DIFERENCIJALNE JEDNADŽBE

DIFERENCIJALNE JEDNADŽBE PRVOGA REDA

Diferencijalna jednadžba: jednadžba u kojoj se javljaju funkcija $y = y(x)$ i njena derivacija

Diferencijalna jednadžba n-tog reda: $F(x, y, y', \dots, y^{(n)}) = 0$

ili $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$

x - varijabla po kojoj deriviramo
$y = y(x)$ - nepoznata funkcija koju tražimo
$y', \dots, y^{(n)}$ - derivacije od y po x

PR 1) $y' = f(x)$ diferencijalna jednadžba prvog reda

$y = \int f(x) + C, \quad C \in \mathbb{R}$

$f(x) = x$
$y = \frac{x^2}{2} + C$

2) $y^{(n)} = 1 \quad / \int$

$$\left.\begin{aligned} y^{(n-1)} &= x + C_1 \quad / \int \\ y^{(n-2)} &= \frac{x^2}{2} + C_1 x + C_2 \\ &\vdots \end{aligned}\right\} n \text{ puta integracija}$$

$y = \frac{x^n}{n!} + C_1 x_{n-1} + \dots + C_{n-1} x + C_n$ — n konstanti

Rješenje dif. jednadžbe je svaka funkcija $y = y(x)$ koja na nekom intervalu ima sve derivacije i zadovoljava diferencijalnu jednadžbu. Rješenje dif. jednadžbe n-tog reda je oblika

① $y = y(x, \underbrace{C_1, \dots, C_n}_{\text{br. konstanti = red jednadžbe}})$ ← opće rješenje dif jednadžbe

② $\Phi(x, y, C_1, \dots, C_n) = 0$ ← integral dif. jednadžbe
implicitno zadana funkcijom Φ

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DEF Cauchyeva zadaća

Određivanje funkcije y koja zadovoljava dif. jednadžbu n-tog reda i n početnih uvjeta

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$$
$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$
$$\vdots$$
$$y^{(n-1)}(x_0) = y_{n-1}$$

Naziva se Cauchyeva zadaća. Tu su $y_0, y_1, \dots, y_{n-1}$ zadani realni brojevi.

ZAD Pronaći konstante C_1 i C_2 takve da je funkcija $y = C_1 e^{3x} + C_2 e^{4x}$ rješenje Cauchyeve zadaće

$$y'' - 7y' + 12y = 0$$
$$y(0) = 4$$
$$y'(0) = 15$$

1) Provjerimo je li to rješenje dif. jednadžbe

$$y = C_1 e^{3x} + C_2 e^{4x}$$
$$y' = 3C_1 e^{3x} + 4C_2 e^{4x}$$
$$y'' = 9C_1 e^{3x} + 16C_2 e^{4x}$$

$$9C_1 e^{3x} + 16C_2 e^{4x} - 21C_1 e^{3x} - 28C_2 e^{4x} + 12C_1 e^{3x} + 12C_2 e^{4x} = 0 \checkmark$$

$$\left.\begin{array}{l} y(0) = C_1 + C_2 = 4 \\ y'(0) = 3C_1 + 4C_2 = 15 \end{array}\right\} \quad \begin{array}{l} C_1 = 1 \\ C_2 = 3 \end{array}$$

$$y = e^{3x} + 3e^{4x}$$

DEF Diferencijalna jednadžba prvog reda

$$F(x, y, y') = 0 \quad , \quad y' = f(x, y)$$

opće rješenje : $y = \varphi(x, c)$

integral : $\Phi(x, y, c) = 0$

DEF Integralna krivulja = krivulja rješenja $y = \varphi(x, c)$ ili $\Phi(x, y, c) = 0$

Cauchyeva zadaća $F(x, y, y') = 0$, $y(x_0) = y_0$ integralna krivulja prolazi točkom (x_0, y_0)

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ZAD $y' = 5x+2$ $\quad y(1) = 5$

$$y = \frac{5x^2}{2} + 2x + c$$

$$5 = \frac{5}{2} + 2 + c \quad / \cdot 2$$
$$10 = 5 + 4 + 2c$$
$$c = \frac{1}{2}$$

integralne krivulje su parabole

Diferencijalna jednadžba familije krivulja

$$F(x, y, y', \dots, y^{(n)}) = 0 \Rightarrow y = y(x, c_1, \dots, c_n)$$

rješenje dif. jedn.

Obrnuti postupak = zadana je familija krivulja $y = y(x, c_1, \dots, c_n)$ ili $\Phi = (x, y, c_1, \dots, c_n)$

Želimo naći dif. jedn. kojoj je familija krivulja opće rješenje.

ZAD Odredi dif. jednadžbu familije krivulja $y^2 + cx = x^2$ / $'$

$$2yy' + c = 2x$$

$$c = \frac{x^2 - y^2}{x}$$

$$2yy' + \frac{x^2 - y^2}{x} = 2x \quad / \cdot x$$

$$y' = \frac{x^2 + y^2}{2xy}$$

ZAD Odredi dif. jednadžbu koje zadovoljavaju krivulje zadane familije

$$y = c_1 e^{2x} + c_2 e^{-x}.$$
$$y' = 2c_1 e^{2x} - c_2 e^{-x}$$
$$y'' = 4c_1 e^{2x} + c_2 e^{-x}$$

$$y'' - y = 3c_1 e^{2x}$$
$$c_1 = \frac{y'' - y}{3e^{2x}}$$

$$y'' - 2y' = 3c_2 e^{-x}$$
$$c_2 = \frac{y'' - 2y'}{3e^{-x}}$$

$$y = \frac{y'' - y}{3} + \frac{y'' - 2y'}{3}$$
$$2y'' = 4y + 2y'$$
$$y'' = 2y + y'$$
$$y'' - y' - 2y = 0$$

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Polje smjerova

ANALITIČKI

$y' = f(x)$

$y_1(x)$ neko rješenje

$y_1'(x) = f(x, y_1(x))$

GEOMETRIJSKI

$\longleftrightarrow$ polje smjerova

$\longleftrightarrow$ integralna krivulja

svakoj točki (x, y) pridružen je koef. smjera u toj točki ($f(x_0, y_0)$ odnosno $y'(y_0)$)

Kako ćemo crtati integralne krivulje:

1. odaberemo koef. smjera tangente c ($tg\,\varphi = c$)
2. nacrtamo $f(x, y) = c$ za različite c (u svim točkama navedenih krivulja polje smjerova ima smjer kao tangenta)

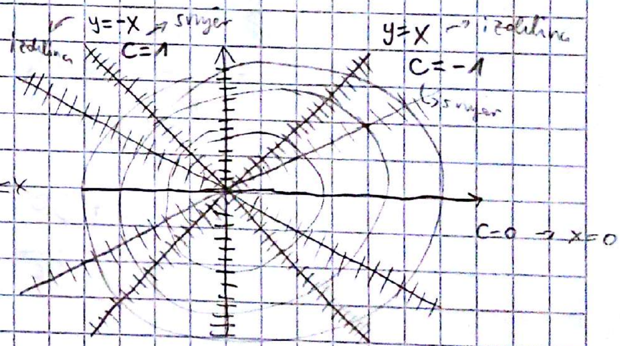
   ↑ izokline

3. Nacrtamo izokline i smjerove
4. Crtati integralne krivulje

PRIMJER $\quad y' = \dfrac{-x}{y}$ $\quad$ Nacrtati polje smjerova i integralne krivulje

izokline $\quad \dfrac{-x}{y} = c$

$y = \dfrac{-1}{c} x$



integralne krivulje su kružnice (koncentrične sa centrom u ishodištu)

PRIMJER $\quad y' = x$

izokline $\quad x = c$

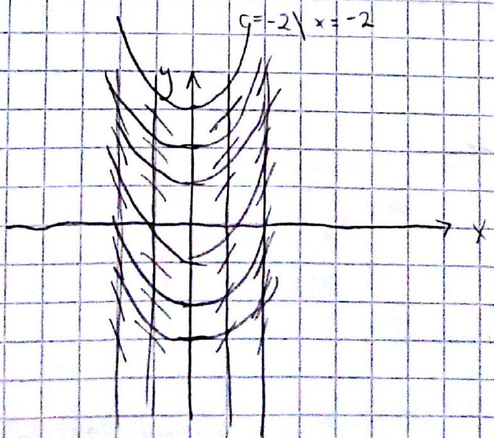
$c = 1 \,/\, x = 1$

$c = -1 \,\backslash\, x = -1$

$c = 0 - x = 0$

$c = 2 \,/\, x = 2$

$c = -2 \,\backslash\, x = -2$



$y' = x \quad / \int$

$y = \dfrac{x^2}{2} + C \quad$ / familija parabola

Jednadžbe sa separiranim varijablama

DEF Dif. jednadžbu 1. reda $F(x, y, y')=0$ zovemo dif. jednadžba sa separiranim varijablama ako se može zapisati u obliku

$$\underbrace{f(y)\,dy = g(x)\,dx}_{\text{svi } y \text{ na jednu, a } x \text{ na drugu stranu}} \quad / \int$$

rješavamo direktnim integriranjem

$$\int f(y)\,dy = \int g(x)\,dx + C$$

Ako je zadan početni $y(x_0)=y_0$ onda izračunamo C.

Mogući zapisi:

$$y' = \frac{g(x)}{f(y)} \quad \text{ili} \quad \frac{dy}{dx} = \frac{g(x)}{f(y)} \quad \text{ili} \quad g(x)\,dx = f(y)\,dy$$

ZAD Riješi $y' = \dfrac{x}{y^2}$ $\quad y \neq 0$

$$\frac{dy}{dx} = \frac{x}{y^2}$$

$$y^2\,dy = x\,dx \;/\int$$

$$\frac{y^3}{3} = \frac{x^2}{2} + C$$

$$y^3 = \frac{3}{2}x^2 + 3C$$

$$y = \sqrt[3]{\frac{3}{2}x^2 + 3C}$$

Cauchyev problem $\quad y(x_0) = y_0$

$$\int_{y_0}^{y} f(y)\,dy = \int_{x_0}^{x} g(x)\,dx + C_1, \quad C_1 \text{ izrač. iz početnog uvjeta } C_1 = 0$$

ZAD Riješi diferencijalnu jednadžbu

$$x(y-1)\frac{dy}{dx} = y, \quad y \neq 0$$

(provjeri lako da $y=0$ je rješenje jer $(0)'=0$)

$$\left(1-\frac{1}{y}\right)dy = \frac{dx}{x}$$

$$y - \ln|y| = \ln|x| + C$$

$$y = \ln|xy| + C$$

opće rješenje dif. jednadžbe, $y=0$ ne pada u njega

ZAD Riješite Cauchyjevu zadaću (početni problem)

$$y' + 4xy^2 = 0$$
$$y(0) = 1$$

$$\frac{dy}{dx} = -4xy^2$$

Za $y=0$ postoji rješenje, ali ono nije rješenje početnog problema ($0 \neq 1$)

$y(x) \neq 0$

$$\frac{dy}{y^2} = -4x\,dx \quad / \int$$

$$-\frac{1}{y} = -2x^2 + C \implies y = \frac{1}{2x^2 - C} \implies C = -1$$

Rješenje početnog problema je $y = \dfrac{1}{2x^2+1}$

ZAD $e^y(1+x^2)\,dy - 2x(1+e^y)\,dx = 0$

$$e^y(1+x^2)\,dy = 2x(1+e^y)\,dx \quad / \cdot \frac{1}{1+x^2} \cdot \frac{1}{1+e^y}$$

$$\frac{e^y}{1+e^y}\,dy = \frac{2x}{1+x^2}\,dx \quad / \int$$

$$\ln(1+e^y) = \ln(1+x^2) + C_1 \to \ln C_1, \; C_1 > 0$$

$$\ln(1+e^y) = \ln C_1(1+x^2)$$

$$1+e^y = C_1(1+x^2)$$

$$e^y = C_1x^2 + C_1 - 1$$

$$y = \ln(C_1x^2 + C_1 - 1), \quad C_1 > 0$$

ZAD Nađi rješenje dif. jednadžbe

$$xy' = \cos^2 y \cdot \ln x$$

koje zadovoljava uvjet: $y(1) = \dfrac{\pi}{4}$

$$x \cdot \frac{dy}{dx} = \cos^2 y \cdot \ln x, \quad x > 0, \quad \text{za } \cos^2 y \neq 0$$

$$\frac{dy}{\cos^2 y} = \frac{\ln x}{x}\,dx \quad / \int$$

$\cos^2 y = 0$

$y = \frac{\pi}{2} + k\pi, \; k \in \mathbb{Z}$

To su rješenja dif. jedn., ali ne i početnog uvjeta

$$\left.\operatorname{tg} y = \frac{1}{2}\ln^2 x + C_1\right\} \text{opće rješenje jednadžbe}$$

$$1 = C_1$$

rješenje početnog problema $\quad \operatorname{tg} y = \frac{1}{2}\ln^2 x + 1$

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Diferencijalne jednadžbe oblika $y' = f(ax + by + c)$

- rješavamo ju supstitucijom $z = ax + by + c$ → svođenje na jednadžbu sa separiranim varijablama (JSV)

$$\frac{dz}{dx} = a + by' = a + b \cdot f(z)$$

$$\frac{dz}{a + f(z)} = dx \quad \leftarrow \text{JSV}$$

ZAD Nađite opće rješenje dif. jednadžbe

$$y' = \sin^2(x - y + 1)$$

$$z = x - y + 1$$

$$\frac{dz}{dx} = 1 - y' = 1 - \sin^2 z = \cos^2 z$$

$$\frac{dz}{dx} = \cos^2 z$$

$$\frac{dz}{\cos^2 z} = dx \quad / \int$$

$$\operatorname{tg} z = x + C$$

$$\operatorname{tg}(x - y + 1) = x + C \quad \leftarrow \text{opće rješenje}$$

→ $\cos^2 z = 0$

$z = \frac{\pi}{2} + k\pi, \; k \in \mathbb{Z}$

$x - y + 1 = \frac{\pi}{2} + k\pi$

$y = x + 1 - \frac{\pi}{2} - k\pi, \; k \in \mathbb{Z}$

jest rješenje dif. jednadžbe

(ne spada u opće)

⇓

ne može ga se dobiti za niti koji C

Homogene jednadžbe

DEF Za diferencijalnu jednadžbu kažemo da je homogena ako se može svesti na oblik $y'(x) = f\left(\frac{y}{x}\right)$

SUPSTITUCIJA $z = z(x) = \frac{y}{x}$

$$y = z \cdot x$$

$$y' = z + xz'$$

$$z + xz' = f(z)$$

$$x\frac{dz}{dx} = f(z) - z \quad \text{(JSV)}$$

$$\frac{dz}{f(z) - z} = \frac{dx}{x}$$

ZAD Riješite dif. jednadžbu $xy' = y \ln \frac{y}{x}$

$$y' = \frac{y}{x} \ln \frac{y}{x} \qquad \frac{y}{x} > 0, x \neq 0$$

$$\underbrace{\frac{y}{x} \ln \frac{y}{x}}_{f(\frac{y}{x})}$$

supst. $z = \frac{y}{x} \Rightarrow y = zx \Rightarrow y' = z + xz'$

$$z'x + z = z \ln z$$

$$z'x = z \ln z - z \qquad \text{Uz } z\ln z - z \neq 0, \ z \neq e, \ x \neq 0$$

$$\frac{dz}{z\ln z - z} = \frac{dx}{x} \quad / \int$$

$$\ln|\ln z - 1| = \ln|x| + C \qquad C \to \ln C_1, \ C_1 > 0$$

$$\ln|\ln z - 1| = \ln C_1|x|$$

$$|\ln z - 1| = C_1|x|$$

$$|\ln \frac{y}{x} - 1| = C_1|x|$$

$$|\ln y - \ln x - 1| = C_1|x|, \quad C_1 > 0$$

opće rješenje

Za $z \ln z - z = 0$

$z = 0, \ \ln z = 1$

$\frac{y}{x} = 0$ — $y = 0$ nije rješenje

$z = e$

$\frac{y}{x} = e$

$y = ex$

je rješenje, ali ne opće zbog $C_1 > 0$

(DEF) Za funkciju $M(x,y)$ kažemo da je homogena stupnja α ako $\forall t > 0$ vrijedi $M(tx,ty) = t^\alpha M(x,y)$

ZAD a) $M(x,y) = \sqrt{x^2+y^2} - 2x$

$$M(tx,ty) = \sqrt{(tx)^2+(ty)^2} - 2tx = t(\sqrt{x^2+y^2} - 2x) = t \cdot M(x,y)$$

b) $M(x,y) = x^3 \ln \frac{x}{y} + xy^2$

$$M(tx,ty) = t^3x^3 \cdot \ln\frac{tx}{ty} + tx \cdot t^2 \cdot y^2 = t^3\left[x^3 \ln\frac{x}{y} + xy^2\right]$$

c) $M(x,y) = x^2y + xy^2 + 3 \quad \to$ nehomogeno

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Diferencijalna jednadžba $M(x,y)\,dx + N(x,y)\,dy = 0$ je homogena

ako M i N homogene istog stupnja

$$y' = -\frac{M(x,y)}{N(x,y)} = -\frac{M\left(x, x\cdot\frac{y}{x}\right)}{N\left(x, x\cdot\frac{y}{x}\right)}$$

$$= -\frac{x^{\alpha}}{x^{\alpha}} \cdot \frac{M\left(1, \frac{y}{x}\right)}{N\left(1, \frac{y}{x}\right)} = f\left(\frac{y}{x}\right)$$

ZAD Riješite diferencijalnu jednadžbu

$$y' = \frac{xy - y^2}{x^2}, \quad \text{uz uvjet} \quad y(1) = 1.$$

$$y' = \frac{y}{x} - \left(\frac{y}{x}\right)^2$$

$M(x,y) = xy - y^2$

$M(tx, ty) = t^2xy - t^2y^2 = t^2 M(x,y)$

$N(x,y) = x^2$

$N(tx, ty) = t^2 N(x,y)$

$$z = \frac{y}{x} \Rightarrow y = zx \Rightarrow y' = z + xz'$$

$$\cancel{z} + xz' = \cancel{z} - z^2$$

$$xz' = -z^2 \qquad z \neq 0,\ x \neq 0$$

$$\frac{dz}{-z^2} = \frac{dx}{x}$$

$$\frac{1}{z} = \ln|x| + C, \quad \ln C_1,\ C_1 > 0$$

$$z = \frac{1}{\ln C_1|x|}$$

$$\frac{y}{x} = \frac{1}{\ln C_1|x|}$$

$$y = \frac{x}{\ln(C_1|x|)} \rightarrow \text{opće rješenje}$$

$y(1) = 1$

$$1 = \frac{1}{\ln C_1}$$

$$\ln C_1 = 1 \Rightarrow C_1 = e$$

$$y = \frac{x}{1 + \ln|x|}$$

$z = 0 \Rightarrow y = 0$ jest rješenje, ali ne Cauchyeve zadaće i ne opće

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DZ. Opće rješenje dif jednadžbe $xy'\cos\left(\frac{y}{x}\right) = y\cos\left(\frac{y}{x}\right) - x$

opće rj: $\sin\left(\frac{y}{x}\right) + \ln|x| + C = 0$

Jednadžbe koje se svode na homogene

Jednadžbe oblika $y' = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right)$ svode se na homogene

1° sustav $\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$ ima rješenje (x_0, y_0).

Uvodi se sljedeća supstitucija $u = x - x_0$, $v = y - y_0$ te se jednadžba svodi na

$$a_1u + b_1v = a_1x - a_1x_0 + b_1y - b_1y_0 = a_1x + b_1y + c_1, \quad c_1 = -a_1x_0 - b_1y_0$$

$$y' = \frac{dy}{dx} = \frac{dv}{du} = f\left(\frac{a_1u + b_1v}{a_2u + b_2v}\right)$$

2° sustav nema rješenje. Uvodi se supstitucija $z = a_1x + b_1y \Rightarrow$ JSV

ZAD Naći opće rješenje od $y' = \frac{2x - y + 1}{2x - y}$

$$\left.\begin{aligned} 2x - y + 1 &= 0 \\ 2x - y &= 0 \end{aligned}\right) \text{ sustav nema rješenja}$$

