13.1. NEPRAVI INTEGRALI

- def. „NEPRAVI INTEGRALI PRVE VRSTE"

Neka je $a \in \mathbb{R}$ i neka je f integrabilna na $[a,b]$, za $\forall b < \infty$. Tada definiramo

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty} \int_a^b f(x)\,dx$$

ako taj limes postoji i konv

$\longrightarrow$ Tada kažemo da integral KONVERGIRA. Inače, ako limes ne postoji ili je ∞, integral DIVERGIRA.

Analogno:

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b f(x)\,dx$$

13. DZ - 1.)

$$\int_0^\infty \frac{x\,dx}{x^2+1} = \left| \begin{array}{l} x^2+1 = t \\ 2x\,dx = dt \\ x=0 \to \alpha = 1 \\ x=\infty \to \beta = \infty \end{array} \right| = \frac{1}{2} \int_1^\infty \frac{dt}{t} = \frac{1}{2} \ln t \Big|_1^\infty = \frac{1}{2}(\ln(\infty) - \ldots$$

$$= \frac{1}{2}(\infty - 0) = \infty \; \longrightarrow \text{DIVERGIRA}$$

DEK. - 11. - 4.)

$$\int_0^\infty \frac{x\,dx}{x^4+1} = \left| \begin{array}{l} x^2 = t \\ 2x\,dx = dt \\ x=0 \to \alpha = 0 \\ x=\infty \to \beta = \infty \end{array} \right| = \frac{1}{2} \int_0^\infty \frac{dt}{t^2+1} = \frac{1}{2} \operatorname{arctg} t \Big|_0^\infty = \frac{\pi}{4}$$

konvergira

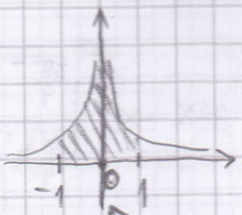
*

Ispravnije je pisati:

$$\lim_{b\to\infty} \operatorname{arctg} t \Big|_0^b = \lim_{b\to\infty} \operatorname{arctg} b - 0$$

U POZADINI PRIČE SU LIMESI !!

$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^{1} = -1-1 = -2 \quad ?$$



TVZ

NIJE U DOMENI!

NE !!

IMAMO PREKID !!

„NEPRAVI INTEGRALI DRUGE VRSTE"

Neka je f integrabilna funkcija na $[a, c-\varepsilon]$ i $[c+\delta, b]$, za $\forall \varepsilon, \delta > 0$, te neka nije omeđena u $c \in [a,b]$. Tada definiramo:

$$\int_a^b f(x)\,dx := \lim_{\varepsilon\to 0} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\delta\to 0} \int_{c+\delta}^{b} f(x)\,dx$$

Ako taj limes postoji i ako je konačan, kažemo da integral konvergira.

Pr.)
$$\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0^-} \frac{dx}{x^2} + \int_{0^+}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^{0^-} + \left(-\frac{1}{x}\right)\Big|_{0^+}^{1} = +\infty - 1 + (-1) + \infty =$$

$= \infty \;\; \rightarrow$ DIVERGIRA !

PAZI !!

LIMES U POZADINI PRIČE:

$$\lim_{x\to 0^-} \left(-\frac{1}{x}\right) = +\infty$$

ZZ-6.)
$$\int_0^{\pi} tg^2(2x)\,dx \overset{TVZ}{=} \left(\frac{1}{2} tg\, 2x - x\right)\Big|_0^{\pi} = -\pi$$

$$\int \frac{\sin^2 2x}{\cos^2 2x}dx = \int \frac{1-\cos^2 2x}{\cos^2 2x}dx =$$
$$= \frac{1}{2} tg\, 2x - x + C$$

NE !

UVIJEK PROVJERITI IMA LI FUNKCIJA PREKID !!!

$D(f)$: $2x \neq \frac{\pi}{2} + k\pi \quad |:2$

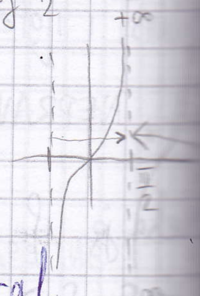
$x \neq \frac{\pi}{4} + \frac{k\pi}{2}$ $\rightarrow$ $\frac{\pi}{4}, \frac{3\pi}{4}$

MUNJA!

$$= \left(\frac{1}{2}\operatorname{tg}2x - x\right)\Big|_{0}^{\frac{\pi}{4}^-} + \left(\frac{1}{2}\operatorname{tg}2x - x\right)\Big|_{\frac{\pi}{4}^+}^{\frac{3\pi}{4}^-} + \left(\frac{1}{2}\operatorname{tg}2x - x\right)\Big|_{\frac{3\pi}{4}^+}^{\pi} =$$

$$= +\infty - \frac{\pi}{4} + \left(+\infty - \frac{3\pi}{4}\right) - \left(-\infty - \frac{\pi}{4}\right) + (-\pi) - \left(-\infty - \frac{3\pi}{4}\right) =$$

$$\lim_{x \to \frac{\pi}{4}^-} \operatorname{tg}2x = \operatorname{tg}\frac{\pi}{2}^- = +\infty$$



$$= \infty - \pi = \infty$$

↳ DIVERGIRA!

$\boxed{\text{TM}}$ „KRITERIJ USPOREDBE"

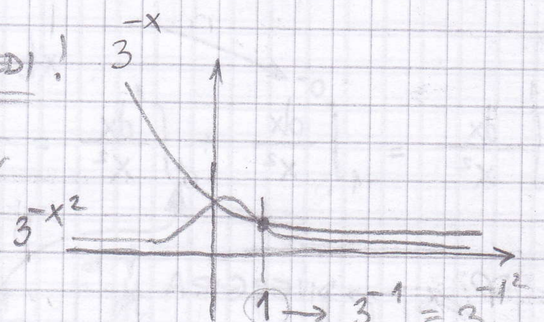
Neka je $|f(x)| \leq g(x)$. Ako konvergira nepravi integral od $g(x)$, tada integral od $f(x)$ koji je također nepravi također konvergira. (za obje vrste)

21-09-07.) Ispitaj konvergenciju:

$$\int_1^{\infty} \cos x \cdot 3^{-x^2}\, dx =$$

VRIJEDI!

$$\Big|\underbrace{\cos x}_{|\cos x| \leq 1}\, 3^{-x^2}\Big| \leq 3^{-x^2} \leq 3^{-x}$$



$1 \to 3^{-1} = 3^{-1^2}$

$$\int_1^{\infty} 3^{-x}\, dx = \frac{-3^{-x}}{\ln 3}\Big|_1^{\infty} = \frac{1}{3\ln 3}$$ konvergira!

