1) $\dot{Q} = 1 - Q$, $Q(0) = 0$

a) $Q = ?$, $Q = Q(t)$

$$\frac{dQ}{dt} = 1 - Q$$

$$\frac{dQ}{1-Q} = dt \qquad \begin{bmatrix} r = 1-Q \\ dr = -dQ \end{bmatrix}$$

$$-\frac{dr}{r} = dt \quad \Big| \int$$

$$-\ln r = t + \ln C$$

$$-\ln(1-Q) = t + \ln C$$

$Q(0) = 0$

$$-\ln(1-0) = 0 + \ln C$$

$$0 = \ln C$$

$$C = e^0$$

$$C = 1$$

$$-\ln(1-Q) = t + \ln 1$$

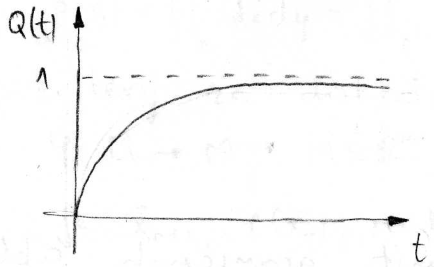
$$t = -\ln(1-Q)$$

$$t = \ln \frac{1}{1-Q}$$

$$e^t = \frac{1}{1-Q}$$

$$1 - Q = e^{-t}$$

$$Q = 1 - e^{-t}$$



b) fazni portret

$$\dot{Q} = 1 - Q = 0$$

$$Q = 1$$

stacionarna točka

$\dot{Q} > 0$ $\qquad$ $\dot{Q} < 0$

$1 - Q > 0$ $\qquad$ $Q > 1$

$-Q > -1$

$Q < 1$

c) max. interval

$t \in \mathbb{R}$

2) $\ddot{\vartheta} + \cos\vartheta = 0$

a)

$$\left.\begin{aligned} \frac{d\vartheta}{dt} &= v \\ \frac{dv}{dt} &= -\cos\vartheta \end{aligned}\right\} \quad \frac{\frac{dv}{dt}}{\frac{d\vartheta}{dt}} = \frac{-\cos\vartheta}{v} \quad \Rightarrow \quad \frac{dv}{d\vartheta} = -\frac{\cos\vartheta}{v}$$

$$v\,dv = -\cos\vartheta\, d\vartheta \quad \Big| \int$$

$$\frac{v^2}{2} = -\sin\vartheta\, d\vartheta + C$$

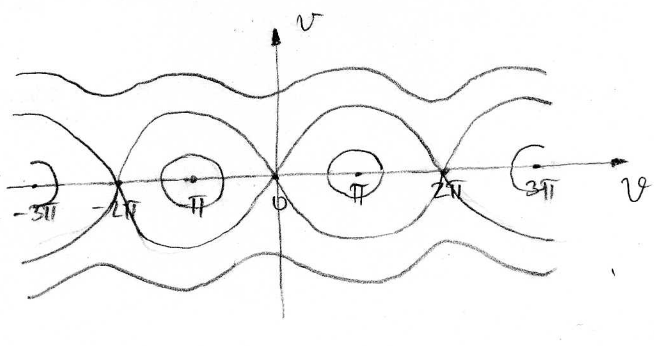
b) fazni portret

$$\dot{\vartheta} = v = 0$$

$$\dot{v} = -\cos\vartheta = 0$$

$$\vartheta = \arccos 0$$

$$\vartheta = \pm k\pi + \frac{\pi}{2}$$



3) $\ddot{y}+4\dot{y}+5y=0$, $y(0)=1$, $\dot{y}(0)=0$

a) $y=?$, $y=y(t)$

$y=e^{\lambda t}$, $\dot{y}=\lambda e^{\lambda t}$, $\ddot{y}=\lambda^2 e^{\lambda t}$

$\lambda^2 e^{\lambda t}+4\lambda e^{\lambda t}+5e^{\lambda t}=0 \quad |: e^{\lambda t}$

$\lambda^2+4\lambda+5=0$

$$\lambda_{1,2}=\frac{-4\pm\sqrt{16-20}}{2}=\frac{-4\pm 2i}{2}=-2\pm i$$

$y(t)=C_1 e^{-(2+i)t}+C_2 e^{(-2+i)t}$

$\dot{y}(t)=-(2+i)C_1 e^{-(2+i)t}+(-2+i)C_2 e^{(-2+i)t}$

$1=C_1+C_2 \Rightarrow C_2=1-C_1 \Rightarrow\Rightarrow C_2=1-C_1=1-\frac{1}{2}-i=\frac{1}{2}-i$

$0=-(2+i)C_1+(-2+i)C_2=-(2+i)C_1+(-2+i)(1-C_1)=\cancel{-2C_1}-iC_1-2\cancel{+2C_1}+i-iC_1=$

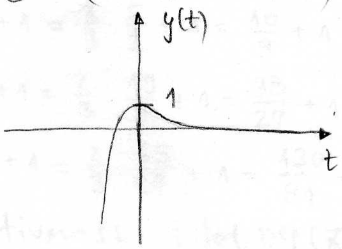
$=-2iC_1-2+i \Rightarrow 2iC_1=-2+i \Rightarrow C_1=\frac{-2+i}{2i}=\frac{1}{2}+i$

$$y(t)=\frac{1+2i}{2}e^{-(2+i)t}+\frac{1-2i}{2}e^{(-2+i)t}$$

$$y(t)=\frac{e^{-2t}}{2}\left[(1+2i)e^{-it}+(1-2i)e^{it}\right]=\frac{e^{-2t}}{2}\left[e^{-it}+2ie^{-it}+e^{it}-2ie^{it}\right]=$$

$$=\frac{e^{-2t}}{2}\left[e^{it}+e^{-it}-2i(e^{it}-e^{-it})\right]=\frac{e^{-2t}}{2}\left[2\cos t+2\cdot 2\sin t\right]=$$

$$=e^{-2t}(\cos t+2\sin t)$$



$y=e^{0}(\cos 0+2\sin 0)=1$

$0=e^{-2t}(\cos t+2\sin t) \Rightarrow \cos t=-2\sin t \Rightarrow$

$\Rightarrow \operatorname{tg} t=-\frac{1}{2}=26{,}565^\circ=-0{,}4636$ rad

$t\to\infty$, $y\to 0$

b) $A=?$

$\ddot{y}+4\dot{y}+5y=0$, $\dot{y}=x$

$\dot{x}+4x+5y=0$

$\dot{x}=-4x-5y$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}=\begin{bmatrix}-4 & -5\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

c) fazni portret

$\dot{y}=x=0$

$\dot{x}=-4x-5y=0$

$5y=-4x \quad |:5$

$y=-\frac{4}{5}x$

4) $\ddot{y} - 4\dot{y} + 3y = 0$, $y(0) = 0$, $\dot{y}(0) = 1$

a) $y = ?$, $y = y(t)$

$y = e^{\lambda t}$, $\dot{y} = \lambda e^{\lambda t}$, $\ddot{y} = \lambda^2 e^{\lambda t}$

$\lambda^2 e^{\lambda t} - 4\lambda e^{\lambda t} + 3 e^{\lambda t} = 0 \quad | : e^{\lambda t}$

$\lambda^2 - 4\lambda + 3 = 0$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 - 12}}{2} = \frac{4 \pm 2}{2} = 2 \pm 1 \qquad \lambda_1 = 1, \; \lambda_2 = 3$$

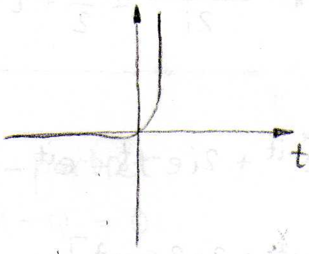
$y(t) = C_1 e^t + C_2 e^{3t}$

$\dot{y}(t) = C_1 e^t + 3C_2 e^{3t}$

$0 = C_1 + C_2 \Rightarrow C_2 = -C_1 \Rightarrow\Rightarrow C_2 = \frac{1}{2}$

$1 = C_1 + 3C_2 = C_1 - 3C_1 = -2C_1 \Rightarrow C_1 = -\frac{1}{2}$

$y(t) = -\frac{1}{2} e^t + \frac{1}{2} e^{3t} = \frac{1}{2}(e^{3t} - e^t)$



$y = \frac{1}{2}(e^0 - e^0) = 0$

$0 = \frac{1}{2}(e^{3t} - e^t) \Rightarrow e^t = e^{3t} \Rightarrow \cancel{t} = 3\cancel{t} \Rightarrow 0 = 3$

