1 Schottkey and the emission surface

1.a

The shift occurs at the location where the Schottkey effect is maximum. So we look at the spatial derivative of the Schottkey effect.

$$\frac{d\Phi_{Schottkey}}{dz} = \frac{e^2}{16\pi\epsilon_0 z^2} - eF_a$$

Setting $\frac{d\Phi_{Schottkey}}{dz} = 0$ and solving for z (and calling this z z_T), we get $z_T^2 = \frac{e}{16\pi\epsilon_0 F_a}$ or $z_T = \frac{1}{4}\sqrt{\frac{e}{\pi\epsilon_0 F_a}}$.

1.b

The Schottkey potential at this location is

$$\begin{aligned}
\Phi_{Schottkey}(z_T) &= -\frac{e^2}{16\pi\epsilon_0 z_T} - eF_a z_T \\
&= -\frac{e^2}{4\pi\epsilon_0\sqrt{\frac{e}{\pi\epsilon_0 F_a}}} - \frac{eF_a}{4}\sqrt{\frac{e}{\pi\epsilon_0 F_a}} \\
&= -\frac{e^{3/2}\sqrt{F_a}}{4\sqrt{\pi\epsilon_0}} - \frac{e^{3/2}\sqrt{F_a}}{4\sqrt{\pi\epsilon_0}} \\
&= -\frac{e}{2}\sqrt{\frac{eF_a}{\pi\epsilon_0}}
\end{aligned}$$

1.c

Notice that we want $W_eff = (1-a)W$ or

$$\Phi_{Schottkey} = aW$$

where $a = 0.1$ in this problem. Using the expression for the Schottkey potential from the previous section, we square both sides and solve the resulting equation for F_a

$$\begin{aligned}
|F_a| &= \frac{4a^2W^2\pi\epsilon_0}{e^3} \\
&= \frac{0.04 \times 4.5^2\pi \times 8.85 \times 10^{-12}}{1.602 \times 10^{-19}}\frac{V^2C}{VmC} \\
&\approx 141\frac{MV}{m}
\end{aligned}$$

1.d

We use the same equation but with $a = 1$. Working this out, we get a factor increase of 100× the previous answer. That is $|F_a| = 14.1\frac{GV}{m}$.

2 Trace ellipse and transport

We are going to use

$$\hat{\gamma}x^2 + 2\hat{\alpha}xx' + \hat{\beta}x'^2 = \epsilon \tag{1}$$

2.a

First we find (x_{int}, x'_{int}). Notice, that these values are when the ellipse crosses the x and x' axes, respectively, i.e. when $x' = 0$ and $x = 0$, respectively. Hence

$$\hat{\gamma}x_{int}^2 = \epsilon$$
$$\hat{\beta}x_{int}'^2 = \epsilon$$

So

$$x_{int} = \pm\sqrt{\frac{\epsilon}{\hat{\gamma}}}$$
$$x'_{int} = \pm\sqrt{\frac{\epsilon}{\hat{\beta}}}$$

We now find (x_{max}, x'_{max}). Notice that x and x' are not necessary independent; that is $\frac{dx}{dx'}$ and $\frac{dx'}{dx}$ are not necessary zero and this relation is described by Eq. (1). We take the derivative of this equation first by x and separately by x'

$$2\hat{\gamma}x + 2\hat{\alpha}(x' + x\frac{dx'}{dx}) + 2\hat{\beta}x'\frac{dx'}{dx} = 0$$
$$2\hat{\gamma}x\frac{dx}{dx'} + 2\hat{\alpha}(x'\frac{dx}{dx'} + x) + 2\hat{\beta}x' = 0$$

However, we are interested in when x and x' are maxima, so $\frac{dx}{dx'} = 0$ and $\frac{dx'}{dx} = 0$, respectively. So

$$2\hat{\gamma}x + 2\hat{\alpha}x'_{max} = 0$$
$$2\hat{\alpha}x_{max} + 2\hat{\beta}x' = 0$$

Solving for the other component, we get

$$x = -\frac{\hat{\alpha}}{\hat{\gamma}}x'_{max}$$
$$x' = -\frac{\hat{\alpha}}{\hat{\beta}}x_{max}$$

