ZI 2009.

$\boxed{\text{Zadatak 1.}}$

a) $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)dx \qquad f(x) = x^2\cos(x) \qquad x_i = -\frac{\pi}{2} + \frac{\pi}{4}i \qquad i = 0,1,2,3,4$

$x_0 = \frac{-\pi}{2} \qquad x_1 = -\frac{\pi}{2} + \frac{\pi}{4} = \frac{-2\pi+\pi}{4} = \frac{-\pi}{4} \qquad x_2 = -\frac{\pi}{2} + \frac{\pi}{2} = 0 \qquad x_3 = \frac{-\pi}{2} + \frac{3\pi}{4} = \frac{-2\pi+3\pi}{4} = \frac{\pi}{4}$

$x_4 = -\frac{\pi}{2} + \frac{\pi}{4}\cdot 4 = -\frac{\pi}{2} + \pi = \frac{\pi}{2}$

→ kompozitna trapezna formula: $\int_{x_{i-1}}^{x_i} f(x)dx = \frac{x_i - x_{i-1}}{2}\left(f(x_i) + f(x_{i-1})\right) + E_i^T(f)$

$$I(f) = \frac{h}{2}\left[f(x_0) + 2\left(f(x_1) + f(x_2) + f(x_3)\right) + f(x_4)\right] + \sum_{i=1}^{4}\left(-\frac{h^3}{12}\right)f''(\eta_i)$$

$h = \frac{b-a}{n}$

→ konačno: $T_n(f) = \frac{h}{2}\left(f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right)$

Rješenje: $T_4(f) = \frac{b-a}{2\cdot 4}\left(f\left(-\frac{\pi}{2}\right) + 2\sum_{i=1}^{3}\left[f(x_1) + f(x_2) + f(x_3)\right] + f\left(\frac{\pi}{2}\right)\right)$

$f\left(-\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2\cos\left(-\frac{\pi}{2}\right) = \frac{\pi^2}{4}\cdot 0 = 0$

$f(x_1) = f\left(-\frac{\pi}{4}\right) = \left(\frac{\pi}{4}\right)^2\cdot\cos\left(-\frac{\pi}{4}\right) = \frac{\pi^2}{16}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}\pi^2}{32}$

$f(x_2) = 0 \qquad f(x_3) = \frac{\pi^2}{16}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}\pi^2}{32}$

$f(b) = f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}\cdot\cos\left(\frac{\pi}{2}\right) = 0$

$\frac{b-a}{8} = \frac{\frac{\pi}{2}+\frac{\pi}{2}}{8} = \frac{\pi}{8}$

$T_4(f) = \frac{\pi}{8}\left(0 + 2\left(\frac{\sqrt{2}\pi^2}{32} + 0 + \frac{\sqrt{2}\pi^2}{32}\right) + 0\right) = \frac{\pi}{8}\left(\frac{\sqrt{2}\pi^2}{8}\right) = \frac{\sqrt{2}\pi^3}{64}$ ✓✓

b) $\max_{x\in[-\frac{\pi}{2},\frac{\pi}{2}]} |f''(x)| = 2\pi$ – apsolutna greška integracije kompozitne trapezne?

$\to \quad |I(f) - T_n(f)| \leq \frac{h^3}{12}\max_{x\in[a,b]}|f''(x)|\cdot n = h^2\|f''\|_\infty\cdot\frac{b-a}{12}$

$|I(f) - T_4(f)| \leq \frac{h^3}{12}\cdot 2\pi\cdot n \Rightarrow |I(f) - T_4(f)| \leq \frac{\left(\frac{b-a}{4}\right)^3}{12}\cdot 2\pi\cdot 4$

$\leq \frac{\frac{\pi^3}{64}}{12}\cdot 2\pi\cdot 4 = \frac{\pi^3}{64\cdot 12}\cdot 8\pi = \frac{\pi^4}{96}$ ✓✓

c) koliko je točaka mreže potrebno da apsolutna greška bude manje od 10^{-1}

$$\frac{h^3}{12}\cdot 2\pi\cdot n \geq 10^{-1} \qquad h^3=\left(\frac{b-a}{n}\right)^3=\left(\frac{\pi}{n}\right)^3$$

$$\frac{\pi^3}{12n^3}\cdot 2\pi\cdot n \geq 10^{-1} \qquad \frac{\pi^4}{6n^2}\geq 10^{-1} \qquad \frac{\pi^4}{10^{-1}\cdot 6}\geq n^2 \qquad n\geq\sqrt{\frac{\pi^4}{6\cdot 10^{-1}}}$$

$$n\geq 12.74$$

$$n=13$$

Zadatak 2.