$t \to \infty \quad ?$

b) $A = ?$

$\ddot{y} - 4\dot{y} + 3y = 0$, $\dot{y} = x$

$\dot{x} - 4x + 3y = 0$

$\dot{x} = 4x - 3y = 0$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

c) fazni portret

$\dot{y} = x = 0$

$\dot{x} = 4x - 3y = 0$

$3y = 4x \quad | : 3$

$y = \frac{4}{3} x$

5) $\dot{x} = 3\sqrt[3]{x^2}$, $x(3) = 0$

a) $x = ?$, $x = x(t)$

$$\frac{dx}{dt} = 3x^{\frac{2}{3}}$$

$$x^{-\frac{2}{3}}dx = 3dt \; \Big| \int$$

$$\frac{x^{\frac{1}{3}}}{\frac{1}{3}} = 3t + C$$

$$\sqrt[3]{x} = t + 3C \; \Big|^3$$

$$x = (t + 3C)^3$$

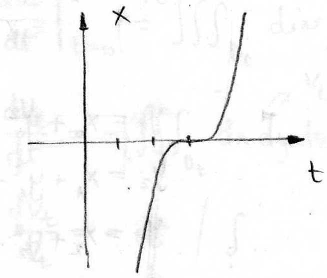
$x(3) = 0$

$0 = (3 + 3C)^3$

$\cancel{3}C = -\cancel{3}$

$C = -1$

$x = (t-3)^3$



$x = (0-3)^3 = -27$

$0 = (t-3)^3 \Rightarrow t = 3$

$t \to \infty \Rightarrow x \to \infty$

b) jednoznačno rješenje = ?

6) $x_{n+1} = \frac{2}{3}x_n + 1$, $x \in \mathbb{Z}$

a) $n = 4$, $x_0 = 1$

$x_1 = \frac{2}{3}x_0 + 1 = \frac{2}{3} \cdot 1 + 1 = \frac{5}{3}$

$x_2 = \frac{2}{3}x_1 + 1 = \frac{2}{3} \cdot \frac{5}{3} + 1 = \frac{10}{9} + 1 = \frac{19}{9}$

$x_3 = \frac{2}{3}x_2 + 1 = \frac{2}{3} \cdot \frac{19}{9} + 1 = \frac{38}{27} + 1 = \frac{65}{27}$

$x_4 = \frac{2}{3}x_3 + 1 = \frac{2}{3} \cdot \frac{65}{27} + 1 = \frac{130}{81} + 1 = \frac{211}{81}$

b) $n = 4$, $x_0 = 3$

$x_1 = \frac{2}{3}x_0 + 1 = \frac{2}{\cancel{3}} \cdot \cancel{3} + 1 = 2 + 1 = 3$

$x_2 = \frac{2}{3}x_1 + 1 = \frac{2}{\cancel{3}} \cdot \cancel{3} + 1 = 2 + 1 = 3$

$x_3 = \frac{2}{3}x_2 + 1 = 3$

$x_4 = \frac{2}{3}x_3 + 1 = 3$

c) disipativnost $|\det D\varphi(\bar{x})| = ?$

$\left| \frac{\partial x_{n+1}}{\partial x_n} \right| = \left| \frac{2}{3} \right| = \frac{2}{3} < 1$ sustav je disipativan

d)

7) $x_{n+1} = y_n + 1, \quad y_{n+1} = x_n + y_n^2 - y_n, \quad x, y \in \mathbb{Z}$

a) $n = 4, \quad x_0 = 0, \quad y_0 = 0$

$x_1 = y_0 + 1 = 0 + 1 = 1 \qquad y_1 = x_0 + y_0^2 - y_0 = 0 + 0^2 - 0 = 0$

$x_2 = y_1 + 1 = 0 + 1 = 1 \qquad y_2 = x_1 + y_1^2 - y_1 = 1 + 0^2 - 0 = 1$

$x_3 = y_2 + 1 = 1 + 1 = 2 \qquad y_3 = x_2 + y_2^2 - y_2 = \cancel{1} + 1^2 \cancel{- 1} = 1$

$x_4 = y_3 + 1 = 1 + 1 = 2 \qquad y_4 = x_3 + y_3^2 - y_3 = 2 \cancel{+ 1^2} \cancel{- 1} = 2$

b) $n = 4, \quad x_0 = 1, \quad y_0 = 1$

$x_1 = y_0 + 1 = 1 + 1 = 2 \qquad y_1 = x_0 + y_0^2 - y_0 = \cancel{1} + 1^2 \cancel{- 1} = 1$

$x_2 = y_1 + 1 = 1 + 1 = 2 \qquad y_2 = x_1 + y_1^2 - y_1 = 2 \cancel{+ 1^2} \cancel{- 1} = 2$

$x_3 = y_2 + 1 = 2 + 1 = 3 \qquad y_3 = x_2 + y_2^2 - y_2 = \cancel{2} + 2^2 \cancel{- 2} = 4$

$x_4 = y_3 + 1 = 4 + 1 = 5 \qquad y_4 = x_3 + y_3^2 - y_3 = 3 + 4^2 - 4 = 15$

c) disipativnost $|\det D\varphi(\vec{x})| = ?$

$$|\det D\varphi(\vec{x})| = \begin{vmatrix} \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial y_n} \\ \frac{\partial y_{n+1}}{\partial x_n} & \frac{\partial y_{n+1}}{\partial y_n} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 2y_n - 1 \end{vmatrix} = |-1| = 1$$

– sustav je konzervativan

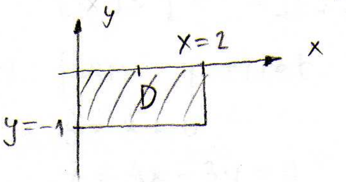
8) $\dot{\vec{x}} = \vec{f}(\vec{x}), \quad \vec{f}(x, y) = (2y - x^3, \; 2 - x - y)$

a) disipativnost $\operatorname{div} \vec{f} = ?$

$$\operatorname{div} \vec{f} = \frac{\partial}{\partial x} f(x, y) + \frac{\partial}{\partial y} f(x, y) = -3x^2 - 1 = -(3x^2 + 1) < 0$$

– sustav je disipativan za sve parametre

b) $\{\varphi^t\}_{t \in \mathbb{R}}, \quad D \subseteq \mathbb{R}^2, \quad D_t = \varphi^t(D), \quad V_t = \operatorname{vol}(D_t), \quad \left.\frac{dV_t}{dt}\right|_{t=0} = ?$



$$\left.\frac{dV_t}{dt}\right|_{t=0} = \iint_{D_t} \operatorname{div} \vec{f}(x, y)\, dx\, dy$$

$$\frac{dV_t}{dt} = -\iint_{D_t} (3x^2 + 1)\, dx\, dy = -3 \iint_{D_t} x^2\, dx\, dy - \iint_{D_t} dx\, dy =$$

$$= -3 \int_0^2 x^2\, dx \int_{-1}^0 dy - \int_0^2 dx \int_{-1}^0 dy = -\cancel{3}\, \frac{x^3}{\cancel{3}}\Big|_0^2 \cdot y\Big|_{-1}^0 - x\Big|_0^2 \cdot y\Big|_{-1}^0 =$$

$$= -2^3 \cdot 1 - 2 \cdot 1 = -8 - 2 = -10$$

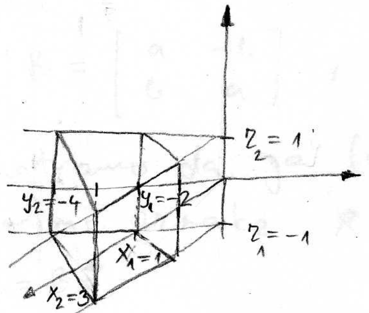
9) $\dot{\vec{x}} = \vec{f}(\vec{x})$, $f(x,y,z) = (yz+1,\ z^2-y^2-2,\ x^3+2yz-2)$

a) disipativnost $\operatorname{div}\vec{f} = ?$

$$\operatorname{div}\vec{f} = \frac{\partial}{\partial x} f(x,y,z) + \frac{\partial}{\partial y} f(x,y,z) + \frac{\partial}{\partial z} f(x,y,z) = 0 \cancel{-2y} \cancel{+2y} - 1 = -1 < 0$$

