Matematika 2 -> MASS (by Lineo)

Diferencijalne jednadžbe

$$y' = \frac{dy}{dx}$$

1) prepoznati jednadžbu:
   1. svođenje na separaciju:
      - homogene $y' = f\left(\frac{y}{x}\right)$
   2. linearna (i one koje se svode na lin.)
      - Beronullijeva
   3. Egzaktne
      - Eulerov multiplikator
   4. općenite prvog reda (opće i sing. rješenje)
   5. općenite višeg reda
   6. Lin. dif. jed. s konstantnim koef. višeg reda
      a) varijacijom konstanti
      b) pogađanjem partikularnog rješenja
   7. Za više bodova: ZADACI S RIJEČIMA
   8. TEORIJA (To je zad. koji Lineo nije rješavao iz ispita :P)

ZADATCI:

HOMOGENA

1.) $(x+y-3)y' + y = 2x$ (izlučimo y')

$$y' = \frac{2x-y}{x+y-3}$$

$$\left.\begin{aligned} 2x-y&=0 \\ x+y-3&=0 \end{aligned}\right\} +$$

$$3x = 3$$

$$\underline{x_0 = 1} \quad \underline{y_0 = 2}$$

$$u = x - x_0 = x - 1$$

$$v = y - y_0 = y - 2$$

$$v' = \frac{2(u+1)-y-2}{u+1+v+2-3} = \frac{2u-v}{u+v} \begin{matrix} /:u \\ /:u \end{matrix} = \frac{2-\frac{v}{u}}{1+\frac{v}{u}}$$

$$v' = f\left(\frac{v}{u}\right)$$

$$\boxed{\begin{aligned} z &= \frac{v}{u} \\ v' &= z + uz' \end{aligned}}$$

$$z + uz' = \frac{2-z}{1+z}$$

$$uz' = \frac{2-2z-z^2}{1+z}$$

$$u\frac{dz}{du} = \frac{2-2z-z^2}{1+z} \quad /\cdot du \; /:u$$

$$\frac{1+z}{2-2z-z^2}dz = \frac{du}{u} \quad /\int$$

$$\boxed{\begin{aligned} t &= 2-2z-z^2 \\ dt &= -2(z+1)\,dz \\ dz &= \frac{dt}{-2(z+1)} \\ &-\frac{1}{2}\int\frac{dt}{t} \end{aligned}}$$

$$-\frac{1}{2}|2-2z-z^2| = \ln u + \ln c \; /\cdot(-2)$$

$$\ln(2-2z-z^2) = \ln\frac{c}{u^2}$$

$$2-2\left(\frac{v}{u} - \frac{v^2}{u^2}\right) = \frac{c}{u^2} \quad /\cdot u^2$$

$$2u^2 - 2vu - v^2 = c$$

$$\underline{2(x-1)^2 - 2(y-2)(x-1) - (y-2)^2 = c}$$

2.) $$3x^2(1+\ln y) = y'\left(2y - \frac{x^3}{y}\right) \qquad \underline{\text{EGZAKTNA}}$$

$$3x^2(1+\ln y) = \frac{dy}{dx}\left(2y - \frac{x^3}{y}\right) \quad /\cdot dx$$

$$\underbrace{3x^2(1+\ln y)}_{P}dx + \underbrace{\left(-2y+\frac{x^3}{y}\right)}_{Q}dy = 0$$

<u>uvjet:</u>

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{3x^2}{y} = \frac{3x^2}{y} \quad \checkmark$$

$$u(x,y) = \int_{x_0}^{x} P\,dx + \int_{y_0}^{y} Q\,dy$$

- pazi da su x_0 i y_0 u domeni
- x_0 uvrstiš u Q

$$= \int_0^x 3x^2(1+\ln y)dx + \int_1^y \left(\frac{0}{y} - 2y\right)dy$$

$$u(x,y)=x^3(1+\ln y)\Big|_0^x - y^2\Big|_1^y$$

$$=x^3(1+\ln y)-y^2+c$$

Rješenje:

$$\underline{x^3(1+\ln y)-y^2=c}$$

3.) $2yy'=\sin 2x - y^2 \operatorname{tg} x$ (uvjet je: $y(\pi)=0$) **LINEARNA** (Bernullijeva)

