Skalarna i vektorska polja

OSNOVNE FORMULE

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$f(r) \rightarrow r \begin{cases} \rightarrow x \\ \rightarrow y \\ \rightarrow z \end{cases}$$

$$\frac{\partial f(r)}{\partial x_{,}y_{,}z} = f'(r)\,\frac{\partial r}{\partial x_{,}y_{,}z} = f'(r)\cdot\frac{x_{,}y_{,}z}{r}$$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

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$$\frac{\partial f}{\partial \vec{s}} = (\vec{s}_0\cdot\nabla)\,f$$

$$\frac{\partial \vec{f}}{\partial \vec{s}} = (\vec{s}_0\,\nabla)\,\vec{f}$$

$$\vec{s}_0 = \frac{\vec{s}}{|\vec{s}|} = s_{01}\vec{i} + s_{02}\vec{j} + s_{03}\vec{k}$$

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① $\frac{\partial f(r)}{\partial \vec{s}} = (\vec{s}_0\cdot\nabla)\,f(r) = *$

$$\vec{s}_0\cdot\nabla = (s_{01}\vec{i} + s_{02}\vec{j} + s_{03}\vec{k})\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) = s_{01}\frac{\partial}{\partial x} + s_{02}\frac{\partial}{\partial y} + s_{03}\frac{\partial}{\partial z}$$

$$* = s_{01}f'(r)\frac{x}{r} + s_{02}f'(r)\frac{y}{r} + s_{03}f'(r)\frac{z}{r} = \frac{f'(r)}{r}\overbrace{(s_{01}x + s_{02}y + s_{03}z)}^{\vec{s}_0\cdot\vec{r}} = \boxed{f'(r)\cdot\vec{s}_0\cdot\vec{r}_0}$$

② $\frac{\partial}{\partial \vec{s}}\left[f(r)(\vec{a}\cdot\vec{r})\right] = (\vec{s}_0\nabla)\big(f(r)\underbrace{(\vec{a}\cdot\vec{r})}_{a_1x+a_2y+a_3z}\big) = \left(s_{01}\frac{\partial}{\partial x} + s_{02}\frac{\partial}{\partial y} + s_{03}\frac{\partial}{\partial z}\right)\big(\underbrace{a_1x f(r) + a_2y f(r) + a_3z f(r)}_{A}\big) = *$

$$\frac{\partial}{\partial x}(A) = a_1\cdot\left(f(r) + x f'(r)\frac{x}{r}\right) + a_2y f'(r)\frac{x}{r} + a_3z f'(r)\frac{x}{r} = a_1 f(r) + f'(r)\frac{x}{r}\underbrace{(a_1x + a_2y + a_3z)}_{\vec{a}\cdot\vec{r}} =$$

$$= a_1 f(r) + f'(r)\frac{x}{r}(\vec{a}\,\vec{r})$$

$$* = s_{01}a_1 f(r) + s_{01}f'(r)\frac{x}{r}(\vec{a}\,\vec{r}) + s_{02}a_2 f(r) + s_{02}f'(r)\frac{y}{r}(\vec{a}\,\vec{r}) + s_{03}a_3 f(r) + s_{03}f'(r)\frac{z}{r}(\vec{a}\,\vec{r}) =$$

$$= f(r)(\vec{s}_0\cdot\vec{a}) + f'(r)\left(\frac{\vec{s}_0\,\vec{r}}{r}\right)(\vec{a}\,\vec{r}) = f(r)(\vec{s}_0\vec{a}) + f'(r)(\vec{s}_0\,\vec{r}_0)(\vec{a}\,\vec{r})$$

(3) $\dfrac{\partial v}{\partial \vec{s}}$, $v = xy + z^2 + \cos x$

$$\frac{\partial v}{\partial \vec{s}} = (\vec{s}_0 \nabla) v = \left(s_{01}\frac{\partial}{\partial x} + s_{02}\frac{\partial}{\partial y} + s_{03}\frac{\partial}{\partial z}\right)\left(xy + z^2 + \cos x\right) =$$

$$= s_{01}\left(y - \sin x\right) + s_{02}(x) + s_{03}(2z)$$

npr. $\vec{s} = \vec{i} + \vec{j} + \vec{k} \Rightarrow s_0 = \dfrac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$

(4) $$\frac{\partial \vec{r}}{\partial \vec{s}} = (\vec{s}_0 \nabla)\vec{r} = \left(s_{01}\frac{\partial}{\partial x} + s_{02}\frac{\partial}{\partial y} + s_{03}\frac{\partial}{\partial z}\right)\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = s_{01}\vec{i} + s_{02}\vec{j} + s_{03}\vec{k} = \boxed{\vec{s}_0}$$

(5) $\dfrac{\partial}{\partial \vec{s}}\left[f(r)\cdot\vec{r}\right]$

$\dfrac{\partial}{\partial \vec{s}}\left((\ln r)\cdot(\vec{r})\right) = ?$

$$= (\vec{s}_0 \nabla)\left(f(r)\vec{r}\right) = \left(s_{01}\frac{\partial}{\partial x} + s_{02}\frac{\partial}{\partial y} + s_{03}\frac{\partial}{\partial z}\right)\cdot\left(f(r)x\vec{i} + f(r)y\vec{j} + f(r)z\vec{k}\right) = *$$

$$\frac{\partial}{\partial x}\left(f(r)x\vec{i} + f(r)y\vec{j} + f(r)z\vec{k}\right) = \left(f'(r)\frac{x}{r}\cdot x + f(r)\right)\vec{i} + f'(r)\frac{x}{r}y\vec{j} + f'(r)\frac{x}{r}z\vec{k} =$$

$$= f'(r)\frac{x}{r}\underbrace{\left(x\vec{i} + y\vec{j} + z\vec{k}\right)}_{\vec{r}} + f(r)\vec{i} = \underbrace{f'(r)\frac{x}{r}\vec{r} + f(r)\vec{i}}_{\text{preslika se na } y, z}$$

