1 Getting to the know the UEM injector

1.a

Let $\sigma = \frac{Ne}{\pi R^2}$ represent the charge density. Since $N = 10^6$ electrons, and $R = 10^{-4}\ m$, the electric field due to the bunch "behind" the bunch is

$$\begin{aligned} E_c &= \frac{\sigma}{2\epsilon_0} \\ &= \frac{10^6 \times 1.602 \times 10^{-19}}{\pi(10^{-4})^2 \times 2 \times 8.85 \times 10^{-12}} \frac{V}{m} \\ &\approx 288 \frac{kV}{m} \end{aligned}$$

This is the critical field, but we are operating at four times this field, so the answer is $E_a = 1.15 \frac{MV}{m}$.

1.b

Let $\mathcal{E}$ represent the energy. An electron in E_a for the distance d has energy $\mathcal{E} = eE_a d \approx 57.6 keV$. The relativistic energy is given by $\mathcal{E} = (\gamma - 1)mc^2$ where m is the mass of the electron, c is the speed of light, and γ is the Lorentz factor. Solving for γ, we get

$$\begin{aligned} \gamma &= 1 + \frac{\mathcal{E}}{mc^2} \\ &\approx 1.11 \end{aligned}$$

Solving for $\beta = \sqrt{1 - \frac{1}{\gamma^2}}$, we get $\beta \approx 0.44$ or roughly 44% the speed of light. This gives a longitudinal velocity of $v_z \approx 1.3 \times 10^8 \frac{m}{s}$.

The arrival time is simply $\frac{z - z_0}{v_z}$ and is filled in in the table below:

Element	Position (z in mm)	Distance ($z - z_0$ in mm)	Arrival time (ns)
Lens 1	20	10	0.076
Lens 2	350	340	2.6
RF cavity	600	590	4.5
Lens 3	700	690	5.2
Object	800	790	6.0

Table 1: Arrival time calculations

1.c

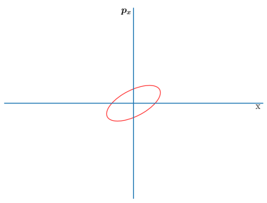
1.d

No. This is quite relativistic although not highly-relativistic. To quantitate this is to look at the ratio of $\frac{\frac{d\beta}{dt}}{\frac{d\gamma}{dt}} = \frac{1}{\gamma^3 \beta}$, which is large ($\beta$ dominated) in the non-relativistic regime and small (γ dominated) in the highly relativsitic regime. Here, this ratio is about 1.7, which is neither small or large indicating that both γ and β are changing almost equally.

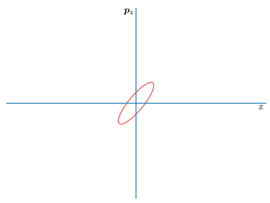
2 The 3-step model

2.a

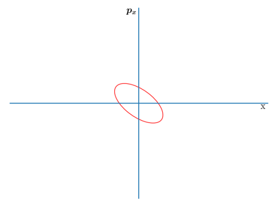
Assume that the electron initially has energy E_i, all in kinetic energy. After photo-collision, the electron now has energy $E_i + \hbar\omega$. Assume the electron has velocity v and direction defined by θ and ϕ using spherical



