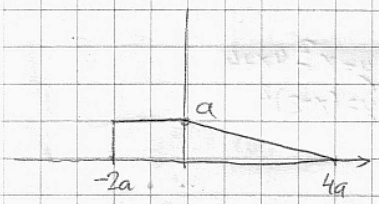
2. MI MASS VIS

(1) Neka je gustoća slučajne varijable x zadana slikom:



a) Odredi konstantu a

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$f(x) = \begin{cases} a & -2a<x<0 \\ a-\frac{1}{4}x & 0<x<4a \end{cases}$$

$$\int_{-2a}^{0} a\,dx + \int_{0}^{4a}\left(a-\frac{1}{4}x\right)dx$$

$$\Rightarrow a = \frac{1}{2}$$

b) očekivanje

$$E(x) = \int_{-\infty}^{\infty} x f(x)dx = \int_{-1}^{0}\frac{1}{2}x\,dx + \int_{0}^{2}\left(\frac{1}{2}-\frac{1}{4}x\right)x\,dx = \frac{1}{12}$$

c) Odredi funkciju razdiobe

$F(x) = ?$

$$f(x) = \begin{cases} \frac{1}{2}, & -1\le x\le 0 \\ \frac{1}{2}-\frac{1}{4}x, & 0\le x\le 2 \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

1.) za $(x<-1) \rightarrow F(x) = \int_{-\infty}^{x} 0\,dx = 0$ //

2.) za $x\in(-1,0)$: $F(x) = \int_{-1}^{x}\frac{1}{2}dx = \frac{1}{2}x+\frac{1}{2}$ //

3.) za $x\in(0,2)$:

$$\cancel{F(x) = \int_{0}^{x}\left(\frac{1}{2}-\frac{1}{4}x\right)dx}$$

Funkcija uvijek ide od $-\infty$!!

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-1}^{0}\frac{1}{2}dx + \int_{0}^{x}\left(\frac{1}{2}-\frac{1}{4}x\right)dx = \frac{1}{2}+\frac{1}{2}x-\frac{1}{8}x^2$$

- za $(x>2)$: $F(x) = 1$ (jer prolazi cijelu funkciju, tj. integral cijelog područja)

d) Izračunaj vjerojatnost da u 14 nezavisnih pokusa, varijabla x točno 3 puta poprimi vrijednost u intervalu $(-a,a)$

1.) Vjerojatnost da u jednom pokusu x poprimi vrijednost iz $(-a,a)$?

$$\Rightarrow P\left(-\frac{1}{2}<x<\frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right)$$

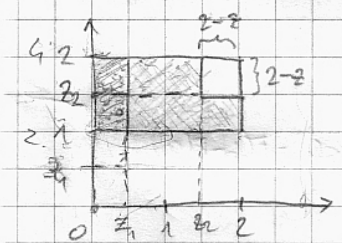
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx = \frac{15}{32}$$

Binomna razdioba jer imamo broj ponavljanja i k ostvarujemo:

$$P = \binom{14}{3}\left(\frac{15}{32}\right)^3\left(\frac{17}{32}\right)^{11}$$

(2) Geometrijski zadatak: Unutar pravokutnika koji je zadan je na sreću odabrana točka t. Neka je slučajna varijabla z manja od koordinata točke t. Odredi funkciju gustoće.

$\{0\le x\le 2,\ 1\le y\le 2\}$ $\quad z = \min(x,y)$



1.) koju vrijednost može poprimiti z? $z\in[0,2]$

2.) odrediti funkciju razdiobe.

$$F(z) = \begin{cases} 0, & \text{za } z<0 \\ 1, & \text{za } z>2 \end{cases}$$

$F(z) = P(Z<z)$ npr. $z=\frac{1}{2}$

→ Gledati da li će interval biti jednak → gdje je prelomna točka? Točka 1

- za $z\in[0,1]$: $F(z) = P(Z<z) = \frac{z\cdot 1}{2\cdot 1} = \frac{z}{2}$

- za $z\in[1,2]$: $F(z) = 1-\frac{(2-z)^2}{\underbrace{2\cdot 1}_{\text{ukupni } \Omega}} = -\frac{1}{2}z^2 - 2z - 1$

