Non-interacting particle kinematics and density evolution

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1 Overview

Consider an electron in a column with the following elements: a dc electron gun used to initially accelerate the electron longitudinally (z-direction), some number of magnetic lenses used to focus the electron radially (x, y, or r-direction), and an RF cavity used to focus the electron longitudinally. What we are interesting in is solving the approximate phase space coordinate of this electron at all times; equivalently obtaining a time-dependent functional form for the position and momentum or velocity of the electron.

We will treat all elements as being at a specific location in z. Most of the description is essentially the electron drifting, which is fairly trivial to analyze. Then, we consider the trajectory of the particle in a scalar electric field that we use to approximate the electron gun. After this constant electric field, the electron will pass through the anode where the electric field "leaks" resulting in a outward, or diverging, radial field component. We will analyze the effect of this portion of the gun separately. Then, we will connect the drift pieces with the remaining lenses and cavities; that is, we will analyze static magnetic lenses, static lens-like portion of the photoelectron gun, and then analyze the rf-cavity.

One note about terminology. We will use the term thin to describe various elements. When we treat an element as thin, we treat it as if an electron traveling through that element has no significant radial evolution; that is, we treat the cylindrical coordinate r as constant. There is a second category that we consider that is not generally discussed in the literature. We call this category "short", and it will be used to refer to elements that have inherent frequencies associated with them. Specifically, particles traveling through the short elements spend much less time in the element than the period of the oscillation associated with the element. Here, the approximation will not be that the radius is constant but that the oscillation can be approximated by the small angle approximation. If we do not call an element thin or short, or if we instead describe it as thick, then the only assumptions made will be the peri-axial assumptions discuss in section 4.

2 Drift

Describe an electron with initial position $\vec{x}_0 = x_0\hat{x} + y_0\hat{y} + z_0\hat{z}$ and some initial velocity, $\vec{v}_0 = c\vec{\beta}_0 = c\beta_{0,x}\hat{x} + c\beta_{0,y}\hat{y} + c\beta_{0,z}\hat{z}$, at time 0. Furthermore, denote the variable sans the subscript $_0$ to be shorthand for the time dependent form of the variable; specifically, $\vec{x} = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$ indicates the trajectory of the particle in time. It should be apparent that

$$\vec{x} = \vec{x}_0 + c\vec{\beta}_0 t \tag{1}$$

$$\vec{p} = \vec{p}_0 \tag{2}$$

3 Electron gun with constant field

We present a fairly straightforward analysis of the equations of motion of a single electron in a scalar accelerating field of strength $\vec{E} = E_a\hat{z}$ to model the electron gun. We adopt the same notation as the previous section with the addition of the momentum, $\vec{p}_0 = p_{x,0}\hat{x} + p_{y,0}\hat{y} + p_{z,0}\hat{z}$, etc. . These equations will form the basis of multiple simple models of the accelerating bunch.

Notice that as the force is a scalar, the momentum at time t may be written as

$$\vec{p}(t) = \vec{p}_0 + qE_a t\hat{z} \tag{3}$$

Likewise, the relativistic energy, E not to be confused with the electric extraction field, E_a, is defined as

$$\begin{aligned}
\frac{1}{m^2c^4}E^2(t) &= 1 + \frac{1}{m^2c^2}p^2(t) \\
&= 1 + \frac{1}{m^2c^2}\left(\vec{p}_0 + qE_a t\hat{z}\right)^2 \\
&= 1 + \frac{1}{m^2c^2}\left(p_0^2 + 2p_{0,z}qE_a t + q^2E_a^2t^2\right) \\
&= 1 + \gamma_0^2\beta_0^2 + 2\gamma_0\beta_{0,z}\frac{qE_a}{mc}t + \frac{q^2E_a^2}{m^2c^2}t^2 \\
&= \gamma_0^2\left(1 + 2\beta_{0,z}\xi t + \xi^2t^2\right) \qquad (4)
\end{aligned}$$

where $\xi = \frac{qE_a}{\gamma_0 mc}$ has units of inverse time. At position, z, we know the change in energy can also be written as $qE_a z$, so the energy can be parameterized by

$$\begin{aligned}
\frac{1}{m^2c^4}E^2(z) &= \frac{1}{m^2c^4}\left(E_0 + qE_a z\right)^2 \\
&= \frac{1}{m^2c^4}\left(E_0^2 + 2E_0qE_a z + q^2E_a^2z^2\right) \\
&= \gamma_0^2 + 2\sqrt{1+\gamma_0^2\beta_0^2}\gamma_0\xi\frac{z}{c} + \gamma_0^2\xi^2\frac{z^2}{c^2} \qquad (5) \\
&= \gamma_0^2\left(1 + 2\xi t_z + \xi^2t_z^2\right) \qquad (6)
\end{aligned}$$

where $t_z = \frac{z}{c}$ has units of time. Setting these two equations equal, we get

$$2\xi t_z + \xi^2t_z^2 = 2\beta_{0,z}\xi t + \xi^2t^2 \tag{7}$$

This is of the form $At_z^2 + Bt_z + C = 0$ with $A = \xi^2$, $B = 2\xi$, and $C = -2\beta_{0,z}\xi t - \xi^2t^2$. Obviously, this has the solution $t_z = \frac{-B\pm\sqrt{B^2-4AC}}{2A}$. Assuming $A, B > 0$ and $C < 0$, only one of these times is positive (and physical), $t_z = \frac{-B+\sqrt{B^2-4AC}}{2A}$. This simplifies to

$$z = \frac{c}{\xi}\left(\sqrt{1 + 2\beta_{0,z}\xi t + \xi^2t^2} - 1\right) \tag{8}$$

Notice that taking the derivative of this monstrosity with respect to t gives

$$\beta_z = \frac{\beta_{0,z} + \xi t}{\sqrt{1 + 2\beta_{0,z}\xi t + \xi^2t^2}} \tag{9}$$

