*ĐURIĆ

12.3 LINEARNE DIF JDŽBE VIŠIH REDOVA

$$A_n(x)\cdot y^{(n)} + A_{n-1}(x)y^{(n-1)} + \dots + A_1(x)y' + A_0 y = f(x)$$

$= 0$ - homogene

$\neq$ nehomogene

Hom.
Lin. dif jdžba 2. reda

$$y'' + p(x)y' + g(x)y = 0$$

Vrijedi sljedeće

i) ako je y_1 rješenje od HLDJ onda je i $c_1 y_1$ rješenje

ii) ako su y_1, y_2 rj. tada je i $y_1 + y_2$ rješenje

$\Rightarrow$ $c_1 y_1 + c_2 y_2 =$ je rješenje

TM: Ako je y_1 bilo koje rj. od HLDJ 2. reda onda svako drugo rješenje ima oblik:

$$y(x) = y_1(x)\left[c_2 \int \frac{1}{y_1(x)^2} e^{-\int p(x)\,dx} dx + c_1\right]$$

Tada je OR. HLDJ 2. reda:

$$y = c_1 y_1(x) + c_2 y_2(x)$$

21-2012-9) Pokaži

a) Pokažite da vrijedi svojstvo aditivnosti HLDJ 2. reda, tj ako su y_1 i y_2 rj onda je i rj. zbroj rj.

$$\left.\begin{aligned} y_1'' + p(x)y_1' + g(x)y_1 = 0 \\ y_2'' + p(x)y_2' + g(x)y_2 = 0 \end{aligned}\right\} +$$

$$y_1'' + y_2'' + p(x)(y_1' + y_2') + g(x)(y_1 + y_2) = 0$$

$$(y_1 + y_2)'' + p(x)(y_1 + y_2)' + g(x)(y_1 + y_2) = 0$$

$\Rightarrow$ možemo uvrstiti u početnu jdžbu i to je dokazano

b) Ako je poznato da su f-je $y_1(x)=e^{x}+x+1$ i $y_2(x)=e^{-x}+x-2$ HLDJ 2 rj nađite ono rj koje siječe os ordinatu u točki $(0,1)$ pod kutom 30°

C.R. $y=c_1(e^{x}+x+1)+c_2(e^{-x}+x-2)$

$1=c_1(1+1)+c_2(-1)$

$1=2c_1-c_2$

$1=\frac{2}{\sqrt{3}}\cdot 2-c_2 \Rightarrow$

$c_2=\sqrt{3}-1$

$y'=c_1(e^{x}+1)+c_2(-e^{-x}+1)$

$y'(0)=tg\,60^\circ=\sqrt{3}$

$=c_1(1+1)+c_2(0)=\sqrt{3}$

$c_1=\frac{\sqrt{3}}{2}$

$\Rightarrow y=\frac{\sqrt{3}}{2}(e^{x}+x+1)+(\sqrt{3}-1)(e^{x}+x-2)$

12. DZ 19 $xy''-(2x+1)y'+(x+1)y=0$ - HLDJ 2. reda

Pronađite param α ako je jedno rj. $y_1=e^{\alpha x}$.

Snižavanjem reda odredite CR ove dif. jednadžbe

$y'=\alpha e^{\alpha x}$, $y''=\alpha^2 e^{\alpha x}$

$x\alpha^2\cancel{e^{\alpha x}}-(2x+1)\alpha\cancel{e^{\alpha x}}+(x+1)\cancel{e^{\alpha x}}=0 \quad |\, e^{\alpha x}$