$$* = \left(s_{01}\left(f(r)\frac{x}{r}\vec{r} + f(r)\vec{i}\right)\right) + \left(s_{02}\left(f(r)\frac{y}{r}\vec{r} + f(r)\vec{j}\right)\right) + \left(s_{03}\left(f(r)\frac{z}{r}\vec{r} + f(r)\vec{k}\right)\right) =$$

$$= \frac{f'(r)}{r}\underbrace{\left(s_{01}x + s_{02}y + s_{03}z\right)}_{\vec{s}_0\vec{r}}\vec{r} + f(r)\left(s_{01}\vec{i} + s_{02}\vec{j} + s_{03}\vec{k}\right) =$$

$$= \boxed{f'(r)\left(\vec{s}_0\cdot\vec{r}_0\right)\vec{r}_0 + f(r)\vec{s}_0}$$

(6) $$\frac{\partial}{\partial \vec{s}}\left(\overset{f(r)}{\ln r}\,\vec{r}\right) = \frac{1}{r}\left(\vec{s}_0\vec{r}_0\right)\vec{r} + \ln r\,\vec{s}_0 = \left(\vec{s}_0\vec{r}_0\right)\vec{r}_0 + \ln r\,\vec{s}_0$$

(7) $\dfrac{\partial(\vec a\times\vec r)}{\partial \vec s} = (\vec s_0 \nabla)(\vec a\times\vec r) = *$

$$\vec a\times\vec r = \begin{vmatrix} \vec i & \vec j & \vec k\\ a_1&a_2&a_3\\ x&y&z\end{vmatrix} = \vec i\overset{A}{(a_2 z - a_3 y)} + \vec j\overset{B}{(a_3 x - a_1 z)} + \vec k\overset{C}{(a_1 y - a_2 x)} = A\vec i + B\vec j + C\vec k$$

$$* = \left(s_{01}\frac{\partial}{\partial x} + s_{02}\frac{\partial}{\partial y} + s_{03}\frac{\partial}{\partial z}\right)\left(A\vec i + B\vec j + C\vec k\right) = s_{01}\left(a_3\vec j - a_2\vec k\right) + s_{02}\left(-a_3\vec i + a_1\vec k\right) +$$
$$+ s_{03}\left(a_2\vec i - a_1\vec j\right) =$$
$$= \vec i\left(-s_{02}a_3 + s_{03}a_2\right) - \vec j\left(-s_{01}a_3 + s_{03}a_1\right) + \vec k\left(-s_{01}a_2 + s_{02}a_1\right) = \begin{vmatrix} \vec i & \vec j & \vec k\\ a_1&a_2&a_3\\ s_{01}&s_{02}&s_{03}\end{vmatrix} =$$
$$= \vec a\times\vec s_0$$

(8) $\nabla f(r) = \left(\frac{\partial}{\partial x}\vec i + \frac{\partial}{\partial y}\vec j + \frac{\partial}{\partial z}\vec k\right)(f(r) = f'(r)\frac{x}{r}\vec i + f'(r)\frac{y}{r}\vec j + f'(r)\frac{z}{r}\vec k = \frac{f'(r)}{r}\vec r = f'(r)\vec r_0$

(9) $\nabla\left[f(r)\vec r\right] = \left(\frac{\partial}{\partial x}\vec i + \frac{\partial}{\partial y}\vec j + \frac{\partial}{\partial z}\vec k\right)\left(f(r)x\vec i + f(r)y\vec j + f(r)z\vec k\right) =$

$$= f'(r)\frac{x}{r}x + f(r) + f'(r)\frac{y}{r}y + f(r) + f'(r)\frac{z}{r} + f(r) = r f'(r) + 3f(r)$$

(10) $\nabla\left[r\,\nabla(r\cdot\vec r)\right] = ?$

$\nabla\left[f(r)\vec r\right] = r f'(r) + 3f(r) = r\cdot 1 + 3r = 4r$

$\nabla(4r^2) = f'(r)\vec r_0 = 8r\,\vec r_0$

(11) $\nabla\left[(\vec a\vec r)\underbrace{\nabla\left(\tfrac{1}{r}\right)}\right] = *$

$\nabla(f(r) = f'(r)\vec r_0 = -\frac{1}{r^2}\vec r_0 = -\frac{1}{r^3}\vec r$

$$\left((\vec a\vec r)\nabla\frac{1}{r}\right) = (a_1x + a_2y + a_3z)\left(-\frac{1}{r^3}\vec r\right) = -\frac{a_1x + a_2x + a_3y}{r^3}x\vec i - \frac{a_1x + a_2y + a_3z}{r^3}y\vec j -$$
$$-\frac{a_1x + a_2y + a_3z}{r^3}z\vec k$$

$$\frac{\partial}{\partial x}\left(-\frac{a_1x + a_2y + a_3z}{r^3}x\right) = -\left(\frac{a_1r^3 - (\vec a\vec r)3r^2\overset{\frac{x}{r}}{\frac{\partial r}{\partial x}}}{r^6}x + \frac{\vec a\vec r}{r^3}\right) = -\left(\frac{a_1r^3 - 3xr(\vec a\vec r)}{r^6}x + \frac{\vec a\vec r}{r^3}\right)$$

$$* = -\frac{a_1r^3x}{r^6} + \frac{3x^2r(\vec a\vec r)}{r^6} - \frac{\vec a\vec r}{r^3} - \frac{a_2r^3y}{r^6} + \frac{3y^2r(\vec a\vec r)}{r^6} - \frac{\vec a\vec r}{r^3} - \frac{a_3r^3z}{r^6} + \frac{3z^2r(\vec a\vec r)}{r^6} - \frac{\vec a\vec r}{r^3} =$$
$$= -\frac{1}{r^3}\big(\underbrace{a_1x + a_2y + a_3z}_{\vec a\vec r}\big) + \frac{3}{r^5}(x^2+y^2+z^2)(\vec a\vec r) - \frac{3\vec a\vec r}{r^3} = \frac{-4\vec a\vec r}{r^3} + \frac{3(\vec a\vec r)}{r^3} = -\frac{\vec a\vec r}{r^3} \text{※}$$