Unfortunately, obtaining the transverse trajectory of the particle is not as straightforward. So we return to Newton's second law, $\frac{d\vec{p}}{dt} = eE_a\hat{z}$. The differential can be separated as

$$\begin{aligned}
d\left(\gamma\vec{\beta}\right) &= \frac{eE_a}{m_ec}\hat{z}dt \\
&= \gamma_0\xi\hat{z}dt \qquad (10)
\end{aligned}$$

Upon integration of (10) we obtain

$$\gamma\vec{\beta} - \gamma_0\vec{\beta}_0 = \gamma_0\xi t\hat{z} \tag{11}$$

Reducing this vector equation to three spatial equations we get

$$\gamma\beta_x - \gamma_0\beta_{x,0} = 0 \tag{12}$$
$$\gamma\beta_y - \gamma_0\beta_{y,0} = 0 \tag{13}$$
$$\gamma\beta_z - \gamma_0\beta_{z,0} = \gamma_0\xi t \tag{14}$$

Solving the last equation for β_z and comparing this to Eq. (9) we can see

$$\frac{\gamma}{\gamma_0} = \sqrt{1 + 2\beta_{0,z}\xi t + \xi^2 t^2} \tag{15}$$

Subbing this into Eqs. (12) and (13), we can obtain the expressions for other velocities

$$\frac{dx}{dt} = c\frac{\beta_{0,x}}{\sqrt{1 + 2\beta_{0,z}\xi t + \xi^2 t^2}} \tag{16}$$

$$\frac{dy}{dt} = c\frac{\beta_{0,y}}{\sqrt{1 + 2\beta_{0,z}\xi t + \xi^2 t^2}} \tag{17}$$

Separating the variables and integrating we get

$$x = x_0 + \frac{c\beta_{x,0}}{\xi} ln\left(\frac{\sqrt{1 + 2\beta_{0,z}\xi t + \xi^2 t^2} + \beta_{0,z} + \xi t}{1 + \beta_{0,z}}\right) \tag{18}$$

$$y = y_0 + \frac{c\beta_{x,0}}{\xi} ln\left(\frac{\sqrt{1 + 2\beta_{0,z}\xi t + \xi^2 t^2} + \beta_{0,z} + \xi t}{1 + \beta_{0,z}}\right) \tag{19}$$

Real-space	**Momentum**
$x = x_0 + \frac{c\beta_{x,0}}{\xi} ln\left(\frac{\sqrt{1+2\beta_{0,z}\xi t+\xi^2 t^2}+\beta_{0,z}+\xi t}{1+\beta_{0,z}}\right)$	$p_x = p_{x,0}$
$y = y_0 + \frac{c\beta_{x,0}}{\xi} ln\left(\frac{\sqrt{1+2\beta_{0,z}\xi t+\xi^2 t^2}+\beta_{0,z}+\xi t}{1+\beta_{0,z}}\right)$	$p_y = p_{y,0}$
$z = \frac{c}{\xi}\left(\sqrt{1 + 2\beta_{0,z}\xi t + \xi^2 t^2} - 1\right)$	$p_z = p_{z,0} + qE_a t$

Table 1: Summary of constant electron gun phase coordinate evolution where $\xi = \frac{qE_a}{\gamma_0 mc}$ has units of inverse time.

4 Peri-axial assumption

After accelerating the bunch to the relativistic regime, we make what is known as the peri-axial assumption. Namely, we assume cylindrical symmetry so that the x- and y-directions, aka the r or transverse directions, are treated identically and the z-direction, longitudinal, is treated separately. Further, we assume that $v_z >> v_x, v_y$ so that $\beta \approx \beta_z$. It is standard to then describe a column not with the time coordinate but the longitudinal, z, coordinate using the replacement $\frac{d}{dt} = \frac{dz}{dt}\frac{d}{dz} \approx \beta c \frac{d}{dz}$. Denote $' \equiv \frac{d}{dz}$. Notice that with this notation, we may reinterpret the momenta in the transverse directions as

$$\begin{aligned} p_i &= \gamma m v_i \\ &= \gamma m \frac{dx_i}{dt} \\ &= \gamma m \frac{dx_i}{dz}\frac{dz}{dt} \\ &\approx \gamma \beta m c x_i' \end{aligned} \tag{20}$$

for either the x or y direction where x_i' is termed the "trace-space" coordinate. We will use this approximation a bunch. As expressions are sometimes easier to work with when they are written in terms of momentum, velocity, or their trace-space equivalent, we will use p_i, v_i, and x_i' interchangeably often hopping back and forth between the representations. Thus we suggest memorizing the relations $p_i = \gamma m v_i$ and $v_i = \beta c x_i'$. Incidentally, these "equivalent" notations are often sources of confusion for emittance definitions, so remain aware of which notation is being used regardless.

Another consequence of the cylindrical symmetry in empty space (lacking any sources) is that the scalar potentials, ϕ and ϕ_M, for the electric and magnetic fields are independent of θ. This results in $E_\theta = 0$ and $B_\theta = 0$. Following Reiser's presentation (section 3.3.1), this also leads to the longitudinal and radial electric and magnetic fields in terms of r are

$$E_z(r,z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \frac{\partial^{2\nu} E_z(0,z)}{\partial z^{2\nu}} \left(\frac{r}{2}\right)^{2\nu}$$
$$E_r(r,z) = \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu!(\nu-1)!} \frac{\partial^{2\nu-1} E_z(0,z)}{\partial z^{2\nu-1}} \left(\frac{r}{2}\right)^{2\nu-1}$$
$$B_z(r,z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \frac{\partial^{2\nu} B_z(0,z)}{\partial z^{2\nu}} \left(\frac{r}{2}\right)^{2\nu}$$
$$B_r(r,z) = \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu!(\nu-1)!} \frac{\partial^{2\nu-1} B_z(0,z)}{\partial z^{2\nu-1}} \left(\frac{r}{2}\right)^{2\nu-1}$$

