13 DZ

1. a)
$$\begin{vmatrix} \sin x & \cos x & \sin 2x \\ \cos x & -\sin x & 2\cos 2x \\ -\sin x & -\cos x & -2\sin 2x \end{vmatrix} = \begin{vmatrix} \sin x & \cos x & \sin 2x \\ \cos x & -\sin x & 2\cos 2x \\ 0 & 0 & -\sin 2x \end{vmatrix}$$

$$= -\sin 2x\left[\sin x(-\sin x) + \cos x \cdot \cos x\right] = -\sin 2x\left[\cos^2 x - \sin^2 x\right]$$

$$= -\sin 2x \cdot \cos 2x = -\tfrac{1}{2}\sin 4x \neq 0 \quad \text{Lin. nez}$$

b)
$$\begin{vmatrix} \cos x & \cos(x-2) & \cos(x+1) \\ -\sin x & -\sin(x-2) & -\sin(x+1) \\ -\cos x & -\cos(x-2) & -\cos(x+1) \end{vmatrix} = \begin{vmatrix} \cos x & \cos(x-2) & \cos(x+1) \\ -\sin x & -\sin(x-2) & -\sin(x+1) \\ 0 & 0 & 0 \end{vmatrix} = 0 \quad \text{Lin. zav}$$

2. a)
$$\begin{vmatrix} e^{x+1} & e^{1-x} & e^{2x} \\ e^{x+1} & -e^{1-x} & 2e^{2x} \\ e^{x+1} & e^{1-x} & 4e^{2x} \end{vmatrix} = \begin{vmatrix} e^{x+1} & e^{1-x} & e^{2x} \\ 0 & -2e^{1-x} & e^{2x} \\ 0 & 0 & 3e^{2x} \end{vmatrix} = e^{x+1}\left[-2e^{1-x}\cdot 3e^{2x} - 0\right]$$

$$= -6e^{\cancel{x}+1+1-\cancel{x}+2x} = -6e^{2x}$$

$$\neq 0 \quad \text{Lin. nez}$$

b)
$$\begin{vmatrix} e^x & e^x\sin x & e^x\cos x \\ e^x & e^x\sin x + e^x\cos x & e^x\cos x - e^x\sin x \\ e^x & \cancel{e^x\sin x} + e^x\cos x + e^x\cos x - \cancel{e^x\sin x} & \cancel{e^x\cos x} - e^x\sin x - e^x\sin x - \cancel{e^x\cos x} \end{vmatrix} = e^{3x}\begin{vmatrix} 1 & \sin x & \cos x \\ 1 & \sin x + \cos x & \cos x - \sin x \\ 1 & 2\cos x & -2\sin x \end{vmatrix}$$

$$= e^{3x}\begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & 2\cos x - \sin x & -2\sin x - \cos x \end{vmatrix} = e^{3x}\left[\cos x(-2\sin x - \cos x) - (2\cos x - \sin x)(-\sin x)\right]$$

$$= e^{3x}\left[-2\sin x\cos x - \cos^2 x - (-2\sin x\cos x + \sin^2 x)\right]$$

$$= e^{3x}\left(-2\cancel{\sin x\cos x} + 2\cancel{\sin x\cos x} - (\cos^2 x + \sin^2 x)\right)$$

$$= e^{3x}(-1) = -e^{3x} \neq 0 \quad \text{Lin. nez}$$

3\. a) $y_1 = e^x,\ y_2 = xe^x,\ y_3 = x^2 e^x \Rightarrow r_1 = r_2 = r_3 = 1$

$(r-1)^3 = 0 \quad r^3 - 3r^2 + 3r - 1 = 0 \Rightarrow \boxed{y''' - 3y'' + 3y' - y = 0}$

b) $y_1 = 1 \Rightarrow r_1 = 0$

$y_2 = \sin 2x,\ y_3 = \cos 2x \Rightarrow r_{2,3} = 0 \pm 2i$

$r\cdot(r+2i)(r-2i) = r\cdot(r^2 - 2ri + 2ri + 4) = r(r^2+4) = r^3 + 4r = 0$

$\Rightarrow \boxed{y''' + 4y' = 0}$

4\. a) $y_1 = e^{3x} \Rightarrow r_1 = 3$

$y_2 = \sin x,\ y_3 = \cos x \Rightarrow r_{2,3} = 0 \pm i$

$(r-3)(r^2+1) = r^3 - 3r^2 + r - 3 = 0 \Rightarrow \boxed{y''' - 3y'' + y' - 3y = 0}$

b) $y_1 = e^{-x}\sin 2x,\ y_2 = e^{-x}\cos 2x \Rightarrow r_{1,2} = -1 \pm 2i$

$y_3 = \sin x,\ y_4 = \cos x \Rightarrow r_{3,4} = 0 \pm i$

$(r + 1 + 2i)(r + 1 - 2i)(r^2+1) = (r^2 + r \cancel{-2ri} + r + 1 \cancel{-2i} \cancel{+2ir} \cancel{+2i} + 4)(r^2+1)$