– sustav je disipativan

b) $\{\varphi^t\}_{t\in\mathbb{R}}$, $D \subseteq \mathbb{R}^3$, $D_t = \varphi^t(D)$, $V_t = \operatorname{vol}(D_t)$, $\left.\frac{dV_t}{dt}\right|_{t=0} = ?$



$$\left.\frac{dV_t}{dt}\right|_{t=0} = \iiint_{D_t} \operatorname{div}\vec{f}(x,y,z)\,dx\,dy\,dz$$

$$\frac{dV_t}{dt} = -\underbrace{\iiint_{D_t} dx\,dy\,dz}_{V_t} = -V_t$$

$$\frac{dV_t}{V_t} = -dt \ \Big/ \int$$

$$\ln V_t = -t + \ln C \ \Rightarrow\ V_t = e^{-t+\ln C} = Ce^{-t}$$

$$\frac{dV_t}{dt} = -\iiint_{D_t} dx\,dy\,dz = -\int_1^3 dx \int_{-2}^{-4} dy \int_{-1}^{1} dz = -(3-1)(-4+2)(1+1) = -2\cdot(-2)\cdot 2 = 8$$

10) Hamiltonov sustav $\dot{x} = -3y$, $\dot{y} = 2x$

a) $H(x,y) = ?$

$$\left.\begin{aligned} \dot{x} &= -3y = H_y' &&\Rightarrow H_y = -3\int y\,dy = -3\frac{y^2}{2} = -\frac{3}{2}y^2 \\ \dot{y} &= 2x = -H_x' &&\Rightarrow H_x = -2\int x\,dx = -\cancel{2}\frac{x^2}{\cancel{2}} = -x^2 \end{aligned}\right\} + \ H(x,y) = -x^2 - \frac{3}{2}y^2$$

b) fazni portret

$\dot{x} = -3y = 0 \ \Rightarrow\ y = 0$

$\dot{y} = 2x = 0 \ \Rightarrow\ x = 0$

11) $\dot{\vec{x}} = A\vec{x}$, $\vec{x} \in \mathbb{R}^2$, $A = \begin{bmatrix} 2 & -3 \\ 1 & -4 \end{bmatrix}$

a) $B = ?$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2 & -3 \\ 1 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2-\lambda & -3 \\ 1 & -4-\lambda \end{bmatrix}\right) =$$

$$= \begin{vmatrix} 2-\lambda & -3 \\ 1 & -4-\lambda \end{vmatrix} = (2-\lambda)(-4-\lambda) - (-3)\cdot 1 = -8 - 2\lambda + 4\lambda + \lambda^2 + 3 =$$

$$= \lambda^2 + 2\lambda - 5 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{-2 \pm \sqrt{4+20}}{2} = \frac{-\cancel{2} \pm \cancel{2}\sqrt{6}}{\cancel{2}} = -1 \pm \sqrt{6}$$

$$B = \begin{bmatrix} -1-\sqrt{6} & 0 \\ 0 & -1+\sqrt{6} \end{bmatrix}$$

b) $\dot{\vec{x}} = A\vec{x}$, skalarni oblik $\qquad \dot{\vec{y}} = B\vec{y}$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{aligned} \dot{x} &= 2x - 3y \\ \dot{y} &= x - 4y \end{aligned}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -1-\sqrt{6} & 0 \\ 0 & -1+\sqrt{6} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{aligned} \dot{x} &= -(1+\sqrt{6})x \\ \dot{y} &= -(1-\sqrt{6})y \end{aligned}$$

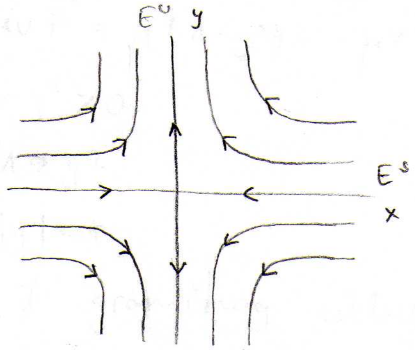
c) fazni portret

$$\frac{dx}{dt} = -(1+\sqrt{6})x$$

$$\frac{dy}{dt} = -(1-\sqrt{6})y$$

$$\frac{dx}{x} = -(1+\sqrt{6})\,dt \quad \Big| \int$$

$$\frac{dy}{y} = -(1-\sqrt{6})\,dt \quad \Big| \int$$



$\lambda_1 < 0$, $\lambda_2 > 0$

$\lambda_1 < 0 < \lambda_2$

$$\ln x = -(1+\sqrt{6})t + \ln C_1$$

$$\ln y = -(1-\sqrt{6})t + \ln C_2$$

$$x = C_1 e^{-(1+\sqrt{6})t} = C_1 e^{-3{,}45t},$$

$$y = C_2 e^{-(1-\sqrt{6})t} = C_2 e^{1{,}45t}$$

$$\left(\frac{x}{C_1}\right)^{\frac{1}{1+\sqrt{6}}} = e^{-t}$$

$$\left(\frac{y}{C_2}\right)^{\frac{1}{1-\sqrt{6}}} = e^{-t}$$

$$\left(\frac{x}{C_1}\right)^{\frac{1}{1+\sqrt{6}}} = \left(\frac{y}{C_2}\right)^{\frac{1}{1-\sqrt{6}}} \quad \Big|^{1-\sqrt{6}}$$

$$\frac{y}{C_2} = \left(\frac{x}{C_1}\right)^{\frac{1-\sqrt{6}}{1+\sqrt{6}}}$$

$$y = C_2 C_1^{\frac{1+\sqrt{6}}{1-\sqrt{6}}} \, x^{\frac{1-\sqrt{6}}{1+\sqrt{6}}}$$

$$y = C\, x^{\frac{1-\sqrt{6}}{1+\sqrt{6}}}$$

d) $E^s = ?$, $E^u = ?$

$(B - \lambda_1 I)\vec{v_1} = 0$

$$\begin{bmatrix} -1-\sqrt6+1+\sqrt6 & 0 \\ 0 & -1+\sqrt6+1+\sqrt6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2\sqrt6\, y = 0$, $y = 0$, $\forall x$

$(1,0)$, $\vec{v_1} = \vec{i}$ stabilni potprostor

$(B - \lambda_2 I)\vec{v_2} = 0$

$$\begin{bmatrix} -1-\sqrt6+1-\sqrt6 & 0 \\ 0 & -1+\sqrt6+1-\sqrt6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2\sqrt6\, x = 0$, $x = 0$, $\forall y$

$(0,1)$, $\vec{v_2} = \vec{j}$ nestabilni potprostor

e) $A(\vec{x}) = A\vec{x}$, $\vec{x} = (x, y)$

12) $\dot{\vec{x}} = A\vec{x}$, $\vec{x} \in \mathbb{R}^2$, $A = \begin{bmatrix} 2 & 0 \\ 5 & 2 \end{bmatrix}$

a) $B = ?$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2 & 0 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \begin{vmatrix} 2-\lambda & 0 \\ 5 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0 \Rightarrow \lambda_{1,2} = 2$$

$$B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

b) $\dot{\vec{y}} = B\vec{y}$, skalarni oblik

$$\begin{bmatrix} \dot x \\ \dot y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{aligned} \dot x &= 2x + y \\ \dot y &= 2y \end{aligned}$$

c) fazni portret

$$\frac{dx}{dt} = 2x + y$$

$$\frac{dy}{dt} = 2y$$

$$\frac{dx}{2x+y} = dt \quad \begin{bmatrix} z = 2x + y \\ dz = 2dx \end{bmatrix} \Rightarrow \frac{dz}{z} = 2dt \;\Big/ \int$$

$$\frac{dy}{y} = 2dt \;\Big/ \int$$

$$\ln z = 2t + \ln c_1$$

$$\ln y = 2t + \ln c_2$$

$$z = c_1 e^{2t} \Rightarrow 2x + y = c_1 e^{2t}$$

$$y = c_2 e^{2t}$$

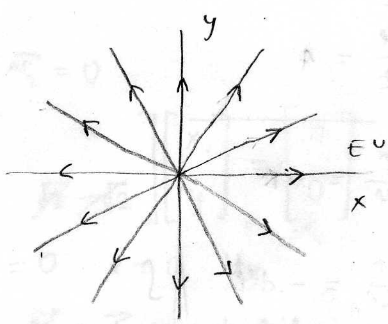
$$x = \frac12 (c_1 e^{2t} - y) = \frac12 (c_1 e^{2t} - c_2 e^{2t})$$

$$y = c_2 e^{2t}$$

$$x = \frac12 (c_1 - c_2) e^{2t} = c' e^{2t}$$

$$y = c_2 e^{2t}$$

$$y = \frac{c_2}{c'} e^{2t} = c e^{2t}$$



$\lambda_1 = \lambda_2 > 0$

d) $E^s = ?$, $E^u = ?$

$(B - \lambda I)\vec{v} = 0$

$$\begin{bmatrix} \cancel{2-2} & 1 \\ 0 & \cancel{2-2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$y = 0$, $\forall x$