Confining the analysis to the region near $r = 0$, the fields to first order in r become

$$E_z(r,z) = E_z(0,z) \tag{21}$$
$$E_r(r,z) = -\frac{\partial E_z(0,z)}{\partial z}\frac{r}{2} \tag{22}$$
$$B_z(r,z) = B_z(0,z) \tag{23}$$
$$B_r(r,z) = -\frac{\partial B_z(0,z)}{\partial z}\frac{r}{2} \tag{24}$$

Using this approximation and the Lorentz force equation, the motion of a charged particle can be written as (see Reisner section 3.3 — this is 3.67)

$$\zeta_L'' + \frac{\gamma'}{\gamma\beta^2}\zeta_L' + \left(\frac{\gamma''}{2\gamma\beta^2} + \left(\frac{\omega_L}{c\beta}\right)^2\right)\zeta_L = 0 \tag{25}$$

where $\zeta_L = (x+iy)e^{-i\theta_r}$ describes the transverse position of the particle as a complex value in the Larmor frame, $\theta_r = -\int_{z_0}^{z} \frac{qB}{2\gamma\beta mc} dz$ is the angle of rotation of the Larmor frame since magnetic field onset, $\omega_L = \frac{qB}{2\gamma m}$ is known as the Larmor frequency, and B is the magnetic field. We'll use this formulation in the next sections.

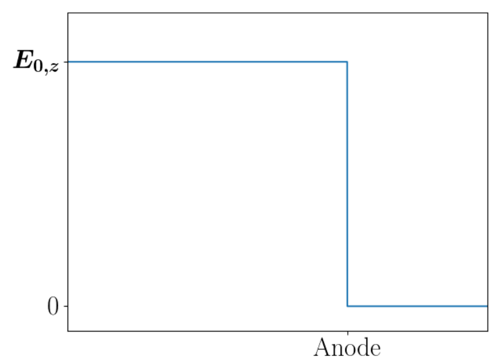

Figure 1: An idealized picture of the electric field inside a photoemission gun as the particle pass through the anode. The electric field inside the gun is solely in the z direction.

5 Anode — thin electric fringe field

In a previous section (section 3), we analyzed the trajectory of an electron in a constant electric field. What we did not discuss is the effect of the electron passing through the anode where the electric field ends. An idealized transition can be seen in Fig. 1 where

$$E_z(0,z) = \begin{cases} E_0, & 0 \le z \le z_a \\ 0, & else \end{cases} \tag{26}$$

where z_a is the z-position of the anode. While this transition appears to be easily handled, i.e. the constant longitudinal field is E_0 before the anode and 0 after, Eq. (22) predicts an infinitely large electric field in the radial direction. This is a true physical results as it is a consequence of $\nabla \cdot \vec{E} = 0$. Therefore, it is not possible to accurately analyze this specific set up naïvely.

Instead we analyze a piece-wise linear field

$$E_z(0,z) = \begin{cases} E_0, & z \leq z_a - \Delta z \\ E_0 - \frac{E_0}{\Delta z}(z - (z_a - \Delta z)), & z_a - \Delta z < z \leq z_a \\ 0, & else \end{cases} \tag{27}$$

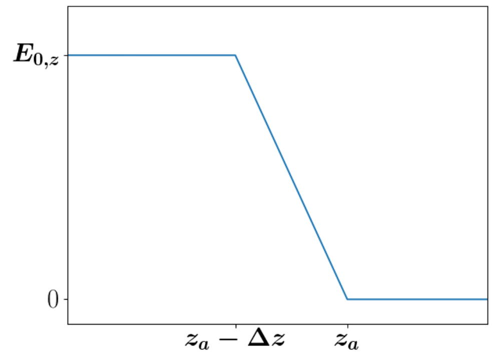

Figure 2: The idealized uniform magnetic field seen in Fig. 1 with fringe that is linear instead of a step function.

This field is visualized in Fig. 2. Notice that in the limit of $\Delta z \to 0$, Eq. (27) converges to Eq. (26). Further notice that applying Eq. (21) and Eq. (22) to Eq(27), we can approximate the off-axis field as

$$E_z(r,z) = E_z(0,z) \tag{28}$$

and

$$\begin{aligned} E_r(r,z) &= -\frac{\partial E_{z,eff}(0,z)}{\partial z}\frac{r}{2} \\ &= \begin{cases} \frac{E_0}{\Delta z}\frac{r}{2}, & z_a - \Delta z < z \leq z_a \\ 0, & else \end{cases} \end{aligned} \tag{29}$$

In the limit of $\Delta z \to 0$, the radial fringe field does in fact go towards infinity as expected.

Of course, what we are interested in is what happens to a particle passing through this fringe field—specifically, "What is the force this particle experiences?" Write the position of a charged particle with charge q as $\vec{x} = x\hat{x} + y\hat{y} + z\hat{z} = r\hat{r} + z\hat{z}$ where $r = \sqrt{x^2 + y^2}$ and $\hat{r} = \frac{x}{r}\hat{x} + \frac{y}{r}\hat{y}$. This is the standard cylindrical coordinate system. The Lorentz force on this particle is then

$$\begin{aligned} \frac{d\vec{p}}{dt} &= q\vec{E} \\ &\approx \frac{qE_0}{\Delta z}\frac{r}{2}\hat{r} + qE_0\hat{z} \end{aligned} \tag{30}$$

From this force, we can calculate the momentum transferred (impulse) to the particle in this field. We assume the peri-axial approximation where $dz \approx c\beta dt$. Thus

$$\begin{aligned} \Delta p_i &= \int_{t_0}^{\Delta t} F_i dt \\ &\approx \int_0^{\Delta z} \frac{F_i}{c\beta} dz \end{aligned} \tag{31}$$

We assume the thin lens formalism here; so while r is time dependent, we treat the radial position as being constant throughout the region. This is especially appropriate when we (eventually) take $\Delta z \to 0$. Thus

$$\Delta p_r = \frac{qE_0}{c\beta}\frac{r}{2} \tag{32}$$

$$\Delta p_z = \frac{qE_0}{c\beta}\Delta z \tag{33}$$

or equivalently $\Delta p_x = \frac{qE_0}{c\beta}\frac{x}{2}$ and $\Delta p_y = \frac{qE_0}{c\beta}\frac{y}{2}$. Notice that as $\Delta z \to 0$, $\Delta p_z \to 0$ but $\Delta p_r \to qE_0\frac{r}{2}$. So despite the "infinite force", there is a finite radial kick (and no change in the longitudinal treatment). This can be summarized in the phase coordinates with as in Table 2 where the initial coordinates are taken at the anode.