$= (r^2 + 2r + 5)(r^2+1) = r^4 + 2r^3 + 6r^2 + 2r + 5 = 0$

$\boxed{y^{IV} + 2y''' + 6y'' + 2y' + 5y = 0}$

5\. $y''' + 6y'' + 11y' + 6y = 0 \Rightarrow r^3 + 6r^2 + 11r + 6 = 0$

$r^3 + 6r^2 + 11r + 6 : r+1 = r^2 + 5r + 6$
$\underline{r^3 + r^2}$
$5r^2 + 11r + 6$
$\underline{5r^2 + 5r}$
$6r + 6$
$\underline{6r + 6}$
0

Racio rješenje je djelitelj slobodnog člana

ovdje je očito negativan: $-1, -2, -3, -6$

-1 zadovoljava

$(r+1)(r^2+5r+6) = 0 \quad \dots \quad (r+1)(r+2)(r+3) = 0$

$\boxed{y = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{-3x}}$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

6\. $y''' - 8y = 0 \Rightarrow r^3 - 8 = 0 \Rightarrow (r-2)(r^2 + 2r + 4) = 0$

$r_1 = 2$

$\dfrac{-2 \pm \sqrt{4-16}}{2}$

$r_{2,3} = -1 \pm \dfrac{2\sqrt{3}}{2} i$

$= -1 \pm \sqrt{3}\, i$

$\boxed{y = C_1 e^{2x} + C_2 e^{-x}\cos(\sqrt{3}x) + C_3 e^{-x}\sin(\sqrt{3}x)}$

7. $y''' - 2y'' + 2y' = 0 \Rightarrow r^3 - 2r^2 + 2r = 0 \quad r(r^2 - 2r + 2) = 0$

$r_1 = 0 \quad r_{2,3} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

$\boxed{y = C_1 + C_2 e^x \cos x + C_3 e^x \sin x}$

8. $y''' - 3y' - 2y = 0 \Rightarrow r^3 - 3r - 2 = 0$ Nultočke rješenja mogu biti samo djelitelji broja -2 : $-1, 1, -2, 2$

$r^3 - 3r - 2 : r + 1 = r^2 - r - 2$
$r^3 + r^2$
$-r^2 - 3r - 2$
$-r^2 - r$
$-2r - 2$
$-2r - 2$
0

$(r+1)(r^2 - r - 2) = 0$ $\quad -1$ je rješenje $(-1 + 3 - 2) = 0$ ✓

$r_1 = -1 \quad r_{2,3} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = -1, 2$

$r_1 = r_2 = -1 \quad r_3 = 2$

$\boxed{y = C_1 e^{-x} + C_2 x e^{-x} + C_3 e^{2x}}$

U zadatcima 9 - 12 nećemo varirati konstante nego pretpostavljati oblik y_p gdje je to lako rečeno

9.

a) $y'' - 4y' + 4y = x^2$ $\quad y_H \ldots \quad r^2 - 4r + 4 = 0 \quad (r-2)^2 = 0$

$y_H = C_1 e^{2x} + C_2 x e^{2x}$

$y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$ $\quad 2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) = x^2$

$y_p'' = 2A$ $\quad 4Ax^2 + (4B - 8A)x + 2A - 4B + 4C = x^2$

$4A = 1$ $\quad 4B - 8A = 0$ $\quad 2A - 4B + 4C = 0 \quad / \cdot 2$

$\boxed{A = \frac{1}{4}}$ $\quad 4B - 2 = 0$ $\quad 4A - 8B + 8C = 0$

$\boxed{B = \frac{1}{2}}$ $\quad 1 - 4 + 8C = 0$

$\boxed{C = \frac{3}{8}}$

$\boxed{y = y_p + y_H = \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{8} + C_1 e^{2x} + C_2 x e^{2x}}$

b) $y'' + y = 4x \cos x$ $\quad y_H \ldots \quad r^2 + 1 = 0 \quad r = \pm i$

$y_H = C_1 \cos x + C_2 \sin x$

$y = C_1(x) \cos x + C_2(x) \sin x$

$C_1' \cos x + C_2' \sin x = 0$

$-C_1' \sin x + C_2' \cos x = 4x \cos x$

$C_2' = 4x + C_1' \frac{\sin x}{\cos x}$

$C_1'\left(\cos x + \frac{\sin^2 x}{\cos x}\right) + 4x \sin x = 0$

$C_1'\left(\frac{\cos^2 x + \sin^2 x}{\cos x}\right) = -4x \sin x$

$C_1' = -4x \sin x \cos x$

$C_1' = -2x \sin 2x$

$C_1(x) = -2\int x \sin 2x \, dx = \left| \begin{array}{ll} u = x & du = dx \\ dv = \sin 2x \, dx & v = -\frac{1}{2}\cos 2x \end{array} \right|$