$(1,0)$ $\vec{v} = i$ nestabilni potprostor

e) $A(\vec{x}) = A\vec{x}$, $\vec{x} = (x, y)$

13) $\dot{\vec{x}} = A\vec{x}$, $\vec{x} \in \mathbb{R}^2$, $A = \begin{bmatrix} 2 & -2 \\ 5 & -4 \end{bmatrix}$

a) $B = ?$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2 & -2 \\ 5 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \begin{vmatrix} 2-\lambda & -2 \\ 5 & -4-\lambda \end{vmatrix} =$$

$$= (2-\lambda)(-4-\lambda) - (-2)\cdot 5 = -8 - 2\lambda + 4\lambda + \lambda^2 + 10 = \lambda^2 + 2\lambda + 2 \Rightarrow$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i \qquad B = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

b) $\dot{\vec{y}} = B\vec{y}$, skalarni oblik

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{aligned} \dot{x} &= -x - y \\ \dot{y} &= x - y \end{aligned}$$

c) fazni portret

promjena koordinata

$a = r\cos\varphi$ $\quad$ $a = -1$ $\quad$ $\dot{r} = ar$ $\quad$ $\dot{r} = -r$

$b = r\sin\varphi$ $\quad$ $b = 1$ $\quad$ $\dot{\varphi} = b$ $\quad$ $\dot{\varphi} = 1$

$$\frac{dr}{dt} = -r$$

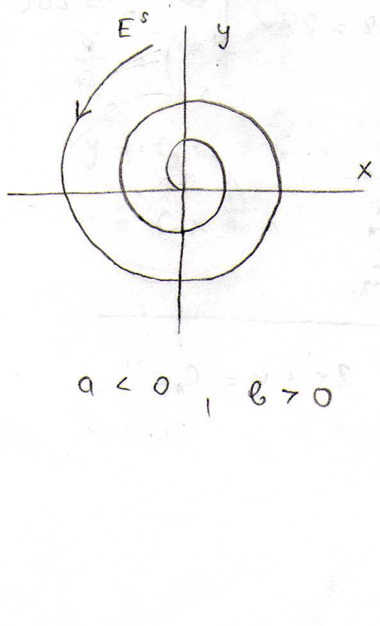
$$\frac{d\varphi}{dt} = 1$$

$$\frac{dr}{d\varphi} = -r$$

$$\frac{dr}{r} = -d\varphi \quad \Big/ \int$$

$$\ln r = -\varphi + \ln C$$

$$r = Ce^{-\varphi}$$



$a < 0$, $b > 0$

d) $E^s = ?$, $E^u = ?$

$x(t) = c_1 e^{-at}\cos(bt) - c_2 e^{-at}\sin(bt)$ $\lim_{t\to\infty} x(t) = 0$

$y(t) = c_1 e^{-at}\cos(bt) + c_2 e^{-at}\sin(bt)$ $\lim_{t\to\infty} y(t) = 0$ stabilan potprostor

$\lambda_{1,2} = -1 \pm i$, $Re < 0$

e) $A(\vec{x}) = A\vec{x}$, $\vec{x} = (x, y)$

14) $\dot{\vec{x}} = A\vec{x}$, $\vec{x} \in \mathbb{R}^2$, $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

a) $B = ?$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \begin{vmatrix} -\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 2 = 0 \Rightarrow \lambda_{1,2} = \pm\sqrt{2}$$

$$B = \begin{bmatrix} -\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

b) $\dot{y} = By$, skalarni oblik

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad \dot{x} = -\sqrt{2}x \qquad \dot{y} = \sqrt{2}y$$

c) fazni portret

$\frac{dx}{dt} = -\sqrt{2}x$ $\qquad x = c_1 e^{-\sqrt{2}t}$

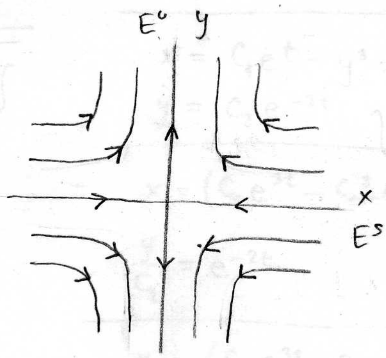
$\frac{dy}{dt} = \sqrt{2}y$ $\qquad y = c_2 e^{\sqrt{2}t}$

$\frac{dx}{dy} = \frac{-\cancel{\sqrt{2}}x}{\cancel{\sqrt{2}}y}$

$\frac{dx}{x} = -\frac{dy}{y} \;/\int$

$\ln x = -\ln y + \ln C$

$x = \frac{C}{y} \Rightarrow y = \frac{C}{x}$



$\lambda_1 < 0$, $\lambda_2 > 0$

$\lambda_1 < 0 < \lambda_2$

d) $E^s = ?$, $E^u = ?$

$(B - \lambda_1 I)\vec{v_1} = 0$

$$\begin{bmatrix} \cancel{-\sqrt{2}+\sqrt{2}} & 0 \\ 0 & \sqrt{2}+\sqrt{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2\sqrt{2}y = 0 \quad y = 0 \quad \forall x$

$(1, 0)$, $\vec{v_1} = \vec{i}$ stabilni potprostor

$(B - \lambda_2 I)\vec{v_2} = 0$

$$\begin{bmatrix} -\sqrt{2}-\sqrt{2} & 0 \\ 0 & \cancel{\sqrt{2}-\sqrt{2}} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

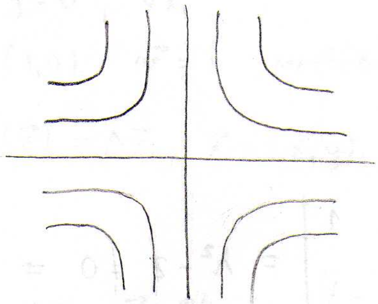
$-2\sqrt{2}x = 0 \quad x = 0 \quad \forall y$

$(0, 1)$, $\vec{v_2} = \vec{j}$ nestabilni potprostor

e) $A(\vec{x}) = A\vec{x}$, $\vec{x} = (x, y)$

15) $\dot{x} = x$, $\dot{y} = -5y$

a) fazni portret



b) $y = ?$

$$\frac{dx}{dt} = x$$

$$\frac{dy}{dt} = -5y$$

$$\frac{dx}{x} = dt \;/\int$$

$$\frac{dy}{y} = -5dt \;/\int$$

$$\ln x = t + \ln C_1$$

$$\ln y = -5t + \ln C_2$$

$$x = C_1 e^{t}$$

$$y = C_2 e^{-5t}$$

$$\frac{x}{C_1} = e^{t}$$

$$y = C_2 \left(\frac{x}{C_1}\right)^{-5} = C_1^{5} C_2 x^{-5} = \frac{C}{x^5}$$

c)

16) $\dot{x} = x + y^3, \quad \dot{y} = -2y$

a) singulariteti = ?