Real-space	Momentum
$x = x_0$	$p_x = p_{x,0} + \frac{qE_0}{c\beta}\frac{x_0}{2}$
$y = y_0$	$p_y = p_{y,0} + \frac{qE_0}{c\beta}\frac{y_0}{2}$
$z = z_0$	$p_z = p_{z,0}$

Table 2: Summary of the effect of the anode of particles exiting the constant electric field.

6 Solenoid

We separate the analysis of electrons inside a solenoid into two pieces. We will treat the magnetic field as uniform and solely in the longitudinal, z, direction. Calling this magnitude B_0, the effective length of the solenoid can be determined from the real field by

$$l = \frac{1}{B_0^2}\int_{-\infty}^{\infty} B^2(z)\ dz \tag{34}$$

If we define $z = 0$ to mean the left-most portion of the solenoid, then we may write

$$B_{z,eff}(0,z) = \begin{cases} B_0, & 0 \leq z \leq l \\ 0, & else \end{cases} \tag{35}$$

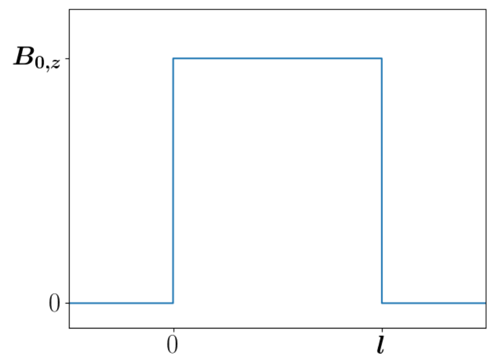

Figure 3: An idealized picture of the magnetic field inside a solenoid with effective length l. The magnetic field inside the solenoid is solely in the z direction.

where this field may be seen in Fig. 3. Exactly analogous to the step function at the anode, considering $\nabla \cdot \vec{B} = 0$ results in infinite radial magnetic fields at both of the ends of the solenoid; however, as this is the magnetic and not electric field, we need to re-analyze these thin fringes. We begin with careful analysis of these fringes, then we incorporate the results of this analysis into the description of the electron inside the uniform longitudinal magnetic field.

6.1 Thin fringes

Just as we did for the anode, we begin by re-defining the magnetic field as

$$B_{z,eff}(0,z) = \begin{cases} \frac{B_0}{\Delta z} z, & 0 \leq z < \Delta z \\ B_0, & \Delta z \leq z \leq l - \Delta z \\ B_0 - \frac{B_0}{\Delta z}(z - (l - \Delta z)), & l - \Delta z < z \leq l \\ 0, & else \end{cases} \tag{36}$$

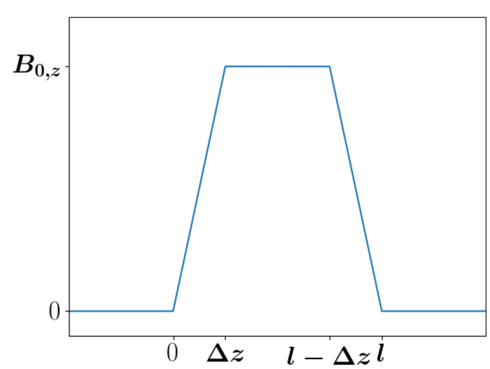

Figure 4: The idealized uniform magnetic field seen in Fig. 3 with fringes that are linear instead of step functions.

Again, notice that in the limit of $\Delta z \to 0$, Eq. (36) converges to Eq. (35). Further notice that applying Eq. (23) and Eq. (24) to Eq(36), we can approximate the off-axis field as

$$B_{z,eff}(r,z) = B_{z,eff}(0,z) \tag{37}$$

and

$$B_{r,eff}(r,z) = -\frac{\partial B_{z,eff}(0,z)}{\partial z}\frac{r}{2} \tag{38}$$

$$= \begin{cases} -\frac{B_0}{\Delta z}\frac{r}{2}, & 0 \leq z < \Delta z \\ \frac{B_0}{\Delta z}\frac{r}{2}, & l - \Delta z < z \leq l \\ 0, & else \end{cases} \tag{39}$$

Just as the anode's radial electric field became infinite in the limit of $\Delta z \to 0$, it is apparent that $\frac{B_0}{\Delta z} \to \infty$ as well. Likewise, we will see that this divergence leads to a finite kick to the particle, but this time the kick will be to angular momentum.

Again we ask "What is the force this particle experiences?" We confine our analysis to the fringe field between 0 and Δz. Following our anode analysis, we write the position of a charged particle with charge q as $\vec{x} = x\hat{x} + y\hat{y} + z\hat{z} = r\hat{r} + z\hat{z}$ where $r = \sqrt{x^2 + y^2}$ and $\hat{r} = \frac{x}{r}\hat{x} + \frac{y}{r}\hat{y}$. However, for the Lorentz force, we are interested in the velocity and are therefore concerned about the time dependence of this unit vector. We define $\hat{\theta} = \frac{-y}{r}\hat{x} + \frac{x}{r}\hat{y}$, $\cos(\theta) = \frac{x}{r}$, and $\sin(\theta) = \frac{y}{r}$ so that $r\dot{\hat{r}} = r\dot{\theta}\hat{\theta}$ where $\dot{} \equiv \frac{d}{dt}$. This is still the standard cylindrical coordinate system. Thus the velocity in the cylindrical coordinate system is

$$\begin{aligned}\vec{v} &= \frac{d\vec{x}}{dt} \\ &= \dot{r}\hat{r} + r\dot{\hat{r}} + \dot{z}\hat{z} \\ &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z} \\ &\approx c\beta\left(r'\hat{r} + r\theta'\hat{\theta} + \hat{z}\right)\end{aligned}$$

and the Lorentz force on this particle is then

$$\begin{aligned}\frac{d\vec{p}}{dt} &= q\vec{v} \times \vec{B}_{eff} \\ &\approx qc\beta\left(r\theta'\frac{B_0}{\Delta z}z\hat{r} + \left(-\frac{B_0}{\Delta z}\frac{r}{2} - r'\frac{B_0}{\Delta z}z\right)\hat{\theta} - r\theta'\frac{B_0}{\Delta z}\frac{r}{2}\hat{z}\right)\end{aligned} \tag{40}$$

