VJEROJATNOST I STATISTIKA

BURIĆEVA PREDAVANJA

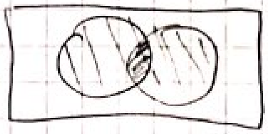
2018.

Kiki G

(1.) $P(A \cup B) = 0.8$

$P(A \cap B) = 0.2$

$P(\bar{A}) = 0.6 \Rightarrow P(A) = 0.4$

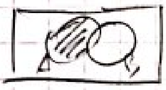


a) $P(B) = ?$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow P(B) = \boxed{0.6}$

b) $P(\bar{A} \cap \bar{B}) = 1 - P(\overline{\bar{A} \cap \bar{B}}) = 1 - P(A \cup B) = 1 - 0.8 = \boxed{0.2}$

c) $P(A \cap \bar{B}) = P(A) - P(A \cap B) = 0.4 - 0.2 = \boxed{0.2}$



1.3. Konačan vj. prostor

def. vjerojatnosti prostor koji ima konačno mnogo elem. događaja

$\Omega = \{w_1, w_2, \dots w_n\} \Rightarrow P(\Omega) = \sum_{i=1} P(w_i) = 1$

Pr.) D'Alambertov problem: bacanje 2 novčića

$w_1 = \{$dvije glave$\}$

$w_2 = \{$dva pisma$\}$ $\quad \cancel{P(w_i) = \frac{1}{3}}$

$w_3 = \{$jedno pismo, jedna glava$\}$

$w_{31} = \{GP\}$ $\quad w_{32} = \{PG\}$ $\quad \Big\}\ P(w_i) = \frac{1}{4}$

$P(w_3) = 0.5$

→ takav vjv. prostor u kojem je vj. svakog elementarnog događaja jednaka, nazivamo KLASIČNI VJV. PROSTOR, tada

$\sum_{i=1}^{N} P(w_i) = N \cdot p = 1 \Rightarrow \boxed{p = \frac{1}{N}}$

Ako imamo događaj $A = \{w_1, \dots w_M\}$ tada

$p(A) = M \cdot \frac{1}{N} \Rightarrow \boxed{p(A) = \frac{M}{N} = \frac{\text{br. povoljnih ishoda}}{\text{ukupan br. ishoda}}}$

(Zad.) Bacamo 2 kocke. Izračunajte vjv.

a) pali isti brojevi: $P(A) = \frac{6}{36} = \frac{1}{6}$

b) zbroj veći od 8: $P(B) = \frac{10}{36} = \frac{5}{18}$

c) barem jedna četvorka: $P(C) = \frac{11}{36}$ − (4,4) samo se jednom broji

$= \left(1 - \frac{25}{36}\right)$

$\boxed{\text{MI - 2014}}$ Bacili smo 4 kocke.

a) svi razl. brojevi $\quad P(A) = \frac{6 \cdot 5 \cdot 4 \cdot 3}{6^4} = \frac{5 \cdot 4 \cdot 3}{6^3} = \frac{60}{216}$

b) točno dvije šestice $\quad P(B) = \frac{\binom{4}{2} \cdot 5^2}{6^4}$

c) barem dvije šestice

$$P(C) = 1 - \underbrace{\frac{5^4}{6^4}}_{0 \text{ š}} - \underbrace{\frac{4 \cdot 5^3}{6^4}}_{1 \text{ š}}$$

d) točno 3 broja manja od 5

$$P(D) = \frac{\binom{4}{3} \cdot \underbrace{4^3}_{\text{4 broja manja od 5}} \cdot \underbrace{2}_{\text{zadnja kocka}}}{6^4}$$

e) dva para jednakih brojeva

$$P(E) = \frac{\underbrace{\binom{6}{2}}_{\text{odabir 2 broja od 6}} \underbrace{\binom{4}{2}\binom{2}{2}}_{\text{odabir mjesta na kocki}}}{6^4}$$

$\boxed{\text{ZAD}}$ U žari se nalaze 5B, 4C. i 2P kuglice. Izvlačimo 4 kuglice.

a) sve kuglice iste boje

$$P(A) = \frac{\binom{5}{4} + \binom{4}{4}}{\binom{11}{4}}$$

↳ ako piše barem i onda ispiše zadnje bilo što ⇒ KRIVO

↳ razdvojiti na disj. slučajeve

b) sve boje su zastupljene

$$P(B) = \frac{\binom{5}{2}\binom{4}{1}\binom{2}{1} + \binom{5}{1}\binom{4}{2}\binom{2}{1} + \binom{5}{1}\binom{4}{1}\binom{2}{2}}{\binom{11}{4}}$$

MI-2016

U bubnju se nalazi 15 kuglica označenih brojevima 1, …, 15. Izvlačimo 5 kuglica. 7P, 8NP

a) točno 2 parna broja
b) barem 2 parna broja

$$P(A) = \frac{\binom{7}{2}\binom{8}{3}}{\binom{15}{5}}$$

$$P(B) = 1 - \underbrace{\frac{\binom{8}{5}}{\binom{15}{5}}}_{0P} - \underbrace{\frac{\binom{7}{1}\binom{8}{4}}{\binom{15}{5}}}_{1P}$$

c) zbroj svih 5 brojeva paran
d) zbroj najveća 2 je veći od 26

$$P(C) = \frac{\binom{7}{5} + \binom{7}{3}\binom{8}{2} + \binom{7}{1}\binom{8}{4}}{\binom{15}{5}}$$

$15+14 \quad \binom{13}{3}$

$15+13 \quad \binom{12}{3}$

$15+12 \quad \binom{11}{3}$

$14+13 \quad \binom{12}{3}$

$$P(D) = \frac{\binom{13}{3} + \binom{12}{3} + \binom{11}{3} + \binom{12}{3}}{\binom{15}{5}}$$

MI-2012

Između 7 bračnih parova biramo 5 osoba.

a) svih 5 osoba istog spola
b) između tih 5 niti jedan bračni par

$$P(A) = \frac{\binom{7}{5} \cdot 2}{\binom{14}{5}}$$

$$P(B) = \frac{\underbrace{\binom{7}{5}}_{\text{koji } 5 \text{ parova}} \cdot 2^5}{\binom{14}{5}}$$

2^5 – za svaki izabrani muža ili ženu

MI-2013

Iz špila od 52 karte izvlačimo 5.

a) barem 1 asa

$\to 4 \times 13$

b) sve karte razl. jačina
c) sve karte iste boje

$$P(A) = 1 - \frac{\binom{48}{5}}{\binom{52}{5}}$$

$$P(B) = \frac{\underbrace{\binom{13}{5}}_{\text{jačine}} \cdot 4^5}{\binom{52}{5}}$$

4^5 – za svaku kartu boju odabrati

$$P(C) = \frac{\underbrace{\binom{4}{1}}_{\text{boju}} \cdot \underbrace{\binom{13}{5}}_{\text{karte}}}{\binom{52}{5}}$$

d) nisu zastupljene sve boje

$$P(D) = 1 - \frac{\binom{4}{1}\binom{13}{2} \cdot 13^3}{\binom{52}{5}}$$

sve jačine

koja boja se 2 puta ponavlja

Uzastopno ponavljanje pokusa

Pr.) Bacamo kocku 3 puta. Izračunajte vjv. da nismo dobili šesticu.

$$P(A) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = 58\%$$

vjv. da nismo prvi put dobili

$\boxed{\text{Zad}}$

Strijelac gađa metu 10 puta. Svaki put je pogađa s vjerov. 0.8

a) Vjerovatnost da je svaki put promašio

b) Točno 2 puta pogodio

c) Barem 2 puta pogodio

d) Najviše 2 puta pogodio

$$P(A) = 0.2^{10}$$

$$P(B) = \underbrace{0.8^2 \cdot 0.2^8}_{\text{pogodio točno 2 puta}} \cdot \binom{10}{2}$$

odabrati koja 2 puta od 10

$$P(C) = 1 - P(1) - P(0) = 1 - 0.8 \cdot 0.2^9 \cdot \binom{10}{1} - 0.2^{10}$$

1 pogodio, nijednom

$$P(D) = P(0) + P(1) + P(2) = 0.2^{10} + 0.8 \cdot 0.2^9 \binom{10}{1} + 0.8^2 \cdot 0.2^8 \cdot \binom{10}{2}$$

M1-2009

U bubnju se nalazi 6B i 4C kuglice. Izvlačimo ih 7. Izračunaj vjr. da smo izvukli barem 5 B.

a) Ako smo svih 7 odjednom izvukli

b) Izvlačimo jednu po jednu s vraćanjem u bubanj.

$$P(A) = \frac{\underbrace{\binom{6}{5}}_{5B}\underbrace{\binom{4}{2}}_{2C} + \underbrace{\binom{6}{6}}_{6B}\underbrace{\binom{4}{1}}_{1C}}{\binom{10}{7}} = \boxed{0.333}$$

$$P(B) = P(5) + P(6) + P(7) = \underbrace{\left(\frac{6}{10}\right)^5}_{\text{bijelo 5 puta}} \underbrace{\left(\frac{4}{10}\right)^2}_{\text{crno 2 puta}} \overbrace{\binom{7}{5}}^{\text{kojih 5 puta bijelo izvukli}} +$$

$$+ \binom{7}{6}\left(\frac{6}{10}\right)^6\left(\frac{4}{10}\right)^1 + \binom{7}{7}\left(\frac{6}{10}\right)^7 = \boxed{0.42}$$

1.4 Beskonačno vj. prostor

Pr: Bacamo kocku dok ne padne šestica.

Kako računamo vjerojatnost?

$$\Omega = \{6, \underbrace{16, 26, 36, 46, 56}_{x6}, xx6, xxx6, \dots\}$$

- def. algebra događaja $\mathcal{F}$ je σ algebra ako vrijedi

$$A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{n=1} A_n \in \mathcal{F}$$

Tada vjr. P mora zadovoljiti uvjet σ-aditivnosti

$$P\left(\bigcup_{n=1} A_n\right) = \sum_{n=1} P(A_n)$$ za disjunktne skupove $A_1, A_2, \dots$

- vrijedi neprekinutost vjerojatnosti:

$$\lim_{n\to\infty} P(A_n) = P\left(\lim_{n\to\infty} A_n\right) = P(A)$$

Vratimo se na prethodni primjer:

$P(1B) = \frac{1}{6}$ — prvo bacanje nismo dobili $= 0.1\dot{6}$

$P(2B) = \frac{5}{6} \cdot \frac{1}{6}$ — drugo jesmo $= 0.13\dot{8}$

$P(3B) = \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6} = 0.116$

$P(n\text{-to } B) = \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6}$

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$$

Naravno, $\sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6} = 1$

Vjv. da je pokus završio u parnom broju bacanja:

$$P(A) = P(x6) + P(xxx6) + P(xxxxx6) + \ldots =$$
$$= \frac{5}{6} \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^3 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^5 \cdot \frac{1}{6} + \ldots$$
$$= \frac{1}{6} \sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^{2n+1} = \frac{1}{6} \cdot \frac{5}{6} \sum_{n=0}^{\infty} \left(\frac{25}{36}\right)^n =$$
$$= \frac{5}{36} \cdot \frac{1}{1 - \frac{25}{36}} = \boxed{\frac{5}{11}}$$

Vjv. da bacanje nikada ne završi:

$\lim_{n \to \infty} \left(\frac{5}{6}\right)^n = 0$ (ekspon. fja s bazom manjom od 1)

$\boxed{\text{ZAD}}$ Dva igrača, Lolek i Bolek naizmjence izvlače kuglicu iz bubnja u kojem su 2B i 6C, a pobjednik je koji prvi izvuče bijelu. Za koga je veća vjv. pobjede

a) s vraćanjem kuglice u bubanj

b) bez vraćanja kuglice

$B \to \frac{1}{4}$

$C = \frac{3}{4}$

$\Omega = \{\boxed{B}, CB, CCB, CCCB, \boxed{CCCCB}, \ldots\}$

$$P(\text{prvi igrač}) = \frac{1}{4} + \left(\frac{3}{4}\right)^2 \frac{1}{4} + \left(\frac{3}{4}\right)^4 \frac{1}{4} + \ldots = \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^{2n}$$
$$= \frac{1}{4} \cdot \frac{1}{1 - \frac{9}{16}} = \boxed{\frac{4}{7}}$$

$$P(\text{drugi}) = 1 - P(\text{prvi}) = \frac{3}{7}$$

b) $\Omega = \{B, CB, CCB, \dots, CCCCCCB\}$ - ukupno 7

$$P(\text{prvi}) = \frac{2}{8} + \frac{6}{8}\cdot\frac{5}{7}\cdot\frac{2}{6} + \frac{6}{8}\cdot\frac{5}{7}\cdot\frac{4}{6}\cdot\frac{3}{5}\cdot\frac{2}{4} +$$

prva crna, druga crna, bijela - 2 povoljne od 6, crne, bijela

$$+ \frac{6}{8}\cdot\frac{5}{7}\cdot\frac{4}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3}\cdot\frac{2}{2} = \frac{4}{7}$$

bijele su samo ostale

$$P(\text{drugi}) = \frac{3}{7}$$

Vježba:

$\boxed{\text{MI - 2015}}$ Poker: 5 karata iz špila od 52 karte.

a) full house (tri karte iste jačine i 2 neke druge)

jačina, 3 boje

$$P(A) = \frac{13\cdot\binom{4}{3}\cdot 12\cdot\binom{4}{2}}{\binom{52}{5}}$$

b) dva para

boja druga, boja prva; zadnja karta

$$P(B) = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\cdot\binom{11}{1}\binom{4}{1}}{\binom{52}{5}}$$

c) jedan par (ostale karte razl.)

$$P(C) = \frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}4^3}{\binom{52}{5}}$$

ostale 3 karte i za svaku boju

d) straight (skala od 5 karata, gdje karte nisu iste boje)

herc, tref, ...

od koje se kreće

$$P(D) = \frac{10\cdot(4^5-4)}{\binom{52}{5}}$$

ZAD Problem rođendana. Vj. barem 2 osobe od n ima rođendan istog dana?

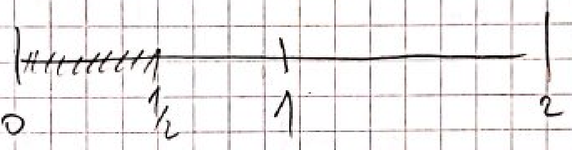
$$P(A) = 1 - \frac{365 \cdot 364 \cdot \ldots \cdot (365-n+1)}{365^n}$$

za svakoga na 365 načina

n	23	30	60	80
p	0.51	0.71	0.994	0.999

1.5. Geometrijska vjerovatnost

- motivacija: biramo broj iz intervala $[0,2]$.
  Izr. vjv. da je broj manji od $1/2$.



$$P(A) = \frac{1}{4}$$

ZAD Unutar jednakostr. Δ upisan je krug. Izračunajte vjv. da slučajno odabrana točka u Δ leži unutar kruga.

$$P(A) = \frac{P_k}{P_\Delta} = \frac{\pi \left(\frac{a\sqrt{3}}{6}\right)^2}{\frac{a^2\sqrt{3}}{4}} = \frac{\frac{\cancel{a^2}\cancel{3}}{\cancel{36}_{12}}\pi}{\frac{\cancel{a^2}\sqrt{3}}{4}} = \boxed{\frac{\pi}{3\sqrt{3}}}$$

- def. geom. vjv. se definira kao $\boxed{P(A) = \frac{m(A)}{m(\Omega)}}$

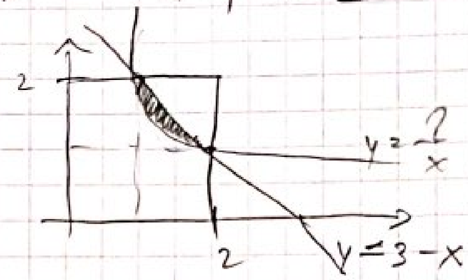
  m - ovisi o koliko se dimenzija radi

  Lako se vidi da ova definicija zadovoljava svojstva vjerovatnosti.

ZAD Biramo 2 točke iz intervala $[0,2]$. Izračunaj vjv. da im je suma < 3, a produkt > 2.

$x, y \in [0,2]$ $\quad x+y<3$ $\quad x \cdot y > 2$

- ekvivalentan vjv. prostor



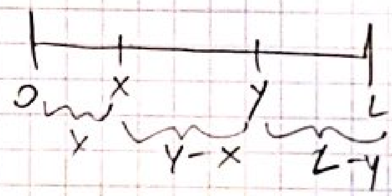
$x+y<3 \rightarrow y<3-x$

$x \cdot y > 2 \rightarrow y > \frac{2}{x}$

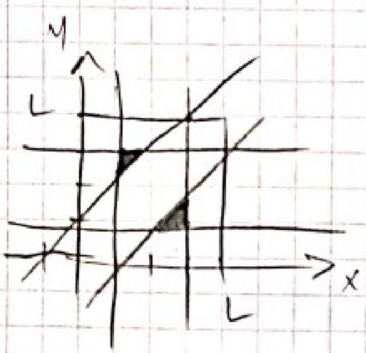
$$P(A) = \frac{\int_1^2 \left(3-x-\frac{2}{x}\right)dx}{4} = \frac{\frac{3}{2}-2\ln 2}{4}$$

MI - 2020

Štap duljine L je prelomljen na 2 mjesta. Izračunajte vjv. da je najkraći dio veći od $\frac{1}{4}L$.



$x, y \in [0, L]$



-> zajedno 1 kvadratić

L> svi su veći od $\frac{1}{4}L$

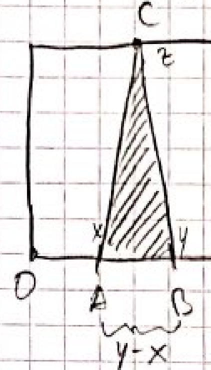
$$y>x \begin{cases} x > \frac{1}{4}L \\ y-x > \frac{1}{4}L \Rightarrow y > x + \frac{1}{4}L \\ L-y > \frac{1}{4}L \Rightarrow y < \frac{3}{4}L \end{cases}$$

$$P(A) = \frac{\left(\frac{1}{4}L\right)^2}{L^2} = \frac{1}{16}$$

$$x>y \begin{cases} y > \frac{1}{4}L \\ x-y > \frac{1}{4}L \\ L-x > \frac{1}{4}L \end{cases}$$

MI - 2011

Na jednoj stranici kvadrata duljine 10 biramo 2 točke A i B, na nasuprotnoj stranici točku C. Izračunaj vjv. da je $P_{\Delta ABC} < 25$



$x, y \in [0, 10]$

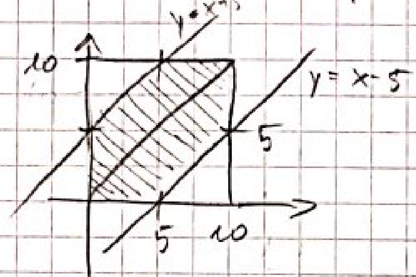
$z \in [0, 10]$

- pov. Δ ovisi o visini stoga odabir točke C nema utjecaja na površinu

$P = \frac{1}{2}(y-x)\,10 < 25$

$y - x < 5$

$y < x + 5$



$P = \frac{1}{2}(x-y)\,10 < 25$

$x - y < 5$

$y > x - 5$

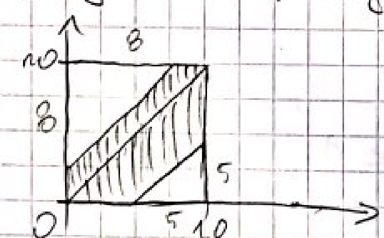
$P(A) = \frac{100 - 25}{100} = \frac{3}{4}$

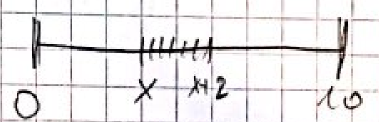
MI - 2009

X - pljačkaš 2:50 - 3:00 → 2 min

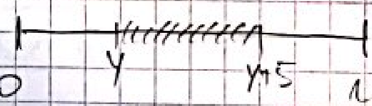
Y - policajac 2:50 - 3:00 → 5 min

a) Vj. da policajac uhvati pljačkaša





$x < y < x + 2$



$P(A) = \frac{100 - \frac{1}{2}8^2 - \frac{1}{2}5^2}{100}$

$= \boxed{\frac{111}{200}}$

$y < x < y + 5$

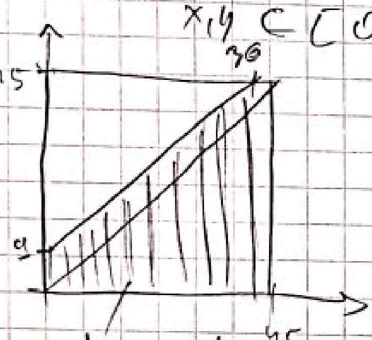
b) Vjr. vj. da prometnik dođe, provjeri i ode prije policajca.

$y > x + 2$ → gornji trokut na slici

$$P(B) = \frac{\frac{1}{2} 8^2}{100} = \boxed{0.32}$$

$\boxed{\text{KPZ-2016}}$ Marko i Pero žive u istoj zgradi i rade u istoj firmi, ali svatko ide svojim autom. Kreću neovisno između 7:00 i 7:45. Marko vozi prosječno 50 km/h, Pero 80 km/h. Firma je udaljena 20 km. Izračunajte vjv. da Pero stigne na posao prije Marka.

$x, y \in [0, 45]$



donja, dionica kada Pero krene prije

$t_M = \frac{2}{5} h = 24$ min

$t_P = \frac{1}{4} h = 15$ min

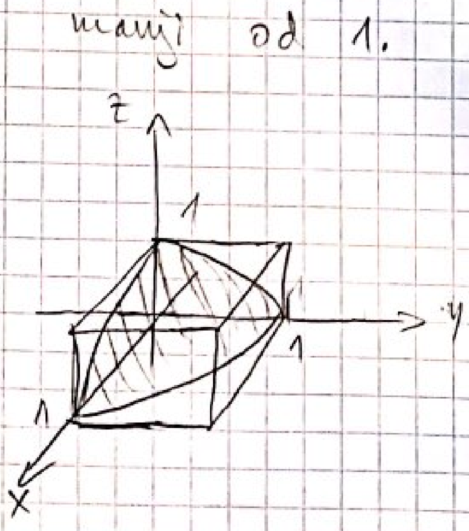
$y + 15 < x + 24$

$y < x + 9$

$$P(A) = \frac{45^2 - \frac{1}{2} 36^2}{45^2} = 0.68$$

$\boxed{210}$ Biramo 3 točke iz $[0,1]$. Vj. da je zbroj kvadrata manji od 1.

$x, y, z \in [0,1]$



$x^2 + y^2 + z^2 < 1$

$$P(H) = \frac{\frac{1}{8} V_k}{1^3} = \frac{1}{8} \cdot \frac{4}{3} r^3 \pi = \boxed{\frac{\pi}{6}}$$

2. UVJETNA VJEROVATNOST

2.1. Definicija

- motivacija: ako znamo da je pao paran broj, vjv. da je pala 6?

$P(A) = \frac{1}{3}$

- def. Neka je B događaj t.d. je $P(B) > 0$
  Uvjetna vjv. događaja A uz uvjet B se definira formulom $\boxed{P(A|B) = \frac{P(AB)}{P(B)}}$

Lako se vidi da ova def. zadovoljava svojstva vjv.

ZAD Bacamo 4 kocke. Kr. vjv. da je zbroj manji od 6 ako su svi brojevi manji od 4.

