Signali i sustavi – Zadaci za aktivnost – Tjedan 09.

Akademska školska godina 2006./2007.

1. Naći odziv mirnog sustava opisanog jednadžbom diferencija:

$$3y(n+2)+6y(n+1)+3y(n)=2u(n+1)-5u(n).$$

Sustav je pobuđen nizom impulsa $u(n)=\{\ldots,\underline{0},0,1,2,1,0,0,\ldots\}$, gdje je podvučena vrijednost amplituda impulsa u koraku $n=0$.

(1)

$$3y(n+2)+6y(n+1)+3y(n)=2u(n+1)-5u(n)$$

$$u(n)=\{\ldots,\underline{0},0,1,2,1,0,0\ldots\}$$

$\rightarrow n=0$

$$y(n+2)=-2y(n+1)-y(n)+\frac{2}{3}u(n+1)-\frac{5}{3}u(n)$$

$y(n)=0$ za $n<0$ $\Rightarrow$ $y(0)=y(1)=\boxed{0}$

$n=0$ $\quad y(2)=-2y(1)-y(0)+\frac{2}{3}u(1)-\frac{5}{3}u(0)=\boxed{0}$

$n=1$ $\quad y(3)=-2y(2)-y(1)+\frac{2}{3}u(2)-\frac{5}{3}u(1)=\boxed{\frac{2}{3}}$

$n=2$ $\quad y(4)=-2y(3)-y(2)+\frac{2}{3}u(3)-\frac{5}{3}u(2)=$

$\quad =-2\cdot\frac{2}{3}-0+\frac{2}{3}\cdot 2-\frac{5}{3}=\boxed{-\frac{5}{3}}$

$n=3$ $\quad y(5)=-2y(4)-y(3)+\frac{2}{3}u(4)-\frac{5}{3}u(3)=$

$\quad =-2\left(-\frac{5}{3}\right)-\frac{2}{3}+\frac{2}{3}-\frac{5}{3}\cdot 2=0$

$n=4$ $\quad y(6)=-2y(5)-y(4)+\frac{2}{3}u(5)-\frac{5}{3}u(4)=0+\frac{5}{3}+0-\frac{5}{3}=0$

$n=5$ $\quad y(7)=-2y(6)-y(5)+\frac{2}{3}u(6)-\frac{5}{3}u(5)=0$

$y(n)=0\,,\quad n>5$

$$y_m(n)=\left\{\ldots\underline{0},0,0,\frac{2}{3},-\frac{5}{3},0,0,0,0\ldots\right\}$$

2. Na ulaz diskretnog sustava narinut je signal *u(n)*. Korištenjem konvolucijske sumacije naći impulsni odziv ako je poznat odziv mirnog sustava *y(n)*. Zadani su ulazni signal $u(n) = \{\ldots, 0, \underline{1}, 2, 3, 4, 5, 6, 7, \ldots\}$ i izlazni signal $y(n) = \{\ldots, 0, \underline{0}, -1, 1, 2, 3, 4, 5, 6, \ldots\}$, gdje je podvučena vrijednost amplituda impulsa u koraku *n=0*.

$$u(n) = (n+1)\mu(n)$$

$$y(n) = (n-1)\mu(n-2) - \delta(n-1)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)u(n-k)$$