Under the assumption of a thin lens, the transverse position is assumed approximately constant. One result from this statement is that $\dot{\hat{r}} \approx 0$ and $\dot{\hat{\theta}} \approx 0$. Thus $\frac{d\vec{p}}{dt} \approx \frac{dp_r}{dt}\hat{r} + \frac{dp_\theta}{dt}\hat{\theta} + \frac{dp_z}{dt}\hat{z}$. From this, we can integrate equation Eq. (40) to obtain obtain the impulse, $\Delta p_i(z)$ where it is understood that $\Delta x(z) = x(z) - z(0)$, for $i \in \{r, \theta, z\}$ and

$$\begin{aligned}\Delta p_i &= \int_{t_0}^{\Delta t} F_i dt \\ &\approx \int_0^{\Delta z} \frac{F_i}{c\beta} dz\end{aligned} \tag{41}$$

Two other consequences from the thin lens assumption are that we have $r(z) \approx r_0$ and $\theta(z) \approx \theta_0$ or equivalently $\Delta r(z) \approx 0$ and $\Delta\theta(z) \approx 0$.

We use these observations to evaluate the 3 impulse integrals described by Eq. (41). First we look at the radial impulse

$$\begin{aligned}\Delta p_r &= \int_0^{\Delta z} \frac{r\theta'\frac{qB_0}{\Delta z}z}{c\beta} dz \\ &= \frac{rqB_0}{c\beta\Delta z}\int_0^{\Delta z} \theta' z dz \\ &= \frac{rqB_0}{c\beta\Delta z}\left(\theta\Delta z - \int_0^{\Delta z} \theta dz\right) \\ &\approx 0\end{aligned} \tag{42}$$

where the integral was evaluated with separation of variables. Similarly, the longitudinal impulse can be shown to be

$$\Delta p_z \approx 0 \tag{43}$$

However, the angular momentum is not zero

$$\begin{aligned}
\Delta p_\theta &= \int_0^{\Delta z} \frac{-\frac{qB_0}{\Delta z}\frac{r}{2} - r'\frac{B_0}{\Delta z}z}{c\beta}dz \\
&= -\frac{qB_0 r}{2\Delta z}\int_0^{\Delta z} dz - \frac{B_0}{c\beta\Delta z}\int_0^{\Delta z} r' z dz \\
&\approx -\frac{qB_0 r}{2} + 0 \\
&= -\frac{qB_0 r}{2}
\end{aligned} \tag{44}$$

where the second integral is approximately zero under similar arguments to the other two impulses. Notice that Δp_θ is independent of Δz, so if we take $\Delta z \to 0$, we still have the same angular impulse despite the fact that the radial force diverges. This will be the model under which we continue to analyzing the solenoid.

So what does this mean, physically? A thin fringe fringe essentially imparts an angular momentum of the order of $-\frac{qB_0}{2}r$ to an electron. That is, the angular momentum impulse is proportional to r similar to the action of a curved lens on light. As relativistic angular momentum is $p_\theta = -\gamma m \omega_L r$ where ω_L is some angular frequency and the negative sign indicates that the kick is in the negative $\hat{\theta}$ direction. Solving for this frequency we find

$$\omega_L = \frac{qB_0}{2\gamma m} \tag{45}$$

This angular frequency is called the Larmor frequency, and analyzing the motion of the electrons in a non-inertial frame rotating at this frequency greatly simplifies the analysis within the solenoid. We will do this analysis in the next section.

On the other hand, this angular momentum can be thought of as "spinning up" a distribution of electrons before they enter the uniform magnetic field; however, care must be taken when we think of the "spinning up" activity. That is, once the electron enters the field, the additional angular momentum corresponds to a helix with radius $\frac{r}{2}$ that is trasverse at the expected cyclotron frequency $\omega_c = \frac{qB_0}{\gamma m} = 2\omega_L$. In other words, the Larmor frequency is a convenience for analysis corresponding to this fringe angular momentum kick, but it is not the frequency at which the electrons move. Specifically, if a group of electrons at different distances have no initial transverse momentum, each electron follows a helical trajectory about the point half way between it's initial position. A projection of such motion onto a transverse plane appears as a bunch of circles, all of which pass through the origin at $t = \frac{1}{\omega_L}$. Fig. (5) shows such a projection, and the mathematics of this behavior will be developed in the next section. However, notice that generally the electrons will have initial transverse momentum, so we will also see that the circles do not converge to a singularity at $t = \frac{1}{\omega_L}$ for such cases.

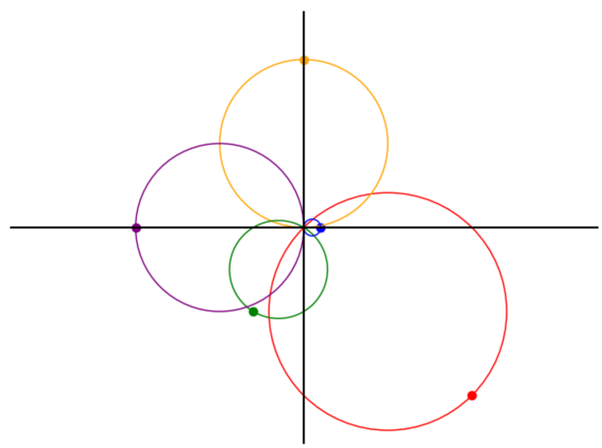

Figure 5: Projection of the motion of 5 (different colored) charged particles inside a solenoid onto the transverse plane. The dot represents the particles' initial position and the circle represents the projection of the helix. All particles are assumed to have no initial transverse momentum. Additionally, all particles move around their respective circle at the cyclotron frequency an therefore pass through the origin at the same time.