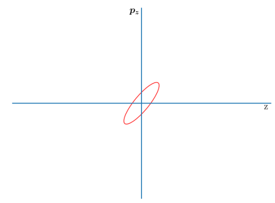
(a) before lens1:x-p_x



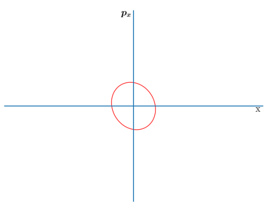
(b) before lens1:z-p_z



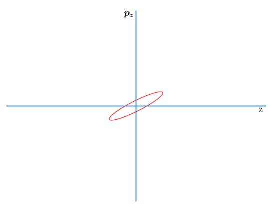
(c) after lens1:x-p_x



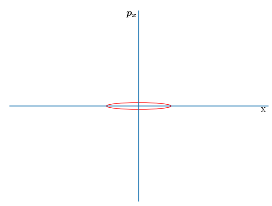
(d) after lens1:z-p_z



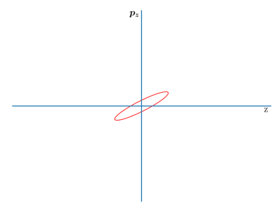
(e) before lens2:x-p_x



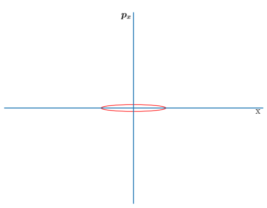
(f) before lens2:z-p_z



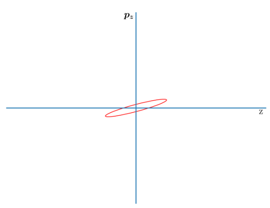
(g) after lens2:x-p_x



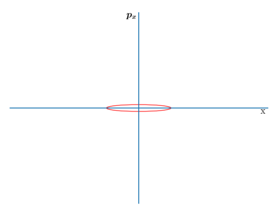
(h) after lens2:z-p_z



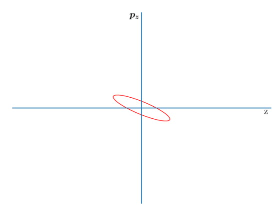
(i) before rf:x-p_x



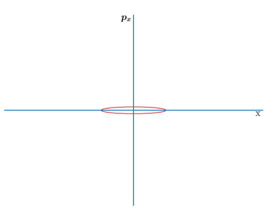
(j) before rf:z-p_z



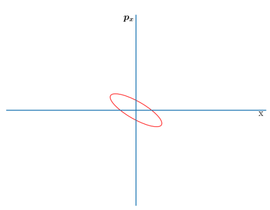
(k) after rf:x-p_x



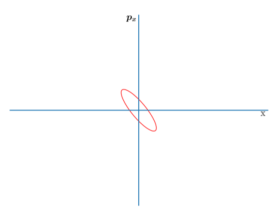
(l) after rf:z-p_z



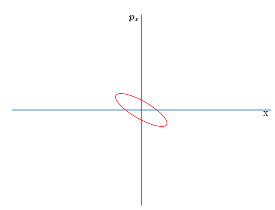
(m) before lens3:x-p_x



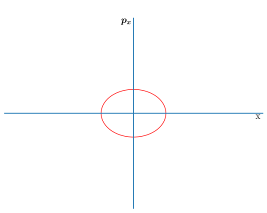
(n) before lens3:z-p_z



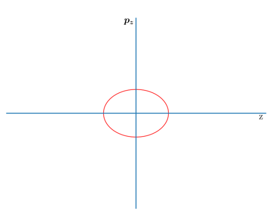
(o) after lens3:x-p_x



(p) after lens3:z-p_z



(q) focus:x-p_x



(r) focus:z-p_z

Figure 1: Explanation for desired phase space. We want there to be no chirp at the object plane so that the widths of the bunch are minimized.

coordinates, where we treat the velocity non-relativistically so

$$\frac{1}{2}mv^2 = E_i + \hbar\omega$$

Solving for v we get $v = \sqrt{2(E_i + \hbar\omega)m}$. The velocity of the electron in Cartesian coordinates inside the metal is given by

$$\begin{aligned}
v_{x,in} &= v\cos\phi\sin\theta \\
v_{y,in} &= v\sin\phi\sin\theta \\
v_{z,in} &= v\cos\theta
\end{aligned}$$

