1 RMS emittance and non-interacting drift

Consider the drift evolution equations for N non-interacting particles

$$\vec{x} = \vec{x}_0 + \frac{\vec{p}_0}{\gamma m} t \tag{1}$$

$$\vec{p} = \vec{p}_0 \tag{2}$$

You are going to analyze the evolution of $m^2 c^2 \epsilon^2_{x,p_x} = s^2_x s^2_{p_x} - s^2_{x,p_x}$ where $s_{a,b} =< ab > - < a >< b >$ and $< a >= \frac{1}{N} \sum a_i$ where a_i is the value of a for the i^{th} particle. Doing this will lead to an understanding of why emittance is important for focussing.

To do this analysis, we need some statistical facts for the random variables a, b, c, and d and the scalar α.

1. $s_\alpha = 0$
2. $s_{\alpha b,c} = \alpha s_{b,c} = s_{b,\alpha c}$
3. $s_{a+b,c+d} = s_{a,c} + s_{a,d} + s_{b,c} + s_{b,d}$

These properties are fairly easy to prove using the definition of standard deviation, but we will not prove these here. Notice that in our description of our ensemble, x_0 and p_0 are random variables while γ, m, and t are scalars.

1.a

Write $s^2_{p_x}$ as a function of the initial statistics and time for non-interacting drift particles.

Hint: This should be trivial.

1.b

Write s_{x,p_x} as a function of the initial statistics and time.

1.c

Write s^2_x as a function of the initial statistics and time.

1.d

Using the results of question 1.c, find the time when s^2_x is minimum.

1.e

What is the value of s^2_x at the time you found in question 1.d? Express the numerator in terms of emittance. How does this relate to focussing?

1.f

Using your answer from question 1.b and the time you found in question 1.d, what is the value of s_{x,p_x} at this time? What is the x-p_x covariance of your ensemble at the focal point?