LINEARNA ALGEBRA

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- RJEŠENJA ZADATAKA -

1. (a) Knjižica 2, Determinante, teorem 2 (iskaz i dokaz).

(b) Računamo

$$a = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{vmatrix} = \left[\begin{array}{l} \text{od } j\text{-tog stupca oduzmemo } (j+1)\text{-vi,} \\ j = 1,2,3,4 \end{array}\right]$$

$$= \begin{vmatrix} -1 & -1 & -1 & -1 & 5 \\ -1 & -1 & -1 & 4 & 1 \\ -1 & -1 & 4 & -1 & 2 \\ -1 & 4 & -1 & -1 & 3 \\ 4 & -1 & -1 & -1 & 4 \end{vmatrix} \begin{array}{l} \mid\cdot(-1) \\ + \\ + \\ + \\ + \end{array}$$

$$= \begin{vmatrix} -1 & -1 & -1 & -1 & 5 \\ 0 & 0 & 0 & 5 & -4 \\ 0 & 0 & 5 & 0 & -3 \\ 0 & 5 & 0 & 0 & -2 \\ 5 & 0 & 0 & 0 & -1 \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 & -1 & 3 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$\begin{array}{ccccc} \mid\cdot\frac{1}{5}\,+ & \mid\cdot\frac{2}{5}\,+ & \mid\cdot\frac{3}{5}\,+ & \mid\cdot\frac{4}{5}\,+ & \end{array} \qquad \begin{array}{ccccc} + & + & + & + & \mid\cdot\frac{1}{3} \end{array}$$

$$= \begin{vmatrix} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix} = \boxed{1875}$$

$$b = \begin{vmatrix} 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{vmatrix} = a = \boxed{1875}$$

$$\det(A+B) = \begin{vmatrix} 6 & 3 & 5 & 7 & 9 \\ 6 & 8 & 5 & 7 & 4 \\ 6 & 8 & 10 & 2 & 4 \\ 6 & 8 & 5 & 7 & 4 \\ 6 & 3 & 5 & 7 & 9 \end{vmatrix} = \left[\begin{array}{l} \text{prvi i posljednji redak} \\ \text{se podudaraju} \end{array}\right] = \boxed{0}$$

Dakle,

$$\frac{\det(-A)}{b} = \frac{(-1)^5 \cdot a}{b} = -\frac{a}{b} = \boxed{-1},$$

$$\frac{\det(A+B)}{ab} = \frac{0}{ab} = \boxed{0}.$$

2. (a) $\det A = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & -1 \end{vmatrix} \begin{matrix} 1\cdot 1 \\ + \\ 1\cdot(-1) \end{matrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{vmatrix} = -4 \neq 0,$

$\det B = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0,$

$\det C = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} \begin{matrix} 1\cdot(-\frac{1}{2}) \\ + \end{matrix} = \begin{vmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6 \neq 0$

$\Rightarrow$ A, B, C su regularne matrice

(b) $X^{-1}A = B \cdot C^{-1} \quad / \cdot A^{-1}$

$X^{-1} = BC^{-1}A^{-1} \quad /^{-1}$

$X = ACB^{-1}$

Računamo

$$\underbrace{\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]}_{B} \begin{matrix} + \\ 1\cdot(-1) \end{matrix} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underbrace{\phantom{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}}}_{=B^{-1}}$$

$$\Rightarrow X = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 0 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & 0 \\ 2 & -1 & -2 \end{bmatrix}$$

3.

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ -3 & -16 & \beta & 2\gamma \\ 2 & 12 & 8 & 2 \end{array}\right] \begin{matrix} \cdot 3 \\ \cdot(-2) \\ \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & -1 & \beta-9 & 2\gamma+3 \\ 0 & 2 & 14 & 0 \end{array}\right] \begin{matrix} \\ \\ :2 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & -1 & \beta-9 & 2\gamma+3 \\ 0 & 1 & 7 & 0 \end{array}\right] \cdot 1 \sim \left[\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & 0 & \beta-2 & 2\gamma+3 \\ 0 & 1 & 7 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & \beta-2 & 2\gamma+3 \end{array}\right]$$

Razlikujemo slučajeve:

1° $\beta - 2 = 0 \iff \beta = 2$

Iz treće jednadžbe tada slijedi $0 \cdot z = 2\gamma + 3$ pa zadani sustav:

1.1° nema rješenja u slučaju $2\gamma + 3 \neq 0$, tj. $\gamma \neq -\frac{3}{2}$,

2.1° ima beskonačno mnogo rješenja u slučaju $\gamma = -\frac{3}{2}$

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \cdot(-5) \sim \left[\begin{array}{ccc|c} 1 & 0 & -38 & 1 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} \Rightarrow x = 1 + 38z \\ \Rightarrow y = -7z \\ \end{matrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+38t \\ -7t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 38 \\ -7 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