$x + y^3 = 0 \Rightarrow x = 0$

$-2y = 0 \Rightarrow y = 0$ $\qquad S(0,0)$

b) $\lambda_1 = ?$, $\lambda_2 = ?$

$f(x,y) = x + y^3$

$g(x,y) = -2y$

$$D\vec{F}(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 3y^2 \\ 0 & -2 \end{bmatrix}$$

$$D\vec{F}(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \qquad \lambda_1 = 1, \ \lambda_2 = -2$$

c) $S(0,0)$ – hiperbolički singularitet

d) topološki tip singulariteta linearnog dijela – sedlo

e) $y = ?$

$\frac{dx}{dt} = x + y^3$

$\frac{dy}{dt} = -2y$

$\frac{dx}{x+y^3} = dt \quad \begin{bmatrix} z = x + y^3 \\ dz = dx \end{bmatrix} \Rightarrow \frac{dz}{z} = dt \ / \int$

$\frac{dy}{y} = -2dt \ / \int$

$\ln z = t + \ln C_1$

$\ln y = -2t + \ln C_2$

$z = C_1 e^t \Rightarrow x + y^3 = C_1 e^t$

$y = C_2 e^{-2t}$

$x = C_1 e^t - y^3 = C_1 e^t - C_2^3 e^{-6t}$

$y = C_2 e^{-2t}$

$x = (C_1 e^{3t} - C_2^3 e^{-4t}) e^{-2t}$

$\frac{y}{C_2} = e^{-2t}$

$x = (C_1 e^{3t} - C_2 e^{-4t}) \frac{y}{C_2}$

$$y = \frac{C_2 x}{C_1 e^{3t} - C_2 e^{-4t}}$$

f) $E^s = ?$, $E^u = ?$

$-3y = 0 \Rightarrow y = 0 \quad \forall x$

$(1,0)$, $\vec{v}_1 = \vec{i}$ $\quad E^s$

$-x = 0 \Rightarrow x = 0 \quad \forall y$

$(0,1)$, $\vec{v}_2 = \vec{j}$ $\quad E^u$

g) $W^s(0) = ?$, $W^u(0) = ?$

$(x(0), y(0)) = (C_1 - C_2^3, C_2)$

$\lim_{t \to \infty} \varphi^t (C_1 - C_2^3, C_2) = 0 \qquad C_1 = 0 \qquad W^s$

$\lim_{t \to -\infty} \varphi^t (C_1 - C_2^3, C_2) = 0 \qquad C_2 = 0 \qquad W^u$

17) $\dot{x} = -x, \quad \dot{y} = y - 2y^2 + y^3$

a) singulariteti = ?

$-x = 0 \Rightarrow x = 0$

$y - 2y^2 + y^3 = 0 \Rightarrow y(1-2y+y^2) = 0 \Rightarrow y(1-y^2) = 0 \Rightarrow y = 0, \ y = \pm 1$

$S_1(0,0), \ S_2(0,-1), \ S_3(0,1)$

b) $\lambda_1 = ?, \ \lambda_2 = ?$

$f(x,y) = -x$

$g(x,y) = y - 2y^2 + y^3$

$$D\vec{F}(x,y) = \begin{bmatrix} -1 & 0 \\ 0 & 1-4y+3y^2 \end{bmatrix},$$

$$D\vec{F}(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D\vec{F}(0,-1) = \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix}, \quad D\vec{F}(0,1) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\lambda_1 = -1, \ \lambda_2 = 1 \qquad \lambda_1 = -1, \ \lambda_2 = 8 \qquad \lambda_1 = -1, \ \lambda_2 = 0$

c) $S_1(0,0), \ S_2(0,-1), \ S_3(0,1)$ – hiperbolički singulariteti

d) topološki tip singulariteta – sedlo, sedlo, sedlo-čvor

e) $y = ?$ Ne

$\frac{dx}{dt} = -x$

$\frac{dy}{dt} = y - 2y^2 + y^3$

$\frac{dx}{x} = -dt \ / \int$

$\frac{dy}{y(1-y)^2} = dt \quad \begin{bmatrix} z = 1-y \\ dz = -dy \end{bmatrix}$

$\left(\frac{1}{y} + \frac{1}{1-y} + \frac{1}{(1-y)^2}\right) dy = dt \ / \int$

$\frac{1}{y(1-y)^2} = \frac{A}{y} + \frac{B}{1-y} + \frac{C}{(1-y)^2}$

$1 = A(1-y)^2 + By(1-y) + Cy$

$1 = A(1-2y+y^2) + B(y-y^2) + Cy$

$1 = A - 2Ay + Ay^2 + By - By^2 + Cy$

$1 = (A-B)y^2 + (-2A+B+C)y + A$

$A - B = 0 \Rightarrow A = B \Rightarrow B = 1$

$-2A + B + C = 0 \Rightarrow C = 2A - B \Rightarrow C = 1$

$A = 1$

$\ln x = -t + \ln C_1 \Rightarrow x = C_1 e^{-t}$

$\ln y - \ln(1-y) + \frac{1}{1-y} = t + \ln C_2 \Rightarrow y - 1 + y + e^{\frac{1}{1-y}} = C_2 e^t$

f) $E^s = ?, \ E^u = ?, \ E^c = ?$

$S_1(0,0) \quad y = 0 \ \forall x \ \vec{v_1} = \vec{i} \ E^s \qquad S_3(0,1) \quad y = 0 \ \forall x \ \vec{v_1} = \vec{i} \ E^s$??

$S_2(0,-1) \quad x = 0 \ \forall y \ \vec{v_2} = \vec{j} \ E^u \qquad x = 0 \ \forall y \ \vec{v_2} = \vec{j} \ E^c$

g) Ne

18) $\dot{x} = y + x^2, \; \dot{y} = -x + y^2$

a) singulariteti = ?

$y + x^2 = 0 \Rightarrow y = -x^2$ $\qquad S_1(0,0)$

$-x + y^2 = 0 \Rightarrow x = y^2$ $\qquad S_2(1,-1)$

b) $\lambda_1 = ?, \; \lambda_2 = ?$

$f(x,y) = y + x^2$

$g(x,y) = -x + y^2$

$$D\vec{F}(x,y) = \begin{bmatrix} 2x & 1 \\ -1 & 2y \end{bmatrix}$$

$$D\vec{F}(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \det(D\vec{F}(0,0) - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$$

$$D\vec{F}(1,-1) = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \qquad \det(D\vec{F}(1,-1) - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} = -(2-\lambda)(2+\lambda) + 1 =$$

$$= \lambda^2 - 4 + 1 = \lambda^2 - 3 = 0 \Rightarrow \lambda_{1,2} = \pm\sqrt{3}$$

c) $S_2(1,-1)$ - hiperbolički singularitet

d) topološki tip singulariteta linearnog dijela - sedlo

e) $y = ?$

$\dot{x} = y + x^2 \qquad x = r\cos\varphi, \quad \dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi$

$\dot{y} = -x + y^2 \qquad y = r\sin\varphi, \quad \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$

$$\left.\begin{aligned} \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi &= r\sin\varphi + r^2\cos^2\varphi \quad /\cdot\cos\varphi \\ \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi &= -r\cos\varphi + r^2\sin^2\varphi \quad /\cdot\sin\varphi \end{aligned}\right\} + \quad \left.\begin{aligned} &/\cdot(-\sin\varphi) \\ &/\cdot\cos\varphi \end{aligned}\right\} +$$

$\dot{r} = r^2(\sin^3\varphi + \cos^3\varphi)$

$\cancel{r}\dot{\varphi} = -\cancel{r} + r^{\cancel{2}}(-\sin\varphi\cos^2\varphi + \sin^2\varphi\cos\varphi) \Rightarrow \dot{\varphi} = r(\sin^2\varphi\cos\varphi - \sin\varphi\cos^2\varphi) - 1$

f) $E^s = ?, \; E^u = ?$

$S_1(0,0) \qquad \lim\limits_{t\to\infty} x(t) = \lim\limits_{t\to\infty}[c_1\cos(bt) - c_2\sin(bt)] = \infty$

$\qquad \lim\limits_{t\to\infty} y(t) = \lim\limits_{t\to\infty}[c_1\cos(bt) + c_2\sin(bt)] = \infty \qquad E^u$

$S_2(1,-1) \quad y = 0 \quad \vec{v}_1 = i \quad E^s$

$\qquad x = 0 \quad \vec{v}_2 = \vec{j} \quad E^u$

g) Ne

19) $\dot{x} = 3y + xy - y^5$, $\quad \dot{y} = -2x - x^2 + 3xy^2$

a) $3y + xy - y^5 = 0 \Rightarrow y(3 + x - y^4) = 0 \qquad S_1(0,0)$

$-2x - x^2 + 3xy^2 = 0 \Rightarrow x(-2 - x + 3y^2) = 0 \qquad S_2(-2,0) \qquad S_3(0, \sqrt[4]{3})$

singulariteti = ?