$z = 2x - y$

$z' = 2 - y' \Rightarrow y' = 2 - z'$

$$2 - z' = \frac{z+1}{z} = 1 + \frac{1}{z} \quad / \cdot (-1)$$

$$z' = 1 - \frac{1}{z}$$

$$z' = \frac{-z+1}{z}$$

$$\frac{dz}{dx} = \frac{1-z}{z} \quad \text{JSV}$$

DZ

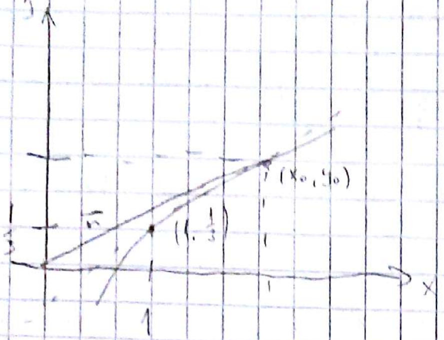
opće rj: $2x - y - \ln|2x - y - 1| = x + c$

posebno rješenje: $y = 2x - 1$

$$\frac{z}{1-z}\,dz = dx$$

9. AUDITORNE

(3.) Odredite jednadžbu krivulje koja prolazi $T(1, \frac{1}{3})$, a kod koje je koeficijent smjera tangente u svakoj točki 3 puta veći od koeficijenta smjera pravca određenog tom točkom



$$y - y_0 = y'(x_0)(x - x_0)$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow y = \frac{y_0}{x_0} x$$

$$y'(x_0) = 3 \cdot \frac{y_0}{x_0}$$

$$y' = 3 \frac{y}{x}$$

$$\begin{cases} \frac{dy}{dx} = 3 \frac{y}{x} \\ y(1) = \frac{1}{3} \end{cases}$$

$$\frac{dy}{y} = 3 \frac{dx}{x} \quad / \int$$

$$\ln|y| = 3 \ln|x| + \underbrace{C}_{\ln C,\ C>0} = 3 \ln C|x| = \ln C|x|^3$$

$$|y| = C|x|^3$$

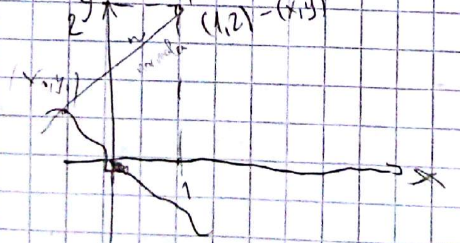
$$y(1) = C \cdot 1^3 = C = \frac{1}{3}$$

$$C = \frac{1}{3}$$

$$\Downarrow$$

$$y(x) = \frac{1}{3} x^3$$

(1.) Nađite krivulju koja prolazi ishodištem, a ima svojstvo da u svakoj točki krivulje pripadna normala prolazi $T(1,2)$



$$t \ldots y - y_0 = y'(x_0)(x - x_0)$$

$$n \ldots \underbrace{y}_{2} - y_0 = \frac{-1}{y'(x_0)}(\underbrace{x}_{1} - x_0)$$

$$2 - y_0 = \frac{-1}{y'(x_0)}(1 - x_0)$$

$$\begin{cases} 2 - y = \frac{-1}{y'}(1 - x) \\ y(0) = 0 \end{cases}$$

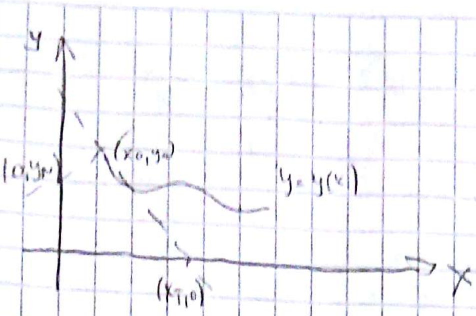
$$(2 - y)dy = -(1 - x)dx$$

$$-\frac{(2-y)^2}{2} = \frac{(1-x)^2}{2} + C$$

$$-1 = \frac{1}{2} + C$$

$$C = -\frac{1}{2}$$

2



$$y - y_0 = y'(x_0)(x - x_0)$$

$$0 - y_0 = y'(x_0)(x_T - x)$$

$$-\frac{y_0}{y'(x_0)} + x_0 = x_T$$

normala.. $y - y_0 = \frac{-1}{y'(x_0)}(x - x_0) \Rightarrow y_N - y_0 = \frac{-1}{y'(x_0)}(0 - x_0) = \frac{x}{y'(x_0)}$

$$y_N = \frac{x_0}{y'(x_0)} + y_0 = -\frac{y_0}{y'(x_0)} + x_0$$

$$\frac{x}{y'} + y = \frac{-y}{y'} + x$$

$$\frac{x+y}{y'} = x - y$$

$$y' = \frac{x+y}{x-y} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}}$$

$$y' = f\left(\frac{y}{x}\right), \quad \frac{y}{x} = z \Rightarrow y = zx$$

$$y' = z'x + z \cdot x' = z'x + z$$

$$z'x + z = \frac{1+z}{1-z}$$

$$z'x = \frac{1+z}{1-z} - z = \frac{1+z - z(1-z)}{(1-z)} = \frac{1+z^2}{1-z}$$

$$\frac{1-z}{1+z^2}\,dz = \frac{dx}{x} \quad /\int$$

$$\int \frac{dz}{1+z^2} - \int \frac{z\,dz}{1+z^2} = \int \frac{dx}{x} + C$$

$$\operatorname{arctg}(z) - \frac{1}{2}\ln(1+z^2) = \ln(Cx)$$

$$\operatorname{arctg}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \ln(Cx)$$

$$\operatorname{arctg}\left(\frac{y}{x}\right) - \frac{1}{2}\ln(x^2+y^2) = C$$

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4.

$$\frac{v_1^2}{2g} + \frac{p_1}{\rho g} + y_1 = \frac{v_2^2}{2g} + \frac{p_2}{\rho g} + y_2$$

$h(0) = h_0 \qquad p_1 = p_2$

$y_1 = h_0$

$y_2 = 0$

$$\frac{v_1^2}{2g} + h_0 = \frac{v_2^2}{2g} \qquad v_1 \ll v_2$$

$$h_0 = \frac{v_2^2}{2g} \Rightarrow \cancel{h_0} \sqrt{2gh}$$ Torricellijev zakon

$$v_2(t) = \sqrt{2gh(t)}$$

$$\Delta V_{van} = \left(\frac{d}{2}\right)^2 \pi \cdot \Delta t \cdot v_2 = \left(\frac{d}{2}\right)^2 \pi \cdot \Delta t \sqrt{2gh(t)}$$

$$\Delta V_{unutra} = \left(\frac{D}{2}\right)^2 \pi \cdot \Delta h$$

$$\Delta V_{van} = -\Delta V_{unutra}$$

$$\left(\frac{d}{\cancel{2}}\right)^2 \cancel{\pi} \cdot \sqrt{2gh(t)} = -\left(\frac{D}{\cancel{2}}\right)^2 \cancel{\pi} \cdot dh$$

$$\Delta h = -\left(\frac{d}{D}\right)^2 \cdot \Delta t \cdot \sqrt{2gh(t)}$$

$$\frac{dh}{dt} = -\sqrt{h} \left(\frac{d}{D}\right)^2 \sqrt{2g}$$

$$\frac{dh}{\sqrt{h}} = -\sqrt{2g}\left(\frac{d}{D}\right)^2 dt \quad / \int$$

$$h(t) = \left[\sqrt{h_0} - \sqrt{\frac{g}{2}} \left(\frac{d}{D}\right)^2 t\right]^2$$

$h(T) = 0$

$$T = \left(\frac{D}{d}\right)^2 \sqrt{\frac{2h_0}{g}}$$

23 cm
1 cm
161 s

ZAD Naći opće rješenje diferencijalne jednadžbe

$$y' = \frac{x+y-3}{x-y-1}$$

$$\left.\begin{aligned} x+y-3 &= 0 \\ x-y-1 &= 0 \end{aligned}\right\} \begin{aligned} x &= 2 \\ y &= 1 \end{aligned}$$ $(2,1)$ je rješenje sustava

Supstitucija: $u = x-2$, $v = y-1$

$$\frac{dv}{du} = \frac{u+v}{u-v}$$ svedemo na homogenu

$$\frac{dv}{du} = \frac{1+\frac{v}{u}}{1-\frac{v}{u}} \rightsquigarrow f\left(\frac{v}{u}\right)$$

supst. $z = \frac{v}{u} \Rightarrow v = zu \Rightarrow v' = z'u + z$

$$z'u + z = \frac{1+z}{1-z}$$

$$z'u = \frac{1+z}{1-z} - z = \frac{1+z^2}{1-z}$$

$$\frac{1-z}{1+z^2}\,dz = \frac{du}{u} \quad \Big/ \int$$

$$\operatorname{arctg}(z) - \frac{1}{2}\ln(1+z^2) = \ln u + C$$

$$\operatorname{arctg}\left(\frac{v}{u}\right) - \frac{1}{2}\ln\left(1+\frac{v^2}{u^2}\right) - \ln u = C$$

$$\operatorname{arctg}\left(\frac{v}{u}\right) - \frac{1}{2}\ln(u^2+v^2) = C$$

$$\operatorname{arctg}\left(\frac{y-1}{x-2}\right) - \frac{1}{2}\ln\left[(x-2)^2+(y-1)^2\right] = C$$

Ortogonalne i izogonalne trajektorije

$\Phi_1(x,y,C) = 0$ ← jednadžba familije krivulja. $\Phi_2(x,y,C) = 0$

Tražimo jednadžbu familije krivulja koje sijeku zadanu familiju pod nekim kutem α (svake dvije krivulje iz pojedinih familija sijeku se pod istim kutem). Krivulje iz te familije nazivamo izogonalnim a za $\alpha = \frac{\pi}{2}$ ortogonalnim trajektorijama polazne familije krivulja.

Postupak određivanja jednadžbe familije ortogonalnih (izogonalnih) trajektorija:

① Tražimo dif. jednadžbu početne familije krivulja

$$\left.\begin{array}{l} \Phi_1(x,y,c)=0 \\ \frac{d}{dx}\left(\Phi_1(x,y,c)\right)=0 \end{array}\right\} \underset{C}{\text{eliminiramo}} \Rightarrow \text{dif. jdb. } F(x,y,y')=0$$

② Veza između krivulja $y_1(x)$ i $y_2(x)$

ortogonalne trajektorije $y_1' = \frac{-1}{y_2'}$ odnosno $y_1' = \Psi(y_2')$

$$tg\alpha = \frac{y_1'-y_2'}{1+y_1'y_2'}$$

③ Uvrstimo vezu u dif. jednadžbu

$$F(x,y,y_1') = F\left(x,y,\Psi(y_2')\right)$$

④ Rješavanjem $F(x,y,\Psi(y_2'))=0$ dobivamo familiju krivulja $\Phi_2(x,y,c)=0$

ZAD Naći ortogonalne trajektorije familije krivulja $x^2+2y^2=a^2$.

① $x^2+2y^2=a^2 \;/\; \frac{d}{dx}$

$2x+4yy'=0$

② veza $y_1' = \frac{-1}{y_2'}$ (ortogonalna trajektorija)

③ Dif. jednadžba ortogonalne familije

$$x - 2y\cdot\frac{1}{y'} = 0 \iff y' = \frac{2y}{x} \quad \frac{dy}{y} = 2\frac{dx}{x}$$

$$\ln|y| = 2\ln|x| + C \to \ln C_1,\ C_1>0$$

$$\ln|y| = \ln C_1 x^2$$

$$|y| = C_1 x^2$$

$$y = C_1 x^2, \quad C\in\mathbb{R}\setminus\{0\} \quad \text{parabole}$$

$$\frac{-2}{Ce^x+e^{-x}} = \frac{-2e^x}{e^x+1}$$

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Linearne diferencijalne jednadžbe prvoga reda

IZVOD!

$$* \quad \boxed{y' + p(x)y = g(x)}$$

pripadna homogena jednadžba $\quad y' + p(x)y = 0$

$$\frac{dy}{y} = -p(x)dx \quad / \int$$

$$\ln y = -\int p(x)\,dx + \ln C, \quad C>0$$

$$y = C e^{-\int p(x)dx}, \quad C \in \mathbb{R}$$

Tražimo rješenje od * u obliku

$$y(x) = C(x)\, e^{-\int p(x)dx}$$

Želimo naći $C(x)$ takav da je rješenje od *

$$y' + p(x)y = C'(x)e^{-\int p(x)dx} + \cancel{C(x)e^{-\int p(x)dx}(-p(x))}$$
$$+ \cancel{p(x)C(x)e^{-\int p(x)dx}} = g(x)$$

$$C'(x) = g(x)\cdot e^{\int p(x)dx}$$

$$C(x) = \int g(x)e^{\int p(x)dx}\,dx + C$$

Opće rješenje * $\quad y(x) = \left[\int\left[g(x)e^{\int p(x)dx}\right]dx + C\right]e^{-\int p(x)dx}$

$$= \underbrace{C\cdot e^{-\int p(x)dx}}_{} + \underbrace{e^{-\int p(x)dx}\cdot\int\left[g(x)e^{\int p(x)dx}\right]dx}_{}$$

rješenje pripadne homogene jednadžbe

y_H

jedno (partikularno) rješenje * (početne jednadžbe)

y_P

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ZAD Riješiti dif. jednadžbu $xy' - 2y = 2x^4 \;/:x \;,\; x \neq 0$

$$y' - \frac{2}{x}y = 2x^3$$

$$y' - 2\frac{y}{x} = 0$$

$$y' = 2\frac{y}{x}$$

$$\frac{dy}{y} = 2\frac{dx}{x}$$

$$y = Cx^2$$

$$y = C(x)\,x^2$$

$$y' = C'(x)x^2 + C(x)\cdot 2x$$

$$y' - 2\cdot\frac{1}{x}\cdot y = C'(x)x^2 + \cancel{C(x)\cdot 2x} - \cancel{2\cdot\frac{1}{x}C(x)\cdot x^2} = 2x^3$$

$$C'(x)\,x^2 = 2x^3$$

$$C'(x) = 2x$$

$$C(x) = x^2 + C$$

$$y = [x^2 + C]\cdot x^2 = Cx^2 + x^4$$

ZAD Riješite Cauchyevu zadaću $y' - tgx\cdot y = \frac{2x}{\cos x} \;,\; y(0) = 1$

$$y' - tgx\, y = 0$$

$$\frac{dy}{dx} = tgx\cdot y$$

$$\frac{dy}{y} = tgx\cdot dx \;/\int$$

$$\ln y = -\ln|\cos x| + C$$

$$y_H = \frac{C}{\cos x}$$

$$y' - tgx\,y = C'(x)\cdot\frac{1}{\cos x} + \cancel{C(x)\frac{\sin x}{\cos^2 x}} - \cancel{tgx\frac{C(x)}{\cos x}} = \frac{2x}{\cos x}$$

$$\Rightarrow C'(x) = 2x$$

$$C(x) = x^2 + C \qquad y = \frac{C(x)}{\cos x} = \frac{x^2 + C}{\cos x} = \underbrace{\boxed{\frac{x^2}{\cos x}}}_{y_P} + \underbrace{\boxed{\frac{C}{\cos x}}}_{y_H}$$

$$y(0) = 1 \qquad 1 = C \quad \Rightarrow \quad y = \frac{x^2+1}{\cos x}$$

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ZAD Riješite jednadžbu

$$y' = \frac{1}{x\cos y + \sin 2y}$$

$x\,\varphi(y) + \psi(y)$

→ nije JSV
→ nije homogena
→ nije linearna

$$\frac{dy}{dx} = \frac{1}{x\cos y + \sin 2y}$$

$$\frac{dx}{dy} = x\cos y + \sin 2y$$

za $x(y)$, $x' = x'(y)$

$$x' - \cos y \cdot x = \sin 2y \rightarrow \text{linearna po } x \;\; x\,p$$

Napomena: Ako imamo dif. jednadžbu prvog reda takvu da je y u funkcijama (argument), a x stoji sam ($x \cdot \varphi(y)$) onda je to linearna diferencijalna jednadžba 1. reda po funkciji $x = x(y)$

$$x' - \cos(y)\,x = \sin(2y)$$ (tražimo $x(y)$ kao rješenje ove jednadžbe)

$$x' - \cos(y) \cdot x = 0$$

$$x' = \cos(y) \cdot x$$

$$\frac{dx}{x} = \cos(y)\,dy \quad / \int$$

$$\ln x = \sin y + \ln C, \quad c > 0$$

$$x = C e^{\sin y}, \quad C \in \mathbb{R}$$

$$x' - \cos(y) \cdot x = C'(y) \cdot e^{\sin y} + \cancel{C(y) \cdot \cos y\, e^{\sin y}} - \cancel{C(y)\cos y\, e^{\sin y}}$$

$$= \sin(2y)$$

$$C'(y) \cdot e^{\sin(y)} = \sin(2y)$$

$$C'(y) = e^{-\sin y} \cdot 2\sin y \cos y \quad / \int$$

$$C(y) = 2\int e^{-\sin y} \sin y \cos y \, dy + C = \left| \begin{matrix} \sin y = t \\ \cos y\, dy = dt \end{matrix} \right|$$

$$= 2\int t e^{-t}\, dt + C =$$

$$C(y) = -2(\sin y + 1)\, e^{-\sin y} + C$$

$$x = \left(-2(\sin y + 1)\, e^{-\sin y} + C\right) e^{\sin y}$$

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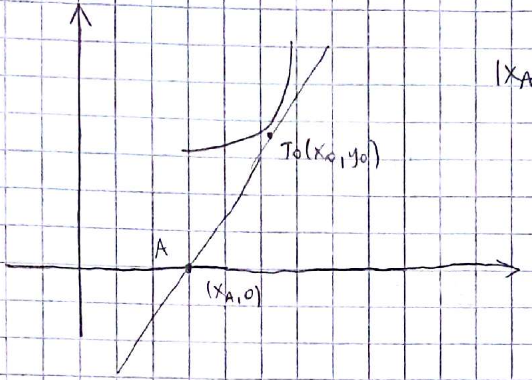
ZAD Odrediti jednadžbu krivulje za koju je odsječak tangente na x-osi jednak kvadratu ordinate dirališta

$$\boxed{\begin{array}{ll} \text{tangenta} & y - y_0 = y'(x_0)(x - x_0) \\ \text{segmetni oblik} & \dfrac{x}{x_0 - \dfrac{y_0}{y'(x_0)}} + \dfrac{y}{y_0 - y'(x_0)\cdot x_0} = 1 \end{array}}$$



$$|x_A| = y_0^2$$

iz segmetnog oblika tangente vidimo:

$$x_A = x_0 - \frac{y_0}{y'(x_0)}$$

$$\left| x_0 - \frac{y_0}{y'(x_0)} \right| = y_0^2$$

$$\Downarrow$$

$$\left| x - \frac{y}{y'} \right| = y^2$$

$$x - \frac{y}{y'} = \pm y^2$$

$$xy' - y = \pm y^2 y'$$

$$y'(x \mp y^2) = y$$

$$y' = \frac{y}{x \mp y^2}, \qquad \frac{1}{y'} = x'$$

$$x - yx' = \pm y^2$$

$$x' - \frac{1}{y}x = \pm y \qquad \text{linearna po } x$$

1) $x' - \frac{1}{y}x = 0$

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\ln|x| = \ln|y| + c$$

$$x = cy, \quad c \in \mathbb{R}$$

2) $x = C(y)\cdot y \qquad x' = C(y)$

$$C'(y)y + \cancel{C(y)} - \frac{1}{y}\cancel{C(y)\cdot y} = \pm y$$

$$C'(y) = \pm 1 \quad /\int$$

$$C(y) = \pm y + c$$

$$x = (c \pm y)\,y \quad \leftarrow \text{opće rješenje}$$

Bernoullijeva diferencijalna jednadžba (svodi se na linearnu)

$$y' + p(x)\cdot y = g(x)\cdot y^n, \quad n \in \mathbb{R} \setminus \{0,1\}$$

$n=0 \to$ LDJ 1. reda
$n=1 \to$ JSV

- Supstitucija $z = y^{1-n}$, $z' = (1-n)\, y^{-n} \cdot y'$

$$\frac{z'}{(1-n)} = \frac{y'}{y^n}$$

$$\frac{y'}{y^n} + p(x)\cdot y^{1-n} = g(x)$$

$$\frac{z'}{(1-n)} + z\cdot p(x) = g(x)$$

ZAD Riješite diferencijalnu jednadžbu $y' + \frac{3}{x}y = x^3 y^2$

$n=2$, $z = y^{1-n} = \frac{1}{y}$, $\frac{z'}{-1} = \frac{y'}{y^2}$

$$-z' + \frac{3}{x}\cdot z = x^3$$

1) $-z' + \frac{3}{x}\cdot z = 0$

$$\frac{z'}{z} = \frac{3}{x}$$

$$\frac{dz}{z} = 3\frac{dx}{x}$$

$$z = C\cdot x^3$$

$z' = x^3 \cdot C'(x) + 3C(x)\cdot x^2$

$-x^3 \cdot C'(x) - \cancel{3C(x)\cdot x^2} + \cancel{3C(x)\cdot x^2} = x^3$

$C'(x) = -1 \; / \int$

$C(x) = -x + C$

$z = (-x + c)\, x^3$

$\frac{1}{y} = (-x + c)\cdot x^3$

ZAD Nađite opće rješenje diferencijalne jednadžbe

$$3xy^2y' + y^3 = x^2$$

$n = -2$ $\quad z' = 3y^2y'$

$z = y^3$

$$y' + \frac{y}{3x} = \frac{x^2}{3xy^2}$$

$$\frac{z'}{3} + \frac{z}{x} = x$$

$$\frac{z'}{3} = \frac{-z}{x}$$

$$\frac{z'}{z} = -3\cdot\frac{1}{x}$$

$$\ln z = \ln|x|^{-3} + c$$

$$z = \frac{c}{x^3}$$

$$z' = \frac{c'(x)}{x^3} - \frac{3c(x)}{x^4}$$

$\frac{c'(x)}{3x^3} - \cancel{\frac{c(x)}{x^4}} + \cancel{\frac{c(x)}{x^4}} = x$

$c'(x) = 3x^4 \; / \int$

$c(x) = \frac{3}{5}x^5 + c$

$y^3 = \frac{3}{5}x^2 + \frac{c}{x^3}$

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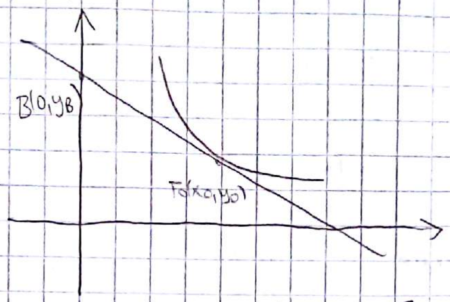
ZAD Nađi krivulje za koje je kvadrat odsječka kojeg na osi ordinata odsijeca tangenta postavljena u bilo kojoj točki krivulje jednak produktu koordinata te točke.



$$y_B = y_0 - y'(x_0)\cdot x_0 =$$

$$(y_0 - y'(x_0)\cdot x_0)^2 = x_0 y_0$$

$$y - xy' = \pm\sqrt{xy}$$

$$xy' = y \mp \sqrt{xy}$$

$$y' = \frac{y}{x} \mp \sqrt{\frac{y}{x}} \rightarrow \text{homogena}$$

DZ

Rj: $2\sqrt{\frac{y}{x}} = C \pm \ln|x|$

ZAD Nađi ortogonalne trajektorije familije kružnica čije je središte na osi y, a prolaze ishodištem.

① $x^2 + (y-r)^2 = r^2$ $\qquad r = \frac{x^2+y^2}{2y}$

$$2x + 2(y-r)y' = 0$$

$$y' = -\frac{x}{y-r} = \frac{-x}{y - \frac{x^2+y^2}{2y}} = \frac{2xy}{x^2-y^2}$$

DZ

② $\alpha = 90^\circ$ $\quad y_1' = \frac{-1}{y_2'}$ $\qquad$ ③ $\frac{-1}{y_2'} = \frac{2xy}{x^2-y^2}$ ← homogena

Rj: $y^2 = c_1x - x^2$

$$\left(x - \frac{c_1}{2}\right)^2 + y^2 = \left(\frac{c_1}{2}\right)^2$$

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11.1. Egzaktne diferencijalne jednadžbe

PR

1. $2x\,dx + 2y\,dy$ → POTPUNI DIFERENCIJAL $u(x,y) = x^2 + y^2$

2. $(3x^2y + 2)dx + x^3 dy$ ← POTPUNI DIFERENCIJAL

$u(x,y) = x^3y + 2x$

3. $x^2y\,dx + y\,dy$ nije potpuni diferencijal

$\frac{\partial u}{\partial x} = x^2y \quad \frac{\partial u}{\partial y} = y$ $\qquad \frac{\partial u}{\partial y \partial x} \neq \frac{\partial u}{\partial x \partial y} \rightarrow$ ne postoji takav potpuni diferencijal

DEF Za diferencijalnu jednadžbu

$$P(x,y)dx + Q(x,y)dy = 0 \qquad (*)$$

kažemo da je jednadžba s potpunim diferencijalom ili egzaktna, ako postoji funkcija dviju varijabla $u(x,y)$ kojoj je potpuni diferencijal lijeva strana jednadžbe $(*)$:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = P\,dx + Q\,dy$$

Onda je $u(x,y) = C$ opće rješenje jednadžbe $(*)$

S1 Uvjet za potpuni diferencijal

ponoviti dokaz

Pretpostavimo da P i Q imaju neprekinute parcijalne derivacije na nekom području $D \subset R^2$. Da bi $P(x,y)dx + Q(x,y)dy$ bio potpuni diferencijal neke funkcije u, nužno je i dovoljno da vrijedi

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}$$

u svakoj točki područja D.