$y'+f(x)y=g(x)y^{\alpha}$

$$y'=\frac{1}{2y}\sin 2x-\frac{1}{2}y\operatorname{tg}x$$

$$y'+\frac{1}{2}\operatorname{tg}x\cdot y=\frac{1}{2}\sin 2x\cdot\frac{1}{y}\Big|\cdot 2y \quad (\alpha=-1)$$

supst.: $z=y^{1-\alpha}$

$$\boxed{\begin{array}{c} z=y^2 \\ z'=2yy' \end{array}}$$

$$2yy'+y^2\operatorname{tg}x=\sin 2x$$

$z'+z\operatorname{tg}x=\sin 2x$ (*) (linearna → nakon supstitucije)

1. $z'+\operatorname{tg}x\cdot z=0$

$$\frac{dz}{dx}=-\operatorname{tg}x\, z$$

$$\frac{dz}{z}=-\operatorname{tg}x\,dx \quad \Big|\int$$

$$\ln z=-\int\frac{\sin x}{\cos x}dx=\int\frac{dt}{t}$$

$$\left|\begin{array}{l}\cos x=t\\ dt=-\sin x\end{array}\right|$$

$$\ln z=\ln|\cos x|+\ln c$$

$z=c\cdot\cos x$ (**) $[\,C=C(x)\,]$

2. (z se uvrsti u poč. linearnu)

u (*)

$$c'(x)\cdot\cos x+c(x)(-\sin x)+\operatorname{tg}x\cdot c(x)\cos x=\sin 2x$$

$$c'(x)\cos x-c(x)\cancel{\sin x}+\cancel{\sin x}\cdot c(x)=2\sin x\cos x \quad \Big/:\cos x$$

$$c'(x)=2\sin x \quad \Big|\int dx$$

$$c(x)=-2\cos x+C_1$$

u (**) $z=(-2\cos x+C_1)\cdot\cos x$

$z = c_1 \cdot \cos x - 2\cos^2 x$

$\underline{y^2 = c_1 \cdot \cos x - 2\cos^2 x}$

za $y(\pi) = 0$

$0 = c_1 \cdot \underbrace{\cos\pi}_{-1} - 2\underbrace{\cos^2\pi}_{1}$

$\underline{c_1 = -2}$

$\underline{y^2 = -2\cos x - 2\cos^2 x = -2\cos x(1 + \cos x)}$

4.) $\underbrace{\frac{2x}{y^3}}_{P} dx + \underbrace{\left(\frac{1}{y^2} + \frac{3x^2}{y^4}\right)}_{Q} dy = 0$ (Eulerov multiplikator)

[moguće i riješiti i preko hom. jed.]

provjera uvjeta:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{-2x \cdot 3\cancel{y^2}}{y^{\cancel{6}4}} = \frac{6x}{y^4}$$

$$-\frac{6x}{y^4} = \frac{6x}{y^4} \quad \#$$

provjera o čem ovisi μ

$\ln \mu(x) = \int \frac{1}{Q}(P_y' - Q_x')\,dx$ (nije)

$\ln \mu(y) = -\int \frac{1}{P}(P_y' - Q_x')\,dy$

$= 6\int \frac{1}{y}\,dy$

$\ln \mu(y) = \ln y^6$

$\underline{\mu(y) = y^6}$

množimo poč. jednadžbu:

