Chapter 7

MAUT – MULTIATTRIBUTE UTILITY THEORY

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Abstract In this chapter, we provide a review of multiattribute utility theory. We begin with a brief review of single-attribute preference theory, and we explore preference representations that measure a decision maker's strength of preference and her preferences for risky alternatives. We emphasize the distinction between these two cases, and then explore the implications for multiattribute preference models. We describe the multiattribute decision problem, and discuss the conditions that allow a multiattribute preference function to be decomposed into additive and multiplicative forms under conditions of certainty and risk. The relationships among these distinct types of multiattribute preference functions are then explored, and issues related to their assessment and applications are surveyed.

Keywords: Multiattribute utility theory, additive value functions, preference modeling.

1. Introduction

In this chapter, we provide a review of multiattribute utility theory. As we shall discuss, multiattribute preference theory would be a more general term for this topic that covers several multiattribute models of choice. These models are based on alternate sets of axioms that have implications for their assessment and use. We begin with a brief review of single-attribute preference theory, and explore preference representations that measure a decision maker's preferences on an ordinal scale, her strength of preference and her preferences for risky alternatives. We emphasize the distinctions among these cases, and then explore their implications for multiattribute preference theory.

In order to differentiate between theories for preference based on the notions of ordinal comparisons and strength of preference versus theories for risky choices, we will use the term value function to refer to the former and utility function to refer to the latter. This distinction was made by Keeney and Raiffa in 1976[1] and has been generally adopted in the literature. Further, we will use the term preference model or multiattribute preference model to include all of these cases.

We describe the multiattribute decision problem, and discuss the conditions that allow a multiattribute preference function to be decomposed into additive and multiplicative forms under conditions of certainty and risk. The relationships between multiattribute preference functions under conditions of certainty and risk are then explored, and issues related to their assessment and applications are surveyed.

There are several important points related to the field of multi-criteria decision analysis that we wish to make. First, multiattribute preference theory provides an axiomatic foundations for choices involving multiple criteria. As a result, one can examine these axioms and determine whether or not they are reasonable guides to rational behavior. Most applications of the methods of multi-criteria decision analysis are developed for individuals who are making decisions on behalf of others, either as managers of publicly held corporations or as government officials making decisions in the best interests of the public. In such cases, one should expect these decision makers to use decision-making strategies that can be justified based on a reasonable set of axioms, rather than some ad hoc approach to decision making that will violate one or more of these axioms.

Often arguments are made that decision makers do not always make decisions that are consistent with the rational axioms of decision theory. While this may be true as a descriptive statement for individual decision making, it is much more difficult to identify situations involving significant implications for other parties where a cavalier disregard for normative theories of choice can be defended.

Second, multiattribute utility theory can be based on different sets of axioms that are appropriate for use in different contexts. Specifically, the axioms that are appropriate for risky choice do not have to be satisfied in order to use multiattribute models of preference for cases that do not explicitly involve risk. Much of the work on multiobjective mathematical programming, for example, does not require the consideration of risk, and many applications of the Analytical Hierarchy Procedure (AHP) are also developed in the context of certainty, Saaty 1980 [35].

The broad popularity of the award-winning textbook on multiattribute utility theory by Keeney and Raiffa (1993) [27] emphasized the use of multiattribute preference models based on the theories of von Neumann and Morgenstern (1947) [41], which rely on axioms involving risk. As a result, this approach has become synonymous in the view of many scholars with multiattribute preference theory. However, this theory is not the appropriate one for decisions involving multiple objectives when risk is not a consideration.[2] Instead, the multiattribute preference theories for certainty are based on ordinal comparisons of alternatives or on estimates of the strength of preference between pairs of alternatives.

Third, many existing approaches to multi-criterion decision analysis can be viewed as special cases or approximations to multiattribute preference models. We shall make this case for the popular methods of goal programming and the AHP as examples. By viewing these seeming disparate methods from this unifying framework, it is possible to gain new insights into the methodologies, recognize ways that these approaches might be sharpened or improved, and provide a basis for evaluating whether their application will result in solutions that are justified by a normative theory.

2. Preference Representations Under Certainty and Under Risk

Preference theory studies the fundamental aspects of individual choice behavior, such as how to identify and quantify an individual's preferences over a set of alternatives, and how to construct appropriate preference representation functions for decision making. An important feature of preference theory is that it is based on rigorous axioms which characterize an individual's choice behavior. These preference axioms are essential for establishing preference representation functions, and provide the rationale for the quantitative analysis of preference.

The basic categories of preference studies can be divided into characterizations of preferences under conditions of certainty or risk and over alternatives described by a single attribute or by multiple attributes. In the following, we will begin with the introduction of basic preference relations and then discuss preference representation under certainty and under risk for the single attribute case. We shall refer to a preference representation function under certainty as a *value function,* and to a preference representation function under risk as a *utility function.*

Preference theory is primarily concerned with properties of a binary preference relation $\succ$ on a choice set *X,* where *X* could be a set of commodity bundles, decision alternatives, or monetary gambles. For example, we might present an individual with a pair of alternatives, say x and y (e.g., two cars) where $x, y \in X$ (e.g., the set of all cars), and ask how they compare (e.g., do you prefer x or y?). If the individual says that x is preferred to y, then we write $x \succ y$, where $\succ$ means strict preference. If the individual states that he or she is indifferent between x and y, then we represent this preference as $x \sim y$. Alternatively, we can define ~ as the absence of strict preference; that is, not $x \succ y$ and not $y \succ x$. If it is not the case that $y \succ x$, then we write $x \succeq y$, where $\succeq$ represents a weak preference (or preference-indifference) relation. We can also define $\succeq$ as the union of strict preference $\succ$ and indifference ~; that is, both $x \succ y$ and $x \sim y$.

Preference studies begin with some basic assumptions (or axioms) of individual choice behavior. First, it seems reasonable to assume that an individual can state her preference over a pair of alternatives without contradiction; that is, the individual does not strictly prefer x to y and y to x simultaneously. This leads to the following definition for *preference asymmetry:* preference is asymmetric if there is no pair x and y in X such that $x \succ y$ and $y \succ x$.

Asymmetry can be viewed as a criterion of preference consistency. Furthermore, if an individual makes the judgment that x is preferred to y, then he or she should be able to place any other alternative z somewhere on the ordinal scale determined by the following: either better than y, or worse than x, or both. Formally, we define *negative transitivity* by saying that preferences are negatively transitive if given $x \succ y$ in *X* and any third element z in *X,* it follows that either $x \succ z$ or $z \succ y$, or both.

If the preference relation $\succ$ is asymmetric and negatively transitive, then it is called a *weak order.* The weak order assumption implies some desirable properties of a preference ordering, and is a basic assumption in many preference studies. If the preference relation $\succ$ is a weak order, then the associated indifference and weak preference relationships are well behaved. The following statements summarize some of the properties of some of these relationships.

If strict preference $\succ$ is a weak order, then

1 strict preference $\succ$ is *transitive* (if $x \succ y$ and $y \succ z$, then $x \succ z$);

2 indifference ~ is transitive, *reflexive* ($x \sim x$ for all x); and *symmetric* ($x \sim y$ implies $y \sim x$);

3 exactly one of $x \succ y, y \succ x, x \sim y$ holds for each pair x and y; and

4 weak preference $\succsim$ is transitive and *complete* (for a pair x and y, either $x \succsim y$ or $y \succsim x$).

Thus, an individual whose strict preference can be represented by a weak order can rank all alternatives considered in a unique order. Further discussions of the properties of binary preference relations are presented in Fishburn (1970) [15, Chapter 2], Kreps (1990) [31, Chapter 2], and by Bouyssou and Pirlot in Chapter 3 of this volume.

2.1 Preference Functions for Certainty (Value Functions)

If strict preference $\succ$ on X is a weak order and X is finite or denumerable, then there exists a numeric representation of preference, a real-valued function $\overset{\circ}{v}$ on X such that $x \succ y$ if and only if $\overset{\circ}{v}(x) > \overset{\circ}{v}(y)$, for all x and y in X (Fishburn, 1970 [15]). Since $\overset{\circ}{v}$ is a preference representation function under certainty, it is often called a *value function* (Keeney and Raiffa, 1993 [27]). This value function is said to be order-preserving since the numbers $\overset{\circ}{v}(x), \overset{\circ}{v}(y), \ldots$ ordered by $>$ are consistent with the order of $x, y, \ldots$ under $\succ$. Thus, any monotonic transformations of $\overset{\circ}{v}$ will also be order-preserving for this binary preference relation. Since such a function only rank orders different outcomes, there is no added meaning of the values of $\overset{\circ}{v}$ beyond the order that they imply.

Notice that we use the symbol "$\circ$" to indicate that $\overset{\circ}{v}$ is an ordinal function. While the notion of an ordinal value function is very important for economic and decision theories, such a function is seldom assessed in practice. For example, if we know that preferences are monotonically increasing for some real-valued attribute x (e.g., more is better), then $\overset{\circ}{v}(x) = x$ is valid ordinal preference function. Therefore, we may choose an objective function of maximizing profits or minimizing costs, and be comfortable assuming implicitly that these objective functions are order-preserving preference functions for a decision maker. However, the notion of an ordinal value function does become important when we speak of multiattribute value functions, as we shall discuss.