(12) $\nabla\left(\frac{1}{\vec{a}\vec{r}}\right)$

(13) $\nabla\left[f(r)(\vec{a}\times\vec{r})\right]$ } za vježbu

(14) $\nabla\left(r^2\cdot\frac{\partial r^2}{\partial\vec{a}}\right) = *$

$$\frac{\partial r^2}{\partial \vec{a}} = (a_0\nabla)(r^2) = 2r\cdot(\vec{a}_0\cdot\vec{r}_0)$$

$r^2 = x^2+y^2+z^2$

$$* = r^2 2r f(r)\cdot a_0 \frac{\vec{r}}{r} = 2r^2(\vec{a}_0\vec{r}) = 2r^2(a_{01}x + a_{02}y + a_{03}z) =$$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left(2r^2 a_{01}x + 2r^2 a_{02}y + 2r^2 a_{03}z\right) =$$

$$= \left(2a_{01}\left(2r\frac{x}{r}x + r^2\right) + 2a_{02}y\cdot 2r\frac{x}{r} + 2a_{03}z\cdot 2r\frac{x}{r}\right)\vec{i} \ldots$$

analogno za $\vec{j}$ i $\vec{k}$

(15) $\nabla\left(\frac{r\,\vec{r}}{\vec{a}\,\vec{r}}\right)$

(16) $\mathrm{div}\left(\frac{\vec{r}}{r^2}\right) = \nabla\left(\frac{\vec{r}}{r^2}\right) = \nabla\left(\frac{1}{r^2}\vec{r}\right) = \nabla\left(f(r)\vec{r}\right)$

(17) $\mathrm{div}(\vec{a}\times\vec{r}) = \nabla(\vec{a}\times\vec{r}) = \nabla(A\vec{i} + B\vec{j} + C\vec{k})\ldots$

(18) $\mathrm{div}\left(\frac{\vec{a}\times\vec{r}}{\vec{a}\vec{r}}\right)$

(19) $\mathrm{rot}(r\vec{r}) = \nabla\times(r\vec{r}) = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ rx & ry & rz\end{vmatrix} = \ldots =$ lijepi oblik

(20) Δr

$$\Delta = \nabla\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\Delta r = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(r) = \frac{r^2-x^2}{r^3} + \frac{r^2-y^2}{r^3} + \frac{r^2-z^2}{r^3} = \frac{3}{r} - \frac{r^2}{r^3} = \frac{2}{r}\ \#$$

$$\frac{\partial^2}{\partial x^2}(r) = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{1\cdot r - x\frac{x}{r}}{r^2} = \frac{r^2-x^2}{r^3}$$

KRIVULJNI INTEGRALI

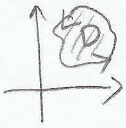
I vrsta $\int_k f\,ds \rightarrow ds = \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt$

parametrizacija

II vrsta $\int_k \vec{f}\,d\vec{r}$, $\int f_1\,dx + f_2\,dy + f_3\,dz$

GREEN → PAZI DA BUDE RAVNINSKA KRIVULJA

$$\oint_\Gamma \vec{f}\,d\vec{r} = \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy$$



$$\vec{f} = f_1\vec{i} + f_2\vec{j}$$

POTENCIJAL

polje potencijalno: $\text{rot}\,\vec{f} = \vec{0}$

$$p(x,y,z) = \int_{x_0}^{x} f_1(x,y,z)\,dx + \int_{y_0}^{y} f_2(x_0,y,z)\,dy + \int_{z_0}^{z} f_3(x_0,y_0,z)\,dz + \boxed{C}$$

$$\int_A^B \vec{f}\,d\vec{r} = p(B) - p(A)$$

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① $\vec{r}(t) = (\underbrace{3\cos t}_{x(t)}, \underbrace{5\sin t}_{y(t)}, \underbrace{4\cos t}_{z(t)})$, $t \in (0, 2\pi]$

$x = 3\cos t$ $\quad x' = -3\sin t\,dt$ $\quad ds = \sqrt{9\sin^2 t + 25\cos^2 t + 16\sin^2 t}\,dt = \sqrt{25}\,dt = 5\,dt$

$y = 5\sin t$ $\quad y' = 5\cos t\,dt$ $\quad \int_A^B ds = \int_0^{2\pi} 5\,dt = 10\pi$

$z = 4\cos t$ $\quad z' = -4\sin t\,dt$

② $k \ldots\ x^2 = 3y \cap 2xy = 9z$

$A(0, y_A, z_A) \rightarrow B(3, y_B, z_B)$

NOTE: onaj koji je pod koordinatom stavi da je $t \Rightarrow x = t$

$dx = dt$

$y = \frac{1}{3}t^2 \quad dy = \frac{2}{3}t\,dt$

$z = \frac{2}{9}xy = \frac{2}{9}t\,\frac{1}{3}t^2 = \frac{2}{27}t^3 \quad dz = \frac{2}{9}t^2\,dt$

$t_A = 0 \quad t_B = 3$

$$\int_0^3 \sqrt{1^2 + \frac{4}{9}t^2 + \frac{4}{81}t^4}\,dt = \int_0^3 \frac{\sqrt{(9+2t^2)^2}}{81}\,dt = \int_0^3 \frac{9+2t^2}{81}\,dt = \ldots$$

$$\sqrt{\frac{81+36t^2+4t^2}{81}}$$

③ $x^2+(y+1)^2+z^2=1$
$(x+1)^2+y^2+z^2=1$ } presjek

$\cancel{x^2}+\cancel{y^2}+2y+\cancel{1}+\cancel{z^2}=\cancel{1}$
$\cancel{x^2}+2x+\cancel{1}+\cancel{y^2}+\cancel{z^2}=\cancel{1}$ } $-$

$$2y-2x=0$$

$\boxed{y=x} \longrightarrow$ u 1. jedn.