$P, Q : D \subseteq R^2$ klase C^1

$P(x,y)dx + Q(x,y)dy$ je potp. dif.

$\Leftrightarrow \frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}, \ \forall (x,y) \in D$

DOKAZ

$(\Rightarrow)$ Moramo dokazat $\underbrace{du = P(x,y)dx + Q(x,y)dy}$ $\Rightarrow \frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial y}(x,y), \ \forall (x,y) \in$

Znači $\frac{\partial u}{\partial x}(x,y) = P(x,y)$

$\frac{\partial u}{\partial y}(x,y) = Q(x,y)$

$\left.\begin{array}{l} \frac{\partial u}{\partial y \partial x}(x,y) = \frac{\partial P(x,y)}{\partial y} \\ \frac{\partial u}{\partial x \partial y}(x,y) = \frac{\partial Q(x,y)}{\partial x} \end{array}\right\} =$ prema Schwarzovom teoremu

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(<=) Dokazujemo tvrdnju

$$\frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y) \Rightarrow \exists u \text{ t.d. je } \quad du = P(x,y)dx + Q(x,y)dy$$

da bi bilo $\frac{\partial u}{\partial x} = P \quad \frac{\partial u}{\partial y} = Q$

$$\frac{\partial u}{\partial x} = P(x,y) \Big/ \int_{x_0}^{x}$$

$$u(x,y) = \int_{x_0}^{x} P(x,y)\,dx + \varphi(y)$$

$$Q(x,y) = \frac{\partial u}{\partial y} = \int_{x_0}^{x} \frac{\partial P(x,y)}{\partial y} dx + \varphi'(y) = \int_{x_0}^{x} \underbrace{\frac{\partial Q(x,y)}{\partial x}}_{Q(x,y) - Q(x_0,y)} + \varphi'(y)$$

$$Q(x,y) = Q(x,y) - Q(x_0,y) + \varphi'(y)$$

$$\varphi'(y) = Q(x_0,y) \Big/ \int_{y_0}^{y}$$

$$\varphi(y) = \int_{y_0}^{y} Q(x_0,y)\,dy$$

Dakle funkcija u za koju je $du = P(x,y)\,dx + Q(x,y)\,dy$

te $u(x,y) = \int_{x_0}^{x} P(x,y)\,dx + \int_{y_0}^{y} Q(x_0,y)\,dy$

DOKAZ

Ako je $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = P\,dx + Q\,dy$, onda vrijedi

$\frac{\partial u}{\partial x} = P$, $\frac{\partial u}{\partial y} = Q$ i zato prema Schwarzovom teoremu vrijedi

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}$$

odnosno slijedi da je to nužan uvjet.

Neka je sada $P'_y = Q'_x$.
Ako je $\frac{\partial u}{\partial x} = P$, onda u mora biti oblika $u(x,y) = \int_{x_0}^{x} P(x,y)\,dx + \varphi(y)$.

Deriviramo po y i koristimo uvjet teorema:

$$Q(x,y) = \frac{\partial u}{\partial y} = \int_{x_0}^{x} P'_y(x,y)\,dx + \varphi'(y) = \int_{x_0}^{x} Q'_x(x,y)\,dx + \varphi'(y)$$

$$= Q(x,y) - Q(x_0,y) + \varphi'(y)$$

Slijedi $\varphi'(y) = Q(x_0,y)$ pa je $\varphi(y) = \int_{y_0}^{y} Q(x_0,y)\,dy$. Dakle postoji funkcija u koja zadovoljava $du = P\,dx + Q\,dy$ i dana je formulom $u(x,y) = \int_{x_0}^{x} P(x,y)\,dx + \int_{y_0}^{y} Q(x_0,y)\,dx$. Slijedi da je uvjet teorema nužan i dovoljan.

DEF Rekonstrukcija funkcije iz potpunog diferencijala

Ako je $P(x,y)dx + Q(x,y)dy$ potpuni diferencijal funkcije u, onda se ta funkcija može dobiti formulama

$$u(x,y) = \int_{x_0}^{x} P(x,y)\,dx + \int_{y_0}^{y} Q(x_0,y)\,dy$$

ili

$$u(x,y) = \int_{x_0}^{x} P(x,y_0)\,dx + \int_{y_0}^{y} Q(x,y)\,dy$$

uz pretpostavku da su sve funkcije definirane u točkama (x_0,y_0) i duž puta integracije.

ZAD Riješi $\underbrace{(3x+y)}_{P(x,y)}dx + \underbrace{(x-2y)}_{Q(x,y)}dy = 0$

egzaktna? $\frac{\partial P(x,y)}{\partial y} \overset{?}{=} \frac{\partial Q(x,y)}{\partial x}$

$1 = 1 \Rightarrow$ jednadžba je egzaktna

1) nađi $u(x,y)$ t.d. $du = P(x,y)dx + Q(x,y)dy$

$\begin{cases} x_0 = 0 \\ y_0 = 0 \end{cases}$ moraju biti u domeni

$$u(x,y) = \int_0^x P(x,y)\,dx + \int_0^y Q(0,y)\,dy$$

$$= \int_0^x (3x+y)\,dx + \int_0^y (0-2y)\,dy$$

$$= \frac{3x^2}{2} + yx - y^2$$

2) Rješenje diferencijalne jednadžbe $u(x,y) = C$

$$\frac{3x^2}{2} + yx - y^2 = C$$

ZAD Nađi opće rješenje dif jednadžbe:

$$\left(\sin y - \frac{y}{x}\right)dx + (x\cos y - \ln x)\,dy = 0$$

$P(x,y) = \sin y - \frac{y}{x}$ $\qquad \frac{\partial P(x,y)}{\partial y} = \cos y - \frac{1}{x}$

$Q(x,y) = x\cos y - \ln x$ $\qquad \frac{\partial Q(x,y)}{\partial x} = \cos y - \frac{1}{x}$ $\quad$ ✓ Egzaktna

$x_0 = 1$

$y_0 = 0$ $\quad u(x,y) = \int_1^x P(x,y)\,dx + \int_0^y Q(1,y)\,dy = x\sin y - y\ln|x|\Big|_1^x + \sin y\Big|_0^y$

$= x\sin y - y\ln x - \sin y + \sin y = x\sin y - y\ln x$

$x\sin y - y\ln x = C$

ZAD Naći opće rješenje diferencijalne jednadžbe

$$\left(\frac{2x}{y} + \frac{y}{x}\right)dx + \left(\ln x - \frac{x^2}{y^2}\right)dy$$

R: $\frac{x^2}{y} + y\ln x - 1 = C$

DZ

Eulerov multiplikator

- svaku jednadžbu možemo prevesti u egzaktnu množenjem sa pomno odabranim $\mu(x,y)$.

Neka je zadana jednadžba

$$P(x,y)dx + Q(x,y)dy = 0$$

koja nije egzaktna. ($P'_y \neq Q'_x$). Pomnožimo ju s $\mu(x,y)$ tako da bude egzaktna.

$$\mu(x,y)\,P(x,y)\,dx + \mu(x,y)\,Q(x,y)\,dy = 0$$

Iz uvjeta za egzaktnost dobivamo:

$$\frac{\partial(\mu P)}{\partial y} = \frac{\partial(\mu Q)}{\partial x} \Rightarrow \mu'_y P + \mu P'_y = \mu'_x Q + \mu Q'_x$$

tj. $\mu'_y P - \mu'_x Q = \mu(Q'_x - P'_y) \Rightarrow$ parcijalna diferencijalna jednadžba

Gledat ćemo specijalne oblike (posebne slučajeve)

1) Ako $\frac{1}{Q}(P'_y - Q'_x)$ ovisi samo o x, Eulerov multiplikator tražimo u obliku $\mu = \mu(x)$.
Tada je $\mu'_y = 0$ $\mu'_x = \mu'$

$$0 - \mu' \cdot Q = \mu(Q'_x - P'_y) = -\mu(P'_y - Q'_x)$$

$$\frac{d\mu}{dx} = \mu \cdot \frac{1}{Q}(P'_y - Q'_x)$$

$$\frac{d\mu}{\mu} = \frac{1}{Q}(P'_y - Q'_x)\,dx \quad / \int$$

$$\ln|\mu(x)| = \int \frac{1}{Q}(P'_y - Q'_x)\,dx$$

2) Ako $\frac{1}{P}(P'_y - Q'_x)$ ovisi samo o y, Eulerov multiplikator tražimo u obliku $\mu = \mu(y)$.
Tada je $\mu'_x = 0$, $\mu'_y = \mu'$

$$\mu' P = -\mu(P'_y - Q'_x)$$

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$$\frac{d\mu}{dy} p = -\mu\, (P'_y - Q'_x)$$

$$\frac{d\mu}{\mu} = -\frac{1}{p} (P'_y - Q'_x)\, dy \quad /\int$$

$$\ln \mu(y) = -\int \frac{1}{p} (p'_y - Q'x)\, dy$$

ZAD Naći opće rješenje dif. jednadžbe $(x \sin y - y)\,dx + (x^2 \cos y - x \ln x)\,dy = 0$

$P'y = x \cos y - 1$
$Q'x = 2x \cos y - \ln x + 1$ } nije egzaktno, tražimo Eulerov multiplikator

$$\frac{1}{Q} (P'_y - Q'x) = \frac{1}{x(x \cos y - \ln x)} (x \cos y - 1 - 2x \cos y + \ln x - 1)$$

$$= \frac{-1}{x} \Rightarrow \text{funkcija koja ovisi samo o } x$$

$$\ln \mu(x) = \int \frac{1}{Q} (P'_y - Q'x)\,dx = -\int \frac{1}{x}\,dx = -\ln|x|$$

$$\mu(x) = \frac{1}{x}$$

$$\left(\sin y - \frac{y}{x}\right) dx + (x \cos y - \ln x)\, dy = 0$$

ZAD Riješi dif. jednadžbu $y(1+xy)\,dx + x\,dy = 0$ uz uvjet $y(2) = 1$.

$P'y = 1 + 2xy$
$Q'x = -1$ } nije egzaktno, tražimo Eulerov multiplikator μ

$$\frac{-1}{p} (p'y - Q'x) = \frac{-1}{y(1+xy)} (2 + 2xy) = \frac{-2}{y} \rightarrow \text{ovisi samo o } y$$

$$\ln \mu(y) = -2 \int \frac{dy}{y}$$

$$\mu(y) = \frac{1}{y^2}$$

DZ

$$\frac{1+xy}{y}\,dx - \frac{x}{y^2}\,dy = \left[\frac{1}{y} + x\right] dx - \frac{x}{y^2}\,dy$$

$P'y = \frac{-1}{y^2}$
$Q'x = \frac{-1}{y^2}$ egzaktna

Rj. $\frac{x}{y} + \frac{1}{2}x^2 + c = 0$
+ uvjet
$c = 4$

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ZAD Nađi opće rješenje dif. jednadžbe

$$(x^2+y^2+1)dx - 2xy\,dy = 0$$

pomoću Eulerovog multiplikatora oblika $\mu = \mu(y^2-x^2)$

$t = y^2 - x^2$

$\frac{\partial t}{\partial x} = -2x$ $\qquad \mu = \varphi(t)$

$\frac{\partial t}{\partial y} = +2y$

$$(x^2+y^2+1)\mu(t)dx - 2xy\,\mu(t)dy = 0$$

$$\frac{\partial (x^2+y^2+1)\mu(t)}{\partial y} = \frac{\partial(-2xy)\mu(t)}{\partial x}$$

Želimo naći μ da jednadžba bude egzaktna.

$$\frac{\partial}{\partial y}(x^2+y^2+1)\mu(t) = +2y\cdot\mu(t) + (x^2+y^2+1)\cdot\mu'(t)\cdot\frac{\partial t}{\partial y}$$

$$\frac{\partial}{\partial x}(-2xy)\mu(t) = -2y\cdot\mu(t) + (-2xy)\cdot\mu'(t)\cdot\frac{\partial t}{\partial x}$$

$$= \frac{2y\cdot\mu(t) + 2y\cdot(x^2+y^2+1)\cdot\mu' t}{-2y\cdot\mu(t) + 2x\cdot 2xy\cdot\mu' t}$$

$2x^2y + 2y^3 + 1 - 2x^2y$

$$\mu'(t)\left[2y(x^2+y^2+1) - 2x\cdot 2xy\right] = -4y\cdot\mu(t)$$

$$\frac{d\mu}{dt}(t+1) = -2\mu$$

$$\frac{d\mu}{\mu} = \frac{-2\,dt}{t+1}$$

$$\ln\mu(t) = -2\ln|t+1|$$

$$\mu(t) = \frac{1}{(t+1)^2}$$

DZ

$$\mu(y^2-x^2) = \frac{1}{(y^2-x^2+1)^2}$$

opće rješenje

$$\frac{x}{y^2-x^2+1} = C$$

Postojanje i jednoznačnost rješenja diferencijalne jednadžbe

DEF Opće rješenje diferencijalne jednadžbe $y' = f(x,y)$ je familija krivulja $y = \varphi(x,c)$ koja zadovoljava uvjete:

1) Za svaku moguću vrijednost od c familija $y = \varphi(x,c)$ zadovoljava jednadžbu:

$$\varphi_x(x,c) = f(x, \varphi(x,c))$$

2) Za bilo koju točku $(x_0, y_0) \in D$ možemo odrediti konstantu $C = C_0$ tako da bude $\varphi(x_0, C_0) = y_0$.

Za rješenje $y = \varphi(x)$ jednadžbe $y' = f(x,y)$ reći ćemo da je na intervalu I:

1) singularno ako je u svakoj točki integralne krivulje narušen uvjet jednoznačnosti (tj. kroz svaku njezinu točku prolazi i neko drugo rješenje)

neregularno

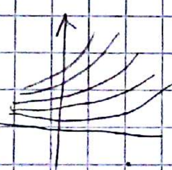
ako postoji točka (x_0,y_0) u kojoj nije ispunjena jedinstvenost $C = C_0$

2) regularno ako je u svakoj njegovoj točki ispunjen uvjet jednoznačnosti (tj. niti kroz jednu njegovu točku ne prolazi neko drugo rješenje)

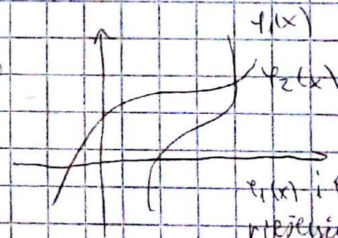
Uvrštavanjem neke konkretne vrijednosti za konstantu $C = C_0$ u opće rješenje dobivamo partikularno rješenje

regularno $\frac{dy}{dx} = y$

$y = e^{x+c}$



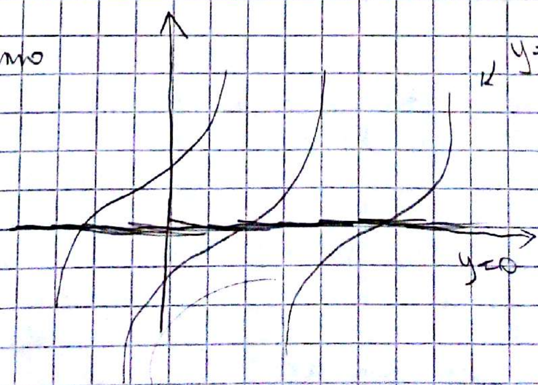
neregularno



$\varphi_1(x)$ i $\varphi_2(x)$ rješenja od $f(x,y)$

singularno

$y = (x+c)^3$



rješenje od

$y' = 3\sqrt[3]{y^2}$

kroz svaku točku pravca možemo naći kubnu parabolu koja ga siječe

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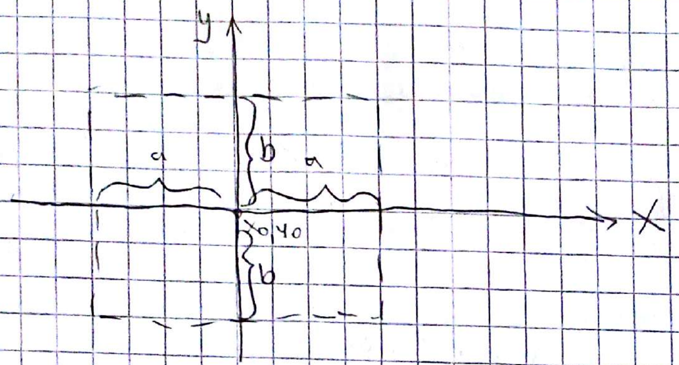
Egzistencija rješenja Cauchyevog problema

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \Rightarrow \text{daje jedno partikularno rješenje.}$$

$f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$, $(x_0, y_0) \in D_f$

$D := \{ (x,y) : |x - x_0| < a, \ |y - y_0| < b \}$

$D \subseteq D_f$



(52) Neka je funkcija f neprekinuta na pravokutniku

$$D = \{ (x,y) : |x - x_0| < a, \ |y - y_0| < b \}$$

oko točke (x_0, y_0). Cauchyeva zadaća

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

ima bar jedno rješenje u nekom okolišu točke x_0.

Jedinstvenost rješenja Cauchyevog problema

(53) Neka je funkcija $f(x,y)$ definirana na pravokutniku D oko točke (x_0, y_0) i ima sljedeća svojstva:

1) f je neprekinuta na D (egzistencija)
2) $\frac{\partial f}{\partial y}$ je omeđena na D (jedinstvenost)

Onda postoji interval $\langle x_0 - h, x_0 + h \rangle$ na kojem postoji jedinstveno rješenje Cauchyeve zadaće.

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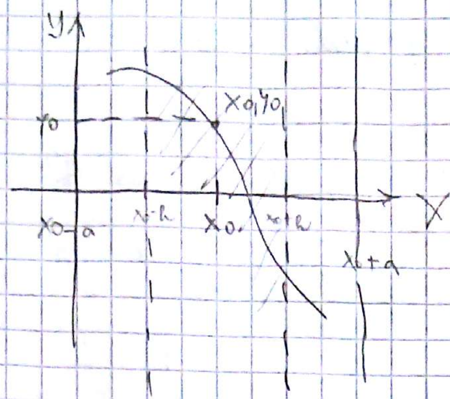


$$m \le \frac{\partial f}{\partial y}(x,y) \le M$$
za $(x,y) \in D$

PR Što možemo zaključiti o jedinstvenosti rješenja dif. jednadžbe:

$$y' = x + y$$

rj. $y = Ce^{x} - x - 1$

$f(x,y) = x + y$ $\Rightarrow$ f neprekinuta $\forall (x,y) \in \mathbb{R}^2$

$\frac{\partial f}{\partial y} = 1$ $\Rightarrow$ uvijek omeđena $\forall (x,y) \in \mathbb{R}^2$

$\forall (x,y) \in \mathbb{R}^2$ ispunjeni uvjeti Picardovog teorema

$\Rightarrow$ Cauchyev problem ima jedinstveno rješenje u okolini svake točke

regularno

PR Što možemo zaključiti o jedinstvenosti rješenja dif. jednadžbe

$$y' = 3\sqrt[3]{y^2}$$

1) $f(x,y) = 3y^{\frac{2}{3}}$ neprekidna $\forall (x,y) \in \mathbb{R}^2$

2) $\frac{\partial f}{\partial y} = 3 \cdot \frac{2}{3} \cdot y^{-\frac{1}{3}} = \frac{2}{\sqrt[3]{y}}$ je omeđena za $y \neq 0$

$\frac{\partial f}{\partial y}(x,0) = \infty$

Uvjeti Picardovog teorema su ispunjeni za (x_0, y_0) gdje je $y_0 \neq 0$.
Za točke na x osi nije zadovoljen uvjet omeđenosti od $\frac{\partial f}{\partial y}$

$\Rightarrow$ ne znamo ništa o jedinstvenosti

Riješimo jednadžbu

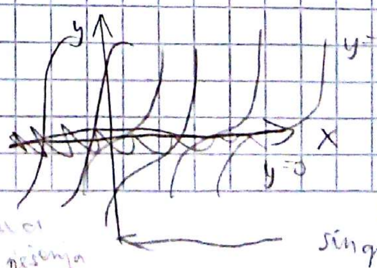
$\frac{y'}{y^{\frac{2}{3}}} = 3$, $y = 0$ je rješenje

$3\sqrt[3]{y} = 3x + C$

$y = (x + C)^3$

ostala rješenja su regularna

ne zadovoljava jedinstvenost rješenja



$y = 0$ je singularno rješenje

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Što možemo zaključiti o jedinstvenosti?

$$y' = -4xy^2$$

1) $f(x,y) = -4xy^2 \Rightarrow f$ neprekinuta na $\mathbb{R}^2$

2) $\frac{\partial f}{\partial y} = -8xy \longrightarrow \frac{\partial f}{\partial y}$ omeđena na $\forall$ pravokutniku D oko svake točke $(x_0,y_0) \in \mathbb{R}^2$

$\Rightarrow$ Cauchyjeva zadaća ima jedinstveno rješenje $\forall (x_0,y_0) \in \mathbb{R}^2$

$y=0$ je regularno jer se ne siječe s $y = \frac{1}{2x^2+C}$

ZAD Odredite neki pravokutnik P na kojem su zadovoljeni uvjeti Picardovog teorema za Cauchyjevu zadaću

$$(x-1)y' = y + \frac{2}{x}$$
$$y(2) = -1$$

$$y' = \underbrace{\frac{y}{x-1} + \frac{2}{(x-1)x}}_{f(x,y)} \qquad \mathcal{D}_f = \{(x,y) \in \mathbb{R}^2 \mid x \neq 0, x \neq 1\}$$

$f(x,y) = \frac{xy+2}{x(x-1)}$ neprekinuta za $x \neq 0, x \neq 1$.

$\frac{\partial f}{\partial y} = \frac{1}{x-1}$ omeđena za $x \neq 1$

Tražimo pravokutnik D oko točke $(x_0,y_0) = (2,-1)$ na kojem su uvjeti Picardovog teorema zadovoljeni.

npr. $|x-2| < \frac{1}{2}$, $|y+1| < 1$ — proizvoljni brojevi $\quad D = \langle \frac{3}{2}, \frac{5}{2} \rangle \times \langle -2, 0 \rangle$

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Singularna rješenja diferencijalne jednadžbe

Lagrangeova i Clairautova diferencijalna jednadžba

DEF Lagrangeova jednadžba

$$y = \varphi(y')x + \psi(y')$$

gdje su φ i ψ zadane diferencijabilne funkcije

Jednadžbu rješavamo sljedećim postupkom uvodeći parametar p:

1) $y' = p$ $\quad (dy = p\,dx)$
2) $y = x\varphi(p) + \psi(p)$ $\longrightarrow$ $dy = \varphi(p)dx + (\varphi'(p)x + \psi'(p))dp$

$$p\,dx = \varphi(p)dx + (\varphi'(p)x + \psi'(p))dp$$

odnosno

$$(\varphi(p) - p)\frac{dx}{dp} + \varphi'(p)\cdot x = -\psi'(p)$$

$\varphi(p) \neq p$

1. slučaj $\quad \varphi(p) - p \neq 0, \ \forall p$

$$\frac{dx}{dp} + \frac{\varphi'(p)}{\varphi(p)-p}x = -\frac{\psi'(p)}{\varphi(p)-p}$$

Rješenje u parametarskom obliku:

$$\begin{cases} x = x(p) \\ y = x(p)\varphi(p) + \psi(p) \end{cases}$$

PR

$y = (1+y')x + (y')^2 \to$ Lagrangeova formula!

