SPUS - DZ
8. tjedan

1. TOČKASTI PROCJENITELJ ($\hat{\Theta}$) nekog parametra Θ je funkcija slučajnih vektora opažanja $x=(x_1, \dots, x_n)$

koja kao rezultat daje estimaciju od Θ

NEPRISTRANOST $\hat{\Theta}$ – ako i samo ako $\underline{E[\hat{\Theta}] = 0}$

KONZISTENTNOST – estimacija je konzistentna ako srednja kv. pogreška

teži u nulu za velike uzorke:

$$\lim_{N\to\infty} E[\varepsilon_N^2] = \lim_{N\to\infty} E[(\hat{\Theta}_N - \Theta)^2] = 0$$

NAJEFIKASNIJI PROCJENITELJ – onaj estimator koji ima minimalnu varijancu $\hat{\Theta}_{MV} = g(x_1, \dots, x_n)$

je najefikasniji nepristrani procjenitelj parametra Θ ako je:

$$Var[\hat{\Theta}_{MV}] \le Var[\hat{\Theta}_N] \quad \text{za sve procjenitelje } \hat{\Theta}_n$$

2. a) pr. 13.23., str. 292

$x(t) \to$ GAUSS, sr. vr. $= 0$ $\qquad E[x^4] = 3\sigma_x^4$

$\le 0.5\%$ sr. vr.

$$\hat{\sigma}^2_{x(t)} = \frac{1}{N}\sum_{i=1}^{N} x_i^2$$

N, pristranost?

$$\sigma^2_{\hat{\sigma}^2_{x(t)}} = E[(\hat{\sigma}^2_{x(t)})^2] - E[\hat{\sigma}^2_{x(t)}]^2$$

$$\underline{E[\hat{\sigma}^2_{x(t)}]} = E\left[\frac{1}{N}\sum_{i=1}^{N} x_i^2\right] = \frac{1}{N}\sum_{i=1}^{N} E[x_i^2] = \frac{1}{N}\sum_{i=1}^{N}\sigma^2_{x(t)} = \underline{\sigma^2_{x(t)}} \qquad \boxed{\text{NEPRISTRAN}}$$

sr. vr. od x je 0, pa je $\boxed{\sigma^2_{x(t)} = E[x_i^2] - 0}$

$$E[(\hat{\sigma}^2_{x(t)})^2] = E\left[\left(\frac{1}{N}\sum_{i=1}^{N} x_i^2\right)^2\right] = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E[x_i^2 x_j^2]$$

$$\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1N} \\ x_{21} & x_{22} & \dots & x_{2N} \\ \vdots & & & \\ x_{N1} & x_{N2} & \dots & x_{NN} \end{bmatrix}$$

ima ih N, a ostalih $N^2 - N$

VRIJEDI ZA GAUSSOVU RAZDIOBU:

$$\begin{cases} i \ne j \Rightarrow E[x_i^2 x_j^2] = E[x_i^2]E[x_j^2] = \sigma^2_{x(t)} \\ i = j \Rightarrow E[x_i^2 x_j^2] = E[x_i^4] = 3\sigma_x^2 \end{cases}$$

$$E[(\hat{\sigma}^2_{x(t)})^2] = \frac{1}{N^2}\left[N \cdot 3\sigma^2_{x(t)} + (N^2 - N)\,\sigma^4_{x(t)}\right] = \frac{2}{N}\sigma^4_{x(t)} + \sigma^4_{x(t)}$$

$$\sigma^2_{\hat{\sigma}^2_{x(t)}} = \frac{2}{N}\sigma^4_{x(t)} + \sigma^4_{x(t)} - \sigma^4_{x(t)} = \frac{2}{N}\sigma^4_{x(t)}$$

$$\sigma_{\hat{\sigma}^2_{x(t)}} = \sqrt{\frac{2}{N}\sigma^4_{x(t)}} \leq 0.005\,\sigma^2_{x(t)}$$

$$\Downarrow$$

$$\frac{2}{N}\cancel{\sigma^4_{x(t)}} \leq 2.5\cdot 10^{-5}\,\cancel{\sigma^4_{x(t)}}$$

$$\boxed{N \geq \frac{2}{2.5\cdot 10^{-5}} = 80000}$$

$\Rightarrow$ POTREBNO JE BAREM 80000 uzoraka da std.dev. ne prelazi 0.5% sr.vr. $\sigma^2_{x(t)}$)

b) zd. 13.6., str. 244.