Dakle, po kriteriju usporedbe konvergira i $\int_1^{\infty} \cos x\, 3^{-x^2}\, dx$.

$\boxed{\text{TM}}$

Neka je $f(x)$ ekvivalentna s $g(x)$ (kada x teži u "kritičnu točku" $c \to c$). Tada je $\int f(x)dx \sim \int g(x)dx$, tj. ili oba konvergiraju ili oba divergiraju.

$$\int_0^1 \frac{\ln(1+x^2)}{\sqrt[3]{x^{10}}}\,dx$$ → veća funkcija od ove koju znamo integrirati ???

ne znamo

→ ČIM VIDIMO ln ODMAH SE SJETITI NEIZMJERNIH VEL.!

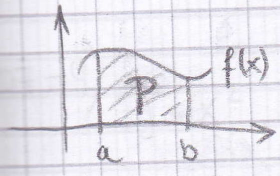
$$\overset{x\to 0}{\sim} \int_0^1 \frac{x^2}{x^{\frac{10}{3}}}\,dx = \int_0^1 x^{-\frac{4}{3}}\,dx = \frac{x^{-\frac{1}{3}}}{-\frac{1}{3}\sqrt[3]{x}}\Bigg|_{0^+}^{1} = -3-(-\infty) = +\infty$$

↓ divergira

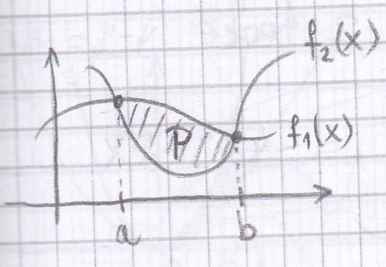
→ pa stoga divergira i početni! (ili oba...) TM

13.2. PRIMJENA INTEGRALA

POVRŠINA LIKA



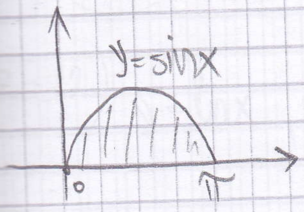
$$\boxed{P = \int_a^b f(x)\,dx}$$



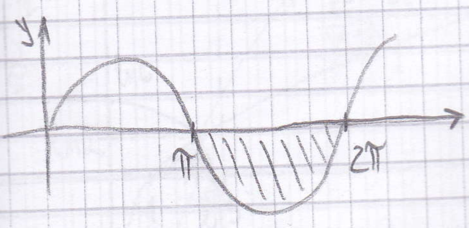
$$\boxed{P = \int_a^b (f_1(x) - f_2(x))\,dx}$$

↳ „GORNJA FUNKCIJA – DONJA FUNKCIJA"

1)



$$P = \int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = 2$$



TVZ

$$P = \int_\pi^{2\pi} -\sin x\,dx \overset{TVZ}{=} \underset{+}{-}2$$ → POVRŠINA NE SMIJE BITI NEGATIVNA!

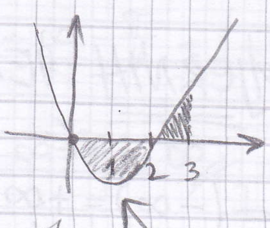
OBAVEZNO NACRTATI!

KADA JE ISPOD X-OSI U INTEG. STAVLJAMO − !!

Zad.) Izračunaj površinu između parabole $y = x^2 - 2x$ i osi x intervalu $[0, 3]$.

TVZ - ovac:

$$\int_0^3 (x^2 - 2x)\,dx = \text{pozitivan !!}$$

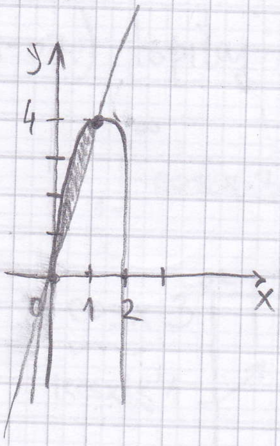


PAZI ! ILKO OBOŽAVA OVAKVE STVARI

$$-\int_0^2 (x^2 - 2x)\,dx + \int_2^3 (x^2 - 2x)\,dx = \dots = DZ$$

ispod x-osi

Z1-09-14.) (2 boda) Izračunaj površinu lika sa slike:



$$P = \int_0^1 (\text{parabola} - \text{pravac})\,dx$$

p.... $y = \underline{4x}$ → pravac kroz 2 točke $y - y_1 =$

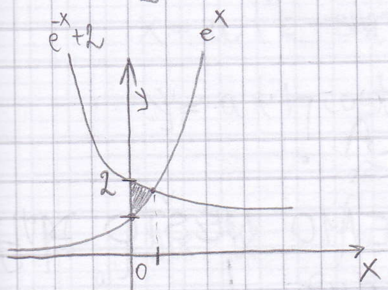
parabola ... $y = \underline{a}(x - x_1)(x - x_2) = -4x^2 + 8x$

$T(1, 4)$

$4 = a(-1)$

$$P = \int_0^1 (-4x^2 + 8x - 4x)\,dx = \frac{2}{3}$$

-8.) Izračunaj površinu lika omeđenog krivuljama $y=e^x$, $y=e^{-x}+2$ i osi y.



$$P=\int_0 (e^{-x}+2-e^{x})\,dx$$

$$e^{x}=e^{-x}+2$$

$$e^{x}-e^{-x}+2=0 \quad / \cdot e^{x}$$

$$e^{2x}-1-2e^{x}=0$$

$$\underline{e^{x}=t}$$

$$t^2-2t-1=0$$

$$t_{1,2}=\frac{2\pm\sqrt{4}}{2}$$

$$t_{1,2}=1\oplus\sqrt{2}$$

e^x ne smije biti negativan!