$$(n-1)\mu(n-2) - \delta(n-1) = \sum_{k=0}^{\infty} h(k)(n+1-k)$$

$n = 0$

$$\boxed{h(0) = 0}$$

$n = 1$

$$-1 = h(0)\cdot 2 + h(1) \Rightarrow \boxed{h(1) = -1}$$

$n = 2$

$$1 = h(0)\cdot 3 + h(1)\cdot 2 + h(2)\cdot 1$$

$$1 = 0 - 2 + h(2)$$

$$\boxed{h(2) = 3}$$

$n = 3$

$$2 = h(0)\cdot 4 + h(1)\cdot 3 + h(2)\cdot 2 + h(3)\cdot 1$$

$$2 = 0 - 3 + 6 + h(3)$$

$$\boxed{h(3) = -1}$$

$\underline{n=4}$

$$3 = 5 \cdot h(0) + h(1) \cdot 4 + h(2) \cdot 3 + h(3) \cdot 2 + h(4) \cdot 1$$

$$3 = 0 - 4 + 9 - 2 + h(4)$$

$$\boxed{h(4) = 0}$$

$\underline{n=5}$

$$4 = 6\,h(0) + 5\,h(1) + 4\,h(2) + 3\,h(3) + 2\,h(4) + h(5)$$

$$4 = 0 - 5 + 12 - 3 + 0 + h(5)$$

$$\boxed{h(5) = 0}$$

Slutnja ... $h(n) = 0,\ n \geq 4$

Pretpostavimo da za $n > 3$ $h(n) = 0$

$$h(0) = 0,\ h(1) = -1,\ h(2) = 3,\ h(3) = -1$$

$$h(n \geq 4) = 0$$

Tada za n-ti korak vrijedi:

$$n-1 = \sum_{k=0}^{n} h(k)(n+1-k)$$

$$= h(0)(n+1) + h(1)n + h(2)(n-1) + h(3)(n-2) + \ldots + h(n)$$

$$= 0 - n + 3(n-1) - 1(n-2) + 0 + 0 + 0 + \ldots + 0 + h(n)$$

$$= -n + 3n - 3 - n + 2 + h(n)$$

$$\cancel{n-1} = \cancel{n-1} + h(n)$$

$$0 = h(n) \Rightarrow \boxed{h(n) = 0,\ n > 3}$$

$$\boxed{h(n) = \{\underline{0}, -1, 3, -1, 0, 0, \ldots\}}$$

3. Diskretan sustav je opisan jednadžbom diferencija

$$y(n) - 6y(n-1) + 8y(n-2) = 4u(n)\ .$$

Ako je ulaz u sustav $u(n) = 2\mu(n) - 3n\mu(n)$, nađite prirodni, prisilni te totalni odziv sustava uz početne uvjete $y(-1)=2$, $y(-2)=1$.

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$$y(n) - 6y(n-1) + 8y(n-2) = 4u(n)$$

$$u(n) = 2\mu(n) - 3n\mu(n)$$

$$y(-1) = 2$$

$$y(-2) = 1$$

$$y(n) - 6y(n-1) + 8y(n-2) =$$

$$= 8\mu(n) - 12n\mu(n) = (8-12n)\mu(n)$$

$$y_h(n) = c \cdot q^n$$

HOMOGENO

$$\cancel{c} \cdot \cancel{q^n} - 6 \cdot \cancel{c} \cdot \cancel{q^n} \cdot q^{-1} + 8 \cdot \cancel{c} \cdot \cancel{q^n} \cdot q^{-2} = 0$$

$$1 - 6q^{-1} + 8q^{-2} = 0 \quad / \cdot q^2$$

$$q^2 - 6q + 8 = 0$$

$$q_{12} = \frac{6 \pm \sqrt{4}}{2}$$

$$q_1 = 4 \qquad q_2 = 2$$

$$\boxed{y_h(n) = c_1 \cdot 4^n + c_2 \cdot 2^n}$$

PARTIKULARNO

$$y_p = (K_0 + K_1 \cdot n) \cdot 1^n \mu(n)$$

$$(K_0 + K_1 n) - 6 \cdot (K_0 + K_1 (n-1)) + 8 \cdot (K_0 + K_1 \cdot (n-2)) =$$

$$= (8 - 12n)$$

$$K_0 + K_1 n - 6K_0 - 6K_1 n + 6K_1 + 8K_0 + 8K_1 n - 16K_1 = 8 - 12n$$

$$(3K_0 - 10K_1) + 3K_1 n = 8 - 12n$$

$$3K_0 - 10K_1 = 8 \qquad 3K_1 = -12$$

$$3K_0 = 8 - 40 \qquad \boxed{K_1 = -4}$$

$$\boxed{K_0 = -\frac{32}{3}}$$

$$\boxed{y_p = \left(-\frac{32}{3} - 4n\right)}$$

$$y = y_h + y_p$$

$$\boxed{y(n) = c_1 \cdot 4^n + c_2 \cdot 2^n - \frac{32}{3} - 4n}$$

Računanje početnih uvjeta:

Zadana je jednadžba diferencija:

$$y(n) - 6y(n-1) + 8y(n-2) = 4(2\mu(n) - 3n\mu(n))$$

$$y(n) = 6y(n-1) - 8y(n-2) + 8\mu(n) - 12n\mu(n))$$

Kako jedinični step počinje djelovati u $n=0$, potrebno je kao početne uvjete računati $y(0)$ i $y(1)$:

$y(0)=8+12-8=12$

$y(1)=8-12+72-16=52$

Konstante C_1 i C_2 se sada računaju iz tih početnih uvjeta:

$$y(0) = C_1 + C_2 - \frac{32}{3} = 12$$

$$y(1) = 4C_1 + 2C_2 - \frac{32}{3} - 4 = 52$$

Rješavanjem ovih jednadžbi dobije se: $C_1 = \frac{32}{3}$, $C_2 = 12$.

Totalni odziv je prema tome:

$$\boxed{y(n) = \left(\frac{32}{3} \cdot 4^n + 12 \cdot 2^n - \frac{32}{3} - 4n\right)\mu(n)}$$

→ TOTALNI ODZIV

$$\boxed{y(n) = \left(-\frac{32}{3} - 4n\right)\mu(n)} \rightarrow \text{PRISILNI ODZIV}$$

$$\boxed{y(n) = \left(\frac{32}{3} \cdot 4^n + 12 \cdot 2^n\right)\mu(n)} \rightarrow \text{PRIRODNI ODZIV}$$

4. Kontinuirani sustav zadan je diferencijalnom jednadžbom

$$y'(t) + 2y(t) = u(t) + u'(t)\,.$$

Provjerite, bez rješavanja zadane diferencijalne jednadžbe, je li impulsni odziv ovog sustava $h(t) = -e^{-2t}\mu(t) + \delta(t)$.

$\boxed{4}$ $y'(t)+2y(t)=u(t)+u(t)$

$\underline{h(t)=-e^{-2t}\mu(n)+\delta(t)}$ $\Rightarrow h(t)=$

$h'(t)+2h(t)=\delta'(t)+\delta(t)$

$h'(t)=+e^{-2t}\cdot 2\mu(n)-e^{-2t}\delta(t)+\delta'(t)$

$\cancel{2e^{-2t}\mu(n)}-e^{-2t}\delta(t)+\delta'(t)\cancel{-2e^{-2t}\mu(n)}+2\delta(t)=\delta'(t)+\delta(t)$

vrijednost funkcije $-e^{-2t}$ u točki $t=0$

$-e^{-2t}\big|_{t=0}=-e^{0}=-1$

$-\delta(t)+\delta'(t)+2\delta(t)=\delta'(t)+\delta(t)$

$\underline{\delta'(t)+\delta(t)=\delta'(t)+\delta(t)}$

5. Kauzalni LTI kontinuirani sustav zadan je diferencijalnom jednadžbom:

$$y''(t) + 6y'(t) + 13y(t) = u'(t) + 4u(t).$$

Nađite impulsni odziv ovog sustava.

5. $y''(t) + 6y'(t) + 13y(t) = u'(t) + 4u(t)$ $\Rightarrow h''(t) + 6h'(t) + 13h(t) = \delta'(t) + 4\delta(t)$