We summarize the effect of the fringe's angular kick on the phase coordinates of an electron with initial phase coordinate indicated by $_0$ in the table below. We switch to velocity space as the effect of the solenoid is more natural in velocity space; however, it is easy to convert to momentum space as the two are simply related by γm. Likewise, this table is related to trace space by a factor of βc.

Real-space	Velocity-space
$x = x_0$	$v_x = v_{x,0} + \omega_L y_0$
$y = y_0$	$v_y = v_{y,0} - \omega_L x_0$
$z = z_0$	$v_z = v_{z,0}$

Table 3: Summary of solenoid fringe angular momentum kick upon entering a "hard-edged" solenoid.

6.2 Solenoid interior

We are now ready to analyze the interior of the solenoid where the magnetic field will be treated as uniform. Treating β and γ as independent of z, i.e. $\gamma' = 0$ and $\gamma'' = 0$, as magnetic fields do not affect the energy of the particle, Eq. (25) reduces to

$$\zeta_L'' = -\left(\frac{\omega_L}{c\beta}\right)^2 \zeta_L \tag{46}$$

This has solution

$$\zeta_L(z) = A_1 \cos\left(\frac{\omega_L}{c\beta}(z - z_0)\right) + A_2 \sin\left(\frac{\omega_L}{c\beta}(z - z_0)\right) \tag{47}$$

$$\zeta_L'(z) = -A_1 \frac{\omega_L}{c\beta} \sin\left(\frac{\omega_L}{c\beta}(z - z_0)\right) + A_2 \frac{\omega_L}{c\beta} \cos\left(\frac{\omega_L}{c\beta}(z - z_0)\right) \tag{48}$$

where A_i are complex constants determined from the initial conditions and $z = 0$ marks the onset of the field.

To determine the constant, care must be taken to include the angular momentum kick from the fringe. Notice that

$$\begin{aligned} \zeta_L'(z) &= (x' + iy')e^{-i\theta_r} - i\frac{d\theta_r}{dz}(x + iy)e^{-i\theta_r} \\ &= (x' + iy')e^{-i\theta_r} + \frac{\omega_L}{\beta c}(-y + ix)e^{-i\theta_r} \end{aligned} \tag{49}$$

Notice that the frame dependent portion of Eq. (49), i.e. the second term exactly cancels out the angular momentum kick induced by the solenoid fringe; thus $\zeta_{L,0} = \zeta_L(0)$ can be seen to be simply $x_0' + iy_0'$. This is precisely the reason for the Larmor frame choice. As a result, the constants in Eqs. (47) and (48) are related to the initial coordinates by

$$A_1 = x_0 + iy_0 \tag{50}$$

$$A_2 = \frac{v_{0,x}}{\omega_L} + i\frac{v_{0,y}}{\omega_L} \tag{51}$$

Therefore, we can summarize the phase coordinate of a particle after entering a "hard edge" solenoid at z_0 and progress through the uniform magnetic field for time t by

Notice that $2\omega_L$ seen in the velocity expressions is the cyclotron frequency as we have discussed previously.

6.3 Exiting the solenoid

Upon leaving the solenoid, another angular momentum kick occurs. This process is exactly analogous to the process of entering the solenoid except for the sign of the radial field — it was negative but now is positive. Therefore, the momentum kick is in the positive $\hat{\theta}$ direction instead of the negative $\hat{\theta}$ direction, i.e. $p_\theta = \gamma m \omega_L r$. The corresponding velocity kick is $-\omega_L y\hat{x} + \omega_L x\hat{y}$, and this is what we will use to determine the velocity coordinate of the electron once it has passed through the thin fringe.

To summarize this effect, though, we are interested in the phase coordinates from the time in which the electron entered the solenoid. As the electron has traveled across l at $c\beta$, we use the phase coordinates in the table from the previous section at time $t_l = \frac{l}{c\beta}$ for our "initial conditions". This leads to

Real-space
$x = x_0 \cos^2(\omega_L t) + y_0 \cos(\omega_L t)\sin(\omega_L t) + \frac{v_{0,x}}{\omega_L}\cos(\omega_L t)\sin(\omega_L t) + \frac{v_{0,y}}{\omega_L}\sin^2(\omega_L t)$
$y = -x_0 \cos(\omega_L t)\sin(\omega_L t) + y_0 \cos^2(\omega_L t) - \frac{v_{0,x}}{\omega_L}\sin^2(\omega_L t) + \frac{v_{0,y}}{\omega_L}\cos(\omega_L t)\sin(\omega_L t)$
$z = z_0 + c\beta_{0,z} t$

Velocity-space
$v_x = -\omega_L x_0 \sin(2\omega_L t) + \omega_L y_0 \cos(2\omega_L t) + v_{0,x}\cos(2\omega_L t) + v_{0,y}\sin(2\omega_L t)$
$v_y = -\omega_L x_0 \cos(2\omega_L t) - \omega_L y_0 \sin(2\omega_L t) - v_{0,x}\sin(2\omega_L t) + v_{0,y}\cos(2\omega_L t)$
$v_z = v_{0,z}$

Table 4: Summary of "hard-edge" solenoid phase coordinate evolution inside the solenoid.