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{\frac{5}{\cancel{6^4}}}{\frac{3^4}{\cancel{6^4}}} = \boxed{\frac{5}{3^4}}$$

1111 -> 1
2111 -> 4

ZAD Vis 0.3 da položi na JIR
Sis 0.45 da položi na SIR
Ako znamo da je položio samo jedan predmet, koliko je vjv. da je to Vis?

$A = \{$vj. da je položio Vis$\}$ $B = \{$položio 1 predmet$\}$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{\overbrace{0.3}^{\text{položio Vis}} \cdot \overbrace{(1-0.45)}^{\text{pao Sis}}}{0.3 \cdot 0.55 + 0.7 \cdot 0.45} = \boxed{0.344}$$

presjek 2 događaja

ZAD 10 puta bacamo novčić. Vjr da je svih 10 puta palo pismo, ako znamo da je pismo palo barem 9 puta.

B - {barem 9 puta P}

A = {svih 10 puta P}

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$= \frac{\left(\frac{1}{2}\right)^{10}}{\left(\frac{1}{2}\right)^{9}\cdot\frac{1}{2}\cdot\binom{10}{9} + \left(\frac{1}{2}\right)^{10}} = \boxed{\frac{1}{11}}$$

— presjek je svih 10 puta P

ZAD

U kutiji se nalaze 4P, 5B i 6C kuglica. Izvlačimo 3 kuglice jednu za drugom.

A = {sve kuglice razl boja}   B = {prva kuglica B}

$P(A|B) = ?$ $P(B|A) = ? = \frac{1}{3}$ samo 2 poretka jer B uvijek namjestimo prva

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{\frac{5}{15}\cdot\frac{4}{14}\cdot\frac{6}{13}\cdot 2!}{\frac{4}{15}\cdot\frac{5}{14}\cdot\frac{6}{13}\cdot 3!} = \boxed{\frac{1}{3}}$$

P B C

poredak

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{\frac{5}{15}\cdot\frac{4}{14}\cdot\frac{6}{13}\cdot 2!}{\frac{5}{15}} = \boxed{\frac{24}{91}}$$

— prva bijela

2.2. Nezavisnost događaja

- motivacija: u kutiji 3B i 7C, izvlačimo 2 kuglice jednu po jednu

A = {prvi put B}   B = {drugi put C}

a) s vraćanjem: $P(A) = \frac{3}{10}$ $P(B) = \frac{7}{10}$

b) bez vraćanja

$P(B|A) = \frac{P(AB)}{P(A)} = \frac{7}{9}$ $P(A) = \frac{3}{10}$ $P(B) = \frac{3}{10}\cdot\frac{7}{9} + \frac{7}{10}\cdot\frac{6}{9}$

- **def.** Za događaje A i B kažemo da su nezavisni ako vrijedi $P(A) = P(A|B)$ ili $P(B) = P(B|A)$. Tada

$$P(A|B) = \frac{P(AB)}{P(B)} = P(A) \Rightarrow \boxed{P(AB) = P(A) \cdot P(B)}$$

nužan i dovoljan uvjet nezavisnosti

ZAD

Ako su događaji A i B nezavisni, dokažite da su tada $\overline{A}$ i $\overline{B}$ također nezavisni.

$$P(AB) = P(A) \cdot P(B)$$

$$P(\overline{A} \cdot \overline{B}) = 1 - P(\overline{\overline{A}\overline{B}}) = 1 - P(A \cup B) =$$

$$= 1 - [P(A) + P(B) - P(AB)] =$$

$$= 1 - P(A) - P(B) + P(A)P(B) =$$

$$= 1 - P(A) - P(B)[1 - P(A)] =$$

$$= (1 - P(A))(1 - P(B)) = P(\overline{A}) \cdot P(\overline{B})$$

- **def.** događaji $A_1, A_2, \ldots, A_n$ su nezavisni ako za svaki izbor $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ navedenih događaja vrijedi $P(A_{i_1}, A_{i_2}, \ldots, A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdot \ldots \cdot P(A_{i_k})$

MI-2009

a) Definiraj nezavisnost događaja A, B i C

$P(ABC) = P(A) \cdot P(B) \cdot P(C)$ i $P(AB) = P(A)P(B)$;

$P(BC) = P(B) \cdot P(C)$ i $P(AC) = P(A) \cdot P(C)$

b) Ako su događaji A, B, C nezavisni, dokažite da su A i $B \cup C$ nezavisni

$$P(A \cdot (B \cup C)) = P(AB \cup AC) = P(AB) + P(AC) - P(ABC) =$$

$$= P(A)P(B) + P(A)P(C) - P(A) \cdot P(B) \cdot P(C) =$$

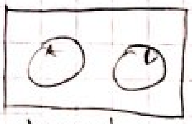
$$= P(A)[P(B) + P(C) - P(B)P(C)] = P(A) \cdot P(B \cup C)$$

$\boxed{\text{MI-2017}}$ a) Ako $P(A|B)=P(A)$ tada je $P(B|A)=P(B)$

$$P(AB)=P(A)\cdot P(B)$$

$$P(B|A)=\frac{P(AB)}{P(A)}=\frac{\cancel{P(A)}P(B)}{\cancel{P(A)}}=P(B)$$

b) Ako su A i B disjunktni, tada su nezavisni



disjunktni

$$P(AB)=P(A)\cdot P(B)$$

$$0=P(A)\cdot P(B) \quad \rightarrow \text{nemoguće}$$

c) Ako vrijedi: $P(ABC)=P(A)P(B)P(C)$, tada su A,B,C nezavisni

↳ NETOČNO

↳ mora dodatno vrijediti i za svaki podskup

2.3. Potpuna vjerojatnost i Bayesova formula

- motivacija: neki izvor emitira 1 s vjr. 0.75, a 0 s vjv. 0.25. Na izlazu iz komunik. kanala, 5% znakova se pogrešno interpretira.

a) Izračunajte vjv. da na izlazu primimo jedinicu?

$$P(1)=\underbrace{0.75}_{P(H_1)}\cdot\underbrace{0.95}_{P(1|H_1)}+\underbrace{0.25}_{P(H_0)}\cdot\underbrace{0.05}_{P(1|H_0)}=0.725$$

- općenito:

$$\Omega=H_1\cup H_2\cup\ldots\cup H_n$$ particija vjv. prostora

$$P(\Omega)=1=\sum_{i=1}^{n}P(H_i)$$

$$A=AH_1\cup AH_2\cup\ldots\cup AH_n \Rightarrow P(A)=P(AH_1)+\ldots+P(AH_n)$$

↳ zbog disjunktnosti

$$=P(H_1)P(A|H_1)+\ldots+P(H_n)P(A|H_n)$$

$\Rightarrow$ zaključak:

$$\boxed{P(A) = \sum_{i=1}^{n} P(H_i)\, P(A|H_i)}$$

FORMULA POTPUNE VJEROJATNOSTI

$\boxed{\text{M1-2014}}$ Student odg. na pitanje s 4 odgovora od kojih je 1 točan. Vjv. da zna odg je 80%, a ako nema nagađa odg na sreću. Kolika je vjv. da točno odgovori na pitanje?

$$P(T) = 0.8 \cdot 1 + 0.2 \cdot 0.25 = \boxed{0.85}$$

$\boxed{\text{ZAD}}$ Iz snopa izvučena jedna karta, a iz drugog 2 karte. Sve 3 karte pomiješali i na sreću 1 okrenuli. Izračunaj vjv. da je okrenuta karta as.

Hipoteze $\rightarrow$ koliko aseva u te 3 karte

$H_i = \{i\text{-aseva}\}$

$$P(H_0) = \frac{48}{52} \cdot \frac{\binom{48}{2}}{\binom{52}{2}} \qquad P(H_1) = \frac{4}{52} \cdot \frac{\binom{48}{2}}{\binom{52}{2}} + \frac{48}{52} \cdot \frac{\binom{4}{1}\binom{48}{1}}{\binom{52}{2}}$$

u drugom ne smijem

$$P(H_2) = \frac{4}{52} \cdot \frac{\binom{4}{1}\binom{48}{1}}{\binom{52}{2}} + \frac{48}{52} \cdot \frac{\binom{4}{2}}{\binom{52}{2}}$$

$$P(H_3) = \frac{4}{52} \cdot \frac{\binom{4}{2}}{\binom{52}{2}} \qquad \left(\sum P(H_i) = 1\right)$$

$P(A|H_0) = 0$ - nemoguće ga okrenut ako ga nema

$P(A|H_1) = \frac{1}{3}$

$P(A|H_2) = \frac{2}{3}$ $\qquad P(A) = \sum P(H_i) \cdot P(A|H_i) = \boxed{\frac{1}{13}}$

$P(A|H_3) = 1$

MI-2012

Kod testiranja na bolest, ako ima bolest test će biti pozitivan u 90% slučajeva.

H_1 - ako ima bolest -> 90% poz?

H_2 - nema bolest -> 5% poz?

: 2% populacije ima promatranu bolest

· a) Ako odaberemo neku osobu, kolika je vjv. da će test biti pozitivan?

$$P(A) = 0.02 \cdot 0.9 + 0.98 \cdot 0.05 = \boxed{0.067}$$

b) Izračunajte vjv. da osoba ima bolest ako je test pozitivan.

$$P(H_1 | A) = \frac{P(H_1 A)}{P(A)} = \frac{0.02 \cdot 0.9}{0.067} = \boxed{0.2687}$$

·BAYESOVA FORMULA

↳ vjv. neke hipoteze ako znamo da se dogodio neki događaj

$$P(H_i | A) = \frac{P(H_i) \cdot P(A|H_i)}{P(A)}$$

$P(A)$ -> najčešće F.P.V.

MI-2010

H_0 - 0 -> 34%

H_A - A -> 37%

H_B - B -> 21%

H_{AB} - AB -> 8%

a) Kolika je vjv. da jedna slučajno primi od slučajno druge?

a) $P(T) = \sum P(H_i) P(T|H_i) = \boxed{0.5738}$

$P(T|H_0) = 0.34$

$P(T|H_A) = 0.37 + 0.34 = 0.71$

$P(T|H_B) = 0.21 + 0.34 = 0.55$

prima B, prima 0

$P(T|H_{AB}) = 1$

b) Ako je osoba uspješno primila krv. Kolika je vjr. da je bila krvne grupe AB?

$$P(H_{AB} | T) = \frac{0.08 \cdot 1}{P(T)} = \boxed{0.1394}$$

p.p.V

$\boxed{\text{MI-2016}}$

4 topa, svaki vjr. 0.4

1 top pogodi - 0.3 da uništi

2 topa pog → 0.6

3 topa → 0.8

4 topa → 0.9

Sva 4 su pucala i cilj je bilo uništen. Vjr. da ga je pogodio samo 1 top.

$H_i = \{ \text{i-topova pogodilo cilj} \}$

$U = \{ \text{uništen cilj} \}$

$P(H_1 | U) = ?$

$P(H_1) = 0.4 \cdot 0.6^3 \binom{4}{1}$

$P(H_2) = 0.4^2 \cdot 0.6^2 \binom{4}{2}$

$P(H_3) = 0.4^3 \cdot 0.6 \binom{4}{3}$

$P(H_4) = 0.4^4$

$P(U | H_1) = 0.3 \quad \dots \quad P(U | H_4) = 0.9$

$$P(H_1 | U) = \frac{P(H_1) P(U | H_1)}{\sum P(H_i) P(U | H_i)} = 0.2269$$

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MI - 2015

1\. kutija $\Rightarrow$ 3 puta više B nego C

ostale 2 kutije = 3 puta više C nego B

Odabire se iz 1 kutije 5 kuglica jednu po jednu s vraćanjem u kutiju.

Ako smo izvukli $\underbrace{3B \text{ i } 2C}_{A}$, izračunajte vjv. da smo izvlačili iz prve kutije.

$P(H_1) = \frac{1}{3}$ $\qquad$ $P(H_2) = \frac{2}{3}$

$P(A|H_1) = \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^2 \cdot \binom{5}{3}$

$P(A|H_2) = \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2 \binom{5}{3}$

$P(H_1|A) = \frac{P(H_1)P(A|H_1)}{P(A)} =$

3. DISKRETNE SLUČAJNE VARIJABLE

3.1 Uvod i definicija

- neka je $S = \{x_1, x_2, \dots\}$ konačan ili prebrojiv skup, te neka varijabla X svakom el. događaju pridružuje neku vrijednost iz S

- def. preslikavanje $X : \Omega \to S$ je diskretna slučajna varijabla ako $\forall x_k \in S$ vrijedi da je skup :

$$A = \{\omega \in \Omega : X(\omega) = x_k\} \text{ događaj } (\text{tj. } A \in \mathcal{F})$$

- označavamo $P(X = x_k) = p_k$, $\sum_k p_k = 1$

Zakon razdiobe od X označavamo sa

$$X \sim \begin{pmatrix} x_1 & x_2 & x_3 & \dots \\ p_1 & p_2 & p_3 & \dots \end{pmatrix}$$

Pr X = broj na kocki

$$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

ZAD Bacamo 3 kocke. Neka je slučajna var. broj šestica
Odredite slučajnu var.

$$X \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ \left(\frac{5}{6}\right)^3 & \frac{1}{6}\cdot\left(\frac{5}{6}\right)^2\binom{3}{1} & \left(\frac{1}{6}\right)^2 \frac{5}{6}\binom{3}{2} & \left(\frac{1}{6}\right)^3 \end{pmatrix}$$

$$P(X > 1) = P(X=2) + P(X=3)$$
$$= \boxed{p_2 + p_3}$$

AUDITORNE

1.
$P(A|B) = 0.4$
$P(B|A) = 0.2$
$P(A \cup B) = 0.26$
$P(A), P(B) = ?$

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$\left.\begin{array}{l} P(AB) = 0.4 \cdot P(B) \\ P(AB) = 0.2\, P(A) \end{array}\right\} P(A) = 2P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$0.26 = 2P(B) + P(B) - 0.4P(B)$$
$$0.26 = 2.6P(B)$$
$$\boxed{\begin{array}{l} P(B) = 0.1 \\ P(A) = 0.2 \end{array}}$$

(2.) U poslovnici A nalazi se 100 srećki.

A 100 srećki 25 dobitnih

B 55 srećki 5 dobitnih

Ako padne 1 na kocki $\rightarrow$ 2 u A

Ako padne 2 na kocki $\rightarrow$ 2 u B

ostalo - 1 iz A i B

Kolika je vjv. da dobije točno jednu dobitnu?

C - { točno jedna dobitna }

H_1 - { pala je jedinica } $P(H_1) = \frac{1}{6}$

H_2 - { pala je dvojka } $P(H_2) = \frac{1}{6}$

H_3 - { palo je 3, 4, 5 ili 6 } $P(H_3) = \frac{4}{6}$

$$P(C) = \sum_{i=1}^{3} P(C|H_i) \cdot P(H_i) =$$

$$P(C|H_1) = \frac{\binom{25}{1}\binom{75}{1}}{\binom{100}{2}} = \frac{25}{66}$$

$$P(C|H_2) = \frac{\binom{5}{1}\binom{50}{1}}{\binom{55}{2}} = \frac{50}{297}$$

$$P(C|H_3) = \frac{25}{100} \cdot \frac{50}{55} + \frac{75}{100} \cdot \frac{5}{55} = \frac{13}{44}$$

(3.)

A = { stigli smo na cilj }

H_i = { pukla je točno i guma } $i = 0, 1, 2, 3, 4$

$$P(A) = \sum_{i=0}^{4} P(A|H_i) P(H_i) = \sum_{i=0}^{4} \frac{1}{2^i} \binom{4}{i} \left(\frac{1}{4}\right)^i \left(\frac{3}{4}\right)^{4-i} = \boxed{0.586}$$

$$P(H_i) = \binom{4}{i} \left(\frac{1}{4}\right)^i \left(\frac{3}{4}\right)^{4-i}$$

točno i guma puklo

da ostale nisu pukle

$$P(A|H_i) = \frac{1}{2^i}$$

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4. Bacamo 3 novčića, one na kojima je pala glava ostavimo sa strane, a preostale bacimo još jednom. Ako su ukupno pale 3 glave, odredite vjv. da su u prvom pale barem 2.

$A = \{\text{pale 3 glave}\}$

$H_i = \{\text{palo } i \text{ glava u prvom bacanju}\}$

$$P(H_2 \cup H_3 | A) = P(H_2|A) + P(H_3|A) =$$

$$= \frac{P(A|H_2)P(H_2)}{P(A)} + \frac{P(A|H_3)P(H_3)}{P(A)} =$$

$$P(A) = \sum_{i=1}^{3} P(A|H_i)P(H_i) =$$

$$P(H_0) = \frac{1}{8} \quad P(H_1) = \frac{1}{8}\binom{3}{1} \quad P(H_2) = \frac{1}{8}\binom{3}{2} \quad P(H_3) = \frac{1}{8}$$

$$P(A|H_0) = \frac{1}{8} \quad P(A|H_1) = \frac{1}{4} \quad P(A|H_2) = \frac{1}{2} \quad P(A|H_3) = 1$$

5. $\left.\begin{array}{ll} n & B \\ n+2 & C \end{array}\right\}$ kuglica U jednoj se izgubila. Ako nakon toga izv. B kuglica je vjv. da je izgubljena bila C.

$H_0 = \{\text{izgubljena je crna kug}\}$

$H_1 = \{\text{izgubljena je bijela kug}\}$

$A = \{\text{izvučena je bijela}\}$

$$P(H_0|A) = \frac{P(A|H_0)P(H_0)}{P(A|H_0)P(H_0) + P(A|H_1)P(H_1)} =$$

$$P(H_0) = \frac{n}{2n+2} \qquad P(A|H_0) = \frac{n}{2n+1}$$

$$P(H_1) = \frac{n+2}{2n+2} \qquad P(A|H_1) = \frac{n-1}{2n+1}$$

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ZAD U urni se nalazi n kuglica numeriranih $1, 2, \dots, n$. Izvlačimo 3. X = najveći od tih 3 broja.

$$X \sim \begin{pmatrix} 3 & 4 & 5 & \dots & n \\ \frac{1}{\binom{n}{3}} & \frac{\binom{3}{2}}{\binom{n}{3}} & \frac{\binom{4}{2}}{\binom{n}{3}} & & \frac{\binom{n-1}{2}}{\binom{n}{3}} \end{pmatrix}$$

Pr Bacamo 2 kocke, X = broj na prvoj, Y = broj na drugoj

$$P(X \leq 2, Y = 3) = \frac{2}{6} \cdot \frac{1}{6} = P(X \leq 2) \cdot P(Y = 3)$$

- def. slučajne var. $X, Y : \Omega \to S$ su nezavisne ako za sve $x_k, y_j \in S$ vrijedi

$$P(X = x_k, Y = y_j) = P(X = x_k) \cdot P(Y = y_j)$$

- def. sluč. var. $X_1, X_2, \dots, X_n$ def. na istom Ω su nezavisne ako za sve $A_1, A_2, \dots, A_n \subset S$ vrijedi

$$P(X_1 \in A_1, \dots, X_n \in A_n) = P(X_1 \in A_1) \dots P(X_n \in A_n)$$

• funkcije diskretnih sluč. varijabli

Neka je X dis. sl. var. i neka je $\Psi : \mathbb{R} \to \mathbb{R}$ neka f-ja

Ako $X \sim \begin{pmatrix} x_1 & x_2 & \dots \\ p_1 & p_2 & \dots \end{pmatrix}$ tada $Y = \Psi(X) \sim \begin{pmatrix} \Psi(x_1) & \Psi(x_2) & \dots \\ p_1 & p_2 & \dots \end{pmatrix}$

ZAD $X \sim \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 0.2 & 0.1 & 0.5 & 0.15 & 0.05 \end{pmatrix}$ a) X^2 b) $X_1, X_2 \sim X$

$X_1 \cdot X_2 = ?$

a) $X^2 \sim \begin{pmatrix} 4 & 1 & 0 & 1 & 4 \\ 0.2 & 0.1 & 0.5 & 0.15 & 0.05 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 4 \\ 0.5 & 0.25 & 0.25 \end{pmatrix}$

b) $X_1 \cdot X_2 \sim \begin{pmatrix} -4 & -2 & -1 & 0 & 1 & 2 & 4 \\ 0.02 & 0.07 & 0.01 & & & & 0.0\dots \end{pmatrix}$

$P(X_1 \cdot X_2 = -4) = P(X_1 = -2, X_2 = 2) + P(X_1 = 2, X_2 = -2)$ = nezavisno presjek

Ako su X_1, X_2 nezavisni:

↳ $P(X_1=-2)P(X_2=2)+P(X_1=2)P(X_2=-2) = 0.2\cdot 0.05 + 0.05\cdot 0.2$

- u primjeru nemamo jesu zavisni ili nezavisni
- za vjv. 0 samo imamo 1-(sve ostalo)

3.2. Diskretni slučajni vektor

- razdiobu sl. vektora (X,Y) definiramo sa

$$p_{ij} = P(X = x_i, Y = y_j)$$

a označavamo tablicom:

X\Y	

ZAD Bacamo 2 kocke. X = broj na prvoj kocki

Y = veći od dvaju brojeva

(X,Y), marginalnost, nezavisnost?

veći od ta 2 ↓ je stupac

X\Y	1	2	3	4	5	6	
1	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	1/6
2	0	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	1/6
3	0	0	$\frac{3}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	1/6
4	0	0	0	$\frac{4}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	1/6
5	0	0	0	0	$\frac{5}{36}$	$\frac{1}{36}$	1/6
6	0	0	0	0	0	$\frac{6}{36}$	1/6
	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$	1

→ na prvoj kocki

✓ Očito su zavisni jer npr.:

$$p_{1\cdot}\, p_{\cdot 1} \neq p_{11}$$

$$\frac{1}{6}\cdot\frac{1}{36} \neq \frac{1}{36}$$

↳ najlakše kada imamo 0

MI-2010

X\Y	0	1	
-1	$\cancel{1/4}$	$\cancel{1/6}$	10/24
0	$\cancel{1/6}$	1/8	7/24
1	1/8	$\cancel{1/6}$	7/24
	$\frac{13}{24}$	$\frac{11}{24}$	1

-> križaci kada ga prebacimo u novu tablicu

a) odredi marginalne razdiobe -> samo sumirati

b) nezavisnost -> očito nisu nezavisni jer: $\frac{10}{24} \cdot \frac{13}{24} \neq \frac{1}{4}$

c) $P(X \geq 0 \mid Y=1) = \frac{P(X \geq 0, Y=1)}{P(Y=1)} = \frac{\frac{1}{8}+\frac{1}{6}}{\frac{11}{24}} = \boxed{\frac{7}{11}}$

↳ F.U.V. - jer su zavisni

d) $(Z, W) = ?$ $Z = X+Y$, $W = X \cdot Y$, odredi razdiobu

Z\W	-1	0	1
-1	0	1/4	0
0	1/6	1/6	0
1	0	$\frac{1}{4}$	0
2	0	0	1/6

$P(-1,-1) = 0$

↳ zavisne, onda oba moraju biti istinita uvjeta

$w = -1 \rightarrow x = -1, y = 1$

↳ $z = 0$

↳ uz ovogu vidimo iz tablice prije

$w = 1 \rightarrow x = 1, y = 1$

$\Rightarrow z = 2$

$w = 0 \Rightarrow x = -1, y = 0 \Rightarrow z = -1$

$x = 0, y = 0 \Rightarrow z = 0$

$x = 0, y = 1 \Rightarrow z = 1$ $\rightarrow$

$x = 1, y = 0 \Rightarrow z = 1$ $\rightarrow$ $\Big\} \frac{1}{8} + \frac{1}{8}$

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3.3. Karakteristike slučajnih varijabli

- motivacija: a) bacili smo kocku 12 puta : očekivani broj šesti?

b) $X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0.05 & 0.15 & 0.6 & 0.15 & 0.05 \end{pmatrix}$ $\quad E(X) = 3$

c) X = broj na kocki

$E(X) = \frac{1+2+3+4+5+6}{6} = 3.5$ $\quad X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$

- def: očekivanje diskr. sl. var. definiramo sumom

$$E(X) = \sum_k x_k \cdot p_k$$

(ako taj red konvergira)

- ostale oznake: $\bar{X}$, m
- nap: $E(X)$ ne mora biti vrijednost od X

$\boxed{\text{MI - 2012}}$ U kutiji se nalazi 10 kuglica označenih brojevima $1, 2, \ldots, n$. Izvučemo 4 kuglice.