$= -2\left(-\frac{1}{2} x \cos 2x + \frac{1}{2}\int \cos 2x \, dx\right)$

$= x \cdot \cos 2x - 2 \cdot \frac{1}{2} \cdot \frac{1}{2} \sin 2x$

$= x \cdot \cos 2x - \frac{1}{2} \sin 2x + C_1$ ↓

$C_2' = 4x + C_1' \frac{\sin x}{\cos x} = 4x + (-4x\sin x\cos x)\cdot\frac{\sin x}{\cos x}$

$= 4x - 4x\sin^2 x$

$C_2 = \int (4x - 4x\sin^2 x)\,dx = 2x^2 - \overbrace{4\int x\sin^2 x\,dx}^{A}$

$A = 4\int x\cdot\sin^2 x\,dx = \overset{2}{\cancel{4}}\int x\cdot\frac{1-\cos 2x}{\cancel{2}}\,dx = 2\int (x - x\cos 2x)\,dx$

$= 2\left(\frac{1}{2}x^2 - \int x\cdot\cos 2x\,dx\right) = x^2 - \overbrace{2\int x\cdot\cos 2x\,dx}^{B}$

$B = 2\int x\cos 2x\,dx = \left|\begin{matrix} u=x & du=dx \\ dv=\cos 2x\,dx & v=\frac{1}{2}\sin 2x \end{matrix}\right| = 2\left(\frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx\right)$

$= x\cdot\sin 2x - \int \sin 2x\,dx = x\cdot\sin 2x + \frac{1}{2}\cos 2x$

$A = x^2 - B = x^2 - x\sin 2x - \frac{1}{2}\cos 2x$

$C_2(x) = 2x^2 - A = 2x^2 - x^2 + x\sin 2x + \frac{1}{2}\cos 2x = x^2 + x\sin 2x + \frac{1}{2}\cos 2x + c_2$

$y = C_1\cos x + C_2\sin x$

$= (x\cos 2x - \frac{1}{2}\sin 2x + c_1)\cos x + (x^2 + x\sin 2x + \frac{1}{2}\cos 2x + c_2)\sin x$

$= c_1\cos x + c_2\sin x + x\cos x\cos 2x - \frac{1}{2}\cos x\sin 2x + x^2\sin x + x\sin x\sin 2x$

$+ \frac{1}{2}\sin x\cos 2x$

$= c_1\cos x + c_2\sin x + x^2\sin x + x\cdot[\cos x\cos 2x + \sin x\sin 2x]$

$+ \frac{1}{2}[\sin x\cos 2x - \cos x\sin 2x]$

$\cos x\cos 2x + \sin x\sin 2x = \cos x(\cos^2 x - \sin^2 x) + \sin x\,2\sin x\cos x$

$= \cos x\,[\cos^2 x - \cancel{\sin^2 x} + 2\sin^2 x]$

$= \cos x\cdot[1] = \cos x$

$\sin x\cos 2x - \cos x\sin 2x = \sin x[\cos^2 x - \sin^2 x] - \cos x\cdot 2\sin x\cos x$

$= \sin x[\cos^2 x - \sin^2 x - 2\cos^2 x]$

$= \sin x\cdot[-\cos^2 x - \sin^2 x]$

$= -\sin x$

$y = \underbrace{C_1\sin x + C_2\cos x}_{y_H} + \underbrace{x^2\sin x + x\cdot\cos x - \frac{1}{2}\sin x}_{y_p}$

ili

$y = k_1\sin x + k_2\cos x + x^2\sin x + x\cos x$

10)

a) $y'' - y' = e^x \sin x$ $\qquad y_H: \; r^2 - r = 0 \quad r(r-1) = 0$

$y_H = C_1 + C_2 e^x$

$y_p = e^x [A \cos x + B \sin x]$

$= A e^x \cos x + B e^x \sin x$

$y_p' = A e^x \cos x - A e^x \sin x + B e^x \sin x + B e^x \cos x = (A+B) e^x \cos x + (B-A) e^x \sin x$

$y_p'' = (A+B) e^x \cos x - (A+B) e^x \sin x + (B-A) e^x \sin x + (B-A) e^x \cos x$

$= e^x \cos x (\cancel{A} + B + B - \cancel{A}) + e^x \sin x (\cancel{B} - A - A - \cancel{B})$

$= 2B e^x \cos x - 2A e^x \sin x$

$2B e^x \cos x - 2A e^x \sin x - (A+B) e^x \cos x - (B-A) e^x \sin x = e^x \sin x$