b) $\lambda_1 = ?, \ \lambda_2 = ?$

$f(x,y) = 3y + xy - y^5$

$g(x,y) = -2x - x^2 + 3xy^2$

$$D\vec{F}(x,y) = \begin{bmatrix} y & 3 + x - 5y^4 \\ -2 - 2x + 3y^2 & 6xy \end{bmatrix}$$

$$D\vec{F}(0,0) = \begin{bmatrix} 0 & 3 \\ -2 & 0 \end{bmatrix}, \quad \det(D\vec{F}(0,0) - \lambda I) = \begin{vmatrix} -\lambda & 3 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 + 3 = 0 \Rightarrow \lambda_{1,2} = \pm\sqrt{3}$$

c) singularitet nije hiperbolički

d) $f(x,y) \overset{?}{=} -f(x,-y), \quad g(x,y) \overset{?}{=} -g(x,-y)$

$-f(x,-y) = -\left[3\cdot(-y) + x\cdot(-y) - (-y)^5\right] = 3y + xy - y^5 = f(x,y)$

$g(x,-y) = -2x - x^2 + 3x\cdot(-y)^2 = -2x - x^2 + 3xy^2 = g(x,y)$

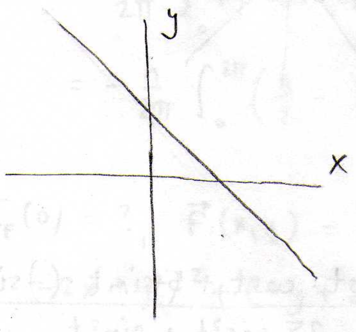
- sustav je reverzibilan

e) $S(0,0)$? centar

ako je sustav reverzibilan, onda u ishodištu ima centar

20) $\dot{x} = 2x - x^2 + xy + y^2$, $\quad \dot{y} = -y + x^2 + xy - \frac{y^2}{2}$

$$\operatorname{div}\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 2 - 2x \cancel{+y} - 1 + x \cancel{-y} = 1 - x \neq 0 \quad \forall x \setminus \{1\}$$



21) $\dot{x} = x, \ \dot{y} = 2y$

a) $I_F(0) = ?, \quad \gamma \ldots \ x = \cos t, \ y = \sin t$

$$I_F(0) = \frac{1}{2\pi}\int_\gamma \frac{f\,dg - g\,df}{f^2+g^2} = \frac{1}{2\pi}\int_\gamma \frac{x\cdot 2dy - 2y\cdot dx}{x^2+4y^2} =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos t\cdot 2\cos t - 2\sin t\cdot(-\sin t)}{\cos^2 t + 4\sin^2 t}\,dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{2}{\cos^2 t + 4\sin^2 t}\,dt$$

$$= \frac{1}{2\pi}\operatorname{arctg}(2\tan t) = 1$$

b) $I_F(0) = ? \quad \gamma \ldots \ x = 2\cos t, \ y = \sin t$

$$I_F(0) = \frac{1}{2\pi}\int_0^{2\pi}\frac{2\cos t\cdot 2\cos t - 2\sin t\cdot(-2\sin t)}{4\cos^2 t + 4\sin^2 t}\,dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{4}{4}\,dt =$$

$$= \frac{1}{2\pi}\,t\,\Big|_0^{2\pi} = 1$$

c) Ne, ali γ mora obuhvaćati singularitet 0

d) $I_0(0) = ?, \quad \vec{F}(x,y) = x\vec{i} + 2y\vec{j}$

$x^2 + 4y^2 = 1$

$T_1(\sqrt{2}, \frac{1}{\sqrt{2}}) \qquad \vec{F}(\sqrt{2}, \frac{1}{\sqrt{2}}) = \sqrt{2}\vec{i} + \sqrt{2}\vec{j}$

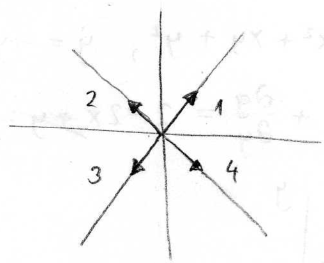
$T_2(-\sqrt{2}, \frac{1}{\sqrt{2}}) \qquad \vec{F}(-\sqrt{2}, \frac{1}{\sqrt{2}}) = -\sqrt{2}\vec{i} + \sqrt{2}\vec{j}$

$T_3(-\sqrt{2}, -\frac{1}{\sqrt{2}}) \qquad \vec{F}(-\sqrt{2}, -\frac{1}{\sqrt{2}}) = -\sqrt{2}\vec{i} - \sqrt{2}\vec{j}$

$T_4(\sqrt{2}, -\frac{1}{\sqrt{2}}) \qquad F(\sqrt{2}, -\frac{1}{\sqrt{2}}) = \sqrt{2}\vec{i} - \sqrt{2}\vec{j}$

1 krug, pozitivan smjer

$I_F(0) = 1$



22) $\dot{x} = -5x, \ \dot{y} = y$

a) $I_F(0) = ?, \quad \gamma \ldots \ x = \cos t, \ y = \sin t$

$$I_F(0) = \frac{1}{2\pi}\int_\gamma \frac{-5x\,dy - y\cdot(-5dx)}{25x^2+y^2} = \frac{1}{2\pi}\int_0^{2\pi}\frac{-5\cos t\cdot\cos t + 5\sin t\cdot(-\sin t)}{25\cos^2 t + \sin^2 t}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{-5}{25\cos^2 t + \sin^2 t}\,dt = \frac{1}{2\pi}\operatorname{arctg}(5\operatorname{ctg} t) = -1$$

b) $I_F(0) = ? \quad \gamma \ldots \ x = \cos t, \ y = 5\sin t$

$$I_F(0) = \frac{1}{2\pi}\int_0^{2\pi}\frac{-5\cos t\cdot 5\cos t + 5\sin t\cdot(-5\sin t)}{25\cos^2 t + 25\sin^2 t}\,dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{-25}{25}\,dt =$$

$$= -\frac{1}{2\pi}\,t\,\Big|_0^{2\pi} = -1$$

c) Ne, ali γ mora obuhvaćati singularitet 0

d) $I_F(0) = ?$, $\vec{F}(x,y) = -5x\vec{i} + y\vec{j}$

$25x^2 + y^2 = 1$

$T_1\left(\frac{1}{\sqrt5}, \sqrt5\right)$ $\quad \vec{F}\left(\frac{1}{\sqrt5}, \sqrt5\right) = -\sqrt5\vec{i} + \sqrt5\vec{j}$

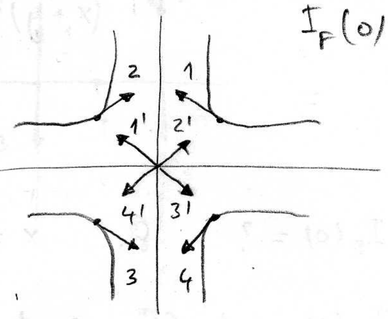
$T_2\left(-\frac{1}{\sqrt5}, \sqrt5\right)$ $\quad \vec{F}\left(-\frac{1}{\sqrt5}, \sqrt5\right) = \sqrt5\vec{i} + \sqrt5\vec{j}$

$T_3\left(-\frac{1}{\sqrt5}, -\sqrt5\right)$ $\quad \vec{F}\left(-\frac{1}{\sqrt5}, -\sqrt5\right) = \sqrt5\vec{i} - \sqrt5\vec{j}$

$T_4\left(\frac{1}{\sqrt5}, -\sqrt5\right)$ $\quad \vec{F}\left(\frac{1}{\sqrt5}, -\sqrt5\right) = -\sqrt5\vec{i} - \sqrt5\vec{j}$

1 krug, negativan smjer

$I_F(0) = -1$



23) $\dot{x} = -2x + y^2$, $\dot{y} = -y + 2x^3$

a) $I_F(0) = ?$ $\quad x = \cos t$, $y = \sin t$

$$I_F(0) = \frac{1}{2\pi}\int_\gamma \frac{(-2x+y^2)\,(-dy) - (-y+2x^3)\cdot(-2dx)}{(-2x+y^2)^2 + (-y+2x^3)^2} = \quad ???,$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(-2\cos t + \sin^2 t)\cdot(-\cos t) - (-\sin t + 2\cos^3 t)\cdot 2\sin t}{(-2\cos t + \sin^2 t)^2 - (-\sin t + 2\cos^3 t)^2}\,dt = ???$$

b) $I_F(0) = ?$ $\quad \gamma \ldots \quad x = \frac{\sin^2 t - \cos t}{2}$, $y = 2\cos^3 t - \sin t$ $\quad$???

$$I_F(0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(-\cancel{2}\,\frac{\cancel{\sin^2 t} - \cos t}{\cancel{2}} + \cancel{\sin^2 t}\right)\cdot(-\cos t) - \left(-\cancel{2\cos^3 t} + \sin t + \cancel{2\cos^3 t}\right)\cdot 2\sin t}{\left(-\cancel{2}\,\frac{\cancel{\sin^2 t} - \cos t}{\cancel{2}} + \cancel{\sin^2 t}\right)^2 + \left(-\cancel{2\cos^3 t} + \sin t + \cancel{2\cos^3 t}\right)^2}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(-\cos^2 t - 2\sin^2 t\right)dt = -\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1+\cos 2t}{2} + \cancel{2}\,\frac{1-\cos 2t}{\cancel{2}}\right)dt =$$

$$= -\frac{1}{2\pi}\int_0^{2\pi}\left(\frac32 - \frac{\cos 2t}{2}\right)dt = -\frac{1}{2\pi}\left(\frac32 t - \frac14 \sin 2t\right)\Big|_0^{2\pi} = -\frac{1}{2\cancel{\pi}}\cdot\frac{\overset{3}{\cancel{6}}\cancel{\pi}}{2} = -\frac32 \approx 2$$

c)

d) $I_F(0) = ?$, $\vec{F}(x,y) = (-2x+y^2)\vec{i} + (-y+2x^3)\vec{j}$ $\quad$???