X = drugi najveći izvučeni broj

Odredi razdiobu i očekivanje od X

$$X \sim \begin{pmatrix} 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \frac{7}{\binom{10}{4}} & \frac{\binom{3}{2}\cdot 6}{\binom{10}{4}} & \frac{\binom{4}{2}\cdot 5}{\binom{10}{4}} & \frac{\binom{5}{2}4}{\binom{10}{4}} & \frac{\binom{6}{2}3}{\binom{10}{4}} & \frac{\binom{7}{2}2}{\binom{10}{4}} & \frac{\binom{8}{2}}{\binom{10}{4}} \end{pmatrix}$$

$E(X) = 6.6$

$\boxed{MI - 2017}$

16 jaja, 4 su čokoladna.

Mali Ivica na sreću uzima 3 jaja, vrati koja nisu čokoladna i ponovno na sreću uzima toliko.

Odredi očekivani broj čokoladnih jaja?

$$X \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ \left(\frac{\binom{12}{3}}{\binom{16}{3}}\right)^2 & & & \end{pmatrix}$$

$\underbrace{\qquad}_{0.154}$

$$P(X=1) = \frac{\binom{4}{1}\binom{12}{2}}{\binom{16}{3}} \cdot \frac{\binom{12}{2}}{\binom{15}{2}} + \frac{\binom{12}{3}}{\binom{16}{3}} \cdot \frac{\binom{4}{1}\binom{12}{2}}{\binom{16}{3}} = 0.482$$

$$P(X=2) = \frac{\binom{4}{2}\binom{12}{1}}{\binom{16}{3}} \frac{\binom{12}{1}}{\binom{14}{1}} + \frac{\binom{12}{3}}{\binom{16}{3}} \cdot \frac{\binom{4}{2}\binom{12}{1}}{\binom{16}{3}} + \frac{\binom{4}{1}\binom{12}{2}}{\binom{16}{3}} \cdot \frac{\binom{3}{1}\binom{12}{1}}{\binom{15}{2}}$$

$= 0.322$

$$P(X=3) = 1 - 0.322 - 0.482 - 0.154 = \boxed{0.042}$$

$$E(X) = \boxed{1.252}$$

$\boxed{MI - 2013}$

1B, 4C. Izvlačimo jedan po jedan dok ne izv. B.

X = broj izvl. $E(X) = ?$

a) bez vraćanja

$$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{1}{5} & \frac{4}{5}\cdot\frac{1}{4} & & & \frac{1}{5} \end{pmatrix} \qquad E(X) = 3$$

$\frac{4}{5}\cdot\frac{\cancel{3}}{\cancel{4}}\cdot\frac{1}{\cancel{3}}$ $\qquad$ $\frac{\cancel{4}}{5}\cdot\frac{\cancel{3}}{\cancel{4}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\frac{1}{\cancel{2}}$

b) s vraćanjem

$$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n & \dots \\ \frac{1}{5} & \frac{4}{5}\cdot\frac{1}{5} & \left(\frac{4}{5}\right)^2\cdot\frac{1}{5} & \dots & & \left(\frac{4}{5}\right)^{n-1}\frac{1}{5} & \end{pmatrix}$$

$$E(X) = \sum_{n=1}^{\infty} n\left(\frac{4}{5}\right)^{n-1}\frac{1}{5} = \frac{1}{5}\cdot\frac{1}{\left(1-\frac{4}{5}\right)^2} = \boxed{5}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \Big/ ' \qquad \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

c) ponavljamo dok drugi put ne izvučemo B

$$X \sim \begin{pmatrix} 2 & 3 & 4 & \dots & n & \dots \\ \left(\frac{1}{5}\right)^2 & \frac{4}{5}\cdot\left(\frac{1}{5}\right)^2\cdot 2 & \left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^2\cdot 3 & & \left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^{n-2}\cdot(n-1) & \end{pmatrix}$$

$$E(X) = \sum_{n=2}^{\infty} \left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^{n-2}\cdot(n-1)\cdot n = \frac{1}{25}\cdot\frac{2}{\left(1-\frac{4}{5}\right)^3} = \boxed{10}$$

TM (svojstva očekivanja)

a) linearnost: $E(sX+tY) = sE(X) + tE(Y), \ \forall s,t \in \mathbb{R}$

b) ako su X i Y nezavisni tada $E(XY) = E(X)\cdot E(Y)$

MI

Dokaz: a) $E(sX) = \sum_k s x_k p_k = s\sum x_k p_k = sE(X)$

$$E(X+Y) = \sum_{k,j}(x_k+y_j)p_{kj} = \sum_{k,j} x_k p_{kj} + \sum_{k,j} y_j p_{kj} =$$

$$= E(X) + E(Y)$$

b) $E(XY) = \sum_{k,j} x_k\cdot y_j p_{kj} = \sum x_k y_j p_k q_j = E(X)\cdot E(Y)$

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- Uoči: ako je $Y=\psi(X)$, onda je $E(Y)=\sum_k \psi(x_k)p_k$

- **def.** ishodišni moment reda n def. s

$$E(X^n)=\sum_k x_k^n p_k$$

Centralni moment reda n:

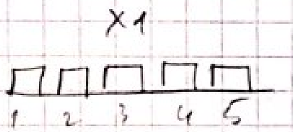
$$E[(X-\underbrace{E(X)}_{\text{broj } \bar{x}})^n]=\sum_k (x_k-\bar{x})^n p_k$$

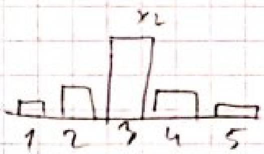
Najvažniji slučaj: $\boxed{n=2}$

- **def.** disperzija ili varijanca je centralni moment reda 2

$$D(X)=E[(X-E(X))^2]$$

$\boxed{Pr}$





$$=\sum_k (X_k-E(X))^2 p_k$$

$E(X_1)=3 \quad \longleftrightarrow \quad E(X_2)=3$

$D(X_1) \quad > \quad D(X_2)$

$\boxed{Pr}$

$$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}$$

$E(X)=3.5, \quad D(X)=2.92, \quad \sigma_X=1.71$

Vrijedi: $D(X)=E(X^2)-E(X)^2=\sum x_k^2 p_k-\bar{x}^2$

Dokaz: $D(X)=E[X^2-2XE(X)+E(X)^2]=$

$=E(X^2)-2E(X)E(X)+E(X)^2=E(X^2)-E(X)^2$

Primjedba: $D(X)\geq 0$ ($D(X)=0$ ako je $X=$ konst.)

$\sigma=\sqrt{D(X)}$ - standardna devijacija ili odstupanje

$$Y \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0.05 & 0.15 & 0.6 & 0.15 & 0.05 \end{pmatrix}$$

$E(Y) = 3 \qquad D(Y) = 0.7 \qquad \sigma_Y = 0.74$

MI - 2016

$$P(X = 2^k) = 6c^{-k}, \quad k = 2, 3, 4 \ldots$$

Odredi $c, E, D, P(X \geq 5)$?

$$X \sim \begin{pmatrix} 4 & 8 & 16 & \ldots \\ \frac{6}{c^2} & \frac{6}{c^3} & \frac{6}{c^4} & \ldots \end{pmatrix}$$

$$\sum_{k=2}^{\infty} \frac{6}{c^k} = 6 \cdot \frac{\left(\frac{1}{c}\right)^2}{1 - \frac{1}{c}} = 1$$ - suma svih vjv. mora biti 1.

$$c^2 - c - 6 = 0$$

$\cancel{c_1 = -2} \qquad \boxed{c_2 = 3}$

jer vjv. nemože biti vjv. negativna

$$P(X \geq 5) = 1 - P(X = 4) = 1 - \frac{6}{3^2} = \boxed{\frac{1}{3}}$$

$$E(X) = \sum_{k=2}^{\infty} 2^k \frac{6}{3^k} = 6 \sum_{k=2}^{\infty} \left(\frac{2}{3}\right)^k = 6 \cdot \frac{\left(\frac{2}{3}\right)^2}{1 - \frac{2}{3}} = \boxed{8}$$

$$D(X) = \sum_{k=2}^{\infty} (2^k)^2 \cdot \frac{6}{3^k} - 8^2 = 6 \sum_{k=2}^{\infty} \left(\frac{4}{3}\right)^k - 64 = \boxed{\infty}$$

↳ ne postoji

veće od 1 i divergira

TM (svojstva disperzije)

a) $D(sX) = s^2 D(X)$

b) ako su X i Y nezavisni

$$D(X+Y) = D(X) + D(Y)$$

Dokaz: a) $D(sX) = E((sX)^2) - (E(sX))^2$

$$= s^2 E(X^2) - s^2 E(X)^2$$
$$= s^2 (E(X^2) - (E(X))^2) = s^2 D(X)$$

b) $D(X+Y) = E((X+Y)^2) - (E(X+Y))^2$

$$= E(X^2 + 2XY + Y^2) - (E(X) + E(Y))^2$$
$$= E(X^2) + \cancel{2E(XY)} + E(Y)^2 - E(X)^2$$
$$- \cancel{2E(X)E(Y)} - E(Y)^2$$
$$= D(X) + D(Y)$$

MI-2011

Ako je $D(X) = 2$, $D(Y) = 4$, X, Y su nezavisni.

a) Koliko je $D(2X - Y) = ?$

b) Dokažite svojstva iz a)?

$$D(2X - Y) = D(2X + (-1)Y) = D(2X) + D(-Y) =$$
$$= 4D(X) + (-1)^2 D(Y) = \boxed{12}$$

$$\boxed{D(X - Y) = D(X) + D(Y)}$$

!

• Međusobni odnos 2. varijable

- def. kovarijacijski moment

$$cov(X,Y) = E((X-E(X))(Y-E(Y)))$$

$$\Rightarrow \boxed{cov(X,Y) = E(XY) - E(X)E(Y)}$$

- def. korelacija

$$r(X,Y) = \frac{cov(X,Y)}{\sigma_X \sigma_Y}$$

- uoči: ako su X i Y nezavisni, $cov(X,Y) = 0 = r(X,Y)$

↳ obrat NE vrijedi

(ako je $r(X,Y)=0$ nemamo kakva je zavisnost)

TM. Disperzija zbroja - općenito

$$\boxed{D(X+Y) = D(X) + D(Y) - 2\,cov(X,Y)}$$

$\boxed{\text{MI-2016}}$

MI OVE GOD

Bacamo 2 tetraedra, neka je X veći od 2 broja ako su različiti, a 0 ako su isti. Y je kvadrat razlike na brojevima.

X|Y	0	1	4	9	
0	$\frac{4}{16}$	0	0	0	$\frac{1}{4}$
2	0	$\frac{2}{16}$	0	0	$\frac{1}{8}$
3	0	$\frac{2}{16}$	$\frac{2}{16}$	0	$\frac{1}{4}$
4	0	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{3}{8}$
	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	1

$E(X) = \frac{5}{2}$ $\quad D(X) = \frac{5}{2}$

$E(Y) = \frac{5}{2}$ $\quad D(Y) = \frac{33}{4}$

$$r(X,Y) = \frac{\frac{23}{8}}{\sqrt{\frac{5}{2} \cdot \frac{33}{4}}} = \boxed{0.633}$$

$$E(XY) = \sum_i \sum_j x_i y_j p_{ij} = \frac{73}{8} \qquad cov(X,Y) = \frac{23}{8}$$

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• Centriranje i normiranje slučajnih var.

- ako znamo razdiobu od X, znamo i od $X-a$, $a \in \mathbb{R}$
tada $E(X-a) = E(X) - a$, $D(X-a) = D(X)$

- ne mijenja se ni cov ni r, ρ.

$$cov(X-a, Y-b) = cov(X, Y)$$

$$r(X-a, Y-b) = r(X, Y)$$

- izaberemo li $a = m_X = E(X)$: $\boxed{\dot{X} = X - m_X}$ zovemo centrirana varijabla

- $E(\dot{X}) = 0$, $D(\dot{X}) = D(X)$

- pogledajmo $aX + b$: želimo

$$D(aX+b) = 1$$
$$a^2 D(X) = 1$$
$$a = \frac{1}{\sigma_X}$$

$$\boxed{X^* = \frac{X - m}{\sigma}}$$ - normirana slučajna varijabla

Dosta, $E(X^*) = \frac{E(X-m)}{\sigma} = 0$

$$D(X^*) = \frac{1}{\sigma^2} D(X-m) = 1$$

Tada:

$$r(X^*, Y^*) = \frac{E(X^* Y^*) - \overbrace{E(X^*) E(Y^*)}^{0}}{\underbrace{\sigma_{X^*} \sigma_{Y^*}}_{1}} =$$

$$= E\left(\frac{X - m_X}{\sigma_X} \cdot \frac{Y - m_Y}{\sigma_Y}\right) = \frac{cov(X, Y)}{\sigma_X \sigma_Y}$$

$$\boxed{r(X^*, Y^*) = r(X, Y)}$$

c) Prethodni zadatak - term edar

$U = X - E(X)$, $V = Y - E(Y)$

$r(U,V) = ?$ → centriranje

$r(U,V) = \boxed{0.633} = r(X,Y)$

- IM: vrijedi $|r(X,Y)| \leq 1$

Dokaz:
M1

$$D(X^* \pm Y^*) = \underbrace{D(X^*)}_{1} + \underbrace{D(Y^*)}_{1} \pm 2cov(X^*, Y^*)$$

$$= 2 \pm 2cov(X^*, Y^*) =$$

$$= 2(1 \pm r(X,Y)) \geq 0 \text{ - disperzija}$$

$$\Rightarrow |r(X,Y)| \geq 1$$

Napomena:

$$r(X,Y) = \pm 1 \text{ ako } \boxed{Y = aX + b}$$

- Karakteristične funkcije

- def. karak. fja slučajne varijable X definirana je s

$$\boxed{\vartheta_X(t) = E(e^{itX})}$$

- za DSV:

$$\vartheta_X(t) = \sum_k e^{itx_k} \cdot p_k$$

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TM. (svojstva): 1) Karakteristična fja jednoznačno određuje sl. varijablu

2) Ako su $X_1, \dots, X_n$ nezavisne, tada

$$\vartheta_{X_1+\dots+X_n}(t) = \vartheta_{X_1}(t)\cdots\vartheta_{X_n}(t)$$

3) $$E(X^n) = \frac{\vartheta^{(n)}(0)}{i^n}$$

Posebno za $n=1$: $E(X) = i\,\vartheta'(0)$

$n=2$: $E(X^2) = -\vartheta''(0)$

RAD

$X \in \{-2,-1,0,1,2\}$ s jednolikom razdiobom

$$X \sim \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{pmatrix}$$

$$\vartheta_X(t) = e^{-2it}\frac{1}{5} + e^{-it}\frac{1}{5} + e^{0}\frac{1}{5} + e^{it}\frac{1}{5} + e^{2it}\frac{1}{5}$$

$$= \frac{1}{5} + \frac{2}{5}\cos t + \frac{2}{5}\cos 2t$$

$E(X) = -i\,\vartheta'(0) = 0$

4. PRIMJERI DISKRETNIH RAZDIOBA

4.1. Geometrijska razdioba

= ponavljamo pokus do prve pojave željenog ishoda

X - broj ponavljanja pokusa; p = vj. željenog ishoda u jednom ponavljanju

$$X \sim \begin{pmatrix} 1 & 2 & 3 & \dots & n & \dots \\ p & (1-p)p & (1-p)^2 p & & (1-p)^{n-1} p & \end{pmatrix}$$

- oznaka za geom. razdiobu

$$X \sim \mathcal{G}(p)$$

$$P(X=k)=(1-p)^{k-1}p, \quad k=1,2,3\dots$$

- očekivanje: $E(X)=\sum_{n=1}^{\infty} n(1-p)^{n-1}p = p\cdot\frac{1}{(1-(1-p))^2}=$

$$=\boxed{\frac{1}{p}}$$

- disperzija: $D(X)=\boxed{\frac{1}{p^2}-\frac{1}{p}}$

- karak. fja:

$$\vartheta_X(t)=\sum_{n=1}^{\infty} e^{itn}\cdot(1-p)^{n-1} =$$

$$=\frac{p}{1-p}\sum_{n=1}^{\infty}\underbrace{\left(e^{it}(1-p)\right)}_{x}{}^{n} =$$

$$=\frac{p}{1-p}\cdot\frac{e^{it}(1-p)}{1-e^{it}(1-p)}=\frac{p\,e^{it}}{1-e^{it}(1-p)}$$

$\boxed{MI-2011}$

c) 52 karte → dok ne izvučemo asa ili trefa

X - broj izvlačenja

$p=\frac{16}{52}$

$E(X)=\frac{1}{p}=\frac{13}{4}$

$$P(X>E(X))=P(X>3.25)=$$

$$=1-P(X=1)-P(X=2)-P(X=3)=\left(\frac{9}{13}\right)^3$$

$$\boxed{P(X>n)=(1-p)^n}, \; n\in\mathbb{N}$$

TM. (odsustvo pamćenja)

X ima geom. razdiobu akko vrijedi:

$$P(X=m+k \mid X>k) = P(X=m)$$

Dokaz: ($\Rightarrow$)

$$P(X=m+k \mid X>k) = \frac{P(X=m+k)}{P(X>k)} =$$

$$= \frac{(1-p)^{m+k-1} p}{(1-p)^k} = (1-p)^{m-1} p = P(X=m)$$

$\boxed{\text{PR.}}$ Ako šestica nije pala u prva 3 bacanja, vj. da padne u četvrtom?

$$P(X=4 \mid X>3) = P(X=1) = \frac{1}{6}$$

$$P(X \leq 5 \mid X>3) = P(X \leq 2) = P(X=1) + P(X=2) =$$

4.2. BINOMNA RAZDIOBA

- ponovili smo pokus n puta, X = koliko puta (od n) se dogodio povoljni ishod

$$X \sim B(n,p), \quad P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k=0,1,\dots,n$$

$\boxed{\text{Pr}}$. Ako smo 12 puta bacili kocku → očekivani broj šestica: 2

- prolaznost VIS-a: 56% → očekivani broj studenata koji su prošli od 200 → 112

- očekivanje: $E(X) = \sum_{k=0}^{n} k\binom{n}{k} p^k (1-p)^{n-k}$

- karakteristična f-ja:

$$\vartheta_X(t) = \sum_{k=0}^{n} e^{itk}\binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (pe^{it})^k (1-p)^{n-k}$$

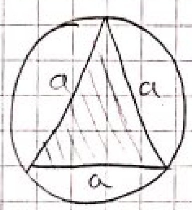
$$= (pe^{it} + 1-p)^n$$

$$\vartheta'(t) = n(pe^{it}+1-p)^{n-1} \cdot pe^{it} \cdot i$$

$$E(X) = -i\vartheta'(0) = np$$

$$D(X) = E(X^2) - E(X)^2 = np(1-p)$$

$\boxed{2A)}$



Od 10 točaka unutar kruga, barem 2 unutar $\triangle$.

$$X \sim B(10, p)$$

$$p = \frac{P_\triangle}{P_O} = \frac{\frac{a^2\sqrt{3}}{4}}{\left(\frac{2}{3}\frac{a\sqrt{3}}{2}\right)^2\pi} = \frac{3\sqrt{3}}{4\pi} = 0.413$$

$$P(X \geq 2) = 1 - P(X=0) - P(X=1) = 1-(1-p)^{10} - \binom{10}{1}p(1-p)^9$$

$$=$$

AUDITORNE

(1.)

X = broj jedinica koje su se pojavile u svim bacanjima

a) $X \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{41}{64} & \frac{22}{64} & \frac{1}{64} \end{pmatrix}$

$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 \qquad \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \binom{2}{1} \qquad \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4}$

$$E(X) = \sum_{i=0}^{2} i\, P(X=i) = \frac{22}{64} + \frac{2}{64} = \frac{24}{64} = \boxed{\frac{3}{8}}$$

$$D(X) = E\left[X - E(X)^2\right] = E[X^2] - (E(X))^2 =$$

$$= \frac{22}{64} + \frac{4}{64} - \frac{9}{64} = \frac{17}{64}$$

(2.)

Negdje plaćaju na 2 od 6 šaltera. Građanin ide po redu dok ne uspije platit.

X ~ broj pokušaja u kojem je uspio platiti račun

$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{2}{6} & \frac{4}{6}\frac{2}{5} & \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} & & \end{pmatrix}$

$\frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \qquad \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{2}$

$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{5}{15} & \frac{4}{15} & \frac{3}{15} & \frac{2}{15} & \frac{1}{15} \end{pmatrix}$

$E(X) = \frac{2}{3} \qquad D(X) = \frac{14}{9}$

(3.)

$$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \ldots \\ p & p^2 & p^3 & p^4 & p^5 \ldots \end{pmatrix}$$

a) Odredite p!

b) Odredite $E(X), D(X) = ?$

$$1 = \sum_{i=1}^{+\infty} p^i = \frac{p}{1-p} \quad \Rightarrow p = \frac{1}{2}$$

$$E(X) = \sum_{i=1}^{+\infty} i p^i = \frac{p}{(1-p)^2} = 2$$

$$\frac{1}{1-x} = \sum x^i \; /'$$

$$\frac{1}{(1-x)^2} = \sum i x^{i-1} = \sum i x^{i-1} \; / \cdot x$$

$$\frac{x}{(1-x)^2} = \sum_{i=1}^{\infty} i x^i$$

$$D(X) = E(X^2) - (E(X))^2 = 6 - 2^2 = 2$$

(4.)

$X \mid Y$	-1	2	
-1	$1/6$	$1/4$	$\frac{10}{24}$
0	$1/6$	$1/8$	$\frac{7}{24}$
2	$1/6$	$1/8$	$\frac{7}{24}$
	$\frac{1}{2}$	$\frac{1}{2}$	1

a) X i Y nezavisne sl. var?

b) raspodjela $(Z, W) = ?$

$$Z = \frac{X}{X+Y}, \quad W = X - Y$$

a) $\forall x, \forall y$

$$P(X = x, Y = y) = P(X = x) P(Y = y)$$

$$\frac{1}{6} \neq \frac{10}{24} \cdot \frac{1}{2} = \frac{5}{24} \quad \rightarrow \text{nisu nezavisne}$$

X	Y	p_{ij}	$\frac{X}{X+Y}$	$X-Y$
-1	-1	1/6	1/2	0
-1	2	1/4	-1	-3
0	-1	1/6	0	1
0	2	1/8	0	-2
2	-1	1/6	2	3
2	2	1/8	1/2	0

z\w	-3	-2	0	1	3
-1	1/4	0	0	0	0
0	0	1/8	0	1/6	0
1/2	0	0	$\frac{7}{24}$	0	0
2	0	0	0	0	1/6
	1/4	1/8	$\frac{7}{24}$	1/6	1/6

d) $$P(z>0 \mid w \geq 0) = \frac{P(z>0, w\geq 0)}{P(w\geq 0)} = \frac{\frac{7}{24}+\frac{1}{6}}{\frac{7}{24}+\frac{1}{6}+\frac{1}{6}}$$

- Stabilnost binomne razdiobe:

$$\left.\begin{matrix} X_1 \sim B(n_1, p) \\ X_2 \sim B(n_2, p) \end{matrix}\right\} \text{ neka su nezavisne}$$

Dokaz:

$$\vartheta_{X_1+X_2}(t) = \vartheta_{X_1}(t)\cdot\vartheta_{X_2}(t) =$$

$$= (pe^{it}+p)^{n_1}\cdot(pe^{it}+1-p)^{n_2} =$$

$$= (pe^{it}+1-p)^{n_1+n_2} \Rightarrow B(n_1+n_2, p)$$

- Bernoullijeva ili indikatorska slučajna var.

$$X_i \sim \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix} \sim B(1,p), \quad E(X_i)=p, \quad D(X_i) = p(1-p)$$

$$\Rightarrow X = X_1+X_2+\ldots+X_n \sim B(n,p)$$

$$E(X) = E(X_1)+\ldots+E(X_n) = n\cdot p, \quad D(X) = np(1-p)$$

4.3. Poissonova razdioba

Pr.