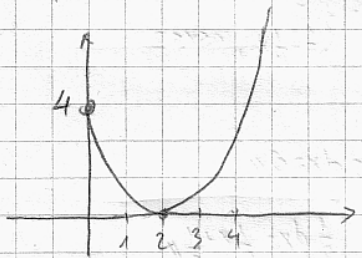
gustoća: $f(z)=F'(z)=\begin{cases}\frac{1}{2}, & z\in(0,1)\\ z+2, & z\in(1,2)\\ 0, & \text{inače}\end{cases}$

3.) X ima eksponencijalnu razdiobu s parametrom 2. Odredi f-ju gustoće i razdiobe od $y=x^2-4x+4$

$y=(x-2)^2$

$X\sim E(2)$ 1.) Nacrtati y

$f(x)=2e^{-2x}$ $(\lambda e^{-\lambda x})$, $x\in(0,\infty)$



$$\boxed{g(y)=f(x)\cdot\left|\frac{dx}{dy}\right|}$$

gustoća $\to \psi^{-1}(y)$

ako je ψ injekcija

Dijelimo na injektivne djelove, napisati koje x poprima vrijednosti

1.) $x\in(0,2)\to y\in(0,4)$

$y=(x-2)^2\to x=2\pm\sqrt{y}$

$\left|\frac{dx}{dy}\right|=\frac{1}{2\sqrt{y}}$

$g_1(y)=2e^{-2(2-\sqrt{y})}\cdot\frac{1}{2\sqrt{y}}$

$\in(0,4)$

2.) $x\in(2,\infty)\to y\in(0,\infty)$

$x=2+\sqrt{y}$

$g_2(y)=2e^{-2(2+\sqrt{y})}\cdot\frac{1}{2y}$

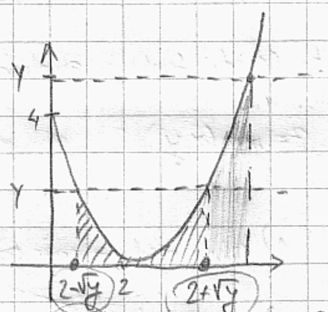
$\in(0,\infty)$

$g(y)=\begin{cases}g_1+g_2, & (0,4)\\ g_2, & (4,\infty)\end{cases}$ ← gdje se preklapaju

→ razdioba - integrirana f-ja gustoće, ili po definiciji (češće)

$G(y)=\int_{-\infty}^{y} g(y)\,dy$

$G(y)=P(Y<y)$



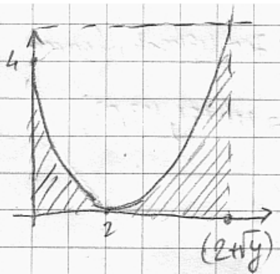
$G(y)=P(Y<y)=P(2-\sqrt{y}<x<2+\sqrt{y})$ ← $P(a<x<b)=F(b)-F(a)$

ne smiju biti fiksne granice

$F(x)=1-e^{-\lambda x}=1-e^{-2x}$

$=F(2+\sqrt{y})-F(2-\sqrt{y})=1-e^{-2(2+y)}-1+e^{-2(2-y)}$

2. interval:



$y \in (4, +\infty)$

$$G(y) = P(Y<y) = P(X< 2+\sqrt{y}) = F(2+\sqrt{y}) = 1-e^{-2(2+\sqrt{y})}$$

(4.) Očekivano vrijeme čekanja filozofa pred Cassandrom iznosi 30 min.

$$E(x)=30=\frac{1}{\lambda} \rightarrow \lambda=\frac{1}{30}$$

a) Vjerojatnost da će čekati više od 15 min? $\rightarrow$ računamo λ

$$P(x>15)=1-F(15)=1-\left(1-e^{-\lambda \times \left(\frac{15}{30}\right)}\right)=e^{-\frac{1}{2}}$$

b) Da će čekati manje od 45 min, ako stoji u redu već 30 min

$$P(x<45 \mid x>30) = \frac{P(30<x<45)}{P(x>30)} = \frac{F(45)-F(30)}{1-F(30)}$$

ILI: $P(x<15)$ zbog odsustva pamćenja $= 1-P(x>15) = 1-e^{-\frac{1}{2}}$

(5.) Raspodjela studentica FERa po grupama $\rightarrow$ normalna razdioba, σ

a) $X \sim N(a, 24)$

$P(x>20)=0,343$

$P(30<x<40)=?$

$P(x>20) = P\left(\tilde{x} > \frac{20-a}{\sqrt{24}}\right)$ Disperzija $=\sigma^2 \rightarrow \sigma=\sqrt{24}$