$$x^2+(x+1)^2+z^2=1$$

$$x^2+x^2+2x+1+z^2=1$$

$$2x^2+2x+z^2=0$$

$$2\left(x^2+x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right)+z^2=0$$

$$2\left(x+\frac{1}{2}\right)^2-\frac{1}{2}+z^2=0$$

$$\frac{\left(x+\frac{1}{2}\right)^2}{\left(\frac{1}{2}\right)^2}+\frac{z^2}{\left(\frac{1}{\sqrt{2}}\right)^2}=1$$

$\rightarrow x+\frac{1}{2}=\frac{1}{2}\cos t$

$x=\frac{1}{2}\cos t-\frac{1}{2} \Rightarrow dx=-\frac{1}{2}\sin t\,dt$

$\rightarrow y=\frac{1}{2}\cos t-\frac{1}{2} \Rightarrow dy=-\frac{1}{2}\sin t\,dt$

$\rightarrow z=\frac{1}{\sqrt{2}}\sin t \Rightarrow dz=\frac{1}{\sqrt{2}}\cos t\,dt$

$t\rightarrow[0,2\pi] \rightarrow$ elipsa

$$\int_0^{2\pi}\sqrt{\frac{1}{4}\sin^2 t+\frac{1}{4}\sin^2 t+\frac{1}{2}\cos^2 t}\,dt=\int_0^{2\pi}\frac{dt}{\sqrt{2}}=\frac{1}{\sqrt{2}}2\pi=\pi\sqrt{2}$$

④ $\int_\Gamma y\cos x \, ds \qquad \Gamma \ldots y=\sin x \,, \; x\in\left[0,\frac{\pi}{2}\right]$

$x=t \quad y=\sin t \,, \quad t\in\left[0,\frac{\pi}{2}\right]$

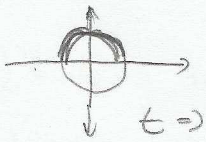
$dx=dt \quad dy=\cos t\, dt$

$ds=\sqrt{1+\cos^2 t}\, dt$

$$\int_0^{\frac{\pi}{2}} \sin t \cos t \sqrt{1+\cos^2 t}\, dt = \left| \begin{matrix} u=1+\cos^2 t \\ du = 2\cos t\,(-\sin t)\, dt \end{matrix} \right| = -\frac{1}{2}\int \sqrt{u}\, du = -\frac{1}{\cancel{2}} \frac{\cancel{2}}{3} u^{\frac{3}{2}} =$$

$$= -\frac{1}{3}\left(1+\cos^2 t\right)^{\frac{3}{2}} \Big|_0^{\frac{\pi}{2}} = \ldots$$

⑤ $\int_\Gamma (1-y^2)\, ds \,, \quad \Gamma \ldots x^2+y^2=1 \,, \; y\geq 0$



$x=\cos t \quad dx=-\sin t\, dt$

$y=\sin t \quad dy=\cos t\, dt \qquad t \Rightarrow [0,\pi]$

$ds=\sqrt{\cos^2 t+\sin^2 t}\, dt = dt$

$$\int_0^{\pi} (1-\sin^2 t)\, dt = \int_0^{\pi} \cos^2 t \, dt = \int_0^{\pi} \frac{1+\cos 2t}{2}\, dt \ldots$$

⑥ $\int_{\overline{AB}} x^2 y\, ds \,, \quad \overline{AB} \ldots \; y=x+2 \,, \; A(1;y_A) \longrightarrow B(x_B,5)$

$x=t \quad dx=dt \qquad t_A=1$

$y=t+2 \quad dy=dt \qquad t_B=3$

$ds=\sqrt{1+1}=\sqrt{2}\, dt$

$$\int_1^3 t^2(t+2)\sqrt{2}\, dt = \sqrt{2}\int_1^3 t^3 dt + \sqrt{2}\int_1^3 2\, dt = \ldots$$

⑦ $\int_\Gamma 2x\, ds \qquad \Gamma \ldots y=x^2 \,, \; x\in(0,1)$

$x=t \quad y=t^2 \quad dx=dt$

$dy=2t\, dt$

$ds=\sqrt{1+4t^2}\, dt \quad t\in(0,1)$

$$\int_0^1 2t\sqrt{1+4t^2}\, dt = \left| \begin{matrix} 1+4t^2=u \\ du=8t\, dt \end{matrix} \right| = \ldots = \frac{1}{4}\int_1^5 \sqrt{u}\, du = \ldots$$

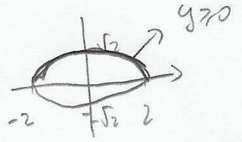
⑧ $\int_\Gamma y\, ds, \qquad \Gamma \; y^2=4x \qquad O(0,0) \to A(2,2\sqrt{2})$

$y=t \quad x=\frac{1}{4}t^2 \qquad t\in[0,2\sqrt{2}]$

$dy=dt \quad dx=\frac{1}{2}t\, dt$

$$\int_0^{2\sqrt{2}} t\sqrt{\frac{1}{4}t^2+1}\, dt \ldots$$

(9) $\int_k (x^2+y^2)\,ds \qquad k \dots x^2+2y^2=4 \ \cap \ z=y \quad , y\geq 0$



$\left(\frac{x}{2}\right)^2+\left(\frac{y}{\sqrt{2}}\right)^2=1$

$t\in[0,\pi]$

$x=2\cos t$

$y=\sqrt{2}\sin t$

$z=\sqrt{2}\sin t$

$dx=-2\sin t\,dt$

$dy=\sqrt{2}\cos t\,dt$

$dz=\sqrt{2}\cos t\,dt$

$ds=\sqrt{4\sin^2 t+2\cos^2 t+2\cos^2 t}\,dt=2\,dt$

$$\int_0^{\pi}\left(4\cos^2 t+2\sin^2 t\right)=\int_0^{\pi}(2+2\cos^2 t)\cdot 2\,dt$$