$h_a''(t) + 6h_a'(t) + 13h_a(t) = \delta(t)$

$h_a(t) = ce^{st}$

$cs^2e^{st} + 6sce^{st} + 13ce^{st} = 0$

$ce^{st}(s^2 + 6s + 13) = 0$

$s^2 + 6s + 13 = 0$

$s_{1,2} = \frac{-6 \pm j4}{2} = -3 \pm j2$

$b_0 = 0$
$b_1 = 1$
$b_2 = 4$

$s_1 = -3 + j2$

$s_2 = -3 - j2$

$h_a(t) = c_1e^{(-3+j2)t} + c_2e^{(-3-j2)t}$

$h_a'(t) = c_1e^{(-3+j2)t} \cdot (-3+j2) + c_2e^{(-3-j2)t} \cdot (-3-j2)$

$h_a(0^+) = 0 \quad h_a'(0^+) = 1$

$h_a(0^+) = c_1 + c_2 = 0 \longrightarrow c_1 = -c_2$

$h_a'(0^+) = c_1(-3+j2) + c_2(-3-j2) = 1$

$-c_2(-3+j2) + c_2(-3-j2) = 1$

$\cancel{3c_2} - j2c_2 - \cancel{3c_2} - j2c_2 = 1$

$-j4c_2 = 1$

$c_2 = \frac{-0{,}25}{j} \cdot \frac{-j}{j}$

$c_2 = 0{,}25j \qquad c_1 = -0{,}25j$

$h_a(t) = -j0{,}25e^{(-3+j2)t} + j0{,}25e^{(-3-j2)t}$

$\rightarrow h_a'(t) = -j0{,}25e^{(-3+j2)t} \cdot (-3+j2) + j0{,}25e^{(-3-j2)t} \cdot (-3-j2)$

$h(t) = b_0h_a''(t) + b_1h_a'(t) + b_2h_a(t)$

$h(t) = -j0{,}25e^{(-3+j2)t}(-3+j2) + j0{,}25e^{(-3-j2)t}(-3-j2) - 4j \cdot 0{,}25e^{(-3+j2)t} + 4j \cdot 0{,}25e^{(-3-j2)t}$

$h(t) = j0{,}75e^{(-3+j2)t} + 0{,}5e^{(-3+j2)t} - j0{,}75e^{(-3-j2)t} + 0{,}5e^{(-3-j2)t} - je^{(-3+j2)t} + je^{(-3-j2)t}$

$$\boxed{h(t) = -j0{,}25e^{(-3+j2)t} + j0{,}25e^{(-3-j2)t} + 0{,}5e^{(-3+j2)t} + 0{,}5e^{(-3-j2)t}}$$

6. Zadan je kontinuirani sustav

$$y'''(t) - y''(t) + y'(t) + 39y = u''(t) + 2u(t)\,.$$

Ispitajte da li je ovaj sustav stabilan.

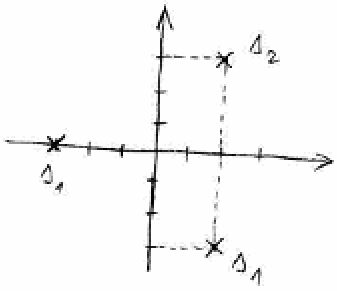
$$y'''(t) - y''(t) + y'(t) + 39\,y(t) = u''(t) + 2u(t)$$

$$y_h(t) = Ce^{st}$$
$$y'_h(t) = Cs\,e^{st}$$
$$y''_h(t) = Cs^2e^{st}$$
$$y'''_h(t) = Cs^3e^{st}$$

$$Ce^{st}\left[s^3 - s^2 + s + 39\right] = 0$$
$$s^3 - s^2 + s + 39 = 0$$
$$\boxed{s_1 = -3}$$

$$(s^3 - s^2 + s + 39) : (s+3) = s^2 - 4s + 13$$
$$\underline{-s^3 - 3s^2}$$
$$-4s^2 + s$$
$$\underline{4s^2 + 12s}$$
$$13s + 39$$
$$\underline{-(13s + 39)}$$
$$=$$

$$(s+3)(s^2 - 4s + 13) = 0$$
$$s_{2,3} = \frac{4 \pm 6j}{2}$$
$$s_2 = 2 + 3j$$
$$s_3 = 2 - 3j$$



$$y_h(t) = C_1e^{-3t} + C_2\,e^{(2+3j)t} + C_3\,e^{(2-3j)t}$$

$$\lim_{t\to\infty} y_h(t) = 0 + \lim_{t\to\infty}\left(C_2\,e^{(2+3j)t} + C_3\,e^{(2-3j)t}\right) = \infty$$

SUSTAV JE NESTABILAN

ILI

Postoje kompleks. frekv. za koje je $Re\{s\} > 0$

SUSTAV JE NESTABILAN

7. Kontinuirani sustav opisan je diferencijalnom jednadžbom čije je homogeno rješenje

$$y_h(t) = \frac{1}{2}e^{3t} + \frac{5}{3}e^{2t}.$$

Sustav nema nula (u diferencijalnoj jednadžbi ne postoje derivacije ulaza). Odredite tu diferencijalnu jednadžbu. Odredite ukupan i odziv mirnog sustava ako se sustav pobudi s $u(t) = \frac{5}{2}e^{2t}$, $t \geq 0$. Odredite početna stanja. Ispitajte stabilnost sustava.