$\varphi(y') = 1+y'$ $\qquad y' = p \quad (dy = p\,dx)$
$\psi(y') = (y')^2$

$$y = (1+p)x + p^2$$

$$dy = (1+p)dx + (x+2p)dp$$

$$p\cdot dx = (1+p)dx + (x+2p)\,dp$$

$$-(x+2)dp = dx$$

$$\frac{dx}{dp} = -x - 2p$$

$$\frac{dx}{dp} + x = -2p \quad \leftarrow \text{LVJ1 po } x \text{ ovisno o } p$$

$$x(p) = C\cdot e^{-p} - 2(p-1)$$

$$\begin{cases} x = c\cdot e^{-p} - 2(p-1) \\ y = (1+p)x + p^2 = (1+p)(ce^{-p} - 2(p-1)) + p^2 \end{cases}$$

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2. slučaj $\varphi(p) = p \quad \forall p$

$$y = xy' + \psi(y') \longrightarrow \text{Clairautova jednadžba}$$

$$y = px + \psi(p) \; / \; \frac{d}{dx}$$

$$p = p + p'x + \psi'(p)\, p' \Rightarrow p'(x + \psi'(p)) = 0$$

Ako je $p' = 0 \Rightarrow p = C$ i $y = Cx + \psi(C)$ ← opće rješenje Clairautove jednadžbe (familija pravaca)

3. slučaj Ako je $\varphi(p_i) = p_i$ za konačno mnogo p_i, onda su

$$y_i(x) = \varphi(p_i)x + \psi(p_i)$$

također rješenja Lagrangeove diferencijalne jednadžbe koja mogu biti i singularna rješenja.

Singularno rješenje $\mathcal{C} \ldots \begin{cases} x = -\psi'(p) \\ y = px + \psi(p) = -p\psi'(p) + \psi(p) \end{cases}$

Neka je (x_0, y_0) točka na $\mathcal{C}$. Točki (x_0, y_0) odgovara parametar p_0 odnosno $\begin{cases} x_0 = -\psi'(p_0) \\ y_0 = p_0 x_0 + \psi(p_0) \end{cases}$

Uočimo da pravac $y = p_0 x + \psi(p_0)$ sadrži točku (x_0, y_0) odnosno traženi pravac dobivamo ako baš uzmemo $C = p_0$

Vrijedi: za svaku točku koja se nalazi na grafu krivulje $\mathcal{C}$ možemo naći pravac iz familije pravaca $y = Cx + \psi(C)$ koji ju sadrži, a to upravo znači da je $\mathcal{C}$ singularno rješenje.

Iz veze $y' = p$ odnosno $y'(x_0) = p_0 = C$ slijedi da je pravac $y = p_0 x + \psi(p_0)$ upravo tangenta na $\mathcal{C}$ u (x_0, y_0).

OVOJNICA

DEF Krivulja koja ima svojstvo da u svakoj točki tangira neku od krivulja familije $\Phi(x, y, C) = 0$ zove se ovojnicom te familije krivulja.

Singularno rješenje Clairautove jednadžbe je ovojnica općeg rješenja tj. familije pravaca

2 i D) 2

$y' = p$

$dy = p\,dx$

$y = xp + \frac{1}{p}$

$dy = p\,dx + \left(x - \frac{1}{p^2}\right)dp = p\,dx$

$p'\left(x - \frac{1}{p^2}\right) = 0$

$p' = 0$

$p = C$

$y = Cx + \frac{1}{C}$

opće rješenje

$x - \frac{1}{p^2} = 0$

$x = \frac{1}{p^2}$

$y = xp + \frac{1}{p} = \frac{1}{p} + \frac{1}{p}$

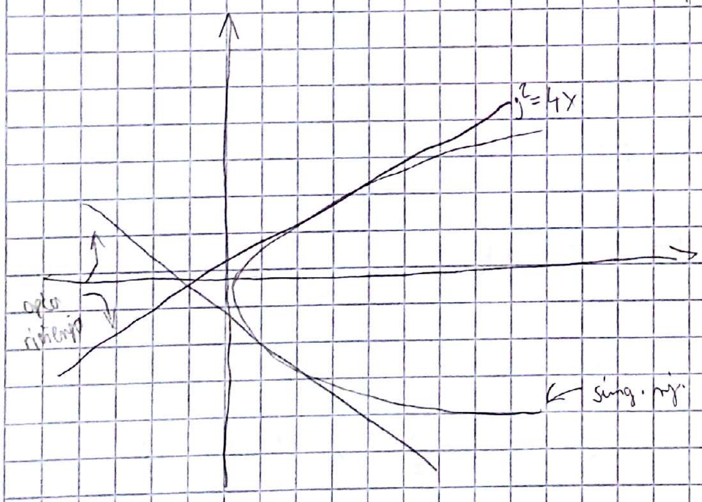
$= \frac{2}{p}$

$\begin{cases} x = \frac{1}{p^2} \\ y = \frac{2}{p} \end{cases}$ → singularno rješenje u parametarskom obliku

$p = \frac{2}{y}$

$x = \frac{y^2}{4}$

$y^2 = 4x$ ← singularno rješenje



opća rješenja

$C = 1$ $y = x + 1$

$C = -1$ $y = -x - 1$

$C = 2$ $y = 2x + \frac{1}{2}$

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10. AUDITORNE

(1.) Odredite λ takav da $\left(\frac{\sin \lambda x}{y} + x\right)dx + \left(y - \frac{\sin^2 x}{y^2}\right)dy = 0$

$u(x,y)$, $du = 0$, $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = P\,dx + Q\,dy$

$u = konst.$

$P = \frac{\sin \lambda x}{y} + x$

$Q = y - \frac{\sin^2 x}{y^2}$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$\frac{\partial P}{\partial y} = \frac{-\sin \lambda x}{y^2} = \frac{\partial Q}{\partial x} = \frac{-2\sin x \cos x}{y^2} = \frac{-\sin 2x}{y^2}$

$\sin \lambda x - \sin 2x = 0$

$\sin \lambda x = \sin 2x$

1° $\lambda x = 2x + 2k\pi$ $\Rightarrow$ $(\lambda - 2)x = 2k\pi$

$\lambda = 2$

2° $\lambda x = \pi - 2x + 2k\pi$

$\Downarrow$

$x(\lambda + 2) = \pi(2k+1)$

$x \neq 0 \Rightarrow \lambda$ ne postoji

$\left(\frac{\sin 2x}{y} + x\right)dx + \left(y - \frac{\sin^2 x}{y^2}\right)dy = 0$ (egzaktna)

$\frac{\partial u}{\partial x} = x + \frac{\sin 2x}{y} \Big/ \int dx$

$u = \int \left(x + \frac{\sin 2x}{y}\right)dx + \varphi(y)$

$u(x,y) = \frac{x^2}{2} - \frac{\cos 2x}{2y} + \varphi(y)$

$\frac{\partial u}{\partial y} = \frac{\cos 2x}{2y^2} + \varphi' = Q = y - \frac{\sin^2 x}{y^2} \Rightarrow \varphi' = y - \frac{\sin^2 x}{y^2} - \frac{\cos 2x}{2y^2}$

$\varphi'(y) = y - \frac{1}{2y^2} \Big/ \int dy$

$\varphi(y) = \frac{y^2}{2} + \frac{1}{2y} + C$

$u(x,y) = \frac{x^2}{2} - \frac{\cos 2x}{2y} + \frac{y^2}{2} + \frac{1}{2y} + C = C$

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{\sin^2 x}{y} = C$

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3. $xy^2(xy' + y) = 1$

$x^2y^2y' + xy^3 = 1 \quad / : x^2y^2$, $y=0$ nije rješenje

$y' + \frac{1}{x}y = \frac{1}{x^2}y^{-2}$ (Bernoullijeva)

$z = y^{1-2} = y^3 \qquad z' = 3y^2y' \Rightarrow y' = \frac{z'}{3y^2}$

$\frac{z'}{3}x^2 + x \cdot z = 1$

$z' + \frac{3}{x} \cdot z = \frac{3}{x^2}$ (homogena)

$z' + \frac{3}{x} \cdot z = 0$

$\frac{z'}{z} = -\frac{3}{x} \quad / \int$

$\ln z = \ln Cx^{-3}$

$z = \frac{C}{x^3} \qquad z' = \frac{1}{x^3} C'(x) - 3C(x) \cdot \frac{1}{x^4}$

$\frac{C'(x)}{x^3} \cancel{- 3C(x) \cdot \frac{1}{x^4}} + \cancel{\frac{3C}{x^4}} = \frac{3}{x^2}$

$C'(x) = 3x \quad / \int dx$

$C(x) = \frac{3x^2}{2} + C$

$z = \frac{\frac{3x^2}{2} + C}{x^3} = \frac{3}{2x} + \frac{C}{x^3}$

$y^3 = \frac{3}{2x} + \frac{C}{x^3}$

4. $e^{y'} = xy' - y \quad / '$

$e^{y'} \cdot y'' = y' + xy'' - y' = xy'' \quad , \quad (y' = p)$

$e^p \cdot p' - xp' = 0 \Rightarrow p'(e^p - x) = 0$

$p' = 0 \Rightarrow y'' = 0 \Rightarrow y = C_1 \cdot x + C_2 \Rightarrow y' = C_1$

$e^{C_1} = x \cdot C_1 - C_1 x - C_2$

$C_2 = -e^{C_1}$

$y = C_1 \cdot x - e^{C_1}$

Singularno: $e^p - x = 0$

$e^p = x$

$p = \ln x = y'$

$y(x) = \int \ln x \, dx + C$

$= x\ln x - x + C$

$e^{\ln x} = \cancel{x\ln x} - \cancel{x\ln x} + x - C$

$e^{\ln x} = x - C$

$C = 0$

$y(x) = x\ln x - x$

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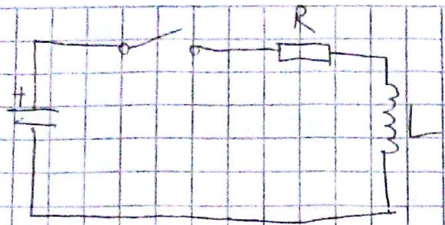
5.



$R = 10\,\Omega$
$L = 100\,mH$

$U = 10\,V$ uključi se u $t = 0$

u $t = 10\,ms$, sklopka se isključi

Za $t < 0$ struja ne teče $i(t<0) = 0$

$$u_L = L\frac{di}{dt} \qquad U_R + U_L = U$$

$$\begin{cases} i \cdot R + L\frac{di}{dt} = U \\ i(t<0) = 0 \end{cases}$$

Lenzovo pravilo $i(0+) = 0 \Rightarrow i(0) = 0$

$$i(0) = 0$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{U}{L}$$

$$i(t) = C \cdot e^{-\frac{R}{L}t}$$

$$i'(t) = C' e^{-\frac{R}{L}t} - \frac{R}{L} C e^{-\frac{R}{L}t}$$

$$C' e^{-\frac{R}{L}t} - \cancel{\frac{R}{L} C e^{-\frac{R}{L}t}} + \cancel{\frac{R}{L} C e^{-\frac{R}{L}t}} = \frac{U}{L}$$

$$C'(t) = \frac{U}{L} e^{\frac{R}{L}t}$$

$$C(t) = \frac{U}{R} e^{\frac{R}{L}t} + C$$

$$i(t) = \frac{U}{R} + \frac{C}{e^{\frac{R}{L}t}}$$

$$i(0) = \frac{U}{R} + C = 0$$

$$i(t) = \frac{U}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$i(t) = \frac{e-1}{e} \cdot e^{-\left(\frac{R}{L}t - 1\right)}$$

2° $\frac{di}{dt} + \frac{R}{L}i = 0$

$$i(10ms) = \frac{e-1}{e}$$

$$i(t) = C \cdot e^{-\frac{R}{L}t}$$

$$i(T) = \frac{C}{e^{\frac{R}{L}T}} = \frac{e-1}{e}$$

$$C = e$$

ZAD. Odredite opće i singularno rješenje diferencijalne jednadžbe

$$y = xy' + y'^2.$$

te nacrtajte nekoliko integralnih krivulja

$y = xy' + y'^2$

$y' = p \quad dy = p\,dx$

$y = xp + p^2$

$dy = p\,dx + (x + 2p)\,dp$

$p\,dx = p\,dx + (x+2p)\,dp$

$p'(x+2p) = 0$

1° $p' = 0$

$p = c$

$\underbrace{y' = c}$

$y = cx + c^2$

familija pravaca

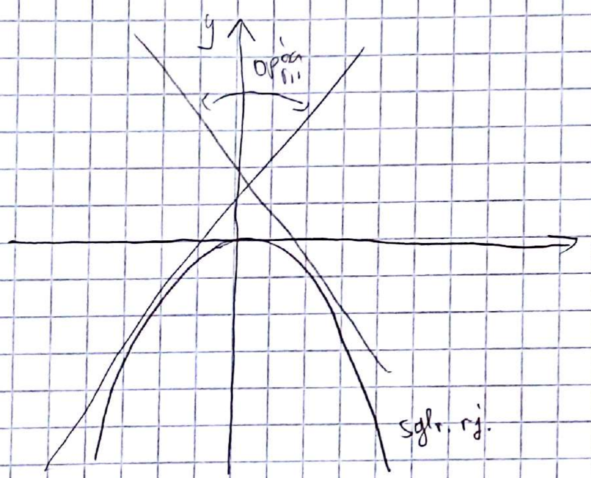
2° $x + 2p = 0$

$x = -2p$

$y = xp + p^2 = -p^2$

$\begin{cases} x = -2p \\ y = -p^2 \end{cases}$ ← singularno rješenje u parametarskom obliku

$y = -\dfrac{x^2}{4}$ ← singularno rješenje



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Diferencijalne derivacije višeg reda

Integriranje snižavanjem reda jednadžbe

- Opći oblik $F(x, y, y', \ldots, y^{(n)}) = 0$
- rješenje : $y = \varphi(x, C_1, \ldots, C_n)$ ili $\Phi(x, y, C_1, \ldots, C_n) = 0$
- metoda integriranja : uzastopno rješavanje dif. jedn. prvog reda

① $y^{(n)} = f(x)$

PR $\quad y'' = 3x^2 \;/\int$

$\frac{d^2y}{dx^2}$

$y' = 3\frac{x^3}{3} + C_1 \;/\int$

$y = \frac{x^4}{4} + C_1x + C_2$

② $F(x, y^{(n)}) = 0$

③ $F(x, y^{(k)}, \ldots, y^{(n)}) = 0$, $1 \leq k \leq n$, supstitucija $y^{(k)} = z\,(z(x))$

PR $\quad (x-3)\,y'' + y' = 0$

$y(4) = 2$

$y'(4) = 1$

$z = y'$

$(x-3)\,z' + z = 0$

$z' + \frac{1}{(x-3)}\,z = 0$ (JSV)

$\frac{dz}{z} = \frac{-dx}{(x-3)}$

$z = \frac{C_1}{x-3}$

$y' = \frac{C_1}{x-3} \;/\int$

$y = C_1 \cdot \ln|x-3| + C_2$

$C_1 = 1$, $C_2 = 2$ $\qquad y = \ln|x-3| + 2$

④ $F(y, y', \ldots, y^{(n)}) = 0$, supstitucija $y' = p$ pri čemu je $p = p(y)$

$$y'' = \frac{dy'}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = p' \cdot p$$

$$y''' = \frac{d(y'')}{dx} = \frac{d(p'p)}{dy} \cdot \frac{dy}{dx} = (p'' \cdot p + p' \cdot p') \cdot p$$

...

$$F(y, y', \ldots, y^{(n)}) \equiv F_1(y, p, p', \ldots, p^{(n-1)}) = 0$$

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PR Naći opće rješenje diferencijalne jednadžbe

$$y' \cdot y'' = \frac{1}{y^2} - \frac{(y')^3}{y} \qquad p^2 \cdot p' = \frac{1}{y^2} - \frac{p^3}{y}$$

$$y' = p = p(y) \qquad p' + \frac{1}{y} \cdot p = \frac{1}{y^2} p^{-2} \quad \text{(Bernoulli'eva)}$$

$$y'' = p' \cdot p \qquad p = p(y)$$

$n = -2$

$z = p^3$

$z' = 3p^2 p'$

$\frac{z'}{3} = p^2 p'$

$$\frac{z'}{3} + \frac{1}{y} z = \frac{1}{y^2}$$

$$z' + \frac{3}{y} z = \frac{3}{y^2}$$

$$z(y) = \frac{3}{2} \cdot \frac{1}{y} + \frac{C}{y^3}$$

$$(y')^3 = \frac{3}{2} \cdot \frac{1}{y} + \frac{C}{y^3}$$

$$y' = \sqrt[3]{\frac{3y^2 + 2C}{2y^3}} \quad \text{(JSV)}$$

$$\frac{dy}{dx} = \sqrt[3]{\ldots}$$

$$\sqrt[3]{\frac{2y^3}{3y^2 + 2C}}\, dy = dx \quad / \int$$

$$\sqrt[3]{2} \int \frac{y^3}{3y^2 + 2C}\, dy = x + C$$

$$2(3y^2 + 2C)^2 = (4x + 4C_1)^3$$

⑤ Jednadžbe homogene u $y, y', \ldots, y^{(n)}$

Ako za $t > 0$ $\quad F(x, ty, ty', \ldots, ty^{(n)}) = t^a F(x, y, y', \ldots, y^{(n)})$

Zamjena: $\quad y = e^{\int z(x)dx}$

$$y' = z e^{\int z(x)dx} = zy$$

$$y'' = z'y + zy' = z'y + z^2 y = (z' + z^2) y$$

PR $\quad y \cdot y'' - 2(y')^2 - y^2 = 0 \quad$ (homogena, provjera s t-ovima!!!)

Zamjena: -||-

$$y(z' + z^2)y - 2z^2 y^2 - y^2 = 0$$

$$y^2 (z' + z^2 - 2z^2 - 1) = 0$$

$y = 0$ je rješenje

$$z' - z^2 - 1 = 0$$

$$z' = z^2 + 1$$

$$\frac{dz}{z^2 + 1} = dx$$

$$z = tg(x + c)$$

$$y = e^{\int tg(x + c) dx}$$

$$y = e^{-\ln|\cos(x + c)| + \ln D}$$

$$= e^{\ln \frac{D}{|\cos(x + c)|}} = \frac{D}{|\cos(x + c)|}$$

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⑥ Jednadžbe s potpunim diferencijalom

$$F(x, y, \dots, y^{(n)}) = \frac{d}{dx} \Phi (x, y, \dots, y^{(n-1)})$$

$$\Phi (x, y, \dots, y^{(n-1)}) = C$$

PR Riješi svodeći na potpuni diferencijal:

$$y' y''' = 2(y'')^2 \quad / : y'' \cdot y'$$

$$\frac{y'''}{y''} = 2 \frac{y''}{y'}$$

$$\frac{d}{dx}(\ln y'') = 2 \frac{d}{dx}(\ln y')$$

$y' = 0$

$y = C$

je rješenje

$$\ln y'' = 2 \ln y' + C$$

$$y'' = C_1 y'^2 \qquad y' = z$$

$$z' = c_1 z^2$$

$$\frac{dz}{z^2} = C_1 dx$$

$$-\frac{1}{z} = c_1 x + C_2$$

$$z = -\frac{1}{c_1 x + c_2}$$

$$z = \frac{1}{c_3 x + c_4} \quad / \int$$

$$c_3 y = \ln|c_3 x + c_4| + c_5 \cdot c_3$$

ZZV 10.