However, the potential outside the metal is $E_F + W$, and the surface is perpendicular to the longitudinal, z-direction. Thus crossing the surface will result in a resistance force in the longitudinal direction that will effectively drain energy, specifically $E_F + W$, from the kinetic energy in the z-direction. Since the kinetic energy in the z-direction is $KE_{z,in} = \frac{1}{2}mv_{z,in}^2 = \frac{1}{2}mv^2\cos^2\theta$, it follows $KE_{z,out} = KE_{z,in} - E_F - W = \frac{1}{2}mv^2\cos^2\theta - E_F - W$. Solving for $v_{z,out} = \sqrt{v^2\cos^2\theta - \frac{2(E_F+W)}{m}}$. That is

$$\begin{aligned}
v_{x,out} &= v\cos\phi\sin\theta \\
v_{y,out} &= v\sin\phi\sin\theta \\
v_{z,out} &= \sqrt{v^2\cos^2\theta - \frac{2(E_F+W)}{m}}
\end{aligned}$$

Notice, there must be sufficient energy in the initial z-direction to overcome this difference. If $\cos^2\theta < \frac{2(E_F+W)}{v^2 m} = \frac{E_F+W}{E_i+\hbar\omega}$, then the solution is imaginary indicating that the kinetic energy was insufficient to overcome the barrier. Thus $\cos^2\theta \geq \frac{E_F+W}{E_i+\hbar\omega}$. Since $\cos\theta$ decreases as θ increases, it follows $\cos\theta_{max} = \sqrt{\frac{E_F+W}{E_i+\hbar\omega}}$.

2.b

There are a number of assumptions made in this derivation. An incomplete list includes that there are no scattering events after the photon is absorbed, the electron is emitted isotropically within the small angle of emission, the occupancies of the continuous band of initial states is uniform, and only a single photon is absorbed. However, despite the severity of many of these assumptions, the model is regarded fairly highly.

2.c

Assuming full isotropic emission (instead of local isotropy which is all that is needed in the derivation), we can find a bound for the Quantum Efficiency (QE). Assuming uniform distribution in the occupancy of the initial states, we see that the denominator of the QE is bound by $denom > 4\pi a^2(E_F + W - \hbar\omega)$ where a is an arbitrary "radius". Introduce $\Delta E = \hbar\omega + E_i - E_F - W$ is the (small) energy the electron has once it comes off the cathode. In this formulation,

$$\begin{aligned}
\cos\theta_{max} &= \sqrt{\frac{E_F+W}{E_F+W+\Delta E}} \\
&= \sqrt{\frac{1}{1+\frac{\Delta E}{E_F+W}}} \\
&\approx 1 - \frac{1}{2}\frac{\Delta E}{E_F+W}
\end{aligned}$$

Thus, $\theta_{max} \approx \sqrt{\frac{\Delta E}{E_F+W}}$. On the other hand, the numerator is

$$\begin{aligned}
num &= \int_0^{E_F+W-\hbar\omega} \int_0^{\theta_{max}} \int_0^{2\pi} a^2 \cos\theta d\phi d\theta d\Delta E \\
&= 2\pi a^2 \int_0^{E_F+W-\hbar\omega} \sin\theta \Big|_0^{\theta_{max}} d\Delta E \\
&\approx 2\pi a^2 \int_0^{E_F+W-\hbar\omega} \theta_{max} d\Delta E \\
&\approx 2\pi a^2 \int_0^{E_F+W-\hbar\omega} \sqrt{\frac{\Delta E}{E_F+W}} d\Delta E \\
&= \frac{4}{3}\pi a^2 \frac{(E_F+W-\hbar\omega)^{\frac{3}{2}}}{\sqrt{E_F+W}}
\end{aligned}$$

So

$$QE < \frac{1}{3}\sqrt{\frac{E_F+W-\hbar\omega}{E_F+W}}$$

Using the values in Table 1, $QE < 0.24$. I think this number is high as that denominator is a really generous estimate.

3 The 1D fluid model

3.a

The evolution of the electron's longitudinal position is

$$z = z_0 + v_0 t + \frac{1}{2}at^2$$

In the non-relativistic limit, $a = q(E_s + E_a) = qE_s + qE_a = a_s + a_a$ where E_s and a_s are the electric field and resulting acceleration due to the interaction between the electrons and E_a and a_a is the applied Electric field and the resulting acceleration. As a_a is a constant, $a' = a'_s$. Thus

$$z' = 1 + v'_0 t + \frac{1}{2}a'_s t^2$$

whether or not you have an extraction field.