7.

$$y_h(t) = \frac{1}{2}e^{3t} + \frac{5}{3}e^{2t}$$

$$u(t) = \frac{5}{2}e^{2t}, \quad t \geq 0$$

$$\boxed{\begin{matrix} s_1 = 3 \\ s_2 = 2 \end{matrix}} \qquad \begin{matrix} c_1 = \frac{1}{2} \\ c_2 = \frac{5}{3} \end{matrix}$$

$$\left.\begin{matrix} s_1 > 0 \\ s_2 > 0 \end{matrix}\right\} \quad \underline{\text{SUSTAV NIJE}} \quad \underline{\text{STABILAN}}$$

$$(s-3)\cdot(s-2) = 0$$

$$s^2 - 5s + 6 = 0$$

$$\boxed{y''(t) - 5y'(t) + 6y(t) = u(t)}$$

– PARTIKULARNO

$$y''(t) - 5y'(t) + 6y(t) = \frac{5}{2}e^{2t}$$

pretpostavimo rješenje:

$$y_p = A\cdot t\cdot e^{2t}$$

$$y'(t) = Ae^{2t} + 2Ate^{2t}$$

$$y''(t) = 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t}$$

$$2A\cancel{e^{2t}} + 2A\cancel{e^{2t}} + 4At\cancel{e^{2t}} - 5A\cancel{e^{2t}} - 10At\cancel{e^{2t}} + 6At\cancel{e^{2t}} = \frac{5}{2}\cancel{e^{2t}}$$

$$-A = \frac{5}{2}$$

$$\boxed{A = -\frac{5}{2}}$$

$$\boxed{y_p = -\frac{5}{2}\,t\,e^{2t}}$$

$$\boxed{y = y_H + y_p = \frac{1}{2}e^{3t} + \frac{5}{3}e^{2t} - \frac{5}{2}\cdot t\cdot e^{2t}}$$

NASTAVAK (7)

$$y(t) = K_1 \cdot e^{3t} + K_2 \cdot e^{2t} - \frac{5}{2} \cdot t \cdot e^{2t}$$

$$y'(t) = 3K_1 e^{3t} + 2K_2 e^{2t} - \frac{5}{2} e^{2t} - 5te^{2t}$$

$y(0) = 0$ $\qquad$ $y'(0) = 0$

$K_1 + K_2 = 0$ $\qquad$ $3K_1 + 2K_2 - \frac{5}{2} = 0$

$K_1 = -K_2$ $\qquad$ $-3K_2 + 2K_2 = \frac{5}{2}$

$$\boxed{K_2 = \frac{-5}{2}}$$

$$\boxed{K_1 = \frac{5}{2}}$$

$$\boxed{y(t) = \frac{5}{2} e^{3t} - \frac{5}{2} e^{2t} - \frac{5}{2} t e^{2t}}$$

$\Rightarrow$ ODZIV MIRNOG SUSTAVA

POČETNA STANJA:

$$\boxed{y(0) = \frac{1}{2} + \frac{5}{3} = \frac{13}{6}}$$

$$y'(t) = \frac{3}{2} e^{3t} + \frac{10}{3} e^{2t} - \frac{5}{2} e^{2t} - 5te^{2t}$$

$$y'(0) = \frac{3}{2} + \frac{10}{3} - \frac{5}{2} = \frac{-6+20}{6} = \frac{\cancel{14}^{7}}{\cancel{6}_{3}} = \frac{7}{3}$$