② 1) $y^2(y - xy') = x^3 y'$

$$y^3 - xy^2 \cdot y' = x^3 \cdot y'$$

$$y'(x^3 + xy^2) = y^3$$

$$y' = \frac{y^3}{x^3 + xy^2} = \frac{1}{(\frac{x}{y})^3 + (\frac{x}{y})} = (\frac{y}{x})^3 \cdot \frac{1}{1 + (\frac{y}{x})^2}$$

$$z = \frac{y}{x} \Rightarrow y = z \cdot x, \quad y' = z + z' \cdot x$$

$$z + z'x = z^3 \cdot \frac{1}{1+z^2}$$

$$z' x = \frac{\cancel{z^3} - z - \cancel{z^3}}{1+z^2}$$

$$\frac{(1+z^2)}{-z} dz = \frac{dx}{x}$$

$$\left(\frac{1}{z} + z\right) dz = \frac{-dx}{x} \quad / \int$$

$$\ln z + \frac{z^2}{2} = \ln\left(\frac{C}{x}\right)$$

$$\ln\left(z \cdot e^{\frac{z^2}{2}}\right) = \ln\left(\frac{C}{x}\right)$$

$$y \cdot e^{\frac{y^2}{2x^2}} = C$$

$$y = C \cdot e^{-\frac{1}{2} \cdot \frac{y^2}{x^2}}$$

2) $y' = 2\left(\dfrac{y+2}{x+y-1}\right)^2$

$u = x-3$, $v = y+2$, $v' = y'$

$\left.\begin{array}{l} y+2=0 \\ x+y-1=0 \end{array}\right\}$ $\begin{array}{l} y=-2 \\ x=3 \end{array}$

$$v' = 2\left(\frac{v}{u+v}\right)^2 = 2\left(\frac{\frac{v}{u}}{1+\frac{v}{u}}\right)^2$$

$$z = \frac{v}{u} \Rightarrow v = z\cdot u, \quad v' = z + z'\cdot u$$

$$z'\cdot u = 2\left(\frac{z}{1+z}\right)^2 - z = \frac{2z^2 - z - 2z^2 - z^3}{1+2z+z^2}$$

$$z'\cdot u = \frac{-z(1+z^2)}{(1+z)^2} = \frac{-z((1+z)^2 - 2z)}{(1+z)^2} = -z + \frac{2z^2}{(1+z)^2}$$

$$2\operatorname{arctg}(z) + \ln|z| = \ln\left(\frac{C}{u}\right)$$

$$\ln(z) = \ln\left(\frac{C}{u}\right) - 2\operatorname{arctg}(z)$$

$$\ln\left(\frac{v}{u}\right) = \ln\left(\frac{C}{u\cdot e^{2\operatorname{arctg}(z)}}\right)$$

$$\frac{y+2}{\cancel{x-3}} = \frac{C}{\cancel{(x-3)}\cdot e^{2\operatorname{arctg}\left(\frac{y+2}{x-3}\right)}}$$

3) $y' = \dfrac{x-2y+5}{2x-y+4}$ $\quad \begin{array}{l} x-2y+5=0 \\ 2x-y+4=0 \end{array}$ $\quad y = 2x+4$ $\quad x-4x-8+5=0$, $-3x=3$, $x=-1$, $y=2$

$u = x+1$
$v = y-2$

$$v' = \frac{u-2v}{2u-v} = \frac{1-2\frac{v}{u}}{2-\frac{v}{u}}, \quad z = \frac{v}{u}, \quad v' = z + z'u$$

$$z + z'\cdot u = \frac{1-2z}{2-z}$$

$$z'u = \frac{1-2z-2z-z^2}{2-z} = \frac{1-4z+z^2}{2-z}$$

$$\frac{(2-z)}{(1-4z+z^2)}dz = \frac{-du}{u}$$

$$\frac{-1}{2}\ln|1-4z+z^2| = \ln\frac{C}{u}$$

$$\frac{u}{C}\sqrt{z^2-4z+1} = 1$$

4) $y' = \dfrac{y}{x}\cdot\ln\dfrac{y}{x}$

$$z + z'x = z\cdot\ln z$$

$$\frac{z'}{z(\ln z - 1)} = \frac{-dx}{x}$$

$$\ln(\ln z - 1) = \ln\frac{C}{x}$$

$$\ln z - 1 = \frac{C}{x}$$

$$\ln\frac{y}{x} = \frac{C}{x} + 1$$

$$y = x\cdot e^{\frac{C}{x}+1}$$

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ZAD Riješi svodeći na potpuni diferencijal 12.2 //

$$yy'' - y'^2 = y^2 y' \quad / : y^2 \qquad y=0 \text{ je rješenje}$$

$$\frac{yy'' - y'^2}{y^2} = y'$$

$$\left(\frac{y'}{y}\right)' = y'$$

$$\frac{d}{dx}\left(\frac{y'}{y} - y\right) = 0$$

$$\frac{y'}{y} - y = C$$

$$y' = C_1 y + y^2 \qquad y = -C_1 \text{ je rješenje}$$

$$\frac{dy}{C_1 y + y^2} = dx \quad / \int$$

$$\ln\left|\frac{y}{y+C_1}\right| = C_1 x + C_2$$

Linearne dif. jednadžbe drugog reda

Linearna dif. jednadžba n-tog reda:

$$y^{(n)} + a_{n-1}(x) y^{(n-1)} + \ldots + a_1(x) y' + a_0(x) y = h(x)$$

Linearna dif. jednadžba drugog reda:

$$y'' + p(x) y' + q(x) y = h(x)$$

Homogena linearna dif. jednadžba drugog reda

$$y'' + p(x) y' + q(x) y = 0$$

1) $y_1(x)$ je rješenje, onda je i $Cy_1(x)$ rješenje, $\forall c$

$$(Cy_1(x))'' + p(x)(Cy_1(x))' + q(x) Cy_1(x) = C\left(y_1''(x) + p(x) y_1' + q(x) \cdot y_1\right) = C \cdot 0 = 0$$

2) $y_1(x)$ i $y_2(x)$ su rješenja, onda je i $y_1(x) + y_2(x)$ također rješenje

DOKAZ: uvrstiti, koristiti tvrdnje da y_1 i y_2 rješenja

$\Rightarrow$ $y_1(x)$ i $y_2(x)$ rješenja, onda je i $y(x) = C_1 y_1(x) + C_2 y_2(x)$ rješenje

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DEF Za par funkcija y_1 i y_2 kažemo da čine bazu rješenja homogene linearne diferencijalne jednadžbe drugog reda ako vrijedi:

1) y_1 i y_2 su rješenja dane jednadžbe
2) te su funkcije linearno nezavisna rješenja (nisu proporcionalne)

S Ako su $y_1(x)$ i $y_2(x)$ dva linearno nezavisna rješenja jednadžbe $y'' + p(x)y' + g(x)y = 0$ onda se opće rješenje može zapisati u obliku:

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

gdje su C_1 i C_2 bilo koje konstante

Postupak snižavanja reda

y_1 jedno poznato rješenje $\rightarrow$ supst. $u = \frac{y}{y_1}$ te kasnije $z = u'$

PR Odredi opće rješenje dif. jednadžbe

$$x^2 y'' + xy' + y = 0$$

ako znamo da je jedno partikularno rješenje $y_p = x$

$$u = \frac{y}{x}, \quad y = u \cdot x, \quad y' = u + u' x, \quad y'' = u' + u'' \cdot x + u'$$

$$2x^2 u' + x^3 \cdot u'' - \cancel{x \cdot u} - u' \cdot x^2 + \cancel{u \cdot x} = 0$$

$$x^3 \cdot u'' + x^2 \cdot u' = 0$$

$$z' \cdot x^3 + x^2 \cdot z = 0 \; / : x^2$$

$$z' \cdot x = -z$$

$$\frac{dz}{z} = \frac{-dx}{x}$$

$$z = \frac{C}{x}$$

$$u' = \frac{C}{x}$$

$$u = C \ln|x| + D$$

$$\frac{y}{x} = C \ln|x| + D$$

$$y = C \underbrace{x \ln|x|}_{y_1} + D \underbrace{x}_{y_2}$$

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Linearne diferencijalne jednadžbe drugog reda s konstantnim koeficijentima

Oblika $y'' + a_1 y' + a_0 y = 0$.

karakteristična jednadžba

Uvođenjem $y = e^{rx}$ dobivamo $r^2 \cdot e^{rx} + a_1 \cdot r \cdot e^{rx} + a_0 \cdot e^{rx} = 0$

Ova jednakost je identički ispunjena onda i samo onda ako je r rješenje jednadžbe

$$r^2 + a_1 r + a_0 = 0 \quad \leftarrow \text{karakteristični polinom}$$

① r_1 i r_2 realna različita rješenja

U tom slučaju $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$ pa opće rješenje ima oblik:

$$y = c_1 y_1 + c_2 y_2 = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

PR Odredimo opće rješenje jednadžbe

$$y'' + y' - 6y = 0. \Rightarrow r^2 + r - 6 = 0 \Rightarrow r_1 = -3,\ r_2 = 2$$

$$y = C_1 e^{-3x} + C_2 e^{2x}.$$

PR Odredimo rješenje Cauchyeve zadaće: $y'' - y' - 2y = 0$, $y(1) = 2$, $y'(1) = 0$

$r^2 - r - 2 = 0$

$r_1 = 2$

$r_2 = -1$

$$y = C_1 \cdot e^{2x} + C_2 e^{-x}$$

$$y' = 2C_1 e^{2x} - C_2 e^{-x}$$

$$\Rightarrow \begin{aligned} 2 &= C_1 \cdot e^2 + C_2 \cdot e^{-1} \\ 0 &= 2C_1 \cdot e^2 - C_2 \cdot e^{-1} \end{aligned}$$

$3C_1 \cdot e^2 = 2$

$C_1 = \frac{2}{3e^2}$

$C_2 = \frac{4}{3} e$

$$y = \frac{2}{3} e^{2x-2} + \frac{4}{3} e^{1-x}$$

② $r_1 = r_2$ dvostruko realno rješenje

onda vrijedi $r_1 = \frac{-a_1}{2}$. $y_1 = e^{r_1 x}$ jedno je rješenje dif. jednadžbe, a $y_2 = x e^{r_1 x}$.

$$y = C_1 y_1(x) + C_2 y_2(x) = C_1 e^{r_1 x} + C_2 x e^{r_1 x}$$

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3. r_1 i r_2 su konjugirano kompleksni

$r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta \quad \Rightarrow \quad \tilde{y}_1 = e^{r_1 x}, \quad \tilde{y}_2 = e^{r_2 x}$

$e^{ix} = \cos x + i \sin x \quad \Rightarrow \quad e^{(\alpha + i\beta)x} = e^{\alpha x} \cdot e^{i\beta x} = e^{\alpha x}(\cos\beta x + i \sin\beta x)$

$$\Rightarrow \quad y = K_1 e^{(\alpha + i\beta)x} + K_2 e^{(\alpha - i\beta)x}$$

$$= K_1 e^{\alpha x}(\cos\beta x + i \sin\beta x) + K_2 e^{\alpha x}(\cos\beta x - i \sin\beta x)$$

$$= (K_1 + K_2) e^{\alpha x} \cos\beta x + i(K_1 - K_2) e^{\alpha x} \sin\beta x$$

$$= C_1 \boxed{e^{\alpha x} \cos\beta x} + C_2 \boxed{e^{\alpha x} \sin\beta x}$$

$$= C_1 y_1(x) + C_2 y_2(x)$$

PR Odredimo rješenje Cauchyeve zadaće $y'' + 4y' + 5y = 0$

$y(0) = 2$

$y'(0) = 0$

$r^2 + 4r + 5 = 0$

$\left.\begin{matrix} r_1 = -2 + 1i \\ r_2 = -2 - 1i \end{matrix}\right\} \begin{matrix} \alpha = -2 \\ \beta = 1 \end{matrix}$

$$y = C_1 \cdot e^{-2x} \cos x + C_2 e^{-2x} \sin x$$

$$y' = -2C_1 e^{-2x} \cos x - C_1 e^{-2x} \sin x - 2C_2 e^{-2x} \sin x + C_2 e^{-2x} \cos x$$

$$\Rightarrow \quad 2 = C_1 \cdot 1 \cdot 1 + C_2 \cdot 1 \cdot 0 \quad \Rightarrow \quad C_1 = 2$$

$$0 = -2 \cdot 2 \cdot 1 \cdot 1 - 2 \cdot 1 \cdot 0 - 2 \cdot C_2 \cdot 1 \cdot 0 + C_2 \cdot 1 \cdot 1$$

$\Rightarrow C_2 = 4$

$$y = 2e^{-2x} \cos x + 4e^{-2x} \sin x = 2e^{-2x}(\cos x + 2 \sin x)$$

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Linearne diferencijalne jednadžbe višeg reda

Oblik: $A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \dots + A_1(x)y' + A_0(x)y = A(x)$

Korisno je promatrati te jednadžbe kao operatorske jednadžbe između dvaju vektorskih prostora:

$$L(y) = f.$$

Ovdje je L diferencijalni operator:

$$L = A_n(x)\frac{d^n}{dx^n} + A_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \dots + A_1(x)\frac{d}{dx} + A_0(x),$$

f zadana funkcija, a y nepoznata funkcija - rješenje dif jednadžbe

Vektorski prostori u ovom će slučaju biti prostori neprekinuto diferencijabilnih funkcija. Skup svih rješenja dif. jednadžbe tvorit će neki potprostor u tim prostorima

Prostor $C[a,b]$ svih neprekinutih funkcija definiranih na intervalu $[a,b]$ je vektorski prostor uz operacije zbrajanja vektora i množenja vektora skalarom definirane na način:

$$(f+g)(x) := f(x) + g(x)$$
$$(\lambda f)(x) := \lambda f(x)$$

(DEF) Vektorski potprostor

Neka je X bilo koji vektorski prostor. Podskup $W \subset X$ vektorskog prostora X je potprostor ako je W vektorski prostor s obzirom na operacije zbrajanja i množenja naslijeđene iz X.
Da bismo utvrdili da je $W \subseteq X$ vektorski potprostor, dovoljno je provjeriti sljedeće:

1) $f_1, f_2 \in W,\ \alpha_1, \alpha_2 \in \mathbb{R} \Rightarrow \alpha_1 f_1 + \alpha_2 f_2 \in W$

2a) $f, g \in W \Rightarrow f+g \in W$
2b) $f \in W,\ \alpha \in \mathbb{R} \Rightarrow \alpha f \in W$

(DEF) Zbroj neprekinuto diferencijabilnih funkcija ponovo je neprekinuto diferencijabilna funkcija. Zato je prostor $C^{(n)}[a,b]$ svih n-puta neprekinuto diferencijabilnih funkcija definiranih na $[a,b]$ vektorski potprostor od $C[a,b]$.

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Vektori $y_1, \dots, y_n$ su linearno nezavisni ako jednakost

$$\alpha_1 y_1 + \dots + \alpha_n y_n = 0$$

vrijedi samo onda kada je $\alpha_1 = \dots = \alpha_n = 0$

Vektori $y_1, \dots, y_n$ čine bazu vektorskog prostora X ako su linearno nezavisni i ako razapinju prostor X, tj. svaki se vektor $y \in X$ može napisati u obliku linearne kombinacije $y = \alpha_1 y_1 + \dots + \alpha_n y_n$. ($n$-dimenz. prostora X)

PR Neka je $\alpha_1 e^{r_1 x} + \alpha_2 e^{r_2 x} + \alpha_3 e^{r_3 x} = 0$, te neka je $\alpha_3 \neq 0$.

$/ : e^{r_1 x}, \; ' \quad \Rightarrow$

$$\alpha_1 + \alpha_2 e^{(r_2 - r_1)x} + \alpha_3 e^{(r_3 - r_1)x} = 0$$

$$\alpha_2 (r_2 - r_1) e^{(r_2 - r_1)x} + \alpha_3 (r_3 - r_1) e^{(r_3 - r_1)x} = 0$$

$/ : e^{(r_2 - r_1)x}, \; ' \quad \Rightarrow$

$$\alpha_2 (r_2 - r_1) + \alpha_3 (r_3 - r_1) e^{(r_3 - r_2)x} = 0$$

$$\alpha_3 (r_3 - r_1)(r_3 - r_2) e^{(r_3 - r_2)x} = 0$$

slijedi da je $\alpha_3 = 0$ što je u proturječju sa pretpostavkom pa su

$$y_1 = e^{r_1 x}, \dots, y_n = e^{r_n x}, \quad (r_i \neq r_j, \text{ za } i \neq j)$$

linearno nezavisne funkcije.

(51) Determinanta Wronskog (Wronskijana) i lin. nezavisnost

Neka su $y_1, \dots, y_n \in C^{(n-1)}[a,b]$. Determinanta Wronskoga definira se s

$$W(y_1, \dots, y_n) := \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

Ako Wronskijana nije identički jednaka nuli, tada su funkcije $y_1, \dots, y_n$ linearno nezavisne.

(52)

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DOKAZ S1

Pretpostavimo da su $y_1, \dots, y_n$ linearno zavisne. Tada postoje skalari $\alpha_1, \dots, \alpha_n$ od kojih je barem jedan različit od nule takvi da je

$$\alpha_1 y_1(x) + \dots + \alpha_n y_n(x) = 0$$

Deriviramo:

$$\alpha_1 y_1'(x) + \dots + \alpha_n y_n'(x) = 0$$

$$\vdots$$

$$\alpha_1 y_1^{(n-1)}(x) + \dots + \alpha_n y_n^{(n-1)}(x) = 0.$$

Ovaj homogeni sustav po nepoznanicama $\alpha_1, \dots, \alpha_n$ ima netrivijalno rješenje za svaki $x \in [a,b]$ onda i samo onda, ako je njegova determinanta jednaka nuli. No njegova determinanta je upravo wronskijana, što je u proturječju s pretpostavkom.

11. AUDITORNE

1. $y'' + (y')^3 y = 0$

$y' = p, \quad y'' = p \cdot p' \quad \Rightarrow \quad p \cdot p' + p^3 \cdot y = 0$

$y = C$

$$\frac{p'}{p^2} = -y$$

$$\frac{dp}{p^2} = -dy \cdot y$$

$$+\frac{1}{p} = +\frac{y^2}{2} + C$$

$$p = \frac{2}{y^2 + C}$$

$C, D \in \mathbb{R}$

$$(y^2 + C)dy = 2dx$$

$$\frac{y^3}{3} + C_1 y = 2x + C_2$$

$\frac{y^3}{6} - Cy = x + D,$

2. $xy'' + 2y' + x = 1$

$y(1) = 2$

$y'(1) = 1$

$y' = p(x), \quad y'' = p',$

$$xz' + 2z = 1 - x \quad / : x$$

$$z' + \frac{2}{x} z = \frac{1}{x} - 1$$

$\frac{1}{2} + \frac{1}{3} =$

$$z' + \frac{2}{x} z = 0$$

$$\frac{dz}{z} = -\frac{2dx}{x} \quad / \int$$

$$\ln z = \ln \frac{C}{x^2}$$

$$z = \frac{C}{x^2}$$

$$z' = \frac{-2C}{x^3} + C' \cdot \frac{1}{x^2}$$

$$\frac{C'}{x^2} - \cancel{\frac{2C}{x^3}} + \cancel{\frac{2C}{x^3}} = \frac{1}{x} - 1$$

$$C' = \frac{1-x}{\cancel{x}} \cdot x^{\cancel{2}} = x - x^2 \; / \int$$

$$C = \frac{x^2}{2} - \frac{x^3}{3} + C$$

$$z = \frac{1}{2} - \frac{x}{3} + \frac{C}{x^2} = y'$$

$$1 = \frac{1}{2} - \frac{1}{3} + C$$

$$C = \frac{5}{6}$$

$$y = \frac{1}{2}x - \frac{x^2}{6} - \frac{5}{6x} + D$$

$$2 = \frac{1}{2} - \frac{1}{6} - \frac{5}{6} + D$$

$$D = \frac{5}{2}$$

$$\Rightarrow y = \frac{1}{2}x - \frac{x^2}{6} - \frac{5}{6x} + \frac{5}{2}$$

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3. Nađite linearnu diferencijalnu jednadžbu trećeg reda i početne uvjete koje zadovoljava rješenje

$$y(t) = e^{t} - 2e^{-2t} + 5te^{-2t}$$

1 jednostruka, −2 dvostruka — multiplicitet $\}$

$$(\lambda-1)(\lambda+2)^2 = 0$$

$$(\lambda-1)(\lambda^2+4\lambda+4) = 0$$

$$\lambda^3 + 4\lambda^2 + 4\lambda - \lambda^2 - 4\lambda - 4 = 0$$

$$\lambda^3 + 3\lambda^2 - 4 = 0$$

$$y''' + 3y'' - 4y = 0$$

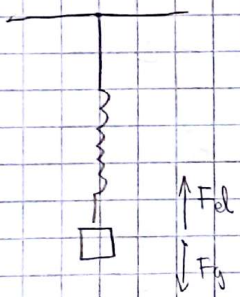
$y(0) = -1$

$y'(0) = 10$

$y''(0) = -27$

4. Izračunajte period titranja utega $m = 0{,}5\,kg$, $k = 19.5\,N/m$

$$T = 2\pi\sqrt{\frac{m}{k}} = 1.006\ s \quad \text{ili} \quad 0{,}8976\ s$$



$$m\ddot{x} = F$$

$$F(x) = \sum_{n=0}^{\infty} a_n (x-l)^n$$

$$= a_0 + a_1(x-l) + a_2(x-l)^2 + \dots$$

$$= a_1(x-l)$$

$$f(x) = -k(x-l)$$

$$m\ddot{x} = mg - k(x-l) = 0$$

$$\Rightarrow k = \frac{mg}{l_0 - l} = 24{,}5\ N/m$$

$$m\ddot{x} + kx = mg + k \cdot l$$

$$\ddot{x} + \frac{k}{m}x = g + \frac{kl}{m}$$

$$x^2 + \frac{k}{m} = 0 \Rightarrow x = \pm i\sqrt{\frac{k}{m}}$$

$$x(t) = A\sin\left(\sqrt{\tfrac{k}{m}}\,t\right) + B\cos\left(\sqrt{\tfrac{k}{m}}\,t\right)$$

$$x_p(t) = C \Rightarrow \frac{k}{m}C = g + \frac{kl}{m}$$

$$C = \frac{m}{k}\left(\frac{mg+kl}{m}\right) = \frac{mg+kl}{k} \text{ ,, } l_0$$

$$x(t) = A\sin(\omega t) + B\cos(\omega t) + l_0$$

ZAD Jesu li funkcije $y_1 = x$, $y_2 = \sin x$ linearno nezavisne?

$$W(x, \sin x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x \not\equiv 0$$

$\exists x_0$ takav da $x\cos x - \sin x \neq 0$

pa su x i $\sin x$ linearno nezavisne

Ako su funkcije $y_1, \dots, y_n$ linearno zavisne ako je Wronskijan identički jednak nuli

Homogene LDJ višeg reda

Oblik: $y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \dots + a_1(x) y'(x) + a_0(x) y(x) = 0$

Nehomogena oblika: $y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \dots + a_1(x) y'(x) + a_0(x) y(x) = f(x)$

Ako je $L = \frac{d^n}{dx^n} + a_{n-1}(x) \frac{d^{n-1}}{dx^{n-1}} + \dots + a_1(x) \frac{d}{dx} + a_0(x)$

tada homogenu jednadžbu možemo zapisati u obliku

$$Ly(x) = 0,$$

a nehomogenu $Ly(x) = f(x)$.

NAPOMENA Ako su funkcije $a_0(x), \dots, a_{n-1}$ neprekinute na nekom intervalu $[a,b]$, tada Cauchyev problem za LDJ n-tog reda ima jedinstveno rješenje (po Picardu)

S3 Prostor rješenja homogene LDJ n-tog reda je n-dimenzionalan vektorski potprostor prostora $C^{(n)}[a,b]$

NAPOMENA Budući da su rješenja homogene LDJ n-tog reda elementi n-dimenzionalnog vektorskog prostora, svako se rješenje može zapisati kao linearna kombinacija vektora iz baze tog prostora

DEF Skup $\{y_1, \dots, y_n\}$ linearno nezavisnih rješenja homogene linearne diferencijalne jednadžbe n-tog reda naziva se BAZA RJEŠENJA ili temeljni sustav rješenja

S4 Rješenja $y_1, \dots, y_n$ homogene LDJ n-tog reda su linearno nezavisna $\iff$ vrijedi:

$$W(y_1, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} \not\equiv 0$$

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DOKAZ $\Leftarrow$ (TM 1)

$W(y_1, \dots, y_n) \neq 0 \Rightarrow$ funkcije $y_1, \dots, y_n$ su linearno nezavisne

(TM 5) $\Rightarrow$ Neka su $y_1, \dots, y_n$ rješenja homogene LDJ n-tog reda.

$y_1, \dots, y_n$ su linearno nezavisna $\Leftrightarrow W(y_1, \dots, y_n) \neq 0$?

$a \Rightarrow b$

$\neg b \Rightarrow \neg a$

odnosno $W(y_1, \dots, y_n) \equiv 0 \Rightarrow$ rješenja $y_1, \dots, y_n$ linearno zavisna

S5 Neka su $y_1, y_2, \dots, y_n$ rješenja homogene LDJ n-tog reda za koje vrijedi $W(y_1, y_2, \dots, y_n)(x_0) = 0$

za neki $x_0 \in \langle a, b \rangle$. Tada su funkcije $y_1, y_2, \dots, y_n$ linearno zavisne

DOKAZ Imamo sustav

$$\alpha_1 y_1(x_0) + \dots + \alpha_n y_n(x_0) = 0$$
$$\alpha_1 y_1'(x_0) + \dots + \alpha_n y_n'(x_0) = 0$$
$$\vdots$$
$$\alpha_1 y_1^{(n-1)}(x_0) + \dots + \alpha_n y_n^{(n-1)}(x_0) = 0$$

Determinanta ovog sustava je $W(y_1, \dots, y_n)(x_0) = 0$ i zato on ima netrivijalno rješenje $\alpha_1^*, \dots, \alpha_n^*$.