$$x_{n+1}=f(x_n),\quad n\in\mathbb{N}\cup\{0\} \quad x_0=0 \quad x_1=f(x_0)=\frac{7}{15} \quad \max_{x\in I}|f'(x)|\leq L \quad L=\frac{3}{10}$$

a) udaljenost od x_n do fiksne točke $\leq 10^{-3}$

ocjena jednostavnih iteracija: $|x_n-x^*|\leq\frac{L^n}{1-L}|x_0-x^*|$

$$|x_n-x^*|\leq\frac{L^n}{1-L}|x_0-x_1|$$

$$|x_n-x^*|\leq 10^{-3} \implies 10^{-3}\leq\frac{\left(\frac{3}{10}\right)^n}{1-\frac{3}{10}}\left|0-\frac{7}{15}\right|$$

$$10^{-3}\leq\frac{\left(\frac{3}{10}\right)^n\cdot 10}{7}\cdot\frac{7}{15_3} \implies 10^{-3}\leq\frac{2}{3}\cdot\left(\frac{3}{10}\right)^n$$

$$\frac{3}{2}\cdot 10^{-3}\leq\left(\frac{3}{10}\right)^n \,/\log$$

$$\implies \log(1.5\cdot 10^{-3})\leq n\log(0.3)$$

$$-2.824\leq n(-0.5228)$$

$$n\geq 5.4 \to n=6$$

b) konvergencija jednostavnih iteracija

$$|f(x)-f(y)|=|f'(\xi)(x-y)|\leq L(x-y) \to L<1 \to |x_1-x^*|=|f(x_0)-f(x^*)|\leq L|x_0-x^*|$$

$$\implies |x_{n+1}-x^*|=|f(x_n)-f(x^*)|\leq L|x_n-x^*|$$

Zadatak 3.

$$f(x)=\begin{cases}\frac{1}{x}e^{-\frac{1}{x^2}} & x\neq 0\\ 0, & x=0\end{cases}$$

a) N.m. prikazati kao metodu jednostavnih iteracija

N.m. za traženje nul točaka x^* $\to$ $x_{n+1}=x_n+\frac{f(x_n)}{f'(x_n)}$

$$-\frac{1}{x^2}=-x^{-2}=2x^{-3}$$
$$\frac{2}{x^3}$$

$x_{n+1}=f(x_n)$ sa grafa $x_0=0$

$$f'(x)=\frac{-1}{x^2}e^{-\frac{1}{x^2}}+\frac{1}{x}\left(+\frac{2}{x^3}e^{-\frac{1}{x^2}}\right)=e^{-\frac{1}{x^2}}\left(-\frac{1}{x^2}+\frac{2}{x^4}\right)=e^{-\frac{1}{x^2}}\left(\frac{-x^2+2}{x^4}\right)$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{\frac{1}{x}e^{-\frac{1}{x^2}}}{e^{-\frac{1}{x^2}}\left(\frac{2-x_n^2}{x_n^4}\right)}=x_n-\frac{\frac{1}{x_n}}{\frac{1}{x_n}\left(\frac{2-x_n^2}{x_n^3}\right)}=x_n-\frac{x_n^3}{2-x_n^2}=x_n\left[1-\frac{x_n^2}{2-x_n^2}\right]$$

$$=x_n\left(\frac{2-x_n^2-x_n^2}{2-x_n^2}\right)=x_n\left(\frac{2-2x_n^2}{2-x_n^2}\right)=2x_n\left(\frac{1-x_n^2}{2-x_n^2}\right)$$

b) odrediti sve $x_0\in\mathbb{R}$

???

Zadatak 4.

4.1. a) $(Ax-b, Ay)=0 \quad \forall y \in \mathbb{R}^2$

$(A^T(Ax-b), y)=0 \quad \forall y \in \mathbb{R}^2$

$A^T(Ax-b)=0 \quad \rightarrow A^TAx = A^Tb$

b) Ima jedinstveno rješenje, matrica A^TA je pozitivno definitna te time i regularna

$A^TA \in \mathbb{R}^{2\times 2}, \quad (A^TA)^T = A^TA \rightarrow (A^TAx, x) = (Ax, Ax) = \|Ax\|_2^2 > 0 \quad \forall x \neq 0$

c) ne, ne koristi. Za numeričko rješavanje je Newtonova metoda.

4.2.

a) N. metoda nultočaka

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \in \mathbb{N}, \quad x_0$ zadano

b) N. m. iteracije za $\sqrt{2}$

$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \quad n \in \mathbb{N}, \quad x_0 = 1$ (npr.)

c) 2 kriterija zaustavljanja N. m.