$P(x>20) = \frac{1}{2}-\frac{1}{2}\phi^*\left(\frac{20-a}{\sqrt{24}}\right)=0,343$

za manji $\rightarrow$ +

$\phi^*\left(\frac{20-a}{\sqrt{24}}\right)=0,314$

$\left(\frac{20-a}{\sqrt{24}}\right)=0,405 \rightarrow a=18$

$P(30<x<40) = P\left(\frac{30-18}{\sqrt{24}}\right) < \tilde{x} < P\left(\frac{40-18}{\sqrt{24}}\right)$

$= \frac{1}{2}\left[\underbrace{\phi^*(4,49)}_{1} - \underbrace{\phi^*(2,45)}_{0,98571}\right] = 0,007$

=

6.) Vjerojatnost da novoupisani student je žensko je 23%. Kolika je vjerojatnost da od 700 upisanih, barem njih 150 bude žensko

$p=23\%$ $n=700$ $\rightarrow X\sim B(700;\,0{,}23) \approx N(np,\, npq) \rightarrow N(161,\, 123{,}97)$

$q \rightarrow (1-p)$

$P(X>150) = \frac{1}{2} - \frac{1}{2}\Phi^*\left(\frac{150-161}{\sqrt{123{,}97}}\right)$

$\doteq$ ako ispod a $(-)$, ide van ispred

$= \frac{1}{2} + \frac{1}{2}\Phi^*(0{,}988) = \frac{1}{2} + \frac{1}{2}\cdot 0{,}67685 = 83{,}8\%$

Slučajni vektori:

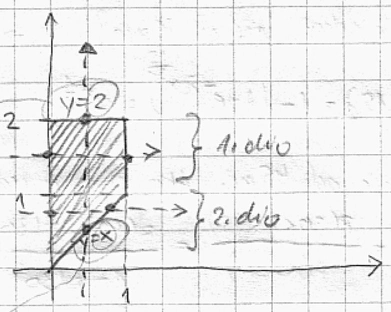
7) Slučajni vektor zadan je funkcijom gustoće $f(x,y) = c(x^2+y)$, $D=\{(x,y) : 0\leq x\leq 1, 0\leq y\leq 2, x\leq y\}$

a) odredi c

$\boxed{\iint f(x,y)\,dx\,dy = 1}$

$\int_0^1 dx \int_x^2 c(y+x^2)\,dy = 1 = c\int_0^1 \left(\frac{y^2}{2} + x^2 y \Big|_x^2\right) dx = \frac{9}{4}c$

$\frac{9}{4}c = 1 \rightarrow c = \frac{4}{9}$



b) Odredi marginalne gustoće

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ ← marginalno x $= \int_x^2 \frac{4}{9}(y+x^2)\,dy = \frac{4}{9}\left(2+\frac{3}{2}x^2+x^3\right)$, za koje x vrijedi !!! $x\in(0,1)$

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$ ← marginalno Y

$f_Y(y) = \int_0^y \frac{4}{9}(y+x^2)\,dx = \frac{4}{9}\left(y^2+\frac{1}{3}y^3\right)$, $y\in(0,1)$

$f_Y(y) = \int_0^1 \frac{4}{9}(y+x^2)\,dx = \frac{4}{9}\left(y+\frac{1}{3}\right)$, $y\in(1,2)$

c) Jesu li varijable nezavisne? Umnožak marginalnih = funkcija gustoće slučajnog vektora

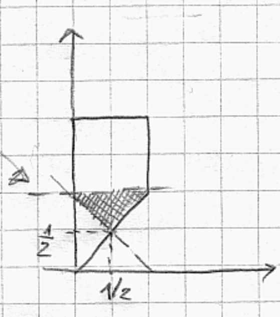
Varijable su zavisne $\rightarrow f(x)\cdot f(y) \neq f(x,y)$