Stavimo $y(x) = \alpha_1^* y_1(x) + \dots + \alpha_n^* y_n(x)$

$$y(x_0) = 0$$
$$\vdots$$
$$y^{(n-1)}(x_0) = 0$$

Istu jednadžbu i iste početne uvjete zadovoljava i trivijalno rješenje $y \equiv 0$. Zbog jednoznačnosti rješenja, mora biti $y(x) \equiv 0$, tj.

$$\alpha_1^* y_1 + \alpha_2^* y_2 + \dots + \alpha_n^* y_n \equiv 0$$

i $y_1, \dots, y_n$ su linearno zavisne

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Nalaženje partikularnog rješenja

Metoda varijacije konstanti

$$y(x)=\sum_{i=1}^{n} C_i(x)\,y_i(x) \quad , \quad C_i \text{ zasad neodređene funkcije}$$

Deriviranjem dobivamo:

$$y'=\sum_{i=1}^{n} C_i(x)\,y_i'(x)+\sum_{i=1}^{n} C_i'(x)\,y(x) \quad \text{te stavljamo uvjet} \quad \sum_{i=1}^{n} C_i'(x)\,y_i(x)=0$$

$$\vdots$$

$$y^{(n-1)}=\sum_{i=1}^{n} C_i(x)\,y_i^{(n-1)}(x)+\sum_{i=1}^{n} C_i'(x)\,y^{(n-2)}(x) \Rightarrow \sum_{i=1}^{n} C_i'(x)\,y^{(n-2)}(x)=0$$

$$y^{(n)}=\sum_{i=1}^{n} C_i(x)\,y^{(n)}(x)+\sum_{i=1}^{n} C_i'(x)\,y^{(n-1)}(x) \Rightarrow \underline{\underline{\sum_{i=1}^{n} C_i'(x)\,y_i^{(n-1)}(x)=f(x)}}$$

Ako su uvjeti ispunjeni, tada je:

$$y^{(n)}+a_{n-1}y^{(n-1)}+\dots+a_0y=\sum_{i=1}^{n} C_i y_i^{(n)}+f+a_{n-1}\sum_{i=1}^{n} C_i y_i^{(n-1)}+\dots+a_0\sum_{i=1}^{n} C_i y_i$$

$$=\sum_{i=1}^{n} C_i\left[y_i^{(n)}+a_{n-1}y_i^{(n-1)}+\dots+a_0y_i\right]+f=f$$

Sustav uvjeta ima Wronskijan za determinantu te budući da $y_1,\dots,y_n$ čine bazu rješenja, sustav uvjeta ima rješenje.

ZAD Naći opće rješenje $y''-2y'+y=\dfrac{e^x}{x^2+1}$ LDJ 2. reda

$$r^2-2r+1=0$$
$$(r-1)^2=0 \quad \Rightarrow \quad y_H=C_1\cdot e^x+C_2\cdot x\cdot e^x$$
$$r_{1,2}=1$$

MVK Pretpostavimo da je rješenje oblika $y(x)=C_1(x)e^x+C_2(x)xe^x$

Moramo odrediti $C_1'(x)$ i $C_2'(x)$

Sustav

$$C_1'(x)e^x+C_2'(x)\,xe^x=0 \quad \Big|\cdot(-1)$$
$$C_1'(x)\,e^x+C_2'(x)(xe^x+e^x)=\frac{e^x}{x^2+1} \quad \Big| +$$

$$C_2'(x)e^x=\frac{e^x}{x^2+1}$$

$$C_2'(x)=\frac{1}{x^2+1} \quad \Big/\int$$

$$C_2(x)=\operatorname{arctg}(x)+C_2$$

$$C_1'(x)e^x=-C_2'(x)\,xe^x$$

$$C_1'(x)e^x=\frac{-xe^x}{x^2+1} \quad \Big/\int$$

$$C_1(x)=\frac{-1}{2}\ln(x^2+1)+C_1$$

opće rješenje

$$y(x)=\left(C_1-\tfrac{1}{2}\ln(x^2+1)\right)e^x+(\operatorname{arctg}x+C_2)\,xe^x$$

$$=\underbrace{C_1e^x+C_2xe^x}_{y_H}+\underbrace{\operatorname{arctg}x\cdot xe^x-\tfrac{1}{2}\ln(x^2+1)e^x}_{y_P}$$

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ZAD Nađi opće rješenje $y'' + 9y = \frac{3}{\sin(3x)}$

$y'' + 9y = 0$

$r^2 + 9 = 0$ $\alpha = 0$

$r_{1,2} = \pm 3i$ $\beta = 3$

$y_1(x) = \cos 3x \cdot e^{\alpha x}$

$y_2(x) = \sin 3x$

$$y_H = C_1 \cos 3x + C_2 \sin 3x$$

MVK pret. rješenja oblika

$$y(x) = C_1(x) \cos 3x + C_2(x) \sin 3x$$

$$\begin{cases} C_1'(x) \cos 3x + C_2'(x) \sin 3x = 0 \\ -C_1'(x) \cdot \cancel{3} \sin 3x + C_2'(x) \cdot \cancel{3} \cos 3x = \frac{\cancel{3}}{\sin(3x)} \end{cases}$$

$$\begin{cases} C_1'(x) \cos 3x + C_2'(x) \sin 3x = 0 \\ -C_1'(x) \sin 3x + C_2'(x) \cos 3x = \frac{1}{\sin(3x)} \end{cases}$$

$$C_1'(x) = -C_2'(x) \frac{\sin 3x}{\cos 3x}$$

$$C_2'(x) \cdot \frac{\sin^2 3x}{\cos 3x} + C_2'(x) \frac{\cos^2 3x}{\cos 3x} = \frac{1}{\sin 3x}$$

$$C_2'(x) = \frac{\cos 3x}{\sin 3x} \quad / \int$$

$$C_2(x) = \frac{1}{3} \ln |\sin 3x| + C_2$$

$$C_1'(x) = -1 \quad / \int$$

$$C_1(x) = -x + C_1$$

$$y = (-x + C_1) \cos 3x + \left(\frac{1}{3} \ln |\sin 3x| + C_2\right) \sin 3x$$

$$= \underbrace{C_1 \cos 3x + C_2 \sin 3x}_{y_H} \underbrace{- x \cos 3x + \frac{1}{3} \ln |\sin 3x| \cdot \sin 3x}_{y_p}$$

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Linearna diferencijalna jednadžba n-tog reda s konstantnim koeficijentima

PR Naći opće rješenje $y^{(IV)} - y = 0$

$$P(r) = r^4 - 1 = (r^2-1)(r^2+1) = (r+1)^1(r-1)^1(r^2+1)^1$$

$r_1 = 1$
$r_2 = -1$
$r_{3,4} = \pm 1i$

$y_1(x) = e^x$
$y_2(x) = e^{-x}$
$y_3(x) = \cos x$
$y_4(x) = \sin x$

$$y = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x$$

ZAD $y^{(V)} + 8y''' + 16y' = 0$

$$P(r) = r^5 + 8r^3 + 16r = r(r^2+4)^2$$

$r_1 = 0, \ n_1 = 1$
$r_{2,3} = \pm 2i, \ n_{2,3} = 2$

$y_1(x) = 1 = e^{0 \cdot x}$
$y_2(x) = \cos 2x$
$y_3(x) = x \cos 2x$
$y_4(x) = \sin 2x$
$y_5(x) = x \sin 2x$

$$y = C_1 + C_2 \cos 2x + C_3 x \cos 2x + C_4 \sin 2x + C_5 x \sin 2x$$

ZAD $y''' - 3y'' + 3y' - y = 0$

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0$$

lin. nez. rješenja

$$y_1(x) = e^x, \quad y_2(x) = x \cdot e^x, \quad y_3(x) = x^2 e^x$$

opće rješenje $\quad y = C_1 e^x + C_2 x e^x + C_3 x^2 e^x$

ZAD $y^{(IV)} - 6y'' + 9y = 0$

$r^4 - 6r^2 + 9 = 0$
$(r^2-3)^2 = 0$
$(r-\sqrt{3})^2 (r+\sqrt{3})^2 = 0$
$r_1 = \sqrt{3} \quad n_1 = 2$
$r_2 = -\sqrt{3} \quad n_2 = 2$

$y_1(x) = e^{\sqrt{3}x} \quad y_2(x) = x e^{\sqrt{3}x}$
$y_3(x) = e^{-\sqrt{3}x} \quad y_4(x) = x e^{-\sqrt{3}x}$

$$y = C_1 y_1 + C_2 y_2 + C_3 y_3 + C_4 y_4$$

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ZAD Nađi rješenje Cauchyevog problema

$$y'' - 5y' + 6y = 13\sin(3x)$$

$y(0) = 0$

$y'(0) = 0$

$$y_H = C_1 e^{2x} + C_2 e^{3x}$$

$f(x) = 13\sin(3x)$

$f(x) = e^{\alpha x} \ldots$

$\alpha = 0$

$\beta = 3$ $\quad p(r) \Rightarrow m = 0$

$Q_1(x) = 0$

$Q_2(x) = 13$

$$y_p = x^0 e^{\alpha x}\left[A_1 \cos(3x) + A_2(\sin(3x))\right]$$

$R_1(x) = A_1$

$R_2(x) = A_1$

$\cancel{y_p = A_1 \cos 3x + A_2 \sin 3x}$

$$y_p = \frac{5}{6}\cos 3x - \frac{1}{6}\sin 3x$$

ZAD Napiši homogenu LDJ najmanjeg stupnja kojoj su rješenja $1, \sin 2x, \cos 2x$

$1 \rightarrow e^{0x} \rightarrow r_1 = 0$

$r_{2,3} = \pm 2i$ $\quad \cos 2x$, $\sin 2x$

$P(r) = r(r-2i)(r+2i) = r(r^2+4) = r^3 + 4r$

$$y''' + 4y' = 0$$

Homogena LDJ s konstantnim koeficijentima:

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = 0$$

Nehomogena (opća) LDJ s konstantnim koeficijentima:

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = f$$

Karakteristični polinom ove jednadžbe:

$$P(r) = r^n + a_{n-1} r^{n-1} + \dots + a_0$$ - polinom n-tog stupnja, n nultočaka

$y = e^{rx}$ će biti rješenje homogene jednadžbe onda i samo onda ako je r nultočka pripadnog karakterističnog polinoma dane jdbe

(S6) Ako karakteristični polinom P ima višestruke nultočke i vrijedi $r_1 = r_2 = \dots = r_k$, onda je

$$L(x^j e^{r_1 x}) = 0, \quad \forall j = 0, \dots, k-1$$

(S7) Ako je kompleksna funkcija y rješenje jednadžbe $Ly = 0$ s realnim koeficijentima, tada su $Re(y)$ i $Im(y)$ realna rješenja te jednadžbe.

DOKAZ: Vrijedi

$$Ly = L(Re\,y + Im\,y) = L(Re\,y) + L(Im\,y) = 0$$

Koeficijenti jednadžbe su realni, pa su zato $L(Re\,y)$ i $L(Im\,y)$ realni. Zato mora biti $L(Re\,y) = 0$ i $L(Im\,y) = 0$

Nalaženje partikularnog rješenja

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = f$$

Ako je desna strana oblika $f(x) = e^{\alpha x}\left(Q_1(x)\cos(\beta x) + Q_2(x)\sin(\beta x)\right)$ gdje su $Q_1(x)$ i $Q_2(x)$ polinomi stupnja $\le p$ tada jednadžba ima partikularno rješenje oblika:

$$y_p = x^m e^{\alpha x}\left[R_1(x)\cos(\beta x) + R_2(x)\sin(\beta x)\right]$$

m - višestrukost broja $\alpha + i\beta$ kao nultočke

R_1, R_2 - polinomi stupnja p čije koeficijente odredujemo uvrštavanjem y_p u jednadžbu

Ako je desna strana $f(x) = f_1(x) + f_2(x) + \dots + f_k(x)$ i y_{p_i} partikularno rješenje jednadžbe $Ly = f_i$, onda je $y = y_{p_1} + y_{p_2} + \dots + y_{p_k}$ partikularno rješenje jednadžbe $Ly = f$

13. DOMAĆA ZADAĆA

1. A) $\sin x, \cos x, \sin 2x$

$$\begin{vmatrix} \sin x & \cos x & \sin 2x \\ \cos x & -\sin x & 2\cos 2x \\ -\sin x & -\cos x & -4\sin 2x \end{vmatrix} \sim \begin{vmatrix} 0 & 0 & -3\sin 2x \\ \cos x & -\sin x & 2\cos x \\ -\sin x & -\cos x & -4\sin 2x \end{vmatrix}$$

$$= 0\,(4\sin x\sin 2x + 2\cos^2 x) - 0(\dots) - 3\sin 2x\,(-\cos^2 x - \sin^2 x)$$

$$= 3\sin 2x \neq 0 \Rightarrow \text{nezavisne}$$

B) $\cos x,\ \cos(x-2),\ \cos(x+1)$

$$\begin{vmatrix} \cos x & \cos(x-2) & \cos(x+1) \\ -\sin x & -\sin(x-2) & -\sin(x+1) \\ -\cos x & -\cos(x-2) & -\cos(x+1) \end{vmatrix} \sim \begin{vmatrix} 0 & 0 & 0 \\ -\sin x & -\sin(x-2) & -\sin(x+1) \\ -\cos x & -\cos(x-2) & -\cos(x+1) \end{vmatrix} = 0 \Rightarrow \text{zavisne}$$

2. A)

$$\begin{vmatrix} e^{x+1} & e^{1-x} & e^{2x} \\ e^{x+1} & -e^{1-x} & 2e^{2x} \\ e^{x+1} & e^{1-x} & 4e^{2x} \end{vmatrix} \sim \begin{vmatrix} 2e^{x+1} & 0 & 3e^{2x} \\ e^{x+1} & -e^{1-x} & 2e^{2x} \\ 2e^{x+1} & 0 & 6e^{2x} \end{vmatrix} \sim \begin{vmatrix} 0 & 0 & -3e^{2x} \\ e^{x+1} & -e^{1-x} & 2e^{2x} \\ 2e^{x+1} & 0 & 6e^{2x} \end{vmatrix}$$

$$= -3e^{2x}\,(2e^{x+1}e^{1-x}) = -6e^{2x+2} \neq 0 \quad \text{nezavisne}$$

B)

$$\begin{vmatrix} e^x & e^x\sin x & e^x\cos x \\ e^x & e^x(\sin x+\cos x) & e^x(\cos x-\sin x) \\ e^x & e^x(\sin x+\cos x+\cos x-\sin x) & e^x(\cos x-\sin x-\sin x-\cos x) \end{vmatrix} \sim \begin{vmatrix} e^x & e^x\sin x & e^x\cos x \\ e^x & e^x(\sin x+\cos x) & e^x(\cos x-\sin x) \\ e^x & e^x\cdot 2\cos x & e^x\cdot(-2)\sin x \end{vmatrix}$$

$$\sim \begin{vmatrix} e^x & e^x\sin x & e^x\cos x \\ 0 & e^x\cos x & -e^x\sin x \\ 0 & e^x(2\cos x-\sin x) & e^x(\cos x-2\sin x) \end{vmatrix} = e^x\left(e^{2x}[\cos^2 x-2\sin x\cos x] + e^{2x}[2\sin x\cos x-\sin^2 x]\right)$$

$$= e^{3x}\cos 2x \Rightarrow \text{nezavisne}$$

3. A) $e^x, xe^x, x^2e^x \rightarrow r_1 = 1,\ n = 3 \qquad (r-1)^3 = 0 \qquad r^3 - 3r^2 + 3r - 1 = 0$

$$y''' - 3y'' + 3y' - y = 0$$

B) $1, \sin 2x, \cos 2x \rightarrow r_1 = 0,\ r_2 = -2i,\ r_3 = +2i$

$$r(r-2i)(r+2i) = 0$$
$$r(r^2+4) = 0 \qquad y''' + 4y' = 0$$
$$r^3 + 4r = 0$$

4. A) $e^{3x}, \sin x, \cos x \rightarrow r_1 = 3,\ r_{2,3} = \pm i$

$$(r-3)(r^2+1) = 0 \qquad y''' - 3y'' + y' - 3y = 0$$
$$r^3 + r - 3r^2 - 3 = 0$$

B) $e^{-x}\sin 2x,\ e^{-x}\cos 2x,\ \sin x, \cos x$

$$r_{1,2} = -1 \pm 2i \qquad (r^2+1)(r^2+2r+5) = 0$$
$$r_{3,4} = \pm i \qquad r^4 + 2r^3 + 6r^2 + 2r + 5 = 0$$

$$(r+1-2i)(r+1+2i)$$
$$= r^2 + r + 2ri + r + 1 + 2i - 2ri - 2i + 4$$
$$= r^2 + 2r + 5$$

$$y^{IV} + 2y''' + 6y'' + 2y' + 5y = 0$$

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9. A) $y'' - 4y' + 4y = x^2$

$r^2 - 4r + 4 = 0$

$y_h = C_1 e^{2x} + C_2 x e^{2x}$

$f(x) = e^{\alpha x}\left[Q_1(x)\cos(\beta x) + Q_2(x)\sin(\beta x)\right]$

$P = P(x) = x^p = x^2 \Rightarrow p = 2$

$f(x) \leq P(x)$

$y_p = x^m e^{\alpha x}\left[R_1(x)\cos\beta x + R_2(x)\sin\beta x\right]$

$\Rightarrow m = 0$ jer $r = \alpha \pm \beta i$ nije nultočka od $P(x)$

R_1 i R_2 su polinomi stupnja $p = 2$

$R_1(x) = R_2(x) = Ax^2 + Bx + C$

$y_p = x^0 e^{0x}\left[R_1(x)\cos 0 + R_2(x)\sin 0\right] = R_1(x)$

$y_p = Ax^2 + Bx + C$

$y'_p = 2Ax + B$

$y''_p = 2A$

$2A - 8Ax - 4B + 4Ax^2 + 4Bx + 4C = x^2$

$\Rightarrow A = \frac{1}{4}$

$\frac{1}{2} - 4B + 4C = 0$ $\quad 4C = -\frac{1}{2} + 1 = \frac{1}{2}$

$-8A + 4B = 0$

$B = A = \frac{1}{4}$ $\quad C = \frac{3}{8}$

$y_p = R_1(x) = \frac{1}{4}x^2 + \frac{1}{4}x + \frac{3}{8}$

B) $y'' + y = 4x\cos x$

MKV

$P(x) = 4x\cos x$

$p = 1$

$y_h = C_1\cos x + C_2\sin x$

$f(x) = e^{\alpha x}\left[Q_1(x)\cos\beta(x) + Q_2(x)\sin(\beta x)\right]$

$f(x) \leq P(x)$ $\quad r = \alpha \pm \beta i$ $\quad \alpha = 0 \quad \beta = 1$

$y_p = x^1 \cancel{e^{0x}}\left[R_1(x)\cos x + R_2(x)\sin x\right]$

$R_1(x) = A_1x + B_1$ $\quad R_2(x) = A_2x + B_2$

$y_p = A_1x^2\cos x + B_1x\cos x + A_2x^2\sin x + B_2x\sin x$

$y'_p = A_1\left[2x\cos x - x^2\sin x\right.$

$C'_1\cos x + C'_2\sin x = 0$

$-C'_1\sin x + C'_2\cos x = 4x\cos$

$C'_1 = -C'_2\frac{\sin x}{\cos x}$

$C'_2\frac{\sin^2 x}{\cos x} + C'_2\frac{\cos^2}{\cos x}$

$C'_2 = 4x\cos^2 x$

$C'_1 = -4x\sin x\cos$

$C_1 = -\int 2x\sin 2x = -2\left(\frac{1}{4}\sin 2x - \frac{1}{2}x\cos 2x\right)$

$y_p =$

$C_2 = 4\left(\frac{1}{2}\left(\frac{1}{16}\cos 4x - \frac{x^2}{2}\right) + \frac{1}{2}x\left(x + \frac{1}{4}\sin 4x\right)\right)$

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(10) A) $y'' - y' = e^x \sin x$

$r^2 - r = 0$

$y = C_1 + C_2 e^x$

$f(x) = e^{\alpha x}\left[Q_1 \cos\beta x + Q_2 \sin\beta x\right]$ $e^x \sin x \Rightarrow r = 0$

$y_p = x^m e^{\alpha x}\left[R_1 \cos\beta x + R_2 \sin\beta x\right]$ R_1, R_2 stupnja 0

rješenja polinoma? $\alpha \pm \beta i$ // $m = 0$, $\alpha = 1$, $\beta = 1$

$y_p = e^x(A\cos x + B\sin x)$

$y'_p = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)$

$y''_p = \cancel{e^x(A\cos x + B\sin x)} + e^x(-A\sin x + B\cos x) + e^x(-A\sin x + B\cos x) + \cancel{e^x(-A\cos x - B\sin x)}$

$y''_p = 2e^x\left[(-A\sin x) + B\cos x\right]$

$2\cancel{e^x}\left[-A\sin x + \cancel{B\cos x}\right] - \cancel{e^x}\left[A\cos x - A\sin x + B\sin x + \cancel{B\cos x}\right] = \cancel{e^x}\sin x$