Putting these back into Eq. (1), we get for x'_{max}

$$\begin{aligned}\epsilon &= \hat{\gamma}\left(-\frac{\hat{\alpha}}{\hat{\gamma}}x'_{max}\right)^2 + 2\hat{\alpha}\left(-\frac{\hat{\alpha}}{\hat{\gamma}}x'_{max}\right)x'_{max} + \hat{\beta}x'^2_{max} \\ &= \frac{\hat{\alpha}^2}{\hat{\gamma}}x'^2_{max} - 2\frac{\hat{\alpha}^2}{\hat{\gamma}}x'^2_{max} + \hat{\beta}x'^2_{max} \\ &= \frac{\hat{\beta}\hat{\gamma} - \hat{\alpha}^2}{\hat{\gamma}}x'^2_{max} \\ &= \frac{1}{\hat{\gamma}}x'^2_{max}\end{aligned}$$

So $x'_{max} = \pm\sqrt{\hat{\gamma}\epsilon}$.

Likewise for x_{max}

$$\begin{aligned}\epsilon &= \hat{\gamma}x^2_{max} + 2\hat{\alpha}x_{max}\left(-\frac{\hat{\alpha}}{\hat{\beta}}x_{max}\right) + \hat{\beta}\left(-\frac{\hat{\alpha}}{\hat{\beta}}x_{max}\right)^2 \\ &= \hat{\gamma}x^2_{max} - 2\frac{\hat{\alpha}^2}{\hat{\beta}}x^2_{max} + \frac{\hat{\alpha}^2}{\hat{\beta}}x^2_{max} \\ &= \frac{\hat{\beta}\hat{\gamma} - \hat{\alpha}^2}{\hat{beta}}x^2_{max} \\ &= \frac{1}{\hat{\beta}}x^2_{max}\end{aligned}$$

So $x'_{max} = \pm\sqrt{\hat{\beta}\epsilon}$.

2.b

Writing down the matrices, we have

$$\begin{bmatrix} x_1 \\ x'_1 \end{bmatrix} = \begin{bmatrix} 1 & s_{0,1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x'_0 \end{bmatrix}$$

Using matrix multiplication, this gives

$$\begin{aligned} x_1 &= x_0 + s_{0,1}x'_0 \\ x'_1 &= x'_0 \end{aligned}$$

which is what we want. So they are equivalent formulations.

2.c

Analogous to the previous problem, we have

$$\begin{bmatrix} x_1 \\ x'_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_t} & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x'_0 \end{bmatrix}$$

Using matrix multiplication, this gives

$$\begin{aligned} x_1 &= x_0 \\ x'_1 &= -\frac{1}{f_t} + x'_0 \end{aligned}$$

which is again what we want. So they are equivalent formulations.

2.d

The drift matrix is

$$\begin{bmatrix} \tau_1 \\ \delta_1 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tau_0 \\ \delta_0 \end{bmatrix}$$

You can see this because $E_0 = \sqrt{p_0^2c^2 + m^2c^4}$, and

$$\begin{aligned} E &= \sqrt{(p_0 + \Delta p)^2c^2 + m^2c^4} \\ &= \sqrt{p_0^2c^2 + m^2c^4 + 2p_0\Delta pc^2 + (\Delta p)^2c^2} \\ &\approx \sqrt{E_0^2 + 2p_0\Delta pc^2} \\ &\approx E_0 + \frac{p_0}{E_0}\Delta pc^2 \end{aligned}$$

where Δp is small. So $\delta = \frac{E-E_0}{E_0} \approx \frac{p_0\Delta pc^2}{\gamma^2m^2c^4} = \frac{v_0}{\gamma mc^2}\Delta p \approx \beta\Delta\beta$. Denote z_{mid} as the center of the pulse, not z_0, which we will reserve for the initial position of the electron. Now notice $z_{mid} = z_{mid,0} + v_0t$ wherease $z = z_0 + (v_0 + \Delta v)t$. So $z - z_{mid} = z_0 - z_{mid,0} + c\Delta\beta t$. So $\frac{z-z_{mid}}{v_0} = \tau_0 + \frac{\Delta\beta}{\beta}t$

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