In order to replicate the preferences of a decision maker with less ambiguity, we may wish to consider a "strength of preference" notion that involves comparisons of preference differences between pairs of alternatives. To do so, we need more restrictive preference assumptions, including that of a weak order over preferences between exchanges of pairs of alternatives (Krantz et al., 1971 [30, Chapter 4]). We use the term *measurable value function* for a value function that may be used to order the differences in the strength of preference between pairs of alternatives or, more simply, the "preference differences" between the alternatives.

Once again, let X denote the set of all possible consequences in a decision situation, $w, x, y, z, w', x', y' \in X$; define X^* as a nonempty subset of $X \times X$, and $\succeq^*$ as a binary relation on X*. We shall interpret $wx \succeq^* yz$ to mean that the strength of preference for w over x is greater than or equal to the strength of preference for y over z. The notation $wx \sim^* yz$ means both $wx \succeq^* yz$ and $yz \succeq^* wx$ and $wx \succ^* yz$ means not $yz \succeq^* wx$.

There are several alternative axiom systems for measurable value functions, including the topological results of Debreu (1960) [7] and the algebraic development by Scott and Suppes (1958) [39]. Some of these systems allow both "positive" and "negative" preference differences and are called algebraic difference structures. For example, the "degree of preference" for x over w would be "negative" if w is preferred to x. Our development is based on an axiom system presented by Krantz et al. (1971) [30, Definition 4.1] that does not allow negative differences; hence it is called a positive difference structure.

This set of axioms includes several technical assumptions that have no significant implications for behavior. However, a key axiom that does have an intuitive interpretation in terms of preferences is the following one: If $wx, xy, w'x', x'y' \in X^*$, $wx \succeq^* w'x'$, and $xy \succeq^* x'y'$, then $wy \succeq^* w'y'$. That is, if the difference in the strength of preference between w and x exceeds the difference between w' and x', and the difference in the strength of preference between x and y exceeds the difference between x' and y', then the difference in the strength of preference between w and y must exceed the difference between w' and y'. Some introspection should convince most readers that this would typically be true for preference comparisons of alternative pairs.

The axioms of Krantz et al. (1971) [30] imply that there exists a real-valued function v on X such that, for all $w, x, y, z \in X$, if w is preferred to x and y to z, then $wx \succeq^* yz$ if and only if

$$v(w) - v(x) \geq v(y) - v(z). \tag{7.1}$$

Further, v is unique up to a positive linear transformation, so it is a *cardinal function* (i.e., v provides an *interval scale of measurement*). That is, if v' also satisfies (7.1), then there are real numbers $a > 0$ and b such that $v'(x) = av(x) + b$ for all $x \in X$ (Krantz et al. [30, Theorem 4.1]).

We define the binary preference relation $\succeq$ on X from the binary relation $\succeq^*$ on X^* in the natural way by requiring $wx \succeq^* yx$ if and only if $w \succeq y$ for all $w, x, y \in X$. Then from (7.1) it is clear that $w \succeq y$ if and only if $v(w) \geq v(y)$. Thus, v is a value function on X and, by virtue of (7.1), it is a *measurable value function.*

The ideas of strength of preference and of measurable value functions are important concepts that are often used implicitly in the implementation of preference theories in practice. Intuitively, it may be useful to think of a measurable value function as the unique preference function in the case of certainty that reveals the marginal value of additional units of the underlying commodity. For example, we would expect that the measurable value function over wealth for most individuals would be concave, since the first million dollars would be "worth" more to the individual than the second million dollars, and so on. This notion would be consistent with the traditional assumption in economics of diminishing marginal returns to scale.

Further, the measurable value function can be assessed using questions for subjects that do not require choices among lotteries, which may be artificial distractions in cases where subjects are trying to choose among alternatives that do not require the consideration of risk. Examples of methods for assessing measurable value functions would include direct rating of alternatives on a cardinal scale, or direct comparisons of preference differences. For a detailed discussion of these approaches, see Farquhar, and Keller (1989) [13], von Winterfeldt and Edwards (1986) [42], and Kirkwood (1997)[29].

In addition, subjects can be asked to make ratio comparisons of preference differences. For example, they might be comparing automobiles relative to a "base case", say a Ford Taurus. Then, they could be asked to compare the improvement in acceleration offered by a BMW over a Taurus to the improvement offered by a Mercedes (relative to the same Taurus) in terms of a ratio. This ratio judgment could be captured and analyzed using the tools of the AHP, and this provides a link between measurable value functions and ratio judgments. This point has been made on numerous occasions, and is worth further exploration (e.g., see Kamenetzky, 1982 [26]; Dyer 1990 [8]; Salo and Hämäläinen 1997 [36]).

2.2 Preference Functions for Risky Choice (Utility Functions)

We turn to preference representation for risky options, where the risky options are defined as lotteries or gambles with outcomes that depend on the occurrence from a set of mutually exclusive and exhaustive events. For example, a lottery could be defined as the flip of a fair coin, with an outcome of $10 if heads occurs and an outcome of -$2 if tails occurs.

Perhaps the most significant contribution to this area of concern was the formalization of *expected utility theory* by von Neumann and Morgenstern (1947) [41]. This development has been refined by a number of researchers and is most commonly presented in terms of three basic axioms (Fishburn, 1970 [15]).

Let P be a convex set of simple probability distributions or lotteries $\{p, q, r, \ldots\}$ on a nonempty set X of outcomes. We shall use p, q and r to refer to probability distributions and random variables interchangeably. For lotteries p, q, r in P and all λ, $0 < \lambda < 1$, the expected utility axioms are:

1 *(Ordering)* $\succ$ is a weak order;

2 *(Independence)* If $p \succ q$ then $(\lambda p + (1-\lambda)r) \succ (\lambda q + (1-\lambda)r)$ for all r in P;

3 *(Continuity)* If $p \succ q \succ r$ then there exist some $0 < \alpha < 1$ and $0 < \beta < 1$ such that $\alpha p + (1-\alpha)r \succ q \succ \beta p + (1-\beta)r$

The von Neumann-Morgenstern expected utility theory asserts that the above axioms hold if and only if there exists a real-valued function u such that for all p, q in P, $p \succ q$ if and only if

$$\sum_{x \in X} p(x)u(x) \geq \sum_{x \in X} q(x)u(x) \tag{7.2}$$

Moreover, such a u is unique up to a positive linear transformation.

The expected utility model can also be used to characterize an individual's risk attitude (Pratt, 1964 [34]; Keeney and Raiffa, 1993 [27, Chapter 4]). If an individual's utility function is concave, linear, or convex, then the individual is risk averse, risk neutral, or risk seeking, respectively.

The von Neumann-Morgenstern theory of risky choice presumes that the probabilities of the outcomes of lotteries are provided to the decision maker. Savage (1954) [38] extended the theory of risky choice to allow for the simultaneous determination of subjective probabilities for outcomes and for a utility function u defined over those outcomes. Deduced probabilities in Savage's model are personal or subjective probabilities. The model itself is a subjective expected utility representation.

The assessment of von Neumann-Morgenstern utility functions will almost always involve the introduction of risk in the form of simple lotteries. For a discussion of these assessment approaches, see Keeney and Raiffa (1993) [27, Chapter 4] or von Winterfeldt and Edwards (1986) [42].

As a normative theory, the expected utility model has played a major role in the prescriptive analysis of decision problems. However, for descriptive purposes, the assumptions of this theory have been challenged by empirical studies (Kahneman and Tversky, 1979 [25]). Some of these empirical studies demonstrate that subjects may choose alternatives that imply a violation of the independence axiom. One implication of the independence axiom is that the expected utility model is "linear in probabilities". For a discussion, see Fishburn and Wakker (1995) [18]. A number of contributions have been made by relaxing the independence axiom and developing some *nonlinear utility models* to accommodate actual decision behavior (Fishburn, 1988 [17] and Camerer, 1995 [3]).

2.3 Comment

Note that both the measurable value function $v(x)$ and the von Neumann and Morgenstern utility function $u(x)$ are cardinal measures, unique up to a positive linear transformation. However, the theory supporting the measurable function is based on axioms involving preferences differences, and it is assessed based on questions that rely on the idea of strength of preference. In contrast, the von Neumann and Morgenstern utility function is based on axioms involving lotteries, and it is assessed based on questions that typically involve lottery comparisons.

Suppose we find a subject and assess her measurable value function $v(x)$ and her utility function $u(x)$ over the same attribute (e.g., over monetary outcomes). Would these two functions be identical, except for measurement error? A quick reaction might be that they would be identical, since they are each unique representations of the subject's preferences, up to a positive linear transformation. However, that is not necessarily the case. Intuitively, a measurable value function $v(x)$ may be concave, indicating decreasing marginal value for the underlying attribute. However, a utility function $u(x)$ may be even more concave, since it will incorporate not only feelings regarding the marginal value of the attribute, but also it may incorporate psychological reactions to taking risks. Empirical tests of this observation are provided by Krzysztofowicz (1983) [32] and Keller (1985) [28] and generally support this intuition. This is an important point, and one that we will emphasize again in the context of multiattribute preference functions (see Ellsberg, 1954 [11]; Dyer and Sarin, 1982 [10]; Sarin, 1982 [37], and Jia and Dyer, 1996 [24]).