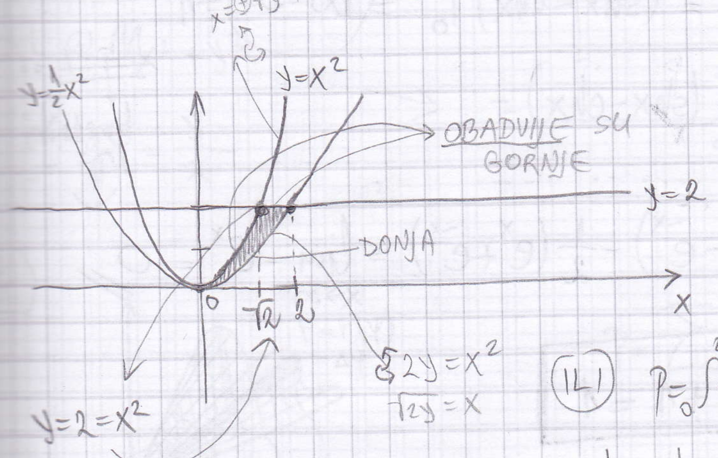
$$\boxed{x=\ln(1+\sqrt{2})}$$

$$P=\int_0^{\ln(1+\sqrt{2})}(e^{-x}+2-e^{x})\,dx=$$

$$=\left(-e^{-x}+2x-e^{x}\right)\Big|_0^{\ln(1+\sqrt{2})}=$$

$$P=2\ln(1+\sqrt{2})+2-2\sqrt{2}$$

-07-9) $y=x^2$, $y=\frac{1}{2}x^2$, $y=2$ (u 1. kvadrantu)



$$P=\int_0^{\sqrt{2}}\left(x^2-\frac{1}{2}x^2\right)dx+\int_{\sqrt{2}}^{2}\left(2-\frac{1}{2}x^2\right)dx$$

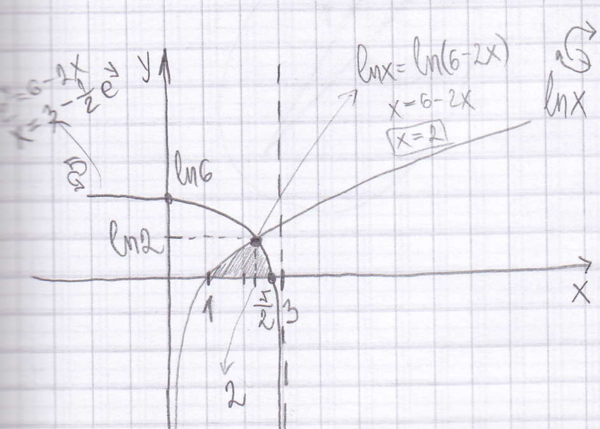
$$\boxed{P=\frac{8}{3}-\frac{4}{3}\sqrt{2}}$$

ILI $P=\int_0^2(\sqrt{2y}-\sqrt{y})\,dy=(\sqrt{2}-1)\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^2=\ldots$

metoda ćorave kokoši?? ne....

DZ-12.) $y=\ln x$, $y=\ln(6-2x)$ i os x

$$y=\ln(-2(3-2x))$$



$$P=\int_1^2\ln x\,dx+\int_2^{\frac{5}{2}}\ln(6-2x)\,dx=3\ln2-\frac{3}{2}$$

parcijala!

ILI OKRENUTI PAPIR!

ROTACIJA OKO y-OSI

$$P=\int_0^{\ln2}\left(3-\frac{1}{2}e^{y}-e^{y}\right)dy=3\ln2-\frac{3}{2}$$

! OKRENUTI CIJELU SLIKU, ROTIRATI I NAĆI INVERZE FUNKCIJA !

→ NE TREBA RASTAVLJATI NA DVA DIJELA !

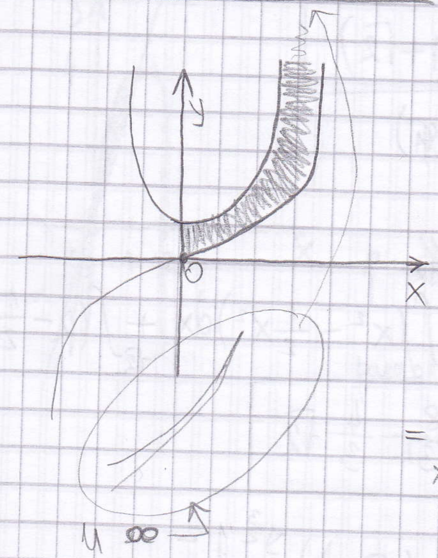
→ UVIJEK AKO NORMALNIM PUTEM DOBIJEMO NEŠTO DIVLJE ONDA OKRENUTI PODRUČJE INTEGRACIJE !

13.DZ-11.) Dokaži da je površina lika omeđena zadanim jednadžbama konačna.

$y \geq shx$

$y \leq chx$

$x \geq 0$



$$P = \int_0^{\infty} (chx - shx)\,dx =$$

$$= (shx - chx)\Big|_0^{\infty} = \infty - \infty + 1$$

NEODREĐENI OBLIK

NLO

limesom

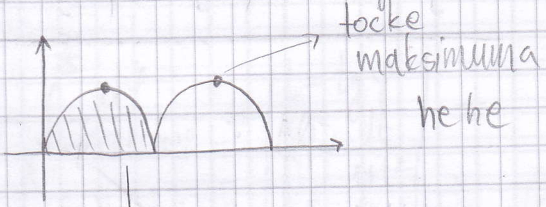
$$\lim_{x \to \infty} (shx - chx) =$$

$$= \lim_{x \to \infty} \frac{1}{2}(e^x - e^{-x}) - \frac{1}{2}(e^x + e^{-x}) = \lim_{x \to \infty} -e^{-x} = 0$$

$$\boxed{P = 0 + 1 = 1}$$

IZ-14.) Izračunaj površinu prvog svoda cikloide ö

$$x = a(t - \sin t)$$
$$y = a(1 - \cos t) \qquad t \in [0, 2\pi]$$



he he

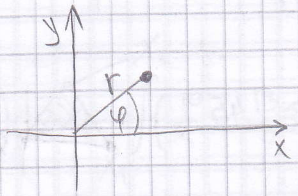
SISOIDA !!!