$\hookrightarrow \frac{1+\cos 2t}{2}$

(10) $\int_{\Gamma} y\,ds \quad , \quad \Gamma \dots r=1+\cos\varphi \quad , \text{I kv.}$

moglo se i preko one formule $\sqrt{(r(\varphi))^2+(r'(\varphi))^2}\,d\varphi$

$x=r\cos\varphi=(1+\cos\varphi)\cos\varphi \qquad dx=\overbrace{(-\sin\varphi\cos\varphi-(1+\cos\varphi)\sin\varphi)}^{(2\cos\varphi+1)(-\sin\varphi)}\,d\varphi$

$y=r\sin\varphi=(1+\cos\varphi)\sin\varphi \qquad dy=\underbrace{(-\sin\varphi\cos\varphi+(1+\cos\varphi)\cos\varphi)}_{-1+\cos\varphi(2\cos\varphi+1)}\,d\varphi$

$$ds=\sqrt{\sin^2\varphi\cdot(2\cos\varphi+1)^2+(-1+\cos\varphi(2\cos\varphi+1))^2}\,d\varphi=\sqrt{\sin^2\varphi(1+2\cos\varphi)^2+1-2\cos\varphi(1+2\cos\varphi)+\cos^2\varphi(1+2\cos\varphi)^2}=$$

$$=\sqrt{(1+2\cos\varphi)^2+1-2\cos\varphi(1+2\cos\varphi)}\,d\varphi=\sqrt{(1+2\cos\varphi)(1+2\cos\varphi-2\cos\varphi+1)}\,d\varphi=\sqrt{2+2\cos\varphi}\,d\varphi$$

$$\int_0^{\frac{\pi}{2}}(1+\cos\varphi)\sin\varphi\sqrt{1+\cos\varphi}\cdot\sqrt{2}\,d\varphi=\left|\begin{matrix}1+\cos\varphi=u\\ -\sin\varphi\,d\varphi=du\end{matrix}\right|=\int\sqrt{2}u^{\frac{3}{2}}(-du)=-\sqrt{2}\,\frac{2}{5}u^{\frac{5}{2}}=$$

$$=\frac{-2\sqrt{2}}{5}(1+\cos\varphi)^{5/2}\Big|_0^{\frac{\pi}{2}}=\dots=-\frac{2\sqrt{2}}{5}+\frac{2\sqrt{2}}{5}\cdot 4\sqrt{2}\dots$$

(11) $\int_{\Gamma}\frac{x}{\sqrt[4]{\underbrace{x^2+y^2}_{r^2}}}\,ds \quad , \quad \Gamma\dots r=1+\cos\varphi, \quad \varphi\in[0,2\pi]$

$ds=\sqrt{2+2\cos\varphi}\,d\varphi$

$$\int_0^{2\pi}\frac{(1+\cos\varphi)\cos\varphi}{\cancel{\sqrt[4]{(1+\cos\varphi)^2}}}\cdot\sqrt{2}\cdot\cancel{\sqrt{1+\cos\varphi}}\,d\varphi=\sqrt{2}\int_0^{2\pi}(\cos^2\varphi+\cos\varphi)\,d\varphi\dots$$

(12) $\int_C\sqrt{xy-z^2+1}\,ds, \quad C\dots x^2+y^2+z^2=1 \ \cap \ y=2x$

$x^2+4x^2+z^2=1$

$5x^2+z^2=1$

$\left(\frac{x}{\frac{1}{\sqrt{5}}}\right)^2+z^2=1$

$t\in[0,2\pi]$

$x=\frac{1}{\sqrt{5}}\cos t \qquad dx=\frac{1}{\sqrt{5}}(-\sin t)\,dt$

$y=\frac{2}{\sqrt{5}}\cos t \qquad dy=\frac{2}{\sqrt{5}}(-\sin t)\,dt$

$z=\sin t \qquad dz=\cos t\,dt$

$ds=\sqrt{\frac{1}{5}\sin^2+\frac{4}{5}\sin^2+\cos^2}=\sqrt{1}\,dt=dt$

$$\cancel{\int_0^{2\pi}\sqrt{\frac{1}{\sqrt{5}}\cos t\cdot\frac{2}{\sqrt{5}}\cos t-\sin^2 t+1}\,dt=}$$

$$=\int_0^{2\pi}\sqrt{\frac{2}{5}\cos^2 t\underbrace{-\sin^2 t+1}_{\cos^2 t}}\,dt=\int_0^{2\pi}\sqrt{\frac{2}{5}\cos^2 t-\cos^2 t}\,dt=\int_0^{2\pi}\sqrt{\frac{7}{5}}\,|\cos t|\,dt$$

rastavi na + od $0\to\pi$

i − od $\pi\to 2\pi$

(13) $\int_\Gamma \sqrt{\frac{1}{2}-3yz}\, ds,\ \Gamma \dots\ (x-1)^2+4y^2+4z^2=4$

$\cap$

$z=y$

$(x-1)^2+8y^2=4 \ /:4$

$\left(\frac{x-1}{2}\right)^2+\left(\frac{y}{\frac{1}{\sqrt{2}}}\right)^2=1$

$ds=\sqrt{4\sin^2 t+\frac{1}{2}\cos^2 t+\frac{1}{2}\cos^2 t}\, dt=$

$=\sqrt{1+3\sin^2 t}\, dt$

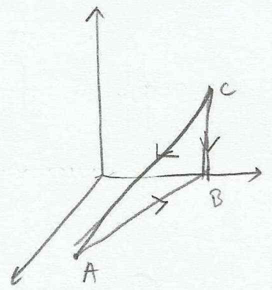
$x=2\cos t+1$

$y=\frac{1}{\sqrt{2}}\sin t$

...