$E[x(t)] = \mu$

$x(t)$ otipkamo u N vrem. tr. t_i, $i=1..N$

$$\hat{\sigma}^2_{x(t)} = \frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2$$

PRISTRANOST)

$$E[\hat{\sigma}^2_{x(t)}] = E\left[\frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2\right] = \frac{1}{N}\sum_{i=1}^{N}E[(x_i-\mu)^2] = \frac{1}{N}\sum_{i=1}^{N}E[x_i^2-2\mu x_i+\mu^2]$$

$$= \frac{1}{N}\sum_{i=1}^{N}\left[E[x_i^2]-2E[\mu x_i]+E[\mu^2]\right]$$

$$= \frac{1}{N}\sum_{i=1}^{N}\left(E[x_i^2]-2\mu\underbrace{E[x_i]}_{\mu}+\mu^2\right)$$

$$= \frac{1}{N}\sum_{i=1}^{N}\underbrace{\left(E[x_i^2]-\mu^2\right)}_{\sigma^2_{x(t)}} = \frac{1}{N}\sum_{i=1}^{N}\sigma^2_{x(t)} = \underline{\underline{\sigma^2_{x(t)}}}$$

$$\Rightarrow \boxed{E[\hat{\sigma}^2_{x(t)}] = \sigma^2_{x(t)}}$$ NEPRISTRAN

c) pr. 13.24, str. 244

$x_1, \dots, x_N$ → UZORCI POISSONOVE SLUČ. VAR. X S NEP. PAR. λ

$$\lambda_1 = \frac{1}{N}\sum_{i=1}^{N} x_i \quad , \quad \lambda_2 = \frac{1}{2}(x_1 + x_2)$$

PRISTRANOST, bolji estimator?

$$E[\lambda_1] = E\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right] = \frac{1}{N}\sum_{i=1}^{N} E[x_i] = \lambda \qquad \boxed{\text{NEPRISTRAN}}$$

$$E[\lambda_2] = E\left[\frac{1}{2}(x_1+x_2)\right] = \frac{1}{2}\left(E[x_1] + E[x_2]\right) = \frac{1}{2}(\lambda+\lambda) = \lambda \qquad \boxed{\text{NEPRISTRAN}}$$

$$\sigma_{\lambda_1}^2 = E[\lambda_1^2] - E^2[\lambda_1] = E\left[\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)^2\right] - \lambda^2$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E[x_i x_j] - \lambda^2 = \frac{1}{N^2}\left[N(\lambda^2+\lambda) + (N^2-N)\lambda^2\right] - \lambda^2$$

$$= \lambda^2 + \frac{\lambda}{N} - \lambda^2 = \frac{\lambda}{N}$$

$$\sigma_{\lambda_2}^2 = E[\lambda_2^2] - E^2[\lambda_2] = E\left[\frac{1}{4}(x_1+x_2)^2\right] - \lambda^2$$

$$= \frac{1}{4}\left[E[x_1^2] + E[x_2^2] + 2E[x_1]E[x_2]\right] - \lambda^2$$

$$= \frac{1}{4}\left[\lambda^2 + \lambda + \lambda^2 + \lambda + 2\lambda\cdot\lambda\right] - \lambda^2 = \frac{\lambda}{2}$$

BOLJI JE ESTIMATOR S MANJOM VARIJANCOM

⇒ λ_1 ESTIMATOR ⟶ bolji $\forall\, N > 2$

3. FUNKCIJA VJEROJATNOSTI $L(\Theta)$ → zajednička f-ja gustoće vj. sluč. var. $X_1, \dots, X_n$!

$$L(\Theta) = f_{X_1 X_2 \dots X_n}(x_1, x_2, \dots, x_n; \Theta)$$

a ako je za neku danu realizaciju $x = (x_1, x_2, \dots x_n)$ $\Theta^*(x_1, x_2, \dots, x_n)$ vr. koja maksimizira $L(\Theta)$ ⇒ onda je Θ^* ML (maximum likelihood) estimacija od Θ, a:

$$\boxed{\hat{\Theta} = \Theta^*(X_1, X_2, \dots, X_n)}$$ ML ESTIMATOR PARAMETRA Θ

$(X_1, X_2, \dots, X_n)$ ⇒ sluč. uzorak norm. sl. var. X

ML est. sr. vr. μ, pristranost?