U 10 min stigne 300 zahtjeva na obradu.

-> prosječan broj zahtjeva po minuti: $E(X)=30=300\cdot\frac{1}{10}=n\cdot p=\lambda$

- def.

Poissonova razdioba se definira:

$$P(X=k)=\frac{\lambda^k}{k!}e^{-\lambda},\quad k=0,1,2,\dots,$$

oznaka $X\sim\mathcal{P}(\lambda)$

X - broj jedinica koje su ušle u sustav u jedinici vremena

- karak. fja:

$$\vartheta_X(t)=\sum_{k=0}^{\infty}e^{itk}\frac{\lambda^k}{k!}e^{-\lambda}=e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^{it})^k}{k!}=e^{-\lambda}\cdot e^{\lambda e^{it}}$$

$$=\boxed{e^{\lambda(e^{it}-1)}}$$

$$E(X)=-i\vartheta'(0)=\dots=\underline{\underline{\lambda}}\quad\text{ili}\quad E(X)=\sum_{k=0}^{\infty}k\frac{\lambda^k}{k!}e^{-\lambda}=\underline{\underline{\lambda}}$$

$$D(X)=\underbrace{E(X^2)}_{-\vartheta''(0)}-\lambda^2=\dots=\lambda$$

MI-2008

180 mailova tijekom 1h. Vj. da u jednoj min stignu barem 3 maila?

$\lambda=180\ m/h=3\ m/min$

$$P(X\geq 3)=1-P(X=0)-P(X=1)-P(X=2)$$

$$=1-e^{-3}-3e^{-3}-\frac{3^2}{2}e^{-3}=0.5768$$

$\boxed{\text{MI-2013}}$

40 a/h. Samo 1 pumpa - 1 min/a, vj. da se pojavi red na pumpi

$\lambda = \frac{40}{60} = \frac{2}{3}$ po min

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$
$$= 1 - e^{-\frac{2}{3}} - \frac{2}{3} e^{-\frac{2}{3}} = 0.144$$

Ako se otvori još jedna pumpa:

$$P(X \geq 3) = 1 - P(X=0) - P(X=1) - P(X=2)$$
$$= 0.03$$

VIS SIS BI

- stabilnost Poissonove razdiobe : $\left.\begin{matrix} X_1 \sim P(\lambda_1) \\ X_2 \sim P(\lambda_2) \end{matrix}\right\}$ ako su nezavisni

$\Rightarrow X_1 + X_2 \sim P(\lambda_1 + \lambda_2)$

Dokaz: $\vartheta_{X_1+X_2}(t) = \vartheta_{X_1}(t) + \vartheta_{X_2}(t) =$

$$= e^{\lambda_1(e^{it}-1)} \cdot e^{\lambda_2(e^{it}-1)} = e^{(\lambda_1+\lambda_2)(e^{it}-1)}$$

$\boxed{\text{MI-2016}}$

b) prvi red 120 z/min, 2 sek za obradu zahtjeva
drugi red 90 z/min
Vjv. da će ukupno barem 2 zahtjeva čekati na obradu

$X_1 \sim P(\lambda_1 = 4 \text{ z}/2\text{s})$

$X_2 \sim P(\lambda_2 = 3 \text{ z}/2\text{s})$

$X = X_1 + X_2 \sim P(\lambda = 7)$

$P(X \geq 3) = 1 - P(0) \ldots$

$= \boxed{0.97}$

$\boxed{\text{TM.}}$ APROKSIMACIJE BINARNE RAZDIOBE POISSONOVOM

Neka je n velik, a p malen, te neka je $\lambda = np$.

Tada $B(n,p) \approx P(\lambda)$, tj. $\binom{n}{k} p^k (1-p)^{n-k} \to \frac{\lambda^k}{k!} e^{-\lambda}$

Dokaz: uvrstimo $p = \frac{\lambda}{n}$

$$\binom{n}{k}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} = \frac{1}{k!} \frac{n(n-1)\cdot\ldots\cdot(n-k+1)}{n^k} \lambda^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{\lambda^k}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\left(1-\frac{\lambda}{n}\right)^n \cdot \left(1-\frac{\lambda}{n}\right)^{-k} \quad / \lim_{n\to\infty}$$

$$\Rightarrow \frac{\lambda^k}{k!} e^{-\lambda}$$

$\boxed{\text{MI-2012}}$ b) 99% ispravnih, 1% neispravnih

Vjv. da od 300 proizvoda budu barem 4 neispravna?

$$X \sim B(300, 0.01) \text{ ili } \approx P(\lambda = np = 3)$$

$$P(X \geq 4) = 1 - P(X=0) - P(X=1) - P(X=2) - P(X=3)$$

$$= 1 - 0.99^{300} - \binom{300}{1} 0.01 \cdot 0.99^{299} - \ldots =$$

$$= \boxed{0.352}$$

S aproksim.: $P(X \geq 4) = 1 - e^{-3} - 3e^{-3} - \frac{3^2}{2!}e^{-3} - \frac{3^3}{3!}e^{-3}$

$$= \boxed{0.35276}$$

MI-2015

1000 projektila, vjer. pogotka je 0.005
Ako je brod pogođen, vj. da bude neutraliziran je 0.05. Vjer. da brod bude neutraliziran?

$n = 1000$ $p = 0.005 \cdot 0.05 \to$ vj. za 1 projektil

$\mathcal{P}(\lambda = np = 0.25) =$

$$P(X \geq 1) = 1 - P(X=0) = 1 - e^{-0.25} = \boxed{0.22}$$

5. NEPREKINUTE SLUČAJNE VAR.

5.1. Definicije i svojstva

- def. preslikavanje $X : \Omega \to \mathbb{R}$ je sl. var. ako $\forall x \in \mathbb{R}$ je skup $A_x = \{\omega \in \Omega : X(\omega) < x\}$ događaj.

Funkcija razdiobe $F : \mathbb{R} \to [0,1]$ je definirana s

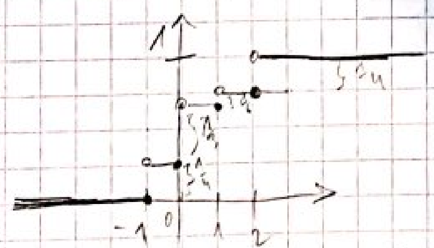
$$\boxed{F(x) = P(X < x)}$$

ZAD

$$X \sim \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1/4 & 1/3 & 1/6 & 1/4 \end{pmatrix}$$

(diskretna)

$F(x)$ je stepenasta fja



$F(-2) = 0$ $F(-1) = 0$ $F(-0.999) = \frac{1}{4}$ $F(0) = \frac{1}{4}$

$F(1) = \frac{7}{12}$

TM (svojstva od F):

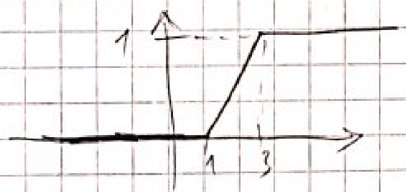
1) $P(a \leq X < b) = F(b) - F(a)$

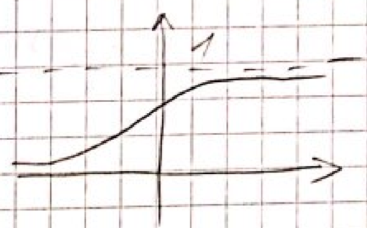
2) $F(x)$ je rastuća fja

3) $\lim\limits_{x \to -\infty} F(x) = 0$ $\lim\limits_{x \to \infty} F(x) = 0$

4) $F(x)$ je neprekinuta s lijeva

Primjeri fja razdiobe:

a) 

b) 

- def. za sl. var. X kažemo da je NEPREKINUTA ako postoji nenegativna fja $f : \mathbb{R} \to \mathbb{R}_0^+$ takva da:

$$P(X < x) = \boxed{F(x) = \int_{-\infty}^{x} f(t)\,dt}$$

funkcija $f(x)$ se naziva fja gustoće i vrijedi

$$\boxed{f(x) = F'(x)}$$

- vjv. je površina ispod grafa fje gustoće

$$P(a < X < b) = \int_a^b f(x)\,dx = F(b) - F(a)$$

Uočimo: $P(a < X < b) = P(a \leq X \leq b) = P(a < X \leq b)$

$\boxed{\text{MI-2016}}$ Mogu li biti jednake?

$F(x)$ ne može.

$f(x)$ može

Uvjet za fju gustoće:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

- očekivanje:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

- disperzija

$$D(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - E(X)^2$$

- karakt. fja

$$\vartheta_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$$

$\boxed{\text{MI-2010}}$

$$f(x) = 2 - Cx, \quad x \in [0,1]$$

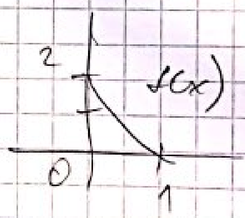
Odredi sve: $C, E, D, F, P\left(\frac{1}{2} < X \leq \frac{3}{2}\right)$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 (2 - Cx)\,dx = 2 - \frac{C}{2} = 1 \Rightarrow \boxed{C = 2}$$

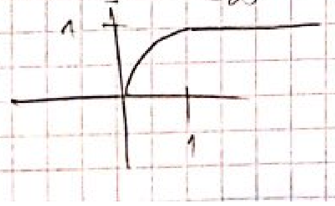
$$E(X) = \int_0^1 x \cdot (2 - 2x)\,dx = \boxed{\frac{1}{3}}$$

$$D(X) = \int_0^1 x^2 (2 - 2x)\,dx - \left(\frac{1}{3}\right)^2 = \boxed{\frac{1}{18}}$$

$$P\left(\frac{1}{2} < X \leq \frac{3}{2}\right) = \int_{1/2}^{1} (2 - 2x)\,dx = \boxed{\frac{1}{4}}$$



$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x (2 - 2t)\,dt = \boxed{2x - x^2, \ x \in [0,1]}$$



$\boxed{\text{MI - 2011}}$

$$f(x) = \begin{cases} x^2, & x \in [0,1\rangle \\ 2x - x^2, & x \in [1,2\rangle \end{cases}$$

$E, F, P(\frac{1}{2} < x < \frac{3}{2})$

$$E(X) = \int_0^1 x \cdot x^2 dx + \int_1^2 x \cdot (2x - x^2) dx = \boxed{\frac{7}{6}}$$

$$P\left(\frac{1}{2} < x < \frac{3}{2}\right) = \int_{1/2}^1 x^2 dx + \int_1^{3/2} (2x - x^2) dx = \boxed{\frac{3}{4}}$$

- za $x \in [0,1\rangle$: $F(x) = \int_{-\infty}^x f(t) dt = \int_0^x t^2 dt = \frac{x^3}{3}$

- za $x \in [1,2\rangle$: $F(x) = \int_0^1 t^2 dt + \int_1^x (2t - t^2) dt =$

  ↳ za drugi interval uvijek pokupiti i prijašnji

$$= \boxed{\frac{-1}{3} x^3 + x^2 - \frac{1}{3}}$$

- za $x > 2$: $F(x) = 1$

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^3}{3}, & x \in [0,1\rangle \\ \frac{-1}{3} x^3 + x^2 - \frac{1}{3}, & x \in [1,2\rangle \\ 1, & x \geq 2 \end{cases}$$

$\boxed{\text{MI - 2017}}$ Odredite a i b

$F(x) = a \frac{e^x - 1}{e^x + 2} + b, \; x \in \mathbb{R}$ bude F razdiobe.

$$P(X > 1) = ? = 1 - P(X \leq 1) = 1 - F(1) = \boxed{0.545}$$

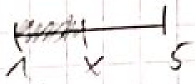
$$\lim_{x \to -\infty} \left(a \frac{e^x - 1}{e^x + 2} + b\right) = \frac{-1}{2} a + b = 0$$

$$\lim_{x \to +\infty} \left(a \underbrace{\frac{e^x - 1}{e^x + 2}}_{\to 1} + b\right) = a + b = 1$$

$$a = \frac{2}{3}, \quad b = \frac{1}{3}$$

Od sada sami do fja gustoće

ZAD)



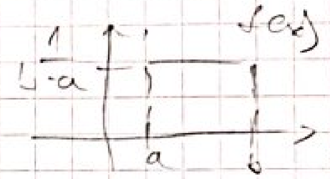
$X \in [1,5]$ - na sreću biramo broj
kont. broj

$$F(x) = P(X < x) = \frac{x-1}{4}$$

$$f(x) = F'(x) = \frac{1}{4}$$

$$E(X) = \int_1^5 x \cdot \frac{1}{4}\, dx = 3$$

Općenito:



$X \sim U[a,b]$
jednolika razdioba

$$\boxed{f(x) = \frac{1}{b-a}}$$

$$E(X) = \frac{a+b}{2}$$

$$D(X) = \frac{(b-a)^2}{12}$$

AUDITORNE

1. 2 igrača naizmjence bacaju simetričan novčić. Pobjednik je onaj kojem prvom padne glava. Nagrada $2^{\text{broj bacanja}}$. Vjv. da dobije više od 200 kn prvi igrač?

X = broj bacanja do pojave glave

$$X \sim G\left(\frac{1}{2}\right)$$

$$P(X = k) = \left(\frac{1}{2}\right)^{k-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^k$$

$$P(X \in \{51, 53, \dots\}) = \sum_{k=25}^{\infty} P(X = 2k+1) = \sum_{k=25}^{\infty} \left(\frac{1}{2}\right)^{2k+1} =$$

$$= \frac{1}{2} \sum_{k=25}^{\infty} \left(\frac{1}{4}\right)^k = \frac{1}{2} \cdot \frac{\left(\frac{1}{4}\right)^{25}}{1 - \frac{1}{4}} = \frac{1}{6 \cdot 4^{24}} =$$

(2.) Konobar počinje s 0 kn. Od svakog dobije napojnicu od 5 ili 10 kn. Vjr. da je 5 kn dobio je duplo veća. Vjr. da je dobio manje od 30 kn od 4 gosta.

X = broj većih napojnica

$X \sim B(4, \frac{1}{3})$

vjv da je dobio veću napojnicu

$$P(10X + 5\cdot(4-X) < 30) = P(5X < 10) = P(X < 2)$$

$$= P(X=0) + P(X=1) = \left(\frac{2}{3}\right)^4 + \binom{4}{1}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3 = \boxed{\frac{16}{27}}$$

(3.)

$X \sim P(\lambda)$ - Poissonova $P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}, k \in N_0$

$P(X=1) = P(X=2)$

$E(X), D(X), P(X \geq 5) = ?$

$$\frac{\lambda^1}{1!} \cancel{e^{-\lambda}} = \frac{\lambda^2}{2!} \cancel{e^{-\lambda}}$$

$$\lambda = \frac{\lambda^2}{2} \Rightarrow \cancel{\lambda = 0} \quad \boxed{\lambda = 2}$$

$E(X) = \lambda = 2 \qquad D(X) = \lambda = 2$

$$P(X \geq 5) = 1 - P(X < 5) = 1 - (P(X=1) + P(X=2) + P(X=3) + P(X=4))$$

$$= 1 - \left(e^{-2} + 2e^{-2} + \frac{2^2}{2}e^{-2} + \dots\right) = \boxed{0.576}$$

(4.)

Mjesta za 300 putnika. Vjv. da će putnik zakasniti 0.01, kompanija prodala 302 karte. Vjv. da će biti mjesta za sve?

X = broj putnika koji su zakasnili

$$X \sim B(302, 0.01) \approx \mathcal{P}(3.02)$$

$$P(X \geq 2) = 1 - P(X<2) = 1 - (P(X=0) + P(X=1))$$

$$= 1 - \left(0.99^{302} + \binom{302}{1} 0.01 \cdot 0.99^{301}\right)$$

$$= \boxed{0.8053}$$

$$P(X \geq 2) = 1 - \left(\frac{3.02^0}{0!} e^{-3.02} + \frac{3.02^1}{1!} e^{-3.02}\right) = \boxed{0.8038}$$

(5.)

$$X \quad f(x) = C e^{-|x|}, \; x \in \mathbb{R}$$

$$C = ?,\; D(X),\; E(X),\; P(-2 \leq X < 2) = ?$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} C e^{-|x|} dx = C \int_{-\infty}^{0} e^{x} dx + C \int_{0}^{\infty} e^{-x} dx =$$

$$= C e^{x}\Big|_{-\infty}^{0} - C e^{-x}\Big|_{0}^{\infty} = C + C = 2C = 1$$

$$\rightarrow \boxed{C = \frac{1}{2}}$$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \frac{1}{2} \int_{-\infty}^{\infty} x e^{-|x|} dx = 0$$

neparna f-ja na simetr. intervalu

$$D(X) = E(X^2) - E(X)^2 = E(X^2) =$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x^2 e^{-|x|} dx = \frac{1}{2} \cdot 2 \int_{0}^{+\infty} x^2 e^{-|x|} dx = \int_{0}^{+\infty} x^2 e^{-x} dx =$$

parna

$$= \left| \begin{matrix} u = x^2 & dv = e^{-x} \\ du = 2x & v = -e^{-x} \end{matrix} \right| = -x^2 e^{-x}\Big|_{0}^{\infty} + 2\int_{0}^{+\infty} x e^{-x} dx =$$

$$= \left| \begin{matrix} u = x & dv = e^{-x}dx \\ du = dx & v = -e^{-x} \end{matrix} \right| =$$

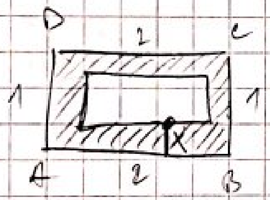
$$= 2\Big[\underbrace{x e^{-x}\Big|_0^{\infty}}_{0} + \underbrace{\int_0^{+\infty} e^{-x}dx}_{1}\Big] = \boxed{2}$$

$$P(-2 \leq X < 2) = \int_{-2}^{2} f(x)dx = \int_{-2}^{2} \frac{1}{2} e^{-|x|} dx =$$

$$= \frac{1}{2} \cdot 2 \int_0^2 e^{-x} dx = -e^{-x}\Big|_0^2 = -e^{-2} + 1 =$$

$\boxed{MI - 2015}$

Biramo točku unutar pravokutnika ABCD stranica 1 i 2.



X = udaljenost do najbliže stranice

$E(X) = ?$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$F(x) = P(X < x) = \frac{2 - (1-2x)(2-2x)}{2} = 3x - 2x^2, \quad x \in (0, 0.5)$$

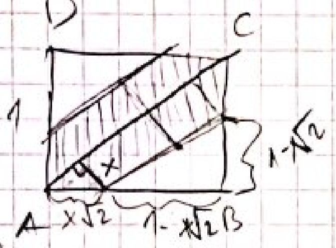
$$f(x) = F'(x) = 3 - 4x, \quad x \in (0, 0.5)$$

$$E(X) = \int_0^{1/2} x f(x) dx = \boxed{\frac{5}{24}}$$

Kada uvrstimo veću treba ispasti 1, a za manju 0

Mora biti rastuća fja

M1-2014



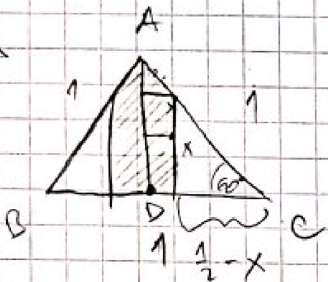
X - udalj. točke do $\overline{AC}$

$F(x) = ?$

$$F(x) = P(X<x) = \frac{1 - 2\cdot\frac{1}{2}(1-x\sqrt2)^2}{1} = 1-(1-x\sqrt2)^2$$

$$= 1-(1-2x\sqrt2+2x^2) = \boxed{2x\sqrt2-2x^2}, \quad x\in\left(0,\frac{\sqrt2}{2}\right)$$

M1-2013



X = udaljenost točke do visine $\overline{AD}$

$$F(x) = P(X<x) = \frac{\frac{\sqrt3}{4} - (\frac12-x)(\frac12-x)\sqrt3}{\frac{1^2\sqrt3}{4}} =$$

$$\operatorname{tg} 60^\circ = \frac{y}{\frac12-x}$$

$$= \boxed{1-4\left(\frac12-x\right)^2}, \quad x\in\left(0,\frac12\right)$$

$$\sqrt3 = \frac{y}{\frac12-x} \Rightarrow y=\left(\frac12-x\right)\sqrt3$$ - visina malog trokuta

$$f(x) = 8\left(\frac12-x\right) = 4-8x, \quad E(X) = \int_0^{1/2} x(4-8x)\,dx = \boxed{\frac16}$$

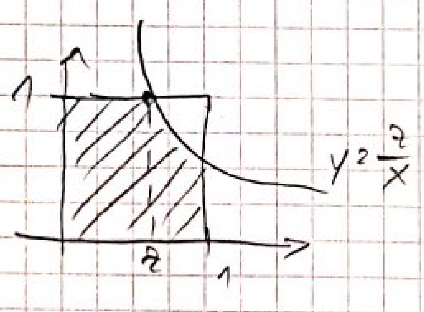
M1-2017

$X, Y \in (0,1)$, $Z = X\cdot Y$, $F(z) = ?$, $E(Z) = ?$

$$E(Z) = E(XY) = \frac12\cdot\frac12 = \boxed{\frac14}$$

konkretna vrijednost neka

$$F(z) = P(Z<z) = P(XY<z) = \left(Y<\frac{z}{x}\right)$$



$$= \frac{z\cdot 1 + \int_z^1 \frac{z}{x}\,dx}{1} = z + z\ln x\Big|_z^1 =$$

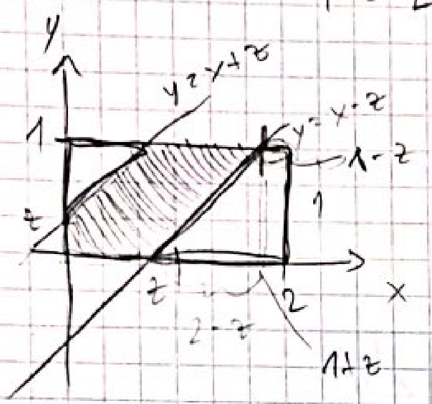
$$= \boxed{z - z\ln z}, \quad z\in(0,1)$$

11-2012

$$\left.\begin{array}{l} X \in [0,2] \\ Y \in [0,1] \end{array}\right\} Z = |X-Y|$$