$z=\frac{1}{\sqrt{2}}\sin t$

(14) $\int_\Gamma yz\, dx+4x\, dz$, $\Gamma \dots\ \triangle ABC$, $A(3,2,0)$, $B(0,5,0)$, $C(0,5,3)$



$AB \dots\ y=-x+5$

$x=t \qquad dx=dt$

$y=-t+5 \qquad dy=-dt$

$z=0 \qquad dz=0$

$I_{AB}=0$

$BC \dots\ x=0 \qquad dx=0$

$y=5 \qquad t\in(0,3)$

$z=t$

$I_{BC}=0$

$CA \dots$

$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}=t$

$\frac{x-0}{3-0}=\frac{y-5}{2-5}=\frac{z-3}{0-3}=t$

$x=3t \qquad dx=3\,dt \qquad t\in[0,1]$

$y=-3t+5 \qquad dy=-3\,dt$

$z=-3t+3 \qquad dz=-3\,dt$

$I_{CA}=\int_0^1(-3t+5)\cdot(-3t+3)\,3\,dt+4\cdot 3t\cdot(-3\,dt)=0$

(15) $\oint_k \vec{a}\, d\vec{r},\ k \dots\ x^2+y^2=4 \cap y+z=3$

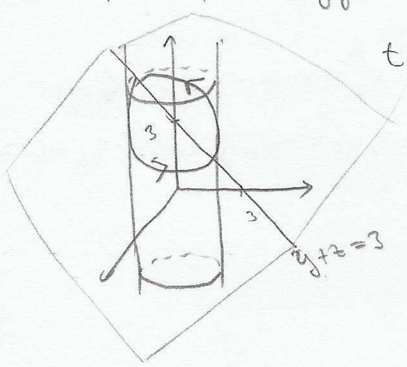
$\vec{a}=z\vec{i}+x\vec{j}$ $\qquad$ ↻ iz $(0,0,4)$

$d\vec{r}=dx\vec{i}+dy\vec{j}+dz\vec{k}$ $\qquad$ $\vec{a}\,d\vec{r}=z\,dx+x\,dy$

$x=2\cos t \qquad dx=-2\sin t\, dt$

$y=2\sin t \qquad dy=2\cos t\, dt$

$z=3-2\sin t \qquad dz=-2\cos t\, dt$



$t\in[0,2\pi)$ → smjer orijentacije u x-y ravnini

$\int_0^{2\pi}(3-2\sin t)(-2\sin t\, dt)+2\cos t\cdot 2\cos t\, dt=\dots$

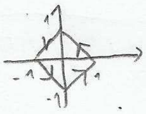
(16) $\int_\Gamma (x+y)\,dx + \ln x\, dy + z\,dz$, Γ ... pravac $A(1,1,1) \to B(2,3,4)$

(17) $\int_\Gamma y\,dx + x\,dy$, $\Gamma \ldots \frac{x^2}{2} + y = 1$

$x = 2\cos t$
$y = \sin t$ ...

(18) $\int_K xy\,dx + yz\,dy + zx\,dz$, $K \ldots x^2+y^2+z^2 = R^2$, $R>0$, $z = x$, $y \geq 0$

$M(0,R,0) \to N\left(\frac{R}{\sqrt{2}}, 0, \frac{R}{\sqrt{2}}\right)$

(19) $\oint_\Gamma \frac{dx+dy}{|x|+|y|}$; Γ ... 

AB ... $y = -x+1$
$x = t \quad dx = dt$
$y = -t+1 \quad dy = -dt$

$\int_{AB} = 0$

BC ... $y = x+1$
$x = t \quad dx = dt$
$y = t+1 \quad dy = dt$

$$\int_{0\,BC}^{-1} \frac{dt+dt}{|t|+|t+1|} = \int_0^{-1} \frac{2dt}{-t+t+1} = -2$$

⋮

CD ... 0

DA ... 2 $\Rightarrow$ 0 sve skupa

(20) $\vec{a} = \frac{1}{x^2}\vec{i} + z\vec{j} + y\vec{k}$

$\vec{a} \overset{?}{=}$ potencijalno ?

$$\operatorname{rot}\vec{a} \overset{?}{=} 0 \to \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{x^2} & z & y \end{vmatrix} = i(0) - j\,0 + k\,0 = \vec{0} \checkmark \quad \vec{a} \to \text{pot}$$

$$p(x,y,z) = \int_{x_0}^{x} \frac{1}{x^2}\,dx + \int_{y_0}^{y} z\,dy + \int_{z_0}^{z} y_0\,dz = -\frac{1}{x}\Big|_{x_0}^{x} + yz - \cancel{y_0 z} + \cancel{y_0 z} - y_0 z_0 + C =$$

$$= -\frac{1}{x} + yz + K$$ → sve zagurano u K

(21) $\int_{(1,0,1)}^{(1,2,2)} (2x+yz)\,dx + xz\,dy + (xy+2z)\,dz$

$$p(x,y,z) = \int_{x_0}^{x} (2x+yz)\,dx + \int_{y_0}^{y} x_0 z\,dy + \int_{z_0}^{z} (x_0 y_0 + 2z)\,dz + C = x^2 - x_0^2 + xyz - \cancel{x_0 yz} + \cancel{x_0 yz} -$$

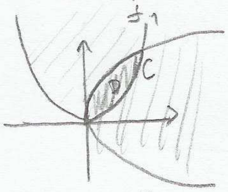
$$- \cancel{x_0 y_0 z} + \cancel{x_0 y_0 z} - x_0 y_0 z_0 + z^2 - z_0^2 + C =$$

$$= x^2 + xyz + z^2 + K$$

$$p(1,2,2) - p(1,0,1) = 9 + K - 2 - K = 7$$

(22) $\displaystyle\int_A^B \frac{yz\,dx + zx\,dy + xy\,dz}{xyz}$, $A(1,1,1)$, $B(2,3,4)$

(23) $\displaystyle\oint_C \underbrace{x^2y}_{f_1}\,dx + \underbrace{(y+xy^2)}_{f_2}\,dy$, $D = \{(x,y)\in\mathbb{R}^2 : y \geq x^2,\ x \geq y^2\}$

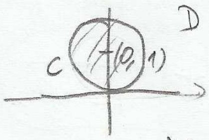


$$\overset{\text{GREEN}}{=} \iint_D (y^2 - x^2)\,dx\,dy = \int_0^1 dx \int_{x^2}^{\sqrt{x}} (y^2 - x^2)\,dy$$