$$\boxed{y'(0) = \frac{7}{3}}$$

$y(0^-)=y(0^+)=y(0)$
$y'(0^-)=y'(0^+)=y'(0)$

8. Riješite diferencijalnu jednadžbu

$$y'(t) + 2y(t) = u(t)\,,$$

ako je ulaz

$$u(t) = A \cdot \cos(\omega_0 t) \cdot \mu(t)\,,$$

pri čemu je A realna konstanta i uzimajući da su početni uvjeti jednaki nula. Bez dodatnog računanja odredite rješenje ove jednadžbe ako je ulaz

$$u(t) = B \cdot \cos\omega_0 (t-1) \cdot \mu(t-1)\,.$$

$y'(t) + 2y(t) = u(t)$

$y(t) = y_h(t) + y_p(t)$

$y_h = ?$ $\qquad u(t) = A\cos(\omega_0 t)\mu(t)$

$h'(t) + 2h(t) = 0 \qquad h(t) = Ce^{st}$

$s\,Ce^{st} + 2Ce^{st} = 0 \qquad Ce^{st}(s+2) = 0 \qquad \boxed{s = -2}$

$h(t) = Ce^{st}$

$y_p = ?$

$y_p(t) = K_1\cos(\omega_0 t) + K_2\sin(\omega_0 t)$

$y_p'(t) = -\omega_0 K_1 \sin(\omega_0 t) + \omega_0 K_2 \cos(\omega_0 t)$

$-\omega_0 K_1 \sin(\omega_0 t) + \omega_0 K_2 \cos(\omega_0 t) + 2K_1 \cos(\omega_0 t) +$
$+ 2K_2 \sin(\omega_0 t) = A\cos(\omega_0 t)$

$$\sin(\omega_0 t)\left[-\omega_0 K_1 + 2K_2\right] + \cos(\omega_0 t)\left[\omega_0 K_2 + 2K_1\right] =$$
$$= A\cos(\omega_0 t)$$

$$-\omega_0 K_1 + 2K_2 = 0 \Rightarrow 2K_2 = \omega_0 K_1 \Rightarrow K_2 = \frac{\omega_0}{2} K_1$$
$$\omega_0 K_2 + 2K_1 = A \qquad \frac{\omega_0^2}{2} K_1 + 2K_1 = A \quad K_1\left(\frac{\omega_0^2}{2} + 2\right) = A$$

$$K_1 = \frac{A}{\frac{\omega_0^2}{2} + 2} \qquad K_2 = \frac{\frac{\omega_0}{2} A}{\frac{\omega_0^2}{2} + 2}$$

$$y_p(t) = K_1 \cos(\omega_0 t) + K_2 \sin(\omega_0 t) =$$
$$= \frac{A}{\frac{\omega_0^2}{2} + 2} \cos(\omega_0 t) + \frac{\frac{\omega_0}{2} A}{\frac{\omega_0^2}{2} + 2} \sin(\omega_0 t) =$$
$$= A\left[\frac{2}{\omega_0^2 + 4} \cos(\omega_0 t) + \frac{\omega_0}{\omega_0^2 + 4} \sin(\omega_0 t)\right]$$

$$y(t) = y_h(t) + y_p(t) = Ce^{-2t} + A\left[\frac{2}{\omega_0^2+4}\cos(\omega_0 t) + \frac{\omega_0}{\omega_0^2+4}\sin(\omega_0 t)\right]$$

$$y(0) = C + A\left(\frac{2}{\omega_0^2+4}\right) = 0 \Rightarrow C = \frac{-2A}{\omega_0^2+4}$$

$$y(t) = \frac{-2A}{\omega_0^2+4} e^{-2t} + A\left[\frac{2}{\omega_0^2+4}\cos(\omega_0 t) + \frac{\omega_0}{\omega_0^2+4}\sin(\omega_0 t)\right]$$

ZA $u(t) = B\cos\omega(t-1)\,\mu(t-1)$

$$\boxed{y(t) = \frac{-2B}{\omega_0^2+4} e^{-2(t-1)} + B\cdot\left[\frac{2}{\omega_0^2+4}\cos\omega_0(t-1) + \frac{\omega_0}{\omega_0^2+4}\sin\omega_0(t-1)\right]}$$