$F(z) = ?$
$E(Z) = ?$

$Z \in [0, 2]$



$F(z) = P(Z < z) = P(|X-Y| < z)$

$|x - y| < z$

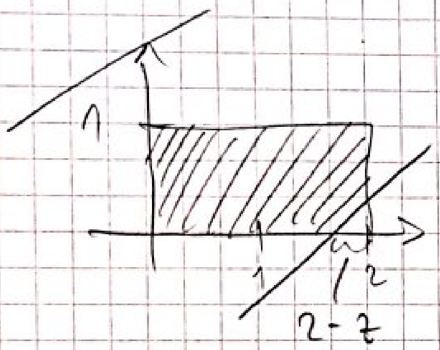
$x - y < z \quad y - x < z$

1. $z \in [0,1]$: trapez

$$F(z) = \frac{2 - \frac{1}{2}(1-z)^2 - \frac{(2-z+1-z)\cdot 1}{2}}{2}$$

$$= \boxed{z - \frac{1}{4}z^2}, \; z \in (0,1)$$

2. $z \in [1,2]$



$$F(z) = \frac{2 - \frac{1}{2}(2-z)^2}{2} = \boxed{z - \frac{1}{4}z^2}, \; z \in (1,2)$$

$f(z) = \boxed{1 - \frac{1}{2}z}, \; z \in (0,2)$

$E(Z) = \int_0^2 z\left(1 - \frac{1}{2}z\right) dz = \boxed{\frac{2}{3}}$

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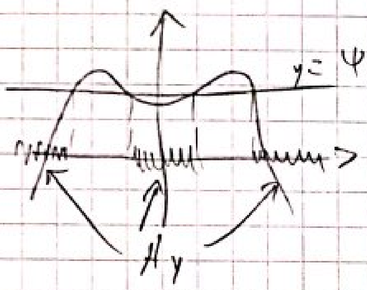
5.2. FUNKCIJE SLUČAJNIH VARIJABLI

- neka je $\psi : \mathbb{R} \to \mathbb{R}$ i X sl. var., te
  neka je $Y = \psi(X)$
- ako znamo razdiobu od X, možemo li odrediti razdiobu od Y?

$$G(y) = P(Y < y) = P(\psi(X) < y) =$$



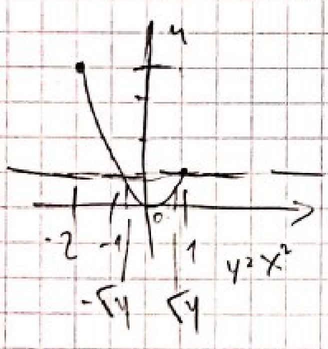
$$= P(X \in A_y)$$

$\boxed{\text{MI-2012}}$

$Y = X^2$, ako X ima jednoliku razdiobu na $[-2, 1]$ $\;\big|\;$ $g(y) = ?$

$f(x) = \frac{1}{3}$, $x \in [-2, 1]$

↳ za jednoliku $f(x)$ je 1 kroz duljina intervala



$$G(y) = P(Y < y) = P(X^2 < y), \quad y \in [0, 4]$$

1. slučaj $y \in [0, 1]$:

$$G(y) = P(-\sqrt{y} < X < \sqrt{y}) =$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{3}\, dx = \boxed{\frac{2\sqrt{y}}{3}}$$

2. slučaj $y \in [1, 4]$: $G(y) = P(-\sqrt{y} < X < 1) =$

$$= \boxed{\frac{1 + \sqrt{y}}{3}}$$

$$g(y) = G'(y) = \begin{cases} \frac{1}{3\sqrt{y}}, & y \in (0,1) \\ \frac{1}{6\sqrt{y}}, & y \in (1,4) \end{cases}$$

- neka je ψ monotono rastuća fja (tj. injekcija)

$$G(y) = P(Y < y) = P(\psi(X) < y) =$$
$$= P(X < \psi^{-1}(y)) = F(\psi^{-1}(y))$$

$$g(y) = G'(y) = f(\psi^{-1}(y)) \cdot \frac{d\psi^{-1}(y)}{dy}$$

- analogno za padajuće fje.

=> dobijemo formulu: ~ u gustoći

$$\boxed{g(y) = f(x)\left|\frac{dx}{dy}\right|}, \quad x = \psi^{-1}(y)$$

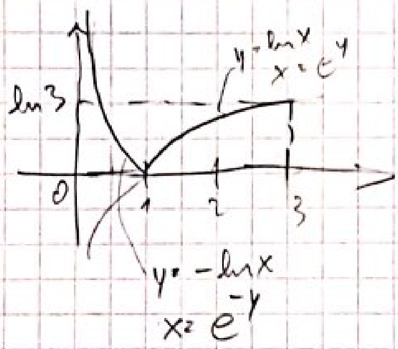
$\boxed{\text{MI-15}}$ $f(x) = \frac{c}{1+x^2}$, $x \in \mathbb{R}$. Gustoća od $Y = \frac{1}{x}$

$$\int_{-\infty}^{\infty} \frac{c}{1+x^2}\,dx = C\,\mathrm{arctg}\,x \Big|_{-\infty}^{\infty} = C\pi = 1 \rightarrow c = \frac{1}{\pi}$$

$$\frac{dx}{dy} = \frac{-1}{y^2} \qquad \underbrace{g(y)}_{x = \frac{1}{y}} = \frac{1}{\pi\left(1 + \frac{1}{y^2}\right)} \cdot \frac{1}{y^2} = \frac{1}{\pi(1+y^2)}$$

$\pi(y^2+1)$

HI - 11 $\quad f(x)=\frac{2}{9}x \; , \; x\in(0,3)$. Odredi gustoću od $Y=|\ln x|$



i) $x\in(0,1)$

$y\in(0,+\infty)$

$x=e^{-y}$

$\left|\frac{dx}{dy}\right| = |-e^{-y}| = e^{-y}$

ii) $x\in(1,3)$

$y\in(0,\ln 3)$

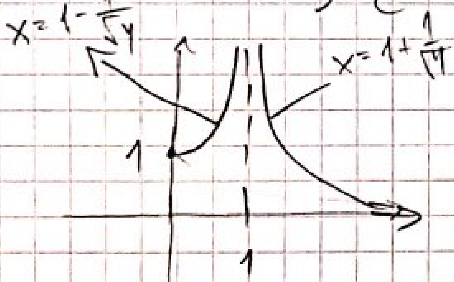
$x=e^{y}$

$\left|\frac{dx}{dy}\right| = e^{y}$

$g_1(y)=\frac{2}{9}e^{-y}\cdot e^{-y}$ $\qquad$ $g_2(y)=\frac{2}{9}e^{y}\cdot e^{y}$

$$g(y)=\begin{cases} g_1+g_2 \, , \; y\in(0,\ln 3) & \Rightarrow \frac{2}{9}\left(e^{-2y}+e^{2y}\right) \\ g_1 \, , \; y\in(\ln 3,+\infty) & \Rightarrow \frac{2}{9}e^{-2y} \end{cases}$$

HI - 13 $\quad f(x)=e^{-x} \; , \; x>0 . \qquad Y=\frac{1}{(x-1)^2}$



$(x-1)^2=\frac{1}{y}$

$x-1=\pm\frac{1}{\sqrt{y}}$

$x=1\pm\frac{1}{y}$

i) $x\in(0,1)$

$y\in(1,+\infty)$

$\left|\frac{dx}{dy}\right| = \frac{1}{2y\sqrt{y}}$

$g_1(y)=e^{-\left(1-\frac{1}{\sqrt{y}}\right)}\frac{1}{2y\sqrt{y}}$

ii) $x\in(1,+\infty)$

$y\in(0,+\infty)$

$\left|\frac{dx}{dy}\right| = \frac{1}{2y\sqrt{y}}$

$g_2(y)=e^{-\left(1+\frac{1}{\sqrt{y}}\right)}\frac{1}{2y\sqrt{y}}$

$$g(y)=\begin{cases} g_2 \, , \; x\in(0,1) \\ g_1+g_2 \, , \; x\in(1,+\infty) \end{cases}$$

Odredi funkciju razdiobe $G(y) = ?$

1. način: $G(y) = \int_{-\infty}^{y} g(t)dt$ - IKAD

2. način: (po definiciji)

i) $G(y) = P(Y < y) = P\left(X > \left(1 + \frac{1}{\sqrt{y}}\right)\right) =$

$$= \int_{1+\frac{1}{\sqrt{y}}}^{+\infty} e^{-x} dx = e^{-1-\frac{1}{\sqrt{y}}}, \quad y \in (0,1)$$

ii) $G(y) = P(Y < y) = P\left(0 < x < 1 - \frac{1}{\sqrt{y}}\right) + P\left(x > 1 + \frac{1}{\sqrt{y}}\right) =$

$$= \int_{0}^{1-\frac{1}{\sqrt{y}}} e^{-x} dx + e^{-1-\frac{1}{\sqrt{y}}} = 1 - e^{-1+\frac{1}{\sqrt{y}}}$$

6. Primjeri neprekinutih razdioba

6.1 Eksponencijalna razdioba

- neka je $Z \sim P(\lambda)$, tada je $Z_x \sim P(\lambda x)$ broj pojavljivanja u $[0, x]$

X = vrijeme do prve pojave promatranog događaja

$$F(x) = P(X < x) = 1 - P(Z_x = 0) = 1 - e^{-\lambda x}$$

⇒ zaključak: pišemo $X \sim \mathcal{E}(\lambda)$, $F(x) = 1 - e^{-\lambda x}$, $x > 0$

$$f(x) = \lambda e^{-\lambda x}$$

- očekivanje $E(X) = \int_0^{\infty} x \lambda e^{-\lambda x} dx = \boxed{\frac{1}{\lambda}}$

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- disperzija:

$$D(X) = \int_0^\infty x^2 \cdot f(x) dx - \left(\frac{1}{\lambda}\right)^2 = \boxed{\frac{1}{\lambda^2}}$$

- karakter. fja:

$$\vartheta_x(t) = \int_0^\infty e^{itx} \lambda e^{-\lambda x} dx = \lambda \frac{e^{-x(\lambda - it)}}{\lambda_{it}} \Big|_0^\infty$$

$$= \boxed{\frac{\lambda}{\lambda - it}}$$

4. MI-2008

$E(X) = 3$ god. $\Rightarrow \lambda = \frac{1}{3}$

a) tijekom prve godine

$$P(X<1) = F(1) = 1 - e^{-\frac{1}{3} \cdot 1} = \boxed{0.28}$$

b) tijekom treće

$$P(2<X<3) = F(3) - F(2) = e^{-\frac{2}{3}} - e^{-1} = \boxed{0.14}$$

c) tijekom prve 3 godine:

$$P(X<3) = F(3) = 1 - e^{-1} = \boxed{0.63}$$

2ZD

$P(X < E(X))$ ne ovisi o λ

$$= F(E(X)) = F\left(\frac{1}{\lambda}\right) = 1 - e^{-\lambda \frac{1}{\lambda}} = 1 - e^{-1} = 0.63$$

· TM (odsustvo pamćenja)

X ima ekspon. raspodjelu akko

$$P(X < x+t \mid X > t) = P(X < x)$$

Dokaz:

$$P(X < x+t \mid X > t) = \frac{P(t < X < x+t)}{P(X > t)} = \frac{F(x+t) - F(t)}{1 - F(t)} =$$

$$= \frac{\cancel{e^{-t\lambda}} - e^{-\lambda x - \cancel{\lambda t}}}{\cancel{e^{-t\lambda}}} = 1 - e^{-\lambda x} = F(x) = P(X < x)$$

$\boxed{ZAD}$ Nastavak prethodnog zadatka:
Vj da se pokvari tijekom treće, ako prve 2 nije bio pokvaren?

$$P(2 < X < 3 \mid X > 2) = P(X < 1) = \boxed{0.28}$$

$\boxed{MI-14}$

$$P(X > 1) = 0.9$$

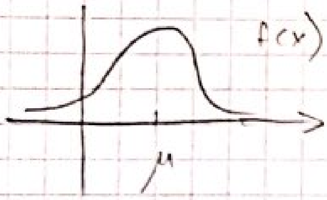
$$1 - F(1) = 0.9$$
$$e^{-\lambda} = 0.9$$
$$\lambda = -\ln 0.9$$

$$P(X > 2) = 1 - F(2) = e^{-2\lambda}$$
$$= 0.9^2 = \boxed{0.81}$$

$x^{1/2} e^{-\lambda x} dx$

6.2. Normalna (Gaussova) razdioba

- definira se fjom gustoće



$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x\in R$$

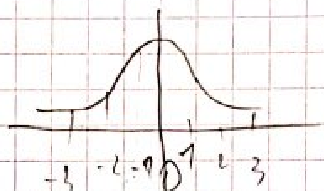
$$\underbrace{E(X)=\mu \qquad D(X)=\sigma^2}$$

normirana
$\nearrow$ $E(X)=0$ $D(X)=1$

$$X \sim N(\mu, \sigma^2)$$

- pogledajmo $N(0,1)$ jedinična normalna razdioba

$$\Rightarrow f(x)=\frac{1}{\sqrt{2\pi}}\, e^{\frac{-x^2}{2}}$$



$$\Rightarrow \Phi(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}t^2}\,dt = P(X<x) \qquad \langle 0,+\infty\rangle$$

$$\Rightarrow \underbrace{\int_{-\infty}^{0}\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}t^2}\,dt}_{0,5} + \int_{0}^{x}\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}t^2}\,dt = \frac{1}{2}+\frac{1}{2}\underbrace{\int_{-x}^{x}\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}t^2}\,dt}_{\Phi^*(x)}$$

$\Phi^*(x)$ zadana tablično

$$\Rightarrow \boxed{\Phi(x)=\frac{1}{2}+\frac{1}{2}\Phi^*(x)}$$

$$\Rightarrow \text{tj. } P(X<a)=\frac{1}{2}+\frac{1}{2}\Phi^*(a)$$

MI-2014

$\mu = 50$ min

8:15, kreće u 7:20

vj. da stigne između 8:05 i 8:15 → 0.383

vj. da će kasniti više od 5 min?

$\rightarrow P(X > 60) = ? = P\left(X^* > \frac{60-50}{10}\right) = P(X^* > 1) =$

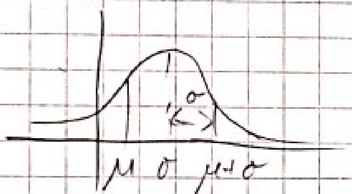
$= \frac{1}{2} - \frac{1}{2}\Phi^*(1) = \boxed{0.159}$

$P(45 < X < 55) = 0.383$

$P\left(\frac{45-50}{\sigma} < X^* < \frac{55-50}{\sigma}\right) = \frac{1}{2}\left[\Phi^*\left(\frac{5}{\sigma}\right) - \Phi^*\left(\frac{-5}{\sigma}\right)\right]$

$= \Phi^*\left(\frac{5}{\sigma}\right) = 0.383$ $\qquad \frac{5}{\sigma} = \frac{1}{2} \Rightarrow \boxed{\sigma = 10}$

• uvodimo: $P(\mu - k\sigma < X < \mu + k\sigma) = P(-k < X^* < k) = \Phi^*(k)$



- za $k=1$: $P(\mu - \sigma < X < \mu + \sigma) = 0.68269$

$k=2$: $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9545$

$k=3$: $P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$

- karakter. fja:

$$\vartheta_{X^*}(t) = \int_{-\infty}^{+\infty} e^{itx} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \,\Big|\, \frac{d}{dt}$$

$$\vartheta'_{X^*}(t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\, ix\, e^{itx} e^{-\frac{1}{2}x^2} dx =$$

$$\vartheta_{X^*}(t) = e^{-\frac{1}{2}t^2}$$

$$\vartheta_X(t) = \vartheta_{\mu + \sigma X^*}(t) = \boxed{e^{it\mu - \frac{1}{2}\sigma^2 t^2}}$$

• TM (stabilnost normalne razdiobe)

$$\left.\begin{array}{l} X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2) \\ X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2) \end{array}\right\} \text{ ako su nezavisne}$$

$1+\Phi^*\left(\frac{-\infty-850}{20}\right)$

tada: $s_1X_1 + s_2X_2 \sim \mathcal{N}(s_1\mu_1 + s_2\mu_2, s_1^2\sigma_1^2 + s_2^2\sigma_2^2)$

Dokaz: $\vartheta_{sX}(t) = e^{its\mu - \frac{1}{2}s^2\sigma^2t^2} = \ldots =$

$$= e^{it\underbrace{(s_1\mu_1 + s_2\mu_2)}_{\text{oček}} - \frac{1}{2}t^2\underbrace{(s_1^2\sigma_1^2 + s_2^2\sigma_2^2)}_{\text{disp.}}}$$

$1 - \Phi^*\left(\frac{60-\infty}{x}\right)$

$\boxed{\text{M1 - 2020}}$

$\left.\begin{array}{l} \mu_1 = 180 \\ \sigma_1 = 20 \end{array}\right\}$ obične $\quad \left.\begin{array}{l} \mu_2 = 220 \\ \sigma_2 = 5 \end{array}\right\}$ ind.

2 obične i 2 ind. jabuke zajedno. Vj. da je masa paketa između 820 i 1000 g.

X

$\cancel{X = 2X_1 + 2X_2 \sim \mathcal{N}\ldots}$

$$X = X_1 + X_1 + X_2 + X_2 \sim \mathcal{N}(180+180+220+220, 20^2+20^2+5^2+5^2)$$

$$\sim \mathcal{N}(800, 850)$$

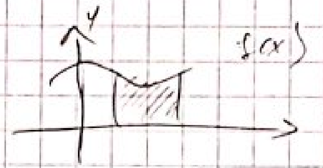
$$P(820 < X < 1000) = P\left(\frac{850-800}{\sqrt{850}} < X^* < \frac{1000-800}{\sqrt{850}}\right) =$$

$$= P(0.686 < X^* < 6.86) = \frac{1}{2}[\underbrace{\Phi^*(6.86)}_{1} - \underbrace{\Phi^*(0.686)}_{0.50724}] =$$

$$= \boxed{0.246}$$

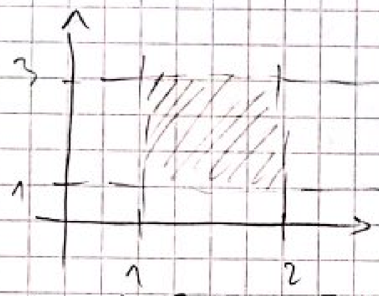
UVOD U DVOSTRUKE INTEGRALE

$\int_a^b f(x)dx = P$



$\iint_D f(x,y)\,dx\,dy = V$

-> postavljanje granica

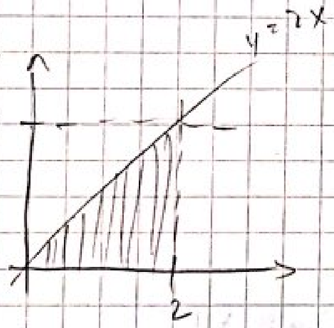


$\int_1^2 dx \int_1^3 f(x,y)\,dy$ ili

$\int_1^3 dy \int_1^2 f(x,y)\,dx$

$(x,y) \in [1,2] \times [1,3]$

$\boxed{240}$

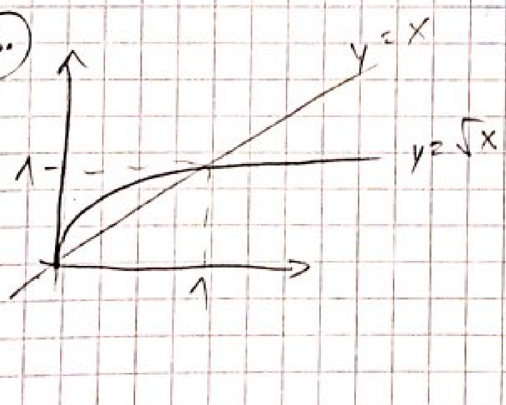


$\int_0^2 dx \int_0^{2x} f(x,y)\,dy$ ili

$\int_0^4 dy \int_{\frac{y}{2}}^{2} f(x,y)\,dx$

područje omeđeno s $y=2x$, $x=2$, $y=0$

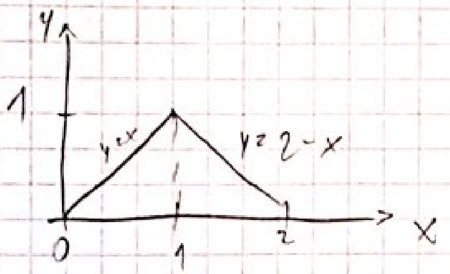
3.



$\int_0 dx \int_x^{\sqrt{x}} f(x,y)\,dy$ ili

$\int_0^1 dy \int_{y^2}^{y} f(x,y)\,dx$

(4.)

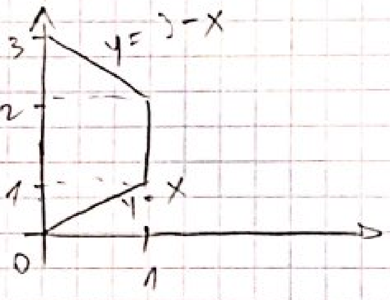


$\int_0^1 dx \int_0^x f(x,y)\,dy + \int_1^2 dx \int_0^{2-x} f(x,y)\,dy$

ili

$\int_0^1 dy \int_y^{2-y} f(x,y)\,dx$

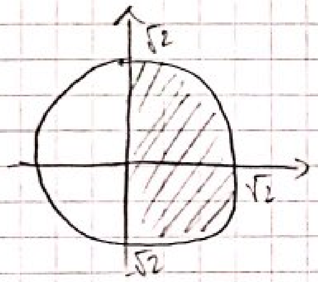
(5.)



$\int_0^1 dx \int_x^{3-x} f(x,y)\,dy$ ili

$\int_0^1 dy \int_0^y f(x,y)\,dx + \int_1^3 dy \int_0^1 f(x,y)\,dx + \int_2^3 dy \int_0^{3-y} f(x,y)\,dx$

(6.)



$x^2+y^2=2$ i $x>0$ $\rightarrow$ $y=\pm\sqrt{2-x^2}$

$\int_0^{\sqrt2} dx \int_{-\sqrt{x^2-2}}^{\sqrt{x^2-2}} f(x,y)\,dy$ ILI

$\int_{-\sqrt2}^{\sqrt2} dy \int_0^{\sqrt{2-y^2}} f(x,y)\,dx$

- računanje integrala

$$\int_0^1 dx \int_x^2 (2x+x^2y)\,dy = \int_0^1 dx \left(2xy + x^2\frac{y^2}{2}\right)\Big|_x^2 =$$

$$= \int_0^1 dx \left(4x + \cancel{2x^2} - \cancel{2x^2} - \frac{x^4}{2}\right) = \int_0^1 \left(4x - \frac{x^4}{2}\right)dx =$$

$$= \left(\frac{4x^2}{2} - \frac{x^5}{2\cdot 5}\right)\Big|_0^1 = \frac{19}{10}$$

Aproksimacija binarne razdiobe normalnom

Tm. (Moivre - Laplace)

$$X \sim B(n,p) \approx N(np, np(1-p))$$

za dovoljno velik n

$\boxed{21D}$

$n = 6000$, vj. 950 i 1050 šestica

$X \sim B(n=6000, p=\frac{1}{6})$

$$P(950 < X < 1050) = \sum_{n=950}^{1050} \binom{n}{k} p^k (1-p)^{n-k} = 0.914$$

$$\approx N(1000, 833.3)$$

$$P\left(\frac{950-\mu}{\sigma} < X^* < \frac{1050-\mu}{\sigma}\right) = P(-1.73 < X^* < 1.73) =$$

$$= \Phi^*(1.73) = 0.916$$

Pazi: $P(X = 1000) = 0$ (vjv. da je točno jedan broj je 0)

- radi $P(999.5 < X < 1000.5) =$

↳ lokalni MLT

$\boxed{M-2011}$ 30 pitanja su T ili N. Ako odg na sreću vjv. da smo barem 16 točno.