$x \to [0,1]$

$y \Rightarrow [x^2, \sqrt{x}]$

(24) $\displaystyle\oint_C (\sin x + y)\,dx + (x^2 + \cos y)\,dy$



green

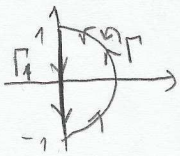
$$\iint_D (2x-1)\,dx\,dy = \int_0^{2\pi} d\varphi \int_0^1 r(2r\cos\varphi - 1)\,dr = \dots$$

$x = r\cos\varphi$

$y = r\sin\varphi + 1$

(25)

$\displaystyle\int_\Gamma (e^x - y^3)\,dx + (\cos y + x^2)\,dy$



$\to$ nije zatvorena, moremo ju zatvorit

$$\int_{\Gamma\cup\Gamma_1} (e^x - y^3)\,dx + (\cos y + x^2)\,dy = \iint (2x + 3y^2)\,dx\,dy =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^1 r\cdot 2(r\cos\varphi) + 3r^2\sin^2\varphi\,dr$$

$$\int_{\Gamma_1} (e^x - y^3)\,dx + (\cos y + x^2)\,dy = \int_1^{-1} \cos t\,dt = \sin t \Big|_1^{-1} = -2\sin 1$$

$x = 0 \quad dx = 0$

$y = t \quad , t \in [1, -1]$

$dy = dt$

$$\int_\Gamma = \int_{\substack{\text{GREEN} \\ \Gamma\cup\Gamma_1}} - \int_{\Gamma_1}$$

$\downarrow$

naš koji tražimo

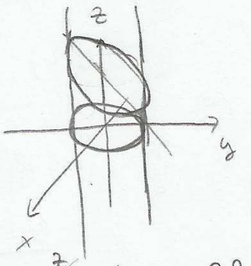
zzv. PLOŠNI INTEGRALI

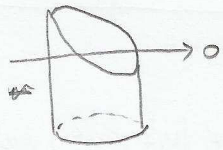
① $\boxed{x^2+y^2=1}$ valjak pov. dijela plohe

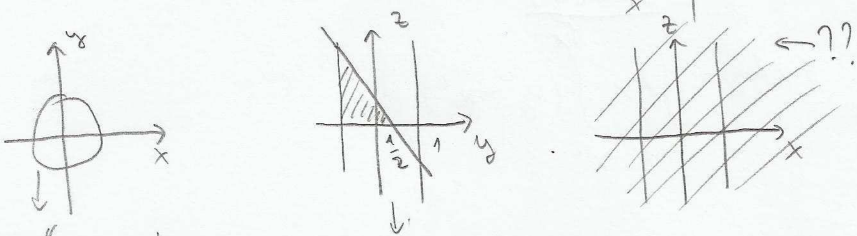
$z=0$

$2y+z=1$

$$\iint_\Sigma dS$$







nećemo tu projicirat jer vidimo samo rub

$x \to$ prikaži preko y, z

$x^{\Sigma} = \pm\sqrt{1-y^2}$

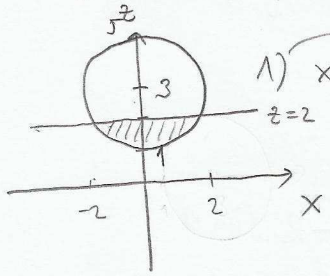
$$dS = \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2} = \sqrt{1+\frac{y^2}{1-y^2}+0}\;dydz = \frac{dy\,dz}{\sqrt{1-y^2}}$$

ako imamo npr. x kao podint fju.

$$\iint_{x>0} dS + \iint_{x<0} dS = 2\iint \frac{dydz}{\sqrt{1-y^2}} = 2\int_{-1}^{1/2} dy \int_0^{1-2y} \frac{dz}{\sqrt{1-y^2}} = \ldots$$

⑦ $\iint_\Sigma z\,dS$, $\Sigma \ldots x^2+y^2+(z-3)^2=4$

$z \le 2$



1) $x^2+(z-3)^2=4$

$y \to$ preko x i z

$y = \pm\sqrt{4-x^2-(z-3)^2}$

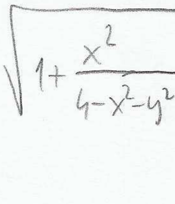
$$dS = \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dxdz$$

krivo odabrali i zapetljali se !

lakše:

2) presjek

$x^2+y^2=3$



$z = \pm\sqrt{4-y^2-x^2}+3$

jer imamo donji dio sfere

$$dS = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy =$$

$$= \sqrt{1+\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}}\,dxdy = \frac{2\,dxdy}{\sqrt{4-x^2-y^2}}$$

$$\iint_D \left(3-\sqrt{4-y^2-x^2}\right)\frac{2\,dxdy}{\sqrt{4-x^2-y^2}} =$$

$$= \iint_D \frac{6\,dxdy}{\sqrt{4-x^2-y^2}} - \iint_D 2\,dxdy$$

u polarne

$x = r\cos\varphi$

$y = r\sin\varphi$

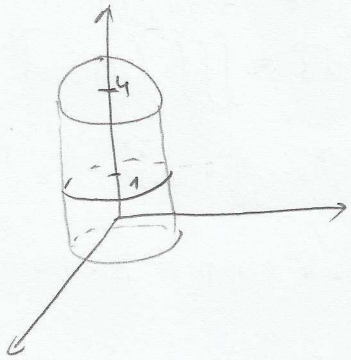
$J = r$

8) $\vec{f} = x^2\vec{i} + y\vec{j} + z\vec{k}$ plošni 2. vrste

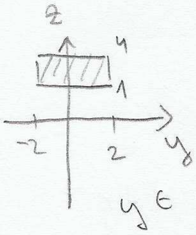
ZZV

$\boxed{x^2+y^2=4}$ ploha

$1 \le z \le 4$



$$\iint_\Sigma x^2\,dydz + y\,dxdz + z\,dxdy$$



z-y ravnina

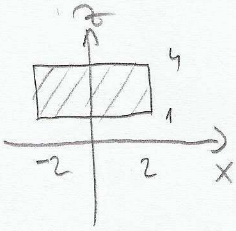
$x = \pm\sqrt{4-y^2}$

$(+)\iint_{x>0} (4-y^2)\,dydz - \iint_{x<0} 4-y^2\,dydz =$

(da je samo $\underline{\underline{x}}$ u fji: $(+)\iint_{x>0} \sqrt{4-y^2}\,dydz\,(-)\iint_{x<0} -\sqrt{4-y^2}\,dydz =$