Real-space
$x = x_0 \cos^2(\omega_L t_l) + y_0 \cos(\omega_L t_l)\sin(\omega_L t_l) + \frac{v_{0,x}}{\omega_L}\cos(\omega_L t_l)\sin(\omega_L t_l) + \frac{v_{0,y}}{\omega_L}\sin^2(\omega_L t_l)$
$y = -x_0 \cos(\omega_L t_l)\sin(\omega_L t_l) + y_0 \cos^2(\omega_L t_l) - \frac{v_{0,x}}{\omega_L}\sin^2(\omega_L t_l) + \frac{v_{0,y}}{\omega_L}\cos(\omega_L t_l)\sin(\omega_L t_l)$
$z = z_0 + l$

Velocity-space
$v_x = -\omega_L x_0 \cos(\omega_L t_l)\sin(\omega_L t_l) - \omega_L y_0 \sin^2(\omega_L t_l) + v_{0,x}\cos^2(\omega_L t_l) + v_{0,y}\cos(\omega_L t_l)\sin(\omega_L t_l)$
$v_y = \omega_L x_0 \sin^2(\omega_L t_l) - \omega_L y_0 \cos(\omega_L t_l)\sin(\omega_L t_l) - v_{0,x}\cos(\omega_L t_l)\sin(\omega_L t_l) + v_{0,y}\cos^2(\omega_L t_l)$
$v_z = v_{0,z}$

Table 5: Electron phase coordinate just after exiting the solenoid.

For a short solenoid where $\omega_L t_l << 1$, it follows $\cos(\omega_L t_l) \approx 1$ and $\sin(\omega_L t_l) \approx \omega_L t_l$. Plugging these approximation into the above coordinates and keeping all terms linear in $\omega_L t_l$ we obtain

Real-space	Velocity-space
$x = x_0 + (\omega_L y_0 + v_{0,x})t_l$	$v_x = v_{0,x} + (-\omega_L x_0 + v_{0,y})\omega_L t_l$
$y = y_0 + (-\omega_L x_0 + v_{0,y})t_l$	$v_y = v_{0,y} + (-\omega_L y_0 - v_{0,x})\omega_L t_l$
$z = z_0 + l$	$v_z = v_{0,z}$

Table 6: Electron phase coordinate just after exiting a short solenoid.

In other words, in real space, there is a minor rotational displacement of the particle as well as a displacement of the particle in the same direction as the initial velocity — this latter term is identical to what would be expected in drift space. It should be apparent here why the term short is used as the change in radius is only zero when $l = 0$, but then there is no effect of the solenoid. As long as the solenoid is at least short, there are two effects in velocity space. First, there is a radial kick. This is why people discuss magnetic lenses as the electron microscope's converging lens. If the initial transverse velocity is zero, this is the only effect, and this spatially motivated radial kick is where most people end their analysis. However, in general, there is also a rotation in velocity space; in a sense, this rotation is in the same direction as the spatial rotation. We will have a problem specifically working on the consequences of this on emittance.

7 RF cavity

Operated in compression mode, the RF cavity may be thought of as a longitudinal lens. However, as the RF cavity has time dependent fields, the standard peri-axial analysis, built upon assumptions of static fields, is not appropriate. In fact, there does not seem to be concerned with the full 3D motions of particles in fields consistent with Maxwell's equations; instead, the traditional approach to understanding the RF cavity is through analysis of electron energies from fields that may or may not be consistent with Maxwell's equations. This approach results in a formula called the Panofsky equation that details the energy kick a particle experiences when passing through a RF cavity. In the next few sections, we detail the tradition derivation of the Panofsky equation both in general as well as under the assumption of a sinusoidally time-varying uniform electric field that incidentally violates Maxwell's equations. We show how this equation leads to a longitudinal momentum kick. Assuming that the evolution of the particle's positions follow the drift evolution, we present a transfer map that approximates some the dynamics of the RF cavity that lead to longitudinal bunching.

7.1 Panofsky equation

7.1.1 General

We examine a particle of charge q moving through a RF cavity of length L along the z-axis. Represent $E_z(r=0,z) = E(0,z)$ as the field along the axis within the RF cavity, and add a time varying component so that $E_z(0,z,t) = E(0,z)\cos(\omega t + \phi)$ where ϕ is some phase. Since the particle is moving, there is some relation between z and t, i.e. we can write time as a function of position, $t(z)$, so we can think of the field as $E_z(0,z,t) = E(0,z)\cos(\omega t(z) + \phi)$. Choosing your zero position, $z = 0$, to correspond to the location in the center of the cavity and your zero time when your electron crosses the center of the cavity, i.e. $t(0) = 0$), we see that $E_z(0,0,0) = E(0,z)\cos(\phi)$; that is the phase, ϕ, provides information on the field strength when the particle is at the center of the cavity. Specifically, if $\phi = \frac{\pi}{2}$, the field strength when the particle crosses the center of cavity is exactly zero. Moreover, notice, defining the time and position in this way results in the relationship between time and velocity of

$$t(z) = \int_0^z \frac{1}{v(z)} dz \tag{52}$$

for $v_z(z) > 0$. This time is negative when $z < 0$ consisted with our choice of the 0 time corresponding to when the particle is at the center of the cavity.

Now let ΔW represent the kinetic energy gain of an electron through an RF gap. Simply writing out the integrals of the kinetic energy change we get

$$\begin{aligned}
\Delta W &= \int_{\text{gap}} q\vec{E}\cdot d\vec{l} \\
&= q\int_{-\frac{L}{2}}^{\frac{L}{2}} E(0,z)\cos(\omega t(z) + \phi) dz \\
&= q\int_{-\frac{L}{2}}^{\frac{L}{2}} E(0,z)\left(\cos(\omega t(z))\cos\phi - \sin(\omega t(z))\sin\phi\right) dz \\
&= q\left(\int_{-\frac{L}{2}}^{\frac{L}{2}} E(0,z)\cos(\omega t(z)) dz - \tan\phi \int_{-\frac{L}{2}}^{\frac{L}{2}} E(0,z)\sin(\omega t(z)) dz\right)\cos\phi
\end{aligned} \tag{53}$$