2° $\beta \neq 2$

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & \beta-2 & 2\gamma+3 \end{array}\right] \Big| :(\beta-2)\neq 0 \sim \left[\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & \frac{2\gamma+3}{\beta-2} \end{array}\right] \begin{array}{l} + \\ \big| \cdot 3 \\ + \\ \big| \cdot (-7) \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & 0 & \frac{\beta+6\gamma+7}{\beta-2} \\ 0 & 1 & 0 & -\frac{14\gamma+21}{\beta-2} \\ 0 & 0 & 1 & \frac{2\gamma+3}{\beta-2} \end{array}\right] \begin{array}{l} + \\ \big| \cdot (-5) \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{\beta+91\gamma+112}{\beta-2} \\ 0 & 1 & 0 & -\frac{14\gamma+21}{\beta-2} \\ 0 & 0 & 1 & \frac{2\gamma+3}{\beta-2} \end{array}\right]$$

Sustav u ovom slučaju ima jedinstveno rješenje za sve $\gamma \in \mathbb{R}$ koje glasi:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{\beta+91\gamma+112}{\beta-2} \\ -\frac{14\gamma+21}{\beta-2} \\ \frac{2\gamma+3}{\beta-2} \end{bmatrix}$$

4. Uočimo da je sjecište S dijagonala tog kvadrata upravo nožište okomice iz točke A na zadani pravac p. Odredimo jednadžbu ravnine π okomite na pravac p, a koja prolazi točkom A:

$$-(x-2)+(y-4)+4(z-5)=0$$

$$-x+y+4z-22=0$$

$$\pi \ldots \ -x+y+4z=22$$

Točka S je presjek ravnine π i pravca p. Parametarske jednadžbe tog pravca su

$$p \ldots \begin{cases} x = 4-t \\ y = 5+t \\ z = 3+4t \end{cases}, \qquad t \in \mathbb{R},$$

pa njihovim uvrštavanjem u jednadžbu od π dobivamo koordinate točke S:

$$-(4-t)+(5+t)+4(3+4t)=22$$

$$18t+13=22$$

$$18t=9$$

$$t=\frac{1}{2} \Rightarrow \boxed{S=\left(\frac{7}{2}, \frac{11}{2}, 5\right)}$$

Budući da je S polovište dužine $\overline{AC}$, odmah slijedi

$$\left.\begin{aligned} \frac{x_c+2}{2} &= \frac{7}{2} \Rightarrow x_c = 5 \\ \frac{y_c+4}{2} &= \frac{11}{2} \Rightarrow y_c = 7 \\ \frac{z_c+5}{2} &= 5 \Rightarrow z_c = 5 \end{aligned}\right\} \Rightarrow \boxed{C=(5,7,5)}$$

Uočimo da je $|AS| = \sqrt{\left(2-\frac{7}{2}\right)^2+\left(4-\frac{11}{2}\right)^2+(5-5)^2} = \frac{3\sqrt{2}}{2} = |BS| = |CS| = |DS|$.

Sve točke na pravcu BD imaju koordinate oblika $(4-t, 5+t, 3+4t)$ za neki $t \in \mathbb{R}$. Zanimaju nas one čija je udaljenost od S jednaka $\frac{3\sqrt{2}}{2}$:

$$\sqrt{\left(\frac{7}{2}-4+t\right)^2+\left(\frac{11}{2}-5-t\right)^2+\left(5-3-4t\right)^2} = \frac{3\sqrt{2}}{2} \quad \Big|^2$$

$$\left(t-\frac{1}{2}\right)^2+\left(t-\frac{1}{2}\right)^2+\left(4t-2\right)^2 = \frac{9}{2}$$

$$18t^2-18t+\cancel{\frac{1}{2}+4} = \cancel{\frac{9}{2}}$$

$$t(t-1)=0$$

$$t_1 = 0, \quad t_2 = 1$$

Ovi parametri odgovaraju točkama B i D pa imamo

$$\boxed{\begin{array}{l} B(4, 5, 3), \\ D(3, 6, 7) \end{array}}$$

(i još jedno rješenje u kojem su koordinate od B i D zamijenjene).

5. (a) Neka su $\alpha, \beta \in \mathbb{R}$ i $p, q \in \mathcal{P}_3$ proizvoljni. Imamo

$$A(\alpha p+\beta q) = ((\alpha p+\beta q)(0), (\alpha p+\beta q)(1), (\alpha p+\beta q)(2))$$

$$= (\alpha p(0)+\beta q(0), \alpha p(1)+\beta q(1), \alpha p(2)+\beta q(2))$$

$$= \alpha (p(0), p(1), p(2)) + \beta (q(0), q(1), q(2))$$

$$= \alpha A(p) + \beta A(q),$$

pa po definiciji slijedi da je A linearni operator.