$2B - (A+B) = 0$ $\qquad -2A - (B-A) = 1$

$2B - A - B = 0$ $\qquad -2A - B + A = 1$

$B - A = 0$ $\qquad -A - B = 1$

$B = A$ $\qquad -A - A = 1$

$A = -\frac{1}{2} = B$

$y_p = -\frac{1}{2} e^x \cos x - \frac{1}{2} e^x \sin x$

$\boxed{y = y_H + y_p = C_1 + C_2 e^x - \frac{1}{2} e^x \cos x - \frac{1}{2} e^x \sin x}$

b) $y''' - y'' + y' - y = x^2 + x$ $\qquad y_H: \; r^3 - r^2 + r - 1 = 0$

$r^2(r-1) + (r-1) = 0$

$y_p = Ax^2 + Bx + C$ $\qquad (r-1)(r^2+1) = 0 \quad r_1 = 1 \quad r_{2,3} = \pm i$

$y_p' = 2Ax + B$ $\qquad y_H = C_1 e^x + C_2 \cos x + C_3 \sin x$

$y_p'' = 2A$

$y_p''' = 0 \qquad 0 - 2A + 2Ax + B - Ax^2 - Bx - C = x^2 + x$

$-Ax^2 + (2A - B)x - 2A + B - C = x^2 + x$

$-A = 1 \qquad 2A - B = 1 \qquad -2A + B = C$

$A = -1 \qquad B = 2A - 1 \qquad C = 2 - 3 = -1$

$B = -3$

$y_p = -x^2 - 3x - 1$

$\boxed{y = -x^2 - 3x - 1 + C_1 e^x + C_2 \cos x + C_3 \sin x}$

11. a) $y''' - y' = \sin x$

y_H: $r^3 - r = 0 \quad r(r^2-1)=0$

$r_1 = 0 \quad r_{2,3} = \pm 1$

$y_H = C_1 + C_2 e^{x} + C_3 e^{-x}$

$y_p = A\sin x + B\cos x$

$y_p' = A\cos x - B\sin x$

$y_p'' = -A\sin x - B\cos x$

$y_p''' = -A\cos x + B\sin x$

$-A\cos x + B\sin x - A\cos x + B\sin x = \sin x$

$-2A = 0 \qquad 2B = 1$

$A = 0 \qquad B = \frac{1}{2}$

$y_p = \frac{1}{2}\cos x$

$\boxed{y = C_1 + C_2 e^{x} + C_3 e^{-x} + \frac{1}{2}\cos x}$

b) $y'' + 4y' = x + e^{-4x}$

y_H: $r^2 + 4r = 0$

$r(r+4) = 0$

$r_1 = 0 \quad r_2 = -4$

$y_H = C_1 + C_2 e^{-4x}$

$y_{p_1} = x\cdot(Ax + B) = Ax^2 + Bx$

$y_{p_1}' = 2Ax + B$

$y_{p_1}'' = 2A \qquad 2A + 8Ax + 4B = x$

$8A = 1 \qquad 2A + 4B = 0$

$A = \frac{1}{8} \qquad B = -\frac{1}{2}A$

$\boxed{B = -\frac{1}{16}}$

$y_{p_1} = \frac{1}{8}x^2 - \frac{1}{16}x$

$y_{p_2} = x\cdot k e^{-4x}$

$y_{p_2}' = k e^{-4x} - 4kx e^{-4x}$

$y_{p_2}'' = -4k e^{-4x} - 4k e^{-4x} + 16kx e^{-4x}$

$-8k e^{-4x} + \cancel{16kx e^{-4x}} + 4k e^{-4x}$

$\cancel{-16kx e^{-4x}} = e^{-4x}$

$-4k = 1$

$k = -\frac{1}{4}$

$y_{p_2} = -\frac{1}{4}x e^{-4x}$

$y_p = y_{p_1} + y_{p_2} = \frac{1}{8}x^2 - \frac{1}{16}x - \frac{1}{4}x e^{-4x}$

$\boxed{y = y_p + y_H = C_1 + C_2 e^{-4x} + \frac{1}{8}x^2 - \frac{1}{16}x - \frac{1}{4}x e^{-4x}}$

12. a) $y'' + y' = x^2 - e^{-x} + e^{x}$

y_H: $r^2 + r = 0$

$r(r+1) = 0$

$r_1 = 0 \quad r_2 = -1$

$\boxed{y_H = C_1 + C_2 e^{-x}}$

$y_{p_1} = x(Ax^2 + Bx + C) = Ax^3 + Bx^2 + Cx$

$y_{p_1}' = 3Ax^2 + 2Bx + C$

$y_{p_1}'' = 6Ax + 2B$

$\Rightarrow 6Ax + 2B + 3Ax^2 + 2Bx + C = x^2$

$3A = 1 \qquad 6A + 2B = 0 \qquad 2B + C = 0$

$A = \frac{1}{3} \qquad B = -3A \qquad C = -2B$

$B = -1 \qquad C = 2$

$\boxed{y_{p_1} = \frac{1}{3}x^3 - x^2 + 2x}$

$\downarrow$

$y_{p2} = x \cdot h e^{-x}$ $\quad y_{p2}' = h e^{-x} - h x e^{-x}$ $\quad y_p'' = -h e^{-x} - h e^{-x} + h x e^{-x}$