- ocjena reziduala : $|f(x_n)| \leq \varepsilon |f(x_0)|$
- ocjena duljine koraka : $|x_{n+1} - x_n| \leq \varepsilon |x_0|$

Zadatak 5.

a) razlog pivotiranja kod rješavanja linearnih sustava
- zbog malih pivotnih elemenata ili $a_{kk}^{(k)}=0$
- radi stabilnosti gaussovih eliminacija

b) $n=1024$
- $\rightarrow$ za DFT $\rightarrow 1024 \times 1024 \Rightarrow$ potrebno je $1024 \rightarrow O(n^2)$
- $\rightarrow$ kod FFT $\rightarrow 5 \times 1024 \Rightarrow$ 200 puta brže $\rightarrow O(\frac{1}{2}N\log N)$

c) uvjeti za interpolaciju kubičnim splajnom
- neprekinost funkcije $s(x_i-0)=s(x_i+0) \quad i=1,2,..n-1$
- interpolaciju $s(x_i)=y_i, \ i=0,1,..n$
- neprekidnost prve derivacije $s'(x_i-0)=s'(x_i+0) \quad i=1,2,..n-1$
- — '' — druge — '' — $s''(x_i-0)=s''(x_i+0) \quad i=1,2,..n-1$

d) 2 matrična rastava:
- $\rightarrow$ LU faktorizacija $\rightarrow$ sustavi linearnih jednadžbi
- $\rightarrow$ QR faktorizacija $\rightarrow$ metoda najmanjih kvadrata
- $\rightarrow$ SVD — '' — $\rightarrow$ — '' —

Zadatak 6.

a) $O\left(\frac{2n^3}{3}\right)$ → TOČNO

b) Uvjetovanost matrice $A \in \mathbb{R}^{n \times n}$ je veća od 1:

Uvjetovanost $\kappa(A) = \frac{\lambda_{max}}{\lambda_{min}}$ $\lambda_{max}, \lambda_{min}$ → max i min svojstvene vrijednosti

$\kappa(A) = \|A\| \|A^{-1}\|$ $\kappa(A) \geq 1$ TOČNO

c) NETOČNO

d) NETOČNO

ZI 2010.

Zadatak 1.

a)

x	0	1	2
y	2	4	6

$y = c_0 + c_1 x$

$$E(c_0, c_1) = \sum_{i=0}^{n} (y_i - c_0 - c_1 x_i)^2$$

$$\frac{\partial(c_0, c_1)}{\partial c_0} = -2 \sum_{i=0}^{n} (y_i - c_0 - c_1 x_i) \Rightarrow \quad 2\sum_{i=0}^{2} y_i = 6c_0 + 2c_1 \sum_{i=0}^{2} x_i$$

$$\frac{\partial(c_0, c_1)}{\partial c_1} = -2 \sum_{i=0}^{n} (y_i - c_0 - c_1 x_i)(x_i) \Rightarrow \quad 2\sum_{i=0}^{2} y_i x_i = 2c_0 \sum_{i=0}^{2} x_i + 2c_1 \sum_{i=0}^{2} x_i^2$$

$\sum_{i=0}^{2} x_i = 0+1+2 = 3$ $\qquad \sum_{i=0}^{2} x_i^2 = 0^2+1^2+2^2 = 1+4 = 5$

$\sum_{i=0}^{2} y_i = 2+4+6 = 12$ $\qquad \sum_{i=0}^{2} y_i x_i = 0\cdot 2 + 1\cdot 4 + 2\cdot 6 = 0+4+12 = 16$

$$\left.\begin{aligned} 2(12) &= 6c_0 + 2c_1(3) \\ 2(16) &= 2c_0(3) + 2c_1(5) \end{aligned}\right| \Rightarrow \quad \left.\begin{aligned} 24 &= 6c_0 + 6c_1 \quad /(-1) \\ 32 &= 6c_0 + 10c_1 \end{aligned}\right| \quad \left.\begin{aligned} -24 &= -6c_0 - 6c_1 \\ 32 &= 6c_0 + 10c_1 \end{aligned}\right| +$$

$8 = 4c_1 \quad c_1 = \frac{8}{4} = 2 \Rightarrow 24 = 6\cdot c_0 + 12 \quad 12 = 6c_0 \quad \boxed{c_0 = 2} \quad \boxed{c_1 = 2}$

$y = 2 + 2x$

$\Rightarrow$ matrični oblik

$$E(c_0, c_1, \ldots c_k) = \left\| \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix} - \begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^k \\ 1 & x_1 & x_1^2 & & x_1^k \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & & x_n^k \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_k \end{bmatrix} \right\|^2$$

$$y = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad c = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

b) $\min \|Ax - b\|_2 \quad A \in \mathbb{R}^{n \times k}, \ b \in \mathbb{R}^n, \ k < n, \ r(A) = k$

sustav normalnih jednadžbi:

$$\|Ax - b\|_2 \to \min \Rightarrow (Ax - b, Ay) = 0 \quad \forall y \in \mathbb{R}^k$$
$$(A^T(Ax - b), y) = 0 \quad \forall y \in \mathbb{R}^k$$
$$A^T(Ax - b) = 0 \quad \underline{A^T A x = A^T b}$$

$\Rightarrow$ ima li jedinstveno rješenje? Objasnite

↳ Ima jedinstveno rješenje: $A^T A \in \mathbb{R}^{k \times k}$, $(A^T A)^T = A^T A$

$(A^T A x, x) = (Ax, Ax) = \|Ax\|_2^2 > 0 \quad \forall x \neq 0 \to$ matrica je pozitivno definitna te time i regularna.