$X \sim B(n=30, p=\frac{1}{2})$

točno: $P(X \geq 16) = \sum_{k=16}^{30} \binom{30}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{30-k} = \boxed{0.4278}$

aproks: $X \approx N(\mu = 15, \sigma = 7.5)$

$$P\left(\frac{X-\mu}{\sigma} \geq \frac{16-\mu}{\sigma}\right) = P(X^* \geq 0.365) =$$

$$= \frac{1}{2} - \frac{1}{2}\Phi^*(0.365) = 0.3575$$

uvodimo korekciju:

$$P(X > 15.5) = P\left(X^* > \frac{15.5-\mu}{\sqrt{7.5}}\right) = P(X^* > 0.18257) =$$

$$= \boxed{0.4276}$$

MI

7. SLUČAJNI VEKTORI

7.1. Neprekinuti slučajni vektori

- def. n-dim. slučajni vektor je uređena n-torka sl. varijabli

$$X = (X_1, X_2, \dots, X_n)$$ ... F razdiobe se definira kao:

$$F(x_1, x_2, \dots, x_n) = P(X_1 < x_1, X_2 < x_2, \dots, X_n < x_n)$$

- mi ćemo gledati samo 2-dim. vektor

ovakva: (X, Y), a njegova F razdiobe:

$$F(x, y) = P(X < x, Y < y)$$

def. za sl. vektor (X, Y) kažemo da je neprekinut ako postoji nenegativna fja

$$f: \mathbb{R}^2 \to [0, +\infty)$$ t.d. za $\forall x, y$ vrijedi:

$$F(x, y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x, y)\,dx\,dy$$; fju $f(x, y)$ zovemo fja gustoće i vrijedi: $f(x, y) = \frac{\partial^2 F}{\partial x \partial y}$

Očito vrijedi

$$\boxed{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = 1}$$

- primijetimo:

$$F_X(x) = P(X<x) = P(X<x, -\infty<Y<\infty)$$

$$= \int_{-\infty}^{x} dx \underbrace{\int_{-\infty}^{+\infty} f(x,y)\,dy}_{} = \int_{-\infty}^{x} \underline{f_X(x)}\,dx$$

- dobili smo marginalne gustoće:

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy$$

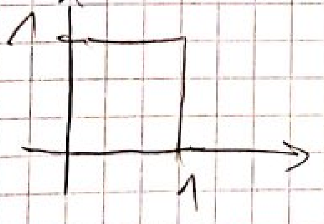
$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx$$

<u>TM.</u> Komponente X i Y vektora (X,Y) su NEZAVISNE ako:

$$\boxed{f(x,y) = f_X(x)\cdot f_Y(y)}$$

ZAD. $f(x,y) = \begin{cases} cxy, & 0<x<1,\ 0<y<1 \\ 0, & \text{inače} \end{cases}$

$c=?$, marginalne, nezavisnost?



$$\iint_D cxy\,dx\,dy = c\int_0^1 x\,dx \int_0^1 y\,dy = c\,\frac{1}{2}\,\frac{1}{2} = \frac{c}{4} = 1$$

$$\boxed{c=4}$$

$f_X(x) = \int_0^1 4xy\,dy = 4x\frac{y^2}{2}\Big|_0^1 = 2x, \; x\in(0,1)$

gledamo od kud do kud ide y

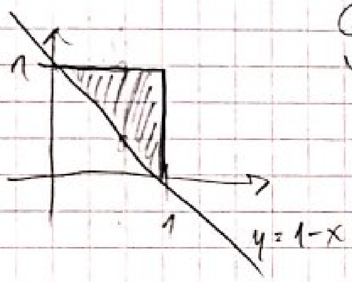
- granice inače neće biti brojevi nego f-je po drugoj varijabli

$f_Y(y) = \int_0^1 4xy\,dx = 4y\frac{x^2}{2}\Big|_0^1 = 2y, \; y\in(0,1)$

$\Rightarrow f_X(x)\cdot f_Y(y) = 2x\cdot 2y = 4xy = f(x,y)$ NEZAVISNI

Vjv. da $P(\underbrace{X+Y>1}_{G}) = \iint_G 4xy\,dx\,dy =$ — NIJE POVRŠ!

G: $y > 1-x$



$= 4\int_0^1 x\,dx\int_{1-x}^1 y\,dy =$

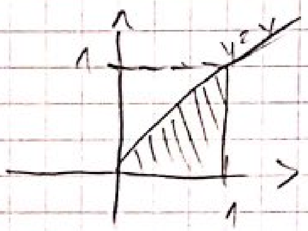
vanjski uvijek konst.

unutarnji krivulje

$= 4\int_0^1 x\cdot\frac{y^2}{2}\Big|_{1-x}^1 dx = 2\int_0^1 x\left(1-(1-x)^2\right)dx = \boxed{\frac{5}{6}}$

$\boxed{\text{MI-2011}}$

$f(x,y) = Cx, \quad 0\le y\le x\le 1$

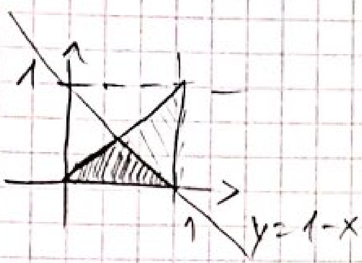


$\iint Cx\,dx\,dy = C\int_0^1 x\,dx\int_0^x dy = C\int_0^1 x^2\,dx =$

$= \frac{C}{3} = 1 \Rightarrow \boxed{C=3}$

$\left.\begin{array}{l} f_X(x) = \int_0^x 3x\,dy = 3x^2, \; x\in(0,1) \\ f_Y(y) = \int_y^1 3x\,dx = \frac{3}{2}(1-y^2), \; y\in(0,1) \end{array}\right\}$ ZAVISNE

$$P(X+Y<1) = P(Y<1-X) =$$



$$= \int_0^{1/2} dy \int_y^{1-y} 3x\,dx = \boxed{\frac{3}{8}}$$

$$(2) \quad \int_0^{1/2} dx \int_0^{x} 3x\,dy + \int_{1/2}^{1} dx \int_0^{1-x} 3x\,dy$$

AUDITORNE

Normalna razdioba

$$X \sim N(a, \sigma^2) \Rightarrow f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}}, \quad x \in R$$

$a \in R, \ \sigma^2 > 0$

$$E(X) = a \qquad D(X) = \sigma^2$$

Jedinična normalna razdioba: $X \sim N(0,1)$

$$X \sim N(a, \sigma^2) \Rightarrow \frac{X-a}{\sigma} \sim N(0,1)$$

ZAD Težina čovjeka je normalno distribuirana s očekivanjem 74 kg. Ako 23% ljudi ima težinu preko 90 kg, izračunajte vjv. da na sreću odabran čovjek teži između 50 i 98 kg.

X - masa čovjeka $\Rightarrow X \sim N(74, \sigma^2)$

$$0.23 = P(X > 90) = P\left(\frac{X-74}{\sigma} > \frac{16}{\sigma}\right) =$$

$$= \frac{1}{2}\left(1 - \Phi^*\left(\frac{16}{\sigma}\right)\right)$$

$$\Rightarrow \Phi^*\left(\frac{16}{\sigma}\right) = 0.54 \Rightarrow \frac{16}{\sigma} = 0.739$$

$$\Rightarrow \sigma = 21.65088$$

$$P(50 < X < 98) = P\left(\frac{50-74}{21.65099} < \frac{X-74}{21.6\ldots} < \frac{98-74}{21.6\ldots}\right)$$

$$= P\left(-1.11 < \frac{X-74}{21.650} < 1.11\right)$$

$$= \Phi^*(1.11) = 0.733$$

ZAD Zadane su normalne sl. var.

$$X \sim N(8,9);\ Y \sim N(3,25),\ Z \sim N(2,1)$$

Izračunajte $P(X-Y > 2Z)$

$$P(X-Y>2Z) = P(X-Y-2Z>0) = P(W>0)$$

$$W := X - Y - 2Z$$

$$W \sim N(8-3-2\cdot 2,\ 9+25+4\cdot 1) = N(1,38)$$

neparna f-ja

$$P(W>0) = P\left(\frac{W-1}{\sqrt{38}} > -\frac{1}{\sqrt{38}}\right) = \frac{1}{2}\left(1-\Phi^*\left(-\frac{1}{\sqrt{38}}\right)\right) =$$

$$= \frac{1}{2}\left(1+\Phi^*\left(\frac{1}{\sqrt{38}}\right)\right) = \frac{1}{2}\left(1+\Phi^*(0.162)\right) = \boxed{0.564347}$$

ZAD Težina građevinskog elementa je norm. distr. s parametrima $a = 0.5t$ i $\sigma = 0.01\ t$. Kolika je vjr. da težina 5 takvih elemenata bude veća od $2.55\ t$?

X_i = težina i-tog elementa, $i = 1,2,3,4,5$

$$\Rightarrow X_i \sim N(0.5,\ 0.01^2)$$

nezavisnost

$$Y = X_1 + X_2 + X_3 + X_4 + X_5 \sim N(2.5,\ 5\cdot 0.01^2)$$

$$P(Y > 2.55) = \ldots = \frac{1}{2}\left(1-\Phi^*(2.24)\right) = \boxed{0.01254546}$$

$\boxed{2AD}$ Vjv. rađanja dječaka jednaka je 0.515. Kolika je vjv. da među 100 novorođene djece bude od 50 do 55 dječaka?

$X \sim$ broj dječaka među 100 novorođenčadi

$$X \sim B(100,\ 0.515) \underset{\text{MCT}}{\approx} N(100\cdot 0.515,\ 100\cdot 0.515\cdot(1-0.515))$$

$$X \approx N(51.5,\ 24.9775)$$

$$P(50 \leq X \leq 55) = \ldots = \frac{1}{2}\left(\Phi^*(0.7) + \Phi^*(0.3)\right) =$$

$$= \boxed{0.375994777}$$

$\boxed{2AD}$ Odredite $\alpha \in \mathbb{R}$ t.d. $f(x) = \alpha e^{-x^2+4x}$, $x \in \mathbb{R}$ bude gustoća neke razdiobe. Izračunajte disperziju i očekivanje te razdiobe.

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \alpha \int_{-\infty}^{\infty} e^{-x^2+4x}\,dx$$ $\rightarrow$ ideja: doći do oblika $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$= \alpha \int_{-\infty}^{\infty} e^{-(x^2-4x+4)+4}\,dx = \alpha e^4 \int_{-\infty}^{\infty} e^{-\frac{(x-2)^2}{2\cdot\frac{1}{2}}}\,dx =$$

$$= \alpha e^4 \int_{-\infty}^{\infty} e^{-\frac{(x-2)^2}{2\left(\frac{1}{\sqrt{2}}\right)^2}}\,dx = \alpha e^4 \frac{1}{\sqrt{2}}\sqrt{2\pi} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\frac{1}{\sqrt{2}}\cdot\sqrt{2\pi}} e^{-\frac{(x-2)^2}{2\left(\frac{1}{\sqrt{2}}\right)^2}}\,dx}_{=1}$$

$$= \alpha e^4 \sqrt{\pi} \Rightarrow \alpha = \frac{1}{e^4\sqrt{\pi}}$$

jer je pod integralom f-ja gustoće $N\left(2, \frac{1}{\sqrt{2}}\right)$ razdiobe

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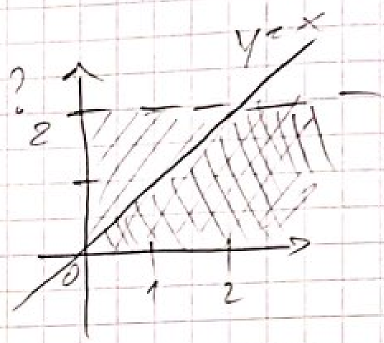
ZAD

$X \sim \mathcal{E}(a=2)$ $\quad E(X) = \frac{1}{2}$ - eksponenc. $\rightarrow x > 0$

$Y \sim U[0,2]$ - jednolika razdioba $\rightarrow y \in [0,2]$

X, Y nezavisni, vjv. Y manji od X?



$$P(Y < X) = \iint_D \underbrace{f(x,y)}_{f_X(x) f_Y(y)} \, dx\,dy =$$

$f_X(x) f_Y(y)$ - jer su nezavisni

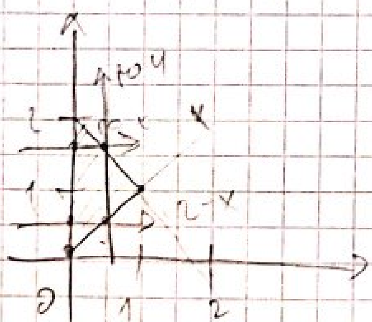
$$= \iint_D 2 e^{-2x} \cdot \frac{1}{2} \, dx\,dy =$$

$$= \int_0^2 dy \int_y^{+\infty} e^{-2x} \, dx = \boxed{\frac{1}{4} - \frac{1}{4} e^{-4}}$$

MI-2020

Sl. vektor ima razdiobu na Δ s vrhovima $(0,0)$ $(1,1)$ $(0,2)$

a) marg. gustoće



$$f(x,y) = \frac{1}{P_\Delta} = 1$$

↳ jedn. razdioba je UVIJEK

1 / mjera (pov. ili volumen)

ako y unutar granica je

$$f_X(x) = \int f(x,y)\,dy = \int_x^{2-x} 1\,dy =$$

$$= 2 - 2x, \; x \in (0,1)$$

imamo 2 raz. gornje granice
moramo 2 integrala

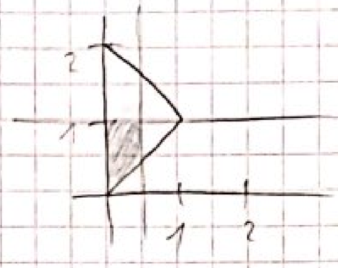
$$f_Y(y) = \int_0^y 1\,dx = y, \; y \in (0,1)$$

$$f_Y(y) = \int_0^{2-y} 1\,dx = 2 - y, \; y \in (1,2)$$

očito zavisni jer $f_X f_Y \neq f(x,y)$

b) $P\left(X<\frac{1}{2} \mid Y<1\right) = \frac{P\left(X<\frac{1}{2}, Y<1\right)}{P(Y<1)} =$

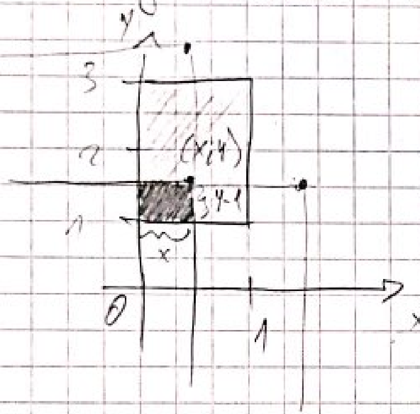
$$= \frac{\int_0^{1/2} dx \int_x^1 dy}{\int_0^1 y\, dy} = \boxed{\frac{3}{4}}$$



$\boxed{240}$ Biramo na sreću točku unutar pravok.

$0 \le x \le 1, \ 1 \le y \le 3$

Odredi f-ju razdiobe sl. vektora $X, Y = ?$



$F(x,y) = P(X<x, Y<y)$ – obična geom. vjv.

$$F(x,y) = \frac{x(y-1)}{2}, \quad 0 \le x \le 1, \ 1 \le y \le 3$$

$$F(x,y) = \frac{\cancel{2}x}{\cancel{2}}, \quad 0 \le x \le 1, \ y > 3$$

$$F(x,y) = \frac{1 \cdot (y-1)}{2}, \quad x > 1, \ 1 \le y \le 3$$

$$F(x,y) = 0, \quad x < 0 \text{ ili } y < 1$$

$$F(x,y) = \frac{2}{2} = 1, \quad x > 1, \ y > 3$$

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

$$f(x,y) = \begin{cases} \frac{1}{2}, & 0 \le x \le 1, \ 1 \le y \le 3 \\ 0, & \text{inače} \end{cases}$$

TM (svojstvo očekivanja)

a) $E(sX+tY) = sE(X) + tE(Y)$

b) $E(XY) = E(X)E(Y)$, ako su X i Y nezavisni

- također vrijedi

$$E(\Psi(X,Y)) = \iint_{-\infty}^{+\infty} \Psi(x,y) f(x,y)\,dx\,dy$$

Zadatak

$f(x,y) = x+y \qquad x \in [0,1], \; y \in [0,1]$

očekivanje od $Z = X \cdot Y$

1. način:

$E(Z) = \int z f_Z(z)\,dz = \ldots$

2. način:

$$E(Z) = E(XY) = \int_0^1\int_0^1 xy(x+y)\,dx\,dy = \boxed{\frac{1}{3}}$$

7.2. Uvjetne razdiobe

- def. neka je $f(x,y)$ gustoća vektora (X,Y) i neka je pomoćna razdioba od varijable X. Tada se uvjetna gustoća od Y uz uvjet $X=x$ definira s:

$$\boxed{f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)}}$$

- uoči tada je: gustoća od Y: $f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx =$

$$= \int_{-\infty}^{+\infty} f_{Y|X=x}(y) \cdot f_X(x)\,dx$$

- **def.** uvjetno očekivanje od Y koji ovisi o realizacijama varijable X definira se s:

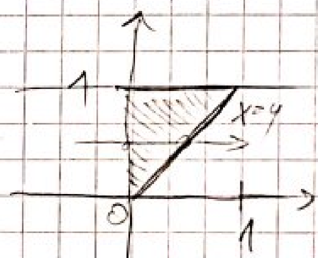
$$E(Y|X=x) = \int_{-\infty}^{+\infty} y\, f_{Y|X=x}(y)\,dy$$

te je tada:

$$E(Y) = \int_{-\infty}^{+\infty} E(Y|X=x)\cdot f_X(x)\,dx$$

ZI-2014 Sl. varijabla X je zadana s $f_X(x)=2x$, $x \in [0,1]$. Potom biramo Y iz intervala $[x,1]$. Odredi gustoću i očekivanje od Y?

$$f_{Y|X=x}(y) = \frac{1}{1-x}$$ duljina intervala



$$f_Y(y) = \int_0^y \frac{1}{1-x}\cdot 2x\,dx =$$

$$= -2\int_0^y \frac{1-x+1}{1-x}\,dx = \dots = -2\left(y+\ln|1-y|\right),\ y\in(0,1)$$

1. način

$$E(Y) = \int y f_Y(y)\,dy = \int y\cdot\left(-2(y+\ln|1-y|)\right)dy = \dots$$

2. način

$$E(Y|X=x) = \int_x^1 y\cdot\frac{1}{1-x}\,dy = \frac{1}{1-x}\,\frac{y^2}{2}\Big|_x^1 = \boxed{\frac{1+x}{2}}$$

granice za uvjetno Y koji ovisi o X

$$E(Y) = \int_0^1 \frac{1+x}{\cancel{2}}\cdot\cancel{2}x\,dx = \boxed{\frac{5}{6}}$$

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31-2016

n brojeva iz $[0,2]$, X najveći od njih
$Y \in [X,2]$. $E(Y)=?$

$X = \max\{X_1, \dots, X_n\} \to f_{X_i}(x) = \frac{1}{2}$ — duljina intervala

$$F(x) = P(X < x) = P(\max\{X_1,\dots,X_n\} < x) =$$

$$= P(X_1 < x, X_2 < x, \dots, X_n < x) =$$

$$= P(X_1 < x) \dots P(X_n < x) = \left(\frac{x}{2}\right)^n = \frac{x^n}{2^n}$$

0 x 2

$\underbrace{\int_0^x \frac{1}{2}\,dx = \frac{x}{2}}_{P(x) = \frac{x}{2}}$

$$f(x) = F'(x) = \frac{n x^{n-1}}{2^n}$$

$$f_{Y|X=x}(y) = \frac{1}{2-x}$$

$$E(Y|X=x) = \int_x^2 y \cdot \frac{1}{2-x}\,dy = \frac{x+2}{2}$$

$$E(Y) = \int E(Y|X=x) f(x)\,dx = \int_0^2 \frac{(x+2)}{2} \cdot \frac{n x^{n-1}}{2^n}\,dx =$$

$$= \frac{n}{2^{n+1}}\left[\int_0^2 x^n\,dx + 2\int_0^2 x^{n-1}\,dx\right] = \dots =$$

$$= \boxed{\frac{2n+1}{n+1}}$$

- def. vjr. događaja A koji ovisi o realizaciji varijable X (koja ima gustoću $f_X(x)$)

$$\Rightarrow \boxed{P(A) = \int_{-\infty}^{+\infty} P(A|X=x) \cdot f_X(x)\,dx}$$

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$\boxed{\text{MI - 2011}}$ Radijus kruga je sl. var. s jednolikom razdiobom

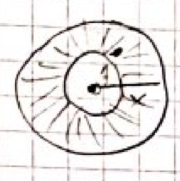
$r \in [1, 2]$ - jednolika var.