$-2he^{-x} + \cancel{hxe^{-x}} + he^{-x} - \cancel{hxe^{-x}} = -e^{-x}$

$-he^{-x} = -e^{-x}$

$\boxed{h = 1}$ $\quad \boxed{y_{p2} = xe^{-x}}$

$y_{p3} = he^{x}$ $\quad y_{p3}' = he^{x}$ $\quad y_{p3}'' = he^{x}$

$he^{x} + he^{x} = e^{x}$ $\quad \boxed{y_{p3} = \frac{1}{2} e^{x}}$

$h = \frac{1}{2}$

$y_p = y_{p1} + y_{p2} + y_{p3} = \frac{1}{3}x^3 - x^2 + 2x + xe^{-x} + \frac{1}{2}e^{x}$

$\boxed{y = y_p + y_H = C_1 + C_2 e^{-x} + \frac{1}{3}x^3 - x^2 + 2x + xe^{-x} + \frac{1}{2}e^{x}}$

1b)

$y''' - 4y' = xe^{2x} + \sin x + x^2$ $\qquad y_H$: $v^3 - 4v = 0$

$v(v^2 - 4) = 0$

$y_{p1} = x \cdot (Ax + B)e^{2x}$ $\qquad v_1 = 0 \quad v_2 = 2 \quad v_3 = -2$

$= Ax^2e^{2x} + Bxe^{2x}$ $\qquad \boxed{y_H = C_1 + C_2 e^{2x} + C_3 e^{-2x}}$

$y_{p1}' = 2Axe^{2x} + 2Ax^2e^{2x} + Be^{2x} + 2Bxe^{2x}$

$= (2A + 2B)xe^{2x} + 2Ax^2e^{2x} + Be^{2x}$

$y_{p1}'' = (2A + 2B)e^{2x} + 2(2A + 2B)xe^{2x} + 4Axe^{2x} + 4Ax^2e^{2x} + 2Be^{2x}$

$= (2A + 4B)e^{2x} + (8A + 4B)xe^{2x} + 4Ax^2e^{2x}$

$y_{p1}''' = 2(2A + 4B)e^{2x} + (8A + 4B)e^{2x} + 2(8A + 4B)xe^{2x} + 8Axe^{2x} + 8Ax^2e^{2x}$

$= (12A + 12B)e^{2x} + (24A + 8B)xe^{2x} + 8Ax^2e^{2x}$

$y_{p1}''' - 4y_{p1}' = xe^{2x} \Rightarrow (12A + 12B - 4B)e^{2x} + (24A + \cancel{8B} - 8A - \cancel{8B})xe^{2x}$

$+ (8A - \cancel{8A})x^2e^{2x} = xe^{2x}$

$16A = 1$ $\qquad 12A + 8B = 0$ $\qquad \boxed{y_{p1} = \frac{1}{16}x^2e^{2x} - \frac{3}{32}xe^{2x}}$

$\boxed{A = \frac{1}{16}}$ $\qquad B = -\frac{3}{2}A = \boxed{-\frac{3}{32}}$

$y_{p2} = A\sin x + B\cos x$

$y_{p2}''' - 4y_{p2}' = \sin x$

$y_{p2}' = A\cos x - B\sin x$

$-A\cos x + B\sin x - 4A\cos x + 4B\sin x = \sin x$

$y_{p2}'' = -A\sin x - B\cos x$ $\qquad 5B = 1 \quad B = \frac{1}{5}$ $\qquad \boxed{y_{p2} = \frac{1}{5}\cos x}$

$-4A = 0$

$y_{p2}''' = -A\cos x + B\sin x$ $\qquad A = 0$

↓

$y_{P3} = x(Ax^2+Bx+C) \qquad y_{P3}' = 3Ax^2+2Bx+C \qquad y_{P3}''' = 6A$

$= Ax^3+Bx^2+Cx \qquad y_{P3}'' = 6Ax+2B$

$6A - 4(3Ax^2+2Bx+C) = x^2$

$-12Ax^2 - 8Bx + 6A - 4C = x^2 \qquad \boxed{y_{P3} = -\frac{1}{12}x^3 - \frac{1}{8}x}$

$-12A = 1 \qquad -8B = 0 \qquad 6A - 4C = 0$

$\boxed{A = -\frac{1}{12}} \qquad \boxed{B=0} \qquad C = \frac{3}{2}A = -\frac{3}{24} = \boxed{-\frac{1}{8}}$

$y_P = y_{P1} + y_{P2} + y_{P3} = \boxed{\frac{1}{16}x^2e^{2x} - \frac{3}{32}xe^{2x} + \frac{1}{8}\cos x - \frac{1}{12}x^3 - \frac{1}{8}x}$