$\Rightarrow$ QR faktorizacija: Definirajte : $Ax \approx b \qquad A = Q \begin{bmatrix} R \\ 0 \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \\ 0 & 0 \end{bmatrix}$

$R \to$ gornje trokutasta matrica $\qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

...

$\Rightarrow \dots \quad d_1 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad d_2 = [b_3]$

Rješavamo sustav: $Rx = d_1 \Rightarrow$ zatim $\|Ax - b\|_2 = \|d_2\|_2$ je minimum reziduala

Zadatak 2.

a) Definirajte New. metodu za traženje (jednostruke) nultočke $f: \mathbb{R} \to \mathbb{R}$.

$$f(x_1) = f(x_0) - \frac{f(x_0)}{f'(x_0)} \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

b) N. m. kao metoda jednostavnih iteracija s $\varphi(x)$

$$N.m. \Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow x_{n+1} = \varphi(x) \Rightarrow \varphi(x) = x_n - \frac{f(x_n)}{f'(x_n)}$$

c) Definirajte kvadratičnu konvergenciju iterativne metode i pokažite da New. metoda konvergira kvadratično.

$\to$ kvadratična konvergencija N. m. $\Rightarrow x_{n+1} - x^* = \frac{1}{2}(x^* - x_n)^2 \frac{f''(\xi_n)}{f'(x_n)}$

$\to$ kvadratična konvergencija: $x^* \in \mathbb{R}$ ako postoji $L > 0$ i $n_0 \in \mathbb{N}$ takvi da za svaki $n \geq n_0$ vrijedi:

$$|x_{n+1} - x^*| \leq L |x_n - x^*|^2$$

d) 2 kriterija zaustavljanja New. metode?

- ocjena reziduala: $|f(x_n)| \leq \varepsilon |f(x_0)|$
- ocjena duljine koraka: $|x_{n+1} - x_n| \leq \varepsilon |x_0|$

Zadatak 3.

$$x_1 + 2x_2 = 2 \qquad x^0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \text{prve dva koraka N. m. } = ?$$
$$x_1^2 + 4x_2^2 = 4$$

$$F(x_1, x_2) = \begin{bmatrix} x_1 + 2x_2 - 2 \\ x_1^2 + 4x_2^2 - 4 \end{bmatrix} \qquad DF(x_1, x_2) = \begin{bmatrix} \frac{d}{dx_1} & \frac{d}{dx_2} \\ \frac{d}{dx_1} & \frac{d}{dx_2} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2x_1 & 8x_2 \end{bmatrix}$$

$$DF(x_1, x_2) S^0 = -F(x_1, x_2)$$

$$F(1,2) = \begin{bmatrix} 1 + 4 - 2 \\ 1 + 16 - 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \end{bmatrix} \qquad DF(1,2) = \begin{bmatrix} 1 & 2 \\ 2 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 16 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} -3 \\ -13 \end{bmatrix} \qquad \begin{matrix} s_0 + 2s_1 = -3 \quad / \cdot (-2) \\ 2s_0 + 16s_1 = -13 \end{matrix} \qquad \begin{matrix} -2s_0 - 4s_1 = 6 \\ 2s_0 + 16s_1 = -13 \end{matrix} \Big/ + \Rightarrow \begin{matrix} 12s_1 = -7 \\ s_1 = \frac{-7}{12} \end{matrix}$$

$$s_0 = -3 - 2s_1 = -3 + \frac{7}{6} = \frac{-6+7}{6} = \frac{1}{6} \Rightarrow S^0 = \begin{bmatrix} \frac{1}{6} \\ -\frac{7}{12} \end{bmatrix} \Rightarrow x^1 = x^0 + s^0 = \begin{bmatrix} \frac{7}{6} \\ \frac{15}{12} \end{bmatrix} = \begin{bmatrix} 7/6 \\ 5/4 \end{bmatrix}$$

korak

$$F(7/6, 5/4) = \begin{bmatrix} 7/6 + \frac{5}{2} - 2 \\ \frac{49}{36} + \frac{25}{4} - 4 \end{bmatrix} = \begin{bmatrix} \frac{7+15-12}{6} \\ \frac{49+225-144}{36} \end{bmatrix} = \begin{bmatrix} 5/3 \\ 65/18 \end{bmatrix} \qquad D(7/6, 5/4) = \begin{bmatrix} 1 & 2 \\ 7/3 & 10 \end{bmatrix}$$

$$DF \cdot S = -F \Rightarrow \begin{bmatrix} 1 & 2 \\ 7/3 & 10 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} -5/3 \\ -65/18 \end{bmatrix}$$