Na sreću biramo točku T unutar kruga.

vjerojatnost Vju. da udaljenost T od središta bude veća od $\frac{1}{2}$?

$$X \sim U[1,2] \rightarrow f_X(x) = \frac{1}{1} = 1$$

$$P(A) = \int_1^2 \frac{x^2\pi - \frac{1}{4}\pi}{x^2\pi} \cdot 1\,dx = \ldots = \boxed{\frac{7}{8}}$$



8. FUNKCIJE SLUČAJNIH VEKTORA

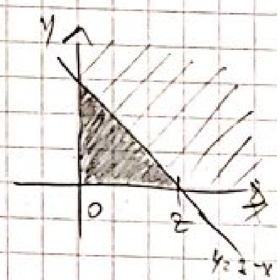
$\boxed{\text{ZAD}}$ (X, Y) ima $f(x,y) = e^{-x-y}$, $x > 0$, $y > 0$

F razdiobe od $Z = X + Y$?

$$F(z) = P(Z < z) = P(\underbrace{X + Y < z}_{y < z - x}) =$$

$$= \iint_D f(x,y)\,dx\,dy$$

$$= \int_0^z dx \int_0^{z-x} f(x,y)\,dy = 1 - e^{-z} - ze^{-z}, \quad z \in [0, +\infty\rangle$$



$$f(z) = F'(z) = ze^{-z}$$

- prisjetimo se: $Y = \psi(X) \Rightarrow g(y) = f(x)\cdot\left|\frac{dx}{dy}\right|,\ x = \psi^{-1}(y)$

- ako imamo $f(x,y)$ od (X,Y), kako odrediti $g(u,v)$ od (U,V) gdje je $(U,V) = \psi(X,Y)$

Vrijedi $P((X,Y)\in D) = P((U,V)\in D')$

$$\iint_D f(x,y)\,dx\,dy = \iint \boxed{\underbrace{f(x,y)}_{\text{preko } u,v}\cdot|J|} = \iint_{D'} \boxed{g(u,v)}\,du\,dv$$

$$\Rightarrow g(u,v) = \underbrace{f(x,y)}_{\text{preko } u,v}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

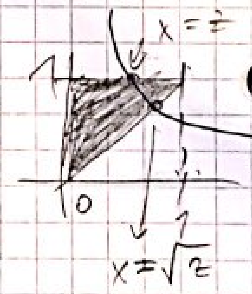
- mi ćemo gledati slučaj: $(X,Y)\to(X,Z)$

$$Z=\psi(X,Y) \Rightarrow \left.\begin{matrix} x = x \\ y = \psi^{-1}(x,z)\end{matrix}\right\}\ J = \frac{\partial(x,y)}{\partial(x,z)} = \begin{vmatrix}1 & 0\\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial z}\end{vmatrix} = \frac{\partial y}{\partial z}$$

$$\Rightarrow g(x,z) = f(x,y)\left|\frac{\partial y}{\partial z}\right| \Rightarrow \boxed{g(z) = \int_{-\infty}^{+\infty} f(x,y)\left|\frac{\partial y}{\partial z}\right| dx}$$

$y = \psi^{-1}(x,z)$

$\boxed{240}$ $f(x,y) = 2(x+y)$, $0\le x\le y\le 1$ → zavisne



$Z = X\cdot Y$, $E(Z)=?$

$Z\in[0,1]$ → $y = \frac{z}{x}$, $\frac{\partial y}{\partial z} = \frac{1}{x}$

$$\Rightarrow g(z) = \int_z^{\sqrt{z}} 2\left(x+\frac{z}{x}\right)\frac{1}{x}\,dx = \boxed{2-2z}$$

→ za određivanje granica

↳ nacrtamo $y=\frac{z}{x}$, i gledamo sjecišta i tamo gdje ulazi u područje je donja granica, a izlazi je gornja

$$E(Z) = \int_0^1 z(2-2z)\,dz = \boxed{\frac{1}{3}}$$

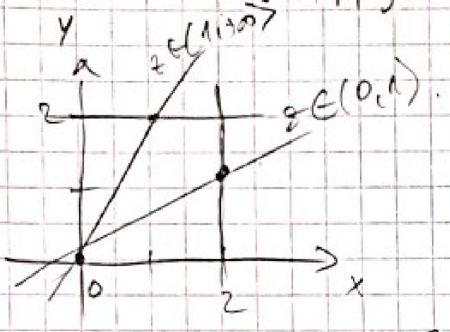
$\boxed{11-2013}$

$z \in (0,+\infty)$

$$f(x,y) = \frac{1}{8}(x+y), \quad 0 \le x \le 2, \quad 0 \le y \le 2, \quad Z = \frac{Y}{X}, \quad g(z) = ?$$

$y = z \cdot x$

$\left|\frac{\partial y}{\partial z}\right| = x$



$$g(z) = \int \frac{1}{8}(x + zx) \cdot x\,dx$$

$$g(z) = \begin{cases} \frac{z+1}{8} \int_0^2 x^2\,dx = \frac{1+z}{3}, & z \in (0,1) \\ \frac{z+1}{8} \int_0^{\frac{2}{z}} x^2\,dx = \frac{1+z}{3z^3}, & z \in (1,+\infty) \end{cases}$$

$\boxed{2KPZ-14}$

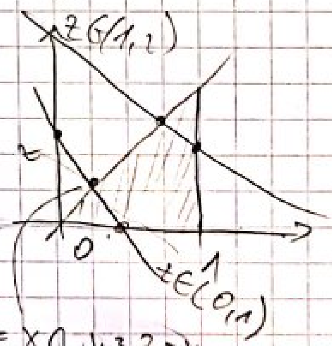
$$f(x,y) = 2(x+y), \quad 0 \le y \le x \le 1, \quad Z = X+Y, \quad z \in (0,2)$$

$y = z - x$

$\frac{\partial y}{\partial z} = 1$

$$g(z) = \int_{-\infty}^{+\infty} 2(\cancel{x} + z - \cancel{x}) \cdot 1\,dx$$

$$= \int 2z\,dx$$



$$g(z) = \begin{cases} 2z \int_{\frac{z}{2}}^{z} dx = z^2, & z \in (0,1) \\ 2z \int_{\frac{z}{2}}^{1} dx = 2z - z^2, & z \in (1,2) \end{cases}$$

$y = x \cap y = z - x$

$x = \frac{z}{2}$

AUDITORNE

(1.)

$$f_{X,Y}(x,y) = \frac{Ay}{x^2+1} \quad \text{za} \quad x>0 \;, y \in [0,1]$$

a) $A = ?$ b) X, Y nez? c) $P(X+Y<1 \mid X<\frac{1}{2})$

a) $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1$

$$\int_0^{+\infty}\int_0^1 \frac{Ay}{x^2+1}\,dx\,dy = A\int_0^{+\infty} \frac{1}{x^2+1} \frac{y^2}{2}\Big|_0^1 dx = \frac{A}{2}\int_0^{+\infty}\frac{1}{x^2+1}\,dx =$$

$$= \frac{A}{2}\operatorname{arctg} x\Big|_0^{+\infty} = \frac{A}{2}\cdot\frac{\pi}{2} = \frac{A\pi}{4} = 1$$

$$\boxed{A = \frac{4}{\pi}}$$

b) X, Y su nezavisni $\Leftrightarrow f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) \;\forall x,y$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_0^1 \frac{Ay}{x^2+1}\,dy = \frac{A}{x^2+1}\cdot\frac{y^2}{2}\Big|_0^1 =$$

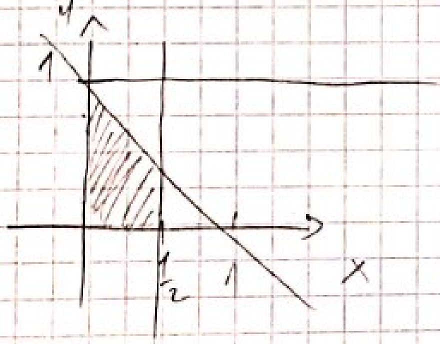
$$= \frac{\frac{4}{\pi}}{x^2+1}\cdot\frac{1}{2} = \boxed{\frac{2}{\pi}\frac{1}{x^2+1}} \;, x>0$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \int_0^{+\infty}\frac{Ay}{x^2+1}\,dx = Ay\operatorname{arctg} x\Big|_0^{+\infty} =$$

$$= \frac{4}{\pi}y\frac{\pi}{2} = 2y \;, y\in[0,1]$$

$y < 1-x$

c) $$P(X+Y<1 \mid X<\tfrac{1}{2}) = \frac{P(X+Y<1, X<\frac{1}{2})}{P(X<\frac{1}{2})}$$



$$P(X+Y<1, X<\tfrac{1}{2}) = \int_0^{1/2}\int_0^{1-x}\frac{Ay}{x^2+1}\,dy\,dx$$

$$= \int_0^{1/2}\frac{A}{x^2+1}\frac{y^2}{2}\Big|_0^{1-x}dx =$$

$$= \frac{A}{2}\int_0^{\frac{1}{2}}\frac{(1-x)^2}{x^2+1}\,dx = \frac{A}{2}\int_0^{\frac{1}{2}}\frac{1-2x+x^2}{x^2+1}\,dx =$$

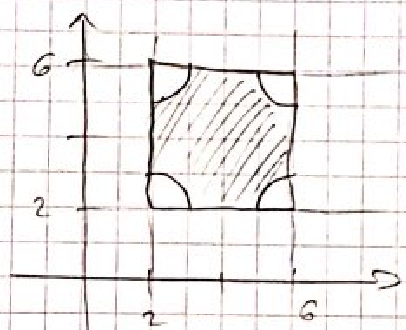
$$= \frac{A}{2}\int_0^{1/2}\left(1-\frac{2x}{x^2+1}\right)dx = \frac{A}{2}\left(\frac{1}{2}-\ln\left(\frac{5}{4}\right)\right)$$

$P(X<\frac{1}{2}) = \int_0^{\frac{1}{2}} f_X(x)\,dx = \frac{A}{2}\arctg\frac{1}{2} =$

rj. konačno:

$$= \frac{\frac{A}{2}\left[-\frac{1}{2}-\ln\left(\frac{5}{4}\right)\right]}{\frac{A}{2}\arctg\frac{1}{2}} = \boxed{0.512}$$

(2.) X - duljina str. kvadrata



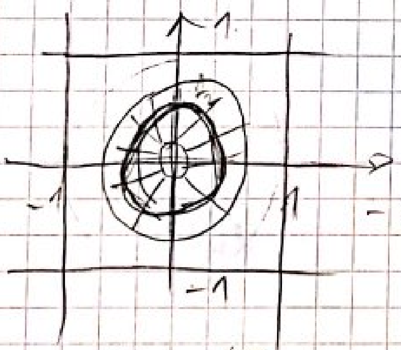
$f_X(x) = \frac{1}{16}x \;,\; x\in[2,6]$

Y - udaljenost od najbližeg vrha

$P(Y>1) = \int_{-\infty}^{+\infty} P(Y>1\,|\,X=x)\, f_X(x)\,dx =$

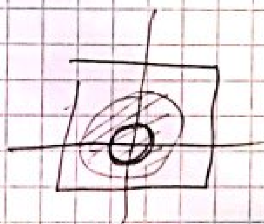
$= \int_2^6 \frac{x^2-\pi}{x^2}\,\frac{1}{16}\,x\,dx = \dots = 1-\frac{\pi\ln 3}{16}$

(3.) R - radijus kružnice, $r\in\langle 0,\frac{1}{2}] \Rightarrow f_R(r)=2$



X - udaljenost nasumično odabrane točke u $[-1,1]^2$ od kružnice

$r>\frac{1}{4}$ $\qquad$ $r<\frac{1}{4}$



$P(X<\frac{1}{4}) = \int_{-\infty}^{+\infty} P(X<\frac{1}{4}\,|\,R=r)\, f_R(r)\,dr =$

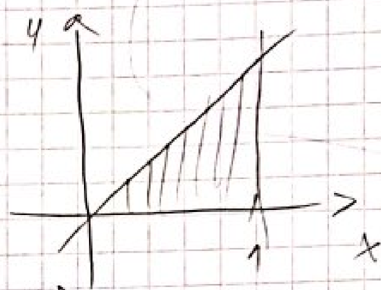
$= \int_0^{\frac{1}{4}} \frac{(r+\frac{1}{4})^2\pi}{4}\,2\,dr + \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{(r+\frac{1}{4})^2\pi-(r-\frac{1}{4})^2\pi}{4}\,2\,dr =$

$= \dots = \frac{25\pi}{384}$

(4.) (X,Y) $f(x,y)=\frac{c}{x+y}$, $0<y\le x\le 1$

$c=?$, $f_Y(y)=?$ $Z=X\cdot Y$, $g_Z(z)=?$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy=$$



$$=\int_0^1\int_0^x \frac{c}{x+y}\,dy\,dx=$$

$$=c\int_0^1 \ln|x+y|\Big|_0^x dx = c\int_0^1\Big(\underbrace{\ln(2x)}_{\ln 2+\ln x}-\ln x\Big)dx=$$

$$=c\ln 2 \Rightarrow c=\frac{1}{\ln 2}$$

$$f_Y(y)=\int_{-\infty}^{+\infty} f(x,y)\,dx=\int_y^1\frac{c}{x+y}\,dx = c\ln|x+y|\Big|_y^1=$$

$$=c\big(\ln(1+y)-\ln(2y)\big)=\frac{1}{\ln 2}\ln\left(\frac{1+y}{2y}\right)$$

$$g_Z(z)=\int_{-\infty}^{+\infty} f(x,y)\cdot\left|\frac{\partial y}{\partial z}\right|dx = \int_{\sqrt z}^1 f\left(x,\frac zx\right)\left|\frac1x\right|dx=$$

$$z=xy \Rightarrow y=\frac zx$$

$$=\frac{1}{\ln 2}\int_{\sqrt z}^1\left(\frac{1}{x+\frac zx}\cdot\frac1x\right)dx=\frac1{\ln2}\int_{\sqrt z}^1\frac{1}{\frac{x^2+z}{x}}\frac1x\,dx=$$

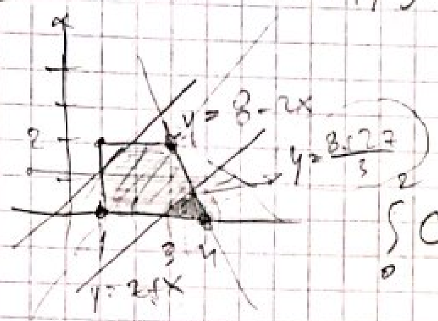
$$=\frac1{\ln2}\int_{\sqrt z}^1\frac{1}{x^2+\sqrt z^2}\,dx=\frac1{\ln2}\frac1{\sqrt z}\operatorname{arctg}\left(\frac{x}{\sqrt z}\right)\Big|_{\sqrt z}^1=$$

$=$

$\frac{z}{x}\cdot x$

$z=x^2 \Rightarrow x=\sqrt z$

$\boxed{21 - 2017}$ $f(x,y) = Cy$, $z = y - x$, $P(z \geq 0) = ?$



$z \in (-4,1)$ F različito od z ?

$$\int_0^2 Cy\,dy \int_1^{\frac{8-y}{2}} dx = \ldots = \frac{14}{3}C = 1 \Leftrightarrow C = \boxed{\frac{3}{14}}$$

$$F(z) = P(Z < z) = P(Y - X < z) = \iint_{y < z + x} f(x,y)\,dx\,dy =$$

$$= \int_0^{\frac{8+2z}{3}} \frac{3}{14} y\,dy \int_{y-z}^{\frac{8-y}{2}} dx = \frac{z^3}{64} + \frac{4z^2}{21} + \frac{16z}{21} + \frac{64}{63},\ z \in (-4,-1)$$

$$F(z) = P(Z < z) = 1 - \int_1^{2-z} dx \int_{z+x}^{2} \frac{3}{14} y\,dy = \frac{23}{28} + \frac{9z}{28} - \frac{3z^2}{28} - \frac{z^3}{28},\ z \in (-1,1)$$

$$P(Z \geq 0) = 1 - F(0) = 1 - \frac{23}{28} = \boxed{\frac{5}{28}}$$

2. način

$$f_1(z) = \int_{-z}^{\frac{8-z}{3}} \frac{3}{14}(z+x)\,1\,dx = \ldots,\ z \in (-4,-1)$$

$$f_2(z) = \int_1^{2-z} \frac{3}{14}(z+x)\,1\,dx = \ldots,\ z \in (-1,1)$$

- bitan princip za sve gustoće

9. TEORIJA VJEROJATNOSTI
9.1. Zakoni velikih brojeva

TM. a) Nejednakost Markova:
Ako X poprima nenegativne vrijednosti onda $\forall \varepsilon > 0$ vrijedi:
$$P(X \geq \varepsilon) \leq \frac{E(X)}{\varepsilon}$$

b) Nejednakost Čebiševa:
Za svaku sl. var. X s konačnim očekivanjem vrijedi:
$$P(|X - E(X)| \geq \varepsilon) \leq \frac{D(X)}{\varepsilon^2}$$

Dokaz: a) $P(X \geq \varepsilon) = \int_{\varepsilon}^{+\infty} f(x)\,dx \leq \int_{\varepsilon}^{+\infty} \frac{x}{\varepsilon} f(x)\,dx$
$$\leq \frac{1}{\varepsilon} \underbrace{\int_0^{+\infty} x f(x)\,dx}_{E(X)} = \frac{E(X)}{\varepsilon}$$

b) $P(\underbrace{|X - E(X)|}_{poz} \geq \varepsilon) = P(\underbrace{|X-E(X)|^2}_{poz} \geq \varepsilon^2) =$
$$\leq \frac{E((X-E(X))^2)}{\varepsilon^2} \leq \frac{D(X)}{\varepsilon^2}$$

$\boxed{21-2019}$ $E(X) = 75$ - br. sunčanih dana u godini.
Vj. da ne bude više od 200 sunč. dana veća od $\frac{5}{8}$.

$$P(X \geq 200) \leq \frac{75}{200}$$
$$P(X < 200) > 1 - \frac{75}{200}$$
$$P(X < 200) > \frac{5}{8}$$

$\boxed{7I - 2012}$

$E(X) = 65 \quad D(X) = 81$, br. bodova na ispitu

tj. da je $55 < X < 75$ veća od 19%.

$P(|X - E(X)| \geq \varepsilon) \leq \frac{D(X)}{\varepsilon^2}$

$P(|X - 65| < 10) > 1 - \frac{D(X)}{\varepsilon^2} = 1 - \frac{81}{100}$

$P(|X - 65| < 10) > \frac{19}{100}$

$\boxed{7I - 2016}$

$P(a - 3\sigma < X < a + 3\sigma) > \frac{8}{9}$

$\Rightarrow P(|X - a| < \overset{\varepsilon}{3\sigma}) > 1 - \frac{D(X)}{\varepsilon^2} \quad \sigma^2$

$P(|X - a| < 3\sigma) > 1 - \frac{\sigma^2}{9\sigma^2} = \frac{8}{9}$

- def.

niz X_n konvergira po vjerojatnosti k slučajnoj varijabli X ako za svaki $\forall \varepsilon > 0$ vrijedi

$$\lim_{n \to \infty} P(|X_n - X| > \varepsilon) = 0$$

Oznaka $X_n \xrightarrow{P} X$

Tm. (slabi zakon velikih brojeva)

$\overline{X} = \frac{\sum X_i}{n} \xrightarrow{P} m \quad (n \to \infty)$

(jaki zako

- def. niz X_n konvergira gotovo sigurno ka sl. var. Y ako vrijedi:

$$P(\lim_{n\to\infty} X_n = Y) = 1$$

Oznaka $X_n \xrightarrow{g.s.} Y$

TM. $X_n \xrightarrow{g.s} Y$, tada $X_n \xrightarrow{P} Y$.

TM. (jaki zakon velikih brojeva)

$$\overline{X} = \frac{\sum_{i=1}^{n} x_i}{n} \xrightarrow{g.s} m$$

, gdje su x_i nezavisne, ident. distr. sl. var.

- def. niz X_n konvergira po distribuciji ka sluč. var. X ako za odgovarajuće F razdiobe vrijedi

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

Oznaka: $X_n \xrightarrow{D} X$

TM (Levy) $X_n \xrightarrow{D} X$ akko $\vartheta_{X_n}(t) \to \vartheta_X(t)$

$\boxed{\text{ZAD}}$ Dokažite $B(n,p) \xrightarrow{D} P(\lambda = np)$ koristeći Levy-ev tm.

$$\lim_{n\to\infty} \vartheta_{X_{n,p}}(t) = (q + pe^{it})^n = \left(1 + p(e^{it}-1)\right)^{\frac{1}{p}\cdot pn} = \left(e^{e^{it}-1}\right)^{\lambda} = e^{\lambda(e^{it}-1)} \qquad e^{\lambda(e^{it}-1)}$$

STATISTIKA

- statistika za procjenu očekivanja $\boxed{\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}}$

- statistika za procjenu disperzije uz poznato očekivanje a

$$\boxed{D^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - a)^2}$$

1) nepristranost:

$$E(D^2) = E\left(\frac{1}{n} \sum (X_i - a)^2\right) = \frac{1}{n} \cdot \sum_{i=1}^{n} E(X_i - a)^2 =$$

$$= \frac{1}{\cancel{n}} \cancel{n} \cdot D(X_i) = \underline{\underline{\sigma^2}}$$

2) valjanost:

$$D(D^2) = D\left(\frac{1}{n} \sum (X_i - a)^2\right) = \frac{1}{n^2} \sum_{i=1}^{n} D(X_i - a)^2 = \frac{1}{n^{\cancel{2}}} \cdot \cancel{n} \cdot D[(X-a)^2]$$

$$= \frac{1}{n}\left[E[(X-a)^4] - [\underbrace{E(X-a)^2}_{\sigma^2}]^2\right] =$$

$$= \frac{\mu_4 - \sigma^4}{n} \xrightarrow{n \to \infty} \underline{\underline{0}}$$

- statistika za procjenu disperzije uz nepoznato očekivanje:

↳ nepristrana procjena: $\boxed{S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2}$

M1

$\boxed{\text{ZI - 2012}}$ $[\alpha, 1]$ - n brojeva $x_1, \dots, x_n$

$Y = \min(X_1, \dots, X_n) - \frac{1}{n+1}$, prov. je li statis. nepristrana i skolikim faktorom pomnožit da bude nepristrana.

- uvjet za nepristranost: $E(Y) \overset{!}{=} \alpha$

↪

$F_X(x) = P(X < x) = P(\min(X_1, \dots, X_n) < x) =$

$1 - P(\min(X_1, \dots, X_n) > x) = 1 - P(X_1 > x_1, \dots, X_n > x_n) \overset{nez.}{=}$

$= 1 - P(X_1 > x) \cdot \ldots \cdot P(X_n > x) = 1 - P(X_1 > x)^n =$

$= 1 - \left(\int_x^{+\infty} f_{X_1}(x) dx \right)^n = 1 - \left(\int_x^1 \frac{1}{1-\alpha} dx \right)^n = 1 - \left(\frac{1-x}{1-\alpha} \right)^n$

$f_X(x) = F_X'(x) = -n \left(\frac{1-x}{1-\alpha} \right)^{n-1} \cdot \frac{-1}{1-\alpha} = \frac{n}{(1-\alpha)^n} \cdot (1-x)^{n-1}$

$E(X) = \int_\alpha^1 x \cdot \frac{n}{(1-\alpha)^n} \cdot (1-x)^{n-1} dx = \left| \begin{matrix} t = 1-x \\ dt = -dx \end{matrix} \right| =$

$= \frac{n}{(1-\alpha)^n} \int_0^{1-\alpha} (1-t) t^{n-1} dt = \frac{n}{(1-\alpha)^n} \int_0^{1-\alpha} (t^{n-1} - t^n) dt =$

$= \frac{n}{(1-\alpha)^n} \left[\frac{t^n}{n} - \frac{t^{n+1}}{1+n} \right] \Big|_0^{1-\alpha} = \frac{n}{(1-\alpha)^n} \left[\frac{(1-\alpha)^n}{n} - \frac{(1-\alpha)^{n+1}}{n+1} \right] =$

$= \boxed{1 - \frac{n}{n+1}(1-\alpha)}$

$E(Y) = 1 - \frac{n}{n+1}(1-\alpha) - \frac{1}{n+1} = \boxed{\frac{n}{n+1} \cdot \alpha \neq \alpha}$

↳ nije nepristrana statistika

- treba pomnožiti sa $\frac{n+1}{n}$ postaje nepristrana statistika

10.2. Kriterij najveće izglednosti

- **def.** neka je $x_1, x_2, \dots, x_n$ realizacija uzorka populacije X čija f gustoće $f(\vartheta, x)$ ovisi o parametru ϑ. Funkcija izglednosti se definira kao umnožak $\boxed{L(\vartheta, x_1, \dots, x_n) = f(\vartheta, x_1) \cdots f(\vartheta, x_n)}$

za procjenu parametra ϑ uzima se vrijednost za koju fja L poprima globalni maksimum.