$$P = \int_a^b y(x)\,dx = \int_0^{2\pi} \underbrace{a(1-\cos t)}_{y}\, \underbrace{a(1-\cos t)}_{x'}\,dt$$

$$P = a^2 \int_0^{2\pi} (1 - \cos 2t + \underbrace{\cos^2 t}_{\frac{1}{2}(1+\cos 2t)})\,dt = 3\pi a^2$$

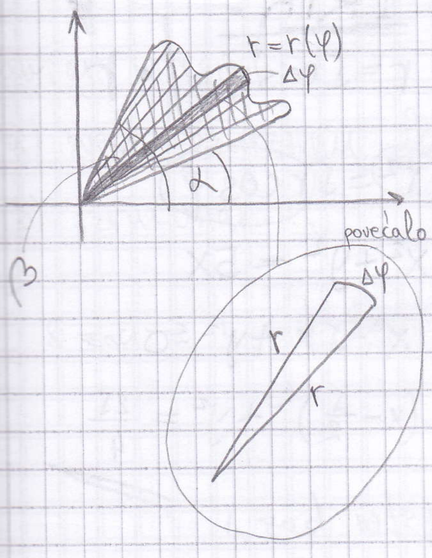
POVRŠINA U POLARNIM KOORDINATAMA

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$



$$r^2 = x^2 + y^2$$

$$\operatorname{tg}\varphi = \frac{y}{x}$$



$$P = \sum P_{k.i.} = \sum \frac{1}{2} r^2 \Delta\varphi \quad \Big| \lim_{x\to\infty}$$

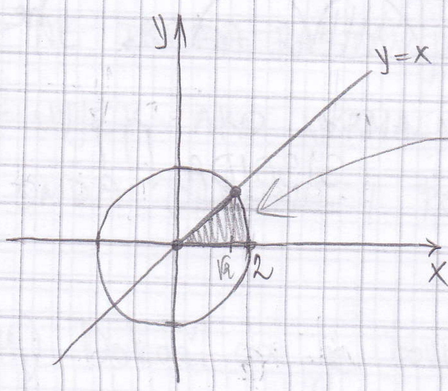
$$P_{k.i.} = \frac{1}{2} r^2 \Delta\varphi$$

kružni isječak

$$\boxed{P = \frac{1}{2}\int_\alpha^\beta r^2 \Delta\varphi}$$

Pr.) Izračunaj površinu lika između krivulja:

$y=\sqrt{4-x^2}$, $y=x$, $y=0$



$y^2=4-x^2$

$x^2+y^2=4 \to$ kružnica, r=2

$y=\sqrt{4-x}$

$\boxed{y>0}$

NE I DOLJE! ZBOG KORJENA

1. NAČIN:

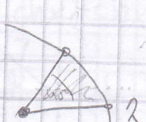
$\longrightarrow$ gornje su: $y=x$ i $y=\sqrt{4-x^2}$

TVZ $\boxed{x=\sqrt{2}}$

$$P=\int_0^{\sqrt{2}} x\,dx+\int_{\sqrt{2}}^{2}\sqrt{4-x^2}\,dx=\left|\begin{array}{c}x=2\sin t\\ \vdots\end{array}\right| \text{ DZ } \quad \frac{\pi}{2}$$

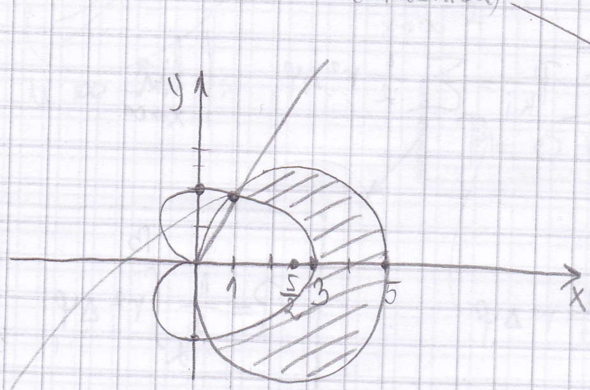
II. NAČIN:

$\frac{\pi}{4} \to 45°$



$$P=\frac{1}{2}\int_0^{\frac{\pi}{4}} 4\,d\varphi=\frac{\pi}{2}$$

13. ZZV – 24.) Ip lok $\overset{\text{unutar}}{}$ $r=5\cos\varphi$ (kružnica) i $\overset{\text{izvan}}{}$ $r=2+\cos\varphi$ (kardioida). Slika!



TRIK:

$r=5\cos\varphi \quad / \cdot r$

$x^2+y^2 = r^2=5\,r\cos\varphi = x$

$x^2+y^2=5x$

$x^2-5x+y^2=0$

$\left(x-\frac{5}{2}\right)^2+y^2=\frac{25}{4}$

$5\cos\varphi=2+\cos\varphi$

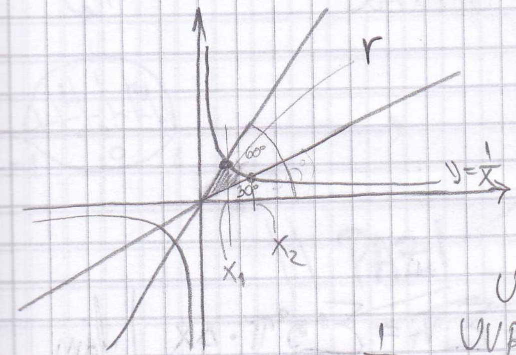
$\cos\varphi=\frac{1}{2}$

$\varphi=\frac{\pi}{3}$

$$P=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \underbrace{(5\cos\varphi)^2}_{\text{DALJNJI } r}-\underbrace{(2+\cos\varphi)^2}_{\text{BLIŽI } r}\,d\varphi$$

$$=\underbrace{\frac{1}{2}\cdot 2\cdot}_{\text{simetrično}}\int_0^{\frac{\pi}{3}}(25\cos^2\varphi-4-4\cos\varphi-\cos^2\varphi)\,d\varphi=\dots=\frac{8\pi}{3}+\sqrt{3}$$