$2xy^3\,dx + (y^4 + 3x^2y^4)\,dy = 0$

$u(x,y) = \int_0^x 2xy^3\,dx + \int_0^y y^4\,dy$

$= x^2y^3 + \frac{1}{5}y^5 + c_1$

5.) $2y - xy' = 1 + \frac{x}{y'}$ $\qquad y' = p(x)$

(*) $2y - xp = 1 + \frac{x}{p} \Big| \frac{d}{dx}$

$2y' - p - xp' = \frac{1 \cdot p - xp'}{p^2}$

$2p - p - xp' = \frac{p - xp'}{p^2} \quad | \cdot p^2$

$p^3 - xp'p^2 - p + xp' = 0$

$xp'(-p^2 - 1) + p(p^2 - 1) = 0$

$\underline{(p^2-1)}(p - xp') = 0$

$p^2 - 1 = 0$

$p^2 = 1$

$\underline{p = \pm 1}$

u (*) uvštavamo p

za $p = 1$: $2y - x = 1 + x$

$\underline{y = \frac{1}{2} + x}$

za $p = -1$: $2y + x = 1 - x$ singularna rješenja

$\underline{y = \frac{1}{2} - x}$

$p - xp' = 0$

$xp' = p$

$x \frac{dp}{dx} = p \quad | \cdot dx$

$x\,dp = p\,dx$

$\frac{dp}{p} = \frac{dx}{x} \quad | \int$

$\ln p = \ln x + \ln c$

$\underline{p = cx}$

u (*) $2y - x^2 c = 1 + \frac{\cancel{x}}{\cancel{x}c}$

$2y - x^2 c = 1 + \frac{1}{c}$

$y = \frac{1}{2} + \frac{1}{2}cx^2 + \frac{1}{2c}$ → opće rješenje

$$\frac{y'}{\cos^2 y} - tg y = 2$$

PRIMJER ZAMJENE

$$z = tg y$$

$$z' = \frac{1}{\cos^2 y} y'$$

$$z' - z = 2$$

Jednadžbe višeg reda

6.) $y \cdot y'' + y'^2 = 2yy'$

$$\boxed{\begin{aligned} y' &= p(y) \\ y'' &= pp' \end{aligned}}$$ u jed. samo y (bez x)

$$ypp' + p^2 = 2yp \;/ : yp$$

$$(*) \quad p' + \frac{1}{y} p = 2$$

① $p' = -\frac{1}{y} p$

$$\frac{dp}{dy} = -\frac{1}{y} p$$

$$-\frac{dy}{y} = \frac{dp}{p} \;/ \int$$

$$\ln p = -\ln y + c$$

$$p = \frac{c}{y}$$

② $p = \frac{c(y)}{y} = c(y) \cdot \frac{1}{y}$

u (*)

$$c'(y) \cdot \frac{1}{y} + \cancel{c(y) \cdot \left(-\frac{1}{y^2}\right)} + \cancel{\frac{1}{y} \cdot \frac{c(y)}{y}} = 2$$

$$c'(y) = 2y \;/ \int$$

$$c(y) = y^2 + c_1$$

$$\boxed{p = \frac{y^2 + c_1}{y}}$$

$$y' = \frac{y^2 + c_1}{y}$$

$$\frac{dy}{dx} = \frac{y^2 + c_1}{y}$$

supst.

$$t = y^2 + c_1$$

$$dt = 2y\,dy$$

$$\int \frac{y\,dy}{y^2 + c_1} = \int dx$$

$$\frac{1}{2} \ln |y^2 + c_1| = x + c_2 \;/ e$$

$$(y^2 + c_1)^{\frac{1}{2}} = e^{x + c_2} = c_2 e^x \;/ ()^2$$

$$y^2 + c_1 = c_2 e^{2x}$$

$$y^2 = c_2 e^{2x} + c_1$$

7.)

$$(1+x^2)y'(yy''-y'^2)-2xyy'^2=0$$

tada imamo u jed. $x \Rightarrow y'=z(x)$ ovisi o x

I. SUPST. $y'=z(x)$

$y''=z'(x)$

$y'''=z''(x)$

$$(1+x^2)z(yz'-z^2)-2xyz^2=0$$

(uglavnom y bi se trebao pokratiti i ostane jed. s x i z)

II. SUPST.

$$y=e^{\int z(x)dx}$$

(*) $\boxed{y'=z\cdot y \quad y''=y(z'+z^2)}$

smije se iskoristiti ako je poč. jed. homogena po varijablama $y, y', y'', \dots$