$s_0 + 2s_1 = -\frac{5}{3}$ $\quad /\cdot(-5)$ $\quad -5s_0 - 10s_1 = \frac{25}{3}$

$\frac{7}{3}s_0 + 10s_1 = -\frac{65}{18}$ $\quad \frac{7}{3}s_0 + 10s_1 = -\frac{65}{18}$

$-\frac{8}{3}s_0 = \frac{150-65}{18}$

$-\frac{8}{3}s_0 = \frac{85}{18}$

$s_0 = -\frac{85}{\cancel{18}_6} \cdot \frac{\cancel{3}^1}{8} = -\frac{85}{48} = -1{,}77$

$-\frac{85}{48} + 2s_1 = -\frac{5}{3}$

$2s_1 = -\frac{5}{3} + \frac{85}{48} = \frac{-80+85}{48} = \frac{5}{48}$ $\quad s_1 = \frac{5}{96}$

$s^1 = \begin{bmatrix} -85/48 \\ 5/96 \end{bmatrix} \rightarrow x^2 = x^1 + s^1 = \begin{bmatrix} 7/6 \\ 5/4 \end{bmatrix} + \begin{bmatrix} -85/48 \\ 5/96 \end{bmatrix}$ $\quad \frac{56-85}{48} = \frac{-29}{48}$

$\frac{5}{4} + \frac{5}{96} = \frac{125}{96}$

$x^2 = \begin{bmatrix} -29/48 \\ 125/96 \end{bmatrix}$

Zadatak 4.

a) Definirajte strojni epsilon $\quad \varepsilon = 2^{-(p-1)}$

b) $Ax = b \quad A \in \mathbb{R}^{n\times n}, \; b \in \mathbb{R}^n$ -> Gaussove eliminacije

$O(\ldots) = ? \longrightarrow O(\frac{2n^3}{3})$

c) Objasniti PLU na $Ax = b$

L-> donjetrokutasta matrica U-> gornjetrokutasta matrica

P-> matrica permutacija

$P/\, Ax = b \qquad PAx = Pb \qquad PA = LU \rightarrow LUx = Pb$ te imamo dva sustava: $Ly = Pb \quad Ux = y$

d) Uvjeti interpolacije kubičnim splajnom

- neprekidnost funkcije $s(x_i - 0) = s(x_i + 0) \quad i = 1, 2 \ldots n-1$
- interpolacija: $s(x_i) = y_i \quad i = 0, 1 \ldots n$
- neprekidnost prve derivacije $s'(x_i - 0) = s'(x_i + 0) \quad i = 1, 2 \ldots n-1$
- neprekidnost druge derivacije $s''(x_i - 0) = s''(x_i + 0) \quad i = 1, 2 \ldots n-1$

e) Uvjetovanost matrice $A \in \mathbb{R}^{n\times n}$, kada je loše uvjetovana?

-> uvjetovanost $\quad \kappa(A) = \frac{\lambda_{max}}{\lambda_{min}}$ ili $\kappa(A) = \|A\|\,\|A^{-1}\|$

-> loša uvjetovanost $\quad \kappa(A) >> 1$

f) Svojstvene vrijednosti simetrične pozitivno definitne matrice su <u>pozitivne</u> i <u>realne</u>

Zadatak (*)

a) Neka je $U \in \mathbb{C}^{4\times4}$ unitarna matrica. Izračunajte $\kappa_2(U)$.

$U \in \mathbb{C}^{4\times4} \rightarrow UU^* = U^*U = I \rightarrow$ svojstvo unitarnosti

$\kappa(I) = 1 \qquad \kappa_2(I) = 1$

b) Konvergira li N.M. za $x^2 + 2x^4 = 0$ kvadratično prema $x^* = 0$. Objasnite

–

c) –

$\boxed{\text{II } 2011.}$

Zadatak 1

a)

x	x_0	x_1	..	x_4
y	y_0	y_1	..	y_4

$\tilde{f}(x) = c_0 + c_1 x$

funkcija minimizirajuća : $E(x,y) = \sum_{i=0}^{4}(y_i - \tilde{f}(x))^2 = \sum_{i=0}^{4}(y_i - c_0 - c_1 x)^2$

b)
$$c_0(n+1) + c_1\sum_{i=0}^{4}x_i = \sum_{i=0}^{4}y_i$$
$$c_0\sum_{i=0}^{4}x_i + c_1\sum_{i=0}^{4}x_i^2 = \sum_{i=0}^{4}x_i y_i$$