$x\alpha^2-2x\alpha-\alpha+x+1=0$

$x\underbrace{(\alpha^2-2\alpha+1)}_{0}+\underbrace{1-\alpha}_{0}=0$

$x\underbrace{(\alpha-1)^2}_{0}+(1-\alpha)=0$

$\boxed{\alpha=1}$

=> jedno rj. je e^x, $y_1 = e^x$

– uvijet za traženje HLDJ 2. reda (2. rješenje)

supstitucija $\boxed{u = \frac{y}{y_1}}$ => $y = u y_1 = u \cdot e^x$

$$y' = u' \cdot e^x + u e^x = e^x (u' + u)$$

$$y'' = e^x (u' + u) + e^x (u'' + u') = e^x (u'' + 2u' + u)$$

$$x \cancel{e^x} (u'' + 2u' + u) - (2x+1) \cancel{e^x} (u' + u) + (x+1) u \cancel{e^x} = 0$$

$$x u'' + \cancel{x 2u'} + \cancel{xu} - \cancel{2xu'} - \cancel{2xu} - u' - \cancel{u} + \cancel{xu} + \cancel{u} = 0$$

$$x u'' - u' = 0$$

$x y'' - y = 0$

ili $y' = z(x)$ - kada imamo x } ali se zapravo jedino pojavljuje na ispitima

ili $y' = p(y)$ - kada imamo y

-> $u' = z(x)$, $u'' = z'$

$$x \cdot z' - z = 0$$

$x \frac{dz}{dx} = z$ $\qquad \int \frac{dz}{z} = \int \frac{dx}{x}$

$\ln z = \ln x + \ln C_1 = \ln C_1 x$

$z = C_1 x$

$u' = C_1 x \quad / \int$

$u = C_1 \frac{x^2}{2} + C_2$

$u = \frac{y}{e^x}$ $\qquad \frac{y}{e^x} = C_1 x^2 + C_2 \quad / e^x$

$$\boxed{y = e^x C_1 x^2 + C_2 e^x}$$

2.4 Hom. lin. dif. jdžbe 2. reda s konst. koef.

$$y'' + a_1 y' + a_0 y = 0$$

ISPIT

$$y = e^{rx}$$

$$y' = r e^{rx}$$

$$y'' = r^2 e^{rx}$$

$$r^2 e^{rx} + r e^{rx} + a_0 = 0$$

$r^2 + a_1 r + a_0 = 0$ - karakteristična jdžba

$r_{12} =$

O.R. HLDJ skh 2. reda ima oblik

1.) $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$, ako su $r_1 \neq r_2$ realni

2.) $y = C_1 e^{r_1 x} + C_2 x e^{r_1 x}$, ako je $r_1 = r_2$

3.) $y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x$, $r_{1,2} = \alpha \pm i\beta$

12. DZ) 7. c) $y'' - 6y' + 9y = 0$

$y'' \to r^2$ $\quad r^2 - 6r - 9 = 0$

$y' \to r$ $\quad r_{12} = 3$

$y \to 1$

$$y = C_1 e^{3x} + C_2 x e^{3x}$$

8. a) $y'' + y' + y = 0$

$$r^2 + r + 1 = 0$$

$$r_{12} = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

$$y = C_1 e^{-\frac{1}{2}x} \cos \frac{\sqrt{3}}{2} x + C_2 e^{\frac{1}{2}x} \sin \frac{\sqrt{3}}{2} x$$

Zad) $y'' - y' - 2y = 0$, $y(1) = 2$, $y'(1) = 0$

$r^2 - r - 2 = 0$

$r_1 = 2$, $r_2 = -1$ $\Rightarrow$ $y = C_1 e^{2x} + C_2 e^{-x}$

$y' = 2C_1 e^{2x} - C_2 e^{-x}$

$2 = C_1 e^2 + C_2 e^{-1}$

$0 = 2C_1 e^2 - C_2 e^{-1}$

$2 = 3C_1 e^2 \Rightarrow C_1 = \frac{2}{3} e^{-2}$

$0 = \frac{4}{3} - C_2 e^{-1}$

$C_2 = \frac{4}{3} e$

$$\boxed{y = \frac{2}{3} e^{-2} e^{2x} + \frac{4}{3} e \cdot e^{-x}}$$

12.07 - 13)

