1 RF cavity as a longitudinal lens

1.a

The "chirp" gives the change in velocity, Δv from the center of mass velocity, $v_{com} = c\beta$, as a function of position from the center of mass, Δz. Specifically

$$\Delta v = C\Delta z$$

Assuming that the chirp after the RF cavity is given by the change in chirp introduced by the RF cavity, it follows

$$\Delta v = -\frac{2eE_0}{mc\beta\gamma}\sin\left(\frac{\pi L}{\beta\lambda_{RF}}\right)\Delta z$$

Now to get back to the focus, we need to have the electron cross the extra $-\Delta z$ (negative because if it is in front (positive Δz, the electron needs to move back (negative Δz), and if it is back (negative Δz), it needs to move forward (positive Δz) as it travels at the additional Δv. In other words,

$$\begin{aligned} t &= \frac{-\Delta z}{\Delta v} \\ &= \frac{mc\beta\gamma}{2eE_0}\csc\left(\frac{\pi L}{\beta\lambda_{RF}}\right) \end{aligned}$$

This time is valid for every point on the chirp. Now the focal length is just this time times the COM speed.

$$\begin{aligned} f_l &= v_{com}t_l \\ &= \frac{mc^2\beta^2\gamma}{2eE_0}\csc\left(\frac{\pi L}{\beta\lambda_{RF}}\right) \end{aligned}$$

1.b

$$\begin{aligned} M_{RF} &= \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_l} & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -\frac{2eE_0}{mc^2\beta^2\gamma}\sin\left(\frac{\pi L}{\beta\lambda_{RF}}\right) & 1 \end{bmatrix} \end{aligned}$$

1.c

We first look at E_{com} and E for a particle with a small difference in its momentum compared to the center of mass momentum, i.e. $\Delta p << p_{com}$.

$$E_{com} = \sqrt{p_{com}^2c^2 + mc^2}$$

and

$$\begin{aligned} E &= \sqrt{(p_{com} + \Delta p)^2c^2 + mc^2} \\ &= \sqrt{E_{com}^2 + 2p_{com}\Delta pc^2 + (\Delta p)^2c^2} \\ &\approx \sqrt{E_{com}^2 + 2p_{com}\Delta pc^2} \\ &= E_{com}\sqrt{1 + 2\frac{p_{com}\Delta pc^2}{E_{com}^2}} \\ &\approx E_{com}\left(1 + \frac{p_{com}\Delta pc^2}{E_{com}^2}\right) \\ &= E_{com} + \frac{p_{com}\Delta pc^2}{E_{com}} \\ &= E_{com} + v_{com}\Delta p \end{aligned}$$

Thus

$$\begin{aligned}\delta &= \frac{E - E_{com}}{Ecom} \\ &= \frac{v_{com}\Delta p}{\gamma mc^2} \\ &\approx \frac{v_{com}\Delta v}{c^2}\end{aligned}$$

Now we look at the longitudinal parameters where we denote the pre-RF parameters with $_0$ and the post with $_1$. Notice $\tau_1 = \frac{z_1 - z_{com}}{v_{com}}$ and $\tau_0 = \frac{z_0 - z_{com}}{v_{com}}$. Further notice that according to M_{RF}, $z_1 - z_{com} = z_0 - z_c om$. So it is trival to see that $\tau_1 = \tau_0$. Moreover, solving for $z_0 - z_{com}$, we also have $z_0 - z_{com} = v_{com}\tau_0$.

On the other hand, $\delta_1 = \frac{v_{com}}{c^2}(v_1 - v_{com})$ and $\delta_0 = \frac{v_{com}}{c^2}(v_0 - v_{com})$. Again, according to M_{RF},

$$\begin{aligned}v_1 - v_{com} &= -\frac{1}{f_l} z_0 + v_0 - v_{com} \\ &= -\frac{1}{f_l}(v_{com}\tau_0) + v_0 - v_{com}\end{aligned}$$

So

$$\begin{aligned}\delta_1 &= \frac{v_{com}}{c^2}(v_1 - v_{com}) \\ &= \frac{v_{com}}{c^2}(-\frac{1}{f_l} v_{com}\tau_0 + v_0 - v_{com}) \\ &= -\frac{v_{com}^2}{f_l c^2}\tau_0 + \delta_0 \\ &= -\frac{\beta^2}{f_l}\tau_0 + \delta_0\end{aligned}$$

Thus the corresponding matrix in this coordinate system is

$$\begin{aligned}M_{RF} &= \begin{bmatrix} 1 & 0 \\ -\frac{\beta^2}{f_l} & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -\frac{2eE_0}{mc^2\gamma}\sin\left(\frac{\pi L}{\beta\lambda_{RF}}\right) & 1 \end{bmatrix}\end{aligned}$$