Notice that the potential applied to the RF cavity, V_0 is

$$V_0 = \int_{-\frac{L}{2}}^{\frac{L}{2}} E(0,z) dz \tag{54}$$

and from this, an average electric field, $E_0 L = \frac{V_0}{L}$ may be defined. Notice, however, that the integrals in (53) are similar to (54) and may be interpreted as the time-modulated potential the charged particle experiences. We normalize this time modulated potential and will call the resulting dimensionless factor the Transit-time factor, T:

$$T = \frac{\int_{-\frac{L}{2}}^{\frac{L}{2}} E(0,z)\cos(\omega t(z)) dz}{\int_{-\frac{L}{2}}^{\frac{L}{2}} E(0,z) dz} - \frac{\int_{-\frac{L}{2}}^{\frac{L}{2}} E(0,z)\sin(\omega t(z)) dz}{\int_{-\frac{L}{2}}^{\frac{L}{2}} E(0,z) dz}\tan\phi \tag{55}$$

Thus, the kinetic energy in (53) may be written in terms of these parameters

$$\Delta W = qE_0 LT \cos\phi \tag{56}$$

which is known as the Panofsky equation. Notice that this equation implies that the phase of the field when the charged particle crosses the middle of the cavity determines the effect of the cavity. For instance, if ϕ is chosen to be $\frac{\pi}{2}$, the charged particle does not have an kinetic energy change across the entire RF cavity.

The above analysis was for a single charged particle; of course with a bunch of charged particles, ϕ will be different for each particle. Specifically, denote $z_{mid}(t)$ to represent the position of the middle of the electron bunch where the 0 time is chosen when this mid-position crosses the center of the RF cavity. Further denote $z_i(t)$ as the position of the i^{th} particle; then $\Delta z_i = z_i(t) - z_{mid}(t)$ is the i^{th} particle's displacement from the bunch's mid-point. Assuming that all the particlea have essentially the same speed, $c\beta$, the time at which the i^{th} particle crosses the RF cavity mid-point is then $-\frac{\Delta z_i}{c\beta}$. The negative indicates that particles in front of the middle of the bunch arrive first, as would be expected. As the phase for the mid-point is $\frac{\pi}{2}$, it follows that the phase for the i^{th} particle can be written as $\phi_i = \frac{\pi}{2} - \frac{\Delta z_i \omega}{c\beta}$.

Putting this into the Panofsky equation, we can see that this results in the longitudinal analog to the magnetic lens. Specifically, the Panofsky equation for the i^{th} particle can be written as $\Delta W_i = qE_0 LTcos(\phi_i) = qE_0 LTcos\left(\frac{\pi}{2} - \frac{\Delta z_i \omega}{c\beta}\right)$. Assuming $\frac{\Delta z_i \omega}{c\beta} << 1$, we get $\Delta W_i \approx -qE_0 LT \frac{\omega}{c\beta} \Delta z_i$. This can be related to a momentum kick by noticing first that $W = E - mc^2 \implies \Delta W = \Delta E$. Also in the periaxial limit $\Delta E \approx \frac{p_z}{E} \Delta p_z = c\beta \Delta p_z$. So $\Delta p_{z,i} \approx -qE_0 LT \frac{\omega}{c^2\beta^2} \Delta z_i$. As this is a longitudinal kick toward the bunch center that is linear in the particle's displacement from the center, we have analogous focussing behavior to that of a magnetic lens.

However, much of the physics is hidden in that transit-time factor, T. To garner further insights, the transit-time factor needs to be further examined; however, it dependents of the details of the electric field in the interior of the RF cavity and is therefore context dependent. In the next section we will examine the special case of a uniform electric field, which, if examined carefully, does not conform to Maxwell's equations. However, such an examination will reveal more about the Panofsky equation.

7.1.2 Transit-time in a uniform sinusoidally varying field

Revisiting the analysis of a particle in the RF cavity, we approximate that the particle has constant velocity to first order. That is (52) becomes

$$\begin{aligned} t(z) &= \int_0^z \frac{1}{c\beta} dz \\ &= \frac{z}{c\beta} \end{aligned} \tag{57}$$

Thus we can rewrite the trigonometric, time-dependent function by noting that $\omega t(z) = \frac{\omega z}{c\beta} = \frac{2\pi c z}{c\beta\lambda_{RF}} = \frac{2\pi z}{\beta\lambda_{RF}}$, where λ_{RF} is the RF wavelength with $\omega\lambda_{RF} = 2\pi c$. Putting this observation into (55), we get

$$\begin{aligned} T &= \frac{\int_{-\frac{L}{2}}^{\frac{L}{2}} E(0,z)\cos(\frac{2\pi z}{\beta\lambda_{RF}})dz}{\int_{-\frac{L}{2}}^{\frac{L}{2}} E(0,z)dz} - \frac{\int_{-\frac{L}{2}}^{\frac{L}{2}} E(0,z)\sin(\frac{2\pi z}{\beta\lambda_{RF}})dz}{\int_{-\frac{L}{2}}^{\frac{L}{2}} E(0,z)dz}\tan\phi \\ &= \frac{\int_{-\frac{L}{2}}^{\frac{L}{2}} E(0,z)\cos(\frac{2\pi z}{\beta\lambda_{RF}})dz}{\int_{-\frac{L}{2}}^{\frac{L}{2}} E(0,z)dz} \end{aligned} \tag{58}$$

where the integral in the numerator of the second term is zero due to symmetry.

In the special case that the electric field is uniform, i.e. $E(0,z) = E_0$, the transit time under this approximation may be evaluated exactly. Namely,

$$T = \frac{\beta\lambda_{RF}}{\pi L}\sin\left(\frac{\pi L}{\beta\lambda_{RF}}\right) \tag{59}$$

and (56) becomes

$$\Delta W = \frac{qE_0 \beta \lambda_{RF}}{\pi} \sin\left(\frac{\pi L}{\beta \lambda_{RF}}\right) \cos\phi \tag{60}$$