U matričnom obliku : $s_e = \sum_{i=0}^{4}x_i^e \quad r_e = \sum_{i=0}^{4}x_i^e \cdot y_i \quad e = 0,1,2$

$$Sc = r \quad S = \begin{bmatrix} s_0 & s_1 \\ s_1 & s_2 \end{bmatrix} \quad c = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} \quad r = \begin{bmatrix} r_0 \\ r_1 \end{bmatrix}$$

ima jedinstveno rješenje: $\det(S) \neq 0$

$\det(S) = s_0 s_2 - s_1^2 = (n+1)\sum_{i=0}^{4}x_i^2 - (\sum_{i=0}^{4}x_i^2) = \|e\|_2^2 \cdot \|x\|_2^2 - |(x,e)|^2 \geq 0$ ako su $x_i \neq x_j$ i $y_i \neq y_j$ $\det(S) > 0$

te samo ako su x i y kolinearni

c) $T_0(\overset{x_0}{0}, \overset{y_0}{-1})$ $T_1(\overset{x_1}{1}, \overset{y_1}{0})$ $T_2(\overset{x_2}{-1}, \overset{y_2}{1})$ $T_3(\overset{x_3}{3}, \overset{y_3}{2})$

$f(x) = ax^2 + c$ $\quad \rightarrow$ pronaći polinom drugog stupnja za najbolju aproksimaciju

$n = 0,1,2,3$

$$E(x,c) = \sum_{i=0}^{3}(y_i - \tilde{f}(x_i))^2 = \sum_{i=0}^{3}(y_i - ax_i^2 - c)^2$$

$$\frac{\partial E(x,c)}{\partial a} = 2\sum_{i=0}^{3}(y_i - ax_i^2 - c)(-x_i^2) \Rightarrow -2\sum_{i=0}^{3}(y_i x_i^2 - ax_i^4 \quad cx_i^2) = 0$$

$$\frac{\partial E(x,c)}{\partial c} = 2\sum_{i=0}^{3}(y_i - ax_i^2 - c)^2 \cdot (-1) \Rightarrow -2\sum_{i=0}^{3}(y_i - ax_i^2 - c) = 0$$

$$2\sum_{i=0}^{3}y_i x_i^2 = 2a\sum_{i=0}^{3}x_i^4 + 2c\sum_{i=0}^{3}x_i^2$$

$$2\sum_{i=0}^{3}y_i = 2a\sum_{i=0}^{3}x_i^2 + 2c(4)$$

$\Rightarrow \sum_{i=0}^{3}x_i^2 = 0^2 + 1^2 + (-1)^2 + (3)^2 = 0+1+1+9 = 11$

$\sum_{i=0}^{3}y_i x_i^2 = (0^2 \cdot (-1)) + (1^2 \cdot 0) + ((-1)^2 \cdot 1) + (9 \cdot 2) = -1+1+1+18 = 19$

$\sum_{i=0}^{3}x_i^4 = 0^4 + 1^4 + (-1)^4 + (3)^4 = 0+1+1+81 = 83$

$\sum_{i=0}^{3}y_i = (-1) + (0) + (1) + (2) = -1+1+2 = 2$

$2(19) = 2a \cdot (83) + 2c(11) \Rightarrow 38 = 166a + 22c \,/\cdot 4$

$2 \cdot (2) = 2a(11) + 2c(\cdot 4) \Rightarrow 4 = 22a + 8c \,(-11)$

$152 = 664a + 88c$
$-44 = -242a - 88c$ $\,/+$ $\quad 108 = 422a \quad a = 0.2559$

$8c = 4 - 22a = 4$
$8c = -1.63 \rightarrow c = -0.20375$

$$f(x) = 0.2559\,x^2 - 0.20375$$

d) $Ax \approx b$ metoda najmanjih kvadrata $A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \\ 0 & 0 \end{bmatrix}$ i $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

R:

$Ax \approx b \to$ QR faktorizacija

$$A = Q \cdot \begin{bmatrix} R \\ 0 \end{bmatrix}$$

R = gornjetrokutasta kvadrica

$\to$ zatim $R \cdot x = d$ $\quad d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ $\quad d_1 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $\quad d_2 = [b_3]$

$\to$ zatim $\|Ax - b\|_2 = \|d_2\|_2$ je minimum rezultata

Zadatak 2.

a) Teorem o dovoljnom uvjetu konvergencije metode jednostavnih iteracija:

Neka je $f \in \mathbb{R} \to \mathbb{R}$, $f(x^*) = \underline{x^*} \to f$ ... te neka vrijedi $\underline{|f'(x)| < 1}$ ...