3. Ordinal Multiattribute Preference Functions for the Case of Certainty

A decision maker uses the appropriate preference function, $\mathring{v}(x)$ or $v(x)$ in the case of certainty or $u(x)$ in the case of risk, to choose among available alternatives. The major emphasis of the work on multiattribute utility theory has been on questions involving conditions for the decomposition of a preference function into simple polynomials, on methods for the assessment of these decomposed functions, and on methods for obtaining sufficient information regarding the multiattribute preference functions so that the evaluation can proceed without its explicit identification with full precision.

Suppose that the alternatives defined for single attribute preference functions are now considered to be vectors. That is, suppose that $X = \prod_{i=1}^{n} X_i$ where X_i represents the performance of an alternative on attribute i. We will be interested in conditions allowing the determination that $(x_1, \ldots, x_n) \succeq (y_1, \ldots, y_n)$ if and only if $\mathring{v}(x_1, \ldots, x_n) \geq \mathring{v}(y_1, \ldots, y_n)$ for example. Essentially, all that is required is the assumption that the decision maker's preferences are a weak order on the vectors of attribute values.

In some cases, methods for multiattribute optimization do not need any additional information regarding a multiattribute preference function, other than perhaps invoking concavity to allow maximization. Geoffrion, Dyer, and Feinberg (1972) [20] provide an example of an early approach to multiattribute optimization that does proceed with only local information regarding the implicit multiattribute preference function. Additional conditions are needed to decompose the multiattribute preference function into simple parts.

3.1 Preference Independence

The most common approach for evaluating multiattribute alternatives is to use an additive representation. For simplicity, we will assume that there exist a most preferred outcome x_i^* and a least preferred outcome x_i° on each attribute $i = 1$ to n. In the additive representation, a real value $\overset{\circ}{v}$ is assigned to each outcome $(x_1, \ldots, x_n)$ by

$$\overset{\circ}{v}(x_1, \ldots, x_n) = \sum_{i=1}^{n} \overset{\circ}{v}_i\,(x_i) \tag{7.3}$$

where the $\overset{\circ}{v}_i$ are single attribute value functions over X_i[3]. When it is convenient, we may choose the scaling $\overset{\circ}{v}_i(x_i^*) = 1, \overset{\circ}{v}_i(x_i^\circ) = 0$, and write $\overset{\circ}{v}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \overset{\circ}{\lambda}_i \overset{\circ}{v}_i(x_i)$ where $\sum_{i=1}^{n} \overset{\circ}{\lambda}_i = 1$.

If our interest is in simply rank-ordering the available alternatives then the key condition for the additive form in (7.3) is *mutual preference independence.* Suppose that we let $I \subset \{1, 2, \ldots, n\}$ be a subset of the attribute indices, and define X_I as the subset of the attributes designated by the subscripts in *I*. Also, we let $\bar{X}_I$ represent the complementary subset of the n attributes. Then,

1 X_I is *preference independent* of $\bar{X}_I$ if $(w_I, \bar{w}_I) \succeq (x_I, \bar{w}_I)$ for any $w_I, x_I \in X_I$ and $\bar{w}_I \in \bar{X}_I$ implies $(w_I, \bar{x}_I) \succeq (x_I, \bar{x}_I)$ for all $\bar{x}_I \in \bar{X}_I$.

2 The attributes $X_1, \ldots, X_n$ are *mutually preference independent* if for every subset $I \subseteq \{1, \ldots, n\}$ the set X_I of these attributes is preference independent of $\bar{X}_I$.

When coupled with a solvability condition and some technical assumptions, mutual preference independence implies the existence of an *additive ordinal multiattribute value function* for $n \geq 3$ attributes. Furthermore, this additive ordinal value function is unique up to a positive linear transformation.

Attributes X_i and X_j are preference independent if the tradeoffs (substitution rates) between X_i and X_j are independent of all other attributes. Mutual preference independence requires that preference independence holds for all pairs X_i and X_j. Essentially, mutual preference independence implies that the indifference curves for any pair of attributes are unaffected by the fixed levels of the remaining attributes. Debreu (1959) [7], Luce and Tukey (1964) [33], and Gorman (1968) [21] provide axiom systems and analysis for the additive form (7.3).

An example may help to illustrate the idea of preference independence. Suppose that a subject is attempting to evaluate automobiles based on the three criteria of cost, horsepower, and appearance. Assume that the subject decides that her preferences between two automobiles differing in cost and horsepower but with identical values for appearance are as follows: ($24,000, 150 hp, ugly) $\succ$($25,000, 170 hp, ugly). If the level of appearance does not affect the subject's indifference curve between cost and horsepower, then she will also prefer ($24,000, 150 hp, beautiful) to ($25,000, 170 hp, beautiful), and will maintain the same preference relation for any common value of appearance.

As a practical matter, it is only necessary for preference independence to hold for the $n - 1$ pairs of criteria involving the first criterion and the other $n - 1$ criteria taken one at a time. See Keeney and Raiffa (1993) [27, Chapter 3] for a discussion.

In Chapter 3 of this volume, Bouyssou and Pirlot provide an excellent discussion of the additive ordinal value function which they present as the use of *conjoint measurement* for multiple criteria decision making. In our development, we use the terminology *ordinal additive value function* instead in order to contrast this preference representation with other additive and non-additive preference models. We also use the term *preference independence* rather than simply *independence* to distinguish this key assumption from other forms of independence conditions that are appropriate for multiple criteria decision making in different contexts.

3.2 Assessment Methodologies

The additive ordinal value function would seem to be an attractive choice for practical applications of multiattribute decision making. However, the resulting additive function is, in general, difficult to assess. The problem arises because the single attribute functions $\mathring{v}_i$ cannot be assessed using the methods appropriate for the single-attribute *measurable* value functions. Instead, these functions can only be assessed through protocols that require tradeoffs between two attributes throughout the process, and these protocols are therefore burdensome for the decision makers. Further, the resulting additive function will only have an ordinal interpretation, rather than providing a measure of the strength of preference.

Keeney and Raiffa (1993) [27, Chapter 3] illustrate two assessment procedures for ordinal additive value functions. However, an example may be helpful to emphasize that the resulting additive value function may only provide an ordinal ranking of alternatives, since this important point is also a subtle one.

Suppose that an analyst is attempting to assess a preference function from a decision maker on three attributes *X, Y,* and *Z* that are related in the mind of the decision maker in a multiplicative form; that is, the decision maker's true preferences are represented by the product xyz where x, y, and z are attribute values. Of course, the analyst is not aware of this multiplicative form, and is attempting to develop an appropriate preference representation from the decision maker based on a verbal assessment procedure. Further, suppose that there is no risk involved, so the analyst would like to consider the use of an additive ordinal multiattribute value function.

An example of a situation that might involve this type of a preference function would be the ranking of oil exploration opportunities based on estimates of their oil reserves. Suppose that the decision maker thinks that these reserves can be estimated by multiplying the area (x) of the structure containing oil by its depth (y) to obtain the volume of the structure, and then multiplying this volume by its rate of recovery per volumetric unit (z). In practice, this is a simplification of the approach actually used in many cases to estimate oil reserves.

This multiplication of the relevant parameters could be done explicitly in this case, but this example should suggest that such a true preference structure could occur naturally. For simplicity, and to avoid complications associated with units of measurement, we will assume that $X = Y = Z = [1, 10]$, which might occur if the analyst rescaled the actual units of measurement.

The analyst does not know the true underlying preference model of the decision maker, and so he might ask a series of questions to determine if mutual preference independence holds in this case. Consider alternative 1, with $x_1 = 2$, $y_1 = 3$, and $z_1 = 4$, versus alternative 2, with $x_2 = 4$, $y_2 = 2$, and $z_2 = 4$. The decision maker would be asked to compare (2,3,4) with (4,2,4), and would reply that she prefers alternative 2 (because $2 \times 3 \times 4 = 24$ and $4 \times 2 \times 4 = 32$, although these calculations are unknown to the analyst). It is easy to see that alternative 2 would remain preferred to alternative 1 for all common values of z_1 and z_2, so attributes X and Y are preference independent of Z. Likewise, a similar set of questions would reveal that X and Z are preference independent of Y, and Y and Z are preference independent of X, so these three attributes are mutually preference independent.

Therefore, the analyst concludes that the preferences of the decision maker can be represented by the ordinal additive multiattribute preference function

$$\mathring{v}(x, y, z) = \mathring{v}_x(x) + \mathring{v}_y(y) + \mathring{v}_z(z)$$

As we shall see, this is not a mistake even though the true preference function is multiplicative, and the assessment procedure will construct the correct ordinal additive function that will result in the same rank ordering of alternatives as the multiplicative preference function.