$(-2x+y^2)^2 + (-y+2x^3)^2 = 1$

$T_1(1,1)$ $\quad \vec{F}(1,1) = -\vec{i} + \vec{j}$

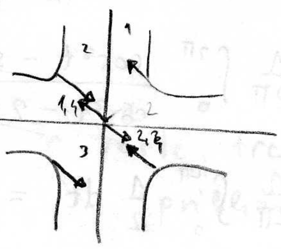
$T_2(-1,1)$ $\quad \vec{F}(-1,1) = 3\vec{i} - 3\vec{j}$

$T_3(-1,-1)$ $\quad \vec{F}(-1,-1) = 3\vec{i} - \vec{j}$

$T_4(1,-1)$ $\quad \vec{F}(1,-1) = -\vec{i} + 3\vec{j}$

2 kruga, negativan smjer

$I_F(0) = 2$



???

24) $\dot{x} = x + y \,,\quad \dot{y} = y$

a) $I_F(0) = ?$ $\quad \gamma \ldots\ x = \cos t \,,\ y = \sin t$

$$I_F(0) = \frac{1}{2\pi} \int_\gamma \frac{(x+y)\,dy - y\,dx}{(x+y)^2 + y^2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos t \cdot \cos t}{(\cos t + \sin t)^2 + \sin^2 t}\, dt = ???$$

$$=$$

b) $I_F(0) = ?$ $\quad \gamma \ldots\ x = \cos t - \sin t \,,\ y = \sin t$

$$I_F(0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos t \cdot \cos t - \sin t(-\sin t - \cos t)}{\cos^2 t + \sin^2 t}\, dt = \frac{1}{2\pi} \int_0^{2\pi} (2\cos^2 t + \sin t)\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(\cancel{2} \cdot \frac{1 + \cos 2t}{\cancel{2}} + \frac{1 - \cos 2t}{2} \right) dt = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{3}{2} + \frac{\cos 2t}{2} \right) dt =$$

$$= \frac{1}{2\pi} \left(\frac{3}{2} t \Big|_0^{2\pi} + \frac{1}{4} \sin 2t \Big|_0^{2\pi} \right) = \frac{3}{\cancel{4}\cancel{\pi}_{2}} \cdot \cancel{2\pi} = \frac{3}{2} \approx 2 \qquad ???$$

c)

d) $I_F(0) = ?$ $\quad \vec{F}(x,y) = (x+y)\vec{i} + y\vec{j}$

$(x+y)^2 + y^2 = 1$

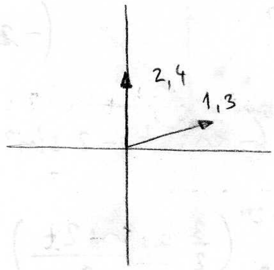
$T_1(1,1) \quad \vec{F}(1,1) = 4\vec{i} + \vec{j}$

$T_2(-1,1) \quad \vec{F}(-1,1) = \vec{j}$

$T_3(-1,-1) \quad \vec{F}(-1,-1) = 4\vec{i} + \vec{j}$

$T_4(1,-1) \quad \vec{F}(1,-1) = \vec{j}$

2 kruga, pozitivan smjer



$I_F(0) = 2$

25) $\dot{x} = x - y \,,\quad \dot{y} = x + y$

a) $I_F(0) = ?$ $\quad \gamma \ldots\ x = \cos t \,,\ y = \sin t$

$$I_F(0) = \frac{1}{2\pi} \int_\gamma \frac{(x-y)\,dy - (x+y)\,dx}{(x-y)^2 + (x+y)^2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{(\cos t - \sin t)\cos t - (\cos t + \sin t)(-\sin t)}{(\cos t - \sin t)^2 + (\cos t + \sin t)^2}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos^2 t - \cancel{\sin t \cos t} + \cancel{\sin t \cos t} + \sin^2 t}{\cos^2 t - \cancel{2 \sin t \cos t} + \sin^2 t + \cos^2 t + \cancel{2 \sin t \cos t} + \sin^2 t}\, dt =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2}\, dt = \frac{1}{4\pi}\, t \Big|_0^{2\pi} = \frac{2\pi}{4\pi} = \frac{1}{2} \approx 1$$

b) $\gamma \ldots\ x = \cos t \,,\ y = \sin t$

c) Ne,

d) $I_F(0) = ?$, $\vec{F}(x,y) = (x-y)\vec{i} + (x+y)\vec{j}$

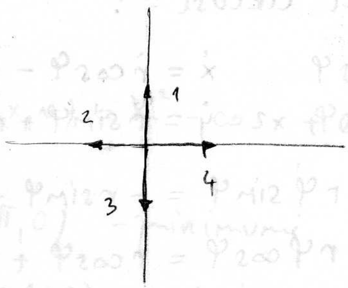
$(x-y)^2 + (x+y)^2 = 1$ 1 krug, pozitivan smjer $I_F(0) = 1$

$T_1(1,1)$ $\vec{F}(1,1) = 2\vec{j}$

$T_2(-1,1)$ $\vec{F}(-1,1) = -2\vec{i}$

$T_3(-1,-1)$ $\vec{F}(-1,-1) = -2\vec{j}$

$T_4(1,-1)$ $\vec{F}(1,-1) = 2\vec{i}$



26) $\dot{x} = -y + x(1-x^2-y^2)^2$

$\dot{y} = x + y(1-x^2-y^2)^2$

a) granični ciklusi = ?

$x = r\cos\varphi$ $\dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi$

$y = r\sin\varphi$ $\dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$

$$\left.\begin{aligned} \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi &= -r\sin\varphi + r\cos\varphi(1-r^2)^2 \quad / \cdot \cos\varphi \\ \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi &= r\cos\varphi + r\sin\varphi(1-r^2)^2 \quad / \cdot \sin\varphi \end{aligned}\right\} + \quad \left.\begin{aligned} &/ \cdot (-\sin\varphi) \\ &/ \cdot \cos\varphi \end{aligned}\right\} +$$

$$\left.\begin{aligned} \dot{r} &= r(1-r^2)^2 \\ \cancel{r}\dot{\varphi} &= \cancel{r} \Rightarrow \dot{\varphi} = 1 \end{aligned}\right\} \quad r = 1 \text{ granični ciklus}$$

b) $\nabla F(\gamma(t)) = ?$

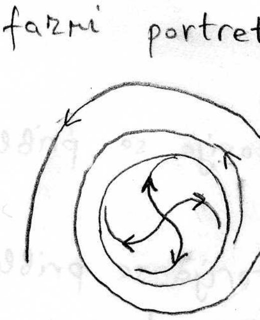
$$\nabla\vec{F}(x,y) = (1-x^2-y^2)^2 + x\cdot 2(1-x^2-y^2)\cdot(-2x) + (1-x^2-y^2)^2 + y\cdot 2(1-x^2-y^2)\cdot(-2y) =$$
$$= 1 + x^4 + y^4 - 2x^2 - 2y^2 + 2x^2y^2 - 4x^2 + 4x^4 + 4x^2y^2 + 1 + x^4 + y^4 - 2x^2 - 2y^2 + 2x^2y^2$$
$$- 4y^2 + 4x^2y^2 + 4y^4 = 2 + 6x^4 + 6y^4 - 8x^2 - 8y^2 + 12x^2y^2$$

$$\int_0^{2\pi} \nabla\vec{F}(\gamma(t))\,dt = \int_0^{2\pi}(2 - 8\cos^2 t - 8\sin^2 t + 12\sin^2 t\cos^2 t + 6\cos^4 t + 6\sin^4 t)\,dt$$

$$= \left[-6t - \frac{3}{8}(\sin(4t) - 4t) + \frac{3}{8}(12t + \sin(4t))\right]_0^{2\pi} = \left(-6t + \frac{3}{2}t + \frac{9}{2}t\right)\Big|_0^{2\pi} =$$

$= 0$ ← polustabilan granični ciklus

c) fazni portret = ?