$$L(\mu) = f_{X_1 X_2 \dots X_n}(x_1, x_2, \dots x_n, \mu) = \prod_{i=1}^{N} f_X(x_i; \mu)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\Rightarrow L(\mu) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \cdot e^{-\frac{\sum_{i=1}^{N}(x_i-\mu)^2}{2\sigma^2}}$$

$$\log(L(\mu)) = \log(2\pi\sigma^2)^{-\frac{N}{2}} + \frac{-\sum_{i=1}^{N}(x_i-\mu)^2}{2\sigma^2}$$

$$= -\frac{N}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2$$

$$\frac{\partial \log L(\mu)}{\partial \mu} = 0 \Rightarrow \sum_{i=1}^{N}(x_i - \mu) = 0$$

$$\sum_{i=1}^{N} x_i - N\mu = 0$$

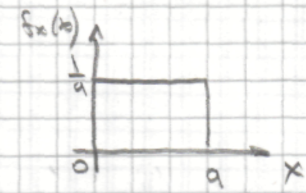
$$\Downarrow$$

$$\hat{\mu} = \mu^* = \frac{1}{N}\sum_{i=1}^{N} X_i$$

$$E[\hat{\mu}] = E\left[\frac{1}{N}\sum_{i=1}^{N} X_i\right] = \frac{1}{N}\sum_{i=1}^{N} E[X_i] = \underline{\underline{\mu}}$$ NEPRISTRAN

4.

$$f_x(x) = \begin{cases} \frac{1}{a}, & 0 \le x \le a \\ 0, & \text{inače} \end{cases}$$



$\hat{a} = \frac{2}{N} \sum_{i=1}^{N} x_i$ $\qquad \sigma_a \le 0{,}001\, a$

$\lim_{N\to\infty} E[\hat{a}] = a$

$\lim_{N\to\infty} Var[\hat{a}] = \lim_{N\to\infty} \left[E[\hat{a}^2] - E^2[\hat{a}] \right] = 0$

NEPRISTRANOST, konzistentnost, N

$E[\hat{a}] = E\left[\frac{2}{N}\sum_{i=1}^{N} x_i\right] = \frac{2}{N}\sum_{i=1}^{N} E[x_i]$ $\qquad \Rightarrow E[\hat{a}] = \frac{2}{N}\sum_{i=1}^{N}\frac{a}{2} = \underline{\underline{a}}$ $\boxed{\text{NEPRISTRAN}}$

$E[x_i] = \int_0^a x \cdot \frac{1}{a}\, da = \frac{1}{a} \cdot \frac{x^2}{2}\Big|_0^a = \frac{1}{a}\cdot\frac{a^2}{2} = \frac{a}{2}$

$\lim_{N\to\infty} E[\hat{a}] = \lim_{N\to\infty} a = \underline{\underline{a}}$ $\qquad \Rightarrow$ $\underline{E[\hat{a}] \text{ ne ovisi o } N}$

$\lim_{N\to\infty} \left[E[\hat{a}^2] - E[\hat{a}]^2 \right] = \lim \left[E[\hat{a}^2] - a^2 \right]$

$E[\hat{a}^2] = E\left[\frac{4}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j\right] = \frac{4}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E[x_i x_j]$

$$\begin{cases} i \ne j & E[x_i x_j] = E[x_i]\, E[x_j] = \frac{a}{2}\cdot\frac{a}{2} = \frac{a^2}{4} \\ i = j & E[x_i x_i] = \int_0^a x^2 \frac{1}{a}\, dx = \frac{1}{a}\frac{x^3}{3}\Big|_0^a = \frac{a^2}{3} \end{cases}$$

$E[\hat{a}^2] = \frac{4}{N^2}\left[N\frac{a^2}{3} + (N^2 - N)\frac{a^2}{4} \right] = \frac{4}{N^2}\left[N\frac{a^2}{12} + \frac{N^2}{4}a^2 \right] = \frac{a^2}{3N} + a^2$

$\lim_{N\to\infty} \left[E[\hat{a}^2] - E^2[\hat{a}] \right] = \lim_{N\to\infty}\left[\frac{a^2}{3N} + \cancel{a^2} - \cancel{a^2} \right] = \lim_{N\to\infty}\frac{a^2}{3N} = \underline{\underline{0}}$ $\boxed{\text{KONZISTENTAN}}$

$\sigma_{\hat{a}}^2 = E[\hat{a}^2] - E^2[\hat{a}] = \frac{a^2}{3N}$

$\sigma_{\hat{a}} = \frac{a}{\sqrt{3N}} \le 0{,}001 \cdot a$

$3N \ge 10^6$

$N \ge 333\,333{,}\dot{3} \quad \Rightarrow \quad \boxed{N \approx 333\,334}$

5.