For this example only, we will abuse the notation and let subscripts of the attributes indicate the corresponding values of the single attribute functions. For examples, we will let x_0 indicate the value of attribute X such that $\mathring{v}_x(x_0) = 0$, and let y_1 indicate the value of attribute Y such that $\mathring{v}_y(y_1) = 1$, and so forth.

Suppose the analyst begins the assessment procedure by letting $x_0 = y_0 = z_0 = 1$, which is allowable given the fact that the function is unique up to a linear transformation. That is, the analyst scales $\overset{\circ}{v}_x$ so that $\overset{\circ}{v}_x(x_0) = \overset{\circ}{v}_x(1) = 0$, and similarly scales $\overset{\circ}{v}_y(1) = \overset{\circ}{v}_z(1) = 0$. Therefore, we would have

$$\overset{\circ}{v}(1,1,1) = \overset{\circ}{v}_x(1) + \overset{\circ}{v}_y(1) + \overset{\circ}{v}_z(1) = 0 + 0 + 0 = 0.$$

The analyst then arbitrarily selects $x_1 = 2$; that is, he sets $\overset{\circ}{v}_x(x_1) = \overset{\circ}{v}_x(2) = 1$, which is also allowable by virtue of the scaling convention. Finally, the analyst involves the decision maker, and asks her to specify a value y_1 so that she is indifferent between the alternative (2,1,1) and the alternative $(1,y_1,1)$. Based on her true multiplicative preference model unknown to the analyst, if she is indifferent between (2,1,1) and $(1,y_1,1)$ it must be the case that $2 \times 1 \times 1 = 1 \times y_1 \times 1$, so she responds $y_1 = 2$. Based on this response, the analyst sets $\overset{\circ}{v}_y(y_1) = \overset{\circ}{v}_y(2) = 1$.

This means that $\overset{\circ}{v}(2,1,1) = \overset{\circ}{v}_x(2) + \overset{\circ}{v}_y(1) + \overset{\circ}{v}_z(1) = 1 + 0 + 0 = 1$, and that $\overset{\circ}{v}(1,2,1) = \overset{\circ}{v}_x(1) + \overset{\circ}{v}_y(1) + \overset{\circ}{v}_z(1) = 0 + 1 + 0 = 1$, which verifies to the analyst that the additive representation indicates that the decision maker is indifferent between the alternatives (2,1,1) and (1,2,1). In addition, the analyst knows that $\overset{\circ}{v}(2,2,1) = \overset{\circ}{v}_x(2) + \overset{\circ}{v}_y(2) + \overset{\circ}{v}_z(1) = 1 + 1 + 0 = 2$.

Now, the analyst asks the decision maker to specify a value for x_2 so that she is indifferent between the alternatives (2,2,1) and $(x_2,1,1)$. This response will determine the value of x_2 such that $\overset{\circ}{v}_x(x_2) = 2$, because indifference between these two alternatives will require $\overset{\circ}{v}(x_2,1,1) = \overset{\circ}{v}_x(x_2) + \overset{\circ}{v}_y(1) + \overset{\circ}{v}_z(1) = 2 + 0 + 0 = 2$ also.

Using her implicit multiplicative preference function for the alternative (2, 2, 1), she obtains $2 \times 2 \times 1 = 4$, and since indifference requires $x_2 \times 1 \times 1 = 4$, she would identify $x_2 = 4$, so $\overset{\circ}{v}_x(4) = 2$. The reader should confirm that similar questions would determine $\overset{\circ}{v}_y(4) = \overset{\circ}{v}_x(4) = 2$, and that $\overset{\circ}{v}_x(8) = 3$, and so forth. Continuing in this fashion, and using similar questions to develop the assessments of $\overset{\circ}{v}_y(y)$ and $\overset{\circ}{v}_z(z)$, the analyst would develop graphs that would indicate $\overset{\circ}{v}_x(x) = \ln x/\ln 2$, $\overset{\circ}{v}_y(y) = \ln y/\ln 2$, and $\overset{\circ}{v}_z(z) = \ln z/\ln 2$, so that the ordinal additive multiattribute value function would be given by the sum of the logs of the variables. Notice that this ordinal value function is an order preserving transformation of the true underlying preference representation of the decision maker, which was never revealed explicitly to the analyst.

As this example illustrates, the assessment procedure will determine an additive ordinal value function that may be an order preserving transformation of a true preference relation that is not additive. The log function provides an example of such a transformation for a multiplicative preference relation, but other non-additive relationships may also be transformed to order preserving additive value functions. See Krantz et al. (1971) [30] for a discussion of other such transformations.

This example also illustrates the fact that the assessment methods required for accurately capturing an additive ordinal multiattribute value function may be tedious, and will require tradeoffs involving two or more attributes. This same point is made by Bouyssou and Pirlot in Chapter 3 of this same volume. Thus, while this approach could be used in practice, it would be desirable to have simpler means of assessing the underlying preference functions. This can be accomplished if some additional preference conditions are satisfied, but the requirement of mutual preference independence will still be common to the preference models that are to follow.

4. Cardinal Multiattribute Preference Functions for the Case of Risk

When $X = \prod_{i=1}^{n} X_i$ in a von Neumann-Morgenstern utility model and the decision maker's preferences are consistent with some additional independence conditions, then $u(x_1, x_2, \ldots, x_n)$ can be decomposed into additive, multiplicative, and other well-structured forms that simplify assessment. In comparison with other sections, our coverage of this topic will be relatively brief since it is perhaps the most well known multiattribute preference model.

4.1 Utility Independence

An attribute X_i is said to be *utility independent* of its complementary attributes if preferences over lotteries with different levels of X_i do not depend on the fixed levels of the remaining attributes. Attributes $X_1, X_2, \ldots, X_n$ are mutually utility independent if all proper subsets of these attributes are utility independent of their complementary subsets. Further, it can be shown that if these same attributes are mutually preference independent, then they will also be mutually utility independent if any pair of the attributes is utility independent of its complementary attributes (Keeney and Raiffa, 1993 [27]).

Returning to the automobile selection example, suppose that a decision maker is considering using the attributes of cost, horsepower, and appearance as before, but there is some uncertainty regarding some new environmental laws that may impact the cost and the horsepower of a particular automobile. Further, assume that the decision maker prefers more horsepower to less, lower costs and more attractive automobiles. The current performance levels of one of the alternatives may be ($25,000, 170 hp, ugly), but if the legislation is passed a new device will have to be fitted that will increase cost and decrease horsepower to ($25,700,150 hp, ugly). An alternative automobile might have possible outcomes of ($28,000, 200 hp, ugly) and ($29,000, 175 hp, ugly) depending on this same legislation, which the decision maker estimates will pass with probability 0.5.

Therefore, the decision maker may consider choices between lotteries such as the one shown in Figure 7.1. For example, the decision maker may prefer Auto 1 to Auto 2 because the risks associated with the cost and the horsepower for Auto 1 are more acceptable to her than the risks associated with the cost and horsepower of Auto 2. If the decision maker's choices for these lotteries do not depend on common values of the third attribute, then cost and horsepower are utility independent of appearance.

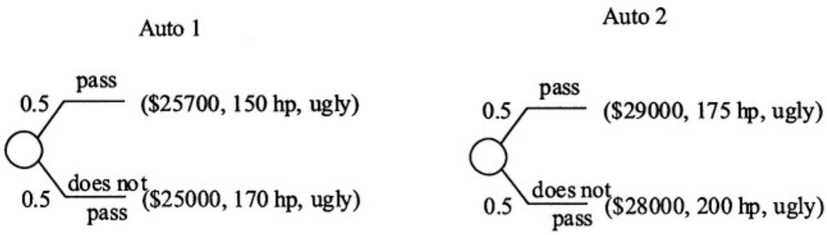

Figure 7.1. Choice between two lotteries.

A multiattribute utility function $u(x_1, x_2, \ldots, x_n)$ can have the multiplicative form

$$1 + ku(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} [1 + kk_i u_i(x_i)] \tag{7.4}$$

if and only if the attributes $X_1, X_2, \ldots, X_n$ are mutually utility independent, where the u_i are single-attribute functions over X_i scaled from 0 to 1, $0 \leq k_i \leq 1$ are positive scaling constants, and k is an additional scaling constant. If the scaling constant k is determined to be 0 through the appropriate assessment procedure, then (7.4) reduces to the additive form

$$\sum_{i=1}^{n} k_i u_i(x_i) \tag{7.5}$$

where $\sum_{i=1}^{n} k_i = 1$.

4.2 Additive Independence

A majority of the applied work in multiattribute utility theory deals with the case when the von Neumann-Morgenstern utility function is decomposed into the additive form (7.5). Fishburn (1965) [14] has derived necessary and sufficient conditions for a utility function to be additive. The key condition for additivity is the marginality condition which states that the preferences for any lotteries

$p, q \in P$ should depend only on the marginal probabilities of the attribute values, and not on their joint probability distributions.

Returning to the automobile example once again, for additivity to hold, the decision maker must be indifferent between the two lotteries shown in Figure 7.2, and for all other permutations of the attribute values that maintain the same marginal probabilities for each.