$0 < r < 1$ $\dot{r} > 0$ r raste, trajektorija se približava

$1 < r < \infty$ $\dot{r} > 0$ r raste, trajektorija se udaljava

27) $\dot{x} = -y + x(1-x^2-y^2)(4-x^2-y^2)$

$\dot{y} = x + y(1-x^2-y^2)(4-x^2-y^2)$

a) granični ciklusi = ?

$$x = r\cos\varphi \qquad \dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi$$
$$y = r\sin\varphi \qquad \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$$

$$\left.\begin{aligned}\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi &= -r\sin\varphi + r\cos\varphi(1-r^2)(4-r^2) \quad &|\cdot\cos\varphi\\ \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi &= r\cos\varphi + r\sin\varphi(1-r^2)(4-r^2) \quad &|\cdot\sin\varphi\end{aligned}\right\}+ \quad \left.\begin{aligned}&|\cdot(-\sin\varphi)\\ &|\cdot\cos\varphi\end{aligned}\right\}+$$

$$\left.\begin{aligned}\dot{r} &= r(1-r^2)(4-r^2)\\ \cancel{r}\dot{\varphi} &= \cancel{r} \Rightarrow \dot{\varphi} = 1\end{aligned}\right\} \quad \begin{aligned}r&=1\\ r&=2\end{aligned} \quad \text{granični ciklusi}$$

b) $\nabla\vec{F}(\gamma(t)) = ?$

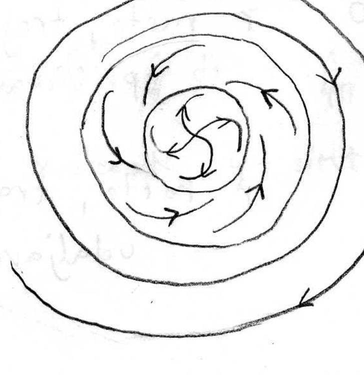
$$\nabla\vec{F}(x,y) = (1-x^2-y^2)(4-x^2-y^2) + x\cdot(-2x)(4-x^2-y^2) + x(1-x^2-y^2)\cdot(-2x) +$$
$$+ (1-x^2-y^2)(4-x^2-y^2) + y\cdot(-2y)(4-x^2-y^2) + y(1-x^2-y^2)\cdot(-2y) =$$
$$= 4 - x^2 - y^2 - 4x^2 + x^4 + x^2y^2 - 4y^2 + x^2y^2 + y^4 - 8x^2 + 2x^4 + 2x^2y^2 -$$
$$- 2x^2 + 2x^4 + 2x^2y^2 + 4 - x^2 - y^2 - 4x^2 + x^4 + x^2y^2 - 4y^2 + x^2y^2 + y^4 -$$
$$- 8y^2 + 2x^2y^2 + 2y^4 - 2y^2 + 2x^2y^2 + 2y^4 =$$
$$= 8 - 20x^2 - 20y^2 + 12x^2y^2 + 6x^4 + 6y^4$$

$$\int_0^{2\pi}\nabla\vec{F}(\gamma(t)) = \int_0^{2\pi}(8 - 20\cos^2 t - 20\sin^2 t + 12\sin^2 t\cos^2 t + 6\cos^4 t + 6\sin^2 t)\,dt =$$
$$= \left[-12t - \frac{3}{8}(\sin(4t) - 4t) + \frac{3}{8}(12t + \sin(4t))\right]\Big|_0^{2\pi} =$$
$$= \left[-12t + \frac{3}{2}t + \frac{9}{2}t\right]\Big|_0^{2\pi} = -6t\Big|_0^{2\pi} = -12\pi < 0$$

– stabilan granični ciklus

$$\int_0^{2\pi}\nabla\vec{F}(\gamma(t)) = \int_0^{2\pi}(8 - 80\cos^2 t - 80\sin^2 t + 192\sin^2 t\cos^2 t + 96\cos^4 t + 96\cos^4 t)\,dt =$$
$$= \left[-72t - 6(\sin(4t) - 4t) + 6(12t + \sin(4t))\right]\Big|_0^{2\pi} =$$
$$= \left[\cancel{-72t} + 24t + \cancel{72t}\right]\Big|_0^{2\pi} = 24t\Big|_0^{2\pi} = 48\pi > 0$$

– nestabilan granični ciklus



$0 < r < 1 \qquad \dot{r} > 0$

r raste, trajektorija se približava

$1 < r < 2 \qquad \dot{r} < 0$

r pada, trajektorija se približava $r = 1$ i udaljava od $r = 2$

$2 < r < \infty \qquad \dot{r} > 0$

r raste, trajektorija se udaljava

28) $\frac{d^2x}{dt^2} + \sin x = 0$, fazni portret = ?

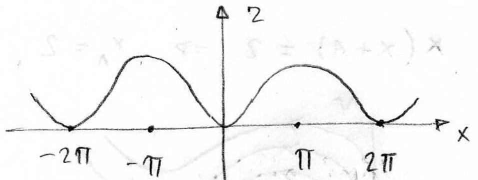
$\left.\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\sin x \end{aligned}\right\}$ $\frac{dv}{dx} = -\frac{\sin x}{v} \Rightarrow v\,dv = -\sin x\,dx \;/\int \Rightarrow \frac{v^2}{2} = \cos x + C$

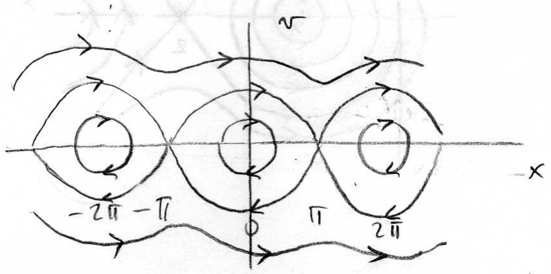
$v = 0$

$-\sin x = 0 \Rightarrow x = k\pi$ $\quad S_k(k\pi, 0)$ $\quad E(x,v) = \frac{v^2}{2} - \cos x + C$, $E(0,0) = 0$

$S_k(2k\pi, 0)$ – minimumi

$S_k(2k+1, 0)$ – maksimumi





29) $\frac{d^2x}{dt^2} - e^x + x^2 - 1 = 0$, fazni portret = ?

$\left.\begin{aligned} \dot{x} &= v \\ \dot{v} &= e^x - x^2 + 1 \end{aligned}\right\}$ $\frac{dv}{dx} = \frac{e^x - x^2 + 1}{v} \Rightarrow v\,dv = (e^x - x^2 + 1)\,dx \;/\int \Rightarrow$

$\frac{v^2}{2} = e^x - \frac{x^3}{3} + x + C \Rightarrow E(x,v) = \frac{v^2}{2} - e^x + \frac{x^3}{3} - x + C$, $E(0,0) = -1$

$v = 0$

$e^x - x^2 + 1 = 0$

30) $\frac{d^2x}{dt} + x^2 + x - 2 = 0$, fazni portret

$$\left.\begin{aligned} \dot{x} &= v \\ \dot{v} &= -x^2 - x + 2 \end{aligned}\right\} \quad \frac{dv}{dx} = \frac{-x^2-x+2}{v} \Rightarrow v\,dv = (-x^2-x+2)\,dx \;\Big/ \int \Rightarrow$$

$$\frac{v^2}{2} = -\frac{x^3}{3} - \frac{x^2}{2} + 2x + C \Rightarrow E(x,v) = \frac{v^2}{2} + \frac{x^3}{3} + \frac{x^2}{2} + 2x + C \quad E(0,0) = 0$$

$v = 0$ $\qquad S_1(1,0)$, $S_2(2,0)$

$-x^2 - x + 2 = 0 \Rightarrow x^2 + x = 2 \Rightarrow x(x+1) = 2 \Rightarrow x_1 = 2,\ x_2 = 1$