$= 2\iint_\Sigma \sqrt{4-y^2}\,dydz$)

z-x ravnina



$y = \pm\sqrt{4-x^2}$

$(+)\iint_{y>0} \sqrt{4-x^2}\,dxdz - \iint_{y<0} -\sqrt{4-x^2}\,dxdz = 2\iint \sqrt{4-x^2}\,dxdz =$

$= 2\int_{-2}^{2} dx \int_{1}^{4} \sqrt{4-x^2}\,dz = 12\pi$

x-y ravnina

vidimo samo crtu → $=0$

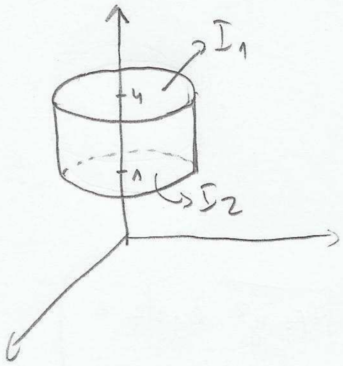
(8) preko TOD

zzv

$$\iint_\Sigma \vec{f}\, d\vec{S} = \iiint_V (\operatorname{div} \vec{f})\, dV = \iiint_V (2x+2)\, dV$$

$$\vec{f} = x^2 \vec{i} + y \vec{j} + z \vec{k}$$

$$\operatorname{div} \vec{f} = 2x + 1 + 1 = 2x + 2$$



$x = r\cos\varphi$

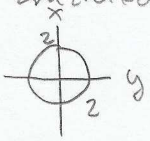
$y = r\sin\varphi$

$z = z$

$$\int_0^{2\pi} d\varphi \int_0^2 r\, dr \int_1^4 2(r\cos\varphi + 1)\, dz = 3\int_0^{2\pi} d\varphi \int_0^2 (2r^2\cos\varphi + 2r)\, dr =$$

$$= 3\left(\int_0^{2\pi} \frac{2}{3} 8\cos\varphi\, d\varphi + \int_0^{2\pi} 4\, d\varphi\right) = 3\cdot\left(\frac{16}{3}\cdot 0 + 8\pi\right) = 24\pi$$

$$I_1 = \iint \underbrace{x^2\, dy\,dz}_{0} + \underbrace{y\, dx\,dz}_{0} + \underbrace{z\, dx\,dy}_{\text{kružnica}} = \iint z\, dx\,dy = (+)\iint 4\, dx\,dy = 16\pi$$



$I_2 =$ isto tako ... 4π

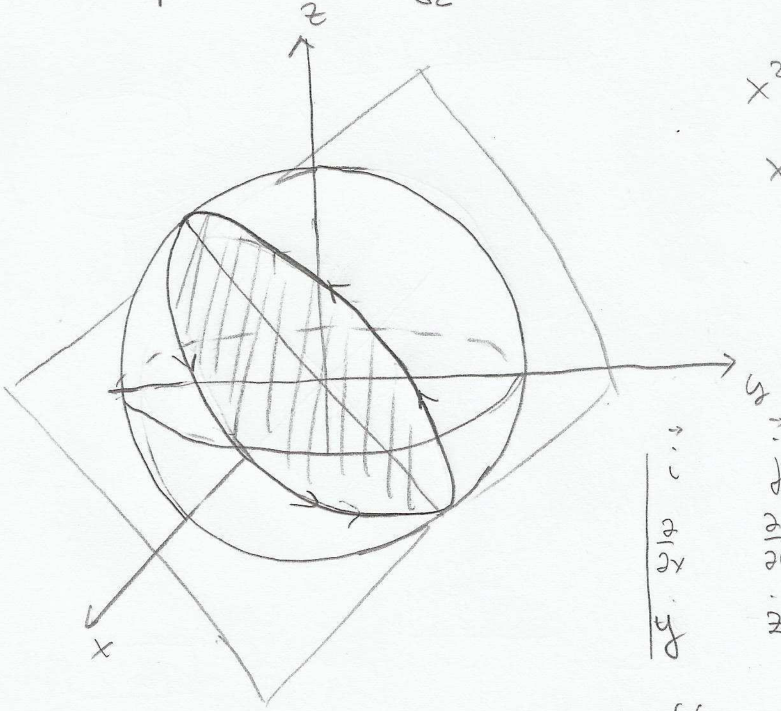
$$\iint_\Sigma \vec{f}\, d\vec{S} = \iiint_V - I_1 - I_2 = 24\pi - 16\pi + 4\pi = 12\pi \checkmark$$

(26) zzv $\oint_\Gamma y\,dx + z\,dy + x\,dz$ $\qquad \Gamma \dots\ x^2+y^2+z^2=a^2,\ a>0$

$z=-y$ $\qquad$ ↺ iz $(0, a, 0)$

STOKES

$$\oint_\Gamma \vec{f}\,d\vec{r} = \iint_\Omega (\mathrm{rot}\,\vec{f})\vec{k}\,dx\,dy = *$$



$$x^2+y^2+z^2=a^2$$

$$x^2+2y^2=a^2$$

$$\frac{x^2}{a^2}+\frac{y^2}{\left(\frac{a}{2}\right)^2}=1$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\vec{i}-\vec{j}-\vec{k}$$

$$* = \iint_\Omega -1\,dx\,dy$$

u $\Omega_{xy} \Rightarrow$ elipsa