13.) Pravci koji sa osi x zatvaraju 30° i 60° sijeku hiperbolu $\frac{1}{x}$ u prvom kvadrantu. Površina?



$$P=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{d\varphi}{\sin\varphi\cos\varphi}=\left|\begin{array}{l} t=\operatorname{tg}\varphi \\ d\varphi=\frac{dt}{1+t^2} \\ \operatorname{tg}\frac{\pi}{6}=\frac{\sqrt{3}}{3}\ldots \end{array}\right|=$$

$$=\frac{1}{2}\int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}}\frac{\frac{dt}{1+t^2}}{\frac{t}{\sqrt{1+t^2}}\cdot\frac{1}{\sqrt{1+t^2}}}=\frac{1}{2}\int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}}\frac{dt}{t}=$$

$$=\frac{1}{2}\ln t\Big|_{\frac{\sqrt{3}}{3}}^{\sqrt{3}}=\frac{1}{2}\ln 3$$

UVIJEK UVRSTI POLARNE

$$y=r\sin\varphi=\frac{1}{r\cos\varphi}$$

$$r^2=\frac{1}{\sin\varphi\cos\varphi}$$

II: {KARTEZIJEV}

$k=\operatorname{tg}\varphi$

$$P=\int_0^{x_1}\left(\sqrt{3}x-\frac{\sqrt{3}}{3}x\right)dx+\int_{x_1}^{x_2}\left(\frac{1}{x}-\frac{\sqrt{3}}{3}x\right)dx$$

→ neće biti uopće golih integrala, sve u primjeni

eventualno kombinirano sa teorijom: a) def. određeni int.
b) izračunaj $\int_1^3$

običan integral:
a) izvedi rekurzivnu formulu
b) izračunaj ...

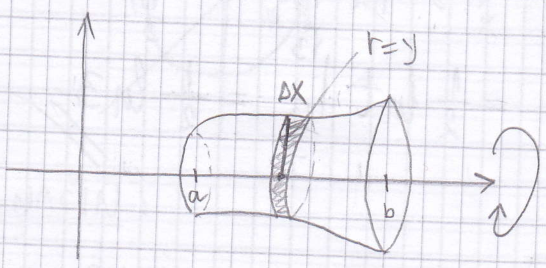
8 zad → 4 zad derivacije → 1. graf, 2. Taylor, 3. ekstremi zadatak s riječima, 4. teorija → ili teorija u komb. sa tangentama

4 zad integrala:
1. teorija (parcij.) Newton-Leibniz
2, 3, 4 → primjena ⇒ površina sigurno, duljina luka ...

pitanja općenito → 1) lin. nezavisnost, rang o parametru → sigurno
→ 2) funkcije – pomaci, crtanje.
→ 1. ciklus → OSNOVNE DEF. neće biti dokaza

B) VOLUMENI ROTACIONIH TIJELA

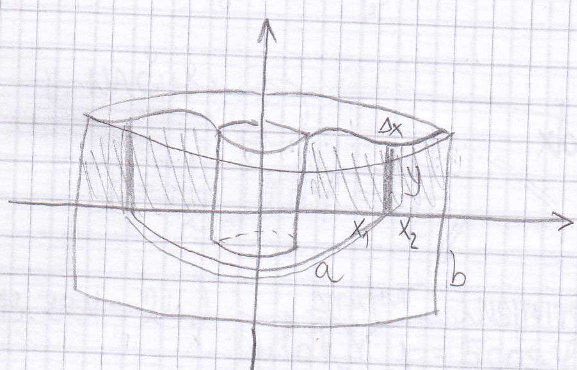
① ROTIRANJE OKO OSI X



$V = \sum V_i = \sum y^2\pi \cdot \Delta x \quad \Big| \lim_{n\to\infty}$

$$\boxed{V = \pi \int_a^b y^2 dx}$$

⑪ ROTIRANJE OKO OSI y

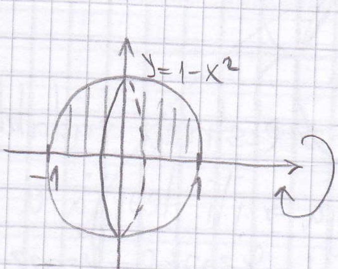


$V = \sum V_i = 2 \sum y \cdot \pi (x_2^2 - x_1^2)^2 \quad \Big| \lim_{n\to\infty}$

$$\boxed{V = 2\pi \int_a^b xy\, dx}$$

13. DZ – 16.) $y = 1 - x^2$ i osi x

a) rotiramo oko osi x, V = ?

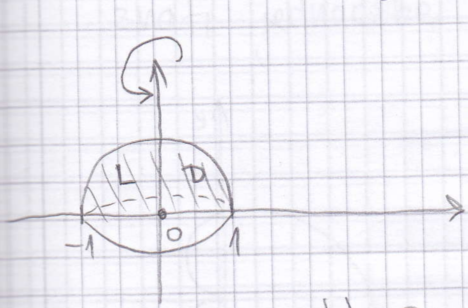


→ POANTA: UVRSTITI U FORMULU

$V = \pi \int_{-1}^{1} (1-x^2)^2 dx = \pi \int_{-1}^{1} (1 - 2x^2 + x^4) dx = \dots$

$$\boxed{V = \frac{16\pi}{15}}$$

rotiramo oko osi y



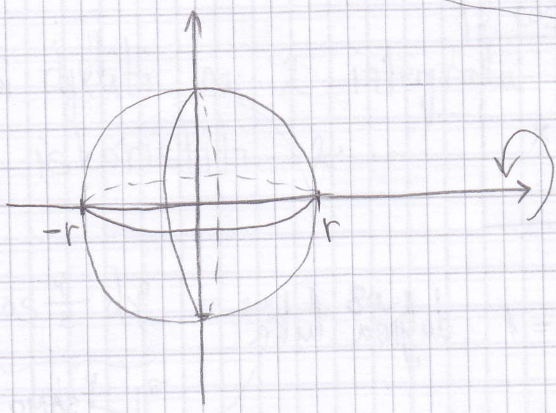
$$V = 2\pi \int_{\cancel{-1}\ 0}^{1} x(1-x^2)\,dx$$

!! PAZI → kada rotira L ide u D i obratno pa od -1 do 1 je zapravo dupli volumen !!