Provjera homogenosti:

$$(1+x^2)y'\cdot t(ty\,ty''-t^2y'^2)-2xty\,t^2y'^2=0$$

$$t^3[(1+x^2)y'(yy''-y'^2)-2xyy'^2]=0 \quad \text{(homogena je)}$$

iz (*) uvrstimo:

$$(1+x^2)zy(y^2(z'+z^2)-z^2y^2)-2xy\cdot z^2y^2=0 \quad /:y^3$$

$$(1+x^2)z(z'+z^2)-z^2-2xz^2=0$$

$$(1+x^2)zz'-2xz^2=0$$

$$\frac{dz}{z}=\frac{2x}{1+x^2}dx \quad /\int$$

$$\ln z=\ln|1+x^2|+\ln C$$

$$z=C_1(1+x^2)$$

$$\underline{y=e^{C_1(x+\frac{x^3}{3})+C_2}}$$

8.)

$$y'''+5y''-3y'+14=bla(x)$$

bilo šta (u ovisnosti o x)

(linearna s konst. koef.)

$$y^{IV}+4y''=2x^2+e^{2x}$$

$y\to r$

$y^{(n)}=r^{(n)}$

① $y^{IV}+4y''=0$

$$r^4+4r^2=0$$

$$r^2(r^2+4)=0$$

$r^2=0$

$\underline{r=0}$ (dva puta se pojavljuje jer je r^2)

$r^2+4=0$

$r^2=-4$ $\qquad \alpha=0 \quad \beta=2$

$\underline{r}=\pm\sqrt{-4}=\underline{\pm 2i}$

$y_h = C_1 \cdot e^{0x} + C_2 x e^{0x} + C_3 e^{0x} \cdot \cos 2x + C_4 \cdot e^{0x} \cdot \sin 2x$

DIGRESIJA!

$r_{1,2}=\alpha \pm i\beta$

$y_h = C_1 e^{\alpha x} \cdot \cos\beta x + C_2 \cdot e^{\alpha x} \cdot \sin\beta x$

$\underline{y_h = C_1 + C_2 x + C_3 \cos 2x + C_4 \sin 2x}$

→ ako fun. smetnje ima polinome, e^x, $\sin x$ ($\cos x$)

onda se partikularno rješenje može pogoditi!

poč. jednadžba:

$y^{IV}+4y''=2x^2+e^{2x}=f_1(x)+f_2(x)$

→ tražimo rješenje (partikularno) za $f_1(x)$ i za $f_2(x)$

1. za $f_1(x)=2x^2$

$y_{p1}=(Ax^2+Bx+C)x^k$

ovdje je $\alpha=0$ i provjerimo je li α rješenje karakteristične jednadžbe.

(kakva je fun. takvo mora biti i njeno partikularno rješenje)

x^k (ako imamo eksp fun.)

$k=$ kratnost rješenja

$k=2$

$\alpha=0$ je rješenje (2 puta)

$y_{p1}=(Ax^2+Bx+C)\cdot x^2$

$y_{p1}=Ax^4+Bx^3+Cx^2$

$y'_{p1}=4Ax^3+3Bx^2+2Cx$

$y''_{p1}=12Ax^2+6Bx+2C$

$y'''_{p1}=24Ax+6B$

$y^{IV}_{p1}=24A$

uvrstimo u $y^{IV}+4y''=2x^2$

$24A+48Ax^2+24Bx+8C=2x^2$

$48A=2 \Rightarrow \boxed{A=\frac{1}{24}}$

$24B=0 \Rightarrow \boxed{B=0}$

$24A+8C=0$

$1+8C=0 \Rightarrow \boxed{C=-\frac{1}{8}}$

$y_{p1}=(Ax^2+Bx+C)x^2$

$y_{p1}=\frac{1}{24}x^4+0-\frac{1}{8}x^2$

$y_{p1}=\frac{1}{24}x^4-\frac{1}{8}x^2$

DIGRESIJA.

ako imamo $f(x)=x^2+e^x+1$

smetujem : 1) x^2+1
2) e^x

ako imamo x^2e^x

$y_p=(Ax^2+Bx+C)\cdot e^x$

(2.) $f_2(x)=e^{2x}$

$y_{p2}=Ae^{2x}\cdot x^0$

$\alpha=2$ (dva nije rješenje)

