07t - 4t)^2 = 0{,}0098t^2$$

Moment inercije kvadra bez provrta s obzirom na os x_{s1} iznosi

$$J_{xs1} = \frac{m_1(a^2+b^2)}{12} = \frac{1{,}07m[(6t)^2+(8t)^2]}{12} =$$

$$= 8{,}9167\,mt^2$$

Slijedi

$$J'_{xs} = 8{,}9167\,mt^2 + 0{,}0098\,t^2 \cdot 1{,}07\,m = 8{,}92715\,mt^2$$

Steinerovim pravilom određujemo i moment inercije valjka mase m_2 s obzirom na os x_s:

$$J''_{xs} = J_{xs2} + d_2^2 \cdot m_2$$

Prema sl. d kvadrat udaljenosti osi x_{s2} i x_s iznosi

$$d_2^2 = (y_s - y_2)^2 + (z_s - z_2)^2 = (3{,}07t - 2t)^2 + (4{,}07t - 3t)^2 =$$

$$= 2{,}2898\,t^2$$

Moment inercije valjka mase m_2 obzirom na težišnu os x_{s2} iznosi

$$J_{xs2} = \frac{m_2 R^2}{2} = \frac{0{,}07m \cdot t^2}{2} = 0{,}035\,mt^2$$

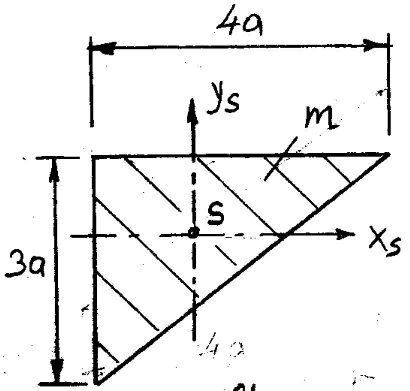
Slijedi $$J''_{xs} = 0{,}035\,mt^2 + 2{,}2898\,t^2 \cdot 0{,}07\,m = 0{,}195286\,mt^2$$

Konačno dobivamo traženi moment inercije zadanog tijela s obzirom na njegovu težišnu os:

$$J_{xs} = J'_{xs} - J''_{xs} = 8{,}92715\,mt^2 - 0{,}195286\,mt^2 =$$

$$= 8{,}731864\,mt^2$$

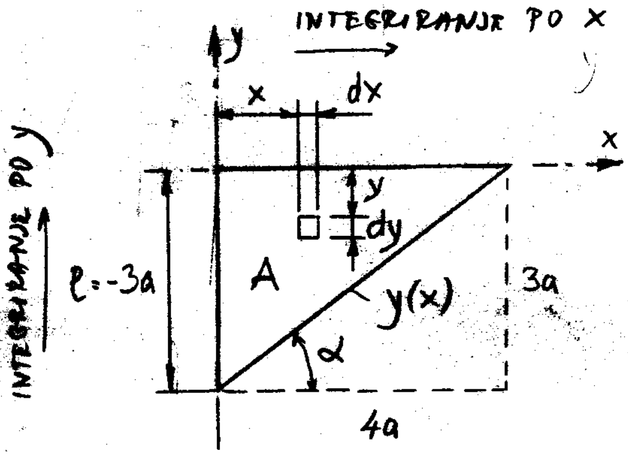
Uočavamo da je ovaj moment inercije samo za oko 2% manji od momenta inercije valjka bez provrta J_{xs1}.



Sl. a

Za ~~homogenu~~ trokutnu ploču debljine t treba odrediti centrifugalni moment inercije s obzirom na težišne osi x_s i y_s.

Zadano: m, a.



Sl. b

Najprije ćemo odrediti centrifugalni moment inercije s obzirom na osi x i y.