2008.–7.) Izvedi formulu za volumen kugle polumjera r.

→ mora rotira kružnica polumjera r

$$x^2 + y^2 = r^2 \rightarrow y^2 = r^2 - x^2$$



$$V = \pi \int_{-r}^{r} (r^2 - x^2)\,dx =$$

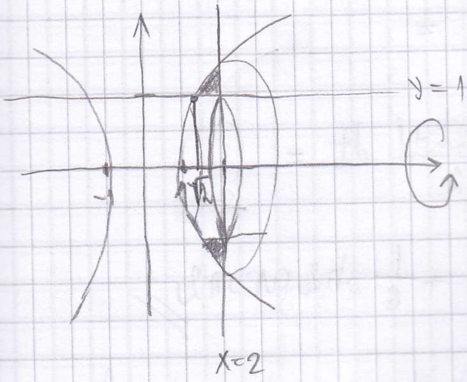
$$V = \pi \left(r^2 x - \frac{x^3}{3} \right) \Big|_{-r}^{r}$$

$$\boxed{V = \frac{4}{3} r^3 \pi}$$

07–9.) Dio krivulje $x^2 - y^2 = 1$, $y=1$, $x=2$ (omeđen, I. kvadrant) rotira oko osi x. Volumen?

$x^2 - y^2 = 1$ → HIPERBOLA !

BITNO JE ZNATI FORMUL



PAZI !

$$V = \pi \int_{\cancel{1}\ \sqrt{2}}^{2} \left((x^2 - 1) - 1 \right) dx \quad \boxed{y^2_{GORNJA} - y^2_{DONJA}}$$

$y = 1$, $y^2 = x^2 - 1$

$1 = x^2 - 1$

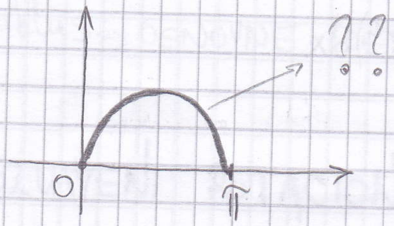
$x = \sqrt{2}$

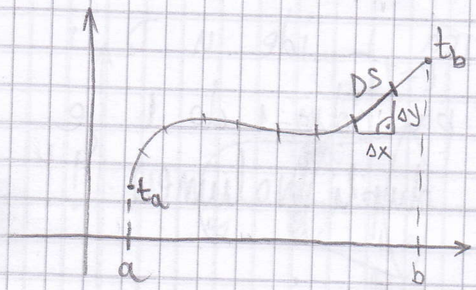
$$V = \pi \left(\frac{x^3}{3} - 2x \right) \Big|_{\sqrt{2}}^{2}$$

$$\boxed{V = \frac{4\pi}{3}(\sqrt{2} - 1)}$$

!

c) DULJINA LUKA





$$S = \sum \Delta S = \sum \sqrt{(\Delta x)^2 + (\Delta y)^2} \cdot \frac{\Delta t}{\Delta t} =$$

$$= \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \cdot \Delta t \quad \Big/ \lim_{n \to \infty}$$

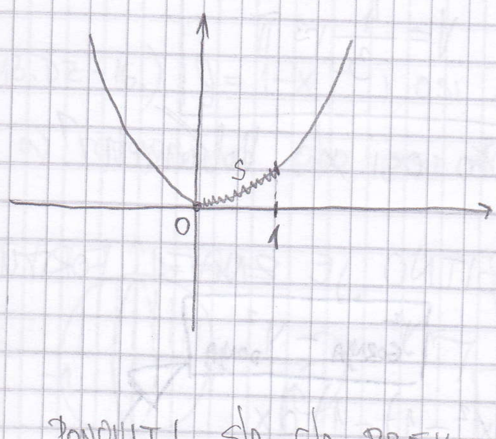
parametarski zadane $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$

$$\boxed{S = \int_{t_a}^{t_b} \sqrt{(\dot{x}^2) + (\dot{y})^2}\, dt}$$

za $y = y(x)$:

$$\boxed{S = \int_a^b \sqrt{1 + (y'(x))^2}\, dx}$$

13. DZ-18.) $y = x^2$ od $x = 0$ do $x = 1$, duljina luka



SAMO UVRŠTAVANJE U FORMULU !!

$$S = \int_0^1 \sqrt{1 + 4x^2}\, dx = \left| \begin{array}{l} x = \frac{1}{2}\operatorname{sh} t \\ dx = \frac{1}{2}\operatorname{ch} t\, dt \\ x = 0 \to \alpha = 0 \\ x = 1 \to \beta = \operatorname{arsh} 2 \end{array} \right|$$

$$S = \frac{1}{2} \int_0^{\operatorname{arsh} 2} \operatorname{ch}^2 t\, dt =$$

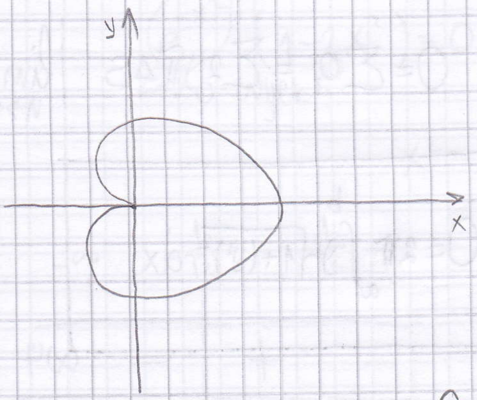
$$= \frac{1}{2} \int_0^{\operatorname{arsh} 2} \frac{\operatorname{ch} 2t + 1}{2}\, dt =$$

$$= \frac{1}{4} \operatorname{arsh} 2 + \frac{1}{8} \operatorname{sh} 2 \operatorname{arsh} 2 \;//$$

↳ u službenim rj. preko ln-ova

PONOVITI sh, ch PREKO ln-ova !