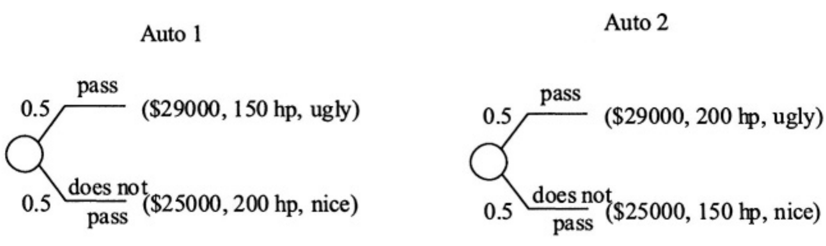

Figure 7.2. Additive Independence Criterion for Risk.

Notice that in either lottery, the marginal probability of receiving the most preferred outcome or the least preferred outcome on each attribute is identical (0.5). However, a decision maker may prefer the right-hand side lottery over the left-hand side lottery if the decision maker wishes to avoid a 0.5 chance of the poor outcome ($29,000, 150 hp, ugly) on all three attributes, or she may have the reverse preference if she is willing to accept some risk in order to have a chance at the best outcome on all three attributes. In either of the latter cases, utility independence may still be satisfied, and a multiplicative decomposition of the multiattribute utility function (7.4) may be appropriate.

Other independence conditions have been identified that lead to more complex non-additive decompositions of a multiattribute utility function. These general conditions are reviewed in Farquhar (1977) [12].

4.3 Assessment Methodologies

The assessment of the multiplicative or additive form implied by the condition of mutual utility independence is simplified by the fact that each of the single-attribute utility functions may be assessed independently (more accurately, while all of the other attributes are held constant at arbitrarily selected values), using the well-known utility function assessment techniques suitable for single attribute utility functions. In addition, the constants k_i and k can be assessed using n relatively simple tradeoff questions. See Keeney and Raiffa (1993) [27] or Kirkwood (1997) [29]for additional details and examples.

5. Measurable Multiattribute Preference Functions for the Case of Certainty

We have delayed the discussion of measurable multiattribute preference functions until after the review of multiattribute utility theory because the latter may be more familiar to the reader. If so, this transposition of a more natural order of presentation may be helpful in providing the opportunity to discuss similarities between these models of preference, and therefore to enhance an intuitive understanding of the relationships among some important concepts.

Again let X denote the set of all possible consequences in a particular decision problem. In the multiattribute problem $X = \prod_{i=1}^{n} X_i$ where X_i is the set of possible consequences for the ith attribute. In this section, we use the letters $w, x, y,$ and z to indicate distinct elements of X. For example, $w \in X$ is represented by $(w_1, \ldots, w_n)$, where w_i is a level in the nonempty attribute set X_i for $i = 1, \ldots, n$. Once again, we let $I \subset \{1, 2, \ldots, n\}$ be a subset of the attribute indices, define X_I as the subset of the attributes designated by the subscripts in I, and let $\bar{X}_I$ represent the complementary subset of the n attributes. We may write $w = (w_I, \bar{w}_I)$ or use the notation $(w_i, \bar{w}_i)$ and $(x_i, \bar{w}_i)$ to denote two elements of X that differ only in the level of the ith attribute. Finally, we also assume that the preference relation $\succsim$ on X is a weak order.

Next we introduce the notation necessary to define preferences based on strength of preference between vector-valued outcomes. We let $X^* = \{wx : w, x \in X\}$ be a nonempty subset of $X \times X$, and $\succsim^*$ denote a weak order on X^*. Once again, we may interpret $wx \succsim^* yz$ to mean that the preference difference between w and x is greater than the preference difference between y and z.

It seems reasonable to assume a relationship between $\succsim$ on X and $\succsim^*$ on X^* as follows. Suppose the attributes $X_1, \ldots, X_n$ are mutually preference independent. These two orders are *difference consistent* if, for all $w_i, x_i \in X_i$, $(w_i, \bar{w}_i) \succsim (x_i, \bar{w}_i)$ if and only if $(w_i, \bar{w}_i)(x_i^\circ, \bar{w}_i) \succsim^* (x_i, \bar{w}_i)(x_i^\circ, \bar{w}_i)$ for some $x_i^\circ \in X_i$ and some $\bar{w}_i \in \bar{X}_i$, and for any $i \in \{1, \ldots, n\}$, and if $w \sim x$ then $wy \sim^* xy$ or $yw \sim^* yx$ or both for any $y \in X$. Loosely speaking, this means that if one multiattributed alternative is preferred to another differing only on the value of attribute X_i, then the preference difference between that alternative and some common reference alternative $(x_i^\circ, \bar{w}_i)$ will be larger than the difference between the alternative that is not preferred and this reference alternative.

5.1 Weak Difference Independence

In this section we identify a condition that we refer to as *weak difference independence*. This condition plays a role similar to the utility independence condition in multiattribute utility theory. We show how this condition can be exploited to obtain multiplicative and other nonadditive forms of the measurable multiattribute value function.

Specifically, the subset of attributes X_I is weak difference independent of $\bar{X}_I$ if, given any $w_I, x_I, y_I, z_I \in X_I$ and some $\bar{w}_I \in \bar{X}_I$ such that the subject's judgments regarding strength of preferences between pairs of multiattributed alternatives is as follows: $(w_I, \bar{w}_I)(x_I, \bar{w}_I) \succeq^* (y_I, \bar{w}_I)(z_I, \bar{w}_I)$ then the decision maker will also consider $(w_I, \bar{x}_I)(x_I, \bar{x}_I) \succeq^* (y_I, \bar{x}_I)(z_I, \bar{x}_I)$ for any $\bar{x}_I \in \bar{X}_I$. That is, the ordering of preference differences depends only on the values of the attributes X_I and not on the fixed values of $\bar{X}_I$.

The attributes are mutually weak difference independent if all proper subsets of these attributes are weak difference independent of their complementary subsets. Further, it can be shown that if these same attributes are mutually preference independent, then they will also be mutually weak difference independent if any pair of the attributes is weak difference independent of its complementary attributes (Dyer and Sarin, 1979 [9]).

Notice the similarity of the definition of weak difference independence to that of utility independence. In the latter case, preferences among lotteries depend only on the values of the attributes X_I and not on the fixed values of $\bar{X}_I$. In the case of certainty, the same notion applies to preference differences. Therefore, it should not be surprising that this condition leads to a decomposition of a measurable value function that is identical to the one implied by utility independence for utility functions.

This intuition may be formalized as follows. A measurable multiattribute value function $v(x_1, x_2, \ldots, x_n)$ on X can have the multiplicative form

$$1 + \lambda v(x) = \prod_{i=1}^{n} [1 + \lambda \lambda_i v_i(x_i)] \qquad (7.6)$$

if and only if $X_1, \ldots, X_n$ are mutually weak difference independent, where v_i is a single-attribute measurable value function over X_i scaled from 0 to 1, the λ_i are positive scaling constants, and λ is an additional scaling constant. If the scaling constant λ is determined to be 0 through the appropriate assessment procedure, then (7.4) reduces to the additive form

$$v(x) = \sum_{i=1}^{n} \lambda_i v_i(x_i) \qquad (7.7)$$

where $\sum_{i=1}^{n} \lambda_i = 1$. Therefore, we obtain either an additive or a multiplicative measurable preference function that is based on notions of strength of preference.

5.2 Difference Independence

Finally, we are interested in the conditions that are required to ensure the existence of an additive multiattribute measurable value function. Recall that mutual preference independence guarantees the existence of an additive preference function for the case of certainty that will provide an ordinal ranking of alternatives, but it may not capture the underlying strength of preference of the decision maker. Further, the appropriate assessment technique will require tradeoffs that simultaneously consider two or more attributes as illustrated in Section 3.2.

Recall the example from Section 3.2 where the decision maker's true preferences were represented by the product of the attributes. If we were to ask the decision maker to express her preferences for the first attribute while holding the other attributes constant at some given values, she would respond in such a way that we would obtain a linear function for each attribute, rather than the correct logarithmic form. We would like to exclude this case, and be assured that the preference function that also measures strength of preference is additive.

Perhaps this point is worth some elaboration. Recall that the true preferences of the hypothetical decision maker introduced in Section 3.2 were consistent with the multiplicative representation xyz. Suppose we set $y = z = 1$, and ask the decision maker to consider the importance of changes in the attribute x while holding these other attribute values constant. Considering the alternatives (1,1,1), (3,1,1), and (5,1,1), she would indicate that the preference difference between (3,1,1) and (1,1,1) would be the same as the preference difference between (5,1,1) and (3,1,1). This is because her true preference relation gives $1 \times 1 \times 1 = 1$, $3 \times 1 \times 1 = 3$, $5 \times 1 \times 1 = 5$, and the preference difference between (3,1,1) and (1,1,1) is $3 - 1 = 2$, which is also the preference difference between (5,1,1) and (3,1,1). If the analyst is not aware of the fact that this assessment approach cannot be used when only preference independence is satisfied, he might mistakenly conclude that $\mathring{v}(x, y, z) = x + y + z$ rather than the appropriate logarithmic transformation that we obtained earlier in Section 3.2.