3.b

Starting from the end of the last problemm... As $a_s = \frac{q}{2\epsilon_0 m}\int_{-z_0}^{z_0} \rho(\tilde{z}_0)d\tilde{z}_0$, it follows $a'_s = \frac{q}{2\epsilon_0 m}2\rho(z_0) = \frac{q}{\epsilon_0 m}\rho_0$ using the fundamental theorem of calculus and the symmetry of the distribution, i.e. $\rho(-z_0) = \rho(z_0)$. Thus

$$z' = 1 + v'_0 t + \frac{q}{2\epsilon_0 m}\rho_0 t^2$$

Due to the conservation of charge,

$$\begin{aligned}
\rho &= \frac{\rho_0}{z'} \\
&= \frac{\rho_0}{1 + v'_0 t + \frac{q}{2\epsilon_0 m}\rho_0 t^2}
\end{aligned}$$

3.c

Starting from cold initial conditions, i.e. $v_0 = 0$, the velocity is

$$v = at$$

and

$$\begin{aligned} z &= z_0 + \frac{1}{2}at^2 \\ &= z_0 + \frac{1}{2}vt \end{aligned}$$

Taking the derivative of this with respect to z, we get

$$\begin{aligned} 1 &= \frac{1}{z'} + \frac{1}{2}\frac{dv}{dz}t \\ &= \frac{1}{1 + \frac{q}{2\epsilon_0 m}\rho_0 t^2} + \frac{1}{2}\frac{dv}{dz}t \end{aligned}$$

Solving for $\frac{dv}{dz}$, the chirp, we get

$$\begin{aligned} \frac{dv}{dz} &= \frac{2}{t}\left(1 - \frac{1}{1 + \frac{q}{2\epsilon_0 m}\rho_0 t^2}\right) \\ &= \frac{2}{t}\left(\frac{1 + \frac{q}{2\epsilon_0 m}\rho_0 t^2 - 1}{1 + \frac{q}{2\epsilon_0 m}\rho_0 t^2}\right) \\ &= \frac{\frac{q}{\epsilon_0 m}\rho_0 t}{1 + \frac{q}{2\epsilon_0 m}\rho_0 t^2} \\ &= \frac{t}{\frac{\epsilon_0 m}{\rho_0 q} + \frac{1}{2}t^2} \end{aligned}$$

Finally, $v = v_{mid} + \frac{dv}{dz}\Delta z = v_{mid} + \frac{t}{\frac{\epsilon_0 m}{\rho_0 q} + \frac{1}{2}t^2}\Delta z$. This is linear only if ρ_0 is uniform; otherwise, the slope is not constant. We can only use the linear chirp assumption if the distribution is linear or at least near-linear.

3.d

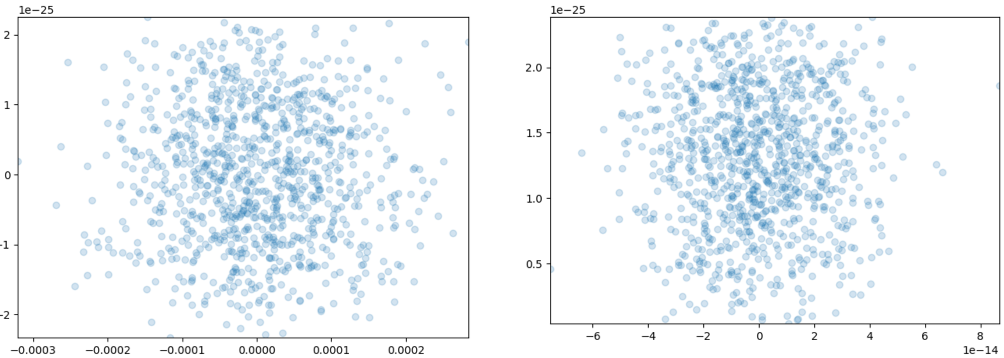
The true emittance is zero.