$\boxed{y = y_H + y_P = C_1 + C_2e^x + C_3e^{-2x} - \frac{1}{12}x^3 - \frac{1}{8}x + \frac{1}{16}x^2e^{2x} - \frac{3}{32}xe^{2x} + \frac{1}{8}\cos x}$

d) $y'' - y' = \frac{1}{e^x+1} \qquad y_H \ldots \quad r^2 - r = 0 \quad r_1 = 0$

$r(r-1) = 0 \quad r_2 = 1 \qquad y_H = C_1 + C_2e^x$

$y = C_1(x) + C_2(x)e^x \qquad C_1'\cdot 1 + C_2'e^x = 0 \qquad C_2' = \frac{1}{e^x(e^x+1)}$

$C_1'\cdot 0 + C_2'e^x = \frac{1}{e^x+1} \qquad C_1' = -C_2'e^x = -\frac{1}{e^x+1}$

$C_1 = -\int \frac{dx}{e^x+1} = \left| \begin{array}{l} t = e^x+1 \\ dt = e^x dx \end{array} \right| = -\int \frac{dt}{(t-1)\cdot t} = -\left[\int \frac{dt}{t-1} - \int \frac{dt}{t}\right]$

$\frac{1}{(t-1)t} = \frac{A}{t-1} + \frac{B}{t} \qquad A = \frac{1}{1} = 1 \qquad B = \frac{1}{0-1} = -1$

metoda "prekrivanja"

$C_1(x) = -\int \frac{dt}{t-1} + \int \frac{dt}{t} = -\ln|t-1| + \ln|t| = -\ln|e^x| + \ln|e^x+1|$

$= \boxed{-x + \ln(e^x+1)} + C_1$

$C_2(x) = \int \frac{dx}{e^x(e^x+1)} = \left| \begin{array}{l} t = e^x \\ dt = t\,dx \end{array} \right| = \int \frac{dt}{t^2(t+1)} = \int \frac{-dt}{t} + \int \frac{dt}{t^2} + \int \frac{dt}{t+1}$

$\frac{1}{t^2(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1} \quad / \cdot t^2(t+1) \qquad = -\ln|t| - \frac{1}{t} + \ln(t+1)$

$1 = At(t+1) + B(t+1) + Ct^2 \qquad = -\ln(e^x) - \frac{1}{e^x} + \ln(e^x+1) + C_2$

$1 = t^2(A+C) + t(A+B) + B \qquad = \ln\left(\frac{e^x+1}{e^x}\right) - e^{-x} + C_2$

$\boxed{B=1} \qquad A+B=0 \qquad A+C=0 \qquad = \boxed{\ln(e^{-x}+1) - e^{-x} + C_2}$

$\boxed{A=-1} \qquad \boxed{C=1}$

$y = -x + \ln(e^x+1) + C_1 + [\ln(e^{-x}+1) - e^{-x} + C_2]e^x$

$= \underbrace{C_1 + C_2e^x}_{y_H} \underbrace{- x + \ln(e^x+1) + e^x\ln(e^{-x}+1) - 1}_{y_P}$

14.

$y'' + y = \frac{1}{\sin x}$ $\quad y_H:\ r^2+1=0 \quad r=\pm i$

$y_H = C_1\cos x + C_2\sin x$

$\boxed{y = C_1(x)\cos x + C_2(x)\sin x}$

$C_1'\cos x + C_2'\sin x = 0$

$-C_1'\sin x + C_2'\cos x = \frac{1}{\sin x}$

$C_2' = \frac{1}{\sin x\cdot\cos x} + C_1'\cdot\frac{\sin x}{\cos x}$

$C_1'\left\{\cos x + \frac{\sin^2 x}{\cos x}\right\} + \frac{1}{\cos x} = 0$

$C_1'\left\{\frac{\cos^2 x + \sin^2 x}{\cos x}\right\} = -\frac{1}{\cos x}$

$C_2' = \frac{1}{\sin x\cos x} - \frac{\sin x}{\cos x}$ $\qquad \boxed{C_1' = -1} \quad C_1(x) = -x + C_1$

$= \frac{1}{\cos x}\left[\frac{1}{\sin x} - \sin x\right]$

$= \frac{1}{\cos x}\left[\frac{1-\sin^2 x}{\sin x}\right] = \frac{\cos^2 x}{\sin x\cdot\cos x} = \frac{\cos x}{\sin x}$ $\quad C_2(x) = \int\frac{\cos x}{\sin x}dx$

$= \int\frac{d(\sin x)}{\sin x} = \ln|\sin x| + C_2$

$\boxed{y = C_1\cos x + C_2\sin x - x\cos x + \ln|\sin x|\cdot\sin x}$

15.