The required condition for additivity that also provides a measurable preference function is called *difference independence* . The attribute X_i is difference independent of $\bar{X}_i$ if, for all $w_i, x_i \in X_i$ such that $(w_i, \bar{w}_i) \succeq (x_i, \bar{w}_i)$ for some $\bar{w}_i \in \bar{X}_i$, $(w_i, \bar{w}_i)(x_i, \bar{w}_i) \sim^* (w_i, \bar{x}_i)(x_i, \bar{x}_i)$ for any $\bar{x}_i \in \bar{X}_i$. Intuitively, the preference difference between two multiattributed alternatives differing only on one attribute does not depend on the common values of the other attributes.

The attributes are mutually difference independent if all proper subsets of these attributes are difference independent of their complementary subsets. Again, it can be shown that if these same attributes are mutually preference independent, then they will also be mutually difference independent if X_1 is difference independent of $\bar{X}_1$ (Dyer and Sarin, 1979[9]). For the case of $n \geq 3$, mutual difference independence along with some additional structural and technical conditions[4] ensure that if $wx, yz \in X^*$, then $wx \succsim^* yz$ if and only if

$$\sum_{i=1}^{n} \lambda_i v_i(w_i) - \sum_{i=1}^{n} \lambda_i v_i(x_i) \geq \sum_{i=1}^{n} \lambda_i v_i(y_i) - \sum_{i=1}^{n} \lambda_i v_i(z_i) \tag{7.8}$$

and $x \succsim y$ if and only if

$$\sum_{i=1}^{n} \lambda_i v_i(x_i) \geq \sum_{i=1}^{n} \lambda_i v_i(y_i), \tag{7.9}$$

where v_i is a single-attribute measurable value function over X_i scaled from 0 to 1, and $\sum_{i=1}^{n} \lambda_i = 1$. Further, if $v_i', i = 1, \ldots, n$ are n other functions with the same properties, then there exist constants $\alpha > 0, \beta_1, \ldots, \beta_n$ such that $v_i' = \alpha v_i + \beta_i, i = 1, \ldots, n$.

Result (7.9) is well known and follows immediately from the assumption that the attributes are mutually preference independent (Section 3.1). The significant result is (7.8), which means that $v(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \lambda_i v_i(x_i)$ also provides difference measurement on X. Note that this latter result is obtained based on the observation that any arbitrarily selected attribute is difference independent of its complementary attributes.

5.3 Assessment Methodologies

Because the notion of a measurable multiattribute value function may not be familiar to many readers, we will briefly consider methods for the assessment of them. Further details and examples are provided by von Winterfeldt and Edwards (1986) [42], and by Kirkwood (1997) [29].

5.3.1 Verification of the Independence Conditions. The first issue to be considered is the verification of the independence conditions. Since methods for verifying mutual preference independence are discussed in Keeney and Raiffa (1993) [27], we focus on the independence conditions involving preference differences .

Difference consistency is so intuitively appealing that it could simply be assumed to hold in most practical applications. The following procedure could be used to verify difference independence. We determine $w_1, x_1 \in X_1$ such that $(w_1, \bar{w}_1) \succsim (x_1, \bar{w}_1)$ for some $\bar{w}_1 \in \bar{X}_1$. We then ask the decision maker to imagine that she is in situation 1: She already has $(x_1, \bar{w}_1)$ and she can exchange it for $(w_1, \bar{w}_1)$. Next, we arbitrarily choose $\bar{x}_1 \in \bar{X}_1$ and ask her to imagine situation 2: She already has $(x_1, \bar{x}_1)$, and she can exchange it for $(w_1, \bar{x}_1)$. Would she prefer to make the exchange in situation 1 or in situation 2, or is she indifferent between the two exchanges? If she is indifferent between the two exchanges for several different values of $w_1, x_1 \in X_1$ and $\bar{w}_1, \bar{x}_1 \in \bar{X}_1$ then we can conclude that X_1 is difference independent of $\bar{X}_1$.

For example, suppose we ask the decision maker to consider exchanging a car described by ($ 25,000, 150 hp, ugly) for a car described by ($ 25,000, 180 hp, ugly). Next, we ask her to consider exchanging ($ 35,000, 150 hp, nice) for ($ 35,000, 180 hp, nice). Would the opportunity to exchange a car with 150 hp for one with 180 hp be more important to the decision maker when the cost and appearance are $25,000 and ugly, or when they are $ 35,000 and nice? If the common values of these two attributes do not affect her judgments of the importance of these exchanges, then horsepower would be difference independent of cost and appearance.

Before using this procedure, we must ensure that the decision maker understands that we are asking her to focus on the exchange rather than on the final outcomes. For example, if she states that she prefers an exchange of $1,000,000 for $1,000,001 to an exchange of $5 for $500, then she undoubtedly is not focusing on the substitution of one outcome for another, but she is focusing instead on the final outcome. Thus, some training may be required before this approach to verification of difference independence is attempted.

To verify weak difference independence, partition X into X_I and $\bar{X}_I$, and choose $w_I, x_I, y_I, z_I \in X_I$ and $\bar{w}_I \in \bar{X}_I$ so that $(w_I, \bar{w}_I) \succ (x_I, \bar{w}_I)$, $(y_I, \bar{w}_I) \succ (z_I, \bar{w}_I)$ and the exchange of $(x_I, \bar{w}_I)$ for $(w_I, \bar{w}_I)$ is preferred to the exchange of $(z_I, \bar{w}_I)$ for $(y_I, \bar{w}_I)$. Then pick another value $\bar{x}_I \in \bar{X}_I$ and ask if the decision maker still prefers the exchange of $(x_I, \bar{x}_I)$ for $(w_I, \bar{x}_I)$ to the exchange of $(z_I, \bar{x}_I)$ for $(y_I, \bar{x}_I)$. This must be true if the subset X_I is weakly difference independent of $\bar{X}_I$. If the decision maker's response is affirmative, we repeat the question for other quadruples of consequences from X_I with the values of the criteria in $\bar{X}_I$ fixed at different levels. Continuing in this manner and asking the decision maker to verbally rationalize her responses, the analyst can either verify that X_I is weakly difference independent of $\bar{X}_I$ or discover that the condition does not hold.

Note that for the multiplicative measurable value function, it would only be necessary to verify weak difference independence for the special case of $I = \{i, j\}$, where i and j indicate the subscripts of an arbitrarily chosen pair of alternatives. This is true so long as the attributes are mutually preference independent.

For example, suppose we establish that the decision maker would prefer the exchange of the car ($25,000, 150 hp, ugly) for the car ($27,000, 200 hp, ugly) to the exchange of the car ($24,000, 130 hp, ugly) for the car ($25,000, 150 hp, ugly). If this preference for the first exchange over the second exchange does not depend on the common value of appearance, and if it also holds true for all other combinations of the values of cost and horsepower, then cost and horsepower are weak difference independent of appearance.

5.3.2 Assessment of the Measurable Value Functions. If difference independence or weak difference independence holds, each conditional measurable value function v_i can be assessed while holding $\bar{x}_i$ constant at any arbitrary value (generally at $\bar{x}_i^\circ$). With the additive value function that does not provide difference measurement, this strategy cannot be used as illustrated above. As a result, any of the approaches for assessing a single attribute measurable value function referenced in Section 2.1 may be used, including the direct rating of attribute values on an arbitrary scale (e.g., from 0 to 100), or direct estimates of preference differences.

If the measurable value function is additive, the scaling constants may be assessed using the same trade-off approach suggested for estimating the scaling constants for the additive ordinal value function (Keeney and Raiffa, 1993 [27, Chapter 3]). In this volume (Chapter 10), Bana e Costa, De Corte, and Vansnick discuss the use of MACBETH to assess a preference scale measuring preference differences based on qualitative judgments about "the difference of attractiveness" between two alternatives. For a discussion of other approaches to the assessment of the scaling constants for the additive and multiplicative cases, see also Dyer and Sarin (1979) [9].

Measurable multiattribute value functions may also be assessed using the ratio judgments and tools provided by the AHP methodology, and used as a basis for relating the AHP to formal preference theories that are widely accepted by economists and decision analysts. This point has been made by several authors, notably Kamenetsky (1982) [26] and Dyer (1990) [8].

Perhaps the best discussion of this important point is provided by Salo and Hämäläinen (1997) [36]. As they observe, once a suitable range of performance $[x_i^\circ, x_i^*]$ has been defined for each attribute, the additive measurable value function representation may be scaled so that the values $v(x^\circ) = v(x_1^\circ, \ldots, x_n^\circ) = 0$ and $v(x^*) = v(x_1^*, \ldots, x_n^*) = 1$ are assigned to the worst and best conceivable consequences, respectively. By also normalizing the component value functions onto the [0,1] range, the additive representation can be written as

$$\begin{aligned} v(x) &= \textstyle\sum_{i=1}^n v_i(x_i) = \sum_{i=1}^n [v_i(x_i) - v_i(x_i^\circ)] &&= \\ &= \textstyle\sum_{i=1}^n [v_i(x_i^*) - v_i(x_i^\circ)] \frac{v_i(x_i) - v_i(x_i^\circ)}{v_i(x_i^*) - v_i(x_i^\circ)} &&= \textstyle\sum_{i=1}^n w_i s_i(x_i), \end{aligned}$$

where $s_i(x_i) = v_i(x_i) - v_i(x_i^\circ)/v_i(x_i^*) - v_i(x_i^\circ) \in [0, 1]$ is the normalized score of x on the ith attribute and $w_i = v(x_i^*) - v(x_i^\circ)$ is the scaling constant or weight of the ith attribute.