- $\boxed{21-2009}$ $f(x) = \dfrac{\lambda}{2\sqrt{x}} e^{-\lambda\sqrt{x}}, \quad x > 0$

Pomoću KNI, odredite procjenu za λ

$$L(\lambda, x_1, \dots, x_n) = f(x_1) \cdots f(x_n) = \frac{\lambda}{2\sqrt{x_1}} e^{-\lambda\sqrt{x_1}} \cdot \dots \cdot \frac{\lambda}{2\sqrt{x_n}} e^{-\lambda\sqrt{x_n}} =$$

$$= \frac{\lambda^n}{2^n\sqrt{x_1 \cdots x_n}} e^{-\lambda(\sqrt{x_1} + \dots + \sqrt{x_n})}$$

- $$\ln L = \ln \lambda^n - \ln 2^n\sqrt{x_1 \cdots x_n} + \ln e^{-\lambda(\sqrt{x_1} + \dots + \sqrt{x_n})} =$$

$$\ln L = n \ln \lambda - \ln(2^n\sqrt{x_1 \cdots x_n}) - \lambda \cdot \sum_{i=1}^{n} \sqrt{x_i}$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} \sqrt{x_i} = 0$$

$$\boxed{\lambda = \frac{n}{\sum_{i=1}^{n} \sqrt{x_i}}}$$

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$\boxed{21\text{-}2010}$

5 puta -> 2P
8 puta -> 3P

Ponovo KNJ, rj. posuda.

$$L(p) = P(X=2)\cdot P(X=3) = \binom{5}{2}p^2(1-p)^3\cdot\binom{8}{3}p^3(1-p)^5$$
$$= \binom{5}{2}\binom{8}{3}p^5(1-p)^8 =$$

$$\ln L(p) = \ln\binom{5}{2}\binom{8}{3} + 5\ln p + 8\ln(1-p)$$

$$\frac{d}{dp}\ln L = \frac{5}{p} + \frac{-8}{1-p} = 0 \Rightarrow \boxed{p = \frac{5}{13}}$$

$\boxed{Z1\text{-}2016}$

a) Poissonova razdioba, $\lambda = ?$

b) Je li procjena nepristrana?

MI za eksp.

$$L(\lambda, x_1, \ldots, x_n) = P(X=x_1)\cdot\ldots\cdot P(X=x_n) =$$
$$= \frac{\lambda^{x_1}}{x_1!}e^{-\lambda}\cdot\ldots\cdot\frac{\lambda^{x_n}}{x_n!}e^{-\lambda} = \frac{\lambda^{x_1+\ldots+x_n}}{x_1!\,x_2!\ldots x_n!}e^{-\lambda n}$$

$$\frac{\ln L}{\partial\lambda} = \frac{\sum x_i}{\lambda} - n = 0 \Rightarrow \boxed{\lambda = \frac{\sum_{i=1}^{n} x_i}{n}} = \bar{X}$$

-> sigurno nepristrana

b) $$E\left(\frac{\sum x_i}{n}\right) = \frac{1}{n}\sum_{i=1}^{n}\underbrace{E(X_i)}_{\lambda} = \frac{1}{n}\cdot n\lambda = \lambda$$

-> procjena je nepristrana

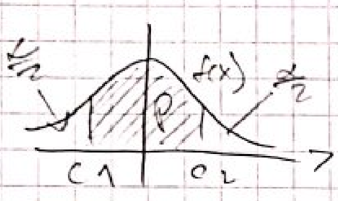
11. Intervalne procjene

11.1. Uvod

- def. interval $[c_1, c_2]$ za koji vrijedi

$$P(c_1 < X < c_2) = p$$ se naziva

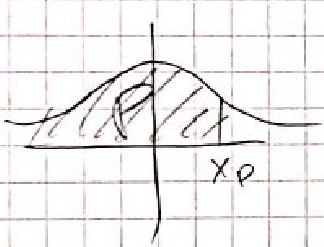
interval procjene (pouzdanosti) reda p.



- def. veličina $\alpha = 1 - p$ se naziva **nivo značajnosti (signifikantnosti)**

- def. redni broj x_p za koji vrijedi $F(x_p) = p$

tj. $P(X < x_p) = \int_{-\infty}^{x_p} f(t)dt = p$ se naziva **kvantil reda p**



- primjeti: $c_1 = x_{\frac{\alpha}{2}}$ $\quad c_2 = x_{1-\frac{\alpha}{2}}$ za simetrične razdiobe

↳ $c_1 = -c_2 = -x_{1-\frac{\alpha}{2}}$

$\boxed{21-2011}$ b) $u_p = u_{0.2} = -u_{0.8} = \boxed{-0.84162}$

c) $u_{0.813} \Rightarrow P(X < u_{0.813}) = 0.813$

$$\frac{1}{2} + \frac{1}{2}\Phi^*(u_{0.813}) = 0.813$$

$$\Phi^*(u_{0.813}) = 0.626$$

$$u_{0.813} = \boxed{0.889}$$

$\boxed{21-2016}$ c) $u_p = 0.484$

$$P(X < u_p) = p$$

$$P(X < 0.484) = p$$ — red kvantila

$$\frac{1}{2} + \frac{1}{2}\Phi^*(0.484) = p$$

$$p = 0.686 \quad \Rightarrow \quad u_{0.686} = 0.484$$

$$u_p = -0.484 \Rightarrow p = 1 - 0.686 = \boxed{0.314}$$

$\boxed{11.2}$ Intervalne procjene za parametre normalne razdiobe

- kratko objašnjenje:
  - za očekivanje uz poznatu disperziju σ^2

$$\bar{x} = \frac{\sum x_i}{n} \quad \Rightarrow P\left(c_1 < \frac{\bar{x}-a}{\sigma/\sqrt{n}} < c_2\right) = p$$ — $u_{1-\frac{\alpha}{2}}$

$$\Rightarrow P\left(\bar{x} - u_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} < a < \bar{x} + u_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) = p$$

$$\Rightarrow \left(a_{1,2} = \bar{x} \pm u_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$$

$\boxed{21-2009}$ a) $\bar{x} = \frac{\sum n_j \cdot x_j}{n} = 25$, $n = \sum n_j = 27$

$$\hat{s}^2 = \frac{1}{n-1}\sum n_j (x_j - \bar{x})^2 \overset{\text{ili}}{=} \frac{1}{n-1}\left(\sum n_j x_j^2 - n\bar{x}^2\right)$$

$$= 8.846$$

b) 95% , $\hat{s} = \sqrt{\hat{s}^2}$

$$t_{n-1, 1-\frac{\alpha}{2}} = t_{26, \alpha=5\%} = \boxed{2.056}$$

$$\Rightarrow P(23.823 < a < 26.177) = \boxed{0.95}$$

$$\chi^2_{n-1;1-\frac{\alpha}{2}} = \chi^2_{26,0.975} = 41.923$$

$$\chi^2_{n-1,\frac{\alpha}{2}} = \chi^2_{26,0.025} = 13.844$$

$$P(5.486 < \sigma^2 < 16.613) = 0.95$$

$\boxed{81 \cdot 2012}$

$\sigma = 0.3 \quad \to 0.3123$

$$\Rightarrow u_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} = \frac{0.3123}{2} \Rightarrow u_{1-\frac{\alpha}{2}} = 1.645$$

$n = 10$

$1 - \frac{\alpha}{2} = 0.95$

$\alpha = 0.1$

Odg: $\boxed{p = 90\%}$

11.3. Intervalne procjene za ostale parametre ($\neq N$)

- za vjerovatnost p događaja

$$\hat{p} = \frac{m}{n} \sim \frac{B(n,p)}{n} \approx \frac{N(np, n>9)}{n} = N\left(p, \frac{pq}{n}\right)$$

ako iz uzorka sa sucodostl. uzrejem

$$\Rightarrow P\left(-u_{1-\frac{\alpha}{2}} \leq \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}} \leq u_{1-\frac{\alpha}{2}}\right) = p$$

$1-p$

$$\to p_{1,2} = \hat{p} \pm u_{1-\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

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$\boxed{2008-4}$

a) $\hat{p} = \frac{3}{100} = 0.03$

α je suprotno od pouzdanja
$\alpha = 5\% \Rightarrow$
$u_{1-\frac{\alpha}{2}} = 1.96$
0.975

$p_{1,2} = 0.03 \pm 1.96\sqrt{\frac{0.03 \cdot 0.97}{100}}$

$\Rightarrow P(\ 0 < p < 0.0634) = 95\%$

b) $\hat{p} + u_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq 0.05$

$1.96\sqrt{\frac{0.03 \cdot 0.97}{n}} \leq 0.02 \Rightarrow n \geq 280$

$\boxed{21-2008}$

a) $\hat{p} = \frac{112}{200} = 0.56$

$\alpha = 5\% \rightarrow u_{1-\frac{\alpha}{2}} = 1.96$, $n = 200$

$p_{1,2} = 0.56 \pm 0.069$

$\Rightarrow P(0.491 \leq p \leq 0.629) = 95\%$

b) $\hat{p} - u_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} > 0.5$

$0.56 - u_{1-\alpha/2}\sqrt{\frac{0.56 \cdot 0.44}{200}} > 0.5$

$u_{1-\alpha/2} < 1.7094$

$u_{1-\frac{\alpha}{2}} = 1.64485 \rightarrow 1 - \frac{\alpha}{2} = 0.95$

$\boxed{\alpha = 10\%}$

$P = 1 - \alpha = \boxed{90\%}$

c) $\alpha = 0.05$, $n = ?$

$\hat{p} - u_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} > 0.5$

$0.56 - 1.96\sqrt{\frac{0.56 \cdot 0.44}{n}} > 0.5 \Rightarrow n \geq 262.93$

$\boxed{Rj: n \geq 263}$

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21-2013

$p = 90\%$, $\alpha = 10\%$, $\bar{x} = 66$ auta/h

$n = 24$ - ? => iz sume jer smo podijelili sumu X sa n a gledalo se ukupno 24 sata

$$\Rightarrow |\bar{x} - \lambda| < u_{1-\alpha/2}\sqrt{\frac{\lambda}{n}}$$

$u_{0.95} = 1.64435$

$$\Rightarrow \lambda \in \langle \lambda_1, \lambda_2 \rangle$$

$$|66 - \lambda| < 1.64435\sqrt{\frac{\lambda}{24}} \quad |^2$$

$$\Rightarrow \lambda^2 - 132.113\lambda + 4356 < 0$$

$\lambda_1 = 63.33$ $\quad \lambda_2 = 68.79$

$$\Rightarrow \boxed{\lambda \in \langle 63.33, 68.79 \rangle}$$

12. Testiranje hipoteza

12.1. Hipoteze i pogreške odlučivanja

Pr 50 stranica/min

$n = 30$ -> dobili samo 47 str./min

- uvodimo hipoteze:

proizv. tvrdi -> H_0 ... proizvođač govori istinu (50 str/min)

H_1 ... proizvođač laže (broj ispisa < 50)

- def. snaga testa se definira s: $S(\vartheta) = P$(prihvaćamo H_1)

idealno $S(\vartheta) = 0$ za $\vartheta \in H_0$

$S(\vartheta) = 1$ za $\vartheta \in H_1$

ali nikada nije tako

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- def. pogreška **prve vrste**: α = vjv. da prihvatimo H_1, a istina je H_0

pogreška **druge vrste**: β = vjv. da prihvatimo H_0, a istina je H_1

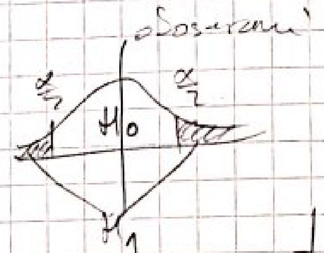
- α je što manji, najčešće 1%, 5%

12.2. Testiranje parametarskih hipoteza

- testiramo neki parametar:

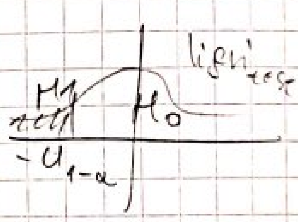
$H_0 \ldots \quad \vartheta = \vartheta_0$

$H_1 \ldots \quad \vartheta \neq \vartheta_0$ dvostrani test $\Rightarrow$



$H_1 \ldots \quad \vartheta < \vartheta_0$ lijevi test

$H_1 \ldots \quad \vartheta > \vartheta_0$ desni test



$\boxed{21-2013}$

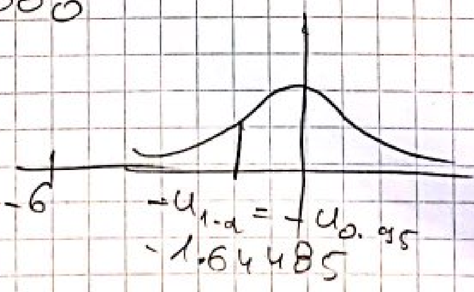
a) $\sigma = 5$ ml

$n = 100 \rightarrow \bar{x} = 997$

$H_0 \ldots \quad a = 1000$

$$\hat{u} = \frac{\bar{x} - a_0}{\sigma/\sqrt{n}} = \frac{997 - 1000}{5/\sqrt{100}} = -6$$

$H_1 \ldots \quad a < 1000$ (lijevi)



$-6 < -1.64485$

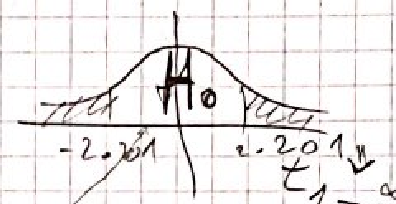
odbacujemo H_0, tj. prihvaćamo H_1

Proizvođač laže uz $\alpha = 5\%$

ZI-2009 (4.)

$H_0 \ldots \; a = 98$

$H_1 \ldots \; a \neq 98$



$t_{1-\frac{\alpha}{2}, \, n-1}$

$\bar{x} = 97.6583 \; (n=12)$

$\hat{s}^2 = 0.6079 \quad \Rightarrow \frac{1}{n-1} \sum (x_i - \bar{x})^2$

$\hat{t} = \frac{\bar{x} - 98}{s/\sqrt{n}} = -1.518$

$t_{11, \alpha = 5\%}$

=> Prihvaćamo H0, tj. rafinerija ne laže. , uz 5%

ZI-2012-6

$H_0 \ldots \; a = 200$

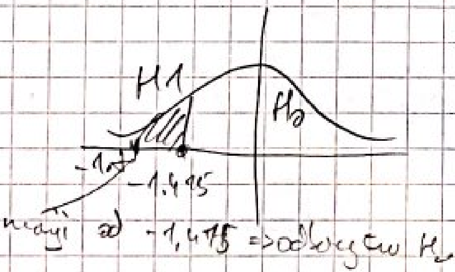
$H_1 \ldots \; a < 200$

$\bar{x} = 190.25$

$n = 8$

$\hat{s}^2 = 259.36$

$\hat{t} = \frac{\bar{x} - 200}{\hat{s}/\sqrt{n}} = -1.712$



manji od -1,415 => odbacujemo H0

$\alpha = 0.1$

$t_{n-1, 1-\alpha} = 1,415$

7, α2 /

Pazi! jer je tablica napisana za $1 - \alpha/2$

=> Odbacujemo H0 tj. proizvođač laže

$\boxed{\text{ZI}-2016-6}$

a) $\hat{p} = \frac{m}{n} = \frac{314}{455} = 0.69$

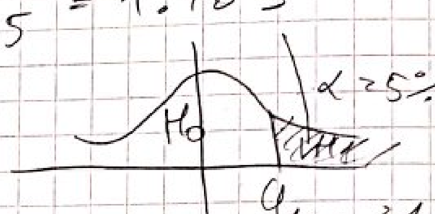
$p_{1,2} = 0.69 \pm 0.042$

b)

$H_0 \ldots \; p = 0.65 \qquad \alpha = 5\%$

$H_1 \ldots \; p > 0.65$

$\hat{u} = (0.69 - 0.65)\sqrt{\frac{455}{0.65 \cdot 0.35}} = 1.789$



$u_{1-\alpha} = 1.64485$

=> Odbacujemo H_0, tj. uz $\alpha = 5\%$ pivovara će povećati udio reklama!

$\boxed{\text{ZI}-14-7}$

$\left.\begin{array}{l} H_0 \ldots \; p = 0.2 \\ H_1 \ldots \; p > 0.2 \end{array}\right\} \quad \hat{u} = (\underbrace{\hat{p}}_{\frac{1}{4}} - \underbrace{p_0}_{0.2})\sqrt{\frac{n}{\underbrace{p_0}_{0.2}\,\underbrace{q_0}_{0.8}}}$

$n = ? \qquad \alpha = 5\%$

$\rightarrow \; (p - p_0)\sqrt{\frac{n}{p_0 q_0}} > u_{1-\alpha}$ prihvaćamo H_1

$0.05\sqrt{\frac{n}{0.16}} > 1.64485$

$n > 172.55$

$\rightarrow$ Moramo ispitati minimalno 173 uzorka.

12.3. Usporedba dviju populacija

$\boxed{Z1 - 2011 - 6}$

$H_0 \dots \; \mu = \nu$

$H_1 \dots \; \mu \neq \nu$

$n = m = 7$

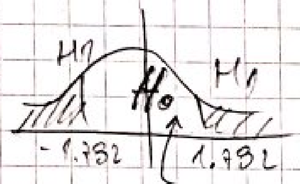
$\bar{x} = 10, \; s_x^2 = \frac{28}{6}$

$\bar{y} = 9, \; s_y^2 = \frac{20}{6}$

$$t_{n+m-2; 1-\alpha/2} = t_{12, \alpha = 10\%} = 1.782$$

$$S_z^2 = \frac{1}{n+m-2}\left[(n-1)s_x^2 + (m-1)s_y^2\right] = 4$$

$$\hat{t} = \frac{\bar{x}-\bar{y}}{\hat{S}_z}\sqrt{\frac{nm}{n+m}} = 0.935$$



=> Prihvaćamo H_0, tj. prosječna prodaja u obje ista

$\boxed{Z1 - 09 - 5}$

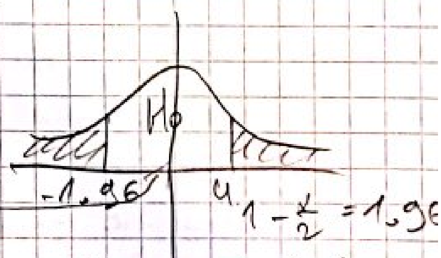
$H_0 \dots \; p_1 = p_2 \; \rightarrow \; \hat{p}_1 = \frac{90}{200}, \; \hat{p}_2 = \frac{69}{150}$

$H_1 \dots \; p_1 \neq p_2$

$\alpha = 5\%$

$$p = \frac{159}{350} = 0.454 \Rightarrow \sigma^2 = p(1-p) = 0.248$$

$$\hat{u} = \frac{\hat{p}_1 - \hat{p}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \boxed{-0.186}$$



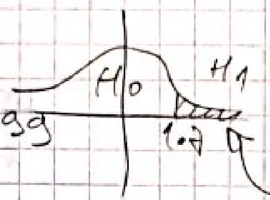
=> Prihvaćamo H_0, tj. možemo zaključiti da se jednako postotak muš. i žena bavi sportom. ($\alpha = 5\%$)

$\boxed{ZI-14-6}$

b) $\alpha = 0.05$, $n = 30$

$$\left.\begin{array}{l} H_0 \ldots \ a = 21 \\ H_1 \ldots \ a > 21 \end{array}\right\} \quad t = \frac{\bar{x} - a_0}{S/\sqrt{n}} = \frac{23-21}{3.2/\sqrt{30}}$$

$$t = 3.423$$



$t_{n-1,1-\alpha} = 1.699$

$29, \alpha = 5\%$

iz rez. tablice

=> Odbacujemo H_0 tj. možemo tvrditi da je novi lijek efikasniji od starog lijeka ($\alpha = 5\%$)

12.4. Test prilagodbe razdiobama (χ^2 - test)

tek nakon što spojimo

$\boxed{ZI-10-6}$

$H_0 \ldots$ ravna se po $B(3, \frac{1}{6})$

x_j	n_j	p_j	np_j (teor. frekv.)	$\frac{(n_j - np_j)^2}{np_j}$
0	61	$(\frac{5}{6})^3$	57.87	0.169
1	29	$\binom{3}{1}\frac{1}{6}(\frac{5}{6})^2$	34.72	0.942
2	8	$\binom{3}{2}(\frac{1}{6})^2\frac{5}{6}$	6.94 } 7.4	0.913
3	2	$(\frac{1}{6})^3$	0.46	
Σ	$n=100$	1		

$$\chi^2_q = 2.024$$

↳ ako je teor. frekv. manja od 5 spojimo ju sa susjednim razredom

$$\chi^2_{m-r-1,1-\alpha} = \chi^2_{2,95\%} = 5.991 > 2.024$$

↳ $\boxed{\text{Prihvaćamo } H_0.}$

br. razreda nakon spajanja $m=3$

$r=0$ nismo ništa procjenili

Z1-12-09

x_j	n_j	p_j	np_j	$\frac{(n_j-np_j)^2}{np_j}$
0	39	0.606	56.36	0.192
1	14	0.303	18.18	0.961
2	5 } 7	0.076	4.56 } 5.34	0.516
3	2	0.013	0.78	
Σ	n=60	1		$\chi^2_e = 1.669$

$$P(X=j) = \frac{\lambda^j}{j!} e^{-\lambda}$$

$$\lambda = E(X) = \bar{X} = \frac{1}{2} \qquad P(X=0) = \frac{\lambda^0}{0!} e^{-\frac{1}{2}}$$

$$\chi^2_{k-r-1,1-\alpha} = \chi^2_{1,95\%} = 3.841 < 1.669$$

$3-1-1, 1-5\%$

procijenili samo λ

r = 2 kada gledamo po N - očekivanje i disperziju

=> Prihvaćamo Ho, tj. auti se ravnaju po Poissonu. $\alpha = 5\%$

Z1-16-8

	n_j	p_j	np_j	$\frac{(n_j-np_j)^2}{np_j}$
O	24	$\frac{1}{3}$	$26.\dot{6}$	0.26
A	39	$\frac{2}{5}$	32	1.53125
B	13	$\frac{1}{5}$	16	0.5625
AB	4	$\frac{1}{15}$	$5.\dot{3}$	$0.\dot{3}$
	n=80			

5:6:3:1

$$\chi^2_e = 2.69 < \chi^2_{3,1-\alpha} = 3.665$$

prvi veći

70%

$$\boxed{\alpha = 30\%}$$

$\boxed{2011-10}$

$\frac{a+b}{2}$	$[a,b]$	n_j	p_j	$n \cdot p_j$	$\frac{(n_j - np_j)^2}{np_j}$
12.5	0 - 25	87	0.613	91.95	0.266
37.5	25 - 50	46	0.237	35.55	3.072
62.5	50 - 75	14	0.092	13.8	0.0029
87.5	75 - 100	3	0.058	8.7	3.734
		$n=150$	≈ 1		$\chi_q^2 = 7.075$

$$P(a < X < b) = e^{-\lambda a} - e^{-\lambda b} \Rightarrow p_j$$

$$\Rightarrow \lambda = \frac{1}{\bar{x}} = \frac{1}{26.3} = 0.038$$

Procjene:
eksp: $\lambda = \frac{1}{\bar{x}}$
poiss: $\lambda = \bar{x}$

H_0 ... eksp $\qquad H_1$ ... nije eksp.

$m - r - 1 = 2$, $\alpha = 5\%$ $\qquad \chi^2_{2,5\%} = 5.991 < \chi_q^2$

$\Rightarrow$ Odbacujemo H_0, tj. nemožemo zaključiti da se ravna po eksponen. razdiobi uz $\alpha = 5\%$

$\boxed{Z1-14-8}$

x_j	n_j	p_j	np_j	$\frac{(n_j - np_j)^2}{np_j}$
0	74	1/10	80	0.45
1	92	1/10	80	1.8
2	83	1/10	80	0.1125
3	79	⋮	⋮	⋮
4	80	⋮		0
5	73	⋮	⋮	⋮
6	77	⋮	⋮	
7	75	⋮	⋮	
8	76	1/10	80	
9	91	1/10	80	1.952
	800	1		$\chi_q^2 = 5.152$

$$\chi^2_{m-r-1, 1-\alpha} = \chi^2_{9, 0.90} = 14.684$$

$\boxed{\text{Prihvaćamo } H_0.}$