... zadana formulom $\underline{x_{n+1} = f(x_n)}$ konvergira prema x^*

b) Definicija konvergencije reda p?

niz $(x_n)_{n \in \mathbb{N}_0}$ konvergira linearno prema $x^* \in \mathbb{R}$, ako postoji $L \in (0,1), n_0 \in$

tako da za svaki $n \geq n_0$ vrijedi: $|x_{n+1} - x^*| \leq L |x_n - x^*|$

c) $x^2 - 6x + 8 = 0$

i) $6x = 8 + x^2 \to x = \frac{1}{6}(x^2 + 8)$ $\qquad$ ii) $x^2 = 6x - 8 \quad /:x$

$\qquad x = 6 - \frac{8}{x_n}$

i) $x_{n+1} = \frac{1}{6}(x_n^2 + 8)$ $\quad x_0 = 2.9$

ii) $x_{n+1} = 6 - \frac{8}{x_n}$ $\quad x_0 = 2.9$ $\qquad x_{n+1} = 6 - \frac{8}{x_n}$

$\to x_{1,2} = \frac{6 \pm \sqrt{36 - 32}}{2} = \frac{6 \pm 2}{2} = \begin{cases} 4 \\ 2 \end{cases}$

$\to$ uvjet jednostavnih iteracija $|f'(x^*)| < 1$

i) $f'(x) = \frac{2x_n}{6} = \frac{x_n}{3}$ $\to f'(2) = \frac{2}{3} < 1$ W

ii) $f'(x) = -(x_n^{-1}) \cdot 8 = 1 \cdot x_n^{-2} \, 8 = \frac{8}{x_n^2} \to f'(4) = \frac{8}{16} = \frac{1}{2} < 1$ W

Zadatak 3.

a) Newtonova metoda za traženje nul točaka

$\rightarrow \quad f(x) = f(x_0) - f'(x_0)(x - x_0) \quad \rightarrow \quad x_n = x_0 - \frac{f(x_0)}{f'(x_0)}$

$\rightarrow$ konvergira kvadratično: $x_{n+1} - x^* = \frac{1}{2}(x^* - x_n)^2 \frac{f''(\varepsilon_n)}{f'(x_n)}$

b) povezanost N.m. i metode sekante

$\rightarrow$ kod metode sekante koristimo aproksimaciju derivacije $f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$

c) prve 2 iteracije Newtonove metode za:

$2x^3 - y^2 = 1 \qquad x^0 = [1, 1.5] \quad \rightarrow \quad F(x,y) = 2x^3 - y^2 - 1 = 0$

$xy^3 - y = 4 \qquad\qquad F(x,y) = xy^3 - y - 4 = 0$

$D(x,y) \cdot S = \begin{bmatrix} \frac{d}{dx_1} & \frac{d}{dy_1} \\ \frac{d}{dx_2} & \frac{d}{dy_2} \end{bmatrix}$

$\frac{d}{dx_1} = 6x^2 \qquad \frac{d}{dy_1} = -2y$

$\frac{d}{dx_2} = y^3 \qquad \frac{d}{dy_2} = 3xy^2 - 1$

$D(x,y) = \begin{bmatrix} 6x^2 & -2y \\ y^3 & 3xy^2 - 1 \end{bmatrix} \Rightarrow D(1, \frac{3}{2}) = \begin{bmatrix} 6 & -3 \\ \frac{27}{8} & \frac{23}{4} \end{bmatrix}$

$F(1, \frac{3}{2}) = 2 \cdot 1 - \frac{9}{4} - 1 = \frac{8 - 9 - 4}{4} = \frac{-5}{4} \qquad 1 \cdot \frac{27}{8} - \frac{3}{2} - 4 = \frac{27 - 12 - 32}{8} = \frac{-17}{8}$

$DF(1, \frac{3}{2}) \cdot s_0 = -F(1, \frac{3}{2}) \Rightarrow \begin{bmatrix} 6 & -3 \\ \frac{27}{8} & \frac{23}{4} \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} \\ -\frac{11}{8} \end{bmatrix}$

$6s_0 - 3s_1 = \frac{5}{4} \quad / \cdot \frac{23}{12}$

$\frac{27}{8}s_0 + \frac{23}{4}s_1 = \frac{17}{8}$

$\Rightarrow \quad \frac{23}{2}s_0 - \frac{23}{4}s_1 = \frac{115}{48}$

$\frac{27}{8}s_0 + \frac{23}{4}s_1 = \frac{17}{8}$

$+ \Rightarrow \left(\frac{23}{2} + \frac{27}{8}\right)s_0 = \frac{115}{48} + \frac{17}{8}$

$\left(\frac{92 + 27}{8}\right)s_0 = \frac{115 + 102}{48}$

$s_0 = \frac{217}{48} \cdot \frac{8}{119} = \frac{217}{714} = 0{,}3039 \qquad 6s_0 - 3 \cdot s_1 = \frac{5}{4} \qquad 1.8235 - 1.25 = 3s_1$

$0{,}5735 = 3s_1$

$\Downarrow$

$s_1 = 0.1911$

$S = \begin{bmatrix} 0.3039 \\ 0.1911 \end{bmatrix} \rightarrow X_1 = X_0 + S = \begin{bmatrix} 1 + 0.3039 \\ 1.5 + 0.1911 \end{bmatrix} = \begin{bmatrix} 1.3039 \\ 1.6911 \end{bmatrix}$