A careful evaluation of this representation leads Salo and Hämäläinen to the conclusion that pair wise comparisons in ratio estimation should be interpreted in terms of ratios of value differences between pairs of underlying alternatives. This, in turn, provides the link between traditional models of preference theory and the AHP, and reveals that the latter can be an alternative assessment technique for measurable multiattribute value functions (with some simple adjustments for normalization and scaling).

5.4 Goal Programming and Measurable Multiattribute Value Functions

Goal programming was originally proposed by Charnes, Cooper, and Ferguson (1955) [6] as an ingenious approach to developing a scheme for executive compensation. As noted by Charnes and Cooper (1977) [5] in a review of the field, this approach to multiple objective optimization did not receive significant attention until the mid-1960's. However, during the past forty years, we have witnessed a flood of professional articles and books (e.g. Ijiri (1965) [23], Ignizio (1986) [22], Trzaskalik and Michnik (2002) [40]) dealing with applications of this methodology.

This discussion is limited to the use of goal programming as a methodology for solving problems with multiple, compensatory objectives. That is, we do not address problems that do not allow tradeoffs among the objectives. These non-compensatory models involve the use of the non-Archimedian, or "preemptive priority", weights. An analysis of these models would be based on the theory of lexicographic orders, summarized by Fishburn (1974)[16]. The conditions that would justify the use of a non-compensatory model are very strict, and are unlikely to be met in a significant number of real-world applications.

5.4.1 Goal Programming as an Approximation to Multiattribute Preferences. Let us begin with a simple example. Suppose a manager has identified a problem that can be formulated as a traditional mathematical programming problem with one exception – there are two criterion functions, $f_1(x)$ and $f_2(x)$ where $x \in X$ is an n-vector of controllable and uncontrollable variables, and the non-empty feasible set X is defined by a set of constraints. For simplicity, and without loss of generality, we assume that our choice of X ensures $0 \leq f_i(x) \leq 1,\ i = 1, 2$.

To use goal programming, we ask the manager if she has any "goals" in mind for the criteria. She replies that she would be happy if $f_1(\cdot)$ were at least as large as b_1, but she does not feel strongly about increasing $f_1(\cdot)$ beyond b_1. However, she would like for $f_2(\cdot)$ to be somewhere *between* b_{2L} and b_{2U}. Finally, we ask her to assign "weights" of relative importance to the deviations of $f_1(\cdot)$ from b_1, and of $f_2(\cdot)$ from b_{2L} and b_{2U}, respectively. After some thought, she responds with the weights w_1, w_2 and w_3.

Now, we can immediately write down this problem as follows:

$$
\begin{array}{rcll}
\min_{x \in X} w_1 y_1^- + w_2 y_2^- + w_3 y_3^+ & & & \\
\text{subject to} f_1(x) - y_1^+ + y_1^- & = & b_1 & \\
f_2(x) - y_2^+ + y_2^- & = & b_{2L} & \\
f_2(x) - y_3^+ + y_3^- & = & b_{2U} & \\
y_i^+, y_i^- & \geq & 0, & i = 1, 2, 3.
\end{array}
\tag{GP}
$$

Notice that (GP) includes a "one-sided" formulation with respect to $f_1(\cdot)$, and a "goal interval" formulation with respect to $f_2(\cdot)$.

Let us pause a moment to reflect on this formulation. First, notice that $y_1^- = b_1 + y_1^+ - f_1(x)$. Suppose we introduce the relationship $f_1(x) - y_{12}^+ = 0$ as a new constraint for (GP). Since b_1 is a constant, minimizing $w_1 y_1^-$ is obviously equivalent to minimizing $w_1 \left(y_1^+ - y_{12}^+\right)$.

Similarly, if we introduce the constraint $f_2(x) - y_{22}^+ = 0$, minimizing $w_2 y_2^-$ is equivalent to minimizing $w_2 \left(y_2^+ - y_{22}^+\right)$, and minimizing $w_3 y_3^+$ is equivalent to minimizing $w_3 \left(y_3^- + y_{22}^+ - b_{2U}\right)$. The constant b_{2U} is maintained in the last expression in order to facilitate a graphical portrayal of the objective function as we shall see. Combining the results and re-writing (GP) as a maximization problem, we have the equivalent problem statement:

$$
\max_{x \in X} w_1 \left(y_{12}^+ - y_1^+\right) + w_2 \left(y_{22}^+ - y_2^+\right) + w_3 \left(y_{22}^+ + y_3^- - b_{2U}\right)
$$

$$
\begin{array}{rcll}
\text{subject to } f_1(x) - y_1^+ + y_1^- & = & b_1 & \\
f_2(x) - y_2^+ + y_2^- & = & b_{2L} & \\
f_2(x) - y_3^+ + y_3^- & = & b_{2U} & \\
f_1(x) - y_{12}^+ & = & 0 & \\
f_2(x) - y_{22}^+ & = & 0 & \\
y_i^+, y_i^- & \geq & 0 & i = 1, 2, 3 \\
y_{12}^+, y_{22}^+ & \geq & 0, &
\end{array}
\tag{VA}
$$

where the objective function may be interpreted as the sum of two piecewise linear functions (e.g. see Charnes and Cooper, 1961 [4, pp. 351-355])

Figures 7.3 and 7.4 illustrate these two piecewise linear functions. Recall that piecewise linear transformations are commonly used to transform additive separable nonlinear programming problems into linear programming problems. The lines labeled $v_1(\cdot)$ and $v_2(\cdot)$ in Figures 7.3 and 7.4 respectively suggest nonlinear preference functions that *might* be approximated by the bold piecewise linear functions.

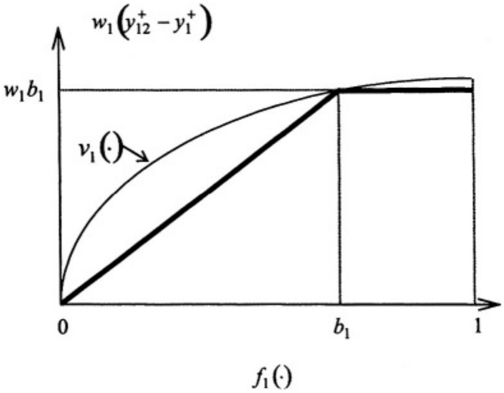

Figure 7.3. Piecewise linear approximation of $v_1(\cdot)$.

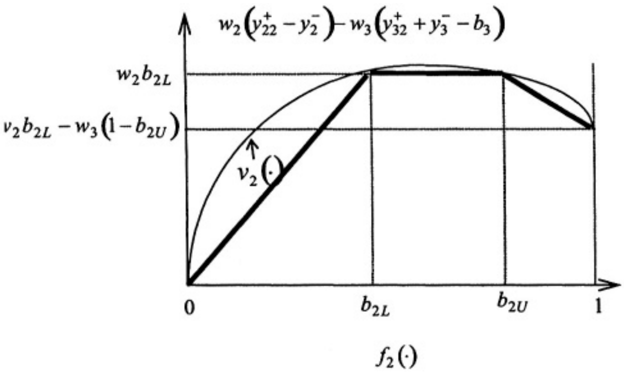

Figure 7.4. Piecewise linear approximation of $v_2(\cdot)$.

Thus since (VA) is equivalent to (GP), and (VA) may be viewed as a piecewise linear approximation to an additive separable nonlinear objective function, we are led to the conclusion that (GP) is an implicit approximation to the problem:

$$\max_{x \in X} v_1(f_1(x)) + v_2(f_2(x)). \tag{V}$$

And how do we interpret (V)? Since the choice of goals and goal intervals in (GP) reflect the decision maker's preferences and no uncertainty is involved in the decision, $v_1(\cdot)$ and $v_2(\cdot)$ are measurable functions, and their sum, $v(f_1(\cdot), f_2(\cdot)) = v_1(f_1(\cdot)) + v_2(f_2(\cdot))$, is an additive separable measurable value function.

Goal programming is generally applied to problems where risk is not explicitly involved in the formulation. Therefore, the additive utility function theory developed for risky choice is not relevant for these applications. Likewise, the ordinal additive theories are not operational here because they require a simultaneous conjoint scaling of the separable terms. Goal programming applications generally allow the selection of each goal or goal interval independent of consideration of the values of the other criteria. This practice implies the existence of a measurable additive utility function under certainty.

This point has been made recently by Bordley and Kirkwood (2004) [2] in a general discussion of the relationship between goals and multiattribute preference models. This perspective provides some insights regarding the nature of goal programming, as well as some challenges. For example, how should the piecewise linear approximations to the nonlinear value functions be selected in order to minimize error? Geoffrion (1977) [19] provides some useful guidelines for choosing "goals" or "goal intervals" for each criterion so that the piecewise linear approximation to the implicit utility function provides the best fit.