7) Izračunaj opseg kardioide ($r=a(1+\cos\varphi)$). (ne treba napamet znati jednadžbe kardioide, astroide... pisat će nam!)



$$\left.\begin{array}{l} x=r\cos\varphi \\ y=r\sin\varphi \end{array}\right\} S=\int\sqrt{(r'\cos\varphi-r\sin\varphi)^2+(r'\sin\varphi+r\cos\varphi)^2}$$

$$\Rightarrow \boxed{S=\int_{\alpha}^{\beta}\sqrt{r^2+(r')^2}\,d\varphi}$$

$$S=\int_0^{2\pi}\sqrt{a^2(1+2\cos\varphi+\cos^2\varphi)+a^2\sin^2\varphi}\,d\varphi=$$

$$=a\int_0^{2\pi}\sqrt{2+2\cos\varphi}\,d\varphi=$$

PAZI !!!

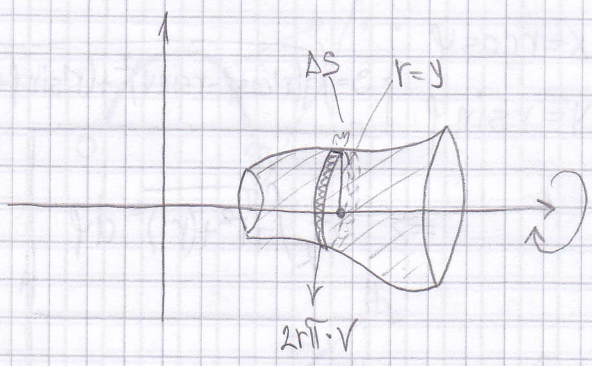
$$=a\sqrt{2}\int_0^{2\pi}\sqrt{1+\cos\varphi}\,d\varphi=2a\int_0^{2\pi}\left|\cos\frac{\varphi}{2}\right|d\varphi=$$

$$\sqrt{1+\cos\varphi}=\sqrt{2\cos^2\frac{\varphi}{2}}$$

→ trebali bi dijeliti na 2 integrala, ali je SIMETRIČNO pa možemo iskoristiti 2·...

$$=2a\cdot 2\underbrace{\int_0^{\pi}\cos\frac{\varphi}{2}\,d\varphi}_{\frac{\sin\frac{\varphi}{2}}{\frac{1}{2}}\Big|_0^{\pi}}=\dots=8a$$

D) OPLOŠJE ROTACIJSKIH TIJELA

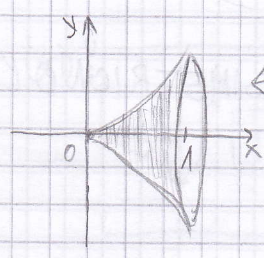


$$O = \sum O_i = \sum 2y\pi \Delta S \quad \Big| \lim_{n\to\infty}$$

$$\boxed{O = 2\pi \int_a^b y\sqrt{1+(y')^2}\,dx}$$

13.ZZV-33.) Izračunaj oplošje tijela koje rotira $y=\frac{x^3}{3}$ za $x\in[0,1]$.

$$O = 2\pi\int_0^1 \frac{x^3}{3}\sqrt{1+x^4}\,dx = \left| \begin{matrix} 1+x^4 = t \\ 4x^3dx = dt \end{matrix} \right| = \frac{\pi}{6}\int_0^1 \sqrt{t}\,dt = \dots = \frac{\pi}{9}(2\sqrt{2}-1)$$



BUREK TORNADO !!!

ZADACI KOJE VRLO VJEROJATNO MOŽEMO OČEKIVATI :

PZI-08-14.) Nacrtaj kvalitativni graf funkcije i izračunaj površinu koju zatvara sa svojim asimptotom : $f(x) = xe^{-x^2}$.

→ ILKO ŽIVI ZA OVO, NA ROKU SIGURNO OVAKAV NEKI ZADATAK KOJI SUMIRA SVE !

1.) $D_f = \mathbb{R}$

2.) asimptote : VA → nema

NLO

HA → $\lim_{x\to+\infty}(xe^{-x^2}) = (\infty\cdot 0) = \lim_{x\to+\infty}\frac{x}{e^{x^2}} = \left(\frac{\infty}{\infty}\right) \overset{L'H}{=}$

$= \lim_{x\to+\infty}\frac{1}{e^{x^2}\cdot 2x} = \frac{1}{\infty} = 0$ $\boxed{y=0 \text{ je HA}}$

$(-)$ ISTO

s obje strane

Ekstremi

$$f'(x) = e^{-x^2} + x \cdot e^{-x^2} \cdot (-2x) = 0$$

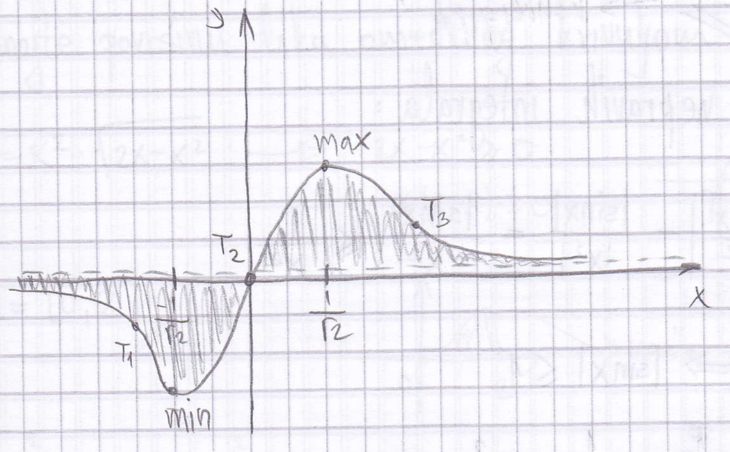
$$e^{-x^2}(1-2x^2) = 0$$

$\hookrightarrow > 0$ uvijek!

$$x^2 = \frac{1}{2} \longrightarrow x = \pm\frac{1}{\sqrt{2}}$$

PAZI!