$y''' + y' = \frac{1}{\cos^2 x}$ $\quad y_H:\ r^3 + r = 0 \quad r(r^2+1)=0$

$r_1 = 0 \quad r_{2,3} = \pm i$

$C_1'\cdot 1 + C_2'\cdot\cos x + C_3'\cdot\sin x = 0$ $\qquad y_H = C_1 + C_2\cos x + C_3\sin x$

$C_1'\cdot 0 - C_2'\sin x + C_3'\cos x = 0$ $\qquad y = C_1(x) + C_2(x)\cos x + C_3(x)\sin x$

$0 \ - C_2'\cos x - C_3'\sin x = \frac{1}{\cos^2 x}$

$C_3' = -\frac{1}{\sin x\cos^2 x} - C_2'\frac{\cos x}{\sin x}$ $\qquad C_3' = -\frac{1}{\sin x\cos^2 x} + \frac{1}{\sin x}$

$-C_2\left\{\sin x + \frac{\cos^2 x}{\sin x}\right\} - \frac{1}{\sin x\cos x} = 0$ $\qquad \frac{1}{\sin x}\left(1 - \frac{1}{\cos^2 x}\right) = \frac{1}{\sin x}\left(\frac{\cos^2 x - 1}{\cos^2 x}\right)$

$C_2'\frac{1}{\sin x} = -\frac{1}{\sin x\cos x}$ $\qquad = \frac{-\sin^2 x}{\sin x\cos^2 x}$ $\quad \boxed{-\frac{\sin x}{\cos^2 x}}$

$\boxed{C_2' = -\frac{1}{\cos x}}$ $\qquad C_1' + -\frac{1}{\cos x}\cdot\cos x + -\frac{\sin x}{\cos^2 x}\cdot\sin x = 0$

$C_1' - 1 - \tan^2 x = 0$ $\qquad \sin^2 x + \cos^2 x = 1$

$C_1(x) = \int\frac{1}{\cos^2 x}dx = \tan x + C_1$ $\qquad C_1' = 1 + \tan^2 x$ $\qquad \tan^2 x + 1 = \frac{1}{\cos^2 x}$

$C_2(x) = -\int\frac{dx}{\cos x} = -\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C_2$

$\boxed{C_1' = \frac{1}{\cos^2 x}}$

$C_3(x) = -\int\frac{\sin x}{\cos^2 x}dx = \int\frac{d(\cos x)}{\cos^2 x}dx = -\frac{1}{\cos x} + C_3$

$y = C_1 + C_2\cos x + C_3\sin x + \cancel{\tan x} - \cos x\left(\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|\right) - \cancel{\tan x}$

$= \boxed{C_1 + C_2\cos x + C_3\sin x - \cos x\ \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|}$ $\qquad \int\frac{-dx}{\cos x}$ $\quad t = \tan\frac{x}{2}$

Wolfram Alpha $\qquad \cos x = \frac{1-t^2}{1+t^2}$

$\boxed{y = C_1 + C_2\cos x + C_3\sin x + 2\cos x\ \mathrm{atanh}\left(\tan\frac{x}{2}\right)}$ $\qquad = -\int\frac{1+t^2}{1-t^2}\cdot\frac{2dt}{1+t^2} = -2\int\frac{dt}{1-t^2}$ $\qquad dx = \frac{2dt}{1+t^2}$

$= -2\,\mathrm{atanh}(t) = -2\,\mathrm{atanh}\left(\tan\frac{x}{2}\right)$

16. $y'' + 4y = \frac{1}{\sin^2 x}$ $\quad y_H \quad r^2+4=0 \quad r=\pm 2i$

$y_H = c_1\cos 2x + c_2 \sin 2x$

$c_1'\cos 2x + c_2'\sin 2x = 0$

$y = c_1(x)\cos 2x + c_2(x)\sin 2x$

$-2c_1'\sin 2x + 2c_2'\cos 2x = \frac{1}{\sin^2 x}$

$c_1' \frac{1}{\cancel{\cos 2x}} = -\frac{\sin 2x}{2\cos 2x \sin^2 x}$

$c_2' = \frac{1}{2\cos 2x \cdot \sin^2 x} + 2c_1' \frac{\sin 2x}{\cos 2x}$

$\boxed{c_1'} = -\frac{\sin 2x}{2\sin^2 x}$

$c_1'\left[\cos 2x + \frac{\sin^2 2x}{\cos 2x}\right] + \frac{\sin 2x}{2\cos 2x\sin^2 x} = 0$

$= -\frac{2\sin x\cos x}{2\sin^2 x}$

$= \boxed{-\frac{\cos x}{\sin x}}$

$c_2' = \frac{1}{2\cos 2x \cdot \sin^2 x} + \frac{-\sin 2x}{2\sin^2 x}\cdot\frac{\sin 2x}{\cos 2x}$

$= \frac{1}{2\cos 2x \sin^2 x}\left[1 - \sin^2 2x\right] = \frac{\cos^{\cancel{2}} 2x}{2\cdot\cancel{\cos 2x}\cdot \sin^2 x} = \boxed{\frac{1}{2}\frac{\cos 2x}{\sin^2 x}}$

$c_1(x) = -\int \frac{\cos x}{\sin x} = -\int \frac{d(\sin x)}{\sin x} = \boxed{-\ln|\sin x| + C_1}$