One important implication of this point of view is that goal programming should not be considered an *ad hoc,* heuristic approach to solving multiple objective problems. Rather, the approach is based on a set of implicit, well-understood assumptions from multiattribute preference theory. Goal programming formulations should be either criticized or justified on the basis of these assumptions.

6. The Relationships Among the Multiattribute Preference Functions

The necessary conditions for the additive and multiplicative measurable value functions and risky utility functions, notably mutual preference independence, are also necessary and sufficient for the ordinal additive value function that does not provide difference measurement. Therefore, it is natural to investigate the relationships among them. The following choice of scaling will be imposed. For $f = \overset{\circ}{v}, v,$ or u, f is normalized by $f(x_1^*, \ldots, x_n^*) = 1$ and $f(x_1^\circ, \ldots, x_n^\circ) = 0$ and $f_i(x_i)$ is a conditional function on X_i scaled by $f_i(x_i^*) = 1$ and $f_i(x_i^\circ) = 0$. Finally, $\overset{\circ}{\lambda}, \lambda$ and k will be used as scaling constants for the ordinal and measurable value functions and the utility function, respectively.

6.1 The Additive Functions

The relationships among the alternative developments of the additive forms of real-valued functions on X follow immediately from their respective uniqueness properties. This may be summarized as follows. Assume $n \geq 3$ and $X_1, \dots, X_n$ are mutually preference independent. Then

1 if $X_1, \dots, X_n$ are difference consistent and X_1 is difference independent of $\bar{X}_1$ then $\mathring{v} = v$;

2 if there exists a utility function u on X and if preferences over lotteries on $X_1, \dots, X_n$ depend only on their marginal probability distributions and not on their joint probability distributions, then $\mathring{v} = u$.

3 if both 1 and, 2 are satisfied , $\mathring{v} = v = u$.

Note the implication of this result. In order for $\mathring{v} = v = u$ for a single decision maker, she must have preferences simultaneously consistent with mutual preference independence, difference independence, and additive independence for risky alternatives. Mutual preference independence will hold in all cases, but it may be the case that difference independence and/or additive independence for risky alternatives will not hold. Further, difference independence may hold for the preferences of a decision maker, implying that an additive measurable value function would provide a valid representation of her preferences, but additive independence for risky alternatives may not be satisfied, implying that an additive utility function would not be a valid representation of her preferences in decision scenarios involving risk.

6.2 The Multiplicative Functions

Throughout this section we assume that the following conditions are satisfied:

1 There are $n \geq 3$ attributes, and $X_1, \dots, X_n$ are mutually preference independent;

2 There exists a measurable value function v on X and X_1 is weak difference independent of $\bar{X}_1$, and

3 There exists a utility function u on X and X_1 is utility independent of $\bar{X}_1$.

Suppose we have assessed the additive value function $\mathring{v}$ and wish to obtain either v or u. Then the following relationships will hold (Dyer and Sarin (1979) [9, Theorem 5]). Either

1 $\mathring{v}(x) = v(x)$ and $\mathring{v}_i(x_i) = v_i(x_i), i = 1, \dots, n,$ or

2 $\overset{\circ}{v}(x)\ln(1+\lambda) = \ln[1+\lambda v(x)]$ and $\overset{\circ}{\lambda}_i \overset{\circ}{v}_i(x_i)\ln(1+\lambda) = \ln[1+\lambda\lambda_i v_i(x_i)], i = 1,\ldots,n$.

Either

1 $\overset{\circ}{v}(x) = u(x)$ and $\overset{\circ}{v}_i(x_i) = u_i(x_i), i = 1,\ldots,n$, or,

2 $\overset{\circ}{v}(x)\ln(1+k) = \ln[1+ku(x)]$ and $\overset{\circ}{\lambda}_i \overset{\circ}{v}_i(x_i)\ln(1+k) = \ln[1+kk_i u_i(x_i)], i = 1,\ldots,n$.

These relationships may be used to simplify the assessment of multiattribute preference functions. For example, suppose we define x_i' as the *equal difference point* for attribute X_i if $(x_i', \bar{x}_i)(x_i^\circ, \bar{x}_i) \sim^* (x_i^*, \bar{x}_i)(x_i', \bar{x}_i)$ for any $\bar{x}_i \in \bar{X}_i$. Notice that $v_i(x_i') = 1/2$ because of our choice of scaling. Given $\overset{\circ}{v}$, the assessment of x_i' for any attribute X_i is enough to completely specify v, because if $v_i(x_i') = 1/2$ for some $i \in \{1,\ldots,n\}$ then $v = \overset{\circ}{v}$. Otherwise, $1+(1+\lambda)^{\overset{\circ}{\lambda}_i} = 2(1+\lambda)^{\overset{\circ}{\lambda}_i \overset{\circ}{v}_i(x_i)}$.

Finally, to derive u from $\overset{\circ}{v}$, find x_i'' for some attribute X_i such that the decision maker is indifferent between x_i'' and an equal chance lottery between x_i^* and x_i° with the other criteria held fixed. A parallel result to the above relationship between ordinal and measurable value functions holds. Specifically, if $\overset{\circ}{v}_i(x_i'') = 1/2$ for some $i \in \{1,\ldots,n\}$, then $u = \overset{\circ}{v}$. Otherwise, $1+(1+k)^{\overset{\circ}{\lambda}_i} = 2(1+k)^{\overset{\circ}{\lambda}_i \overset{\circ}{v}_i(x_i)}$.

These results can also be used to derive v after u has been assessed, or vice versa. For example, suppose u has been assessed using appropriate procedures. To obtain v, we find the equal difference point x_i' for some criterion X_i. The second result above is used to obtain $\overset{\circ}{\lambda}_i$ and $\overset{\circ}{v}_i$ for each criterion, and we can obtain v. In a similar manner, u can be obtained from v after assessing x_i'' for some criterion X_i.

Since the AHP can be interpreted as ratios of preference differences, this relationship also allows the results from assessments based on the AHP to be suitably transformed into multiattribute utility functions appropriate for use in risky situations. This completes the circle required to synthesize ordinal multiattribute value functions, measurable multiattribute value functions, multiattribute utility functions, and multiattribute functions based on ratio judgments. As a result, the analyst is justified in choosing among a variety of assessment tools, and making the appropriate adjustments in order to calibrate the results into a coherent and theoretically sound representation of preferences.

7. Concluding Remarks

In this chapter, we have presented an informal discussion of "mutiattribute utility theory". In fact, this discussion has emphasized that there is no single version of multiattribute utility theory that is relevant to multicriteria decision analysis. Instead, there are three distinct theories of multiattibute preference functions that may be used to represent a decision maker's preferences.

The ordinal additive multiattribute preference model requires the assumption of mutual preference independence, and is appropriate for use in the case of certainty. Most of the applications and methods of multicriteria decision analysis are presented in the context of certainty, and so this would seem to be an appealing theory to use for framing these approaches. However, as we have emphasized, the ordinal additive multiattribute preference model requires assessment techniques that are cumbersome in practice, and that force the decision maker to make explicit tradeoffs between two or more criteria in the assessment of the value functions defined on the individual criteria.

The measurable value functions also require the assumption of mutual preference independence, along with the stronger assumptions of weak difference independence or difference independence in order to obtain convenient decompositions of the model that are easy to assess. The assessment of these preference models is relatively easy, and they can be interpreted intuitively as providing a measure of strength of preference. In addition, the ratio judgments of the AHP can be interpreted as ratios of preference differences based on this theory, linking the AHP methodology to traditional models of preference accepted in the decision analysis and economics literatures.

Finally, multiattribute utility theory is an elegant and useful model of preference suitable for applications involving risky choice. The brilliant work of Keeney and Raiffa (1993) [27] has made this theory synonymous to many scholars with multiple criterion decision making, and the ordinal and measurable theories are often overlooked or ignored as a result. In fact, these latter approaches may provide more attractive and appropriate theories for many applications of multicriteria decision analysis.

Notes

1. The classic book *Decisions with Multiple Objectives* by R. L. Keeney and H. Raiffa was originally published by Wiley in 1976. The Cambridge University Press version was published in 1993 [27].

2. "The important addition since 1976 concerns value functions that address strength of preference between pairs of consequences (see Dyer and Sarin, 1979 [9]; Bell and Raiffa, 1988 [1])." A quote from the Preface to the Cambridge University Press Edition, R. L. Keeney and H. Raiffa, *Decisions with Multiple Objectives,* Cambridge University Press, 1993 [27].

3. Note that the $\overset{o}{v}_i$ are called partial value functions by Bouyssou and Pirlot in Chapter 3 of this volume.

4. Specifically, we assume restricted solvability from below, an Archimedian property, at least three attributes are essential, and that the attributes are bounded from below. If $n = 2$, we assume that the two attributes are preferentially independent of one another and that the Thomsen condition is satisfied (see Krantz et al. (1971) (30] and the discussion by Bouyssou and Pirlot in this volume).

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