$y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 1$

$r^2 + 4 = 0$

$r^2 = -4$

$r_{1,2} = \pm 2i$, $\alpha = 0$, $\beta = 2$

$y = C_1 \cos 2x + C_2 \sin 2x$

$\alpha = 0$, $\beta = 2$

$1 = C_1$

$y' = -2C_1 \sin 2x + 2C_2 \cos 2x$

$1 = 2C_2 \rightarrow \boxed{C_2 = \frac{1}{2}}$

2I - 2010 - 3)

$$y^2 y'' = \frac{1}{y^2} - \frac{y'^3}{y}$$

zupst $\left.\begin{array}{l} y' = p(y) \\ y'' = p'p \end{array}\right\}$ $\quad p^2 p' = \frac{1}{y^2} - \frac{p^3}{y} \quad /: p^2$

$y' + f(x)y = g(x)y^\alpha$ - Bernoullijeva

$p' + f(y) \cdot p = g(y) p^\alpha$

$$p' + \frac{p}{y} = \frac{1}{y^2 p^2} \quad / \cdot 3p^2$$

$\boxed{-2}$

zupst $z = p^{1-\alpha} = p^3$

$z' = 3p^2 p'$

$$3p^2 p' + \frac{3p^3}{y} = \frac{3}{y^2}$$

$$z' + \frac{3}{y} z = \frac{3}{y^2} \quad \text{Lin.}$$

$z' + \frac{3}{y} z = 0$

$\frac{dz}{dy} = -\frac{3z}{y}$

$\frac{dz}{z} = -\frac{3}{y}$

$\int \frac{dz}{z} = -\int \frac{3}{y} dy$

$\ln z = -3 \ln y + \ln C_1 = \ln C_1 y^{-3}$

$z = \frac{C_1}{y^3}$ $\quad$ VKR zupst $\quad = \frac{C_1(y)}{y^3} \Rightarrow$ u poč. Lin

$$z' = \frac{C_1'(y) y^3 - C_1(y) \cdot 3y^2}{y^6} + \frac{3}{y} \cdot \frac{C_1(y)}{y^3} = \frac{3}{y^2}$$

$$C_1'(y) \frac{1}{y^3} = \frac{3}{y^2} \quad / \cdot y^3$$

$$C_1'(y) = 3y \quad / \int dy$$

$$C_1(y) = \frac{3y^2}{2} + C_1$$

$$z = \frac{\frac{3}{2}y^2 + C_1}{y^3} = \frac{3}{2y} + \frac{C_1}{y^3}$$

$$p^3 = \frac{3}{2y} + \frac{c_1}{y^3}$$

$$\boxed{p = \sqrt[3]{\frac{3}{2y} + \frac{c_1}{y^3}}} - \boxed{\text{rj. BERN. DIF. J.}}$$

$$y' = \sqrt[3]{\frac{3}{2y} + \frac{c_1}{y^3}}$$

$$\frac{dy}{dx} = \sqrt[3]{\frac{3}{2y} + \frac{c_1}{y^3}}$$

$$\int \frac{dy}{\sqrt[3]{\frac{3}{2y} + \frac{c_1}{y^3}}} = \int dx \quad \Rightarrow \quad x + C_2$$

$$\int \frac{dy}{\sqrt[3]{\frac{3y^2 + c_1}{2y^3}}} = \int \frac{y\,dy}{\sqrt[3]{\frac{3}{2}y^2 + c_1}} = \left| \begin{array}{l} \frac{3}{2}y^2 + c_1 = t \\ 3y\,dy = dt \end{array} \right| = \frac{1}{3}\int \frac{dt}{\sqrt[3]{t}} = \frac{1}{3}\frac{t^{\frac{2}{3}}}{\frac{2}{3}} = \frac{1}{2}\sqrt[3]{t^2}$$

$$\Rightarrow \frac{1}{2}\left(\sqrt[3]{\frac{3}{2}y^2 + c_1}\right)^2$$

CR. preč. jdžbe

$$\frac{1}{2}\sqrt[3]{\left(\frac{3}{2}y^2 + c_1\right)^2} = x + c_2 \quad / \cdot 2 \quad /^3$$

$$\left(\frac{3}{2}y^2 + c_1\right)^2 = (2x + C_2)^3$$