$c_2(x) = \frac{1}{2}\int \left(\frac{\cos^2 x}{\sin^2 x} - \frac{\sin^2 x}{\sin^2 x}\right)dx = -\frac{1}{2}\int dx + \frac{1}{2}\int \frac{1-\sin^2 x}{\sin^2 x}$

$= -\frac{1}{2}x + \frac{1}{2}\int \frac{1}{\sin^2 x}dx - \frac{1}{2}\int dx = -x + \frac{1}{2}\int\frac{dx}{\sin^2 x} = -x - \frac{1}{2}\cot x + C_2$

$y = C_1\cos 2x + C_2\sin 2x - \cos 2x\ln|\sin x| - x\sin 2x - \sin 2x\cdot\frac{1}{2}\frac{\cos x}{\sin x}$

$\boxed{y = C_1\cos 2x + C_2\sin 2x - \cos 2x\ln|\sin x| - x\sin 2x - \cos^2 x}$

$= -2\sin x\cos x\cdot\frac{\cos x}{2\sin x}$

$= -\cos^2 x$

17. $x^2y'' + xy' - y = 0$ Eulerova jednadžba

$y_1 = x^k$

$x^2 k(k-1)x^{k-2} + x\, k x^{k-1} - x^k = 0$

$k(k-1)\cancel{x^k} + kx^k - x^k = 0 \quad /: x^k$

$k^2 - \cancel{k} + \cancel{k} - 1 = 0$

$\boxed{y = C_1 x + \frac{C_2}{x}}$

$k^2 = 1 \quad k = \pm 1$

18. $x^2y'' + 2xy' + 6y = 0$ Eulerova jednadžba

$y = x^k$

$k(k-1)\cancel{x^k} + 2k\,\cancel{x^k} + 6\,\cancel{x^k} = 0$

$k^2 - k + 2k + 6 = 0$

$y_1 = x^{-\frac{1}{2}}\cos\left(\frac{\sqrt{23}}{2}\ln x\right) \quad y_2 = x^{-\frac{1}{2}}\sin\left(\frac{\sqrt{23}}{2}\ln x\right)$

$k^2 + k + 6 = 0$

$\boxed{y = \frac{C_1}{\sqrt{x}}\cos\left(\frac{\sqrt{23}}{2}\ln(x)\right) + \frac{C_2}{\sqrt{x}}\sin\left(\frac{\sqrt{23}}{2}\ln x\right)}$

$k = \frac{-1\pm\sqrt{1-24}}{2} = -\frac{1}{2} \pm \frac{i\sqrt{23}}{2}$

19. $x^2 y''' - 3xy'' + 3y' = 0$ $\quad y = x^k$ Eulerova jednadžba

$$k(k-1)(k-2)\cancel{x^{k-1}} - 3k(k-1)\cancel{x^{k-1}} + 3k\cancel{x^{k-1}} = 0$$

$$k\left[k^2 - 3k + 2 - 3k + 3 + 3\right] = 0$$

$$k(k^2 - 6k + 8) = 0$$

$$k(k-4)(k-2) = 0$$

$$\boxed{y = C_1 + C_2 x^2 + C_3 x^4}$$

20 $(x+1)^2 y''' - 12y' = 0$ Eulerova jednadžba + zamjena varijable

$$\boxed{x+1 = e^t} \quad x = e^t - 1$$

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{e^t} = e^{-t}\dot{y}$$

$$y'' = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\ddot{y}e^{-t} - e^{-t}\dot{y}}{e^t} = e^{-2t}(\ddot{y} - \dot{y}) = e^{-2t}\ddot{y} - e^{-2t}\dot{y}$$

$$y''' = \frac{dy''}{dx} = \frac{\frac{dy''}{dt}}{\frac{dx}{dt}} = \frac{\dddot{y}e^{-2t} - 2\ddot{y}e^{-2t} - \ddot{y}e^{-2t} + 2\dot{y}e^{-2t}}{e^t}$$

$$= e^{-3t}\left[\dddot{y} - 3\ddot{y} + 2\dot{y}\right]$$

$$\cancel{e^{2t}} \cdot \cancel{e^{-3t}}\left[\dddot{y} - 3\ddot{y} + 2\dot{y}\right] - 12\cancel{e^{-t}}\dot{y} = 0 \quad / \cdot e^{t}$$

$$\dddot{y} - 3\ddot{y} - 10\dot{y} = 0 \quad \Rightarrow \quad r^3 - 3r^2 - 10r = 0$$

$$r(r^2 - 3r - 10) = 0$$

$$r_1 = 0 \quad r_2 = 5 \quad r_3 = -2$$

$$y = C_1 + C_2 e^{5t} + C_3 e^{-2t} \qquad t = \ln(x+1)$$

$$\boxed{y = C_1 + C_2 (x+1)^5 + \frac{C_3}{(x+1)^2}}$$