$(-A - B)\sin x - A\cos x = \sin x$

$A = -\frac{1}{2}$

$B = -\frac{1}{2}$

$y_p = e^x\left(-\frac{1}{2}\cos x - \frac{1}{2}\sin x\right)$

B) $y''' - y'' + y' - y = x^2 + x$

$r^3 - r^2 + r - 1 = 0$

$(r^2 + 1)(r - 1) = 0$

$r = 1, \ r = \pm i$

$y_H = C_1 e^x + C_2 \cos x + C_3 \sin x$

$f(x) = e^{\alpha x}\left(Q_1(x)\cos\beta x + Q_2(x)\sin\beta x\right)$

$P(x) = x^2 + x \rightarrow p = 2$

$y_p = x^m e^{\alpha x}\left(R_1(x)\cos\beta x + R_2(x)\sin\beta x\right)$

R_1, R_2 stupnja 2 $R_1 = A_1x^2 + B_1x + C_1$

$R_2 = A_1x^2 + B_1x + C_1$

$m = 0, \ \alpha = 0, \ \beta = 0$

$y_p = R_1(x) = Ax^2 + Bx + C$

$y'_p = 2Ax + B$

$y''_p = 2A$

$y'''_p = 0$

$0 - 2A + 2Ax + B - Ax^2 - Bx - C = x^2 + x$

$-2A + B - C = 0$

$2A - B = 1$

$-A = 1$

$y_p = -x^2 - 3x - 1$

$A = -1$ $C = -1$

$B = -3$

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11. A) $y''' - y' = \sin x$

$y = C_1 + C_2 e^{x} + C_3 e^{-x}$

$y_p = A\cos x + B\sin x$

$y'_p = -A\sin x + B\cos x$

$y''_p = -A\cos x - B\sin x$

$y'''_p = A\sin x - B\cos x$

$A\sin x - B\cos x + A\sin x - B\cos x = \sin x$

$A = \frac{1}{2}$

$B = 0$

$y_p = \frac{1}{2}\cos x$

B) $y'' + 4y = x + e^{-4x}$

$y = C_1 + C_2 e^{-4x}$

$y_p = x(Ax+B) = Ax^2 + Bx$

$y'_p = 2Ax + B$

$y''_p = 2A$

$2A + 8Ax + 4B = x$

$A = \frac{1}{8}$

$4B = \frac{-1}{4}$

$B = \frac{-1}{16}$

$y_{p_1} = \frac{1}{8}x^2 - \frac{1}{16}x$

13. $y'' - y' = \frac{1}{e^x+1}$

$y = C_1 + C_2 e^{x}$

$C_1' + C_2' e^{x} = 0$

$C_2' e^{x} = \frac{1}{e^x+1}$

$C_2' = \frac{1}{e^{2x}+e^x}$ $\quad C_1' = \frac{-1}{e^x+1}$

$C_2 = -x + \ln|e^x+1| - e^{-x}$

$C_1 = -x + \ln|e^x+1|$

$y = C_1 + C_2 e^{x} - x + \ln|e^x+1| - e^{-x} - xe^{x} + e^{x}\ln|e^{x}+1|$

14. $y'' + y = \frac{1}{\sin x}$

$y = C_1\cos x + C_2 \sin x$

$C_1'\cos x + C_2'\sin x = 0$

$-C_1'\sin x + C_2'\cos x = \frac{1}{\sin x}$

$C_1' = -C_2'\frac{\sin x}{\cos x}$

$C_2'\frac{\sin^2 x}{\cos x} + C_2'\frac{\cos^2 x}{\cos x} = \frac{1}{\sin x}$

$C_2' = \frac{\cos x}{\sin x}$

$C_1' = -1$

$C_1 = -x + C_1$

$C_2(x) = \ln|\sin x| + C_2$

$y = C_1\cos x + C_2 \sin x - x\cos x + \sin x \ln|\sin x|$

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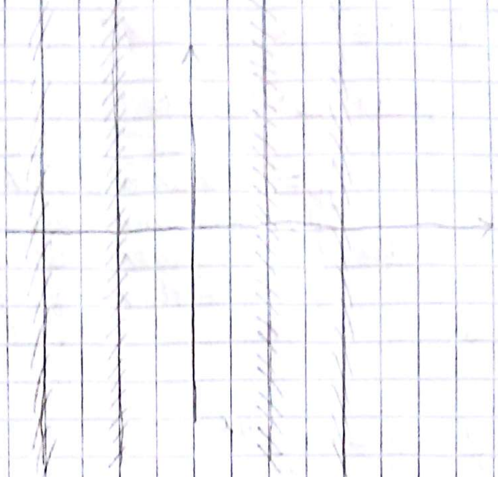
10. DOMAĆA ZADAĆA

① $y' = \frac{-1}{2}x, \quad y(-1) = 3$



$\frac{-1}{2}x = c \to c = \frac{13}{8}$

$x = 2, \; c = -1$

$x = -2, \; c = 1$

$x = 4, \; c = -2$

$x = -4, \; c = 2$

$dy = \frac{-1}{2}x\,dx \,/\int$

$y = \frac{-x^2}{4} + c \Rightarrow c = \frac{13}{4}$

$y = \frac{-x^2}{4} + \frac{13}{4}$

② $y' = -y, \quad y(1) = 2$

$-y = c$

$y = -c$

$\ln|y| = -x + C$

$y = Ce^{-x}$

$C = 2e$

$y = 2e^{1-x}$

$y = 0$ partikularno rješenje

③ $y' = -xy, \quad y(0) = 1$

$\frac{dy}{y} = -x\,dx$

$\ln y = \frac{-x^2}{2} + \ln C \Rightarrow c = 1$

$y = e^{\frac{-x^2}{2}}$

④ $y' = \frac{-x}{y}, \quad y(2) = 2$

$\frac{y^2}{2} = \frac{-x^2}{2} + C$

$x^2 + y^2 = C$

$x^2 + y^2 = 8$

$y = 0$ spada u opće rješenje

⑤ $y' + y^2 = 0, \quad y(5) = \frac{1}{4}$

$y' = -y^2$

$\frac{+1}{y} = +x + C$

$y = \frac{1}{x + C} \Rightarrow C = -1$

⑥ $y' = \frac{x^2}{y(1+x^3)}$

$yy' = \frac{x^2}{(1+x^3)} \,/\int$

$\frac{y^2}{2} = \frac{1}{3}\ln(1+x^3) + C$

⑦ $y' = \frac{x - e^{-x}}{y + e^y}$

$(y + e^y)y' = x - e^{-x} \,/\int$

$\frac{y^2}{2} + e^y = \frac{x^2}{2} + e^{-x} + C$

⑧ $xy' = \sqrt{1-y^2}$

$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{x} \,/\int$

$\arcsin(y) = \ln x + C$

$y = \sin(\ln|Cx|)$

⑨ $y' = \frac{2x}{y + x^2y}, \quad y(0) = 2$

$y \cdot y' = \frac{2x}{1+x^2} \,/\int$

$\frac{y^2}{2} = \ln|1+x^2| + |C|$

$C = 2$

$y^2 = 4 + 2\ln(1+x^2)$

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(10.) $y' = \frac{2x}{1+2y}$, $y(2)=0$

$y^2 + y = x^2 + C$

$y^2 + y = x^2 - 2 \cdot 4$

(11.) $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} = 1 + \frac{y}{x} + \frac{y^2}{x^2}$

$z = \frac{y}{x}$, $y' = z + z'x$

$\cancel{z} + z'x = 1 + \cancel{z} + z^2$

$\frac{dz'}{1+z^2} = \frac{dx}{x} \; /\int$

$\operatorname{arctg}(z) = \ln|x| + C$

$\operatorname{arctg}\left(\frac{y}{x}\right) = \ln|x| + C = \ln|Cx|$

$y = x \cdot tg(\ln|Cx|)$

(12.) $\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy} = \frac{x}{2y} + \frac{3}{2}\frac{y}{x}$

$z + z' \cdot x = \frac{x}{2y} + \frac{3}{2} z$

$z' \cdot x = \frac{1}{2z} + \frac{1}{2} z$

$z'x = \frac{1+z^2}{2z}$

$\frac{2z}{1+z^2} dz = \frac{dx}{x}$

$\ln|1+z^2| = \ln|Cx|$

$1 + z^2 = Cx$

$1 + \frac{y^2}{x^2} = Cx$

$y^2 = Cx^3 - x^2$

(13.) $\frac{dy}{dx} = \frac{4y - 3x}{2x - y} = \frac{4\frac{y}{x} - 3}{2 - \frac{y}{x}}$

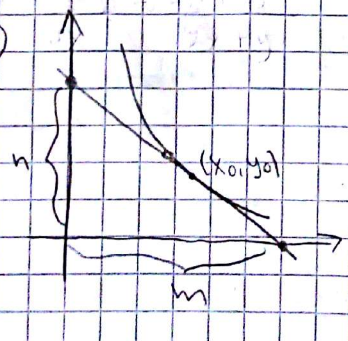
$z + z' \cdot x = \frac{4z - 3}{2 - z}$

$z' x = \frac{4z - 3 - 2z + z^2}{2 - z}$

$\frac{(2-z)}{(z^2 + 2z - 3)} dz = \frac{dx}{x}$

$\frac{1}{4} \ln|z-1| - \frac{5}{4} \ln|z+3| = \ln|Cx|$

(16.)



$x_0 = \frac{m}{2}$

$y_0 = \frac{n}{2}$

$(m, 0)$

$(0, n)$

tangenta

$y - y_0 = y'(x_0)(x - x_0)$

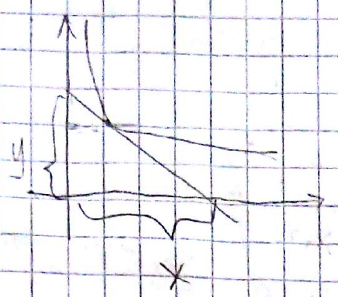
$y'(x_0) = \frac{y - y_0}{x - x_0} = \frac{\frac{n}{2}}{-\frac{m}{2}} = \frac{-n}{m} = \frac{-y}{x}$

$\frac{dy}{y} = \frac{-dx}{x} \; /\int$

$\ln|y| = -\ln|x|$

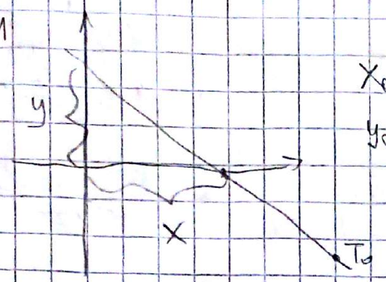
$y = \frac{1}{Cx}$

* (17.) normala $y - y_0 = \frac{-1}{y'(x_0)}(x - x_0)$



$$y'(x_0) = -\frac{(x - x_0)}{(y - y_0)}$$

(18.) $y(4) = 1$



$x_0 = 2x$

$y_0 = -y$

$$2y = y' \cdot (-x)$$

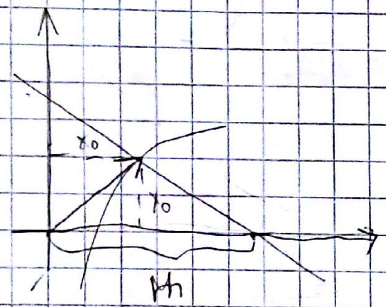
$$\frac{dy}{2y} = \frac{-dx}{x} \quad / \int$$

$$\frac{1}{2}\ln|y| = -\ln|Cx|$$

$$y = \frac{1}{C^2 \cdot x^2}$$

$$y = \frac{16}{x^2}$$

(19.)



normala

$$y - y_0 = \frac{-1}{y'(x_0)}(x - x_0)$$

$$y'(x_0) = \frac{-(x - x_0)}{(y - y_0)} = \frac{-\frac{x}{2}}{-y} = \frac{x}{2y}$$

$$\sqrt{x_0^2 + y_0^2} = \sqrt{(x_0 - x + y_0 y_0')^2 + (y_0)^2}$$

$$x_0^2 + \cancel{y_0^2} = y_0^2 y_0'^2 \cancel{+ y_0^2}$$

$$N_T - y_0 = \frac{+x_0}{y'(x_0)}$$

$$y' = \pm\frac{x}{y}$$

$$x^2 \pm y^2 = C$$

(20.)

$$\sqrt{x_0^2 + y_0^2} = \sqrt{x_0^2 + \left(y_0 - y_0 - \frac{x_0}{y'(x_0)}\right)^2}$$

$$\sqrt{x_0^2 + y_0^2} = \left|y_0 + \frac{x_0}{y'(x_0)}\right|^2$$

$$x_0^2 + y_0^2 = y_0^2 + \frac{2x_0 y_0}{y'(x_0)} + \frac{x_0^2}{y'^2(x_0)}$$

$$y'^2(x_0) \cdot x_0^2 = 2x_0 y_0 y'(x_0) + x_0^2$$

$$y'(x_0) = \frac{2x_0 y_0 \pm \sqrt{4x_0^2 y_0^2 + 4x_0^4}}{2x_0^2}$$

$$y'(x_0) = \frac{y_0 \pm \sqrt{y_0^2 + x_0^2}}{x_0}$$

$$y' = \frac{y}{x} \pm x\sqrt{\left(\frac{y}{x}\right)^2 + 1}$$

$$z + z'x = z \pm x\sqrt{z^2 + 1}$$

$$\frac{z'}{\sqrt{z^2 + 1}}$$

ZZV 10.

③ 1) $y^2(y-xy')=x^3y'$

$y^3-y^2xy'=x^3y'$

$\frac{y^3}{x^3}-\frac{y^2}{x^2}y'=y'$

$y'=\frac{\frac{y^3}{x^3}}{1+\frac{y^2}{x^2}}$

$z+z'x=\frac{z^3}{1+z^2}$

$z'x=\frac{\cancel{z^3}-z-\cancel{z^3}}{1+z^2}=\frac{-z}{1+z^2}$

$\left(\frac{-1}{z}-z\right)dz=\frac{dx}{x} \quad /\int$

$-\ln|z|-\frac{z^2}{2}=\ln|x|+c$

$-\ln\left|\frac{y}{x}\right|-\frac{y^2}{2x^2}=\ln|x|+C$

$y=C\cdot e^{-\frac{y^2}{2x^2}}$

2) $y'=2\left(\frac{y+2}{x+y-1}\right)^2$

$y+2=0$
$y=-2$
$x=3$

$u=x-3$
$v=y+2$

$y'=2\left(\frac{v}{u+v}\right)^2=2\left(\frac{\frac{v}{u}}{1+\frac{v}{u}}\right)^2$

$z+z'u=2\left(\frac{z}{1+z}\right)^2$

$z'u=\frac{2z^2}{(1+2z+z^2)}-z=\frac{\cancel{2z^2}-z-\cancel{2z^2}-z^3}{(1+z)^2}$

$\frac{(1+z)^2}{-z-z^3}dz=\frac{du}{u}$

$-2\operatorname{arctg}(z)-\ln|z|=\ln|Cu|$

4) $xy'=y\ln\frac{y}{x}$

$z+z'x=z\ln z$

$\frac{dz}{z(\ln z-1)}=\frac{dx}{x}$

$\ln|\ln|z|-1|=\ln|Cx|$

$\ln|z|-1=Cx$

$\ln|z|=Cx+1$

$\frac{y}{x}=e^{Cx+1}$

$y=xe^{Cx+1}$

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③ 1) $y' - \frac{y}{x} = x\sin x$

$y' - \frac{y}{x} = 0$

$\frac{y'}{y} = \frac{dx}{x}$

$\ln|y| = \ln|Cx|$

$y = Cx$

$y' = C'(x)\cdot x + C$

$C'(x)\cdot x + \cancel{C} - \cancel{C} = x\sin x$

$C'(x) = \sin x \,/\int$

$C(x) = -\cos x$

$y = -\cos x \cdot x$

2) $y = xy' + y'\ln y$

$\frac{dx}{dy} - \frac{x}{y} = \frac{\ln y}{y}$

$\ln|x| = \ln|Cy|$

$x = Cy$

$C'\cdot y + C = x'$

$C'\cdot y + \cancel{C} - \cancel{C} = \frac{\ln y}{y}$

$C' = \frac{\ln y}{y^2}$

$C = -\frac{\ln y}{y} - \frac{1}{y} + C$

$x = -\ln y - 1 + Cy$

* 3) $dx + (e^y - x)\,dy = 0$

$e^y - x = \frac{-dy}{dx}$

$x = y' + e^y$

$-e^{-y} = -x + C$

$e^{-y} = x + C \,/\ln$

$-y = \ln(x+c)$

$y = \ln\left(\frac{1}{x+c}\right)$

* 4) $(x+1)(yy' - 1) = y^2$

$xyy' - x + yy' = y^2$

$y'(xy + y) - x = y^2$

$y' - \frac{x}{xy+y} = \frac{y^2}{xy+y}$

$y' - \frac{1}{\frac{y}{x}(x+1)} = \frac{y}{\ }$

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(4.) tangenta $y - y_0 = y'(x_0)(x - x_0)$

$$N_T - y_0 = y'(x_0)(-x_0)$$

$$NT = y_0 - x_0 \cdot y'(x_0) = y - xy'$$

$$NT = \sqrt{(x_0)^2 + (NT - y_0)^2}$$

$$y - xy' = \sqrt{x^2 + x^2y'} \;/^2$$

$$y^2 - 2xyy' + \cancel{x^2y'} = x^2 + \cancel{x^2y}$$

$$2xy \cdot y' = y^2 - x^2$$

$$y' = \frac{y}{2x} - \frac{x}{2y} = \frac{y}{2x} - \frac{1}{2\frac{y}{x}}$$

$z = \frac{y}{x}$, $y' = z + z'x$

$$z + z'x = \frac{z}{2} - \frac{1}{2z}$$

$$z'x = \frac{\cancel{z^2} - 1 - \cancel{2z^2}}{2z}$$

$$\frac{2z \cdot dz}{+z^2 - 1} = + \frac{dx}{x}$$

$$-\ln|z^2 - 1| = \ln\left|\frac{1}{Cx}\right|$$

$$z^2 + 1 = Cx$$

$$y^2 + x^2 = Cx^3$$

ZZV 12.

(1.) 1) $y \cdot y'' + y'^2 = y'^3$

$y' = p$

$y'' = p' \cdot y' = p' \cdot p$

$p(y \cdot p' + p - p^2) = 0$

$p = 0$

$y' = 0 \;/\int$

$y = C$

$y \cdot p' + p - p^2 = 0$

$y \cdot p' = p^2 - p$

$$\frac{dp}{p^2 - p} = \frac{dy}{y}$$

$$\ln\left|\frac{p-1}{p}\right| = \ln|Cy|$$

$$1 - \frac{1}{y'} = Cy$$

$$1 - Cy = \frac{1}{y'}$$

$$y'(1 - Cy) = 1 \;/\int$$

$$y - \frac{Cy^2}{2} = x + D$$

$$y = \frac{Cy^2}{2} + x + D$$

(2.) $y'y'' = \frac{1}{y^2} - \frac{y'^3}{y}$, $y' = p$, $y'' = p' \cdot p$

$$p^2 \cdot p' = \frac{1}{y^2} - \frac{p^3}{y}$$

$$p' = \frac{1}{y^2 p^2} - \frac{1}{y} \cdot p$$

$$p' + \frac{1}{y} p = \frac{1}{y^2} p^{-2}$$

$$z = p^{1-n} = p^3$$

$$z' = (1-n) p^{-n} \cdot p'$$

$$\frac{z'}{3} + \frac{1}{y} z = \frac{1}{y^2}$$

$$\frac{z'}{3} = \frac{-z}{y}$$

$$\frac{z'}{3z} = \frac{-1}{y}$$

$$\frac{1}{3} \ln|z| = -\ln|y|$$

$$z = \frac{1}{(Cy)^3}$$

$$z' =$$

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12. DOMAĆA ZADAĆA

(1.) $1 + y^3 y'' = 0$

$y' = p$

$y'' = p' \cdot p$

$1 + y^3 \cdot p' \cdot p = 0$

$p' \cdot p = \frac{-1}{y^3}$

$\frac{p^2}{2} = \frac{+1}{2y^2} + C$

$y'^2 = \frac{+1}{y^2} + C$

$y' = \pm\sqrt{\frac{+1}{y^2} + C} \quad / \int$

$\frac{y'}{\pm\sqrt{\frac{1}{y^2} + C}} \quad / \int \rightarrow \quad \pm\frac{1}{C}\sqrt{1 + Cy^2} = \pm x + K$

(2.) $y^3 y'' = 1$

$y' = p$

$y'' = p' \cdot p$

$p' \cdot p = \frac{1}{y^3}$

$\frac{p^2}{2} = \frac{-1}{2y^2} + C$

$p^2 = \frac{-1}{y^2} + C$

$y' = \pm\sqrt{\frac{-1}{y^2} + C}$

$\frac{y\,dy}{\sqrt{Cy^2 - 1}} = \pm dx \quad / \int$

$\pm x + D = \frac{1}{2C} \cdot 2 t^{\frac{1}{2}} = \frac{1}{C}\sqrt{Cy^2 - 1}$

(3.) $y''(1+y) = y'^2 + y'$

$y' = p$

$y'' = p' \cdot p$

$p' \cdot p(1+y) = p^2 + p$

$p\left[p'(1+y) - p - 1\right] = 0$

$p = 0$

$y' = 0$

$y = C$

$\frac{dp}{p+1} = \frac{dy}{1+y}$

$\ln|p+1| = \ln|C(1+y)|$

$y' + 1 = C(1+y)$

$y' = C + Cy - 1$

$\frac{dy}{Cy + C - 1} = dx \quad / \int$

$\frac{1}{C}\ln\left|y + 1 - \frac{1}{C}\right| = x + D$

$\ln\left|y + 1 - \frac{1}{C}\right| = Cx + D$

$y + 1 - \frac{1}{C} = e^{Cx + D}$

$\ln x$

$\frac{\ln^2|x|}{2}$

(4.) $xy'' + y' = \frac{1}{x}$

$y' = z(x)$

$x \cdot z' + z = \frac{1}{x}$

$z' + \frac{z}{x} = \frac{1}{x^2}$

$z' = \frac{-z}{x}$

$\frac{z'}{z} = \frac{-dx}{x}$

$z = \frac{C}{x}$

$z' = \frac{C'}{x} - C \cdot \frac{1}{x^2}$

$\frac{C'}{x} = \frac{1}{x^2}$

$C' = \frac{1}{x} \quad / \int$

$C = \ln|x| + C$

$z = \frac{\ln|x|}{x} + \frac{C}{x}$

$y' = \frac{\ln|x|}{x} + \frac{C}{x} \quad / \int$

$y = \frac{\ln^2|x|}{2} + C\ln|x| + D$

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5.
$$\frac{y''}{x+y'} = \frac{2}{x} \qquad y'' = 2 + \frac{2y'}{x}$$

$y' = z$

$$z' - \frac{2z}{x} = 2$$

$$\frac{dz}{z} = \frac{2dx}{x}$$

$$z = C_1 x^2$$

$$z' = C_1' x^2 + 2xC_1$$

$$C_1' x^2 + 2xC_1 - \cancel{2C_1 x} = 2$$

$$C_1' = 2x^{-2} \,/\int$$

$$C = \frac{-2}{x} + C_2$$

$$y' = -2x + C_2 x^2 \,/\int$$

$$y = -x^2 + \frac{C_2 x^3}{3} + C_3$$

$$= -x^2 + C_2 x^3 + C_3$$

6.
$$x^2 y y'' + y'^2 = 0$$

$$y = e^{\int z(x)dx}$$

$$y' = z \cdot y$$

$$y'' = z' \cdot y + z \cdot y' = z' \cdot y + z^2 \cdot y$$

$z = 0$

$y' = 0$

$y = C$

$$x^2 \cdot \cancel{y} \cdot (z' \cdot \cancel{y} + z^2 \cdot \cancel{y}) + z^2 \cdot \cancel{y^2} = 0$$

$$z' \cdot x^2 + x^2 \cdot z^2 + z^2 = 0$$

$$\frac{z'}{z^2} = \frac{-(1+x^2)}{x^2} = \frac{-1}{x^2} - 1$$

$$\frac{-1}{z} = \frac{1}{x} - x + C$$

$$z = \frac{-1}{\frac{1}{x} - x + C} = \frac{-x}{1 - x^2 + Cx}$$

$$y = e^{\int z dx} = e^{\int \frac{-x}{1-x^2+Cx}} = e^{\int \frac{x}{x^2-1+Cx}dx}$$

$$\int \frac{x}{x^2+Cx-1}dx = \left| \begin{array}{l} x^2+Cx-1 = t \,/' \\ 2x + C = dt \end{array} \right.$$