Uvjetna gustoća

d) $f_{Y|X=x}(y) = \frac{f(x,y)}{f(x)} = \frac{\frac{4}{9}(y+x^2)}{\frac{4}{9}\left(2+\frac{3}{2}x-x^3\right)}$

e) Vjerojatnost slučajnog vektora:

$P(X+Y>1 \mid Y<1) = \frac{P(Y>1-X,\, Y<1)}{P(Y<1)}$ ← vjerojatnost presjeka



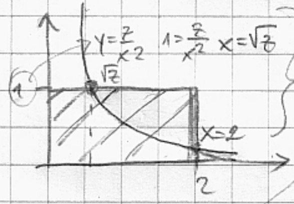
$\boxed{P=\iint_A f(x,y)\,dx\,dy}$ $= \frac{\int_{\frac{1}{2}}^{1} dy \int_{1-y}^{y} \frac{4}{9}(y+x^2)\,dx}{\int_0^1 f_Y(y)\,dy} = \ldots$

$1<Y$

najmanji mogući Y

8.) Slučajni vektor (X,Y) ima razdiobu na pravokutniku $x \in (0,2)$, $y \in (0,1)$. Odredi gustoću varijable $\underline{Z = YX^2}$

$$f(x,y) = \frac{1}{\text{mjera}(D)} = \frac{1}{\text{površina}} = \frac{1}{2}$$



$y = \psi(x,z)$

$$g(z) = \int_{-\infty}^{\infty} f(x,y) \cdot \left|\frac{\partial y}{\partial z}\right| dx$$

$y = \frac{z}{x^2}$

$$\frac{\partial y}{\partial z} = \frac{1}{x^2} \qquad z \in (0,4)$$

$$g(z) = \int_2^{\sqrt{z}} \frac{1}{2} \cdot \frac{1}{x^2}\, dx = -\frac{1}{4} + \frac{1}{2\sqrt{z}}, \quad z \in (0,4)$$

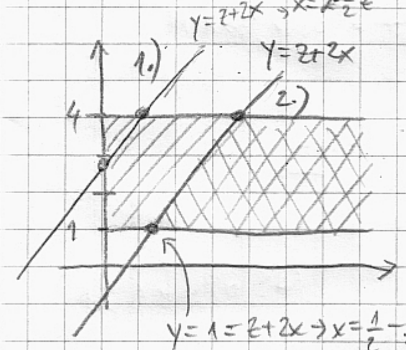
9.) X, Y nezavisne, slučajne varijable. X - eksponencijalna, parametar 3. Y - normalna jednolika (uniformna) $\in (1,4)$

$G(z)$, $g(z) = ?$

$Z = Y - 2X$

$$\left.\begin{aligned} f(x) &= \lambda e^{-\lambda x} = 3e^{-3x}, \quad x>0 \\ f(y) &= \frac{1}{b-a} = \frac{1}{3} \end{aligned}\right\} f(x|y) = f(x)\cdot f(y) = e^{-3x}$$

$Z = Y - 2X \to Y = Z + 2X$
$\to Z \in (-\infty, 4)$



$z \in (1,4)$

1.) $$g(z) = \int_0^{2-\frac{1}{2}z} e^{-3x} \cdot 1\, dx = \underline{\frac{1}{3} - \frac{1}{3} e^{-6+\frac{3}{2}z}}$$

$\left|\frac{\partial y}{\partial z}\right|$

2.) $$\int_{\frac{1}{2}-\frac{1}{2}z}^{2-\frac{1}{2}z} e^{-3x} \cdot 1 \cdot dx = \frac{1}{3} e^{-\frac{3}{2}+\frac{3}{2}z} - \frac{1}{3} e^{-6x}$$

$G(z) = ?$ (po definiciji)

$$G(z) = P(Z<z) = P(Y-2X<z) = \int_1^4 dy \int_{\frac{1}{2}y - \frac{1}{2}z}^{\infty} \underbrace{e^{-3x}}_{f(x,y)} dx = \ldots, \quad z \in (-\infty, 1)$$

$y < z + 2x$