Po definiciji je

$$J_{xy} = \int_m xy \, dm \qquad (1)$$

Kako je ploča homogena, vrijedi

$$dm = \rho \, dV = \frac{m}{V} dV \qquad (2)$$

Ako se (2) uvrsti u (1), integriranje po masi prelazi u integriranje po volumenu:

$$J_{xy} = \frac{m}{V} \int_V xy \, dV$$

Diferencijal volumena ploče, je (sl. b)

$$dV = dx\,dy\,t$$

Slijedi

$$J_{xy} = \frac{m}{V} \int_V xy \, dx\,dy\,t = \frac{m}{V} t \int_A xy \, dx\,dy$$

Integriranje po volumenu prelazi u integriranje po površini ploče A (sl. b). Granice integriranja po x su x = 0 i x = 4a, a po y su pravac y(x) i y = 0 (0 ≤ x).

~~Postavimo~~ jednadžbu pravca y(x): (sl. b)

$$y = l + kx\,;$$

$$l = -3a\,;$$

$$k = \operatorname{tg}\alpha = \frac{3a}{4a} = \frac{3}{4}\,,$$

$$y = -3a + \frac{3}{4}x$$

Slijedi

$$J_{xy} = \frac{m}{V}\cdot t\int_{x=0}^{x=4a} x\,dx \int_{y=-3a+\frac{3}{4}x}^{y=0} y\,dy =$$

$$= \frac{m}{V}t\int_{x=0}^{x=4a} x\,dx \left|\frac{y^2}{2}\right|_{-3a+\frac{3}{4}x}^{0} =$$

$$= \frac{m}{V}t\int_{x=0}^{x=4a} x\cdot\frac{1}{2}\left[0^2-\left(-3a+\frac{3}{4}x\right)^2\right] =$$

$$= -\frac{m}{V}t\int_{0}^{4a}\left(\frac{9}{2}a^2x-\frac{9}{4}ax^2+\frac{9}{32}x^3\right)dx =$$

$$= -\frac{m}{V}t\left|\frac{9}{2}a^2\cdot\frac{x^2}{2}-\frac{9}{4}a\frac{x^3}{3}+\frac{9}{32}\frac{x^4}{4}\right|_0^{4a} =$$

$$= -\frac{m}{V}t\left(\frac{9}{4}\cdot a^2\cdot 16a^2-\frac{3}{4}a\cdot 64a^3+\frac{9}{128}\cdot 256a^4\right) =$$

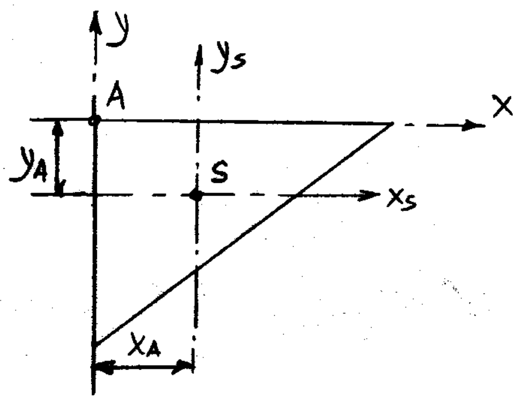
$$= -\frac{m}{V}\cdot ta^4\,(36-48+18) = -6\,\frac{m}{V}a^4t \qquad (3)$$

Ako u (3) uvrstimo

$$V = A \cdot t = \frac{4a \cdot 3a}{2} \cdot t = 6a^2 t$$

dobit ćemo

$$J_{xy} = -6\,\frac{m}{6a^2 t}\,a^4 t = \cancel{m}\,ma^2$$



Primijenimo Steinerovo pravilo:

$$J_{xy} = J_{x_s y_s} + x_A \cdot y_A \cdot m$$

Ovdje su x_A i y_A koordinate ishodišta A koordinatnog sustava xy u koordinatnom sustavu $x_s y_s$:

$$x_A = -\frac{4a}{3}, \quad y_A = \frac{3a}{3}$$

Koordinate x_A i y_A jednakog su iznosa i suprotnog predznaka od koordinata težišta S u koordinatnom sustavu xy.

Konačno dobivamo

$$J_{x_s y_s} = J_{xy} - x_A \cdot y_A \cdot m =$$

$$= -ma^2 - \left(-\frac{4a}{3}\right) \cdot \frac{3a}{3} \cdot m =$$

$$= -ma^2 + \frac{4}{3}a^2 m = \frac{1}{3}ma^2$$