7. a)
$$y'' - 2y' = 0$$

$$r^2 - 2r = 0$$

$$r = 0, \; r = 2$$

$$y = C_1 + C_2 e^{2x}$$

c)
$$y'' - 6y' + 9y = 0$$

$$r^2 - 6r + 9 = 0$$

$$(r-3)^2 = 0$$

$$y = C_1 e^{3x} + C_2 x e^{3x}$$

8. a)
$$y'' + y' + y = 0$$

$$r^2 + r + 1 = 0$$

$$r_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm \sqrt{3} \cdot i$$

$$y = C_1 \cdot e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + C_2 \cdot e^{-\frac{1}{2}} \cdot \sin\left(\frac{\sqrt{3}}{2}x\right)$$

b)
$$y'' + 4y = 0$$

$$r^2 + 4 = 0$$

$$r_{1,2} = \pm 2i$$

$$y = C_1 \cdot \cos(2x) + C_2 \cdot \sin(2x)$$

c)
$$y'' - 4y' + 4y = 0$$

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$y = C_1 e^{2x} + C_2 x e^{2x}$$

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(9.) $y'' - 3y' + 2y = 2$

$r^2 - 3r + 2 = 0$

$r_{1,2} = \frac{3 \pm \sqrt{9-8}}{2} = 1, 2$

$y = 1 + c_1 \cdot e^x + c_2 e^{2x}$

$y' = c_1 \cdot e^x + 2c_2 e^{2x}$

$y'' = c_1 \cdot e^x + 4c_2 e^{2x}$

$c_1 e^x + 4c_2 e^{2x} \cancel{- 3c_1 e^x} - 6c_2 e^{2x} + \cancel{2} + \cancel{2c_1 e^x} + 2c_2 e^{2x} = \cancel{2}$

(11.) $y'' + 2y' + y = x$

$y = c_1 e^{-x} + c_2 x e^{-x} + x - 2$

(12.) $y'' - 5y' + 6y = 2e^x$

$(r-2)(r-3) = 0$

$y_h = c_1 e^{2x} + c_2 e^{3x} + e^x$

(13.) $y'' + 4y = 0, \ y(0) = 1, \ y'(0) = 1$

$r_{1,2} = \pm 2i$

$y = c_1 \cos(2x) + c_2 \sin(2x)$

$1 = c_1$

$1 = -c_1 \sin(0) + 2c_2 \cos(0) \Rightarrow c_2 = \frac{1}{2}$

$y = \cos(2x) + \frac{1}{2} \sin(2x)$

(15.) $y'' + 2y' = 0, \ y(0) = 2, \ y'(0) = 1$

$r^2 + 2r = 0$

$r = 0, \ r = -2$

$y = c_1 + c_2 e^{-2x}$

$2 = c_1 + c_2$

$1 = -2c_2$

$c_2 = \frac{-1}{2}$

$c_1 = \frac{5}{2}$

(17.) $x^2 y'' - 4xy' + 6y = 0, \ y_1 = x^\alpha$

$y = x^\alpha$

$y' = \alpha (x)^{\alpha - 1}$

$y'' = \alpha(\alpha - 1) x^{\alpha - 2}$

$\alpha(\alpha-1)x^\alpha - 4\alpha x^\alpha + 6x^\alpha = 0$

$x^\alpha (\alpha^2 - \alpha - 4\alpha + 6) = 0$

$\alpha^2 - 5\alpha + 6 = 0$

$\alpha_1 = 2, \ \alpha_2 = 3$

$y_1 = x^2$

$y_2 = x^3$

$y = c_1 x^2 + c_2 x^3 = x^2 (c_1 + c_2 x)$

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(18.) $y = (2x+1)y'' + 4xy' - 4y = 0, \quad y_1 = x+k$

$y' = 1$ $\quad 4x - 4x - 4k = 0$ $\quad y'' + \frac{4x}{2x+1}y' - \frac{4}{2x+1}y = 0$

$y'' = 0$ $\quad k = 0$

$y_1 = x \quad y_1' = 1 \quad y_1'' = 0$

$u = \frac{y}{y_1} \Rightarrow y = u \cdot y_1, \quad y' = u' \cdot y_1 + u \cdot y_1'$

$y'' = u'' \cdot y_1 + y_1' u' + u' \cdot y_1' + u \cdot y_1''$

$y = u \cdot x$

$y' = u' \cdot x + u$

$y'' = 2u' + u''$

$(2x+1)(2u' + u'') + 4x(u' \cdot x + u) - 4u \cdot x = 0$

$(2x+1)u'' + 4u' \cdot x + 2u' +$

$$y_2 = y_1\left[C_2 \int \left(\frac{1}{y_1^2} e^{-\int p\,dx}\right) dx + C_1\right]$$

$$y_2 = x\left[C_2 \int \left(\frac{1}{x^2} e^{-\int \frac{4x}{2x+1} dx}\right) dx + C_1\right]$$

$$\int \frac{4x}{2x+1} dx = 2x+1 - \ln|2x+1|$$

$$y_2 = x\left[C_2 \int \frac{1}{x^2} e^{\ln \frac{|2x+1|^2}{e^{2x+1}}} + C_1\right]$$

$$y_2 = y_1\left[C_2 \int \left(\frac{1}{y_1^2} e^{-\int p\,dx}\right) dx + C_1\right]$$

$$y_2 = x\left[C_2 \int \frac{2x+1}{x^2} \cdot e^{-(2x+1)} dx + C_1\right]$$

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(19.) $xy'' - (2x+1)y' + (x+1)y = 0$, $y_1 = e^{\alpha x}$

$y'' - \frac{2x+1}{x} y'$

$y_1' = \alpha e^{\alpha x}$

$y_1'' = \alpha^2 e^{\alpha x}$

$x\alpha^2 \cancel{e^{\alpha x}} - (2x+1)\alpha \cancel{e^{\alpha x}} + (x+1)\cancel{e^{\alpha x}} = 0$

$\alpha^2 \cdot x - 2\alpha \cdot x - \alpha + x + 1 = 0$

$x \cdot \alpha^2 - (2x+1)\alpha + (x+1) = 0$

$$\alpha_{1,2} = \frac{2x+1 \pm \sqrt{\cancel{4x^2} + \cancel{4x} + 1 - \cancel{4x^2} - \cancel{4x}}}{2x}$$

$$= 1 + \frac{1}{x} \ , \ 1$$

$y_1 = e^{x+1}$

$y_2 = e^x$ linearno zavisna

$y_1 = e\, y_2$

$y_1 = e^x$

$$y_2 = y_1 \int \left(\frac{1}{y_1^2} \cdot e^{+\int (2 + \frac{1}{x}) dx} \right) dx = e^x \cdot \int \frac{1}{e^{2x}} \cdot x \cdot e^{2x} dx$$

$$= e^x \int x\, dx = e^x \cdot \frac{x^2}{2}$$

$$y = C_1 \cdot e^x + C_2 e^x \cdot \frac{x^2}{2} = C_1 \cdot e^x + C_2 e^x x^2 = e^x (C_1 + C_2 x^2)$$

(20.) $x(x-1)y'' - xy' + y = 0$, $y_1 = x + k$ $\quad y_1 = x$

$y' = 1$

$y'' = 0$

$-x + x + k = 0$

$k = 0$

$$y_2 = y_1 \int \left(\frac{1}{y_1^2} \cdot e^{-\int p dx} \right) dx = x \int \frac{1}{x^2} \cdot (x-1) dx$$

$$= x \ln x + 1 + C = x \ln x + 1$$

$$y = C_1 x + C_2 x \ln x = x(C_1 + C_2 (\ln|x| + \tfrac{1}{x}))$$

ZZV12.

(4.) 1) $yy''' + 3y'y'' = 0$

$$\frac{y'''}{y''} = -3\frac{y'}{y}$$

$$\frac{d(\ln y'')}{dx} = \frac{d}{dx}(\ln y^{-3})$$

$$y'' = \frac{C}{y^3}$$

(3.) $y'' = xy' + y + 1$

$$\frac{d}{dx}(y' - xy - y) = 0 \;/\int$$

$$y' - xy - y = C_1$$

(10.) $xy'' + 2y' + xy = 0$ $\qquad y_1 = \frac{\sin x}{x}$

$$y'' + \frac{2}{x}y' + xy = 0$$

$$y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int p(x)dx} dx$$

$$-\int p(x)dx = -\int \frac{2}{x} = -\ln|x^2| = \ln\left(\frac{1}{x^2}\right)$$

$$\int \frac{x^2}{\sin^2 x} \cdot \frac{1}{x^2} \cdot dx = \int \frac{dx}{\sin^2 x} = -\mathrm{ctg}\, x$$

$$y_2 = \frac{\cos x}{x}$$

$$y = C_1 \frac{\sin x}{x} + C_2 \frac{\cos x}{x}$$

(8.) $y^2 \cdot y'' + 2y \cdot y'^2 - y' = 0$, $\qquad y(0) = 2$, $y'(0) = 1$

$y' = z$ $\qquad y^2 \cdot z \cdot z' + 2y \cdot z^2 - z = 0$ $\qquad y' = 0$

$y'' = z' \cdot z$ $\qquad z \cdot z' + \frac{2}{y}z^2 - \frac{z}{y^2} = 0$ $\qquad y = c$ ✗

$$z' + \frac{2}{y}z - \frac{1}{y^2} = 0$$

$\frac{z'}{z} = \frac{-2}{y}$ $\qquad z' + \frac{2}{y}z = \frac{1}{y^2}$

$z = \frac{C}{y^2}$ $\qquad \frac{C'}{y^2} = \frac{1}{y^2}$

$z' = \frac{C'}{y^2} - 2\frac{C}{y^3}$ $\qquad C' = 1$ $\qquad z = \frac{1}{y} + \frac{C}{y^2}$

$C = y + C$

$$y' = \frac{1}{y} + \frac{C}{y^2}$$

$$\frac{y^2 \cdot y'}{y + C} = 1$$

$$C_1^2\left(\frac{1}{2}\left(\frac{y}{C_1}+1\right)^2 - 2\left(\frac{y}{C_1}+1\right) + \ln\left|\frac{y}{C_1}+1\right|\right) = x + C_2$$

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IZV. 10.

6.

$$y - y_0 = y'(x_0)(x - x_0)$$

$$N_T = y_0 - y'(x_0)x_0$$

$$x \cdot y = (y - y' \cdot x)^2$$

$$z = \frac{y}{x}$$

$$y' = z + z' \cdot x$$

$$y - y'x = \pm\sqrt{xy}$$

$$\cancel{x} \cdot \frac{y}{x} - y'\cancel{x} = \pm x\sqrt{\frac{y}{x}}$$

$$-y' = \pm\sqrt{\frac{y}{x}} - \frac{y}{x}$$

$$y' = \pm\sqrt{\frac{y}{x}} + \frac{y}{x}$$

$$\cancel{z} + z' \cdot x = \pm\sqrt{z} + \cancel{z}$$

$$\frac{dz}{\pm\sqrt{z}} = \frac{dx}{x} \quad / \int$$

$$\pm 2\sqrt{z} = \ln|cx|$$

$$\pm 2\sqrt{\frac{y}{x}} = \ln|cx|$$

$$\pm\sqrt{y} = \frac{\sqrt{x}}{2}\ln|cx|$$

LOWRY	7
IRVING	10
BUTLER	8
THOMPSON	11
DETROPAN	9
DURANT	5
MELO	15
GEORGE	13
BARNES	14
GREEN	4
COUSINS	12
JORDAN	6

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11. DOMAĆA ZADAĆA

(1.) $(2x^3-xy^2)dx+(2y^3-x^2y)\,dy=0$

$P(x,y)=2x^3-xy^2$

$Q(x,y)=2y^3-x^2y \qquad Q(x_0,y)=2y^3$

$x_0=0,\ y_0=0$

$$\mu(x,y)=\int_{x_0}^{x}P\,dx+\int_{y_0}^{y}Q\,dy=\frac{x^4}{2}-\frac{x^2y^2}{2}+\frac{y^4}{2}=C$$

(2.) $(2xy+3y^2)dx+(x^2+6xy-3y^2)dy=0$

$x_0, y_0=0 \qquad P(x,y_0)=2xy+3y^2$

$Q(x,y)=x^2+6xy-3y^2$

$$\mu(x,y)=\int_{x_0}^{x}P(x,y_0)dx+\int_{y_0}^{y}Q(x,y)dy=x^2y+3xy^2-y^3=C$$

(3.) $\left(\frac{y}{x^2+y^2}-1\right)dx-\frac{x}{x^2+y^2}dy=0$

$x_0=0,\ y_0=1$

$$\mu(x,y)=\int_{x_0}^{x}P\,dx+\int_{y_0}^{y}Q(x_0,y)dy$$

$$=\operatorname{arctg}\left(\frac{x}{y}\right)-x=C$$

(4.) $\frac{dy}{dx}=\frac{e^y}{2y-xe^y} \implies e^y\cdot dx+(xe^y-2y)dy=0$

$x_0=0,\ y_0=1 \qquad P(x,y)=e^y$

$Q(x_0,y)=-2y$

$$\mu(x,y)=\int_0^x P(x,y)dx+\int_1^y Q(x_0,y)dy=xe^y-y^2+1$$

$xe^y-y^2=C$

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(5.) $\frac{x}{x^2+y}dx + \left(\frac{1}{y}\ln(x^2+y) + \frac{1}{2(x^2+y)}\right)dy = 0$

$P'_y = \frac{-x}{(x^2+y)^2}$ $\quad Q'_x = \frac{1}{(x^2+y)\cdot y}\cdot 2x - \frac{1}{2(x^2+y)^2}\cdot 2x$

$P'_y - Q'_x = \frac{-2x}{(x^2+y)\cdot y}$

$\ln \mu(y) = -\int \frac{1}{P}\cdot(P'_y - Q'_x)\,dy$

$= -\int \frac{-2}{y}dy = 2\ln|y|$

$\mu(y) = y^2$

$\frac{xy^2}{x^2+y}dx + \left(y\ln(x^2+y) + \frac{y^2}{2(x^2+y)}\right)dy = 0$

$u(x,y) = \int_0^x \frac{xy^2}{x^2+y}dx + \int_1^y \left(y\ln y + \frac{y}{2}\right)dy$

$= \frac{1}{2}y^2\ln(x^2+y)\Big|_0^x + \frac{1}{2}y^2\ln y - \cancel{\frac{y^2}{4}} + \cancel{\frac{y^2}{4}}\Big|_1^y$

$= \frac{1}{2}y^2\ln(x^2+y) - \cancel{\frac{1}{2}y^2\ln(y)} + \cancel{\frac{1}{2}y^2\ln(y)}$

$= \frac{1}{2}y^2\ln(x^2+y) = C$

(6.) $(x^2+\ln y)y\,dx + x\,dy = 0$

$P'_y = x^2 + \ln y + 1$ $\quad P'_y - Q'_x = x^2 + \ln y$

$Q'_x = 1$

$\ln \mu(y) = -\int \frac{1}{P}(x^2+\ln y)\,dy = -\ln|y|$

$\mu(y) = \frac{1}{y}$

$(x^2+\ln y)\,dx + \frac{x}{y}dy = 0$

$\int P\,dx = \frac{x^3}{3} + \ln y\cdot x$ $\quad u(x,y) = \frac{x^3}{3} + x\ln y = C$

$\int Q\,dy = x\ln y$

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(8.) $(3xy^2 - 3y^3)\,dx + (7 - 3xy^2)\,dy = 0$

$P'_y = 6xy - 9y^2 \qquad P'_y - Q'_x = 6xy - 6y^2$

$Q'_x = -3y^2$

$$\ln \mu(y) = -\int \frac{1}{P}(6xy - 6y^2)\,dy = -2\int \frac{1}{y}\,dy$$

$$\mu(y) = \frac{1}{y^2}$$

$$(3x - 3y)\,dx + \left(\frac{7}{y^2} - 3x\right)dy = 0$$

$x_0 = 0$

$y_0 = 1 \qquad u(x,y) = \frac{3x^2}{2} - 3xy - \frac{7}{y} = C$

(9.) $(x^2 + y^2 + 1)\,dx - 2xy\,dy = 0$

$\mu = \mu(y^2 - x^2) = \varphi(t) \qquad \frac{\partial(\varphi(t)\cdot P)}{\partial y} = \frac{\partial(\varphi(t)\cdot Q)}{\partial x}$

$y^2 - x^2 = t$

$\frac{\partial t}{\partial x} = -2x \quad \frac{\partial t}{\partial y} = 2y \qquad 2y\cdot\varphi'(t)\cdot P + 2y\cdot\varphi(t) = -2x\cdot\varphi'(t)\cdot Q - 2y\cdot\varphi(t)$

$$\varphi'(t)(x^2 + y^2 + 1) + \varphi(t) = 2x^2\varphi'(t) - \varphi(t)$$

$$\varphi'(t)(t + 1) = 2\varphi(t)$$

$$\ln(\varphi(t)) = -2\ln|t+1|$$

$$\varphi(t) = \frac{1}{(t+1)^2}$$

$$\mu = \frac{1}{(y^2 - x^2 + 1)^2}$$

$$\frac{x^2 + y^2 + 1}{(y^2 - x^2 + 1)^2}\,dx - \frac{2xy}{(y^2 - x^2 + 1)^2}\,dy = 0$$

$x_0 = 0$

$y_0 = 0 \qquad u(x,y) = \int_0^x P\,dx + \int_0^y Q\,dy = \int_0^x \frac{x^2+1}{1-x^2}\,dx - \int_0^y \frac{2xy}{y^2 - x^2 + 1}\,dy$

$$= 2\,\mathrm{arcth}(x) - x - x\cdot\frac{1}{\sqrt{1-x^2}}\cdot \mathrm{arctg}\left(\frac{y}{\sqrt{1-x^2}}\right)$$

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(11.) $x^4 y'^2 - xy' = y, \quad y' = p$

$x^4 p^2 - xp = y \quad / \frac{d}{dx}$

$p = 4x^3p^2 + 2x^4 \cdot p \cdot p' - p - xp'$

$4x^3p^2 + 2x^4 pp' - xp' - 2p = 0$

$xp'(2x^3p - 1) + 2p(2x^3p - 1) = 0$

$(xp' + 2p)(2x^3p - 1) = 0$

1° $xp' = -2p$

$\frac{p'}{2p} = \frac{-1}{x} \quad / \int$

$\frac{1}{2}\ln|p| = \ln\left|\frac{c}{x}\right|$

$\sqrt{p} = \frac{c}{x}$

$p = \frac{c}{x^2}$

$x^4 \cdot \frac{c^2}{x^4} - x\frac{c}{x^2} = y$

$y = c^2 - \frac{c}{x}$

2° $2x^3p = 1$

$p = \frac{1}{2x^3}$

$y' = \frac{1}{2x^3}$

singularno

$x^4 \cdot \frac{1}{4x^6 2} - \frac{x}{2x^3 2} = \frac{2-1}{4x^2} = \frac{1}{4x^2}$

(12.) $5y + y'^2 = x(x + y')$ $\quad y' = p$

$5y = x(x + y') - y'^2$

$5y = x(x + p) - p^2 \quad / \frac{d}{dx}$

$5p = 2x + p + x \cdot p' - 2p \cdot p'$

$x(2 + p') - 2p(2 + p') = 0$

$(x - 2p)(2 + p') = 0$

1° $p = \frac{x}{2}$ $\quad 5y = \frac{3}{2}x^2 - \frac{x^2}{4} = x^2$

$5y = \frac{5}{4}x^2 \quad y = \frac{x^2}{4}$

singularno

2° $2 + p' = 0$

$p' = -2 \quad / \int$

$p = -2x + c$

$5y = x(c - x) - 4x^2 + 4cx + c^2$

$5y = cx - \cancel{x^2} - 4x^2 - 4cx + c^2 + \cancel{x^2}$

$y = cx - x^2 + \frac{c^2}{5}$

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13.

$y - y_0 = y'(x_0)(x - x_0)$

$M_T \ldots \quad -y_0 = y'(x_0)(M_T - x_0)$

$M_T = x_0 - \frac{y_0}{y'(x_0)} \sim x - \frac{y}{y'}$

$N_T \ldots \quad N_T = y_0 - y'(x_0) \cdot x_0 \sim y - y' \cdot x$

$x = \frac{M_T}{2} \qquad 2x = x - \frac{y}{y'} \qquad \frac{y}{y'} = -x$

$y = -xy'$

$x^2 = \frac{N_T}{2} \qquad 2x^2 = y - y' \cdot x$

$y = 2x^2 + x \cdot y'$

$m = 2x \qquad \frac{x}{m} + \frac{y}{n} = 1$

$n = \frac{m^2}{2} \qquad y = \frac{-n}{m}x + n$

$y = \frac{-m}{2}x + \frac{m^2}{2}$

$k = y' = \frac{-m}{2} \Rightarrow m = -2y'$

$-y = y'(-2y' - x)$

$y = 2y'^2 + xy' \qquad y' = p$

$y = 2p^2 + xp \quad / \frac{d}{dx}$

$\cancel{p} = 4p \cdot p' + \cancel{p} + x \cdot p'$

$p'(4p + x) = 0$

$p = \frac{-x}{4} \qquad y = \frac{2x^2}{16} - \frac{x^2}{4} = \frac{2x^2}{16} - \frac{4x^2}{16} = \frac{-2}{16}x^2$

$y = \frac{-1}{8}x^2$

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15.

$x = \frac{m}{2}$ $\quad m = 2x$

$y = \frac{n}{2} \ /|^2$

$y^2 = \frac{n^2}{4}$ $\quad n^2 = 4y^2 = 4x^3 = \frac{4}{8} m^3 = \frac{m^3}{2}$

$$\frac{x}{m} + \frac{y}{n} = 1$$

$$y = \frac{-n}{m} x + n$$

$$y = \frac{-\sqrt{m}}{\sqrt{2}} x + \frac{m\sqrt{m}}{2}$$

$$k = y' = \frac{-\sqrt{m}}{\sqrt{2}} \Rightarrow m = 2(y')^2$$

$$-y = y'(2(y')^2 - x)$$

$$y = y' \cdot x - 2(y')^3$$

17.

$y - y' \cdot x + x - \frac{y}{y'} = 4 \ / \cdot y'$

$y \cdot y' - y'^2 \cdot x + x \cdot y' - y = 4y'$

$y = y \cdot y' - y'^2 + xy' - 4y' \ / \frac{d}{dx}$ $\quad y' = p$

$p =$