2. korak

$F(1.3039, 1.6911) \rightarrow \begin{bmatrix} 0.57387 \\ 0.61487 \end{bmatrix} \qquad DF(1.3039, 1.6911) = \begin{bmatrix} 10.2 & -3.3822 \\ 4.8362 & 10.187 \end{bmatrix}$

$10.2s_0 - 3.3822s_1 = -0.5738 \quad / \cdot (-4.8362) \rightarrow \quad 16.357s_1 = 2.775$

$4.8362s_0 + 10.187s_1 = -0.61487 \quad / \cdot (10.2) \qquad 103.93s_1 = -6.27167$

$120.257s_1 = -3.496$

$s_1 = -0.029$

$10.2s_0 = -0.6721$

$s_0 = -0{,}0659$

$s' = \begin{bmatrix} -0.0659 \\ -0.029 \end{bmatrix} \rightarrow X^2 = x' + s' = \begin{bmatrix} 1.3039 - 0.0659 \\ 1.6911 - 0{,}029 \end{bmatrix} = \begin{bmatrix} 1.2380 \\ 1.6621 \end{bmatrix}$

4 Pitanja iz cijelog gradiva

a) fp brojevi u IEEE formatu dvostruke preciznosti

64-bita => 1 bit predznak ; 11 bitova eksponent ; 52 bita mantisa

b) LU faktorizacija $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$

$$\begin{array}{ccc|c} 2 & 1 & 3 & \\ \hline 4 & 4 & 5 & (-2) \\ 6 & 7 & 8 & (-3) \end{array} \quad \begin{array}{ccc} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 4 & -1 \end{array}$$

$$\begin{array}{cc|c} 2 & -1 & (-2) \\ \hline 4 & -1 & \end{array} \Rightarrow \begin{array}{cc} 2 & -1 \\ 0 & 1 \end{array}$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

c) Definirajte uvjetovanost simetrične pozitivno definitne matrice. Kada kažemo da je matrica loše uvjetovana?

→ ocjena uvjetovanosti: $\kappa(A) = \frac{\lambda_{max}}{\lambda_{min}}$ ili $\kappa(A) = \|A\| \|A^{-1}\|$

→ loša uvjetovanost → $\kappa(A) \gg 1$

d) Thomasov algoritam zahtjeva O(...) operacija

→ $O(8n)$

Zadatak 5.

a) prednost Lagrangeove baze u odnosu na standardnu?

→ Izbjegavanje loše uvjetovane Vandermondeove matrice

b)
$$S(x) = \begin{cases} x^2 + 2x + 3, & x \in [0, 1] \\ x^2 - x + 6, & x \in [1, 2] \end{cases}$$ → je li kvadratični splajn? Objasnite

$S(x)$ da bi bila kvadratični mora biti $S_1'(1) = S_2'(1)$ tj. da nema prelaska izvoda

→ $S_1'(x) = 2x + 2$ $\quad S_2'(x) = 2x - 1$
$S_1'(0) = 2$ $\quad S_2'(1) = 2 - 1 = 1$
$S_1'(1) = 4$ $\quad S_2'(2) = 3$

$S_1'(1) = 4 \neq S_2'(1) = 1$

c) $x_j = \frac{2\pi j}{N} \rightarrow$ periodički

$\rightarrow$ trig. interp. polinom $\Rightarrow$ $\varphi(x) = a_0 + \sum_{k=1}^{M} [a_k \cos(kx) + b_k \sin(kx)]$ $|a_M| + |b_M| \neq 0$

$\rightarrow$ fazni $\Rightarrow$ $p(x) = \sum_{k=0}^{N-1} c_k e^{ikx}$, $c_k \in \mathbb{C}$

$\rightarrow$ veza koeficijenata: $a_0 = c_0$ $a_k = c_k + c_{N-k}$ $b_k = i(c_k - c_{N-k})$ $k = 1..M$

Zadatak 6.

a) $A \in \mathbb{R}^{n \times k}$ $b \in \mathbb{R}^n$ $\rightarrow$ $\min_{x \in \mathbb{R}^k} \|Ax - b\|_2$

$\rightarrow$ pripadni sustav normalnih jednadžbi $\Rightarrow$ $A^T A x = A^T b$

b) ortogonalna transformacija vektora:

Qx vektor $\rightarrow$ matrica Q ortogonalna $\rightarrow$ $Q^T Q = I \rightarrow \|Qx\|_2^2 = (Qx)^T Q x = x^T Q^T Q x =$
$= x^T x = \|x\|_2^2$

c) U kojem slučaju skraćujemo korak kod globalno konvergentne Newtonove metode?

$\rightarrow$ U slučaju $|f(x_{n+1})| \geq |f(x_n)|$