	$-\infty$	$-\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$	
$f'(x)$	$-$		$+$		$-$
$f(x)$	$\searrow$	min	$\nearrow$	max	$\searrow$



$$P = 2\int_0^\infty x e^{-x^2}\,dx = \left| \begin{array}{c} x^2 = t \\ 2x\,dx = dt \\ \text{granice iste} \end{array} \right| = \int_0^\infty e^{-t}\,dt = -e^{-t}\Big|_0^\infty$$

$$\boxed{P = 1}$$

konzultacije $\rightarrow$ četvrtak!

AUDITORNE VJEŽBE

Zad. Izračunajte ili ustvrdite konvergenciju sljedećeg integrala:

$$\int_a^{\infty} \frac{dx}{x \ln^2 x}, \quad a > 1 .$$

$$\int_a^{+\infty} \frac{dx}{x\ln^2 x} = \lim_{B \to +\infty} \int_a^B \frac{dx}{x \ln^2 x} =$$

$$\int_a^B \frac{dx}{x\ln^2 x} = \left| \begin{array}{l} t = \ln x \\ dt = \frac{1}{x} dx \\ a \to \ln a \\ B \to \ln B \end{array} \right| = \int_{\ln a}^{\ln B} \frac{dt}{t^2} = \frac{-1}{t} \Big|_{\ln a}^{\ln B}$$

$$= -\frac{1}{\ln B} + \frac{1}{\ln a}$$

$$= \lim_{B \to +\infty} \left(\frac{-1}{\ln B} + \frac{1}{\ln a} \right) = \frac{1}{\ln a} \longrightarrow \text{konvergira!}$$

Zad. Ispitajte konvergenciju nepravih integrala:

a)

$$\int_{\frac{\pi}{2}}^{+\infty} \frac{\sin x}{x^2} dx \qquad \left| \frac{\sin x}{x^2} \right| = \frac{|\sin x|}{|x^2|} = \frac{|\sin x|}{x^2} \leq \frac{1}{x^2}$$

$$-1 \leq \sin x \leq 1 \implies |\sin x| \leq 1$$

$$\int_{\frac{\pi}{2}}^{+\infty} \frac{1}{x^2} dx = -\frac{1}{x} \Big|_{\frac{\pi}{2}}^{+\infty} = -\frac{1}{+\infty} + \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

$\longrightarrow$ konvergira!

b)

$$\int_1^2 \frac{1}{\ln x} dx = \int_1^2 \frac{dx}{\ln(1+x-1)} = \left| \begin{array}{l} x-1 = t \\ dx = dt \\ 1 \to 0 \\ 2 \to 1 \end{array} \right| = \int_0^1 \frac{dt}{\ln(1+t)} \sim \int_0^1 \frac{dt}{t} = \ln t \Big|_0^1 =$$

$$= \ln(1+t) \sim t \longrightarrow \text{divergira}$$

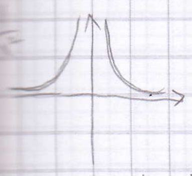
$\ln 0 \to$ NE POSTOJI!

Izračunajte površinu lika omeđenog krivuljom $y=\frac{1}{x^2+2x+3}$ i njezinom asimptotom.

$\longrightarrow$ računamo HA:

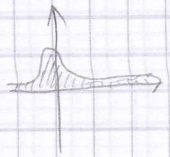
$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{1}{x^2+2x+3} = \dots = 0$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} \frac{1}{(x+1)^2+2}\,dx = \left| \begin{array}{l} x+1=t \\ dx=dt \\ -\infty \to -\infty \\ \infty \to \infty \end{array} \right| = \int_{-\infty}^{+\infty} \frac{dt}{t^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}} \cdot \operatorname{arctg} \frac{t}{\sqrt{2}} \Big|_{-\infty}^{+\infty} =$$



$$= \frac{1}{\sqrt{2}} \left(\operatorname{arctg}(+\infty) - \operatorname{arctg}(-\infty) \right) =$$

$x^2+2x+3=(x+1)^2+2 \longrightarrow$



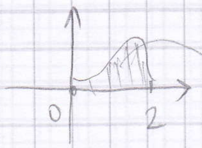
$$= \frac{1}{\sqrt{2}} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = \frac{\pi}{\sqrt{2}}$$

Izračunajte površinu lika omeđenog krivuljom $y=x^2 \cdot \sqrt{2x-x^2}$ i osi x.

$f(x)=x^2 \cdot \sqrt{2x-x^2} \longrightarrow 2x-x^2 \geq 0$

$x(2-x) \geq 0$

$D(f)=[0,2]$



$\rightarrow$ naša funkcija („nebitno" kako izgleda)

$$P = \int_0^2 f(x)\,dx = \int_0^2 \left(x^2 \cdot \sqrt{2x-x^2}\right) dx = \dots = \int_0^2 x^2 \cdot \sqrt{1-(x-1)^2}\,dx = \left| \begin{array}{l} x-1=\sin t \\ dx=\cos t\,dt \\ 0 \to -\frac{\pi}{2} \\ 2 \to \frac{\pi}{2} \end{array} \right| =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t+1)^2 \cdot \cos t \cdot \cos t\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{(\sin t+1)^2}_{\sin^2 t+2\sin t+1} \cos^2 t\,dt =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^2 t\cos^2 t + 2\sin t\cos^2 t + \cos^2 t)\,dt =$$

$\longrightarrow$ neparna na simetričnoj domeni

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t\cos^2 t\,dt + \underbrace{2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin t\cos^2 t\,dt}_{=0} + \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\,dt}_{=DZ=\frac{\pi}{2}} = \frac{\pi}{8} + \frac{\pi}{2} = \frac{5\pi}{8}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t\cos^2 t\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t\sin t)^2\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2}\sin 2t\right)^2 dt = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 2t\,dt = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\cos 4t}{2}\,dt =$$

$$= \frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos 4t)\,dt = \frac{1}{8}\left[t - \frac{1}{4}\sin 4t\right]\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{8}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{\pi}{8}$$

Zad.

$r = a(1+\cos\varphi)$

$P = \frac{1}{2}\int_{\alpha}^{\beta} r^2(\varphi)\,d\varphi = \dots\dots = \frac{3a^2}{2}\pi$

$\int_{-\pi}^{\pi}$ ili od $\int_{0}^{2\pi}$