The rms emittance is a different story, though. Note that in this cold beam formalism with no external field, the expectation is

$$\begin{aligned} < b(t) > &= \int b(t)\rho\ dz \\ &= \int b(t)\frac{\rho_0}{z'}\ z' dz_0 \\ &= \int b(t)\rho_0\ dz_0 \end{aligned}$$

where $dz = z' dz_0$. So $< z >= 0$ and

$$\begin{aligned} < z^2 > &=< (z_0 + \frac{1}{2}a_s t^2)^2 > \\ &=< z_0^2 > + \frac{1}{4}t^4 < a_s^2 > + t^2 < z_0 a_s > \end{aligned}$$



(a) initial x-p_x

(b) intital t-p_z

Figure 2: The phase space coming out of the photoemission code. The macroparticle size was assumed to be 1.

and $< p_z >= 0$ and $< p_z^2 >= m^2 < a_s^2 > t^2$ and $< zp_z >= mt < z_0 a_s > + \frac{m}{2} t^3 < a_s^2 >$ Thus

$$
\begin{aligned}
\frac{1}{m^2c^2}\epsilon_{z,p_z}^2 &= \frac{1}{m^2c^2}((< z^2 > - < z >^2)(< p_z^2 > - < p_z >^2) - (< zp_z > - < z >< p_z >)^2) \\
&= \frac{1}{m^2c^2}((< z_0^2 > + \frac{1}{4}t^4 < a_s^2 > + t^2 < z_0 a_s >)(m^2 < a_s^2 > t^2) - (mt < z_0 a_s > + \frac{m}{2}t^3 < a_s^2 >)^2) \\
&= \frac{1}{c^2}(t^2 < z_0^2 >< a_s^2 > + \frac{1}{4}t^6 < a_s^2 >^2 + t^4 < z_0 a_s >< a_s^2 > \\
&\quad - t^2 < z_0 a_s >^2 - \frac{1}{4}t^6 < a_s^2 >^2 - t^4 < z_0 a_s >< a_s^2 >) \\
&= \frac{t^2}{c^2}(< z_0^2 >< a_s^2 > - < z_0 a_s >^2)
\end{aligned}
$$

or

$$
\frac{1}{mc}\epsilon_{z,p_z} = \frac{t}{c}\sqrt{< z_0^2 >< a_s^2 > - < z_0 a_s >^2}
$$

This depends on your initial distribution.

4 Figures of merit

Chung-Yu, can you answer this?

5 Simulations

5.a

Skip.

5.b

6 Coherence length and brightness

6.a Transverse phase space

The dose, D, is defined as the number of electrons per area, or equivalently, $D = \frac{N}{\pi \sigma_x^2}$. Thus $\sqrt{D} = \frac{\sqrt{N}}{\sqrt{\pi} r}$. Likewise, the coherent length, L, is $L = \frac{\lambda}{2\alpha} = \frac{\frac{h}{p_0}}{2\left(\frac{\sigma_{v_x}}{\sigma_{v_0}}\right)} = \frac{h}{2\sigma_{v_{x,int}}\gamma m} = \frac{h\gamma m}{2\sigma_{p_{x,int}}}$. Finally, $B_{4D} = \frac{N}{\epsilon_{x,px}^2} = \frac{N}{\sigma_x^2 \sigma_{p_{x,int}}^2}$, so

$$
\begin{aligned}
\sqrt{B_{4D}} &= \frac{\sqrt{N}}{\sigma_x \sigma_{p_{x,int}}} \\
&= \frac{\sqrt{N}}{\sigma_x} \frac{1}{\sigma_{p_{x,int}}} \\
&= \frac{2}{\gamma m h} \sqrt{D} L
\end{aligned}
$$

6.b

Notice $E^2 = p^2c^2 + m^2c^4$, so $2E\Delta E = 2p\Delta p c^2$ and $\Delta E = v\Delta p$. On the other hand, $\Delta t = \frac{\Delta z}{v}$. So $\Delta E \Delta t \approx \Delta p \Delta z$ which is the longitudinal rms emittance when no chirp is present.