(b) $A(1) = (1,1,1)$, $A(t) = (0,1,2)$, $A(t^2) = (0,1,4)$, $A(t^3) = (0,1,8)$

$$\Rightarrow [A] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix}$$

(c) $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix} \begin{matrix} \\ 1\cdot(-1) + \\ 1\cdot(-1) + \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 4 & 8 \end{bmatrix} \begin{matrix} \\ 1\cdot(-2) \\ + \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 7 \end{bmatrix}$$

$\Rightarrow r(A) = 3$

$\Rightarrow d(A) = \dim \mathcal{P}_3 - r(A) = 4-3 = 1$

(po teoremu o rangu i defektu)

Neka je $p(t) = at^3+bt^2+ct+d \in \mathcal{P}_3$ proizvoljan polinom.
Elementi slike od A su oblika

$$A(p) = (p(0), p(1), p(2))$$

$$= (d, a+b+c+d, 8a+4b+2c+d)$$

$$= a(0,1,8)+b(0,1,4)+c(0,1,2)+d(1,1,1), \quad a,b,c,d \in \mathbb{R}$$

Budući da je

$$(0,1,8) = 3(0,1,4) - 2(0,1,2),$$

vidimo da skup $\{(0,1,4), (0,1,2), (1,1,1)\}$ razapinje $\mathrm{Im} A$, a budući da je $\dim \mathrm{Im} A = r(A) = 3$, slijedi da je taj skup baza za $\mathrm{Im} A$.

Nadalje, jezgru od A čine svi polinomi $p(t) = at^3 + bt^2 + ct + d \in \mathcal{P}_3$ za koje je

$$A(p) = (0,0,0)$$

$$\Rightarrow (d, a+b+c+d, 8a+4b+2c+d) = (0,0,0)$$

Rješavamo pripadni homogeni sustav

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 8 & 4 & 2 & 1 & 0 \end{array}\right] \begin{array}{l} \\ | \cdot (-1) \\ | \cdot (-1) \end{array} \sim \left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 8 & 4 & 2 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ | : 2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} \\ | \cdot (-1) \\ + \end{array} \sim \left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ + \\ | \cdot (-1) \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow d = 0 \\ \Rightarrow c = 2a \\ \Rightarrow b = -3a \end{array} \qquad a = \alpha, \ \alpha \in \mathbb{R}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \alpha \\ -3\alpha \\ 2\alpha \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -3 \\ 2 \\ 0 \end{bmatrix}, \qquad \alpha \in \mathbb{R}$$

Budući da je $\dim \mathrm{Ker} A = d(A) = 1$, (jedna) baza za $\mathrm{Ker} A$ je skup

$$\{t^3 - 3t^2 + 2t\}.$$

(d) Uočimo da za polinom $p_0(t) = t+1$ vrijedi

$$A(p_0) = (0+1,\ 1+1,\ 2+1) = (1,2,3).$$

Dakle, svi traženi polinomi su zbroj p_0 i linearne kombinacije elemenata baze od $\operatorname{Ker} A$, tj.

$$\begin{aligned} p(t) &= p_0(t) + \alpha\left(t^3 - 3t^2 + 2t\right) \\ &= \alpha t^3 - 3\alpha t^2 + (2\alpha+1)t + 1, \qquad \alpha \in \mathbb{R} \end{aligned}$$

6. (a) $\|C\| = \sqrt{\langle C | C \rangle} = \sqrt{tr(CC^T)} = \sqrt{tr\begin{bmatrix} 5 & -4 \\ -4 & 4 \end{bmatrix}} = \sqrt{5+4} = \boxed{3}$

(b) Za proizvoljnu matricu $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$ imamo

$$X \in W \iff \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \ \& \ tr\left(\begin{bmatrix} 2 & -1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = 0$$

$$\iff b = c \ \& \ tr\begin{bmatrix} 2a-b & 2b-d \\ -2a & -2b \end{bmatrix} = 0$$

$$\iff b = c \ \& \ 2a - 3b = 0$$

$$\iff b = c \ \& \ a = \frac{3}{2}b$$

$$\iff X = \begin{bmatrix} \frac{3}{2}b & b \\ b & d \end{bmatrix} = b\underbrace{\begin{bmatrix} \frac{3}{2} & 1 \\ 1 & 0 \end{bmatrix}}_{=:D_1} + d\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{=:D_2}$$

Budući da direktnom provjerom po definiciji vidimo $D_1, D_2 \in W$ te da je skup $\{D_1, D_2\}$ linearno nezavisan, taj je skup baza za W i $\dim W = 2$.

W nije ortogonalni komplement prostora $L(C)$ zbog dodatnog uvjeta da sadrži samo simetrične matrice (on je zapravo potprostor vektorskog prostora $L(C)^\perp$).