$y_{p2}=Ae^{2x}$

uvrštavamo u $y^{IV}+4y''=e^{2x}$

$$\left.\begin{aligned} y'_{p2}&=2Ae^{2x}\\ y''_{p2}&=4Ae^{2x}\\ y'''_{p2}&=8Ae^{2x}\\ y^{IV}_{p2}&=16Ae^{2x}\end{aligned}\right\}$$

$16Ae^{2x}+4(4Ae^{2x})=e^{2x}$

$16A+16A=1$

$\boxed{A=\frac{1}{32}}$

$y_{p2}=\frac{1}{32}e^{2x}$

$y_p=y_{p1}+y_{p2}=\frac{1}{24}x^4-\frac{1}{8}x^2+\frac{1}{32}e^{2x}$

9.) (*) $y'' - 2y' + y = \frac{e^x}{x^2+1}$ M (ovo ne možemo pogađati jer se M dijeli)

Rješavamo VARIJACIJOM KONSTANTI

① $r^2 - 2r + 1 = 0$

$r_{1,2} = 1$

$k = 2$

$y_h = c_1 e^x + c_2 x e^x$

②
$$(**)\ \begin{cases} c_1'(x)\, e^x + c_2'(x)\, x e^x = 0 \\ c_1'(x)\, e^x + c_2'(x)\,(e^x + x e^x) = \frac{e^x}{x^2+1} \end{cases}$$

fun. smetnje iz (*)

derivacije

$-\cancel{e^x} c_2'(x) = -\frac{e^x}{x^2+1}$

$c_2'(x) = \frac{1}{x^2+1} \quad / \int$

$c_2(x) = \int \frac{dx}{x^2+1}$

$c_2(x) = \operatorname{arctg} x + D_2$

uvrstimo u (**)

$c_1'(x) e^x + c_2'(x) e^x x = 0$

$c_1'(x) + c_2'(x)\, x = 0$

$c_1'(x) = -\frac{1}{x^2+1} \cdot x \quad / \int$

$\left| \begin{array}{l} t = x^2+1 \\ dt = 2x\,dx \end{array} \right|$

$c_1(x) = -\int \frac{x}{x^2+1}$

$c_1(x) = -\frac{1}{2} \ln(x^2+1) + D_1$

$y = c_1 e^x + c_2 x e^x$

$y = -\left[\frac{1}{2}\ln(x^2+1) + D_1\right] e^x + \left[\operatorname{arctg} x + D_2\right] x e^x$

10.

10.) $y'' - y = (2\cos^2 x - 1)\, e^x$

$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

① $r^2 - 1 = 0$

$r^2 = 1$

$r_{1,2} = \pm 1$

$y_h = c_1 e^x + c_2 e^{-x}$

$y'' - y = (\cancel{2} \cdot \frac{1}{\cancel{2}} + \cancel{2} \cdot \frac{1}{\cancel{2}} \cos 2x - 1)\, e^x = e^x \cdot \cos x$

$y_p = e^x (A \cos 2x + B \sin 2x)$

β

$\alpha = 1$

$\beta = 2$

$r = \alpha \pm \beta = 1 \pm 2i$ (je li to rješenje?) NIJE

$y_p' = e^x(A\cos 2x + B \sin 2x) + e^x(-2A \sin 2x + 2B \cos 2x)$

FUNKCIJA SMETNJE

a) $f(x) = \operatorname{sh} x = \frac{1}{2} e^x - \frac{1}{2} e^{-x}$ (POGAĐANJE)

b) $f(x) = \ln x \cdot x$ (VARIJACIJA)

c) $f(x) = \operatorname{tg} x$ (VARIJACIJA)

d) $f(x) = \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ (POGAĐANJE)

e) $f(x) = \sin x \cdot \cos x$ (POGAĐANJE)

$= \frac{1}{2} \sin 2x$

f) $f(x) = \frac{1}{x} \cdot e^x$ (VARIJACIJA)

$= x^{-1} \cdot e^x$

TEORIJA : 13. KNJIŽICA (Teoremi → DETERMINANTE)
linearna zavisnost...

SRETNO NA ISPITU

butterfly :D